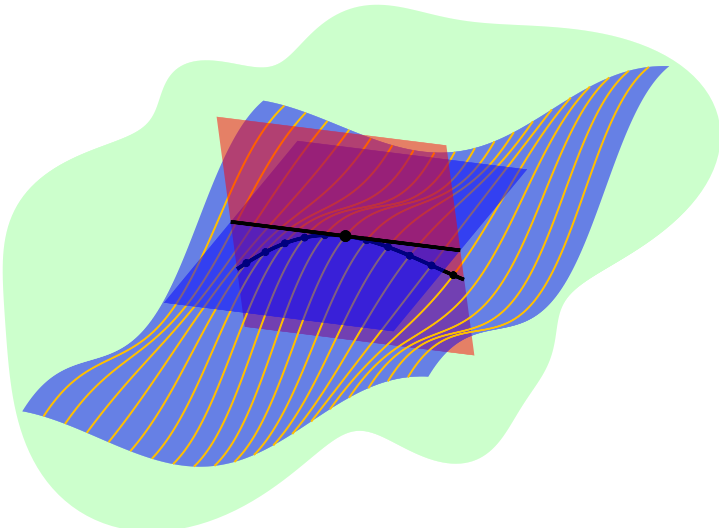
On Monopoles with Arbitrary Symmetry Breaking and their Moduli Spaces

Jaime Mendizabal



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I, Jaime Mendizabal Roche, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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1. **For a research manuscript that has already been published:**

   (a) **What is the title of the manuscript?** A hyper-Kähler metric on the moduli spaces of monopoles with arbitrary symmetry breaking

   (b) **Please include a link to or doi for the work:** `https://doi.org/10.1007/s10455-024-09954-z`

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Candidate: Jaime Mendizabal Roche
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Abstract

The subject of this thesis is monopoles – solutions to the Bogomolny equations – on the Euclidean 3-space $\mathbb{R}^3$, with arbitrary gauge group, mass and charge, and hence symmetry breaking. We start by providing an overview of monopoles themselves and discussing their asymptotics in relation to mass and charge. These are used to define a framing, using an asymptotic model for each such choice. With this we set up an analytic framework appropriate for the construction of moduli spaces of framed monopoles. The construction is then carried out as a quotient of infinite-dimensional spaces, which requires a careful analysis of the differential operators involved and their Fredholmness and other mapping properties. More specifically, a combination of the b and scattering calculuses is used to define appropriate Sobolev spaces and analyse the partial differential equations. The resulting framed moduli spaces are constructed as smooth manifolds and we see that they also carry hyper-Kähler metrics, obtained through a hyper-Kähler quotient construction. Lastly we consider how our results fit into some of the pre-existing knowledge in this area. In particular, we discuss the relationship between the mass and the charge and the symmetry breaking, and expand upon these concepts in the cases of special unitary and orthogonal groups.

Impact Statement

In this thesis, a construction of the framed moduli spaces of monopoles with arbitrary symmetry breaking is carried out. Aside from the resulting smooth hyper-Kähler structure, it provides an analytical framework which can be used to continue studying these spaces, and hence it can be of interest to researchers who are investigating these moduli spaces and their properties. The approach differs from many other lines of research in the area in that it is done from the perspective of monopoles themselves, without relying on the correspondence with other mathematical objects like Nahm's equations, rational maps or spectral data. Therefore, it can provide a complementary outlook to better understand the subject.

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Chapter 1

Background and overview

Monopoles are defined on an oriented Riemannian 3-manifold as pairs (A, Φ), where A is a connection on a principal G-bundle and Φ, the *Higgs field*, is a section of its adjoint bundle, which satisfy the Bogomolny equations

$$\star F_A = \mathrm{d}_A \Phi \,, \tag{1.0.1}$$

as well as a finite energy condition.

There are several perspectives which motivate their definition and study. Perhaps the first motivation comes from physics, whose language pervades the topic of monopoles, as well as the more general area of gauge theory. However, our interest in this thesis lies in the mathematical aspects of monopoles and their moduli spaces, so we will centre our attention on this.

From a geometric perspective, monopoles make a compelling subject of study. They are related to other objects in gauge theory, like anti-self-dual Yang–Mills connections, and similarly form interesting moduli spaces, which have interesting properties like a hyper-Kähler metric, and they furthermore have correspondences with a variety of other mathematical objects: Nahm's equations, rational maps and spectral data.

Monopoles on $\mathbb{R}^3$ for the simplest non-Abelian gauge group, $G = \mathrm{SU}(2)$, have been studied extensively. In this case, we have a significant knowledge of their asymptotic behaviour [JT80; Gro84]. Firstly, it is known that the size of the Higgs field must tend to a constant called the *mass*, which, if not 0 – which doesn't produce non-trivial monopoles–, can be assumed through rescaling to be 1. This means that, near infinity, the Higgs field defines a map between the 2-sphere of directions in $\mathbb{R}^3$ and the unit 2-sphere in the Lie algebra $\mathfrak{su}(2)$. The degree of this map is called the *charge*, and it rules the asymptotic behaviour of the monopole. More specifically, it determines the behaviour of the monopole to order $\frac{1}{r}$ near infinity, with more precise behaviour of the lower order terms following from the differential equations.

Another established result in this case is that SU(2)-monopoles form moduli spaces [AH88]. Since the charge provides a topological quantity, we restrict ourselves to monopoles of a fixed charge. It is furthermore convenient to consider *framed* moduli spaces, which means that a specific asymptotic behaviour is fixed near infinity and only gauge transformations which are the identity at infinity are considered. This yields a complete hyper-Kähler manifold for any non-negative charge, whose dimension is four times the value of this charge. This construction can yield interesting spaces, like the Atiyah–Hitchin manifold. Additionally, its metric has been interpreted as providing the evolution of low-energy monopoles [Man82; Stu94].

These moduli spaces and their metrics have been studied, and their asymptotic regions have been interpreted as parametrising monopoles of a the given charge separating into several monopoles of lower charge [Wei79; Bie95; Bie98a; FKS18].

For other gauge groups we don't have such a detailed picture. Firstly, it is important to note that the asymptotic conditions are no longer so straightforward: The mass – the limit of the Higgs field at infinity – is no longer determined only by its size. Rather, it is given by an adjoint orbit inside the Lie algebra of the gauge group. Therefore, it plays a more prominent role, and, in particular, its symmetries determine a property of the monopoles called *symmetry breaking*.

The case in which the mass is a regular element, like for SU(2), is known as *maximal symmetry breaking*, and exhibits a number of properties which make its study simpler. For example, the behaviour of order $\frac{1}{r}$ is once again ruled by a topological (or *magnetic*) charge: the homotopy class of the map given by the Higgs field at infinity from the 2-sphere of directions in $\mathbb{R}^3$ to the adjoint orbit of the mass, which is given by a number of integers.

If, however, the symmetry breaking is non-maximal, we have some additional features. An important observation is that in this case the topological charge does no longer truly determine the behaviour of order $\frac{1}{r}$, so we must furthermore consider *holomorphic charges*. Then, if we hope to build hyper-Kähler moduli spaces, we must fix all of the charge components.

For an arbitrary gauge group, there has been work to interpret monopoles as superpositions of SU(2) monopoles [Wei80; Wei82; LWY96], which has been used to study their moduli spaces and their metrics [GRG97; Bie98b].

A remarkable feature of monopoles, as pointed out above, is the multitude of correspondences with other mathematical objects, which have been used to study their properties employing the advantages of each correspondence.

A fruitful correspondence, known as the *Nahm transform*, exists between monopoles and solutions to Nahm's equations. If the Bogomolny equations can be viewed as the dimensional reduction of the anti-self-dual Yang–Mills equation on $\mathbb{R}^4$ to 3 dimensions, Nahm's equations are the dimensional reduction to 1 dimension. This can be viewed as an instance of a family of such correspondences between different gauge theory problems related to these instantons [Jar04], which includes the ADHM construction relating instantons with algebraic data. In the case of the Nahm transform for monopoles, the other side of the correspondence is not purely algebraic, but it *is* a set of ordinary differential equations.

Another equivalence links monopoles with rational maps from $\mathbb{CP}^1$ into another complex space, which depends on the gauge group and the symmetries of the mass and charge. An interesting feature of this approach is that one can build moduli spaces in which only the topological component of the charge is fixed.

Lastly, we note that one can also establish a correspondence with *spectral data*, which consists of algebraic-geometric data on the minitwistor space $T\mathbb{CP}^1$.

These correspondences have been investigated in many different cases, often relating them to one another, and have facilitated the study of many aspects of monopoles and their moduli spaces. The correspondences for SU(2) have been studied in great detail [Hit82; Hit83; Don84; Hur85; Nak93], and they have also been extended to monopoles with arbitrary gauge groups [Mur83; Mur84; Hur89; HM89; Mur89; HM90; Jar98a; Jar98b; Jar00; CN22]. They have then been used to study the parameters of general monopoles [Bow85], specific cases of SU(3)-monopoles [Dan92; DL93; Dan93; Dan94; DL97; Irw97], and to produce some examples with non-maximal symmetry breaking [CDLNY22].

Although the equivalences outlined above provide powerful tools in the study of monopoles, it is also possible to take a more direct approach. The advantage of this is that the properties of monopoles become more apparent, and that certain elements, like the hyper-Kähler metric on the moduli space, appear more explicitly.

Our main aim is therefore to carry out the construction of the moduli space of monopoles without relying on these correspondences. In particular, we define a configuration space of pairs whose asymptotic conditions are adapted to a choice of mass and charge, and we define the moduli space as the quotient of the monopoles inside that configuration space modulo a group of gauge transformations. Our main result can be summarised in the following way:

Theorem (Theorem 5.1.12)**.** *The moduli space $\mathcal{M}_{\mu,\kappa}$ of framed monopoles of mass μ and charge κ is either empty or a smooth hyper-Kähler manifold whose dimension is four times the sum of the integer charges.*

The approach we take to this construction is relatively straightforward, and in many ways mirrors the construction of moduli spaces of anti-self-dual Yang–Mills connections on 4-manifolds [DK90], as well as of SU(2)-monopoles [AH88], which relies on an infinite-dimensional version of the hyper-Kähler quotient [HKLR87]. A crucial aspect is the utilisation of the analytical tools developed by Kottke [Kot15a], which combined Melrose's b and scattering calculuses [Mel93; Mel94] in a way which is particularly well-suited for monopoles – in fact they were used by the same author to study SU(2)-monopoles on other 3-manifolds [Kot15b], carrying out an analysis of the linearised problem similar to the one here. Melrose's formulation provides powerful results and a convenient setting to combine these formalisms, with the b calculus being analogous to the analysis on cylindrical ends studied in other works [Can75; LM85] and the scattering calculus in this case simply corresponding to the usual analysis on a Euclidean space, where Callias's index theorem can be applied [Cal78; Kot11].

Similar techniques were employed by Sánchez Galán in his PhD thesis [Sán19]. In it, a combination of the b and scattering calculuses is applied to the construction of the moduli spaces of SU(n)-monopoles with arbitrary symmetry breaking and their smooth and hyper-Kähler structures, and an index theorem from Kottke's work [Kot15a] is applied to the computation of the dimension for maximal symmetry breaking.

Although many of the features are already present in the case of SU(n), our setting is a more general class of gauge groups. We similarly apply a combination of the b and scattering calculuses, mainly following Kottke's work, but make use of slightly different Sobolev spaces. In particular, we completely fix the decay parameters and we allow the regularity parameters to be arbitrarily large integers. We also make some different choices in the definition of the configuration space and framing, as well as of the space of small gauge transformations. Furthermore, we carry out a detailed analysis of the linearised operator and its indicial roots for arbitrary symmetry breaking. This allows us to compute the dimension of the moduli space directly, and is important in order to choose the initial decay parameters as well as to deduce asymptotic properties of the monopoles.

Aside from obtaining the moduli space itself, this construction provides analytical tools which can be used to further study monopoles. In particular, we also obtain a regularity and decay result in Theorem 4.2.12.

We also discuss some consequences of our framework and results and aim to put them into a wider context.

Much of the work in this thesis is also contained in a previous work of the candidate [Men24], as noted in the declaration above. In particular, the paper contains the construction of the moduli spaces, although here we provide additional details. On the other hand, it did not contain the most refined version of the decay result, or most of the discussion of Chapters 5 and 6.

In Chapter 2 we begin by introducing monopoles relying on general geometric concepts and we establish convenient notions of mass and charge to use throughout the construction.

We then discuss the moduli space itself in Chapter 3. In particular, we set up the formal construction from the analytical tools discussed in the appendices.

Chapter 4 contains the construction of the moduli space. This involves studying the setup from the previous chapter and applying the necessary analytical results.

In Chapter 5 we expand upon certain aspects of our construction and put them in the context of previous work in the area.

Finally, in Chapter 6 we explain how our results translate into certain families of Lie groups and cases studied previously.

The appendices provide a few tools which have been separated from the main body of this thesis to provide a more straightforward exposition.

Appendix A does not contain any particularly novel material, and is mainly intended to fix the notation used regarding spinor bundles and Dirac operators.

Similarly, Appendix B is mostly a review of the analytical tools from the the literature that we aim to apply in the moduli space construction, with the aim of providing a self-contained account of the elements which will be utilised.

In Appendix C we provide a link between the analytical results of the previous appendix and our setting by defining the function spaces which are used throughout, additionally providing some technical properties necessary for the proofs.

Chapter 2

Monopoles

We will start by establishing our setting and defining monopoles in Section 2.1. Then, in Section 2.2 we explain how monopoles are related to anti-self-dual Yang–Mills connections in 4 dimensions. Lastly, we establish a definition of mass and charge and study the asymptotics of monopoles in Section 2.3, establishing an asymptotic model on which we will base the moduli space construction.

2.1 Setting and definitions

In order to define monopoles we must start by choosing an underlying manifold and a gauge group.

The underlying manifold must be an oriented Riemannian 3-manifold. Here, we choose the simplest possibility: the Euclidean 3-space $\mathbb{R}^3$.

On the other hand, the choice of gauge group is much more general. For simplicity, we start by allowing any real, compact, connected, simply connected, semisimple Lie group G, which will be referred to as the *gauge group.* However, in Section 5.4 we discuss how these results are applicable to any compact Lie group.

Note that any such group G admits an inner product $\langle \bullet, \bullet \rangle_{\mathfrak{g}}$ on its Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action. We consider this inner product fixed along with the group.

The monopoles will then be constructed on a principal G-bundle P over $\mathbb{R}^3$. Since the Euclidean space is contractible all such bundles are isomorphic, so we are not making any additional choice. In particular, P must be trivial, but it will not necessarily be convenient to always consider it as such. Indeed, in Section 2.3 we will build P in a way which exhibits its structure more clearly with respect to the monopoles we will consider.

Let us recall that for a given principal G-bundle P we can construct associated bundles through actions of G. Importantly, through the conjugation action of G on itself we obtain the *automorphism bundle* $\mathrm{Aut}(P)$, whose fibres are the automorphism groups of the fibres of P. An automorphism of the bundle P, called a *gauge transformation*, can then be viewed as a section of $\mathrm{Aut}(P)$. The group of these gauge transformations is denoted by $\mathscr{G}$, so we can write

$$\mathscr{G} = \Gamma(\mathrm{Aut}(P))\,. \tag{2.1.1}$$

Another relevant associated bundle is the *adjoint bundle* $\mathrm{Ad}(P)$, obtained through the adjoint action of G on $\mathfrak{g}$, which is denoted by Ad. Note that the Ad-invariant inner product on $\mathfrak{g}$ carries over fibrewise to a metric on this bundle. Furthermore, the adjoint action Ad of G on $\mathfrak{g}$, and the adjoint action of $\mathfrak{g}$ on itself, denoted by ad or by the Lie bracket, carry over fibrewise to the automorphism and adjoint bundles as well. Sections of this adjoint bundle can also be regarded as *infinitesimal gauge transformations*. With the Lie bracket they form a Lie algebra

$$\mathfrak{G} = \Gamma(\mathrm{Ad}(P))\,, \tag{2.1.2}$$

which will be the Lie algebra of the group $\mathscr{G}$ of gauge transformations when the appropriate conditions are added to make it into a Lie group.

Note that the metric on the underlying Euclidean space $\mathbb{R}^3$ induces metrics on its exterior bundle, which we denote simply as $\bigwedge^\bullet$. We can combine this with the metric on the adjoint bundle to obtain metrics on the bundles $\bigwedge^j \otimes \mathrm{Ad}(P)$, and hence, using the Euclidean measure, to obtain L^p norms on the spaces of $\mathrm{Ad}(P)$-valued j-forms $\Omega^j(\mathrm{Ad}(P))$.

On these bundles, we will denote the fibrewise inner product on the adjoint bundle by $\langle \bullet, \bullet \rangle_{\mathfrak{g}}$ and its combination with the Riemannian metric on forms by $\langle \bullet, \bullet \rangle_{\mathbb{R}^3,\mathfrak{g}}$. The L^2 inner product of sections is denoted by $\langle \bullet, \bullet \rangle_{L^2}$, where the fibrewise product is usually understood. Similarly, $\|\bullet\|_{\mathfrak{g}}$, $\|\bullet\|_{\mathbb{R}^3,\mathfrak{g}}$ and $\|\bullet\|_{L^2}$ are used to denote the corresponding (possibly fibrewise) norms, with $\|\bullet\|_Z$ used to denote the norm with respect to any other normed vector space Z.

Remark 2.1.3. Here we are using Γ and Ω to denote spaces of sections and forms,[1] which we can initially think of as being smooth. However, in Chapter 3 we will see how, in fact, it will be necessary to consider sections (and forms) with other regularity and asymptotic conditions.

Given such a principal bundle P, the basic objects which we consider are the following.

Definition 2.1.4. We define the *configuration space (of pairs)* as

$$\mathscr{C} := \mathscr{A}(P) \oplus \Gamma(\mathrm{Ad}(P)) \,, \tag{2.1.5}$$

where $\mathscr{A}(P)$ is the space of principal connections on P. A pair $M = (A, \Phi) \in \mathscr{C}$ is called a *configuration pair*, and its constituent parts are referred to as the connection and the *Higgs field*, respectively.

This configuration space is an infinite-dimensional affine space over the vector space

$$\Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) = \Gamma((\textstyle\bigwedge^1 \oplus \bigwedge^0) \otimes \mathrm{Ad}(P)) \,. \tag{2.1.6}$$

Furthermore, there is a natural action of the group $\mathscr{G}$ of gauge transformations on this space: it acts on the connections in the usual manner, and through the adjoint action Ad on the adjoint bundle. We denote the action of a gauge transformation $g \in \mathscr{G}$ on a pair $M = (A, \Phi)$ as $g \cdot M = (g \cdot A, g \cdot \Phi)$.

On this configuration space we can define the following maps.

Definition 2.1.7. We define the *Bogomolny map* as

$$\begin{aligned} \mathcal{B} \colon \mathscr{C} &\to \Omega^1(\mathrm{Ad}(P)) \\ (A, \Phi) &\mapsto \star F_A - \mathrm{d}_A \Phi \end{aligned} \tag{2.1.8}$$

[1] Along with $\bigwedge$, the notation omits the underlying manifold when it is understood – usually $\mathbb{R}^3$ or a subset of it. When it is necessary to specify the underlying space we write it as a subscript.

and the *energy* map as

$$\begin{aligned}\mathcal{E}\colon \mathscr{C} &\to \mathbb{R}_{\geq 0} \cup \{\infty\} \\ (A, \Phi) &\mapsto \frac{1}{2}(\|F_A\|^2_{L^2} + \|\mathrm{d}_A\Phi\|^2_{L^2}) \,. \end{aligned} \tag{2.1.9}$$

Naturally, the gauge group also acts on the codomain of $\mathcal{B}$ through the adjoint action on its $\mathrm{Ad}(P)$ component – as it does on any space $\Gamma(E \otimes \mathrm{Ad}(P))$ for any vector bundle E. Considering this, we have the following properties.

Proposition 2.1.10. *With respect to the group $\mathscr{G}$ of gauge transformations, the Bogomolny map is equivariant and the energy map is invariant.*

Proof. This follows from the facts that

$$F_{g\cdot A} = g \cdot F_A \tag{2.1.11}$$

and

$$\mathrm{d}_{g\cdot A}(g \cdot \Phi) = g \cdot \mathrm{d}_A\Phi \,, \tag{2.1.12}$$

together with the Ad-invariance of the metric on $\mathrm{Ad}(P)$. □

We can now define monopoles in our setting.

Definition 2.1.13. A *monopole* is a configuration pair $M \in \mathscr{C}$ which satisfies the Bogomolny equations

$$\mathcal{B}(M) = 0 \tag{2.1.14}$$

and has finite energy, that is,

$$\mathcal{E}(M) < \infty \,. \tag{2.1.15}$$

From Proposition 2.1.10 we deduce that the action of $\mathscr{G}$ preserves the condition of being a monopole.

2.2 As dimensional reduction

There is a strong relationship between monopoles and anti-self-dual Yang–Mills connections on the Euclidean 4-space $\mathbb{R}^4$ [Jar04].

Recall that a connection M on a principal G-bundle P over an oriented Riemannian manifold is a *Yang–Mills connection* when

$$\mathrm{d}_M^* F_M = 0. \tag{2.2.1}$$

However, if the underlying manifold is 4-dimensional we have some additional features. We start by noting that the Hodge star operator preserves 2-forms, and squares to the identity. Considering its ± 1 eigenspaces yields a decomposition

$$\Omega^2_{\mathbb{R}^4} = \Omega^+_{\mathbb{R}^4} \oplus \Omega^-_{\mathbb{R}^4} \tag{2.2.2}$$

of 2-forms into self-dual and anti-self-dual components. We then say that a M is an *anti-self-dual Yang–Mills connection* when

$$F_M^+ = 0, \tag{2.2.3}$$

that is, the self-dual component of the curvature vanishes.

Taking the manifold to be $\mathbb{R}^4$, let us write $\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}^3$, and assume that the connection M is invariant under translations of the first summand. This connection is then given by a connection on $\mathbb{R}^3$, representing M along the directions of $\mathbb{R}^3$, together with a section of the adjoint bundle, again over $\mathbb{R}^3$, representing the connection along the direction of $\mathbb{R}$. Calling these components A and Φ, respectively, we have precisely a monopole configuration pair.

More precisely, let us consider the coordinates x_0 and x_1, x_2, x_3 on the space $\mathbb{R} \oplus \mathbb{R}^3$. With respect to a trivialisation of P, we can write a connection M on this space as

$$M = \mathrm{d} + m_0 \mathrm{d}x_0 + m_1 \mathrm{d}x_1 + m_2 \mathrm{d}x_2 + m_3 \mathrm{d}x_3\,, \tag{2.2.4}$$

where

$$m_j \in \Gamma_{\mathbb{R}^4}(\mathrm{Ad}(P))\,. \tag{2.2.5}$$

If M is invariant with respect to translations in x_0, then so are the sections m_j, which can hence be considered sections on $\mathbb{R}^3$. We can then rename

$$A = \mathrm{d} + m_1 \mathrm{d}x_1 + m_2 \mathrm{d}x_2 + m_3 \mathrm{d}x_3\,, \tag{2.2.6a}$$

$$\Phi = m_0\,, \tag{2.2.6b}$$

which define a connection and a Higgs field on $\mathbb{R}^3$.

The crucial fact is that the self-dual part of the curvature of the original connection now becomes precisely the Bogomolny map on (A, Φ), that is,

$$\mathcal{B}(A, \Phi) = F_M^+\,, \tag{2.2.7}$$

where

$$\begin{aligned} \Omega^1_{\mathbb{R}^3} &\cong \Omega^+_{\mathbb{R}^4} \\ \sigma &\leftrightarrow \mathrm{d}x_0 \wedge \sigma + \star_{\mathbb{R}^3}\sigma\,. \end{aligned} \tag{2.2.8}$$

Therefore anti-self-dual Yang-Mills connections invariant in one direction correspond to solutions of the Bogomolny equations.

Throughout the study of monopoles, analogies can be drawn with this 4-dimensional setting. For example, the bundle $\bigwedge^1_{\mathbb{R}^3}$ would be analogous to $\bigwedge^+_{\mathbb{R}^4}$, as seen in (2.2.8), and the bundle $\bigwedge^1_{\mathbb{R}^3} \oplus \bigwedge^0_{\mathbb{R}^3}$ encountered before would be analogous to $\bigwedge^1_{\mathbb{R}^4}$.

This indicates that the study of monopoles will involve some of the many interesting properties found in the study of anti-self-dual Yang–Mills connections. We will remark further on these analogies when relevant throughout the thesis.

2.3 Asymptotic models

The finite energy condition (2.1.15) constitutes a constraint on the asymptotic behaviour of monopoles. However, a more precise picture might be possible, or at least hoped for.

The idea is that the behaviour up to a certain order should be determined by two elements $\mu, \kappa \in \mathfrak{g}$ in the Lie algebra of the gauge group, which are called the *mass* and the *charge*, respectively. Specifically, the monopoles (A, Φ) considered satisfy that, along any ray from the origin, they can be written in some gauge such that

$$\Phi = \mu - \frac{1}{2r}\kappa + o(r^{-1})\,, \tag{2.3.1a}$$

$$\star F_A = \mathrm{d}_A\Phi = \frac{1}{2r^2}\kappa \otimes \mathrm{d}r + o(r^{-2})\,, \tag{2.3.1b}$$

where r is the radial variable. The conditions imposed on the lower order terms vary between different works.

For $G = \mathrm{SU}(2)$, the simplest non-Abelian gauge group, we know that all monopoles fall into this classification, with the lower order term in (2.3.1a) being of order r^{-2} [JT80], but for a general gauge group we don't have such a clear picture.

Our approach largely sidesteps the issue of proving such behaviour for all monopoles, opting instead for directly considering monopoles which satisfy the desired conditions.

For a given choice of μ and κ, we will construct a "model" configuration pair near infinity which satisfies the desired asymptotic conditions without lower order terms. In other words, a pair $(A_{\mu,\kappa}, \Phi_{\mu,\kappa})$ such that

$$\Phi_{\mu,\kappa} = \mu - \frac{1}{2r}\kappa\,, \tag{2.3.2a}$$

$$\star F_{A_{\mu,\kappa}} = \mathrm{d}_{A_{\mu,\kappa}}\Phi_{\mu,\kappa} = \frac{1}{2r^2}\kappa \otimes \mathrm{d}r\,, \tag{2.3.2b}$$

near infinity in some gauge along each ray.

We will then consider monopoles of mass μ and charge κ to be those which are sufficiently close to the model pair near infinity. To be more accurate, this will result in *framed* monopoles, since monopoles gauge equivalent to these ones will also be considered to have the same mass and charge. In Chapter 3 we provide a more precise definition.

Alongside this model pair we will establish, near infinity, a decomposition of the adjoint bundle $\mathrm{Ad}(P)$. The behaviour of the model pair will be very simple with respect to this decomposition, and it will be used to understand the behaviour of the monopoles we consider.

We start the construction by observing that the mass μ and charge κ must commute for such a model pair to exist. To see this, let us take a gauge locally around a ray near infinity with respect to which we have the form (2.3.2). Then, taking the covariant derivative of the Bogomolny map yields

$$
\begin{aligned}
0 &= \mathrm{d}_{A_{\mu,\kappa}}(\mathcal{B}(A_{\mu,\kappa}, \Phi_{\mu,\kappa})) \\
&= \mathrm{d}_{A_{\mu,\kappa}}(\star F_{A_{\mu,\kappa}}) - \mathrm{d}^2_{A_{\mu,\kappa}}\Phi_{\mu,\kappa} \\
&= \mathrm{d}_{A_{\mu,\kappa}}(\star F_{A_{\mu,\kappa}}) - [F_{A_{\mu,\kappa}}, \Phi_{\mu,\kappa}] \\
&= \mathrm{d}_{A_{\mu,\kappa}}\left(\frac{1}{2r^2}\kappa \otimes \mathrm{d}r\right) - \frac{1}{2r^2}[\kappa, \mu] \otimes (\star \mathrm{d}r) \\
&= \left(\mathrm{d}_{A_{\mu,\kappa}}\left(\frac{1}{2r^2}\kappa\right)\right) \wedge \mathrm{d}r - \frac{1}{2r^2}[\kappa, \mu] \otimes (\star \mathrm{d}r)\,.
\end{aligned}
\tag{2.3.3}
$$

By observing the 1-form components of the two resulting summands we can see that they must be linearly independent if they are non-zero, and hence they must both be zero, implying that $[\mu, \kappa] = 0$ as desired.

Therefore, to simplify the construction, we take a maximal torus subgroup T in G whose Lie algebra $\mathfrak{t}$ contains μ and κ, and we will build the pair on a principal T-bundle before taking it to the principal G-bundle P through an associated bundle construction.

Note that, since T is Abelian, the adjoint bundle of any principal T-bundle is canonically identified with the trivial bundle with fibre $\mathfrak{t}$. In particular, μ and κ can now be used as sections without needing to choose a local gauge.

We first construct the connection. Restricted to the unit sphere S^2, the curvature of this connection must be equal to $\frac{1}{2}\kappa \otimes \mathrm{dvol}_{S^2}$ – note that on other spheres we would have the same expression once we rescale. To have such a connection we in fact need an additional integrality condition on the charge.

Proposition 2.3.4. *There exists a principal T-bundle Q on the unit sphere with a connection A_Q with curvature $F_{A_Q} = \frac{1}{2}\kappa \otimes \mathrm{dvol}_{S^2}$ if and only if* $\exp(2\pi\kappa) = 1_T$.

Proof. On S^2, let us write θ_1 for the polar coordinate and θ_2 for the azimuthal coordinate (so the north pole corresponds to $\theta_1 = 0$ and the south pole to $\theta_1 = \pi$). In these coordinates, the curvature expression becomes $F_{A_Q} = \frac{1}{2}\kappa \otimes \sin(\theta_1)\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$.

Now suppose we had such a bundle Q and connection A_Q. Over the unit sphere with the south pole removed we trivialise the connection along meridians from the north to the south pole. Since the group is Abelian, the connection 1-form can be obtained by integrating the curvature. We get

$$a_N = \frac{1 - \cos(\theta_1)}{2}\kappa \otimes \mathrm{d}\theta_2 \,. \tag{2.3.5}$$

Trivialising from the south pole along meridians (excluding the north pole), we likewise get

$$a_S = \frac{-1 - \cos(\theta_1)}{2}\kappa \otimes \mathrm{d}\theta_2 \,. \tag{2.3.6}$$

Using the commutativity once again we deduce that the transition function g between the two charts (which takes values in T) must satisfy

$$(\mathrm{d}g)g^{-1} = a_N - a_S = \kappa \otimes \mathrm{d}\theta_2 \,. \tag{2.3.7}$$

and hence must be of the form

$$g = \exp(\theta_2\kappa) \tag{2.3.8}$$

(up to a multiplicative constant). But in order for this to be well defined, we must have $\exp(2\pi\kappa) = 1_T$.

Conversely, if we have this condition, the construction described provides the desired bundle and connection. □

Of course, since T is a subgroup of G, the integrality condition on the charge can also be written as $\exp(2\pi\kappa) = 1_G$.

Now, this bundle and connection can be extended to the punctured space

$\mathbb{R}^3 \setminus \{0\}$ radially, where we denote them still as Q and A_Q. Note that now we have

$$F_Q = \frac{1}{2r^2} \kappa \otimes \star \mathrm{d}r \,, \tag{2.3.9}$$

as desired. Furthermore, the adjoint bundle $\mathrm{Ad}(Q)$ is the trivial bundle with fibre $\mathfrak{t}$, so we can simply define

$$\Phi_Q \coloneqq \mu - \frac{1}{2r} \kappa \tag{2.3.10}$$

as a section.

Now we can simply define the bundle P outside of the origin as the principal G-bundle associated to Q, since T is a subgroup of G. Given that G is simply connected, this can be completed to a bundle over the entire $\mathbb{R}^3$.[2] Note that, given that the base manifold $\mathbb{R}^3$ is contractible, the bundle P must be trivial in any case, so this construction does not constitute a choice in this respect. However, the relationship

$$P|_{\mathbb{R}^3 \setminus \{0\}} = Q \times_T G \tag{2.3.11}$$

will provide a convenient link with the mass and the charge.

Recall that, for a maximal torus such as T, we have a root space decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra of G. If $R \subset (\mathfrak{t}^{\mathbb{C}})^*$ denotes the set of roots, we can write this as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \,, \tag{2.3.12}$$

where $\mathfrak{g}_\alpha$ is the root space corresponding to α. But the adjoint action of T preserves this decomposition, and outside the origin the bundle $\mathrm{Ad}(P)$ can also be viewed as an associated bundle

$$\mathrm{Ad}(P)|_{\mathbb{R}^3 \setminus \{0\}} = Q \times_T \mathfrak{g} \tag{2.3.13}$$

[2] For a general compact group this will require a stronger integrality condition on the charge, as discussed in Section 5.4.

through the adjoint action. Hence, the root space decomposition can be carried over to this adjoint bundle.

Note that this will include the bundle $\mathrm{Ad}(Q)$ as the trivial subbundle associated to $\mathfrak{t} \subset \mathfrak{t}^{\mathbb{C}}$.

Definition 2.3.14. Near infinity, we define the *root subbundle decomposition* as the decomposition

$$\mathrm{Ad}(P)^{\mathbb{C}} = \underline{\mathfrak{t}^{\mathbb{C}}} \oplus \bigoplus_{\alpha \in R} \utilde{\mathfrak{g}_\alpha} \tag{2.3.15}$$

of $\mathrm{Ad}(P)^{\mathbb{C}}$ associated to the root space decomposition (2.3.12) through (2.3.13). The subbundles $\utilde{\mathfrak{g}_\alpha}$ associated to each root space $\mathfrak{g}_\alpha$ are referred to as a *root subbundles.*

Note that, as is reflected in the notation, although $\underline{\mathfrak{t}^{\mathbb{C}}}$ is trivial, the other subbundles $\utilde{\mathfrak{g}_\alpha}$ might not be. We can also observe that this complex bundle has a natural real structure inherited from $\mathfrak{g}^{\mathbb{C}}$.[3] Furthermore, this real structure preserves $\underline{\mathfrak{t}^{\mathbb{C}}}$, but not necessarily the root subbundles $\utilde{\mathfrak{g}_\alpha}$. However, it *does* preserve the subbundles $\utilde{\mathfrak{g}_\alpha} \oplus \utilde{\mathfrak{g}_{-\alpha}}$. This means that $\mathrm{Ad}(P)$ can be thought of as being decomposed into $\underline{\mathfrak{t}}$ and the real parts of the spaces $\utilde{\mathfrak{g}_\alpha} \oplus \utilde{\mathfrak{g}_{-\alpha}}$.

Remark 2.3.16. In (2.3.12), we could substitute $\mathfrak{t}^{\mathbb{C}}$ with $(\mathfrak{g}_0)^{\mathrm{rank}(G)}$, where $\mathfrak{g}_0 \cong \mathbb{C}$ and 0 is interpreted as an element of $(\mathfrak{t}^{\mathbb{C}})^*$, analogous to a "root equal to zero". Analogously, in (2.3.15) we could substitute $\underline{\mathfrak{t}^{\mathbb{C}}}$ with $(\utilde{\mathfrak{g}_0})^{(\mathrm{rank}(G)}$, where $\utilde{\mathfrak{g}_0}$ is the (trivial) line bundle corresponding to $\mathfrak{g}_0$ through (2.3.13).

This will be notationally convenient, since many properties of the root subbundles $\utilde{\mathfrak{g}_\alpha}$ will be true of $\underline{\mathfrak{t}^{\mathbb{C}}}$ when changing α to $0 \in (\mathfrak{t}^{\mathbb{C}})^*$ and taking the multiplicity into account.

Hence, unless specified otherwise, we will understand all the properties deduced for the subbundles $\utilde{\mathfrak{g}_\alpha}$ to extend to $\underline{\mathfrak{t}^{\mathbb{C}}}$ in this manner (like in Proposition 2.3.21).

Naturally, the adjoint action behaves as expected in this decomposition. That

[3]This can be formalised as a fibrewise conjugate-linear involution whose fixed points make up the real part.

is, if $X \in \Gamma(\underline{\mathfrak{t}^{\mathbb{C}}})$ and $Y \in \Gamma(\underset{\sim}{\mathfrak{g}_\alpha})$, then

$$\operatorname{ad}_X(Y) = \alpha(X)Y\,. \tag{2.3.17}$$

Now, through this associated bundle construction we can also carry over A_Q and Φ_Q, making them smooth over the origin with a cutoff function.

Definition 2.3.18. We define $A_{\mu,\kappa}$ as the connection on P associated to A_Q through (2.3.11) and smoothed over the origin. Likewise, we define $\Phi_{\mu,\kappa}$ as the section of $\operatorname{Ad}(P)$ given by the inclusion of $\operatorname{Ad}(Q)$ in $\operatorname{Ad}(P)$, similarly smoothed over the origin. We refer to

$$M_{\mu,\kappa} \coloneqq (A_{\mu,\kappa}, \Phi_{\mu,\kappa}) \tag{2.3.19}$$

as the *model pair*.

This definition provides a pair $(A_{\mu,\kappa}, \Phi_{\mu,\kappa})$ which, near infinity, satisfies the desired asymptotic conditions. Furthermore, its behaviour with respect to the root subbundle decomposition will provide the basis for much of the analysis throughout the rest of this thesis.

To begin understanding this, let us start by fixing more precise notation.

Definition 2.3.20. We write $\underline{\mu}$ and $\underline{\kappa}$ for the sections of $\underline{\mathfrak{t}}$ inside $\operatorname{Ad}(P)$ near infinity which are constant of values μ and κ, respectively.

We can now write out the behaviour of the model pair near infinity along the decomposition.

Proposition 2.3.21. *Near infinity, we have*

$$\Phi_{\mu,\kappa} = \underline{\mu} - \frac{1}{2r}\underline{\kappa}\,, \tag{2.3.22a}$$

$$\star F_{A_{\mu,\kappa}} = \mathrm{d}_{A_{\mu,\kappa}}\Phi_{\mu,\kappa} = \frac{1}{2r^2}\underline{\kappa} \otimes \mathrm{d}r\,, \tag{2.3.22b}$$

as well as

$$\mathcal{B}(A_{\mu,\kappa}, \Phi_{\mu,\kappa}) = 0\,. \tag{2.3.23}$$

Furthermore, the connection $A_{\mu,\kappa}$ and the adjoint action of $\Phi_{\mu,\kappa}$ decompose along the root subbundle decomposition (2.3.15)*. Restricted to each root subbundle we have*

$$\operatorname{ad}_{\Phi_{\mu,\kappa}}|_{\underset{\sim}{\mathfrak{g}_\alpha}} = \alpha(\mu) - \frac{\alpha(\kappa)}{2r}\,, \tag{2.3.24a}$$

$$\star F_{A_{\mu,\kappa}|_{\underset{\sim}{\mathfrak{g}_\alpha}}} = \frac{\alpha(\kappa)}{2r^2}\mathrm{d}r\,. \tag{2.3.24b}$$

Proof. The form (2.3.22) follows from the construction, and (2.3.23) follows from the fact that the connection $A_{\mu,\kappa}$ is trivial on $\underline{\mathfrak{t}^{\mathbb{C}}}$, so its sections $\underline{\mu}, \underline{\kappa} \in \Gamma(\underline{\mathfrak{t}^{\mathbb{C}}})$ are covariantly constant.

The second part of the proposition follows from the general properties of associated bundles, together with the properties of the root subbundle decomposition, such as (2.3.17). □

This, in turn, provides a better understanding of the subbundles in the decomposition in terms of the complex line bundles described in Section A.4.

Corollary 2.3.25. *Near infinity, each complex line subbundle $\underset{\sim}{\mathfrak{g}_\alpha}$ has degree $i\alpha(\kappa)$ over each sphere centred at the origin. Furthermore, the restriction of $A_{\mu,\kappa}$ to each of these subbundles is homogeneous on these spheres. In other words, considered with the connection $A_{\mu,\kappa}|_{\underset{\sim}{\mathfrak{g}_\alpha}}$, we have*

$$\underset{\sim}{\mathfrak{g}_\alpha} \cong \mathscr{L}^{i\alpha(\kappa)}\,. \tag{2.3.26}$$

Proof. This follows from (2.3.24b) and the construction of $A_{\mu,\kappa}$. This is because the connection is radially constant and satisfied the desired curvature condition, which in the case of unitary line bundles on a simply connected manifold like the 2-sphere determines the connection up to gauge equivalence. □

Note that, although not all of the notation built up in this section makes explicit reference to the choice of mass μ and charge κ they are still used to carry out the constructions. Therefore in Chapters 3 and 4 we will assume that, along with the connected, simply connected, semisimple compact group G, we have fixed such two elements $\mu, \kappa \in \mathfrak{g}$ satisfying that

$$[\mu, \kappa] = 0\,, \tag{2.3.27a}$$

$$\exp(2\pi\kappa) = 1_G\,, \tag{2.3.27b}$$

as well as the model pair $M_{\mu,\kappa}$. We will then study the moduli space of monopoles framed by using this construction.

Remark 2.3.28. Let us lastly remark upon some of the choices made here.

Firstly, we know that the mass μ and charge κ are only relevant up to a joint transformation under the adjoint action of the gauge group G, since that same action applied to the entire construction would yield an equivalent setting.

However, for a given pair of mass and charge, the specific construction carried out is a priori not unique, although it is not difficult to see that some of the choices, like the use of different cutoff functions, will not ultimately affect the results. Regardless, since we fix these choices here for the rest of the thesis, we will continue referring to the resulting constructions as *the* model pair, *the* root subbundle decomposition, and so on.

Chapter 3

The framed moduli space

One of the interesting features of monopoles is that they can be assembled into moduli spaces.

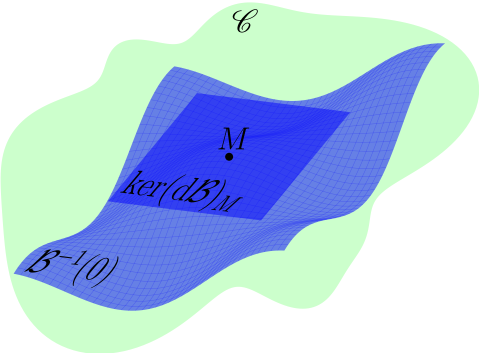
Our aim here is to construct the moduli space of framed monopoles with a fixed mass and charge. This means that we fix a mass μ and a charge κ and we consider monopoles which, near infinity, approach the model pair $(A_{\mu,\kappa}, \Phi_{\mu,\kappa})$ constructed in Section 2.3. Then, we quotient by a group of gauge transformations which fixes the asymptotic behaviour.

We start by giving an outline of the construction in Section 3.1. In Section 3.2 we take a closer look at the linearised operator, whose kernel models the moduli space to linear order, and serves to motivate the formal analytic setup. This is then established in Section 3.3, based on Appendices B and C.

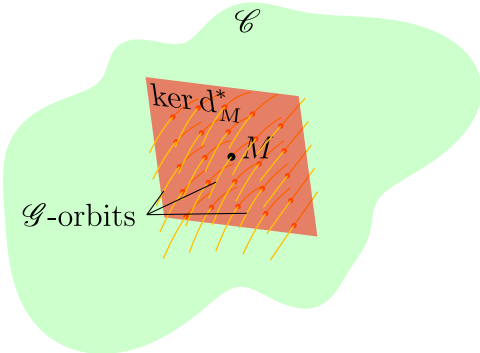
3.1 Construction outline

Here we give an outline of the moduli space construction. Figure 3.1.1, which we will reference throughout, gives a qualitative picture of this construction, although many of the spaces involved are in fact infinite-dimensional. Furthermore, we don't define these spaces formally now, postponing the precise definitions to Section 3.3.

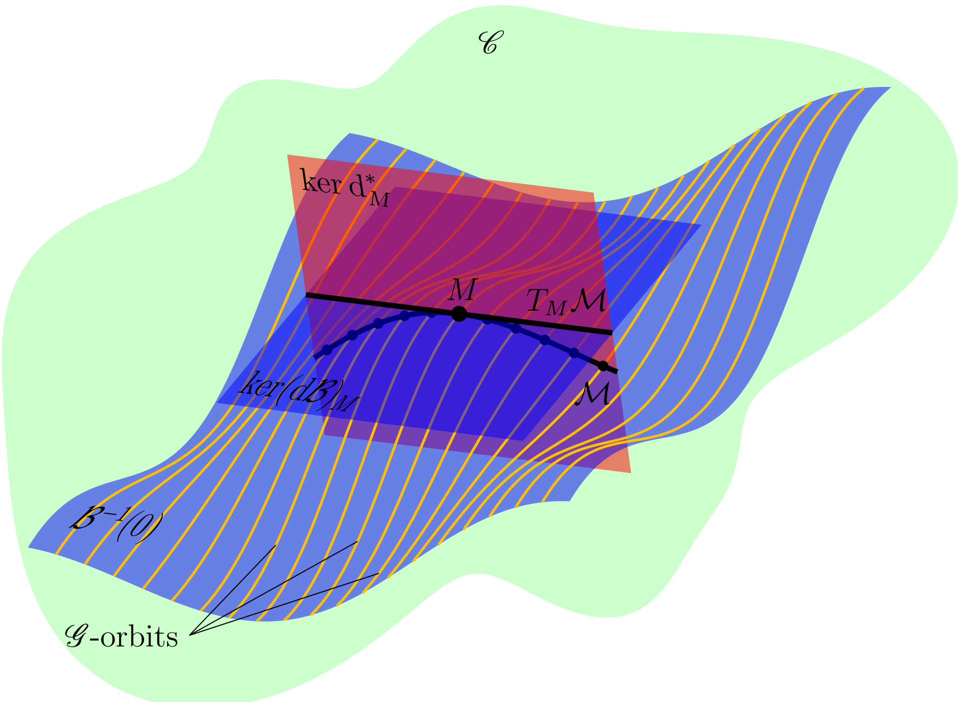
The starting point of our construction is a configuration space of pairs which includes not only monopoles but all those pairs which satisfy the desired asymptotic behaviour (but not necessarily the Bogomolny equations). In particular, we



a: Within the configuration space $\mathscr{C}$, the monopole M is inside the space of monopoles $\mathcal{B}^{-1}(0)$. Its tangent space is given by $\ker(\mathrm{d}\mathcal{B})_M$, the kernel of the derivative of the Bogomolny map.



b: The group $\mathscr{G}$ acts on $\mathscr{C}$, resulting in $\mathscr{G}$-orbits through every point. The space of pairs in Coulomb gauge with respect to M is given by $\ker \mathrm{d}^*_M$, and intersects each nearby orbit locally at a single point.



c: The intersection of the space of monopoles and the space of pairs in Coulomb gauge represents the moduli space $\mathcal{M}$ near the monopole, since it intersects each nearby $\mathscr{G}$-orbit locally exactly once. Its tangent space $T_M\mathcal{M}$, given by intersecting $\ker(\mathrm{d}\mathcal{B})_M$ and $\ker \mathrm{d}^*_M$, will provide a chart for the moduli space through the implicit function theorem.

Figure 3.1.1: Qualitative representation of the moduli space construction

define the configuration space as the space of all pairs which differ from the model pair by a decaying element of the space

$$\Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P))\,. \tag{3.1.2}$$

The specific decay condition will be given by requiring these elements to be in a specific subspace of such forms. This provides the configuration space with the structure of an affine space of infinite dimension.

Within this space we can now consider the subspace of monopoles, that is, those pairs which satisfy the Bogomolny equations. This space, represented in Figure 3.1.1a, can be written as $\mathcal{B}^{-1}(0)$. Hence, under appropriate conditions, it will be a smooth submanifold (still of infinite dimension), and its tangent space at a given monopole $M = (A, \Phi) \in \mathscr{C}$ will be the kernel of the derivative of the Bogomolny map at that monopole. This derivative is given by

$$\begin{aligned}(\mathrm{d}\mathcal{B})_{(A,\Phi)} \colon \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) &\to \Omega^1(\mathrm{Ad}(P)) \\ (a, \varphi) &\mapsto \star \mathrm{d}_A a + \mathrm{ad}_\Phi(a) - \mathrm{d}_A \varphi\,.\end{aligned} \tag{3.1.3}$$

We then consider the quotient of this space under the action of the group of gauge transformations. Locally around a monopole, this quotient will be modelled on a slice of the action of the group of gauge transformations. This slice will be given by a gauge fixing condition which, at least locally, must guarantee that every gauge orbit is represented by a single monopole.

In order to find such a condition, let us begin by observing that the infinitesimal action of the group of gauge transformations on the configuration space is given, for a pair $(A, \Phi) \in \mathscr{C}$ and an infinitesimal gauge transformation $X \in \mathfrak{G}$, by

$$(X^{\#})_{(A,\Phi)} = -(\mathrm{d}_A X, \mathrm{ad}_\Phi X) \in T_{(A,\Phi)}\mathscr{C}\,. \tag{3.1.4}$$

For notational convenience, and inspired by the analogy with instantons on $\mathbb{R}^4$, let us define

$$\begin{aligned}\mathrm{d}_{(A,\Phi)} \colon \Omega^0(\mathrm{Ad}(P)) &\to \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) \\ X &\mapsto (\mathrm{d}_A X, \mathrm{ad}_\Phi(X))\,,\end{aligned} \tag{3.1.5}$$

so that for a pair $M \in \mathscr{C}$ we can simply write

$$(X^{\#})_M = -\mathrm{d}_M X\,. \tag{3.1.6}$$

This implies that the tangent space to the orbit of M will be given by the image of d_M.

We can similarly define

$$\begin{aligned} \mathrm{d}^*_{(A,\Phi)} \colon \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) &\to \Omega^0(\mathrm{Ad}(P)) \\ (a,\varphi) &\mapsto \mathrm{d}^*_A a - \mathrm{ad}_\Phi(\varphi)\,, \end{aligned} \tag{3.1.7}$$

the formal adjoint of d_M. Then, once again under the appropriate conditions, the kernel of d^*_M will be L^2-orthogonal to the image of d_M inside the tangent space $T_M\mathscr{C}$, and hence, considered as a subspace of the affine space $\mathscr{C}$, it will be transverse to the orbit through M. In this way, the kernel of d^*_M could define a slice which, locally, intersects nearby orbits only once. Pairs in this space are said to be in *Coulomb gauge* with respect to M. Figure 3.1.1b represents this gauge fixing condition.

Putting both steps together, as seen in Figure 3.1.1c, the moduli space $\mathcal{M}$ near a given monopole $M \in \mathscr{C}$ is given by intersecting the space of monopoles with the space $\ker \mathrm{d}^*_M$ (as a subspace of $\mathscr{C}$, that is, with its origin on M). This intersects orbits of monopoles near M locally exactly once. The tangent space $T_M\mathcal{M}$ to the moduli space will be analogously given by intersecting the tangent space to the space of monopoles with the same linear space $\ker \mathrm{d}^*_M$.

To describe this more concisely, let us define, for a pair $M \in \mathscr{C}$, the function

$$f_M(\bullet) \coloneqq (-\mathcal{B}(\bullet), \mathrm{d}^*_M(\bullet - M)) \colon \mathscr{C} \to \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P))\,. \tag{3.1.8}$$

Then, the moduli space is given locally around a monopole M by $f_M^{-1}(0)$. In order to prove that this is a smooth manifold we will apply the implicit function theorem, so it is important to understand its derivative

$$(\mathrm{d}f_M)_M = (-(\mathrm{d}\mathcal{B})_M, \mathrm{d}^*_M) \tag{3.1.9}$$

at the monopole, which we will refer to as the *linearised operator*. As we will see in the next section, it will be mainly written as $\not{D}_M$.

Remark 3.1.10. These maps fit into the complex

$$\Omega^0(\mathrm{Ad}(P)) \underset{\mathrm{d}_M^*}{\overset{\mathrm{d}_M}{\rightleftarrows}} \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) \overset{-(\mathrm{d}\mathcal{B})_M}{\to} \Omega^1(\mathrm{Ad}(P))\,. \tag{3.1.11}$$

which, following the analogies of Section 2.2, would correspond to the complex

$$\Omega^0_{\mathbb{R}^4}(\mathrm{Ad}(P)) \underset{\mathrm{d}_M^*}{\overset{\mathrm{d}_M}{\rightleftarrows}} \Omega^1_{\mathbb{R}^4}(\mathrm{Ad}(P)) \overset{-\mathrm{d}_M^+}{\to} \Omega^+_{\mathbb{R}^4}(\mathrm{Ad}(P)) \tag{3.1.12}$$

of forms on $\mathbb{R}^4$.

3.2 The linearised operator

The analysis of the linearised operator is crucial for our construction, and hence it plays an important role in motivating the analytic setup that we will use. We begin by recasting the operator as a Dirac operator.

Definition 3.2.1. Let $(A,\Phi) \in \mathscr{C}$. We define its *linearised operator* as the operator

$$\not{D}_{(A,\Phi)} \colon \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) \to \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) \tag{3.2.2}$$

given by

$$\not{D}_{(A,\Phi)} \coloneqq \begin{pmatrix} -(\mathrm{d}\mathcal{B})_{(A,\Phi)} \\ \mathrm{d}^*_{(A,\Phi)} \end{pmatrix} = \begin{pmatrix} -\star\mathrm{d}_A & \mathrm{d}_A \\ \mathrm{d}_A^* & 0 \end{pmatrix} - \mathrm{ad}_\Phi\ . \tag{3.2.3}$$

We denote the first summand by $\not{D}_A$.

As indicated, this is simply the derivative (3.1.9) of the function Ψ_M defined above (the change of notation between row and column vectors notwithstanding).

Remark 3.2.4. Although the setup involves mainly real spaces, it will be convenient to complexify some of them. This will allow us to make use of the theory of complex spinor bundles laid out in Appendix A, as well as the Fredholm theory for elliptic operators laid out in Appendix B.

It is relevant to note, therefore, that the complexified spaces carry a real structure, and that (most of) the operators involved preserve it. Hence, if a complexified operator is Fredholm, the real operator between the real parts is also Fredholm, and its kernel will have the same real dimension as the complex dimension of the kernel of the complexification.

We will specify complexified spaces with a superscript $\mathbb{C}$, but we will denote the operators in the same way regardless of whether they act between the real or the complexified spaces.

Now, as pointed out, (the complexification of) the operator (3.2.3) can also be viewed as a Dirac operator, which will facilitate much of the analysis. This is laid out in the following proposition, which is explained in more detail in Appendix A.

Proposition 3.2.5. *The bundle*

$$((\bigwedge^1 \otimes \mathrm{Ad}(P)) \oplus (\bigwedge^0 \otimes \mathrm{Ad}(P)))^{\mathbb{C}} \tag{3.2.6}$$

is isomorphic to

$$\not{S} \otimes \underline{\mathbb{C}^2} \otimes \mathrm{Ad}(P)^{\mathbb{C}}. \tag{3.2.7}$$

Under this isomorphism, $\not{D}_A$, the first summand of (3.2.3)*, is the Dirac operator twisted by the bundle $\underline{\mathbb{C}^2} \otimes \mathrm{Ad}(P)^{\mathbb{C}}$, with the connection on the second factor being induced by A.*

Proof. This is a consequence of Proposition A.3.4, considering that on $\mathbb{R}^3$ the bundle $\not{S}^*$ is trivial of rank 2. □

In (3.2.7), we write $\underline{\mathbb{C}^2}$ for the bundle $\not{S}^*$ to simplify notation. This factor will in fact not have much relevance beyond duplicating the bundle and the operator. We will see later that, for example, this also duplicates its index. Note that the notation for $\not{S}$ is preserved to emphasise that it carries the Clifford representation, as explained in the appendix.

This characterisation will provide a simple picture for these operators. This picture is even simpler for the model pair, since its connection splits along the root subbundles. Near infinity, let us write

$$\not{D}_\alpha := \not{D}_{M_{\mu,\kappa}}|_{\not{S}\otimes \underset{\sim}{\mathfrak{g}_\alpha}} . \tag{3.2.8}$$

We can completely understand the linearised operator near infinity in the following way.

Lemma 3.2.9. *Near infinity, the operator* $\not{D}_{M_{\mu,\kappa}}$ *splits as*

$$\not{D}_{M_{\mu,\kappa}} = \not{D}_0^{\oplus 2\,\mathrm{rank}(G)} \oplus \bigoplus_{\alpha\in R} \not{D}_\alpha^{\oplus 2} , \tag{3.2.10}$$

following the decomposition (2.3.15)*. On each root subbundle, we have*

$$\not{D}_\alpha = \not{D}_{i\alpha(\kappa)} - \alpha(\mu) + \frac{\alpha(\kappa)}{2r} . \tag{3.2.11}$$

Proof. This follows from the properties of the model pair and the root subbundle decomposition explained in Proposition 2.3.21 and Corollary 2.3.25 and the notation of Appendix A. Note that the operators are duplicated due to the factor $\underline{\mathbb{C}^2}$. □

For any other configuration pair, we can write the linearised operator in relation to the model.

Lemma 3.2.12. *Let* $M, M' \in \mathscr{C}$*. Then,*

$$\not{D}_{M'} - \not{D}_M = (\mathrm{cl}\otimes \mathrm{ad})_{M'-M} . \tag{3.2.13}$$

In particular, this is an algebraic term proportional to

$$M' - M \in \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P)) . \tag{3.2.14}$$

Here, cl ⊗ ad *denotes the combination of the Clifford action of* 1*-forms in the sense of Proposition 3.2.5 with the adjoint action of sections of* $\mathrm{Ad}(P)^{\mathbb{C}}$ *(with a sign change where necessary).*

Proof. This follows from the properties of Dirac operators. □

Corollary 3.2.15. *If $M = M_{\mu,\kappa} + m$ for some $m \in \Omega^1(\mathrm{Ad}(P)) \oplus \Omega^0(\mathrm{Ad}(P))$, then*

$$\not{D}_M = \not{D}_{M_{\mu,\kappa}} + (\mathrm{cl} \otimes \mathrm{ad})_m \,. \tag{3.2.16}$$

Lastly, we show the following property, which will be useful later on. It is stated in Nakajima's work [Nak93] in a similar setting but we prove it here for completeness, following from the exposition in Appendix A. It concerns the composition of the linearised operator with its formal adjoint $\not{D}^*_{(A,\Phi)} = \not{D}_{(A,-\Phi)}$.

Lemma 3.2.17. *Let $(A, \Phi) \in \mathscr{C}$ satisfy the Bogomolny equations. Then*

$$\not{D}_{(A,\Phi)} \not{D}_{(A,-\Phi)} = \nabla^*\nabla - \mathrm{ad}^2_\Phi \,, \tag{3.2.18}$$

where ∇ denotes the covariant derivative with respect to A and the Levi-Civita connection.

Proof. Writing out $\not{D}_{(A,\Phi)}$ and $\not{D}_{(A,-\Phi)}$, we see that the claim is equivalent to

$$\not{D}_A \not{D}_A - \nabla^*\nabla = \mathrm{ad}_\Phi \not{D}_A - \not{D}_A \mathrm{ad}_\Phi \,. \tag{3.2.19}$$

To prove this, let $u_{\not{S}} \in \Gamma(\not{S})$ and $u_{\mathrm{Ad}} \in \Gamma(\underline{\mathbb{C}^2} \otimes \mathrm{Ad}(P)^{\mathbb{C}})$, and let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the tangent space of $\mathbb{R}^3$ at a point. Then we can apply the Lichnerowicz–Weitzenböck formula [LM89, Thm. 8.17] to obtain

$$\begin{aligned}
(\not{D}_A \not{D}_A - \nabla^*\nabla)(u_{\not{S}} \otimes u_{\mathrm{Ad}}) &= \sum (\mathrm{cl}_{e_{j_1}} \mathrm{cl}_{e_{j_2}} u_{\not{S}}) \otimes (\mathrm{ad}_{F_A(e_{j_1}, e_{j_2})} u_{\mathrm{Ad}}) \\
&= \sum (\mathrm{cl}_{\star e_{j_3}} u_{\not{S}}) \otimes (\mathrm{ad}_{\nabla_{e_{j_3}} \Phi} u_{\mathrm{Ad}}) \\
&= -\sum (\mathrm{cl}_{e_{j_3}} u_{\not{S}}) \otimes (\mathrm{ad}_{\nabla_{e_{j_3}} \Phi} u_{\mathrm{Ad}}) \\
&= -\sum (\mathrm{cl}_{e_{j_3}} u_{\not{S}}) \otimes (\nabla_{e_{j_3}} (\mathrm{ad}_\Phi u_{\mathrm{Ad}})) \\
&\quad + \mathrm{ad}_\Phi \sum (\mathrm{cl}_{e_{j_3}} u_{\not{S}}) \otimes (\nabla_{e_{j_3}} u_{\mathrm{Ad}}) \\
&= -\not{D}_A (\mathrm{ad}_\Phi (u_{\not{S}} \otimes u_{\mathrm{Ad}})) \\
&\quad + \mathrm{ad}_\Phi (\not{D}_A (u_{\not{S}} \otimes u_{\mathrm{Ad}})) \,,
\end{aligned} \tag{3.2.20}$$

as desired. □

3.3 Analytic setup

A crucial observation about the behaviour of the linearised operator is the different asymptotics exhibited along different subbundles of $\mathrm{Ad}(P)^{\mathbb{C}}$. In particular, we are interested in separating the cases in which the adjoint action of the mass does and does not degenerate. Near infinity, we write

$$\mathrm{Ad}(P)_C \coloneqq \ker(\mathrm{ad}_{\underline{\mu}})\,, \tag{3.3.1a}$$

$$\mathrm{Ad}(P)_{C^\perp} \coloneqq \ker(\mathrm{ad}_{\underline{\mu}})^\perp\,, \tag{3.3.1b}$$

which we refer to as the C and $C^\perp$ parts of the adjoint bundle, respectively, since the former corresponds to the centraliser of μ in the Lie algebra $\mathfrak{g}$, whereas the latter corresponds to its orthogonal complement.

Note that if we complexify this decomposition we have

$$\mathrm{Ad}(P)_C^{\mathbb{C}} = \underline{\mathfrak{t}}^{\mathbb{C}} \oplus \bigoplus_{\substack{\alpha\in R\\ \alpha(\mu)=0}} \underset{\sim}{\mathfrak{g}_\alpha}\,, \tag{3.3.2a}$$

$$\mathrm{Ad}(P)_{C^\perp}^{\mathbb{C}} = \bigoplus_{\substack{\alpha\in R\\ \alpha(\mu)\neq 0}} \underset{\sim}{\mathfrak{g}_\alpha}\,. \tag{3.3.2b}$$

The properties of the operator along the C and $C^\perp$ parts require different tools to study. These are the b and *scattering calculuses*, respectively, whose main concepts are summarised in Appendix B. These two approaches are then combined, and in Appendix C we consider spaces of functions which are adapted to our specific setting. We are particularly interested in the spaces

$$\mathscr{H}_0^{s,2} = \mathcal{H}^{-1,1,s,2}(\textstyle\bigwedge^0 \otimes \mathrm{Ad}(P))\,, \tag{3.3.3}$$

$$\mathscr{H}^{s,1} = \mathcal{H}^{0,1,s,1}((\textstyle\bigwedge^0 \oplus \bigwedge^1) \otimes \mathrm{Ad}(P))\,, \tag{3.3.4}$$

$$\mathscr{H}_1^{s,0} = \mathcal{H}^{1,1,s,0}(\textstyle\bigwedge^1 \otimes \mathrm{Ad}(P))\,, \tag{3.3.5}$$

defined in said appendices.

Here, $s \in \mathbb{Z}_{\geq 1}$ is a regularity parameter which we will assume fixed for now. The constructions will a priori depend on this parameter, but in Proposition 4.3.16

we show that the resulting moduli space is independent.

This allows us to finally define the necessary elements to construct the moduli space. Let us begin with the configuration space, which must be restricted to pairs which approach the model $M_{\mu,\kappa}$ at infinity.

Definition 3.3.6. We define the *configuration space of framed pairs of mass* μ *and charge* κ as the affine space

$$\mathscr{C}^s_{\mu,\kappa} := M_{\mu,\kappa} + \mathscr{H}^{s,1}\,. \tag{3.3.7}$$

Note that the decay conditions of sections in $\mathscr{H}^{s,1}$ guarantee that these configuration spaces are disjoint for different masses and charges. Furthermore, the L^2 norm is finite by Lemma C.2.3. The regularity will also play a role as we consider different maps built from these configuration pairs.

Let us prove now some important properties of such maps.

Lemma 3.3.8. *For any* $(A,\Phi)\in\mathscr{C}^s_{\mu,\kappa}$*, the maps*

$$\mathrm{d}_A\colon \mathscr{H}^{s,k}_j \to \mathscr{H}^{s,k-1}_{j+1} \tag{3.3.9}$$

and

$$\mathrm{ad}_\Phi\colon \mathscr{H}^{s,k}_j \to \mathscr{H}^{s,k-1}_j \tag{3.3.10}$$

are continuous for $k\in\{1,2\}$ *and any (appropriate)* j*.*

Proof. The continuity of the two maps for the pair $(A_{\mu,\kappa},\Phi_{\mu,\kappa})$ follows from Lemma C.2.4. For any $(A,\Phi)+(a,\varphi)$, where $(a,\varphi)\in\mathscr{H}^{s,1}$, we only have to add the continuity of the maps (C.2.12) and (C.2.14). □

Corollary 3.3.11. *For any* $(A,\Phi)\in\mathscr{C}^s_{\mu,\kappa}$*, the linearised operator*

$$\not{D}_{(A,\Phi)}\colon \mathscr{H}^{s,1} \to \mathscr{H}^{s,0} \tag{3.3.12}$$

is continuous.

Proof. This follows from Lemma 3.3.8 for $k=1$ and $j\in\{0,1,2\}$. □

Lemma 3.3.13. *Let $(A, \Phi) \in \mathscr{C}^s_{\mu,\kappa}$. Then we can perform integration by parts between $\mathscr{H}_0^{s,2}$ and $\mathscr{H}_0^{s,1}$ using the covariant derivative of A.*

Proof. If $\nabla_{\frac{\partial}{\partial x_j}}$ denotes the covariant derivative of A in the direction of a coordinate x_j of $\mathbb{R}^3$, then from Lemmas 3.3.8 and C.2.3 we deduce that

$$\begin{aligned} \mathscr{H}_0^{s,2} \times \mathscr{H}_0^{s,1} &\to \mathbb{R} \\ (u, u') &\mapsto \langle \nabla_{\frac{\partial}{\partial x_j}} u, u' \rangle_{L^2} + \langle u, \nabla_{\frac{\partial}{\partial x_j}} u' \rangle_{L^2} \end{aligned} \tag{3.3.14}$$

is continuous. Furthermore, it is 0 for smooth compactly supported functions. Since these are dense due to Lemma B.2.16, the map must be identically 0. □

We can now consider the Bogomolny map on this configuration space.

Definition 3.3.15. We write

$$\mathcal{B}^s_{\mu,\kappa} := \mathcal{B}|_{\mathscr{C}^s_{\mu,\kappa}} \tag{3.3.16}$$

for the restriction of the Bogomolny map to our configuration space.

Proposition 3.3.17. *The Bogomolny map is smooth as a map*

$$\mathcal{B}^s_{\mu,\kappa} \colon \mathscr{C}^s_{\mu,\kappa} \to \mathscr{H}_1^{s,0}\,, \tag{3.3.18}$$

and the energy map is finite on $\mathscr{C}^s_{\mu,\kappa}$.

Proof. Firstly, we observe that $\mathcal{B}(A_{\mu,\kappa}, \Phi_{\mu,\kappa})$ is smooth and compactly supported, and hence in $\mathscr{H}_1^{s,0}$. Secondly, if $(A, \Phi) = (A_{\mu,\kappa}, \Phi_{\mu,\kappa}) + (a, \varphi)$, then

$$\mathcal{B}(A, \Phi) - \mathcal{B}(A_{\mu,\kappa}, \Phi_{\mu,\kappa}) = \star \mathrm{d}_{A_{\mu,\kappa}} a + \mathrm{ad}_{\Phi_{\mu,\kappa}} a - \mathrm{d}_{A_{\mu,\kappa}} \varphi + \frac{1}{2} \star [a \wedge a] - [a, \varphi]\,. \tag{3.3.19}$$

But this is also in $\mathscr{H}_1^{s,0}$ due to the continuity of (3.3.9), (3.3.10) and (C.2.14).

Similarly, it is clear that the energy map is finite on the model pair, and the same argument as above shows that this is also the case for any other configuration pair. □

Lastly, we will define the group of gauge transformations which approach the identity in the appropriate manner. We aim to make this group a Lie group, with the structure of a Banach manifold. In order to simplify the definition, let us consider the Lie group G as a compact subgroup inside an algebra of matrices, and let us denote by E^{Mat} the vector bundle (in fact, bundle of algebras) over $\mathbb{R}^3$ associated to P with fibres modelled on this space of matrices.

The bundles $\mathrm{Aut}(P)$ and $\mathrm{Ad}(P)$ can then be regarded as subbundles of E^{Mat}, with matrix operations providing the fibrewise group and Lie algebra structures. Furthermore, this bundle can be split near infinity into C and $C^{\perp}$ parts similarly to the adjoint bundle: E^{Mat}_C is the subbundle which commutes with $\underline{\mu}$ and $E^{\mathrm{Mat}}_{C^{\perp}}$ its fibrewise orthogonal complement (with respect to any metric which extends the metric on $\mathrm{Ad}(P)$). Considering the splitting $E^{\mathrm{Mat}} = E^{\mathrm{Mat}}_C \oplus E^{\mathrm{Mat}}_{C^{\perp}}$, we can make the definition of the group of gauge transformations.

Definition 3.3.20. We define the *group of small gauge transformations* as

$$\mathscr{G}^s_{\mu,\kappa} := \{g \in \underline{1_G} + \mathscr{H}^{s,2}(E^{\mathrm{Mat}}) \mid g \text{ takes values in } \mathrm{Aut}(P)\}\,. \tag{3.3.21}$$

Here it is crucial to choose a space with enough regularity obtain a Lie group. In particular, note that from Remark B.2.27 we deduce that these sections are continuous.

Proposition 3.3.22. *The set $\mathscr{G}^s_{\mu,\kappa}$ is a Lie group of gauge transformations and its Lie algebra is given by*

$$\mathfrak{G}^s_{\mu,\kappa} := \mathrm{Lie}(\mathscr{G}^s_{\mu,\kappa}) = \mathscr{H}^{s,2}_0\,. \tag{3.3.23}$$

Proof. Since the group G is an embedded submanifold of the space of matrices, it can be locally defined as the zero locus of a smooth function. Similarly, locally around a section in $\mathscr{G}^s_{\mu,\kappa}$ we can define a fibrewise smooth function which takes values in a transverse bundle, whose zero locus defines $\mathscr{G}^s_{\mu,\kappa}$ locally. The Sobolev structures on the spaces provide the manifold structure.

The multiplication, which is smooth, is an internal operation due to the continuity of the map (C.2.15). □

All of these spaces have been chosen so that we can carry out a moduli space construction.

Proposition 3.3.24. *The group $\mathscr{G}^s_{\mu,\kappa}$ acts on the space $\mathscr{C}^s_{\mu,\kappa}$ smoothly.*

Proof. This is a consequence of the continuity of the maps (3.3.9) and (C.2.12). □

Therefore we can now define the moduli space of framed monopoles.

Definition 3.3.25. We define the *moduli space of framed monopoles of mass μ and charge κ* as

$$\mathcal{M}^s_{\mu,\kappa} \coloneqq (\mathcal{B}^s_{\mu,\kappa})^{-1}(0)/\mathscr{G}^s_{\mu,\kappa}\,. \tag{3.3.26}$$

Thanks to the above setup this is well defined as a set – although a priori depending on the regularity parameter s – but our aim in the next chapter is to prove that this is in fact a smooth hyper-Kähler manifold and to compute its dimension. In order to do this, we will follow the outline described in Section 3.1 applying the analytical results in Appendices B and C within this setup. As pointed out, we will furthermore prove that this structure is independent of s.

Chapter 4

Moduli space construction

We are now ready to carry out the construction of the framed moduli space. We start by studying the linearised problem in Section 4.1, which will provide a local model for the moduli space. In fact, this study will only be valid for monopoles with a certain regularity, but we see that any monopole in our configuration space is gauge equivalent to one with such regularity in Section 4.2, where we also obtain some asymptotic properties for our framed monopoles.

In Section 4.3 we complete the proof of the smoothness of the moduli spaces, and we finish by explaining how our construction can be viewed as an infinite-dimensional hyper-Kähler quotient, yielding a hyper-Kähler metric, in Section 4.4.

This chapter draws from the analytic results and function spaces laid out in Appendices B and C.

4.1 The linearised problem

Using the setup established in the previous chapter, we can now study the linearised operator $\not{D}_M$ for a given monopole $M = (A, \Phi) \in \mathscr{C}^s_{\mu,\kappa}$.

As we saw, $\not{D}_M - \not{D}_{M_{\mu,\kappa}}$ is an algebraic term proportional to $M - M_{\mu,\kappa} \in \mathscr{H}^{s,1}$. If we furthermore require this term to be bounded polyhomogeneous, in the sense of Section B.1, we will be able to deduce very strong mapping properties for $\not{D}_M$.

The index will be computed using the results explained in Appendix B and relying on the root subbundle decomposition (2.3.15), as well as on the decomposition into the C and $C^\perp$ parts (3.3.2). Along the former part, the linearised operator will behave like a weighted elliptic b operator, while along the latter it will behave like a fully elliptic scattering operator, which means that we will be able to apply properties similar to those described in Section B.3 and Section B.4, respectively. Of course, it will also have off-diagonal terms, which are taken into account when unifying both approaches in Section B.5.

Now, to compute the index of the b part of the operator we will need to rely on both relative index formulas explained in Theorem B.3.6. However, the b part of $\not{D}_M$ is not, in general, self-adjoint. This means that we will need to consider a family of operators connecting $\not{D}_M$ to one which is self-adjoint in the appropriate sense. This will be achieved by considering a continuous family of modifications to the Higgs field.

Definition 4.1.1. Let $(A, \Phi) = (A_{\mu,\kappa}, \Phi_{\mu,\kappa}) + (a, \varphi)$. For $t \in \mathbb{R}$, define

$$\Phi^{(t)} = \underline{\mu} - \frac{t}{2r}\underline{\kappa} + t\varphi \,. \tag{4.1.2}$$

Furthermore, we define the operators

$$D^{(t)} := \not{D}_{(A, -\frac{t}{2r}\underline{\kappa} + t\varphi)} \,, \tag{4.1.3a}$$

$$\Psi := -\operatorname{ad}_{\underline{\mu}} \,, \tag{4.1.3b}$$

which make up the operator

$$\not{D}_{(A,\Phi^{(t)})} = D^{(t)} + \Psi \,, \tag{4.1.4}$$

Remark 4.1.5. Recall that, near infinity, we have $\Phi_{\mu,\kappa} = \underline{\mu} - \frac{1}{2r}\underline{\kappa}$, but these constant sections are not well defined at the origin, where the term $\frac{1}{r}$ is also not defined. However, we can interpret both summands as having been smoothed out near the origin as in the construction of $\Phi_{\mu,\kappa}$ itself, since their behaviour will only be important near infinity.

The most relevant cases are $t = 1$, which represents the operator $\not{D}_{(A,\Phi)}$ we want to study, and $t = 0$, which represents an operator whose b part satisfies the necessary self-adjointness property. However, all the operators in between will also be Fredholm for the right spaces.

The analytical framework we want to apply is the one described in Section B.5. Therefore, near infinity, we write

$$D^{(t)} = \begin{pmatrix} D^{(t)}_{00} & D^{(t)}_{01} \\ D^{(t)}_{10} & D^{(t)}_{11} \end{pmatrix} \tag{4.1.6}$$

along the decomposition

$$(\textstyle\bigwedge^1 \oplus \bigwedge^0) \otimes \mathrm{Ad}(P)^{\mathbb{C}} = ((\bigwedge^1 \oplus \bigwedge^0) \otimes \mathrm{Ad}(P)^{\mathbb{C}}_C) \oplus ((\bigwedge^1 \oplus \bigwedge^0) \otimes \mathrm{Ad}(P)^{\mathbb{C}}_{C^\perp})\,, \tag{4.1.7}$$

as well as $\widetilde{D}^{(t)}_{00} = x^{-2}\, D^{(t)}_{00}\, x$. Furthermore, we consider the space $\mathrm{spec}_b(\widetilde{D}^{(t)}_{00})$, along with the definition of order for its elements, and the operators $I(\widetilde{D}^{(t)}_{00}, \lambda)$ and $\eth^+_+$, as defined in Sections B.3 and B.4. Note that x is now used for the boundary defining function of the radial compactification $\overline{\mathbb{R}^3}$, which is equal to $\frac{1}{r}$ near infinity, as described in Section C.1.

We can now formulate the main Fredholmness and index result for our setting.

Lemma 4.1.8. *Let $(A, \Phi) \in \mathscr{C}^s_{\mu,\kappa}$ be bounded polyhomogeneous*[4] *and $t \in \mathbb{R}$, and assume that the elements in $\mathrm{spec}_b(\widetilde{D}^{(t)}_{00})$ are real and of order 1. Then, for any $\delta \in \mathbb{R} \setminus \mathrm{spec}_b(\widetilde{D}^{(t)}_{00})$ the operator*

$$\not{D}_{(A,\Phi^{(t)})} \colon (\mathcal{H}^{\delta-\frac{1}{2},\delta+\frac{1}{2},s,1})^{\mathbb{C}} \to (\mathcal{H}^{\delta+\frac{1}{2},\delta+\frac{1}{2},s,0})^{\mathbb{C}} \tag{4.1.9}$$

is Fredholm.

Furthermore, its index is given by

$$\mathrm{ind}(\not{D}_{(A,\Phi^{(t)})}, \delta) = \mathrm{ind}(\eth^+_+) + \mathrm{def}(\not{D}_{(A,\Phi^{(t)})}, \delta)\,. \tag{4.1.10}$$

Here, the defect $\mathrm{def}(\not{D}_{(A,\Phi^{(t)})}, \delta)$ remains constant if t or δ is varied continuously (as long as the condition $\delta \notin \mathrm{spec}_b(\widetilde{D}^{(t)}_{00})$ is preserved throughout the variation). Furthermore, if $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \cap \mathrm{spec}_b(\widetilde{D}^{(t)}_{00}) = \{\lambda_0\}$, then

$$\mathrm{def}(\not{D}_{(A,\Phi^{(t)})}, \lambda_0 - \varepsilon) = \mathrm{def}(\not{D}_{(A,\Phi^{(t)})}, \lambda_0 + \varepsilon) + \dim \mathrm{Null}(I(\widetilde{D}^{(t)}_{00}, \lambda_0))\,. \tag{4.1.11}$$

[4]By this we mean that $(A, \Phi) - (A_{\mu,\kappa}, \Phi_{\mu,\kappa})$ is bounded polyhomogeneous.

Additionally,

$$\operatorname{def}(\not{\mathbb{D}}_{(A,\Phi^{(0)})}, \delta) = \operatorname{def}(\not{\mathbb{D}}_{(A,\Phi^{(0)})}, -\delta)\,. \tag{4.1.12}$$

Lastly, the elements in the kernel of this operator are in $(\mathscr{B}^{\lambda_1+1,\lambda_1+2})^{\mathbb{C}}$*, where* λ_1 *is the smallest indicial root in* $\operatorname{spec}_b(\widetilde{D}^{(t)}_{00})$ *larger than* δ*, provided that* $\lambda_1 \geq 0$*.*

Proof. This is a consequence of Theorem B.5.7, where we are taking $E = E_0 \oplus E_1$ to be (4.1.7) with the connection $A_{\mu,\kappa}$.

Then, $D^{(t)}$ is a Dirac operator twisted by this connection plus an algebraic term. This term consists of $\operatorname{ad}_{\frac{t}{2r}\underline{\kappa}}$ plus a part proportional to the bounded polyhomogeneous pair (a, φ), which is in $\mathscr{H}^{s,1}$ and hence must be of order $x^{\frac{3}{2}}$.

The endomorphism is simply $-\operatorname{ad}_{\underline{\mu}}$, and we can check that the remaining conditions are satisfied – considering that the condition that the elements of the b spectrum be real and of order one is incorporated into the statement as an assumption.

Lastly, to justify (4.1.12) we only need to observe that for $t = 0$ the operator $D^{(t)}$ is self-adjoint, since it is simply a Dirac operator. □

This allows us to prove that the linearised operator is Fredholm between the appropriate spaces and to compute its index. We start by computing the contribution to the index from the scattering part of the operator, which is independent of the weight.

Lemma 4.1.13. *The term* $\operatorname{ind}(\not{\partial}^+_+)$ *in* (4.1.10) *in Lemma 4.1.8 is given by*

$$\operatorname{ind}(\not{\partial}^+_+) = 2 \sum_{\substack{\alpha \in R \\ i\alpha(\mu) > 0}} i\alpha(\kappa)\,. \tag{4.1.14}$$

Proof. The operator $\not{\partial}^+_+$ is the operator induced at infinity by the scattering part of the linearised operator, $D^{(t)}_{11}$, which is simply the Dirac operator associated to the connection A restricted to the bundles $\not{S} \otimes \underset{\sim}{\mathfrak{g}_\alpha}$ for $\alpha(\mu) \neq 0$ – with two copies for each such root. Furthermore, we must restrict ourselves to the positive imaginary eigenspaces of $\Psi = -\operatorname{ad}_{\underline{\mu}}$, which leaves the bundles for which $i\alpha(\mu) > 0$.

Since the operator decomposes at infinity, we can treat each of these subbundles independently, where the induced operator $\eth^+_+$ simply becomes the Dirac operator $\not{D}^+_{i\alpha(\kappa)}$ on each corresponding line bundle over the sphere at infinity. As seen in Proposition A.4.4, the index of this operator is $i\alpha(\kappa)$, so putting all of them together we get precisely (4.1.14). $\square$

The b part is somewhat more involved. Recall that we are interested in the linearised operator, which arises when $t = 1$. Furthermore, we will be interested in the spaces which result from taking $\delta = \frac{1}{2}$, but in fact the operator will be Fredholm for a certain interval of weights around this value.

Lemma 4.1.15. *If $(A, \Phi) \in \mathscr{C}^s_{\mu,\kappa}$ is bounded polyhomogeneous, all the elements in $\operatorname{spec}_b(\widetilde{D}^{(t)}_{00})$ are real and of order* 1, *and*

$$\left(-\frac{1}{2}, 1\right) \cap \operatorname{spec}_b(\widetilde{D}^{(1)}_{00}) = \varnothing\,. \tag{4.1.16}$$

Furthermore, if $\delta \in (-\frac{1}{2}, 1)$, then

$$\operatorname{def}(\not{D}_{(A,\Phi^{(1)})}, \delta) = -2 \sum_{\substack{\alpha\in R\\ \alpha(\mu)=0\\ i\alpha(\kappa)\geq 0}} i\alpha(\kappa)\,. \tag{4.1.17}$$

Proof. We actually compute the b spectrum $\operatorname{spec}_b(\widetilde{D}^{(t)}_{00})$ for any $t \in \mathbb{R}$, and, once more, we look at each root subbundle individually. In this case, we must consider $\not{S} \otimes \underset{\sim}{\mathfrak{g}_\alpha}$ for $\alpha(\mu) = 0$ – again considering two copies for each such root α, and additionally $2\operatorname{rank}(G)$ copies for $\alpha = 0$ to account for $\underline{\mathfrak{t}^{\mathbb{C}}}$.

Now we consider, for each subbundle, the decomposition

$$\not{S} \otimes \underset{\sim}{\mathfrak{g}_\alpha} = (\not{S}^+ \otimes \underset{\sim}{\mathfrak{g}_\alpha}) \oplus (\not{S}^- \otimes \underset{\sim}{\mathfrak{g}_\alpha})\,, \tag{4.1.18}$$

as described in Section A.4. With respect to this decomposition, the operator can be written as

$$\widetilde{D}^{(t)}_{00}|_{\underset{\sim}{\mathfrak{g}_\alpha}} = \begin{pmatrix} -i\Big(x\dfrac{\partial}{\partial x} + \dfrac{it\alpha(\kappa)}{2}\Big) & \not{D}^-_{i\alpha(\kappa)} \\ \not{D}^+_{i\alpha(\kappa)} & i\Big(x\dfrac{\partial}{\partial x} - \dfrac{it\alpha(\kappa)}{2}\Big) \end{pmatrix} + \psi\,, \tag{4.1.19}$$

where the first summand combines (A.4.13) with the action of the charge component of the Higgs field and ψ is an algebraic term proportional to $tx^{-1}(a,\varphi)$.

Given that the lower order term ψ vanishes at infinity, we obtain the operators

$$I(\widetilde{D}^{(t)}_{00}|_{\underset{\sim}{\mathfrak{g}}_\alpha},\lambda) = \begin{pmatrix} -i\Big(\lambda + \dfrac{it\alpha(\kappa)}{2}\Big) & \not{D}^{-}_{i\alpha(\kappa)} \\ \not{D}^{+}_{i\alpha(\kappa)} & i\Big(\lambda - \dfrac{it\alpha(\kappa)}{2}\Big) \end{pmatrix} \tag{4.1.20}$$

on each bundle

$$(\not{S}^{+} \otimes \mathscr{L}^{i\alpha(\kappa)}) \oplus (\not{S}^{-} \otimes \mathscr{L}^{i\alpha(\kappa)}) \tag{4.1.21}$$

over the sphere at infinity.

We must therefore determine the values of λ for which this operator has a kernel. To do so, let us suppose that $u^{+} \in \Gamma(\not{S}^{+} \otimes \mathscr{L}^{i\alpha(\kappa)})$ and $u^{-} \in \Gamma(\not{S}^{-} \otimes \mathscr{L}^{i\alpha(\kappa)})$ are such that

$$\not{D}^{-}_{i\alpha(\kappa)}\, u^{-} = i\Big(\lambda + \frac{it\alpha(\kappa)}{2}\Big)u^{+}\,, \tag{4.1.22a}$$

$$\not{D}^{+}_{i\alpha(\kappa)}\, u^{+} = -i\Big(\lambda - \frac{it\alpha(\kappa)}{2}\Big)u^{-}\,, \tag{4.1.22b}$$

that is, (u^{+}, u^{-}) is in this kernel.

Applying $\not{D}^{-}_{i\alpha(\kappa)}$ to (4.1.22b) and using (4.1.22a) to substitute we deduce that

$$\not{D}^{-}_{i\alpha(\kappa)}\, \not{D}^{+}_{i\alpha(\kappa)}\, u^{+} = \Big(\lambda^2 - \Big(\frac{it\alpha(\kappa)}{2}\Big)^2\Big)u^{+}\,. \tag{4.1.23}$$

If u^{+} is not identically 0, from Proposition A.4.4, we have

$$\Big(\lambda^2 - \Big(\frac{it\alpha(\kappa)}{2}\Big)^2\Big) = j(j + |i\alpha(\kappa)|) \tag{4.1.24}$$

for some $j \in \mathbb{Z}_{\geq 1}$, or $j = 0$ if $i\alpha(\kappa) > 0$. For $j > 0$, λ will take real values outside of the interval $(-1,1)$, which leaves the case $j = 0$, and hence $\lambda = \pm\frac{it\alpha(\kappa)}{2}$, for $i\alpha(\kappa) > 0$.

In an analogous way we can deduce that, if u^{-} is not identically zero, then, excluding the cases $\lambda \notin (-1,1)$, we must have $\lambda = \pm\frac{it\alpha(\kappa)}{2}$ and $i\alpha(\kappa) < 0$.

Although not relevant for the current computation, we note that for $t = 1$ the indicial roots outside $(-1, 1)$ must be of the form $\lambda = \pm(j + \frac{|i\alpha(\kappa)|}{2})$ for $j \in \mathbb{Z}_{\geq 1}$.

Let us now investigate the case of $\lambda = \pm\frac{it\alpha(\kappa)}{2}$ more carefully, using once more the results from Appendix A.

Let us first assume that $\lambda = \frac{it\alpha(\kappa)}{2}$. Then, the equations (4.1.22) become

$$\not{D}^{-}_{i\alpha(\kappa)}\, u^{-} = -t\alpha(\kappa)u^{+}\,, \tag{4.1.25a}$$

$$\not{D}^{+}_{i\alpha(\kappa)}\, u^{+} = 0\,. \tag{4.1.25b}$$

We once again make use of Proposition A.4.4. If $i\alpha(\kappa) < 0$, the second equation implies that $u^{+} = 0$. The first equation then has a space of solutions of dimension $-i\alpha(\kappa)$. On the other hand, if $i\alpha(\kappa) \geq 0$, then $u^{-} = 0$, as we saw above. The first equation then implies that $u^{+} = 0$, unless $t = 0$, in which case we have a space of solutions of dimension $i\alpha(\kappa)$ from the second equation.

If we now assume that $\lambda = -\frac{it\alpha(\kappa)}{2}$, the equations (4.1.22) become

$$\not{D}^{-}_{i\alpha(\kappa)}\, u^{-} = 0\,, \tag{4.1.26a}$$

$$\not{D}^{+}_{i\alpha(\kappa)}\, u^{+} = -t\alpha(\kappa)u^{-}\,. \tag{4.1.26b}$$

Analogously to above we deduce that if $i\alpha(\kappa) > 0$ we have a space of solutions of dimension $i\alpha(\kappa)$, and that otherwise can only have non-trivial solutions in a space of dimension $-i\alpha(\kappa)$ when $t = 0$.

Putting both cases together, we deduce that if $\lambda \in (-1, 1)$, we only have non-trivial solutions, in a space of dimension $|i\alpha(\kappa)|$, when $\lambda = -\frac{t|i\alpha(\kappa)|}{2}$. In other words, the only indicial root in $\operatorname{spec}_b(\widetilde{D}^{(t)}_{00}|_{\underset{\sim}{\mathfrak{g}_\alpha}}) \cap (-1, 1)$ can be $-\frac{t|i\alpha(\kappa)|}{2}$, and with a nullspace of dimension $|i\alpha(\kappa)|$. Note that when $\alpha(\kappa) = 0$ this is not really an indicial root, but treating it as one doesn't change our results, since the nullspace is 0-dimensional.

By computing the formal nullspaces of the indicial operator we can additionally see that all the indicial roots have order 1.

We can now deduce from (4.1.11) that

$$\operatorname{def}(\widetilde{D}^{(0)}_{00}|_{\underset{\sim}{\mathfrak{g}_\alpha}}, -\varepsilon) = \operatorname{def}(\widetilde{D}^{(0)}_{00}|_{\underset{\sim}{\mathfrak{g}_\alpha}}, \varepsilon) + |i\alpha(\kappa)| \tag{4.1.27}$$

and from (4.1.12) that

$$\operatorname{def}(\widetilde{D}_{00}^{(0)}|_{\underset{\sim}{\mathfrak{g}_\alpha}}, \varepsilon) = -\operatorname{def}(\widetilde{D}_{00}^{(0)}|_{\underset{\sim}{\mathfrak{g}_\alpha}}, -\varepsilon)\,, \tag{4.1.28}$$

so we have

$$\operatorname{def}(\widetilde{D}_{00}^{(0)}|_{\underset{\sim}{\mathfrak{g}_\alpha}}, \varepsilon) = -\frac{|i\alpha(\kappa)|}{2}\,. \tag{4.1.29}$$

Furthermore,

$$(0,1) \cap \operatorname{spec}_b(\widetilde{D}_{00}^{(t)}|_{\underset{\sim}{\mathfrak{g}_\alpha}}) = \varnothing\,, \tag{4.1.30}$$

so

$$\operatorname{def}(\widetilde{D}_{00}^{(t)}|_{\underset{\sim}{\mathfrak{g}_\alpha}}, \varepsilon) = -\frac{|i\alpha(\kappa)|}{2} \tag{4.1.31}$$

for any $t \geq 0$. Since

$$\left(-\frac{1}{2}, 1\right) \cap \operatorname{spec}_b(\widetilde{D}_{00}^{(1)}|_{\underset{\sim}{\mathfrak{g}_\alpha}}) = \varnothing\,, \tag{4.1.32}$$

we deduce that

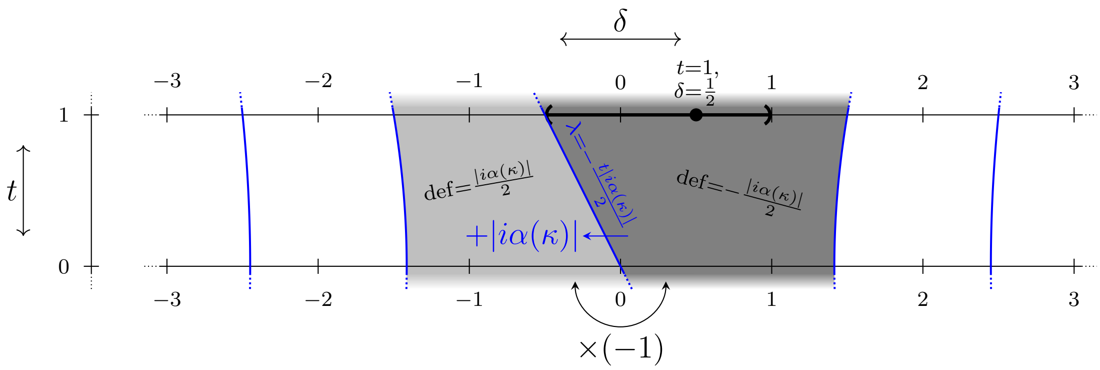
$$\operatorname{def}(\widetilde{D}_{00}^{(1)}|_{\underset{\sim}{\mathfrak{g}_\alpha}}, \delta) = -\frac{|i\alpha(\kappa)|}{2} \tag{4.1.33}$$

for any $\delta \in (-\frac{1}{2}, 1)$.

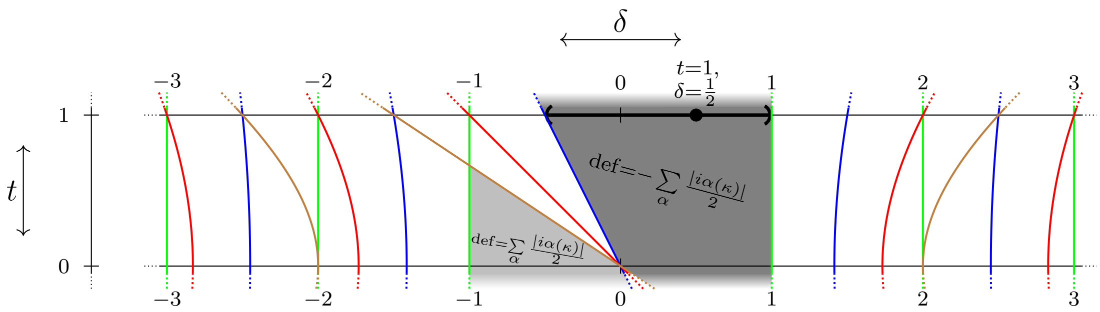
Putting this together for all the relevant subbundles we obtain the formula (4.1.17).

The indicial roots and the computation of the defect are represented in Figure 4.1.34. □

Putting these lemmas together we can deduce the desired properties for the linearised operator. Since the index will provide the dimension of the moduli



a: Here we see the lines depicting the indicial roots for a specific root subbundle $\underset{\sim}{\mathfrak{g}}_\alpha$, in this case drawn for $|i\alpha(\kappa)| = 1$. As discussed above, the relative index formula (4.1.11) implies that crossing the smallest indicial root $\lambda = -\frac{t|i\alpha(\kappa)|}{2}$ from right to left adds $|i\alpha(\kappa)|$ to the defect, and (4.1.12) implies that for $t = 0$ the defect changes sign when reflected around the origin. This implies that in the dark grey region the defect must be $-\frac{|i\alpha(\kappa)|}{2}$.



b: In this diagram we include indicial roots corresponding to several line bundles, in particular those for which $|i\alpha(\kappa)|$ is equal to 0 (where there is no indicial root in $(-1, 1)$), 1, 2 and 3. We then add all the contributions to the defect in the dark grey region, which will always include the interval $\delta \in (-\frac{1}{2}, 1)$ for $t = 1$.

Figure 4.1.34: We represent the indicial roots of the operator $\widetilde{D}_{00}^{(t)}$ as lines as the parameter t varies. Outside of these lines the operator is Fredholm, and its defect is constant on each of the resulting regions. We then compute the defect of the relevant region by using the relative index formula across the smallest indicial roots and the self-adjointness of the operator for $t = 0$. Note that all the indicial roots in the range are depicted, but only the smallest one for each subbundle is relevant. We then obtain our result for the **relevant interval** in $t = 1$, and particularly for $\delta = \frac{1}{2}$, which will represent our actual linearised operator.

space, let us write

$$\dim_{\mu,\kappa} := 2 \sum_{\substack{\alpha\in R \\ i\alpha(\mu)>0}} i\alpha(\kappa) - 2 \sum_{\substack{\alpha\in R \\ \alpha(\mu)=0 \\ i\alpha(\kappa)\geq 0}} i\alpha(\kappa)\,, \tag{4.1.35}$$

which is simply the sum of (4.1.14) and (4.1.17).

Proposition 4.1.36. *Let* $(A,\Phi)\in\mathscr{C}^s_{\mu,\kappa}$ *be bounded polyhomogeneous. Then the operator*

$$\not{D}_{(A,\Phi)}\colon \mathscr{H}^{s,1}\to\mathscr{H}^{s,0} \tag{4.1.37}$$

is Fredholm, and its index is given by

$$\operatorname{ind}\not{D}_{(A,\Phi)} = \dim_{\mu,\kappa}\,. \tag{4.1.38}$$

Proof. Once complexified, the operator (4.1.37) is simply the operator (4.1.9) for $t=1$ and $\delta=\frac{1}{2}$, so we can apply Lemma 4.1.8. For these values of t and δ, Lemma 4.1.15 implies that the operator is Fredholm and provides the defect. Putting it together with the index of the scattering part computed in Lemma 4.1.13 completes the proof. □

Another important property of the linearised operator is its surjectivity, since it will also be necessary to apply the implicit function theorem. Together with the above results, this provides a significantly detailed picture of its kernel.

Theorem 4.1.39. *If* $(A,\Phi)\in\mathscr{C}^s_{\mu,\kappa}$ *is bounded polyhomogeneous and satisfies the Bogomolny equations, the linearised operator* (4.1.37) *is surjective. Hence, its kernel has dimension* $\dim_{\mu,\kappa}$*. Furthermore, the elements in this kernel are polyhomogeneous sections in* $\mathscr{B}^{2,3}$*.*

Proof. In order to prove that $\not{D}_{(A,\Phi)}$ is surjective we will prove that its formal L^2-adjoint $\not{D}_{(A,-\Phi)}$ is injective between the L^2-dual spaces, that is, as an operator

$$\not{D}_{(A,-\Phi)}\colon (\mathscr{H}^{s,0})^*\to(\mathscr{H}^{s,1})^*\,. \tag{4.1.40}$$

Similarly to the operator $\not{D}_{(A,\Phi)}$ we can deduce from Kottke's parametrix construction [Kot15a] that if $u \in (\mathscr{H}^{s,0})^*$ is in the kernel of $\not{D}_{(A,-\Phi)}$, then it must be polyhomogeneous of order $x^{1+\lambda_1}$, where λ_1 is the smallest indicial root larger than a certain weight, which we can compute to be $-\frac{1}{2}$ by comparing it to the weight of $\mathscr{H}^{s,0}$ (since we are following a notation analogous to the one in Lemma 4.1.8, where our spaces correspond to $\delta = \frac{1}{2}$). Since the indicial roots of $\not{D}_{(A,-\Phi)}$ are the opposite of the ones for $\not{D}_{(A,\Phi)}$, we deduce that $\lambda_1 \geq \frac{1}{2}$, so u must be polyhomogeneous of order $x^{\frac{3}{2}}$.

Now, for such u we can apply Lemma 3.2.17 to obtain

$$0 = \not{D}_{(A,\Phi)} \not{D}_{(A,-\Phi)} u = \nabla^* \nabla u - \mathrm{ad}_\Phi^2 u \,. \tag{4.1.41}$$

From the polyhomogeneity of u we can deduce that ∇u and $\nabla^* \nabla u$ are also bounded polyhomogeneous, of orders $x^{\frac{5}{2}}$ and $x^{\frac{7}{2}}$ respectively. Hence, $\mathrm{ad}_\Phi^2 u$ is also polyhomogeneous of order $x^{\frac{7}{2}}$.

We can then write

$$0 = \langle \not{D}_{(A,\Phi)} \not{D}_{(A,-\Phi)} u, u \rangle_{L^2} = \langle \nabla^* \nabla u, u \rangle_{L^2} - \langle \mathrm{ad}_\Phi^2 u, u \rangle_{L^2} = \|\nabla u\|_{L^2}^2 + \|\mathrm{ad}_\Phi\|_{L^2}^2 \,, \tag{4.1.42}$$

we were have used the decay conditions to apply integration by parts in the first summand and the pointwise anti-symmetry of ad_Φ on the second. This implies that $\|\nabla u\|_{L^2} = 0$, so u must be identically 0.

Therefore, $\not{D}_{(A,-\Phi)}$ is injective and hence $\not{D}_{(A,\Phi)}$ is surjective, as desired.

The dimension of the kernel follows from Proposition 4.1.36. The polyhomogeneity of its elements follows from the final statement in Lemma 4.1.8, since the smallest element in $\mathrm{spec}_b(\widetilde{D}_{00}^{(1)})$ larger than $\frac{1}{2}$ must be at least equal to 1, as a consequence of (4.1.16). □

4.2 Regularity

In the previous section we have seen that, for a given monopole $(A, \Phi) \in \mathscr{C}^s_{\mu,\kappa}$, the linearised operator $\not{D}_{(A,\Phi)}$ satisfies the properties needed to apply the implicit function theorem, but we had to add the regularity condition that (A, Φ) be bounded polyhomogeneous. As it turns out, this condition will not impose a significant restriction, since any monopole will be gauge equivalent to another with such regularity. This is essentially a consequence of the ellipticity of the Bogomolny equations together with the Coulomb gauge fixing condition. Let us recall this condition from Section 3.1 and see some of its properties.

Definition 4.2.1. Let $M_0 \in \mathscr{C}^s_{\mu,\kappa}$. We say that another pair $M \in \mathscr{C}^s_{\mu,\kappa}$ is *in Coulomb gauge with respect to* M_0 if

$$\mathrm{d}^*_{M_0}(M - M_0) = 0\,. \tag{4.2.2}$$

Lemma 4.2.3. *The Coulomb gauge fixing condition is symmetric and gauge-invariant.*

Proof. Let $M_0 = (A_0, \Phi_0)$, $M = (A, \Phi)$ and $(a, \varphi) = (A, \Phi) - (A_0, \Phi_0)$. Then,

$$\begin{aligned}
\mathrm{d}^*_{M_0}(M - M_0) + \mathrm{d}^*_M(M_0 - M) &= \mathrm{d}^*_{A_0} a - \mathrm{ad}_{\Phi_0}\varphi + \mathrm{d}^*_A(-a) - \mathrm{ad}_\Phi(-\varphi) \\
&= -\star(\mathrm{d}_{A_0} - \mathrm{d}_A)\star a - (\mathrm{ad}_{\Phi_0} - \mathrm{ad}_\Phi)\varphi \\
&= \star\,\mathrm{ad}_a \star a + \mathrm{ad}_\varphi\,\varphi \\
&= \star[a \wedge \star a] + [\varphi, \varphi] \\
&= 0\,,
\end{aligned} \tag{4.2.4}$$

proving that the condition is symmetric.

The gauge invariance is a consequence of the gauge invariance of d^* (for connections) and ad. □

We start with an important property of this gauge fixing condition: that it selects locally unique representatives.

Proposition 4.2.5. *Let $M_0 \in \mathscr{C}^s_{\mu,\kappa}$. Then, if U is a sufficiently small neighbourhood of M_0 inside $\mathscr{C}^s_{\mu,\kappa}$, there exists another neighbourhood U' of M_0 inside $\mathscr{C}^s_{\mu,\kappa}$ with the following property: for any $M \in U'$ there exists a gauge transformation $g \in \mathscr{G}^s_{\mu,\kappa}$ such that $g \cdot M$ is inside U and in Coulomb gauge with respect to M_0, and this gauge transformation is unique if required to be sufficiently small.*

Proof. Consider the map

$$\begin{aligned} f\colon \mathscr{C}^s_{\mu,\kappa} \times \mathscr{G}^s_{\mu,\kappa} &\to \mathscr{H}^{s,0}_0 \\ (M,g) &\mapsto \mathrm{d}^*_{M_0}(g\cdot M - M_0)\,. \end{aligned} \tag{4.2.6}$$

We have that $f(M_0, \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}) = 0$, and its derivative with respect to the second argument is

$$(\mathrm{d}f)_{(M_0,\mathbf{1}_{\mathscr{G}_{\mu,\kappa}})}(0,\bullet) = -\mathrm{d}^*_{M_0}\mathrm{d}_{M_0}\colon \mathscr{H}^{s,2}_0 \to \mathscr{H}^{s,0}_0\,. \tag{4.2.7}$$

Then, if this an isomorphism, we can apply the implicit function theorem to obtain the result.

Hence, we must prove that $\mathrm{d}^*_{M_0}\mathrm{d}_{M_0}$ an isomorphism for any pair $M_0 \in \mathscr{C}^s_{\mu,\kappa}$, which will be achieved by showing that it is Fredholm, has index 0, and is injective.

To see that it is Fredholm of index 0 we start by considering the problem for a pair M_c which splits along the decomposition of the adjoint bundle into C and $C^\perp$ parts. In general, $\mathrm{Ad}(P)_C$ and $\mathrm{Ad}(P)_{C^\perp}$ can not be extended over $\mathbb{R}^3$. However, their complexifications are trivial, so they *can* be extended. Then, we can take a unitary connection A_c which differs from $A_{\mu,\kappa}$ smoothly in a compact set, and a Higgs field Φ_c which is simply $\Phi_{\mu,\kappa}$ cut off smoothly to be 0 in a compact set. Note that this extension of $\mathrm{Ad}(P)^{\mathbb{C}}_C$ and $\mathrm{Ad}(P)^{\mathbb{C}}_{C^\perp}$ does not necessarily preserve the same properties with respect to the adjoint action in the region where it is extended near the origin (for example, it might no longer be true that the Lie bracket is internal in the extension of $\mathrm{Ad}(P)^{\mathbb{C}}_C$), but this will not be important, since Φ_c can be taken to be 0 in that region.

Now, by construction, the operator $\mathrm{d}^*_{M_c}\mathrm{d}_{M_c}$ decomposes along the two subbundles over the entire $\mathbb{R}^3$. Since near infinity the pair M_c is identical to $M_{\mu,\kappa}$, we can see that the operator decomposes completely as a weighted elliptic b operator on the C part and a fully elliptic scattering operator on the $C^\perp$ part, and that it is furthermore formally self-adjoint. The fully elliptic scattering part is therefore Fredholm of order 0.

To see that the same applies to the weighted elliptic b part we carry out a similar procedure as the one applied to the linearised operator: By taking the different weights into consideration once more, we conclude that the b part of the operator is Fredholm between the spaces

$$x^{\delta-1}H_b^2(\mathrm{Ad}(P)_C^{\mathbb{C}}) \to x^{\delta+1}H_b^2(\mathrm{Ad}(P)_C^{\mathbb{C}}) \tag{4.2.8}$$

when δ is not an indicial root of the elliptic b operator $x^{-\frac{5}{2}}\,\mathrm{d}_{M_c}^*\mathrm{d}_{M_c}\,x^{\frac{1}{2}}$. We can compute its spectrum from the spectrum of the Laplacian on line bundles on the unit sphere, and check that 0 is not an indicial root – all the indicial roots must in fact be half-integers with absolute value greater than or equal to $\frac{1}{2}$. Given the self-adjointness, this implies that the b part of $\mathrm{d}_{M_c}^*\mathrm{d}_{M_c}$ is Fredholm of index 0 between $(\mathscr{H}_0^{s,2})^{\mathbb{C}}$ and $(\mathscr{H}_0^{s,0})^{\mathbb{C}}$.

But, since $M_{\mu,\kappa} - M_c$ is smooth and compactly supported, $\mathrm{d}_{M_{\mu,\kappa}}^*\mathrm{d}_{M_{\mu,\kappa}} - \mathrm{d}_{M_c}^*\mathrm{d}_{M_c}$ is a compact operator, so $\mathrm{d}_{M_{\mu,\kappa}}^*\mathrm{d}_{M_{\mu,\kappa}}$ is also Fredholm of index 0 between $(\mathscr{H}_0^{s,2})^{\mathbb{C}}$ and $(\mathscr{H}_0^{s,0})^{\mathbb{C}}$, and equivalently between their real parts.

For a general pair $M_0 = M_{\mu,\kappa} + (a_0, \varphi_0)$, we have

$$\begin{aligned}&(\mathrm{d}_{M_0}^*\mathrm{d}_{M_0} - \mathrm{d}_{M_{\mu,\kappa}}^*\mathrm{d}_{M_{\mu,\kappa}})X \\ &\quad = -\star[a_0 \wedge \star\mathrm{d}_{A_{\mu,\kappa}}X] + \mathrm{d}_{A_0}^*[a_0 \wedge X] - \mathrm{ad}_{\varphi_0}\mathrm{ad}_{\Phi_{\mu,\kappa}}X - \mathrm{ad}_{\Phi_0}\mathrm{ad}_{\varphi_0}X.\end{aligned} \tag{4.2.9}$$

From Lemmas 3.3.8 and C.2.11 we can therefore see that $\mathrm{d}_{M_0}^*\mathrm{d}_{M_0} - \mathrm{d}_{M_{\mu,\kappa}}^*\mathrm{d}_{M_{\mu,\kappa}}$ is also a compact operator from $\mathscr{H}_0^{s,2}$ to $\mathscr{H}_0^{s,0}$, and hence $\mathrm{d}_{M_0}^*\mathrm{d}_{M_0}$ must be Fredholm and have index 0 as well.

Lastly, the operator $\mathrm{d}_{M_0}^*\mathrm{d}_{M_0}$ is injective. To see this, suppose that $u \in \mathscr{H}_0^{s,2}$ satisfies $\mathrm{d}_{M_0}^*\mathrm{d}_{M_0}u = 0$. Then, from Lemma 3.3.13 and the properties of the adjoint action we deduce that

$$0 = \langle \mathrm{d}_{M_0}^*\mathrm{d}_{M_0}u, u\rangle_{L^2} = \langle \mathrm{d}_{M_0}u, \mathrm{d}_{M_0}u\rangle_{L^2}\,, \tag{4.2.10}$$

which implies that $\mathrm{d}_{M_0}u = 0$. This means that the covariant derivative of u must be 0, which implies that u is identically 0 due to the decay conditions on $\mathscr{H}_0^{s,2}$. □

We are particularly interested in having arbitrary monopoles in Coulomb gauge with respect to pairs which are very regular.

Corollary 4.2.11. *Let $M \in \mathscr{C}_{\mu,\kappa}^s$. Then, there exists a configuration pair*

$M_c \in \mathscr{C}^s_{\mu,\kappa}$ such that $M_c - M_{\mu,\kappa}$ is smooth and compactly supported, and a gauge transformation $g \in \mathscr{G}^s_{\mu,\kappa}$, which satisfy that $g \cdot M$ is in Coulomb gauge with respect to M_c.

Proof. From Lemma 4.2.3, we know that the Coulomb gauge fixing condition is symmetric and gauge invariant, so our statement is equivalent to the existence of M_c and g which satisfy that $g^{-1} \cdot M_c$ is in Coulomb gauge with respect to M.

Now, from Lemma B.2.16 we can find such a pair M_c arbitrarily close to M. Then, from Proposition 4.2.5 we can find a gauge transformation satisfying the desired condition. □

We can now combine the gauge fixing condition with the Bogomolny equations. The regularity of the above M_c provides smooth coefficients for the differential operator, which are in fact completely known near infinity. This can be used to deduce strong regularity results through the usual "bootstrapping" techniques.

Theorem 4.2.12. *Let $M \in \mathscr{C}^s_{\mu,\kappa}$ be a monopole. Then there exists a gauge transformation $g \in \mathscr{G}^s_{\mu,\kappa}$ such that*

$$g \cdot M \in M_{\mu,\kappa} + \mathscr{B}^{2,(\kappa),\infty} . \tag{4.2.13}$$

Proof. We apply Corollary 4.2.11 to obtain a pair $M_c = (A_c, \Phi_c)$ which differs from $M_{\mu,\kappa}$ by a smooth compactly supported element and a gauge transformation which takes M to a monopole in Coulomb gauge with respect to M_c. For ease of notation, we simply write $M = (A, \Phi)$ for the transformed monopole, so we assume that it is in Coulomb gauge with respect to M_c, that is,

$$\mathrm{d}^*_{M_c}(M - M_c) = 0 , \tag{4.2.14}$$

and aim to prove that

$$M \in M_{\mu,\kappa} + \mathscr{B}^{2,(\kappa),\infty} . \tag{4.2.15}$$

We know that $\mathcal{B}(M_c)$ – the result of applying the Bogomolny map to M_c – is compactly supported, since M_c, like $M_{\mu,\kappa}$, satisfies the Bogomolny equations near infinity. It must furthermore be smooth, since M_c is. Additionally, since M is a monopole, we have $\mathcal{B}(M) = 0$. Therefore $\mathcal{B}(M) - \mathcal{B}(M_c)$ is also smooth and compactly supported. If we write $(a, \varphi) = (A, \Phi) - (A_c, \Phi_c)$, we have

$$\begin{aligned} \mathcal{B}(M) - \mathcal{B}(M_c) &= \star F_A - \mathrm{d}_A \Phi - \star F_{A_c} + \mathrm{d}_{A_c} \Phi_c \\ &= \star(F_{A_c+a} - F_{A_c}) - \mathrm{d}_{A_c+a}(\Phi_c + \varphi) + \mathrm{d}_{A_c}\Phi_c \\ &= \star(\mathrm{d}_{A_c} a + \frac{1}{2}[a \wedge a]) - \mathrm{d}_{A_c}\varphi + \mathrm{ad}_{\Phi_c} a - [a, \varphi] \,. \end{aligned} \tag{4.2.16}$$

Now, if $m = (a, \varphi)$, we write

$$\{m, m\} = \left(\frac{1}{2}\star[a \wedge a] - [a, \varphi], 0\right), \tag{4.2.17}$$

as well as

$$v = (\mathcal{B}(M) - \mathcal{B}(M_c), 0)\,, \tag{4.2.18}$$

which are pairs consisting of a 1-form and a 0-form (valued in $\mathrm{Ad}(P)$). Putting (4.2.16) together with (4.2.14) is equivalent to

$$\not{D}_{M_c} m + \{m, m\} = v\,. \tag{4.2.19}$$

We note that v is smooth and compactly supported, and that the bilinear product $\{\bullet, \bullet\}$ preserves the C part of the adjoint bundle, and hence satisfies the conditions of Lemma C.2.11.

The idea is to now use (4.2.19) to carry out a bootstrapping argument on the regularity of m, which we will do by induction with the induction hypothesis for any $j \in \mathbb{Z}_{\geq 0}$ that

$$m \in \mathcal{H}^{j\eta, 1+j\eta, s, 1+j} + \mathscr{B}^{2,(\kappa),\infty}\,, \tag{4.2.20}$$

where $0 < \eta < \frac{1}{4}$ is any fixed irrational number. We observe that due to the Sobolev embeddings – including embeddings into Hölder spaces – the above property being true for every j implies that $m \in \mathscr{B}^{2,(\kappa),\infty}$, which will complete the proof.

The initial case, for $j = 0$, is implied by the fact that $m \in \mathscr{H}^{s,1} = \mathcal{H}^{0,1,s,1}$.

For the induction step, we combine two properties.

Firstly, we observe the multiplication property

(4.2.21)
$$m \in \mathcal{H}^{j\eta,1+j\eta,s,1+j} + \mathscr{B}^{2,(\kappa),\infty} \implies \{m,m\} \in \mathcal{H}^{1+(j+1)\eta,1+(j+1)\eta,s,1+j} + \mathscr{B}^{4,(\kappa),\infty}\,.$$

This relies on the continuity of the multiplication map (C.2.16), which implies the continuity of

$$\{\bullet,\bullet\}\colon \mathcal{H}^{j\eta,1+j\eta,s,1+j} \times \mathcal{H}^{jn,1+jn,s,1+j} \to \mathcal{H}^{1+(j+1)\eta,1+(j+1)\eta,s,1+j} \tag{4.2.22}$$

by taking $k = 1+j$ and adding $j\eta$ to the weight of the spaces on the left and $j\eta+(\eta-\frac{1}{4})$ to the space on the right. Note that, since $\eta < \frac{1}{4}$, we are always adding more to the weights on the left than to those on the right. From the more straightforward multiplication properties of the spaces of bounded polyhomogeneous functions, together with the product properties of the Lie bracket with respect to root spaces, we have that

$$\{\mathscr{B}^{2,(\kappa),\infty}, \mathscr{B}^{2,(\kappa),\infty}\} \subseteq \mathscr{B}^{4,(\kappa),\infty}\,, \tag{4.2.23}$$

and we obtain (4.2.21).

Secondly, we need to apply the regularity property

$$\not{D}_{M_c} m \in \mathcal{H}^{\delta_0,\delta_1,s,k} + \mathscr{B}^{4,(\kappa),\infty} \implies m \in \mathcal{H}^{\delta_0-1,\delta_1,s,k+1} + \mathscr{B}^{2,(\kappa),\infty}\,, \tag{4.2.24}$$

which holds when δ_0 is not an indicial root of the operator $\not{D}_{M_c}$. This is an elliptic regularity result adapted to the analytic framework we have laid out, and can be deduced from Kottke's work and the general theory of b and scatering calculuses. Importantly, the operator D_{M_c} decomposes near infinity not only along the C and $C^\perp$ parts of the adjoint bundle, but also along the root subbundles. For each of those, as we saw in the proof of Lemma 4.1.15, the smallest relevant indicial root was $1 + \frac{|i\alpha(\kappa)|}{2}$, explaining the specific weights of the space $\mathscr{B}^{2,(\kappa),\infty}$.

Now, from the equation (4.2.19), the induction hypothesis (4.2.20) for j, and the multiplication property (4.2.21) we deduce that

$$\not{D}_{M_c} m = -\{m,m\} + v \in \mathcal{H}^{1+(j+1)\eta,1+(j+1)\eta,s,1+j} + \mathscr{B}^{4,(\kappa),\infty}\,. \tag{4.2.25}$$

Recalling that the indicial roots are always half-integers and that η is irrational, we can take $\delta_0 = \delta_1 = 1 + (j+1)\eta$ and $k = 1 + j$ in (4.2.24) to obtain

$$m \in \mathcal{H}^{(j+1)\eta, 1+(j+1)\eta, s, 1+j+1} + \mathscr{B}^{2,(\kappa),\infty}, \tag{4.2.26}$$

which is the induction hypothesis for $j+1$, finishing the proof. □

This result does not only guarantee that every monopole is gauge equivalent to a bounded polyhomogeneous one, which was the necessary step to be able to apply Theorem 4.1.39, it also shows significantly stronger decay properties.

We recall that, although the space $\mathscr{B}^{2,(\kappa),\infty}$ relies on the root subbundle decomposition of the complexification $\mathrm{Ad}(P)^{\mathbb{C}}$, the real subbundle $\mathrm{Ad}(P)$ can also be decomposed along the real parts of the subbundles $\underset{\sim}{\mathfrak{g}_\alpha} \oplus \underset{\sim}{\mathfrak{g}_{-\alpha}}$. Along such a subbundle, elements in this space will be bounded polyhomogeneous of order $x^{2+\frac{|i\alpha(\kappa)|}{2}}$ (or will banish to infinite order if $\alpha(\mu) \neq 0$). Of course, along $\underline{\mathfrak{t}}$ they will simply be of order x^2.

The asymptotic properties of monopoles are discussed further in Section 5.3.

4.3 The smooth structure

With the above regularity result we can make use of the properties deduced in Section 4.1 in order to construct the moduli space near any specific monopole.

We begin with a technical lemma with several important consequences. An analogous lemma can be found in Donaldson and Kronheimer's book [DK90] over a compact manifold, and here we provide a proof in our setting taking the decay conditions into account.

Lemma 4.3.1. *Let $\{M_j\}$ and $\{M'_j\}$ be two sequences of pairs in $\mathscr{C}^s_{\mu,\kappa}$, and $\{g_j\}$ a sequence of gauge transformations in $\mathscr{G}^s_{\mu,\kappa}$ such that $g_j \cdot M_j = M'_j$ for every j. Assume that $\{M_j\}$ and $\{M'_j\}$ have limits M_∞ and M'_∞, respectively, in $\mathscr{C}^s_{\mu,\kappa}$. Then, the sequence $\{g_j\}$ has a limit g_∞ in $\mathscr{G}^s_{\mu,\kappa}$ such that $g_\infty \cdot M_\infty = M'_\infty$.*

Proof. Let us write $M_j = M_{\mu,\kappa} + m_j$ and $M'_j = M_{\mu,\kappa} + m'_j$. Then, the condition

$g_j \cdot M_j = M_j'$ can then be rewritten as

$$\mathrm{d}_{M_{\mu,\kappa}} g_j = g_j m_j - m_j' g_j \,, \tag{4.3.2}$$

where we are viewing g_j, m_j and m_j' as sections of the bundle E^{Mat} used in Definition 3.3.20 to define the group of gauge transformations.

The proof relies on a series of bootstrapping arguments using (4.3.2). In particular, we rely on two finite sequences of Sobolev spaces $\{Z_k^{(1)}\}$ and $\{Z_k^{(2)}\}$ defined as

$$Z_{-3}^{(1)} := L^2 \,, \tag{4.3.3a}$$

$$Z_{-2}^{(1)} := \mathcal{H}^{0,\frac{1}{2},0,0} \,, \tag{4.3.3b}$$

$$Z_{-1}^{(1)} := \mathcal{H}^{0,1,0,0} \,, \tag{4.3.3c}$$

$$Z_k^{(1)} := \mathscr{H}^{k,1} \qquad \text{for } k = 0, 1, \ldots, s \,, \tag{4.3.3d}$$

and

$$Z_{-4}^{(2)} := L^\infty \,, \tag{4.3.4a}$$

$$Z_{-3}^{(2)} := L^6 \,, \tag{4.3.4b}$$

$$Z_{-2}^{(2)} := \mathcal{W}^{0,\frac{1}{2},0,0,6} \,, \tag{4.3.4c}$$

$$Z_{-1}^{(2)} := \mathcal{H}^{-1,1,0,1} \,, \tag{4.3.4d}$$

$$Z_k^{(2)} := \mathscr{H}^{k,2} \qquad \text{for } k = 0, 1, \ldots, s \,. \tag{4.3.4e}$$

The spaces involved in this proof are over the bundle E^{Mat}, but we omit this from the notation for simplicity.

Now, these sequences are chosen so that, for $k = -3, -2, \ldots, s-1$, the map

$$Z_{k-1}^{(2)} \times \mathscr{H}^{s,1} \to Z_k^{(1)} \tag{4.3.5}$$

is continuous,

$$\mathscr{H}^{s,1} \subseteq Z_k^{(1)} \,, \tag{4.3.6}$$

and

$$u \in \mathscr{H}_0^{1,2} \text{ and } \mathrm{d}_{M_{\mu,\kappa}} u \in Z_k^{(1)} \implies u \in Z_k^{(2)} \text{ and } \|u\|_{Z_k^{(2)}} \preccurlyeq \|\mathrm{d}_{M_{\mu,\kappa}} u\|_{Z_k^{(1)}}\,, \tag{4.3.7}$$

where $\preccurlyeq$ indicates that the right-hand side is larger than the left-hand side when multiplied by a constant which does not depend on u. Note that $Z_s^{(2)} = \mathscr{H}^{s,2}$.

Let us prove these conditions, starting with the continuity of the multiplication maps (4.3.5). For $k \in \{-3, -2, -1\}$ we can simply apply Hölder's inequality after a Sobolev embedding of $\mathscr{H}^{s,1}$ into an appropriate space, obtaining the continuous maps

$$L^\infty \times \mathscr{H}^{s,1} \subseteq L^\infty \times L^2 \to L^2\,, \tag{4.3.8a}$$

$$L^6 \times \mathscr{H}^{s,1} \subseteq L^6 \times \mathcal{W}^{\frac{1}{2},\frac{1}{2},0,0,3} \to \mathcal{H}^{0,\frac{1}{2},0,0}\,, \tag{4.3.8b}$$

$$\mathcal{W}^{0,\frac{1}{2},0,0,6} \times \mathscr{H}^{s,1} \subseteq \mathcal{W}^{0,\frac{1}{2},0,0,6} \times \mathcal{W}^{\frac{1}{2},1,0,0,3} \to \mathcal{H}^{0,1,0,0}\,. \tag{4.3.8c}$$

For $k = 0$ this the map is (C.2.17), and for $k \geq 1$ it is (C.2.18).

Condition (4.3.6) is straightforward to verify.

The last condition (4.3.7) follows from the Gagliardo–Nirenberg–Sobolev inequality [Nir59] for $k \in \{-3, -2\}$.[5] For $k \geq -1$ it is a consequence of the elliptic regularity results of the b and scattering calculuses, similarly to (4.2.24).

Now, we know that the sequences $\{m_j\}$ and $\{m'_j\}$ are uniformly bounded in $\mathscr{H}^{s,1}$, since they converge to $m_\infty = M_\infty - M_{\mu,\kappa}$ and $m'_\infty = M'_\infty - M_{\mu,\kappa}$, respectively. Furthermore, the sequence $\{g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ is uniformly bounded in L^∞, since G is a compact group.

Our first aim is to show that the sequence $\{g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ is uniformly bounded in $\mathscr{H}^{s,2}$, which we will do by finite induction with the hypothesis that the sequence is uniformly bounded in $Z_k^{(2)}$ for $k = -4, -3, \ldots, s$. The initial step has already been seen. To see the induction step, suppose that it is uniformly bounded in

[5]For $k = -2$ we need to use the fact that $\mathrm{d}_{M_{\mu,\kappa}}$ includes the term $\mathrm{ad}_{\Phi_{\mu,\kappa}}$, which is bounded below in the $C^\perp$ subbundle near infinity. We can then obtain a weighted version of this inequality on this subbundle as long as we have chosen the boundary defining function to be equal to 1 on a sufficiently large compact set.

$Z^{(2)}_{k-1}$. Rewriting (4.3.2) as

$$\mathrm{d}_{M_{\mu,\kappa}}(g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}) = (g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}})m_j - m'_j(g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}) + m_j - m'_j \tag{4.3.9}$$

and applying properties (4.3.5) and (4.3.6) we deduce that $\{\mathrm{d}_{M_{\mu,\kappa}}(g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}})\}$ is uniformly bounded inside $Z^{(1)}_k$. Applying property (4.3.7) we conclude that $\{g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ is uniformly bounded in $Z^{(2)}_k$, as desired.

Now, since $\{g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ is uniformly bounded in $\mathscr{H}^{s,2}$ it must have a weak limit g_∞, which must be in $\mathscr{G}_{\mu,\kappa}$ and satisfy

$$\mathrm{d}_{M_{\mu,\kappa}} g_\infty = g_\infty m_\infty - m'_\infty g_\infty \,. \tag{4.3.10}$$

It remains to prove that g_∞ is the strong limit of $\{g_j\}$ in $\mathscr{G}^s_{\mu,\kappa}$. We once again prove this by finite induction, with the hypothesis that $\{g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ is strongly convergent in $Z^{(2)}_k$ for $k = -4, -3, \ldots, s$. The initial step follows from the fact that $\{g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ is weakly convergent, and hence bounded, in $\mathscr{H}^{s,2}$, together with the compact embedding $\mathscr{H}^{s,2} \Subset Z^{(2)}_{-4}$, which implies that it must be strongly convergent in the latter space.

For the induction step, assume that $\{g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ is strongly convergent in $Z^{(2)}_{k-1}$ and subtract (4.3.10) from (4.3.2) to get

$$\begin{aligned}\mathrm{d}_{M_{\mu,\kappa}}(g_j - g_\infty) = {} & (g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}})(m_j - m_\infty) + (g_j - g_\infty)m_\infty + (m_j - m_\infty) \\ & - (m'_j - m'_\infty)(g_j - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}) - m'_\infty(g_j - g_\infty) - (m_j - m'_\infty)\,.\end{aligned} \tag{4.3.11}$$

Using (4.3.5) and (4.3.6), the fact that $\{m_j\}$ and $\{m'_j\}$ (strongly) converge to m_∞ and m'_∞ in $\mathscr{H}^{s,1}$ we deduce that the right-hand side of (4.3.11) converges to 0 strongly in $Z^{(1)}_k$, and hence so does $\{\mathrm{d}_{M_{\mu,\kappa}}(g_j - g_\infty)\}$. Using (4.3.7) completes the proof. □

A consequence of this lemma is the following strengthening of Proposition 4.2.5.

Corollary 4.3.12. *In Proposition 4.2.5, if the neighbourhood U is required to be small enough, then the gauge transformation g is unique in the entire group $\mathscr{G}^s_{\mu,\kappa}$.*

Proof. Let us take $M_0 \in \mathscr{C}^s_{\mu,\kappa}$ and any U and U' which satisfy the statement of Proposition 4.2.5. We aim to prove that there exists a neighbourhood U_0 such that there are no two gauge equivalent pairs inside U_0 which are in Coulomb gauge with respect to M_0. This will complete are proof, since it will be enough to require U to be inside U_0 in the statement of Proposition 4.2.5.

Suppose that such a U_0 did not exist. Then, there would exist a sequence of pairs $\{M_j\}$ and a sequence of gauge transformations $\{g_j\}$ such that $\{M_j\}$ and $\{g_j \cdot M_j\}$ tend to M_0 and are in Coulomb gauge with respect to it. We can assume that the sequences are inside U and U', so applying Proposition 4.2.5 we deduce that the sequence $\{g_j\}$ is bounded away from the identity transformation $\mathbf{1}_{\mathscr{G}_{\mu,\kappa}}$. But from Lemma 4.3.1 we know that g_j must have a limit $g_\infty \in \mathscr{G}_{\mu,\kappa} \setminus \{\mathbf{1}_{\mathscr{G}_{\mu,\kappa}}\}$ such that $g_\infty \cdot M_0 = M_0$. If we view g_∞ as a section of the bundle E^{Mat}, this would mean that $d_{A_0}(g_\infty - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}) = 0$. Since $g_\infty - \mathbf{1}_{\mathscr{G}_{\mu,\kappa}} \in \mathscr{H}^{s,2}(E^{\mathrm{Mat}})$ must decay at infinity, this would imply that $g_\infty = \mathbf{1}_{\mathscr{G}_{\mu,\kappa}}$, arriving at a contradiction. □

We can also use the lemma to prove that the moduli space is Hausdorff.

Corollary 4.3.13. *The quotient* (3.3.26) *is Hausdorff.*

Proof. Suppose, the contrary. Then, there would be two monopoles M and M', a sequence $\{M_j\}$ in $(\mathcal{B}^s_{\mu,\kappa})^{-1}(0)$, and a sequence $\{g_j\}$ in $\mathscr{G}^s_{\mu,\kappa}$, such that $\{M_j\}$ tends to M and $\{g_j \cdot M_j\}$ tends to M'. But then, by Lemma 4.3.1, M and M' must be gauge equivalent. □

We now have all the ingredients to show that the moduli space is a smooth manifold.

Theorem 4.3.14. *The moduli space $\mathcal{M}^s_{\mu,\kappa}$ is either empty or a smooth manifold of dimension* $\dim_{\mu,\kappa}$.

Proof. If it is not empty, let $M_0 \in \mathscr{C}^s_{\mu,\kappa}$ be a monopole, which we can assume to be bounded polyhomogeneous by Theorem 4.2.12, and consider the function

$$\begin{aligned} f_{M_0} \colon \mathscr{C}_{\mu,\kappa} &\to \mathscr{H}^{s,0} \\ M &\mapsto (-\mathcal{B}^s_{\mu,\kappa}(M), \mathrm{d}^*_{M_0}(M - M_0)) \,. \end{aligned} \tag{4.3.15}$$

It satisfies $f_{M_0}(M_0) = 0$, and at this point its derivative is $\not{D}_{M_0}$, which we know by Theorem 4.1.39 to be surjective and have a kernel of dimension $\dim_{\mu,\kappa}$. Applying the implicit function theorem we obtain a neighbourhood U of M_0 and a smooth embedding from an open set in $\mathbb{R}^{\dim_{\mu,\kappa}}$ to U such that pairs in U are in the image of this embedding if and only if they are monopoles in Coulomb gauge with respect to M_0. We refer to the image of this embedding as the slice $\mathcal{S}$.

We can furthermore take U to be as small as we want, so in particular we assume that it is small enough that the slice is transverse to gauge orbits, since this is true at M_0 and an open condition.

We furthermore assume that U is small enough for Proposition 4.2.5 to be true with g being unique in $\mathscr{G}^s_{\mu,\kappa}$, which is possible by Corollary 4.3.12, and take U' as in the statement of that result.

Then, consider the open set of monopoles $U'' = \mathscr{G}^s_{\mu,\kappa} \cdot (U' \cap (\mathcal{B}^s_{\mu,\kappa})^{-1}(0))$. Any orbit of a monopole in U'' will have a representative in U', and hence will have a unique representative in U which is in Coulomb gauge with respect to M_0. This unique representative must be in the constructed slice. Since the slice is smoothly embedded and transverse to the gauge orbits, the map from U'' choosing this unique representative is a submersion onto $\mathcal{S} \cap U''$. It hence provides a chart for the quotient near M_0.

To see that the transition functions between different such charts must be smooth suppose we have two open sets of orbits U''_1 and U''_2 as above. Then, their intersection has a smooth submersion to two different smooth slices of the gauge action. The restriction of the first submersion to the second slice provides a smooth map from the second slice to the first, and we can analogously provide a smooth map from the first slice to the second. Both of these maps are the identity on the quotient, so the charts are compatible. □

Importantly, this smooth structure does not actually depend on the regularity parameter.

Proposition 4.3.16. *The smooth manifold $\mathcal{M}^s_{\mu,\kappa}$ is independent of the regularity parameter s.*

Proof. Let $s_1 < s_2$ be in $\mathbb{Z}_{\geq 1}$.

We first prove that the identity map between $\mathcal{M}^{s_1}_{\mu,\kappa}$ and $\mathcal{M}^{s_2}_{\mu,\kappa}$ is well defined. To do so, let $[M] \in \mathcal{M}^{s_1}_{\mu,\kappa}$. From Theorem 4.2.12 we know that there exists at least one $g \in \mathscr{G}^{s_1}_{\mu,\kappa}$ such that $g \cdot M \in \mathscr{C}^{s_2}_{\mu,\kappa}$, and hence we can define the identity map as taking $[M] \in \mathcal{M}^{s_1}_{\mu,\kappa}$ to $[g \cdot M] \in \mathcal{M}^{s_2}_{\mu,\kappa}$ for any such $g \in \mathscr{G}^{s_1}_{\mu,\kappa}$. This is well defined because if $g \cdot M$ and $g' \cdot M$ are both in $\mathscr{C}^{s_2}_{\mu,\kappa}$ for $g, g' \in \mathscr{G}^{s_1}_{\mu,\kappa}$, then in fact $g'g^{-1} \in \mathscr{G}^{s_2}_{\mu,\kappa}$, as can be seen by applying the same bootstrapping argument used in Lemma 4.3.1, and hence $[g \cdot M] = [g' \cdot M]$ in $\mathcal{M}^{s_2}_{\mu,\kappa}$. The definition is also independent of the choice of representative M in $[M]$, so the map is well defined.

Since $\mathscr{G}^{s_2}_{\mu,\kappa} \subset \mathscr{G}^{s_1}_{\mu,\kappa}$, the map is injective, and since $\mathscr{C}^{s_2}_{\mu,\kappa} \subset \mathscr{C}^{s_1}_{\mu,\kappa}$, it is surjective, so this identity map is a bijection. We must now prove that it is a diffeomorphism.

To do so, let M_0 be a bounded polyhomogeneous monopole, and construct slices $\mathcal{S}_{s_1}$ and $\mathcal{S}_{s_2}$ for $\mathcal{M}^{s_1}_{\mu,\kappa}$ and $\mathcal{M}^{s_2}_{\mu,\kappa}$ as in the proof of Theorem 4.3.14. When close enough to M_0, any point of $\mathcal{S}_{s_2}$ must also be in $\mathcal{S}_{s_1}$, since such a point must be a monopole, in Coulomb gauge with respect to M_0, and can be made to be close enough to M_0 in $\mathscr{C}^{s_1}_{\mu,\kappa}$ due to the continuity of the embedding of $\mathscr{C}^{s_2}_{\mu,\kappa}$ in $\mathscr{C}^{s_1}_{\mu,\kappa}$. Therefore, this embedding, which is smooth due to being continuous and linear, restricts near M_0 to a smooth map from $\mathcal{S}_{s_2}$ to $\mathcal{S}_{s_1}$. Since the identity map has an injective derivative and the slices have the same dimension this must be a diffeomorphism near M_0. Given that it corresponds to the identity map between $\mathcal{M}^{s_2}_{\mu,\kappa}$ and $\mathcal{M}^{s_1}_{\mu,\kappa}$, this completes the proof. □

This justifies defining the moduli space independently of s.

Definition 4.3.17. We write

$$\mathcal{M}_{\mu,\kappa} \tag{4.3.18}$$

for $\mathcal{M}^{s}_{\mu,\kappa}$ with its smooth structure with respect to any $s \in \mathbb{Z}_{\geq 1}$.

4.4 The hyper-Kähler metric

As we saw in Section 2.2, there is a strong analogy between monopoles and instantons on $\mathbb{R}^4$. One of the consequences is the possibility of considering a relationship with the quaternions.

To see this, let us fix an identification

$$(\mathbb{R}^3)^* \cong \operatorname{Im}\mathbb{H}\,. \tag{4.4.1}$$

This induces identifications on related objects, like

$$\textstyle\bigwedge^1 \cong \underline{\operatorname{Im}\mathbb{H}}\,, \tag{4.4.2}$$

$$\Omega^1(\operatorname{Ad}(P)) \cong \Omega^0(\operatorname{Ad}(P)) \otimes \operatorname{Im}\mathbb{H}\,, \tag{4.4.3}$$

$$\mathscr{H}_1^{s,k} \cong \mathscr{H}_0^{s,k} \otimes \operatorname{Im}\mathbb{H}\,. \tag{4.4.4}$$

Furthermore, it induces the structure of a left $\mathbb{H}$-module on the space

$$(\mathbb{R}^3)^* \oplus \mathbb{R} \cong \operatorname{Im}\mathbb{H} \oplus \operatorname{Re}\mathbb{H} \cong \mathbb{H} \tag{4.4.5}$$

through left multiplication. By considering this structure fibrewise on the corresponding bundle $\bigwedge^1 \oplus \bigwedge^0$, we also obtain a left $\mathbb{H}$-module structure on $\mathscr{H}^{s,1}$. Since the L^2 metric on this space is compatible with this structure, this confers the structure of a hyper-Kähler manifold to the configuration space $\mathscr{C}^s_{\mu,\kappa}$, which is just an affine space over this vector space. If we bundle the triple of symplectic forms into an $\operatorname{Im}\mathbb{H}$-valued form we can write it out with a relatively simple expression.

Proposition 4.4.6. *The configuration space $\mathscr{C}^s_{\mu,\kappa}$ is hyper-Kähler. The triple of symplectic forms is given on any tangent space $T_{(A,\Phi)}\mathscr{C}^s_{\mu,\kappa} \cong \mathscr{H}^{s,1}$ by the form*

$$\begin{aligned} \omega\colon \mathscr{H}^{s,1} \quad &\times \mathscr{H}^{s,1} \quad \to \operatorname{Im}\mathbb{H} \\ ((a_1,\varphi_1), \quad &(a_2,\varphi_2)) \mapsto \int_{\mathbb{R}^3} \star\langle a_1 \wedge a_2\rangle_{\mathfrak{g}} + \langle \varphi_1, a_2\rangle_{\mathfrak{g}} - \langle a_1, \varphi_2\rangle_{\mathfrak{g}}\,. \end{aligned} \tag{4.4.7}$$

The integral in (4.4.7) is interpreted in the following way: The integrand is a 1-form, so, since the bundle $\bigwedge^1$ is trivial, the form can be integrated to obtain an element of the fibre, which is $(\mathbb{R}^3)^*$ and hence isomorphic to $\operatorname{Im}\mathbb{H}$ through (4.4.1).

Proof. The metric and the $\operatorname{Im}\mathbb{H}$-valued symplectic form on a hyper-Kähler manifold are equivalent to a bilinear form which takes values in $\mathbb{H}$, is $\mathbb{H}$-left-linear in the first variable, and is conjugate-symmetric. This bilinear form corresponds to the metric minus the symplectic form.

On the space $\mathbb{H}$ of quaternions such a form can be constructed as the product of a quaternion and the conjugate of another. To mimic our notation, suppose that $\varphi_1, \varphi_2 \in \mathbb{R}$ and $a_1, a_2 \in \operatorname{Im}\mathbb{H}$. Then, this product is given by

$$(a_1 + \varphi_1)\overline{(a_2 + \varphi_2)} = (-a_1 \times a_2 - \varphi_1 a_2 + a_1\varphi_2) + (a_1 \cdot a_2 + \varphi_1\varphi_2)\,, \tag{4.4.8}$$

where the right-hand side is separated into the imaginary and real parts.

Since the fibres of $\bigwedge^1 \oplus \bigwedge^0$ are isomorphic to $\mathbb{H}$, we can extend this product by integrating over $\mathbb{R}^3$. Combining with the inner product on $\mathfrak{g}$ we obtain the bilinear form

$$((a_1, \varphi_1), (a_2, \varphi_2)) \mapsto \int_{\mathbb{R}^3} (-\star\langle a_1 \wedge a_2\rangle_{\mathfrak{g}} - \langle \varphi_1, a_2\rangle_{\mathfrak{g}} + \langle a_1, \varphi_2\rangle_{\mathfrak{g}}, \langle a_1, a_2\rangle_{\mathbb{R}^3,\mathfrak{g}} + \langle \varphi_1, \varphi_2\rangle_{\mathfrak{g}}) \tag{4.4.9}$$

on sections $(a_1, \varphi_1), (a_2, \varphi_2)$ of $(\bigwedge^1 \oplus \bigwedge^0) \oplus \operatorname{Ad}(P)$, and in particular on elements of $\mathscr{H}^{s,1}$. Here, the integral is interpreted like the previous one, this time using the isomorphism (4.4.5), and the expression is simply derived from (4.4.8), where the Hodge dual of the wedge product provides the equivalent of the cross product.

Since the real part of this form is the L^2 norm on $\mathscr{H}^{s,1}$, we obtain the desired expression for the symplectic form as the negative of the imaginary part. $\square$

Then, as it turns out, the group of gauge transformations $\mathscr{G}^s_{\mu,\kappa}$ not only preserves this structure, but in fact acts in a tri-Hamiltonian way.[6] We can once again see the analogy with the 4-dimensional case, since the moment map will be given by the Bogomolny map.

Here it is crucial that we are building the moduli space of *framed* monopoles, since the decay conditions are important to not only define the L^2 metric, but to show that the gauge action has a moment map.

Proposition 4.4.10. *The action of the group $\mathscr{G}^s_{\mu,\kappa}$ on $\mathscr{C}^s_{\mu,\kappa}$ is tri-Hamiltonian and its moment map is given by*

$$\mathcal{B}^s_{\mu,\kappa} \colon \mathscr{C}^s_{\mu,\kappa} \to \mathscr{H}^{s,0}_1 \subseteq (\mathfrak{G}^s_{\mu,\kappa})^* \otimes \operatorname{Im}\mathbb{H}\,. \tag{4.4.11}$$

[6] We do not go here into the background of hyper-Kähler geometry and quotients, which can be found in the literature [HKLR87; Hit92]

Proof. First let us look at the inclusion $\mathscr{H}_1^{s,0} \subseteq (\mathfrak{G}_{\mu,\kappa}^s)^* \otimes \operatorname{Im}\mathbb{H}$. At the level of bundles, the $\operatorname{im}\mathbb{H}$ factor simply comes from $\bigwedge^1$. Then, from Lemma C.2.3, we see that the L^2 pairing between $\mathscr{H}_0^{s,0}$ and $\mathfrak{G}_{\mu,\kappa}^s = \mathscr{H}_0^{s,2}$ is continuous.

Now, the moment map equation, for $(A,\Phi) \in \mathscr{C}_{\mu,\kappa}^s$, $(a,\varphi) \in T_{(A,\Phi)}\mathscr{C}_{\mu,\kappa}^s$ and $X \in \mathfrak{G}_{\mu,\kappa}^s$, is the $\operatorname{Im}\mathbb{H}$-valued expression

$$\langle (\mathrm{d}\mathcal{B}_{\mu,\kappa})_{(A,\Phi)}(a,\varphi), X\rangle_{L^2} = \omega((X^{\#})_{(A,\Phi)}, (a,\varphi))\,. \tag{4.4.12}$$

We once again integrate by parts using Lemma 3.3.13 to get

$$\begin{aligned}\langle (\mathrm{d}\mathcal{B}_{\mu,\kappa})_{(A,\Phi)}(a,\varphi), X\rangle_{L^2} &= \langle \star \mathrm{d}_A a - \mathrm{d}_A\varphi + \mathrm{ad}_\Phi(a), X\rangle_{L^2} \\ &= \int_{\mathbb{R}^3} \star\langle \mathrm{d}_A a, X\rangle_{\mathfrak{g}} - \langle \mathrm{d}_A\varphi, X\rangle_{\mathfrak{g}} + \langle \mathrm{ad}_\Phi(a), X\rangle_{\mathfrak{g}} \\ &= \int_{\mathbb{R}^3} \star\langle a \wedge \mathrm{d}_A X\rangle_{\mathfrak{g}} + \langle \varphi, \mathrm{d}_A X\rangle_{\mathfrak{g}} - \langle a, \mathrm{ad}_\Phi(X)\rangle_{\mathfrak{g}} \\ &= \omega((-\mathrm{d}_A X, -\mathrm{ad}_\Phi X), (a,\varphi)) \\ &= \omega((X^{\#})_{(A,\Phi)}, (a,\varphi))\,,\end{aligned} \tag{4.4.13}$$

as desired. □

With this, we can deduce that the moduli spaces are hyper-Kähler – with the metric being independent of s.

Theorem 4.4.14. *The L^2 norm on the configuration space descends to a hyper-Kähler metric on the smooth manifold $\mathcal{M}_{\mu,\kappa}$.*

Proof. Proposition 4.4.10 allows us to interpret the moduli space construction formally as a hyper-Kähler quotient, since

$$\mathcal{M}_{\mu,\kappa}^s = (\mathcal{B}_{\mu,\kappa}^s)^{-1}(0)/\mathscr{G}_{\mu,\kappa}^s = \mathscr{C}_{\mu,\kappa}^s /\!/\!/ \mathscr{G}_{\mu,\kappa}^s\,. \tag{4.4.15}$$

In order for this to actually yield a hyper-Kähler metric we need the quotient to be a smooth manifold. But this is true by Theorem 4.3.14.

Furthermore, in the construction of the moduli space the tangent spaces were modelled on subspaces of $\mathscr{H}^{s,1}$ which contained only bounded polyhomogeneous sections, so they are independent of s and hence so is the metric. □

Chapter 5

Further Observations

We now discuss in more detail some aspects of our construction and its relationship to other work.

In Section 5.1 we provide an alternative perspective on the conditions imposed on the mass and the charge. In particular, we explain how the charge can be viewed as a collection of integers which produce a simpler formula for the dimension of the moduli space, and discuss the concept of symmetry breaking.

In Section 5.2 we connect this with the correspondence between monopoles and rational maps, in particular in relation to non-maximal symmetry breaking.

We then revisit some of the concepts surrounding the asymptotic properties of monopoles in Section 5.3.

We finish in Section 5.4 by explaining how our construction can be used to build moduli spaces for any real compact Lie group, despite the conditions imposed originally on G.

5.1 Symmetry breaking and integer charges

One of the aims of the approach we have followed was to carry out the construction with independence of the symmetries of the mass and the charge. Let us now examine this with more detail in order to put it into the wider context of previous research on monopoles and their moduli spaces. We particularly follow Murray and Singer's work [MS03].

General facts about Lie groups and Lie algebras can be found in the literature, like in Hall's book [Hal15].

Remark 5.1.1. Recall that if the pair of mass μ and charge κ are (simultaneously) related by the adjoint action of an element of G, then all the constructions carried out in the previous chapters can be taken to be isomorphic. We will keep this in mind when we are discussing possible choices which are related by these transformations.

One of the most relevant properties of monopoles is related to the symmetries of the mass μ.

Definition 5.1.2. Let C_μ be the centraliser group of the mass μ, that is, the subgroup of G of elements which leave μ invariant under the adjoint action. Then, we say that for this choice of mass *the symmetry breaks from G to C_μ*.

In many ways, the simplest case happens when the mass is as generic as possible.

Definition 5.1.3. If the mass μ is a regular element of the Lie algebra $\mathfrak{g}$, we say that the symmetry breaking is *maximal*.

Note that in the case of maximal symmetry breaking we are breaking the symmetry to the smallest possible centraliser subgroup, which will be a (maximal) torus $T^{\mathrm{rank}(G)}$.

One of the features of this case is that we cannot have a root $\alpha \in R$ such that $\alpha(\mu) = 0$. In particular, the subbundle $\mathrm{Ad}(P)_C^{\mathbb{C}}$ is simply $\underline{\mathfrak{t}^{\mathbb{C}}}$. We will see how this makes several matters simpler, but we can start by noticing, for example, that in the dimensional formula (4.1.35) the second summand no longer appears – with the corresponding index calculation becoming more straighforward.

Remark 5.1.4. If we recall the definitions of the maximal torus and roots in Section 2.3, we can see that for the case of maximal symmetry breaking there is a single possible maximal torus subgroup T. However, in non-maximal symmetry breaking, depending on the symmetries of the charge κ we might have some ambiguity in this choice. Of course, all choices are related by the adjoint action of the group, so will produce equivalent results. However, we will see how this ambiguity plays out below.

The symmetry breaking is very closely related to the understanding of the dimensional formula for the moduli spaces and the role of the charge in it.

In order to see this, let us take a look at the definition (4.1.35) of the dimension $\dim_{\mu,\kappa}$, which seems somewhat artificial. Indeed, it can be defined in a more natural way.

We begin by choosing a set R^+ of positive roots of R such that

$$\left\{\alpha \in R \mid \frac{1}{i}\alpha(\mu) > 0\right\} \cup \left\{\alpha \in R \mid \alpha(\mu) = 0 \text{ and } \frac{1}{i}\alpha(\kappa) < 0\right\} \subseteq R^+ \,. \tag{5.1.5}$$

Then, we can rewrite the dimension as

$$\dim_{\mu,\kappa} = 2 \sum_{\alpha \in R^+} \frac{1}{i}\alpha(\kappa)\,. \tag{5.1.6}$$

Note that we will from now on often consider the functions $\frac{1}{i}\alpha$ more often that $i\alpha$. This simply changes some signs, but will make our notation more consistent with the usual notation for specific groups in the next chapter.

Remark 5.1.7. When the symmetry breaking is maximal, the first set that appears in the condition (5.1.5), which depends only on the mass, determines R^+ completely. However, when the symmetry breaking is non-maximal, we must use the charge to define the set of positive roots. This will play an important role in the definition of the holomorphic integer charges below.

Note that in some cases there might remain some ambiguity even when taking into account the charge. This is similar to the ambiguity encountered in the choice of maximal torus, as observed in Remark 5.1.4, and will not change the results.

From this set of positive roots we obtain the corresponding set of simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{\mathrm{rank}(G)}$. Furthermore, associated to this set of positive roots we have a set of fundamental weights $w_1, w_2, \ldots, w_{\mathrm{rank}(G)}$. As it turns out, the integrality condition $\exp(2\pi\kappa) = 1_G$ is equivalent to the numbers $\frac{1}{i}w_j(\kappa)$ being integers for $j = 1, 2, \ldots, \mathrm{rank}(G)$, and we will use these numbers to provide an alternative understanding of the charge.

Definition 5.1.8. The integers

$$\frac{1}{i}w_1(\kappa), \frac{1}{i}w_2(\kappa), \ldots, \frac{1}{i}w_{\mathrm{rank}(G)}(\kappa) \in \mathbb{Z} \tag{5.1.9}$$

are the *(integer) charges* of the monopole. Furthermore, if $\alpha_j(\mu) = 0$, then we say that $\frac{1}{i}w_j(\kappa)$ is a *holomorphic* charge, and otherwise we say that it is *magnetic*.

Remark 5.1.10. One of the important distinctions between magnetic and holomorphic charges is that the former determine some topological information. In particular, if we trivialise the adjoint bundle $\mathrm{Ad}(P)$, the Higgs field defines near infinity a map from the sphere at infinity to the adjoint orbit of the mass μ, and the magnetic charges determine the homotopy type of this map.

We can now use these integers to rewrite the dimensional formula, since half the sum of the positive roots is equal to the sum of the fundamental weights:

$$\dim_{\mu,\kappa} = 4 \sum_{j=1}^{\mathrm{rank}(G)} \frac{1}{i}w_j(\kappa)\,. \tag{5.1.11}$$

That is, the dimension of the moduli space is four times the sum of all the integer charges. Let us now compactly write the main result of this thesis, putting together Theorem 4.3.14, Proposition 4.3.16, and Theorem 4.4.14.

Theorem 5.1.12. *The moduli space $\mathcal{M}_{\mu,\kappa}$ of framed monopoles of mass μ and charge κ is either empty or a smooth hyper-Kähler manifold whose dimension is four times the sum of the integer charges.*

This dimensional formula coincides with the computation of the dimension carried out by Murray and Singer [MS03][7] based on the analogous space of rational maps, which will be discussed in Section 5.2. Furthermore, it provides more insight into the significance of the charge, by differentiating between holomorphic and magnetic components, as well as into the possible charges that may appear. Murray and Singer's work suggests the following conjecture.

Conjecture 5.1.13. The moduli space $\mathcal{M}_{\mu,\kappa}$ is non-empty if and only if all the integer charges are non-negative.

[7]Different conventions account for the different factors of i.

Note that the combinations of integer charges that could appear for a given choice of mass μ are not so straightforward to determine, since the definition of the fundamental weights used the charge κ itself.

In order to better understand the relationship between the different choices we have made, let us make the following definition.

Definition 5.1.14. Let $\mu \in \mathfrak{g}$, and let (T, R^+) be a choice of a maximal torus T in G such that $\mu \in \mathfrak{t}$ and a set of positive roots R^+ inside the set R of roots of $\mathfrak{t}$ such that

$$\left\{\alpha \in R \mid \frac{1}{i}\alpha(\mu) > 0\right\} \subseteq R^+ . \tag{5.1.15}$$

Then, we say that an element $\kappa \in \mathfrak{g}$, with $[\mu, \kappa] = 0$, is *compatible* with the choice of pair (T, R^+) when $\kappa \in \mathfrak{t}$ and

$$\frac{1}{i}\alpha(\kappa) \leq 0, \quad \forall \alpha \in R^+ \text{ such that } \alpha(\mu) = 0 . \tag{5.1.16}$$

Condition (5.1.16) can be rewritten as

$$\left\{\alpha \in R \mid \alpha(\mu) = 0 \text{ and } \frac{1}{i}\alpha(\kappa) < 0\right\} \subseteq R^+ , \tag{5.1.17}$$

so, of course, together with (5.1.15) it is simply equivalent to (5.1.5).

Remark 5.1.18. A choice of positive roots R^+ is equivalent to choosing a fundamental Weyl chamber in $\mathfrak{t}$, with condition (5.1.15) requiring this chamber to contain μ in its closure. Furthermore, let us consider the facets of this Weyl chamber which contain μ, and let us extend them linearly into hyperplanes of $\mathfrak{t}$. Then, the compatibility condition on κ is equivalent to requiring that it be (non-strictly) on the opposite side of the chosen Weyl chamber with respect to any such hyperplane.

These conditions allow us to reframe the choices by starting with a mass and fixing a maximal torus and set of positive roots, to then investigate which charges can appear. Importantly, this does not leave out any possibilities, as seen from Proposition 4.1 in Murray and Singer's work [MS03].

Proposition 5.1.19. *Let* $\mu \in \mathfrak{g}$ *and* (T, R^+) *be as above. Then, if* $\kappa \in \mathfrak{g}$ *satisfies* $[\mu, \kappa] = 0$*, there is a unique element* $\kappa' \in \mathfrak{g}$ *which is compatible with* (T, R^+) *and is related to* κ *through the adjoint action of an element of* C_μ*.*

As observed in Remark 5.1.1, applying the adjoint action of an element of G produces equivalent results, so we can conclude that, after fixing a μ and a pair (T, R^+) as above, considering only the charges κ that satisfy (5.1.16) will provide all the possible monopoles (without repeating pairs of mass and charge related through the adjoint action).

Of course, for there to exist monopoles, according to Conjecture 5.1.13, the integer charges – which are defined with respect to the first chosen chamber – will have to be non-negative. This can also be expressed as belonging to a certain region of the Lie algebra $\mathfrak{t}$. The intersection of this region with the Weyl chamber for R^+ will define a conical subset of $\mathfrak{t}$, whose integral elements will be the charges for which we expect to have non-empty monopole moduli spaces.

Remark 5.1.20. We can also consider this approach from the perspective of the integer charges. The first thing to observe is that after fixing $\mu \in \mathfrak{g}$ and (T, R^+) as above we can always find at least one compatible charge $\kappa \in \mathfrak{g}$ with any combination of non-negative magnetic charges. This can be done, for example, by setting all the holomorphic charges equal to 0. However, there might be other possible choices of holomorphic charges, which will need to satisfy a set of constraints. As it turns out, these constrains will allow only a finite amount of combinations of holomorphic charges for each choice of magnetic charges [MS03, Prop. 4.3].

In Chapter 6 we put this discussion into the context of specific groups, masses and charges. However, let us briefly discuss here a trivial case which appears for any group.

Remark 5.1.21. Suppose that $\mu = 0$. Then, we do not actually have any interesting monopoles. Indeed, we first see that

$$
\begin{aligned}
\mathrm{d}^*\mathrm{d}\|\Phi\|_{\mathfrak{g}}^2 &= -\star\mathrm{d}\star\mathrm{d}\langle\Phi,\Phi\rangle_{\mathfrak{g}} \\
&= -2\star\mathrm{d}\star\langle\Phi,\mathrm{d}_A\Phi\rangle_{\mathfrak{g}} \\
&= -2\star\mathrm{d}\langle\Phi,\star\mathrm{d}_A\Phi\rangle_{\mathfrak{g}} \\
&= -2\star\langle\mathrm{d}_A\Phi\wedge\star\mathrm{d}_A\Phi\rangle_{\mathfrak{g}} - 2\star\langle\Phi,\mathrm{d}_A F_A\rangle_{\mathfrak{g}} \\
&= -2\|\mathrm{d}_A\Phi\|_{\mathbb{R}^3,\mathfrak{g}}^2 \\
&\leq 0\,.
\end{aligned}
\tag{5.1.22}
$$

Then, since the Higgs field of a monopole with mass 0 must decay, it must in fact be constantly 0 due to the maximum principle. The connection must then be flat, and hence gauge equivalent to the identity.

Of course, in this case, all the charges are holomorphic. However, following Definition 5.1.14, any charge compatible with a choice (T, R^+) must satisfy $\frac{1}{i}\alpha(\kappa) \leq 0$ for all $\alpha \in R^+$. But this implies that we must also have $\frac{1}{i}w(\kappa) \leq 0$ for all fundamental weights w associated to this choice – in other words, all the charges must be non-positive. This is consistent with Conjecture 5.1.13, which leaves only the case of all the charges being 0. Naturally, it is also consistent with our dimensional formula, since the resulting moduli space would have dimension 0.

5.2 Rational maps

There is an important relationship between monopoles and rational maps, which has been explored in different contexts.

More specifically, this correspondence relates framed monopoles with algebraic maps from $\mathbb{CP}^1$ to certain varieties, with a fixed point. Let us remark on two specific cases, which are once again discussed in Murray and Singer's work [MS03]. The correspondences themselves can be found in the references therein, particularly in Jarvis's work [Jar98a; Jar98b], as well as in the references mentioned in Chapter 1. The case of $G = \mathrm{SU}(3)$ is discussed in this context at the end of Section 6.3.

It is important to note here that the precise definition of framing used in this correspondence differs from ours. Therefore, a rigorous relationship with our work would require a more detailed study of this matter (on which we comment in Section 5.3). However, here we point to some of the ideas regarding what this relationship could look like, since this correspondence provides another way of building these moduli spaces.

Firstly, directly related to the moduli spaces we have constructed, let us fix a mass μ and a charge κ a consider the orbit space $O_{\mu,\kappa}$ of the pair (μ, κ) under the adjoint action of the gauge group. Then, the correspondence establishes a natural bijection between framed monopoles with mass μ and charge κ and based rational maps from $\mathbb{CP}^1$ to $O_{\mu,\kappa}$ of a particular type determined by the charge.

However, there is another correspondence in which we only fix the magnetic charges. Here, the target variety is O_μ, the orbit of the mass μ, and we consider a framing of the monopoles which fixes the mass but only the topological aspect of the charge (corresponding to only fixing the magnetic integer charges). We then have a bijection between such framed monopoles and based rational maps from $\mathbb{CP}^1$ to O_μ of a specific degree, this time given by the magnetic component of the charge.

Interestingly, the latter moduli spaces gather monopoles with all possible holomorphic charges for a set of magnetic ones. The idea is that this is organised into a stratification, and monopoles with different holomorphic charges would correspond to points in different strata. The strata, however, don't necessarily correspond exactly to our framed moduli spaces, since the framing in these moduli spaces of rational maps leave a certain ambiguity when the centraliser of the mass does not preserve the charge: in this case the choice of charge $\kappa \in \mathfrak{g}$ is not uniquely determined by the choice of mass μ and the integer charges. In this situation, the strata would correspond to a fibration, where the base parametrises these possible choices for the charge (given by the orbit of any choice under the action of the centraliser of the mass), which determine a full framing, and the fibres are the hyper-Kähler moduli spaces we have constructed in this thesis corresponding to such full framings.

This suggests the possibility of building such a moduli space from a differential-geometric perspective, and of studying how the hyper-Kähler metric would behave inside this stratified space.

5.3 Asymptotic behaviour of monopoles

Our approach to the construction of monopole moduli spaces has been to fix a notion of mass and charge and to study the monopoles which fit into this definition. However, there remain some interesting questions about the different asymptotic conditions of monopoles.

Perhaps the most natural one that arises from our approach is whether every monopole falls into this classification. That is, whether can we deduce from the finite energy condition that in fact the asymptotics must be determined by a mass and charge as we have described them.

Another question, which we have answered to some extent, is what asymptotic conditions can be deduced about a monopole once we know that it follows our definition of mass and charge. The regularity theorem Theorem 4.2.12 provides significantly strong asymptotic properties in this case.

Lastly, as we have noted, different authors have used slightly different definitions for monopoles with a specific mass, charge and framing [Jar98a; CN22]. The regularity theorem implies that once a monopole fits our definition it will also fit essentially any other reasonable definition, but the converse is not necessarily so clear. Of course, this converse would be resolved from a positive answer to the first question, that is, if the finite energy condition were enough to guarantee that monopoles fit our framework. Rigorously establishing the equivalence of these definitions would help to strengthen the connection between the results of this thesis and other research in the area.

Note that for the gauge group SU(2) these matters have been largely resolved [JT80; Gro84; AH88].

5.4 General compact groups

In Section 2.1 we restrict our study to monopoles with connected, simply connected, semisimple gauge groups, but in fact the theory for any real compact Lie group can be related to the one studied here. Let us discuss this more general setting now, informally considering how the framed moduli spaces would be constructed.

We start by observing that since that base manifold $\mathbb{R}^3$ is contractible, the adjoint bundle must be trivial. Therefore the configuration space only depends on the connected component of the gauge group which contains the identity. Furthermore, the same is true for the group of small gauge transformations, since these must tend to the identity at infinity. Hence, the framed moduli spaces for a given gauge group only depend on the connected component of the identity, and so we will restrict our attention to connected compact groups.

We also recall that a connected compact Lie group is always a product of a simply connected, semisimple Lie group and a torus, modulo a finite subgroup of the centre. In particular, its Lie algebra is the product of a semisimple and an Abelian Lie algebra.

We will now take two approaches, which largely mirror each other: firstly we will see how our constructions would be adapted to a general (connected) compact group, and secondly we will see how the moduli spaces for the general groups can be built from from the simply connected, semisimple case.

For the first approach, let us revisit the construction of the asymptotic model from Section 2.3. In order to build it on a sphere, we required the charge to satisfy the integrality condition $\exp(2\pi\kappa) = 1_G$. We then relied on the simple connectedness of the group to extend it radially over $\mathbb{R}^3$. If the group is not simply connected, however, the above integrality condition is not enough, and we have to require the path

$$\begin{aligned} [0,1] &\to G \\ \theta &\mapsto \exp(2\pi\theta\kappa) \end{aligned} \tag{5.4.1}$$

to be contractible.

In particular, this condition implies that the charge must be entirely inside the semisimple component of the Lie algebra. Furthermore, the exponential of $2\pi\kappa$ will be the identity in the simply connected semisimple group factor of the above decomposition.

Therefore, the choices of mass and charge are equivalent to choosing any mass, and then choosing a charge in the semisimple component of the Lie algebra with the integrality condition coming from the simply connected semisimple group factor in the decomposition. If we establish a choice of positive roots as in Section 5.1, this integrality condition is simply that the fundamental weights divided by i produce integers when applied to the charge, which is a condition that only depends on the Lie algebra.

In this case, the Lie algebra will have an extra Abelian factor in addition to the root space decomposition, which will be included in the C part of the adjoint bundle. However, the dimensional formula would only depend on the semisimple part, and would coincide with the dimension of the moduli space for the choice of mass and charge restricted to this factor. Notice, in particular, that it does not depend on the subgroup of the centre we are quotienting by.

Let us now look at the second approach by considering the three additional elements involved: torus groups, products and quotients.

If G is an Abelian group, the connection becomes trivial on the adjoint bundle. Then, we have

$$\mathrm{d}^*\mathrm{d}\Phi = \mathrm{d}^*\star F_A = 0\,, \tag{5.4.2}$$

that is, the Higgs field is harmonic on $\mathbb{R}^3$. Therefore it must be constant, and the connection must be flat, and hence gauge equivalent to the trivial one. Given this, the charge must be 0, and for any mass the moduli space would consist only of the monopole with a flat connection and a Higgs field constant and equal to the mass. Note the consistency with the integrality condition described above, which requires $\kappa = 0$ on Abelian groups, as well as with the dimension computations, which suggest a moduli space of dimension 0.

Suppose now that G is a product of two other groups. In this case, the connection and the Higgs field decompose along this product, and so do the gauge transformations. Therefore, the moduli space would simply be the product of the moduli spaces for the factor groups, with the masses and the charges simply being the components of each factor. This is also consistent with the above discussion.

Lastly, suppose that G is a finite quotient of a group by a finite subgroup of its centre. Since the base manifold is contractible, the configuration space will be the same for G and its cover. Furthermore, the group of small gauge transformations will also be the same. To see this, let us view these transformations as based maps from S^3 – the one-point compactification of $\mathbb{R}^3$ – into the (quotient) gauge group. Since the 3-sphere is simply connected such maps can always be lifted to a cover. This lift will provide a small gauge transformation for the cover gauge group. Furthermore, the action of this gauge transformation will remain unchanged, given that the quotient was carried out with respect to a central subgroup. Noting once again that the integrality condition and dimension formula depend only on the Lie algebra we see that the framed moduli spaces are not affected by this quotient operation.

To summarise, framed moduli spaces for any real compact group could be built from the framed moduli spaces for connected, simply connected, semisimple, compact groups, and in fact most of the interesting phenomena occur in these cases.

We end by noting that, of course, we could also restrict ourselves to simple groups, since compact, connected, simply connected, semisimple groups are always a product of simple groups.

Chapter 6

Special unitary and orthogonal groups

We now explain how our results apply to monopoles for specific gauge groups. This is done by studying the Lie algebra structures and the possible masses, charges and symmetry breaking types.

Firstly we look at general special unitary groups in Section 6.1, with special emphasis on SU(2) in Section 6.2 and SU(3) in Section 6.3.

We then move on to the special orthogonal groups in Sections 6.4 and 6.5. These are not simply connected, but offer a more familiar setting than the spin groups, their simply connected covers. Recall that in Section 5.4 we discuss how the framed moduli spaces for both groups would be the same.

A similar approach could be followed to study the symplectic and exceptional Lie groups, which would cover all the necessary building blocks for any real compact group, as explained in Section 5.4.

Many of the Lie algebra results can be found in the standard literature, like Hall's book [Hal15].

6.1 General $\mathrm{SU}(N)$-monopoles

Let us start by considering $\mathrm{SU}(N)$, the group of unitary $N \times N$ complex matrices of determinant one, with its Lie algebra $\mathfrak{su}(N)$, which is the space of anti-Hermitian traceless ones. We now have some preferred choices.

Firstly, the maximal toral subalgebra $\mathfrak{t}$ can be taken to be the subspace of diagonal matrices inside $\mathfrak{su}(N)$. These matrices will have N imaginary terms which add up to zero, that is, they are of the form

$$\operatorname{diag}(z_1, z_2, \ldots, z_N) = \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_N \end{pmatrix} \tag{6.1.1}$$

for some $z_1, z_2, \ldots, z_N \in i\mathbb{R}$ such that $z_1 + z_2 + \cdots + z_N = 0$.

The roots then correspond to the $N(N-1)$ linear functionals which subtract one diagonal term from another (different) one, that is, maps of the form

$$\operatorname{diag}(z_1, z_2, \ldots, z_N) \mapsto z_{j_1} - z_{j_2}\,, \tag{6.1.2}$$

for $j_1 \neq j_2$. But there is furthermore a preferred set of positive roots R^+: those which subtract a diagonal term further down from one which is higher up, that is, maps of the form (6.1.2) but only for $j_1 < j_2$. The simple roots corresponding to this choice are then the ones which subtract a diagonal term from the term immediately above. We denote these by

$$\alpha_j \colon \operatorname{diag}(z_1, z_2, \ldots, z_N) \mapsto z_j - z_{j+1}\,, \tag{6.1.3}$$

for $j = 1, 2, \ldots, N-1 = \operatorname{rank}(\mathrm{SU}(N))$. The fundamental weights associated to this choice are given by

$$w_j \colon \operatorname{diag}(z_1, z_2, \ldots, z_N) \mapsto z_1 + z_2 + \cdots + z_j\,. \tag{6.1.4}$$

Recall that, given that all maximal toral subalgebras and choices of positive roots are related through the adjoint action of the group, we are not restricting ourselves by making these choices. Therefore, let us now consider a mass μ and a charge κ which are compatible.

The mass must be any element in the closure of the fundamental Weyl chamber, that is, we must have $\frac{1}{i}\alpha(\mu) \geq 0$ for all the positive roots $\alpha \in R^+$. This means that it must be of the form

$$\mu = i\,\mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_N)\,, \tag{6.1.5}$$

subject to the condition

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \tag{6.1.6}$$

(and satisfying $\mu_1 + \mu_2 + \cdots + \mu_N = 0$).

Then, the charge must satisfy that $\frac{1}{i}\alpha(\kappa) \leq 0$ when $\alpha(\mu) = 0$. This means that it must be of the form

$$\kappa = i\,\mathrm{diag}(\kappa_1, \kappa_2, \ldots, \kappa_N)\,, \tag{6.1.7}$$

subject to the condition

$$\mu_j = \mu_{j+1} \implies \kappa_j \leq \kappa_{j+1} \tag{6.1.8}$$

(and satisfying $\kappa_1 + \kappa_2 + \cdots + \kappa_N$). The $N-1$ integer charges for this choice of μ and κ are therefore the integers

$$\frac{1}{i} w_j(\kappa) = \kappa_1 + \kappa_2 + \cdots + \kappa_j, \tag{6.1.9}$$

for $j = 1, 2, \ldots, N-1$. The dimension formula yields

$$\dim_{\mu,\kappa} = 4\sum_{j=1}^{N-1}(N-j)\kappa_j\,. \tag{6.1.10}$$

As we have seen, the eigenvalues of the mass must be decreasing in their imaginary parts, and the eigenvalues of the charge must be increasing in their imaginary parts within each block of equal eigenvalues in the mass. In terms of the forms (6.1.5) and (6.1.7), supposing that μ has B blocks of sizes $N_1, N_2, \ldots, N_B$, these conditions can be rewritten as

$$\mu_1 = \mu_2 = \cdots = \mu_{N_1} > \mu_{N_1+1} = \mu_{N_1+2} = \cdots = \mu_{N_1+N_2} > \cdots, \tag{6.1.11a}$$

$$\kappa_1 \le \kappa_2 \le \cdots \le \kappa_{N_1}, \quad \kappa_{N_1+1} \le \kappa_{N_1+2} \le \cdots \le \kappa_{N_1+N_2}, \quad \cdots. \tag{6.1.11b}$$

Naturally, we must have $N_1 + N_2 + \cdots N_B = N$. The magnetic charges are then those which include only entire blocks in which the eigenvalues of the mass are equal, that is, the numbers

$$\frac{1}{i} w_{N_1+N_2+\cdots+N_j}(\kappa) = \kappa_1 + \kappa_2 + \cdots + \kappa_{N_1+N_2+\cdots+N_j} \tag{6.1.12}$$

for $j = 1, 2, \ldots, B-1$. The rest of the charges are holomorphic.

In this case the symmetry of the monopole breaks to the group

$$\mathrm{S}(\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_B)) \tag{6.1.13}$$

of block-diagonal special unitary elements, of dimension

$$N_1^2 + N_2^2 + \cdots + N_B^2 - 1\,. \tag{6.1.14}$$

Maximal symmetry breaking corresponds to the case

$$B = N \iff N_1 = N_2 = \cdots = N_B = 1 \iff \mu_1 > \mu_2 > \cdots > \mu_N\,. \tag{6.1.15}$$

Naturally, here the symmetry breaks to the maximal torus

$$\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)) \cong T^{N-1}\,. \tag{6.1.16}$$

6.2 SU(2)-monopoles

For SU(2) monopoles, the mass must be of the form $\mu = i\,\mathrm{diag}(\mu_1, -\mu_1)$. But in fact, when the mass is not zero, the monopole is usually rescaled over $\mathbb{R}^3$ so that we only consider

$$\mu = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{6.2.1}$$

The charge is then given by a single integer κ_1, which is itself usually referred to simply as the charge, so that

$$\kappa = i \begin{pmatrix} \kappa_1 & 0 \\ 0 & -\kappa_1 \end{pmatrix} . \tag{6.2.2}$$

In these conditions, the moduli space of framed monopoles of a given charge $\kappa_1 \geq 0$ is a hyper-Kähler manifold of dimension $4\kappa_1$, as we expect from our formula.

Note that, in the case of non-zero mass, the single integer charge is magnetic.

6.3 SU(3)-monopoles

For SU(3) the situation is a bit more interesting, since we can now have different types of symmetry breaking.

Throughout this section we represent several elements in $\mathfrak{t}$ and its (complexified) dual in diagrams, where $\mathfrak{t}$ is a maximal toral subalgebra of the Lie algebra $\mathfrak{su}(3)$.

In particular, we represent them on a 2-(real-)dimensional plane, which we can think of as the plane given by the equation $x_1+x_2+x_3 = 0$ inside the 3-dimensional space $\mathbb{R}^3$, where all the points with integer coordinates are marked with small dots. However, we also admit triples of coordinates which do not add up to 0, which simply represent their orthogonal projections from $\mathbb{R}^3$ onto the plane. Note that this allows points with integer coordinates (not adding up to 0) which are not marked, since the projections might not have integer coordinates.

In these diagrams, we represent elements of two different but closely related spaces: $\mathfrak{t}$ and $i\mathfrak{t}^*$. In the former case, the coordinates (x_1, x_2, x_3), with $x_1+x_2+x_3 = 0$, will represent the matrix

$$i \operatorname{diag}(x_1, x_2, x_3) , \tag{6.3.1}$$

and will be marked with dots. In the latter case, the coordinates (ξ_1, ξ_2, ξ_3) – even if they do not add up to 0 – will represent the map

$$i\,\mathrm{diag}(x_1, x_2, x_3) \mapsto \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3\,, \tag{6.3.2}$$

and will be marked with arrows.

In Figure 6.3.3a we demostrate the coordinate system. In Figure 6.3.3b we show the root system of $\mathfrak{su}(3)$, following the notation of the previous section for α_1 and α_2, and furthermore shade the preferred Weyl chamber and show the associated fundamental weights w_1 and w_2. We also mark the lines $(\frac{1}{i}w_j)^{-1}(\mathbb{Z})$, for $j = 1, 2$ on whose intersections lie precisely the marked integer points, which are those which satisfy the integrality condition required of charges.

We will now study the possible charges that can appear for different types of symmetry breaking. These are represented in Figure 6.3.4.

Let us firstly assume that we have maximal symmetry breaking, that is,

$$\mu = i \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}\,. \tag{6.3.5}$$

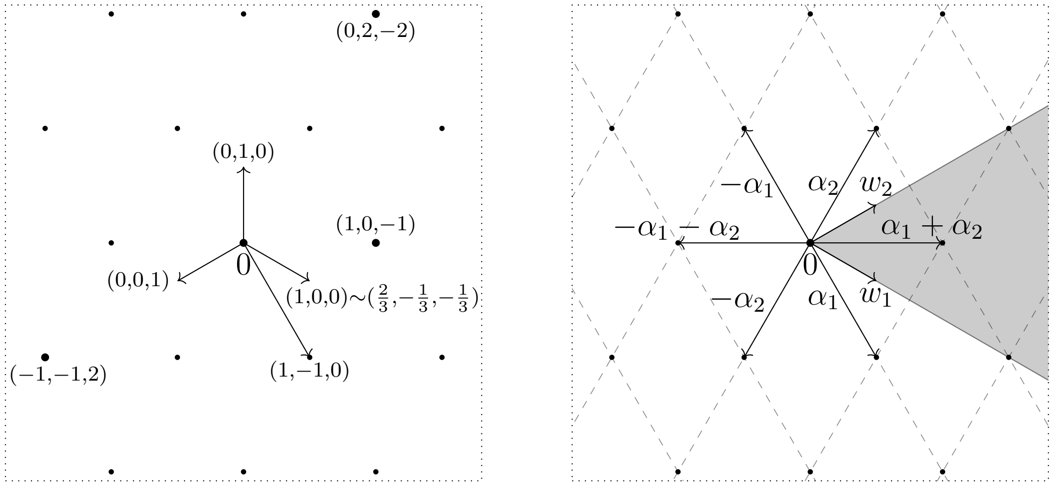
with $\mu_1 > \mu_2 > \mu_3$ and $\mu_1 + \mu_2 + \mu_3 = 0$.

Then, the two integer charges are magnetic, there is no compatibility condition on the charge, and we expect to have monopoles with any non-negative integer values for these two magnetic charges. In Figure 6.3.4a we see the region of non-negative charges, marked with dense diagonal lines, and some possible charges κ are annotated with the pair of corresponding magnetic charges $(\frac{1}{i}w_1(\kappa), \frac{1}{i}w_2(\kappa))$.

The case of non-maximal symmetry breaking is slightly more involved. Let us suppose that $\mu_2 = \mu_3$, so that

$$\mu = i \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}\,, \tag{6.3.6}$$

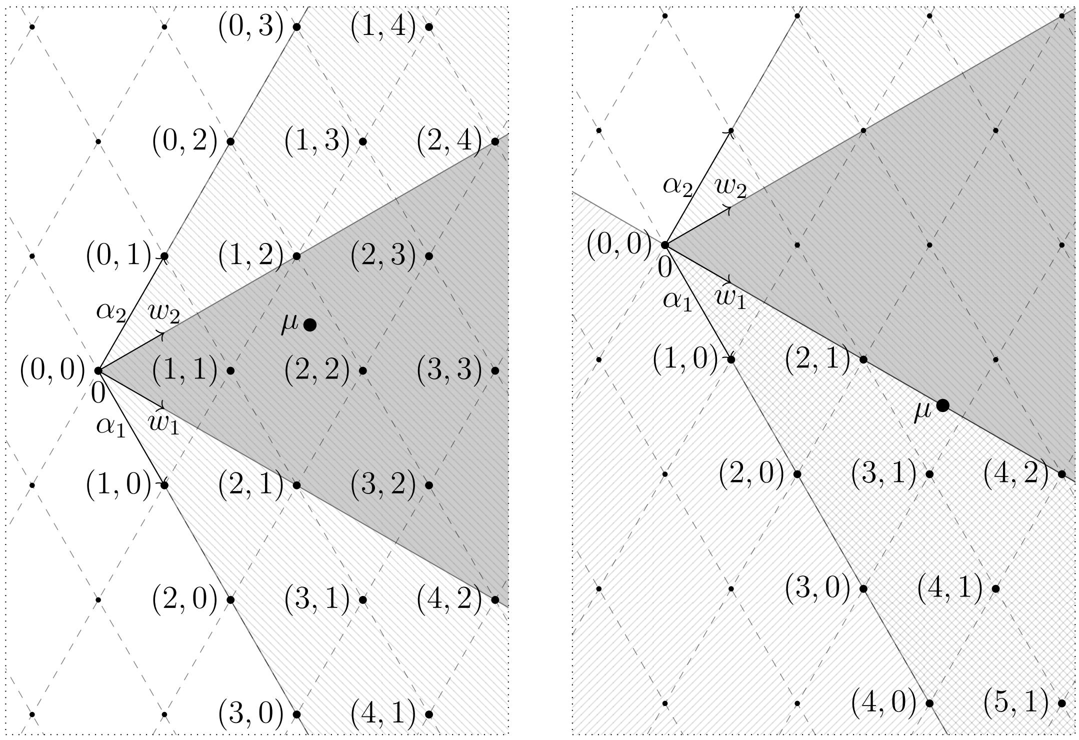
with $\mu_1 > \mu_2$ and $\mu_1 + 2\mu_2 = 0$.



a: Diagram with integer points marked and examples of elements with coordinates

b: Root system, preferred Weyl chamber, fundamental weights and integral lines

Figure 6.3.3: Diagram of $\mathfrak{t}$ and $i\mathfrak{t}^*$



a: Maximal symmetry breaking

b: Non-maximal symmetry breaking

Figure 6.3.4: Possible charges for different symmetry breaking types

Then, the first charge $\frac{1}{i}w_1(\kappa)$ is still magnetic, but $\frac{1}{i}w_2(\kappa)$ is in this case holomorphic. In Figure 6.3.4b, aside from the region of non-negative charges, we see the compatibility condition added to the diagram, marked with perpendicular dense diagonal lines. The intersection of these two regions contains the possibilities for which we expect to have monopoles. Once again, some of these possible charges are annotated with their integer charges, the first one being the magnetic one and the second one being the holomorphic one. Notice that for each choice of the magnetic charge we only have finitely many choices of the holomorphic one, which can also be deduced from the fact that the charge must be of the form

$$\mu = i \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}, \tag{6.3.7}$$

with $\kappa_2 < \kappa_3$ (compatibility), $\kappa_1, \kappa_1 + \kappa_2 \geq 0$ (non-negativity) and, of course, $\kappa_1 + \kappa_2 + \kappa_3 = 0$. Using this, given any non-negative choice of the magnetic charge $\frac{1}{i}w_1(\kappa) = \kappa_1$, for the holomorphic charge we must have

$$\frac{1}{i}w_2(\kappa) = -\kappa_3 \in \left\{0, 1, \ldots, \left\lfloor \frac{\kappa_1}{2} \right\rfloor \right\}. \tag{6.3.8}$$

This covers all the interesting possibilities for SU(3), since the only other possible symmetry breaking for non-zero mass would be for $\mu = i\,\mathrm{diag}(\mu_1, \mu_1, \mu_2)$. However, by changing the sign of the Higgs field and reflecting the underlying space $\mathbb{R}^3$ (thereby changing the orientation) we obtain a case equivalent to the last one discussed.

The case of non-maximal symmetry breaking has been studied already in some detail, especially for magnetic charge 2 [Dan92; DL93; Dan94; BS98]. In the interpretation of the stratified spaces of rational maps pointed at in Section 5.2, the moduli space for magnetic charge 2 has an open stratum of dimension 12 corresponding to holomorphic charge 1 and a lower stratum of dimension 10 corresponding to holomorphic charge 0. In terms of rational maps, these are made up of based maps from $\mathbb{CP}^1$ to $\mathrm{SU}(3)/\mathrm{U}(2) \cong \mathbb{CP}^2$ of degree 2, with the lower stratum consisting of the maps which fall into a specific line $\mathbb{CP}^1 \subset \mathbb{CP}^2$.

Since for holomorphic charge 1 the charge κ has the same centraliser as the mass, the open stratum would correspond directly to the moduli space we have constructed (which, of course, also has dimension 12). However, for holomorphic charge 0 the orbit of the charge κ under the adjoint action of the centraliser of the mass is a 2-sphere, so, in terms of the fibration picture, the lower 10-dimensional stratum would be a fibration over this sphere, with the 8-dimensional fibres corresponding to the moduli spaces of this thesis. The base of this fibration could also be viewed as parametrising the line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ which forms the image of the maps.

Bais and Schroers give an account of some properties of SU(3) moduli spaces beyond charge 2. In particular, it is worth remarking on the similarity and relationship between Figure 6.3.4 and Figure 1 in their work. In their case, they point out the dimension of the strata of the stratified moduli space, which differs from our dimension when the stratum is a fibration, as explained. Note that the orbit of the charge under the action of C_μ is always a 2-sphere in $\mathfrak{su}(3)$ when the holomorphic charge is not one half of the magnetic charge.

6.4 SO(2N)-monopoles

Let us now consider the group SO($2N$) (for $N \geq 2$) of orthogonal, orientation-preserving $2N \times 2N$ real matrices, and its Lie algebra $\mathfrak{so}(2N)$ of anti-symmetric matrices.

A maximal toral subalgebra can be written as the space of matrices of the block diagonal form

$$\text{(6.4.1)} \qquad \operatorname{bdiag}(z_1, z_2, \ldots, z_N) := \left(\begin{array}{cc|cc|c|cc} 0 & z_1 & 0 & 0 & \multirow{2}{*}{$\cdots$} & 0 & 0 \\ -z_1 & 0 & 0 & 0 & & 0 & 0 \\ \hline 0 & 0 & 0 & z_2 & \multirow{2}{*}{$\cdots$} & 0 & 0 \\ 0 & 0 & -z_2 & 0 & & 0 & 0 \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline 0 & 0 & 0 & 0 & \multirow{2}{*}{$\cdots$} & 0 & z_N \\ 0 & 0 & 0 & 0 & & -z_N & 0 \end{array}\right),$$

with $z_1, z_2, \ldots, z_N \in \mathbb{R}$.

The root system in this case has rank N, with the roots corresponding to the $2N(N-1)$ maps of the form

$$\text{(6.4.2a)} \qquad \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto i(z_{j_1} + z_{j_2})\,,$$

$$\text{(6.4.2b)} \qquad \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto i(z_{j_1} - z_{j_2})\,,$$

$$\text{(6.4.2c)} \qquad \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto -i(z_{j_1} + z_{j_2})\,,$$

$$\text{(6.4.2d)} \qquad \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto -i(z_{j_1} - z_{j_2})\,,$$

for $j_1 < j_2$.

We can choose the roots of the forms (6.4.2a) and (6.4.2b) to constitute the set of positive roots R^+, among which the N simple roots are

$$\text{(6.4.3a)} \qquad \alpha_j\colon \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto i(z_j - z_{j+1}) \qquad \text{for } j = 1, 2, \ldots, N-1\,,$$

$$\text{(6.4.3b)} \qquad \alpha_N\colon \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto i(z_{N-1} + z_N)\,.$$

The corresponding fundamental weights are then

$$\text{(6.4.4a)} \qquad w_j\colon \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto i(z_1 + z_2 + \cdots + z_j) \qquad \text{for } j = 1, 2, \ldots, N-2\,,$$

$$\text{(6.4.4b)} \qquad w_{N-1}\colon \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto \frac{i}{2}(z_1 + z_2 + \cdots + z_{N-1} - z_N)\,,$$

$$\text{(6.4.4c)} \qquad w_N\colon \operatorname{bdiag}(z_1, z_2, \ldots, z_N) \mapsto \frac{i}{2}(z_1 + z_2 + \cdots + z_{N-1} + z_N)\,.$$

A mass μ for these choices must therefore be of the form

$$\text{(6.4.5)} \qquad \mu = \operatorname{bdiag}(\mu_1, \mu_2, \ldots, \mu_N)\,,$$

satisfying

$$\text{(6.4.6)} \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{N-1} \geq |\mu_N|\,,$$

and compatible charges must be of the form

$$\kappa = \operatorname{bdiag}(\kappa_1, \kappa_2, \ldots, \kappa_N)\,, \tag{6.4.7}$$

where

$$\mu_j = \mu_{j+1} \implies \kappa_j \leq \kappa_{j+1} \qquad \text{for } j = 1, \ldots, N-1\,, \tag{6.4.8a}$$

$$\mu_{N-1} = -\mu_N \implies \kappa_{N-1} \leq -\kappa_N\,. \tag{6.4.8b}$$

The integer charges in this case are the numbers

$$\frac{1}{i}w_j = \kappa_1 + \kappa_2 + \cdots + \kappa_j \qquad \text{for } j = 1, \ldots, N-2\,, \tag{6.4.9a}$$

$$\frac{1}{i}w_{N-1} = \frac{1}{2}(\kappa_1 + \kappa_2 + \cdots + \kappa_{N-1} - \kappa_N)\,, \tag{6.4.9b}$$

$$\frac{1}{i}w_N = \frac{1}{2}(\kappa_1 + \kappa_2 + \cdots + \kappa_{N-1} + \kappa_N)\,. \tag{6.4.9c}$$

Note here how these are integers precisely when all the numbers κ_j are integers and additionally they add up to an even number. Without this last condition, we would still have $\exp(2\pi\kappa) = 1_{\mathrm{SO}(2N)}$, but not $\exp(2\pi\kappa) = 1_{\mathrm{Spin}(2N)}$ (see Section 5.4 for a discussion of integrality conditions on groups which are not simply connected).

From the integer charges we can deduce that the dimension of the corresponding moduli space of framed monopoles is

$$\dim_{\mu,\kappa} = \sum_{j=1}^{N-1} (N-j)\kappa_j\,. \tag{6.4.10}$$

In order to understand the possible symmetry breaking types, let us assume, once again, that the mass element is split into blocks[8]. However, this time we consider blocks in which the elements μ_j are equal *up to sign*. Given (6.4.6), this is only relevant in the last block, since all elements but μ_N must be non-negative, and μ_N can only be equal in absolute value to the elements in the last block. In

[8] By this we mean “large” blocks made up of the 2×2 blocks in (6.4.1).

other words, we have

$$\begin{aligned}\mu_1 = \mu_2 = \cdots = \mu_{N_1} > \mu_{N_1+1} = \mu_{N_1+2} = \cdots = \mu_{N_1+N_2} > \cdots \\ > \mu_{N-N_B+1} = \mu_{N-N_B+2} = \cdots = \mu_{N-1} = \pm\mu_N ,\end{aligned} \tag{6.4.11}$$

with the charge hence satisfying

$$\begin{aligned}\kappa_1 \le \kappa_2 \le \cdots \le \kappa_{N_1}, \quad \kappa_{N_1+1} \le \kappa_{N_1+2} \le \cdots \le \kappa_{N_1+N_2}, \quad \cdots \\ \kappa_{N-N_B+1} \le \kappa_{N-N_B+2} \le \cdots \le \kappa_{N-1} \le \pm\kappa_N .\end{aligned} \tag{6.4.12}$$

The magnetic charges will always include those of the form $\frac{1}{i}w_{N_1+N_2+\cdots+N_j}(\kappa)$ for $j = 1, 2, \ldots, B-1$. However, we also have $\frac{1}{i}w_{N-1}(\kappa)$, given by (6.4.9b), if $\mu_{N-1} \neq \mu_N$, and $\frac{1}{i}w_N(\kappa)$, given by (6.4.9c), if $\mu_{N-1} \neq -\mu_N$. Note that both of these last two conditions hold simultaneously precisely when $N_B = 1$ (in which case $N_1 + N_2 + \cdots + N_{B-1} = N-1$), and neither will be satisfied if $N_B > 1$ and the last block is 0. The remaining charges are holomorphic.

Let us now study the symmetry breaking groups. By looking at the root spaces we can deduce that the elements which commute with the mass must be block diagonal, following the blocks of sizes $2N_1, 2N_2, \ldots, 2N_B$ outlined above.

Let us therefore consider the case of any such $2N_j \times 2N_j$ block, and we assume that $\mu_{N_1+N_2+\cdots+N_j} \neq 0$. After dividing by a positive number, the corresponding block of the mass element will be of the form

$$\left(\begin{array}{cc|cc|c|cc} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & & 0 & 0 \\ \hline \vdots & & \vdots & & \ddots & \vdots & \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & & -1 & 0 \end{array}\right), \tag{6.4.13}$$

where the sign of the last 2×2 block might have the opposite sign if $j = B$. By identifying $\mathbb{R}^{2N_j}$ with $\mathbb{C}^{N_j}$ we can see that the subgroup of $\mathrm{SO}(2N_j)$ which preserves this is precisely $\mathrm{U}(N_j)$.

On the other hand, if $\mu_{N_1+N_2+\cdots+N_j} = 0$, the subgroup preserving it is simply $\mathrm{SO}(2N_j)$. This can only happen when $j = B$ and $\alpha_{N-1}(\mu) = \alpha_N(\mu) = 0$.

We therefore have two slightly different cases. Firstly, if

$$\alpha_{N-1}(\mu) = \alpha_N(\mu) = 0\,, \tag{6.4.14}$$

then the symmetry breaks to

$$\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_{B-1}) \times \mathrm{SO}(2N_B)\,. \tag{6.4.15}$$

Otherwise it breaks to

$$\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_{B-1}) \times \mathrm{U}(N_B)\,. \tag{6.4.16}$$

The case of maximal symmetry breaking here corresponds to

$$B = N \iff N_1 = N_2 = \cdots = N_B = 1 \iff \mu_1 > \mu_2 > \cdots > \mu_{N-1} > |\mu_N|\,, \tag{6.4.17}$$

where the symmetry breaks to the maximal torus

$$\mathrm{U}(1) \times \mathrm{U}(1) \times \cdots \times \mathrm{U}(1) \times \mathrm{U}(1) \cong T^N\,. \tag{6.4.18}$$

6.5 $\mathrm{SO}(2N+1)$-monopoles

Lastly, let us consider the group $\mathrm{SO}(2N+1)$ (for $N \geq 1$) of orthogonal, orientation-preserving $(2N+1) \times (2N+1)$ real matrices, and its Lie algebra $\mathfrak{so}(2N+1)$ of anti-symmetric matrices.

Its maximal toral subalgebra can be identified with the one for $\mathfrak{so}(2N)$, since it can be written as the space of matrices of the form

$$\operatorname{bdiag}_0(z_1, z_2, \dots, z_N) \coloneqq \left(\begin{array}{cc|cc|c|cc|c} 0 & z_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -z_1 & 0 & 0 & 0 & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & z_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -z_2 & 0 & & 0 & 0 & 0 \\ \hline \vdots & & \vdots & & \ddots & \vdots & & \vdots \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & z_N & 0 \\ 0 & 0 & 0 & 0 & & -z_N & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right), \tag{6.5.1}$$

with $z_1, z_2, \dots, z_N \in \mathbb{R}$.

Through the identification

$$\operatorname{bdiag}(z_1, z_2, \dots, z_N) \leftrightarrow \operatorname{bdiag}_0(z_1, z_2, \dots, z_N)\,, \tag{6.5.2}$$

the roots here are the same as for $\mathfrak{so}(2N)$, which were given by (6.4.2), with the addition of the $2N$ maps of the form

$$\operatorname{bdiag}_0(z_1, z_2, \dots, z_N) \mapsto iz_j\,, \tag{6.5.3a}$$

$$\operatorname{bdiag}_0(z_1, z_2, \dots, z_N) \mapsto -iz_j\,, \tag{6.5.3b}$$

for $j = 1, 2, \dots, N$, having a total of $2N^2$ roots.

A set of positive roots R^+ can then be defined from those of the forms (6.4.2a), (6.4.2b) and (6.5.3a), and the corresponding simple roots are

$$\alpha_j\colon \operatorname{bdiag}_0(z_1, z_2, \dots, z_N) \mapsto i(z_j - z_{j+1}) \qquad \text{for } j = 1, 2, \dots, N-1\,, \tag{6.5.4a}$$

$$\alpha_N\colon \operatorname{bdiag}_0(z_1, z_2, \dots, z_N) \mapsto iz_N\,. \tag{6.5.4b}$$

The fundamental weights are then

$$w_j\colon\ \mathrm{bdiag}(z_1, z_2, \ldots, z_N) \mapsto i(z_1 + z_2 + \cdots + z_j) \quad \text{for } j = 1, 2, \ldots, N-1\,, \tag{6.5.5a}$$

$$w_N\colon\ \mathrm{bdiag}(z_1, z_2, \ldots, z_N) \mapsto \frac{i}{2}(z_1 + z_2 + \cdots + z_N)\,. \tag{6.5.5b}$$

We now consider a mass μ and a charge κ compatible with these choices. The mass must be of the form

$$\mu = \mathrm{bdiag}_0(\mu_1, \mu_2, \ldots, \mu_N)\,, \tag{6.5.6}$$

with

$$\mu_1 \geq \mu_2 \geq \cdots \mu_N \geq 0\,, \tag{6.5.7}$$

and the charge must be

$$\kappa = \mathrm{bdiag}_0(\kappa_1, \kappa_2, \ldots, \kappa_N)\,, \tag{6.5.8}$$

where

$$\mu_j = \mu_{j+1} \implies \kappa_j \leq \kappa_{j+1} \qquad \text{for } j = 1, \ldots, N-1\,, \tag{6.5.9a}$$

$$\mu_N = 0 \implies \kappa_N \leq 0\,. \tag{6.5.9b}$$

The integer charges are then

$$\frac{1}{i}w_j = \kappa_1 + \kappa_2 + \cdots + \kappa_j \qquad \text{for } j = 1, \ldots, N-1\,, \tag{6.5.10a}$$

$$\frac{1}{i}w_N = \frac{1}{2}(\kappa_1 + \kappa_2 + \cdots + \kappa_N)\,. \tag{6.5.10b}$$

We once again note how the integrality condition requires the exponential of $2\pi\kappa$ to be the identity in the 2-to-1 cover $\mathrm{Spin}(2N+1)$.

The dimension of the moduli space of framed monopoles of mass μ and charge

κ in this case is

$$\dim_{\mu,\kappa} = \sum_{j=1}^{N} \Big(N + \frac{1}{2} - j\Big)\kappa_j \,. \tag{6.5.11}$$

To study the possible symmetry breaking types we once again divide the matrices into B "large" blocks, each corresponding to distinct values of the numbers μ_j. For notational convenience, we always leave the last diagonal element outside of these blocks, so they have sizes $2N_1, 2N_2, \dots, 2N_B$. In this case, the values μ_j will be exactly the same in each block, since they must all be non-negative (as opposed to the case of SO($2N$) where a change of sign was allowed). In other words, the conditions (6.1.11) are satisfied (with the addition of $\mu_N \geq 0$ and (6.5.9b)).

The magnetic charges, once more, include those of the form $\frac{1}{i} w_{N_1+N_2+\cdots+N_j}(\kappa)$ for $j = 1, 2, \dots, B-1$, as well as the charge $\frac{1}{i} w_N(\kappa)$ when $\mu_N \neq 0$, with the rest being holomorphic.

Lastly, we can once again look at the root spaces to see that the centraliser of the mass element is also block-diagonal. Also similarly, the centraliser of a block of size N_j with $\mu_{N_j} > 0$ is U(N_j). The only block which can be zero is the last one, when $\alpha_N(\mu) = 0$, in which case the centraliser must be considered adding the last 1×1 block in (6.5.1), yielding the group SO($2N_B + 1$).

Summarising, if

$$\alpha_N(\mu) = 0 \,, \tag{6.5.12}$$

then the symmetry breaks to

$$\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_{B-1}) \times \mathrm{SO}(2N_B + 1) \,. \tag{6.5.13}$$

Otherwise it breaks to

$$\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_{B-1}) \times \mathrm{U}(N_B) \,. \tag{6.5.14}$$

The case of maximal symmetry breaking now corresponds to

$$B = N \iff N_1 = N_2 = \cdots = N_B = 1 \iff \mu_1 > \mu_2 > \cdots > \mu_N > 0\,, \tag{6.5.15}$$

where the symmetry breaks to the maximal torus

$$\mathrm{U}(1) \times \mathrm{U}(1) \times \cdots \times \mathrm{U}(1) \times \mathrm{U}(1) \cong T^N\,. \tag{6.5.16}$$

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Appendix A

Dirac operators

In this appendix we work out some details regarding Dirac operators and spinor bundles which are relevant to us. Although they are not particularly original or surprising results – in fact, they are used in the cited literature –, we write them out to have a clear picture of the elements involved and the notation and conventions followed.

We shall not dwell on the general theory, which can be found in many references, like Lawson and Michelsohn's book [LM89], and instead focus on the particular case at hand.

We will start by looking at the pointwise picture of the relevant bundles in Sections A.1 and A.2, and then we will translate this to actual bundles and see how it relates to certain differential operators in Section A.3. Lastly, in Section A.4 we consider the case of the Euclidean 3-space which is relevant to us.

A.1 Linear algebra in dimension 3

Let $\mathbb{R}^3 = \operatorname{span}(e_1, e_2, e_3)$ be the oriented Euclidean 3-space. Later, we will think about this space as the fibre of the tangent bundle of an oriented Riemannian 3-manifold. We will consider several other vector spaces and their relationships. These spaces will then yield relevant bundles on the manifold.

Let us therefore also consider the space $\mathbb{C}(2)$ of complex 2×2 matrices. A useful basis for this space is given by

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{A.1.1}$$

where we notice that the matrices τ_1, τ_2, τ_3 are obtained from the Pauli matrices, and that they satisfy the quaternionic relations

$$\tau_1^2 = \tau_2^2 = \tau_3^2 = \tau_1\tau_2\tau_3 = -\tau_0\,. \tag{A.1.2}$$

Now let $\mathbb{C}\mathrm{l}(\mathbb{R}^3)$ denote the complex Clifford algebra of $\mathbb{R}^3$. As a vector space, this is isomorphic to the complexification of the exterior algebra of $\mathbb{R}^3$, that is,

$$\mathbb{C}\mathrm{l}(\mathbb{R}^3) \cong (\textstyle\bigwedge(\mathbb{R}^3))^{\mathbb{C}}\,, \tag{A.1.3}$$

where

$$\begin{aligned} \textstyle\bigwedge(\mathbb{R}^3) &= \textstyle\bigwedge^0(\mathbb{R}^3) \oplus \bigwedge^1(\mathbb{R}^3) \oplus \bigwedge^2(\mathbb{R}^3) \oplus \bigwedge^3(\mathbb{R}^3) \\ &= \mathrm{span}(1, e_1, e_2, e_3, \star e_1, \star e_2, \star e_3, \star 1)\,. \end{aligned} \tag{A.1.4}$$

Furthermore, as an algebra, we have the isomorphism

$$\mathbb{C}\mathrm{l}(\mathbb{R}^3) \cong \mathbb{C}(2) \oplus \mathbb{C}(2)\,, \tag{A.1.5}$$

which is given by

$$1 \leftrightarrow (\tau_0, \tau_0)\,, \tag{A.1.6a}$$

$$e_j \leftrightarrow (\tau_j, -\tau_j)\,, \tag{A.1.6b}$$

$$\star e_j \leftrightarrow (\tau_j, \tau_j)\,, \tag{A.1.6c}$$

$$\star 1 \leftrightarrow (\tau_0, -\tau_0)\,, \tag{A.1.6d}$$

for $j = 1, 2, 3$, where the basis elements are interpreted through the isomorphism (A.1.3).

The spin group Spin(3) is defined as the unit-norm elements of the real subspace $\mathrm{span}_{\mathbb{R}}(1, \star e_1, \star e_2, \star e_3) \subset \mathbb{C}\mathrm{l}(\mathbb{R}^3)$. If we regard it as a subspace of $\mathbb{C}(2)$ through either of the two (in this case, equivalent) components of the isomorphism (A.1.5), we obtain an isomorphism

$$\text{Spin}(3) \cong \text{SU}(2)\,. \tag{A.1.7}$$

Furthermore, since Spin(3) is inside the Clifford algebra, its elements can act on the Clifford algebra itself through conjugation. This action preserves the space $\mathbb{R}^3 \subset \mathbb{C}\mathrm{l}(\mathbb{R}^3)$, and in fact is orthogonal and preserves the orientation. This provides a spin representation $\rho^{\mathbb{R}^3}$ on $\mathbb{R}^3$ and in fact defines a 2-to-1 cover

$$\text{Spin}(3) \overset{2:1}{\to} \text{SO}(3)\,. \tag{A.1.8}$$

We now define several additional spaces with Clifford and spin actions.

We start with the space $\mathbb{C}^2$, which will define the spinor bundle.

Definition A.1.9. On the space $\mathbb{C}^2$ we define the Clifford action as

$$\mathrm{cl}^{\mathbb{C}^2}_{(\tau,\tau')} := \tau'\,, \tag{A.1.10}$$

that is, as the action of the matrix given by the second component of the isomorphism (A.1.5). We define the spin action as the restriction of this action to the group Spin(3),

$$\rho^{\mathbb{C}^2}_{\tau} := \tau\,, \tag{A.1.11}$$

that is, the defining SU(2) representation through the isomorphism (A.1.7).

Remark A.1.12. Note that there are two inequivalent irreducible Clifford actions, given by the action of the matrix given by either component of (A.1.5). Here we take the second component, but making the other choice would simply change some signs. Note that the spin representation would not change in either case.

We now consider the space $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$. It will be convenient to view it as a space of matrices through the isomorphism

$$\mathbb{C}^2 \otimes (\mathbb{C}^2)^* \cong \mathbb{C}(2)\,. \tag{A.1.13}$$

Definition A.1.14. On the space $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$ we define the Clifford action as the action defined on its first component,

$$\mathrm{cl}^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{(\tau,\tau')}(\bullet) := \tau'\bullet\,, \tag{A.1.15}$$

that is, left matrix multiplication on $\mathbb{C}(2)$ by the second component of the isomorphism (A.1.5). The spin representation is defined as the representation associated to this space from the one on $\mathbb{C}^2$,

$$\rho^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{\tau}(\bullet) := \rho^{\mathbb{C}^2}_{\tau}(\bullet) \otimes (\rho^{\mathbb{C}^2}_{\tau})^*(\bullet) = \tau\bullet\tau^{-1}\,, \tag{A.1.16}$$

that is, matrix conjugation on $\mathbb{C}(2)$ through the isomorphism (A.1.7).

Definition A.1.17. On the space $(\mathbb{R}^3)^* \oplus \mathbb{R}$ we define a spin action $\rho^{(\mathbb{R}^3)^*\oplus\mathbb{R}}$ as the dual of $\rho^{\mathbb{R}^3}$ together with the trivial action on $\mathbb{R}$.

Now, if $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is the dual basis of $\{e_1, e_2, e_3\}$, we have an isomorphism

$$((\mathbb{R}^3)^* \oplus \mathbb{R})^{\mathbb{C}} \cong \mathbb{C}^2 \otimes (\mathbb{C}^2)^* \tag{A.1.18}$$

given by

$$\hat{e}_j \leftrightarrow -\tau_j\,, \tag{A.1.19a}$$

$$1 \leftrightarrow \tau_0\,, \tag{A.1.19b}$$

where we are once again we are viewing the latter space as $\mathbb{C}(2)$. Crucially, the spin representations on both spaces coincide.

Proposition A.1.20. *The isomorphism* (A.1.18) *exchanges the spin representations* $(\rho^{(\mathbb{R}^3)^*\oplus\mathbb{R}})^{\mathbb{C}}$ *and* $\rho^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}$.

Proof. We can see this by looking at the representations induced on the Lie algebra

$$\mathfrak{spin}(3) = \mathrm{span}_{\mathbb{R}}(\star e_1, \star e_2, \star e_3) \cong \mathrm{span}_{\mathbb{R}}(\tau_1, \tau_2, \tau_3)\,. \tag{A.1.21}$$

If (j_1, j_2, j_3) is a cyclic permutation of $(1, 2, 3)$, the representation on $(\mathbb{R}^3)^* \oplus \mathbb{R}$

is given by

$$\rho^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{\star e_{j_1}}(\hat{e}_{j_1}) = 0\,, \tag{A.1.22a}$$

$$\rho^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{\star e_{j_1}}(\hat{e}_{j_2}) = 2\hat{e}_{j_3}\,, \tag{A.1.22b}$$

$$\rho^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{\star e_{j_1}}(\hat{e}_{j_3}) = -2\hat{e}_{j_2}\,, \tag{A.1.22c}$$

$$\rho^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{\star e_{j_1}}(1) = 0\,. \tag{A.1.22d}$$

This can be computed from the representation $\rho^{\mathbb{R}^3}$ making use of the fact that the group action is orthogonal.

Similarly, on $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$ we can deduce from (A.1.2) that the representation is given by

$$\rho^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{\tau_{j_1}}(\tau_0) = 0\,, \tag{A.1.23a}$$

$$\rho^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{\tau_{j_1}}(\tau_{j_1}) = 0\,, \tag{A.1.23b}$$

$$\rho^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{\tau_{j_1}}(\tau_{j_2}) = 2\tau_{j_3}\,, \tag{A.1.23c}$$

$$\rho^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{\tau_{j_1}}(\tau_{j_3}) = -2\tau_{j_2}\,, \tag{A.1.23d}$$

Comparing through the necessary isomorphisms shows that these representations are the same. □

This isomorphism, lastly, provides a Clifford action on the former space.

Definition A.1.24. On the space $((\mathbb{R}^3)^* \oplus \mathbb{R})^{\mathbb{C}}$ we define the Clifford action $\mathrm{cl}^{((\mathbb{R}^3)^*\oplus\mathbb{R})^{\mathbb{C}}}$ as the action $\mathrm{cl}^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}$ pulled back through the isomorphism (A.1.18).

We are particularly interested in the Clifford action of $\mathbb{R}^3 \subset \mathrm{Cl}(\mathbb{R}^3)$ on the space $((\mathbb{R}^3)^* \oplus \mathbb{R})^{\mathbb{C}}$.

Proposition A.1.25. *If* (j_1, j_2, j_3) *is a cyclic permutation of* $(1, 2, 3)$, *we have*

$$\mathrm{cl}^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{e_{j_1}}(\hat{e}_{j_1}) = -1\,, \tag{A.1.26a}$$

$$\mathrm{cl}^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{e_{j_1}}(\hat{e}_{j_2}) = -\hat{e}_{j_3}\,, \tag{A.1.26b}$$

$$\mathrm{cl}^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{e_{j_1}}(\hat{e}_{j_3}) = \hat{e}_{j_2}\,, \tag{A.1.26c}$$

$$\mathrm{cl}^{(\mathbb{R}^3)^*\oplus\mathbb{R}}_{e_{j_1}}(1) = \hat{e}_{j_1}\,. \tag{A.1.26d}$$

Proof. Once more, we can use (A.1.2) to see that

$$\mathrm{cl}^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{(\tau_{j_1},-\tau_{j_1})}(\tau_0) = -\tau_{j_1}\,, \tag{A.1.27a}$$

$$\mathrm{cl}^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{(\tau_{j_1},-\tau_{j_1})}(\tau_{j_1}) = \tau_0\,, \tag{A.1.27b}$$

$$\mathrm{cl}^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{(\tau_{j_1},-\tau_{j_1})}(\tau_{j_2}) = -\tau_{j_3}\,, \tag{A.1.27c}$$

$$\mathrm{cl}^{\mathbb{C}^2\otimes(\mathbb{C}^2)^*}_{(\tau_{j_1},-\tau_{j_1})}(\tau_{j_3}) = \tau_{j_2}\,, \tag{A.1.27d}$$

and the result follows from the appropriate isomorphisms. □

A.2 Linear algebra in dimension 2

We now summarise the the basic structures in dimension 2, paying special attention to their relationship with the constructions seen above for dimension 3.

Let us hence consider $\mathbb{R}^2 = \mathrm{span}(e_1, e_2) \subset \mathbb{R}^3$. We have an embedding of the corresponding Clifford algebras, so, preserving the notation of the previous section, we can write

$$\mathbb{C}\mathrm{l}(\mathbb{R}^2) \cong \mathrm{span}_{\mathbb{C}}(1, e_1, e_2, \star e_3) \subset \mathbb{C}\mathrm{l}(\mathbb{R}^3) \tag{A.2.1}$$

(where the Hodge star $\star$ still refers to the structure on $\mathbb{R}^3$). Through the second component of the isomorphism (A.1.5) we obtain the isomorphism

$$\mathbb{C}\mathrm{l}(\mathbb{R}^2) \cong \mathbb{C}(2)\,, \tag{A.2.2}$$

given by

$$1 \leftrightarrow \tau_0\,, \tag{A.2.3a}$$

$$e_1 \leftrightarrow -\tau_1\,, \tag{A.2.3b}$$

$$e_2 \leftrightarrow -\tau_2\,, \tag{A.2.3c}$$

$$\star e_3 \leftrightarrow \tau_3\,. \tag{A.2.3d}$$

(A.2.3e)

The spin group Spin(2) is then made up of the unit-norm elements of the real subspace $\mathrm{span}_{\mathbb{R}}(1, \star e_3)$, and as such is a subgroup of Spin(3).

Let us consider the spin representation on $\mathbb{C}^2$ given in the previous section. If it is restricted to Spin(2) it splits into two 1-dimensional components, corresponding to each component of $\mathbb{C}^2$. We refer to the first component as $\rho^{\mathbb{C}^+}$ and to the second as $\rho^{\mathbb{C}^-}$. Both are faithful, and they are the inverse of one another. By declaring $\rho^{\mathbb{C}^-}$ to be the fundamental representation of $U(1)$ we obtain an isomorphism

$$\mathrm{Spin}(2) \cong \mathrm{U}(1)\,. \tag{A.2.4}$$

The group Spin(2) also acts on $\mathbb{R}^2 \subset \mathbb{C}\mathrm{l}(\mathbb{R}^2)$ through conjugation, providing the 2-to-1 cover

$$\mathrm{Spin}(2) \to \mathrm{SO}(2)\,. \tag{A.2.5}$$

If we identify $\mathbb{R}^2 \cong \mathbb{C}$, then this spin representation is the square of $\rho^{\mathbb{C}^-}$.

Lastly, we note that if we consider the Clifford representation $\mathrm{cl}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ defined in the previous section, and we restrict it to $\mathbb{C}\mathrm{l}(\mathbb{R}^2)$, then we can observe that the action of $\mathbb{R}^2 \subset \mathbb{C}\mathrm{l}(\mathbb{R}^2)$ exchanges the two components of $\mathbb{C}^2$.

A.3 Associated bundles

Our interest in the spaces and representations discussed in the previous section stems from the associated bundles we can obtain. In particular, if we have an oriented, Riemannian 3-manifold with a spin structure, we can use any spin representation to associate a bundle. If we furthermore had a Clifford representation on it this will also carry over to the bundle to provide a Clifford action of the tangent bundle. In particular we will be able to define a Dirac operator on it, defined at each point of the manifold by the formula

$$\sum_{j=1}^{3} \mathrm{cl}_{e_j} \nabla_{e_j}\,, \tag{A.3.1}$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis for the tangent space at the given point.

The two spaces we are most interested in are $((\mathbb{R}^3)^* \oplus \mathbb{R})^{\mathbb{C}}$ and $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$, with the representations described above. If we write $\not{S}$ for the bundle associated to the spin representation on $\mathbb{C}^2$, then the two relevant bundles on the manifold are $(\bigwedge^1 \oplus \bigwedge^0)^{\mathbb{C}}$ and $\not{S} \otimes \not{S}^*$.

Proposition A.3.2. *We have an isomorphism of bundles*

$$(\bigwedge^1 \oplus \bigwedge^0)^{\mathbb{C}} \cong \not{S} \otimes \not{S}^* \tag{A.3.3}$$

Proof. Proposition A.1.20 shows that the spin representations used to define the bundles are isomorphic, so the bundles must be as well. □

The aim of this construction is to understand a Dirac operator on the former bundle. We further tensor with a vector bundle E with a connection A.

Proposition A.3.4. *Through the isomorphism* (A.3.3)*, the operator*

$$\begin{pmatrix} -\star \mathrm{d}_A & \mathrm{d}_A \\ \mathrm{d}_A^* & 0 \end{pmatrix} : (\Omega^1(E) \oplus \Omega^0(E))^{\mathbb{C}} \to (\Omega^1(E) \oplus \Omega^0(E))^{\mathbb{C}} \tag{A.3.5}$$

is the Dirac operator

$$\not{D}_A \colon \Gamma(\not{S} \otimes \not{S}^* \otimes E) \to \Gamma(\not{S} \otimes \not{S}^* \otimes E) \tag{A.3.6}$$

associated to the Clifford multiplication on the factor $\not{S}$ (and the connection made up of the Levi-Civita connection and A). In other words, it is the Dirac operator associated to the spinor bundle $\not{S}$ on the manifold, twisted by the bundle $\not{S}^* \otimes E$.[9]

Proof. This is equivalent to proving that (A.3.5) is the Dirac operator obtained from the Clifford action $\mathrm{cl}^{((\mathbb{R}^3)^* \oplus \mathbb{R})^{\mathbb{C}}}$ from Definition A.1.24.

Let x_1, x_2, x_3 be normal coordinates around any given point and let us write elements of $(\Omega^1(E) \oplus \Omega^0(E))^{\mathbb{C}}$ as column vectors with 4 entries, corresponding to the coefficients for $\mathrm{d}x_1$, $\mathrm{d}x_2$ and $\mathrm{d}x_3$, and the element of $(\Omega^0)^{\mathbb{C}}(E)$, in that order.

[9]Here, the bundle $\not{S}^* \otimes E$ is considered simply as a bundle with a connection, that is, with no Clifford action on it.

If ∇ denotes de covariant derivative of E, from Proposition A.1.25 we can deduce that the Dirac operator at the point $(0,0,0)$ must have the form
(A.3.7)

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \nabla_{\frac{\partial}{\partial x_1}} + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \nabla_{\frac{\partial}{\partial x_2}} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \nabla_{\frac{\partial}{\partial x_3}} ,$$

which is simply

$$\text{(A.3.8)} \qquad \left(\begin{array}{ccc|c} 0 & \nabla_{\frac{\partial}{\partial x_3}} & -\nabla_{\frac{\partial}{\partial x_2}} & \nabla_{\frac{\partial}{\partial x_1}} \\ -\nabla_{\frac{\partial}{\partial x_3}} & 0 & \nabla_{\frac{\partial}{\partial x_1}} & \nabla_{\frac{\partial}{\partial x_2}} \\ \nabla_{\frac{\partial}{\partial x_2}} & -\nabla_{\frac{\partial}{\partial x_1}} & 0 & \nabla_{\frac{\partial}{\partial x_3}} \\ \hline -\nabla_{\frac{\partial}{\partial x_1}} & -\nabla_{\frac{\partial}{\partial x_2}} & -\nabla_{\frac{\partial}{\partial x_3}} & 0 \end{array}\right) = \left(\begin{array}{c|c} -\star \mathrm{d}_A & \mathrm{d}_A \\ \hline \mathrm{d}_A^* & 0 \end{array}\right) ,$$

as desired. □

For dimension 2 we can similarly associate bundles to spin representations. In this case, the representation described on $\mathbb{C}^2$ splits into two 1-dimensional representations, so we write the associated bundle as $\not{S}^+ \oplus \not{S}^-$, corresponding to the representations $\rho^{\mathbb{C}+}$ and $\rho^{\mathbb{C}-}$. Recall that, although the spin representation reduces into these two components, the Clifford action does not. In fact, Clifford multiplication by elements in the original space $\mathbb{R}^2$ interchanges these two subspaces. In the associated bundle, this means that the corresponding Dirac operator interchanges these subbundles, so we can write its components as $\not{D}^\pm$, where

$$\text{(A.3.9)} \qquad \not{D}^\pm \colon \Gamma(\not{S}^\pm) \to \Gamma(\not{S}^\mp) .$$

A.4 On S^2 and $\mathbb{R}^3 \setminus \{0\}$

Let us consider the unit sphere S^2. The complex unitary line bundles on it can be classified by their degree, which can be any integer. Furthermore, for any degree d, we can define a homogeneous unitary connection on the corresponding line bundle (which is unique up to automorphisms). We refer to this line bundle with its connection as

$$\mathscr{L}^d \,. \tag{A.4.1}$$

As we saw, a spin structure on S^2 – which exists and is unique – provides a spinor bundle $\$^+ \oplus \$^-$ as described above. The bundles $\$^\pm$ are complex line bundles, and with their spin connections they satisfy

$$\$^\pm \cong \mathscr{L}^{\mp 1} \,. \tag{A.4.2}$$

On these bundles we have the operators $\not{D}^\pm$, but it will be useful to understand how these objects behave when twisting by additional line bundles. Therefore, let us take $d \in \mathbb{Z}$, and let us write $\not{D}_d^\pm$ for the components of the Dirac operator twisted by $\mathscr{L}^d$, which act as

$$\not{D}_d^\pm \colon \Gamma(\$^\pm \otimes \mathscr{L}^d) \to \Gamma(\$^\mp \otimes \mathscr{L}^d) \,. \tag{A.4.3}$$

Some crucial facts for us are the following.

Proposition A.4.4. *The operator $\not{D}_d^\pm$ is Fredholm of index $\pm d$. Furthermore,*

$$\dim \ker \not{D}_d^\pm = \begin{cases} d & \text{if } d \gtreqless 0 \\ 0 & \text{otherwise} \,. \end{cases} \tag{A.4.5}$$

Furthermore, the eigenvalues of

$$\not{D}_d^\mp \not{D}_d^\pm \colon \Gamma(\$^\pm \otimes \mathscr{L}^d) \to \Gamma(\$^\pm \otimes \mathscr{L}^d) \tag{A.4.6}$$

are

$$j(j + |d|) \,, \tag{A.4.7}$$

for $j \in \mathbb{Z}_{\geq 1}$, and additionally for $j = 0$ when $d \gtrless 0$.

Proof. The first part is easily deduced from the identification of S^2 with $\mathbb{CP}^1$ and the general facts about holomorphic line bundles on this space.

For the second part, from the Lichnerowicz–Weitzenböck formula one obtains

$$\not{D}_d^{\mp}\,\not{D}_d^{\pm} = \mathrm{d}_{d\mp1}^*\mathrm{d}_{d\mp1} + \frac{1\mp d}{2}\,, \tag{A.4.8}$$

where $\mathrm{d}_{d\mp1}$ is covariant derivative on the bundle $\not{S}^{\pm}\otimes\mathscr{L}^d \cong \mathscr{L}^{d\mp1}$. Using the known spectrum of this Laplacian [Kuw82, Thm. 5.1], we deduce that the spectrum of $\not{D}_d^{\mp}\,\not{D}_d^{\pm}$ is

$$\Big(\frac{|d\mp1|}{2}+j\Big)\Big(\frac{|d\mp1|}{2}+j+1\Big) - \frac{(d\mp1)^2}{4} + \frac{1\mp d}{2}\,, \tag{A.4.9}$$

for $j\in\mathbb{Z}_{\geq0}$, from which we obtain our result. □

We lastly consider $\mathbb{R}^3\setminus\{0\}$ with the Euclidean metric, which we view as $\mathbb{R}_{>0}\times S^2$ with the cone metric.

On each sphere $\{r\}\times S^2$ we can consider its spinor bundle, which we write as $\not{S}^+\oplus\not{S}^-$. This bundle can be identified with the bundle on the unit sphere regardless of the scale. Furthermore, we use $\not{D}^{\pm}$ to refer to the components of the Dirac operator specifically on the unit sphere. On other spheres $\{r\}\times S^2$, due to scaling, the Dirac operator will instead be given by $\frac{1}{r}\not{D}^{\pm}$.

On $\mathbb{R}^3\setminus\{0\}$, the spinor bundle $\not{S}$ restricted to each sphere $\{r\}\times S^2$ can be identified with the spinor bundle of this submanifold. This provides a decomposition of the spinor bundle on $\mathbb{R}^3\setminus\{0\}$ as

$$\not{S} = \not{S}^+\oplus\not{S}^-\,, \tag{A.4.10}$$

where $\not{S}^{\pm}$ restrict to the corresponding bundles on each sphere centred at the origin. We likewise carry over the notation for the Dirac operator on the spheres as described above. Then, the Dirac operator $\not{D}$ on $\not{S}$ can be written as

$$\not{D} = \begin{pmatrix} i\Big(\dfrac{\partial}{\partial r}+\dfrac{1}{r}\Big) & \dfrac{1}{r}\,\not{D}^- \\ \dfrac{1}{r}\,\not{D}^+ & -i\Big(\dfrac{\partial}{\partial r}+\dfrac{1}{r}\Big)\end{pmatrix} \tag{A.4.11}$$

with respect to the decomposition (A.4.10) [Nak93].

Note that the subbundles $\not{S}^{\pm}$ of $\not{S}$ can also be defined as the $\pm i$ eigenspaces of the Clifford action of $\frac{\partial}{\partial r}$.

Extending the notation from the unit sphere, for any $d \in \mathbb{Z}$ we can consider the complex line bundle of degree d on each sphere $\{r\} \times S^2$. Extending the homogeneous connection radially over $\mathbb{R}^3 \setminus \{0\}$ provides complex line bundles with unitary connections over this 3-manifold, which we still refer to as $\mathscr{L}^d$.

The formula (A.4.11) still holds when twisting by any such line bundle $\mathscr{L}^d$. That is, if $\not{D}_d$ is the Dirac operator on the bundle

$$\not{S} \otimes \mathscr{L}^d = (\not{S}^+ \otimes \mathscr{L}^d) \oplus (\not{S}^- \otimes \mathscr{L}^d)\,, \tag{A.4.12}$$

we can write

$$\not{D}_d = \begin{pmatrix} i\Big(\dfrac{\partial}{\partial r} + \dfrac{1}{r}\Big) & \dfrac{1}{r}\,\not{D}_d^- \\ \dfrac{1}{r}\,\not{D}_d^+ & -i\Big(\dfrac{\partial}{\partial r} + \dfrac{1}{r}\Big) \end{pmatrix}. \tag{A.4.13}$$

Appendix B

Analytical tools

The main analytic results used in this thesis make use of the b and scattering calculuses, as well as of polyhomogeneous expansions. We summarise here the main concepts which we will need to apply. Our exposition and notation largely follow Kottke's work [Kot15a] (with some slight differences, like defining Sobolev spaces for general p), since we aim to apply the Fredholmness and index theorems found in it. This article and the references therein [Cal78; LM85; Mel93; Mel94; Kot11] provide a more extensive and rigorous treatment of most of the contents of this appendix.

The main setup will be a compact n-manifold K with boundary and a smooth vector bundle E over it with an inner product and a connection which preserves the inner product and whose covariant derivative is denoted by ∇. We furthermore assume we have a *boundary defining function* x, that is, a smooth non-negative function on K which is zero precisely on its boundary ∂K and such that $\mathrm{d}x|_{\partial K} \neq 0$.

The bundle E can be assumed to be real or complex in the first two sections of this appendix, but must be complex to be able to apply the Fredholm theory later on.

Note that in this appendix and the next, the term *smooth* refers to being smooth in the entire compact manifold, that is, up to the boundary. When only smoothness in the interior is required this will be specified. An intermediary condition is introduced in Section B.1.

In Section B.2 we introduce the b and scattering calculuses. We then summarise some of the Fredholm theory for these frameworks in Sections B.3 and B.4 to provide the main elements involved in the Fredholm theory of the hybrid calculus explained in Section B.5.

In Appendix C and in Chapters 3 and 4 we will see how these results adapt to the study of monopoles.

B.1 Polyhomogeneous expansions

Polyhomogeneous expansions provide a framework to better understand different possible asymptotic behaviours near the boundary of sections which are smooth in the interior of K but not necessarily smooth up to the boundary. This is done by considering expansions in terms of powers of the boundary defining function x and its logarithm.

These powers will be indexed by the following sets.

Definition B.1.1. Let $\delta \geq 0$. We say a $\mathcal{I} \subset \mathbb{C} \times \mathbb{Z}_{\geq 0}$ is a *index set* if it is discrete, for any $k \in \mathbb{Z}_{\geq 0}$, the set

$$\{(\lambda, \nu) \in \mathcal{I} \mid \operatorname{Re} \lambda \leq k\} \tag{B.1.2}$$

is finite, and

$$(\lambda, \nu) \in \mathcal{I} \implies (\lambda + j_1, \nu - j_2) \in I \tag{B.1.3}$$

for all $j_1 \in \mathbb{Z}_{\geq 0}$ and $j_2 \in \{0, 1, \ldots, \nu\}$.

These conditions guarantee that these indices can be used to define an expansion.

Definition B.1.4. Let u be a smooth section of E on the interior of K. We say that u has a *polyhomogeneous expansion in x with index set $\mathcal{I}$* if for every $(\lambda, \nu) \in \mathcal{I}$ there exists a section $u_{(\lambda,\nu)}$ which is smooth on K up to the boundary such that, for any $k \in \mathbb{Z}_{\geq 0}$, the section

$$u - \sum_{\substack{(\lambda,\nu)\in\mathcal{I} \\ \operatorname{Re}(\lambda)\leq k}} x^{\lambda} \log(x)^{\nu} u_{\lambda,\nu} \tag{B.1.5}$$

and its first k derivatives vanish at the boundary of K to order x^k.

We are particularly interested in functions which are polyhomogeneous with certain index sets. In particular, we restrict to real exponents, and we will be mainly interested in the leading term, which must be of the form x^δ for $\delta \geq 0$.

Definition B.1.6. Let $\delta \in [0, \infty]$. We say u is *bounded polyhomogeneous of order* x^δ if it is polyhomogeneous with an index set $\mathcal{I}$ which satisfies

$$\begin{aligned} &\mathcal{I} \subset ((\delta, \infty) \times \mathbb{Z}_{\geq 0}) \cup \{(\delta, 0)\} && \text{if } \delta < \infty\,, \\ &\mathcal{I} = \varnothing && \text{if } \delta = \infty\,. \end{aligned} \tag{B.1.7}$$

We write

$$\mathscr{B}^\delta(E) \tag{B.1.8}$$

for the space of such sections, and refer to the sections in $\mathscr{B}^0(E)$ simply as *bounded polyhomogeneous.*

Of course, sections in $\mathscr{B}^\infty(E)$ are those which vanish with all derivatives to infinite order at the boundary. Furthermore,

$$\mathscr{B}^\delta(E) \subseteq \mathscr{B}^{\delta'}(E) \tag{B.1.9}$$

when $\delta \geq \delta'$, and

$$\mathscr{B}^\infty(E) = \bigcap_{\delta \in [0,\infty)} \mathscr{B}^\delta(E)\,. \tag{B.1.10}$$

B.2 B and scattering calculuses

The b and scattering calculuses are based on considering differential operators with different asymptotic behaviours near the boundary. Given their similarities, we will here define the most important concepts of both side by side, usually denoting the relevant objects with the subscript b for the b calculus and the subscript sc for the scattering calculus.

Throughout, will write $y_1, y_2, \dots, y_{n-1}$ to denote a set of local coordinates on the boundary, which together with the boundary defining function x provide local coordinates on the manifold near a boundary point.

We start by defining the corresponding spaces of vector fields (derivations) on the compact manifold K, from which we will build differential operators.

Definition B.2.1. If $\mathcal{V}$ denotes the space of smooth vector fields on K, we define the spaces of *b* and *scattering vector fields* as

$$\mathcal{V}_b := \{V \in \mathcal{V} \mid V \text{ is tangent to } \partial K\}\,, \tag{B.2.2}$$

$$\mathcal{V}_{sc} := x\mathcal{V}_b\,. \tag{B.2.3}$$

These vector fields can be viewed as sections of certain bundles on K, called the *b* and *scattering tangent bundles.* Their dual bundles are the *b* and *scattering cotangent bundles.* Local frames for these bundles near a boundary point are given by

$$\left\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_{n-1}}, x\frac{\partial}{\partial x}\right\}, \quad \left\{\mathrm{d}y_1, \mathrm{d}y_2, \dots, \mathrm{d}y_{n-1}, \frac{\mathrm{d}x}{x}\right\}, \tag{B.2.4}$$

for the b tangent and cotangent bundles, respectively, and

$$\left\{x\frac{\partial}{\partial y_1}, x\frac{\partial}{\partial y_2}, \dots, x\frac{\partial}{\partial y_{n-1}}, x^2\frac{\partial}{\partial x}\right\}, \quad \left\{\frac{\mathrm{d}y_1}{x}, \frac{\mathrm{d}y_2}{x}, \dots, \frac{\mathrm{d}y_{n-1}}{x}, \frac{\mathrm{d}x}{x^2}\right\}, \tag{B.2.5}$$

for the scattering tangent and cotangent bundles, respectively.

We can now define differential operators.

Definition B.2.6. For $k \in \mathbb{Z}_{\geq 0}$, we define the *b* and *scattering differential operators* on E of order k as

$$\begin{aligned}&\mathrm{Diff}_b^k(E)\\ &\quad := \mathrm{span}_{\Gamma(\mathrm{End}(E))}\{\nabla_{V_1}\nabla_{V_2}\cdots\nabla_{V_\ell} \mid V_1, V_2, \dots, V_\ell \in \mathcal{V}_b, 0 \leq \ell \leq k\}\,,\end{aligned} \tag{B.2.7}$$

$$\begin{aligned}&\mathrm{Diff}_{sc}^k(E)\\ &\quad := \mathrm{span}_{\Gamma(\mathrm{End}(E))}\{\nabla_{V_1}\nabla_{V_2}\cdots\nabla_{V_\ell} \mid V_1, V_2, \dots, V_\ell \in \mathcal{V}_{sc}, 0 \leq \ell \leq k\}\,,\end{aligned} \tag{B.2.8}$$

where $\Gamma(\mathrm{End}(E))$ is the space of sections of $\mathrm{End}(E)$ smooth up to the boundary, and a composition of 0 covariant derivatives is understood as the identity map.

In order to define Sobolev spaces from this we will assume that we have a *scattering metric* h_{sc} on the interior of K, that is, a metric which in a tubular neighbourhood of ∂K has the form

$$h_{sc} = \frac{(\mathrm{d}x)^2}{x^4} + \frac{h_{\partial K}}{x^2}\,, \tag{B.2.9}$$

where $h_{\partial K}$ is a symmetric 2-tensor which is smooth up to the boundary and restricts to a metric on ∂K. We will use the measure on K provided by this scattering metric.

As its name indicates, this metric is particularly suited to the scattering calculus, since it in fact defines an inner product on the scattering tangent bundle. However, we will also use it for the b calculus to make the combination of both calculuses – and the notation involved – simpler.

A metric adapted to the b calculus would simply be the scattering metric weighted by x^2, that is,

$$h_b = x^2 h_{sc}\,. \tag{B.2.10}$$

The resulting measure would hence differ from the scattering one in a weighting by x^n, so some results for the b calculus will involve tailoring the weights to this situation.

Furthermore, in order to make sure that the Sobolev spaces we define satisfy the necessary properties we will add the assumption that the interior of K has bounded geometry with respect to both h_{sc} and h_b, by which we mean that it has positive injectivity radius and that the curvature tensor and all its derivatives are bounded.

Since we are interested in spaces which combine b and scattering derivatives, we will directly define such Sobolev spaces. Additionally, we will consider them with weights, since, as we will see, these will play an important role throughout the analysis. Note that we consider spaces of sections of E on the interior of K.

Definition B.2.11. Let $\delta \in \mathbb{R}$, $k, \ell \in \mathbb{Z}_{\geq 0}$ and $p \in [1, \infty]$. We define

$$x^\delta W^{k,\ell,p}_{b,sc}(E) \coloneqq \{u \in L^p_{loc}(E) \mid x^{-\delta} D_b D_{sc} u \in L^p(E), \forall D_b \in \mathrm{Diff}^k_b(E), \forall D_{sc} \in \mathrm{Diff}^\ell_{sc}(E)\} \tag{B.2.12}$$

and

$$x^\delta H^{k,\ell}_{b,sc}(E) \coloneqq x^\delta W^{k,\ell,2}_{b,sc}(E)\,. \tag{B.2.13}$$

When $k = 0$ we omit it as a superscript, together with the subscript b, and similarly with $\ell = 0$ and sc. The weight is also omitted when trivial.

Remark B.2.14. Note that in (B.2.12) the order of the three terms in $x^{-\delta} D_b D_{sc}$ is not important. This can be checked by studying the commutators of these terms.

Remark B.2.15. It is important to note that the spaces $W^{k,p}_{sc}(E)$ are just the standard Sobolev spaces on the interior of K with respect to the scattering metric h_{sc}. If we instead consider the metric h_b we obtain the spaces $x^{\frac{n}{p}} W^{k,p}_b(E)$.

These Sobolev spaces can be equipped with norms in the usual way, with respect to which they acquire the structure of Banach spaces – and Hilbert spaces if $p = 2$. Importantly, with this topology we have the following property.

Lemma B.2.16. *If $p < \infty$, the space of smooth compactly supported sections is dense in $x^\delta W^{k,\ell,p}_{b,sc}(E)$.*

Proof. This is a consequence of the bounded geometry [Heb96, Thm. 2.8]. □

We now provide some embedding properties for these spaces, many of them shared with the usual Sobolev spaces. The notation $\subseteq$ will be used to indicate that the identity map is an inclusion between the spaces which is continuous with respect to their Banach space topology.

We start by noting the following property.

Lemma B.2.17. *If*

$$x^\delta W^{k,\ell,p}_{b,sc}(E) \subseteq x^{\delta'} W^{k',\ell',p}_{b,sc}(E) \tag{B.2.18}$$

and we have $\delta'' \in \mathbb{R}$ *and* $k'', \ell'' \in \mathbb{Z}_{\geq 0}$*, then*

$$x^{\delta+\delta''} W_{b,sc}^{k+k'',\ell+\ell'',p}(E) \subseteq x^{\delta'+\delta''} W_{b,sc}^{k'+k'',\ell'+\ell'',p}(E)\,. \tag{B.2.19}$$

Furthermore, if (B.2.18) *is compact, so is* (B.2.19).

Proof. This is a consequence of the property noted in Remark B.2.14. □

This means that we can state embedding results which only involve some of the parameters and then combine them in the way we would expect.

Furthermore, b and scattering derivatives can be exchanged by taking the weights into account in the following way.

Lemma B.2.20. *We have*

$$x^k W_{sc}^{k,p}(E) \subseteq W_b^{k,p}(E) \subseteq W_{sc}^{k,p}(E)\,. \tag{B.2.21}$$

Proof. This follows from the fact that

$$x^{-1}\mathcal{V}_{sc} \supseteq \mathcal{V}_b \supseteq \mathcal{V}_{sc}\,. \tag{B.2.22}$$

□

Lastly, we state the Sobolev embedding theorems adapted to these spaces. One of the advantages of considering weights is that in some cases we can obtain compact embeddings.

Lemma B.2.23. *Assume that*

$$k > k'\,, \tag{B.2.24a}$$

$$k - \frac{n}{p} > k' - \frac{n}{p'}\,, \tag{B.2.24b}$$

$$p \leq p'\,, \tag{B.2.24c}$$

$$\delta \geq \delta'\,. \tag{B.2.24d}$$

Then

$$x^{\delta} W_b^{k,p}(E) \subseteq x^{\delta' + \frac{n}{p} - \frac{n}{p'}} W_b^{k',p'}(E) \tag{B.2.25}$$

and

$$x^{\delta}W^{k,p}_{sc}(E) \subseteq x^{\delta'}W^{k',p'}_{sc}(E)\,. \tag{B.2.26}$$

If, furthermore, $\delta > \delta'$, then the embeddings are compact.

Proof. The continuous embeddings are simply the usual Sobolev embeddings, which follow from the bounded geometry [Aub82, Thm. 2.21], and the interpretation noted in Remark B.2.15.

The compactness follows by relying on the compactness of the equivalent spaces over compact subsets, similarly to Proposition 1.2 in Kottke's work [Kot15a]. □

Remark B.2.27. Following the same argument we can also obtain embeddings into Hölder spaces. We don't need such precise statements for this case, and it is enough to observe that sections in $H^2_{sc}(E)$ are continuous and bounded.

B.3 Fredholm theory for the b calculus

If we have a b operator $D \in \mathrm{Diff}^k_b(E)$ on K, we can define its principal symbol in the usual way on the b tangent bundle. Then, if it is elliptic we can expect it to be Fredholm between Sobolev spaces with appropriate weights, and we can obtain certain information about its index.

In order to determine the appropriate weights, let us look more closely at its behaviour at the boundary. Suppose that locally near the boundary we have

$$D = \sum_{j+|\beta|\leq k} b_{j,\beta}(x,y)\Big(x\frac{\partial}{\partial x}\Big)^j\Big(\frac{\partial}{\partial y}\Big)^{\beta}, \tag{B.3.1}$$

where y represents the coordinates on the boundary, β is a multiindex, and $(\frac{\partial}{\partial y})^{\beta}$ is interpreted in the usual way. Then, we can define the *indicial operator*

$$I(D) = \sum_{j+|\beta|\leq k} b_{j,\beta}(0,y)\Big(\zeta\frac{\partial}{\partial \zeta}\Big)^j\Big(\frac{\partial}{\partial y}\Big)^{\beta}, \tag{B.3.2}$$

which is a differential operator on the inward-pointing normal bundle to the boundary ∂K of K inside the b tangent bundle. The fibres of this bundle are generated by $x\frac{\partial}{\partial x}$ and parametrised by the variable $\zeta \geq 0$, and the bundle can be thought of as a model for K near its boundary.

From this, for any $\lambda \in \mathbb{C}$, we can consider the operator

$$I(D,\lambda) = \sum_{j+|\beta|\leq k} b_{j,\beta}(0,y)\lambda^j \left(\frac{\partial}{\partial y}\right)^\beta, \tag{B.3.3}$$

which is an elliptic differential operator on the boundary ∂K. We are interested in the values of $\lambda \in \mathbb{C}$ for which this operator is not invertible, which form the *b spectrum* of the operator,

$$\operatorname{spec}_b(D) \coloneqq \{\lambda \in \mathbb{C} \mid I(D,\lambda) \text{ is not invertible}\}. \tag{B.3.4}$$

The real parts of elements in this b spectrum are called *indicial roots*, although in our case these elements will be real and hence will coincide with the indicial roots.

The idea is that if $\lambda \in \operatorname{spec}_b(D)$, then an element $u \in \operatorname{Null}(I(D,\lambda))$ represents an asymptotic section in the kernel of D which is homogeneous of order λ, in the sense that $I(D)\zeta^\lambda u(y) = 0$. Of course, its possible that this does not correspond to an actual element in the kernel of D over the entire manifold K, but it will nonetheless affect the index of the operator. In fact, an indicial root λ may have an order $\operatorname{ord}(\lambda) \in \mathbb{Z}_{\geq 1}$, which represents elements in the kernel of the indicial operator of the form

$$\sum_{j=0}^{\operatorname{ord}(\lambda)-1} \zeta^\lambda \log(\zeta)^j u_j(y). \tag{B.3.5}$$

The space of such elements is referred to as the *formal nullspace at* λ, and its dimension is simply $\operatorname{ord}(\lambda)\operatorname{Null}(I(D,\lambda))$. In our case the order of elements of the b spectrum will always be 1.

If we avoid the indicial roots, then the operator is Fredholm. Furthermore, the elements in the b spectrum and the operators $I(D,\lambda)$ can provide information about the index with respect to different weights, as well as about the elements in in the kernel of D.

Theorem B.3.6. *Let $D \in \operatorname{Diff}_b^\ell(E)$ be elliptic, and suppose that elements in its b* *spectrum are real and of order* 1*. Then, for any* $\delta \notin \operatorname{spec}_b(D)$*, the map*

$$D\colon x^{\delta-\frac{n}{2}} H_b^k(E) \to x^{\delta-\frac{n}{2}} H_b^{k-\ell}(E) \tag{B.3.7}$$

is Fredholm for any k*.*

The indices $\operatorname{ind}(D,\delta)$ *for different values of* δ *satisfy*

$$\operatorname{ind}(D,\lambda_0-\varepsilon) = \operatorname{ind}(D,\lambda_0+\varepsilon) + \dim \operatorname{Null}(I(D,\lambda_0)) \tag{B.3.8}$$

when $[\lambda_0-\varepsilon,\lambda_0+\varepsilon] \cap \operatorname{spec}_b(D) = \{\lambda_0\}$*. Furthermore, if* D *is self-adjoint, then*

$$\operatorname{ind}(D,-\delta) = -\operatorname{ind}(D,\delta)\,. \tag{B.3.9}$$

Lastly, let λ_1 *be the smallest indicial root larger than* δ*. Then, if* $\lambda_1 \geq 0$*, the elements in the kernel of* D *are bounded polyhomogeneous sections in* $\mathscr{B}^{\lambda_1}(E)$*.*

B.4 Fredholm theory for the scattering calculus

For the scattering calculus, the usual notion of ellipticity is not enough to guarantee Fredholmness: we need an additional non-degeneracy condition near the boundary. Operators which satisfy this are referred to as *fully elliptic*.

Essentially, on top of the usual principal symbol, we can define a scattering symbol on the boundary ∂K which takes into account the terms of all orders, rather than only the highest order. This symbol is then required to be invertible everywhere (even on the zero section).

With these conditions, elliptic operators will be Fredholm, and furthermore elements in their kernels will decay to infinite order with all derivatives.

The case of most interest to us is that of Dirac operators, where we essentially follow the line of Callias's index theorem and Kottke's adaptation to this context. Here, the full ellipticity is provided by an algebraic term which is non-degenerate at the boundary.

Assume K is odd-dimensional and that we have a Clifford action of the scattering tangent bundle of K (with respect to the scattering metric) on the bundle E. Then, our setting is an operator $D+\Psi$, where D is the associated Dirac operator on E and Ψ is an algebraic, skew-Hermitian term which commutes with the Clifford action and is non-degenerate on the boundary ∂K.

Now, let E_+ denote the subbundle of $E|_{\partial K}$ formed of the positive-imaginary eigenspaces of E with respect to Ψ, and let $E_+^{\pm}$ denote the $\pm i$ eigenspace of the Clifford action of $x^2 \frac{\partial}{\partial x}$ on E_+. Then, we have a Dirac operator

$$\eth_+^+ \colon \Gamma(E_+^+) \to \Gamma(E_+^-) \tag{B.4.1}$$

induced on the boundary of K. This operator can be viewed as (one part of) the operator associated to the Clifford action of the scattering tangent bundle restricted to the boundary considered along the boundary directions, or equivalently the operator associated to the metric $h_{\partial K}$ from (B.2.9) restricted to the usual tangent bundle of the boundary. Note that the Clifford relations guarantee that the Clifford action of the vectors tangent to the boundary will exchange E_+^+ and E_+^-.

This operator provides the index of $D+\Psi$, which is Fredholm for any weight. Importantly, elements in its kernel vanish to infinite order.

Theorem B.4.2. *Let $D+\Psi$ be as above. Then, for any $\delta \in \mathbb{R}$, the operator*

$$D+\Psi \colon x^\delta H^k_{sc}(E) \to x^\delta H^{k-1}_{sc}(E) \tag{B.4.3}$$

is Fredholm for any k, and

$$\operatorname{ind}(D+\Psi) = \operatorname{ind}(\eth_+^+)\,. \tag{B.4.4}$$

Furthermore, elements in the kernel of $D+\Psi$ are in $\mathscr{B}^\infty(E)$.

B.5 Hybrid calculus

The hybrid calculus combines the b and scattering calculuses to study operators whose behaviour along different subbundles fits into these two formalisms.

To be more precise, let us assume that the bundle E decomposes as $E = E_0 \oplus E_1$ into two subbundles in a neighbourhood of the boundary, and suppose that the connection decomposes along this splitting. Our aim is to study operators which behave like weighted elliptic b operators along E_0 and as fully elliptic scattering operators along E_1.

We start by defining Sobolev spaces adapted to this situation. Here, we let Π denote the projection from E onto E_0 near the boundary, and χ a cutoff function which is identically 0 everywhere except near the boundary, and 1 when sufficiently close to it.

Definition B.5.1. Let $\delta_0, \delta_1 \in \mathbb{R}$ and $s, k \in \mathbb{Z}_{\geq 0}$. Then, we define

$$\begin{aligned} \mathcal{H}^{\delta_0,\delta_1,s,k}(E) \coloneqq \{u \in L^2_{loc}(E) \mid \quad & \Pi\chi u \in x^{\delta_0} H_b^{s+k}(E_0), \\ & (1-\Pi)\chi u \in x^{\delta_1} H_{b,sc}^{s,k}(E_1), \\ & (1-\chi)u \in H_{loc}^{s+k}(E) \qquad \}\,. \end{aligned} \tag{B.5.2}$$

The parameter k accounts for b derivatives along the subbundle E_0 and scattering derivatives along the subbundle E_1, and hence an operator with the behaviour described above will decrease k. The parameter s simply adds b derivatives on the entire bundle, and will be used in most of the thesis as a fixed parameter.

We also define spaces of bounded polyhomogeneous sections with different orders along the different subbundles.

Definition B.5.3. Let $\delta_0, \delta_1 \in \mathbb{R}_{\geq 0}$. Then, we define

$$\mathscr{B}^{\delta_0,\delta_1}(E) \coloneqq \{u \in \mathscr{B}^0(E) \mid \Pi\chi u \in \mathscr{B}^{\delta_0}(E_0), (1-\Pi)\chi u \in \mathscr{B}^{\delta_1}(E_1)\}\,. \tag{B.5.4}$$

We now restrict ourselves once again to the case of a Dirac operator D on an odd-dimensional manifold with an additional algebraic term Ψ. However, unlike for the pure scattering calculus, we no longer require this term to be non-degenerate. Instead, we assume that it is non-degenerate only along E_1, but degenerate along E_0.

Then, along E_1, it will behave similarly to the fully elliptic scattering operators considered in the previous section. Furthermore, if the algebraic term degenerates fast enough along E_0, we will be able to weight the operator on this subbundle to produce one which is elliptic in the sense of the b calculus. Note that this weighting is necessary to account for the weight relating the b and scattering tangent bundles.

If we write the Dirac operator on E as

$$D = \begin{pmatrix} D_{00} & D_{10} \\ D_{01} & D_{11} \end{pmatrix} \tag{B.5.5}$$

near the boundary along the decomposition $E_0 \oplus E_1$, then D_{11} is a scattering operator which we can expect to be fully elliptic with the additional algebraic term, while

$$\widetilde{D}_{00} := x^{-\frac{n+1}{2}}\, D_{00}\, x^{\frac{n-1}{2}} \tag{B.5.6}$$

is a b operator, where it is necessary to multiply by x^{-1}, and we additionally conjugate by $x^{-\frac{n-1}{2}}$ to make the notation simpler later on.

Let us formulate the relevant result, adapted to our case to some extent, where we once again assume that there is a Clifford action of the scattering tangent bundle on E. This time we allow D to also include some algebraic terms, and decompose it as Equation (B.5.5) in the same way. We use $\eth_+^+$ to denote the Dirac operator induced on the boundary by D_{11} as in Section B.4, as well as the notions surrounding the b calculus from Section B.3.

Importantly, the following result does not require the operators involved to be smooth up to the boundary, only to be bounded polyhomogeneous.

Theorem B.5.7. *Suppose that*

- *D is the Dirac operator associated to the bundle E plus a bounded polyhomogeneous algebraic term of order x,*
- *Ψ is a bounded polyhomogeneous skew-Hermitian endomorphism of E which commutes with the Clifford action and satisfies $\ker \Psi = E_0$ near the boundary,*
- *the endomorphism Ψ, the Clifford action and the connection preserve the decomposition $E = E_0 \oplus E_1$ near the boundary, and*
- *the elements of $\operatorname{spec}_b(\widetilde{D}_{00})$ are real and of order 1.*

Then, for any $\delta \notin \operatorname{spec}_b(\widetilde{D}_{00})$,

$$D + \Psi \colon \mathcal{H}^{\delta-\frac{1}{2},\delta+\frac{1}{2},s,1}(E) \to \mathcal{H}^{\delta+\frac{1}{2},\delta+\frac{1}{2},s,0}(E) \tag{B.5.8}$$

is Fredholm.

Furthermore, its index is given by

$$\operatorname{ind}(D + \Psi) = \operatorname{ind}(\not{\partial}_+^+) + \operatorname{def}(\widetilde{D}_{00}, \delta)\,, \tag{B.5.9}$$

where $\operatorname{def}(\widetilde{D}_{00}, \delta)$ *satisfies*

$$\begin{aligned} &(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap \operatorname{spec}_b(\widetilde{D}_{00}) = \{\lambda_0\} \\ &\qquad \implies \operatorname{def}(\widetilde{D}_{00}, \lambda_0 - \varepsilon) = \operatorname{def}(\widetilde{D}_{00}, \lambda_0 + \varepsilon) + \dim \operatorname{Null} I(\widetilde{D}, \lambda_0) \end{aligned} \tag{B.5.10}$$

and

$$D \text{ is self-adjoint} \implies \operatorname{def}(\widetilde{D}_{00}, -\delta) = -\operatorname{def}(\widetilde{D}_{00}, \delta)\,. \tag{B.5.11}$$

Lastly, elements in its kernel are in the space $\mathscr{B}^{\lambda_1+1,\lambda_1+2}(E)$*, where* λ_1 *is the smallest indicial root in* $\operatorname{spec}_b(\widetilde{D}_{00})$ *larger than* δ*, provided that* $\lambda_1 \geq 0$*.*

Proof. This follows from Theorems 2.4 and 3.6 in Kottke's work [Kot15a]. □

Appendix C

Function spaces

We now put the results of the previous appendix into our context and prove some useful properties of the related spaces. The specific spaces chosen are motivated by our requirements in this thesis and adapted to the analytical tools explained in the last appendix, particularly Theorem B.5.7. The lemmas proved are simply technical conditions needed to carry out our construction.

In Section C.1 we explain how we view the Euclidean 3-space as a manifold with boundary to be able to apply the results from the previous appendix.

Then we define the most relevant spaces that we will use and show some of their properties in Section C.2.

C.1 The radial compactification

The b and scattering calculuses were formulated in terms of a compact manifold with boundary. The base manifold we are considering in this thesis, $\mathbb{R}^3$, will therefore be regarded as the interior of its *radial compactification*.

This compactification of the Euclidean space adds to it the *sphere at infinity*, which consists of a point for every (oriented) direction. Topologically, the result is a compact ball, whose interior is the original Euclidean space and whose boundary is the sphere at infinity.

More precisely, consider the space $\mathbb{R}^4$, with coordinates x_0, x_1, x_2, x_3, and the hyperplane $\{x_0 = 1\} \subset \mathbb{R}^4$, which we identify with $\mathbb{R}^3$ with the coordinates

x_1, x_2, x_3. The projection through the origin to the unit sphere is then a diffeomorphism from the hyperplane to the open hemisphere

$$S^3_+ = \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1, x_0 > 0\} \subset \mathbb{R}^4 . \tag{C.1.1}$$

The closure of this hemisphere is a compact manifold with boundary, and the function

$$\frac{x_0}{\sqrt{1 - x_0^2}} , \tag{C.1.2}$$

appropriately smoothed out near $x_0 = 1$, provides a boundary defining function which we will write simply as x. This will be the setting for the b and scattering calculuses in our case.

Note that its interior is diffeomorphic to $\mathbb{R}^3$, and, when pulled back, the boundary defining function is

$$x = \frac{1}{r} \tag{C.1.3}$$

near infinity. The boundary of this compactification, as intended, is a 2-sphere with a point corresponding to each direction in $\mathbb{R}^3$. This provides the radial compactification $\overline{\mathbb{R}^3}$.

If we write $\mathbb{R}^3 \setminus \{0\} = \mathbb{R}_{>0} \times S^2$, the Euclidean metric can be written near infinity as

$$\mathrm{d}r^2 + r^2 h_{S^2} = \frac{\mathrm{d}x^2}{x^4} + \frac{h_{S^2}}{x^2} , \tag{C.1.4}$$

where h_{S^2} is the metric on the unit 2-sphere. Hence, the Euclidean metric is of the form (B.2.9), and hence a scattering metric, and will be the one used throughout.

Lastly, we know that the Euclidean metric has bounded geometry. Furthermore, if we weight it by $x^2 = \frac{1}{r^2}$, near infinity it will become a cylindrical metric, which also satisfies this property.

C.2 Hybrid spaces

Let us assume that we have a vector bundle E on $\overline{\mathbb{R}^3}$ with a connection A. This will usually be a combination of bundles associated to P with the connection $A_{\mu,\kappa}$ and tensor bundles with the Levi-Civita connection. From the constructions in Section 2.3 we can see that P and $A_{\mu,\kappa}$ can be extended smoothly up to the boundary of the radial compactification, and the same is true for the tensor bundles and the Levi-Civita connection. We furthermore assume, as in Section B.5, that we have a splitting $E = E_0 \oplus E_1$ preserved by the connection.

We start by fixing some combinations of parameters which will be particularly useful for our constructions. Here we once again use the regularity parameter $s \in \mathbb{Z}_{\geq 0}$, which in Chapters 3 and 4 will be mostly fixed to a value in $\mathbb{Z}_{\geq 1}$.

Definition C.2.1. Let $k \in \{1, 2, 3\}$. We define

$$\mathscr{H}^{s,k}(E) := \mathcal{H}^{1-k,1,s,k}(E)\,. \tag{C.2.2}$$

Note how this definition is adapted to the operators we are interested in, which behave like weighted b operators along E_0 and like scattering operators along E_1.

We now prove some properties which will be useful for us.

Lemma C.2.3. *The L^2 pairing is continuous on the pair $\mathscr{H}^{s,k}(E) \times \mathscr{H}^{s,2-k}(E)$ for $k \in \{0, 1, 2\}$.*

Proof. This is a consequence of the fact that $\mathscr{H}^{s,k}(E) \subseteq x^{1-k}L^2(E)$. □

Lemma C.2.4. *The map*

$$\mathrm{d}_A \colon \mathscr{H}^{s,k}(\textstyle\bigwedge^j \otimes E) \to \mathscr{H}^{s,k-1}(\textstyle\bigwedge^{j+1} \otimes E) \tag{C.2.5}$$

is continuous.

Additionally, suppose that Ψ is a smooth endomorphism of E which preserves the decomposition $E_0 \oplus E_1$ and satisfies that $x^{-1}\,\Psi|_{E_0}$ is smooth and bounded. Then,

$$\Psi \colon \mathscr{H}^{s,k}(E) \to \mathscr{H}^{s,k-1}(E) \tag{C.2.6}$$

is countinuous

Proof. To prove the continuity of the first map we observe that d_A is a scattering operator of order 1, and that $x^{-1}\mathrm{d}_A$ is a b operator of order 1.

The continuity of the second map follows from the definitions of the spaces and the condition imposed on the endomorphism, observing that reducing the number of b or scattering derivatives of a Sobolev space by 1 is continuous. □

In order to prove some multiplication properties for the Sobolev spaces, let us assume that $E' = E'_0 \oplus E'_1$ and $E'' = E''_0 \oplus E''_1$ are bundles like the above, and that we have a smooth fibrewise multiplication map

$$\gamma\colon E \times E' \to E'' \tag{C.2.7}$$

which satisfies

$$\gamma(E_0, E'_0) \subseteq E''_0\,. \tag{C.2.8}$$

This will induce products on the hybrid Sobolev spaces we have defined, so that if all the paramenters are chosen appropriately, we will have maps

$$\gamma\colon \mathcal{H}^{\delta_0,\delta_1,s,k} \times \mathcal{H}^{\delta'_0,\delta'_1,s',k'} \to \mathcal{H}^{\delta''_0,\delta''_1,s'',k''} \tag{C.2.9}$$

which are continuous (and bilinear). We will furthermore be interested in maps which are compact in the first argument, by which we mean that for any element $u' \in \mathcal{H}^{\delta'_0,\delta'_1,s',k'}$ the map

$$\gamma(\bullet, u')\colon \mathcal{H}^{\delta_0,\delta_1,s,k} \to \mathcal{H}^{\delta''_0,\delta''_1,s'',k''} \tag{C.2.10}$$

is a compact linear map.

The relevant properties are the following.

Lemma C.2.11. *Let $s, k \in \mathbb{Z}_{\geq 1}$. Then, the maps*

$$\gamma\colon \mathscr{H}^{s,2}(E) \times \mathscr{H}^{s,1}(E') \to \mathscr{H}^{s,1}(E''), \tag{C.2.12}$$

$$\gamma\colon \mathscr{H}^{s,2}(E) \times \mathscr{H}^{s,0}(E') \to \mathscr{H}^{s,0}(E''), \tag{C.2.13}$$

$$\gamma\colon \mathscr{H}^{s,1}(E) \times \mathscr{H}^{s,1}(E') \to \mathscr{H}^{s,0}(E''), \tag{C.2.14}$$

$$\gamma\colon \mathscr{H}^{s,2}(E) \times \mathscr{H}^{s,2}(E') \to \mathscr{H}^{s,2}(E''), \tag{C.2.15}$$

$$\gamma\colon \mathcal{H}^{0,1,s,k}(E) \times \mathcal{H}^{0,1,s,k}(E') \to \mathcal{H}^{\frac{5}{4},\frac{5}{4},s,k}(E''), \tag{C.2.16}$$

$$\gamma\colon \mathcal{H}^{-1,1,0,1}(E) \times \mathscr{H}^{s,1}(E') \to \mathscr{H}^{0,1}(E''), \tag{C.2.17}$$

and, if $k \leq s$, the map

$$\gamma\colon \mathscr{H}^{k-1,2}(E) \times \mathscr{H}^{s,1}(E') \to \mathscr{H}^{k,1}(E''), \tag{C.2.18}$$

are continuous. Furthermore, in the first three cases they are compact in the first argument.

Proof. This can be deduced by applying Hölder's inequality and the Sobolev embeddings from Lemma B.2.23. It will be important to take advantage of the weight improvement of the Sobolev embedding (B.2.25) for b Sobolev spaces.

We start by observing that the decay conditions imposed on the subbundle E_0 are always weaker than those imposed on the subbundle E_1 (with the local regularity conditions being the same). Together with (C.2.8), this means that to prove the continuity of a multiplication map of the form

$$\gamma\colon \mathcal{H}^{\delta_0,\delta_1,s,k}(E) \times \mathcal{H}^{\delta_0',\delta_1',s',k'}(E') \to \mathcal{H}^{\delta_0'',\delta_1'',s'',k''}(E'') \tag{C.2.19}$$

we only need to prove the continuity of

$$\gamma\colon x^{\delta_0} H_b^{s+k}(E_0) \times x^{\delta_0'} H_b^{s'+k'}(E_0') \to x^{\delta_0''} H_b^{s''+k''}(E_0''), \tag{C.2.20a}$$

$$\gamma\colon x^{\delta_0} H_b^{s+k}(E_0) \times x^{\delta_1'} H_{b,sc}^{s',k'}(E_1') \to x^{\delta_1''} H_{b,sc}^{s'',k''}(E''), \tag{C.2.20b}$$

$$\gamma\colon x^{\delta_1} H_{b,sc}^{s,k}(E_1) \times x^{\delta_0'} H_b^{s'+k'}(E_0') \to x^{\delta_1''} H_{b,sc}^{s'',k''}(E''), \tag{C.2.20c}$$

and the same applies to the compactness property.

In the rest of the proof we omit the bundles from the Sobolev spaces to avoid overburdening the notation.

Let us now demonstrate the proof for the map (C.2.12), and taking $s = 0$. The above observation means that we only have to consider the maps

(C.2.21a) $$\gamma\colon x^{-1}H_b^2 \times H_b^1 \to H_b^1\,,$$

(C.2.21b) $$\gamma\colon x^{-1}H_b^2 \times xH_{sc}^1 \to xH_{sc}^1\,,$$

(C.2.21c) $$\gamma\colon xH_{sc}^1 \times H_b^1 \to xH_{sc}^1\,.$$

Now, let us take u and u' to be smooth compactly supported sections. For the first map, we write

(C.2.22)
$$\begin{aligned}
&\|uu'\|_{H_b^1} \\
&\preccurlyeq \|x^{-1}\nabla(uu')\|_{L^2} && + \|uu'\|_{L^2} \\
&\preccurlyeq \|x^{-1}(\nabla u)u'\|_{L^2} + \|ux^{-1}\nabla u'\|_{L^2} && + \|uu'\|_{L^2} \\
&\preccurlyeq \|x^{-1}\nabla u\|_{x^{-\frac{1}{2}}L^4}\|u'\|_{x^{\frac{1}{2}}L^4} + \|u\|_{L^\infty}\|x^{-1}\nabla u'\|_{L^2} && + \|u\|_{L^\infty}\|u'\|_{L^2} \\
&\preccurlyeq \|u\|_{x^{-\frac{1}{2}}W^{1,4}}\|u'\|_{x^{\frac{1}{2}}L^4} + \|u\|_{L^\infty}\|u'\|_{H_b^1} && + \|u\|_{L^\infty}\|u'\|_{L^2}\,,
\end{aligned}$$

where $\preccurlyeq$ denotes that the right-hand side is greater than the left-hand side after multiplying by a constant which is independent of u and u'. Note that when taking the covariant derivative ∇ we obtain a 1-form, and the metric used to define the corresponding Sobolev space of such forms uses the scattering metric. Therefore, the derivative $x^{-1}\nabla$ must be used for b derivatives.

We can now deduce from Lemma B.2.23 that

(C.2.23) $$x^{-1}H_b^2 \Subset x^{-\frac{1}{4}}W^{1,4}, L^\infty$$

and

(C.2.24) $$H_b^1 \subseteq x^{\frac{1}{2}L^4}, L^2\,,$$

where $\Subset$ denotes a compact embedding. Observing that smooth compactly supported functions are dense in $x^{-1}H_b$ and H_b^1 by Lemma B.2.16, we are done.

Applying the same procedures to (C.2.21b) and (C.2.21c) finishes the proof of the properties of (C.2.12) for $s = 0$.

The same method will yield proofs of the properties of (C.2.13), (C.2.14) and (C.2.15) with $s = 0$, and of (C.2.16), (C.2.17) and (C.2.18) with $s = k = 1$.

It then only remains to observe that if we take higher values of s and k the same proofs apply. This is because Hölder's inequality can also be used with Sobolev spaces with b and scattering derivatives, so the L^p spaces used in (C.2.22) can be substituted with Sobolev spaces $W^{\bullet,\bullet,p}_{b,sc}$ as appropriate. $\square$

Lastly, to simplify notation, we introduce the following conventions for Sobolev spaces over the bundles we will use most typically. Here, we take any bundle $E \otimes \mathrm{Ad}(P)$ to be decomposed as $(E \otimes \mathrm{Ad}(P)_C) \oplus (E \otimes \mathrm{Ad}(P)_{C^\perp})$ for the relevant definitions.

Definition C.2.25. We write

$$\mathcal{H}^{\delta_0,\delta_1,s,k}_j := \mathcal{H}^{\delta_0,\delta_1,s,k}(\textstyle\bigwedge^j \otimes \mathrm{Ad}(P)) \,, \tag{C.2.26}$$

$$\mathcal{H}^{\delta_0,\delta_1,s,k} := \mathcal{H}^{\delta_0,\delta_1,s,k}((\textstyle\bigwedge^1 \oplus \bigwedge^0) \otimes \mathrm{Ad}(P)) \,, \tag{C.2.27}$$

and similarly

$$\mathscr{H}^{s,k}_j := \mathscr{H}^{s,k}(\textstyle\bigwedge^j \otimes \mathrm{Ad}(P)) \,, \tag{C.2.28}$$

$$\mathscr{H}^{s,k} := \mathscr{H}^{s,k}((\textstyle\bigwedge^1 \oplus \bigwedge^0) \otimes \mathrm{Ad}(P)) \,. \tag{C.2.29}$$

Likewise, we simplify the notation for the space of bounded polyhomogeneous sections which we will use most often.

Definition C.2.30. We write

$$\mathscr{B}^{\delta_0,\delta_1} := \mathscr{B}^{\delta_0,\delta_1}((\textstyle\bigwedge^1 \oplus \bigwedge^0) \otimes \mathrm{Ad}(P)) \,. \tag{C.2.31}$$

We also define a space of bounded polyhomogeneous sections with more granular decay conditions. To do so, assume that χ is a cutoff function which is only non-zero near infinity and 1 when sufficiently close to it, and for any $\alpha \in R$, let Π_α be the projection of $\mathrm{Ad}(P)^{\mathbb{C}}$ to $\underset{\sim}{\mathfrak{g}_\alpha}$ – where Π_0 is simply the projection onto $\underline{\mathfrak{t}}^{\mathbb{C}}$.

Definition C.2.32. If $\kappa \in \mathfrak{g}$, we write

$$\begin{aligned} &\mathscr{B}^{\delta,(\kappa),\infty} \\ &:= \{u \in \mathscr{B}^{\delta,\infty} \mid \Pi_\alpha \chi u \in \mathscr{B}^{\delta + \frac{|i\alpha(\kappa)|}{2}}((\bigwedge^1 \oplus \bigwedge^0) \otimes \Pi_\alpha(\mathrm{Ad}(P))), \forall \alpha\}. \end{aligned} \tag{C.2.33}$$

Note that the spaces described in this section have been defined as real spaces of sections. Their complexifications will simply be notated with a subscript $\mathbb{C}$.

In the last definition in particular, note that it is well defined because the projections Π_α can be restricted to the real bundle $\mathrm{Ad}(P)$, although they will project into sections which are not necessarily real.