Analysing the educative potential of mathematics curriculum materials for formative assessment practices

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I, Sumeyra Tutuncu, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

ABSTRACT

In response to mathematics teachers' critical need for accessible curriculum materials that can enhance their formative assessment practices, this thesis focuses on two research-informed aspects of these materials: (1) inherent mathematics-specific formative assessment messages, and (2) educative features that can facilitate teachers' effective practices. The key research output of this thesis is the development of a framework that can guide the analysis of the curriculum materials for their educative potential, with a specific emphasis on the potential to enhance mathematics teachers' in-the-moment formative assessment practices.

A collective case study was conducted by using well-designed teacher guides from five sets of curriculum materials from England and the United States as instrumental cases. The data analysis was completed in two phases, where the key findings of the first phase informed the methods employed in the second phase. In particular, the clarification of three key elements of in-the-moment classroom formative assessment practices – identifying learning intentions, noticing students' mathematical thinking, and creating feedback situations – and proposing three educative features of teacher guides – alert, equip, and guide – informed the second phase. This first phase contributes to the existing literature by intersecting two important issues in mathematics education: enhancing teachers' formative assessment practices and using curriculum materials to enhance teachers' practices.

In the second phase, the analyses were targeted at the educative potential of the teacher guides. During this process, the framework that was aimed to be developed was improved. The final version of the developed framework provides new insights in terms of three key aspects. First, it clearly distinguishes the roles of curriculum materials and teachers. Second, it highlights the importance of identifying the pedagogical messages that can directly influence practice beyond the knowledge teachers should possess. Third, it contributes to the research realm that predominantly focuses on the cognitive elements of learning mathematics when analysing curriculum materials, with its inclusion of productive disposition.

IMPACT STATEMENT

This thesis aims to contribute to mathematics teachers' practical need for well-designed and accessible resources that can facilitate their effective formative assessment practices. Teachers' in-the-moment formative assessment practices are the primary focus. As reported in academic and governmental reports, in both primary and secondary schools worldwide, teachers have found these practices to be challenging. That is, effective in-the-moment formative assessment practices that require actively engaging with students' thinking and providing them autonomy for their learning could challenge the traditional way of teaching that a substantial number of teachers may tend to practice. This proves the practical need for supporting teachers' in-the-moment formative assessment practices.

In this thesis, the educative curriculum materials are acknowledged as tools that can support teachers' in-the-moment formative assessment practices. These materials have the potential to support teacher development in the absence of other support mechanisms, such as professional development sessions. This is important because substantial professional development support is usually not available for the majority of teachers, especially within under-resourced schools in developing countries. In this thesis, I identified a framework to guide the analysis of mathematics curriculum materials for their educative potential. This framework could be used to evaluate the quality of existing mathematics curriculum materials and could be extended to inform the design of future materials. The well-designed and accessible materials could, as a result, provide an inclusive opportunity for teachers lacking access to high-quality professional development.

This doctoral research will have an impact on Turkish educational policy and practice. After completing my studies, I will work in the Assessment Department of the Ministry of National Education in Türkiye, where I will have an opportunity to use the insights I gained throughout this research. This department's recent agenda includes improving the quality of teachers' classroom formative assessment practices. To achieve this purpose, there has been an attempt to provide teachers with curriculum materials they can use in the classroom. The framework developed through this research could contribute to the development of further high-quality resources to support teachers' better formative assessment practices.

In addition to contributing to the development of curriculum materials, using the outputs of my doctoral research, I plan to initiate projects that focus on how to improve Turkish teachers' in-the-moment formative assessment practices. I have already initiated a small-scale project with a colleague in Türkiye that aims to investigate Turkish middle school mathematics teachers' use of technology for classroom assessment, and I bring the insights gained through this doctoral research to this project. In the future, I plan to expand this project in collaboration with in-service middle school mathematics teachers in Türkiye, to develop effective formative assessment lessons for their use.

Finally, to maximise the potential impact of this research, I will disseminate the findings to both academic and professional audiences worldwide. In addition to presenting my findings in international academic conferences and publishing my findings in international academic journals, I will produce briefings targeted at practitioners, as well as publishers and developers of curriculum materials.

DEDICATION

I dedicate this thesis with heartfelt gratitude to the memory of Vural Üçgül, my beloved mathematics teacher in middle school, whose passion for teaching influenced my pursuits in teaching and research. I extend my dedication to all teachers who, like him, positively influence their students' lives.

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I am grateful to my family for their constant support. I cannot imagine how this journey would be without them, especially, my beloved mother Nurşen, my sisters Tuba and Selma, my niblings Zeren and Eymen, and my brother-in-law Nihat. I should particularly thank Tuba for her patience in listening to my research ideas although she has no interest in mathematics education and for her generous financial support in the final year. And, surely, I would like to thank my late father Ömer. Remembering his enduring values has inspired me to maintain my persistence through challenging times. Furthermore, I would like to thank my aunts, uncles, and cousins for their constant trust and compassion.

I am grateful for many friends in Türkiye because they have always been there for me. Especially, I would like to express my gratitude to Elif, who consistently checked on my well-being and encouraged me. Additionally, I want to thank Şerefnur, who inspired me by sharing memories from her PhD journey and stories from Turkish literature, especially for reminding me of the “carrying wood” metaphor from Yunus Emre’s life story. Finally, I would like to thank Harun for his constant support.

I met many amazing friends in London who are a significant part of my training and will be life-long friends. Thanks to meeting her in several training courses I took in the first year, Frances has been an important supporter for me in London from the earlier stages to the end of my journey. Refika helped me a lot to shift from a practitioner mindset to a researcher mindset, generously sharing her knowledge and experience in academia. Also, although we have different research interests she kindly listened to my rough ideas for hours and encouraged me to keep on working on these ideas. Furthermore, I would like to thank Stella for her friendship, constructive comments on some parts of my thesis, and constant encouragement, and Sondos for kindly sharing her experiences regarding the ways to improve academic writing skills, especially by reviewing my very early drafts in the early years of my studies. Last but not least, I would like to thank my non-academic friend Beatrice, a valuable member of my supporting crew in London, for her sincere friendship.

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LIST OF ABBREVIATIONS

AfL: Assessment for learning

BSRLM: British Society for Research into Learning Mathematics

CCSSM: Common Core State Standards for Mathematics

CCSSO: Council of Chief State School Officers

CERME: Congress of the European Society for Research in Mathematics Education

CMP: Connected Mathematics Project

CMP1: The first iteration of the Connected Mathematics Project

CMP2: The second iteration of the Connected Mathematics Project

CMP3: The third iteration of the Connected Mathematics Project

CMP4: The fourth iteration of the Connected Mathematics Project

DNL: Double number line

GCSE: The General Certificate of Secondary Education

ICCAMS: Increasing Competence and Confidence in Algebra Structures

KS3: Key stage 3

MKT: Mathematical knowledge for teaching

PD: Professional development

PDC: Pedagogical design capacity

PM1: The first pedagogical message identified for vertical analysis

PM2: The second pedagogical message identified for vertical analysis

PM3: The third pedagogical message identified for vertical analysis

PM4: The fourth pedagogical message identified for vertical analysis

PM5: The fifth pedagogical message identified for vertical analysis

RQ1: The first research question

RQ2: The second research question

RQ3: The third research question

SoL: Scheme of learning

UCL: University College London

US: The United States

CHAPTER 1 - INTRODUCTION

1.1. Background

In an era where external high-stakes testing dominates teaching and learning in many countries, effectively integrating formative assessment into school teaching practices offers a promising remedy for the potential negative impacts of these tests on learning. Formative assessment shifts the focus from merely measuring learning outcomes to engaging with students' learning processes, adapting teaching to meet their needs (Shepard, 2000; Wiliam & Thompson, 2007). This form of assessment is highly valued among practitioners and researchers for its potential to enhance the quality of both teaching and learning (Heritage & Harrison, 2020). However, implementing formative assessment, while staying true to its core principles remains a challenge. There is a possibility this mode of assessment can be misinterpreted as a result of teachers and students having a superficial understanding of its purpose, hindering students' learning, despite teachers' good intentions.

This thesis therefore aims to explore the characteristics of teacher resources in England and the United States (US), identifying ways to enhance teachers' understanding of the key principles of formative assessment, so they can design their teaching practices accordingly. I aim to contribute significantly to ongoing academic discourse by focusing on the potential challenges teachers might encounter in diverse educational settings, proposing nuanced insights into how to support teachers effectively.

This background section sets the stage for this doctoral thesis by detailing the pressing need for this research. Namely, that teachers require well-designed and accessible resources to facilitate their integration of formative assessment in their practices. This section also presents driving motivations transformed from the earlier stages of this research to the final stages.

1.1.1. Problematising the current situation with formative assessment in research and practice

Vague conceptualisations

One challenge faced by mathematics teachers when implementing formative assessment is the weak conceptualisations of the concept. These weak conceptualisations might result in a lack of explicit comprehension of the elements that comprise formative assessment and a poor understanding of the interactions between these elements (Bennett, 2011). In particular, discrepancies that affected a large number of studies on this topic in both academic and policy contexts in the early 2000s led to confusion in terms of the function and scope of formative assessment. More recently, however, various researchers in the field of educational assessment and policy-informing organisations have sought to present clearer conceptualisations identifying the elements of formative assessment (e.g., Black & Wiliam, 2009; CCSSO, 2018; Cizek et al., 2018). These conceptualisations are beneficial in understanding the essence of formative assessment as planned classroom practice, identifying its functions, such as sharing disciplinary learning intentions with students, supporting students to become self-directed learners, and assisting teachers in planning and developing better instructional decisions; also specifying the key actors involved (teachers, peers, and students).

Wiliam and Thompson (2007) proposed an actionable framework comprising five strategies to guide teachers' practices. These strategies involved: (1) identifying learning intentions and communicating them effectively to students, (2) eliciting students' learning through tasks and questioning, (3) providing constructive feedback to shift students' learning forward, (4) utilising students as resources to scaffold each other's learning, and (5) empowering students to take an active role in their own learning. This framework provides actionable strategies to benefit teachers. Moreover, Wiliam and Thompson highlight all teachers, peers and students as agents of formative assessment, identifying each agent's role.

Despite the potential benefits of this framework, contextualising and adapting these strategies to specific teaching environments still represent a significant challenge for teachers. In his critical review of formative assessment, Bennett

(2011) drew attention to the requirement for discipline-specific adaptations of the framework. Explicitly, learning indications can vary across different disciplines, as observed by Hodgen and Marshall (2005) who contrasted different requirements when teaching English and mathematics. Supporting Bennett's critique, this suggests a need for a thorough examination of the meaning of learning in specific disciplines, exemplifying strategies corresponding with this meaning.

Teachers' tendency towards conventional practices

In both educational research and policy contexts, considerable attention has been directed towards fostering a paradigm shift (e.g., DfE, 2015; Kilpatrick et al., 2001). This shift aims to identify learning intentions and develop classroom practices that align with these intentions. Practices associated with these new paradigms include communicating learning intentions that emphasise reasoning and problem-solving alongside procedural skills. They also involve fostering student autonomy in learning, enhancing their thinking processes beyond merely categorising their responses as correct or incorrect, and promoting collaborative work among students. However, the implementation and sustainability of these practices have proven challenging and have only been gradually undertaken (Lithner, 2017).

Crucially, effective formative assessment practices necessitate promoting learners' autonomy (Wiliam & Thompson, 2007). Notably, however, teachers may tend to teacher-centred practices that can inhibit students' active involvement in assessing their learning (Antoniou & James, 2014; Marshall & Drummond, 2006). Paradoxically, in those instances where teachers declare a commitment to supporting students' autonomy, their practices remain inconsistent with the "spirit" of promoting student autonomy, as exemplified by Marshall and Drummond's empirical findings.

Teachers might struggle to identify the learning goals they expect from their students when implementing formative assessment (Antoniou & James, 2014). This can be especially affected by the tendency to prioritise mastering skills, and delivery of content focused primarily on transmitting facts to students (Dayal, 2021). Additionally, although teachers may prioritise skills such as students' reasoning and problem-solving, they may require further support identifying instances of these skills (Barham, 2020). Furthermore, teachers exhibit negative attitudes towards formative assessment practices, finding them both time-consuming and demanding (Yan et al., 2021).

Teachers' critically understanding of what students already know is another important requirement when implementing formative assessment practices (Wiliam & Thompson, 2007). Such understanding requires the use of appropriate tasks and questioning techniques, coupled with listening attentively to students' responses and interpreting those responses (Bennett, 2011; Wiliam & Thompson, 2007). Notably, although such practices arise organically in many classrooms, assuring their quality may require nuanced expertise. For example, teachers may use unstructured questioning or short-answer questioning as a tool to elicit correct answers, thereby neglecting the opportunity to uncover and assess students' thinking processes (Antoniou & James, 2014; Furtak et al., 2016). Moreover, interpreting students' responses could be a challenging task for teachers, as illustrated in Furtak et al.'s (2016) research.

An enactment of productive feedback practices is another aspect that demands refined teacher expertise. In essence, teachers may be inclined to rely on feedback for merely correction or judgment and as short-term rewards (Antoniou & James, 2014).

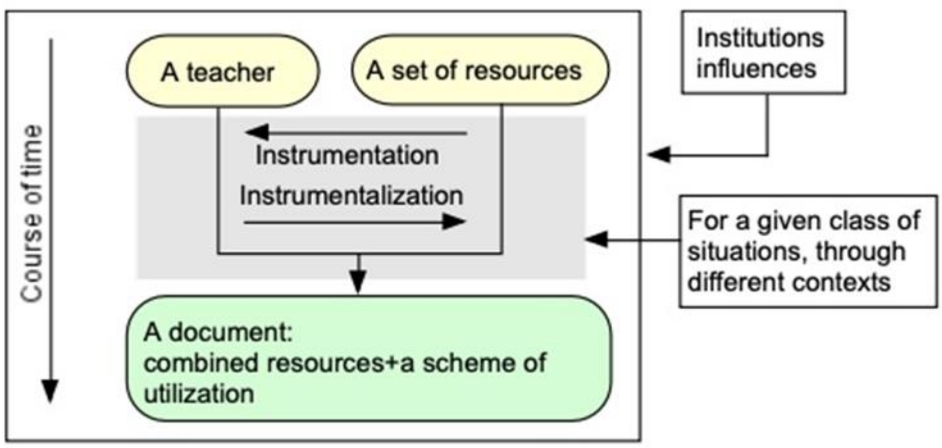
As established above, there are many challenges to implementing formative assessment due to the vague conceptualisations of it and teachers' reluctance to change their practices. To mitigate such challenges and navigate the teachers' need for further support, this thesis investigates how best to support teachers through accessible materials. The following section discusses the use of curriculum materials for this purpose.

1.1.2. Educative role of curriculum materials

Curriculum materials are artefacts designed to support teachers in embracing and enacting specific educational policies (Valverde et al., 2002). The shift from policy to practice is often referred to using the notions of a "written curriculum" and an "enacted curriculum" (Stein et al., 2007). The written elements of the curriculum are often directly accessible to teachers, and as such, are used as opportunities to convey the intended policy intentions to teachers. However, for over two decades, research has provided evidence of the barriers affecting teachers' understanding of the written aspects of the curriculum and enacting the effective suggested practices in classrooms (e.g., Askew, 1996). These arise for several reasons linked to their expertise in the subject and the level of attention they pay to engaging with such documents (Askew, 1996).

The source of the potential inconsistency between the intention of the curriculum and teachers' practices can be variously explained relative to several mechanisms: teachers' experience, knowledge, and beliefs; the specific features of curriculum materials; specific classroom context, or the external pressures such as high-stakes exams and school policy (Charalambous et al., 2012; Van Steenbrugge & Ryve, 2018). The documentational approach, proposed by Gueudet and Trouche (2009), theorises the relationship between teachers and curriculum materials by embracing various mechanisms that can influence teachers' practices. As shown in Figure 1.1, two directions, illustrating the interactions between teachers and curriculum materials are positioned at the centre of this model.

Figure 1.1 Representation of the documentational approach



Note. Reprinted from Gueudet & Trouche (2009, p. 206)

In this thesis, the direction away from curriculum materials, titled "a set of resources" in the original model, to a teacher was examined. During this examination, the potential "educative" function of the curriculum materials was explored. The notion "educative" has been present in the literature, since Ball and Cohen (1996) explicitly highlighted the potential of curriculum materials to develop teachers' learning alongside students' learning. They emphasised the importance of considering educative potential when designing curriculum materials. Building on Ball and Cohen's recommendation, a body of research has arisen, directly focusing on the issue of designing curriculum materials with educative potential in the domains of science and mathematics (e.g., Davis & Krajcik, 2005; Davis et al., 2017; Quebec-Fuentes & Ma, 2018). Despite this valuable endeavour, the identification of the specific features with the capacity to enhance the educative potential of curriculum materials is an ongoing process.

In this thesis, which is informed by the body of literature on the relationships between teachers and curriculum materials, and specifically on the educative potential of these materials, the educative potential of curriculum materials is considered a key phenomenon. This potential is considered as a mediator to enhance mathematics teachers' effective in-the-moment formative assessment practices.

1.1.3. Opportunities and limitations of curriculum material analysis studies in the current literature

Research on mathematics curriculum materials has been of interest for decades (e.g., Haggarty & Pepin, 2002; Herbel-Eisenmann, 2007; Stylianides, 2009; Van Steenbrugge & Remillard, 2023). A considerable body of research was undertaken to reveal the current state of textbooks, focusing on messages concerning mathematics as a discipline and pedagogy when teaching this discipline. A large number of these studies aimed to reveal the overall characteristics of certain sets of textbooks, so as to be able to feature understanding in specific contexts, such as country and designers (e.g., Haggarty & Pepin, 2002). More recently, the focus of this research has been explicitly expanded to explore the relationships between textbooks, teachers and students (e.g. Van Steenbrugge & Remillard, 2023).

As detailed in Section 3.4, the current body of research focusing on the analysis of curriculum materials can offer valuable insights into the analysis of the educative potential of mathematics curriculum materials for teachers' effective formative assessment practices. Importantly, the theoretical frameworks and analytical approaches employed in these studies can form the basis of rigorous study (e.g., Choppin et al., 2021; Quebec-Fuentes & Ma, 2018).

However, despite the opportunities offered, one limitation of this body of research is the lack of deeper insight into the features of the materials. This limitation can stem from the analytical approaches commonly employed in these studies, which prioritise the feasibility of handling large volumes of data. Such approaches allow researchers to attain a broad overview and make comparisons within different educational settings. The emphasis on sketching an overall picture has led to certain analytical strategies influenced by quantitative methods, even in qualitative studies (e.g., Machalow et al., 2020). Consequently, this focus can potentially thwart in-depth understanding of specific features of the materials. As will be discussed in detail later in this thesis (Sections 3.5, 4.1.1 and 4.5.1), this study contributes to this body of research through a pure qualitative data analysis, inspired by Braun and Clake's approach to the thematic analysis (2021).

1.1.4. Significance of the research

The findings of this doctoral research are expected to have significance for both educational assessment and mathematics curriculum resources research fields. In the educational assessment field, the key contribution is related to addressing the need for a discipline-specific understanding of classroom formative assessment that extends beyond generic definitions and strategies. In particular, this research rigorously considered the specific features of teaching and learning mathematics by expanding five formative assessment strategies, using notions in mathematics education literature such as strands of mathematical proficiency (Kilpatrick et al., 2001) and noticing students' mathematical thinking (van Es and Sherin, 2021).

In relation to the mathematics curriculum resources field, this doctoral research endeavours to advance ongoing research on the use of curriculum materials to mediate teachers' productive practices by theorising and exemplifying the educative features of well-designed curriculum materials.

Significantly, this research also seeks to establish a connection between the theoretical and rather abstract concepts of formative assessment and the teacher guides which are considered everyday aspects of teaching practice.

In conclusion, this research introduces complementary aspects from the educational assessment field with a focus on formative assessment, the mathematics education field in terms of teaching and learning, and mathematics curriculum resources with focusing on the educative features of curriculum materials. Applying this approach, this research endeavours to explore the implications for mathematics teachers' classroom practices, by proposing features of mathematics curriculum materials that can mediate teachers' effective formative assessment practices through enhancing their professional growth.

1.1.5. Personal motivation

I had two personal motivations to conduct this research. The first motivation comes from my teaching background. Before starting this PhD, I taught mathematics in several schools in Türkiye. Although I had the opportunity to take a high-quality pre-service teacher training in a well-acknowledged institution in Türkiye, I encountered several challenges in my almost 10 years spent teaching. This led me to seriously consider the inclusive ways to support in-service teachers' continuing professional development by providing accessible tools.

The second motivation emerged as my academic focus shifted during my MA studies in the Educational Assessment field in England in the 2016-2017 academic year. Although I had started the MA to specialise in large-scale exams, this one-year experience shifted my interest to classroom formative assessment. I was excited about this new topic and enthusiastic to understand it well and introduce the ideas I encountered to Türkiye where this topic was under-researched.

When I started this research with these motivations in the academic year 2018-2019, I was concerned about finding techniques that could be used to contextualise and clarify formative assessment strategies for mathematics teachers. Throughout the research, I adapted my focus away from techniques and towards pedagogical messages. Although I still value contextualised and practical classroom practices for formative assessment in mathematics classrooms, I prioritise identifying the pedagogical messages to be conveyed to teachers to enable them to enact these messages in their classrooms.

The sharp shifts in my approaches throughout the research were unexpected in the initial stages, but I consider the processes I underwent to have been a valuable learning experience for a developing researcher. In particular, these shifts helped me to be prepared for unexpected changes in future research endeavours and to embrace potential changes as opportunities for growth as a researcher.

1.1.6. My position on students' learning of mathematics and teachers' professional development

Before starting this PhD, I taught mathematics to early secondary students in several institutions in Türkiye (see Appendix B). My experiences respectively involved preparing students for national exams, teaching basic mathematics in mainstream schools and leading workshops with students who were diagnosed as gifted. During these years, I had the opportunity to engage with various mathematics curriculum materials and observe how these materials impacted my teaching and my students' learning. In particular, I had the opportunity to teach both low and high-privileged students with varying attainment levels. This enriched my understanding of the benefits of specific curriculum materials on students' learning in different contexts.

When I was teaching in Türkiye, my strong motivation to improve my teaching practices led me to pursue postgraduate studies in mathematics education and the completion of a master's degree in this area. After getting funding from the Ministry of National Education in Türkiye to continue postgraduate studies abroad, my research focus shifted to the issues of educational assessment. Within this studentship, I completed a master's in the MA Educational Assessment program at University College London (UCL). During this master's I had an opportunity to gain insights into the issues related to the effective classroom formative assessment. When I reflected on my experiences as a student and teacher in Türkiye with the help of these insights, I thought that there was a need for experts in this topic in Türkiye. This has changed my research interest in effective formative assessment practices in secondary mathematics classrooms. My postgraduate studies in mathematics education and educational assessment, along with my teaching experiences, encouraged me to pursue this research topic.

Reflecting on my experiences as both a student and teacher, and informed by my readings on mathematics education literature, I consider socio-cultural approaches of learning to guide my interpretations in this study. This influenced my interpretations in two aspects: students' learning of mathematics and teachers' professional development. In this research, it is a key aspect to contextualise broad formative assessment principles in the mathematics context. This requires a reflexive positioning of what learning mathematics means to me, which influenced my choices of theoretical positions and the analysis of teacher guides. When seeking the answer to the question of what learning mathematics is, I considered Sfard's (1998) two metaphors of learning: acquisiting intended knowledge and skills versus participating in a learning community.

While I am sceptical of the idea of considering the components of these metaphors as two distinct approaches, I position myself closer to the participation metaphor. As will be discussed in more detail in Section 2.2.1, Kilpatrick and his colleagues' (2001) model of mathematical proficiency was chosen in this research as it aligns with the participation metaphor of learning. Namely, this comprehensive model involves five interrelated elements as signs of students' being competent in mathematics. These elements are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Importantly, the element of productive disposition explicitly refers to students' becoming mathematics learners. Multiplicative reasoning was chosen as a context to study for its important role in students' learning of mathematics. Namely, multiplicative reasoning is an essential skill for students' shift from concrete procedures to abstract thinking as well as being a foundation for further mathematics (Lamon, 2007).

For teachers' professional development, the key idea that has guided my choices in this research is that teachers' professional learning should directly inform their classroom practices (Rowland et al., 2005; Schoenfeld, 2020). This led me to explore the theories that examined the relationship between curriculum resources and teachers' practices from a socio-cultural perspective (e.g., Gueudet & Trouche, 2009; Remillard, 2005). Section 2.3 provides a discussion of these approaches.

1.2. Research context

1.2.1. Formative assessment in England and the US: Research, policy and practice

In both England and the US, the notion of formative assessment has captivated researchers, practitioners and policymakers for several decades. In the earlier years, formative assessment was linked to tests, the results of which were used for formative purposes rather than to award students' grades (Bloom, 1969). This understanding of formative assessment evolved into teachers' assessment practices during teaching to enhance students' learning (Sadler, 1989). In particular, the seminal review by Black and Wiliam (1998) highlighted the role of formative assessment in the classroom environment and triggered a reinterpretation of the term formative assessment in both England and the US. This was followed by a scholarly discussion about what formative assessment really is, and how best to integrate it into teaching. The key shift here was from evaluating the outcomes of students' learning to assessing students' learning processes and encouraging them to take responsibility for their own learning.

In England, the practical implications of enhancing learning through effective formative assessment practices have been of interest since the publication of the aforementioned paper. In the early 2000s, this significant research topic received considerable funding to support collaborations among well-acknowledged educational researchers at highly recognised institutions, including King's College London and Cambridge University. The project, led by Paul Black and Dylan Wiliam and known as the King's-Medway-Oxfordshire Formative Assessment Project (KMOFAP), was one of the most influential of these collaborations. This project involved teacher training for effective formative assessment practices, observing changes in teachers' practices as a result of the training and examining the effect of this change on students' learning for subjects English, Mathematics and Science. Alongside several academic publications, the research team published practical guides for teachers (e.g., Hodgen & Wiliam, 2006). Moreover, in 2002, informed by the findings of this research project, the Assessment Reform Group (ARG) – a group of assessment researchers that guided educational policies and practice in England between 1989 and 2010 – prepared and sent a leaflet to all public schools in England. This leaflet proposed the introduction of formative assessment principles for teachers.

Even after these years, when formative assessment was at the peak of its popularity in England, the interest in leveraging formative assessment to enhance learning has remained strong. Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) project, which was conducted between 2008 and 2018, aimed to explore how to integrate formative assessment into mathematics lessons. As a result of this project, a set of formative assessment integrated lessons on multiplicative reasoning and algebra topics were developed. Following this, more recently, a large-scale project was conducted, including 140 English secondary schools, to investigate the effects of formative assessment integrated teaching on students' learning in English and mathematics (Anders et al., 2022).

Similar to the English context, considerable interest in formative assessment began in the US in the early 2000s after Black and Wiliam's seminal review paper was published. However, the governmental support for integrating formative assessment in teaching there was rather limited compared to the English context (Heritage & Harrison, 2020). In particular, the dominant testing culture across the US hindered the understanding and implementation of this promising assessment approach. Test publishers in this country have contributed to misinterpretations of formative assessment by marketing the tests they developed as formative assessment tools.

Despite such misinterpretations, several well-acknowledged researchers in the educational assessment field in the US embraced the essence of formative assessment as introduced by Black and Wiliam (e.g., Sheppard, 2000; Stiggins, 2002). Particularly, Lorrie Shepard contributed to this area of research with her publications since 2000. She has drawn attention to the need for a shift from testing culture to learning culture through formative assessment, highlighting the potential of formative assessment for providing equity and high-quality learning as opposed to testing (Shepard, 2000; Shepard, 2021; Shepard et al., 2018).

In addition to the individual efforts of researchers in integrating formative assessment in teaching in the US, in recent years collaborative endeavours for this purpose have arisen. Notably, partnerships between academic institutions and non-profit organisations, as well as joint research projects, have played a pivotal role in advancing the understanding of formative assessment. For example, The Council of Chief State School Officers (CCSSO) is a nationwide organisation in the US which works collaboratively with states to improve the quality of education, publish research-based reports to inform policy and practice for formative assessment (Popham, 2006; CCSSO, 2018). Moreover, collaborative research projects that involve integrating formative assessment principles into curriculum materials have shown ongoing interest in this topic, as exemplified by the Mathematics Assessment Project (MAP). MAP was conducted in collaboration with researchers from the University of California and the University of Nottingham. As a result of this project, mathematics lessons that aim specifically to assess students' learning were designed.

The positive developments in formative assessment over the years in both countries are apparent. These include a deeper understanding and support for the "spirit" of formative assessment and the increased accessibility of formative assessment resources. However, implementing such principles in their classrooms continues to be a challenge for teachers. Indeed, a recent report from the Office for Standards in Education (Ofsted) in England (2023) highlighted how teachers' assessment practices have shifted from identifying students' learning processes and the need to prepare them for tests from the late stages of primary school onwards.

1.2.2. Multiplicative reasoning as a critical case

Multiplicative reasoning represents a pivotal shift in students' mathematical thinking, marking the transition from concrete thinking, as characterised by counting and grouping objects to the development of a more abstract level of thinking. Such abstract thinking is essential for grasping the multiplicative relationships between quantities (Lamon, 2007). That is, it enables the learner to use reasoning beyond the repeated addition, which is often considered the initial stage when learning multiplication. Such reasoning is not only fundamental but also serves as a cornerstone for mastering further mathematical concepts. It underpins students' learning in areas ranging from fractions and ratios to algebra and geometry, thereby laying the groundwork for a comprehensive understanding of mathematics.

Despite its critical importance, acquiring proficiency in multiplicative reasoning poses significant challenges to both teachers and students (Hilton & Hilton, 2019). This difficulty often stems from the process of moving from additive to multiplicative thinking (Askew, 2018). Formative assessment related practices have the potential to address the complexity of this transition. However, considering that even teachers might have difficulties with such thinking, they require additional support when implementing such practices in the multiplicative reasoning context.

1.3. Research questions

In Section 1.1.4, the significance of this doctoral research was explained. That is, it aims to address the need to conceptualise generic formative assessment strategies in the context of secondary mathematics teaching. Additionally, this research aims to focus on educative curriculum materials as tools that can facilitate teachers to deliver effective in-the-moment formative assessment practices. In this section, I will outline the precise research questions that governed the research procedures.

1.3.1. Research questions that address the aim of developing a framework that can guide the analysis of the educative potential of curriculum materials

In Section 1.1, the need to contextualise formative assessment strategies and the potential role of curriculum materials when conveying key principles to teachers were highlighted. Recognising the intersection of these concerns with the methodological gap in the research on curriculum material analysis, this doctoral study specifically aimed to analyse well-designed curriculum materials to reveal nuanced features of these materials. This aim is beyond the provision of overall generalisations within certain curriculum materials or making comparative judgments. As a result of this analysis, it was aimed to produce a framework to guide analysis of the educative potential of the mathematics curriculum materials focusing on teachers' in-the-moment formative assessment practices.

Adopting a qualitative approach towards Braun and Clarke's (2022) approach, two gaps within the existing research were investigated to inform the development of this framework: operationalising formative assessment strategies and identifying educative features. The following two sub-sections present the specific research questions that guided this investigation.

Operationalising five formative assessment strategies

In order to operationalise formative assessment strategies, this research focused on analysing well-designed curriculum materials from England and the US. As described in Section 1.2., formative assessment has been valued within research and policy contexts in both countries, making it possible to access high-quality examples of formative assessment practices within the curriculum materials developed in these countries.

Beyond the country context, multiplicative reasoning was chosen as a critical case and a subject-specific topic for two reasons. First, multiplicative reasoning has been acknowledged as fundamental to mathematical development because it is required when learning various topics in mathematics, such as ratios, percentages and statistics (Kilpatrick et al., 2001). Second, there exists a large body of research that provides the challenges teachers and students face when teaching and learning multiplicative reasoning (Lamon, 2007).

As a result, the initial research question was identified as the following:

RQ1: How can the five strategies of formative assessment suggested by Wiliam and Thompson (2007) be operationalised so that they guide mathematics teachers' in-the-moment formative assessment practices?

This research question was supported by further questions as a result of the three key challenges encountered during the initial stages of the data analysis: overlaps among the five formative assessment strategies, identifying the five strands of mathematical proficiency in suggested classroom practices, and limited references for detailing classroom feedback practices (elaborated on further in Chapter 5). Driven by these challenges, and in order to provide a comprehensive understanding of five formative assessment strategies in the secondary mathematics context, this study explored several complementary inquiries, each focusing on a different aspect of classroom formative assessment. More explicitly, in the subsequent stages of the research, the following complementary question was investigated.

Complementary question for RQ1: What are the example classroom practices suggested in the well-designed teacher guides that can be associated with identifying the five strands of learning mathematics, eliciting students' learning and providing feedback that can move their learning forward?

Identifying educative features

Existing conceptualisations of educative features of curriculum materials provide insights into the intention of curriculum materials to support teachers' learning. These conceptualisations may create confusion about the teachers' and materials' roles, which can hinder the identification of the relationship between teachers and curriculum materials as well as the real function of the materials as a result. In order to address this issue, the second aspect of the developed framework was to identify educative features that highlight the function of curriculum materials in enhancing teachers' in-the-moment practices. Specifically, the following research question guided this investigation.

RQ2: What educative features of curriculum materials can be suggested?

1.3.2. Research questions that address the aim of employing the developed framework

Once the operationalisation of the five formative assessment strategies and identification of the educative features were completed, the next aim was to employ the constructs developed to guide the analysis of the educative potential of teacher guides. During this exploration, an issue arose. The overall educative potential of curriculum materials for formative assessment might not reflect the educative potential of these materials to convey specific pedagogical messages. As a result, the educative potential of curriculum materials was analysed from two perspectives as explored in the following research question and sub-questions.

RQ3: How does the developed framework contribute to the understanding of the educative potential of teacher guides to facilitate mathematics teachers' in-the-moment formative assessment practices?

- *What are the characteristics of the educative potential inherent to the teacher guides to facilitate teachers' in-the-moment formative* *assessment practices, specifically when approached from a horizontal perspective?*

- *How can teacher guides facilitate teachers' formative assessment practices for specific pedagogical messages?*

1.4. Research approach

Drawing on examples from earlier studies that combine different paradigms in a single research (Goldkuhl, 2012), pragmatism and a qualitative approach towards "big Q" (Braun & Clarke, 2021) were employed as guiding approaches. Explicitly, while the knowledge to be created and the choice of methods were aligned with pragmatism, the process of producing the knowledge, mainly data analysis, was guided by the qualitative approach (Braun & Clarke, 2021).

First, the knowledge I intended to produce as a result of this research was aligned with the philosophy of pragmatism. In the pragmatist approach, knowledge is considered as a grounding for action (Goldkuhl, 2012). It is also anticipated that the knowledge created as a result of this research would usefully serve as a guide for textbook designers and teachers. Pragmatism views advanced understanding as an instrument that can result in change (ibid.). This research aims to understand and illustrate the educative features present in curriculum materials initially. The ultimate goal is to develop a framework to guide analysis of the educative potential of existing curriculum materials, with a specific focus on mathematics teachers' in-the-moment formative assessment practices.

Second, when choosing methods to employ and when collecting and analysing data, instead of staying loyal to specific methodologies such as design-based research or grounded theory, the research questions informed the decisions made. More specifically, I conducted a collective case study, wherein five sets of curriculum materials served as instrumental cases. The aim was to identify a diverse range of educative features and explore their synergies, providing valuable insights to develop the intended framework. This purpose aligns with the pragmatist paradigm, which emphasises creating knowledge to ensure change and improvement (ibid.).

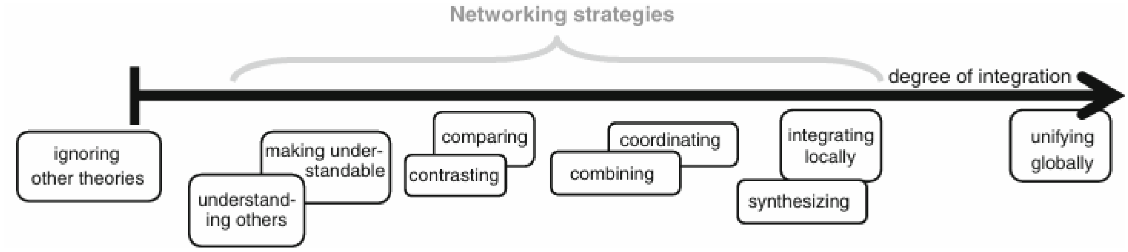
A pragmatist paradigm enabled me to choose techniques from different methodological approaches and create an original working analytic strategy to address my research questions. However, it has been a challenge to ensure that these techniques align together. After using five formative assessment strategies (Wiliam & Thompson, 2007) as a guiding framework in the explorative phase, during the critical phase, I combined several data analysis techniques. These techniques include theoretical coding by using different frameworks such as five strands of mathematical proficiency for the first formative assessment strategy and three elements of noticing (Van Es & Sherin, 2021) and openness of tasks for the second formative assessment strategy (Yeo, 2015). While combining these various theoretical frameworks, the networking theories was the key approach to aligning the approaches (Prediger et al., 2008).

In the networking theories approach, the diversity of theories is considered as a potential opportunity for more insightful research (ibid.). More explicitly, the networking theories approach is a systematic approach in order to identify the use of more than one theory in a study by enabling communication among various theories by preserving the specific features of these theories. Networking theories can serve various research purposes such as understanding the theories better and giving the rationale for choosing particular theories, understanding the contribution of the theories, understanding the empirical data, and creating new theoretical frameworks by building upon the existing theories (ibid.).

Prediger and her colleagues proposed a conceptual framework that can guide networking theories systematically. In this framework, two extreme strategies were identified as ignoring other theories and unifying globally as presented in Figure 1.1. The four pairs of strategies between these two extremes were considered as various ways of connecting more than one theoretical approach. Within these strategies, I used coordinating. Namely, this strategy involves identifying the fitting elements of different complementary theoretical frameworks and creating a coherent framework with these elements. In particular, I coordinated the first formative assessment strategy, identifying and sharing learning intentions, with five strands of mathematical proficiency, and the second strategy, engineering tasks and questioning to elicit students' learning, with the three components of noticing (Van Es & Sherin, 2021).

Furthermore, I coordinated the shaping element of the noticing framework with the framework of task openness (Yeo, 2015).

Figure 1.2 Networking strategies



Note. Reprinted from Prediger et al. (2008, p. 170)

It should be noted that, although my research follows a pragmatist paradigm, not all the elements of research are wholly aligned with practices commonly associated with pragmatism. For example, pragmatist research is often associated with mixed methods research and research tools originally developed within positivist paradigms (Morgan, 2014). However, in my research, I predominantly employ qualitative techniques. Importantly, I view the researchers' subjectivity not as a threat to the quality of the research, but rather as an opportunity for in-depth analysis (Ball, 1990). Embracing subjectivity allows for a more comprehensive exploration of the nuances and complexities inherent in the study. While I occasionally incorporate quantitative techniques (as in Section 6.1), my emphasis remains on qualitative methods.

1.5. Impact of the COVID-19 pandemic on the research design: From a design-based research idea to the collective case study

In the initial stages of this doctoral research, design-based research was acknowledged to be the leading methodological approach due to its suitability for the intended research purposes. Although design-based research presents practical challenges, it is widely recognised and valued as a methodological approach by mathematics education researchers, such as Cobb (2003), Ruthven et al. (2009), and Swan (2007). The approach is particularly beneficial, as it can generate practical solutions while also offering theoretical insights (Bakker, 2018). The driving force behind my decision to engage in design-based research was to address the difficulties teachers face when practising formative assessment in secondary mathematics classrooms. My goal was to develop design principles for future lesson plans and provide exemplary lessons as a solution to resolve these obstacles.

One drawback of conducting design-based research in the classrooms is the challenge of mitigating ethical issues that impact both teachers and students. In this study, the plan was to design six formative assessment lessons and implement and revise these lessons within three design cycles in a secondary school in London. It was significant to consider the extra workload this might impose upon teachers and the potential harm to students' learning as a result of the implementation of these lessons. BERA's ethics guideline (2018) was revisited in view of these issues and ethical approval was sought and received from the ethical committee at UCL in April 2020. However, the unpredictable pandemic situation raised additional ethical issues regarding the researcher's physical well-being, as well as increasing the potential harm for the teachers from participating in the research. Consequently, this research plan was withdrawn in May 2020.

The second plan, which was initiated after November 2020, was to conduct iterative interviews inspired by the iterative design cycles in design-based research (Bakker, 2018). In this research plan, lesson implementations were to be replaced by three to four iterative interviews with at least six mathematics teachers in England. These interviews included discussions with teachers concerning the materials they already used, as well as materials designed by other researchers that teachers are unfamiliar with. The enactment of this plan was slower than expected for three reasons. First, it proved a challenge to access teachers, mostly because the pandemic situation did not improve. The uncertainties and unexpected extra workload were still a concern for teachers. Although I approached a large number of teachers via different channels, such as social media, school websites, and circulated the research invitation letter among the existing contacts, and more than 10 teachers showed interest initially, only two iterative interviews with three teachers had been conducted by the end of April 2021. This speed of data collection seriously risked the completion of my studies.

Second, the second interview required teachers to devote extra time for the preparation before the interview by reviewing two formative assessment lessons with which they were not familiar. It was observed that only one of the three teachers truly engaged with these lessons. At the time, the teachers were still under pressure due to the extra workload in their schools and some of them were also home schooling their own children. Continuing the study was determined to potentially violate the ethical consideration of avoiding making excessive demands on participants (BERA, 2018), or seeking to exploit the participants' good intentions in order to speed up the process. Finally, as a novice researcher, I could not predict the time needed for tasks such as preparing, transcribing and analysing the interviews. The latter challenge could be managed by piloting every step of the interviews; however, the time restrictions and limited access to teachers, as mentioned previously, excluded this possibility. As a result, in order to be able to complete a rigorous research study within the scheduled time constraints of my PhD studies, I shifted my focus from interviewing teachers to curriculum materials analysis. This shift required a change in research design.

Ultimately, I conducted a collective case study, which involved analysing well-designed curriculum materials as instruments to develop a framework as a guide to analyse educative potential of curriculum materials with a focus on formative assessment. The focus shifted from concentrating on developing design principles to developing a framework to guide the analysis of the educative potential of curriculum materials. This shift mainly resulted from the methodological gap identified regarding this process. The details of this methodological gap will be elaborated upon in Section 3.5 and the details of the final research design will be covered in Chapter 4.

1.6. Overview of the research procedures

This study was conducted in two main phases, employing distinct research methods by using the same set of data but focusing on different aspects of this data set in each phase. While the details of the research procedures and the findings will be elaborated on in the later chapters of the thesis, this section only provides an overview to assist in providing a preliminary understanding of the research.

1.6.1. Explorative phase

Overview of purposes and methods

This phase was conducted to address three purposes: (1) to become familiarised with the literature concerning the key constructs explored in this research including formative assessment, educative curriculum materials, teacher learning and multiplicative reasoning; (2) to explore the educative features inherent in curriculum materials, which can potentially facilitate mathematics teachers' formative assessment practices when teaching multiplicative reasoning; (3) to inform research questions and research methods in the second phase of the research.

In order to address these aims, alongside a continuous review of the literature to deeply understand the key constructs of the research, I conducted a reflexive thematic analysis following the six steps suggested by Braun and Clarke (2006). The procedures of this analysis will be detailed in Section 4.5.

Overview of the results that informed the second phase

As a result of the reflexive thematic analysis, it was found that materials had the potential to provide opportunities to enhance teachers' effective noticing practices. This rich potential guided me to expand the focus on the second strategy of formative assessment – engineering tasks and questioning to elicit students' learning – by exploring and integrating noticing literature, which was widely studied by mathematics education researchers (e.g., Jacobs et al., 2009; van Es & Sherin, 2002; 2021).

In addition to the results of the reflexive thematic analysis, the challenges faced during this analysis informed the second phase. One challenge faced related to the overlaps in the five formative assessment strategies. In order to overcome this challenge, the function of each strategy was reconsidered by working on examples of classroom practices suggested in teacher guides. Therefore, in order to reorganise five formative assessment strategies so that they could be easily separated and identified, four decisions were made. First, teachers' identifying learning intentions was thought to be a prerequisite for their productive formative assessment practices. Second, two of the five formative assessment strategies comprised the key classroom formative assessment practices: eliciting students' learning and providing feedback that can move students' learning forward. Third, the fourth strategy, activating students for their peers' learning can facilitate the enactment of the two key classroom formative assessment practices. Fourth, the final strategy, activating students for their own learning, was acknowledged to be another expected function of formative assessment, alongside informing teachers' decision-making. These decisions encouraged a specific emphasis on the three formative assessment strategies implemented during subsequent phases of the research.

Alongside operationalising formative assessment strategies, three educative features were identified during the explorative phase: *alerting* teachers to the rationale and purpose to inform the suggested pedagogical practices; *equipping* teachers with useful tools and knowledge to facilitate teachers' enactment; and *guiding* teachers to use these tools and knowledge in correspondence with the rationale and the purpose of the suggested pedagogical practice. These educative features were employed when analysing the educative potential of the teacher guides in the following stages.

1.6.2. Critical phase

The second phase of the research aimed to analyse the educative potential of the curriculum materials using the framework proposed as a result of the first phase. In this phase, deeper analysis was conducted with chosen lessons that showed rich educative potential. The purpose of this phase was to avoid making comparisons across all the projects, while identifying various references of educative features and comparing these references. This process of comparison made it possible to exemplify the target educative features and to inform the future design of curriculum materials with educative potential.

While this thesis delineates a clear separation between the explorative and critical phases, it is essential to acknowledge that the practical implementation involved a dynamic interplay, characterised by iterative exchanges and mutual influences across the explorative and critical approaches. Significantly, certain aspects of the analysis during the critical phase enhanced some of the explorations undertaken in the initial phase.

1.7. Structure of the thesis

The structure of this thesis was finalised through an iterative process that involved multiple revisions of early drafts, reviews of existing literature, and the collation of findings from the analysed teacher guides. This final structure was designed to convey the main arguments comprehensively and understandably, rather than reflecting the order of the research procedures in terms of developing the conceptualisations and analysing the data.

Chapter 2 explored the theoretical foundation elucidating the underlying models and constructs that are instrumental to shaping this study. By presenting these models and constructs at an earlier stage in the thesis, readers can obtain a comprehensive understanding of the fundamental basis supporting the arguments.

Chapter 3 presents a synthesis of empirical studies, focusing on the relationship between curriculum materials and teachers' classroom practices; formative assessment integrated within the curriculum materials; and key methodological considerations that informed the methods in this study. This chapter mainly aims to justify the research questions and methodological procedures that underpin the study.

Chapter 4 elaborates on the methodological approaches used in this research and the methods employed.

Chapters 5, 6, and 7 provide the findings of the analysis of the teacher guides, situating them within the current body of literature. That is, Chapter 5 presents the framework developed during this research. Chapter 6 presents the findings that provide an understanding of the educative potential within the materials for formative assessment using a horizontal approach. Chapter 7 presents the findings of the analysis, demonstrating the educative potential of the teacher guides, focusing on specific pedagogical messages with a vertical approach.

Chapter 8 wraps up the thesis by reiterating its contribution to the literature and its implications for policy and practice.

CHAPTER 2 - THEORETICAL AND CONCEPTUAL UNDERPINNINGS OF THE STUDY

In this chapter, I introduce the conceptual and theoretical underpinnings of my research. These underpinnings were identified through a critical analysis of relevant key concepts, theories, and models. These contribute to the conceptualisations and theorisations that justify and partly address RQ1 and RQ2 below.

RQ1: How can the five strategies of formative assessment suggested by Wiliam and Thompson (2007) be operationalised so that they guide mathematics teachers' in-the-moment formative assessment practices?

RQ2: What educative features of curriculum materials can be suggested?

The discussions in this chapter will position my interpretations of formative assessment and the use of curriculum materials within wider scholarly discourse, as well as clarify the terminology to be used in this thesis. Consequently, three issues will be central to my research: exploring formative assessment (Section 2.1), implementation of formative assessment in mathematics teaching (Section 2.2) and examining teachers' interactions with curriculum resources (Section 2.3). The related conceptual and theoretical grounding to approach these issues will be introduced in this chapter.

2.1. Exploring formative assessment

2.1.1. Conceptualising formative assessment

The term "formative" has been part of the educational context since the 1960s, signifying its role in enhancing the quality of education. Originally referred to as the evaluation of educational programs (Scriven, 1967), it has evolved to be closely linked with enhancing students' learning through assessment evidence (Black & Wiliam, 2009; Sadler, 1989; Wiliam, 2011). Wiliam and Thompson (2007) assert that formative assessment shifts the focus from merely controlling or monitoring learning quality to actively fostering it. This involves assessing learning as it happens and adjusting the learning environment accordingly.

The terms “Assessment for Learning” (AfL) and “formative assessment” are either used interchangeably (Hodgen & Marshall, 2005) or nuanced differences are suggested between the two terms. Although most researchers agree that both formative assessment and AfL refer to assessment practices that improve students’ learning, some have aimed to highlight specific aspects in making distinctions. To illustrate, while Black and Wiliam prefer the term “formative” to highlight the function of assessment practices in improving students’ learning by adjusting learning environments beyond merely gathering information about students’ learning, Baird et al. (2017) prefer the term AfL, to highlight the role of fostering students’ self-regulation. As Bennett (2011) highlighted, merely choosing specific terminology may not solve the confusion in this research area without terms being conceptualised properly.

Definitional issues regarding formative assessment or AfL might result in difficulties in communicating related ideas in both research and practice. One communication challenge for research is to access and engage with the targeted studies. In two earlier reviews, researchers encountered this challenge: Black and Wiliam’s (1998) seminal comprehensive review which examined relations between assessment and classroom learning, and Kingston and Nash’s (2011) meta-analysis exploring the effectiveness of formative assessment practices. In both reviews, researchers encountered the challenge of there being no agreed definition of the term formative assessment. On one hand, Black and Wiliam (1998) addressed this challenge in their review by conceptualising formative assessment. In their account, an assessment was formative whenever the comparison of actual learning and intended learning informed the strategies to be used to address the gap therein. Although this approach enabled them to identify studies that are related to classroom formative assessment comprehensively, this approach could result in a significantly large number of studies to review. Kingston and Nash (2011), on the other hand, included studies that explicitly referred to either formative assessment or AfL. This approach could have resulted in a relatively manageable number of studies to review. However, not reviewing what these terms specifically refer to in each study might have resulted in comparing studies that highlight different aspects of these assessment practices. This makes it hard to rely on the conclusions of this meta-analysis.

With these potential challenges in mind, the initial purpose of this research was to choose one of the terms, either formative assessment or assessment for learning, for consistency and to conceptualise this term to enable a focused and rigorous review of the literature. In this research, inspired by Black and Wiliam (1998), the term formative assessment was chosen to highlight its function of improving students' learning by regulating learning environments according to assessment information.

Following this choice, formative assessment was conceptualised by examining three existing definitions, as presented in Table 2.1. These definitions were chosen because of their characteristics of addressing assessment in the classroom context, featuring students' participation in the formative assessment process, and highlighting the function of formative assessment in improving learning. Despite their common features, these definitions have some distinguishing characteristics, as presented below. For example, CCSSO (2018) and Cizek et al. (2019) highlighted that formative assessment is a planned process including discipline-specific features, while Black and Wiliam (2009) highlight the role of peers in the process. In this research, in line with these three definitions, formative assessment is considered a planned classroom practice that helps teachers and students improve discipline-specific learning outcomes. Formative assessment practices support students' understanding of disciplinary learning intentions, encourage them to take responsibility for their learning, support teachers' planning activities, and build a foundation for better decisions to be made during the instruction. As a result, three functions of formative assessment have been considered in this research: sharing disciplinary learning intentions with students, supporting students to take responsibility for their learning, and enhancing teachers' planning and decision-making.

In practice, formative assessment can be implemented in various ways. According to Wiliam (2018), formative assessment can be categorised into three types based on its scope. The first category is long-cycle formative assessment, which spans across terms and teaching units for a duration longer than a month. Its primary objectives are to monitor student achievement and ensure curriculum alignment. The second category is medium-cycle formative assessment, which takes place within teaching units lasting less than a month. Its purpose is to involve students in classroom assessment and to facilitate teachers' improvement. The third category is short-cycle formative assessment, which occurs within and between lessons, on either a minute-by-minute or a day-by-day basis. It aims to enhance student engagement and improve teacher responses. This research specifically focuses on short-cycle formative assessment with a focus on teachers' minute-by-minute assessment that occurs within lessons. To ensure consistency with mathematics education literature, which is the key context of this research, I use the notion of "in-the-moment" formative assessment practices throughout this thesis.

Table 2.1 Definitions inspired the conceptualisation of formative assessment in this research

Definition	Nature	Actors	Function	Elements
"Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited." (Black & Wiliam, 2009, p. 9)	Classroom practice	Teachers, students and their peers	Being a foundation for better instructional decisions by teachers, students, and peers	Eliciting, interpreting and using the evidence about student achievement
"A planned, ongoing process used by all students and teachers during learning and teaching to elicit and use evidence of student learning to improve student understanding of intended disciplinary learning outcomes and support students to become self-directed learners." (CCSSO, 2018, p. 2)	A planned, ongoing process used during learning and teaching	Teachers and students	To improve student understanding of intended disciplinary learning outcomes To support students to become self-directed learners	To elicit and use evidence of student learning
"As part of a planned assessment system, formative assessment supports teachers' and students' inferences about strengths, weaknesses, and opportunities for improvements in learning. It is a source of information that educators can use in instructional planning and students can use in deepening their understandings, improving their achievement, taking responsibility for, and self-regulating, their learning. Formative assessment includes both general principles, and discipline-specific elements that comprise the formal and informal materials, collaborative processes, ways of knowing, and habits of mind particular to a content domain." (Cizek et al. 2019, p. 14)	A part of a planned assessment system	Teachers and students	To support teachers in planning To support students to be agents of their learning	General principles and discipline-specific elements that comprise formal and informal materials, collaborative processes, ways of knowing, and habits of mind particular to a content domain

Note. Created by the author

2.1.2. Five strategies of formative assessment (Wiliam & Thompson, 2007)

In this research, five strategies of formative assessment were chosen as a guiding framework for two reasons. First, these strategies are readily adaptable to in-the-moment formative assessment practices: they can potentially facilitate teachers making timely decisions to enhance students' learning. Second, these strategies involve various aspects of formative assessment by offering a holistic view of formative assessment which allows for a more nuanced and multifaceted understanding of in-the-moment formative assessment. These strategies were originally introduced by Leahy et al. (2005) in a paper that primarily addresses practitioners rather than the academic community. Shortly after, Wiliam and Thompson (2007) brought these strategies into the scholarly conversation by briefly mentioning them in a chapter in the book "Future of Assessment", which is a collection of chapters from acknowledged scholars in the educational assessment field. More recently, Wiliam elaborated on these strategies by providing example classroom techniques in his book "Embedded Formative Assessment" (2018). The following table presents these strategies as originally proposed in these publications.

Table 2.2 Five strategies of formative assessment

	Where the learner is going	Where the learner is right now	How to get there
Teacher	Clarifying and sharing learning intentions and criteria for success	Engineering effective classroom discussions and tasks that elicit evidence of learning	Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	Activating students as instructional resources for one another	
Learner	Understanding learning intentions and criteria for success	Activating students as the owners of their own learning	

Note. Adapted from Wiliam & Thompson (2007, p. 63)

The model presented by Wiliam and Thompson (2007) distinctly emphasises three key agents of formative assessment: the teacher, peer, and learner. In alignment with this framework, this research will specifically concentrate on the role of teachers. As a result, the first three strategies – "clarifying and sharing learning intentions and criteria for success", "engineering effective classroom discussions and tasks that elicit evidence of learning", and "providing feedback that moves learners forward" – are the key foci. In this research, the focus is teachers' in-the-moment formative assessment practices, which mainly focus on the process of students' developing multiplicative reasoning rather than the outcome of learning. This led me to discard the success criteria from the focus of the study. In Section 2.2, further exploration of these three strategies will be undertaken within the context of mathematics teaching.

2.2. Formative assessment in mathematics teaching

In this section, a theoretical investigation was carried out to partly address RQ1. In this research, one key aim is to better understand the unique discipline- and subject-specific requirements of formative assessment strategies in mathematics teaching, in the context of multiplicative reasoning. In order to address this purpose, five formative assessment strategies (Wiliam & Thompson, 2007) were reconsidered using theoretical and conceptual models from the teaching and learning mathematics field, namely strands of mathematical proficiency (Kilpatrick et al., 2001) and the noticing framework (van Es & Sherin, 2021). Moreover, the requirements of multiplicative reasoning were examined, drawing on the existing literature. The following sub-sections present these explorations in terms of three formative assessment strategies.

2.2.1. Learning intentions in mathematics

Theoretical basis for learning mathematics

In this thesis, as an approach to learning, both cognitive and situated approaches are embraced when considering the learning intentions of students. In the cognitive approach, the learning intention is to internalise concepts and to improve reasoning skills by making connections among the concepts (Greeno, 1998). Conceptual field theory, grounded in the epistemology and psychology of mathematics, is an example of the cognitive approach (Vergnaud, 1994). This theory argues that concepts in a particular field are formed by recognising the interactions among schemes, concepts, and symbols (ibid.). While schemes refer to students' previous knowledge relevant to these concepts, symbols help to represent abstract mathematical concepts. More explicitly, when introducing multiplication to the students, teachers facilitate students making connections with their knowledge of addition, which already exists in the students' schema. Further, in order to enrich their knowledge of multiplication in their schema, these students need to have the opportunity to experience different representations of multiplication. Namely, while students only make sense of repeated addition by using existing addition knowledge in their schema, different representations including scaling and area-producing will help them to contextualise the notion of multiplication in their existing schema (Davis & Renert, 2009). In order to promote the learning of the concepts and make connections among them an interactive learning environment can be created where the teacher is a facilitator (Greeno, 1998).

Sfard (1998) used the metaphor of "participation" to highlight an alternative to the cognitive approach to learning. In this so-called situated approach, the learning intention is to build identities through participating in a learning culture (Sfard, 1998). Participation refers to two ways in which individuals interact with their community: individual learners learn the norms of the community by practising these norms and making a contribution to the community as a result. In some contexts, the common purposes and individuals' contributions to achieving these purposes are more explicit than they are in others. For example, in a musical ensemble, the shared purpose is performing synchronously to execute a musical composition. Both musicians' individual performance and their synchronous performances have an impact on this intended purpose (Greeno, 1998). In the classroom context, some researchers have shown that minor steps in the classroom can encourage students to learn socio-mathematical norms (Lampert, 1990; Yackel & Cobb, 1996). These steps can contribute to students' building mathematics learner identities.

As Greeno (1998, p.19) suggested, "all teaching and learning are situated; the question is what their situated character is". As well as creating situations in which to teach discipline-specific values, these can also be created to teach subject-specific skills.

In summary, while the cognitive approach can help with learning concepts and making connections among them, the situated learning approach can help with learning those values specific to the discipline as well as dealing with common misconceptions when learning a particular subject. In the cognitive approach, students' reasoning skills are more important, the teacher is the facilitator, and the learning process is more interactive. In the situated approach, on the other hand, the focus on practices and the teacher's role works to develop appropriate learning environments. As Sfard (1998) has highlighted participation refers to being involved in the activities relevant to the discipline, or even to the subject. Social learning is only one way of enacting this participation. Situations should also include subject-specific elements such as appropriate tasks and the use of discourse that encourage students to be engaged in this particular learning community. Moreover, situated approaches can be effective in motivating students.

Models that can represent learning intentions in mathematics

Although it is an old model developed in the 1950s, Bloom's taxonomy has been influential in education in terms of identifying learning goals and designing assessment tasks (Krathwohl, 2002). The original model involves six hierarchical levels of cognitive skills, from low-level skills, recalling facts, and understanding, to higher-level skills, application, analysis, synthesis and evaluation. More recently, learners' metacognitive knowledge was included in this model. This model can help identify various skills in a subject, and in particular, considering skills beyond recalling and understanding facts. However, relying on this framework might come with several limitations. First, it suggests a linear path of learning without explicitly demonstrating the relations between levels. Second, this model might not be helpful for specific disciplines and content without a thorough understanding of these.

Mathematics education literature offers two models which could be alternatives to Bloom's widely accepted framework. Namely, the mathematical competency model and five strands of mathematical proficiency were developed at similar times, in the early 2000s, by American and Danish researchers respectively (Kilpatrick et al., 2001; Niss & Jensen, 2002, as cited in Niss & Hojgaard, 2019). Both frameworks consider learners' mathematical thinking beyond procedural skills in specific topics and identify several elements of mathematical thinking necessary for learners' mastery of mathematics. However, a salient distinction between these frameworks relates to their scope for learning mathematics in terms of cognitive and affective domains. While the mathematical competency framework proposes eight constructs with a focused and detailed understanding of learners' cognitive mastery of mathematics only, the mathematical proficiency framework highlights learners' dispositions towards mathematics as an ultimate target of learning the subject. On one hand, the eight distinct but interconnected cognitive constructs involved in the mathematical competency framework are: mathematical thinking, representation, symbols and formalism, communication, aids and tools, reasoning, modelling and problem handling. On the other hand, mathematical proficiency involves five components, which are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. The most distinguishing aspect is the productive disposition in the mathematical proficiency framework. Productive disposition focuses on the mindsets, beliefs, and behaviours necessary for success in mathematics. It requires the learner to have a positive mental attitude, to be motivated, and to have confidence in their mathematics skills. Students with a productive disposition are willing to face difficult tasks, see the significance of mathematics, and view mistakes and challenges as opportunities to learn.

In this thesis, drawing on its comprehensive nature to include both cognitive and affective domains, five strands of mathematical proficiency were chosen as a guiding framework to identify learning intentions in mathematics teaching.

2.2.2. Teachers' noticing of students' learning of multiplicative reasoning

The notion of noticing has been a subject of interest among mathematics education researchers for over two decades. Mason (2002) emphasised that most of the teaching practices, including observing and recognising students' actions and responses, evaluating their performance based on specific criteria, and making informed decisions for the next steps, are dependent upon teachers' noticing. A growing body of research has focused on conceptualising teachers' noticing and exploring teachers' existing noticing skills and strategies to enhance these skills in varying contexts, such as different teaching levels from early years to secondary, a range of teaching experiences from pre-service teachers to experienced teachers, and in different geographical locations in different parts of the world. Whilst empirical studies on noticing will be discussed later in Chapter 3, the following sub-sections focus on linking the second strategy of formative assessment to noticing students' learning of multiplicative reasoning.

Conceptualisation of noticing as an expansion of the second formative assessment strategy, "engineering tasks and questions to elicit students' learning"

In the original framework proposed by Wiliam and Thompson (2007) that introduces the formative assessment strategies, these strategies are not described in detail. Later, in his practical book, Wiliam (2018) elaborated on this second strategy by providing examples of classroom techniques. Through these techniques, he examined the means of eliciting students' learning by highlighting aspects such as enabling student engagement, listening to students' answers interpretatively rather than evaluatively, interrogating the sources that result in students' thinking, and asking questions to activate students' thinking. Although these techniques are already helpful for noticing students' mathematical thinking, they can be enhanced by the conceptualisation of noticing in terms of two aspects. First, the role of these techniques in revealing students' learning is prioritised above teachers' engagement with students' learning. The notion of noticing, however, highlights teachers' skills in engaging with student responses. Second, presenting these techniques separately can overlook the continuous nature of classroom teaching, by associating noticing students' learning with certain techniques, where noticing is actually practiced in any student-teacher interaction.

As a theoretical grounding, three key papers on noticing in mathematics teaching which propose the components of noticing were considered, mainly because these papers specifically focused on classroom interactions when conceptualising noticing (Jacobs et al., 2010; van Es & Sherin, 2002, 2021). In their earlier paper, van Es and Sherin proposed two key components of noticing: attending to and interpreting students' thinking. Teachers attend to students' thinking by identifying noteworthy situations in the classroom and they interpret these situations through their previous experience, knowledge, and beliefs. Although this early conceptualisation helps identify the components of noticing, it can lead to underestimation of the teachers' role in creating situations that can enhance noticing outcomes. Later on, Jacobs et al. (2010), and more recently van Es and Sherin (2021), considered expanding this conceptualisation by involving the decisions teachers made, e.g. setting further tasks and questions, as a result of attending to and interpreting students' thinking. Van Es and Sherin (2021) took a more targeted approach to this decision-making process, emphasising the creation of mathematical interactions for further attending to and interpreting students' thinking. They proposed the additional component of shaping. In this study, I prefer van Es and Sherin's (2021) notion of shape, instead of making a decision or responding, to maintain the focus on the interactions that can facilitate noticing students' thinking. More importantly, "responding" as suggested by Jacobs et al. involves some aspects of feedback, which will be examined separately in my research.

Even though van Es and Sherin's (2021) conceptualisation of noticing is embraced in this study, I approach it slightly differently. Three elements of noticing – attending to, interpreting and shaping – were originally developed using video data which shows teacher-student interactions in the classroom and teachers' interpretations of these interactions. More importantly, in this conceptualisation, teachers' noticing begins when they interact with students, attend to noteworthy interactions, and interpret them using their previous experience, beliefs, and knowledge. The process of shaping follows, facilitating deeper exploration of students' learning. Nevertheless, in my research, the key focus is on how to support teachers' preparation before interacting with students and how curriculum resources that are used during this interaction can enhance teachers' noticing. This focus led me to approach the element of shaping as an earlier step of noticing.

Attending to and interpreting students' multiplicative reasoning

Teachers inevitably attend to specific aspects of students' responses in lessons. However, the aspects they attend to can be far from students' learning of mathematics and may not contribute to the quality of teaching and learning as a result. More specifically, they may attend to students' behaviours and learning profiles (Ebby, 2015), or pay less attention to the mathematical strategies of students they know (Goldsmith & Seago, 2011). Sufficient knowledge of students' mathematical thinking can help teachers to focus their attention on students' learning.

One way of supporting teachers' noticing of multiplicative reasoning can be increasing their awareness of students' potential misconceptions in this conceptual field. However, overly relying on students' misconceptions in teaching may result in neglecting to notice different paths in students' thinking processes (Sfard, 2008; Smith et al., 1994). In the multiplicative reasoning context, commonly expected student misconceptions are using additive reasoning in multiplicative situations and the belief that multiplication always makes the number bigger (Bell et al., 1981). Awareness of these misconceptions can help teachers to attend to them, but can limit student perception of incorrect answers these two misconceptions result in. Instead of solely focusing on students' misconceptions, multiplicative reasoning needs to be explored through a specific focus on the process of students' learning in this field.

Multiplicative reasoning refers to how students comprehend and use multiplicative structures, which requires abstract thinking, beyond using only concrete operations (Lamon, 2007). Multiplicative reasoning has received robust interest over the course of several decades from Piaget's clinical interviews that aimed to reveal how students construct their knowledge, and to identify students' reasoning and strategies, to recent developmental studies aiming to design lessons to improve students' multiplicative reasoning (Hodgen et al., 2014).

From a theoretical perspective, Vergnaud (1994) discussed multiplicative structures in the light of conceptual field theory, which combines the epistemology of mathematics and the psychology of learning mathematics. In the context of multiplicative reasoning, this refers to combining the knowledge of the concept of multiplication itself and how students can learn this concept. Instead of approaching the concept in solitude, conceptual field theory considers that concepts in a particular field are formed by recognising the interactions among schemes, concepts and symbols (ibid.).

Earlier empirical studies from the 1990s have provided rich insights into students' learning of multiplicative reasoning. From a cognitive perspective, Steffe (1992) explored students' schemes and how they adapt these when engaged in specific tasks. Moreover, how students identified units was of interest. Steffe and Olive (2010) contributed to this aspect by identifying various ways of students' constructing units such as "composing and decomposing, iterating, partitioning and measuring."

Multiplicative reasoning is a difficult skill to develop. An extensive body of research has proven that these challenges result from the long and difficult process of shifting from additive reasoning to multiplicative reasoning (Fischbein et al., 1985). Smith and Confrey (1994) highlighted that awareness of the historical development of the concept of multiplication can help researchers to understand learners' conceptualisations. Formerly, the use of numbers for only counting and measuring led to multiplication being conceived as repeated addition (ibid.). This understanding can lead to numbers being considered to be discrete, as well as leading to the misconception that multiplication makes numbers bigger and division makes them smaller (ibid.). On the other hand, numbers have come to be appreciated as continuous, including rationals and irrationals, later in history (ibid). To some extent, students' use of multiplication can follow the same order with their experiences out of school as well as their experiences in primary school.

Similar to the historical development of the concept of multiplication, students' additive reasoning can be shaped from their very early years with the impact of learning multiplication as repeated addition and dominantly using intuitive thinking, as highlighted by Confrey and Harel (1994). The instruction and curriculum resources used in the early years of learning multiplication can lead to challenges in attempting the adaptation of additive reasoning to various multiplicative situations in later years in secondary school (Askew, 2018; Dooren et al., 2010; Sowder et al., 1998). This reasoning can result in students' difficulty with understanding multiplicative relations in various situations such as when the scale factor is not a whole number (Fischbein et al., 1985), the misconception that multiplication always makes numbers bigger (Bell et al., 1981), and difficulty in identifying the scale factor in functional relationships (Askew, 2018).

The level of activity and attention to research in multiplicative reasoning decreased in 2000s in comparison to the robust interest and focus it received during the 1990s (Lamon, 2007). However, recently, an interest in this topic has been apparent as the references used in the previous paragraph indicate. Some interesting evidence relating to students' ongoing difficulties comes from Hodgen et al.'s study (2009). Hodgen and his colleagues compared test results for students' multiplicative reasoning test results in the 1970s with a group of students' test results in the 2000s and they found that students' multiplicative reasoning skills had not improved over the years.

Identifying the multiplicative structures in the following missing value problem can be challenging for some students.

> *The photograph is enlarged to make a poster. The photograph is 10 cm wide and 16 cm high. The poster is 25 cm wide. How high is the poster? (Swan, 2005).*

To find the height of the poster, two directions of multiplicative relation can be identified, scalar versus functional. Scalar relation is the multiplicative relation between widths and functional relation is the multiplicative relation between width and height. Drawing on evidence from earlier research, it is expected that identifying the scalar relation can be easier than identifying the functional relation for some students. In this example, the ratio of the width of the poster to the width of the photograph is 2.5. As a result, the height of the poster can be calculated by multiplying the height of the photograph by 2.5, which gives 40. Identifying the functional relationship between the lengths of height and width is expected to be more challenging for some students, not only because of the direction of the multiplicative relationship but also because the quantity of scale factor 1.6 can be more difficult to identify with intuitive additive thinking, compared to 2.5.

Shaping interactions for further noticing

The second strategy of formative assessment proposed by Wiliam and Thompson (2007) highlights the role of classroom tasks and discussions in eliciting students' learning. In his manuscript providing guidance to teachers for formative assessment practices, Wiliam (2018) elaborated on the strategy of eliciting students' learning by presenting example classroom practices that can help teachers of any discipline grasp the core ideas behind this strategy. Through these techniques, he highlighted teachers' responsibility for enabling students to think and gathering evidence about students' learning which will help teachers make decisions about the next steps. However, the example techniques provided seem generic and might require further consideration within specific subjects, and even topics at times. For example, among these techniques, Wiliam advises teachers to listen to students to learn about their thinking rather than focusing on the correctness of the responses. Although this advice can fit any subject, the specifics of the listening practice such as the content to concentrate on and the content-specific probes to provide students with, will vary.

This strategy aligns with the shaping element of noticing, as proposed by van Es and Sherin (2021). Specifically, Wiliam and Thompson (2007) highlight the technique of "engineering tasks and questions" to facilitate student learning. Van Es and Sherin advocate for shaping interactions to enhance opportunities to notice. These approaches are complementary, suggesting that thoughtfully designed or chosen tasks and interactive engagements can synergise to deepen teacher engagement with students' learning processes. The use of task analysis literature in this thesis will be discussed further in Section 3.5.1.

2.2.3. An examination of the concept of feedback as a core element of formative assessment

Feedback plays a significant role in distinguishing formative assessment from other classroom assessment methods, including continuous and diagnostic assessments (Wiliam & Thompson, 2007). This distinction goes beyond the mere presence of feedback; it lies in the specific purpose it serves in moving students' learning forward (ibid.). Nevertheless, putting this purpose into practice can be a challenging endeavour. Indeed, the existing literature provides evidence that feedback can have both positive and negative consequences with respect to students' learning (Kluger & DeNisi, 1996; Smit et al., 2023). In this section, feedback will be examined as a core element of formative assessment through discussing the conceptualisations of feedback alongside evidence of the factors that can influence its effectiveness. In order to gain insight into the factors that can influence the effectiveness of feedback, as well as examining the findings of four review studies (Hattie & Timperley, 2007; Kluger & DeNisi, 1996; Shute, 2008; Van der Kleij et al., 2015), recent empirical evidence that involves explorative case studies and interventions was examined. It should be noted that three of these reviews have formed the basis for more recent empirical studies (Hattie & Timperley, 2007; Kluger & DeNisi, 1996; Shute, 2008). One of these reviews is a recent meta-analysis which reviewed experimental studies on the effect of feedback types on expected student outcomes in computer-based environments (Van der Kleij et al., 2015). The result of this examination reveals the complicated nature of providing and receiving feedback, which supports the need for a thorough consideration of creating feedback situations in the classroom.

Conceptualisations of feedback

Understanding of feedback has changed over the years from being solely a response to students' answers to an integrated part of the teaching and learning process. Originally feedback referred to the information that was provided to learners or systems with the intention of moving them forward by addressing the gap between intended and actual learning (Ramaprasad, 1983; Sadler, 1989). More recently, there has been increased emphasis on the pedagogy of feedback provision and reception. Feedback has recently been acknowledged as a process that requires students to actively engage with the information provided in order to enhance their work (Hattie & Timperley, 2007; Henderson et al., 2019; Kluger & DeNisi, 1996). This process can be present in diverse forms depending on specific circumstances and via the use of a variety of sources.

Feedback can occur in different forms with respect to three dimensions. The first dimension is the target of feedback. This dimension was named the "level" of feedback by Hattie and Timperley (2007). They identified four levels: whether the task was accomplished (task-level); how to accomplish the task or how to improve the intended product (process-level); highlighting students' skills as learners (self-regulation level) and comments on students' personalities (self-level). The second dimension is the level of detail in the information provided through the feedback. This dimension involves providing information only on whether a student's response is correct; providing information as to the expected correct answers; and providing information beyond correctness, such as hints, explanations, and additional instruction (Shute, 2008). The third dimension of feedback is timing. The literature has widely categorised feedback types in relation to timing as immediate and delayed feedback. While immediate feedback is associated with the one provided immediately following the task completion, delayed feedback is associated with that provided within a certain time after task completion.

Beyond categorising types of feedback within these three dimensions, the inevitable convergence of feedback types within each dimension creates even more types. For instance, feedback can take on an immediate-corrective form, where it is delivered promptly and addresses students' mistakes, or can be immediate-elaborated, combining quick delivery with rich contextual information. These intersections within feedback categories highlight the complexity of feedback provision and its potential to take on multifaceted roles in the learning process.

In addition to the types of feedback, identifying the sources of feedback and students' use of feedback that comes from these sources is part of the conceptualisation of feedback. Students can receive feedback from diverse resources such as teachers, peers, self, tasks, and books (Hattie & Timperley, 2007; Quinlan & Pitt, 2021; Wiliam & Thompson, 2007). According to Wiliam and Thompson (2007), in line with the essence of formative assessment they propose, students should be actively using the feedback that comes from these resources, and it should either explicitly or implicitly guide their next steps.

Quinlan and Pitt (2021) proposed four categories of feedback sources in the professional development context: self, disciplinary colleagues, service users, and objects. They suggest that these sources can provide two types of feedback: evaluative and consequential. While self and disciplinary colleagues can provide evaluative feedback by directly providing information related to performance, service users and objects can result in consequential feedback. More explicitly, in the latter category, the learner receives feedback whilst they are engaged with the work rather than receiving external feedback. In the mathematics classroom context, this can relate to mathematical tasks which can inherently provide feedback.

Wiliam and Thompson (2007) explicitly mention that in order to achieve its goal feedback should involve guidance for future action, either by explicitly indicating the specific actions to undertake or implicitly creating a classroom culture that can enable students to decide how to use feedback (ibid.). The key point here is to enable students' engagement with their learning and their taking responsibility for future steps. Wiliam and Thompson's point is that teachers should "engineer" the environment for students and that learners should form their own learning. In his practical book, Wiliam (2018) highlights a misunderstanding relating to feedback, that any information about students' performance can be feedback, which contradicts the key idea of feedback that it should move students' learning forward. He highlights that feedback should encourage students to think cognitively through productive challenges rather than emotional reactions.

In traditional secondary classrooms, teachers are the most powerful source of evaluative feedback. The reconceptualisation of feedback offered earlier in this section aims to share this power with students and their peers. As a result, in this framework teachers, self, and peers are considered to be the main sources of feedback. The ultimate goal of feedback should be to give the learner the opportunity to reflect on their learning in order to improve it.

Conditions for the effectiveness of feedback

Types of feedback

Literature suggests that different sorts of feedback can have varying effects on students' learning under different circumstances. Intriguingly, the research evidence points to the unfavourable potential consequences of teachers' feedback on students' learning (Kluger & DeNisi, 1996). More specifically, among the studies Kluger and DeNisi explored in their comprehensive review, more than one-third of the feedback interventions harmed students' learning, mainly due to the attention to student self rather than task-related feedback. Three more recent reviews provided further insights into the conditions that result in the positive or negative effects of feedback (Hattie & Timperley, 2007; Shute, 2008; Van der Kleij et al., 2015). Similarly to Kluger and DeNisi's findings, Hattie and Timperley's (2007) review has suggested the potential harmful effects of feedback focused on the self, that can influence students' ego in either a positive or a negative way.

Literature offers consistent results in terms of corrective and elaborated feedback, arguing in favour of elaborated feedback. Van der Kleij et al.'s (2015) more recent meta-analysis also supports these results, suggesting that elaborated feedback is more effective than corrected feedback for both lower- and higher-order learning.

According to Shute (2008), while immediate feedback can be useful for enabling students' engagement with difficult tasks in the early stages and consolidating procedural and conceptual knowledge, delaying feedback can prevent student distraction when working on easy tasks and can be beneficial for long-term high-level skills.

The more recent meta-analysis conducted by Van der Kleij et al. (2015) provided evidence that while immediate feedback was more effective with respect to lower-order learning delayed feedback is more effective for higher-order learning. However, as Shute mentioned it is difficult to find consistent results in terms of the effects of immediate and delayed feedback. One reason for this inconsistency is that these terms have been considered differently in different research studies. On the whole, while immediate feedback was described as given straight after the student completed a task, delayed feedback was given after a time.

Learners' prior knowledge

Following Shute's (2008) and Hattie and Timperley's (2007) seminal works, researchers have interrogated the effects of learners' prior knowledge on the effectiveness of different feedback types, and the impact of feedback on learners' retention levels. In what follows, I will discuss some of these studies.

The literature offers intervention studies that tested the effects of varying feedback conditions in primary and further education contexts. In the primary context, a group of researchers in the US ran a series of experiments (Fyfe et al., 2012; Fyfe & Rittle-Johnson, 2016, 2017). These studies were conducted with a similar experimental design: a large group of elementary students aged 8 and 9 as a sample using a problem-solving intervention that includes 12 to 14 mathematical equivalence problems. Also, these studies all investigated the effect of students' knowledge prior to intervention, as well as that of different conditions of feedback such as timing and feedback type on learning experiences. In what follows, I will present the details of these studies in relation to varying feedback conditions interrogated therein.

In the first study, Fyfe et al. (2012) conducted experiments to test the effects of three different feedback conditions during an exploratory problem-solving practice prior to instruction. The first condition was providing feedback related to the correctness of students' strategies; the second was providing feedback related to the correctness of students' numerical answers; the third was not giving any feedback. The overall results showed that while providing any type of feedback was more beneficial for students with little prior knowledge, exploring without feedback was more useful for students with moderate prior knowledge. However, this study did not show a significant effect of feedback type on students' performance.

In the second study, Fyfe and Rittle-Johnson (2016) looked at the impact of the timing of feedback. They designated the feedback conditions as no feedback, immediate feedback and summative feedback. In the no-feedback condition, when students answered a problem correctly, they moved to the next question, and when they answered incorrectly, they were given repeated instructions before moving to the next problem. In the immediate feedback condition, students were informed whether their answer was correct, with the correct answer given if they made a mistake. In the summative feedback condition, students were given the same information once they had solved all the problems.

In the previous study (Fyfe et al., 2012), the moderator variable was students' prior knowledge. However, in this study (Fyfe & Rittle-Johnson, 2016), the moderator variable was determined by a brief instruction provided before the problem-solving intervention. Namely, a group of students were provided with strategies for solving four problems, before the problem-solving intervention. This study validated the findings of previous research regarding the moderating role of prior knowledge. It showed that students with prior instruction achieved better learning outcomes in both procedural and conceptual knowledge when they were not given immediate feedback.

In the third study, Fyfe and Rittle-Johnson (2017) investigated the effects of types of feedback on students' retention of knowledge. In that design, all students received instruction prior to the problem-solving intervention. The students were allocated to one of three feedback conditions in relation to timing: no feedback, immediate feedback, or summative feedback. Unlike the previous two studies, students were given a retention test one week after the post-test. According to the results, students who did not receive feedback during a problem-solving intervention showed higher retention levels than those who received either immediate or summative feedback.

A study with similar aims to those outlined above, but carried out in a different context, with learners who take high-stakes tests after completing secondary school (similar to the further education context in England), was conducted by Attali and Van Der Kleij (2017). This study investigated the effects of feedback types (providing correct responses versus providing elaboration such as hints, explanations, or worked examples) and feedback timing (immediately after each item versus delayed until after the whole test was complete) on test takers' performance in a web-based test. In this experimental design, the online tests included isomorphic items that require the same mathematical skills to solve but are different on the surface. With the help of these items, researchers estimated the possibility of participants' answering the second item correctly in relation to the variables of feedback type and feedback timing. The findings suggest that the effectiveness of feedback varied depending on its type and timing, as well as the accuracy of the initial response. Specifically, individuals who initially answered incorrectly showed significant improvement when they received detailed feedback, whereas this effect was not observed in those who initially answered correctly. Regarding the timing of feedback, providing it immediately was beneficial for enhancing the performance of those who had initially answered incorrectly, but it did not have the same positive impact on test takers who had initially provided a correct response.

The studies presented above focused on the feedback provided in a test taken after learning occurred. My research has a diverse focus on examining potentially effective recommendations for teachers' feedback practices in teacher guides. The findings from the existing experimental studies presented above were used to interpret identified feedback recommendations within the teacher guides. Importantly, these studies demonstrated that students' prior knowledge can be a moderator for the effectiveness of the type and timing of feedback. That is, students who have limited prior knowledge can benefit from immediate and elaborated feedback more than those with stronger prior knowledge.

Peer feedback

Beyond the four feedback types presented above, another type of feedback in the classroom can be peer feedback. Literature offers some benefits of peer feedback on students' learning. In an experimental research design, Roschelle et al. (2019) compared the effects of both individual and group feedback on students' learning. They found that students in group feedback environments showed higher learning outcomes but with a small effect size.

In another study, Kapur and Bielaczyc (2012) compared two conditions. In the first condition, students worked on complex mathematics problems collaboratively without instructional support from a teacher, whereas in the second a teacher provided instructional support, scaffolding, and feedback. Although the former group failed in their problem-solving efforts, they performed better at solving complex problems in the post-test. However, the size effect was small.

Although the effect sizes in both studies are small, it is worth considering peer feedback as a different way of delivering feedback in the classroom.

Students' use of feedback

Wiliam (2018) proposed four types of student response to feedback in two situations, when the feedback suggests that a student is outperforming and where they are seemingly underperforming: changing behaviour by either showing less or more effort; changing the goal by either increasing or reducing the aspiration; abandoning the goal by deciding that it is too easy or too difficult; or ignoring feedback in each situation. He highlights that the possibility of students taking feedback negatively is higher than the possibility of them taking it positively.

Take-aways from the literature on different types of feedback

Drawing on Wiliam and Thompson's (2007) third strategy, providing feedback that moves learners forward, in this thesis, feedback is conceptualised as part of teachers' in-the-moment decisions which are integrated into learning process. This conceptualisation requires teachers to engineer feedback situations in order to encourage student reflection on their learning, aligned with specific goals. These situations may include peer interaction as well as employing diverse strategies tailored to enable students to reflect on their learning aligned with disciplinary or pedagogical goals (Quinlan & Pitt, 2021).

It is crucial to acknowledge the potential unintended consequences that may arise from the delivery of feedback, while also remaining mindful of the diverse factors that can influence its impact on students' learning (Hattie & Timperley, 2007; Shute, 2008). Providing a definitive guide for teachers on effectively delivering feedback in the classroom can be a challenge. Consequently, it becomes worthwhile to explore how various forms of feedback can influence the development of distinct strands of mathematical proficiency. Furthermore, it is worthwhile considering the potential adverse effects of feedback on students' self-perception and recognising the limitations associated with immediate-corrective feedback. Given the potential inclination of teachers towards these two types of feedback (Antoniou & James, 2014), exploring alternative approaches becomes a worthwhile endeavour.

2.3. Teachers' learning through curriculum resources

2.3.1. Mathematics teachers' professional knowledge: From possessing knowledge to enactment

Shulman (1986) is the pioneer of studies related to the knowledge teachers need when teaching. He is interested in how teachers who are experts in their subject can use their knowledge appropriately to teach students who are novices with respect to the subject. He highlighted that pedagogical content knowledge and curriculum knowledge are required beyond basic content knowledge.

These elements of teacher knowledge were enhanced further specifically relating to mathematics teachers' knowledge. Rowland et al. (2005) proposed a framework for conceptualising teacher knowledge that encompasses the knowledge applied in teaching, extending beyond the passive knowledge that teachers possess. This framework, known as the knowledge quartet, has four components: foundation, transformation, connection, and contingency. While the first component of this framework, foundation, refers to a kind of passive knowledge teachers have, the other three are relevant to lesson preparation and enacting. Foundation refers to teachers' theoretical equipment and their beliefs with respect to maths and teaching maths; transformation refers to teachers' capacity to put their theoretical background into practice; connection refers to making connections between different concepts and awareness of the cognitive demands of these concepts; contingency refers to teachers' readiness to alter their plans according to student responses. It can be said that foundation involves content knowledge, pedagogical content knowledge, and curriculum knowledge, as suggested by Shulman. This framework highlights the importance of the further knowledge that is required to put the passive knowledge into action.

More recently, Schoenfeld (2020) proposed a framework with respect to teacher knowledge which highlights that teachers require skills beyond passive knowledge. That is to say, instead of teacher knowledge, he used the term teacher proficiency, which can involve skills beyond knowledge. According to Schoenfeld's framework teacher proficiency must involve five dimensions: 1) supporting rich mathematics, 2) supporting productive struggle, 3) equitable access to rich content, 4) agency, ownership, and identity, and 5) formative assessment.

It can be said that the most striking components of this framework are the third and fourth. The third component draws attention to how to enable equity in terms of students' access to content. That is to say, it is not merely about letting all students speak in the classroom, but also providing an opportunity to think. In the example Schoenfeld gives, a student did not give the correct answer to the question their teacher asked. The teacher gave the student time to think and returned to them after a while. When the teacher asked the question again the student was able to give the correct answer. Unlike in common practice, this teacher did not move on from the student when she did not answer, instead helping her to move one step further in her thinking. Although this is a useful example to illustrate how to enable equity in terms of access to mathematical thinking, as Schoenfeld already mentioned, the challenge here is to determine the level of cognitive demand. Furthermore, even though the teacher can identify the appropriate cognitive demand level for the student, it is still a challenge to address this. If the teacher focuses solely on receiving the correct answer from the students, the teacher's questions may not align with the learning intentions or expected cognitive demand. If the teacher is not confident with digging into high cognitive demand as required by the students, then the teacher can hesitate to further the discussion.

The fourth component brings a sociocultural insight into traditional understandings of teacher knowledge, which are largely influenced by behaviourist or cognitive approaches. That is to say, this dimension can allow teachers to encourage students to be active participants when learning mathematics.

2.3.2. Relationship between teachers and curriculum materials

This section will start by outlining the theoretical approaches that explicitly define the relationship between teachers and curriculum materials. Initially, three models which are drawn on sociocultural perspectives – the participatory relationship (Remillard, 2005), the documentational approach (Gueudet & Trouche, 2009) and the didactical tetrahedron (Rezat, 2012) – will be introduced. Subsequently, teachers' role as designers of teaching practice will be highlighted. Finally, the educative role of curriculum materials will be elucidated, drawing on existing research on "educative curriculum materials." This research highlights the contribution of materials to teachers' professional development, alongside their role in improving student learning.

Theoretical approaches underpinning the relationship between teachers and curriculum materials

Ball and Cohen's (1996) seminal paper is the grounding for more recent approaches that focus on the relationship between curriculum materials and teacher practices. Ball and Cohen (1996) explicitly draw attention to the potential role of curriculum materials in teachers' learning by suggesting a reconsideration of the design of such materials, pointing out their limited contribution to teachers' learning.

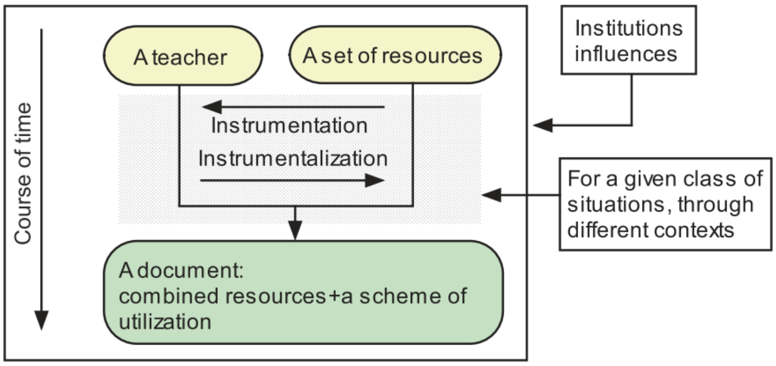
10 years after Ball and Cohen (1996) drew attention to curriculum materials' potential contribution to teachers' professional development, Remillard reviewed empirical studies that provide insights into the relationship between teachers and curriculum materials and proposed a participatory relationship as a result (2005). Remillard categorised these empirical studies in terms of their

theoretical approach, such as positivist, interpretivist, and socio-cultural presenting the varying roles of teachers and curriculum materials within each approach. In the studies classified under the positivist category, teachers are expected to passively follow the curriculum materials as prescribed. In contrast, the interpretivist category views teachers as constructors of their own meanings through the use of these materials. Meanwhile, within the socio-cultural category, teachers are anticipated to autonomously decide on the utilisation of the materials.

The difference between the positivist approach and the other two is readily apparent, while interpretivist and socio-cultural approaches can overlap because they both require teachers to make their own interpretations. The notion of participation, which is a core concept in Remillard’s participatory model, can help to distinguish the socio-cultural approach from other interpretivist approaches. In order to utilise the materials teachers must interpret them; however, interpretation may not be adequate for participation. Participation requires teachers to actively consider the extent to which these materials meet their needs and to decide how best to use, or not use, the materials in their local context.

Gueudet and Trouche (2009) move this participatory relationship forward by elaborating on the two-way relationship between teachers and curriculum materials. Namely, an instrumental approach, which explains the relationship between artefact and human activity, is used to theorise this mediating process. The artefact in the instrumental approach refers to the resources in documentational genesis. Apparently, resource involves other elements alongside curriculum materials, such as student work, software, and discussions among teachers. According to documentational genesis, teachers’ relationships with a set of resources result in the creation of documents, or the operationalised versions of the resources (see Figure 2.1). The relationship between teachers and resources is conceived as a two-way relationship that includes instrumentation and instrumentalisation. While instrumentation refers to the features of the resources that shape the schemes developed by the teacher, instrumentalisation refers to teachers’ modifications of their schemes. As a result of the instrumentation and instrumentalisation processes, teachers produce documents in line with their intentions.

Figure 2.1 Schematic representation of a documentational genesis



Note. Reprinted from Gueudet & Trouche (2009, p. 206)

European scholars have used the notion of "resources", as opposed to their American counterparts who focus on "materials". With this approach, European scholars brought further theoretical insights to the relationship between teachers and curriculum materials by highlighting additional mechanisms that can influence this relationship. That is to say, the broader term "resource" or "resource system" involves various aspects of teaching resources such as knowledge, interaction with colleagues, students' work, and time, alongside curriculum materials, which mostly refers to textbooks and accompanying resources (Adler, 2000; Gueudet & Trouche, 2009).

Ruthven's (2009) framework for the structuring features of the classroom approach identifies five key aspects: the working environment, resource system, activity format, curriculum script, and time economy. Within this framework, resources represent just one of the elements shaping teachers' practices. Given its specific focus, this widely recognised model concerning curriculum resources was not applied in this thesis.

Teachers as designers of their teaching practice

In recent years, particularly due to the transformation of curriculum materials onto digital and online platforms providing accessible tools that can be adopted flexibly, an interest in teachers' design capacity has emerged.

Brown (2009) can be considered the pioneer of this relationship, exploring three science teachers' different uses of curriculum resources and suggesting the term "design" to refer to teachers' usage of curriculum materials. That is to say, teachers design their own practices as a result of their interaction with the materials. Teachers can interact with the materials in three forms: offload, adapt, and improvise. *Offload* refers to teachers' use of the materials wherein they rely largely on the material with limited agency, *adapt* refers to teachers' use of curriculum materials alongside their own personal resources, and *improvise* refers to creating spontaneous strategies with minimal use of existing resources. Although improvising seems to require more competent teachers than offloading and adapting do, and this may be closer to the notion of participation, Brown does not claim that improvising is superior to offloading and adapting. A teacher can offload for different reasons. They might not be very confident in their subject knowledge with respect to that particular topic and may feel safer using the material without changing it (Noh & Webb, 2015). Alternatively, a teacher could find the curriculum material to be useful in her classroom context and prefer to offload the material. When it comes to improvising, a teacher can have different intentions again. For example, the teacher can improvise the material in line with the purpose of the designer or they can improvise due to not appreciating the designer's message, thereby teaching with different learning intentions.

Brown (2009) suggested the term "pedagogical design capacity" (PDC) to refer to teachers' potential to make these decisions appropriately. More specifically, a high PDC requires understanding and using curriculum resources for instructional purposes by identifying the useful aspects of these resources and making instructional decisions by using these aspects and enacting decisions. By defining PDC, Brown contributes to the discussion on how to enhance the quality of participation between teachers and curriculum materials. More importantly, this suggests that teachers possessing knowledge and skills and these teachers' levels of commitment may not be indicators of high PDC.

Beyond acknowledging that teachers are designers of their practices, more recent research was conducted that encouraged teachers to be involved in this design process in collaboration with researchers (e.g., Pepin et al., 2019). Pepin et al. proposed three levels of teacher design: micro, meso, and macro. At the micro level teachers design their own practices; at the meso level they design collaboratively with their colleagues, and at the macro level they are involved in the curriculum design process.

The “educative” role of curriculum materials

In this thesis, the term "educative" is specifically chosen to highlight the role of curriculum materials in enhancing teachers’ classroom practices. It signifies the potential of curriculum materials to foster transferable learning experiences for teachers. This terminology has already been used by researchers like Remillard (2005) and Davis and Krajcik (2005), who describe curriculum materials designed to support both teachers’ and students’ learning processes as “educative curriculum materials”.

Recent studies on educative curriculum materials often reference Ball and Cohen’s influential 1996 paper, which outlines five key expectations of curriculum materials with respect to the enhancement of teachers’ learning: (1) anticipate student thinking, (2) develop teachers’ content knowledge, (3) connect content over time, (4) make pedagogical rationales explicit, and (5) support decision-making. Following this foundational work, further research – such as Davis and Krajcik (2005), Quebec-Fuentes and Ma (2018), and Remillard (2012) – expanded upon these ideas, focusing on integrating specific types of teacher knowledge into curriculum materials and exploring the features of curriculum materials which enables the transfer of this knowledge to teachers. This body of literature collectively emphasises the importance of providing rationales, knowledge, tools, and support for teacher adaptation of materials. These elements of curriculum materials will be discussed in the following subsections.

Providing rationale

Researchers have recently demonstrated consensus calling for the inclusion of the rationale behind the design of curriculum materials within these materials (Davis et al., 2017; Machalow et al., 2020; Quebec-Fuentes & Ma, 2018). This rationale encompasses the aims of the suggested practices and the mathematical goals.

Several researchers proposed the transparency of curriculum materials as an essential element of educative curriculum materials (Reinke et al., 2020; Stein & Kim, 2009). This transparency involves providing rationales to teachers. Remillard et al. (2019) found that teachers’ practices aligned closely with the presented mathematical goals. Specifically, teachers followed practices that were deeply elaborated on within the materials. Moreover, Davis et al. (2014) and Arias et al. (2016) found a positive relationship between providing rationale and the possibility of teachers enacting these goals.

As part of a large-scale ICUBIT project led by Janine Remillard in the US, Reinke et al. (2020) conducted a detailed analysis of five curriculum materials in the US with respect to their explicit inclusion of the mathematical goals and the rationale behind the tasks and practices therein. They categorised the type of rationale provided into five categories: identifying the mathematical objectives or goals for a lesson; providing the rationale for those objectives; explaining the purpose of an instructional activity or a particular aspect of that activity (representations or tools used); explaining the purpose of a recommended teacher or student action; and providing the rationale for a universal design decision. Moreover, a doctoral thesis at Michigan State University (Males, 2011), which analysed the educative potential of middle school mathematics curriculum materials, showed that curriculum materials involved rationale for several aspects of teaching mathematics. The author of this thesis provided a long list of the ways of providing rationale, including the rationale for experiences students engage with; rationale for using mathematical representations; rationale for the tools used; rationale for questions asked; rationale for justification, reasoning and proof; and that for developing mathematical terminology.

The design principles for educative curriculum materials in elementary science, developed by Davis et al. (2017), emphasise providing teachers with rationale through various forms of support grounded in their practice. This approach contrasts with Reinke et al.'s and Male's, focusing instead on integrating the rationale for several aspects of the curriculum into the materials to foster teacher engagement directly. Davis et al. advocate for a design that positions understanding the rationale behind recommendations as central, aiming to enhance teachers' interactions with the curriculum materials by making the reasons for their recommendations understandable to teachers.

Providing knowledge and tools

Alongside the explicit focus on providing rationale to teachers, the research on "educative curriculum materials" widely draws on the knowledge teachers should possess for effective teaching. Davis and Krajcik (2005) pioneered the design heuristics aimed at supporting teachers' learning through curriculum materials, focusing on essential knowledge categories and the supportive educative content required. They proposed that such materials should cover subject matter knowledge, and pedagogical content knowledge, both topic-specific and discipline-specific. Subsequent research expanded on these principles, incorporating educative features into materials to facilitate teacher learning. In the context of mathematics, Quebec-Fuentes and Ma (2018) similarly outlined the knowledge required for mathematics teachers, integrating educative features into curriculum materials to assist in learning this knowledge, marking a continuity in focusing on teacher knowledge and its delivery through educative features.

Acknowledging that teachers adapt materials

In the previous section, theoretical approaches to the relationship between teachers and curriculum materials suggest that teachers may not use materials as they are. This is suggested by the participatory approach (Remillard, 2005), documentational approach (Gueudet & Trouche, 2009), and Brown's (2009) metaphor of "design".

In the earlier section, it was noted that one foundation of the design principles proposed by Davis and her team (2017) focuses on enhancing teacher engagement with materials by understanding the rationale behind recommendations. Additionally, they identified a second foundation: acknowledging teachers' tendency to adapt materials and ensuring the curriculum supports productive adaptation. Consequently, they advocate for designing curriculum materials with an educative potential that anticipates teachers' adaptations, providing support that benefits students' learning.

2.4. Summary of the chapter

This chapter served as a theoretical basis for RQ1 and RQ2 by providing an explanation of the foundational concepts and theoretical frameworks to be used within this research study. Initially, attention was drawn to the various terms associated with formative assessment (e.g., AfL) and conceptualisations of these terms, highlighting distinct features such as its function or the role of students. In order to introduce the scope of this study, formative assessment was conceptualised within this study in terms of its nature, actors, and function. In terms of its nature, it is acknowledged as a practice integrated into teaching and learning; the actors are the teachers, students, and peers; and its function is empowering students as agents of their learning and guiding teachers in making decisions.

Wiliam and Thompson's (2007) five strategies were chosen as the guiding framework for this research, as they provide elements of formative assessment that can guide teachers' practice. However, these strategies need to be reconsidered in the context of teachers' real-time practices with respect to mathematics. This need leads to the first research question, which focuses on the operationalisation of these five strategies in the secondary mathematics context.

This operationalisation was initiated in this chapter by linking the five strategies to important aspects of the mathematics education literature, such as the five strands of mathematical proficiency (Kilpatrick et al., 2001) and the noticing framework (van Es & Sherin, 2021). This was furthered through analysis of teacher guides, as will be presented in Chapter 5.

Following the conceptualisation of formative assessment and the operationalisation of five formative assessment strategies, the focus of the chapter turned to theoretical approaches to the relationship between teachers and curriculum materials. Importantly, among socio-cultural approaches to this relationship, Gueudet and Trouche's (2009) documentation system for mathematics was chosen as an umbrella theorisation, for its feature of distinguishing the roles of teachers and materials within their relationship. Although earlier research regarding educative features of curriculum materials provides insights into the features within the curriculum materials that highlight the expected teacher learning and teacher tendencies and needs, a gap for educative features that highlights how curriculum materials can facilitate teachers' learning was identified. This gap justifies RQ2.

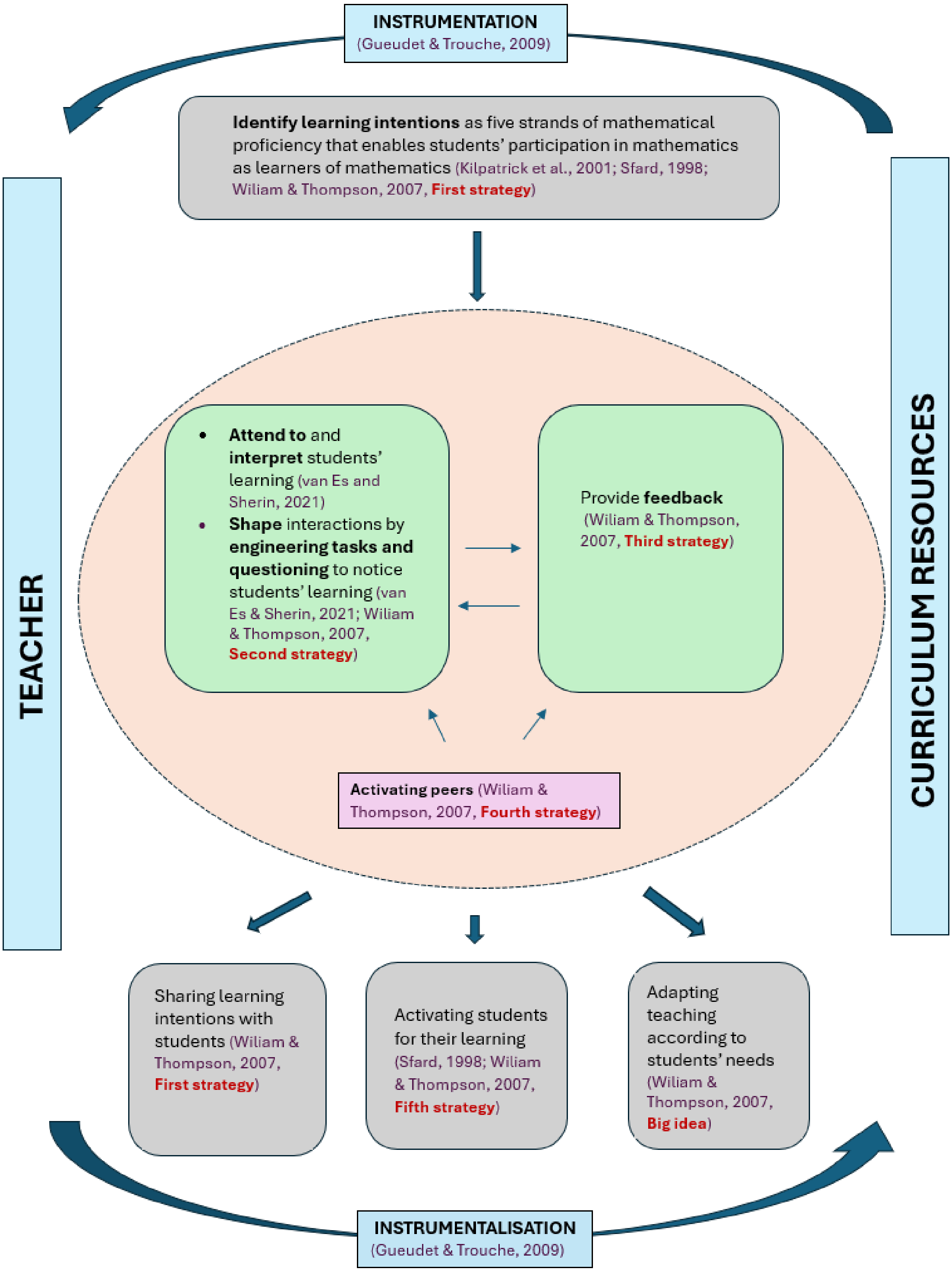
As mentioned in Section 1.4 and elaborated on in this chapter, several theoretical framings were considered for this study. Figure 2.2. presents the chosen theoretical framings and links between them. Namely, this figure includes five formative assessment strategies and one big idea as the skeleton for framing formative assessment in the classroom. These elements are presented in red. The grey area at the top of the figure introduces “identifying learning intentions” as a prerequisite for teachers’ effective formative assessment practices. In order to present the learning intentions for this study Sfard’s metaphor of participation was used (1998). That is to say, in this study learning was considered as students’ participation as mathematics learners. In order to consider this participation in the mathematics context, Kilpatrick et al.’s (2001) five strands of mathematical proficiency were chosen.

The circular area presents teachers’ key formative assessment practices in the classroom. Namely, the second formative assessment strategy, engineering tasks and questioning to elicit students’ learning, and the third formative assessment strategy, providing feedback that moves students’ learning forward are considered as the key two formative assessment practices. Here, it is important to reiterate that the second strategy of formative assessment was networked with the three elements of noticing as introduced by van Es and Sherin (2021). The fourth strategy, activating students for their peers’ learning is considered as a supporting strategy for effective noticing and feedback practices.

Three grey areas at the bottom of the figure were considered as the expected outcomes of effective formative assessment practices. While in the original framework only the big idea, adapting teaching according to students’ needs is the expected outcome of formative assessment, in this contextualised version sharing learning intentions with students and activating students for their own learning are also considered as expected outcomes.

The curriculum resources were considered as the artefacts that can facilitate teachers’ effective formative assessment practices. In that sense, the terms instrumentation and instrumentalisation were chosen to present the relationships between teachers and curriculum resources (Gueudet & Trouche, 2009).

Figure 2.2 Theoretical foundation for the study



Note. Created by the author

CHAPTER 3 - REVIEW OF THE EMPIRICAL EVIDENCE

Having presented the theoretical and conceptual underpinnings of this study in Chapter 2, this chapter turns to the review of empirical studies related to the research constructs. By exploring the current state of research, I aim to present insights that justify my three research questions and the research methods employed to address them.

This chapter starts with presenting the methods used to review the literature to identify the empirical studies to be considered in this chapter (Section 3.1). It then presents a comprehensive review and synthesis of existing empirical studies that shed light on the mediating role of curriculum materials in mathematics teachers' classroom formative assessment practices (Sections 3.2 to 3.4). Existing methodological approaches that can be adopted for the analysis of curriculum materials with respect to their educative role in teachers' formative assessment practices are also examined (Section 3.5). This chapter finally presents how this body of literature led to the research questions and research methods (Section 3.6).

3.1. Methods for the review

3.1.1. Strategies for a comprehensive literature search

The studies in this literature review were accessed and identified through four key strategies throughout this doctoral study. The relevant studies identified were collected using the reference software Endnote, as well as through recording the key aspects of the studies in various forms (e.g., overview summaries and synthesis tables for certain search questions). The first strategy to access studies was to conduct a Boolean search by using the ERIC EBSCHO database with respect to certain aspects of the research. Appendix A presents example questions and keywords used for the Boolean searches. Boolean search was mostly considered a starting point to scope the studies comprehensively for specific questions and was complemented by other strategies. The second strategy was visiting the recent issues (2018 onwards) of certain leading academic journals in the fields of education (e.g., Review of Education), mathematics education (e.g., ZDM) and educational assessment (e.g., Assessment in Education: Principles, Policy & Practice). This strategy updated my knowledge of the topic. The third strategy was the snowball approach, which involves checking the reference lists of recent papers, and was especially helpful for accessing influential studies. Finally, I accessed some papers on digital online platforms such as ResearchGate and Twitter, by connecting with leading researchers in the education field. Although this approach significantly contributed to my being updated on the field, I remained mindful of the potential for bias, as it primarily provided access to researchers who are active users of these platforms.

The studies generated from these four procedures served as a foundation for accessing relevant studies. Importantly, these strategies helped me gain insights into mathematics education and educational assessment literature more broadly, as well as mathematics curriculum materials specifically, and teachers' professional development and formative assessment literature. This also positioned my research within the academic landscape. However, the studies in this review were chosen purposefully among those accessed studies. The section that follows introduces the inclusion and exclusion criteria for my focused review.

3.1.2. Inclusion and exclusion criteria

In this review of empirical studies, the key approach is to review the literature *for research*, instead of conducting a review *of research* (Maxwell, 2006). That is, the inclusion and exclusion criteria were employed in order to thoroughly meet two key purposes of this review: to justify the research questions and methodological decisions. First, in justifying the research questions, prior research can help understand the role of curriculum materials in teachers' formative assessment practices, identify areas needing further investigation, and position this PhD research within the context of the existing literature.

The focus of the research on the educative features of curriculum materials that can enhance teachers' formative assessment practices initially led me to review studies that explicitly examine empirical evidence surrounding the relationship between curriculum materials and teachers' classroom formative assessment practices, and the integration of formative assessment into curriculum materials.

In order to critically assess the relevance of the studies, I examined whether the conceptualisations of the terms “formative assessment” and “educative features” aligned with those used in this thesis (see Chapter 2). This examination was especially crucial with respect to formative assessment, on which a large amount of research, utilising various conceptualisations, has been conducted. In several studies, formative assessment has been conceptualised as a continuous assessment tool conducted upon the completion of a topic or unit. This approach offers feedback on student learning at the end of each topic or unit taught (e.g., Kobiela & Lehrer, 2019). In this review, I excluded such studies but included those focusing on at least one of the three elements of formative assessment identified in Section 2.1. To reiterate, these elements are identifying learning intentions, noticing students’ thinking, and providing feedback that moves students’ learning forward.

Similarly, a considerable amount of literature has been published on the role of curriculum materials in mathematics education. This was considered at the macro level, with respect to using these resources as tools to put policy into practice (Valverde et al., 2002), as well as at the micro level, considering the role of the materials in classroom practices (e.g., Drake & Remillard, 2019). This PhD research particularly examines the latter role in teachers’ formative assessment practices from a socio-cultural perspective. First, due to the limited availability of studies that directly focus on the relationship between curriculum materials and teachers’ classroom formative assessment practices, in order to understand this relationship more fully, I expanded the scope of this review to broader classroom practices beyond formative assessment. Gueudet and Trouche’s (2009) approach to a two-way relationship guided the scope of this search, wherein. I included studies that work in both directions: from teacher to curriculum material and from curriculum material to teacher.

The second purpose of this review is to justify the methodological decisions taken in this research. Previously utilised methodological approaches in analysing mathematics curriculum materials were examined. As the key emphasis of my research is on the analysis of teacher guides, I primarily sought studies with the same focus. However, I have also included textbook analysis research as this body of research has a longstanding presence with an extensive body of studies (e.g., Fan et al., 2013). While a significant portion of these studies has focused primarily on task analysis, this extensive body of literature can provide insights into specific aspects of formative assessment, i.e. tasks can act as tools for noticing student thinking. Furthermore, in addition to studies primarily focused on curriculum material analysis, several studies which incorporated this type of analysis as a supplementary component of their research were included.

3.2. The relationship between curriculum materials and teachers' classroom practices

Several research studies have focused on how to design classroom tasks, textbooks and accompanying teacher guides with the assumption of these resources' mediating role in teachers' practices from a sociocultural approach (e.g., Prediger et al., 2021). While there is evidence that various features of curriculum materials can have a mediating role in teachers' practices (Collopy, 2003; Fan & Kaeley, 2000), the intended change is expected to be challenging due to several mechanisms that may influence teachers' practices (Potari et al., 2019). In the following four sub-sections, this process will be examined using evidence from empirical research studies.

3.2.1. The mediating role of curriculum materials on teachers' knowledge and practices

The existing body of literature provides evidence from several studies that demonstrate the potential of curriculum materials to guide and enhance teachers' classroom practices. In their investigations, these studies encompass both teachers' self-reports regarding the support of curriculum materials (Fan et al., 2021) and researchers' evaluations of teachers' shifts when they use curriculum materials, gained through observations and task-based interviews (Charalambous et al., 2012; Noh & Webb, 2015). These studies highlight the potential support provided by curriculum materials to teachers' mathematical knowledge, including a comprehensive understanding of mathematical concepts, their interconnections, procedural fluency, mathematical reasoning, mathematical representations, and cross-disciplinary connections (e.g., Fan et al., 2021; Noh & Webb, 2015). Additionally, the potential support of curriculum materials for teachers' knowledge of students' mathematical thinking was considered (Fan et al., 2021). Alongside teachers' knowledge, there is evidence for support for teachers' direct actions such as lesson planning and enactment (Choppin et al., 2021; Fan et al., 2021; Hill & Charalambous, 2012; Van Steenbrugge & Ryve, 2018). I will now examine these studies in more detail.

Numerous studies have demonstrated the potential of curriculum materials to enhance teachers' mathematical knowledge. In the context of Chinese secondary mathematics teachers, Fan et al. (2021) conducted a comprehensive survey coupled with follow-up interviews with a representative sample from Shanghai. The findings of this research showed that teachers acknowledge the mediating role of textbooks and accompanying teacher guides in supporting their content knowledge. In particular, teachers reported that these materials facilitated their understanding of mathematical concepts, the interconnections of these concepts, mathematical procedures, mathematical reasoning, mathematical representations and broader cross-disciplinary connections.

Similarly, task-based interviews conducted by Noh and Webb (2015) provided further insights into the potential support of curriculum materials for teachers' mathematical knowledge. Their analysis showed that teachers with extensive experience in using educative curriculum materials demonstrated enhanced confidence in tackling complex problems. Moreover, these teachers recognised shared and contrasting characteristics of diverse mathematical representations across varying contexts.

Beyond enhancing teachers' mathematical knowledge, research findings also offer insight into the potential of curriculum materials to deepen teachers' understanding of students' mathematical thinking. In Fan et al.'s study (2021), the survey results on a 4-point Likert scale showed a mean response of 3.18, indicating that teachers perceived teacher guides as beneficial in understanding students' characteristics. However, a nuanced perspective emerged during the follow-up interviews, pointing out some limitations of teacher guides in this regard. One teacher emphasised that while textbooks provide information of expected students' thinking and prior knowledge, the assumptions underlying such information may not necessarily align with the diverse realities of students within the classroom context.

This section now turns to studies that provide evidence for the mediating role of curriculum materials on teachers' practices. In 2012, in The Journal of Curriculum Studies, four connected case studies focusing on the interrelation between the features of curriculum materials, teachers' knowledge, and their classroom practice were published (Charalambous et al., 2012; Hill & Charalambous, 2012; Lewis et al., 2012; Sleep & Eskelson, 2012). Each study examined two or three US middle school teachers with varying levels of mathematical knowledge for teaching (MKT). One purpose of these studies was to examine how the curriculum materials mediated teachers' practices in relation to different levels of MKT. Jointly, the findings of these studies concluded that curriculum materials with integrated educative features leveraged the quality of teachers' practices. In particular, these materials mitigated teachers' relatively low MKT. The observed benefits of curriculum materials in these teachers' classroom practices were as follows: decreasing mathematical mistakes; staying focused on the topic; effective use of representations; providing effective explanations; using language and notation accurately; and defining concepts properly.

Furthermore, Fan et al.'s (2021) findings revealed teachers' recognition of the facilitative role of textbooks in classroom enactment. The survey responses showed mean scores ranging from 3.25 to 3.40 for items related to the role of curriculum materials in teachers' enactment. These items specifically focused on the following classroom practices: effective questioning, assessing for student understanding, addressing common misconceptions, integrating curriculum reform, fostering student motivation, and assisting students in overcoming learning difficulties.

Finally, the literature contains studies suggesting that teachers use curriculum materials in their lesson planning. Choppin et al. (2021) investigated the extent to which curriculum materials can influence teachers' planning activities. They found indications that teachers' planning aligned with the curriculum materials they used. Similarly, Van Steenbrugge and Hyve (2018) found that Swedish teachers used teacher guides for planning.

In summary, the studies discussed above collectively suggest the multifaceted potential of curriculum materials in mediating teachers' professional knowledge and classroom practices. Further investigation is required to understand the mechanisms influencing the interplay of specific features of curriculum materials and teachers' practices. Inspired by the sociocultural models that theorise a two-way relationship between teachers and curriculum materials (Gueudet & Trouche, 2009; Remillard, 2005), I synthesised the empirical evidence in the existing literature in line with this relationship. The following three sections present this synthesis.

3.2.2. Teacher-centric mechanisms that can influence their use of curriculum materials

The way teachers interpret curriculum materials plays a crucial role in shaping their classroom practices (Askew, 1996; Chowdhuri, 2021; Thompson & Senk, 2014). For instance, Askew's (1996) research highlights that teachers might interpret curriculum documents in a way that aligns with their existing practices, rather than challenging or enhancing them. To understand this phenomenon better, it is essential to explore the mechanisms contributing to this diversity of approaches. Three primary teacher-centric mechanisms have been identified through empirical studies: teachers' experience, knowledge, and attitudes towards the utility of materials (Charalambous et al., 2012; Chowdhuri, 2021; Haggarty & Pepin, 2002). These mechanisms are central to understanding the complexities of teacher interpretation and its impact on teaching practices. Furthermore, by delving into these mechanisms, it can be possible to work towards preventing misinterpretations of curriculum materials and fostering an environment where teachers can effectively use these materials to improve their classroom practices.

Studies exploring the relationship between teacher experience and their use of curriculum materials have shed light on the differences between how experienced and novice teachers employ these resources, as well as the varying degrees of effectiveness in their usage among teachers with different levels of familiarity with respect to specific aspects of the curriculum materials. First, novice teachers often rely heavily on curriculum materials to shape their practices, while experienced teachers tend to use these materials as a source of ideas (Haggarty & Pepin, 2002; Remillard & Bryans, 2004). Second, teachers who are not familiar with either the content or curriculum materials might need additional support (e.g., professional development) to use the curriculum materials effectively. Moreover, it is expected that teachers with strong MKT are in a position to use curriculum materials more productively compared to their colleagues who have a less comprehensive knowledge base (Charalambous et al., 2012).

Teachers' attitudes towards the utility of curriculum materials might be another mechanism influencing their use. Chowdhuri (2021) explored teachers' tendency to use textbooks, by analysing interviews conducted with 10 primary teachers who use the same mathematics textbook, alongside observations of these teachers' lessons. She used Brown's (2009) notions of *offloading*, *adapting,* and *improvising*, to explore patterns in each participant teacher's use of textbooks. The findings of this research suggest that some teachers consistently use curriculum materials in a specific manner, either by *offloading*, *adapting*, or *improvising*. As Chowdhuri highlighted, this constant use of certain curriculum materials may stem from teachers' attitudes. As an example, three teachers in this study reported that they neglected to use the textbook as they did not consider it useful.

3.2.3. The educative features of mathematics curriculum materials that can mediate teachers' practices

The leading conceptualisations regarding educative curriculum materials were already discussed in Section 2.3.2. Beyond these conceptualisations, mathematics education literature offers further large-scale studies that provide evidence for the educative role of specific features of curriculum materials. Importantly, the findings of these studies suggest a piece of strong evidence for the value of providing the rationales of the suggested pedagogical practices, providing tools that can be directly used during teaching, and guiding teachers to implement these practices.

According to the quantitative results of Fan et al.'s (2021) survey, there was a statistically significant difference between teachers' conceiving of the facilitating role of student textbooks and teacher guides in favour of teacher guides in terms of understanding mathematics but not for formative assessment and understanding students' prior knowledge. Charalambous et al.'s study (2012) supports Fan et al.'s finding and provides further insights into the certain mediating features of curriculum materials. That is to say, according to

Charalambous et al.’s findings, teacher guides that involve detailed scripts that guide teachers in the implementation of the suggested task and worked examples can influence the quality of teachers’ enactment even when the mathematical knowledge for teaching was not very strong. In that paper, while the detailed script was not elaborated on, it was mentioned that these scripts enhanced a teacher’s appropriate use of representations, language and notations, and mathematical explanations. Moreover, the examples provided in the materials enhanced a focused and coherent lesson.

Remillard’s study (2012) suggests that providing the rationale for suggested practices can be helpful. More specifically, teachers’ not fully embracing the rationale behind certain suggested practices could lead to counterproductive modifications of suggested lessons. Within the context of that study, an illustrative example emerged from a participant teacher’s use of the textbook. In this material, a starting warm-up activity was suggested to prepare students for the subsequent main task. The participant teacher was observed to begin the lesson with the main task by omitting the warm-up activity and the students experienced challenges with the following main task. As Remillard emphasised, the reason for that teacher’s omitting an essential aspect of the suggested lesson could be because the rationale of this aspect was not effectively communicated to them.

Similarly, Remillard et al.’s (2019) research findings also provide insights into the benefits of detailed explanations for teachers in the curriculum materials. In this research, they examined the relationship between the features of the teacher guides and teachers’ practices by analysing these materials and lesson observations. The analysis of the teacher guides showed that only half of the learning goals mentioned in the teacher guides were identified and elaborated for teachers. The lesson observations revealed that participant teachers were inclined to pursue the goals identified and elaborated in teacher guides.

In addition to the rationale of the materials and detailed scripts that can guide teachers for how to enact the suggested practices, literature provide evidence for teachers’ tendency to use tools which can be directly used in the lessons such as slides (Van Steenbrugge & Ryve, 2018).

3.2.4. An alternative contemporary approach: Teachers as co-designers of curriculum materials

The view of involving teachers in the design and analysis process of curriculum materials is an emerging view in the field, especially with the wide accessibility and use of digital resources (Pepin et al., 2017). Although this part of the literature is not directly relevant to the analysis of my study, I'll give part to these studies with paying respect to their influence in the field.

Pepin's research (2012) focused on the teachers' learning opportunities from textbooks. This research involved four stages each of which involve collaboration among teachers and the researcher. In the first stage, teachers were provided with the opportunity to engage with mathematics and students' learning literature. In the second stage, teachers developed a framework that can help them analyse tasks. In the third stage, teachers analysed the mathematics tasks. In the fourth stage, teachers planned lessons with their peers, implemented these lessons and observed each other's lessons. As a result of this research, it was offered that engaging with the tasks helped teachers develop their professional development.

Importantly, in this research, Pepin showed that teachers' engagement with critically analysing tasks encouraged them to reflect on their pedagogical practice and enhance it. It should be noted that this engagement was structured by the researcher and this process might have had a considerable impact on teachers' change.

In this project, teachers were involved in developing the task analysis framework. During this stage, these teachers had an opportunity to clarify the purpose of the tasks and whether the task could meet this purpose. They then had an opportunity to consider alternative ways to develop a task and its implications for classroom practices. They considered the assessment pressure too when developing these tasks. Finally, they had an opportunity to work collaboratively when planning the lessons, they observed each other, and they encouraged each other to try new practices. This project enhanced teachers' pedagogical design capacity.

Analysing and enriching tasks enabled them to analyse their own practices and enhance their practices. In Pepin's research, the focus is the opportunity that is given to teachers to reflect on their beliefs and practices through analysing tasks.

3.3. Formative assessment: Teachers' practices and development

As highlighted in Chapter 1, teachers can adopt specific practices regarding formative assessment which may not result in effective outcomes. This chapter further elaborates on this argument, providing empirical evidence from the literature. Subsequently, it will discuss the characteristics of professional development programs that enhance teachers' formative assessment practices.

3.3.1. Teachers' beliefs and practices for formative assessment

A review of empirical studies on teachers' beliefs and practices regarding formative assessment suggests that while teachers often value the positive role of formative assessment related practices in students' learning, their classroom practices may not reflect this (Antoniou & James, 2014; Marshall & Drummond, 2006). The following paragraphs will elaborate on the findings of these studies.

Antoniou and James (2014) conducted a study in Cyprus, exploring the beliefs and practices of four classroom teachers regarding formative assessment. The researchers employed 24 classroom observations and conducted pre- and post-lesson interviews with teachers for each observed lesson. Furthermore, teachers' written feedback on students' work was analysed. The findings of this study suggest that although these teachers acknowledge the benefit of formative assessment their classroom practices showed weakly enacted formative assessment practices. First, the study revealed that teachers faced challenges in identifying success criteria that would guide them in assessing students' learning. Moreover, the teachers tended to adopt teacher-centred practices without facilitating student engagement and active involvement in the assessment process. The study also highlighted that while the teachers used a variety of practices to collect information about student learning, these practices predominantly involved unstructured observation and questioning. The nature of the questioning was primarily aimed at evaluating whether students knew the correct answers.

In terms of feedback, the teachers' practices primarily involved correction and judgment rather than providing constructive guidance for improvement. Additionally, short-term rewards and the use of grades were found as common feedback strategies, which may undermine students' intrinsic motivation and hinder their long-term development. The teachers' focus on groups over individual attention was also evident, potentially neglecting the specific needs and progress of individual students. This stemmed from teachers' avoiding losing other students' attention while focusing on one student. Finally, the study identified a perception among teachers that formative assessment is time-consuming, which could lead to a perceived burden of record-keeping. This perception may impact teachers' willingness and ability to fully embrace and implement formative assessment practices consistently.

Marshall and Drummond (2006) conducted a comprehensive analysis exploring the relationship between the classroom practices of four teachers regarding formative assessment and their beliefs about learning. They posited that the essence of formative assessment lies in promoting learner autonomy, characterising it as the "spirit" of formative assessment. Conversely, they described formative assessment practices and intentions that overlook learners' autonomy as the "letter" of formative assessment. As a result of the analysis, they suggest relations between teachers' beliefs and practices. Unsurprisingly, teachers' beliefs and practices were consistent. However, interestingly, the teachers who did not prioritise their students' autonomy claimed that they valued students' autonomy. The researchers argued that the possible reason for conflicting practices could be teachers' attitudes about their own professional learning. That is to say, teachers who possess the spirit of formative assessment and enact practices with this spirit consider classroom teaching experiences as professional learning opportunities as opposed to teachers who possess the letter of formative assessment and consider several constraints, such as school culture, the exams and students' ability, against enabling students' autonomy.

3.3.2. Teachers' professional development for formative assessment

Having discussed the research evidence for teachers' common beliefs and practices with respect to formative assessment, this section turns to the characteristics of professional development interventions and their impact on teachers' beliefs and practices. Overall, these studies suggest that changing teachers' beliefs and practices could be a challenging and long-term endeavour. In this section, these studies are discussed in two categories in the following sub-sections. First, studies that focus on formative assessment holistically will be discussed. As mentioned in Chapter 2, in this thesis, noticing was considered to be a complimentary practice for the second formative assessment strategy, engineering tasks and discussions that elicit students' learning. Drawing on this, as a second focus, a review of empirical studies, examining the characteristics and impact of specific professional development interventions with respect to noticing will be discussed.

A holistic approach

The existing literature offers strong evidence that teachers' takeaways from professional development interventions could vary. First, these takeaways could be influenced by their prior beliefs about learning and assessment, as well as by their beliefs in their own self-efficacy (Dayal, 2021; Dixon & Haigh, 2009). Second, the positive impact of interventions might enhance only specific aspects of teachers' practices (Furtak et al., 2016). The following section elaborates on the studies.

Dayal's study (2021) examined change in two mathematics teachers' formative assessment beliefs and practices in Fiji. In this study, teachers' portfolio assessment was examined focusing on three formative assessment strategies: clarifying and sharing learning intentions and success criteria, activating students as instructional resources, and providing feedback that moves learners forward. These teachers were interviewed prior to the professional development to explore their initial beliefs about assessment and teaching. After providing teachers with training for the use of portfolio assessment, seven lessons carried out by these teachers were observed and post-observation interviews were conducted. The researchers identified two polar beliefs with respect to assessment and teaching: traditional versus constructivist. According to this categorisation, teachers with traditional beliefs viewed teaching as mastering skills, focusing on content, and transmitting facts to students, and they considered assessment primarily as testing students' knowledge. On the other hand, teachers with constructivist beliefs viewed teaching as learner-focused and centred on actively constructing knowledge. Unlike their traditional counterparts, these teachers viewed assessment as a process that involves various modes and aims to enhance students' learning. The findings of this research suggested that both traditional and constructivist teachers showed signs of formative assessment in their lessons as a result of the professional development sessions. However, while the teacher with traditional beliefs showed only an emerging understanding of formative assessment practices the teacher with constructivist beliefs enacted formative assessment more productively. This difference in the takeaways of two teachers from professional development strongly suggests that the variation can be attributed to their individual beliefs.

Similarly, in their longitudinal study, Dixon and Haigh (2009) led professional development interventions for participant teachers. This professional development intervention involved teachers conducting their own action research focusing on their students' needs and developing a community of practice among teachers and researchers through regular meetings, which aimed to provide feedback to teachers and plan for the next step of action research. One aim of this implementation was to encourage a shift in teachers' practices from a teacher-centred to a student-centred approach. The findings suggested that teachers' beliefs and practices changed to varying degrees: while some teachers' beliefs and practices altered considerably, the change for some was more limited. The authors argued that this variety stemmed from teachers' self-efficacy beliefs: while those teachers whose practices and beliefs altered considerably in a positive direction showed high self-efficacy beliefs and were persistent and resilient when they faced setbacks, those whose change was limited had weakened their expectations following setbacks.

A three-year project was conducted by Furtak et al. (2016) with secondary biology teachers to test the effectiveness of a formative assessment professional development intervention involving guiding teachers to design formative assessment tasks for a certain learning goal, enacting these tasks, and reflecting on them. They focused on three formative assessment practices: asking questions to elicit students' thinking, interpreting students' thinking and providing feedback. They distinguished between traditional and formative assessment integrated approaches with respect to these activities. Traditional ways involved closed-ended questions that might not reveal students' thinking, as opposed to authentic, open-ended questions. Moreover, in the traditional approach, teachers might focus on the correctness of students' answers rather than digging into their understanding, and they might offer evaluative responses to students rather than encouraging them to think further and guiding them towards improving their learning. In professional development, the researchers aimed to enable teachers' shift from traditional practices to formative assessment-integrated practices. The findings of the study demonstrated that for teachers this shift happened in terms of the quality of questioning, and interpreting students' ideas and feedback, but not in terms of the quality of tasks teachers designed. This finding also contributes to the understanding that, although making practical shifts away from traditional teaching practices is possible, it is a practical challenge and can be a slow and gradual undertaking.

A focus on teachers' noticing

In this literature review, large number of studies involving the term "formative assessment" were considered. However, the refinement process of these studies with respect to specific aspects resulted in limited studies being selected. The topic noticing, on the other hand, which is embraced as the complementary practice for second formative assessment strategy in this thesis, provided relatively rich results. The following paragraphs will focus on some of these studies.

The review of the literature showed that the number of empirical studies with in-service teachers appears to be lower than those with pre-service teachers. That is to say, the vast majority of this body of research was conducted with pre-service teachers, focusing on the impact of various training interventions on these teachers' noticing skills (e.g., Caylan-Ergene & Isiksal-Bostan, 2022; Lee & Lee, 2023). Although these studies could also provide insights into strategies that can improve teachers' noticing skills, in this review the studies conducted with in-service teachers were explored. This choice was made to focus the review on in-service teachers' takeaways from various professional development interventions and to interpret their needs.

I divided these studies into three categories. Those in the first category involve intervention studies for in-service teachers that resulted in positive changes in teachers' noticing skills and practices. These interventions include various forms of support such as providing teachers with deeper pedagogical content knowledge (e.g., Haj-Yahya, 2022), enabling teachers' collaboration with their peers (e.g., Suh et al., 2020), and their reflection on their own practices as well as other teachers' (e.g., Gonzalez & Vargas, 2020). In the second category, studies that assess teachers' noticing skills and can suggest teachers' needs in terms of specific aspects of noticing were involved (e.g., LaRochelle & Nickerson, 2019). In the final category, studies that suggest the relationship between teachers' noticing skills and their adaption of tasks are involved (e.g., Choppin, 2011). Overall, the findings of these studies will be interpreted for teachers' professional development needs to develop their noticing skills. Here I will now elaborate on these studies.

First, studies involving interventions with in-service teachers offer various strategies that might shift teachers' attention to students' learning in the intended direction. In Gonzalez and Vargas' (2020) study, a two-year professional development intervention was conducted with five teachers with a range of teaching experience, from 4 years to 26 years. The intervention took place after school hours, comprising a total of 20 sessions, each lasting three hours, across two research cycles. In the first cycle, teachers had the opportunity to watch animations that involved problems related to mathematical concepts, discuss the lessons they would develop, review instructional materials, and develop their own lessons. This was followed by teaching the lessons they developed and engaging with video clips from these recorded lessons, as chosen by the research team, to enable productive discussions with respect to the two aspects of noticing, attending and interpreting. In the second cycle, teachers watched different animations that included examples of students' thinking revealed in the lessons during the first cycle. Teachers revised the lessons and taught these updated versions. They then engaged with the videos focusing on analysing students' thinking. The findings of this study indicated a change in how teachers attended to and interpreted students' thinking, evident even in the sessions of the first cycle. Furthermore, a statistical comparison between the first and second research cycles revealed a significant increase in teachers' noticing of students' thinking in the lessons they implemented. Teachers focused more on understanding students' thought processes rather than simply seeking a specific answer.

This professional development intervention provides teachers with comprehensive support which includes active engagement with mathematics and students' thinking, working on lesson design, peer support, and reflective practices. The combination of these supports, coupled with the long-term nature of the intervention, likely facilitated substantial improvement in teachers' practices. However, it may not be practical to offer such extensive and long-term opportunities to teachers in many contexts. In fact, literature presents evidence of shorter-term interventions that are more practical whilst still enhancing teachers' noticing skills.

As an example, an unusual intervention was conducted with 22 secondary mathematics teachers. Teachers were given the opportunity to employ a coding scheme to analyse transcripts from recordings of secondary mathematics classrooms (Scherrer & Stein, 2013). These lessons involved cognitively demanding and open-method tasks. The coding scheme included codes that directed teachers' focus to the references for following instances in transcriptions: initiating a discussion, furthering the discussion to elaborate on students' thinking, and eliciting additional information about students' thinking. The intervention took four weeks, with two-hour sessions each week, involving introducing the coding scheme and enabling teachers to employ it both in groups and independently. Participant teachers' noticing was assessed through pre- and post-tasks. These tasks involved teachers' reflecting on a short video involving a teacher's discussion with their students about creating an equation for a real-life situation. The findings suggested that the interventions were effective in facilitating a focus on interactions between teachers and students, rather than on teachers only. However, the intervention did not evidence a considerable change in terms of teachers' recognising the opportunity for students' thinking that comes from specific teacher-student interactions.

In this research, teachers becoming familiar with the coding scheme might help them shift their attention from teachers' actions to interactions between teacher and student. Although this seems to be a limited improvement in teachers' skills, as teachers did not recognise how they could use these interactions for students' learning, it can be an important first step.

Another interesting teacher training intervention was employed in Israel with 41 in-service teachers over the course of two months (Haj-Yahya, 2022). This training involved teachers engaging with mathematical tasks and identifying potential difficulties inherent in these tasks, engaging with theoretical and empirical research studies in the mathematics education field, and engaging with a recording of a mathematics lesson as a final activity. The researchers categorised the aspects teachers noticed as difficulties specific to the task and more general geometrical difficulties which might not be specific to the task. The findings showed that as a result of the teacher training intervention teachers focused on the difficulties specific to the task and could thereby interpret these difficulties and respond to them more specifically.

The most striking aspect of this intervention is its provision of insightful MKT to teachers. This includes engaging with mathematics tasks, understanding mathematical concepts, and engaging with mathematics education literature. It appears that enhancing teachers' MKT in specific topics is valuable for noticing students' difficulties related to these topics and making informed decisions to address these difficulties.

The studies I will discuss in the second category offer evidence for teachers' existing noticing skills. Compared to studies aiming to demonstrate the effectiveness of specific professional development interventions, those providing an overview of teachers' current noticing tendencies are rare. A large-scale study from China comparing novice and experienced teachers' noticing skills is an example of these studies (Yang et al., 2021). This study involves 152 pre-service, 162 early-career, and 123 experienced mathematics teachers. In order to assess teachers' noticing skills teachers watched three different video clips for around four minutes that involved teaching situations on the topics of functions, volumes, and surface areas of geometric solids. The participants completed Likert scale items after each video clip. The findings showed sharp differences between novice and experienced teachers' noticing skills, in favour of experienced teachers, in terms of attending to and interpreting students' thinking.

These findings strongly suggest that novice teachers might need more support to develop their noticing skills. This might stem from their relative lack of experience with various student-teacher interactions. Although experienced teachers in this study showed better noticing skills compared to novice teachers there is not enough evidence to make inferences about the specific needs of experienced teachers. However, when the intervention studies presented earlier are considered it can be said that teachers might need to shift their attention to student-teacher interactions and students' thinking in specific mathematics contents.

In addition, LaRochelle and Nickerson's (2019) study, examined in-service secondary mathematics teachers' noticing skills qualitatively, suggested that while teachers could strongly attend to different aspects of students' mathematical thinking, their interpreting and responding skills were weaker. Considering the more demanding nature of interpreting students' responses and responding to them, which might require extensive experience with interacting with students and a strong MKT, it can be argued that teachers might need additional support with these aspects of noticing, compared to the aspect of attending.

In the third category, studies that explore the pedagogical implications of teachers' noticing will be presented. Choppin's study (2011) suggested an important relationship between teachers' noticing skills and their adaptation of classroom tasks. The data was collected through observing and interviewing teachers who implemented curriculum materials involving challenging tasks for three or more years. The findings showed that participant teachers who attended to students' thinking deeply developed conjectures about how students developed their thinking during instruction, and these conjectures informed their adaptations of tasks. On the contrary, teachers who focused on the correctness of their students' responses rather than attending to their thinking deeply tended to simplify the tasks when adapting these tasks. This relationship may suggest that teachers' better noticing skills could enhance their use of mathematics tasks.

In addition, in their paper, Wickstrom and Lamgrall (2020) focused on an elementary teacher's use of a certain learning trajectory for area measurement, which involves moving from merely understanding areas as individual units to comprehending multiplicative relations, to inform their teaching practice. The participant teacher conducted task-based interviews with students, designed and implemented lessons, and reflected on her experiences through written reflections and interviews with researchers. An analysis of this in-depth data suggested that the use of the learning trajectory facilitated this teacher's attending to students' thinking and adaptation of tasks to develop students' thinking.

Choppin's, and Wickstrom and Lamgrall's studies offer an implication for the *shaping* element of noticing. That is to say, when teachers attend to students' thinking and interpret this, they may better shape interactions to further notice students' thinking by adapting tasks appropriately.

To conclude this review, the findings of these studies might suggest that various opportunities teachers are given can enhance their noticing skills. These opportunities are engaging with various student responses and student-teacher interactions; the provision of various aspects of learning mathematics; and reflecting on insights gained from experiences. Moreover, it can be said that improving even one of the aspects of noticing can be valuable as a step towards positive shifts in other aspects. More explicitly, the studies showed that it was easier to provide a shift in teachers' attending to students' thinking than other aspects of noticing. Teachers' attending to students' thinking might not be sufficient but can be an essential step towards enhancing *interpreting* students' thinking and *shaping* interactions for further noticing.

Although this literature provides insights into teachers' noticing practices and strategies to improve these practices, a research gap has been identified with respect to two issues. First, these interventions may not be practical and accessible to most in-service teachers. There is a need for more research on strategies accessible to most teachers that can facilitate enhancing their noticing practices. Second, although there is variety in terms of the mathematics topics involved in existing studies, they mostly focus on the conceptual understanding aspect of mathematics learning. There is a need for more comprehensive studies that focus on the aspect of learning mathematics more fully. This doctoral research aims to address both gaps by focusing on noticing opportunities in teacher guides and examining these opportunities by using five strands of mathematical proficiency as a framework. Moreover, a focus on accessible tools such as teacher guides could be a more practical way to access more teachers, considering that well-designed and high-quality professional development sessions may not be accessible to most teachers.

3.4. Formative assessment in curriculum materials

In this section, I present a synthesis of studies involving analysis of commonly used mathematics curriculum materials, with a focus on the integration of three elements of formative assessment. To identify these studies, as introduced in Chapter 2, the three elements of classroom formative assessment for teaching mathematics – identifying learning intentions, noticing students' mathematical thinking and creating feedback situations – were carefully pursued. These studies can provide insights into how formative assessment elements are interwoven with current curriculum materials. However, it is important to note that the insights gained from these studies might be limited to specific elements of formative assessment, potentially hindering a comprehensive understanding of potential formative assessment support. In the following subsections, I will highlight the limitations of these studies in terms of how they incorporate aspects of formative assessment into the curriculum materials, as well as the insights they provide.

3.4.1. Identifying learning intentions

In the research studies explored, the references for teachers' identifying learning intentions were observed in two respects: (1) the existence of learning goals for lessons, (2) indications of the mediating role of curriculum materials in teachers' grasping of learning intentions.

While studies that uncover learning goals within mathematics curriculum materials provide insights into the learning intentions linked to the cognitive dimensions of mathematical proficiency, they offer only limited and indirect insights about learning intentions related to productive disposition, one of the five strands of mathematical proficiency suggested by Kilpatrick et al. (2001). In Choppin et al.'s study (2022), curriculum materials were analysed along a spectrum, with delivery mechanisms and thinking devices at opposite ends. This analysis aimed to understand the perceived value of learning mathematics. Yet, both extremes of the spectrum mainly concern cognitive skills, with providing limited insights for productive disposition. Specifically, the delivery mechanism aligns more with the conceptual understanding and procedural fluency strands of mathematical proficiency, whereas the thinking device aligns with strategic competence and adaptive reasoning. Similarly, another study by Choppin et al. (2021) categorised mathematical experiences in curriculum materials into being, sensing, and doing, all of which primarily address cognitive aspects and offer limited details on integrating productive disposition again. Contrasting these studies, which primarily focus on cognitive aspects of learning intentions in mathematics curriculum materials, Koljonen et al.'s research (2018) provides a nuanced insight into learning intentions linked to productive disposition, albeit indirectly. This study reviewed norms potentially established by prevalent curriculum materials in Finnish elementary mathematics classrooms, highlighting that problem-solving tasks encouraging connections to everyday life were notable in textbooks. Such tasks could foster a productive disposition in students towards mathematics learning.

In conclusion, current research offers limited insights into incorporating learning intentions related to productive disposition within curriculum materials. This limitation may stem from two main factors: the absence of this aspect in the curriculum materials themselves, and the frameworks employed to analyse these materials. The latter issue will be explored further in Section 3.5.

When it comes to communicating learning intentions to teachers, the findings of several studies suggest that the use of some curriculum materials helped teachers understand the mathematical and pedagogical intentions of lessons; however, these studies provide limited insights into what specific features facilitate this communication. In Fan et al.'s large-scale study (2021), mentioned earlier, teachers reported that teacher guides facilitated their identifying lesson objectives; however, there is no information about how lesson objectives were included in the materials. In Charalambous et al.'s case study (2012), in which a more thorough examination of teachers' practices was conducted, the findings suggest that the use of curriculum materials helped teachers gain insights into learning intentions and integrate them into their practices beyond just mentioning them. The findings of these two studies could be evidence for the potential of curriculum materials to communicate learning intentions to teachers.

3.4.2. Noticing students' mathematical thinking

As in the case of learning intentions, research studies that directly focus on noticing opportunities through curriculum materials are limited; however, a comprehensive review of the literature has revealed several key aspects of teachers' noticing students' mathematical thinking within the existing curriculum materials. The curriculum materials in these studies have the potential to serve as facilitators for teachers in attending to their students' mathematical thinking, albeit to varying degrees.

One aspect of Charalambous et al.'s (2012) analysis of teachers' practices involved teachers attending to students' mathematical thinking. Within that study, the teacher with mathematical knowledge for teaching, yet using curriculum materials that were notably educative and involved information about potential student difficulties, displayed indications of attending to students' thinking to a certain extent. Notably, the authors hypothesised that this observed productive practice might have been influenced, at least in part, by the use of curriculum materials. It is worth noting that while these observations raise the possibility of curriculum materials playing a mediating, concrete empirical evidence validating such mediation remains limited.

In the study by Koljonen et al. (2018) that I previously mentioned, which analysed curriculum materials in Finland, the researchers aimed to understand the type of classroom culture these materials might help to create. They suggested that these materials could play a key role in encouraging teachers to attend to students' mathematical thinking. Specifically, they found that the curriculum materials included mathematical tasks at different levels of difficulty. This variety is seen as indicative of an acknowledgement within these curriculum materials that students' understanding and difficulties also vary. Consequently, these tasks can serve as effective tools for uncovering and addressing these varying levels of understanding and difficulties. Koljonen et al. interpreted this aspect of the teacher guide as addressing all students' learning needs; however, their analysis does not offer a deeper insight into either the characteristics of these tasks or how this variety in these tasks will address students' needs.

The teacher who mentioned support for students' mathematical thinking in Fan et al.'s (2021) follow-up interviews highlighted the limitation of this support as the assumptions made in the textbooks might not always be relevant to real classrooms. More importantly, the prior knowledge of students, as assumed in the teacher guides, was not helpful for this teacher as it was not relevant to his students.

3.4.3. Creating feedback situations

Compared to learning intentions and noticing aspects of classroom formative assessment, it has been relatively challenging to find elements of feedback being directly addressed in studies related to curriculum materials. However, a close examination of these studies reveals these implicit feedback elements in line with the conceptualisation of feedback in this study, which basically refers to providing students with a space to reflect on their mathematical thinking. As an example, in Charalambous et al.'s study (2012) researchers' evaluations suggest that teachers mostly think for their students. This tendency of teachers might refer to the curriculum materials not providing teachers with the opportunity to enable their students to reflect on their thinking.

The conceptualisation of feedback in this PhD research requires the active involvement of students and their peers in the process of having the opportunity to rethink and reflect on their understandings. One step towards this should be enabling students' engagement in lessons and creating an environment for them to work collaboratively. Although earlier studies have not analysed the opportunity for this sort of feedback, inferences can be made as opposed to opportunities for students' active engagement and collaborative work. As an example, in Koljonen et al.'s study (2018), one of the norms of the classroom was identified as students' active engagement in the lesson.

3.5. Methodological approaches to the analysis of teacher guides

In this section, I provide an overview of the methodological strategies employed in studies that involve the analysis of mathematics curriculum materials. This examination involves a critical consideration of the theoretical approaches guiding these analyses and the analytical tools employed. Sections 3.5.1 and

3.5.2 respectively involve a synthesis of theoretical and analytical approaches that have been used in the literature to date. The discussion in these sections is a grounding for the justification of the theoretical and analytical decisions that were made in this doctoral research, which will be discussed in Section 3.6 separately.

Overall, this section of the literature review aims to provide a basis for the methodological decisions made in this research by locating them within existing literature and articulating the potential methodological gaps my research addresses. The studies selected were chosen regardless of the key purpose of these studies, but rather due to their potential to inform the analysis of teacher guides in this study.

3.5.1. A synthesis of theoretical approaches that can guide the analysis of the educative potential of teacher guides for formative assessment

In Chapter 2, I presented the overarching theoretical approaches that guided the design of this doctoral research. Namely, in order to conceptualise classroom formative assessment practices, I considered three key elements of teachers' classroom formative assessment practices: identifying learning intentions, noticing students' mathematical thinking in line with these learning intentions, and providing feedback that can help students achieve the intended learning. These elements were derived from Wiliam and Thompson's five formative assessment strategies, as discussed in detail in Section 2.1.2. In addition to this, in order to guide the exploration of educative features I used the design principles developed by Davis and her colleagues over the course of 10 years (Davis & Krajcik, 2005; Davis et al., 2017), as well as the insights I gained from other studies focused on the educative aspect of curriculum materials (e.g., Quebec-Fuentes & Ma, 2018).

In this section, the focus shifts to the theoretical approaches employed to analyse mathematics curriculum materials in the existing literature, which directly informed the methodological decisions made in this PhD research. The review of theoretical approaches that guided the analysis of curriculum materials in existing studies revealed intriguing trends within these guiding frameworks.

These trends revealed specific aspects of the curriculum materials whilst constraining access to other dimensions. With respect to the formative assessment aspect, while dominantly employed frameworks can help reveal the cognitive aspects within curriculum materials (e.g., Choppin et al., 2021), these frameworks may limit the uncovering of the aspects of learning mathematics, which is related to the *productive disposition* strand of mathematical proficiency in this thesis. For the educative features aspect, the frameworks employed can lead to revealing the potential of curriculum materials which are related to passively equipping teachers with specific aspects of teacher knowledge (e.g., Quebec-Fuentes & Ma, 2018).

In the subsequent subsections, I will examine the theoretical approaches identified, discussing their implications for the analysis of curriculum materials in two key aspects of this research: formative assessment and educative features. Moreover, within these sections, I will provide justification for my selection of guiding theoretical frameworks when analysing the teacher guides and shed light on the potential methodological gaps that my study can address.

Formative assessment aspect

Prevalent focus on the cognitive aspects of learning mathematics

One aspect of the theoretical frameworks commonly used in the existing literature was a tendency to focus on the cognitive aspects of learning mathematics, such as students' conceptual understanding, procedural fluency, problem-solving, and reasoning. However, these theoretical approaches have limitations in revealing the references for productive disposition (Kilpatrick et al., 2001).

A commonly used framework is the one suggested by Stein and her colleagues (1996) for categorising the cognitive demand level of tasks according to four categories: doing mathematics, procedures with connections, procedures without connections, and memorisation. These four categories can strongly relate to four aspects of mathematical proficiency, with less emphasis on productive disposition.

More recently, Janine Remillard, one of the influential researchers in mathematics curriculum materials research, led a research study that involved the analysis of textbooks with respect to the potential influence of curriculum materials on students' learning (Remillard et al., 2014). One key element of their analysis was the level of mathematical emphasis within the materials. They built their analysis of mathematical emphasis on the assumption that both procedural fluency and conceptual understanding are key elements in a high-quality mathematics lesson. As a result, they looked at three elements of the mathematical content: the conceptual level of mathematical tasks, opportunities for students to engage with concepts, facts and procedures, and opportunities for developing procedural fluency. As with the elements of Stein et al.'s (1996) framework, although these elements of the analysis can help in identifying the learning intentions inherent in the materials, they can also potentially limit access to the learning intentions relating to productive disposition.

Choppin and his colleagues (2021) employed a linguistic framework as a theoretical framework in their analysis of curriculum materials. Based on this framework, they considered language to be a tool that creates meaning in terms of the prevalent learning intentions in the curriculum materials. More explicitly, the framework they employed involved three key guiding verbs: being, sensing, and doing. Being involved verbs that refer to more descriptive activities, such as classifying and identifying; sensing involved verbs that refer to reasoning such as explaining and justifying; and doing involved verbs that refer to applying mathematical knowledge, such as multiplying, simplifying, and doubling. This approach can extend beyond a mere focus on procedural fluency and conceptual understanding, with its additional focus on mathematical reasoning and strategic competence; however, the possibility of revealing references for productive disposition will remain limited.

In addition to this focus on the general elements of learning mathematics, there exist studies that only focus on specific mathematical topics or skills. As an example, Stylianides (2009) focused on the adaptive reasoning strand of mathematical proficiency in his analysis. The elements of his framework involved identifying a pattern, making a conjecture, providing proof, and providing a non-proof argument, which can aid in understanding the elements related to adaptive reasoning.

In contrast to these studies, the framework established by Dole and Shield (2008) for analysing mathematics textbooks involved elements of productive disposition. The main difference with this framework is its focus on specific mathematics skills, namely multiplicative reasoning. Alongside the cognitive elements such as multiplicative structures, various representations, and making connections to topics such as ratio, contextualising the comparison of additive and multiplicative reasoning was an element of their analysis. This element has potential to reveal the element of productive disposition which involves finding mathematics meaningful.

In conclusion, the synthesis of various theoretical approaches used to analyse curriculum materials has provided valuable insights into the strategies used for identifying the different strands of mathematical proficiency. However, the prevailing frameworks employed can potentially hinder identifying the references for the productive disposition strand of mathematical proficiency. An explicit focus on elements of productive disposition within curriculum materials can significantly contribute to the understanding of how these materials can facilitate students' identity development as learners of mathematics.

Insights from task-analysis research that can inform the analysis of noticing

Task analysis research offers valuable insights into the effective categorisation of mathematical tasks and facilitates a comprehensive understanding of the potential implications associated with incorporating these different types of tasks into teaching practices. This thesis places a particular emphasis on the analysis of mathematical tasks due to their pivotal role as fundamental components of formative assessment in mathematics lessons, enabling the noticing of students' proficiency in multiplicative reasoning. Within the existing body of literature, praxeological analysis, the identification of task openness, and the categorisation of problem-posing tasks emerge as prevalent approaches (Cai & Ciang, 2017; Miyakawa, 2017; Yeo, 2015). When deciding which approach to employ for analysing and interpreting mathematical tasks in the teacher guides, I focused on the potential of these approaches to reveal students' transition from additive thinking to multiplicative reasoning (This was the critical mathematical context of this thesis as elaborated on in Section 2.2.2). Consequently, I opted to analyse the openness of tasks.

Praxeological analysis, an aspect of ATD, has been widely adopted in analysing and comparing textbooks in recent years (e.g., Miyakawa, 2017; Solis & Isoda, 2023; Wang et al., 2023; Wijayati & Winslow, 2017). The key benefit of this approach is enabling an understanding of the factors that influence mathematics teaching and learning in specific contexts, referred to as institutions within this theory and making comparisons between distinct contexts. Task analysis is considered a crucial component of this approach, providing a high-level understanding of tasks that may facilitate a comprehensive grasp of the institutional context. However, it does not necessarily uncover the opportunities within the task that could reveal detailed insights into students' thinking. Namely, in this framework, praxeology involves four interrelated elements: task, technique, technology, and theory (Winslow, 2011). The elements of task and technique enable to identify the types of tasks and techniques that are expected from students. Theory refers to the rationale of the expected techniques to be used and technology refers to the practical tools required to solve the tasks. Analysis of the relations between these four elements can provide a broader picture of teaching and learning mathematics in specific educational contexts at the institutional level (e.g., school level or country level). However, this sort of analysis may not provide insights into the finely grained pedagogical opportunities of using these tasks, which is the key focus of task analysis in this thesis.

Problem-posing tasks in the curriculum materials seem to be of interest in the existing literature. Problem-posing tasks, by their nature, are expected to require high-level cognitive skills that may involve all five strands of learning mathematics. Involving these tasks in teaching mathematics can provide an opportunity for students to improve these skills. Probably because it is an emerging research topic, researchers focused on the analysis of problem posing tasks focus on their presence, proportion, and typology in existing curriculum materials (e.g., Cai & Ciang, 2017).

Finally, another important aspect of mathematical tasks of interest among mathematics education researchers is the openness of tasks. Several researchers suggested several explanations for the openness of mathematical tasks. Yeo (2015) proposed a framework to identify the openness of tasks in terms of five different aspects: goal, method, task complexity, answer, and extension. Among these elements the openness of both the method and the answer can provide opportunities for revealing students' mathematical thinking. That is to say, open method tasks can reveal students' various ways of thinking about the solution and open answer tasks can provide students with the opportunity to create innovative answers.

As a result, in this thesis, the openness of tasks was chosen as a focus to analyse and interpret the potential of the opportunities teacher guides provide teachers with to facilitate their noticing of students' mathematical thinking.

Educative aspect

Building upon the seminal research of Davis and Krajcik (2005) and Ball and Cohen (1996), extensively discussed in Section 2.3.2, this section aims to present studies that have further advanced their work, specifically by analysing the existing mathematics curriculum materials. Their frameworks, pivotal in guiding the content of educative curriculum materials, have laid the groundwork for the subsequent research explored here. In this section, I do not only acknowledge the foundational contributions of two group of researchers, but also examine how their insights have been expanded upon in more recent studies to analyse the educative potential of existing curriculum materials.

Lewis et al. (2011) compared the potential educative support for teachers' teaching of the area of quadrilaterals in Japanese and US teacher guides. They used Ball and Cohen's (1996) framework which proposes five educative features. Lewis et al.'s bottom-up approach, alongside the top-down approach, enabled them to identify further features and to thereby extend Ball and Cohen's framework. As a result, they proposed sub-categories for three educative features originally proposed by Ball and Cohen. First, the anticipation of student thinking can be supported by either providing single, correct answers or multiple student responses, including potential misunderstandings. Second, connection of content can be facilitated by providing prerequisite skills or standards or drawing on the instructional implications of these connections. Third, teachers' decision-making can be supported by providing support either for specific student responses or for a certain group of students.

Stein and Kim (2009) analysed two curriculum materials by building on Davis and Krajcik's framework which proposes supporting teachers to enhance their pedagogical content knowledge and content knowledge. Stein and Kim adapted these elements, which were originally suggested and exemplified in the science context, to mathematics. As a result, they focused on three aspects of the materials: level of cognitive demand for students, teacher materials for transparency, and anticipating students' responses.

In their analysis of educative features, Remillard and her team (2019) analysed the type of guidance and the message the guidance involves separately. Their analysis involved elements of curriculum materials that aim to strengthen teachers' mathematical knowledge, communicate the rationale behind the suggested practices, strengthen teachers' pedagogical content knowledge that involves anticipating students' thinking, adapting the content according to students' needs, and general pedagogical knowledge that involves classroom organisation and management.

Quebec-Fuentes and Ma (2018) developed a framework to evaluate the educative features of curriculum materials for mathematics teachers. This framework involves two dimensions. The first dimension is seven categories of teacher knowledge, which they suggested as a result of a comprehensive literature review. These categories are content knowledge, students' thinking, disciplinary discourse, assessment, differentiated instruction, use of technology, and mathematical community. The second dimension of the framework involves three levels of educative features to be considered for each knowledge category. Basically, these features question the extent to which rationale and guidance are provided with respect to potential educative support for any of these teacher knowledge categories. Although this framework can provide insights into the appearance of educative materials, rationale and guidance might be too broad to guide the design of effective educative features. Moreover, informing and guiding teachers might not ensure teachers' implementation of the recommended practice.

Unlike Quebec-Fuentes and Ma's approach, some recent research studies investigated educative features as a means of supporting specific aspects of mathematics teacher practice rather than merely focusing on broad teacher knowledge. As part of the project led by Janine Remillard and Ok-Kyeong Kim, leading researchers in curriculum material research in the mathematics context, Machalow et al. (2020) analysed teacher guides in five curriculum resources in the US to examine whether and how curriculum materials could facilitate teachers' noticing practices. They developed a framework informed by noticing literature. More specifically, they explored how materials support teachers to attend to, interpret, and respond to student thinking, and evaluated the mathematical quality of these supports.

A PhD thesis at Michigan State University (Males, 2011) focused on an analysis of mathematics curriculum materials for educative support. The framework used in this research involved two aspects: content and language. For content, the researcher adapted Beyer et al.'s framework (2009) which was originally developed to analyse educative support in biology curriculum materials. This analysis mainly focuses on teacher knowledge. Males identified four aspects of teacher knowledge: mathematics subject matter knowledge, pedagogical content knowledge for mathematics topics, pedagogical content knowledge for mathematics practices, and mathematics curricular knowledge.

The studies previously discussed offer valuable insights into the essential elements that should be incorporated into curriculum materials, such as providing rationales for teachers, guiding them, and enhancing their professional knowledge. However, these elements primarily focus on the knowledge teachers should possess and the practices they should implement. This focus on teachers' expected knowledge and practices can hinder a direct focus on the role of curriculum materials in enhancing teachers' knowledge and practices. Despite specific features emphasising the content of educative curriculum materials, such as providing a rationale, there remains a lack of detailed understanding as to how to effectively deliver this rationale or what specific characteristics can facilitate it for teachers. Consequently, further theoretical exploration of the educative features of curriculum materials is essential to address the gap of "how" to incorporate content for educative purposes beyond "what" content to include.

3.5.2. A synthesis of analytical approaches to the analysis of teacher guides

The general tendency of representative sampling: Using curriculum materials to make inferences for the broader contexts to which they belong

The key function of the textbook analysis in this body of research seems to be making generalisable inferences either with the purpose of comparison between different contexts, such as different historical times, and educational systems (e.g., Hemmi et al., 2018) or revealing the prevailing messages within a set of curriculum materials (e.g., Reinke et al., 2020; Van Steenbrugge & Remillard, 2023; Van Steenbrugge & Yolcu, 2023).

As an example, to analyse the intended support for giving a particular lesson in terms of provided content and the nature of communication, in the study with Finnish schools (Hemmi et al., 2018), researchers aimed to access most of the teacher guides used by teachers in the country. This sampling strategy enabled them to make inferences about general tendencies in Finland. They used a similar approach to make inferences about the overview of these teacher guides by focusing on topics represented in most of them.

My sampling approach differs from the approach used in these studies. In my research, I use the curriculum materials instrumentally in order to reveal high-quality and varying ways of integrating formative assessment elements into secondary mathematics lessons. As a result, my purpose is not to make generalisations with respect to either the broader country context or the full set of the material. I elaborated on these strategies in Section 4.3.

Holistic-implicit versus fragmented-explicit focus on the features of curriculum materials

The synthesis of the analytical approaches to the analysis of curriculum materials revealed two poles of identifying the unit of analysis: while some studies focused on fragmented and explicit references to the features of curriculum materials others focused on the implicit and holistic references. It was observed that when the purpose is to make inferences beyond identifying explicit features of the materials, a holistic approach was taken.

The study by Koljonen et al. (2018) that analysed Finnish elementary mathematics teacher guides in order to identify mathematics classroom norms in teacher guides involves both aspects. The distinct feature of the analysis in their research was employing a framework that reveals both explicit surface features of the teacher guides and implicit messages. For explicit surface features they looked at descriptions, visualisations, locations of these descriptions and visualisations, and how teachers' and students' roles were located within these descriptions and visualisations. These units of analysis are fragmented but explicit aspects of the curriculum materials. For implicit features, they examined four aspects of suggested practices: intention, action, expected outcome and inferred meaning. In this case, implicit references were identified through a more holistic approach. Similarly, Choppin and his colleagues (2021) used holistic and implicit references for the unit of analysis to identify the mathematical experiences involved in curriculum materials used by teachers in their planning.

In this doctoral research, the aim is to identify those pedagogical messages which may not necessarily be found through fragmented references, requiring me to examine the teacher guides holistically.

Horizontal versus vertical analysis of the curriculum materials

Charalambous et al. (2010) proposed a framework that enabled a holistic approach to the analysis of curriculum materials, building on the critique that earlier studies focused only on specific aspects fragmentedly. They divided the approaches to analysing curriculum materials into three categories: horizontal, vertical, and contextual. In the horizontal approach, a textbook is examined as a whole, in the vertical approach a single mathematical concept is the focus, and in the contextual approach the focus is on the use of textbooks by students and teachers. The authors argue that a strong analysis should involve all three aspects.

I have made similar observations and critiques to Charalambous and his colleagues. However, I think it is necessary to address the same issue differently. The horizontal analysis they suggest involves certain descriptive and quantitative analyses of the materials to understand the overall picture regarding the materials, such as topics involved, number of tasks, etc. Then, they suggest moving to the vertical and contextual analyses. Although I have similar concerns, I replace the horizontal analysis they suggest with familiarisation with materials and thick descriptions that will inform both explorative and critical analysis. That is to say, these descriptions will help the reader to make sense of the main analysis. The difference of my approach from descriptive horizontal analysis is my focus on specific aspects of the materials that can feed further analysis, rather than stopping at a general independent horizontal analysis.

3.6. Justifying research questions and research methods

3.6.1. Justifying the research questions

This doctoral study embraces the two-way documentational approach as the theoretical foundation for studying the relationship between teachers and curriculum materials. This approach highlights the interrelated relationship between teachers and curriculum materials, using the concepts of *instrumentation* (from curriculum material to teacher) and *instrumentalisation* (from teacher to curriculum material) (Gueudet & Trouche, 2009). In this research, I have placed a special emphasis on the instrumentation element of this relationship, with a specific focus on exploring the features of curriculum materials that can enhance teachers' effective formative assessment practices.

Curriculum materials can have a mediating role in enhancing teachers' knowledge and classroom practices (e.g., Charalambous et al., 2012; Fan et al., 2021). However, it is crucial to acknowledge that mediating materials are significantly influenced by teacher-centric mechanisms, such as their experience, knowledge, and beliefs (Fan et al., 2021; Noh & Webb, 2015). As presented in Chapter 2, the design principles proposed for designing educative curriculum materials aim to address the differing needs of teachers by providing them with knowledge, tools, and the rationale underlying suggested practices, and consider that teachers often adapt materials rather than using them as they are (Ball & Cohen, 1996; Davis et al., 2017). The review of empirical studies relating to mathematics teachers' use of curriculum materials provided the distinctive characteristics of mathematics curriculum materials that can mediate teachers' effective practices, such as detailed scripts of guidance, high-quality tasks, clear communication of the rationale behind suggested practices, and providing teachers with opportunities to analyse tasks (e.g., Remillard, 2012).

Improving teachers' formative assessment practices has proven to be a challenging and long-term process (e.g., Furtak et al., 2016). As discussed in Section 3.3.1, although teachers seem to appreciate the usefulness of formative assessment, it can still be a challenge to change their practices for the better. In the following section, 3.3.2, I argue that existing professional development interventions could be time-consuming, as well as inaccessible to many teachers. Instead of these interventions, well-designed curriculum materials that are easily accessible could have a more direct and wider ranging effect on teachers' practices. However, the existing research provides limited insight into the integration of elements of formative assessment into prevailing curriculum materials. Namely, a number of studies have provided insights into the integration of learning intentions but even these studies focus only on limited aspects of mathematical proficiency, mainly conceptual understanding (e.g. Choppin et al., 2021).

Mathematics teachers need professional support to improve their formative assessment practices. Curriculum materials should be accessible resources to address this need. The review of empirical studies has shown a gap for research that specifically focuses on the educative potential of curriculum materials to improve mathematics teachers' formative assessment practices. RQ2 and RQ3 in this research, as introduced in Section 1.3, address this gap.

3.6.2. Justifying the research methods

In Section 3.5, the examination of early studies in terms of their methodological approaches served two functions. First, insights were gained with respect to making decisions about analysing teacher guides in my research, and methodological gaps were identified. The methodological approaches were examined in terms of two aspects: the theoretical approaches that guided the analysis and the analytical approaches that were employed.

The theoretical approaches that can inform the analysis of teacher guides in this research have been considered for their formative assessment aspect and the educative potential of materials. In terms of the analysis of formative assessment, the review of existing studies has informed two decisions in this study. First, it was concluded that there was a prevalent focus on cognitive aspects of learning mathematics, as presented in Section 3.5.1. In order to have a comprehensive view of the learning intentions within the curriculum materials beyond cognitive aspects of learning mathematics, in this research I employed five strands of mathematical proficiency as a guiding framework (Kilpatrick et al., 2001). Second, mathematics task analysis research can inform the analysis of opportunities for shaping mathematical interactions with students, including noticing their mathematical thinking. In particular, Yeo's (2015) approach to the analysis of the openness of tasks was chosen to be employed.

In addition to this, in terms of the elements of formative assessment, the insights from textbook analysis research could only inform the approaches to learning intentions in teaching mathematics and shaping the element of noticing, providing limited insights into other elements of noticing and feedback. In order to complete this picture, I combined the insights I gained from these studies with those gained about research elaborates on noticing and feedback, as already discussed in Sections 2.2.3. and 2.2.4.

In terms of the analysis of educative potential, it was concluded that there is a need to develop a framework that can reveal the educative potential of curriculum materials according to the pedagogical messages to practice formative assessment in the classroom effectively. In order to develop this framework, previous studies that identified educative features of curriculum materials were used (Davis et al., 2017; Quebec-Fuentes & Ma, 2018; Remillard, 2012). More specifically, the ideas of providing the rationale for the suggested pedagogical practices, guiding teachers, and example practices being situated in teachers' practices can be used as a reference in the initial stages of developing the framework.

The synthesis of analytical approaches in existing studies involving the analysis of curriculum materials informed three decisions in my analysis of teacher guides. First, rather than representative sampling, a purposive sampling approach was chosen. In this research, the purpose of the analysis of curriculum materials is to identify educative features that can enhance teachers' effective formative assessment practices rather than making generalisations for the specific educational context the curriculum materials belong to. As a result, well-designed materials that can provide variety were chosen, as will be detailed in Chapter 4.

Second, identifying implicit messages within the curriculum materials was another focus of my research. The three analytic approaches in earlier studies informed the analysis for implicit messages: a top-down approach by using specific theories to enable a focus (Stein & Kim, 2009), focusing on the features of the curriculum materials holistically (Choppin et al., 2021), and conducting vertical analysis (Charalambous et al., 2010).

CHAPTER 4 - METHODOLOGY AND METHODS

The research questions in this doctoral research were introduced in Section 1.4 and justified through Chapters 2 and 3. This chapter aims to provide a transparent and rigorous account of the process of addressing these research questions by elaborating on the research design, sampling, and analytical approaches employed to analyse the data. Table 4.1 presents an overview of the methodological decisions made in this research, linking them to the research questions.

This chapter starts by introducing the overarching methodological decisions such as research design (Section 4.1), strategies employed to enhance the rigour of the research (Section 4.2), and sampling and introduction of the cases (Sections 4.3 and 4.4). Then it presents the specific decisions made in each explorative and critical phase of the research (in Sections 4.5 and 4.6).

4.1. Research design

4.1.1. Qualitative approach

In the design of this doctoral research, a big qualitative research approach (big Q) recently proposed by Braun and Clarke (2021) guided the methodological decisions. The key idea of “big Q” centres around the adherence to the principles and values of qualitative ethos as opposed to the mere application of procedures and techniques associated with the research approach known as “small q”, as described by Braun and Clarke (2021). Commonly, the studies that are closer to the “small q” pole are influenced by the positivist paradigm by considering researchers’ subjectivity as a threat to the quality of the data analysis. In contrast, in the other pole, “big Q", researchers’ subjectivity is considered an opportunity for an in-depth analysis.

Table 4.1 The overview of the research design

Phase	Research questions	Sampling	Data analysis
Explorative	RQ1: How can five strategies of formative assessment suggested by Wiliam and Thompson (2007) be operationalised so that they guide mathematics teachers' in-the-moment formative assessment practices? RQ2: What educative features of curriculum materials can be suggested?	Teacher guides in five sets of curriculum materials	Reflexive thematic analysis (Braun & Clarke, 2022) Supplementary feedback analysis
Critical	RQ3: How does the developed framework contribute to the understanding of the educative potential of teacher guides to facilitate mathematics teachers' in-the-moment formative assessment practices?	Teacher guides for chosen one or two lessons from each set	Deductive oriented analyses inspired by the big qualitative approach (Braun & Clarke, 2021) and constant comparison technique (Charmaz, 2014) Horizontal and vertical analysis (Charalambous et al., 2010)

Note. Created by the author

While none of the “big Q” or “small q” studies is inherently superior to the other, it is important to approach one of these methods with caution, depending on the specific research questions at hand. More explicitly, both approaches have their strengths and limitations. On the one hand, the “small q” approach, which entails clear systematic procedures, allows for making generalisations among large datasets and facilitates consistent work by a group of researchers within the same dataset (e.g., in the ICUBIT project, Remillard & Kim, 2020). However, this approach could potentially limit access to the implicit patterns inherent in the data. On the other hand, while the “big Q” approach allows for a deeper analysis by using the researchers’ subjectivity as an opportunity (e.g., in Van Steenbrugge & Remillard, 2023), it may present challenges in managing large data sets and communicate the analytical decisions that are taken by the researcher.

In this doctoral research, an approach closer to the "big Q" approach facilitated the identification of nuances within the analysed teacher guides in terms of educative features and recommended classroom practices associated with formative assessment. This approach specifically allowed for a focus on nuances, prioritising the recognition of subtle details over the identification of more common patterns.

4.1.2. Collective case study

As previously stated in Section 1.5, the research design for my PhD study has changed twice due to the COVID-19 pandemic. While engaging with these changes, I prioritised the original research purpose: to explore the characteristics of curriculum materials that can encourage teachers to engage with the key principles of formative assessment and facilitate their implementation in practice. The limitations in terms of accessing the classroom data forced a shift to a case study design. This design involved analysing various curriculum materials to identify their features that can potentially encourage and facilitate teachers’ effective formative assessment practices.

Stake (1995) suggests three distinct types of case studies: intrinsic, instrumental, and collective. Intrinsic case studies involve a researcher’s focused exploration of specific cases to gain deeper insights. Instrumental case studies use a particular case as an instrument to explore a broader phenomenon (ibid.). In the context of my research, if the aim was to explore various features within each material to draw generalisations within the materials themselves, an intrinsic case study would be considered. However, the primary purpose of this study was to investigate the phenomenon of how features in curriculum materials can potentially promote and support teachers' effective formative assessment practices, extending beyond the understanding of individual cases. This approach falls into the third category of case studies, a collective case study (ibid.), as it examines the synergy among different cases rather than concentrating deeply on a single case. In the following sections, I will provide detailed explanations of the sampling and data analysis methods employed in conducting this case study.

4.2. Strategies employed to enhance the rigour of the research

4.2.1. Positioning myself

Researchers' subjectivity plays an important role in research practices that are close to the "big Q" approach (Braun & Clarke, 2021). To enhance the rigour of a research design which incorporates subjectivity, additional techniques are necessary. It is important to note that these techniques differ from the methods employed for validating and ensuring reliability in quantitative research (Braun & Clarke, 2022).

Reflexivity is an approach that can enhance the rigour of qualitative research that embraces subjectivity. Reflexivity refers to the researcher's awareness and transparency about the potential influence of their subjectivity throughout the research procedures from collecting the data to analysing the data and interpreting the findings (Ball, 1990; Charmaz, 2014). For rigorous reflexivity practices, it can be beneficial to be sensitised to the elements that can lead to subjective decisions through research (Duffy et al., 2021).

In this research, researchers' positionality was considered as the key element to be considered. My position in the research influenced my decisions in terms of two aspects: (1) being an insider or an outsider, and (2) the power relations that can influence my decisions. Considering that the main research instrument in qualitative research is the researcher (Ball, 1990; Jaworski, 1997), it is worth presenting my position in this research before articulating the methodological decisions I made.

In line with Merriam et al.'s (2001) explanation of being an insider or outsider researcher, I do not consider these positions as dichotomous categories and position myself merely in one of these categories. Instead, I argue that I am both an insider and outsider to some extent. First, I am insider as a former early secondary mathematics teacher whose educational background is mostly relevant to teaching mathematics (Appendix B presents my background). Explicitly, I had an opportunity to teach a variety of early secondary students in Türkiye, from different socio-economic backgrounds and having different attainer levels. Also, I taught both to be dependent on the specific curriculum materials that were provided by the Ministry of National Education in the state schools and to be more flexible to choosing or designing curriculum materials as a teacher by following a generic guideline in the centres for gifted students. Inevitably, during my research, I had assumptions about how the materials that I analysed would access both teachers and students. These assumptions could both enhance and hinder an insightful analysis. As a result, I was aware of these assumptions when engaging with the data. Second, I am an outsider because most of my educational and professional background is neither in the US nor in England. Being an outsider made understanding the real situations in these contexts difficult and slowed down the process, as I needed to invest extra time and effort to become familiar with the context.

The power relation between designers and myself as a researcher might have led to assumptions when I am engaging with the data. While some of the designers are experienced practitioners (White Rose Maths), some designers are researchers who are respected all around the world (e.g., Mathematics Assessment Project). When engaging with those materials I might have inclined to assume that while the materials designed by the practitioners considered usefulness for teachers, the materials designed by the researchers considered exemplifying theories. This remains a question to keep in mind.

4.2.2. Peer debriefing

In this research, peer debriefing was used as a way to enhance the rigour of the analysis (Creswell & Miller, 2000). In peer debriefing, the person who is debriefing should be familiar with the phenomenon when someone familiar with the research or phenomenon provides support as well as challenges the researchers' assumptions, methods and interpretations.

I discussed my methodological decisions with three other PhD researchers in England and one researcher who recently completed a PhD in mathematics education in Türkiye. Two of the PhD researchers were not familiar with the phenomena of formative assessment in mathematics education but were familiar with case studies and qualitative data analysis. With them, I had an opportunity to discuss general broader issues such as the rationale for thematic analysis and the strategies to code and analyse qualitative data. The other PhD student was an experienced mathematics teacher who conducted PhD research on formative assessment. I discussed issues raised related to formative assessment and mathematics in particular with them. Especially, we met during the coding process to discuss the credibility of the codes. The fourth researcher is an experienced mathematics teacher and has recently completed a PhD in mathematics education with a focus on textbook analysis. The frequency of meetings with the fourth researcher was rare compared to other three researchers; however, discussing different strategies to analyse textbooks and present the findings helped me gain insights. Except these regular meetings, I used the opportunity to discuss about my research with other PhD students and senior researchers, particularly during the 12th Congress of the European Society for Research in Mathematics Education (CERME12), the 13th Congress of the European Society for Research in Mathematics Education (CERME13), and British Society for Research into Learning Mathematics (BSRLM) New Researchers' Day.

4.2.3. Thick description

In the realm of qualitative research, the concept of *thick description*, as introduced by Clifford Geertz (1973), highlights the significance of providing rich and detailed context when interpreting social phenomena. In this research, the depth of exploration of the educative potential of well-designed teacher guides independently and investigation of the synergy among them is crucial. Thick descriptions enhance presenting this exploration to the reader. As a result, when presenting the analysis for formative assessment opportunities and educative potential, I provided detailed descriptions of mathematical tasks and the pedagogical practices in this thesis. By employing this approach, I intend to provide syntheses beyond superficial interpretations and contribute to a more nuanced and holistic comprehension of the educative potential of the teacher guides with a particular focus on teachers' classroom formative assessment related practices (Geertz, 1973; Morgan et al., 2014).

4.2.4. Reflexive journals

During this doctoral research, I kept reflexive journals at different times and in different forms as a tool to enhance the methodological rigour of my research by critically reflecting on the research process (Ortlipp, 2008). In the early stages, I kept physical notebooks to reflect on my progress and jot down my initial research ideas, linking them to my own teaching practices, experiences I had during my doctoral training, and initial informal observations regarding the education system in England. These early journals helped me identify the research problem and develop the research design. When I started interviewing teachers in May 2020, I took digital notes related to the interviews before and after these interviews. Although I have not used the interview data in this thesis, my notes about the interviews helped my familiarisation with potential tendencies among teachers regarding formative assessment practices and the potential effects of external mechanisms on their practices.

The most effective reflexive journals that influenced and shaped this thesis were taken after December 2021. Between December 2021 and July 2022, I kept a digital journal that particularly focused on the data analysis process during the explorative phase of the research. This journal initially involved familiarisation notes regarding the curriculum materials. These initial notes guided me in sampling aspects of the curriculum materials for analysis in later stages. This journal also involved my reflections on challenges I encountered when using techniques and tools for coding the data. These reflections aided in the refinement of codes and the identification of effective coding strategies. After that, from August 2022 to the submission of the thesis, I largely used physical notebooks when focusing on the critical phase of the research. These notebooks involved in-depth interpretations of the suggested practices within teacher guides, as well as links to the existing literature. These later journals helped me reflect on the initial versions of the framework I developed and finalise it.

Alongside helping me engage with the research process, these journals helped me to reflect on my subjectivity which potentially influenced my choices during several aspects of my research. In the following piece from my journal, written on May 22, 2022, it is explicit that my experiences and observations as a teacher influenced my focus on the literature search.

> *The literature I read so far can be misleading in terms of teachers' needs for formative assessment and noticing students' mathematical proficiency. It might be assumed that these are old-fashioned problems. However, from practice, I know that teachers still face challenges to truly implement formative assessment and support students' mathematical proficiency. Even a quite current well-accepted material, WRM, includes fragmented mathematics learning objectives. I should really find further literature that shows teachers' tendencies and practices in the classroom.*

The following piece, written on December 22, 2022, shows the impact of my background as a researcher on a qualitative analysis by mentioning my earlier experiences with analysing quantitative data.

> *In their book Braun and Clarke started with a section about reflexivity, and they suggest not to start data analysis before reading that section. They used a concept of "qualitative sensibility" to highlight the difficulty of qualitative analysis as it requires a kind of maturity in thinking. They also said that it might take time to have qualitative sensitivity, especially when we first learnt quantitative methods as researchers. I think I really have had this challenge throughout my PhD.*

Finally, the following piece, written on December 5, 2023, reflects my perspective as a teacher, feeling discontented with the notion of researchers being regarded as superior to teachers.

> *I need to distinguish my framework from Davis and Krajcik's. They focus on teachers' needs but I feel that they consider themselves superior to teachers when suggesting the framework.*

4.3. Sampling strategies

As mentioned in Section 4.1.2, this collective case study positions the curriculum materials as instrumental cases that serve as strategic resources to identify features of the curriculum materials, which have potential to encourage and facilitate mathematics teachers' formative assessment-related practices. Thus, a thorough purposive sampling process (Creswell, 2003) was conducted to select curriculum materials with specific characteristics that could facilitate identifying these intended features. I completed the selection of sets of curriculum materials and the aspects of these materials for analysis in two major steps. The following sections present these steps, detailing the choice of sets of materials (Section 4.3.1) and the decision to focus on teacher guides within these materials (Section 4.3.2).

4.3.1. Initial two steps: Sampling the sets of the curriculum materials

The curriculum materials for this research were selected purposively in two steps. In the first step, the materials, which were written in English and included multiplicative reasoning at the secondary school level as well as aspects of formative assessment, were identified. Another initial criterion was to choose materials that incorporate supplementary resources that can have educative potential, such as teacher guides. In this step, I identified the materials that I came across on mathematics teachers' social media networks and that I heard from teachers during my informal school visits in my first year of the PhD, as well as during the interviews with teachers as a part of the previous research design. Moreover, I identified the curriculum materials that I found in the research papers. To exemplify, while I heard Diagnostic Questions and White Rose Maths (WRM) from teachers, I found Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) and Connected Mathematics Project (CMP) in the previous literature. Furthermore, I conducted an online investigation and discovered additional resources. An illustration of such materials is the Mathematics Formative Assessment System (MFAS) originating from the US. Following the initial phase of the material search, I have identified a total of 12 materials: three from the US and nine from England, as indicated in Table 4.2.

Table 4.2 The curriculum materials identified in the first step of sampling

Country	Curriculum materials
United States	Connected Mathematics Project (CMP)
	Mathematics Formative Assessment System (MFAS)
	Mathematics Assessment Project (MAP)
England	Corner Stone Maths (CSM)
	Diagnostic Questions
	Dr Frost Maths
	Formative Assessment in Science and Mathematics Education (FaSMED)
	Graded Assessment in Mathematics (GAIM)
	Hegarty Maths
	Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS)
	Standard Box
	White Rose Maths (WRM)

Note. Created by the author

When sampling the set of materials, the purpose was to maximise the variety in the data, which enabled access to a variety of examples of educative features and recommended formative assessment-related practices. With this purpose in mind, in the second step of sampling, these 12 materials were examined more closely in order to identify similarities and differences among them at the surface.

The examination yielded three evident categories within the materials. First, these materials could be classified based on the designers' profile and the intended purpose of the design. In essence, some materials were thoroughly designed by researchers specifically for research purposes, such as ICCAMS, while others were developed by practitioners with the primary aim of supporting mainstream teachers, exemplified by resources like WRM.

Second, materials were categorised by the educational context of which they were part. Here, the prominent difference among the materials in this research is the country: the US versus England. While CMP, MAP, and MFAS are from the US, ICCAMS, WRM, and CSM are from England. Among these materials, MAP has a specific situation. These materials are upgraded versions of the Standards Unit that was developed by mathematics education researchers in England. In the development of MAP resources, researchers from England and the US collaborated. The main difference in the educational context of these two countries is the mathematics curriculum they follow. While England-based materials depend on the National Curriculum, US-based materials depend on the Common Core Standards.

Finally, the third category is the way of including formative assessment. Three features were observed in this category: (1) materials include complete lessons that are particularly developed for formative assessment; 2) materials include lessons that acknowledge formative assessment as part of pedagogy; and 3) formative assessment tools that can be used as supplementary tools in the lessons. When categorising the materials in terms of these criteria, I considered how designers targeted the use of formative assessment in these materials. In this research, I only included materials encompassing complete lessons. My choice is based on Wiliam and Thompson's (2007) five formative assessment strategies that I employed for this study, as introduced in Section 2.1.2. That is, these strategies can be observed in complete lessons more effectively. At the end of this second step of sampling, I identified five sets of materials to be used in this research, namely, CMP, CSM, ICCAMS, MAP and WRM (Appendix C presents how I accessed these materials). I excluded other materials that were only for teachers' supplementary use rather than including complete lessons, such as MFAS.

4.3.2. Third step: Sampling the teacher guides

After choosing five sets of curriculum materials for this research, I chose the aspects that are relevant to the research context within each set of materials in the third step of sampling. I first identified the lessons that involved topics relevant to multiplicative reasoning. Geometric similarity and ratio are the common topics that are involved in almost all sets of materials. ICCAMS and CSM are slightly different from other materials in terms of including topics relevant to multiplicative reasoning. On the one hand, since the particular focus of the CSM is teaching geometry at the secondary level, the materials of this project only include geometric similarity that is relevant to multiplicative reasoning. On the other hand, teaching multiplicative reasoning is one of two core mathematical areas – the other is algebra – in the ICCAMS project. As a result, this project included more detailed lessons on teaching multiplicative reasoning, such as models of multiplication and rational numbers (Table 4.3).

Second, I identified the educative tools present within the materials. In my research, these tools are considered instruments used to convey designers' messages to teachers, either explicitly or implicitly. As educative tools, all the materials included written teacher guides in various forms for each lesson. In addition to these teacher guides some materials included additional videos, exemplar student work, power points to be used in the classroom, and written professional development documents (Table 4.3).

In the initial stages, I familiarised myself with various elements of the materials that hold educative potential to facilitate a profound engagement with each set. However, later, teacher guides were determined to serve as the primary data source. This decision is grounded in the fact that these teacher guides often provide comprehensive references to other educative elements within the materials. Moreover, while teacher guides across different materials exhibited similarities, other educative elements were in diverse forms. This variability poses a potential challenge to the quality of the analysis, as the presence of such diverse formats may result in inconsistency and hinder a thorough examination. However, although these additional resources were not involved in the sample, they were examined and included in the analysis in cases they help better interpreting the pedagogical message underlying chosen recommended practice and the educative potential to convey this message.

Table 4.3 Overview of the curriculum materials chosen

The name of the set of materials	Country	Number of the lessons chosen	The topics of the lessons chosen	Educative tools
CMP	US	8 lessons	Geometric similarity Ratio and Proportion	Teacher training videos Sample student works Lesson plans
CSM	England	12 lessons (some of them are shorter than others)	Geometric similarity	A short document that gives an overview of the unit Lesson plans including teacher guides Exemplar student work 3 minutes video that introduces the unit A guide for planning the lessons
ICCAMS	England	12 lessons	Models of multiplication Multiplicative structure Scale factor Geometric similarity Fractions Rational numbers Multiplicative relations Percentages	Lesson plans including teacher guides and assessment tasks A plan for 9 professional development sessions to be enacted in two years

MAP	England	4 full lessons	Using proportional reasoning (Grade 6) Classifying proportion and non-proportion situations (Grade 7) Comparing strategies for proportion problems Identifying similar triangles (Grade 8)	Lesson plans including teacher guides Self-PD resources that include guide, teacher tasks and short classroom videos
WRM	England	18 short lessons	Year 8-Proportional reasoning (6 weeks unit-10 lessons) Year 9-Reasoning with proportion (6 weeks unit-8 lessons)	Teacher guides for each lesson 3 pages teacher guide for lower attainers PowerPoint slides include exemplar questions to be used in the classroom Assessment tasks

Note. Created by the author

4.4. Introducing the cases

In Section 4.3, I outlined the sampling strategies and the sample identified for this thesis. I will now present brief contextual information about the five selected sets of curriculum materials. This will facilitate an understanding of these materials within their respective contexts. The descriptions in this section also justify the selection of these materials for the thesis, meeting the criteria through their well-designed nature, integrating formative assessment elements – the key pedagogical practice under examination in this thesis – and inclusion of comprehensive teacher guides for complete lesson plans, which justifies their educative potential.

4.4.1. Connected Mathematics Project (CMP)

The curriculum materials developed for the Connected Mathematics Project (CMP) are recognised as well-designed because the members of the project team include experienced researchers and practitioners, and these materials have been tested and improved over 30 years. That is, this project has been led by experienced researchers at Michigan State University in the US, collaborating with practitioners. The materials in this project have been updated since it was launched in 1991. Since it was launched, three editions of the materials have been developed: CMP1 (1991–1996), CMP2 (2000–2006), and CMP3 (2010–2014). The fourth edition, CMP4, is expected to be ready for implementation in schools starting in the 2024–2025 academic year. The version used in this thesis is the third edition of it – known as CMP3 and was published in 2018 – which justifies its potential high quality.

The main aim of the CMP project is to promote both students' and teachers' learning in terms of mathematical knowledge, understanding, and skills, as well as connections within mathematics and other disciplines. The key pedagogical approach in this project is identified as inquiry-based teaching and learning. The curriculum is grounded in the Common Core State Standards for Mathematics (CCSSM) in the US.

This project also meets the criteria of including the elements of formative assessment. As introduced on the project website (Connected Mathematics

Project, n.d.), in this project, the formative assessment approach was based on Wiliam and Thompson's (2007) five formative assessment strategies. In line with Wiliam and Thompson's framework, in this project, formative assessment is considered integrated into teaching and learning. Thus, it involves anticipating students' thinking, understanding students' current thinking, and adapting the practice based on students' thinking. Instead of a sequential process, these three aspects are considered to be integrated into any phase of the lesson.

The CMP3 materials demonstrate educative elements that can potentially communicate key principles to teachers. Importantly, the website for the project provides rich teacher training resources, including videos from previously held lessons. The videos aim to teach aspects of teaching and exemplar student responses.

In this thesis, the teacher guides for two units were chosen to be analysed. These units are entitled "Stretching and Shrinking" and "Comparing and Scaling", and include the topics of understanding similarity and ratios, rates, and proportions, respectively. For each unit, teachers are provided with physical books that include an introduction to the unit, lesson plans, lesson overview pages, and keys for the tasks. The Comparing and Scaling unit involves 270 pages, and the Stretching and Shrinking unit involves 324 pages. Other than these physical books, teachers are provided with a set of digital resources that include student worksheets and videos to be used in the lesson.

4.4.2. Corner Stone Maths (CSM)

Similar to CMP, CSM materials are considered well-designed as they are informed by research. This project was led by researchers at UCL in England with a focus on the use of technology in teaching geometry in early secondary mathematics.

Formative assessment is not the key focus of this project. However, designers articulate one of the functions of these lessons as providing an opportunity to assess students' understanding of mathematical concepts. Specifically, for the unit "Geometric Similarity", one aim of the designers is to facilitate accurately assessing students' understanding of scale factors, relations between similar shapes, and the relationship between corresponding angles in similar shapes (University College London, n.d.).

This set of materials includes a variety of educative tools. First, they include an overview of the units, which includes suggested time, key mathematical ideas, and key technology experiences for each lesson. Second, they provide a teacher guide for each unit. For example, for the similarity unit, a 118-page teacher guide is provided. This teacher guide compiles an overview of each lesson and the mathematical goals of the lessons as a whole. It outlines the aspects of the National Curriculum that are addressed and provides overall suggestions for implementing the lessons. Additionally, it includes student editions for each lesson, featuring two pages of brief notes for teachers before each lesson, and concludes with a two-page guide for using the software. Thirdly, the set of materials encompasses PowerPoint slides designed for use in the lessons. Fourth, the set of materials provides a set of resources for school-based professional development. These materials can be useful for both professional development as a group and teachers' individual use. More specifically, this set includes a short, three-minute introductory video, tasks to be used in PD sessions, a guide to support teachers' lesson planning, a template for a lesson plan, and exemplar student work.

The overall materials include the topics of linear functions, geometric similarity, and algebraic patterns and expressions. In my research, the geometric similarity module, with 12 lessons, has been chosen to be analysed because it is a sub-topic of multiplicative reasoning.

4.4.3. Increasing Competence and Confidence in Algebra Structures (ICCAMS)

ICCAMS materials have been considered as well-designed as they were developed by experienced researchers in England. Namely, this project was conducted in collaboration of researchers in The University of Nottingham, Durham University and the University of Manchester and funded by the Economic and Social Research Council (2008–2012) and the Education Endowment Foundation in England (2015-2018). Similar to the CMP, the materials of this project were developed and tested for a decade.

The main purpose of this project is to enhance students' learning of mathematics by using formative assessment as the key pedagogical practice, which justifies its direct relevance to this thesis. For this purpose, the researchers developed and tested a set of lessons for productive formative assessment practices in the context of multiplicative reasoning and algebra. The framework to design the lessons includes the aspects of formative assessment, collaborative working, realistic contexts, connectionist approach, multiple representations and models (Increasing Competence and Confidence in Algebra and Multiplicative Structures, n.d.).

Although the project involves PD sessions for teachers, the main educative tools of the materials are the guides embedded in the lesson plans. Namely, ICCAMS include 12 sets of lessons for multiplicative reasoning and 12 sets for algebra. Teacher guides for these lessons start with a pre-assessment test that includes both multiplicative reasoning and algebra questions. Then, it includes mini-assessment tasks before a group of lessons, aiming to assess students' prior learning. Each separate lesson guides include six pages and each page include the following: (1) the main task and the rationale for the lesson; (2) an overview of mathematical ideas, the mathematical experiences students will potentially have, exemplar questions to be asked to students, suggestions for assessment and feedback, and suggestions to adapt the lesson to different difficulties; (3) one-page lesson outline; (4) one-page background for maths; (5) additional resources (e.g., various diagrams for arithmetic operations; and, (6) one blank page for notes. In this thesis, 12 multiplicative reasoning lessons were chosen to be analysed.

4.4.4. Mathematics Assessment Project (MAP)

MAP is another research-informed well-designed project. It was funded by Bill and Melinda Gates Foundation and conducted by researchers at the University of California at Berkeley (US) and The University of Nottingham (England) between 2007 and 2015. The project team includes well-acknowledged researchers in mathematics education worldwide, such as Alan Schoenfeld, Hugh Burkhardt and Malcolm Swan.

The main purpose of the project is to provide teachers with supporting resources to enact qualified formative and summative assessments. In order to support formative assessment, two types of lessons are provided. These lessons are concept development lessons and problem-solving lessons. On the one hand, concept development lessons aim to identify students' existing knowledge, to develop students' understanding of the concepts building on this knowledge and help students make connections between these concepts and other mathematical knowledge. On the other hand, problem-solving lessons aim to assess and develop students' skills to apply their knowledge and reasoning to solve problems.

This set of materials includes a variety of educative tools. First, it includes a separate 16 pages guide including an overview of formative assessment lessons, the rationale for these lessons, a brief guideline to use these lessons and design principles underlying these lessons. Second, it includes five sets of professional development modules. These modules present general information for formative assessment, a guideline to use concept development and problem-solving lessons, the role of questioning to improve learning and students' collaborative work. Third, teacher guides are provided for each lesson.

Three concept development and one problem-solving lesson are chosen to be analysed in this research. Concept development lessons are "Using proportional reasoning" from Grade 6, "Classifying proportion and non-proportion situations" and "Comparing strategies for proportion problems" from Grade 7. The problem-solving lesson is "Drawing to scale: A garden".

4.4.5. White Rose Maths (WRM)

White Rose Maths is a commercial teaching resource for teachers in England. Unlike the other four sets of curriculum materials, these materials are designed and developed in collaboration with Trinity MAT, an educational trust in England. The curriculum materials of WRM are commonly used by mathematics teachers in England.

Formative assessment is not a direct focus of these materials. However, the materials include resources that can support teachers' classroom assessment practices. Teachers are provided with Scheme of Learning (SoL) documents.

These documents include both learning objectives and a suggested timeline to teach these learning objectives. The teacher guides emphasise a “small step” approach as a key pedagogy. That is, instead of teaching multiple concepts simultaneously, the guides recommend focusing on teaching a single concept, dividing each main part into 10, 8, and 8 steps, respectively. Teachers are advised to allocate time for each step based on their students’ learning needs. Moreover, in the SoL documents, guidelines for teachers and exemplar questions to be used in the lessons are provided.

In this thesis, two parts of these SoL documents were chosen to be analysed, which included topics relevant to multiplicative reasoning.

4.5. Methods employed during the explorative phase

The explorative phase of this research aimed to identify examples of formative assessment-related practices and educative features. The primary data analysis strategy employed during this phase was reflexive thematic analysis, following the six steps outlined by Braun and Clarke (2006) in their seminal paper: familiarisation, initial coding, searching for themes, reviewing themes, defining and naming themes, and producing the report. Wiliam and Thompson's five formative assessment strategies (2007) and Quebec-Fuentes and Ma’s (2018) two educative features (i.e., rationale and guide) served as theoretical guides in this analysis. However, the findings were limited for the third formative assessment strategy (i.e., providing feedback that moves learning forward). This limitation prompted a follow-up analysis focused specifically on feedback, guided by a review of the feedback literature, presented in Section 2.2.3. The subsequent sub-sections provide detailed insights into the methodological strategies employed to ensure a rigorous analysis in this explorative phase.

4.5.1. Rationale for conducting reflexive thematic analysis

In the process of determining the appropriate analytical approach during the explorative phase, I considered several alternatives. However, I ultimately decided to conduct a reflexive thematic analysis which has been adopted by a large number of researchers following Braun and Clarke’s introduction of this specific type of thematic analysis in their seminal paper that was published in

2006. This approach suits this research for two reasons. First, the six steps of conducting the reflexive thematic analysis provide a practical structure to progress the analysis of the data while allowing flexibility in choosing theoretical approaches (Braun & Clarke, 2021). Second, in this data analysis, reflexivity is the key feature, which is considered the way of ensuring rigour where the researchers' subjectivity is centred. Braun and Clarke's detailed guidance enabled me to critique and change my earlier habits that were influenced by positivist paradigms, whenever necessary. As examples provided in Section 4.2, reflexive journals were used as a technique to increase my awareness of my subjectivity which potentially influenced my decisions throughout the research. Moreover, I used strategies such as peer debriefing and thick descriptions as alternatives to commonly used strategies to ensure the quality of research as detailed in Section 4.2.

Before making this decision, I discarded certain analytic approaches for two primary reasons. First, the compatibility of these methods with the overarching purpose of this research phase was insufficient. Specifically, these approaches did not align well with the objectives and scope of this study. Second, the procedural requirements of these methods presented significant challenges. The complexity and rigidity of the required procedures made it impractical to adhere to them strictly within the constraints of this research. The following paragraphs respectively elaborate on these two reasons.

The previous research on analysing curriculum materials commonly followed a structured deductive approach in which a theoretical or conceptual framework is identified before the data analysis by drawing on the existing literature and the materials were analysed according to those frameworks (e.g., Beyer et al., 2009; Quebec-Fuentes & Ma, 2018). Although following those structured frameworks can help to conduct the analysis systematically, my judgement was that these approaches would risk limiting the identification of new insights into the data, which would not suit the explorative purpose of the first phase of this research. Moreover, these commonly used approaches have been informed by the post-positivist paradigm which considers researchers' subjectivity as a threat against the rigour of the research as opposed to my approach that considers researchers' subjectivity as an opportunity for an insightful analysis (Ball, 1990).

After discarding this option, two other common qualitative data analysis traditions, grounded theory and content analysis were considered. Grounded theory was discarded in the first place as it would not be practical to conduct a complete grounded theory within the data and time restrictions of this research. The second option that was discarded was content analysis, which was considered more seriously and given a try. In particular, an evaluative content analysis was considered (Kuckartz, 2014). According to this approach, two evaluative categories were determined in the beginning and then it was aimed to code the data based on two evaluative categories. The evaluative categories in this research were formative assessment strategies and educative features of curriculum materials. This approach was not adopted for two reasons. First, this approach has not allowed much space to describe and explore the data before starting the analysis (Kuckartz, 2014). However, since the five materials analysed in this research have different characteristics in terms of involving assessment and the places they spoke to teachers, it was important to be truly familiarised with the data initially (Braun & Clarke, 2006). Although I have been familiarised with the materials during the sampling process, this familiarisation was limited to making general comparisons among the surface characteristics of the materials in terms of identifying their developers, their approach to formative assessment and the national curriculum they referred to. Second, the strategy of identifying evaluative categories did not work well. When coding the data, some codes were linked to both evaluative categories. For example, tasks could be relevant to either formative strategies or educative features. This overlap made it difficult to use these evaluative categories accurately. Moreover, although both categories have strong foundations in the literature, formative assessment strategies are more established than educative features. More specifically, while it was not expected that five formative assessment strategies (Wiliam & Thompson, 2007) would require significant modification at the end of this research, there would be a need to explore educative features further.

4.5.2. Familiarising with the materials

In accordance with the guidelines delineated by Braun and Clarke (2022), the initial stage of familiarisation requires the incorporation of three specific procedural components. The first component, immersion, requires the analyst to profoundly understand the data. In my doctoral research, this component was especially crucial compared to the research practices involving the acquisition of interview or observational data by the analyst. Essentially, while in those cases analysts can start getting to be familiarised with the data while they collect it, in my study, this familiarisation process was initiated only when the data analysis started. Hence, due to the distinctive nature of my study, a more extended duration was essential for me to thoroughly familiarise myself with the data during the analysis phase.

This immersion process started when sampling the curriculum materials, as elaborated on in Section 4.3. During this process, I familiarised myself with the general structure of the data in terms of how they approached teaching multiplicative reasoning and formative assessment as well as the sections those aiming to communicate to teachers. In order to go beyond this earlier and on surface familiarisation process, I chose lessons that focus on the same specific topic from each material, which is involved in each of the set of materials and explored further how these lessons communicate to teachers in relation to formative assessment practices.

In the second practice of familiarisation, the analyst should critically be engaged with the data. This critical engagement was supported by asking questions such as, “What are the differences among the materials in terms of the approach to formative assessment and delivering this approach to teachers?”, “To what extent does my interpretation of the data rely on my theoretical knowledge, my interactions with teachers in England and my own teaching experiences?”, “What can be the different ways to make sense of these materials?”. Initial coding of the selected lessons helped me to address these questions.

The third practice of familiarisation is taking written notes throughout the familiarisation process. I kept reflexive diaries while engaging with the data and wrote a structured familiarisation section that included my initial observations about the materials. The reflexive diary that supported my familiarisation with the data and completing the explorative phase I kept between 6th December 2021 and 29th July 2022, which involved 28,277 words in total. Moreover, I printed down the lessons and jot down notes on these documents to be able to engage with the data deeply.

The familiarisation process enabled me to deeply engage with the curriculum materials by identifying key features of each set, including their layout, primary pedagogical approaches, and elements relevant to five formative assessment strategies (Wiliam & Thompson, 2007) and educative features (Quebec-Fuentes & Ma, 2018). This process also shaped the rest of the research, particularly guiding the focus of the data through a series of analyses undertaken in both explorative and critical phases.

4.5.3. Coding

Coding is one of the essential elements of a credible qualitative data analysis which brings both insight and rigour to the data analysis process (Braun & Clarke, 2022). In order to ensure the quality of coding, a number of decisions should be made. The following sections elaborate on the decisions made in each stage of coding.

Credibility and reliability of codes

In order to ensure the rigorous analysis, the credibility of the codes was considered. The credibility was largely ensured by discussing codes with another PhD student who is an experienced mathematics teacher and studying for a PhD with a focus on the formative assessment in the mathematics context. Moreover, I revisited the codes several times by following the process of refining codes as Braun and Clarke (2022) suggested. Occasionally, when I needed an overall comparison, namely in the analysis of teacher guides and educative features, in order to ensure the reliability of the codes, I recoded the teacher guides after distancing myself from the data for a couple of days.

Initial coding decisions: deductive versus inductive and latent versus semantic

In the initial explorative stages of the analysis of teacher guides, the first decision I made was to employ either inductive or theoretical coding. While in an inductive approach, data is coded without locating the data in a pre-determined framework or researchers' analytic preconceptions, in a deductive approach, the data is coded according to the researchers' theoretical or analytic interest (Braun & Clarke, 2006).

Theoretical coding was chosen to identify fine-grained codes in relation to formative assessment and educative features. For formative assessment, the five formative assessment strategies (Wiliam & Thompson, 2007), already presented in Section 2.1.2, served as a guiding framework. This framework was chosen as it is a well-recognised framework that promises comprehensive consideration of multiple aspects of formative assessment. Example codes that were identified in relation to this framework are “eliciting students’ conceptual understanding”, “questioning”, and “informing students about learning objectives”.

To code for educative features, I initially sensitised myself to the literature on educative curriculum materials. As previously mentioned in Section 2.3.2, in the existing literature, “providing rationale”, “guiding”, and “providing knowledge” have been widely accepted as the grounding of educative features (e.g., Ball & Cohen, 1996; Davis et al., 2017; Quebec-Fuentes & Ma, 2018). In this doctoral research, these features were used to identify fine-grained codes regarding educative features such as “background subject knowledge”, “general pedagogy”, “rationale for an assessment task”, “guidance for implementation of group work” and “guidance for differentiation”.

The second decision was whether to employ semantic coding, which focuses on explicit meaning, or latent coding, which focuses on implicit meaning. I integrated both coding strategies into my analysis. The process of refining codes can be delineated into three parts when utilising these coding strategies. In the initial stages, I began by employing latent coding, utilising theoretical codes such as "pedagogical content knowledge" to exemplify formative assessment procedures, and "providing rationale" to exemplify educative features. During the intermediate stages, I transitioned to employing semantic codes such as "knowledge of students’ misconception" and "rationale for questioning". This transition allowed for a detailed and precise characterisation of the teacher guides. In subsequent stages, this transitioned to latent coding again, specifically referred to as "alerting" and "equipping". Instead of using abstract phrases from the literature, using latent codes I developed drove the educative features proposed (these features will be elaborated on in Section 5.1.3).

4.5.4. Developing themes

In this section, I will provide the process of initial coding, refining coding and developing themes in detail. This iterative process significantly influenced the development of educative features I used as analytical terms in the following steps of the analysis. As a result, I decided to include details of the theme development process in this section.

Stage 1: Coding to explore the initial themes

When familiarising with the curriculum materials, I examined the general structure of the materials and identified the places these materials communicate to teachers. When I moved to the coding phase in December 2021, I started with open coding by using the latest version of the software Atlas.ti. Also, I started to keep a regular reflexive research diary that mainly includes the ideas I come up with and the challenges I face during coding. In this coding phase, I interrogated any instance that has the potential to provide formative assessment techniques or educative features. This process helped me to move one step further from familiarisation and identify fine-grained features in the dataset. In that stage, I created a manual codebook in order to record the codes and labels for these codes. Some examples of code labels in this codebook are "classroom task", "differentiation", "example question", "instruction to teacher", "lesson objectives", "pre-assessment" and "student difficulty".

This coding strategy in the first stage helped me be more engaged with the curriculum materials because I focused on particular instances after familiarising myself with these materials' overall structure and purpose. However, the strategy of distinguishing whether a data extract is a formative assessment technique, or an educative feature was not a productive way of coding. Also, I had labelled the codes with broad labels, which did not help me identify various features in the materials. In order to overcome these limitations of the initial explorative coding process, I decided to change the coding strategy. The following section presents this change.

Stage 2: Refining codes and developing themes

In the second stage, the key shift in the data coding strategy was to search for the data segments that include a particular educative feature and label this segment to give clues about that educative feature. As a result of the observations during familiarisation and explorative coding process, initially, I divided the data extracts into four general groups: classroom material (CM), teacher instruction (TI), teacher knowledge (TK) and learning objectives (LO). The plan was to start labelling the codes with abbreviations of one of these groups and then give detail about the extract, such as TI-group work. When coding in that stage, the group of teacher knowledge did not contribute to the data coding. More explicitly, when teacher knowledge was distinguished from other educative features, this led me to consider only the background information provided in the materials as a part of teacher knowledge. As a result, instead of considering teacher knowledge as a separate category, I decided to approach teacher knowledge as integrated with the educative features. Later, I consider the data extracts to be involved in three categories (classroom materials, learning objectives, and teacher instruction). I coded the extracts that did not fit one of these categories with a label that started with "other". In order to decide the accompanying relatively fine-grained label to these groups, three questions were answered: 1) Does this data extract fit one of the groups I identified earlier: teacher instruction, learning objective, or classroom materials? 2) How can this data extract serve my data analysis? 3) What educative function does this data extract serve?

Answering these questions served as a convenient coding process for most data extracts. Figure 4.1. shows one of these data extracts. This data extract was coded with the label of "TI-pre-assessment-informing-student thinking". "TI-pre-assessment" provides information about the location of the data extract. This data extract was located in the teacher instructions for a pre-assessment task. "Informing" means that the educative function of this data extract is to inform teachers. The last part of the label "student thinking" presents that this data extract informs teachers about student thinking.

Figure 4.1 A data extract from ICCAMS with the code label "TI-pre-assessment-informing-student thinking"

When Jay has walked 10 m, Kim has walked 12 m.

When Jay has walked 25 m, Kim has walked ....

Data from the *Mini Ratio Test* suggests that students are likely to answer this task in a variety of ways.

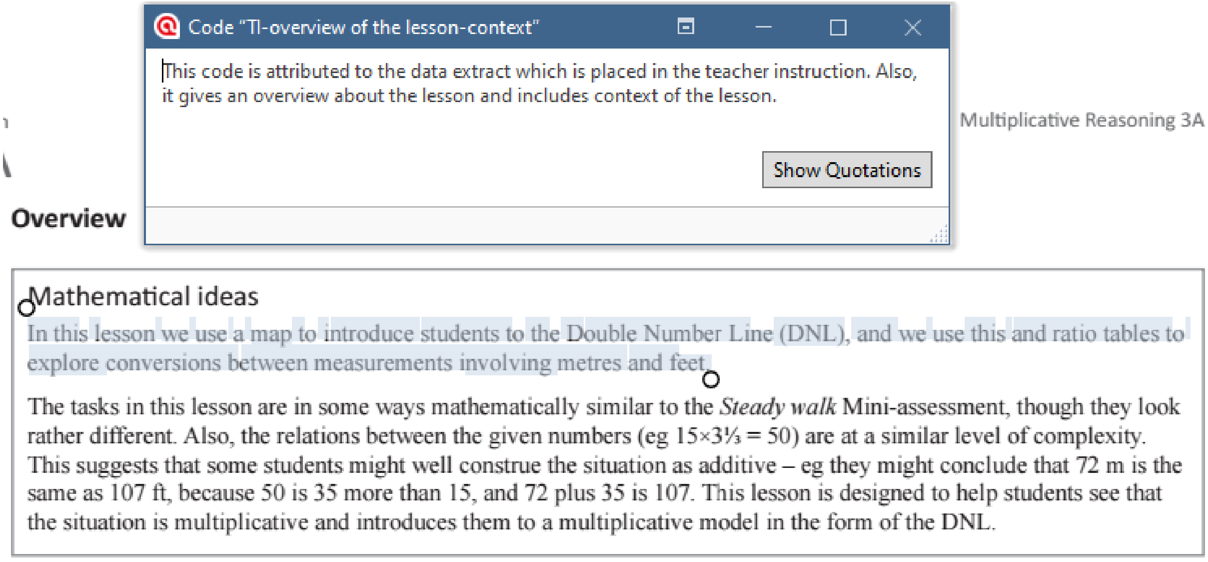
Some might say that Kim has walked 27 m, using an additive argument of the sort, "Kim is always 2 m ahead of Jay", or "Jay has walked another 15 m, so Kim will have walked another 15 m, making 27 m in all".

Some might give the correct distance, 30 m, on the basis that Kim walks 1.2 times as far as Jay, or because Jay's distance has increased by a factor ×2.5, or by using some form of rated addition (10 + 10 + half of 10 = 25, 12 + 12 + half of 12 = 30), or scaling ("When Jay has walked 5 m, Kim has walked 6 m, 5×5=25, 6×5 = 30").

Note. Reprinted from ICCAMS, Steady walk mini-assessment, p. 89

However, this strategy did not work in all instances. Figure 4.2. shows one of these instances. That data extract is labelled as "TI-overview of the lesson-context", which means that that data extract was found in the group of teacher instruction. It gives an overview of a particular lesson and provides the context of the lesson. The highlighted section in Figure 4.2 shows the data extract to which this code is attributed and the box above this extract explains this code. Reminding the three questions I asked when coding the data extract (location, function for the analysis and educative function), I could not answer how this data extract will serve the analysis and I did not add a clue about its function to the code label. I left refining these codes to the later stages of the data analysis. In the later stage, these codes did not serve to identify themes. However, when analysing the data extracts related to the developed themes, I used these sections in the materials. When an educative feature was being analysed, the complete lesson plan was checked for evidence of this feature. That is to say, although these codes did not serve theme development, the extracts they are attributed to were used in the data analysis.

Figure 4.2 A data extract from ICCAMS with the code label TI-overview of the lesson-context

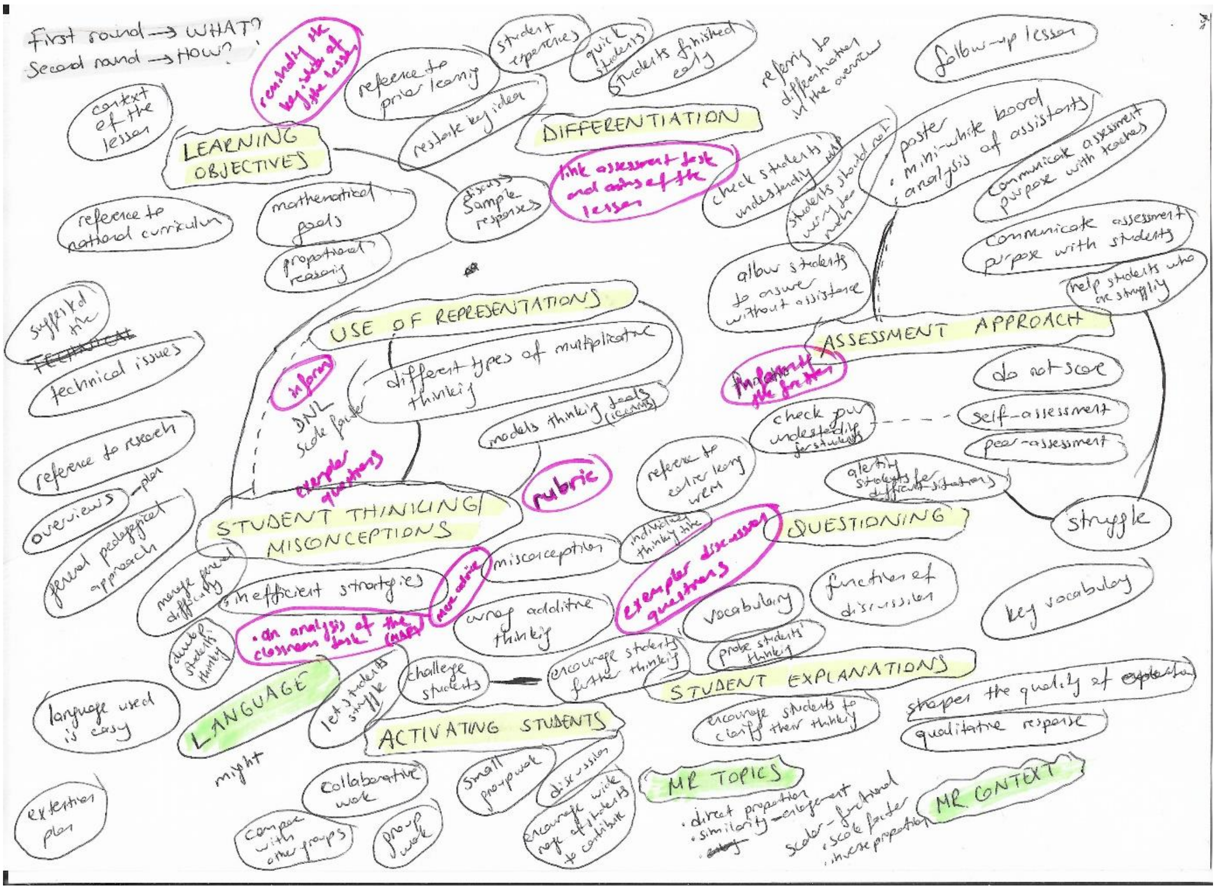


Note. Created by the author

Another strategy I used to refine the codes was to group the documents to be coded in a row by selecting documents from each different project instead of coding the documents in the same project in a row. When coding the first group of documents, I checked the consistency of the codes and code labels across the projects. When I moved to coding the second group of documents, I checked the consistency of the codes and code labels in a document in the same project that was already coded in the previous group. I continued this process until the coding did not provide new insights. The list of the aspects of the teacher guides coded for the explorative thematic analysis is presented in Appendix D.

In order to move from coding to the data analysis process, I revisited codes to identify candidate themes. Initially, I came up with eight candidate themes regarding educative features: learning objectives, differentiation, use of representations, assessment approach, student thinking/misconceptions, questioning, student explanations and activating students. Also, language, multiplicative reasoning topics and multiplicative reasoning context were the themes that emerged from the data. In that stage, in order to consider the overall picture and relations between the candidate themes, I created a thematic map (see Figure 4.3). On this map, the themes in yellow are directly relevant to educative features and the themes in green are other emerging themes. Other small writings are more specific features that emerged from the codes. Some of them are direct quotes from the data extract (e.g., “allow students to answer without assistance” and “do not score”). The pink writings present more explicit educative features that come from the codes. On the right top of the map, I remind that while the first iteration of the coding focuses on what educative support exists in the curriculum materials.

Figure 4.3 Initial thematic map



Note. Created by the author

In order to develop themes, I considered how all the candidate themes can tell a story of the existing educative features in the materials for productive formative assessment practices. The following questions suggested by Braun and Clarke (2022) are used in this process in the context of this research. “Does this provisional theme capture something meaningful? Is it coherent, with a central idea that meshes the data and codes together? Does it have clear boundaries?”

When answering these questions, although all the candidate themes were considered as capturing something meaningful and coherent with a central idea, it was a challenge to define clear boundaries between some of these themes. More explicitly, students’ misconceptions, use of representations, questioning, language and student explanations were closely related. I developed two themes by using these five themes, which have boundaries. These themes are students' difficulties and misconceptions and students' participation. I decided to consider students' difficulties and misconceptions separately because there are plenty of instances in the materials. Although the main purpose of this analysis was to elicit educative features instead of identifying the most recurring instances, these instances provided the potential to access rich educative features. I began the data analysis with a theme relevant to students' difficulties and misconceptions, and this theme evolved during the analysis. A close look at data extracts resulted in the sense that these extracts could be evidence for support for teachers to notice students' difficulties and misconceptions and use these difficulties and misconceptions as tools to further students' learning. As a result, this theme was named as noticing students' difficulties and misconceptions. In that stage, other themes were learning goals, differentiation and assessment norms. In Table 4.3, I demonstrated the candidate themes that are related to these themes as well as the formative assessment strategies relevant to these themes.

Table 4.4 The relation of the themes with candidate themes and formative assessment strategies

Initial Themes	Related candidate themes	Related formative assessment strategy (Wiliam and Thompson, 2007, p. 63)
Learning goals	Learning objectives Language Multiplicative reasoning topics Multiplicative reasoning context	Clarifying and sharing learning intentions
Noticing students' difficulties and misconceptions	Use of representations Student thinking /misconceptions Student explanations Questioning	Engineering effective classroom discussions and tasks that elicit evidence of learning Providing feedback that moves learners forward
Students' participation	Student explanations Activating students Questioning Language	Activating students as instructional resources for one another Activating students as the owners of their own learning Providing feedback that moves learners forward
Differentiation	Differentiation Use of representations	Regulating learning

Assessment norms	Assessment approach	Clarifying and sharing criteria for success

Note. Created by the author

These five themes enabled me to discern the overview of the educative features that are involved in the materials analysed. In the next step of the analysis, all these themes were reconsidered by examining the function of the attributed data extracts within the whole lesson they are involved in. As a result, it was revealed that there were no boundaries between some of these five themes. More importantly, the theme “assessment norms” overlapped with other themes, which created the need to refine the themes. Finally, I decided to use the theme regarding assessment norms as the overarching theme of the thematic analysis and focus mainly on how the curriculum materials can challenge teachers’ traditional classroom practices in order to afford productive formative assessment.

Following this iterative process, an overarching theme was developed: the teacher guides can challenge teachers’ conventional and potentially ineffective formative assessment practices. After completing the follow-up supplementary feedback analysis, themes were refined in the final stages of the explorative phase. The following section elaborates on the process of this supplementary analysis. The finalised versions of themes will be presented in Chapter 5.

4.5.5. Complementary targeted analysis for feedback

As a result of the initial reflexive thematic analysis, it was found that the references for feedback were limited. As feedback is one of the core elements of formative assessment, I conducted a complementary analysis targeted at feedback. Thus, I analysed the teacher guides to identify recommended practices that can be associated with feedback situations. Initially, in order to take a nuanced approach to identifying implicit references for feedback, the literature was reviewed in order to identify a conceptualisation of feedback to be used in the analysis, as presented in Section 2.2.3. (Appendix A includes further reflections on this literature review process). Depending on this review, in this analysis, feedback was considered as a situation in which the ultimate goal is to encourage students to reflect on their learning towards specific learning intentions, which can involve interaction with peers and employing diverse strategies and techniques based on the intended disciplinary or pedagogical goals. Three components of feedback have been suggested to enable productive practices: learning aims, sources and types of feedback. Learning aims were acknowledged as five strands of mathematical proficiency; sources of feedback were considered as teacher, peers and students; the types of feedback were considered as immediate corrective, immediate elaborated, delayed corrective and delayed elaborated. These components of feedback guided coding the teacher guides for feedback analysis.

The software Atlas.ti was used for coding with the same sample in reflexive thematic analysis. One challenge in this analysis was that the coded extracts might not provide insightful meaning when they are isolated from their context. In order to handle this challenge, alongside the teacher guides, available professional development and general project introduction resources were examined in order to position the instances in the lessons better. Additionally, I followed the template presented in Figure 4.4 for the coded lessons to gain deeper insights into the dominant approach for feedback within a lesson or project.

Figure 4.4 A template for feedback analysis

Date of analysis		
Project		
Lesson		
Feedback aims		
Feedback sources		
Feedback types		
Feedback techniques		
Notes/Summary/Reflection		

Note. Created by the author

4.6. Methods employed during the critical phase

The findings of the explorative phase guided the decisions in the critical phase. After operationalising five formative assessment strategies and proposing three educative features (which will be elaborated on in Chapter 5), I applied these findings to critically analyse the educative potential of teacher guides for formative assessment. This phase dominantly involved deductive-oriented analyses which were informed by existing frameworks such as mathematical proficiency (Kilpatrick et al., 2001), noticing (Van Es & Sherin, 2021) and openness of mathematics tasks (Yeo, 2015). Moreover, the analyses in this phase were inspired by the "big Q" approach (Braun & Clarke, 2021) and constant comparison technique (Charmaz, 2014), in order to identify nuanced examples of formative assessment-related practices and educative features among the teacher guides.

4.6.1. Horizontal and vertical analysis

In the critical phase of the research, Charalambous et al.'s (2010) analytic approach of horizontal versus vertical analysis, as outlined in Section 3.5.2, was employed. In this analytic approach, horizontal analysis involves a broad examination of curriculum materials for their general characteristics, whereas vertical analysis concentrates on specific aspects, such as focusing on a single mathematics topic. This approach was adapted in this thesis to align with the unique characteristics of this study. That is, in order to examine the educative potential inherent in teacher guides, two distinct analyses were conducted. The first analysis was conducted to reveal a variety of recommended in-the-moment formative assessment practices among teacher guides with a focus on *noticing* and *feedback*. This analysis also involved an overview of presented learning intentions within teacher guides, as presented in Section 6.1. The analysis of the inclusion of learning intentions enabled interpretation of the findings in relation to the alignment between the intended aspects of learning and the educative support provided, to notice the indications of these learning intentions and feedback practices to enhance students' learning in line with these learning intentions. During the second analysis, the focus was shifted to the educative potential of teacher guides for specific pedagogical messages. This enabled a focus on examining how a set of educative features could effectively communicate these messages to teachers. The findings of the horizontal analysis will be presented in Chapter 6 and the findings of the vertical analysis will be presented in Chapter 7.

4.6.2. Coding for learning intentions

In Section 2.2.1, it was mentioned that five strands of mathematical proficiency were chosen as a guiding framework to identify mathematics-specific learning intentions in this thesis (Kilpatrick et al., 2001). These strands were chosen as pre-determined codes when coding for learning intentions. After familiarising myself with teacher guides, these five strands were operationalised for the analysis of these teacher guides. Initially, certain words that were typically used when communicating learning intentions were identified and these words served as initial cues for potential codes. After identifying these potential codes, the data extracts were examined more thoroughly by referring to the definitions of each strand of mathematical proficiency. The following table presents the definition of each mathematical proficiency and the indicator words that were used as cues.

Table 4.5 Coding guide for learning intentions

The strand of mathematical proficiency	Definition by Kilpatrick et al. (2001)	Indicator words
Conceptual understanding	"Conceptual understanding refers to an integrated and functional grasp of mathematical ideas." (p. 118)	Understand, recognise, identify and distinguish
Procedural fluency	"Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently." (p. 121)	Convert, draw, multiply, enlarge, calculate and precise
Strategic competence	"Strategic competence refers to the ability to formulate mathematical problems, represent them, and solve them." (p. 124)	Solve, represent, work out, model and strategy
Adaptive reasoning	"Adaptive reasoning refers to the capacity to think logically about the relationships among concepts and situations and to justify and ultimately prove the correctness of a mathematical procedure or assertion. Adaptive	Explore, interpret, explain reasoning and compare

	reasoning also includes reasoning based on pattern, analogy, or metaphor." (p. 170).	
Productive disposition	"Productive disposition refers to the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics." (p. 131).	Make sense, persevere, everyday life and reasonable

Note. Created by the author

4.6.3. Coding for educative features

The content and the questions that are specified in Table 4.6 guided the coding for educative features.

Table 4.6 Coding guide for educative features

Code	Content	Questions to ask
Equip	Information or tool	What knowledge do teachers require? What tools do teachers need?
Alert	Link to the purpose of the suggested pedagogical practice	Why would teachers practice this?
Guide	Guidance that can help teachers to make better use of tools and knowledge or to implement the practice better	How would teachers practice this?

Note. Created by the author

4.6.4. Task analysis

The mathematical tasks were considered as main sources in lessons that can shape interactions with students to notice their thinking. Thus, the analysis of tasks was a part of the analysis of the educative potential of teacher guides for noticing opportunities. In order to analyse tasks for this purpose, I used the framework suggested by Yeo (2015) that examines the openness of the tasks in terms of five aspects: goal, method, answer, complexity and extension, as well as the five strands of mathematical proficiency. In order to focus on the opportunities for students' responses other than other aspects, I examined the openness of the tasks for methods and answers. When the task requires an application of known procedures, it is a closed-method task; otherwise, it is an open-method task. When it is possible to find certain correct answers for a task, it is a closed-answer task; otherwise, it is an open answer task.

4.6.5. Identifying pedagogical messages

The idea of identifying pedagogical messages as the focus of analysis is the unique characteristic of the analysis conducted in this phase. Insights gained from the earlier attempts to analyse the educative potential of teacher guides suggested that while a set of materials' educative potential could be high for some specific messages, this level of this potential could change for other messages. Thus, for the final step of the analysis, six pedagogical messages were identified. These messages were informed by the horizontal analysis initially. Following that, a closer examination of the learning goals and recommended practices in chosen teacher guides helped finalising these messages. In Chapter 7, the details for the identification of each of the five pedagogical messages will be presented.

4.6.6. Constant comparison

In the critical phase, after identifying the pedagogical messages, the constant comparison technique was employed to reveal nuanced insights into the existence and quality of educative features. This analytical approach, borrowed from grounded theory methodology (Charmaz, 2014), involves a systematic and iterative process of comparing certain aspects of the data to identify patterns, relationships and variations. The constant comparison technique was particularly useful during the critical phase of analysis. In this phase, I particularly used this technique in order to reveal variety among the teacher guides. It allowed to reveal the nuanced educative potential of teacher guides.

More explicitly, rather than identifying patterns, this technique was used to detail two spectrums. The key problem focused on this research is mathematics teachers' potential need for formative assessment support for better in-the-moment practices. Through the research process for each three key elements of in-the-moment formative assessment practices two ends of a spectrum that present the expected potentially unhelpful practices and the ideal practices were identified. Constant comparison technique enabled the identification of the path between these two ends. The following table presents these spectrums for each aspect of the research.

Table 4.7 Two ends of the spectrum that guided the analysis for teachers' in-the-moment formative assessment practices

Construct	Two ends of the spectrum
Identifying learning intentions	From hierarchical-fragmented to relational learning intentions
Noticing	From merely eliciting the correctness of the answers to noticing students’ thinking
Feedback	From corrected immediate feedback to providing feedback that moves students’ learning forward

Note. Created by the author

CHAPTER 5 - THE FRAMEWORK TO ANALYSE THE EDUCATIVE POTENTIAL OF CURRICULUM MATERIALS FOR FORMATIVE ASSESSMENT

Addressing the first and second research questions of the research, one of the key outcomes of this doctoral research is the development of a comprehensive framework that can be employed to analyse the educative potential inherent to the curriculum materials which can facilitate teachers' effective formative assessment practices. Chapters 2 and 3 presented a synthesis of the existing theoretical approaches and the empirical research respectively that highlights the necessity for the proposed framework as well as significantly contributing to the evolution of this framework. Chapter 4, on the other hand, provided a detailed account of the analytical process involved when shaping this framework.

This chapter turns to elaborating on the insights from the review of the literature and the findings of the thematic analysis that fed the development of the framework providing example data extracts in Section 5.1. Subsequently, in Section 5.2, this framework will be situated within the broader landscape of existing literature by highlighting the new insights it offers.

5.1. Findings that informed the development of the framework

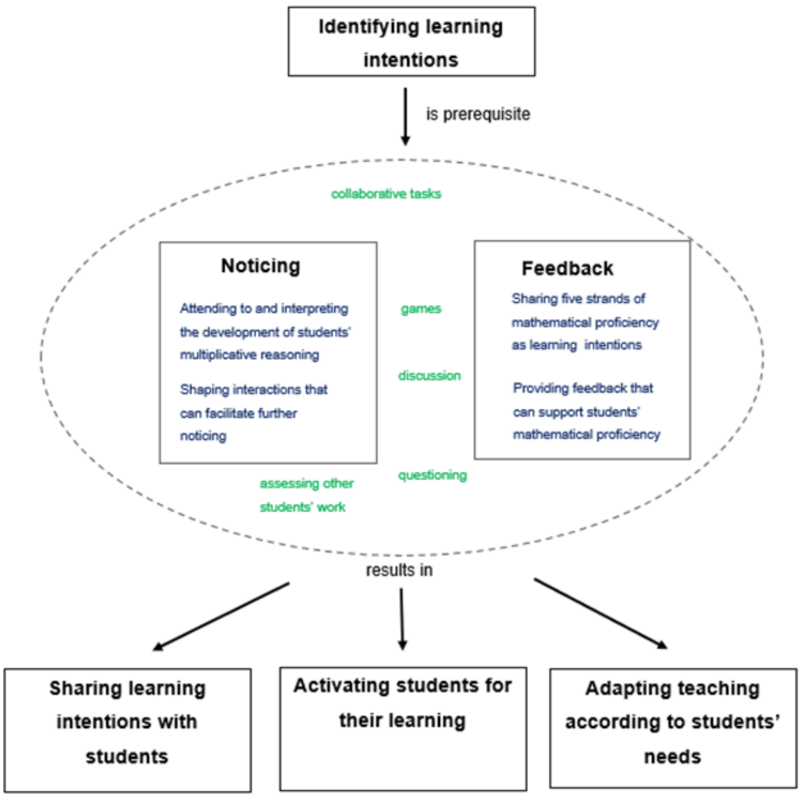
5.1.1. Operationalising five formative assessment strategies (Wiliam & Thompson, 2007) in mathematics teaching

Modifications and adaptations to the framework

In the initial stages of this study, the formative assessment strategies proposed by Wiliam and Thompson (2007) provided a foundational framework for identifying formative assessment practices during the explorative phase. While these strategies were instrumental in comprehensively addressing various aspects of formative assessment, their overlap presented a challenge to the analysis.

Capitalising on this challenge and drawing on the conceptualisation of formative assessment as mentioned in Chapter 2, a new framework was developed to operationalise formative assessment in secondary mathematics lessons that informed subsequent analyses. In contrast to Wiliam and Thompson's approach, which treats each strategy as distinct and of equal importance within the formative assessment framework, this operationalisation process involves a more nuanced examination. By recognising the overlaps in the application of these strategies, I have taken a deliberate approach to understanding the unique functions of each formative assessment strategy both before and during lessons. Importantly, this approach aims to distinguish the function of formative assessment, core in-the-moment formative assessment practices and supporting strategies based on their specific roles in the overall formative assessment enactment process. Figure 5.1 visualises this operationalisation.

Figure 5.1 Operationalising formative assessment strategies suggested by Wiliam and Thompson (2007) in teachers' in-the-moment formative assessment practices



Note. Created by the author

Initially, three expected outcomes of formative assessment guided the operationalisation process. These outcomes are: (1) sharing disciplinary learning intentions with students, (2) activating students for their learning, and (3) adapting teaching according to students' learning needs. Drawing on this, the five formative assessment strategies were thoroughly examined, and three methodological foci were identified.

The initial decision was to identify teachers' noticing students' mathematical thinking and providing feedback that can enhance students' mathematical proficiency as the two pillars of teachers' in-the-moment classroom formative assessment practices. This strategic choice introduces two nuanced aspects that differ from Wiliam and Thompson's framework. First, it centres on two of the five strategies, underscoring their role as crucial. Second, it expands the second strategy, engineering tasks and questions to elicit students' learning, by integrating the notion of noticing (van Es & Sherin, 2021). This integration not only provides deeper insights into the effective implementation of the second strategy but also highlights the role of teachers in engaging with students' thinking beyond merely expecting high-quality tasks to elicit students' learning, as opposed to the expected role of tasks within the large-scale summative assessment practices.

The next methodological decision was related to operationalising the first formative assessment strategy, clarifying and sharing learning intentions and criteria for success (Wiliam & Thompson, 2007). Clarifying learning intentions was identified as a prerequisite for enacting the two core classroom formative assessment practices. This element is considered essential for their implementation. This decision separates the two elements of the first strategy suggested by Wiliam and Thompson. Specifically, the original strategy involves both teachers' identifying learning intentions and sharing these intentions with students. In the conceptualisation I propose, teachers are expected to identify learning intentions before engaging in noticing and feedback practices and they are expected to share learning intentions with students throughout noticing and feedback practices. This deliberate sequencing highlights the essential nature of establishing clear learning goals as a foundational and integral aspect that precedes and informs subsequent formative assessment practices.

While positioning identifying and sharing learning intentions within my conceptualisation, I acknowledge that I put aside the element of directly sharing learning intentions with students by certain techniques such as writing the learning goals of the lesson on the board or use of rubrics in the initial stages of the lesson (Wiliam, 2018). In my conceptualisation, I consider these aspects of sharing learning intentions as elements that were naturally involved in noticing as a part of encouraging students' engagement with the task. As already discussed in Section 2.2.2, grounding on the existing literature, and will be elaborated on Section 6.1.2, with illuminating examples from the teacher guides analysed, shaping interactions with students that can prompt students to think and reveal their mathematical thinking is an essential element of noticing. In my conceptualisation, I view sharing intentions of the specific lessons as elements of shaping interactions with students, which can facilitate student engagement with the main task.

Another aspect of the original first formative assessment strategy is identifying and sharing success criteria with students according to which their performance will be judged. Whilst this aspect of the first strategy may be important for enabling students to understand what constitutes "success" in a lesson or sequence of lessons, this is not in the scope of this doctoral study. Instead, the main focus in this study is teachers' in-the-moment formative assessment decisions that prioritise enhancing students' learning process.

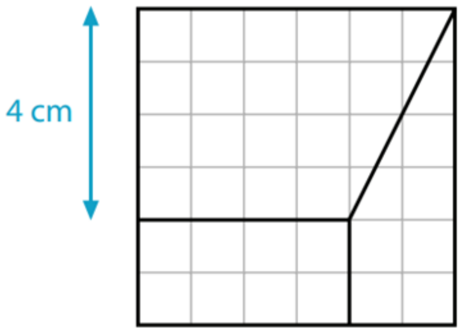
The third big decision for the conceptualisation I suggest is acknowledging the fourth strategy of formative assessment in the original framework, activating students for their peers' learning, as a supporting strategy that can enhance two core classroom formative assessment practices, noticing and feedback.

Elaborating on teachers' formative assessment practices through illuminating examples

In this section, I use example extracts from the teacher guides that exemplify the two pillars of in-the-moment formative assessment practices respectively.

The first example, in Figure 5.2, was taken from the ICCAMS project to elaborate on the implementation of the noticing element of the framework.

Figure 5.2 Tangram task



This tangram consists of three pieces. We want a larger version of the tangram where the 4 cm length becomes a 7 cm length. Work in a group of three. Choose one piece each. Draw the larger version of your piece on 1 cm squared paper. Carefully cut it out. Check that the 3 new pieces again fit together.

(ICCAMS, Tangram, p. 288)

This task can facilitate noticing students' adaptive reasoning. More specifically, this task encourages the realisation that to preserve the original shape using correct reasoning is essential and incorrect additive reasoning does not result in the enlargement of the original shape without distortion. In order to attend to students' initial correct or incorrect reasoning about how to enlarge the original shape without distortion and focus their shaping practices, it is essential that teachers have already identified this intended learning outcome. Teachers can attend to students' incorrect thinking when they observe that the larger versions of the tangram pieces students produced do not fit and they can interpret this as potentially incorrect additive reasoning. This can initiate teachers' further shaping interactions by asking probing questions to engage with students' actual reasoning.

This second example is taken from a MAP lesson that elaborates on the implementation of feedback.

> *When Sam and his friends get together, Sam makes a fizzy orange drink by mixing orange juice with soda. On Friday, Sam makes 7 liters of fizzy orange by mixing 3 liters of orange juice with 4 liters of soda. On Saturday, Sam makes 9 liters of fizzy orange by mixing 4 liters of orange juice with 5 liters of soda. Does the fizzy orange on Saturday taste the same as or different to Friday's fizzy orange? If you think it tastes the same, explain how you can tell. If you think it tastes different, does it taste more or less orangey? Explain how you know.*
>
> *(MAP, Using Proportional Reasoning, p. T-2)*

This task serves as a valuable tool for teachers to notice students' adaptive reasoning in the context of comparing part-to-part ratios. Within the lesson guide, teachers are provided with example student responses along with suggested responses that guide teachers in addressing their students' responses effectively. Illustrating this, a potential incorrect additive reasoning was presented. That is to say, the teacher guide highlights that some students might think that the fizzy orange tastes the same in both situations as the amount of orange juice is one litre less than the amount of soda in each situation. In order to address this incorrect reasoning teachers are advised to ask questions to students such as:

> *How could you use math to check that the addition of a litre of orange and a litre of soda has no effect on the taste?*
>
> *What would happen to the taste if a litre of orange and a litre of soda were added to 1 litre of soda?*
>
> *If 3 litres of fizzy orange was made in the same way, by mixing 1 litres of orange with 2 litres of soda, would this taste the same also?*
>
> *(MAP, Using Proportional Reasoning, p. T-3)*

By asking these questions, teachers can create feedback situations in which students are prompted to consider the mathematical procedures they know when answering the original fizzy orange task and consider variations of this original task that can encourage them to reflect on their responses and revise these responses.

In addition to offering students an opportunity for reflecting on their thinking, this feedback scenario has the potential to enhance students' understanding of the learning intentions in this lesson. Crucially, by encouraging students to ponder three questions instead of merely informing them that their answer is incorrect, they may gain valuable insights into the learning intention of adaptive reasoning. This involves establishing connections among mathematical ideas and adapting them to various contexts, emphasising the importance of the reasoning process over a mere focus on finding the correct answer.

Two core classroom formative assessment practices, noticing and feedback, could be facilitated by activating students for their peers' learning. In the teacher guides, collaborative work among students was one of the explicitly suggested classroom practices. As an example, the teacher guide for the lesson that aims to students' discovery of scale factor in the CSM, Investigation 3 in the Geometric Similarity Unit, encourages teachers to devote time for peer discussions in many of the tasks. The purpose of the main mathematical task in this lesson was to facilitate students' discovery that the scale factor represents a multiplicative relationship by observing the changes in both numbers and shapes with the help of the digital dynamic tool. Specifically, students are expected to work on the digital dynamic tool that provides students an opportunity to observe various similar copies of an original shape by changing the magnitude of the scale factor. Teachers are advised to encourage students to work with their peers and then lead a large group discussion on whether this software can always produce proper enlargements or not. Working with their peers is likely to prompt students' thinking, creating opportunities for interactions that reveal their understanding (Walshaw & Anthony, 2008). This approach can enable teachers to observe multiple students sharing their thoughts, which would be challenging or impractical to achieve in a one-on-one setting.

5.1.2. Identified themes and underlying assumptions

The overall purpose of the reflexive thematic analysis was to grasp an overview of the educative support of teacher guides for enacting formative assessment in secondary mathematics classrooms. As a result of an iterative coding and theme development process as detailed in Section 4.5, an overarching and four sub-themes were developed.

The overarching theme is “challenging teachers’ daily practices towards more effective formative assessment practices” and sub-themes are moving beyond content and procedure-oriented learning goals, leading students’ productive collaborative work, noticing students’ mathematical thinking rather than merely measuring what they know and creating feedback situations that can encourage students to reflect on their learning beyond correcting their responses.

These themes were developed drawing on the assumption of that it could be a challenge for a large number of teachers to enact the four highlighted practices in the sub-themes. The first assumption is that a considerable proportion of mathematics teachers might have a tendency to focus on content and procedure-oriented learning goals. Over three decades, research in mathematics education stresses this tendency by interrogating the reasons and developing alternative teaching practices (Cohen & Hill, 2000; Kilpatrick et al., 2001; Koedinger et al., 2013). Although recent policies encourage teachers to involve other aspects of learning mathematics such as problem-solving and making connections amongst various types of mathematical knowledge (e.g., DfE, 2013), changing the past traditional practices is expected to be a slow process. Indeed, a more recent large-scale empirical study conducted with 237 secondary mathematics teachers in Sweden (Lithner, 2017) provides evidence for this slow process. The findings of this study demonstrated that although these teachers used tasks in which the intended focus was conceptual understanding and strategic competence, the majority of these teachers’ delivery of these tasks promoted students’ procedural fluency.

Findings from the reflexive thematic analysis strongly suggest that the curriculum materials have the potential for either reinforcing or challenging teachers’ tendencies. In some instances, teachers were presented fragmented and procedure-oriented learning goals, which is assumed to be in line with the majority of mathematic teachers’ experiences. For example, listing a set of learning goals such as, “enlarge a shape by a positive scale factor” and “solve problems with similar triangles” (WRM) can encourage teachers to focus on the procedures separately. In other instances, however, the authors of the curriculum materials mentioned learning goals that emphasise skills beyond procedures, such as “estimating”, “talking about problems”, “solving problems using efficient methods”, “interpreting and using scale drawings” and “use proportional reasoning to solve a real-world problem". These learning goals can challenge teachers to change their understanding of teaching mathematics by involving a variety of skills.

The second theme was built upon the assumption that leading students' collaborative work in the classroom is challenging for most teachers. Collaborative work in the classroom can be a challenge for teachers due to a number of reasons such as lack of knowledge or experience in how to lead effective collaborative work and hesitancy due to time constraints in classroom (Stigler & Hiebert, 2004; Watson & Mason, 2007). The analysis of teacher guides revealed clear guidance aimed at enhancing teachers' facilitating of collaborative work among students in the classroom. This involves anticipating potential challenges for teachers and offering recommendations on how to address these challenges. For instance, in a MAP lesson, teachers are advised to incorporate group work that involves students visiting other groups and sharing their thoughts. Some teachers may be hesitant to implement this aspect of the activity due to potential chaos in the classroom. However, the teacher guide provides specific hints to facilitate a smoother implementation, as the following example illustrates.

> *It may be helpful for students to jot down on their mini-whiteboards their agreed order of the cards before they visit another group.*
>
> *(MAP, Using Proportional Reasoning, p. T-6)*

The next assumption about teachers' common practices to be challenged by the curriculum materials is that teachers tend to focus on the accuracy of students' answers rather than noticing their mathematical thinking. Curriculum materials can challenge teachers by informing them about common students' misconceptions and their potential thinking and provide techniques for teachers to be able to find out their students' mathematical thinking. Mainly, it is common in the materials to inform teachers about students' potential additive misconceptions. This is an example of students' potential additive misconception from a MAP lesson, "…adding the same amount to both sides of a rectangle will create a similar rectangle.". Another example from ICCAMS highlights that students can think additively in multiplicative situations depending on the context.

They might recognise a recipe task as being multiplicative, but resort to additive strategies in a task involving geometric enlargement. It is thus worth helping students to extend the range of contexts that they 'see' as multiplicative.

(ICCAMS, Post Shadows, p. 121)

The final assumption about teachers' common practices is their inclination to provide immediate corrective feedback in the lesson. The materials can challenge teachers to delay feedback, when necessary, and provide feedback that will encourage students to reflect on their learning and share the responsibility for providing feedback with students. For example, the following extract from CMP is an exemplar question that is provided for teachers to be used as feedback when students give a non-sense answer to the question which asks for a teacher's height.

What would you expect the range of possible heights for the heights for the mystery teacher to be? If an answer is over 7 ft, is that reasonable? What about an answer under 4 ft?

(CMP, Stretching and Shrinking, p. 48)

In this question, rather than providing corrective feedback by telling students that their answer is incorrect, teachers are encouraged to ask questions that will help students reflect on their answers and hopefully find that their answer does not make sense because no one can be at this height.

5.1.3. A model of the educative potential of curriculum materials

As a result of the explorative phase, I developed a model for analysing the educative potential of curriculum materials. Within this framework, the initial step involves the identification of specific pedagogical messages intended to be communicated to teachers. The model posits that these pedagogical messages can effectively reach teachers through three educative features: *alert*, *equip*, and *guide*. I contend that the integration of all these educative features completes the cycle of educative potential for conveying specific pedagogical messages to teachers. By incorporating all three features, the likelihood of effectively communicating messages to teachers and activating their responses could be enhanced. The subsequent sub-sections elaborate on these components of the proposed model individually to provide a comprehensive understanding. However, in Chapter 7, the application of this model will be presented by analysing the educative potential of teacher guides for identified pedagogical messages.

Three educative features

The educative features proposed in this thesis were developed based on the codes and categories identified through the thematic analysis and insights drawn from the literature on educative curriculum materials. As already mentioned in Chapter 4, during the thematic analysis phase, the codes were divided into five categories. The codes within the category “learning objectives” led me to suggest the educative feature of “alert”, the codes within the categories “classroom material” and “teacher knowledge” led me to suggest the educative feature of “equip” and the codes within the category “teacher instruction” led me to suggest the educative feature of “guide”.

This section will elaborate on these features through example references from teacher guides.

Alert

The alert feature extends the characteristic of the suggested educative curriculum materials that were discussed in Section 2.3.2. This characteristic involves uncovering the design rationale of materials for teachers, a concept widely explored in earlier studies focusing on the mathematics knowledge upon which the curriculum is based or the rationale behind suggested pedagogical practices. In this thesis, I deliberately use the word “alert” instead of “providing rationale” to emphasise the curriculum materials’ role in prompting teachers towards the practices aligned with the pedagogical messages intended to be conveyed to teachers.

The notion of alert involves both the provision of the rationale provided to teachers and the way to provide the rationale to them. In order to distinguish two aspects of the rationale, I use alerting *to* the rationale of specific pedagogical practice and alerting *by* utilising specific ways. While alerting to includes what earlier studies refer to the rationale to be provided, alerting by includes the way of providing the rationale.

The variety in terms of alerting to and alerting by was summarised in Table 5.1. Further explanations that elaborate on these examples were provided in the following paragraphs.

Table 5.1 Variations of "alerting to" and "alert by" within the teacher guides

Alert to	Alert by
The rationale of the lesson	Explicit narratives
The rationale of specific pedagogical practice	Implicitly, using context
The rationale for using specific mathematics tasks	Implicitly, using tasks
The rationale of example questions	
The rationale for using specific tools (e.g., digital tools)	
The link between the lesson and national standards	
Content storyline	

Note. Created by the author

As an example, on MAP, with the explanations in Figure 5.3, teachers can be *alerted to* one of the overall rationales of the lesson as addressing specific Common Core State Standards for Mathematics (CCSSM) *by* direct explanations.

Figure 5.3 Linking lesson goal to the CCSSM

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.G: Draw, construct, and describe geometrical figures and describe the relationships between them.
Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.EE: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

7.RP: Analyze proportional relationships and use them to solve real-world and mathematical problems.

Note. Reprinted from MAP, Drawing to a Scale: Garden lesson, p. T-1

In the second example in the same lesson, teachers could be *alerted to* one of the overall rationales of the lesson by direct explanations as in the following.

This lesson unit is intended to help assess how well students are able to interpret and use scale drawings to plan a garden layout. This involves using proportional reasoning and metric units.

(MAP, Drawing to a Scale: Garden, p. T-1)

Beyond providing the overall purpose of the lesson, the analysis showed references for alerting teachers to the rationale of certain classroom practices by direct explanations.

This task gives students an opportunity to evaluate different approaches to the task.

(MAP, Drawing to a Scale: Garden, p. T-7)

The following data extract from CSM is an example of alerting teachers to the purpose of the lesson by implicitly using context. In contrast to the conventional way of providing rationale, this teacher guide does not feature direct references intended to alert teachers to the purpose of the specific lesson. Instead, teachers are provided with contextual information and an understanding of the mathematical concept underlying the lesson, without explicitly linking it to pedagogical purposes. To illustrate this, one example of context is presented as follows.

One of our artists, Eileen, found free software without a scale factor slider but with two strange other sliders. She says that the software can still be used to create mathematically similar copies.

(CSM, Investigation 4: Broken Scale Factor, p. TN-23a)

Similarly, the key mathematical idea is presented as,

Scaling a shape so that it creates a mathematically similar copy requires that all lengths of the shape be scaled by the same number.

(CSM, Investigation 4: Broken Scale Factor, p. TN-23a)

These instances are not included with the intention of alerting teachers explicitly, as the information is presented without providing a specific purpose. However, teachers are alerted to the purpose of the specific tasks in the following notes. As an example, it was stated that the digital tool was expected to prompt students to address incorrect additive reasoning by comparing the original shape and the copy of this shape.

A lesson guide from CMP, the Wump family, alerts teachers to the rationale of a mathematics task by providing a content storyline and explicitly stating the role of this problem in students' developing an accurate definition of mathematical similarity.

> *Students continue to work with the Wump family as they investigate side lengths, angles, perimeters, and areas of similar rectangles and triangles. This problem helps students to form a more precise definition of the meaning of similar in mathematics.*
>
> *(CMP, Mouthing Off and Nosing Around, p. 98)*

More explicitly, this lesson is a part of a teaching unit, which includes a series of linked lessons. After exploring the relationships between measures of similar polygons, students are expected to develop formal definitions by working on questions that ask for mathematical explanations for these relationships, such as, "Does the same relationship between the scale factor of similar rectangles and their area apply for similar triangles? Explain."

Although the language used in this extract does not explicitly establish a connection between using the same context and the focus of this lesson, presenting the purpose of the lesson after reminding teachers that students were already familiar with the context and had worked on measurements such as side lengths, angles, perimeters and areas, the message that working with the same context can enable a focus on the concept similarity can be conveyed to teachers. More explicitly, in this lesson, students can delve into the concept and relationships better, without being distracted by calculations.

In this teacher guide, it was also found that teachers were alerted to the function of some suggested tools. As an example, a video was suggested to be shown to students in the early stages of the lesson. In the teacher guide it was explicitly stated that "this animation gives students a dynamic representation of similar figures" (p. 99).

Equip

Equipping refers to providing teachers with sufficient knowledge and tools to implement suggested practices towards intended pedagogical messages. This feature has two theoretical bases. First, the leading literature on educative curriculum materials suggests that, to be educative, curriculum materials should integrate teachers' knowledge for teaching (Davis & Krajcik, 2005; Quebec-Fuentes & Ma, 2018). Second, one of Davis et al.'s (2017) design principles indicates that teachers are likely to better engage with suggested practices when they are provided with tools. Table 5.2 presents the various types of knowledge and tools found in the teacher guides.

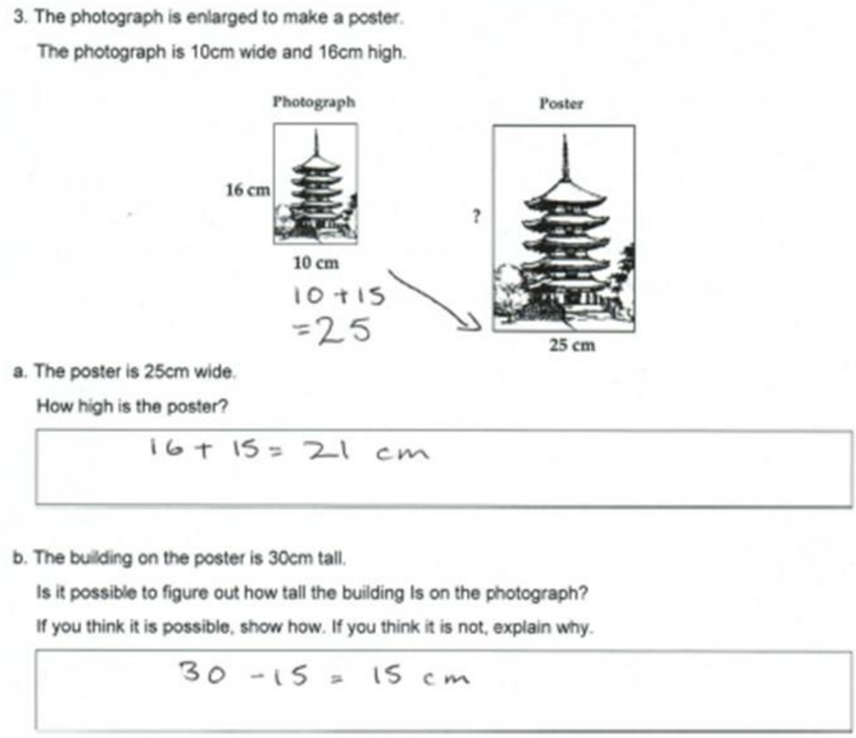
Table 5.2 Variations of equipping with knowledge and tools within the teacher guides

Equipped with knowledge	Equipped with tools
Knowledge of student thinking	Videos
Mathematical knowledge	Digital tools
Underlying pedagogical knowledge for suggested practices	Projector resources
Curricular knowledge	Student worksheets
	Expected correct answers
	Example student answers
	Example student reasoning
	Mathematics tasks
	Variations of mathematics tasks
	Example prompts and questions

Note. Created by the author

One example for equipping with knowledge of student thinking by providing example student work, from MAP, is presented in Figure 5.4. In this data extract, teachers are equipped with students' potential incorrect additive reasoning.

Figure 5.4 Example student work



3. The photograph is enlarged to make a poster.
The photograph is 10cm wide and 16cm high.

a. The poster is 25cm wide.
How high is the poster?

16 + 15 = 21 cm

b. The building on the poster is 30cm tall.
Is it possible to figure out how tall the building Is on the photograph?
If you think it is possible, show how. If you think it is not, explain why.

30 - 15 = 15 cm

Note. Reprinted from MAP, Comparing Strategies for Proportion Problems, p. T-2

This example student work was supported by equipping teachers with the explanation of potential student thinking when making this sort of mistake, as following.

> *Gavin incorrectly uses an additive strategy. He is not considering proportion, but using the difference between known lengths to calculate unknown ones. This is a common error in ratio problems…Students may find it difficult to solve the second, reverse question. Some students are not convinced that lengths within pictures scale by the same factor.*
>
> *(MAP, Comparing Strategies for Proportion Problems, p. T-8)*

As opposed to equipping teachers with potential student thinking, teacher guides can merely involve the expected correct answers, as illustrated with an example from CMP. The question to be asked is as following.

> *I want to grow a new Wump from Wump 1 (Mug). The scale factor is 9. What are the dimensions and perimeter of the new Wump's mouth?*
>
> *(CMP, Mouthing off and Nosing around, p. 103)*

For this question, teachers were equipped with only expected correct answer, 36 x 9, p=90, without providing any information about students' potential difficulties or incorrect answers.

One reference for students' thinking in the same lesson was found as following. "Note that this activity uses prime notation, which students may not be familiar with." This information is far from providing insights into making sense of how students learn similarity.

Guide

Guiding teachers refers to the assistance provided to teachers, enabling them to effectively utilise the tools and knowledge with which they have been equipped. This aspect is rooted in Davis et al.'s (2017) design principle, emphasising that teachers can adapt curriculum materials based on constraints like limited time. Educative curriculum materials, accordingly, should offer instructional support to help teachers make productive adaptations. In my approach, the term "guide" encompasses a broader meaning, encompassing the effective utilisation of knowledge and tools. This includes not only leveraging them efficiently but also anticipating potential adaptations by teachers and providing support for more effective adjustments. Table 5.3 provides the variety found within the materials for guiding teachers. Following paragraphs exemplify some of these variations.

Table 5.3 Variation of the references for the educative feature "guide"

Guide for	Guide by
How to effectively implement suggested pedagogical practice	Instructions
How to help students organise their work	
How to introduce and explain content such as context and mathematical conventions	
How to extend tasks	
How to involve national standards	
How to use mathematical representations	

Note. Created by the author

Teachers can be guided for effectively using the suggested pedagogical practice. As an example, in a MAP lesson, teachers were guided to observe students carefully and attend to the variety of thinking among students and incorrect responses. This guidance can be vague for teachers as it does not involve specific alternative ways to engage with students' thinking.

> *Listen and watch students carefully. Note different approaches to the task and any incorrect solutions.*
>
> *(MAP, Comparing Strategies for Proportion Problems, p. T-5)*

On MAP lesson, in several places teachers are guided to give some extension tasks as homework in case they have limited time for these tasks.

Following references can exemplify an explicit and substantial emphasis on mathematics-specific guidance rather than generic pedagogical approaches and organisational guidance. For instance, teachers are guided to use Cartesian graphs in the context of measuring post heights and their corresponding shadows, as demonstrated by the following instruction.

> *To establish a connection between the Cartesian graph of post heights and shadow lengths (below, right) and the side elevation of posts and their shadows (below, upper left), might be beneficial.*
>
> *(ICCAMS, Post Shadows, p. 121)*

Furthermore, the guide offers recommendations for teachers to adapt the level of tasks based on students' readiness, as exemplified by the following guidance.

> *In addition to selecting appropriate slides for your class, you may consider incorporating posts with heights chosen by either yourself or the students to increase or decrease the level of difficulty.*
>
> *(ICCAMS, Post Shadows, p. 119)*

From these examples, it can be implied that when guiding teachers, this teacher guide places significant emphasis on mathematics specific practices over generic pedagogical practices.

A shift on focus from classroom practices to pedagogical messages

The distinguishing character of the framework I developed for educative features places the intended pedagogical messages at the heart of the framework. In the earlier frameworks, educative features widely draw on professional knowledge teachers should possess (Ball & Cohen, 1996; Davis et al., 2017; Quebec-Fuentes & Ma, 2018). These frameworks can be interpreted as if the role of the educative curriculum materials is to teach several aspects of teacher knowledge widely to teachers. This differs in my aim as I directly focus on formative assessment support.

As opposed to these earlier frameworks, Machalow et al.'s framework (2020) focuses on noticing support for teachers specifically. However, this framework guides to analyse mainly the content the materials involve regarding noticing without providing insights how these materials can communicate these elements to teachers. While this framework involves an element "foundational guidelines that support noticing", the explicit link between this element and other elements directly related to noticing, which are "attending to and anticipating student thinking" and "analysing and evaluating their thinking", is not explicit. My focus on pedagogical messages rather than the content helps considering the role of teachers and the curriculum materials distinctly but also making link between these roles. That is to say, in my framework, the role of curriculum materials is to communicate specific pedagogical messages to teachers and activate them to implement their classroom practices in line with these messages, beyond only providing the required content or pedagogical content knowledge.

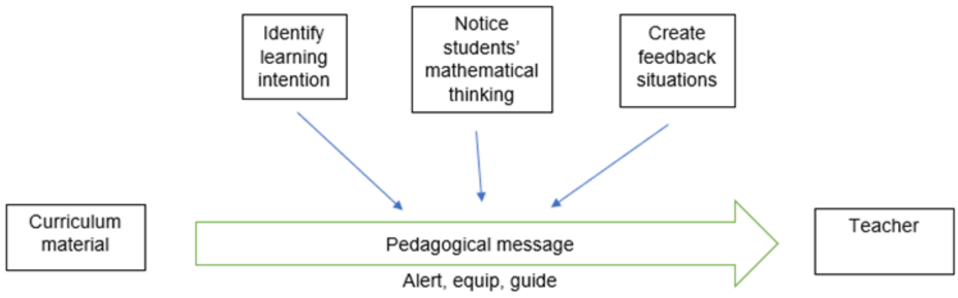
In the context of this doctoral research, these pedagogical messages are related to enacting formative assessment effectively in the classroom. Considering the possibility of confusing the learning intentions with the purpose of tasks, as raised by Wiliam (2018), these messages are identified beyond a focus on the task or classroom practice. Tasks or classroom practices were considered as tools that can enable to effectively communicate the pedagogical messages to teachers.

Rather than identifying rationale broadly for any aspect of the curriculum materials I focus on the rationale for formative assessment practices. The key difference in my approach is to see each feature as complementary.

5.2. Positioning the proposed framework within the existing literature

In this chapter, so far, I introduced the overall framework I developed during the explorative phase of my doctoral research. This framework aims to reveal the educative potential of curriculum materials for effective enactment of formative assessment in classroom setting. The key characteristic of this framework is its focus on the pedagogical messages that are aimed to be conveyed to teachers. In the context of this research, these pedagogical messages are related to noticing students' mathematical thinking and providing feedback that can move students' learning forward. I suggest that after identifying the pedagogical messages, three educative features, alert, equip and guide, should be integrated in the curriculum materials to facilitate teachers' enactment of practices aligned with these messages. Importantly, I suggest that incorporating all these three features for specific pedagogical messages can increase the possibility of communicating the pedagogical messages to teachers.

Figure 5.5 The relationship between the constructs of the framework to analyse the educative potential of teacher guides



Note. Created by the author

Although there is no discrete line between formative assessment aspects and educative features, I prefer to present them separately in order to be able to distinguish the role of teachers and the role of the materials. Distinguishing these roles can allow a focus on the educative potential of curriculum materials when analysing them. When making the distinction I consider the following rule:

when a technique or practice is for teachers' use then it is associated with the elements of formative assessment, when a technique or practice is for designers' use then it is associated with educative features. For example, when considering use of mathematics tasks, it is teachers' responsibility to use these tasks effectively as formative assessment practice; when considering the involvement of appropriate tasks for students within the teacher guides then this is designer's responsibility, which makes this an element of educative feature. In other words, while contextualised formative assessment elements involve the messages to be conveyed to teachers, educative features involve ways to convey these messages. While formative assessment features will be contextual, educative features will be more generalisable.

This in-depth exploration of formative assessment and educative features, coupled with an examination of the interrelations among the specific components of these two phenomena, represents a valuable contribution to the existing literature. While prior studies have predominantly treated formative assessment strategies in a generic manner without truly operationalising them for specific subjects (Bennett, 2011), this doctoral study specifically delves into these strategies' connection with curriculum materials, a dimension that has yet to be thoroughly investigated (as presented in Section 3.3.).

Beyond investigating the link between formative assessment strategies and educative features, this framework adds to the existing body of research that considered examining the educative potential of curriculum materials through suggested frameworks. As mentioned earlier in Section 3.4.1, educative features Quebec-Fuentes and Ma (2018), Remillard's team (e.g., Machallow et al., 2020) and Davis et al. (2017) suggested have been the theoretical basis for the development of the framework in my research. More specifically, for the framework introduced in this thesis, the acknowledgment of the importance of the following aspects comprised a basis: providing guidance and rationale as educative elements of curriculum materials (Quebec-Fuentes & Ma, 2018), addressing teachers' tendencies and practical needs when they adapt materials, multiple forms of support that may address teachers' individual needs and discipline-specific support rather than generic guidance (Davis et al., 2017).

The first unique aspect of my proposed framework is the focus on the pedagogical messages inherent in the curriculum materials and identifying these messages as the unit of analysis. This aspect brings a methodological new insight into the assessment of curriculum material analysis which can contribute to the attempts in line with the “big Q” approaches discussed in Section 4.1.1 (Braun & Clarke, 2021).

The second unique aspect of this framework is the notion of alerting teachers to the suggested pedagogical messages. Although the notion of alert was informed by the earlier frameworks that highlighted the importance of including the purpose and the rationale of the suggested pedagogical practices (e.g., Quebec-Fuentes & Ma, 2018), this notion explicitly highlights the need for activating teachers towards convincing them for the usefulness of the suggested practices beyond merely informing them.

Finally, this framework extends the documentational approach proposed by Gueudet and Trouche (2009), which theorises the dynamic two-way relationship between teachers and curriculum resources through notions instrumentation and instrumentalisation. Notably, the transmission of pedagogical messages to teachers through curriculum materials, by alerting, equipping, and guiding, enriches the concept of instrumentation.

CHAPTER 6 - THE EDUCATIVE POTENTIAL OF THE TEACHER GUIDES: A FOCUS ON THE IN-THE-MOMENT FORMATIVE ASSESSMENT PRACTICES

In Chapter 5, I introduced the framework developed during the explorative phase, which serves as a guide for analysing teacher guides in terms of their educative potential to support teachers with their in-the-moment formative assessment practices. This chapter presents an analysis using this framework, alongside an adapted Charalambous et al.'s (2010) analytic approach, as discussed in Sections 3.5.2 and 4.6.1. Namely, I conducted a horizontal analysis of teacher guides, focusing on recommended practices related to in-the-moment formative assessment that could enhance teachers' practices. This analysis directly addresses the first sub-question of RQ3, as follows:

What are the characteristics of the educative potential inherent to the teacher guides to facilitate teachers' in-the-moment formative assessment practices, specifically when approached from a horizontal perspective?

In this critical phase of the research, I chose one lesson or unit from each of the five sets of materials for a deeper focus. The lessons are "Mouthing off and Nosing Around: Scale Factor" in CMP, "Broken Scale Factor" in CSM, "Post Shadows" in ICCAMS, and "Drawing to a Scale: A Garden" in MAP. The unit is "Enlargement and Similarity" in WRM. WRM SoL has a different characteristic than the other four teacher guides. Rather than detailed teacher guides as in other sets of materials, WRM SoL includes key topics as a unit and provides a one-page teacher guide for each learning goal of these topics. In order to have sufficient content from WRM, I chose one unit from this set. When I moved to the analysis of educative potential for noticing and feedback practices, I discarded WRM resources as they provided only limited educative potential for these pedagogical practices.

6.1. Varying and conflicting approaches to learning intentions: Integrating five strands of mathematical proficiency

In Section 2.2.2, I introduced five strands of mathematical proficiency as a guiding framework to identify learning intentions in mathematics in this thesis. Later, in Section 4.6.2, I presented the coding scheme that was used as a guide to identify learning intentions in the teacher guides. This coding guide was used when identifying the distribution of each strand within chosen aspects of curriculum materials, as presented in the following section.

When analysing the learning intentions within these materials, I used a sampling strategy specific to this analysis. That is to say, the analyses during the explorative phase helped me to identify aspects of the materials that include the learning intentions. More explicitly, CMP involves learning intentions explicitly in the overview pages alongside the lesson plans; CSM involves learning intentions in both introductory overview pages and the lesson plans; ICCAMS involves learning intentions in various sections within the lesson plans; MAP involves learning intentions on the first page of the lesson plans as well as involving references for learning intentions through the list of suggested example questions to be asked in the lessons; and finally WRM involves learning intentions on a separate page that present small steps for each topic. For the analysis for identifying the distribution of learning intentions, I coded all these aspects and extended the sample to more than one lesson to increase the representativeness of the sample.

6.1.1. Distribution of five strands of mathematical proficiency within five sets

Table 6.1. illustrates the distribution of references for each strand of mathematical proficiency in each set of materials. It should be noted that that table was created to show an indication of the general tendencies among the materials according to the sections that were analysed in each set of materials. This table should not be accepted as an absolute representation of the distribution of the five strands of mathematical proficiency across these materials; rather, it should serve as an overview, forming the basis for the subsequent discussions in this chapter. The volume of teacher guides was not

consistent with the different materials coded in this analysis. As a result, the numbers are not comparable across different sets of materials. In this section, my interpretations rely on a comparison between the percentages calculated for both within and across sets of materials.

Table 6.1 Frequencies of each strand of mathematical proficiency within teacher guides

Set of Materials	Procedural fluency	Conceptual understanding	Strategic competence	Adaptive reasoning	Productive disposition	Total
CMP	5 (~5%)	31 (~33%)	28 (~30%)	24 (~26%)	5 (~5%)	93 (~100%)
CSM	19 (~21%)	25 (~28%)	17 (~19%)	17 (~19%)	11 (~12)	89 (~100)
MAP	9 (~10%)	15 (~16%)	28 (~30%)	29 (~31%)	13 (~14%)	94 (~100)
ICCAMS	4 (~8%)	14 (~27%)	20 (~39%)	8 (~16%)	5 (~10)	51 (~100)
WRM	15 (~33%)	11 (~24%)	11 (~24%)	9 (~20)	0	46 (~100)

Note. Created by the author

One striking finding is that while in some materials the distribution of the strands of mathematical proficiency were balanced, in other materials, specific strands had high proportion compared to other strands. To illustrate, MAP and CSM exhibited a balanced approach. In the MAP, there was a balance in the distribution of five strands, with strategic competence and adaptive reasoning each incorporating approximately 30% of all identified references for the strands of mathematical proficiency. On the other hand, while the CSM also demonstrated a balanced approach, a unique distribution pattern was found in the teacher guide analysed. More specifically, conceptual understanding exhibited the highest proportion. Despite these differences, these findings suggest that teacher guides in both MAP and CSM can provide an opportunity to involve all five strands of mathematical proficiency as learning intentions in classroom teaching.

In the teacher guides in other three projects, some strands of mathematical proficiency were found to be dominant than others. More specifically, in the CMP, the references for procedural fluency and productive disposition were limited to an approximate percentage of 5 individually. However, the references for conceptual understanding, strategic competence and adaptive reasoning had much higher proportion with between 26 and 33.

In ICCAMS, strategic competence was found as the dominant strand, with a notable 39% of the references. Conceptual understanding followed closely behind with a 27% proportion. This distribution indicates a deliberate focus on the learning opportunity for students' problem-solving skills and understanding concept. The comparatively smaller proportions for the other three strands suggest a targeted approach within the ICCAMS lessons.

A unique case, the WRM framework stands out with its emphasis on procedural fluency, representing the highest proportion of references, with closer proportion with other three strands, conceptual understanding, strategic competence and adaptive reasoning. Notably, productive disposition seems to be absent within the identified references. This suggests a specialised focus on developing procedural skills, possibly in contexts where application and disposition may not be the primary intentions.

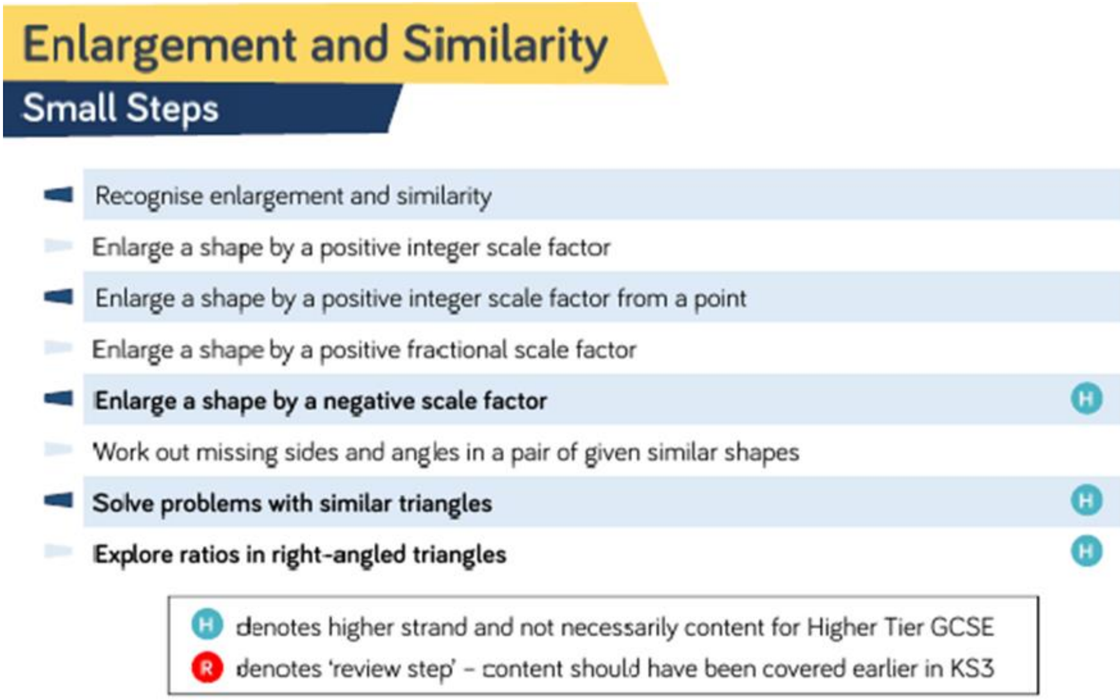
6.1.2. Hierarchical-fragmented versus relational learning intentions

Two conflicting approaches in presenting learning intentions were found in different teacher guides: fragmented versus relational. In this aspect, White Rose Maths lessons were sharply differed from other lessons with its tendency to presenting learning intentions fragmentedly.

Figure 6.1. presents a typical way of introducing learning intentions to teachers in WRM teacher guides. The specific learning goals related to enlargement and similarity are presented fragmentedly and hierarchically. First, the list involves learning goals fragmentedly so that these goals can be easily associated with only one strand of mathematical proficiency. To illustrate, the learning goal "recognise enlargement and similarity" can be associated with conceptual

understanding and “solve problems with similar triangles” can be associated with strategic competence. Second, the learning goals in this list are ordered progressing from less demanding to more challenging ones. That is to say, three of the learning goals introduced towards the end of the list were signed as “higher standard” goals.

Figure 6.1 An example representation of hierarchical and fragmented learning goals



Note. Reprinted from WRM, Year 9, Summer term SoL

As opposed to this approach, MAP features the relationships among different strands of mathematical proficiency. The following extract is from the MAP project. In that extract, the overarching learning goal was presented as reasoning proportionally, which relates to adaptive reasoning. Rather than listing the learning goals from less demanding to more demanding ones, the learning goals that can relate to other strands of mathematical proficiency were linked to this overarching goal. More explicitly, elements of this goal were presented as “describe a ratio relationship between two quantities”, “compare ratios expressed in different ways”, and “use proportional reasoning to solve a real-world problem”, which can be associated with conceptual understanding and strategic competence.

Figure 6.2 An example representation of relational learning goals

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to reason proportionally when comparing the relationship between two quantities expressed as unit rates and/or part-to-part ratios. In particular, it will help you assess how well students are able to:

- Describe a ratio relationship between two quantities.
- Compare ratios expressed in different ways.
- Use proportional reasoning to solve a real-world problem.

Note. Reprinted from MAP, Using Proportional Reasoning lesson, p. T-1

6.1.3. Interpreting the spectrum

The variation in the presentation of learning intentions can be implications of certain acknowledged frameworks that explains learning. Specifically, introducing the fragmented and hierarchical learning intentions can be an implication of Blooms taxonomy that have had considerable impact on the curriculum and assessment design since 1950s (Krathwohl, 2002). In contrast, introducing the relational learning intentions can be an implication of relatively recent models such as mathematical proficiency (Kilpatrick et al., 2001).

In classroom context, embracing either hierarchical-fragmented or relational learning goals can result in certain implications for the quality of formative assessment practices. Specifically, engaging with learning intentions in two contradictory ways can influence what aspects of students' existing learning teachers attend to, how these teachers shape interactions to reveal students' thinking further and how they create feedback situations. Broadly, hierarchical-fragmented learning goals can hinder teachers' engaging with students' mathematical thinking holistically during these processes.

However, considering the long-term impact of embracing hierarchical and fragmented learning intentions in schools, it is expected that teachers might tend to engage with these learning intentions easier. Rather than involving approaches with which teachers are not familiar with, the approaches that look familiar to teachers can facilitate teachers' engagement with the curriculum materials.

6.2. Noticing students' learning

In Section 2.2.2, I contended that the noticing framework, recently proposed by van Es and Sherin (2021), and the second strategy of formative assessment (Wiliam & Thompson, 2007), complement each other. To recap, the second strategy involves *engineering effective classroom discussions, questions, and learning tasks that elicit evidence of learning*, while the three elements of noticing include *attending to noteworthy features of classroom interactions, interpreting the observed interactions,* and *shaping interactions with students that can enable further noticing*. The ultimate objective of the second strategy is to establish a learning environment wherein teachers can actively engage with students' learning, and teachers have the responsibility of meticulously forming and managing this environment (Wiliam, 2018). Noticing is regarded as a component of teacher expertise that facilitates the achievement of this goal. Specifically, the three elements of noticing play a crucial role in breaking down this process into actionable steps for teachers, thereby cumulatively supporting teachers in effectively engaging with students' thinking processes.

Expanding upon the connection between the second strategy of formative assessment and the noticing framework, the analysis of recommended teacher practices related to the second strategy within the teacher guides was guided by three interrelated elements of noticing. In Section 6.2.1, the focus is the analysis of the references for attending to and interpreting elements of noticing. These two elements of noticing are associated with the teachers' required knowledge of students' thinking that can facilitate teachers' noticing of students' multiplicative reasoning. The analysis in this section was guided by the knowledge of students' learning of multiplicative reasoning, which was discussed in Section 2.2.2 and the five strands of mathematical proficiency (Kilpatrick et al., 2001).

In Section 6.2.2, the shaping element was given a specific attention by expanding the original notion proposed by van Es and Sherin (2021). In the original framework, shaping refers to teachers' exploring students' learning further when they are interacting with students "in the midst of noticing" (p.23). These shaping practices play an essential role in teachers' in-the-moment formative assessment practices. Notably, the original framework primarily emphasises the teacher's role in shaping interactions, but it demonstrates limitations in addressing the tools and knowledge that can enhance these purposeful and productive interactions. The focus of Section 6.2.2 shifts to fill this gap, exploring the pedagogical practices that can facilitate shaping.

6.2.1. Attending to and interpreting students' multiplicative reasoning

As presented in Chapter 3, Lewis and her colleagues' (2011) comparative analysis of teacher guides from the US and Japan employed Ball and Cohen's (1996) well-acknowledged framework that presents five features of curriculum materials that can facilitate teachers' learning. Importantly, in this comparative analysis, they expanded on the feature of "anticipating students' thinking" by suggesting two sub-categories: anticipating a single correct answer and anticipating multiple responses and misunderstandings. In Section 2.2.2, students' challenges when learning multiplicative reasoning were discussed, and the main challenge to be focused on this thesis was identified as the shift from additive reasoning to multiplicative reasoning. The analysis in this section initially combines Lewis et al.'s approach of anticipating students' various responses and the knowledge of the development of students' multiplicative reasoning. Furthermore, to advance the notion of eliciting students' learning to noticing their mathematical thinking, as discussed in Section 2.2.2, the analysis presented in this section focused on the spectrum ranging from eliciting the correctness of students' answers to noticing the development of their multiplicative reasoning. The primary emphasis of the analysis in this section shifted towards examining the references in teacher guides positioned between these two ends and exploring the variations among these references. More explicitly, the references within the teacher guides that can facilitate teachers' attending to and interpreting various steps of the development of students' multiplicative reasoning were identified and discussed.

Understanding of the concept scale factor

Within the analysed teacher guides, teachers were equipped with example student responses, which can facilitate their attending to and interpreting students' understanding of the concept of scale factor. It was striking that while in some instances, teachers were only equipped with the expected correct responses from students, in other instances, they were equipped with a range of example student responses. In this section, examples for both instances will be presented and discussed in terms of the opportunities they potentially provide for teachers' attending to and interpreting students' understanding of the scale factor. Namely, these extracts were taken from the lesson "Mouthing Off and Nosing Around", in the CMP, and "Investigation 3: Scale Factor", in the CSM.

The following example instance is from CMP. In the original task, students are expected to work on the relationships between angles, lengths, perimeters and areas of similar rectangles and triangles. Namely, they are initially asked to identify similar polygons; and find scale factors, perimeters, and areas for these similar polygons. This is followed by further questions that require higher level reasoning such as exploring the relationships between the scale factor and the perimeters and the areas of the polygons.

Through this teacher guide, it was common to alert and equip teachers with expected correct student thinking and responses with limited opportunities for expected student thinking and responses which may not be fully correct. The following two extracts alert teachers for the expected student understanding of the concept scale factor.

> *As students analyse the diagram of noses and mouths from the Wump family, they will notice that scale factor only applies to similar figures. Scale factor does not apply to nonsimilar figures, as pairs of corresponding sides do not grow or shrink by the same factor.*
>
> *(CMP, Stretching and Shrinking, p. 98)*

> *For rectangles J and L, students may talk about the width growing by 2 and the length growing by 2. The perimeters also grow by a factor of 2. This gives you an opportunity to help students describe the growth in a different way. We say that the widths, lengths, and perimeters grow by a scale factor of 2.*
>
> *(CMP, Stretching and Shrinking, p. 102)*

These two extracts can facilitate teachers' attending to student responses that demonstrates their correct understanding of the scale factor. While the extract above introduced the expected reasoning from students, the extract below can facilitate teachers' attending to students' responses from which the formal definition of scale factor can be derived. This can help teachers make the instructional decision of introducing or reiterating the concept of "scale factor".

The following data extracts present provided student responses for specific questions in the same teacher guide. The original task is as following.

> *Describe the relationship between the perimeters of two similar rectangles and the scale factor.*
>
> *(CMP, Stretching and Shrinking, p. 117)*

In the teacher guide, teachers were not provided with example student responses to this original task. However, they were provided with a follow-up probing question and potential student responses to this probing question, as following.

> *Why does the perimeter grow the same way as the lengths of the sides of a rectangle? (Students should be able to explain that the perimeter is really a length, so it behaves like the width and lengths. Some might say that the perimeter =2(l+w), and if the scale factor is 2, then the new perimeter =2(2l+2w) and this is just double the original perimeter. A few students might recognise that 2(2l+2w) = 2x2(l+w) or that in the expression 2(2l+2w), the factor 2l+2w is the perimeter of the original rectangle).*
>
> *(CMP, Stretching and Shrinking, p. 103)*

In this data extract, teachers were alerted to the response expected from all students, which demonstrates their reasoning that the perimeter behaves as a length. This can facilitate teachers' attention to such responses. Moreover, it explicitly alerts teachers to two different algebraic expressions that some students might use to express their correct reasoning. These expressions would potentially prevent teachers from overlooking these responses. However, providing only examples for the algebraic expression of the expected reasoning might hinder teachers' attending to different variations of correct reasoning.

Furthermore, in this teacher guide, the opportunity for attending to students' incorrect reasoning and making instructional decisions based on this incorrect reasoning seems to be limited. This might lead teachers to attend to only students' correct responses and hinder their use of students' mistakes as learning opportunities.

The data extracts presented above closely align with the spectrum of anticipating students' correct reasoning. Nonetheless, effectively attending to and interpreting the development of students' multiplicative reasoning requires a deeper understanding of students' incorrect or insufficient reasoning as well as the steps of their correct reasoning.

This section now turns to the interpretation of the teacher guide from CSM that similarly focuses on students' understanding of the concept scale factor in geometrical similarity context. In contrast to the CMP lesson, the teacher guide in the CSM provided commonly expected student mistakes as in the following extract.

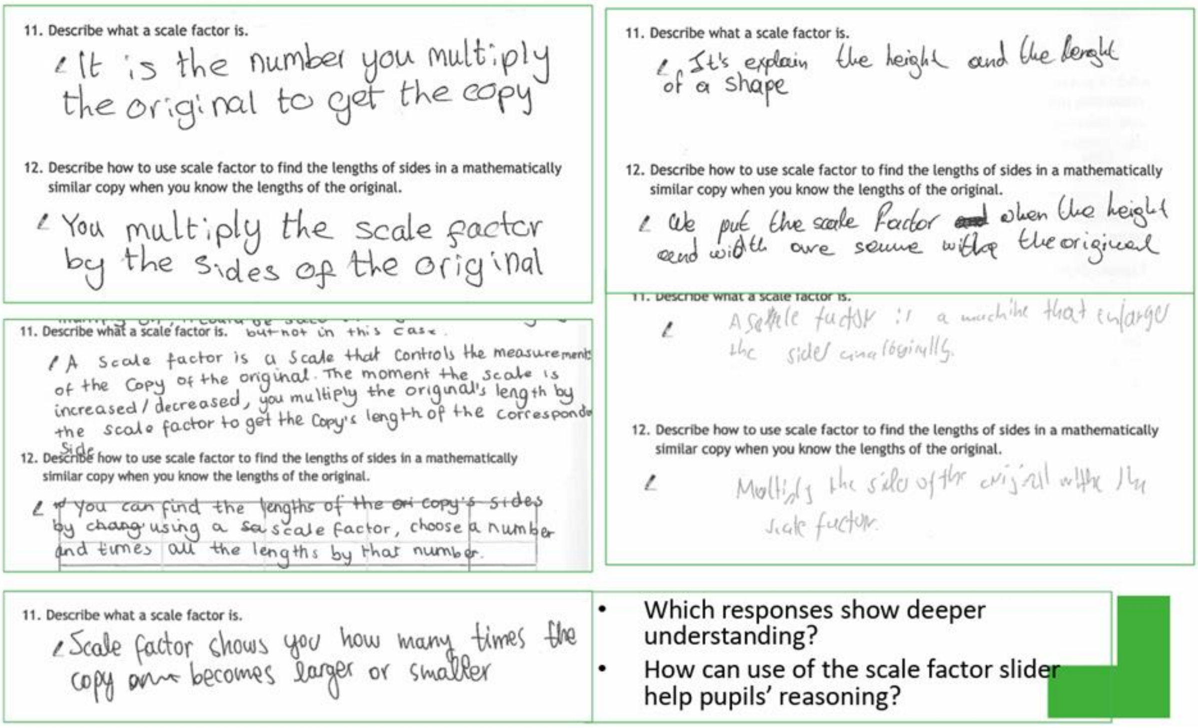
> *A common error is for pupils to do the setup ratios correctly but not in the correct order to see the relationship between the original and the copy, thus deriving the inverse scale factor (i.e., a scale factor of 2 instead of ½).*
>
> *(CSM, London trending: A module on similarity, Investigation 3: Scale factor, p. TN-15b)*

In addition to this knowledge of student mistake, in an additional material for teachers, teachers were equipped with example student responses to two questions, as following.

> *Describe what a scale factor is. Describe how to use scale factor to find the lengths of slides in a mathematically similar copy when you know the lengths of the original.*
>
> *(CSM, London trending: A module on similarity, Investigation 3: Scale factor, p. 18)*

In Figure 6.3, a range of student responses were provided in a complementary slide for use of teachers' professional development within schools. These slides are accessible to teachers as the teacher guides which are part of main sample of this research. These responses can help teachers attend to different levels of students' progress.

Figure 6.3 Example of varying student answers



Note. Reprinted from CSM, Geometric Similarity: Examples of pupil's work, Slide 4

The shift from additive reasoning to multiplicative reasoning

The key challenge students face when learning multiplicative reasoning is shifting from additive to multiplicative reasoning, as highlighted in Chapter 2. Teachers need to notice students' thinking processes during this shift beyond only attending to their correct or incorrect responses. In the teacher guides analysed, opportunities for teachers to attend to specific challenges students face during this process, such as identifying and applying decimal scale factors and functional relationships, were found. These opportunities will be exemplified in the following.

In the following extract, teachers could be alerted that while some students might be able to apply multiplicative reasoning with integer scale factors, they might resort to additive reasoning when faced with decimal scale factors.

> *Some students might recognise that the situation is multiplicative where it involves doubling but might resort to an additive approach where underlying multiplier is 2.5.*
>
> *(ICCAMS, Pre-test, p. 22)*

As discussed in Section 2.2.2 and illustrated in this extract, challenges with decimal scale factors might arise from an underlying additive reasoning. More than simply informing teachers about students' potential challenges with decimal scale factors and their incorrect responses, the extract offers deeper insights into the development of students' multiplicative reasoning. It suggests that intuitive strategies, such as doubling and halving, may represent an initial step towards developing multiplicative reasoning, as evidenced by Hart (1981). The teachers' noticing that students may recognise the multiplicative relationship when the scale factor is 2 offers an insight into this developmental reasoning process.

Students may face challenges in recognising the multiplicative relationship in functional as opposed to scalar relationships, as presented in the literature (e.g., Askew, 2018). This issue may signify a distinct stage in the development of multiplicative reasoning. The subsequent extract in Figure 6.4 potentially provides teachers with insights into this specific area of student challenge. Just as noticing students' struggles with identifying decimal scale factors, recognising their difficulties with functional relationships might enhance teachers' noticing and support the progression of students' multiplicative reasoning.

Figure 6.4 An example of students' difficulty in identifying functional relationship

The items below are on the *Mini Ratio Test*. We gave the items to comparable samples of secondary school students (mostly Year 8, $N = 77$ and $N = 74$). Item A turned out to be much easier than item B (with facilities of 91% and 51% respectively). This supports the conjecture that students prefer scalar relations, since this relation is much simpler for item A ($\times 3$) than for item B ($\times 2\frac{3}{11}$).

A

Ant is making a spicy soup for 11 people.
He uses 25 ml of tabasco sauce.

Bea is making the same soup for 33 people.
How much tabasco sauce should she use?

B

Ant is making a spicy soup for 11 people.
He uses 33 ml of tabasco sauce.

Bea is making the same soup for 25 people.
How much tabasco sauce should she use?

Note. Reprinted from ICCAMS, Post Shadows lesson, p.122

Moreover, teachers can attend to students' adaptive reasoning in the following extract with the knowledge that they may not transfer their conceptual understanding of multiplicative reasoning to different contexts.

Figure 6.5 An example of students' difficulty in reasoning in unusual contexts

Using different contexts

Students might recognise that mathematical relationships are multiplicative in a task set in one context, but not in another. Thus, they might recognise a recipe task as being multiplicative, but resort to additive strategies in a task involving geometric enlargement.

It is thus worth helping students to extend the range of contexts that they 'see' as multiplicative.

Note. Reprinted from ICCAMS, Post Shadows lesson, p. 121

6.2.2. Pedagogical practices that can shape interactions to uncover students' learning

The second strategy of formative assessment, "engineering effective classroom discussions and tasks that elicit evidence of learning" (Wiliam & Thompson, 2007), itself suggests classroom discussions and tasks as the pedagogical practices to elicit evidence for students' learning. The teacher guides provided references for both practices. Beyond these two practices, in the teacher guides, references for specifically monitoring and facilitating students' engagement with these tasks and discussions were found. I argue that this aspect is an essential element in order to enhance teachers noticing of students' mathematical thinking. In the following sub-sections, the pedagogical practices that can empower teachers' noticing of students' mathematical thinking were discussed in three categories: task choice, use of variations of tasks and probing questions, and monitoring and facilitating students' engagement.

Task choice: Open versus close tasks

The analysis of mathematical tasks revealed a spectrum of diversity, considering the openness of tasks based on both expected answers and methods (Yeo, 2015). As a result of the task analysis, I suggest that while open-ended tasks can potentially provide an opportunity to elicit various strands of mathematical proficiency and students' development of multiplicative reasoning, the tasks with close-ended nature can potentially enable teachers to focus on specific aspects of learning.

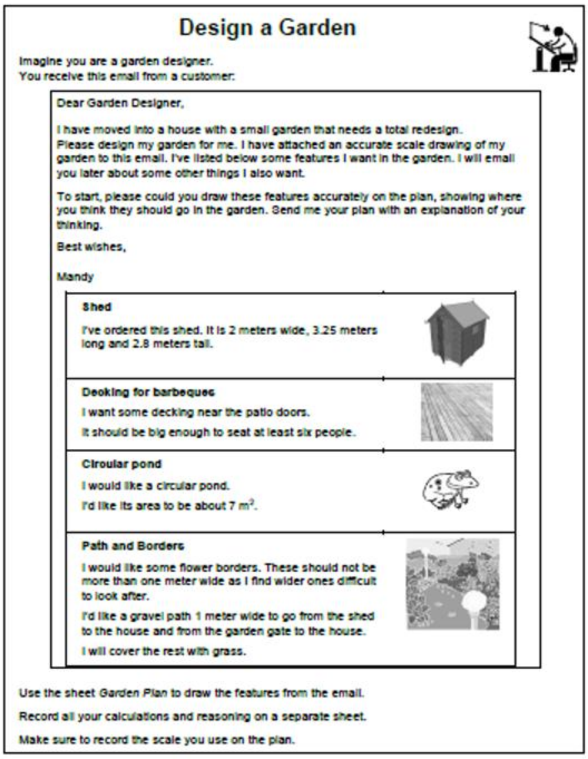
In this section, I present an in-depth analysis of the openness of selected tasks from each of the four sets of projects (i.e., MAP, ICCAMS, CMP and CSM) aiming to present the diverse noticing opportunities that can be supported by the inherent characteristics of the chosen mathematical tasks. For this reason, in this section, examples for four different types of tasks were chosen: open-answer and open-method task; open-answer and closed-method task; closed-answer and open-method task; and closed-answer and closed-method task. While choosing these tasks, the purpose was not to choose the tasks that can be representative of the project they belong but to choose tasks that can enable an analysis of the noticing opportunities inherent to various types of tasks.

The shaping opportunities for these tasks will be examined in three categories moving from open to closed tasks: open for both method and answer; open either for method or answer; and closed for both method and answer.

Open for both method and answer

The first example was chosen from the MAP project, as an example of an open-answer and open-method task. In the task presented in Figure 6.6, students are asked to design a garden according to a customer's request from a garden designer. In this task, students are expected to use scale factor in order to be able to position the four items with specific features in the garden plan appropriately.

Figure 6.6 An example open method and open answer task



Design a Garden

Imagine you are a garden designer.
You receive this email from a customer:

Dear Garden Designer,

I have moved into a house with a small garden that needs a total redesign.
Please design my garden for me. I have attached an accurate scale drawing of my garden to this email. I've listed below some features I want in the garden. I will email you later about some other things I also want.

To start, please could you draw these features accurately on the plan, showing where you think they should go in the garden. Send me your plan with an explanation of your thinking.

Best wishes,

Mandy

Shed

I've ordered this shed. It is 2 meters wide, 3.25 meters long and 2.8 meters tall.

Decking for barbeques

I want some decking near the patio doors.

It should be big enough to seat at least six people.

Circular pond

I would like a circular pond.

I'd like its area to be about 7 m^2.

Path and Borders

I would like some flower borders. These should not be more than one meter wide as I find wider ones difficult to look after.

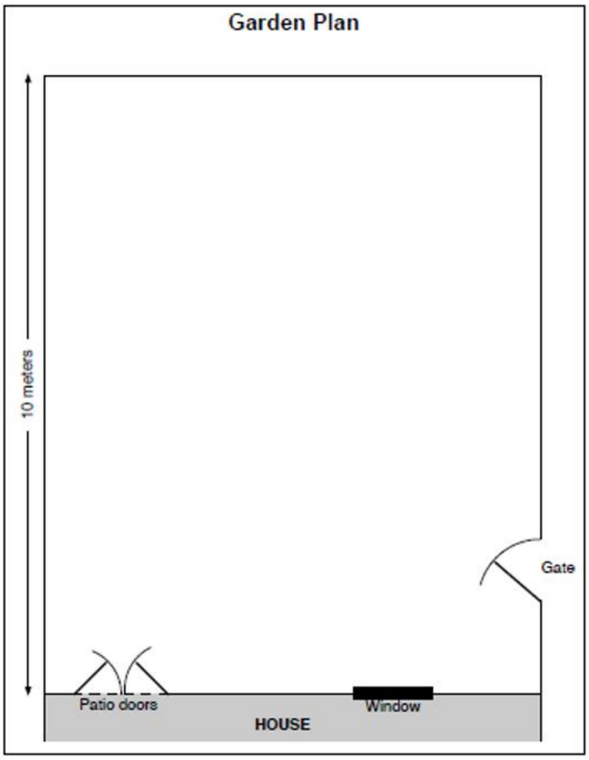
I'd like a gravel path 1 meter wide to go from the shed to the house and from the garden gate to the house.

I will cover the rest with grass.

Use the sheet *Garden Plan* to draw the features from the email.

Record all your calculations and reasoning on a separate sheet.

Make sure to record the scale you use on the plan.



Note. Reprinted from MAP, Drawing to Scale: Garden lesson, p. T-2 & T-3

This task is an open method and open answer task as it does not have only one correct answer or one specific method to get an acceptable answer. That is to say, different students might create different garden designs by using different strategies and reasoning.

A closer examination of this task uncovered insightful opportunities to facilitate classroom interactions that might reveal students' multiplicative reasoning across various aspects of mathematical proficiency. Initially, this task might be instrumental in shaping interactions that have the potential to demonstrate students' strategic competence. To clarify, students were presented with a task that required a considerable amount of time, but specific instructions on which procedures to follow were not provided. Importantly, students were required to formulate their own plans and decide on the procedures to follow. During this planning process, teachers might potentially engage with students' planning and observe the extent of their strategic competence.

This task could also facilitate shaping interactions that reveal students' adaptive reasoning as it requires making connections among concepts and situations and most importantly students are expected to report their reasoning in detail. When teachers engage with students' written or oral responses, it is likely that they would be able to observe students' logical thinking process that demonstrates how students link the concept of scale factor and related mathematical procedures to the specific context of garden designing and express their reasoning by using mathematical language. These elements would potentially enable teachers to attend to students' adaptive reasoning for the concept scale factor.

In addition to shaping interactions that reveal students' strategic competence and adaptive reasoning in relation to the scale factor, this task has the potential to enable students to demonstrate their conceptual understanding, procedural fluency and strategic competence in specific topics that may not be directly linked to applying the concept of scale factor but may be required to solve problems related to the scale factor in various contexts. More explicitly, in the garden design context in this task, students are expected to use procedures that will enable them to identify the areas of the items to be located in the designed garden. Specifically, for the shed to be located in the garden, the dimensions were given, and students are expected to calculate or estimate the area the shed will cover; for the decking area, students are expected to consider an area that will be enough for sitting six people; for the circular pond area, students are expected to calculate or estimate the diameter of the pond. Involving these various aspects of area measurement can enhance shaping interactions that can reveal students' conceptual understanding of the area of geometric shapes, procedural fluency by using the area formulas and strategic competence by formulating or modelling the area that is required for six sitting people.

The second example was chosen from CMP as a part of a series of linked questions, from the Unit "Stretching and Shrinking" and lesson "Mouthing Off and Nosing Around". As opposed to the approach in the MAP task that asks for a big task and expects students to create their own plan and strategies to find the answer, in this example, students are expected to work on 12 sequential small tasks that belong to the same context. While the majority of these 12 small tasks are closed-answer tasks (11 of them), more than half of them are open-method tasks (7 of them). By providing sequential and linked questions rather than expecting students to make their own plans to solve a big task, these small tasks are expected to provide limited opportunities to shape interactions to reveal students' strategic competence; however, they can provide opportunities to attend to other strands of mathematical proficiency. The following is an example of an open method and open answer task among these questions.

> *Draw three right triangles such that exactly two of the right triangles are similar. Explain how each triangle is similar or not similar to the other two.*
>
> *(CMP, Stretching and Shrinking, p. 117)*

This task can also have the potential to reveal students' strategic competence as they are asked to apply their conceptual understanding of similarity in a different context. In particular, in both tasks, students will have more than one strategy to follow, which can provide the opportunity to reveal students' various strategies.

However, it can be said that the "Drawing to a Scale" task can reveal higher-level strategic competence as students need to make a detailed plan for a bigger task.

The two tasks presented above could provide different opportunities in terms of students' productive disposition towards mathematics. These tasks show differences in terms of the use of context and the possible duration required to complete the task. That is to say, the drawing a scale of garden task uses a context from daily life while the second task is a question from pure mathematics. Moreover, this task involves multiple steps to solve, which may require a longer time to complete. These two distinguishing characteristics of that task have the potential to shape interactions to reveal students' dispositions towards mathematics. That is to say, teachers could identify whether students are able to patiently work on this long task and their perceptions of the usefulness of mathematics in daily life.

Open for either method or answer

The third example was chosen from an open method and closed answer task as follows.

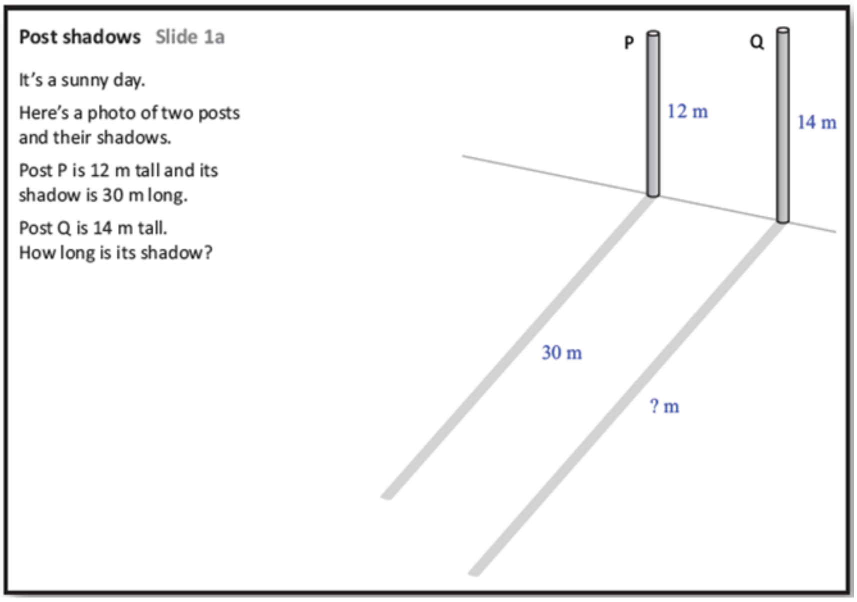
After studying the mouths in the diagram, Marta and Zack agree that Rectangles J and L are similar. Marta says the scale factor is 2. Zack says it is 0.5. Is either of them correct? How would you describe the scale factor so there is no confusion?

(CMP, Stretching and Shrinking, p. 117)

In this task, students are asked to move beyond the context of certain rectangles and triangles by providing the generalisations of the meaning of the scale factor and the procedure to find it. This task potentially reveals students' conceptual understanding of the scale factor. Moreover, in order to move to the abstraction to make generalisations, they would also need to use adaptive reasoning. Also, when they are explaining how to find the scale factor, they might reveal how appropriately they chose the procedures.

The fourth task was taken from the ICCAMS project (Figure 6.7). This task is a closed answer but an open method task. It potentially reveals students' conceptual understanding of similarity, procedural fluency which involves finding the length of the shadow and strategic competence which involves making a plan to solve this question.

Figure 6.7 An example of a closed answer open method task



Note. Reprinted from ICCAMS, Post Shadows lesson, p. 118

The fifth task was chosen from the CSM project, "Broken Scale Factor". During the lesson, students were expected to work with digital tools to observe the changes in the lengths of similar rectangle. With a similar approach to CMP lesson, students were asked a series of questions related to the key context. The task chosen among these questions is a closed-method and open-answer task, as following.

> *Use the two sliders to make a mathematically similar copy that is not the same size as the original. Explain how you know your copy is similar to the original. Write down the scale factor.*
>
> *(Investigation 4: Broken Scale Factor, CMP, p. 23)*

In this task, the method to solve the task was determined for students. That is, students were expected to use the digital tool to create similar polygons. This might restrict shaping interactions to reveal students' strategic competence. Moreover, as students will use digital tools to create similar shapes, there may be limited opportunities for shaping interactions to reveal students' procedural fluency. However, this restriction would potentially enable teachers to focus on the other three strands without being distracted by students' procedures or strategies.

Closed for both method and answer

The sixth task is an example for the closed-answer and closed-method task from CMP lesson. It was asked after students identify the similar rectangles along the series of questions.

> *For each pair of similar rectangles, find the scale factor and the perimeter and area of each rectangle.*
>
> *(CMP, Stretching and Shrinking, p. 117)*

This task could potentially reveal students' conceptual understanding regarding the relationships among the scale factor, and the perimeter and area of polygons. In addition, teachers might notice students' procedural fluency of using scale factor with the help of this task.

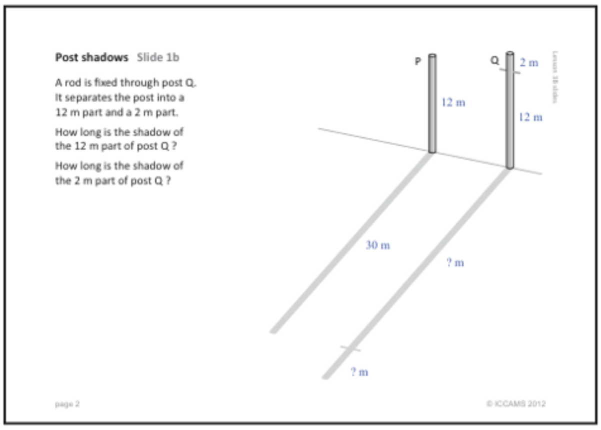
This task could also reveal students' strategic competence subject to students' prior learning. That is to say, in case students were not familiar with the calculations of perimeter and area, students would need to think strategically to answer the question.

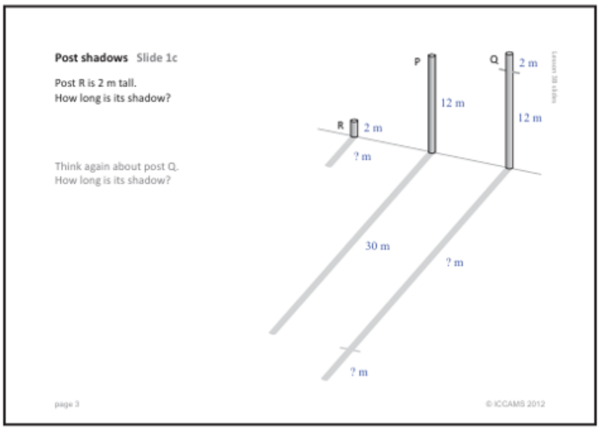
Using variations of tasks for further noticing

In this section, two examples of using variations of tasks will be discussed. These examples are considered as providing the potential to reveal students' development of multiplicative reasoning processes beyond merely eliciting the correctness of their answers.

In ICCAMS, in the "Post Shadows" lesson teachers are equipped with the variations of the main task that can show various thinking steps of students. The two tasks in Figure 6.8 are variations of the main task in Figure 6.7. In the teacher guide, these tasks were recommended to be used when students answer 32 which would possibly result from their incorrect additive reasoning. A potential reason for this additive reasoning might be the decimal scale factors between the length of the post and its shadow and between the length of two posts. The two variations of the main task separate 2-metre part of the longer post. This separation could encourage students to consider the scale factor between the 2-metre part of the post and the 12-metre length post. In this way, teachers might attend to whether students are able to identify a scale factor when it is a whole number. This could facilitate them attending to whether students cannot identify and use the scale factor at all, or they have difficulty because the scale factor is decimal.

Figure 6.8 An example of using variations of the main task





Note. Reprinted from ICCAMS, Post Shadows lesson, p. 123

In CMP in the "Mouthing Off and Nosing Around" lesson, teachers were advised to use the variations of the original tasks. In this lesson, a series of questions that encourage students' reasoning for the relationships between the lengths and areas of similar triangles (or rectangles) and the scale factor were provided as the main material. Teachers are advised to ask similar questions for quadrilaterals. These questions would potentially help teachers shape interactions that reveal whether students transfer their reasoning with triangles and rectangles to similar quadrilaterals.

Although both examples above might facilitate teachers' shaping interactions that reveal students' thinking further, each example shows differences in terms of the aspect of students' thinking. On the one hand, in the example in ICCAMS, the tasks provided might facilitate shaping interactions that reveal the source of students' mistakes. On the other hand, in the example in CMP, the tasks might facilitate shaping interactions that reveal whether students reason further in different contexts. While the former example might facilitate teachers' making decisions for dealing with students' incorrect additive reasoning, the latter example can facilitate teachers' eliciting students' correct reasoning.

Monitoring and facilitating students' engagement

In the previous sections, the potential shaping opportunities inherent to tasks were analysed. This section turns to an analysis of the opportunities for students' engagement which is an ordinary but potentially overlooked aspect of classroom practices that might enhance shaping. The students initially need to be engaged with the tasks to be able to demonstrate their learning. Another suggested practice involved enabling students' active engagement with the task. When students are disengaged from the task, it might become challenging for teachers to shape interactions that elicit students' understanding and effectively attend to students' thinking. By actively participating in the task, students would be more likely to demonstrate their thoughts, engage in discussions, and contribute to the collaborative process. This, in turn, might provide teachers with valuable opportunities to attend to students' thinking.

The analysis showed that materials might help teachers to facilitate students' engagement with tasks through six practices: familiarise students with the context, familiarise students with the mathematical concept informally, enable students to activate their previous formal and informal knowledge related to the task and make sure that students understand what is required in the task. In the following paragraphs, these practices will be exemplified.

The first practice is to familiarise students with the context. It is possible that when students are not familiar with the context, they might have difficulty engaging with the task. Students might be familiar with contexts they might commonly come across in daily life such as the similarity between the objects and their shadows. However, they might not be familiar with certain contexts such as planning a garden and designing computer games. In such cases, it would be important for teachers to make sure that their students are familiar with the context.

In the MAP lesson, Drawing to a Scale, at the beginning of the teacher guide, teachers present pictures of example garden plans to be shown to students. Additionally, they use prompts and questions to capture students' attention, such as asking, "Does anyone have a nice-looking garden?". These introductory practices might facilitate engaging students and encouraging them to share their relevant experiences, thereby engaging them with the problem at hand. These questions might prompt the discussion and help students who are not familiar with designing a garden.

The inclusion of visuals and thought-provoking prompts demonstrates a deliberate effort to foster students' connection to the topic. By drawing upon their own experiences and interests, students would be more likely to become actively involved in the task and exhibit a deeper level of engagement.

Figure 6.9 An example visual that can support students' engagement with the main task

Note. Reprinted from MAP, Drawing to Scale: A Garden lesson, p. T-2

The second example of practice for familiarising students with the context was found in the CSM lesson. In that lesson, an introductory activity that might help students familiarise themselves with the broken slider that they would use in the subsequent activities was involved. Although involving this activity will not ensure students' engagement with the activity, the lack of this activity might make students' engagement with the subsequent tasks more difficult. That is to say, if the first question asked to the students is to create similar copies of a shape by using the scale factor, students might feel overwhelmed as the scale factor does not work and they could leave the task incomplete.

The second practice was familiarising students with the mathematical concept informally. This is an example from the CMP lesson. In the initial stages of the lesson, teachers were encouraged to use a video that introduced the concept of similarity to students. On the CMP, in the "Mouthing off and nosing around" lesson, a launch video was provided to help students become familiar with similar figures. This video might help teachers shape students' engagement with similar shapes.

The third practice is enabling students to activate their previous formal and informal knowledge related to the task. As an example, on the CMP, the teacher guides for each lesson involved a section that aims to help students connect the current task with their prior knowledge. For example, in the “Mouthing off and Nosing Around” lesson, teachers are expected to have a generic discussion on similar rectangles, linking students’ prior knowledge about rectangles and their new learning about similarity.

In the materials, some tasks are related to these instances. For example, in the CMP, in a main task, students are expected to identify similar triangles and rectangles by using corresponding sides and angles. In the teacher guide, teachers are advised to challenge students to use corresponding sides and angles while identifying similar shapes as opposed to choosing similar shapes based on their appearance. In that example, students might not figure out how to identify corresponding sides and angles, which can hinder their thinking about similar shapes.

The fourth practice is making sure that students understand what is required in the task. In line with this, in the following page, teachers are equipped with questions to be asked to students who have difficulty to start the task. This is also related to student engagement. Teachers are also equipped with a set of questions that can inspire teachers for asking questions.

The following example questions and instructions are given to teachers in the beginning of the lesson that follows the assessment task. These questions and instructions can equip teachers to enable students’ engagement with the task.

Figure 6.10 Instruction to facilitate students’ engagement with the task

Begin the lesson by briefly reintroducing the problem:

Recall your work on the design task. What was the task about?

Today we are going to improve your work on this task.

I have looked over your papers and have some questions for you.

Work individually. Read my questions. Use the questions to figure out how to improve your work. Write notes on the sheet, or on the blank paper, about what you think will improve your work.

Note. Reprinted from MAP, Drawing to a Scale lesson, p. T-5

Another example can be given from a MAP lesson which requires students to draw a garden plan by using the concept of scale factor. As discussed already in previous section, this is an open task which requires students to use their problem-solving skills and make connections between different concepts. This can result in students' difficulty in starting the task. The teacher guide takes attention to this possibility.

6.3. Providing feedback that can move students' learning forward

In Section 4.5.4, I presented the challenges of identifying feedback references and the strategies when analysing teacher guides. As a reminder, the frequency of feedback references within the teacher guides was low, and some of these references were implicit. This resulted in even fewer references found in the teacher guides for the selected four lessons, as introduced at the beginning of this chapter. In order to identify more references for feedback, I revisited teacher guides for lessons other than these four selected ones. Thus, an analysis of a broader sample revealed more variety in references that can exemplify mathematics teachers' effective feedback practices.

Some of the references identified for feedback practices provided exemplifications of alternative feedback practices for teachers' commonly expected feedback practices. Namely, despite its potential limitations on students' long-term and high-level learning (Van der Kleij, 2015), teachers might be dominantly using immediate-corrective feedback (ibid.). The analysis of alternative feedback practices that can eliminate the limitations of immediate-corrective feedback will be presented in Section 6.3.1.

In Chapter 2, I presented Wiliam's (2018) argument that teachers play a crucial role in shaping the learning environment, while students themselves serve as the primary agents responsible for their own learning. Within the teacher guides, the implicit references for feedback exemplified this argument for mathematics teaching. These references were presented in Section 6.3.2. by highlighting the use of peers and mathematics tasks as sources for shaping environments for feedback for students. These implicit feedback references are linked to the idea of consequential feedback that was proposed by Quinlan and Pitt (2021).

Finally, the teacher guides provided references to share learning intentions regarding productive disposition. The references in the teacher guides related to this aspect of feedback will be presented in Section 6.3.3.

6.3.1. Questions and prompts as an alternative to immediate-corrective feedback

Immediate and corrective feedback is commonly expected in teachers' feedback practices (e.g., Antoniou & James, 2014). While these practices have potential to support students' procedural fluency and conceptual understanding, they could hinder other mathematical skills that require students' making connections deeper connections such as strategic competence and adaptive reasoning. In the teacher guides, teachers were equipped with questions and prompts that can be used as alternative practices to immediate corrective feedback. These questions and prompts could support students' strategic competence and adaptive reasoning beyond a mere focus on correcting students' answers.

As an example, rather than directly correcting the wrong answer students give, questions that help students reflect on their answers could be asked as follows.

> *How can you determine that your scale is/measures are reasonable? What's your evidence?*
>
> *(MAP, Drawing to Scale: Garden, p. T-4)*

In the teacher guide, this example question was recommended to teachers to ask students when they make a technical error. Asking this question still holds the feature of being immediate, but it could provide students an opportunity to find the reason for their mistakes on their own. Importantly, this question can encourage students to activate their adaptive reasoning when focusing on the reason for a technical error. Teachers, however, might find it difficult to use questioning for feedback if they and their students are not familiar with these practices. In the same teacher guide, as a way to mitigate this potential teacher challenge, teachers were equipped with direct instructions to use the questioning technique as feedback practice as follows:

> *Work individually. Read my questions. Use the questions to figure out how to improve your work. Write notes on the sheet, or on the blank paper, about what you think will improve your work.*
>
> *(MAP, Drawing to Scale: Garden, p. T-5)*

The second example for using questioning as a feedback practice was found in the CMP. In this teacher guide, teachers were equipped with a pre-lesson task to use before teaching geometrical similarity. This task involved students' sharing their estimations about a person's height whose photo was provided. Although teachers were guided not to judge students' estimations in this task, they were guided to act immediately once students gave answers that could not be a human's height. Similar to the example from MAP lesson, the suggested question in this teacher guide could encourage students to reflect on their answer, which cannot be a human's height.

> *If an answer is over 7 ft, is that reasonable? What about an answer under 4 ft?*
>
> *(CMP, Stretching and Shrinking, p. 48)*

These examples of using questioning for feedback are both alternatives to immediate and corrective feedback. The most important opportunity these questions provide could be encouraging students to reflect on their answers rather than directly receiving the correct answer. However, the two questions presented above could provide nuances. The example from CMP could be closer to traditional corrective feedback a large number of teachers might be used to. That is to say, the question is more structured by giving the clue for students that a human's height would usually be between 4 ft and 7 ft. The question provided on MAP lesson, however, has a more open approach by asking the evidence for reasonable answer to students. Both approaches could have benefits and limitations for students' learning in relation to students' prior experiences. As discussed in Section 2.2.3, the variation in students' prior knowledge influences how feedback affects their learning (e.g., Fyfe & Rittle-Johnson, 2016). Drawing on this evidence from literature, the nuances between questions on the MAP and CMP could have different effects on students with different prior experiences. On the one hand, the structured question on the CMP teacher guide could enable teachers to direct students' attention to the warrant related to a human's height. Although this could eliminate students' thinking deeply the warrant themselves, it can encourage students who do not have prior experience with this kind of reasoning to engage with the task. On the other hand, the open question on the MAP teacher guide could encourage students' who have prior experience with this sort of reasoning practices to think deeply about their responses and reflect on it.

6.3.2. Engineering feedback situations by using peers and tasks as feedback sources

As introduced in Section 2.2.3, Quinlan and Pitt (2021) proposed that specific sources such as the product and service users can provide consequential feedback for apprentices and these feedback opportunities may not be as solid as the information about students' performance that is directly provided. As mentioned earlier in this section, in the teacher guides, I identified explicit and implicit references for feedback. In the analysis that is presented in this section, I adopted Quinlan and Pitt's approach which considers the sources of feedback as a distinguishing aspect of the explicit and implicit feedback. As a result, the references that can provide consequential feedback for students when they are communicated with their peers and engaged with the mathematical tasks were identified. More explicitly, these resources can have potential to indirectly encourage students to reflect on their learning and refine their mathematical thinking.

Using peers as source of feedback

Peers can serve as valuable sources of feedback. The most common practice within the analysed teacher guides that could enable this was to create situations such as peer work and group discussion in which students can interact and engage with each other's thinking. In Figure 6.11, a collaborative task was presented as an example of these practices. In this task, when listening to each other carefully students can compare their thinking with their peers' thinking and reflect on their thinking as a result.

Figure 6.11 Example task where peers can be a source of feedback

Collaborating With Your Partner

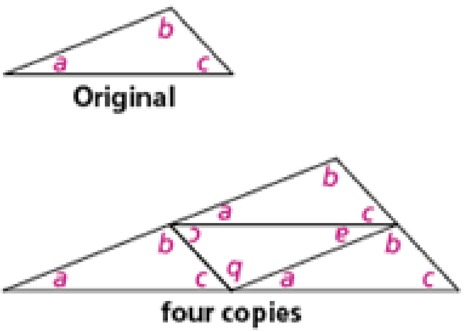
- Take turns explaining your *Garden Plan* to your partner. Explain how you would improve your solution. Listen to each other carefully.
- Ask 'clarifying questions' that will help you understand your partner's reasoning.
- When you have both taken a turn, decide how to design a new, better garden together.
- Draw your plan on the *Garden Plan* sheet and stick it in the middle of the poster paper.
- Use the space around the edge to write your reasoning, decisions, and calculations.

Note. Reprinted from MAP, Drawing to Scale: A Garden lesson, p. T-5

Using mathematics tasks as a source of feedback

In the teacher guides, it was found that teachers were equipped with mathematical tasks that could be feedback sources for students. These tasks could inherently provide feedback. The task in Figure 6.10 is an example for these tasks. In this task, students were expected to create an enlarged version of a triangle where the scale factor between the enlarged triangle and the original triangle would be 2. In this task, students might expect that when the scale factor between the lengths of the two triangles is two, the area of the enlarged triangle would be as twice as the original one. When students doubled the length and found that the enlarged triangle involves four of the original one, this may contradict their potential early thinking of that the enlarged triangle should involve two of the original one.

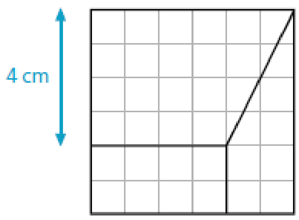
Figure 6.12 An example task that can be used as feedback source



Note. Reprinted from CMP, Stretching and Shrinking unit, p. 20

The task presented in Figure 6.11 is the second example for mathematics tasks that could be source of feedback. In this task from ICCAMS teacher guides, students were given a tangram that forms a square 6 x 6 and they are expected to enlarge this tangram so that 4 cm length will become 7 cm in the enlarged version. While working on this task, it is likely that some students use an inappropriate additive reasoning by adding 3 cm to each length rather than identifying and using the exact scale factor. The students who followed this wrong path would experience cognitive conflict when the pieces they formed do not fit in the larger square. This situation might encourage these students to assess the reasons for failing of their additive reasoning and reflect on their earlier thinking.

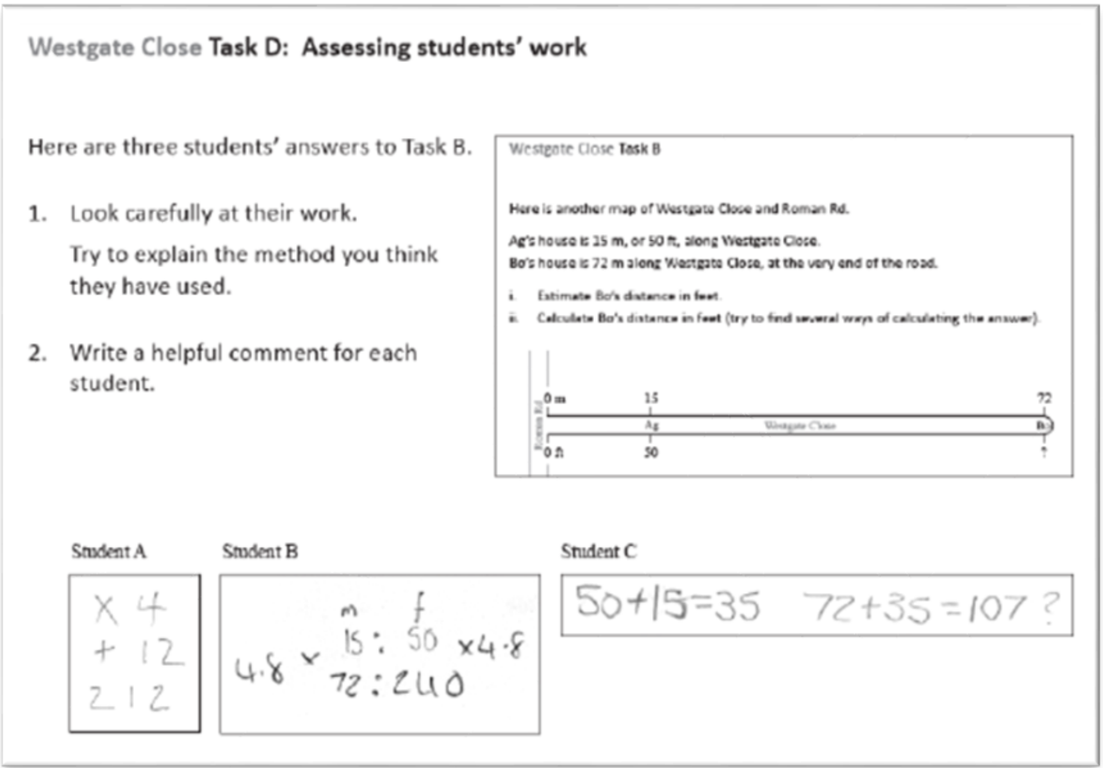
Figure 6.13 An example task that can be used as feedback source



Note. Reprinted from ICCAMS, Tangram lesson, p. 288

In addition to mathematics tasks that inherently provide source for feedback, teacher guides equipped teachers with task that can be a source of feedback for students by using scenarios that involve peers' thinking. Such tasks were found on the MAP, ICCAMS and WRM lessons. Within these tasks, students are allowed to work on example student work that involves several mistakes. After working on these examples, students can assess their learning by making comparisons with other imaginary works.

Figure 6.14 An example task that uses peer thinking as a feedback source



Note. Reprinted from ICCAMS, Westgate Close lesson, p. 94

In conclusion, all these three presented tasks might encourage students to reflect on their learning by using the characteristics of tasks.

6.3.3. Feedback for productive disposition

Within the materials beyond providing feedback regarding students' cognitive understanding of multiplicative reasoning, the teacher guides equipped teachers with example practices that can support students' productive disposition. As an example, in the ICCAMS lesson, after a potentially challenging test, teachers were guided to encourage students to work on the pre-assessment task even though the item in this assessment seems difficult to them as following:

> *The items vary in difficulty and you should alert students to this: they shouldn't worry if some items appear strange or difficult, and they should have a go nonetheless.*
>
> *(ICCAMS, Pre-test, p. 22)*

This guidance might not provide an explicit reference for feedback, as it does not explicitly refer to responding to students' answers. However, it could be an implicit reference for the recommended feedback approach in this teacher guide. Specifically, through this guidance, teachers could be alerted to the importance of students' perseverance in working on the task rather than getting the right answers in the first place. This could potentially enable teachers to avoid providing evaluative and corrective feedback, which could potentially discourage students from engaging with the main tasks.

The example guidance from a lesson guide on MAP can also exemplify an implicit feedback recommendation that can help teachers encourage students' perseverance rather than a focus on the correct answer quickly.

> *It is important that, as far as possible, students are allowed to answer the questions without your assistance. Explain to students that they should not be concerned if they cannot complete everything in the task. In the next lesson they will work on this material, which should help them to make progress.*
>
> *(MAP, Comparing Strategies for Proportion Problems, p. T-2)*

6.4. Summary and conclusion

The key purpose of this chapter was to identify and interpret the educative potential of teacher guides, focusing on the effective in-the-moment formative assessment practices. In particular, it was aimed to reveal recommended practices in the teacher guides as alternatives to teachers commonly used ineffective practices. With a focus on three identified elements of in-the-moment formative assessment practices a variety of opportunities for teachers were found.

In Section 6.1, two key findings in relation to identifying learning intentions were presented. First, it was found that while in CMP, WRM and ICCAMS the references for specific five strands of mathematical proficiency were dominant, in MAP and CSM these references seemed to be balanced. Second, it was found that as opposed to other sets of curriculum materials that involve relational learning intentions, WRM involved hierarchical and fragmented learning intensions. These findings suggested that these teacher guides could provide opportunities for teachers in terms of noticing students' learning with respect to different aspects of learning mathematics and providing feedback to enhance these aspects of learning.

This expectation was mostly met for noticing. The teacher guides presented explicit opportunities for teachers to notice students' conceptual understanding of scale factor, their reasoning, procedural fluency and strategic competence. The specific references were found in the teacher guides, addressing to all three elements of noticing with respect to four aspects of learning mathematics. These references include knowledge of students learning of multiplicative reasoning to attend to and interpret students' learning and using different features of mathematics tasks to create interactions for further noticing opportunities. Teacher guides also presented implicit references for noticing productive disposition by providing opportunities to monitor students' engagement with task.

Furthermore, for opportunities for feedback, it was found that the teacher guides offer alternatives to immediate corrective feedback. Questioning was found as a strong way of providing immediate feedback to students. However, it was found that the character of the questions can be closer to corrective feedback in some instances. Drawing on the literature regarding the relation between students' prior knowledge and how they benefit from feedback, this finding suggests that while open questions can be more beneficial for students who are experienced with such questions, the structured questions can be beneficial to encourage students' engagement who have limited experience with this sort of practices. Moreover, it was found that the instances that encourage students to delay feedback could be used opportunities for supporting students' productive disposition towards mathematics.

In conclusion, various opportunities for formative assessment support were found in the materials. When the findings were considered cumulatively for three aspects of formative assessment, it was interesting that while productive disposition was explicitly found as a learning intention within the teacher guides, the support for teachers' in-the-moment formative assessment practices for this learning intention was implicit.

CHAPTER 7 - THE EDUCATIVE POTENTIAL OF THE TEACHER GUIDES: A FOCUS ON COMMUNICATING PEDAGOGICAL MESSAGES TO TEACHERS

In Chapter 6, I presented the findings of a horizontal analysis of the educative potential of the teacher guides (Charalambous et al., 2010), focusing on the in-the-moment formative assessment practices recommended in these guides. This chapter shifts to a vertical analysis, which focused on the educative potential of the teacher guides for specific pedagogical messages. In particular, this chapter focuses on how teacher guides communicate specific pedagogical messages to teachers. This exploration addresses the second sub-question of RQ3, as follows:

How can teacher guides facilitate teachers' formative assessment practices for specific pedagogical messages?

The analysis in this chapter builds upon conclusions drawn from the findings discussed in Chapters 5 and 6. In Chapter 5, I argued that notwithstanding the inherent overall educative potential of the curriculum materials, there can exist a significant opportunity or limitation to effectively communicate specific pedagogical messages to teachers. This opportunity or limitation depends on the extent to which the three educative features are integrated with respect to the pedagogical messages. Namely, when all three educative features are integrated effectively within the materials, it is expected that the intended pedagogical message will be conveyed to teachers more efficiently. In Sections 7.1 and 7.2, this argument will be elaborated further by presenting the analysis of educative potential of selected teacher guides with respect to specific pedagogical messages.

The pedagogical messages identified in this chapter were initially derived from the recommended noticing and feedback practices that were already presented in Chapter 6. In order to refine the initially identified pedagogical messages, two elements in the teacher guides were double-checked: the key mathematical intention of the lesson and suggested practices linked to these intentions. As a result, three pedagogical messages for noticing and two pedagogical messages for feedback were identified to be examined in this chapter.

The focus of this chapter is the educative potential inherent in the teacher guides to communicate underlying specific pedagogical messages with respect to noticing and feedback. The pedagogical messages identified were used instrumentally in order to reveal variety in terms of the ways of including three educative features within teacher guides. This variety was analysed to interpret the varying potential of these teacher guides to communicate underlying pedagogical messages to teachers and facilitate their aligned classroom practices. That is, the degree of including three educative features for specific pedagogical messages varied in teacher guides.

7.1. Educative potential for the pedagogical messages with respect to noticing students' multiplicative reasoning

In the analyses presented in the subsequent sections, I focused on two lessons: "Drawing to a Scale: A Garden" from MAP and "Post-shadows" from ICCAMS. These lessons were chosen because they strongly provide opportunities to examine educative potential of the teacher guides related to teachers' noticing. They serve as instrumental cases to apply the framework I identified in Chapter 5, offering a nuanced understanding of educative potential rather than treating these lessons as cases for examination. I identified two different pedagogical messages on MAP and one on ICCAMS. This enabled me to compare similarities and differences in the educative potential with respect to different messages within the same teacher guide as well as those in different teacher guides.

In the remainder of this chapter, identified pedagogical messages will be referred to with the abbreviation "PM". Each subsection will start with a brief introduction of the pedagogical message, providing references from the teacher guide that informed identifying this message. This will be followed by an analysis of the educative potential inherent in the teacher guide with respect to this message.

PM 1: The open-method and open-answer tasks can facilitate teachers' noticing of students' strategic competence and adaptive reasoning in multiplicative reasoning.

Identifying the message

This message was identified on a MAP lesson, Drawing to Scale: A Garden. It was derived from the stated goal of the lesson, as following.

> *This lesson unit is intended to help assess how well students are able to interpret and use scale drawings to plan a garden layout. This involves using proportional reasoning and metric units.*
>
> *(MAP, Drawing to Scale: A Garden, page T-1)*

The coding scheme presented in Section 4.6.2 guided identifying the learning intentions referred to in this stated goal of the lesson. According to this, the goal of students' interpreting scale drawings was linked to adaptive reasoning and using scale drawings to plan a garden layout with strategic competence. The presented aim of this lesson was to assess students' abilities with respect to these learning intentions. Drawing on the recommended tasks and guidance throughout this teacher guide, I interpreted this intended assessment as a focus on noticing students' abilities. These will be detailed in the analysis part.

Alongside these explicitly stated goals of the lesson, the characteristics of the main task in this lesson informed the pedagogical message. That is to say, the main task of the lesson is designed as an open-answer and open-method task, allowing for multiple correct answers and applying more than one strategy. Although the authors do not explicitly state it, the implicit assumption in the teacher guide might be that this open task presents an opportunity to attend to students' strategic competence and adaptive reasoning and shape interactions that can reveal these skills. In the following paragraphs, an analysis of this teacher guide will be presented to critically examine its educative potential in conveying the pedagogical message to teachers and supporting them to use this open task to notice students' strategic competence and adaptive reasoning effectively.

Analysis of the educative potential

On the first page of this teacher guide, as already mentioned, the aim of the lesson has been explicitly presented by involving references for the learning intentions of strategic competence and adaptive reasoning. This is the first element that could potentially alert teachers to the overarching learning intentions of the lesson. This was supported by explicitly presenting the curriculum standards that this lesson addresses, as in Figure 7.1. The majority of these listed standards closely relate to strategic competence and adaptive reasoning. More explicitly, while the standards that highlight the skills of solving problems, modelling with mathematics, and using appropriate tools strategically relate to the strand of strategic competence, the standards that highlight skills such as abstract reasoning and analysing proportional relationships relate to the strand of adaptive reasoning.

Figure 7.1 The Common Core State Standards for Mathematics linked to the lesson

This lesson relates to the following *Standards for Mathematical Practices* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 3, 4, 5, and 6:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.G: Draw, construct, and describe geometrical figures and describe the relationships between them.
Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.EE: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

7.RP: Analyze proportional relationships and use them to solve real-world and mathematical problems.

Note. Reprinted from MAP, Drawing to a Scale: Garden lesson, p. T-1

In this lesson, teachers were equipped with an open-method and open-answer task as the main task of the lesson [1] (see Figure 6.6). It is expected that open-method and open-answer tasks might be challenging to be used by teachers. The first challenge could be to manage uncertainties that stem from various expected student responses. In this teacher guide, teachers were guided to identify students' difficulties before the lesson by giving the main task to students and attending to students' thinking before the lesson. This can assist teachers to shape interactions focusing on what they attended to during this pre-lesson assessment process. This practice can also assist teachers to attend to students' real difficulties beyond merely anticipating these difficulties relying on the experiences with those or other students.

When using an open method and open answer task it can also be challenge for teachers to enable students' engagement with the task. The teacher guide guided teachers to spend a certain time to help students engage with the task. Teachers were guided to attend to the students who have difficulties in terms of beginning the task, identifying the scale factor and making sense of the plan view.

Moreover, in this teacher guide, teachers were equipped with exemplary warm-up statements and questions as following.

> *Here are some pictures of gardens that have been designed. They are drawn carefully to scale so that the customer can get a good idea of what the finished garden will look like.*
>
> *Does anyone have a nice-looking garden?*
>
> *What do you have in it?*
>
> *Has anyone ever used a garden designer?*
>
> *(MAP, Drawing to Scale: A Garden, page T-2)*

---

[1] In Section 6.2.2, this task was already analysed for its potential to elicit students' mathematical proficiency. In this section, the focus is on the potential of the teacher guide to convey the message to teachers in relation to the benefit of using this sort of open task to notice students' mathematical proficiency as well as its potential to facilitate teachers' implementation in line with this pedagogical message.

Furthermore, this task requires students to work on their own without assistance to reveal their strategic planning. With this in mind, teachers are guided to monitor whether students do not begin the task because they do not understand what is asked. In this case, teachers are equipped with example questions that can help students understand the task, such as the following.

> *What useful information are you given? Underline this. What do you need to find out? How can you use the information you know to do this?*
>
> *(MAP, Drawing to Scale: A Garden, page T-4)*

Overall, this lesson guide is expected to have a high potential to communicate the message that open-method and open-answer tasks can enhance teachers' noticing of students' thinking and activate them to use these tasks. More importantly, this lesson guide considers teachers' potential difficulties and resistance to using these tasks. Equipping them with sufficient tools and knowledge and guiding them through how to utilise these tools and knowledge can activate teachers to use open methods and open-answer tasks.

PM 2: Students' collaborative work can facilitate teacher's noticing of students' mathematical thinking

Identifying the message

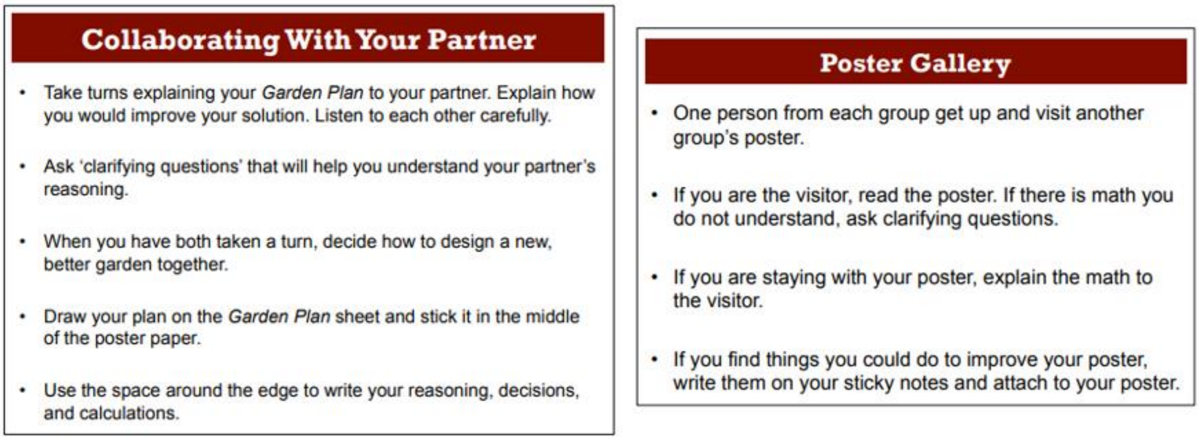
PM2 shares the same learning goal as PM1, which includes references to two strands of mathematical proficiency and teachers' noticing, as previously mentioned. In the teacher guide "Drawing to Scale: A Garden", a strong emphasis is placed on students' collaborative work, equipping teachers with example tasks and instructions, and guidance on facilitating effective collaborative work. This emphasis underlies the consideration of students' collaborative work as a key pedagogical practice in PM2.

Analysis of the educative potential

In this lesson, references to equip and guide teachers in encouraging students' collaborative work were explicit; however, the opportunities to alert teachers to the rationale of using collaborative work were implicit.

In this lesson, teachers were equipped with a two-step task prioritising students' collaborative work. The first step of the task requires students to share their solutions with their peers, collaborate to develop an improved solution, and prepare a poster to present their final solution. In the second step, students exchange their posters with other groups of students and engage with the posters prepared by other groups. Teachers were equipped with potential instructions to be used in the lesson through example slides, as in Figure 7.2.

Figure 7.2 An example collaborative task



Note. Reprinted from MAP, Drawing to Scale: Garden lesson, p. T-5 & T-7

The teacher guide does not explicitly express the purpose or underlying rationale of this collaborative task, which relates to the alerting feature; however, it provides narratives that direct teachers to focus on three aspects during task implementation: noticing students' strengths and difficulties, supporting students' reasoning, and fostering collaborative work. This guidance could implicitly suggest that one intended purpose of this collaborative work might be to enable teachers to notice students' thinking.

Like using open mathematical tasks, managing students' collaborative tasks could be a challenge for teachers, which might result in their hesitancy to enact these tasks. In this guide, teachers were provided with detailed guidance for leading students' collaborative work effectively. First, teachers were guided to promote active participation from all students within the group, which might help teachers notice more students' reasoning about the scale factor and its application. In order to promote students' active participation, strategies such as reminding them that they should all talk, actively listen to their peers, and take responsibility for each other's understanding are recommended. Additionally, teachers were guided to check that all students in the group contributed to the collaborative work by checking that different students' handwriting existed on the poster and that each student could explain the solutions that were written by other students. By promoting active participation from all students, teachers can create an environment that allows them to attend to the thinking of a diverse range of students rather than focusing on certain students.

The teacher guide emphasised specific risks and difficulties in leading group work. One risk is students potentially not working on their own but copying from their peers' thinking. Teachers were guided to move students to different seats when they were working individually. Teachers were also equipped with the information that the students who sit together usually have similar thoughts to say. Thus, by separating students and then letting them sit together again and discuss, it is more likely that they will have more productive discussions, which can reveal their thinking. Furthermore, teachers were guided to give students individual thinking time in order to facilitate all group members' active participation, as follows:

> *Before students work collaboratively, it can be helpful to give students individual 'thinking time'. This allows everyone to have time to construct ideas to share and avoids the conversation being dominated by one student.*
>
> *(MAP, Using Proportional Reasoning, p. T-4)*

Overall, this teacher guide shows strong elements of the educative features "equip" and "guide". This can facilitate teachers' effectively managing students' collaborative work. More importantly, this teacher guide demonstrated strong references for anticipating teachers' potential challenges with managing students' collaborative work and guided teachers to eliminate these challenges.

While references for the alerting feature for this pedagogical message seem to be implicit within the teacher guide, the rationale for encouraging students' collaborative work through the project was provided on the project website separately as a part of professional development resources (Mathematics Assessment Project, n.d.), as follows:

If students are to make sense of mathematical concepts, then they will need opportunities to share, discuss and work together. Research has shown that cooperative small group work has positive effects on learning, but that this is dependent on the existence of shared goals for the group and individual accountability for the attainment of these goals. It has also been seen to have a positive effect on social skills and self-esteem (Askew & Wiliam, 1995).

(Students' Working Collaboratively, Professional Development Module, MAP)

This statement suggests that students' collaborative work can provide opportunities for students' learning of mathematical concepts, social skills, and self-esteem if used effectively; however, it provides limited insights into the benefits of students' collaborative work for specific elements of teaching and learning mathematics. In fact, the benefits of these collaborative works can go beyond the surface. Effective implementation of these collaborative tasks can potentially facilitate students' active engagement with tasks and shape interactions that can provide further information about students' thinking. Alerting teachers to the specific benefits of these practices could enhance the possibility of triggering teachers to implement these challenging tasks.

PM 3: Use of diverse contexts and variations of the main task can enable noticing students' multiplicative reasoning

Identifying the message

This pedagogical message, PM3, was identified in the "Post Shadows" lesson on ICCAMS. It was derived from the explicitly expressed goals of the lesson, as follows:

In this lesson we consider ratio in another geometric context, this time involving shadows. By presenting students with appropriate variants of the basic task, students are likely on occasion to adopt (or at least consider) an additive or a mixed additive/multiplicative approach.

(ICCAMS, Post Shadows, p. 119)

In this statement, the rationale for using variations of the main task was clearly articulated: to reveal students' reasoning processes, which may involve additive reasoning, multiplicative reasoning, or a combination of the two, as they engage with these tasks.

Analysis of the educative potential

This teacher guide encourages teachers to use two types of variations when teaching multiplicative reasoning: altering the context and using variations of the main mathematics task. In this section, I will examine the educative potential of this teacher guide for communicating the message to teachers that has the potential to trigger them for these practices and facilitate their effective implementation.

First, teachers were alerted to the deliberate change of context in teaching multiplicative reasoning by statements such as, “this lesson considers ratio in a fresh context” (p. 118) and “in this lesson we consider ratio in another geometric context” (p. 119). Moreover, teachers were alerted to the rationale of the change of context by following the narrative.

> *Students might recognise that mathematical relationships are multiplicative in a task set in one context, but not in another. Thus, they might recognise a recipe task as being multiplicative, but resort to additive strategies in a task involving geometric enlargement.*
>
> *(ICCAMS, Post Shadows, p. 121)*

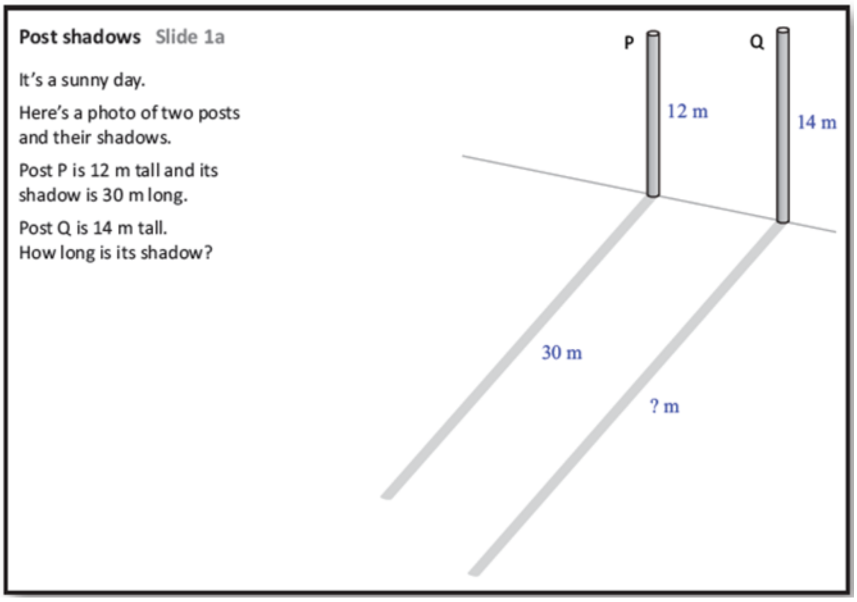
This statement also equips teachers with knowledge of student thinking. That is, it was explicitly stated that students could potentially face difficulty in recognising multiplicative relations in contexts other than the recipe context.

Teachers were equipped with specific characteristics of the main task and they were alerted to the rationale of using these variants, as in the following statement:

> *The tasks include situations where such approaches come into conflict with normal expectations (e.g., by producing a shadow that is much too long) or with other salient relations (e.g., trebling), and students are encouraged to discuss their methods and solutions, to allow such conflicts to emerge.*
>
> *(ICCAMS, Post Shadows, p. 119)*

This statement highlights the main task, in Figure 7.3, involve situations students may not expect. Authors highlighted that this character of the task provided an opportunity for conflicts emerged among students. Teachers are guided to encourage students to discuss their methods and solutions. These discussions can potentially maximise the possibility of students' attending to a variety of solutions and reasoning among students.

Figure 7.3 The main task in "Post Shadows" lesson in ICCAMS

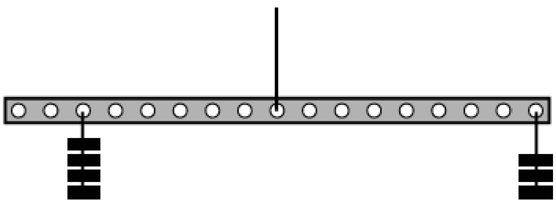


Note. Reprinted from ICCAMS, Post Shadows lesson, p. 118

The pedagogical message in relation to change of context was even strengthened by equipping teachers with further different contexts, where multiplicative reasoning could be used. The Figure 7.3 presents only one example for these contexts.

Figure 7.4 An example alternative context for applying multiplicative reasoning

A context that might seem more remote from 'shadows' is the see-saw balance, where the independent variable could be the left-hand mass or its distance from the fulcrum, and the dependent variable the right-hand mass or its distance from the fulcrum.



Note. Reprinted from ICCAMS, Post shadows lesson, p. 121

In addition to the practice of altering context, this teacher guide presents educative potential for the practice of using variations of main tasks to notice students' multiplicative reasoning. Alongside being alerted to the rationale of using variations of tasks by the statements presented earlier, teachers were equipped with a main task that requires multiplicative reasoning to be solved. The noticing opportunities this task and its variants provided were already discussed in Section 6.2.2. In this section, the focus is the educative potential of the teacher guide with respect to conveying the underlying pedagogical message to teachers rather than re-examining this task.

The teacher guide equipped teachers with seven variants of the main task, as two of them already presented in Figure 6.8 in Section 6.2.2. Teachers were also guided on the effective use of these variants. Namely, they were guided to use the results of a pre-assessment test to anticipate the appropriate variation of the tasks for their students. This pre-assessment test was part of the set of teacher guides. It involved tasks that required using multiplicative reasoning to solve problems in different contexts including recipe, currency and geometry. It was expected that the use of this test could reveal the strategies students used when solving multiplicative reasoning problems in different contexts and the challenges they faced in particular. Through equipping teachers with these tools (the pre-assessment test and the variations of tasks) and guiding them to link these tools, teachers' attending to their students' thinking and interpreting this thinking is potentially facilitated.

Overall, this lesson demonstrates strong educative potential for PM3. In particular, it demonstrates explicit references that can alert teachers to the message and equip them with tools and knowledge that can facilitate their enactment.

7.2. Educative potential for the pedagogical messages for creating feedback situations

As mentioned earlier, in the teacher guides, the references for recommended feedback practices were not as explicit as the ones for teachers' noticing. As a result, in this section, only two pedagogical messages were identified.

PM 4: Using questioning rather than evaluative feedback can contribute to students' productive disposition

Identifying the message

In the "Drawing to Scale: A Garden" lesson in MAP, it was an explicit message that teachers should not provide evaluative feedback for formative assessment. The evaluative feedback was exemplified as scoring students' work. This message was identified through the statement underneath the section "assessing students' responses" in this teacher guide. In this section, teachers were explicitly advised not to score their students' work in the provided pre-assessment task but to use questioning instead. Moreover, I linked this message to productive disposition. The reference that informed this is the following statement:

> *The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.*
>
> *(MAP, Drawing to Scale: A Garden, p. T-3)*

The focus on the issue of distracting students' attention from improving their work might be an implicit reference for the potential negative impact of evaluative feedback on students' productive disposition.

Analysis of the educative potential

First, in this teacher guide, teachers were explicitly alerted to the rationale of choosing questions rather than providing evaluative feedback by explicitly stating that when teachers score students' work students would likely compare their scores with their peers' scores, and this could switch their attention from improving their learning of mathematics to comparing their scores with their peers' scores. Furthermore, teachers were equipped with research evidence that can facilitate conveying this message to teachers.

Beyond alerting teachers to the potential harm of scoring students' work and equipping them with research evidence to support this message, teachers were equipped with example questions to be asked in various situations as an alternative to evaluative feedback, as in Figure 7.5.

Figure 7.5 Example questions suggested as an alternative to scoring students' work

Common issues	Suggested questions and prompts
Has little or no written work	• Which direction are you looking from when you make a plan view? • What useful information are you given? Underline this. What do you need to find out? How can you use the information you know to do this?
Lacks precision in their work For example: The student writes the scale in words or as an equality '2 cm is/= 1 m'.	• What do you mean by 'is' / 'equals'? Can you be more precise?
Has difficulty in expressing a scale For example: The student does not calculate the scale ratio or has calculated it incorrectly or drawn objects with no apparent calculations of scale. For example: The student has the scale ratio as 1 : 200 or 1cm : 4m. Or: The student writes the figures in reverse order, for example 50 : 1 instead of 1 : 50. Or: The student uses fractions in the scale ratio, for example 1 : 0.5. Or: The student has not calculated a unitary scale ratio, for example 2 cm : 1 m.	• How long is the real-life garden in centimeters? How long is the drawing in centimeters? Can you use this to figure out the scale? • If an object on the plan is 1cm long, how long is the real object? How do you know? • Usually a scale ratio is written in the order length on the plan : length in real life. What is your scale ratio when written this way? • Can you write the scale ratio more simply, without using fractions or decimals?

Note. Reprinted from MAP, Drawing to Scale: A Garden lesson, p. T-4

Moreover, this teacher guide anticipated teachers' potential difficulties with questioning. That is to say, teachers might think this way of providing feedback is time consuming. Teachers were guided to implement questioning in more practical way when they do not have sufficient time for asking these questions to students individually. To be more precise, they were guided to identify a set of questions based on students' responses on the pre-assessment task. The suggested practical ways of using these questions were either to write one or two questions on each student's assignment or to give each student the list of questions, highlighting the questions relevant to their particular responses. This might address the possibility of teachers' rejecting using questioning rather than evaluative feedback when they consider this alternative practice as time consuming.

In summary, this teacher guide is considered as having strong educative potential for PM3, which may increase the possibility of communicating this message to teachers and activating them to employ practices in line with this pedagogical message. However, it should be noted that the reference that can be linked to productive disposition is implicit.

PM 5: Immediate feedback can prevent students' mislearning and encourage their engagement with the main task

Identifying the message

PM5 was an implicit message compared to other identified pedagogical messages in this chapter. The reference for this message was found as "…answers that are obviously unreasonable should be examined closely and efforts should be made to figure out why they are incorrect" (p. 48), in "Mouthing off and Nosing around" lesson in CMP.

Analysis of the educative potential

In this teacher guide, teachers were advised to address students' mistakes that are unreasonable immediately. The task asks students to estimate a person's height. Although teachers were advised not to judge students' estimations in this task, they were advised to act immediately once students gave answers that could not be a human's height.

Teachers were equipped with example questions to be asked to students in that case such as, "If an answer is over 7 ft, is that reasonable? What about an answer under 4 ft?" This question can help students to reflect on their previous answer, which cannot be a human's height. In this example, students might have an opportunity to assess and correct their mislearning at the beginning of the lesson. Moreover, that question can help students assess their belief in the connection between daily life practice and what they will learn in this class.

In this example, an explicit reference for alerting teachers to the importance of preventing students' mislearning or the function of immediate feedback was limited. Thus, this message is considered as an implicit message for which educative potential could be limited.

7.3. Summary and conclusion

In this chapter, a vertical analysis focused on the educative potential of teacher guides for specific messages rather than overall educative potential of these teacher guides was presented. The findings suggest that educative potential of a teacher guide for different pedagogical messages can vary. Moreover, I argue that in case all three educative features are present the possibility of communicating the pedagogical message to teachers and facilitating their aligned practices could be high.

In Section 7.1, two lessons were deliberately chosen as they provided educative potential to trigger teachers for specific pedagogical messages with respect to noticing: “Drawing to Scale: A Garden” from MAP and “Post Shadows” from ICCAMS. The analysis demonstrated that the educative potential for identified two messages on the MAP lesson could vary. The stronger existed educative feature was found as *guide* for the pedagogical message “Students’ collaborative work can facilitate noticing students’ multiplicative reasoning. However, for this strongly recommended pedagogical practice, underlying pedagogical message with respect to its benefit was not clear.

A strong reference to the educative feature of “guide” could be linked to Davis et al.’s (2017) design principles that were based on the idea of that educative materials could anticipate how teachers adapt materials and help them with productive adaptations. The analysis in this chapter showed that MAP lesson anticipated teachers’ potentially rejecting using questions instead of evaluative feedback due to the time restrictions. This teacher guide could help teachers’ productive adaptations by anticipating the potential reason to reject the suggested practice

CHAPTER 8 - DISCUSSION AND CONCLUSION

This final chapter of the thesis serves to provide an overview of the research process and synthesise the key findings; articulate the contributions to knowledge; reflect on the limitations of this study in order to guide readers towards a more precise interpretation of the findings; and suggest the implications for policy, practice, and future research.

In order to address these aims, this chapter starts by briefly reiterating the research problem, analytical procedures, and the key findings of the research (Section 8.1). This is followed by presenting a synthesis of the key findings and an articulation and discussion of the contribution these findings make to the fields of formative assessment and curriculum materials within mathematics education (Section 8.2). Moreover, this chapter presents a critical review of the research methods employed in this study to highlight its potential limitations for further interpretation (Sections 8.3). Finally, the implications for policy and practice, as well as suggested avenues for future research, are discussed (Sections 8.4 and 8.5).

8.1. An overview of the study

8.1.1. Research problem and related research questions

In this section, I will reiterate the central research problem that has been discussed in previous chapters of this thesis. As introduced in Section 1.1.4, and further elaborated on in Chapter 2 and Chapter 3, this research addresses the need to enhance the in-the-moment formative assessment practices of mathematics teachers. In Section 1.1.1, the key research problem was introduced: the vague conceptualisations of formative assessment in the literature. This vagueness can result in misinterpretations and, thus, hinder the effectiveness of teaching practices. Although this was identified as problem by Bennett (2011) over a decade ago, it remains prevalent in the more recent literature. Indeed, the problem can even be exacerbated by teachers' potential resistance to changing their conventional practices, such as a tendency to focus on evaluating correctness of students' responses rather than the development of their mathematical thinking. The problem of vagueness was further explored in Section 2.1 by reflecting on the existing conceptualisations and models for formative assessment, which may not specifically guide teachers' discipline-specific needs. Then, in Section 2.2, teachers' potential tendencies regarding two discipline-specific elements of formative assessment, noticing and feedback, were discussed. Specifically, it was highlighted and explained that an emphasis on the correctness of students' responses can hinder teachers' ability to notice students' transition from additive reasoning to multiplicative reasoning, and lead to immediate-corrective feedback (Antoniou & James, 2014).

Wiliam and Thompson's (2007) five formative assessment strategies represent a well-acknowledged framework that has considerably contributed to addressing the vagueness problem. Acknowledging this framework as a useful theoretical guide and drawing on Bennett's (2011) critique of the need for a discipline focus in guiding formative assessment practices, I focused on possible ways to operationalise these strategies. This was the main concern of the first research question, which was introduced in Section 1.4.1, as follows:

RQ1: How can the five strategies of formative assessment suggested by Wiliam and Thompson (2007) be operationalised so that they can guide mathematics teachers' in-the-moment formative assessment practices?

The problem of teachers' resistance to changing their potentially ineffective formative assessment habits was explored through a synthesis of the theorisations on the relationship between teacher and curriculum material, with a specific focus on educative curriculum materials literature and socio-cultural perspectives on this relationship. In the existing literature, educative curriculum materials are conceptualised as materials that can have the potential to enhance teachers' professional learning alongside students' learning (Davis & Krajcik, 2005). The ongoing research has already proposed specific educative features (Ball & Cohen, 1996; Davis et al., 2017; Remillard et al., 2020). These features highlight the aspects of the materials that encompass expected teacher practices, such as "anticipating students' responses". However, they do not explicitly differentiate between the roles of teachers and of the materials themselves. The two current dominant socio-cultural approaches that theorise the relationship between curriculum resources and teachers, namely the participatory approach and documentary approach, both identify a two-way relationship between teachers and curriculum materials (Gueudet & Trouche, 2009; Remillard, 2005). Gueudet and Trouche's documentational approach provides a conceptualisation of the two directions of this relationship by suggesting two concepts: *instrumentation*, the direction from the resource to the teacher, and *instrumentalisation*, the direction from the teacher to resources. Drawing on the notion of instrumentation, educative features that directly refer to a participatory relationship between teachers and curriculum materials were investigated in the present research. The second research question, as introduced in Section 1.4 and reiterated below, guided this investigation:

RQ2: What educative features of curriculum materials can be suggested?

The third research question arose from the cumulative insights obtained through the investigation of the first two research questions. In the initial explorative phase, a framework was developed to facilitate the analysis of curriculum materials, with a focus on their potential for guiding in-the-moment formative assessment practices. The next research problem emerged in relation to the practical application of the proposed framework to unveil the educative potential inherent in the materials. The following research questions guided the investigation in this second, critical phase of the study, determining the trajectory of the subsequent investigation.

RQ3: How does the developed framework contribute to the understanding of the educative potential of teacher guides to facilitate mathematics teachers' in-the-moment formative assessment practices?

- *What are the characteristics of the educative potential inherent to the teacher guides that facilitate teachers' in-the-moment formative assessment practices, specifically when approached from a horizontal perspective?*

- *How can teacher guides facilitate teachers' formative assessment practices for specific pedagogical messages?*

8.1.2. Research design, analytical procedures and key findings

The research was conducted in two major phases, as summarised in Table 8.1. Each of these phases involved smaller steps that emerged throughout the research as a need to address overarching research questions and iterative processes intended to enhance the rigour of the analyses. The following two sub-sections will elaborate on the methodological strategies and the findings of each phase.

Table 8.1 Summary of the research phases

Phase	Related research questions	Analytical approach	Related literature	Key findings
Explorative	RQ1 and RQ2	Reflexive thematic analysis Follow-up analysis with a focus on feedback	Formative assessment Teaching and learning mathematics Teachers' professional development Educative curriculum materials Feedback Noticing	Operationalising five formative assessment strategies for mathematics teachers' in-the-moment formative assessment practices Proposing three educative features: alert, equip, and guide
Critical	RQ3	Employing the framework that was developed in the first phase Adapting the constant comparison technique to evaluate educative potential for formative assessment in the	Revisiting more recent literature with a focus on teacher guides A focus on the relationship between teachers and curriculum materials A focus on the formative assessment inherent to the existing teacher guides	The analysis of the teacher guides uncovered varying and conflicting learning intentions in terms of five strands of mathematical proficiency. The analysis of teacher guides uncovered a range of references for noticing, spanning from eliciting the correctness of students' reasoning to noticing the

	teacher guides	development of multiplicative reasoning. The analysis of teacher guides uncovered a range of references, spanning from immediate-corrective feedback to creating feedback situations that can move students' learning forward. The presence of the three educative features – alert, equip, and guide – varied in the teacher guides across different pedagogical messages. While for some messages all features were present, for others only one or two of them were present.

Note. Created by the author

Explorative phase (RQ1 & RQ2)

The primary methodological approach employed to address the first two research questions involved a reflexive thematic analysis (Braun & Clarke, 2021) complemented by an analysis of the feedback component integrated within the mathematics curriculum materials for formative assessment. My emphasis on the teacher guides stems from the assumption that teachers' utilisation of these resources plays a pivotal role in shaping their instructional practices (Matic et al., 2021). Specifically, I selected teacher guides from high-quality materials that were either research-based and designed for mainstream teachers (CSM, ICCAMS and MAP); widely recognised and used by mainstream teachers (WRM); or that embodied both features (CMP). The deliberate choice of these materials was made with the expectation that they would offer illuminating examples of mathematics-specific formative assessment practices and educative features.

Two key outcomes of the explorative phase informed the procedures in the second critical phase. The first key outcome of the explorative phase, addressing RQ1, was the operationalisation of five formative assessment strategies for mathematics teachers' in-the-moment formative assessment practices. With the help of explorative thematic analysis, the teacher guides were examined with reference to each of Wiliam and Thompson's five strategies (2007). Following this, noticing students' learning and providing feedback that can move students' learning forward were identified as the core in-the-moment formative assessment practices. Moreover, teachers identifying learning intentions was considered as a prerequisite of these two core practices. These three identified elements of teachers' in-the-moment formative assessment practices were designated as focal points for the analysis in the critical phase.

The second key outcome of the explorative phase, addressing RQ2, which also informed the critical phase, was the identification of three educative features: alert, equip, and guide. These features built on the two educative features that had already been identified in the literature, namely rationale and guidance (Quebec-Fuentes & Ma, 2018), and the codes that emerged during the reflexive thematic analysis in this study (as presented in Section 5.1.3). To reiterate, *alert* refers to triggering teachers to enact the suggested practices by providing the rationale behind these practices; *equip* refers to providing sufficient tools and knowledge to enact these practices; and *guide* refers to providing instructions that can facilitate teachers' effective use of these tools and knowledge. These features guided the analysis of the educative potential inherent to the teacher guides in the critical phase of the research.

Critical phase (RQ3)

In the critical phase, I adopted a qualitative approach aligned with Braun and Clarke's (2021) notion of "big Q". As opposed to the "small q" approach influenced by a positivist paradigm, where the subjectivity of the researcher is viewed as a potential threat to the quality of data analysis, the "big Q" approach considers researcher subjectivity as providing an opportunity for in-depth analysis. To complement this qualitative approach, I integrated Charalambous et al.'s (2010) analytic method, which involves analysing curriculum materials from both horizontal and vertical perspectives. This combined approach guided the analysis of the educative potential of curriculum materials. While the analysis from a horizontal perspective involved an overview of the patterns in teacher guides and distinct examples of the elements of teachers' in-the-moment formative assessment practices, the vertical analysis involved a focus on the educative potential for specific pedagogical messages. In terms of the educative potential of specific pedagogical messages, I argue that the possibility of communicating pedagogical messages to teachers and prompting them towards intended practices is potentially higher in the case that all three educative features are present in specific pedagogical messages. The vertical analytical approach is grounded in this argument. The overview of the findings is presented in Table 8.1 and a detailed presentation of the findings was provided in Chapters 6 and 7.

To summarise, first, the analysis from a horizontal perspective revealed a spectrum in terms of the integration of formative assessment. Specifically, there was variation from hierarchical and fragmented learning intentions to relational learning intentions. For the two core in-the-moment formative assessment practices, this spectrum encompasses a range of approaches, from a focus on the correctness of students' answers to engaging with their thinking. The interpretation of data was grounded in the five strands of mathematical proficiency (Kilpatrick et al., 2001), the literature on students' learning of multiplicative reasoning (e.g., Lamon, 2007), the noticing framework (van Es & Sherin, 2021), and feedback literature (e.g., Hattie & Timperley, 2007). Second, the analysis from a vertical perspective demonstrated that the presence of three educative features in different pedagogical messages varied, which suggests varying educative potential.

8.2. Contribution to knowledge

Having provided an overview of this study, this section turns to the significant contributions this research makes to the field of mathematics education, highlighting how the findings extend existing knowledge and open new avenues for investigation. Namely, this thesis contributes to knowledge in terms of three aspects. First, the framework proposed can be used to guide analysing curriculum materials for their educative potential. Second, five formative assessment strategies were thoroughly contextualised for learning mathematics in general and multiplicative reasoning specifically. Finally, the approach to analyse curriculum materials extends beyond positivist approaches found in the existing literature. Following three sections elaborate on these contributions.

8.2.1. Proposed model to guide the analysis of curriculum materials for their educative potential

A shift from broader teacher knowledge to specific pedagogical messages

In the existing literature, the educative features of curriculum materials have been identified based on the knowledge teachers should possess, including their pedagogical content knowledge, content knowledge and curriculum knowledge. The approach in this doctoral research stands out for its distinctive emphasis on tailoring educative features to specific pedagogical messages, rather than adopting a broad focus on teacher knowledge. In Chapters 2 and 3, it was discussed that the educative features proposed were originally guided by the knowledge teachers are expected to possess (Ball & Cohen, 1996; Davis & Krajcik, 2005). Quebec-Fuentes and Ma's (2018) framework for analysing educative features within mathematics curriculum materials then expanded on these early proposed educative features by incorporating a mathematical dimension. Specifically, they identified seven categories of mathematics teachers' knowledge and analysed two key features, namely rationale and guidance, for each knowledge category.

In this thesis, I extend Quebec-Fuentes and Ma's framework to focus on pedagogical messages. While their framework involves broad categories for mathematics knowledge without providing details of what this knowledge entails, in this thesis, specific and direct pedagogical messages to teachers are examined. For example, "knowledge of assessment in mathematics" is one of the seven mathematics teacher knowledge categories identified by Quebec-Fuentes and Ma. However, in this thesis (Chapter 7), specific pedagogical messages for two key aspects of teachers' classroom formative assessment, noticing and feedback, were identified and the educative potential of each pedagogical message was analysed.

As a result, in the final step of the critical phase, the pedagogical messages explicitly or implicitly integrated in the teacher guides were chosen as the focus of the analysis rather than the explicit references that can be associated with specific area of teaching. I identified these pedagogical messages by examining each teacher guide, using the references to the rationale given for the lesson, curriculum, and suggested pedagogical practice holistically, as well as the practices prevalent within the teacher guide even where the rationale was not explicitly stated. This approach enabled me to reveal the pedagogical messages that are considered to be directly communicated to teachers but that may not be explicitly articulated.

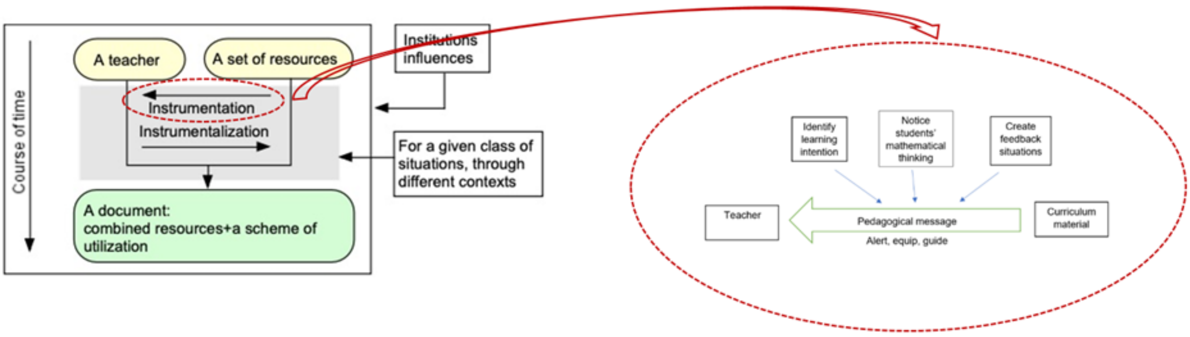
Distinguishing the roles of teachers and curriculum materials

Having presented the distinct approach of focusing on pedagogical messages, in this section, I elaborate on the potential contributions of the three educative features I propose – alert, equip and guide – elucidating how they serve as means to effectively convey the identified pedagogical messages to teachers. I argue that identifying these features makes two contributions to the existing literature.

First, these features emphasise the role of curriculum materials in not only informing teachers about various aspects of the curriculum, but also in activating them to enact specific practices. This goes beyond a broad and fragmented understanding, encouraging a direct focus on specific pedagogical practices. The existing body of literature on educative features predominantly emphasises the provision of a rationale for several aspects of the curriculum within curriculum materials, as a means to enhance teachers' learning. These primarily involve providing a rationale for recommended practices, as exemplified in Quebec-Fuentes and Ma's (2018) framework. Davis et al. (2017) approached to the phenomenon of providing rationale to teachers differently, with an emphasis on enabling teachers to realise the rationales for the recommended practices through specific features of curriculum materials, such as situating the recommendations in teachers' practices and providing multiple forms of educative support. This doctoral research takes Davis et al.'s attempt even further, by directly integrating this "activating teachers" aspect in the educative features of alert and equip. In other words, rather than providing rationale, I propose alerting teachers; rather than providing tools and knowledge, I propose equipping them.

Second, these features explicitly distinguish the difference between the roles of curriculum materials and teachers. As a theoretical base, I employed the documentational approach which conceptualises the roles of teachers and curriculum materials as a two-way relationship: *instrumentation* (from curriculum material to teacher) and *instrumentalisation* (from teacher to curriculum material) (Gueudet & Trouche, 2009). Importantly, I locate the three educative features I suggest within the notion of instrumentation, as presented in the following figure. As Figure 8.1 shows, three core elements of classroom formative assessment inform the pedagogical messages. The pedagogical messages identified can be communicated to and activate teachers by alerting them to the importance of the pedagogical message, equipping them with sufficient tools and knowledge for enactment of that message, and guiding them to use these tools and knowledge effectively. I argue that these features bring new insights to the existing body of research by highlighting the direct role of curriculum materials in communicating pedagogical messages to teachers and prompting them to act in line with these messages, beyond a focus merely on of the content of these materials.

Figure 8.1 Locating three educative features in the documentational approach



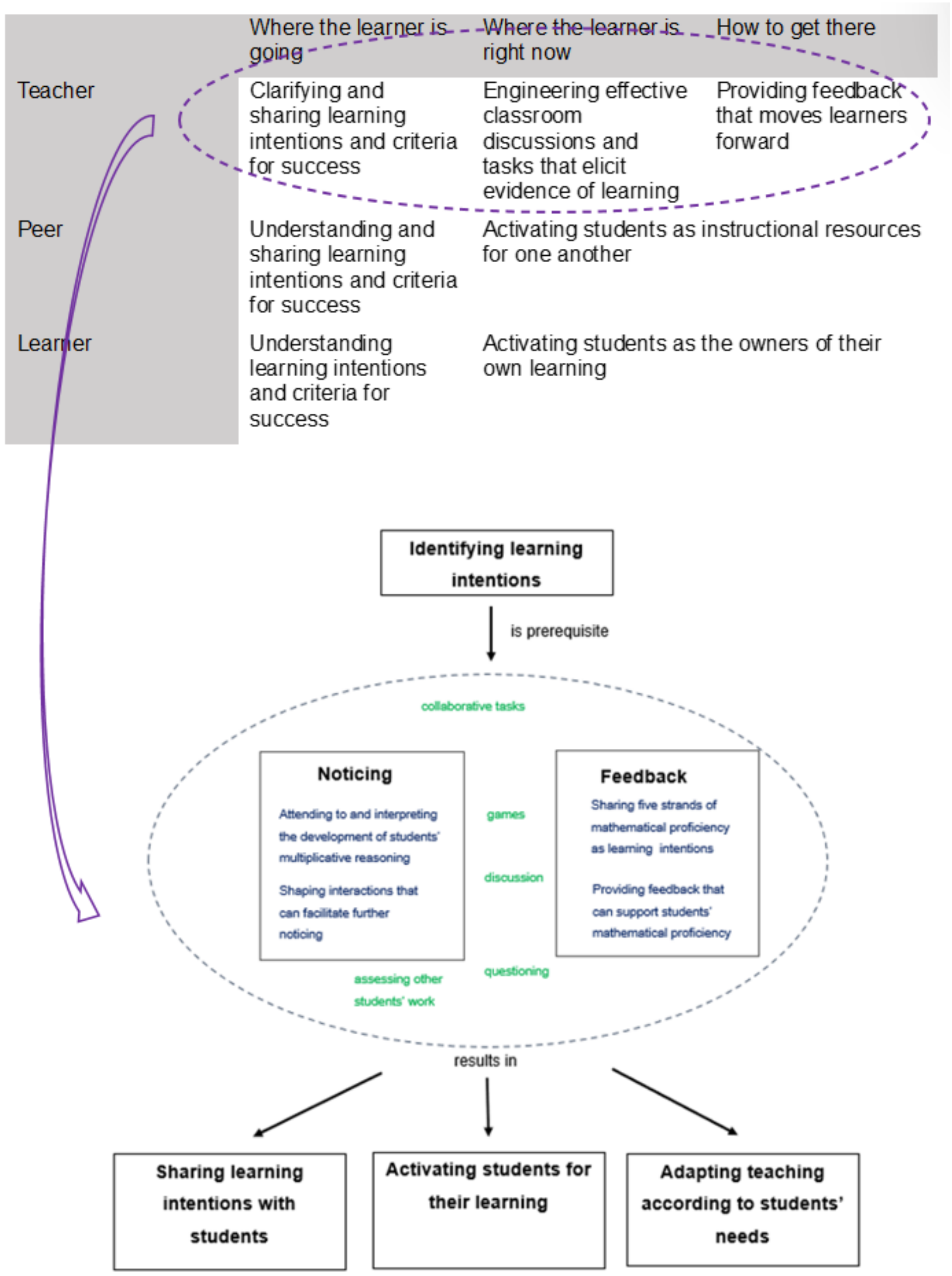
Note. Adapted from Gueudet and Trouche (2009, p. 206)

8.2.2. Operationalising five formative assessment strategies for mathematics teachers' in-the-moment practices

In the context of this thesis, the exploration of teachers' effective formative assessment practices has been guided by the five strategies proposed by Wiliam and Thompson (2007), which have widely informed research on formative assessment in the educational field. However, it is noteworthy that while these strategies have been recognised as providing valuable insights into the general principles of formative assessment, they have been subject to critique for not sufficiently addressing the nuanced needs within specific academic disciplines, as pointed out by Bennett (2011).

In this thesis, I present an adaptation of Wiliam and Thompson's (2007) five strategies of formative assessment, tailored specifically to mathematics teachers' in-the-moment practices, as visualised in Figure 8.2. In the figure, the table above presents the original framework proposed by Wiliam and Thompson, and the figure beneath it shows the adapted version used in this thesis. As indicated by the purple circular area in the table representing the original framework, the adaptation in this thesis involves a focus on the teachers' role in effective formative assessment, which relates to three instructional processes: where the learner is going; where the learner is right now and how they will get there.

Figure 8.2 Operationalising formative assessment strategies



Note. Adapted from Wiliam & Thompson (2007, p. 63)

This adaptation goes beyond a mere strategy-focused approach, advocating for a more nuanced understanding of the core practices that underpin effective formative assessment in the context of mathematics education. Through a critical examination of these practices, I have identified noticing students' learning and providing feedback that can move students' learning forward as core in-the-moment practices of teachers. Moreover, I consider teachers' identification of learning intentions as a foundational prerequisite that enhances the effectiveness of the noticing and feedback practices. This adaptation also introduces “activating students for peer learning” as a supporting technique, thereby integrating these practices into a cohesive framework that is more aligned with the dynamic realities of classroom teaching.

The practical implications of this adaptation are profound, offering a more actionable and context-sensitive framework that can inform teachers’ formative assessment practices. By grounding formative assessment strategies in in-the-moment classroom interactions, this thesis enables a significant reframing of formative assessment strategies within the field of mathematics education.

The following three sub-sections will present more detailed aspects of this adaptation.

Expanding the second strategy with the concept of noticing

The contribution to the field extends beyond the surface, providing a nuanced understanding of how the notion of noticing can help guide practice to elicit students’ mathematical thinking. As Bennett (2011) highlights, for an effective implementation of the second strategy, teachers first need support to obtain evidence of how, and the extent to which, students understand the subject, rather than simply whether or not students can implement procedures correctly in class. Teachers then need help to make inferences about students’ current and future learning based on this evidence. However, recent studies suggest the persistence of a focus on procedures in mathematics education (Lithner, 2017).

As exemplified and discussed in Sections 6.2 and 7.1, incorporation of the three elements of noticing proposed by van Es and Sherin (2021) – i.e., attend to, interpret and shape – can support teachers in more effectively eliciting the development of students’ multiplicative reasoning beyond merely identifying their correct and incorrect answers. The *shape* element of noticing suggested by van Es and Sherin is associated with the original second strategy, engineering effective classroom discussions and tasks that can elicit students’ learning. The elements of *attend to* and *interpret* expand this strategy by highlighting teachers’ active engagement with students’ thinking.

Reconceptualising feedback

The second contribution of this research related to contextualising formative assessment strategies in mathematics is the reconceptualisation of feedback for teachers' in-the-moment decisions. As discussed in Section 2.2.3, the concept of feedback has evolved over several years from merely informing students about their performance with the purpose of helping to close the learning gap, to the practices integrated in teaching practices that require students' engagement and active reflection on their learning (Hattie & Timperley, 2007; Henderson et al., 2019; Ramaprasad, 1983; Sadler, 1989; Wiliam, 2018). As highlighted in Section 1.1.1.2, despite this shift in the understanding of feedback that embraces students' engagement with it, the conceptualisations regarding feedback may lead researchers and teachers to separate feedback practices from teaching practices.

In this research, feedback is considered as a situation that is engineered by teachers as an integrated aspect of teaching, whereby students are given opportunities to reflect on their learning by using a variety of tools and resources, such as questioning, peer engagement, and mathematical tasks. In Section 1.1.1.2, it was highlighted that teachers might tend to provide feedback that can centre the student, such as praise, or immediate-corrective feedback, which may not enhance students' long-term learning (Antoniou & James, 2014; Hattie & Timperley, 2007; Kluger & DeNisi, 1996). The existing literature highlights different was of providing feedback and their potentially different impacts on students' learning, depending on conditions such as students' prior knowledge and learning intentions. More specifically, the impact of a particular type of feedback can vary, being either beneficial or detrimental depending on individual students' needs and learning styles. The conceptualisation of feedback in this thesis enabled the identification of references in the teacher guides that can provide alternatives to typical feedback types proposed in the literature, such as immediate-corrective, immediate-elaborated, delayed-corrective, and delayed-elaborated (Hattie & Timperley, 2007), as discussed in Section 6.3.

Insights into the identification and sharing of productive disposition as a learning intention

In order to operationalise these elements in teaching mathematics, five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) were selected as learning intentions in teaching mathematics (Kilpatrick et al., 2001). To date, the studies that investigate the impact of feedback on students' learning of mathematics or the content of existing mathematics curriculum materials primarily focus on the cognitive aspects of learning mathematics, placing less emphasis on enhancing students' productive disposition (e.g., Choppin et al., 2021).

Examining the teacher guides for the opportunities to shape students' productive disposition revealed valuable insights regarding the potential current situation of existing curriculum materials. For example, as a result of the analyses in Chapter 6, it was concluded that while references for "productive disposition" as a learning intention may be present in teacher guides, the educative support that can potentially facilitate enhancing students' productive disposition was limited, with very few references to sharing this learning intention with students through feedback, as exemplified in Section 6.3.3.

8.2.3. A novel approach to curriculum materials analysis

As outlined in Chapter 3, the prevalent methodology in curriculum material analysis research typically encompasses comparisons across a broad spectrum of materials from different contexts, such as by country or educational philosophy (e.g., Li et al., 2008), alongside examinations into the distribution of content and underlying curriculum philosophies (e.g., Hemmi et al., 2018). While these approaches, particularly those focusing on the analysis of teacher guides, have significantly advanced the understanding of curriculum materials, they may not always examine in-depth the specific features of these materials. This observation suggests that, while valuable, there might be opportunities for more detailed analyses that could uncover additional insights into the nuanced aspects of curriculum materials.

Furthermore, the reliance on frequency analysis within these studies warrants a critical examination. Frequency analysis, by quantifying the existence of certain elements within the curriculum materials, offers tangible insights into their composition. However, this method may overlook the qualitative depth and pedagogical richness inherent in these materials. As such, this approach risks simplifying complex educational content into mere numerical data, potentially neglecting the contextual and instructional nuances that give a curriculum its full meaning and effectiveness. This critique highlights the need for methodological diversity in an approach that includes both quantitative and qualitative elements to provide a more holistic understanding of curriculum materials.

In this research, I have aimed to shift the focus from making inferences for the overall materials to revealing certain features in the teacher guides. In order to support this, an approach closer to the “big Q” approach proposed by Braun and Clarke (2021) guided the analysis of the teacher guides. This provided an opportunity to gain nuanced insights into the features of curriculum materials by identifying fewer but richer examples within the materials, rather than a focus on frequent references, which provides a shallower understanding of the features. When analysing the references for feedback and noticing opportunities in Chapter 6, these references were located between two ends of a spectrum, which included the expected but potentially unhelpful teacher practices and ideally intended practices. Using the qualitative approach enabled me to show the variety between two points on a spectrum, rather than making quantitative comparisons, as earlier studies have done (e.g., Machalow et al., 2020).

8.3. Limitations of the study

While this thesis has strengths in enhancing the understanding of formative assessment within the context of teaching mathematics and elucidating the features of curriculum materials that can potentially improve teachers’ practices, it is crucial to recognise and address the inherent limitations in the interpretation and applicability of the findings. The following four sections will elaborate on four potential limitations and the strategies utilised to mitigate them.

8.3.1. Sampling

First, as I am fluent in English and Turkish, only curriculum materials written in these languages were accessible for the analysis. As curriculum materials that meet the criteria of this research (integrating formative assessment and educative features) are limited in Türkiye, I chose materials written in English. Moreover, as a part of my funding agreement, I was not permitted to study the Turkish context in this research. This led me to select curriculum materials from England and United States, excluding potentially well-designed materials that might meet the sampling criteria in other languages. Including more variety in the sample could help to reveal more nuanced references for educative features and formative assessment practices. This limitation was addressed by incorporating diversity among the chosen teacher guides concerning other aspects, such as the designer profile and design purpose of the materials. Regarding the designer profile, ICCAMS, MAP, and CSM were designed by researchers, while WRM was developed by experienced teachers. CMP serves as a unique case, engaging both experienced teachers and researchers in the design team. Concerning the design purpose, ICCAMS and MAP were specifically designed as formative assessment lessons, whereas the remaining three materials integrated formative assessment as a component of pedagogy.

Unsurprisingly, as MAP and ICCAMS were specifically designed as formative assessment lessons, some aspects of the analysis revealed more variety in these materials. Particularly in the critical phase of the study, richer examples for noticing and feedback practices were largely derived from these two sets. This resulted in a dominant influence of these sets in findings regarding noticing and feedback.

8.3.2. Subjectivity

Subjectivity in the analysis can be either a limitation or a strength, depending on the procedures followed to ensure analytical rigour. While a possible limitation is the production of biased findings, the potential strength is the production of deeply insightful findings (Braun & Clarke, 2021).

Importantly, I conducted the analyses as a sole researcher with a particular background different from that of the teachers who are the target audience for the analysed teacher guides.

The first strategy I employed was to conduct several iterative cycles. Between these cycles, I distanced myself from the data, revisited the related literature and then returned to analyse the data (Braun & Clarke, 2021). The second strategy I employed was peer debriefing (Creswell & Miller, 2000) by contacting with another PhD student who is familiar with both formative assessment and mathematics education. More specifically, I discussed coding the data with them when identifying the references for five strands of mathematical proficiency and feedback. When discussing the references for the five strands of mathematical proficiency, there were instances where we disagreed, which required further thought about this aspect. When discussing the references for feedback, we both agreed that references for feedback instances were limited in the teacher guides.

Lastly, I presented my early ideas at CERME12 in February 2022, BSRLM in June 2022, and in CERME13 in July 2023. During the presentation at the BSRLM day conference, participants challenged me to consider the practicality of open mathematical tasks in teaching maths in large classrooms, and the use of wording when presenting the themes I developed in the early stages of reflexive thematic analysis of the teacher guides. At CERME12, I received feedback on the earlier methodological decisions, such as sampling curriculum materials and strategies to analyse data, and at CERME13 I had the opportunity to discuss the three educative features I developed throughout this research.

Alongside being open to the critiques of other early career and experienced researchers, I used the notion of researchers' reflexivity as a transparency tool (Ball, 1990). That is to say, I have remained aware of my potential biases by considering the academic and professional background I might have brought to this research. Appendix B presents my academic and professional background, which can influence my interpretation of data, in order to provide a transparent account of my own possible bias to readers.

8.3.3. Evidence of practice

Although this research closely examined the issue of how to enhance teachers' classroom formative assessment practices, I was not able to use primary classroom data as planned, due to the limited access to this sort of data following the onset of the COVID-19 pandemic (as presented in Section 1.5). In particular, the development of the themes emerging from the thematic analysis included consideration of four assumptions concerning the highlighted research problem, and specifically the potentially expected conventional practices of teachers. These assumptions are: (1) that teachers might tend to prioritise procedures over other mathematical skills, such as reasoning and problem-solving; (2) that teachers might face difficulties in leading effective student collaborations; (3) that teachers might tend to focus on the correctness of students' responses over noticing students' mathematical thinking; and (4) that teachers might tend to provide immediate-corrective feedback. These assumptions were validated by the existing literature, including both academic and governmental resources (e.g., Antoniou & James, 2014; Lithner, 2017; Ofsted, 2023), as well as my initial interactions with secondary mathematics teachers in England in the early stages of the research.

In order to imagine formative assessment practices in a real classroom, teacher guides, which involve designers' anticipations for real classroom practices, were used. It would have been preferable to use actual classroom data, and it should be noted that this thesis does not involve direct inputs from teachers. These features could be tested and developed further through participatory (Pepin et al., 2019) and collaborative lesson study designs (Wake et al., 2016) that involve teachers' real practices and interviews to capture their voice.

Importantly, beyond teacher-centric mechanisms, there might be external mechanisms that can influence teachers' use of the materials, their preferences, and what is useful for them. As this thesis has particular focus – that is, the features of the curriculum materials – I have not addressed this important issue. However, it should be noted that these external mechanisms can influence what the educative features suggested in this thesis look like.

Van Steenbrugge and Ryve (2018) highlight the importance of context in the effectiveness of the educative features of curriculum materials, depending on the characteristics of classroom practices, teachers' role in classrooms, the level of explicit/implicit support teachers are used to receiving, and teachers' experiences with using teacher guides. Acknowledging this, I suggest that it would be worthwhile to study the three features I propose in different cultural contexts. More explicitly, rather than being universal, these educative features may need to be operationalised differently in different educational contexts.

Although I haven not used primary classroom data in this research, in the earlier stages of the research, in the first year of my PhD, I visited two secondary schools in London, where I informally observed five mathematics lessons and had informal conversations with the head of the mathematics department regarding the need to improve formative assessment practices. Following that, I conducted seven interviews with four Key Stage 3 mathematics teachers in London, focusing on their experiences related to formative assessment practices when teaching multiplicative reasoning. Although these interviews did not contribute to the main data analysed in this thesis, they enabled me to gain insights into the problem as a researcher and sensitised me to specific challenges faced by teachers. One such challenge was a lack of high-quality tasks that can support their classroom assessment in line with current GCSE mathematics objectives, which requires high-level problem-solving and reasoning skills, and leading effective group work. These experiences help to mitigate the limitation of not directly collecting any classroom data.

8.3.4. Focus on sharing learning intentions with students alongside feedback

In this research, sharing learning intentions with students was considered a part of two core in-the moment teacher formative assessment practices, namely, noticing and feedback. Although the materials could provide instances related to sharing learning intentions in different places, these instances were not included in this thesis as the key aspects of formative assessment were noticing and feedback. As a result, the arguments related to sharing learning intentions should not be considered within this scope.

8.4. Implications for policy and practice

Having outlined the limitations of this research, it is imperative to consider the broader implications of the findings for policy and practice. The following section explores how the findings of this research can be applied in practical settings and the potential they have for informing policy-making and educational practice in the field of mathematics education.

8.4.1. Implications for the design of curriculum materials

As introduced in Chapter 5 and applied in Chapters 6 and 7, this thesis presents a framework for the operationalisation of the generic five formative assessment strategies suggested by Wiliam and Thompson (2007) in teaching mathematics, along with three educative features guiding the analysis of the educative potential of curriculum materials for specific pedagogical messages. As reiterated throughout this thesis, effective formative assessment practices remain a challenge for mathematics teachers, given limited resources such as knowledge and tools (Bennett, 2011). This framework is adaptable to design principles that can guide the development of future accessible mathematics curriculum materials, thereby contributing to the enhancement of effective formative assessment practices for mathematics teachers.

8.4.2. The need to focus on holistic and relational mathematics learning intentions

One of the noteworthy but unsurprising findings in this research was the identification of instances within the teacher guides where teachers may be inclined to prioritise fragmented and hierarchical learning intentions, as presented in Section 6.1.1. I argue that this can be an implication of the prevailing use of Bloom's taxonomy (Krathwohl, 2002) in curriculum design. This taxonomy involves hierarchical and generic levels of learning intentions, which may not be straightforwardly adapted in different disciplines. Although a relational and holistic learning approach has long been applied in the literature (e.g., Kilpatrick et al., 2001), this finding was unsurprising as change in policy and practice can take a long time, as discussed in Chapter 1. As a result, it can be valuable for curriculum designers to consider incorporating more hierarchical and relational mathematics learning intentions.

8.5. Implications and suggestions for future research

While addressing the research questions, this study unveiled the need for further investigations. The subsequent two sub-sections will outline suggestions for future research that were identified in the course of this research.

8.5.1. Empirical studies that involve teachers' voice

As highlighted in Section 8.3.3, this research could be expanded by including teachers' voice, which would represent a significant contribution to the literature. This involvement could be achieved through a multifaceted research approach, incorporating interviews with teachers to gather firsthand insights into the practicality and impact of the strategies identified. Additionally, engaging in participatory design processes for curriculum materials can ensure that these educative features are seamlessly integrated and align with teachers' real-world needs. Finally, undertaking longitudinal studies to explore the long-term effects of curriculum materials designed to include these features will provide comprehensive evidence regarding their efficacy and sustainability. Together, these methods would offer a robust framework for assessing the potential of the proposed approach to fundamentally improve teachers' in-the-moment formative assessment practices with the help of curriculum materials.

8.5.2. Comparative studies of England and US mathematics curriculum materials

The scope of this study did not include any direct comparisons among contexts; rather, it aimed to leverage a diverse range of contexts to access rich examples of formative assessment practices and educative potential. Nevertheless, the research revealed significant distinctions in the approaches to teaching mathematics between England and US contexts. As Knipping (2003) highlights, comparative research between different educational context can serve as a method to better understand educational issues, as well as having comparison as the key product of the research. Thus, making systematic comparisons between these two countries can enhance the understanding of specific educational issues.

While existing literature presents research that compares curriculum materials of the US and Asia (e.g., Lewis et al., 2011), the curriculum materials among European countries (e.g., Haggarty & Pepin, 2002), and those in the US and European countries (Van Steenbrugge & Yolcu, 2023), to my knowledge, studies focusing specifically on comparisons between the mathematics curriculum materials in the US and England are limited. Future research endeavours have the potential to fill this gap and provide valuable insights into the nuanced differences between these educational contexts.

8.5.2. Investigating formative assessment practices for developing students' productive disposition towards mathematics

The review of the literature on mathematics curriculum materials analysis revealed that the scope of this body of research is constrained to the cognitive aspects of learning mathematics such as conceptual understanding and problem solving, as presented in Section 3.4.1. While mathematics education researchers specialising in the affective domain, which encompasses students' attitudes and beliefs toward learning mathematics (e.g., Drodge & Reid, 2001), can acknowledge productive disposition, this extensive body of research is inadequately reflected in curriculum material analysis studies.

In this study, productive disposition was taken as one of the learning intentions to address this research gap. In particular, it was found that although the curriculum materials might state learning goals explicitly related to productive disposition, the suggested noticing and feedback practices that could potentially enhance students' productive disposition are implicit. Future research endeavours might consider how to design curriculum materials that can facilitate teachers' noticing and feedback practices to enhance students' productive disposition towards mathematics.

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APPENDICES

Appendix A- Some accounts for the strategies to review the literature

A.1. Guiding Questions and Keywords for the Boolean Search

Guiding questions for the search	Keywords
What does research tell about the intersection of formative assessment and curriculum materials?	“curriculum materials” AND “formative assessment”
What is the range of studies that used the “communities of practice” as a framework?	“communities of practice” AND “mathematics learning”
What are the studies that approach classroom assessment practices from a socio-cultural perspective?	“assessment” AND “classroom” AND “sociocultural”
What are the studies that consider learning mathematics as developing a learner identity?	“identity” AND “community” AND “mathematics education”
What are the studies that focus on feedback practices in mathematics classrooms?	“feedback” AND “classroom” AND “mathematics”
What are the studies that provide insights into noticing opportunities in mathematics textbooks?	“noticing” AND “mathematics” AND “textbook”
What are the studies that provide insights into the noticing opportunities that come from mathematics tasks?	“noticing” AND “mathematics” AND “tasks” NOT “kindergarten”
What are the studies that provide insights into the noticing opportunities in teacher guides for mathematics teachers?	“noticing” AND “mathematics” AND “teacher guides”
What are the studies that directly focused on teacher guides in mathematics education?	“teacher guides” AND “mathematics”

A.2. Reflections on literature search for feedback (July, 2022)

In order to develop a review of feedback literature, I accessed the recent literature in two ways. First, I visited the issues in academic journals after 2020 for some of the key journals in educational assessment by collecting the journals I accessed since I started my PhD. These journals are "Assessment in Education: Principles, Policy, and Practice", "Educational Measurement: Issues and Practice", "Educational Assessment", "Applied Measurement in Education" and "Studies in Educational Evaluation". I revisited the most recent issues of these journals, primarily because I expected them to involve a significant number of feedback-related papers that could shed light on emerging trends in feedback research. When selecting the papers, as a first step, I identified all the papers that involved the keyword feedback in their title and read the abstracts for each of these papers. While looking for the papers that involve the keyword feedback in the title, I have given attention to the papers that refer to classroom interventions (although they do not refer to feedback in the title), and I read the abstracts of these papers to decide whether they should be considered for this review.

Most intervention studies on feedback focus on testing different feedback conditions during a post-instruction test, rather than examining feedback during the learning process. In order to access research studies that study feedback that is prevalent in learning and teaching practices, I conducted a Boolean search on the EBSCO database for peer-reviewed papers by using the keywords "feedback", "classroom" and "mathematics" within the abstracts of the papers. The search yielded 199 results. After reading abstracts for each result, I selected papers addressing at least one of the following questions: 1) What are the examples of feedback types to be used in secondary mathematics classrooms? 2) What evidence does the current literature offer for effective feedback practices in the classroom? 3) What evidence does the current literature offer about teachers' tendencies when providing feedback in the classroom? Therefore, I selected 58 out of the 199 results for further examination.

After identifying the research papers to look at closer, I faced some challenges when using the research findings and arguments in those papers. One challenge was making generalisations and concrete suggestions using the existing literature. That is to say, varying conceptualisations can be attributed to the same terms, or similar conceptualisations were entitled to different terms. For example, Doster and Cuevas (2021) stated that one of the purposes of their research was to compare the effect of scaffolding and feedback in a computer-based environment on students' problem-solving skills and motivation. The feedback conditions they designed involve immediate and elaborated feedback. To some extent, the scaffolding condition they proposed can also be considered a feedback intervention, as it began with instruction and then provided students with additional instructions based on the correctness of their answers.

This sort of challenge can be resolved not only by studies that explicitly investigate feedback efficiency but also by examining the studies that investigated feedback-related practices such as scaffolding and questioning. In order to use time efficiently, I stopped working on this review when I was accomplished to sensitise myself with feedback literature in terms of feedback types and was able to suggest conjectures about the productivity of these types of feedback in different conditions. Once the second iteration of data analysis is completed for feedback, the literature will be revisited to reconceptualise feedback for mathematics, which will likely be the contribution to the literature.

Reference:

Doster, H., & Cuevas, J. (2021). Comparing Computer-Based Programs' Impact on Problem Solving Ability and Motivation. *International Journal on Social and Education Sciences*, *3*(3), 457–488.

Appendix B- My background that can have an impact on my decisions in this research

Table B. 1. My background

My background	Academic Experiences	Experiences in practice/Field work
In Türkiye	Studied in Teacher Training High School in the Science-Mathematics department between 2000-2003. Studied in the Faculty of Education with a particular focus on teaching lower secondary mathematics between 2003-2008. Completed a Master of Science degree in Mathematics Education between 2009-2013. For the thesis, I surveyed elementary teachers' and lower secondary maths teachers' views on characteristics of mathematically gifted students. Started a PhD in Education in the 2014-2015 academic year, completed only two modules. My intention was to continue researching about mathematically gifted students.	Both one-to-one and group tutoring lower secondary maths students in private centres preparing students for national exams. Teaching maths in three different early secondary state schools between 2010 and 2015. Teaching maths in two different centres, state organisations, that aim to provide extra-curricular activities for students who are diagnosed as gifted (2013-2014 complete academic year; 2015-2016 academic year second term; 2017-2018 complete academic year)
In England	Completed a Master of Arts degree between 2016-2017, in Educational Assessment.[2] For the thesis, I explored GCSE maths teachers' and year 11 students' views on the GCSE maths. Started a PhD in 2018 with a particular focus on formative assessment in the context of secondary mathematics.	Observed two lessons of a mathematics teacher in a secondary school in London as a part of a master's module in the autumn term of 2016-2017 academic year. Interviewed four GCSE mathematics teachers and seven Year 11 students, as a part of MA dissertation in the summer term of 2016-2017 academic year.

[2] The main motivation to make this movement from a quitted PhD to a second master's degree in a different context was the abroad studentship I earned from the Turkish Government in 2014, right after I started a PhD. My motivation was to engage with educational assessment theories to explore how to identify mathematically gifted students by building on my previous experiences. However, my interest turned to classroom assessment when I started to explore formative assessment during the master's program in England. Moreover, since in Türkiye, there was a need for experts in formative assessment at the time, I was encouraged to continue with formative assessment research by my supervisor in the Ministry of National Education in Türkiye.

	Trainings I attended that are relevant to qualitative research methods.	I visited two secondary schools in London and had informal conversations with mathematics teachers. I observed four mathematics teachers in a secondary school in London in the summer term of 2018-2019 academic year. Interviewed four mathematics teachers online (pilot interview & main interview) in England between May 2020 and February 2021. Followed social media accounts/groups where plenty of mathematics teachers in England share posts. I volunteered as a mathematics tutor for primary (Spring term, 2022) and secondary students (Autumn term, 2019).

Appendix C- How the materials were accessed

The materials	The way to access the materials	The person who gave the permission for the use of the materials
CSM	Full public access on the project website	Professor Alison Clarke-Wilson (Co-Principal Investigator of the project)
CMP	The materials were bought by the researcher	Elizabeth Lozen (Consortium coordinator of the project)
ICCAMS	Lessons and professional development documents were given access by Jeremy Hodgen.	Professor Jeremy Hodgen (Principal Investigator of the project and primary supervisor of this PhD research)
MAP	Full public access on the project website	Professor Hugh Burkhardt (Co-Principal investigator of the project)
White Rose Maths	Partly public free access, access to premium materials is given by Tony Staneff	Tony Staneff (One of the members of leader team of the organisation)

Appendix D- List of materials coded for the reflexive thematic analysis during explorative phase

Number	The project	Title of the document	Type of the document	How the designers categorised the lesson	When I coded
1	MAP	Classifying proportion and non-proportion situations	Lesson plan	Concept Development Classroom challenges-A formative assessment lesson	Between 6th and 14th February 2022
2	MAP	Drawing to Scale	Lesson plan	Problem Solving Classroom challenges-A formative assessment lesson	
3	CMP	London trending-Preface to teacher edition	Overview of the unit	Teacher guide for geometric similarity unit	
4	CSM	Introduction: welcome to the graphics department	An introductory short lesson	Teacher guide for geometric similarity unit	
5	CSM	Investigation 1: Mathematical similarity	Lesson plan	Teacher guide for geometric similarity unit	
6	CSM	Investigation 2: In the grid	Lesson plan	Teacher guide for similarity unit	
7	ICCAMS	24AB Mini assessment	Pre-assessment	Multiplicative reasoning 12AB	
8	ICCAMS	24A Tangram	Lesson plan	Multiplicative reasoning 12A	
9	ICCAMS	24B Stars	Lesson plan	Multiplicative reasoning 12B	
10	WRM	Enlargement and similarity	Lesson Plan	Summer Term-Scheme of learning- Reasoning with proportion (Year 9)	

11	ICCAMS	Pre-test	Pre-assessment before the unit	ICCAMS pre-test	
12	ICCAMS	Mini-assessment-1AB	Pre-assessment before a set of lesson	Multiplicative reasoning 1AB	
13	ICCAMS	6AB Mini assessment: Steady Walk	Pre-assessment before a set of lesson	Multiplicative reasoning 3AB	
14	ICCAMS	6A Westgate Close	Lesson plan	Multiplicative reasoning 3A	
15	ICCAMS	6B Westgate Close revisited	Lesson plan	Multiplicative reasoning 3B	
16	MAP	Comparing strategies for proportion problems	Lesson plan	Concept Development Classroom challenges-A formative assessment lesson (Grade 7)	14th February 2022, 15th February 2022
17	ICCAMS	11AB Mini assessment: Elastic strip	Pre-assessment	Multiplicative reasoning 8AB	15th February 2022
18	ICCAMS	11A Stretched ruler	Lesson plan	Multiplicative reasoning 8A	15th February 2022
19	ICCAMS	11B Snowmen	Lesson plan	Multiplicative reasoning 8B	17th February 2022
20	WRM	Solving ratio and proportion problems	Lesson plan	Summer term- Scheme of Learning-Reasoning with proportion (Year 9)	17th February 2022

21	WRM	Rates	Lesson plan	Summer Term-Scheme of learning- Reasoning with proportion (Year 9)	18th February 2022
22	CSM	Investigation 3: Scale factor	Lesson plan	Teacher guide for geometric similarity unit	18th February 2022
23	CSM	Investigation 4: Broken scale factor	Lesson plan	Teacher guide for geometric similarity unit	18th February 2022
24	CSM	Investigation 5: More than lenghts of sides	Lesson plan	Teacher guide for geometric similarity unit	18th February 2022
25	CSM	Investigation 6: Ratios	Lesson plan	Teacher guide for geometric similarity unit	18th February 2022
26	WRM	Ratio and Scale	Lesson plan	Year 8 Autumn term-Scheme of Learning-Proportional reasoning	23rd February 2022
27	MAP	Using proportional reasoning	Lesson plan	Concept Development Classroom challenges-A formative assessment lesson	23rd February 2022
28	ICCAMS	7AB Mini assessment: Double number line rules	Pre-assessment	Multiplicative reasoning 4AB	24th February 2022
29	ICCAMS	7A Converting pounds to Leva	Lesson plan	Multiplicative reasoning 4A	24th February 2022
30	ICCAMS	7B Potato pancakes	Lesson plan	Multiplicative reasoning 4B	24th February 2022
31	CMP	Unit Planning	Overview of the unit	Unit planning, Stretching and Shrinking	

32	CMP	Investigation 2: Similar Figures	Lesson plan	Teacher guide, Stretching and Shrinking	Between 28th February and 4th March 2022
33	CMP	Investigation 3: Scaling Perimeter and Area	Lesson plan	Teacher guide, Stretching and shrinking	
34	CMP	Unit Planning	Overview of the unit	Teacher guide, Comparing and scaling	
35	CMP	Investigation 1: Ways of Comparing: Ratios and Proportions	Lesson plan	Teacher guide, Comparing and Scaling	