EPSRC CENTRE FOR DOCTORAL TRAINING IN GEOMETRY AND NUMBER THEORY
LSGNT London School of Geometry & Number Theory

# Birational geometry of fibrations in positive characteristic

## On the canonical bundle formula and on the Iitaka conjectures


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Submitted in partial fulfilment of the requirements for the degree of Doctor in Philosophy

*University College London - UCL*

July 2024

# Abstract


This thesis concerns the study of fibrations between algebraic varieties over fields of positive characteristic. These are fundamental objects used to study the classification of algebraic varieties. In particular, my thesis focuses on two problems: the canonical bundle formula and the Iitaka conjectures.

Let $f\colon X \to Z$ be a fibration between normal projective varieties over a perfect field of positive characteristic.

Assume the Minimal Model Program and the existence of log resolutions. Then, we prove that, if $K_X$ is $f$-nef, $Z$ is a curve and the general fibre has nice singularities, the moduli part is nef, up to a birational map. As a corollary, we prove nefness of the moduli part in the $K$-trivial case. In particular, if $X$ has dimension 3 and is defined over a perfect field of characteristic $p > 5$, the canonical bundle formula holds unconditionally.

We also study an Iitaka-type inequality $\kappa(X, -K_X) \leq \kappa(X_z, -K_{X_z}) + \kappa(Z, -K_Z)$ for the anticanonical divisors, where $X_z$ is a general fibre of $f$. We conclude that it holds when $X_z$ has good $F$-singularities. Furthermore, we give counterexamples in characteristics 2 and 3 for fibrations with non-normal fibres, constructed from Tango–Raynaud surfaces.

## Declaration statement

I, Benozzo Marta, confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

## Impact statement

The projects discussed in this thesis fit in the framework of the classification of varieties over fields of positive characteristic. These results will have an impact in algebraic geometry, in commutative algebra and in number theory. In fact, the study of algebraic varieties over fields of positive characteristic has already had many applications in number theory, opening an interplay between the three subjects. To achieve this impact, I have submitted my work to peer-reviewed journals and I have given several research talks to both algebraic geometry and number theory international conferences and seminars.

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  (ii) **Uploaded to a preprint server:** https://doi.org/10.48550/arXiv.2204.02045

  (iii) **Work intended to be published at:** Michigan Mathematical Journal

  (iv) **Author:** Benozzo Marta

  (v) **Stage of publication:** Provisionally accepted for publication

(b) **This material can be found in:** §1.3.3., §2.2.2., §5.2., §5.4.3., §6.1.

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(a) **For a research manuscript prepared for publication but that has not yet been published:**

(i) **Title:** On the canonical bundle formula in positive characteristic

(ii) **Uploaded to a preprint server:** https://doi.org/10.48550/arXiv.2305.19841

(iii) **Work intended to be published at:** TBD

(iv) **Author:** Benozzo Marta

(v) **Stage of publication:** not yet submitted

(b) **This material can be found in:** §1.2., §2.1., §2.2.1., §2.3. (excluding §2.3.5.), §3.3., Chapter 4.

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(a) **For a research manuscript prepared for publication but that has not yet been published:**

(i) **Title:** Superadditivity of anticanonical Iitaka dimension for fibrations with F-split fibres

(ii) **Uploaded to a preprint server:** https://doi.org/10.48550/arXiv.2309.16580

(iii) **Work intended to be published at:** Advances in Mathematics

(iv) **Authors:** Benozzo Marta, Brivio Iacopo, Chang Chi-Kang

(v) **Stage of publication:** submitted

(b) **Statement of contribution:** The three authors all contributed to the project.

(c) **This material can be found in:** §1.3., §2.2.2., §2.3.5., §3.4.2., §5.1., §5.2., §5.4.1., §5.4.2., §6.2.

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## Acknowledgements

First of all, I would like to thank my PhD advisor Paolo Cascini for all the encouragement he gave me during these years, for teaching me the techniques used in birational geometry, for suggesting the problems I worked on, for his guidance throughout and for his comments. I thank him for having listened to my stream of thoughts, for having helped me to understand if there was some sense in them and for all the chitchats.

A special thank to the friends that have shared with me a big or a small part of their PhD journey in the birational geometry group in London, they have been a great source for references, I have learnt so much by discussing with them and laughing about our mistakes when trying to compute canonical divisors of blow-ups. In particular, I thank Aurore Boitrel, Federico Bongiorno, Samuele Ciprietti, Roktim Mascharak, Liam Stigant, Fredrik Vaeng Rotnes, Pascale Voegtli and Stefania Vassiliadis.

I thank Iacopo Brivio with his overwhelming enthusiasm, for proposing me a project to work on together, for teaching me to pay attention to the details and for getting crazy together trying to fix them. I thank Fabio Bernasconi for the continuous support and the helpful suggestions.

I thank Jefferson Baudin, Chi-Khang Chang, Przemysław Grabowski for the insightful discussions and Martin Ortiz for our "number theory" exchanges in the office. I am grateful to Enrica Floris, Cristopher Hacon, Mihnea Popa, Karl Schwede, Calum Spicer, Jakub Witaszek for the precious conversations and suggestions. I am grateful to my examiners Cecilia Busuioc and Ivan Cheltsov and to the anonymous referees of my papers for the detailed comments and suggestions, my work has been corrected and greatly improved thanks to them. I additionally thank Hamid Abban, Jérémy Blanc, Jungkai Chen, Zsolt Patakfalvi, Ed Segal and Susanna Zimmermann for their support.

For the encouragement they gave me when I needed it the most, I thank Federico Barbacovi, Dario Beraldo, and also Ilaria Di Dedda and Sara Veneziale, who were always there and even learnt my research proposal better than me.

I add a thank to Eurydice Costopoulos, Inés Garcia Redondo and Arne Wolf, who have made a huge difference in my daily life in London. There are many more people that have supported me in these years with their shining friendship, they are very dear to me and I thank them immensely. In particular, my family, my friends in Italy, in Montréal and in London, my friends from the LSGNT and the Imperial lunch group.

I express my gratitude to the LSGNT, to Imperial College London and to University College London for providing a great environment for discussions. This work was supported by the Engineering and Physical Sciences Research Council

[EP/S021590/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London.

*"It's the little big things that stay on my mind,*
*it's the little big things we leave behind,*
*and when all is said, and when all is done*
*they can change the course of the falling sun.*
*It's the little things*
*that mean the big things to me."*

*The little big things* from "The little big things: the musical"

# Contents

# Introduction

Algebraic geometry is the study of *algebraic varieties*, objects defined as the zero locus of polynomial equations. One of the guiding problems in the field is their classification. Curves are completely classified by their genus and, for each genus, we can parametrise them up to isomorphism (see for example [HM98]). In higher dimension, the situation is much more complicated. In birational geometry, a promising approach has been to classify varieties up to *birational maps*, which are isomorphisms on a dense open subset. Much progress has been done in this direction using the Minimal Model Program (MMP) ([BCHM10]). The MMP is an algorithm which allows to find a "nice" representative in each birational class of varieties and its guiding principle is that the positivity of the *canonical divisor* $K_X$ of a variety, defined using the top degree differential forms, carries information about its geometry, for example its Ricci curvature. *Fibrations* are a fundamental tool in the classification problem as they allow to decompose each variety into simpler pieces, the base and the fibres. An important problem arising in this context is to relate the canonical divisors of the total space, the fibres and the base of a fibration. We will make this more precise later, when focusing on our problems of interest.

Recently, there has been a lot of developments in the MMP over fields of positive characteristic, opening an interplay between birational geometry and number theory. For example, the study of rational points has benefited from applications of MMP techniques ([GNT19], [BF23]). Over fields of positive characteristic, many results in birational geometry become open problems and in some cases they fail to hold at all. Despite this, the MMP has been established in dimension at most 3 over perfect fields of characteristic $p > 5$. It is therefore natural to ask to which extent results in birational geometry in characteristic 0 carry over to fields of positive characteristic.

The work presented in this thesis falls into this context. In particular, we focus on two main directions in the study of fibrations in positive characteristic: related to a canonical bundle formula and to the Iitaka conjecture for anticanonical divisors.

### Canonical bundle formula

One of the possible outputs of the MMP are *K-trivial* fibrations $f\colon X \to Z$, for which the fibres are Calabi–Yau. For such fibrations, we study the relation between $K_X$ and $K_Z$. This formula is often used in induction processes since it allows to infer geometric properties of $X$ from geometric properties of $Z$, which is lower dimensional. It has had many important applications in birational geometry, ranging from adjunction on log canonical centres ([Kaw98], [DS17]), to finite generation of the log canonical ring ([FM00, §5]) and to the Iitaka conjecture for anticanonical divisors ([Cha23], [Ben22], [BBC23]).

### Iitaka conjecture for anticanonical divisors

An important birational invariant of algebraic varieties is their *Kodaira dimension* $\kappa(X, K_X)$ (Definition 5.1.1), which is a first measure of the positivity of the canonical divisor. The Iitaka conjectures give geometric constraints on the varieties that can appear in a fibration, by studying the relation between their Kodaira dimensions. Even though it is not a birational invariant, it is useful to study also the negative counterpart of the Kodaira dimension: the *Iitaka dimension* of the anticanonical divisor, $\kappa(X, -K_X)$. This number gives information on the geometry of varieties for which $mK_X$ does not have global sections for any natural number $m$. Inspired by the Iitaka conjectures, we study a relation between Iitaka dimensions of anticanonical divisors in fibrations.

## Minimal Model Program

A notable progress in classifying algebraic varieties has been the development of the Minimal Model Program. The MMP consists of an algorithm that, starting from a variety $X$, performs two types of birational operations for which the exceptional locus consists of "negative curves": *divisorial contractions* and *flips*. Divisorial contractions are birational morphisms $\mu\colon X \to Y$ such that the curves in $X$ that are mapped to points in $Y$ cover a locus of codimension 1. Flips are birational maps $\mu\colon X \dashrightarrow Y$ such that the exceptional locus of $\mu$ has codimension at least 2 and the negative curves contained in it are "flipped" to curves in $Y$ which have positive intersection with the canonical divisor. The MMP predicts that, after finitely many divisorial contractions and flips, this process terminates with a variety $Y$, birationally equivalent to $X$, which satisfies one of the following properties.

(1) There exists a fibration $f\colon Y \to Z$ such that $\dim(Z) < \dim(Y)$ and the general fibres are *Fano*, i.e. varieties with ample anticanonical divisor.

(2) There exists a fibration (called *$K$-trivial*) $f\colon Y \to Z$ such that $\dim(Z) < \dim(Y)$ and the general fibres are *Calabi–Yau*, i.e. varieties with trivial canonical divisor.

(3) The variety $Y$ is *of general type*, i.e. its canonical divisor is a perturbation of an ample divisor.

By further running an MMP on the base of the resulting fibration, we can conjecturally decompose every variety into building blocks that are either Fano, Calabi–Yau, or general type varieties. Over fields of characteristic 0, divisorial contractions and flips have been proven to exist and the MMP has been proven to terminate in full generality in low dimensions by the work of Kawamata, Kollár, Mori, Reid, Shokurov and others, and in many important cases in higher dimension ([BCHM10]).

## MMP in positive characteristic

Recently, there has been growing interest in the possibility of using birational geometry tools on varieties defined over fields of positive characteristic. However, some of the foundational results used in birational geometry are proven with analytic techniques and they are known to fail over fields of positive characteristic. An example is Kodaira Vanishing theorem ([KM98, §2.4]), fundamental to prove the MMP (e.g. for the proof of the Base Point Free theorem) since it allows to lift sections from lower dimensional varieties. Over fields of positive characteristic Kodaira Vanishing is known to not hold ([Xie06, Example 3.7]). A powerful tool that has been revealed to be fundamental in order to overcome these issues is the Frobenius morphism, which acts as the identity on the topological space of varieties and as the $p^{\text{th}}$-power on their structure sheaf. For instance, it is possible to prove vanishing results by applying the Frobenius morphism multiple times and using Serre Vanishing theorem (an algebraic result, see [Har77, Theorem 5.2, Chapter III]) instead of Kodaira vanishing. Thanks to this tool, it has been possible to prove the MMP in dimension at most 3 over fields of characteristic $p > 5$ ([HX15], [Bir16], [BW17], [HNT20]), and to have partial results in low characteristics ([HW22]).

## Fibrations in positive characteristic

Another problem encountered when trying to mimic the classification process in characteristic 0 is that fibrations behave very differently over fields of positive characteristic. More specifically, over fields of characteristic 0, we have that, given a fibration $f\colon X \to Z$ between smooth varieties, the general fibre of $f$ is also smooth (*generic smoothness*, [Sta22, Tag 056V]). This is no longer true in positive characteristic, where regularity of the generic fibre $X_\eta$ does not imply smoothness of the geometric generic fibre and, as a consequence, of the general fibres (see Example 2.1.6, Remark 2.1.7). In particular, the general fibres may be non-normal or even non-reduced. This failure of generic smoothness stems from the fact that the generic fibre of a fibration $f\colon X \to Z$ over a field of positive characteristic is defined over an *imperfect* field. However, this is somehow the only obstacle and, after a purely inseparable base change, all the singularities of the general fibres appear also on the total space. More precisely, we consider the base change:

$$\begin{tikzcd} X^{(e)} \arrow[r] \arrow[d, "f_e"'] & X \arrow[d, "f"] \\ Z^e \arrow[r, "F^e"] & Z, \end{tikzcd} \tag{❉}$$

where $F^e$ is the $e^{\text{th}}$-power of the Frobenius morphism and $X^{(e)}$ is the normalisation of the reduction of the fibre product. The resulting fibration $f_e$ is (universally) homeomorphic to $f$, but its fibres are normal for $e \gg 0$.

Later, in Chapter 2, we will study in details how we can control the singularities of the fibres and the properties of this base change. This is a key ingredient that will be used both in the study of the canonical bundle formula and of the Iitaka conjecture for anticanonical divisors.

## Canonical bundle formula

A natural question arising in birational geometry is whether we can meaningfully relate the canonical divisors of the source and of the target of a fibration. The canonical bundle formula tackles this problem. Kodaira's result on elliptic fibrations (see, for example, [Cor07, Theorem 8.2.1, Chapter 8]) is the first instance of a formula in this direction. It states that, given an elliptic fibration $f\colon X \to Z$ from a normal projective surface over an algebraically closed field of characteristic 0

$$K_X = f^*(K_Z + B_Z + M_Z),$$

where $B_Z$ is an effective divisor which has an explicit description in terms of the singularities of the fibres, while $M_Z$ is a divisor defined via the $j$-invariant of the fibres, a number that classifies elliptic curves. More precisely, if $j\colon Z \to \mathbb{P}^1$ is the map that generically sends $z \in Z$ to the $j$-invariant of the elliptic curve $f^{-1}(z)$, then $M_Z$ is a positive rational multiple of $j^*\mathcal{O}_{\mathbb{P}^1}(1)$. Over fields of positive characteristic, a similar formula holds with an additional term that takes into account the possible presence of *wild fibres*, i.e. fibres with multiplicity divisible by the characteristic.

Later, a canonical bundle formula has been proven in any dimension for $K$-trivial fibrations in characteristic 0. More precisely, if $(X, B)$ is a log canonical pair and $f\colon X \to Z$ is a fibration such that $K_X + B \sim_{\mathbb{Q}} f^*L_Z$, for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L_Z$ on $Z$, then $L_Z = K_Z + B_Z + M_Z$. The divisor $B_Z$ is called *discriminant part* and it is defined by measuring the singularities of $f$, whereas $M_Z$ is called *moduli part* and it measures its *variation*, i.e. how far the fibration is from being a product. In general, it is difficult to construct a moduli space for the fibres, so $M_Z$ does not have an explicit description as in the elliptic curves case. However, we can at least study whether it defines a meaningful map from $Z$. The first step is to look at the positivity properties of $M_Z$.

*Questions* 1. The aim is to answer the following.

(1) Does $M_Z$ define a map towards a space that classifies the fibres of $f$?

(2) Is $M_Z$ semiample up to a birational base change?

(3) Is $M_Z$ nef up to a birational base change?

Using variation of Hodge structures, it is possible to give an affirmative answer to the third question (see [Kaw98], [FM00], [Amb04], [Amb05], [Cor07], [FL20]). Unfortunately, these techniques are not available over fields of positive characteristics.

## Canonical bundle formula in positive characteristic

With the development of the theory of $F$*-splitting* singularities, it is possible to prove a canonical bundle formula using more algebraic techniques. In fact, over fields of positive characteristic, it is useful to study not only the geometric properties, but also the "arithmetic" properties of the varieties we are considering. These are both encoded on how the Frobenius morphism $F$ acts on the structure sheaf (see for instance [PST17, §4]). A famous result of Kunz ([Sta22, Tag 0EC0]) states that a local ring $R$ is regular if and only if $F_*R$ is free. More generally, one is interested in defining $F$*-singularities*, i.e. singularities for which $R \to F_*R$ is a split map. Similarly, in the global case, one is led to the study of *globally* $F$*-split* varieties; that is, varieties $X$ for which $\mathcal{O}_X \to F_*\mathcal{O}_X$ is a split map. For example, *ordinary* elliptic curves are globally $F$-split, while *supersingular* elliptic curves are not.

In [DS17, Theorem 5.2] the authors prove that, given a fibration $f: X \to Z$ with globally $F$-split fibres, the splitting map allows to descend effectiveness on the base. More precisely, if $f: X \to Z$ is a $K$-trivial fibration with globally $F$-split general fibres,

$$K_X \sim_{\mathbb{Q}} f^*(K_Z + B^Z),$$

where $B^Z$ is an effective $\mathbb{Q}$-divisor.

In our project [BBC23], joint with Brivio and Chang, we use similar techniques to prove a canonical bundle formula over perfect fields of characteristic $p > 0$ for surjective morphisms whose finite part in the Stein factorisation has degree coprime with $p$.

**Proposition 2** (See Proposition 3.4.10)**.** *Let $f: X \to Y$ be a surjective projective morphism of normal varieties such that its Stein degree* $\mathrm{St.deg}(f)$ *is not divisible by $p$. Assume $X$ is globally $F$-split and $(1-p^e)K_X \sim f^*L$ for some Cartier divisor $L$ on $Y$. Then, there exists a canonically determined effective $\mathbb{Z}_{(p)}$-divisor $B^Y$ on $Y$ such that* *(i)* $(1-p^e)K_X \sim f^*((1-p^e)(K_Y+B^Y))$*;*

*(ii)* $(Y, B^Y)$ *is globally* $F$*-split.*

## An MMP approach

In [ACSS21], the authors develop a new approach to study positivity properties of the moduli part over fields of characteristic 0, using techniques coming from the MMP rather than variation of Hodge structures. In order to have the necessary flexibility, they consider general fibrations $f\colon X \to Z$, not only $K$-trivial ones. In this context, it is still possible to define a discriminant part $B_Z$ on $Z$ measuring the singularities of $f$ and then the moduli part $M_X$ is a $\mathbb{Q}$-divisor defined on $X$. The problem is again to prove that the moduli part has some geometric meaning linked to the variation in moduli of the fibres.

*Question* 3. Let $f\colon X \to Z$ be a fibration between normal projective varieties over an algebraically closed field and $B$ an effective $\mathbb{Q}$-divisor such that the pair $(X, B)$ is log canonical and $K_X + B$ is $f$-nef, then, is the moduli part $M_X$ is nef, up to birational maps?

The paper [ACSS21] gives an affirmative answer over fields of characteristic 0. Conversely, over fields of positive characteristic, the result is known to be false in general [Wit21, Example 3.5]. One of the main problems to follow the strategy in characteristic 0 is that, due to the failure of generic smoothness, it is not even possible to define a discriminant part of the fibration as all the fibres may be very singular. However, if we ask for the fibration to have log canonical fibres, in [Ben23], we give a positive answer to the question above over perfect fields of positive characteristic when the base is a curve.

**Theorem 4** (See Theorem 4.4.6)**.** *Assume the log MMP and the existence of log resolutions in dimension up to* $n$*. Let* $f\colon X \to Z$ *be a fibration from a normal projective variety* $X$ *of dimension* $n$ *onto a normal projective curve* $Z$*. Let* $B$ *be an effective* $\mathbb{Q}$*-divisor such that both* $(X, B)$ *and the pair induced on the general fibre have at most log canonical singularities. Suppose that* $K_X + B$ *is* $f$*-nef. Then, there exist a pair* $(Y, C)$ *and a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, dashed, "b"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Z, \end{tikzcd}$$

*where* $b$ *is a birational map such that*

*(i) if* $p_1\colon \widetilde{X} \to X$ *and* $p_2\colon \widetilde{X} \to Y$ *resolve the indeterminacies of* $Y \dashrightarrow X$*, then the difference* $p_1^*(K_X+B) - p_2^*(K_Y+C)$ *is vertical with respect to the induced fibration* $\widetilde{X} \to Z$*;*

*(ii) the moduli part $M_Y$ of $(Y/Z, C)$ is nef.*

In [Wit21, Proposition 3.2], the author proves a weak canonical bundle formula for fibrations of relative dimension 1 with smooth log canonical fibres. This result, together with the above theorem, completes the picture in dimension 3 for fibrations with log canonical general fibres over perfect fields of characteristic $p > 5$.

### Property $(*)$

The proof of [ACSS21] that the moduli part is nef follows two main steps: first, the statement is proven for a specific class of fibrations with particularly good singularities (satisfying *Property* $(*)$, as in Definition 3.3.2), and then the general case is reduced to this class with a birational base change (a $(*)$-*modification*, as in Theorem 3.3.4). Using Weak Semistable reduction ([AK00, Theorem 2.1, Proposition 4.4]), over fields of characteristic 0, it is possible to construct $(*)$-modifications in any dimension.

Over fields of positive characteristic, Weak Semistable reduction has not been proven in such generality. Nonetheless, if the base of the fibration is a curve, we construct $(*)$-modifications using log resolutions and the MMP. Moreover, we are able to control the singularities of the fibres after a base change on the base with a high power of the Frobenius morphism as in diagram (❋).

**Theorem 5** (Existence of geometric $(*)$-modifications, see Theorem 4.2.11)**.** *Assume the log MMP and the existence of log resolutions in dimension $n$. Let $f\colon X \to Z$ be a fibration from a normal projective variety $X$ of dimension $n$ onto a normal projective curve $Z$, such that $X_{\bar{\eta}}$ is normal, where $\bar{\eta}$ is the geometric generic point of $Z$. Then, for $e \gg 0$, there exists a $(*)$-modification of $X^{(e)}$.*

### Foliations

The guiding principle of [ACSS21] is that, for fibrations satisfying Property $(*)$, we can compare the moduli part of a fibration with the canonical divisor of the *foliation* induced by the fibration. Positivity of the moduli part then follows from the *foliated Minimal Model Program*, i.e. the MMP for foliations, in particular from the Cone theorem for foliations [ACSS21, Theorem 3.9].

Over fields of characteristic 0, to each equidimensional fibration $f\colon X \to Z$ between normal varieties, we can associate a foliation $\mathcal{F}$, defined as the kernel of the differential map $df\colon T_X \to f^*T_Z$. The canonical divisor of the foliation, $K_{\mathcal{F}}$, can be explicitly computed as $K_X - f^*K_Z - R(f)$, where $R(f)$ is the *ramification divisor*, supported on fibres that have multiplicity at least 2. The moduli part, under Property $(*)$ assumptions, has a similar description. On the other hand, over fields of positive characteristic, the foliation $\mathcal{F}$ could behave very differently. For example, if $f$ is not *separable*, the rank of $\mathcal{F}$ is bigger than expected. Moreover, even if we require the fibration $f$ to be separable and to have normal general fibres, the divisor $K_{\mathcal{F}}$ has a different description due to the presence of possible *wild ramification*, i.e. vertical divisors whose multiplicity is divisible by the characteristic. All in all, we prove that, over perfect fields of positive characteristic,

$$K_{\mathcal{F}} = K_X - f^*K_Z - R(f) - W(f),$$

where $R(f)$ is the ramification divisor defined as in the characteristic 0 case, while $W(f)$ is an effective divisor supported on the wild fibres (see Theorem 2.3.6). Therefore, even if the moduli part will not coincide with the canonical of the foliation, under Property $(*)$ assumptions, we have a good control of the difference between the two divisors.

Additionally, over fields of positive characteristic the foliated MMP is known to fail (see [Ber24]). To overcome this, we exploit a correspondence between foliations and purely inseparable maps in positive characteristic ([PW22, Proposition 2.9]). In fact, since the differential of the Frobenius morphism vanishes, purely inseparable maps define foliations as the kernel of their differential. Analysing this correspondence in the diagram (❋), we relate the canonical divisor of $X^{(e)}$ to the moduli part of $f$ and then use the "standard" MMP on $X^{(e)}$ to show positivity of the moduli part.

**Theorem 6** (See Corollary 4.2.2)**.** *Let $f\colon X \to Z$ be a separable equidimensional fibration between normal varieties and $B$ an effective $\mathbb{Q}$-divisor such that $(X/Z, B)$ satisfies Property $(*)$. Let $M_X$ be the associated moduli part. Let $\alpha_e\colon X \to X^{(e)}$ be the purely inseparable morphism such that the composition $X \xrightarrow{\alpha_e} X^{(e)} \to X$ is $F^e$. Then,*

$$\alpha_e^* K_{X^{(e)}} = (p^e - 1)(M_X - B^h) + K_X - W_e,$$

*where $W_e$ is an effective divisor supported on the wild fibres of $f$.*

# Iitaka conjectures

For varieties defined over the complex numbers, Iitaka proposed a conjecture (called $C_{n,m}$) which motivated Mori to start the theory that led to the MMP. It states that, given a fibration $f\colon X \to Z$, the Kodaira dimensions of $X, Z$ and the general fibre $X_z$, satisfy a subadditivity formula.

**Conjecture 7** ($C_{n,m}$, [Iit72])**.** *Let $f\colon X \to Z$ be a fibration of smooth projective varieties over $\mathbb{C}$, of dimensions $n$ and $m$ respectively, and let $X_z$ be a general fibre. Then*

$$\kappa(X, K_X) \geq \kappa(X_z, K_{X_z}) + \kappa(Z, K_Z).$$

In the same spirit, we can ask if there is a similar relation between the Iitaka dimensions of the anticanonical divisors in fibrations.

*Question* 8. Let $f\colon X \to Z$ be a fibration between normal projective varieties over an algebraically closed field such that $X$ is klt. Let $X_z$ be a general fibre. Does the inequality

$$\kappa(X, -K_X) \leq \kappa(X_z, -K_{X_z}) + \kappa(Z, -K_Z)$$

hold?

It was recently shown in [Cha23, Theorem 1.1] that, under some additional conditions to control the singularities of the anticanonical linear system, the above superadditivity statement holds. We refer to this inequality as $C^-_{n,m}$.

However, over fields of positive characteristic it is known that both $C_{n,m}$ ([CEKZ21]) and $C^-_{n,m}$ (Section 5.4.3) do not hold. On the other hand, some positive results on $C_{n,m}$ have been obtained for fibrations whose general fibres have good $F$-split singularities ([Eji17], [BCZ18], [EZ18], [Zha19], [Bau24]).

We prove $C^-_{n,m}$ for low dimensional fibrations whose fibres have good $F$-singularities.

**Theorem 9** (See Theorem 6.1.10, Theorem 6.1.14, Theorem 6.1.17)**.** *Let $f\colon X \to Z$ be a fibration between normal projective varieties over a perfect field of characteristic $p > 0$. Assume that $-K_X$ is $\mathbb{Z}_{(p)}$-Cartier. Suppose that the general fibre $X_z$ is regular, that the stable base locus of $-\overline{m}K_X$ does not dominate $Z$ for some integer $\overline{m} \geq 1$ not divisible by $p$ and that one of the following holds:*

*(a) $Z$ is a curve ($C^-_{n,1}$);*

*(b) $X$ is a threefold, $p \geq 5$ and $Z$ has at worst canonical singularities ($C^-_{3,m}$);*

*(c) $\kappa(Z, -K_Z) = 0$ and $f$ has relative dimension 1 ($C^-_{n,(n-1)}$).*

*Then,*

$$\kappa(X, -K_X) \leq \kappa(X_z, -K_{X_z}) + \kappa(Z, -K_Z).$$

*Moreover, if $\kappa(Z, -K_Z) = 0$, equality holds.*

In a joint work with Brivio and Chang ([BBC23]), we study $C^-_{n,m}$ for higher dimensional fibrations. In this case, with our strategy we need to assume global $F$-splitting conditions on the general fibres ($K$*-globally* $F$*-regular*, as in Definition 6.2.1).

**Theorem 10** (Tame $C^-_{n,m}$, see Theorem 6.2.19)**.** *Let $f\colon X \to Z$ be a fibration of smooth projective varieties over a perfect field of characteristic $p > 0$, and let $X_z$ be a general fibre. Assume that $X_z$ is $K$-globally $F$-regular. Moreover, suppose that there exists an integer $m \geq 1$ not divisible by $p$ such that the base locus of $-mK_X$ does not dominate $Z$ and let $\phi$ be the rational morphism induced by $-mK_X$. If $p$ does not divide the Stein degree of $\phi|_{X_z}$, then*

$$\kappa(X, -K_X) \leq \kappa(X_z, -K_{X_z}) + \kappa(Z, -K_Z).$$

*Furthermore, if $\kappa(Z, -K_Z) = 0$, equality holds.*

### Counterexamples

Over fields of positive characteristic, there exist ruled surfaces $g\colon P \to C$ which contain a subvariety $B$ that set-theoretically is a section of $g$, but $g|_B$ is the Frobenius morphism. These surfaces were the first examples of varieties where Kodaira Vanishing theorem does not hold. By taking a suitable cover $S \to P \to C$ and the fibre product of $S$ with itself over $C$ multiple times, in [CEKZ21], the authors found counterexamples to $C_{n,m}$ in positive characteristic. By choosing $S$ appropriately, we find counterexamples to $C^-_{7,6}$ in characteristics 2 and 3 (see Theorem 5.4.10, Corollary 5.4.12).

### Positivity descent

The main technical ingredient that is used in the proof of $C^-_{n,m}$ in characteristic 0 is a positivity descent result, which relies on Hodge theoretic techniques which are not available in positive characteristic. More in details, given a fibration $f\colon X \to Z$ such that there exists an effective $\mathbb{Q}$-divisor $\Gamma \sim_{\mathbb{Q}} -K_X - f^*E$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $E$ on $Z$, we want to find an effective $\mathbb{Q}$-divisor that is $\mathbb{Q}$-equivalent to $-K_Z - \epsilon E$ for small $\epsilon > 0$. Over fields of positive characteristic, this result can be achieved for fibrations in low dimensions whose fibres have controlled $F$-singularities, using [Eji17, Theorem 5.1] or the canonical bundle formula result for fibrations of relative dimension 1 proven in [CTX15, Lemma 6.6, Lemma 6.7] and [Wit21, Theorem 3.2].

In higher dimension, in order to get the positivity descent result, we use the canonical bundle formula Proposition 2 for globally $F$-split varieties. However, asking for global $F$-splitting does not give us enough flexibility since this class is not stable under small perturbations of the boundary. We therefore introduce a new class of varieties, namely *$K$-globally $F$-regular varieties*, which interpolates between globally $F$-split and globally $F$-regular varieties. We conjecture that, for these varieties, a *Weak Ordinarity* statement holds.

**Conjecture 11** (Relative Weak Ordinarity, see Conjecture 6.2.6)**.** *Let $X$ be a projective klt pair over $\mathbb{C}$ such that $-K_X$ is semiample. Assume that $X$ can be defined over $\mathbb{Z}$ and let $X_p$ be the reduction modulo a prime $p$ of $X$. Then, $X_p$ is $K$-globally $F$-regular for infinitely many primes $p$.*

Moreover, these singularities satisfy a Bertini-type theorem for semiample anticanonical linear sub-series.

**Theorem 12** (See Theorem 6.2.12)**.** *Let $X$ be a normal projective variety that is $K$-globally $F$-regular, and let $\{V_m \subseteq H^0(X, -mK_X)\}_{m\geq 0}$ be a graded linear subseries. Let $\phi_m$ be the rational morphism induced by $V_m$. Assume there exists an integer $m \geq 1$ not divisible by $p$ such that $\phi_m$ is a morphism defined everywhere and* *such that $p$ does not divide the Stein degree of $\phi_m$. Then, there exists $n \geq 1$ and $D_n \in |V_n|$ such that $\left(X, \frac{1}{n}D_n\right)$ is globally $F$-split.*

As a direct consequence of this positivity descent result, we prove $C^-_{n,m}$ for fibrations whose base has 0 anticanonical Iitaka dimension (see Corollary 6.1.9, Theorem 6.1.14, Theorem 6.2.15).

#### Singularities of the Iitaka fibration

To conclude the proof of Theorem 9 and Theorem 10, given a fibration $f\colon X \to Y$, we reduce to the case of fibrations whose base has 0 anticanonical Iitaka dimension, by exploiting the Iitaka fibration $g\colon Y \dashrightarrow Z$ induced by $-K_Y$. The idea is to apply the results obtained with the positivity descent we discussed above to the induced fibration $f|_{X_y}\colon X_y \to Y_z$, where $X_y$ is a general fibre of $f$ and $Y_z$ a general fibre of $g$. However, since over fields of positive characteristic generic smoothness fails, $Y_z$ may be highly singular. When $Y_z$ is a curve, this "bad behaviour" happens only in characteristics 2 and 3, but when $Y_z$ is higher dimensional it is not even known if $Y_z$ is normal for $p \gg 0$. To overcome this and recover the regularity we need in higher dimension, we perform a high enough Frobenius base change as in diagram (❋).

## Content outline

The original work in this thesis is based on three projects: [Ben22], [Ben23] and [BBC23]. The latter is a collaboration with Brivio and Chang. We outline here the subdivision of the chapters, highlighting the original material in them.

In Chapter 1, we set some notation that will be used throughout the thesis and we recall the definitions of the main types of singularities that are considered in the theory of the Minimal Model Program. We then focus on $F$-singularities, defined over fields of positive characteristic. We recall their definition and discuss some of their properties.

In Chapter 2, we set some more notation concerning fibrations. Then, we discuss the properties of the fibration obtained after a Frobenius base change as in diagram (❋). In particular, in Section 2.3, we exploit the correspondence between foliations and purely inseparable maps to analyse how the canonical divisors are related after this base change. The material presented there comes from [Ben23, §2, §3], with some modifications, and from [BBC23, §2.4]. Lastly, we study how singularities of the general fibres behave over fields of positive characteristic. The results in Section 2.2.1 follow [Ben23, §1, §4], and they discuss some properties of fibrations with geometric generic log canonical fibre. On the other hand, in Section 2.2.2, we present results from [BBC23, §2.3] and from [Ben22], concerning fibrations with globally $F$-split and sharply $F$-pure geometric generic fibre.

In Chapter 3, we give an overview of some of the literature on the canonical bundle formula both over fields of characteristic 0 and $p > 0$. In particular,in Section 3.3, we introduce the approach to the canonical bundle formula studied in [ACSS21], which inspires the work presented in the following chapter. At the end we discuss a canonical bundle formula for morphisms of Stein degree not divisible by $p > 0$, which we studied in [BBC23, §3].

The material in Chapter 4 covers the core of [Ben23], and includes some modifications. In particular, the aim of the chapter is the proof of a canonical bundle formula for separable fibrations onto curves over fields of positive characteristic.

In Chapter 5, we start by recalling some properties of the Iitaka dimension, as presented in [BBC23, §2.2]. We recall the Easy Additivity theorems, with material from [Ben22] and from [BBC23, §2.2, §5]. We then discuss the Iitaka conjectures in characteristic 0 and $p > 0$, with some remarks that we studied in [BBC23, §8] and a heuristic presented in [BBC23, Introduction]. At the end, we describe some counterexamples to the Iitaka conjecture for anticanonical divisors over fields of positive characteristic, that we studied in [Ben22, §5].

In Chapter 6, we present some positive results on the Iitaka conjecture for anticanonical divisors over fields of characteristic $p > 0$. In particular, Section 6.1 covers the results in low dimensions, which form the core of [Ben22]; whereas in Section 6.2, we present the results in higher dimension and we introduce $K$-globally $F$-regular varieties, as studied in [BBC23].

In particular, readers primarily interested in the results on the canonical bundle formula in [Ben23], may concentrate on Section 2.1, Section 2.2.1, Section 2.3.1, Section 2.3.2, Section 2.3.3, Section 2.3.4, Section 3.3 and Chapter 4.

Readers that prefer focusing on the Iitaka conjectures for anticanonical divisors, should follow the path: Section 1.3, Section 2.1, Section 2.2.2, Section 2.3.2, Section 2.3.5, Section 3.4.2, Chapter 5 and Chapter 6.

# Chapter 1

# Singularities in birational geometry

In this chapter we discuss the main classes of singularities that are considered in birational geometry both in characteristic 0 and in positive characteristic. Singularities arise naturally because the class of smooth varieties is not stable under the birational operations of the Minimal Model Program. Over fields of positive characteristic, furthermore, it is natural to define new types of singularities that take into account also the "arithmetic" of the variety, not only its geometry.

- In the whole thesis, a variety $X$ over a field $k$ is an integral and separated scheme of finite type over $k$.
- We denote the function field of a variety $X$ by $k(X)$.
- If a variety $X$ is non-normal, we denote by $\nu\colon X^\nu \to X$ its normalisation morphism.
- If $X$ is a non-reduced scheme, we denote by $X_{\mathrm{red}}$ its reduced structure.

## 1.1. Divisors

In this section we introduce some notation regarding divisors that will be used throughout the thesis.

- By $\mathbb{Z}_{(p)}$ we denote the localisation of $\mathbb{Z}$ at the prime ideal generated by $p$.
- A $\mathbb{K}$-*divisor* $D$ on a scheme $X$ is a formal finite linear combination $D = \sum_i a_i D_i$, where $D_i$ are irreducible closed subsets of codimension one in $X$ and $a_i \in \mathbb{K}$. We will take $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Z}_{(p)}, \mathbb{Q}\}$. If $\mathbb{K} = \mathbb{Z}$ we refer to $D$ as an *integral divisor* or simply a *divisor*. We define the *positive part* (resp. *negative part*) of $D$ to be $D^+ := \sum_{a_i>0} a_i D_i$ (resp. $D^- := \sum_{a_i<0}(-a_i) D_i$).

- A $\mathbb{Q}$-divisor $D$ on a scheme $X$ is $\mathbb{Q}$-*Cartier* if $mD$ is Cartier for some integer $m$. If there exists such $m \geq 1$ not divisible by $p$, then we say $D$ is a $\mathbb{Z}_{(p)}$-*Cartier* $\mathbb{Z}_{(p)}$-divisor.

- If $D_1, D_2$ are $\mathbb{Q}$-divisors on a scheme $X$ such that $mD_i$ is integral for $i = 1, 2$ and $mD_1 \sim mD_2$ for some positive integer $m$, then we say $D_1$ and $D_2$ are $\mathbb{Q}$-*linearly equivalent* $\mathbb{Q}$-divisors, denoted by $D_1 \sim_{\mathbb{Q}} D_2$. If $m$ is not divisible by $p$, we say $D_1$ and $D_2$ are $\mathbb{Z}_{(p)}$-*linearly equivalent* $\mathbb{Z}_{(p)}$-divisors, denoted $D_1 \sim_{\mathbb{Z}_{(p)}} D_2$.

- Let $f\colon X \to Y$ be a morphism of schemes and let $D$ be a divisor on $X$: we write $D \sim_Y 0$ if $D \sim f^*M$ where $M$ is a Cartier divisor on $Y$. If $D$ is a $\mathbb{Q}$-divisor (resp. a $\mathbb{Z}_{(p)}$-divisor) we write $D \sim_{\mathbb{Q},Y} 0$ (resp. $D \sim_{\mathbb{Z}_{(p)},Y} 0$) if for some integer $m$ (resp. for some integer $m$ not divisible by $p$) we have that $mD$ is integral and $mD \sim_Y 0$. In particular, we have that $D$ is Cartier (resp. $\mathbb{Q}$ or $\mathbb{Z}_{(p)}$-Cartier).

- Let $D$ be a $\mathbb{Q}$-divisor on a scheme $X$: we say $D$ is *effective* ($D \geq 0$) if all of its coefficients are non-negative. We say $D$ is $\mathbb{Q}$-*effective* (resp. $\mathbb{Z}_{(p)}$-*effective*) if, for some integer $m \geq 1$ (resp. for some integer $m \geq 1$ not divisible by $p$) $mD$ is integral and $H^0(X, mD) \neq 0$.

- Let $D$ and $D'$ be $\mathbb{Q}$-divisors on a scheme $X$: we write $D \leq D'$ if $D' - D$ is an effective $\mathbb{Q}$-divisor, and $D \leq_{\mathbb{Q}} D'$ if $D' - D$ is a $\mathbb{Q}$-effective $\mathbb{Q}$-divisor.

- Let $D$ be a divisor on a normal projective variety $X$. We say $D$ is *base point free* if for every $x \in X$ there is $s \in H^0(X, D)$ such that $s(x) \neq 0$. If $D$ is a $\mathbb{Q}$-divisor, we say it is *semiample* if there exists an integer $m \geq 1$ such that $mD$ is base point free.

- Let $D$ be a divisor on a normal projective variety $X$. We define the *base locus of* $D$, $\mathrm{Bs}(D)$, as the set of points $x \in X$ such that for all $s \in H^0(X, D)$, $s(x) = 0$.

- If $D$ is a $\mathbb{Q}$-Cartier divisor on a normal projective variety, we say it is *nef* if for every curve $\xi \subseteq X$, $D \cdot \xi \geq 0$. Given $f\colon X \to Z$ a proper morphism between normal projective varieties, we say $D$ is $f$-*nef* if $D \cdot \xi \geq 0$ for all curves $\xi \subseteq X$ such that $f(\xi)$ has dimension 0.

### Divisors on normal varieties

Let $X$ be a normal variety.

- A coherent sheaf $\mathcal{F}$ on $X$ is called *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism, where $\mathcal{F}^{**}$ denotes the double dual.

- We say that a coherent sheaf $\mathcal{L}$ on $X$ is *divisorial* if it is of rank one and reflexive. Weil divisors on $X$ correspond bijectively to divisorial sheaves via the assignment $L \mapsto \mathcal{O}_X(L)$. The set of divisorial sheaves forms a group with the reflexified tensor product $\mathcal{L}_1[\otimes]\mathcal{L}_2 := (\mathcal{L}_1 \otimes \mathcal{L}_2)^{**}$. In particular, if $\mathcal{L}_i = \mathcal{O}_X(L_i)$ for some divisor $L_i$, then $\mathcal{L}_1[\otimes]\mathcal{L}_2 = \mathcal{O}_X(L_1 + L_2)$, and $\mathcal{L}_1 \simeq \mathcal{L}_2$ if and only if $L_1 \sim L_2$. Throughout the rest of the thesis we will often confuse between a divisorial sheaf and its associated Weil divisor; for example we will write $H^i(X, L)$ rather than $H^i(X, \mathcal{O}_X(L))$.

- The *canonical divisor* of $X$, denoted by $K_X$, is the divisor corresponding to $\omega_X$, the reflexification of the determinant of the sheaf of 1-forms, $\Omega^1_X$.

- If $U \subseteq X$ is an open subset such that $\operatorname{codim}_X(X \setminus U) \geq 2$, we say that $U$ is a *big open of* $X$.

- If $X$ is a normal variety and $\mathcal{F}$ a sheaf on $X$, then $\mathcal{F}$ is reflexive if and only if it is determined in codimension one. More explicitly, if $U \subseteq X$ is a big open and $j$ denotes the natural inclusion, then $\mathcal{F}$ is reflexive if and only if the natural map $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism.

- When $X$ is a normal projective $k$-variety and $L$ is a Weil divisor, then $H^0(X, L)$ is a finite dimensional $k$-vector space. We denote by $|L|$ the associated projective space. We can naturally identify $|L|$ as the set of effective Weil divisors $L'$ which are linearly equivalent to $L$, and we refer to $|L|$ as the *linear system associated to* $L$.

- Let $X$ be a normal projective variety and $V$ a subspace of $H^0(X, L)$ for some Weil divisor $L$. We denote by $|V| \subseteq |L|$ the natural associated projective subspace, which is called the *linear subsystem associated to* $V$. This notation extends naturally to $\mathbb{Q}$-divisors: given a $\mathbb{Q}$-divisor $L$ we denote by $|L|_{\mathbb{Q}}$ the set of all effective $\mathbb{Q}$-divisors $L'$ such that $L' \sim_{\mathbb{Q}} L$. Similarly, if $L$ is a $\mathbb{Z}_{(p)}$-divisor, we denote by $|L|_{\mathbb{Z}_{(p)}}$ the set of all effective $\mathbb{Z}_{(p)}$-divisors $L'$ such that $L' \sim_{\mathbb{Z}_{(p)}} L$. We refer to $|L|_{\mathbb{Q}}$ and $|L|_{\mathbb{Z}_{(p)}}$ as the $\mathbb{Q}$- and the $\mathbb{Z}_{(p)}$-linear system of $L$, respectively. We say that a collection of subspaces $V_\bullet := (V_m \subseteq H^0(X, mL))_{m \in \mathbb{N}}$ forms a $\mathbb{Q}$-*linear subsystem* if $V_m \cdot V_{m'} := \{\sigma\tau \text{ s.t. } \sigma \in V_m \text{ and } \tau \in V_{m'}\} \subseteq V_{mm'}$.

- Let $X$ be a normal projective variety and $V$ a subspace of $H^0(X, L)$ for some Weil divisor $L$. We say $|V|$ is *base point free* if, for every $x \in X$ there is $s \in V$ such that $s(x) \neq 0$. We define the *base locus* of $|V|$ as the set of points $x \in X$ such that $s(x) = 0$ for every $s \in V$.

## 1.2. Singularities of the Minimal Model Program

If $\mu\colon X \to Y$ is a proper birational map, $K_X$ and $K_Y$ may differ by some exceptional divisors. In order to have flexibility in handling this *discrepancy*, it is often useful to consider a log version of the canonical divisor instead. This is why in birational geometry it is more natural to work with *pairs*. The main classes of singularities studied in birational geometry are classified according to the discrepancy that appears under birational maps. For a more detailed discussion, see [KM98, Chapter 2].

In this section we consider varieties defined over a perfect field of any characteristic.

**Definition 1.2.1.** A **sub-couple** $(X, B)$ over a field $k$ consists of a normal variety $X$ and a $\mathbb{Q}$-divisor $B$. If $B \geq 0$ we say $(X, B)$ is a **couple**. If $B$ is a $\mathbb{Z}_{(p)}$-divisor, we call $(X, B)$ a $\mathbb{Z}_{(p)}$**-(sub)-couple**. A **sub-pair** is a sub-couple $(X, B)$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. If $B \geq 0$ we say $(X, B)$ is a **pair**. If $K_X + B$ is $\mathbb{Z}_{(p)}$-Cartier, we call $(X, B)$ a $\mathbb{Z}_{(p)}$**-(sub)-pair**.

**Definition 1.2.2.** A pair $(X, B)$ is said to be **log smooth** if $X$ is regular, every prime divisor in $\mathrm{Supp}(B)$ is regular and, for every closed point $x \in X$, a local equation of $B$ around $x$ is given by $x_1 \cdot \ldots \cdot x_\ell$ for $x_1, \ldots, x_\ell \in \mathcal{O}_{X,x}$ independent local parameters with $\ell \leq \dim(X)$. This condition on $B$ is also called **simple normal crossing** or **snc**.

**Definition 1.2.3.** Let $(X, B)$ be a sub-pair. Given a proper birational morphism from a normal variety $\mu\colon Y \to X$, we denote by $\mathrm{Exc}(\mu)$ the union of all exceptional divisors of $\mu$. We write:

$$K_Y + (\mu^{-1})_* B = \mu^*(K_X + B) + \sum_{i \in I} a(E_i, X, B) E_i,$$

where $(\mu^{-1})_* B$ is the strict transform of $B$ and the $E_i$'s are all the prime exceptional divisors in $\mathrm{Exc}(\mu)$. For every exceptional divisor $E$, $a(E, X, B)$ does not depend on the chosen morphism $\mu$, but only on the valuation that $E$ induces on the function field of $X$ ([KM98, Remark 2.23]). The quantity $a(E, X, B)$ is called the **discrepancy** of $E$ with respect to $(X, B)$. The divisor $E$ is called a **place** over $X$, while its image in $X$ is called the **centre** of $E$. We say $(X, B)$ is:

- **terminal** if $a(E, X, B) > 0$ for all possible exceptional divisors $E$ over $X$;
- **canonical** if $a(E, X, B) \geq 0$ for all possible exceptional divisors $E$ over $X$;
- **Kawamata log terminal** or **klt** if $a(E, X, B) > -1$ for all possible exceptional divisors $E$ over $X$ and $\lfloor B \rfloor \leq 0$;

- **divisorial log terminal** or **dlt** if there exists a dense open subset $U \subseteq X$ such that $(U, B|_U)$ is log smooth, and $a(E, X, B) > -1$ for every $E$ whose centre is not contained in $U$;

- **log canonical** or **lc** if $a(E, X, B) \geq -1$ for all possible exceptional divisors $E$ over $X$.

*Remark* 1.2.4. If a sub-pair $(X, B = \sum_i a_i D_i)$ is log canonical, where the $D_i$'s are distinct prime divisors, then all $a_i$ are $\leq 1$ (see [KM98, Corollary 2.31(1)]).

*Remark* 1.2.5. Given a sub-pair $(X, B)$, we can check whether it is terminal/ canonical/ klt or lc on a log resolution. More precisely, if there exists a birational map $\mu \colon Y \to X$ such that $Y$ is regular and $\mathrm{Supp}(\mu_*^{-1}(B)) \cup \mathrm{Exc}(\mu)$ is simple normal crossing, then $(X, B)$ is terminal/ canonical/ klt or lc if and only if all the exceptional divisors on $Y$ satisfy the above inequalities.

Each of the above notions play an important role in the Minimal Model Program.

- Terminal singularities are the smallest class that is necessary to consider to run the Minimal Model Program for smooth varieties $X$ with $B = 0$.

- Canonical singularities appear on the canonical models of varieties of general type if $B = 0$.

- Kawamata log terminal singularities are the natural setting where vanishing theorems hold. They are preserved by the Minimal Model Program.

- Divisorial log terminal singularities are useful for inductive purposes.

- Log canonical singularities are the largest class where the notion of discrepancy makes sense. Indeed, if there exists an exceptional divisor $E$ over a pair $(X, B)$ such that $a(E, X, B) < -1$, then for every $a \in \mathbb{Z}$ there exists $E_a$ exceptional over $X$ such that $a(E_a, X, B) < -a$.

**Lemma 1.2.6** ([BCHM10, Lemma 3.6.3, Lemma 3.6.9])**.** *Let $(X, B)$ be a log canonical (resp. divisorial log terminal or Kawamata log terminal) pair and $\varphi \colon X \dashrightarrow Y$ a step of the $(K_X + B)$-MMP. Then, $(Y, \varphi_* B)$ is log canonical (resp. divisorial log terminal or Kawamata log terminal).*

Sometimes it is useful to consider a weakening of the notion of log canonical singularities for varieties that are not necessarily normal (for a more detailed discussion see [Kol13, Chapter 5]).

**Definition 1.2.7.** Let $X$ be a variety, we say it is **demi-normal** if it is $S_2$ and its codimension one points are either regular or nodal. Note that we can define the canonical divisor $K_X$ also for demi-normal varieties. Given $B$ effective $\mathbb{Q}$-divisor on a demi-normal variety $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier, we say $(X, B)$ is **semi-log**

**canonical** or **slc** if $(X^\nu, B^\nu)$ is log canonical, where $\nu\colon X^\nu \to X$ is the normalisation morphism and $B^\nu$ is defined by log pullback, i.e. $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$.

We end the section with some Bertini-type results. These are well-known over fields of characteristic 0, while they fail in general in positive characteristic. However, in [Tan17], the author proves that we can perturb log canonical and klt pairs with divisors coming from a semiample linear system without changing the singularities also in positive characteristic.

**Definition 1.2.8.** A field $k$ is called ***F*-finite** if the field obtained by adding all $p^{\text{th}}$-roots, $k^{\frac{1}{p}}$, is a finite extension of $k$.

**Theorem 1.2.9** ([Tan17, Theorem 1])**.** *Let $k$ be an $F$-finite field of characteristic $p > 0$ and $k_0$ a perfect field contained in it. Let $X$ be a projective variety over $k$ and $(X, B)$ a log canonical (resp. klt) pair, where $B$ is an effective $\mathbb{Q}$-divisor. Assume there exists a log resolution of $(X, B)$. Let $M$ be a semiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Then, for $m \gg 0$, there exists an effective $\mathbb{Q}$-divisor $\Gamma_m \sim mM$ such that $\left(X, B + \frac{1}{m}\Gamma_m\right)$ is log canonical (resp. klt).*

**Corollary 1.2.10.** *Let $k$ be an $F$-finite field of characteristic $p > 0$ and $k_0$ a perfect field contained in it. Let $X$ be a projective variety over $k$ and $(X, B)$ a log canonical (resp. klt) pair, where $B$ is an effective $\mathbb{Q}$-divisor. Assume the existence of log resolutions in dimension $n := \dim(X)$. Let $M$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ and $|V_\bullet| := (|V_m| \subseteq |mM|)_{m\in\mathbb{N}}$ a sub-linear system. Fix a positive integer $\overline{m} \geq 1$, choose a basis $\mathcal{B} := \{s_1, ..., s_\ell\}$ of $V_{\overline{m}}$ and define $V_{\overline{m}}(k_0) := \left\{\sum_{i=1}^{\ell} a_i s_i \text{ s.t. } a_i \in k_0\right\}$.*

*(i) Suppose that $V_{\overline{m}}$ is base point free. Then, for $m \gg 0$ and sufficiently divisible, there exists an effective $\mathbb{Q}$-divisor $\Gamma_m \in |V_{\overline{m}m}|$ such that $\Gamma_m$ can be decomposed as $\sum_{i=1}^m D_i$ with $D_i \in |V_{\overline{m}}(k_0)|$ and $\left(X, B + \frac{1}{\overline{m}m}\Gamma_m\right)$ is log canonical (resp. klt).*

*(ii) Suppose that $\operatorname{Bs}(V_{\overline{m}}) = W \subseteq X$. Then, for $m \gg 0$, there exists an effective $\mathbb{Q}$-divisor $\Gamma_m \in |V_{\overline{m}m}|$ such that $\Gamma_m$ can be decomposed as $\sum_{i=1}^m D_i$ with $D_i \in |V_{\overline{m}}(k_0)|$, and $\left(X, B + \frac{1}{m}\Gamma_m\right)$ is log canonical (resp. klt) outside $W$ and has a non-klt centre at every irreducible component of $W$.*

*Proof.* Point (i) follows directly from the proof of Theorem 1.2.9 noting the following. In the notation of the proof of [Tan17, Proposition 2, Proposition 3], we consider $T_1 := \mathbb{A}^\ell$ with basis $\mathcal{B}$ and then we choose $\mathcal{B}^{\dim(X)}$ as basis of $T := T_1^{\dim(X)}$. With this choice, we get the statement.

As for point (ii), let $\mu\colon Y \to X$ be a birational model such that $\mu^*|V_{\overline{m}}| = |M_{\overline{m}}| + \Phi$, where $|M_{\overline{m}}|$ is base point free and $\Phi$ is the fixed divisor. In particular, $\Phi$ contains a place over every component of $W$. By possibly passing to a higher model, we can assume $\mu$ is a log resolution of $(X, B)$. Define an effective $\mathbb{Q}$-divisor $B_Y$ as

$$K_Y + B_Y = \mu^*(K_X + B) + E,$$

where $E$ is an effective $\mu$-exceptional divisor with no common components with $B_Y$. Let $\varphi$ be the rational function defining $\Phi$, then, for all $i = 1, ..., \ell$, there is a rational function $t_i$ such that $\mu^* s_i = \varphi t_i$. Define $\mathcal{B}' := \{t_1, ..., t_\ell\}$, $W_1(k_0) := \left\{\sum_{i=1}^{\ell} a_i t_i \text{ s.t. } a_i \in k_0\right\}$ and, for all $m \in \mathbb{N}$, let $W_m := \mathrm{Sym}^m M_{\overline{m}}$. Note that $|W_1(k_0)| + \Phi = \mu^*|V_{\overline{m}}(k_0)|$. By point (i), for some integer $m \geq 1$, we find an effective $\mathbb{Q}$-divisor $\Gamma_{Y,m}$ that admits a decomposition as $\sum_{i=1}^{m} D_i$ with $D_i \in W_1(k_0)$ and such that $\left(Y, B_Y + \frac{1}{m}\Gamma_{Y,m}\right)$ is log canonical. Therefore, $\left(Y, B_Y + \frac{1}{m}\Gamma_{Y,m} + \Phi\right)$ is log canonical outside $\mathrm{Supp}(\Phi)$ and has a non-klt place over every component of $W$. Since $D_i + \Phi \in \mu^*|V_{\overline{m}}(k_0)|$, by the projection formula, there exists an effective $\mathbb{Q}$-divisor $\Gamma_m \in |V_{\overline{m}m}|$ on $X$ which pulls-back to $\Gamma_{Y,m} + m\Phi$ and satisfies the required properties. qed

## 1.3. $F$-singularities

In this section, we overview the definitions of the main types of $F$-singularities over fields of positive characteristic. For a more detailed discussion, we refer to [PST17].

**Definition 1.3.1.** Let $k$ be a field of characteristic $p > 0$ and let $k \subseteq L$ be a field extension. Let $l \in L$ be an element of $L$. We say $l$ is **separable** over $k$ if its minimum polynomial over $k$ has distinct roots. We say $l$ is **purely inseparable** over $k$ if there exists $e \in \mathbb{N}_{>0}$ such that $l^{p^e} \in k$. The extension $L$ is called **separable** (resp. **purely inseparable**) over $k$ if all its elements are separable (resp. purely inseparable) over $k$. If $L$ is neither separable nor purely inseparable over $k$, it is called **inseparable** over $k$. Let $\alpha\colon X \to Y$ be a finite morphism between normal varieties over a field $k$. It is called **separable** (resp. **purely inseparable** or **inseparable**) if the induced extension of function fields $k(Y) \subseteq k(X)$ satisfies the corresponding property.

The most fundamental purely inseparable map is the *Frobenius morphism*. Indeed, every purely inseparable map factorises a suitable power of it.

**Definition 1.3.2.** Let $X$ be a variety over a field of characteristic $p > 0$. We will denote by $F_X\colon X^1 \to X$ the **absolute Frobenius morphism of** $X$ or simply **Frobenius morphism**. It is the identity on the underlying topological space, and it acts on regular functions by raising them to the $p^{\text{th}}$ power. For all $e \geq 1$ we denote by $F^e\colon X^e \to X$ the $e^{\text{th}}$ power of the absolute Frobenius. Note that $X$ and $X^e$ are the same scheme abstractly, although they are not isomorphic over $k$, hence we will often simply write $F^e\colon X \to X$, when it does not cause ambiguity.

*Remark* 1.3.3. Let $X$ be a variety over a perfect field $k$ of characteristic $p > 0$. The *geometric Frobenius morphism* of $X$ is a variant of the Frobenius morphism that is $k$-linear (see Definition 2.3.12). However, since $k$ is perfect, it differs from the absolute Frobenius only by an automorphism of $k$, therefore we will not distinguish them.

### 1.3.1. Traces of Frobenius map

A fundamental tool in the theory of $F$-singularities is the following duality statement, that we will apply to the Frobenius morphism.

Given a variety $X$, we denote by $\mathscr{H}om_{\mathcal{O}_X}(-,-)$ the sheaf of $\mathcal{O}_X$-homomorphisms, whereas $\mathrm{Hom}_{\mathcal{O}_X}(-,-)$ denotes its space of global sections.

**Theorem 1.3.4** (Grothendieck–Verdier duality, [Huy06, Theorem 3.34])**.** *Consider $f\colon X \to Y$ a morphism between smooth schemes over any field of relative dimension $d := \dim(X) - \dim(Y)$. For any $\mathcal{F}$ and $\mathcal{G}$ coherent sheaves on $X$ and $Y$ respectively, there exists a functorial isomorphism in the bounded derived category of coherent sheaves on $Y$:*

$$Rf_* R\,\mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{F}, Lf^*\mathcal{G} \otimes \omega_X \otimes f^*\omega_Y^{-1}[d]\right) \simeq R\,\mathscr{H}om_{\mathcal{O}_Y}(Rf_*\mathcal{F}, \mathcal{G}),$$

*where the $R$ and $L$ in the formula above mean that we are taking the right and left derived functors and $[d]$ is the shift in the derived category. In particular, if we apply it to $F^e\colon X \to X$ and $\mathcal{F}$ and $\mathcal{G}$ locally free sheaves, we obtain the isomorphism:*

$$F^e_* \mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{F}, F^{e*}\mathcal{G} \otimes \omega_X^{\otimes(1-p^e)}\right) \simeq \mathscr{H}om_{\mathcal{O}_Y}(F^e_*\mathcal{F}, \mathcal{G}).$$

*Remark* 1.3.5. We can apply the above isomorphism also to $F^e\colon X \to X$, where $X$ is a normal variety and to reflexive sheaves $\mathcal{F}$ and $\mathcal{G}$ on $X$ by first restricting to the regular locus of $X$ and then extending the isomorphism everywhere using the $S_2$ property of $X$.

Recall also that, if $\mathcal{F}$ is a coherent sheaf on $X$ and $\mathcal{G}$ is reflexive, then $\mathscr{H}om_X(\mathcal{F}, \mathcal{G})$ is again reflexive, and so are the sheaves $\mathcal{O}_X(D)$ for any Weil divisor $D$. In particular, as $X$ is $R_1$, we have that $\mathcal{O}_X(D)$ always restricts to a line bundle on $X \setminus \mathrm{Sing}(X)$, thus we have isomorphisms

$$\mathscr{H}om_X(\mathcal{F}(-D), \mathcal{G}) \cong \mathscr{H}om_X(\mathcal{F}, \mathcal{G}(D)).$$

This fact will be tacitly used when applying Grothendieck–Verdier duality for a finite morphism.

From now on, throughout the section we will denote by $(X, B)$ a sub-couple over a perfect field of characteristic $p > 0$, such that $K_X + B$ is a $\mathbb{Z}_{(p)}$-divisor, unless otherwise stated.

We introduce notation relative to trace maps of Frobenius morphisms. Given $(X, B)$, let $a \geq 1$ be the smallest integer such that $aB$ is integral, and let $d \geq 1$ be the smallest integer such that $a$ divides $(p^d - 1)$. As $(p^e - 1)B$ is integral for all $e \in d\mathbb{N}$, we have divisorial sheaves

$$\mathcal{L}^{(e)}_{X,B} := \mathcal{O}_X((1 - p^e)(K_X + B)).$$

When $X$ is clear from the context, we will simply write $\mathcal{L}_B^{(e)}$. In particular, $\mathcal{L}^{(e)} = \mathcal{O}_X((1-p^e)K_X)$. We have a trace map $F^e_*\mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X)$ by Grothendieck–Verdier duality. When $(X, B)$ is a $\mathbb{Z}_{(p)}$-couple, i.e. $B \geq 0$, then twisting by $-K_X$ gives a map

$$T^e_B \colon F^e_*\mathcal{L}^{(e)}_{X,B} \subseteq F^e_*\mathcal{L}^{(e)}_X \to \mathcal{O}_X.$$

By further twisting the above map by any integral divisor $L$ we then obtain

$$T^e_B(L) \colon F^e_*\mathcal{L}^{(e)}_{X,B} \otimes_{\mathcal{O}_X} \mathcal{O}_X(L) \to \mathcal{O}_X(L).$$

**Definition 1.3.6.** We define the space of **Frobenius stable sections of $\mathcal{O}_X(L)$** as

$$S^0(X, B; L) := \bigcap_{e \in d\mathbb{N}} \operatorname{Im}(H^0(X, T^e_B(L))) \subseteq H^0(X, \mathcal{O}_X(L)).$$

Note that, when $X$ is proper, we have $S^0(X, B; L) = \operatorname{Im}(H^0(X, T^e_B(L)))$ for some sufficiently large $e \in d\mathbb{N}$ ([PST17, Definition 3.10]).

We now extend the above construction to the case where $B$ is not necessarily effective. As $B$ is $\mathbb{Z}_{(p)}$-Weil, we have an $\mathcal{O}_X$-linear twisted trace map $T^e_B$, fitting in the following commutative diagram

$$\begin{tikzcd}
F^e_*\mathcal{L}^{(e)}_{B^+} \arrow[r, "T^e_{B^+}"] \arrow[d, hook] & \mathcal{O}_X \arrow[d, hook] \\
F^e_*\mathcal{L}^{(e)}_{B} \arrow[r, "T^e_{B}"] & k(X)
\end{tikzcd}$$

for all $e \in d\mathbb{N}$. To construct it we work locally over the regular locus of $X$. Write $(p^e - 1)B = E^+ - E^-$, and let $f$ be a regular function such that $E^- = (f = 0)$. If $\sigma$ is a section of $\mathcal{L}^{(e)}_B$, then locally $\sigma = s/f$, where $s \in \mathcal{L}^{(e)}_{B^+}$. Then, by $\mathcal{O}_X$-linearity we must have

$$T^e_B(F^e_*(\sigma)) := \frac{T^e_{B^+}(F^e_*(f^{p^e-1}s))}{f}.$$

In particular, $\operatorname{Im}(T^e_B) \subseteq \mathcal{O}_X(E^-)$.

**Proposition 1.3.7** ([DS17, §2])**.** *The assignment $B \mapsto (T^e_B \colon F^e_*\mathcal{L}^{(e)}_B \to k(X))$ defines bijections*

$$\begin{tikzcd}
\left\{\begin{array}{c}\mathbb{Q}\textit{-divisors } B\geq 0 \textit{ such that}\\ (1-p^e)(K_X+B) \textit{ is integral}\end{array}\right\} \arrow[r, leftrightarrow] \arrow[d, hook] & \left\{\begin{array}{c}\textit{divisorial sheaves } \mathcal{L} \textit{ and}\\ \mathcal{O}_X\textit{-linear maps}\\ \psi : F^e_*\mathcal{L} \xrightarrow{\neq 0} \mathcal{O}_X\end{array}\right\}\Big/\sim \arrow[d, hook] \\
\left\{\begin{array}{c}\mathbb{Q}\textit{-divisors } B \textit{ such that}\\ (1-p^e)(K_X+B) \textit{ is integral}\end{array}\right\} \arrow[r, leftrightarrow] & \left\{\begin{array}{c}\textit{divisorial sheaves } \mathcal{L} \textit{ and}\\ \mathcal{O}_X\textit{-linear maps}\\ \psi : F^e_*\mathcal{L} \xrightarrow{\neq 0} k(X)\end{array}\right\}\Big/\sim,
\end{tikzcd}$$

*where $\psi_1 \sim \psi_2$ if the two maps agree up to multiplication by a unit of $H^0(X, \mathcal{O}_X)$.*

*Sketch of proof.* We construct the top horizontal maps. For the general case, we refer the reader to [DS17, 2.1.1]. Given $B \geq 0$ we set $\mathcal{L} := \mathcal{L}^{(e)}_{X,B}$ and $\psi := T^e_B$. Conversely, given $\psi$, by Grothendieck–Verdier duality (Theorem 1.3.4) we have

$$\begin{aligned}\psi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{L}, \mathcal{O}_X) &\simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X((1-p^e)K_X))\\ &\simeq H^0(X, \mathcal{L}^{-1}((1-p^e)K_X)).\end{aligned}$$

We identify $\psi$ with an element $D_\psi \in H^0(X, \mathcal{L}^{-1}((1-p^e)K_X))$, up to multiplication by a unit in $H^0(X, \mathcal{O}_X)$, and we set $B := D_\psi/(p^e-1)$. qed

### 1.3.2. Global $F$-singularities

In this section, we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Definition 1.3.8.** Let $(X, B)$ be a $\mathbb{Z}_{(p)}$-sub-couple and $d$ the smallest integer such that $(p^d-1)B$ is integral. Then $(X, B)$ is **globally sub-$F$-split** (GsFS) if, for some $e \in d\mathbb{N}$, there exists a map $\sigma^e \colon \mathcal{O}_X \to F^e_*\mathcal{L}^{(e)}$ such that $T^e_B \circ \sigma^e$ is the identity on $\mathcal{O}_X$. If $B \geq 0$, we say $(X, B)$ is **globally $F$-split** (GFS).

*Remark* 1.3.9 ([Eji17, Remark 2.2]). If $(X, B)$ is globally sub-$F$-split, then $T^e_B$ is split surjective for *all* $e \in d\mathbb{N}$.

**Lemma 1.3.10.** *A $\mathbb{Z}_{(p)}$-couple $(X, B)$ is globally $F$-split if and only if $S^0(X, B; \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$.*

*Proof.* If $T^e_B$ is split surjective for all $e \in d\mathbb{N}$, then clearly $S^0(X, B; \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. Conversely, let $\sigma^e \in H^0(X, \mathcal{O}_X((1-p^e)(K_X+B)))$ such that $H^0(X, T^e_B)(\sigma^e) = 1$. The induced map of sheaves $\sigma^e \colon \mathcal{O}_X \to F^e_*\mathcal{L}^{(e)}_B$ gives a splitting of $T^e_B$. qed

**Definition 1.3.11.** A $\mathbb{Z}_{(p)}$-couple $(X, B)$ is **globally $F$-regular** (GFR) if, for every divisor $E \geq 0$, the map $T^e_{B+E/(p^e-1)}$ is split surjective for all $e \in d\mathbb{N}$ sufficiently large.

Globally $F$-regular couples behave similarly to klt pairs with respect to perturbations of the boundary.

**Lemma 1.3.12** ([SS10, Corollary 6.1])**.** *Let $(X,B)$ be a globally $F$-regular couple and let $D \geq 0$ be a divisor. Then $(X, B+\varepsilon D)$ is globally $F$-regular for all sufficiently small and positive $\varepsilon \in \mathbb{Z}_{(p)}$.*

**Lemma 1.3.13** ([SS10, Proposition 3.8(i)])**.** *Let $(X,B)$ be a $\mathbb{Z}_{(p)}$-couple. Then $(X,B)$ is globally $F$-regular if and only if for all divisors $D \geq 0$ and all sufficiently small positive $\varepsilon \in \mathbb{Z}_{(p)}$ the couple $(X, B+\varepsilon D)$ is globally $F$-split.*

The following examples show how the arithmetic comes into play when we consider these singularities.

*Example* 1.3.14. When $B = 0$, a variety $X$ is GFS if and only if the morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits. This is equivalent to asking the map

$$\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

to be surjective. If $X$ is regular of dimension $n$, by Serre duality, this is in turn equivalent to asking

$$H^n(X, \omega_X) \to H^n(X, \omega_X \otimes F_*\mathcal{O}_X)$$

to be injective. Let $E$ be an elliptic curve over an algebraically closed field of characteristic $p > 0$, then it is GFS if and only if it is *ordinary*, i.e. its subgroup of $p$-torsion points $E\left[p\right]$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. This fact is well-known to the experts, anyway we sketch a proof here. Since $\omega_E$ is trivial, by the above discussion, the GFS condition is equivalent to $F$ acting injectively on $H^1(E, \mathcal{O}_E)$. Note that $H^1(E, \mathcal{O}_E)$ can be identified with the tangent space to $\mathrm{Pic}^0(E) \simeq E$ and the action of the Frobenius on $H^1(E, \mathcal{O}_E)$ under this identification corresponds to the action of the Verschiebung morphism $V$ on the tangent space to $E$. This action is injective if and only if it is an isomorphism and this happens if and only if it is étale. But $V$ being étale is equivalent to asking that its kernel on $E$ is a constant group scheme (of order $p$). Since the multiplication by $p$ can be written as $[p] = F \circ V$, $E[p] = E[F]E[V]$. Since $F$ is purely inseparable, $E[p]$ contains the constant group scheme $\mathbb{Z}/p\mathbb{Z}$ if and only if $E[V]$ does.

*Example* 1.3.15. Let $B := \sum_{i=1}^{4} \frac{1}{2}(p_i)$, where the $p_i$'s are distinct points on $\mathbb{P}^1$. Consider the pair $(\mathbb{P}^1, B)$ over a field of characteristic $p > 2$. Let $\pi\colon E \to \mathbb{P}^1$ be the natural cover by an elliptic curve ramified at the four chosen points. The pair $(\mathbb{P}^1, B)$ is GFS if and only if $E$ is GFS, thus ordinary. To prove this, apply [ST14, Theorem 6.28] to the map induced on the cones over $E$ and $\mathbb{P}^1$ and conclude by [SS10, Proposition 5.3].

Globally $F$-split and globally $F$-regular pairs should be thought of as pairs of log Calabi–Yau type, resp. log Fano type, with arithmetically well-behaved Frobenius. This is made more precise in the next statements.

**Definition 1.3.16.** Let $(X, B)$ be a projective pair over a perfect field of any characteristic. We say $(X, B)$ is **log Fano** if $K_X + B$ is anti-ample and $(X, B)$ is klt. We say it is **log Calabi–Yau** if $K_X + B$ is $\mathbb{Q}$-linearly trivial and $(X, B)$ is log canonical. We say $(X, B)$ is of **log Fano type** (resp. of **log Calabi–Yau type**) if there exists an effective $\mathbb{Q}$-divisor $\Delta$ such that $(X, B + \Delta)$ is log Fano (resp. log Calabi–Yau).

**Definition 1.3.17.** Let $(X, B)$ be a projective sub-couple over $\mathbb{C}$. A **model** of $(X, B)$ is a normal, integral, separated, projective scheme of finite type over a finitely generated $\mathbb{Z}$-algebra $A$, $\mathcal{X} \to \operatorname{Spec}(A)$, together with a $\mathbb{Q}$-divisor $\mathcal{B}$ such that $(X, B) = (\mathcal{X}, \mathcal{B}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(\mathbb{C})$. Let $\mathfrak{p}$ be a prime ideal of $A$ and $k(\mathfrak{p})$ the corresponding residue field. We denote by $(X_{\bar{\mathfrak{p}}}, B_{\bar{\mathfrak{p}}})$ the fibre product $(\mathcal{X}, \mathcal{B}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(\overline{k(\mathfrak{p})})$ and we call it the **reduction modulo** $\mathfrak{p}$ of $(X, B)$.

**Theorem 1.3.18** ([SS10, Theorem 5.1])**.** *Let $(X, B)$ be a projective pair over $\mathbb{C}$ of log Fano type. Then $(X, B)$ has open globally $F$-regular type; that is, for every model $(\mathcal{X}, \mathcal{B}) \to \operatorname{Spec}(A)$ of $(X, B)$, the set of primes $\mathfrak{p} \subseteq A$ such that $(X_{\bar{\mathfrak{p}}}, B_{\bar{\mathfrak{p}}})$ is globally $F$-regular is open and dense in* $\operatorname{Spec}(A)$.

**Theorem 1.3.19** ([CGS16, Corollary 4.1])**.** *Let $I \subseteq (0, 1) \cap Q$ be a finite subset. Let*

$$I_+ := \left\{ \sum_{j=1}^{m} a_j i_j \mid i_j \in I,\, a_j \in \mathbb{N},\, m \in \mathbb{N} \right\} \cap [0, 1]$$

*and*

$$D(I) := \left\{ \frac{m - 1 + f}{m} \mid m \in \mathbb{N},\, f \in I_+ \right\} \cap [0, 1] .$$

*Then there exists a positive integer $p_0$ such that, if $(\mathbb{P}^1, B)$ is a log Fano pair defined over a perfect field of characteristic $p > p_0$ such that the coefficients of $B$ belong to $D(I)$, then $(\mathbb{P}^1, B)$ is globally $F$-regular.*

A similar result is expected to hold for log Calabi–Yau pairs.

**Conjecture 1.3.20** (Weak Ordinarity, [HW02, Problem 5.1.2], [SS10, Remark 5.2])**.** *Let $(X, B)$ be a projective pair over $\mathbb{C}$ of log Calabi–Yau type. Then $(X, B)$ has dense globally $F$-split type; that is, for every model $(\mathcal{X}, \mathcal{B}) \to \operatorname{Spec}(A)$ of $(X, B)$, the set of primes $\mathfrak{p} \subseteq A$ such that $(X_{\bar{\mathfrak{p}}}, B_{\bar{\mathfrak{p}}})$ is globally $F$-split is dense in* $\operatorname{Spec}(A)$.

*Remark* 1.3.21. Let $(X, B)$ be a $\mathbb{Z}_{(p)}$-couple over $k$ such that $X$ is geometrically normal, then it is globally $F$-regular (resp. globally $F$-split) if and only if the base change $(\overline{X}, \overline{B}) := (X, B) \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\overline{k})$ over the algebraic closure $\overline{k}$ is globally $F$-regular (resp. globally $F$-split). To see this, apply Lemma 1.3.10 and Lemma 1.3.13 and use the fact that $S^0(X, B; \mathcal{O}_X) \times_k \overline{k} = S^0(\overline{X}, \overline{B}; \mathcal{O}_{\overline{X}})$.

Given $(\mathcal{X}, \mathcal{B}) \to \mathrm{Spec}(A)$, model of a projective pair $(X, B)$ over $\mathbb{C}$, by [Gro67, Proposition 9.9.4], the set of primes such that $X_{\bar{\mathfrak{p}}}$ is normal, is open and dense. Therefore, asking for the existence of an open and dense set of primes $\mathfrak{p}$ of $\mathrm{Spec}(A)$ for which $(X_{\bar{\mathfrak{p}}}, B_{\bar{\mathfrak{p}}})$ is globally $F$-regular is equivalent to asking that there exists such set for which $(X_{\mathfrak{p}}, B_{\mathfrak{p}}) := (\mathcal{X}, \mathcal{B}) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k(\mathfrak{p}))$ is globally $F$-regular. In the same spirit, we can rephrase Conjecture 1.3.20 by asking for the set of primes $\mathfrak{p} \subseteq A$ such that $(X_{\mathfrak{p}}, B_{\mathfrak{p}}) := (\mathcal{X}, \mathcal{B}) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k(\mathfrak{p}))$ is globally $F$-split to be dense in $\mathrm{Spec}(A)$.

### 1.3.3. Local $F$-singularities

In this section, we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Definition 1.3.22.** A couple $(X, B)$ is **sharply F-pure** (resp. **strongly $F$-regular** or **SFR** for short) if $X$ is covered by a finite number of open subsets $U$ such that the pairs $(U, \Delta|_U)$ are globally $F$-split (resp. globally $F$-regular).

*Example* 1.3.23. If $X$ is a regular variety, then it is SFR. Indeed, Kunz' theorem states that $X$ is regular if and only if $F^e_* \mathcal{O}_X$ is locally free for all $e \in \mathbb{N}$. Therefore, on a regular variety, we can always construct the splitting maps locally.

*Remark* 1.3.24. If $(X, B)$ is sharply F-pure (resp. SFR), then it is log canonical (resp. klt) ([HW02, Theorem 3.3]). A surface $S$ over a field of characteristic $p > 5$ is SFR if and only if it is klt (with no boundary divisor). If we allow boundary divisors, we get the same conclusion for every $p \geq p_0$ for a fixed $p_0 \in \mathbb{N}_{>0}$, which depends on the coefficients of the boundary (see [CGS16, Theorem 1.1]).
Furthermore, if $X$ is a normal variety over $\mathbb{C}$ and we choose $\mathcal{X}$ a model of $X$ over $\mathrm{Spec}(\mathbb{Z})$, then $X$ is klt if and only if $X_p$ is SFR for infinitely many primes $p$ (see [Har98, Theorem 5.2], [HW02, Theorem 3.7]).

*Remark* 1.3.25. If $(X, B)$ is SFR and $D$ is an effective divisor, then for any sufficiently small $\varepsilon > 0$, $(X, B + \varepsilon D)$ is SFR as well ([CTX15, Remark 2.8]).

The next results are Bertini-type theorems for semiample linear systems. This problem has been studied in [Tan17]. We will use a slight variation of those results, proven as a corollary below.

**Definition 1.3.26.** Let $X$ be a normal projective variety. Let $|V_\bullet| := (|V_m|)_{m \in \mathbb{N}} \subseteq (|mM|)_{m \in \mathbb{N}}$ be a semiample graded linear system on $X$, where $M$ is a Cartier divisor on $X$. We say $|V_\bullet|$ is **$\mathbb{Z}_{(p)}$-semiample** if there exists an integer $m \geq 1$ not divisible by $p$ such that $V_m$ is base point free.

**Proposition 1.3.27** ([Tan17, Proposition 2])**.** *Let $k$ be an $F$-finite field of characteristic $p > 0$ and $k_0$ a perfect field contained in it. Let $X$ be a projective regular* *variety over $k$ and let $(X, B)$ be a strongly $F$-regular pair, where $B$ is an effective $\mathbb{Q}$-divisor. Let $M$ be a semiample $\mathbb{Q}$-divisor on $X$. Then, for $m \gg 0$, there exists an effective divisor $\Gamma_m \sim mM$ such that $\left(X, B + \frac{1}{m}\Gamma_m\right)$ is strongly $F$-regular.*

**Corollary 1.3.28.** *Let $k$ be an $F$-finite field of characteristic $p > 0$ and $k_0$ a perfect field contained in it. Let $X$ be a projective regular variety over $k$ and $(X, B)$ a strongly $F$-regular pair, where $B$ is an effective $\mathbb{Q}$-divisor. Let $|V_\bullet| := (|V_m|)_{m\in\mathbb{N}} \subseteq (|mM|)_{m\in\mathbb{N}}$ be a semiample graded linear system on $X$, where $M$ is a Cartier divisor on $X$. Then, for $m \gg 0$, there exists an effective divisor $\Gamma_m \in |V_m|$ such that $\left(X, B + \frac{1}{m}\Gamma_m\right)$ is strongly $F$-regular. Moreover, if $|V_\bullet|$ is $\mathbb{Z}_{(p)}$-semiample, $\Gamma_m$ can be chosen in $|V_m|$, for some $m \gg 0$ not divisible by $p$.*

*Proof.* The proof of Proposition 1.3.27 carries over in the same way, taking divisors inside $|V_\bullet|$ instead of divisors $\mathbb{Q}$-equivalent to $L$. This can be done since $|V_\bullet|$ is semiample. qed

# Chapter 2

# Fibrations in positive characteristic

Fibrations are a fundamental tool in birational geometry, used to “split” varieties into simpler pieces. In fact, the Minimal Model Program predicts that we can decompose each variety by means of fibrations into building blocks that are either log Fano varieties, log Calabi–Yau or varieties of general type.

In this section we study properties of fibrations that appear specifically in positive characteristic.

**Definition 2.0.1.** Let $f\colon X \to Z$ be a projective morphism between normal varieties over any field. It is called a **fibration** if $f_*\mathcal{O}_X = \mathcal{O}_Z$.

*Remark* 2.0.2. In characteristic 0, the above definition is equivalent to asking $f$ to be surjective with connected fibres ([Har77, §11, Chapter III]). On the other hand, purely inseparable morphisms are surjective with connected fibres, but they are not fibrations.

### Notation

Let $f\colon X \to Z$ be a fibration between normal varieties over a field $k$ of any characteristic.

- Given a prime divisor $D \subseteq X$, we say $D$ is *horizontal* if $f(D) = Z$, we say it is *vertical* otherwise.
- Given a curve $\xi \subseteq X$, we say $\xi$ is *horizontal* if $f(\xi)$ is a curve, we say it is *vertical* otherwise.
- Given a $\mathbb{Q}$-divisor $D$ on $X$, we can decompose it into its horizontal and vertical parts, denoted by $D^h$ and $D^v$, respectively, so that $D = D^h + D^v$.
- A *general* (resp. *very general*) *fibre of* $f$ is $X_z := f^{-1}(z)$ where $z$ is a closed point over $\bar{k}$ belonging to a dense open subset of $Z$ (resp. to a countable intersection of dense open subsets of $Z$).

- If $\eta$ is the generic point of $Z$, we denote by $\overline{\eta}$ its geometric generic point and by $X_\eta$ (resp. $X_{\overline{\eta}}$) the generic (resp. geometric generic) fibre of $f$. Note that $X_\eta$ may be defined over an imperfect field.

- Let $D$ be a $\mathbb{Q}$-divisor on $X$, and let $z, \eta \in Z$ denote a general point and the generic point, respectively. We denote by $D_\eta$ the base change of $D$ to the generic fibre and $D_{\overline{\eta}}$ the base change to the geometric generic fibre. If $X_z$ is normal, $D$ is $\mathbb{Q}$-Cartier along any codimension 1 point of $X_z$, hence the restricted divisor $D_z := D|_{X_z}$ is well-defined.

- If $\delta$ is a prime divisor on $Z$ whose preimage under $f$ is of pure codimension 1 in $X$, we denote by $f^{-1}(\delta)$ the induced divisor on $X$ with its reduced structure.

- Assume the general fibre $X_z$ is normal and $X$ is projective. Let $D$ be a divisor on $X$ and let $V \subseteq H^0(X, D)$ be a subspace. We denote by $V_z$ the image of $V$ under the natural restriction map $H^0(X, D) \to H^0(X_z, D_z)$ and by $|V|_z \subseteq |D_z|$ the linear subsystem generated by $V_z$. In other words, the divisors in $|V|_z$ are exactly the divisors in $|D_z|$ which extend to divisors in $|V| \subseteq |D|$.

*Remark* 2.0.3. Let $f\colon X \to Z$ be an equidimensional morphism of normal varieties, and let $D$ be a $\mathbb{Q}$-divisor on $Z$. Then we can define $f^*D$ even if $D$ is not $\mathbb{Q}$-Cartier. Let $Z^0 \subseteq Z$ denote the regular locus and let $f^0\colon X^0 := f^{-1}(Z^0) \to Z^0$ be the induced morphism: then we define $f^*D$ as the closure of $f^{0*}(D|_{Z^0})$ inside $X$; it is canonically determined since $\operatorname{codim}(X \setminus X_0) \geq 2$.

Up to a birational base change, we can always assume our fibration is equidimensional.

**Lemma 2.0.4** (Flattening lemma, [AO00, §3.3], [RG71, Théorème 5.2.2])**.** *Consider $f\colon X \to Z$, a projective dominant morphism of normal varieties, and let $\eta$ be the generic point of $Z$. Then, there exists a projective birational morphism $Z' \to Z$ such that, if $\widetilde{X} \subset X \times_Z Z'$ is the Zariski closure of the generic fibre $X_\eta \times_{Z'} Z$, the induced morphism $\tilde{f}\colon \widetilde{X} \to Z'$ is flat. In particular, if $X' := \widetilde{X}^\nu$, the morphism $f'\colon X' \to Z'$ is equidimensional.*

As a consequence of the Flattening lemma we have the following.

**Lemma 2.0.5** ([JW21, Lemma 2.19])**.** *Let $f\colon X \to Z$ be a projective dominant morphism of normal varieties. Then, there is an open subset $U \subseteq Z$ with* $\operatorname{codim}(Z \setminus U) \geq$ 2 *such that $X_U := f^{-1}(U)$ is flat over $U$.*

We will often use the following straightforward extension of the projection formula.

**Lemma 2.0.6.** *Let $f\colon X \to Z$ be an equidimensional projective morphism between normal varieties, let $\mathcal{L}$ be a divisorial sheaf on $Z$ and $\mathcal{F}$ a reflexive sheaf on $X$. Then we have a natural isomorphism*

$$f_*\mathcal{F}[\otimes]\mathcal{L} \xrightarrow{\simeq} f_*(\mathcal{F}[\otimes]f^*\mathcal{L}).$$

*Proof.* By the usual projection formula, we have an isomorphism as above over the regular locus of $Z$. By [Har80, Corollary 1.7], if $\mathcal{G}$ is a coherent reflexive sheaf on $X$, $f_*\mathcal{G}$ is a coherent reflexive sheaf on $Z$. Therefore, the sheaves on both sides of the equation are reflexive. As $Z$ is normal, we conclude by restricting on the smooth locus of $Z$. qed

## 2.1. Separable fibrations

In characteristic 0, given a fibration, it is automatic that the generic fibre is geometrically reduced. In positive characteristic, this is no longer true and it is equivalent to asking that the fibration is *separable*.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Definition 2.1.1.** Let $K \subseteq L$ be a field extension. It is called **separable** if there exists a transcendence basis $t_1, ..., t_\ell$ such that $L$ is a finite separable extension of $K(t_1, ..., t_\ell)$.
Let $f\colon X \to Z$ be a morphism between integral varieties. We say that $f$ is **separable** if the field extension $k(Z) \subseteq k(X)$ is separable; otherwise, $f$ is called **inseparable**.

**Proposition 2.1.2** ([Liu02, Proposition 2.15, Chapter 3])**.** *Let $K$ be a field. A variety $X$ over* $\mathrm{Spec}(K)$ *is geometrically reduced if and only if $f\colon X \to \mathrm{Spec}(K)$ is separable.*

*Remark* 2.1.3. In particular, a fibration $f\colon X \to Z$ is separable if and only if the geometric generic fibre $X_{\bar{\eta}}$ is reduced.

When $Z$ is a curve, a theorem of MacLane [Mac40] allows us to compare the notion of separability of a surjective morphism with its Stein factorisation. In this case it is therefore easier to check this condition. We write here a version of it restated in geometric terms.

**Theorem 2.1.4** ([Sch10, Corollary 2.5])**.** *Let $f\colon X \to Z$ be a fibration onto a curve $Z$. Then $f$ is separable.*

**Definition 2.1.5.** Let $f\colon X \to Z$ be a surjective projective morphism between normal varieties and let $\varphi \circ g$ be its Stein factorisation, where $\varphi$ is finite and $g$ is a fibration. We denote by $\mathrm{St.deg}(f)$ the degree of $\varphi$ and we call it the **Stein degree**

**of** $f$. We further decompose $\varphi = \psi \circ \varphi'$, where $\psi$ is purely inseparable and $\varphi'$ is finite separable. We say that the degree of $\psi$ is the **purely inseparable degree** of $f$.

*Example* 2.1.6. In general, the condition $f_*\mathcal{O}_X = \mathcal{O}_Z$ is not enough to ensure separability. Consider, for example, the threefold $X = V(sx^p+ty^p+z^p) \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{A}^2_{(s,t)}$. Let $f$ be the fibration induced by the natural projection onto $Z := \mathbb{A}^2_{(s,t)}$. Then $f$ satisfies $f_*\mathcal{O}_X = \mathcal{O}_Z$, but it is not separable.

*Remark* 2.1.7. Even if a fibration is separable, its fibres may be highly singular. For example, in characteristics 2 and 3, there exist elliptic surfaces whose structural fibration has general fibre being a cuspidal curve. In dimension 2 this phenomenon happens only for $p = 2, 3$, whereas in higher dimension it is not even known if Calabi–Yau fibrations have normal general fibre for $p \gg 0$.

## 2.2. Singularities of fibrations

### 2.2.1. Log canonical singularities in families

Over fields of positive characteristic, heuristically, the generic fibre of a fibration $f\colon X \to Z$ reflects properties of $X$, while the geometric generic fibre is strictly related to the general fibres of $f$. Here, we make this philosophy more precise, in particular studying the property of being log canonical.

In this section we consider varieties defined over a perfect field of any characteristic, unless otherwise stated.

**Definition 2.2.1.** We denote by $(X/Z, B)$ the data of a a sub-pair $(X, B)$ and a fibration between normal varieties $f\colon X \to Z$.

We say $(X/Z, B)$ is **generically log canonical** or **GLC** if the sub-pair $(X_\eta, B_\eta)$ is log canonical, where $\eta$ is the generic point of $Z$ and $B_\eta$ is defined by restriction. We say $(X/Z, B)$ is **geometrically generically log canonical** or **GGLC** if $Z$ is irreducible and the sub-pair $(X_{\bar\eta}{}^\nu, B_{\bar\eta}{}^\nu)$ is log canonical, where $X_{\bar\eta}{}^\nu$ is the normalisation of the geometric generic fibre and $B_{\bar\eta}{}^\nu$ is the divisor defined on it by restriction.

*Remark* 2.2.2. In particular, given a fibration $f\colon X \to Z$ between normal varieties, if there exists a $\mathbb{Q}$-divisor $B$ on $X$ such that $(X/Z, B)$ is GGLC, then $f$ is separable.

*Remark* 2.2.3. If $f\colon X \to Z$ is a fibration between normal varieties over a field of characteristic 0 and $B$ is a $\mathbb{Q}$-divisor on $X$, then $(X/Z, B)$ is GLC if and only if it is GGLC.

**Definition 2.2.4.** Let $K$ be a field and $\overline{K}$ its algebraic closure. Let $K \subseteq L \subseteq \overline{K}$ be a finite field extension of $K$. Let $X$ be a variety over $\overline{K}$. We say that $X$ is **defined over** $L$ if there exists a variety $X_L$ over $L$ such that $X_L \times_{\mathrm{Spec}(L)} \mathrm{Spec}(\overline{K}) = X$.

**Lemma 2.2.5.** *Let $f\colon X \to Z$ be a fibration between normal varieties. Assume that $B$ is a $\mathbb{Q}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Suppose that the geometric generic fibre $X_{\bar\eta}$ is integral and let $X_{\bar\eta}{}^\nu$ be its normalisation. Let $B_{\bar\eta}{}^\nu$ be the boundary divisor on $X_{\bar\eta}{}^\nu$ defined by restriction. If $(X_{\bar\eta}{}^\nu, B_{\bar\eta}{}^\nu)$ is log canonical, then $X_{\bar\eta}$ is normal. In particular, the geometric generic fibre of a GGLC pair is normal.*

*Proof.* The pair $(X_{\bar\eta}, B_{\bar\eta})$ is slc. In fact, the $S_2$ property is invariant by flat base change and $X_\eta$ is normal. Moreover, if $X_{\bar\eta}$ had singularities worse than nodal in codimension 1, $B_{\bar\eta}{}^\nu$ would have coefficients strictly bigger than 1 coming from the conductor over those singularities, contradicting the log canonical assumption.

Furthermore, the normalisation of the geometric generic fibre is a universal homeomorphism by [Tan18, Lemma 2.2]. Thus, nodal singularities cannot appear. qed

*Remark* 2.2.6. If the characteristic of the base field is $p > 2$, the following is an alternative proof. Since $(X_{\bar\eta}, B_{\bar\eta})$ is slc, the divisor $B_{\bar\eta}{}^\nu$ can be written as $\overline{C} + \overline{B}$, where $\overline{C}$ is the conductor of the normalisation (see [Kol13, 5.7]). By [PW22, Theorem 1.2], the coefficients of $\overline{C}$ are divisible by $p-1$. When $p > 2$, this contradicts the assumption that $(X_{\bar\eta}{}^\nu, B_{\bar\eta}{}^\nu)$ is log canonical. Hence, $\overline{C} = 0$ and the normalisation of the geometric generic fibre is an isomorphism.

**Proposition 2.2.7** ([PW22, Proposition 2.1, Lemma 2.2])**.** *Let $f\colon X \to Z$ be a morphism of varieties. Then the geometric generic fibre is normal (resp. regular, reduced) if and only if a general fibre is normal (resp. regular, reduced). Let $Y \to X$ be the normalisation of $X$. If for a general point $z \in Z$, $Y_z$ is normal, then $Y_z$ is the normalisation of $X_z$.*

**Lemma 2.2.8.** *Let $f\colon X \to Z$ be a separable fibration between normal varieties and let $\varphi\colon Z' \to Z$ be a generically finite map. Let $Y'$ be the normalisation of the main component of the fibre product $X' := X \times_Z Z'$. If the geometric generic fibre $X_{\bar\eta}$ is normal, the conductor of $Y' \to X'$ is vertical. In particular, $Y'_{\bar\eta} = X_{\bar\eta}$.*

*Proof.* Note that $X_{\bar\eta} = X'_{\bar\eta}$, therefore $X'_\eta$ is geometrically normal. This implies that there exists an open dense subset $U \subseteq Z'$ such that $V := f'^{-1}(U)$ is normal, whence $Y' \to X'$ is an isomorphism over $V$. Indeed, by the universal property of the normalisation, $Y'_{\bar\eta}$ is isomorphic to $X'_{\bar\eta}$. qed

**Lemma 2.2.9.** *Let $f\colon X \to Z$ be a separable fibration between normal varieties. Assume that the geometric generic fibre $X_{\bar\eta}$ is normal and let $\bar\sigma\colon \overline{Y} \to X_{\bar\eta}$ be a proper birational morphism between normal varieties. Then, there exist a generically finite map $\varphi\colon Z' \to Z$ and a proper birational morphism $\sigma\colon Y \to Y'$, where:*

*(i) $Y'$ is the normalisation of the main component of $X \times_Z Z'$;*

*(ii) $Y_{\bar\eta} = \overline{Y}$.*

*If $X$ and $Z$ are projective, we can choose $Z'$ and $Y$ projective.*

*Proof.* There exists $L$, finite extension of $k(Z)$ such that $\overline{Y}$ and $\overline{\sigma}$ are defined over $L$. By "spreading out techniques" (see [DW22, Proof of Corollary 1.10] and [Bri22, Lemma 2.25]), there exist $U \subseteq Z$ dense open subset and a finite map $\varphi\colon U' \to U$ such that, if $W'$ is the normalisation of the main component of $f^{-1}(U) \times_U U'$, there is a proper birational map $s\colon W \to W'$ with $W_{\overline{\eta}} = \overline{Y}$. In general, we take $Z' := U'$. If $Z$ is projective, let $\overline{U}'$ be a projective closure of $U'$ and define $\varphi\colon Z' \to Z$ generically finite, as a resolution of the indeterminacies of $\overline{U}' \dashrightarrow Z$. Let $Y'$ be the normalisation of the main component of $X \times_Z Z'$ and $Y''$ a projective closure of $W$, it has an induced rational map $\tau\colon Y'' \dashrightarrow Y'$, which is well-defined over $U'$. We take $\sigma\colon Y \to Y'$ to be a resolution of indeterminacies of $\tau$ that is an isomorphism over $U'$. qed

**Proposition 2.2.10.** *Assume the existence of log resolutions of singularities in dimension $d$. Let $f\colon X \to Z$ be a fibration between normal varieties such that $\dim(X) - \dim(Z) = d$. Let $B \geq 0$ be a $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. The pair $(X/Z, B)$ is GGLC if and only if:*

*(i) the general fibre $X_z$ of $f$ is reduced and normal, and*

*(ii) the pair $(X_z, B_z)$ is log canonical.*

*Proof.* Note that, by Proposition 2.2.7, condition $(i)$ is equivalent to asking that $X_{\overline{\eta}}$ is reduced and normal, where $\overline{\eta}$ is the geometric generic point of $Z$. Thus, by Lemma 2.2.5, the GGLC condition implies $(i)$.

Let $\overline{Y}$ be a log resolution of $(X_{\overline{\eta}}, B_{\overline{\eta}})$. By the above Lemma 2.2.9, there exist a generically finite map $\varphi\colon Z' \to Z$ and a birational map $\sigma\colon Y \to Y'$, where $Y'$ is the normalisation of the main component of $X' := X \times_Z Z'$ and $Y_{\overline{\eta}} = \overline{Y}$. By Proposition 2.2.7, the general fibre of $Y \to Z$ is a log resolution of the general fibre of $Y' \to Z$. Let $\psi\colon Y' \to X$ be the induced generically finite map and define $B'$ on $Y'$ by log pullback, so that $K_{Y'} + B' = \psi^*(K_X + B)$.

Let $X_z$ be the fibre of $f$ over a general point $z \in Z$, $z' \in Z'$ a point mapping to $z$ and $Y'_{z'} := f'^{-1}(z')$. By Lemma 2.2.8, since $X_{\overline{\eta}}$ is reduced, the restriction of $\psi$ to $Y'_{z'}$ is an isomorphism and the pairs $(X_z, B_z)$ and $(Y'_{z'}, B'_{z'})$ defined via adjunction from $(X, B)$ and $(Y', B')$ respectively, coincide. So, $(X_z, B_z)$ is log canonical if and only if $(Y'_{z'}, B'_{z'})$ is. Moreover, $X_{\overline{\eta}} = Y'_{\overline{\eta}}$. Now, let $E$ be a horizontal exceptional divisor of $\sigma$. Since $\sigma^{-1}(Y'_{z'})$ is a log resolution of $(Y'_{z'}, B'_{z'})$, by construction the restriction of $E$ to $\sigma^{-1}(Y'_{z'})$ is an irreducible exceptional divisor, call it $E_{z'}$. The same holds true for the restriction of $E$ to the geometric generic fibre, say $E_{\overline{\eta}}$. Then, by adjunction, we see that the discrepancies of $E, E_{z'}$ and $E_{\overline{\eta}}$ coincide for $z' \in Z'$ general. qed

### 2.2.2. $F$-singularities in families

Here, we state some openness results for globally $F$-split and sharply $F$-pure singularities.

In this section we consider varieties defined over a perfect field $k$ of characteristic $p > 0$, unless otherwise stated.

*Remark* 2.2.11. Let $(X, B)$ be a sub-couple such that $K_X + B$ is a $\mathbb{Z}_{(p)}$-divisor. Then all the different classes of $F$-singularities can be given by replacing the absolute Frobenius $F_X^e$ by the $k$-linear Frobenius $F_{X/k}^e$, as the two differ by the automorphism $F_k^e$.

**Lemma 2.2.12.** *Let $R$ be a smooth local $k$-algebra essentially of finite type, and let $0, \eta \in \operatorname{Spec}(R)$ be its closed and generic point, respectively. Let $(X, B = \sum_i a_i B_i)$ be a $\mathbb{Z}_{(p)}$-pair and let $\pi\colon X \to \operatorname{Spec}(R)$ be a fibration with normal fibres such that $\pi$ and $\pi|_{B_i}$ are flat for all $i$. Suppose that $(1-p^e)(K_X + B) \sim 0$ and that $(X_0, B_0)$ is globally $F$-split. Then $(X_{\overline{\eta}}, B_{\overline{\eta}})$ is globally $F$-split.*

*Proof.* Consider the Grothendieck trace map $F^e_{X/R*}\mathcal{O}_{X^e}(K_{X^e/R^e}) \to \mathcal{O}_{X_{R^e}}(K_{X_{R^e}/R^e})$ of the $R$-linear Frobenius. Twisting by $-K_{X_{R^e}/R^e}$ we obtain

$$T^e_{X/R,B}\colon F^e_{X/R*}\mathcal{O}_{X^e}((1-p^e)(K_{X^e/R^e}+B)) \subseteq F^e_{X/R*}\mathcal{O}_{X^e}((1-p^e)K_{X^e/R^e}) \to \mathcal{O}_{X_{R^e}}. \tag{❀}$$

As all sheaves involved are reflexive and all varieties normal, we can replace $X$ with the complement of some closed subset $Z$ such that $\operatorname{codim}_X(Z), \operatorname{codim}_{X_t}(Z \cap X_t) \geq 2$ for all $t \in \operatorname{Spec}(R)$. In particular, we may assume that $\pi$ is smooth. Taking global sections of (❀) yields a map of finitely generated $R^e$-modules

$$H^0(X, T^e_{X/R,B})\colon H^0(X^e, \mathcal{O}_{X^e}((1-p^e)(K_{X^e/R^e}+B))) \to R^e.$$

Note that $H^0(X^e, \mathcal{O}_{X^e}((1-p^e)(K_{X^e/R^e}+B))) \simeq R^e$, since $(1-p^e)(K_X+B) \sim 0$.

By [PSZ18, Lemma 2.18] the trace map (❀) is compatible with base change. As $T^e_{X/R,B} \otimes k(0)$ is surjective by the GFS hypothesis, then $T^e_{X/R,B} \otimes k(\overline{\eta})$ is also surjective by Nakayama's lemma. Thus, we have $H^0(X_{\overline{\eta}}, \mathcal{O}_{X_{\overline{\eta}}}) = S^0(X_{\overline{\eta}}, B_{\overline{\eta}}; \mathcal{O}_{X_{\overline{\eta}}})$, and we conclude by Lemma 1.3.10. qed

We state the next result in the assumptions in which we need to use it later. In the original paper it is proven in greater generality.

**Theorem 2.2.13** ([PSZ18, Corollary 3.31])**.** *Let $f\colon X \to Z$ be a fibration between normal varieties such that the geometric generic fibre $X_{\overline{\eta}}$ is normal and the variety $Z$ is $\mathbb{Q}$-Gorenstein. Let $B$ be an effective $\mathbb{Z}_{(p)}$-Weil divisor on $X$. Suppose that $K_X + B$ is $\mathbb{Q}$-Cartier and $(X_z, B_z)$ is sharply F-pure, where $X_z$ is a normal fibre over a closed point over $\overline{k}$ and $B_z$ is defined by $(K_X + B)|_{X_z} = K_{X_z} + B_z$. Then, $(X_{\overline{\eta}}, B_{\overline{\eta}})$ is sharply F-pure.*

# 2.3. Foliations and purely inseparable morphisms

## 2.3.1. Foliations

Every separable fibration $f\colon X \to Z$ defines a *foliation* $\mathcal{F}$ whose general leaves consist of the fibres of $f$ and those subvarieties on which $f$ is inseparable. Here, we give a formula for the canonical divisor of $\mathcal{F}$ assuming that the general fibres of $f$ are normal.

In this section we consider varieties defined over a perfect field of any characteristic, unless otherwise stated.

**Definition 2.3.1.** Let $X$ be a normal variety. A subsheaf $\mathcal{F} \subseteq T_X$ is said to be **saturated** if the quotient $T_X/\mathcal{F}$ is torsion free. A **foliation** on $X$ is a subsheaf of the tangent sheaf, $\mathcal{F} \subseteq T_X$, which is saturated, closed under Lie brackets, and, if the characteristic of the base field is $p > 0$, closed under $p$-powers.
Let $\omega_{\mathcal{F}} := \det(\mathcal{F})^*$, the dual of the reflexified top exterior power of $\mathcal{F}$. The canonical divisor of a foliation $\mathcal{F}$ is any Weil divisor $K_{\mathcal{F}}$ such that $\mathcal{O}_X(K_{\mathcal{F}}) = \omega_{\mathcal{F}}$.

*Remark* 2.3.2. Over fields of characteristic $p > 0$, closure under Lie brackets follows from closure under $p$-powers by [Ger64].

Fibrations naturally induce foliations by considering the relative tangent bundle.

**Lemma 2.3.3.** *Let $f\colon X \to Z$ be a separable flat fibration between normal varieties. The kernel of $df\colon T_X \to f^*T_Z$ is saturated in $T_X$.*

*Proof.* Note that $T_X/\ker(df) \subseteq f^*T_Z$. Since $Z$ is normal, $T_Z$ is torsion free and since $f$ is flat, by [Sta22, Tag 0AXV] $f^*T_Z$ is torsion free. Therefore, $T_X/\ker(df)$ is torsion free as well. qed

**Definition 2.3.4.** Let $f\colon X \to Z$ be a separable fibration between normal varieties. It defines an induced foliation $\mathcal{F}$ as the saturation of the kernel of $df\colon T_X \to f^*T_Z$.

**Definition 2.3.5.** Let $f\colon X \to Z$ be a separable equidimensional fibration between normal varieties. If $f$ is equidimensional, we define the **ramification divisor of** $f$ to be

$$R(f) := \sum(f^*\delta - f^{-1}(\delta)) = \sum(\ell_D - 1)D,$$

where the first sum is taken over all prime divisors $\delta$ of $Z$, while the second sum is taken over all vertical prime divisors $D$ on $X$ and $\ell_D$ is their multiplicity with respect to $f$. Assume now the characteristic of the base field is $p > 0$ and let $D$ be a vertical prime divisor. If $p$ divides $\ell_D$, we call $D$ a **wild fibre**. If $f\colon X \to Z$ is an equidimensional separable fibration without wild fibres, we call it a **tame fibration**.

**Theorem 2.3.6.** *Let $f\colon X \to Z$ be an equidimensional separable fibration between normal varieties and let $\mathcal{F}$ be the foliation induced by $f$. Assume that the geometric generic fibre $X_{\overline{\eta}}$ is normal. Then*

$$K_{\mathcal{F}} = K_X - f^*K_Z - R(f) - W(f),$$

*where $R(f)$ is the ramification divisor and $W(f) \geq 0$ is supported on the wild fibres. More precisely, for every wild fibre $D$, there exists an integer $a_D \geq 0$ such that*

$$W(f) = \sum_{D \text{ wild}} (a_D + 1)D.$$

*Proof.* **Step 1:** Since $f$ is equidimensional and $X$ and $Z$ are normal, by Lemma 2.0.5, by restricting to big open subsets, we can assume $f$ is a flat morphism between smooth varieties. Therefore, by Lemma 2.3.3, we can assume $\mathcal{F} = \ker(df)$.
**Step 2:** We claim that there exists a dense open subset $U \subseteq X$ such that $U_\eta$ is a big open subset of $X_\eta$ and the sequence

$$0 \to \mathcal{F}|_U \to T_X|_U \to f^*T_Z|_U \to 0$$

is exact. The sequence $0 \to \mathcal{F} \to T_X \to f^*T_Z$ is always exact, therefore we only need to show surjectivity at the generic fibre. Since $X_{\overline{\eta}}$ is normal, its singular locus $\Sigma$ has codimension $\geq 2$. Let $L$ be a finite Galois extension of $k(Z)$ over which $\Sigma$ is defined. Let $G := \mathrm{Gal}(L/k(Z))$, then $\Sigma^G := \sum_{g \in G} g(\Sigma)$ descends to a cycle of codimension $\geq 2$ defined on $X_\eta$. Let $T$ be the Zariski closure of $\Sigma^G$ in $X$, $U := X \setminus \mathrm{Supp}(T)$ and $V := f(U)$. By [Sta22, Tag 01V8], up to possibly restricting $V$ further (and restricting $U$ accordingly), $f|_U\colon U \to V$ is a flat smooth fibration. By [Sta22, Tag 02G1], the sheaf $T_{U/V}$ is locally free and by [Sta22, Tag 02K4], the sequence

$$0 \to T_{U/V} \to T_U \to f^*T_V \to 0$$

is exact, whence the claim.
**Step 3:** By the first step, the difference between $K_{\mathcal{F}}$ and $K_X - f^*K_Z$ is supported on vertical divisors. Therefore, to conclude, we compute explicitly the image $I$ of $df\colon T_X \to f^*T_Z$ around codimension 1 points corresponding to prime vertical divisors. Note that $I$ is locally free around points of codimension 1.

Let $D$ be a prime vertical divisor. By localising at the generic point of $D$, we can assume $f\colon X \to Z$ is a fibration onto a curve $Z$. Around a general point of $D$, we can further assume that the fibres of $f$ (with their reduced structure) are smooth. Therefore, we can choose local étale coordinates $x, y_1, ..., y_d$ around $D$ such that $D = V(x)$ and $f$ is the map $(x, y_1, ..., y_d) \mapsto x^{\ell_D}u$, where $u \in k[x, y_1, ..., y_d]$ is a unit around $D$ and $\ell_D$ is the multiplicity of $D$ with respect to $f$. Let $\partial_x, \partial_{y_1}, ..., \partial_{y_d}$ be the derivations in $T_X$ corresponding to the coordinate directions. Let $t := x^{\ell_D}u$

be a local coordinate of $Z$ and $\partial_t$ the corresponding derivation which generates $T_Z$.

Tame case: Assume that either the characteristic of the base field is 0 or $p > 0$ and $p$ does not divide $\ell_D$. Then,

$$\begin{cases} df(\partial_x) = x^{\ell_D - 1}(v_x \partial_t); \\ df(\partial_{y_i}) = x^{\ell_D}(v_{y_i}\partial_t), \end{cases}$$

where $v_x$ is a unit around $D$ and $v_{y_i}$ is a function. In particular, in this étale neighbourhood, $I = x^{\ell_D - 1} f^* T_Z$. Therefore, at the generic point of $D$, the sequence

$$0 \to \mathcal{F} \to T_X \to x^{\ell_D - 1} f^* T_Z \to 0$$

is exact. By taking determinants, we get that $K_{\mathcal{F}} = K_X - f^* K_Z - (\ell_D - 1)D$ around $D$.

Wild case: Assume that the characteristic of the base field is $p > 0$ and $p$ divides $\ell_D$. Then,

$$\begin{cases} df(\partial_x) = x^{\ell_D}(x^{a_0} v_0 \partial_t); \\ df(\partial_{y_i}) = x^{\ell_D}(x^{a_i} v_i \partial_t), \end{cases}$$

where, for $j = 0, ..., d$, $a_j \geq 0$ and, since $f$ is separable, there exists an index $j$ such that $0 \neq v_j$ is a unit around $D$. In particular, in this étale neighbourhood, $I = x^{\ell_D + a_D} f^* T_Z$, for some $a_D \geq 0$ and the sequence

$$0 \to \mathcal{F} \to T_X \to x^{\ell_D + a_D} f^* T_Z \to 0$$

is exact. By taking determinants, we obtain $K_{\mathcal{F}} = K_X - f^* K_Z - (\ell_D + a_D)D$ around $D$. qed

*Remark* 2.3.7. The formula in Theorem 2.3.6 is well-known over fields of characteristic 0, where all fibrations are tame. See for example [Dru17, §2.6].

*Example* 2.3.8. Let $g \colon \mathbb{A}^2 \to \mathbb{A}^1$ be the fibration defined by $(x, y) \mapsto t := xy$. Let $X$ be the blow-up at the origin of $\mathbb{A}^2$ and call $E$ the exceptional divisor. Let $f \colon X \to \mathbb{A}^1$ be the induced fibration. Then, $f$ is tame if and only if the characteristic of the base field is $\neq 2$, in which case $K_{\mathcal{F}} = K_X - f^* K_{\mathbb{A}^1} - E$. If the characteristic is 2, the wild fibre is exactly $E$ and $K_{\mathcal{F}} = K_X - f^* K_{\mathbb{A}^1} - 2E$.

*Example* 2.3.9. The assumption on the normality of the geometric generic fibre in Theorem 2.3.6 is essential. Indeed, let $k$ be a field of characteristic 3 and $S := V(y^2 - x^3 - t) \subseteq \mathbb{A}^3_{(x,y,t)}$. Let $f \colon S \to C := \mathbb{A}^1_t$ be the projection onto the third coordinate. The fibration $f$ is a quasi-elliptic fibration, the geometric generic fibre is reduced, but it has a cusp. Let $\mathcal{F} := \ker(df)$ and $D := V(y) \subseteq S$. We compute

$$\begin{cases} df(\partial_x) = 0; \\ df(\partial_y) = 2y\partial_t. \end{cases}$$

Therefore, we obtain the short exact sequence $0 \to \mathcal{F} \to T_S \to 2yf^*T_C \to 0$, whence

$$K_{\mathcal{F}} = K_S - f^*K_C - D.$$

### 2.3.2. Frobenius base change

Note that the differential of the Frobenius morphism is 0. This easy observation plays a key role in the study of fibrations. In fact, to each finite purely inseparable morphism corresponds a foliation and vice-versa. In particular, given a fibration $f$, its induced foliation corresponds to the foliation induced by any power of the relative Frobenius. Using this correspondence, we provide formulas to relate the canonical divisors obtained from a Frobenius base change of a fibration. In Chapter 4, we perform this base change to overcome two issues: the fact that in positive characteristic we do not have a Cone theorem for foliations and to deal with fibrations with normal, but non-log canonical fibres. In Section 6.2, we use it to control fibrations with non-normal fibres.

In this section we consider varieties defined over a perfect field $k$ of characteristic $p > 0$, unless otherwise stated.

**Definition 2.3.10.** Let $X$ and $X'$ be schemes over a field of characteristic $p > 0$. A purely inseparable morphism $a\colon X' \to X$ is called of **height one** if there exists $\alpha\colon X \to X'$ such that $a \circ \alpha = F$.

**Proposition 2.3.11** ([PW22, Proposition 2.9])**.** *Let $X$ be a normal variety. There is a 1-to-1 correspondence*

$$\left\{ \begin{matrix} \textit{Height one morphisms} \\ X \to X' \textit{ with } X' \textit{ normal} \end{matrix} \right\} \longleftrightarrow \left\{ \textit{Foliations } \mathcal{F} \subseteq T_X \right\}$$

*given by:*

*($\leftarrow$) $X' := \mathrm{Spec}_X(\mathcal{O}_X^{\mathcal{F}})$, where $\mathcal{O}_X^{\mathcal{F}} \subseteq \mathcal{O}_X$ is the subsheaf of $\mathcal{O}_X$ that is taken to zero by all the sections of $\mathcal{F}$;*

*($\rightarrow$) $\mathcal{F} := \{\partial \in T_X \text{ s.t. } \partial\mathcal{O}_{X'} = 0\}$.*

*Moreover, morphisms of degree $p^r$ correspond to foliations of rank $r$.*

We now define the relative version of the Frobenius morphism and set some notation that will be used throughout the thesis, unless otherwise stated.

**Definition 2.3.12.** Let $\pi\colon X \to V$ be a morphism of $k$-schemes. We have the following commutative diagram

$$\begin{tikzcd} X^e \arrow[r, "F^e_{X/V}"] \arrow[rr, bend left, "F^e_X"] \arrow[dr, "\pi^e"'] & X_{V^e} \arrow[r, "(F^e_V)_X"] \arrow[d, "\pi_{V^e}"] & X \arrow[d, "\pi"] \\ & V^e \arrow[r, "F^e_V"] & V, \end{tikzcd}$$

where the square is Cartesian and in particular:

- $(F^e_V)_X$ and $\pi_{V^e}$ are the natural morphisms induced on the fibre product $X_{V^e} := X \times_V V^e$;
- $\pi^e$ is exactly the map $\pi$;
- since $(F^e_V)_X$ factorises the Frobenius morphism $F^e$, there is an induced morphism $F^e_{X/V}$, which is called the $e^{\mathbf{th}}$**-relative Frobenius of X over V**, or $V$**-linear Frobenius**.

In the rest of the thesis, we will often omit the superscript $e$ on the source of $F^e$ when it is clear from the context.

*Remark* 2.3.13. Note that $F^e\colon X^e \to X$ is not $k$-linear. On the other hand, if $\mathrm{Spec}(k^e) \to V^e$ is a $k^e$-point, the base change

$$F^e_{X^e/V^e} \otimes_{V^e} k^e\colon X^e_{k^e} \to X_{k^e}$$

coincides with the $k$-linear Frobenius of $X_k := X \times_V \mathrm{Spec}(k)$.

(❋)<u>Construction</u>

Given a fibration $f\colon X \to Z$ between normal varieties, we consider the following diagram

$$\begin{tikzcd} X^e \arrow[r, "\alpha_e"] \arrow[rr, bend left, "F^e"] \arrow[dr, "f"'] & X^{(e)} \arrow[r, "\beta_e"] \arrow[d, "f_e"] & X \arrow[d, "f"] \\ & Z^e \arrow[r, "F^e"] & Z, \end{tikzcd}$$

where:

- $X^{(e)}$ is the normalisation of the reduction of $X_{Z^e}$ and $f_e$ is the induced fibration;
- $\alpha_e$ and $\beta_e$ are the induced maps, so that $\beta_e \circ \alpha_e = F^e$. When $X_{Z^e}$ is reduced, $\alpha_e$ generically coincides with the $e^{\mathrm{th}}$-power of the relative Frobenius of $X$ over $Z$;

- if $D \subseteq X$ is a prime divisor, denote by $D^{(e)} \subseteq X^{(e)}$ its reduced image in $X^{(e)}$;
- when $f$ is separable, $\mathcal{F}$ denotes the foliation induced by $f$ and $\mathcal{F}_e$ the foliation induced by $f_e$, unless otherwise stated.

Next, we study some properties of the maps and the varieties involved in diagram (❋). In particular, we determine the relations between the canonical divisors $K_X$, $K_{X^{(e)}}$ and the canonical divisors of the foliations induced by $f$ and $f_e$.

**Lemma 2.3.14.** *If $f\colon X \to Z$ is a flat separable fibration between normal varieties, $X_{Z^e}$ is integral and $X^{(e)}$ is its normalisation.*

*Proof.* By [Wit21, Remark 2.5], if in $X_{Z^e}$ there are some non-reduced components, they must dominate $Z$ since $f$ is flat. Thus, we can check reducedness at $\eta$, the generic point of $Z$. By Proposition 2.1.2, $X_\eta$ is geometrically reduced. Moreover, since purely inseparable morphisms are homeomorphisms, $X_{Z^e}$ is irreducible as $X$. All in all, $X_{Z^e}$ is integral. qed

**Lemma 2.3.15.** *Let $f\colon X \to Z$ be a separable fibration between normal varieties such that the geometric generic fibre $X_{\bar\eta}$ is normal. Let $\xi$ be a horizontal curve in $X$ and $p^{e_0}$ the purely inseparable degree of $f|_\xi$. Let $e \in \mathbb{N}$ and assume that $\alpha_e(\xi)$ is not contained in the conductor of $X^{(e)} \to X_{Z^e}$ (note that this is automatically satisfied if $Z$ is a curve). Then, $\deg(\alpha_e|_\xi) = \min\{p^e, p^{e_0}\}$.*

*Proof.* Let $np^{e_0}$ be the degree of $f|_\xi$, where $n \in \mathbb{N}$ is coprime with $p$. In particular, $f|_\xi$ factors through $F^{e_0}$, but not through $F^{e_0+1}$. Let $\xi_e := \alpha_e(\xi)$ and $\zeta := f(\xi)$. Around $\xi_e$ we can assume that $X^{(e)} \to X_{Z^e}$ is an isomorphism. Note that, when $Z$ is a curve, since the normalisation of $X_{Z^e}$ is not an isomorphism only on a vertical subset by Lemma 2.2.8, we can always assume this. By the universal properties of the fibre product the purely inseparable part of $f_e|_{\xi_e}$ has degree $p^{e_0-e}$ if $e \le e_0$, while $f_e|_{\xi_e}$ is separable otherwise. Consider the diagram:

$$\begin{tikzcd} \xi^\nu \arrow[r, "\alpha_e|_{\xi^\nu}"] \arrow[dr, "f|_{\xi^\nu}"'] & \xi_e^\nu \arrow[d, "f_e|_{\xi_e^\nu}"] \\ & \zeta^\nu. \end{tikzcd}$$

The purely inseparable parts of $f|_{\xi^\nu}$ and of $f_e|_{\xi_e^\nu} \circ \alpha_e|_{\xi^\nu}$ have the same degree, whence the conclusion. qed

*Example* 2.3.16. In general, we cannot conclude that $e_0$ in the above Lemma 2.3.15 is 0. Below an example where $e_0 = 1$, on Tango–Raynaud surfaces (for more details about their construction, see Section 5.4.3). A **Tango–Raynaud curve** is a normal projective curve $C$ of genus $\ge 2$ on which we can find a rational map $r$ such that the divisor defined by $dr$ is $pD \sim K_C$ for some $D$ effective integral divisor. This determines a non-zero element of $H^1(C, \mathcal{O}_C(-D))$ which is mapped to zero by the Frobenius morphism. Hence $dr$ determines a (non-split) short exact sequence

$$0 \to \mathcal{O}_C(-D) \to \mathcal{E} \to \mathcal{O}_C \to 0,$$

which becomes split after applying the Frobenius morphism:

$$0 \to \mathcal{O}_C(-pD) \to F^*\mathcal{E} \to \mathcal{O}_C \to 0.$$

Let $P := \mathbb{P}(\mathcal{E})$ be the $\mathbb{P}^1$-bundle defined by $\mathcal{E}$, $P' := \mathbb{P}(F^*\mathcal{E})$ the one defined by $F^*\mathcal{E}$, $f\colon P \to C$ and $g\colon P' \to C$ the structural maps. Thus, we have a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[rr, bend left, "F_P"] \arrow[dr, "f"'] & P' \arrow[r, "\beta"] \arrow[d, "g"] & P \arrow[d, "f"] \\ & C \arrow[r, "F_C"] & C, \end{tikzcd}$$

where the lower square is a fibre product diagram and $\alpha$ is the relative Frobenius. The splitting of the last short exact sequence defines a section of $g$, $T'$. Let $T := \alpha^*T'$. The morphisms $f|_T$ and $\alpha|_T$ both coincide with the Frobenius morphism, while $g|_{T'}$ is separable.

**Lemma 2.3.17.** *If $f\colon X \to Z$ is a separable fibration between normal varieties, the foliations induced by $f$ and $\alpha_e$ coincide for every $e > 0$.*

*Proof.* Fix $e > 0$. By [CS23, Lemma 2.2], we can check whether the foliation induced by $f$ and the one induced by $\alpha_e$ coincide on a dense open subset of $X$. Therefore, since $f$ is separable, we may assume that $X_{Z^e}$ is normal, $\alpha_e$ is the $e^{\text{th}}$ relative Frobenius of $X$ over $Z$, $f$ is smooth, and both $\ker(d\alpha_e)$ and $\ker(df)$ are saturated.

Let $\xi \subseteq X$ be a curve. If $\xi$ is vertical, both $df|_\xi$ and $d\alpha_e|_\xi$ are 0. On the other hand, if $\xi$ is horizontal, by Lemma 2.3.15, $\alpha_e|_\xi$ is purely inseparable if and only if $f|_\xi = g \circ F$ for some surjective morphism $g$. Therefore, $d\alpha_e|_\xi = 0$ if and only if $df|_\xi = 0$.

Given a general point $x \in X$, if $v \in \ker(df)_x$, there exists a curve $\xi$ passing through $x$ with tangent vector $v$ and the same holds for $\ker(d\alpha_e)$. This implies that $\ker(df)$ and $\ker(d\alpha_e)$ coincide on an open subset $U \subseteq X$. qed

*Remark* 2.3.18. Let $X$ be a normal variety and $\mathcal{F}$ a foliation on $X$. A subvariety $W$ is said to be **tangent** to $\mathcal{F}$ if $T_W \subseteq \mathcal{F}$. We point out that, over fields of positive characteristic, tangent subvarieties behave differently than in the characteristic 0 case.

Indeed, let $\mathcal{F}$ be a the foliation induced by a separable fibration $f\colon X \to Z$ with normal general fibres. Then, there may be curves that are tangent to $\mathcal{F}$, but that are horizontal. Moreover, their image in $X^{(e)}$ may be not tangent to $\mathcal{F}_e$, the foliation induced by $f_e$. In fact, if $\xi$ is a curve such that the purely inseparable degree of $f|_\xi$ is $p^{e_0}$, for some $e_0 > 0$, then $T_\xi \subseteq \ker(df)$, whereas, for $e \geq e_0$, $T_{\xi_e} \not\subseteq \ker(df_e)$, where $\xi_e$ is the image of $\xi$ in $X^{(e)}$.

Furthermore, there may be curves that are vertical, but not tangent to $\mathcal{F}$. For example, if $f\colon \mathbb{A}^2 \to \mathbb{A}^1$ is defined by $(x, y) \mapsto x^p(x+y)$, then the induced foliation is generated by $\partial_x - \partial_y$ and the curve defined by $x = 0$ is a fibre which is not tangent to $\mathcal{F}$.

### 2.3.3. Wild multiplicities

We now set the ground for the base change formula that we prove in Section 2.3.4. In particular, we study how the above base change (❋) of a fibration with a power of the Frobenius morphism modifies the "multiplicities" of horizontal and vertical divisors.

In this section we consider varieties defined over a perfect field $k$ of characteristic $p > 0$, unless otherwise stated. We use the notation of (❋) in Section 2.3.2.

**Lemma 2.3.19.** *Let $f\colon X \to Z$ be a fibration between normal varieties such that the geometric generic fibre $X_{\bar\eta}$ is normal. Let $D \subset X$ be a horizontal prime divisor. Let $\eta$ be the generic point of $Z$ and assume that $D_{\bar\eta} = p^{e_0}(D_{\bar\eta})_{\mathrm{red}}$ at the generic point of $D_{\bar\eta}$ for some integer $e_0 \geq 0$. Then, at the generic point of $D_{\bar\eta}$, we have:*

$$D_{\bar\eta} = \begin{cases} p^e D_{\bar\eta}^{(e)} & \textit{if } e \leq e_0 \\ p^{e_0} D_{\bar\eta}^{(e)} & \textit{otherwise,} \end{cases}$$

*and*

$$\beta_e^* D = \begin{cases} p^e D^{(e)} & \textit{if } e \leq e_0 \\ p^{e_0} D^{(e)} & \textit{otherwise,} \end{cases} \qquad \alpha_e^* D^{(e)} = \begin{cases} D & \textit{if } e \leq e_0 \\ p^{e-e_0} D & \textit{otherwise.} \end{cases}$$

*Moreover, $D_{\bar\eta}^{(e_0)}$ is reduced at its generic point.*

*Proof.* First of all, note that, by Lemma 2.2.8, around the generic point of $D$, we can assume $X_{Z^e}$ is normal. By cutting $Z$ with general hyperplanes, we can assume that $f\colon X \to Z$ is a fibration onto a curve. Let $f_e|_D\colon D^{(e)} \to Z$ be the induced map on $D^{(e)}$. By the universal property of the fibre product and since $Z$ is a curve, $f_{e_0}|_{D^{(e_0)}}$ is separable, thus $D_{\bar\eta}^{(e_0)}$ is reduced. We will prove the lemma by induction on $e_0$. If $e_0 = 0$, $f_e|_{D^{(e)}}$ is separable for each $e \in \mathbb{N}$, so $D_{\bar\eta}^{(e)}$ is reduced and it coincides with $D_{\bar\eta}$. In particular, $\beta_e^* D = D^{(e)}$ and $\alpha_e^* D^{(e)} = p^e D$ for all $e > 0$. If $e_0 > 0$, consider the natural maps $X^{(e_0)} \to X^{(1)} \to X$. By the universal properties of the fibre product and since $Z$ is a curve, $D$ and $D^{(e)}$ are isomorphic at their generic points for all $e \leq e_0$, thus $f_e|_{D^{(e)}}$ has purely inseparable degree $p^{e_0-e}$ for $e \leq e_0$. In particular, $f_{e_0}|_{D^{(e_0)}}$ is separable and $f_1|_{D^{(1)}}$ has purely inseparable degree $p^{e_0-1}$. The map $X^{(1)} \to X$ is purely inseparable of degree $p$ and $D^{(1)} \to D$ is an isomorphism. Thus $\beta_1^* D = pD^{(1)}$. Then, we conclude by the inductive assumption. qed

*Remark* 2.3.20. Let $f\colon X \to Z$ be a fibration between normal varieties and $(X/Z, B)$ a GGLC pair associated with it. Let $D$ be a horizontal prime divisor contained in the support of $B$. If $D_{\overline{\eta}} = p^{e_0}(D_{\overline{\eta}})_{\mathrm{red}}$, the coefficient of $D$ in $B$ is at most $\frac{1}{p^{e_0}}$.

*Remark* 2.3.21. Let $f\colon X \to Z$ be a tame separable equidimensional fibration between normal varieties. Then, $f_e$ is tame as well. Indeed, if $\delta$ is a prime divisor in $Z$,

$$p^e f_e^* \delta = \beta_e^* f^* \delta = \sum_{D \text{ over } \delta} \ell_D \beta_e^* D = \sum_{D \text{ over } \delta} \ell_D p^{d_D} D^{(e)},$$

where the divisors $D$ and $D^{(e)}$ are prime and reduced, $\ell_D$ is coprime with $p$ by the tameness assumption on $f$ and $d_D \leq e$ since $\beta_e$ is a purely inseparable morphism factorising $F^e$. Hence, we have $d_D = e$ and the multiplicity of the vertical divisor $D^{(e)}$ with respect to $f_e$ is again $\ell_D$.

**Proposition 2.3.22.** *Let $f\colon X \to Z$ be a separable equidimensional fibration between normal varieties and $D$ a vertical prime divisor with multiplicity $\ell = np^{e_0}$, for some $n$ coprime with $p$ and $e_0 \geq 0$. Then, the multiplicity of $D^{(e)}$ with respect to $f_e$ is*

$$\ell_e = \begin{cases} np^{e_0 - e} & \text{if } e \leq e_0 \\ n & \text{if } e \geq e_0. \end{cases}$$

*In particular,*

$$\beta_e^* D = \begin{cases} D^{(e)} & \text{if } e \leq e_0 \\ p^{e-e_0} D^{(e)} & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \alpha_e^* D^{(e)} = \begin{cases} p^e D & \text{if } e \leq e_0 \\ p^{e_0} D & \text{otherwise.} \end{cases}$$

*Proof.* **Step 1:** If $f$ is tame, i.e. $e_0 = 0$, $\ell_e = n$ for all $e \geq 0$ by Remark 2.3.21, whence the conclusion.

**Step 2:** Now, suppose $e_0 > 0$. Since $f$ is equidimensional, by Lemma 2.0.5, up to restricting the fibration to a big open subset, we can assume $f\colon X \to Z$ is a flat fibration between smooth varieties. Let $d := \dim(X) - \dim(Z)$. By localising around the generic point of $f(D)$, we can assume that $Z$ is a curve. Then, around $D$, by [Sta22, Tag 039P] there exist étale morphisms $\varphi$ and $\psi$ and a fibration $f^{\mathrm{et}}$, fitting in the following diagram:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] & A := \operatorname{Spec}(k[x, y_1, ..., y_d]) \arrow[d, "f^{\mathrm{et}}"] \\ Z \arrow[r, "\varphi"] & B := \operatorname{Spec}(k[t]). \end{tikzcd}$$

We can assume $D = V(x) \subseteq A$ and $t = x^{np^{e_0}} u$, where $u \in k[x, y_1, ..., y_d]$ is a unit around $D$ and, since $f$ is separable, its Jacobian has rank 1.

**Step 3:** Let $e \leq e_0$. In this step, we prove the proposition for the morphism $f^{\mathrm{et}} \colon A \to B$.

Consider the diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha_e^{\mathrm{et}}"] \arrow[dr, "f^{\mathrm{et}}"'] & A^{(e)} \arrow[r, "\beta_e^{\mathrm{et}}"] \arrow[d, "f_e^{\mathrm{et}}"] & A \arrow[d, "f^{\mathrm{et}}"] \\ & B := \mathrm{Spec}(k[\tau]) \arrow[r, "F^e"] & B, \end{tikzcd}$$

where $\tau^{p^e} = t$ and $A^{(e)}$ is the normalisation of $\mathrm{Spec}(k[x, y_1, ..., y_d, \tau]/(\tau^{p^e} - x^{np^{e_0}}u))$. We compute $A^{(e)}$ explicitly: it is constructed by adding an element $z$ such that $zx^{np^{e_0-e}} = \tau$. The map $f_e^{\mathrm{et}}$ is described as:

$$A^{(e)} = \mathrm{Spec}\left(\frac{k[x, y_1, ..., y_d, z]}{(z^{p^e} - u)}\right) \to B; \qquad (x, y_1, ..., y_d, z) \mapsto zx^{np^{e_0-e}}.$$

Therefore, the multiplicity of the divisor $D^{(e)} \subseteq A^{(e)}$ with respect to $f_e^{\mathrm{et}}$ is $np^{e_0-e}$.
**Step 4:** We claim that, around $D^{(e)} \subseteq X^{(e)}$, there are étale morphisms $\varphi_e$ and $\psi_e$ and a fibration $f_{\mathrm{et}}^e$, fitting in the following diagram:

$$\begin{tikzcd} X^{(e)} \arrow[r, "\psi_e"] \arrow[d, "f_e"'] & A^{(e)} \arrow[d, "f_e^{\mathrm{et}}"] \\ Z \arrow[r, "\varphi_e"] & B. \end{tikzcd}$$

Since multiplicities can be computed étale locally, Step 3 shows that $\ell_e = np^{e_0-e}$.

Now, let us show the claim. Note that, by commutativity of the Frobenius morphism, we can choose $\varphi_e := \varphi$. Moreover, by the universal properties of the fibre product and since $X^{(e)}$ is normal, there is a map $X^{(e)} \xrightarrow{\psi_e} A^{(e)}$. We need to show that it is étale. Let $C := X \times_A A^{(e)} \xrightarrow{\chi} A^{(e)}$. Since $\psi$ is étale and $A^{(e)}$ is normal, $\chi$ is étale and $C$ is normal as well. By construction, there is a map $X^{(e)} \to C$. By the universal properties of the normalisation and of the fibre product, we construct also a map $C \to X^{(e)}$. With a diagram chasing, we conclude that $C$ and $X^{(e)}$ are isomorphic and $\psi_e = \chi$ is étale.
**Step 5:** Let $e \geq e_0$. By the previous steps and Remark 2.3.21, $f_e$ is tame around $D$ and $\ell_e = n$.

As for the "In particular" part, we compute $\beta_e^* D$ and $\alpha_e^* D^{(e)}$ by using the fact that $f = f_e \circ \alpha_e$ and $\beta_e \circ \alpha_e = F^e$. qed

**Corollary 2.3.23.** *Let $f \colon X \to Z$ be a separable equidimensional fibration between normal varieties. Define $e_f$ to be the maximum of the integers $e$ such that there exists a wild fibre with multiplicity $\ell = np^e$ for some $n \in \mathbb{N}$ coprime with $p$. Then, for $e \geq e_f$, $f_e$ is tame.*

*Proof.* First of all, note that $f$ has at most finitely many wild fibres, therefore $e_f$ is a well-defined natural number. If $e \geq e_f$, by Proposition 2.3.22, for every vertical prime divisor in $X^{(e)}$, its multiplicity with respect to $f_e$ is coprime with $p$. qed

### 2.3.4. Base change formula 1

In the papers [Eke87, Corollary 3.4], [PW22] and [JW21], the authors give a very explicit description of the relative canonical bundle of a purely inseparable base change of height 1. In the next two sections, we study similar relations for the base change described in (❄) in Section 2.3.2.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated. We use the notation of (❄) in Section 2.3.2.

**Proposition 2.3.24** ([PW22, Proposition 2.10], [SB92, Proposition 9.1.2.3], [Eke87, Corollary 3.4])**.** *Let $X \to X'$ be a purely inseparable morphism of height one between normal varieties and let $\mathcal{F}$ be the corresponding foliation. Then*

$$\omega_{X/X'} \simeq (\det \mathcal{F})^{[\otimes](p-1)}.$$

**Lemma 2.3.25.** *Let $f\colon X \to Z$ be a separable equidimensional fibration between normal varieties. If $D$ is a wild fibre, denote by $e_D$ the integer such that the multiplicity of $D$ is $\ell_D = n_D p^{e_D}$, for some $n_D$ coprime with $p$. Then,*

$$\alpha_e^* R(f_e) = R(f) - \sum_D (p^{e_D} - 1)D,$$

*for all $e \gg 0$. If $f$ is tame, for any $e \geq 1$,*

$$\alpha_e^* R(f_e) = R(f).$$

*Proof.* Let $D \subseteq X$ be a vertical prime divisor and $\delta := f(D) \subseteq Z$. We do local computations around $D$. Denote $f^*\delta = \ell D$, $f_e^*\delta = \ell_e D^{(e)}$. By Proposition 2.3.22, if $\ell = np^{e_0}$, with $e_0 \in \mathbb{N}$ and $n$ coprime with $p$, for all $e \geq e_0$, $\ell_e = n$. Moreover, $\alpha_e^* D^{(e)} = p^{e_0} D$. Hence,

$$\alpha_e^* R(f_e) = (n-1)\alpha_e^* D^{(e)} = R(f) - (p^{e_0} - 1)D.$$

qed

**Corollary 2.3.26.** *If $\mathcal{F}$ is the foliation induced by a separable equidimensional tame fibration $f\colon X \to Z$ between normal varieties such that the geometric generic fibre $X_{\overline{\eta}}$ is normal, then:*

$$\alpha_e^* K_{X^{(e)}} = (p^e - 1)K_{\mathcal{F}} + K_X \quad \textit{and} \quad \alpha_e^* K_{\mathcal{F}_e} = p^e K_{\mathcal{F}}.$$

*Proof.* We prove the statement by induction on $e$. For $e = 1$,

$$\alpha_1^* K_{X^{(1)}} = (p-1)K_{\mathcal{F}} + K_X,$$

by Proposition 2.3.24 and Lemma 2.3.17. Thus, since $f$ is tame, by Theorem 2.3.6 and Lemma 2.3.25, $\alpha_1^* K_{\mathcal{F}_1} = pK_{\mathcal{F}}$.

If $e > 1$, factorise the diagram in (❋) in Section 2.3.2, in the following way:

$$\begin{tikzcd}
X \arrow[dr, "\alpha_{e-1}"'] \arrow[drr, "\alpha_e"] \arrow[ddrr, bend right, "f"'] & & & \\
& X^{(e-1)} \arrow[r, "\delta"] \arrow[dr, "f_{e-1}"'] & X^{(e)} \arrow[r] \arrow[d, "f_e"] & X^{(e-1)} \arrow[d, "f_{e-1}"] \\
& & Z \arrow[r, "F"] & Z.
\end{tikzcd}$$

Let $\mathcal{F}_{e-1}$ be the foliation induced by $f_{e-1}$. By Proposition 2.3.24 and Lemma 2.3.17 applied to the lower part of the diagram above,

$$\alpha_e^* K_{X^{(e)}} = \alpha_{e-1}^* \delta^* K_{X^{(e)}} = (p-1)\alpha_{e-1}^* K_{\mathcal{F}_{e-1}} + \alpha_{e-1}^* K_{X^{(e-1)}}.$$

Since $f_{e-1}$ is tame by Remark 2.3.21, we get that $K_{\mathcal{F}_{e-1}} = K_{X^{(e-1)}} - f_{e-1}^* K_Z - R(f_{e-1})$ by Theorem 2.3.6.

By induction, we know that

$$\alpha_{e-1}^* K_{X^{(e-1)}} = (p^{e-1} - 1)K_{\mathcal{F}} + K_X.$$

Using that $K_{\mathcal{F}} = K_X - f^* K_Z - R(f)$ by Theorem 2.3.6 and that $\alpha_{e-1}^* R(f_{e-1}) = R(f)$ by Lemma 2.3.25, we get the result. qed

**Theorem 2.3.27.** *Let $f\colon X \to Z$ be a separable equidimensional fibration between normal varieties such that the geometric generic fibre $X_{\overline{\eta}}$ is normal. Then,*

$$\alpha_e^* K_{X^{(e)}} = (p^e - 1)(K_X - f^* K_Z - R(f)) + K_X - \sum_{D \text{ wild}} w_{D,e} D,$$

*where $w_{D,e} \geq 0$ for all $e$ and all $D$ wild fibres.*

*More precisely, if the multiplicity of $D$ with respect to $f$ is $\ell = np^{e_0}$, with $n$ coprime with $p$, then $w_{D,e} \geq p^e - 1$ if $e \leq e_0$ and $w_{D,e} \geq p^{e_0} - 1$ otherwise.*

*Proof.* Let $D \subseteq X$ be a vertical prime divisor. We prove the statement locally around $D$. By Corollary 2.3.26, if $f$ is tame around $D$, the theorem holds. Therefore, we can suppose $D$ is a wild fibre of multiplicity $\ell = np^{e_0}$.

**Claim 2.3.28.** *If $e \leq e_0$,*

$$\alpha_e^* K_{X^{(e)}} = (p^e - 1)(K_X - f^* K_Z - R(f)) + K_X - w_e D,$$

*where* $w_e \geq p^e - 1$.

We prove the claim by induction. If $e = 1$, by Proposition 2.3.24, Lemma 2.3.17 and Theorem 2.3.6,

$$\alpha_1^* K_{X^{(1)}} = (p-1)(K_X - f^*K_Z - R(f)) + K_X - (p-1)(a_D + 1)D,$$

for some integer $a_D \geq 0$. If $1 < e \leq e_0$, factorise the diagram in (❋) in Section 2.3.2, in the following way:

$$\begin{tikzcd}
X \arrow[dr, "\alpha_{e-1}"'] \arrow[drr, "\alpha_e"] \arrow[ddrr, bend right, "f"'] & & & \\
& X^{(e-1)} \arrow[r, "\delta"] \arrow[dr, "f_{e-1}"'] & X^{(e)} \arrow[r] \arrow[d, "f_e"] & X^{(e-1)} \arrow[d, "f_{e-1}"] \\
& & Z \arrow[r, "F"] & Z.
\end{tikzcd}$$

Then, by Proposition 2.3.22, the multiplicity of $D^{(e)}$ is $\ell_e = np^{e_0 - e}$. Then,

$$\begin{aligned}
&\alpha_e^* K_{X^{(e)}} = \alpha_{e-1}^* \delta^* K_{X^{(e)}} \\
&= \alpha_{e-1}^*((p-1)K_{\mathcal{F}_{e-1}} + K_{X^{(e-1)}}) \qquad \text{by Proposition 2.3.24 and Lemma 2.3.17} \\
&= \alpha_{e-1}^*((p-1)(K_{X^{(e-1)}} - f_{e-1}^* K_Z - (np^{e_0 - e + 1} - 1)D^{(e-1)}) \\
&- (p-1)(b_D + 1)D^{(e-1)} + K_{X^{(e-1)}}) \qquad \text{by Theorem 2.3.6} \\
&= p(p^{e-1} - 1)(K_X - f^*K_Z - R(f)) + pK_X - (p-1)f^*K_Z - (p-1)(np^{e_0} - 1)D \\
&- pw_{e-1}D + (p-1)(p^{e-1} - 1)D - p^{e-1}(p-1)(b_D + 1)D \\
&\qquad \text{by induction and Proposition 2.3.22} \\
&= (p^e - 1)(K_X - f^*K_Z - R(f)) + K_X - w_e D,
\end{aligned}$$

where $b_D \geq 0$ and $w_{e-1} \geq p^{e-1} - 1$ by the inductive step. Therefore,

$$w_e = pw_{e-1} - (p-1)(p^{e-1} - 1) + p^{e-1}(p-1)(b_D + 1) \geq p^e - 1.$$

Now, let $e \geq e_0$ and consider the factorisation

$$\begin{tikzcd}
X \arrow[r, "\alpha_{e_0}"] \arrow[rr, bend left, "\alpha_e"] & X^{(e_0)} \arrow[r, "A_e"] & X^{(e)} \arrow[r, "B_e"] \arrow[rr, bend left, "\beta_e"] & X^{(e_0)} \arrow[r, "\beta_{e_0}"] & X.
\end{tikzcd}$$

By Proposition 2.3.22, $f_e$ is tame around $D^{(e)}$ for $e \geq e_0$, therefore we can apply Corollary 2.3.26 to the morphisms $X^{(e_0)} \xrightarrow{A_e} X^{(e)} \xrightarrow{B_e} X^{(e_0)}$. After that, we apply

Claim 2.3.28 to the morphisms $X \xrightarrow{\alpha_{e_0}} X^{(e_0)} \xrightarrow{\beta_{e_0}} X$. All in all,

$$\begin{aligned}
\alpha_e^* K_{X^{(e)}} &= \alpha_{e_0}^* A_e^* K_{X^{(e)}} \\
&= \alpha_{e_0}^*((p^{e-e_0}-1)(K_{X^{(e_0)}} - f_{e_0}^* K_Z - (n-1)D^{(e_0)}) + K_{X^{(e_0)}}) \\
&\qquad\qquad\qquad\qquad \text{by Corollary 2.3.26} \\
&= p^{e-e_0}(p^{e_0}-1)(K_X - f^*K_Z - R(f)) + p^{e-e_0}K_X - (p^{e-e_0}-1)f^*K_Z \\
&- (p^{e-e_0}-1)R(f) - p^{e-e_0}w_{e_0}D + (p^{e-e_0}-1)(p^{e_0}-1)D \\
&\qquad\qquad\qquad\qquad \text{by Claim 2.3.28 and Proposition 2.3.22} \\
&= (p^e-1)(K_X - f^*K_Z - R(f)) + K_X - w_e D,
\end{aligned}$$

where

$$w_e = p^{e-e_0}w_{e_0} - (p^{e-e_0}-1)(p^{e_0}-1) \geq p^{e_0}-1.$$

qed

*Remark* 2.3.29. Let $f \colon X \to Z$ be an equidimensional separable fibration between normal varieties with normal geometric generic fibre and let $\mathcal{F}$ be the induced foliation. For every $e \geq 0$, let $\mathcal{F}_e$ be the foliation induced by $f_e$. The sequence $(\mathcal{F}, \mathcal{F}_1, ..., \mathcal{F}_e, ...)$ (resp. $(\mathcal{F}, \mathcal{F}_1, ..., \mathcal{F}_e)$) is an *$\infty$-foliation* (resp. an *$e$-foliation*) in the sense of [Gra23, Definition 2.19, Definition 4.6]. Combining the above Theorem 2.3.27 with Theorem 2.3.6, we see that

$$\alpha_e^* K_{X^{(e)}} = (p^e-1)K_{\mathcal{F}} + K_X + \sum_{D \text{ wild}} ((p^e-1)(a_D+1) - w_{D,e})D.$$

We compare this with the results in [Gra24], which state that

$$\alpha_e^* K_{X^{(e)}} = (p^e-1)K_{\mathcal{F}} + K_X + E,$$

for $E$ effective $\mathbb{Q}$-divisor such that $E = 0$ if and only if $(\mathcal{F}, \mathcal{F}_1, ..., \mathcal{F}_e)$ is *Ekedahl* in the sense of [Gra24]. Therefore, we conclude that $E$ is supported on the wild fibres, $w_{D,e} \leq (p^e-1)(a_D+1)$ and $(\mathcal{F}, \mathcal{F}_1, ..., \mathcal{F}_e)$ is Ekedahl if and only if $f$ is a tame fibration. We expect that, if the geometric generic fibre of $f$ is not normal, the divisor $E$ is supported on the wild fibres *and* on the (horizontal) singular locus of the fibres.

### 2.3.5. Base change formula 2

Now, we study a base change formula for fibrations whose geometric generic fibre is not necessarily normal.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Lemma 2.3.30** ([JW21, Lemma 2.4], [PW22, Proposition 2.1]). *Let $f\colon X \to Z$ be a fibration between normal varieties over a perfect field of characteristic $p > 0$. Let $\overline{\eta}$ be the geometric generic point of $Z$. There exists an integer $e \geq 1$ such that $X^{(e)}_{\overline{\eta}^e} = (X_{\overline{\eta},\mathrm{red}})^\nu$. In particular, for $e \gg 0$, the general fibres of $f_e$ are reduced and normal.*

**Theorem 2.3.31** ([PW22, Theorem 3.1]). *Let $f\colon X \to Z$ be a morphism between normal varieties. Let $a\colon Z' \to Z$ be a purely inseparable morphism of height one from a normal variety, let $X'$ be the normalisation of the reduction of $X \times_Z Z'$ and let $f'\colon X' \to Z'$ be the induced morphism. Set $\mathcal{A}$ to be the foliation induced by $a$. Then:*

*(i) $K_{X'/X} \sim (p-1)D$ for some Weil divisor $D$ on $X'$;*

*(ii) there is a non-empty open subset $U \subseteq Z'$ and an effective divisor $C$ on $f'^{-1}(U)$ such that $C \sim -D|_{f'^{-1}(U)}$.*

*Moreover, assume that the geometric generic fibre $X_{\overline{\eta}}$ is reduced, and $f$ is equidimensional. Then:*

*(iii) $f'^*(\det \mathcal{A}) - D \sim C'$ for some effective divisor $C'$ on $X'$.*

*Proof.* Points (i) and (ii) correspond to [PW22, Theorem 3.1(a),(b)]. We prove (iii).
**Step 1.** Suppose first $X \times_Z Z'$ is reduced. Since $f$ is equidimensional and $f'$ is universally homeomorphic to $f$, $f'$ is equidimensional as well. As $Z$ and $Z'$ are $R_1$ we can replace $Z$ by $Z^0 := Z \setminus (\mathrm{Sing}(Z) \cup a(\mathrm{Sing}(Z')))$, $Z'$ by $Z'^0 := a^{-1}(Z^0)$, $X$ by $f^{-1}(Z^0)$ and $X'$ by $f'^{-1}(Z'^0)$. Then point (iii) follows from point (d) of [PW22, Theorem 3.1].
**Step 2.** Suppose now $X \times_Z Z'$ is not reduced. By [JW21, Lemma 2.19] we can find an open $U \subseteq Z$ with $\mathrm{codim}(Z \setminus U) \geq 2$ such that $f|_{X_U}\colon X_U \to U$ is flat, where $X_U := f^{-1}(U)$. Let $U' := a^{-1}(U)$ and $X_{U'} := f'^{-1}(U')$. By [Wit21, Remark 2.5] we have that $X_{U'}$ is reduced, since $f|_{X_U}$ is flat and $X_{\overline{\eta}}$ is reduced. Let $X^\nu_{U'} \subseteq X'$ be the normalisation of $X_{U'}$. By applying Step 1 to $X^\nu_{U'} \to U'$, we conclude that $f'^*(\det \mathcal{A})|_{X^\nu_{U'}} - D \sim C'$ for some effective divisor $C'$ on $X^\nu_{U'}$. Since $\mathrm{codim}_{Z'}(Z' \setminus U') \geq 2$ and $f'$ is equidimensional, we have $\mathrm{codim}(X' \setminus X'_{U'}) \geq 2$, therefore, by normality of $X'$, we can extend the above linear equivalence on all of $X'$. qed

We will need to consider base changes with purely inseparable maps that are not necessarily of height one. Theorem 2.3.31 extends to this situation by induction on the height.

**Corollary 2.3.32.** *Let $f\colon X \to Y$ be an equidimensional fibration between normal varieties and let $g\colon Y \to Z$ be a morphism between normal varieties. Let $Y^{(e)}$ be the normalisation of the reduction of $Y \times_Z Z^e$, and $X^{(e)}$ the normalisation of the reduction of $X \times_Y Y^{(e)}$. Assume that the geometric generic fibre $X_{\overline{\eta}}$ is reduced, where*

*$\bar{\eta}$ is the geometric generic point of $Y$. Let $f_e\colon X^{(e)} \to Y^{(e)}$ and $g_e\colon Y^{(e)} \to Z^e$ be the induced morphisms. Then:*

*(i) $K_{X^{(e)}/X} - f_e^* K_{Y^{(e)}/Y} \sim (1-p)C$ for some effective Weil divisor $C$ on $X^{(e)}$;*

*(ii) $K_{Y^{(e)}/Y} \sim (p-1)D$ for some Weil divisor $D$ on $Y^{(e)}$ and there is a non-empty open subset $U \subseteq Z^e$ with an effective divisor $C'$ on $g_e^{-1}(U)$ such that $-D|_{g_e^{-1}(U)} \sim C'$.*

*Proof.* We proceed by induction. When $e = 1$, let $\mathcal{A}$ be the foliation on $Y^{(1)}$ corresponding to $Y^{(1)} \to Y$. By Proposition 2.3.24, $K_{Y^{(1)}/Y} \sim (\det \mathcal{A})^{[p-1]}$ and, by Theorem 2.3.31(i, iii),

$$K_{X^{(1)}/X} - (p-1)f_1^*(\det \mathcal{A}) \sim (1-p)C$$

for some effective divisor $C$ on $X^{(1)}$, giving point (i). Point (ii) follows from Theorem 2.3.31(i, ii). If $e > 1$, consider the diagram:

$$\begin{array}{ccccc}
X^{(e)} & \xrightarrow{\pi_2} & X^{(e-1)} & \xrightarrow{\pi_1} & X \\
\downarrow{\scriptstyle f_e} & & \downarrow{\scriptstyle f_{e-1}} & & \downarrow{\scriptstyle f} \\
Y^{(e)} & \xrightarrow{p_2} & Y^{(e-1)} & \xrightarrow{p_1} & Y \\
\downarrow{\scriptstyle g_e} & & \downarrow{\scriptstyle g_{e-1}} & & \downarrow{\scriptstyle g} \\
Z^e & \xrightarrow{F} & Z^{e-1} & \xrightarrow{F^{e-1}} & Z,
\end{array}$$

where $\pi_1$, $\pi_2$, $p_1$ and $p_2$ are the induced maps. By the inductive assumptions, there exist $C_1$, $C_2$, $D_1$ and $D_2$ Weil divisors on $X^{(e-1)}$, $X^{(e)}$, $Y^{(e-1)}$ and $Y^{(e)}$ respectively, such that:

- $K_{X^{(e-1)}/X} - f_{e-1}^* K_{Y^{(e-1)}/Y} \sim (1-p)C_1$ and $C_1 \geq 0$;
- $K_{X^{(e)}/X^{(e-1)}} - f_e^* K_{Y^{(e)}/Y^{(e-1)}} \sim (1-p)C_2$ and $C_2 \geq 0$;
- $K_{Y^{(e-1)}/Y} \sim (p-1)D_1$ and there exist a dense open $U_1 \subseteq Z^{e-1}$ and an effective divisor $C_1'$ on $g_{e-1}^{-1}(U_1)$ such that $-D_1|_{g_{e-1}^{-1}(U_1)} \sim C_1'$;
- $K_{Y^{(e)}/Y^{(e-1)}} \sim (p-1)D_2$ and there exist a dense open $U_2 \subseteq Z^e$ and an effective divisor $C_2'$ on $g_e^{-1}(U_2)$ such that $-D_2|_{g_e^{-1}(U_2)} \sim C_2'$.

Setting $C := \pi_2^* C_1 + C_2$, $D := p_2^* D_1 + D_2$, $U := U_1 \cap U_2$ and $C' := (p_2|_{g_e^{-1}(U)})^* C_1 + C_2|_{g_e^{-1}(U)}$ we get the claim. qed

# Chapter 3

# Overview on the canonical bundle formula

In this chapter we give an overview of the main results concerning the canonical bundle formula, presenting the different approaches that have been taken to address the problem.

The first instance of such result is Kodaira's theorem on elliptic surfaces. It states that, given a relatively minimal fibration $f\colon S \to C$ from a smooth surface $S$ to a smooth curve $C$ with elliptic fibres, the canonical divisor of $S$ can be computed in terms of $K_C$, the multiplicities of the singular fibres and the Euler characteristics $\chi(\mathcal{O}_S)$, $\chi(\mathcal{O}_C)$ in a very explicit way. In Section 3.1 we present this and we highlight the main difference between the formula over fields of characteristic 0 and of positive characteristic: the possible presence of *wild fibres*.

Ideally, we would like to have similar formulae for $K$-trivial fibrations $f\colon X \to Z$, i.e. fibrations equipped with a pair $(X, B)$, such that $K_X + B$ is the pullback of a line bundle $L_Z$ from $Z$. More specifically, the goal is to write $L_Z$ as the sum of $K_Z$, a divisor $B_Z$ measuring the singularities of the fibres (the *discriminant part*) and another divisor $M_Z$ which induces a map measuring the *variation* of $f$, determined by how "different" the fibres are between each other or how they vary in the moduli space if this is known to exist (the *moduli part*). The first property the moduli part should have in order to even being able to define such a map is being semiample. Unfortunately, there is no numerical criterion to prove whether a divisor is semi-ample, and the closest property that can be checked numerically is whether it is nef.

The main tool that has then been used to study positivity of the moduli part is variations of Hodge structures. In fact, given a smooth fibration $f\colon X \to Z$ over the complex numbers, the sheaf $f_*\mathcal{O}_X(K_X - f^*K_Z)$ naturally measures how the Hodge data of the fibres vary. Using Hodge theory, this sheaf has been proven to be *weakly positive*. In Section 3.2, we present the results obtained with this approach. Note that the Hodge theoretic input is not available over fields of positive characteristic.

More recently, in [ACSS21], the authors adopt a different approach to the subject. In fact, they use the Minimal Model Program for foliations to get the desired positivity properties of the moduli part. We will talk more about this in Section 3.3. In the next Chapter 4 we discuss a canonical bundle formula result in positive characteristic that builds on these ideas.

Over fields of positive characteristic, weaker results on the canonical bundle formula have been obtained for fibrations of relative dimension 1, exploiting existence and properness of the moduli space of curves. Moreover, in [Wit21] and [CWZ23] the authors prove a canonical bundle formula result for fibrations of relative dimension 1 whose general fibres are non-normal. We discuss them in Section 3.4.1.

In the last Section 3.4.2, we discuss a canonical bundle formula for fibrations in positive characteristic with $F$-split fibres. Using the technique of Frobenius splittings, in [Eji17] and [DS17], the authors "descend" effective boundaries along a $K$-trivial fibration. We discuss also a generalisation of these results to morphisms such that the degree of the finite part of their Stein factorisation is not divisible by the characteristic, proven in [BBC23].

## 3.1. Elliptic surfaces

Kodaira's canonical bundle formula was reformulated in terms of the $j$-invariant and the singularities of the fibres by Fujita. We write this latter version since it is more suited for higher dimensional generalisations.

In this section we consider varieties defined over an algebraically closed field, we will specify the characteristic in each result.

**Definition 3.1.1.** Let $f\colon X \to Z$ be a surjective morphism between normal varieties and $B$ a $\mathbb{Q}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. For each divisor $\delta \subseteq Z$, define

$$\gamma_\delta := \sup\{t \in \mathbb{R} \text{ s.t. } (X, B + tf^*\delta) \text{ is log canonical at the generic point of } \delta\}.$$

Note that the pullback is always well-defined around the generic point of $\delta$.

**Theorem 3.1.2** ([Cor07, Theorem 8.2.1, Chapter 8])**.** *Let $f\colon S \to C$ be a fibration from a smooth surface $S$ to a smooth curve $C$ over an algebraically closed field of characteristic 0. Assume there are no $(-1)$-curves in the fibres and that the general fibres are smooth elliptic curves. Then,*

$$K_S \sim_{\mathbb{Q}} f^*\left(K_C + B_C + M_C\right),$$

*with*

*(i)* $B_C = \sum_{c\in C}(1 - \gamma_c)(c)$;

*(ii)* $M_C = \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1)$, *where* $j\colon C \to \mathbb{P}^1$ *is defined by extending the natural map given by the* $j$*-invariant on the smooth fibres.*

*The* $\mathbb{Q}$*-divisor* $B_C$ *is called* ***discriminant part***, *while* $M_C$ *is called* ***moduli part***.

There are similar versions of this theorem when $S$ is only normal or when the characteristic of the base field is $p > 0$ ([BM77]). The main difference between the situation in characteristic 0 and in positive characteristic is the possible presence of *wildly ramified fibres*.

**Definition 3.1.3.** Let $f\colon S \to C$ be a fibration between a smooth projective surface $S$ and a smooth projective curve $C$ over an algebraically closed field of characteristic $p > 0$. The sheaf $R^1 f_* \mathcal{O}_S$ has rank 1 and it can be decomposed into $\mathcal{O}(L) \oplus \mathcal{T}$, where $L$ is a divisor on $C$ and $\mathcal{T}$ is torsion. In positive characteristic $\mathcal{T}$ may be non-trivial. The points in the support of $\mathcal{T}$ are called **wildly ramified**. All the other points over which the fibre is multiple are called **tamely ramified** (see [KU85, §1] for some equivalent definitions). Remark that wild ramification is a more "arithmetic" phenomenon, rather than "geometric".

**Theorem 3.1.4** ([BM77, Theorem 2]). *Let* $f\colon S \to C$ *be a fibration from a smooth projective surface* $S$ *to a smooth projective curve* $C$ *over an algebraically closed field of characteristic* $p > 0$. *Assume there are no* $(-1)$*-curves in the fibres and that the general fibres are smooth elliptic curves. Then,*

$$K_S \sim_{\mathbb{Q}} f^*(K_C - L) + \sum_{i=1}^{\ell} a_i S_{z_i},$$

*where:*

*(i)* $m_i S_{z_i} = f^*(z_i)$ *for* $i = 1, \ldots, \ell$ *are the multiple fibres, and* $S_{z_i}$ *is defined so that, if* $S_{z_i} = \sum_j d_j D_j$ *is the decomposition in prime divisors, the greatest common divisor of the* $d_j$*'s is* $1$;

*(ii)* $a_i = m_i - 1$ *if the fibre is tame and* $0 \le a_i \le m_i - 1$ *otherwise.*

*Example* 3.1.5. Let $f\colon S := E \times C \to C$, where $E$ is an elliptic curve, $C$ is any normal projective curve and $f$ is the second projection over any algebraically closed field. Then, $K_S = f^* K_C$. In this case all the fibres are smooth and isomorphic, there is no variation in moduli, thus $B_C = M_C = 0$.

*Example* 3.1.6. Let $S$ be the projective closure of $V(y^2 - x(x-1)(x-\lambda))$ inside $\mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^1_{[\lambda:\mu]}$ over an algebraically closed field of characteristic 0 and let $f\colon S \to C$, where $C := \mathbb{P}^1$, be the induced projection.

We compute the discriminant part of $f$. Note that $S$ is normal. Indeed, it is a complete intersection and it is singular at the nodal points of the fibres over $\lambda = 0, 1$ and at the intersection point of the three lines composing the fibre over $[1:0] =: \infty$.

Let $S'$ be a resolution of $S$ and let $g\colon S'' \to C$ be the fibration obtained after a relative $K_{S'}$-MMP over $C$. By Kodaira's canonical bundle formula Theorem 3.1.2, the moduli part of $g$ is $\frac{1}{12}j^*\mathcal{O}_{\mathbb{P}^1}(1)$, where $j$ is the $j$-invariant map. We remark that $S$ is isomorphic to $S''$ everywhere but at its three singular points, therefore the moduli part of $f$ can also be expressed as

$$M_C = \frac{1}{12}j^*\mathcal{O}_{\mathbb{P}^1}(1).$$

Since the $j$-map has degree 6 on the Legendre family, we conclude that the discriminant part has degree $1/2$ since the canonical bundle of $S$ is $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(0,-1)|_S$. Using [GS], we see that the reduction of $S$ modulo 5 is strongly $F$-regular, thus, by [MS18, Theorem 7.9], $S$ is klt. Using [GS], we check that the $F$-pure threshold of the fibres over 0 and 1 at the reduction of $f$ modulo 5 is exactly 1. Therefore, $(S, f^*(0) + f^*(1))$ is log canonical. All in all, we get that the discriminant part is

$$B_C = \frac{1}{2}(\infty).$$

*Example* 3.1.7 ([KU85, Example 4.7]). This is an example of a wildly ramified fibration. Let $E$ be an ordinary elliptic curve over an algebraically closed field of characteristic $p > 0$. Fix $P_0 \in E$ a point of order $p$. Define $g$ to be the automorphism of $\mathbb{P}^1 \times E$ given by

$$g\colon (t, P) \mapsto (t + 1, P + P_0).$$

The group $G = \langle g\rangle \simeq \mathbb{Z}/p\mathbb{Z}$ acts freely on $\mathbb{P}^1 \times E$. Therefore, we have an elliptic surface structure $f\colon S := (\mathbb{P}^1 \times E)/G \to \mathbb{P}^1/G = \mathbb{P}^1$ given by the natural projection. This surface has only one multiple singular fibre $pE_\infty$ over $\infty \in \mathbb{P}^1$ of multiplicity $p$, it is a wild fibre and $K_S = f^*\mathcal{O}_{\mathbb{P}^1}(-1) + (p-2)E_\infty$.

## 3.2. Hodge theoretic approach

The biggest evidence of nefness of the moduli part comes from Hodge theoretic inputs. Under the standard normal crossing assumptions, variations of Hodge structures can be used to prove *semipositivity* of $f_*\omega_{X/Z}$. The moduli part is closely related to this bundle, and, in particular, it is nef.

In this section we consider varieties defined over an algebraically closed field of characteristic 0.

**Definition 3.2.1.** Let $f\colon X \to Z$ be a fibration between normal projective varieties, let $B$ and $\Sigma$ be $\mathbb{Q}$-divisors on $X$ and $Z$ respectively. We say that $(f, B, \Sigma)$ satisfies the **standard normal crossing assumptions** if:

(a) $X$ and $Z$ are smooth,

(b) $\mathrm{Supp}(B + f^*\Sigma)$ and $\Sigma$ are log smooth,

(c) $f$ is smooth over $Z \setminus \Sigma$,

(d) $B$ is simple normal crossing.

**Theorem 3.2.2** ([Cor07, Theorem 8.3.7, Chapter 8])**.** *Let $f\colon X \to Z$ be a fibration between smooth projective varieties and $B$ and $\Sigma$ $\mathbb{Q}$-divisors such that $(f, B, \Sigma)$ satisfies the standard normal crossing assumptions. Assume that $K_X + B \sim_{\mathbb{Q}} f^*L_Z$ for some $\mathbb{Q}$-divisor $L_Z$ on $Z$. Let $B = B^h + B^v$ be the decomposition of $B$ into its horizontal and vertical parts and assume that $B^h \geq 0$. Then,*

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z),$$

*and the following holds.*

*(i) The moduli part $M_Z$ is nef. Moreover, it depends only on $Z$ and $(X_\eta, B|_{X_\eta})$, where $X_\eta$ is the generic fibre of $f$.*

*(ii) The discriminant part $B_Z$ is the unique smallest $\mathbb{Q}$-divisor supported on $\Sigma$ such that*

$$B^v + f^*(\Sigma - B_Z) \leq f^{-1}(\Sigma).$$

*Moreover, it depends only on $f$ and $B^v$.*

*(iii) $(Z, B_Z)$ is log canonical if and only if $(X, B)$ is log canonical;*

*(iv) if $\lfloor B^h \rfloor = 0$, then $(Z, B_Z)$ is klt if and only if $(X, B)$ is klt;*

*(v) an irreducible divisor $B_i$ of $Z$ appears with negative coefficient in $B_Z$ if and only if $f^*B_i - f^{-1}(B_i) < -B^v$.*

This formula was then generalized to other types of fibrations (see for instance [FM00], [Amb04], [Amb05], [Cor07]). In particular, the divisor $B_Z$ is described in terms of the singularities of the fibres using log canonical thresholds.

**Definition 3.2.3.** Let $f\colon X \to Z$ be a surjective proper morphism between normal projective varieties and $(X/Z, B)$ a GLC pair on it (see Definition 2.2.1). Applying generic smoothness and Bertini-type theorems, for all but finitely many divisors $\delta \subseteq Z$, the log canonical threshold (as in Definition 3.1.1) is $\gamma_\delta = 1$. The **discriminant part** of the fibration $f$ is

$$B_Z := \sum_{\delta \subseteq Z} (1 - \gamma_\delta)\delta,$$

where the sum is taken over all prime divisors of $Z$.

**Definition 3.2.4.** Let $f\colon X \to Z$ be a surjective proper morphism between normal projective varieties and $B$ a $\mathbb{Q}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. We say $(X/Z, B)$ is **generically klt** if $(X_\eta, B_\eta)$ is klt, where $\eta$ is the generic point of $Z$. Given a GLC (resp. generically klt) sub-pair $(X/Z, B)$, we say that $f$ is an **lc-trivial fibration** (resp. **klt-trivial fibration**) if:

(a) there exists a $\mathbb{Q}$-Cartier divisor $D$ on $Z$ such that $K_X + B \sim_{\mathbb{Q}} f^*D$;

(b) there exists a log resolution $\pi\colon X' \to X$ of $(X, B)$ such that, if $E := \pi^*(K_X + B) - K'_X = \sum_i a_i E_i$ and $E^{<1} := \sum_{a_i<1} a_i E_i$, then

$$\operatorname{rank}(f \circ \pi)_* \mathcal{O}_{X'}(\lceil -E^{<1} \rceil) = 1.$$

We will denote these fibrations as $f\colon (X, B) \to Z$.

*Remark* 3.2.5. With Ambro's approach it is useful to include sub-pairs $(X, B)$ with $B$ not effective. In fact, consider an lc-trivial fibration $(X/Z, B)$ and $\nu\colon Z' \to Z$ proper birational. Let $X'$ be the normalization of the main component of $X \times_Z Z'$ and let $\mu\colon X' \to X$ be the induced morphism. Define $B' := \mu^*(K_X + B) - K_{X'}$. Then the induced fibration $f'\colon X' \to Z'$ with sub-pair structure $(X'/Z', B')$ is lc-trivial as well. Note that, even if $B$ is effective, $B'$ needs not be. Condition (b) in Definition 3.2.4 is then needed in order to control the singularities of a possible negative part of the boundary $B$. What it is saying is that $B^-$ is a *rigid divisor* over $Z$, i.e. the only global section (over $Z$) of $B^-$ is $B^-$ itself.

**Definition 3.2.6.** Let $f\colon X \to Z$ with pair structure $(X/Z, B)$ be an lc-trivial fibration between normal projective varieties. In particular, $K_X + B \sim_{\mathbb{Q}} f^*D$ for some $\mathbb{Q}$-Cartier divisor $D$ on $Z$. Define the **moduli part** of $f$ to be $M_Z := D - (K_Z + B_Z)$. Note that this is defined only up to $\mathbb{Q}$-linear equivalence.

**Theorem 3.2.7** ([Amb04, §1, §2, §3])**.** *Let $f\colon X \to Z$ with pair structure $(X/Z, B)$ be an lc-trivial fibration between normal projective varieties, then there exists a proper birational map $\varphi\colon Z' \to Z$ such that, in the notation of the construction of Remark 3.2.5:*

*(i) $M_{Z'}$ is $\mathbb{Q}$-Cartier and nef;*

*(ii) for every $\pi\colon Z'' \to Z'$ birational, considering the corresponding lc-trivial fibration constructed as in Remark 3.2.5, then $M_{Z''} = \pi^* M_{Z'}$;*

*(iii) if $(X, B)$ is klt (resp. log canonical), then $(Z', B_{Z'})$ is klt (resp. log canonical).*

*Such $Z'$ is called an **Ambro model** of the fibration.*

Even if we still do not know whether the moduli part gives a map to some moduli space, Ambro proved that it does give a measure of the variation of the fibres.

**Theorem 3.2.8** ([Amb05, Theorem 3.3, Proposition 4.4])**.** *Let $f\colon (X,B) \to Z$ be a klt-trivial fibration between normal projective varieties such that $B \geq 0$ over the generic point of $Z$. Then, there exists a diagram:*

$$\begin{tikzcd} (X,B) \arrow[d, "f"'] & & (X^+,B^+) \arrow[d, "f^+"] \\ Z & \hat{Z} \arrow[l, "\tau"'] \arrow[r, "\rho"] & Z^+, \end{tikzcd}$$

*such that:*

*(i) $\tau$ is generically finite and $\rho$ is surjective;*

*(ii) $f^+$ is klt-trivial;*

*(iii) $(X,B)\times_Z \hat{Z}$ and $(X^+,B^+)\times_{Z^+}\hat{Z}$ are isomorphic over an open subset $U \subseteq \hat{Z}$;*

*(iv) $M_{Z^+}$ is big and, after possibly a proper birational base change, $\tau^*M_Z = \rho^*M_{Z^+}$.*

*Remark* 3.2.9. The upshot of the above Theorem 3.2.8 is that $M_Z$ measures the dimension of a variety (i.e. $Z^+$) over which the fibration is a product. We call $\dim(Z^+)$ the **variation** of $f$.

## 3.3. An MMP approach

In the paper [ACSS21], the authors study the canonical bundle formula in characteristic 0 using tools of the Minimal Model Program. In this setting it is useful to extend our focus on more general fibrations, not only the $K$-trivial ones. This will give us the necessary flexibility to perform birational transformations. In order to do that, we need to redefine the moduli part, while the definition of the discriminant part remains the same (see Definition 3.2.3).

In this section we consider varieties defined over a perfect field and we will specify when we need the characteristic to be 0 or when we assume it is algebraically closed.

**Definition 3.3.1.** Let $f\colon X \to Z$ be a fibration between normal varieties and $(X/Z,B)$ a GGLC pair on it. Let $B_Z$ be the discriminant of $f$. Suppose that $f$ is equidimensional or that $K_Z+B_Z$ is $\mathbb{Q}$-Cartier. Then, the **moduli part** of $f$ is

$$M_X := K_X + B - f^*(K_Z+B_Z).$$

Note that $M_X$ is defined on the total space $X$ and only up to linear equivalence. In general, by the Flattening lemma (see Lemma 2.0.4), there exists an equidimensional fibration $f'\colon X' \to Z'$ with proper birational maps $a\colon Z' \to Z$ and $b\colon X' \to X$. Define $B' := b^*(K_X+B) - K_{X'}$ and $M_{X'}$ on $X'$ accordingly. The moduli part $M_X$ is then defined as $b_*M_{X'}$.

Likewise the previous approach, the idea is to first find a class of fibrations for which it is easier to get positivity properties and then we want to reduce to this case. Instead of standard normal crossing assumptions, the authors of [ACSS21] introduce the notion of Property $(*)$: when a fibration satisfies this property, the moduli part coincides with the canonical divisor of the foliation associated with it. At this point, we can use the birational geometry of the foliation to get the desired positivity.

**Definition 3.3.2** ([ACSS21, Definition 2.13])**.** Let $f\colon X \to Z$ be a fibration between normal varieties and $(X/Z, B)$ a GGLC sub-pair on it. We say it satisfies **Property** $(*)$ if:

(a) there exists a reduced divisor $\Sigma_Z$ on $Z$ such that $(Z, \Sigma_Z)$ is log smooth and $B^v = f^{-1}(\Sigma_Z)$;

(b) for any closed point $z \in Z$ and any divisor $\Sigma \geq \Sigma_Z$ such that $(Z, \Sigma)$ is log smooth around $z$, then $(X, B + f^*(\Sigma - \Sigma_Z))$ is log canonical around $f^{-1}(z)$.

In the next proposition, we list some useful features that Property $(*)$ pairs enjoy.

**Proposition 3.3.3** ([ACSS21, Lemma 2.14, Proposition 2.18])**.** *Let $f\colon X \to Z$ be a fibration between normal varieties and $(X/Z, B)$ a GGLC pair on it satisfying Property $(*)$. Then the following properties hold.*

*(i) The pair $(X, B)$ is log canonical and the discriminant part $B_Z$ coincides with $\Sigma_Z$. Moreover,if $B \geq 0$, the map $f$ is equidimensional outside $\Sigma_Z$.*

*(ii) Suppose $B \geq 0$ and let $\varphi\colon X \dashrightarrow Y$ is a sequence of steps of the $(K_X + B)$-MMP over $Z$. Let $C := \varphi_* B$ and $g\colon Y \to Z$ be the induced fibration. Then, $(Y/Z, C)$ satisfies Property $(*)$ and for any closed point $z \in Z$, the map $\varphi^{-1}$ is an isomorphism along the generic point of any irreducible component of $g^{-1}(z)$. In particular, the discriminant part of $(Y/Z, C)$ coincides with the discriminant part of $(X/Z, B)$.*

Over fields of characteristic 0, it is possible to construct $(*)$-modifications using the Weak Semistable Reduction results in [AK00].

**Theorem 3.3.4** (Existence of $(*)$-modifications, [ACSS21, Proposition 2.17])**.** *Let $f\colon X \to Z$ be a fibration between normal varieties over an algebraically closed field of characteristic 0 and $(X/Z, B)$ a GLC pair on it such that $M_X$ is $f$-nef. Then, there exist a $\mathbb{Q}$-factorial pair $(X', B')$ satisfying Property $(*)$ and a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "b"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "a"] & Z, \end{tikzcd}$$

*where $a$ and $b$ are projective birational morphisms and $f'$ is equidimensional. Moreover,*

$$K_{X'} + B' + R = b^*(K_X + B) + G,$$

*where $G$ and $R$ are effective $\mathbb{Q}$-divisors that are vertical with respect to $f'$ and $b(R)$ is supported on the non-log canonical locus of $(X, B)$. The fibration $f'$ together with the pair $(X'/Z', B')$ is called a $(*)$-**modification** of $(X/Z, B)$.*

**Definition 3.3.5.** Let $X$ be a normal variety. Let $\mathcal{F}$ be a foliation on $X$ and $\Delta$ a $\mathbb{Q}$-divisor on $X$ such that $K_{\mathcal{F}} + \Delta$ is $\mathbb{Q}$-Cartier. We call $(\mathcal{F}, \Delta)$ a **foliated pair**.

We say that $(\mathcal{F}, \Delta)$ satisfies **Property** $(*)$ if there exists a $\mathbb{Q}$-divisor $B$ on $X$ such that $\Delta = B^h$ and $(X/Z, B)$ satisfies Property $(*)$.

**Proposition 3.3.6** ([ACSS21, Proposition 3.6])**.** *Let $(\mathcal{F}, \Delta)$ be a foliated pair on a normal variety $X$ induced by an equidimensional separable tame fibration $f \colon X \to Z$ between normal varieties. Assume that $(\mathcal{F}, \Delta)$ satisfies Property $(*)$ and let $B$ be the divisor such that $(X/Z, B)$ has Property $(*)$ and $B^h = \Delta$. Let $M_X$ be the moduli part of $(X/Z, B)$. Then,*

*(i) $K_{\mathcal{F}} + \Delta \sim_{\mathbb{Q}} M_X$ and*

*(ii) $K_{\mathcal{F}} + \Delta \sim_{\mathbb{Q},Z} K_X + B$.*

Given any foliated pair on a normal projective variety $X$, using Theorem 3.3.4, it is possible to birationally modify it in order to get a foliated pair which satisfies Property $(*)$. The existence of $(*)$-modifications, together with the Cone theorem for foliated pairs are the two main ingredients to prove positivity of the moduli part.

**Definition 3.3.7.** Let $f \colon X \to Z$ be a fibration between normal varieties. If $D$ is a horizontal divisor, we define $\epsilon(D) := 1$, while, if $D$ is a vertical divisor, we define $\epsilon(D) := 0$. Given a proper birational morphism $\pi \colon X' \to X$, let $U$ be the open subset on which $\pi$ is an isomorphism. By [CS23, Lemma 2.2], there is a unique saturated sheaf $\mathcal{F}'$ which extends $\mathcal{F}|_U$, call it the pullback of $\mathcal{F}$ on $X'$. Then, we write:

$$K_{\mathcal{F}'} + \pi_*^{-1}\Delta = \pi^*(K_{\mathcal{F}} + \Delta) + \sum a(E, \mathcal{F}, \Delta)E,$$

where the sum is taken over all exceptional divisors. We call the foliated pair $(\mathcal{F}, \Delta)$ **F-log canonical** if $a(E, \mathcal{F}, \Delta) \geq -\epsilon(E)$ for all exceptional divisors over $X$. If there exists $E$ such that $a(E, \mathcal{F}, \Delta) = -\epsilon(E)$ (resp. $a(E, \mathcal{F}, \Delta) < -\epsilon(E)$), then the image of $E$ in $X$ is called a **log canonical centre** (resp. **non-log canonical centre**) of $(\mathcal{F}, \Delta)$. We denote by $\mathrm{Nlc}(\mathcal{F}, \Delta)$ the union of all non-log canonical centers of $(\mathcal{F}, \Delta)$.

**Theorem 3.3.8** ([ACSS21, Theorem 3.9])**.** *Let $(\mathcal{F}, \Delta)$ be a foliated pair on a normal projective variety over an algebraically closed field of characteristic* 0*, where $\mathcal{F}$ is* *induced by a fibration $f\colon X \to Z$ between normal varieties and $\Delta \geq 0$. Then,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_{\mathcal{F}+\Delta\geq 0}} + Z_{-\infty} + \sum_i \mathbb{R}_{\geq 0}\,[\xi_i]\,,$$

*where:*

*(i) the sum on the RHS is countable;*

*(ii) the $\xi_i$'s are vertical rational curves;*

*(iii) for each $\xi_i$, $0 \leq -(K_{\mathcal{F}} + \Delta)\cdot\xi_i \leq 2n$;*

*(iv) all the curves in $Z_{-\infty}$ are contained in $\mathrm{Nlc}(\mathcal{F}, \Delta)$.*

**Definition 3.3.9.** Let $(X/Z, B)$ and $(X'/Z', B')$ be GGLC pairs over a perfect field of any characteristic such that the associated fibrations $f\colon X \to Z$ and $f'\colon X' \to Z'$ are birationally equivalent. We say that $(X/Z, B)$ and $(X'/Z', B')$ are **crepant over the generic point of** $Z$ if, given birational morphisms $p_1\colon \widetilde{X} \to X$ and $p_2\colon \widetilde{X} \to X'$ which resolve the indeterminacies of $X \dashrightarrow X'$, we have that

$$p_1^*(K_X + B) - p_2^*(K_{X'} + B')$$

is vertical with respect to the induced fibration $\widetilde{X} \to Z$.

**Theorem 3.3.10** ([ACSS21, Theorem 1.1])**.** *Assume termination of flips in dimension $n$. Let $(X, B)$ be a projective log canonical pair of dimension $n$ and let $f\colon X \to Z$ be a fibration between normal projective varieties over an algebraically closed field of characteristic $0$. Assume that $K_X + B$ is $f$-nef and $B \geq 0$. Then, there exist a projective log canonical pair $(Y, C)$ with $C \geq 0$ and a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, dashed, "b"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "a"] & Z, \end{tikzcd}$$

*where $a$ is a projective birational morphism and $b$ is a birational map such that*

*(i) $(X/Z, B)$ and $(Y/Z', C)$ are crepant over the generic point of $Z$;*

*(ii) the moduli part $M_Y$ of $(Y/Z', C)$ is nef.*

We outline the idea of the proof. Given a fibration $f\colon X \to Z$ and a log canonical pair $(X, B)$ such that $K_X + B$ is $f$-nef, we take a $(*)$-modification $(X'/Z', B')$ with induced fibration $f'\colon X' \to Z'$ constructed with Theorem 3.3.4. Possibly after running an MMP, we can suppose the new moduli part $M_{X'}$ is $f'$-nef and it coincides with the canonical bundle of the induced foliation by Proposition 3.3.6. Therefore, by Theorem 3.3.8, we conclude that $M_{X'}$ is nef.

In the case of a $K$-trivial fibration, we recover the previous canonical bundle formula Theorem 3.2.7.

**Theorem 3.3.11** ([ACSS21, Theorem 1.3]). *Let $f\colon X \to Z$ be a fibration between normal projective varieties over an algebraically closed field of characteristic 0. Let $B \geq 0$ be a $\mathbb{Q}$-divisor on $X$ such that $(X/Z, B)$ is a GLC pair. Assume that $M_X \sim_{\mathbb{Q}} f^*L_Z$ for some line bundle $L_Z$ on $Z$. Then, there exists a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "b"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "a"] & Z, \end{tikzcd}$$

*where $a$ and $b$ are proper birational, such that the moduli part $M_{X'}$ of the pair induced by crepant pullback, is nef.*

# 3.4. Canonical bundle formula in positive characteristic

## 3.4.1. Canonical bundle formula for fibrations of relative dimension 1

In [CTX15, Lemmas 6.6, Lemma 6.7] and [Wit21] the authors prove a canonical bundle formula for fibrations of relative dimension 1 using properties of the moduli space of curves. In fact, if $f\colon X \to Z$ has relative dimension 1 with fibres of genus 0, under reasonable assumptions, there is a natural map from $Z$ to the moduli space of curves of genus 0. When the fibres are elliptic curves, the result is a consequence of the subadditivity of Kodaira dimensions for fibrations of relative dimension 1 (see [CZ15]).

In this section we consider varieties defined over fields of characteristic $p > 0$.

**Theorem 3.4.1** ([Wit21, Proposition 3.2]). *Let $(X, B)$ be a quasi-projective log pair over a perfect[1] field of characteristic $p > 0$ and let $f\colon X \to Z$ be a fibration between normal varieties whose geometric generic fibre is a smooth curve. Let $\overline{\eta}$ be the geometric generic point of $Z$. Assume that $(X_{\overline{\eta}}, B_{\overline{\eta}})$ is log canonical and $K_X + B \sim_{\mathbb{Q}} f^*L_Z$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L_Z$ on $Z$. Then,*

$$L_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$$

*for some $\mathbb{Q}$-effective divisor $\Delta_Z$ on $Z$.*

[1] In the original paper, the result is proven over algebraically closed fields, but we can reduce the statement to that situation with a base change.

A similar theorem has been proven for fibrations with general fibres that are not necessarily log canonical by allowing to take purely inseparable covers of the base.

**Theorem 3.4.2** ([Wit21, Theorem 3.4], [CWZ23, Theorem 6.2])**.** *Let $(X, B)$ be a quasi-projective log pair over an algebraically closed field of characteristic $p > 0$ and let $f\colon X \to Z$ be a separable fibration between normal varieties of relative dimension 1. Assume that $(X_\eta, B_\eta)$ is log canonical, where $\eta$ is the generic point of $Z$, and $K_X + B \sim_{\mathbb{Q}} f^*L_Z$ for some $\mathbb{Q}$-Cartier divisor $L_Z$ on $Z$.*

*Then, there exist finite purely inseparable morphisms $\tau_1\colon T \to Z$ and $\tau_2\colon T' \to T$, an effective $\mathbb{Q}$-divisor $E$ on $T'$ and rational numbers $a, b, c \geq 0$ such that*

$$\tau_2^*\tau_1^*L_Z \sim_{\mathbb{Q}} aK_{T'} + b\tau_2^*K_T + c\tau_2^*\tau_1^*K_Z + E.$$

### 3.4.2. Canonical bundle formula for $F$-split fibrations

Over fields of positive characteristic, a similar canonical bundle formula has been established for $K$-trivial fibrations with globally $F$-split general fibres (see [DS17, Theorem 5.2] and [Eji17, Theorem 3.17]). Here, we recall its statement and extend it to morphisms with Stein degree not divisible by the characteristic $p$.

Throughout this section $(X, B)$ will denote a $\mathbb{Z}_{(p)}$-sub-pair over a perfect field of characteristic $p > 0$ and we will use the notation set in Section 1.3.1.

**Proposition 3.4.3.** *Let $f\colon X \to Z$ be a fibration of normal varieties and let $B$ be a $\mathbb{Z}_{(p)}$-divisor on $X$. Assume that $(1 - p^e)(K_X + B) \sim_Z 0$ for some $e \geq 1$, and that $(X_{\bar{\zeta}}, B_{\bar{\zeta}})$ is globally $F$-split, where $\zeta \in Z$ is the generic point. Then there exists a canonically defined effective $\mathbb{Z}_{(p)}$-divisor $B^Z$ on $Z$ such that*

*(i)* $(1 - p^e)(K_X + B) \sim f^*((1 - p^e)(K_Z + B^Z))$;

*(ii)* *if $B \geq 0$ then $B^Z \geq 0$;*

*(iii)* *$(X, B)$ is globally (sub-)$F$-split if and only if $(Z, B^Z)$ is globally (sub-)$F$-split;*

*(iv)* *if $\Lambda \geq 0$ is a $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor on $Z$, then $(B + f^*\Lambda)^Z = B^Z + \Lambda$.*

*Proof.* Point (i) follows by [DS17, Theorem 5.2][2]. This boils down to the following fact: write $(1 - p^e)(K_X + B) \sim f^*M$ for some Cartier divisor $M$ on $Z$, so that $T^e_B\colon F^e_*\mathcal{O}_X(f^*M) \to k(X)$. By pushing forward via $f$ and using the projection formula we obtain $f_*T^e_B\colon F^e_*\mathcal{O}_Z(M) \to k(Z)$. As $(X_{\bar{\zeta}}, B_{\bar{\zeta}})$ is GFS we have $f_*T^e_B \neq 0$ ([Eji17, Observation 3.19]), and Proposition 1.3.7 yields a canonically defined $\mathbb{Z}_{(p)}$-divisor $B^Z$ such that $M \sim (1 - p^e)(K_Z + B^Z)$ and $f_*T^e_B = T^e_{B^Z}$. As for point (ii), if $B \geq 0$ then the images of $T^e_B$ and $f_*T^e_B$ are contained in $\mathcal{O}_X$ and $\mathcal{O}_Z$, respectively, hence by Proposition 1.3.7, the induced $\mathbb{Q}$-divisor $B^Z$ is effective.

[2] If $(X_{\bar{\eta}}, B_{\bar{\eta}})$ is GFS, then so is $(X_\eta, B_\eta)$.

As for point (iii), first assume $(X, B)$ is GsFS, i.e. there is $\sigma \colon \mathcal{O}_X \to F^r_* \mathcal{L}^{(r)}_{X,B}$, a map such that $T^r_B \circ \sigma = \mathrm{id}_{\mathcal{O}_X}$, for some $r \geq 1$. Without loss of generality, we may assume $r$ is a multiple of $e$. In particular, we have the following commutative diagram

$$\begin{tikzcd} f_*\mathcal{O}_X \arrow[r, "g_*\sigma"] & f_*F^r_*\mathcal{L}^{(r)}_{X,B} \arrow[r, "f_*T^r_B"] & k(Z) \\ \mathcal{O}_Z \arrow[u] \arrow[r, "\tau"] & F^r_*\mathcal{L}^{(r)}_{Z,B^Z} \arrow[u] \arrow[r, "T^r_{B^Z}"] & k(Z). \arrow[u] \end{tikzcd}$$

As the vertical arrows are isomorphisms by the projection formula, then $T^r_{B^Z} \circ \tau = \mathrm{id}_{\mathcal{O}_Z}$, i.e. $(Z, B^Z)$ is GsFS. The GFS case follows by the same argument. Conversely, suppose $(Z, B^Z)$ is GsFS, and let $\tau \colon \mathcal{O}_Z \to F^r_* \mathcal{L}^{(r)}_{Z,B^Z}$ such that $T^r_{B^Z} \circ \tau = \mathrm{id}_{\mathcal{O}_Z}$. As $\tau$ corresponds to a global section of $\mathcal{L}^{(r)}_{Z,B^Z}$ and we have an isomorphism $H^0(Z, \mathcal{L}^{(r)}_{Z,B^Z}) \simeq H^0(Z, \mathcal{L}^{(r)}_{X,B})$ by the projection formula, we obtain a section $\sigma$, which satisfies $\sigma \circ T^r_{X,B} = \mathrm{id}_{\mathcal{O}_X}$. Again, the GFS case follows by the same argument.

As for point (iv), note that the rightmost half of the above diagram can be completed to

$$\begin{tikzcd} f_*F^r_*\mathcal{L}^{(r)}_{X,B+f^*\Lambda} \arrow[r] \arrow[rr, bend left, "f_*T^r_{B+f^*\Lambda}"] & f_*F^r_*\mathcal{L}^{(r)}_{X,B} \arrow[r, "T^r_B"] & k(Z) \\ F^r_*\mathcal{L}^{(r)}_{Z,B^Z+\Lambda} \arrow[u] \arrow[r] \arrow[rr, bend right, "T^r_{B^Z+\Lambda}"'] & F^r_*\mathcal{L}^{(r)}_{Z,B^Z} \arrow[u] \arrow[r, "T^r_{B^Z}"] & k(Z) \arrow[u] \end{tikzcd}$$

where the leftmost vertical arrow is also an isomorphism by the projection formula and $f_*T^r_{B+f^*\Lambda} = T^r_{B^Z+\Lambda}$. Note that the latter map is non-zero, as it coincides with $H^0(X_{\bar{\zeta}}, T^r_{B_{\bar{\zeta}}})$ at the geometric generic point of $Z$. We then conclude by the same argument as in point (i). qed

*Remark* 3.4.4 ([DS17, Proposition 5.7]). The divisor $B^Z$ constructed in Proposition 3.4.3 can be also described in terms of the singularities of the fibration, similarly to the definition of the discriminant part for $K$-trivial fibrations over fields of characteristic 0. More precisely, if $\delta$ is a prime divisor of $Z$, let

$$d_\delta := \sup\{t \text{ s.t. } (X, B + tf^*\delta) \text{ is globally sub-}F\text{-split over the generic point of } \delta\}.$$

Then,

$$B^Z = \sum_\delta (1 - d_\delta)\delta,$$

where the sum is taken over all prime divisors $\delta$ of $Z$.

**Definition 3.4.5.** Let $f\colon X \to Z$ be a fibration between normal varieties and $B$ a $\mathbb{Z}_{(p)}$-divisor satisfying the hypotheses of Proposition 3.4.3. The $\mathbb{Z}_{(p)}$-divisor $B^Z$ is called the **$F$-discriminant of the fibration**.

*Example* 3.4.6. Let $S$ be the projective closure of $V(y^2 - x(x-1)(x-\lambda))$ inside $\mathbb{P}^2_{\mathbb{Z},[x:y:z]} \times \mathbb{P}^1_{\mathbb{Z},[\lambda:\mu]}$ over $\operatorname{Spec}(\mathbb{Z})$ and let $f\colon S \to C = \mathbb{P}^1_{\mathbb{Z}}$ be the induced projection. We will denote by $f_0\colon S_0 \to C_0$ the fibration over $\operatorname{Spec}(\mathbb{C})$ and by $f_p\colon S_p \to C_p$ the fibration over $\operatorname{Spec}(\overline{\mathbb{F}}_p)$. In Example 3.1.6 we studied the canonical bundle formula for $f_0$. Now, let us consider what happens over $\operatorname{Spec}(\overline{\mathbb{F}}_p)$ for $p \geq 3$. In particular, we compute $B_p^{C_p}$, the $F$-discriminant defined as in Proposition 3.4.3.

Let $X \to \mathbb{A}^1_p := \operatorname{Spec}(\overline{\mathbb{F}}_p[t])$ be the family of cones over elliptic curves defined by $zy^2 - x(x-z)(x-tz)$ with a section $\gamma\colon \mathbb{A}^1_p \to X$ mapping to the vertices of the cones. Call $Z$ the image of $\gamma$. Let $\widetilde{X} \to X$ be the birational map obtained by blowing-up $Z$ and let $E$ be the exceptional divisor. Note that $E \to Z = \mathbb{A}^1_p$ is exactly the family $f_p$ restricted to $C_p \setminus \{\infty\}$. By the computations in [DS17, Example 3.2], the $F$-discriminant here is given by $B_p^{C_p}|_{\mathbb{A}^1_p} = \frac{1}{p-1}\sum_{\lambda\in\Lambda_p}(\lambda)$. Over $\infty$ we know by Example 3.1.6 that the coefficient is $\frac{1}{2}$. Therefore, the $F$-discriminant, for $p \geq 3$ is

$$B_p^{C_p} = \frac{1}{2}(\infty) + \frac{1}{p-1}\sum_{\lambda\in\Lambda_p}(\lambda),$$

where $\Lambda_p$ is the set of those $\lambda$'s corresponding to supersingular elliptic curves. Remark that the cardinality of $\Lambda_p$ is $\frac{p-1}{2}$ by [Sil09, Theorem 4.1(b), Chapter V], thus giving the right degree for $B_p^{C_p}$.

**Proposition 3.4.7.** *Let $h\colon Z \to Y$ be a surjective separable finite morphism of normal varieties such that $\deg(h)$ is not divisible by $p$, and let $B \geq 0$ be a $\mathbb{Z}_{(p)}$-divisor on $Z$. Assume $(Z, B)$ is globally $F$-split and $(1-p^e)(K_Z+B) \sim_Y 0$ for some $e \geq 1$. Then there exists a canonically defined effective $\mathbb{Z}_{(p)}$-divisor $B^Y$ on $Y$ such that*

*(i)* $(1-p^e)(K_Z+B) \sim h^*((1-p^e)(K_Y+B^Y))$;

*(ii)* $(Y, B^Y)$ *is globally $F$-split;*

*(iii) if $\Lambda \geq 0$ is a $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor on $Y$ such that $(Y, B^Y + \Lambda)$ is globally $F$-split, then $(Z, B + h^*\Lambda)$ is globally $F$-split too, and $(B + h^*\Lambda)^Y = B^Y + \Lambda$;*

*(iv) if $(Z, B)$ is globally $F$-regular then $(Y, B^Y)$ is globally $F$-regular.*

*Proof.* Let $M$ be a Cartier divisor such that $(1-p^e)(K_Z+B) \sim h^*M$, set $\mathcal{L} :=$

$\mathcal{O}_Y(M)$ and $d := \deg(h)$. By [ST14, Corollary 4.2] we have a commutative diagram

$$\begin{tikzcd}
h_*F^e_*h^*\mathcal{L} \arrow[r, "h_*T^e_B"] \arrow[d, "F^e_*\left(\frac{\mathrm{Tr}_{Z/Y}}{d}\otimes\mathcal{L}\right)"'] & h_*\mathcal{O}_Z \arrow[d, "\frac{\mathrm{Tr}_{Z/Y}}{d}"] \\
F^e_*\mathcal{L} \arrow[r, "\psi_Y"] & \mathcal{O}_Y,
\end{tikzcd}$$

where the vertical maps are split surjective via $F^e_*(h^\sharp \otimes \mathcal{L})\colon F^e_*\mathcal{L} \to h_*F^e_*h^*\mathcal{L}$ and $h^\sharp\colon \mathcal{O}_Y \to h_*\mathcal{O}_Z$, respectively, and $\psi_Y := \frac{\mathrm{Tr}_{Z/Y}}{d} \circ h_*T^e_B \circ F^e_*(h^\sharp \otimes \mathcal{L})$. As $(Z, B)$ is GFS, by taking global sections we obtain $1 \in \mathrm{Im}(H^0(Y, \psi_Y))$. In particular, $\psi_Y \neq 0$, thus Proposition 1.3.7 yields an effective $\mathbb{Z}_{(p)}$-divisor $B^Y$ such that $\psi_Y = T^e_{B^Y}$, $(Y, B^Y)$ is GFS and $(1-p^e)(K_Z + B) \sim h^*((1-p^e)(K_Y + B^Y))$, thus proving points (i) and (ii).

To show point (iii), up to replacing $e$ with a multiple, we may assume $(1-p^e)\Lambda$ is also Cartier. Denote by $\lambda\colon \mathcal{L}((1-p^e)\Lambda) \to \mathcal{L}$ the natural map, and observe that the above diagram can be completed to

$$\begin{tikzcd}
h_*F^e_*h^*\mathcal{L}((1-p^e)\Lambda) \arrow[r, "h_*F^e_*h^*\lambda"] \arrow[d, "F^e_*\left(\frac{\mathrm{Tr}_{Z/Y}}{d}\otimes\mathcal{L}((1-p^e)\Lambda)\right)"'] \arrow[rr, bend left, "\varphi_Z"] & h_*F^e_*h^*\mathcal{L} \arrow[r, "h_*T^e_B"] \arrow[d, "F^e_*\left(\frac{\mathrm{Tr}_{Z/Y}}{d}\otimes\mathcal{L}\right)"] & h_*\mathcal{O}_Z \arrow[d, "\frac{\mathrm{Tr}_{Z/Y}}{d}"] \\
F^e_*\mathcal{L}((1-p^e)\Lambda) \arrow[r, "F^e_*\lambda"] \arrow[rr, bend right, "\varphi_Y"'] & F^e_*\mathcal{L} \arrow[r, "\psi_Y"] & \mathcal{O}_Y.
\end{tikzcd}$$

As $(Y, B^Y + \Lambda)$ is GFS, we have $1 \in \mathrm{Im}(H^0(Y, \varphi_Y))$, hence $1 \in \mathrm{Im}(H^0(Z, \varphi_Z))$. By Proposition 1.3.7 the maps

$$F^e_*h^*\mathcal{L}((1-p^e)\Lambda) \xrightarrow{F^e_*h^*\lambda} F^e_*h^*\mathcal{L} \xrightarrow{T^e_B} \mathcal{O}_Z \quad \text{and} \quad F^e_*\mathcal{L}((1-p^e)\Lambda) \xrightarrow{F^e_*\lambda} F^e_*\mathcal{L} \xrightarrow{\psi_Y} \mathcal{O}_Y$$

correspond to the divisors $B + h^*\Lambda$ and $B^Y + \Lambda$, respectively.

As for point (iv), let $D \geq 0$ be a divisor on $Y$. By Lemma 1.3.13 it is enough to show that $(Y, B^Y + D/(p^r-1))$ is GFS whenever $r$ is large enough. As $(Z, B)$ is GFR, we have that $(Z, B + h^*(D/(p^r-1)))$ is GFS provided $r \gg 0$, again by Lemma 1.3.13. Arguing as in points (i) and (ii), we can pushforward the splitting to $Y$ via the trace map of $h$

$$\begin{tikzcd}
h_*\mathcal{O}_Z \arrow[r] \arrow[d, "\frac{\mathrm{Tr}_{Z/Y}}{d}"'] & h_*F^e_*\mathcal{L}^{(e)}_{B+h^*(D/(p^r-1))} \arrow[r, "h_*T^e_{B+h^*(D/(p^r-1))}"] \arrow[d, "F^e_*\left(\frac{\mathrm{Tr}_{Z/Y}}{d}\otimes\mathcal{L}^{(e)}_{B+h^*(D/(p^r-1))}\right)"] & h_*\mathcal{O}_Z \arrow[d, "\frac{\mathrm{Tr}_{Z/Y}}{d}"] \\
\mathcal{O}_Y \arrow[r] & F^e_*\mathcal{L}^{(e)}_{B^Y+D/(p^r-1)} \arrow[r, "T^e_{B^Y+D/(p^r-1)}"] & \mathcal{O}_Y,
\end{tikzcd}$$

hence $(Y, B^Y + D/(p^r-1))$ is GFS. qed

*Example* 3.4.8. It might be tempting to try extending this result to the case of a split finite morphism $h$ (i.e. such that the natural map $\mathcal{O}_Y \to h_*\mathcal{O}_Z$ is split). However, this fails already when $Z = Y = \mathbb{P}^1$ and $h = F$ over a field of characteristic $p = 2$. As explained in [ST14], this is because the splitting of $\mathcal{O}_Y \to h_*\mathcal{O}_Z$ is *not* given by the trace map. Indeed, let $h := F\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Frobenius morphism over a perfect field of characteristic $p = 2$. Then, $\mathcal{O}_{\mathbb{P}^1}(-K_{\mathbb{P}^1}) = F^*\mathcal{O}_{\mathbb{P}^1}(1)$. By a direct calculation, $F_*\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, we have a diagram:

$$\begin{tikzcd}
h_*\mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d, equal] & h_*F_*F^*\mathcal{O}_{\mathbb{P}^1}(1) \arrow[r] \arrow[d, equal] & h_*\mathcal{O}_{\mathbb{P}^1} \arrow[d, equal] \\
\boxed{\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)} \arrow[r] & F_*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \boxed{F_*\mathcal{O}_{\mathbb{P}^1}} \arrow[r] & \boxed{\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)} \\
\mathcal{O}_{\mathbb{P}^1} \arrow[r, "0"] \arrow[u] & F_*\mathcal{O}_{\mathbb{P}^1}(1) \arrow[r, "0"] \arrow[u] & \mathcal{O}_{\mathbb{P}^1}. \arrow[u]
\end{tikzcd}$$

*Example* 3.4.9 (See Example 1.3.15). On the other hand, when the splitting comes from the trace, we then have the familiar formula $B = h^*B^Y - \mathrm{Ram}(h)$, where $\mathrm{Ram}(h)$ denotes the ramification divisor of $h$.

For example, consider $h\colon E \to \mathbb{P}^1$, the cyclic cover of degree 2 of $\mathbb{P}^1$ ramified over four points $p_1, ..., p_4$ over an algebraically closed field of characteristic $p > 2$. Assume that $E$ is an ordinary elliptic curve (therefore it is GFS). By Proposition 3.4.7 and the above observation, we have that $\left(\mathbb{P}^1, \frac{1}{2}\sum_{i=1}^4(p_i)\right)$ is GFS.

**Proposition 3.4.10.** *Let $f\colon X \to Y$ be a surjective projective morphism of normal varieties such that its Stein degree* $\mathrm{St.deg}(f)$ *is not divisible by $p$, and let $B \geq 0$ be a $\mathbb{Z}_{(p)}$-divisor on $X$. Assume $(X, B)$ is globally $F$-split and $(1-p^e)(K_X + B) \sim_Y 0$. Then, there exists a canonically determined effective $\mathbb{Z}_{(p)}$-divisor $B^Y$ on $Y$ such that*

*(i)* $(1-p^e)(K_X + B) \sim f^*((1-p^e)(K_Y + B^Y))$;

*(ii)* $(Y, B^Y)$ *is globally $F$-split;*

*(iii) if $\Lambda \geq 0$ is a $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor on $Y$ such that $K_Y + B^Y + \Lambda$ is $\mathbb{Z}_{(p)}$-Cartier, then $(B + f^*\Lambda)^Y = B^Y + \Lambda$;*

*(iv)* $(X, B + f^*\Lambda)$ *is globally $F$-split if and only if* $(Y, B^Y + \Lambda)$ *is globally $F$-split.*

*Proof.* Apply Proposition 3.4.3 and Proposition 3.4.7 to the Stein factorisation of the morphism $f\colon X \xrightarrow{g} Z \xrightarrow{h} Y$. qed

# Chapter 4

# On the canonical bundle formula for GGLC morphisms

This chapter is the core of [Ben23]. We remark that the proofs have been corrected and the methods improved compared to the original version of the paper. The aim of the chapter is a canonical bundle formula result over perfect fields of positive characteristic for morphisms onto curves, conjecturally to the LMMP and the existence of log resolutions. In particular, when we say "**assume the LMMP in dimension** $n$", we mean that:

(a) we can run a Minimal Model Program for log canonical pairs $(X, B)$ with $B \geq 0$ of dimension $n$ and this terminates;

(b) inversion of adjunction holds for log canonical pairs $(X, B)$ with $B \geq 0$ (from dimension $n-1$ to dimension $n$).

*Remark* 4.0.1. If $X$ is a normal threefold over a perfect field of characteristic $p > 5$, existence of log resolutions is proven and we can run the LMMP for log canonical pairs (see [HX15], [Bir16], [BW17] and [HNT20]). Moreover, inversion of adjunction holds by [BMP$^+$23, Corollary 10.1].

## 4.1. The problem

Following the techniques used in [ACSS21], we work with more general fibrations than $K$-trivial ones. Therefore, we define the discriminant part measuring the singularities as in Definition 3.2.3 and the moduli part on the source of the fibration as in Definition 3.3.1. However, in order to get well-defined divisors, in positive characteristic we need to assume the GGLC condition (see Definition 2.2.1) on our fibration.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Proposition 4.1.1.** *Assume the LMMP and the existence of log resolutions of singularities in dimension $n$. Let $X$ be a variety of dimension $n$. Let $f\colon X \to Z$ be an equidimensional fibration between normal varieties and $(X/Z, B)$ a GGLC pair associated with it with $B_\eta \geq 0$, where $\eta$ is the generic point of $Z$. Then, the discriminant part $B_Z$ is well-defined. Consequently, the moduli part $M_X$ is also well-defined.*

*Proof.* By Proposition 2.2.10, the general fibre $X_z$ is normal and the pair defined by adjunction $(X_z, B_z)$ is log canonical. Let $U \subseteq Z$ be the dense open subset of $Z$ such that $(X_z, B_z)$ is log canonical with $B_z \geq 0$ for all $z \in U$ and $f^*\delta$ is integral for every $\delta$ prime divisor not contained in $Z \setminus U$. Let $\delta$ be a prime divisor in $Z$ which is not contained in $Z \setminus U$ and let $X_\delta$ be the fibre over $\delta$. We claim that $\gamma_\delta = 1$. If this was not the case, there would exist a non-log canonical place $E$ of $(X_\delta, B_\delta)$ such that its centre contains the generic point of $\delta$. In particular, by adjunction, this would imply that, for $z \in \delta$ general, there exists a non-log canonical place of $(X_z, B_z)$, contradiction. By inversion of adjunction, we conclude that $(X, B + f^*\delta)$ is log canonical around the generic point of $\delta$.

In particular, $\gamma_\delta = 1$ for all, but finitely many prime divisors $\delta \subsetneq Z$, therefore $B_Z$ and consequently $M_X$ are well-defined $\mathbb{Q}$-divisors. qed

*Remark* 4.1.2. If we only assume the GLC condition, the canonical bundle formula fails to hold, as shown in [Wit21, Example 3.5].

Recall that our objective is to study positivity properties of the moduli part. In particular, the aim is to understand the following question in the case of fibrations onto curves.

*Question* 4.1.3. Assume the LMMP and the existence of log resolutions in dimension up to $n$. Let $f\colon X \to Z$ be a fibration between normal projective varieties over a perfect field of characteristic $p > 0$, where $\dim(X) = n$, and $(X/Z, B)$ a GGLC pair associated with it with $B \geq 0$ and such that $(X, B)$ is log canonical. Assume that $K_X + B$ is $f$-nef. Is the moduli part $M_X$ nef up to a birational base change that is crepant over the generic point of $Z$, as in Theorem 3.3.10?

*Example* 4.1.4. With this example we illustrate the main idea of the approach we take to prove nefness of the moduli part in positive characteristic.

Let $X$ be the projective closure of $V(y^2 - x^4 - x^2y^2t - t)$ inside $\mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^1_{[s:t]}$ over a perfect field of characteristic $p \neq 2$ and let $f\colon X \to \mathbb{P}^1$ be the induced projection. The fibration $f$ is separable and there are no multiple fibres. Let $\mathcal{F}$ be the foliation induced by $f$, then

$$K_X = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1, -1)|_X \quad \text{and} \quad K_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1, 1)|_X.$$

Now, let us perform a Frobenius base change as in diagram (❋): the resulting variety $X^{(e)}$ is the projective closure of $V(y^2 - x^4 - x^2y^2\tau^{p^e} - \tau^{p^e})$ inside $\mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^1_{[\sigma:\tau]}$, therefore

$$K_{X^{(e)}} = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1, p^e - 2)|_X.$$

We observe that the divisor $K_X$ is not nef, while both $K_{\mathcal{F}}$ and $K_{X^{(e)}}$ are nef. When performing a Frobenius base change, the positivity of $K_{X^{(e)}}$ somehow reflects the positivity of $K_{\mathcal{F}}$.

Under Property $(*)$ assumptions, the moduli part coincides with the canonical bundle of the foliation, up to a term supported on the wild fibres. Therefore, we exploit the phenomenon shown in the above example to relate the positivity of the moduli part of $f\colon X \to Z$ to the positivity of the canonical divisor of the variety $X^{(e)}$ obtained after a high enough Frobenius base change. The advantage of doing so is that, although we do not have a foliated MMP in positive characteristic, we are able to study the positivity of the moduli part using the "standard" tools of the MMP on $K_{X^{(e)}}$.

## 4.2. Property $(*)$ in positive characteristic

### 4.2.1. Moduli part

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated. We use the notation of (❋) in Section 2.3.2.

For GGLC pairs, we define Property $(*)$ as in Definition 2.2.1. When the fibration satisfies Property $(*)$, the moduli part has an easy description.

**Proposition 4.2.1.** *Assume the LMMP and the existence of log resolutions of singularities in dimension $n$. Let $f\colon X \to Z$ be a separable equidimensional fibration between normal varieties, where $\dim(X) = n$, and $(X/Z, B)$ a GGLC pair associated with it which satisfies Property $(*)$. Then*

$$M_X = K_X + B^h - f^*K_Z - R(f),$$

*where $R(f) := \sum_{\delta\subseteq Z}(f^*\delta - f^{-1}(\delta))$ is as in Definition 2.3.5.*

*Proof.* The proof follows [ACSS21, Proposition 3.6], which states a similar equality over fields of characteristic 0. We need to show that, if $B_Z$ is the discriminant part, $B^v - f^*B_Z = -R(f)$. Since $(X/Z, B)$ has Property $(*)$, $B^v$ and $B_Z$ are reduced divisors and $B^v = f^{-1}(B_Z)$. Let $D$ be a vertical divisor such that its multiplicity with respect to $f$ is $\ell_D = 1$. Then, the coefficient of $D$ in $B^v - f^*B_Z$ is 0, as in $R(f)$. On the other hand, if $\ell_D > 1$, its coefficient in $B^v - f^*B_Z$ is $1 - \ell_D$, as in $-R(f)$. qed

**Corollary 4.2.2.** *Assume the LMMP and the existence of log resolutions of singularities in dimension $n$. Let $f : X \to Z$ be an equidimensional fibration between* *normal varieties, where* $\dim(X) = n$*, and* $(X/Z, B)$ *a GGLC pair on it satisfying Property* $(*)$*. For every* $e \geq 0$*:*

$$\alpha_e^* K_{X^{(e)}} = (p^e - 1)(M_X - B^h) + K_X - \sum_{D \text{ wild}} w_{D,e} D,$$

*where* $w_{D,e} \geq 0$ *for every wild fibre* $D$*.*

*Proof.* Combine Theorem 2.3.27 with Proposition 4.2.1. qed

### 4.2.2. Existence of $(*)$-modifications

Given any GLC pair in characteristic 0, we can construct a birationally equivalent model that satisfies Property $(*)$ thanks to the existence of toroidal modifications as proven in [AK00]. In characteristic $p > 0$ however, we cannot always use this construction. In fact, in one of the steps, the authors consider a quotient by the action of a group and that does not have good enough properties if the order of the group is divisible by $p$ ([AdJ97, Remark 0.3.2]). Nonetheless, when $Z$ is a curve, we construct $(*)$-modifications by using log resolutions. Indeed, since in this case fibres are divisors, we can resolve them. This is one of the key reasons why we restrict our main results (Theorem 4.4.5, Theorem 4.4.6 and Theorem 4.4.7) to the case of $Z$ being a curve.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Theorem 4.2.3** (Existence of $(*)$-modifications)**.** *Assume the LMMP and the existence of log resolutions in dimension* $n$*. Let* $f \colon X \to Z$ *be a fibration from a normal projective variety* $X$ *of dimension* $n$ *onto a normal curve* $Z$ *and* $(X/Z, B)$ *a GGLC pair associated with it with* $B \geq 0$*. Then, there exists a GGLC pair* $(Y/Z, C)$ *satisfying Property* $(*)$*, with* $Y$ $\mathbb{Q}$*-factorial,* $C \geq 0$*, and* $(Y, C)$ *dlt, together with a commutative diagram:*

$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Z, \end{tikzcd}$$

*where* $\mu$ *is a proper birational map and* $g$ *is a fibration. Moreover, there exist a vertical effective exceptional* $\mathbb{Q}$*-divisor* $R$ *on* $Y$*, whose image in* $X$ *is supported in the non-log canonical locus of* $(X, B)$*, and a vertical effective* $\mathbb{Q}$*-divisor* $G$ *on* $Y$*, such that*

$$K_Y + C + R = \mu^*(K_X + B) + G.$$

*Additionally, the divisor* $K_Y + C$ *is* $\mu$*-nef.*

*Proof.* **Step 1.** First, we take a log resolution $\rho \colon X' \to X$ of $(X, B)$. Then, we write $K_{X'} + D = \rho^*(K_X + B) + E$, where $D$ and $E$ are both effective with no common components and $E$ is exceptional. If $D = \sum_i a_i D_i$, define $B' := \sum_i \min\{a_i, 1\} D_i$ and $R' := D - B'$, so that

$$K_{X'} + B' + R' = \rho^*(K_X + B) + E.$$

Note that $R'$ is supported on the non-log canonical locus of $(X, B)$, thus it is a vertical divisor. Since $(X_{\bar\eta}, B_{\bar\eta})$ is log canonical, so is $(X'^{\nu}_{\bar\eta}, B'^{\nu}_{\bar\eta})$, where $B'^{\nu}_{\bar\eta}$ is defined by restriction. In particular, the general fibre of the induced map $f' \colon X' \to Z$ is normal and log canonical by Lemma 2.2.5 and Proposition 2.2.10.

**Step 2.** Let $T \subset Z$ be the finite set of points over which the fibre is not normal or not log canonical. Consider now a further log resolution $\sigma \colon X^* \to X$ so that the strict transform of $\operatorname{Supp}(B') \cup f'^{-1}(T)$, together with the support of the exceptional divisors of $\sigma$, is simple normal crossing. We can choose $X^* \to X'$ to be an isomorphism over $Z \setminus T$. Define effective $\mathbb{Q}$-divisors $\overline{B}, \overline{R}, \overline{E}$ in the same way we defined $B', R', E$ respectively, so that

$$K_{X^*} + \overline{B} + \overline{R} = \sigma^*(K_X + B) + \overline{E}.$$

Let $h \colon X^* \to Z$ be the induced map, then $(X^*/Z, \overline{B})$ is GGLC. Let

$$G^* := \left( \sum_{z \in T} h^{-1}(z) \right) - \overline{B}^v.$$

Finally, let $B^* := \overline{B} + G^*$ and let $\Sigma_Z$ be the discriminant part of $(X^*/Z, B^*)$. Note that $\Sigma_Z = \sum_{z \in T} (z)$, where $(z)$ denotes the divisor corresponding to the point $z \in Z$.

We claim that $(X^*/Z, B^*)$ satisfies Property (∗).

- The pair is GGLC because $G^*$ is vertical, so adding it does not affect the singularities at the general fibre.
- Since $Z$ is a curve, $(Z, \Sigma_Z)$ is log smooth.
- If $z \in Z \setminus \operatorname{Supp}(\Sigma_Z)$, $(X^*_z, B^*_z)$ is log canonical, where $X^*_z := h^{-1}(z)$ is normal and $B^*_z$ is defined via adjunction. Thus, by inversion of adjunction, the pair $(X^*, B^* + X^*_z)$ is log canonical around $X^*_z$.
- If $z \in \operatorname{Supp}(\Sigma_Z)$, by construction, $\operatorname{Supp}(\overline{B}) \cup X^*_z$ is simple normal crossing. Thus, $(X^*, B^*)$ is log canonical (around $X^*_z$).

**Step 3.** Now, run a $(K_{X^*} + B^*)$-MMP over $X$ and let $Y$ be the resulting variety, so that it fits in the following diagram:

$$\begin{tikzcd} X^* \arrow[r, dashed, "\varphi"] \arrow[dr, "h"'] & Y \arrow[r, "\mu"] \arrow[d, "g"] & X \arrow[dl, "f"] \\ & Z, & \end{tikzcd}$$

where $\mu$ is the induced birational map and $g := f \circ \mu$ is a fibration. Let $C := \varphi_* B^*$. Note that $X^* \dashrightarrow Y$ is also a sequence of steps of the $(K_{X^*} + B^*)$-MMP over $Z$, thus, by Proposition 3.3.3, $(Y/Z, C)$ satisfies Property $(*)$. Moreover, it is dlt and $\mathbb{Q}$-factorial since these properties are preserved under the MMP. We have $K_{X^*} + B^* + \overline{R} = \sigma^*(K_X + B) + \overline{E} + G^*$. Define $R := \varphi_* \overline{R}$, let $Q$ be the horizontal part of $\varphi_*(\overline{E} + G^*)$ and $G$ its vertical part. We claim that $Q = 0$. In fact

$$Q + G - R = K_Y + C - \mu^*(K_X + B)$$

is $\mu$-nef, and the horizontal part of this divisor is $\mu$-exceptional, therefore $-\mu_*(Q|_{Y_{\bar{\eta}}}) = 0$. By the Negativity lemma [KM98, Lemma 3.39] applied to $Y_{\bar{\eta}}$, since $R|_{Y_{\bar{\eta}}} = 0$, we conclude that $-Q|_{Y_{\bar{\eta}}} \geq 0$. All in all, we get:

$$K_Y + C + R = \mu^*(K_X + B) + G.$$

qed

### 4.2.3. Geometric $(*)$-modifications

In order to do induction on the dimension, we will perform adjunction on log canonical centres of the geometric generic fibre. For this, we need to extract divisors with discrepancy $-1$ over them. However, this cannot always be done over the total space $X$. In the next examples, we see that it may happen that $X$ does not have any log canonical centre, even if the geometric generic fibre does. To overcome this issue, we base change our fibration with a high enough power of the Frobenius morphism as in (❋) in Section 2.3.2: in this way the singularities of the geometric generic fibre appear on the total space.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated. We use the notation of (❋) in Section 2.3.2.

*Example* 4.2.4. Let $X$ be the projective closure of $V(x^3s + y^3s + xyzs + v^3t)$ inside $\mathbb{P}^3_{[x:y:z:v]} \times \mathbb{P}^1_{[s:t]}$ and let $f\colon X \to \mathbb{P}^1_{[s:t]}$ be the fibration induced by the projection onto the second factor. On the open set where $s \neq 0, v \neq 0$, $X$ is regular everywhere. If the characteristic is $\neq 3$, $X_{\bar{\eta}}$ is also smooth. Only the fibre over $t = 0$ has a singularity at the origin. On the other hand, if the characteristic of the base field is 3, the general fibre is a deformation of a cone and it has two singularities that come together over $t = 0$. On the geometric generic fibre they can be described by the equations $W := V(x, z, y^3 + t)$ and $W' := V(y, z, x^3 + t)$. These are canonical centres of the geometric generic fibre. Note that $f|_W$ and $f|_{W'}$ are the Frobenius morphism.

Now, consider the base change with the Frobenius morphism $F\colon Z \to Z$ and let $X^{(1)}$ be the normalisation of the fibre product. Let $\tau$ be an element of $k(X^{(1)})$ such that $\tau^3 = t$. Then, the total space $X^{(1)}$ is no longer regular, but has singularities at $W^{(1)} := V(x, z, y + \tau)$ and $W'^{(1)} := V(y, z, x + \tau)$.

*Example* 4.2.5. Let $X := \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^1_{[s:t]}$, $D = V(x^p s - y^p t) \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^1_{[s:t]}$ and $B := \frac{1}{p} D$ over an algebraically closed field of characteristic $p > 0$. Let $f\colon X \to \mathbb{P}^1_{[s:t]}$ be the natural projection. Consider the base change with $F\colon \mathbb{P}^1 \to \mathbb{P}^1$, let $X^{(1)}$ be the normalisation of the fibre product and $\beta_1\colon X^{(1)} \to X$. Then $(X, B)$ is klt, while$(X^{(1)}, \beta_1^* B)$ is strictly log canonical.

*Remark* 4.2.6. Let $f\colon X \to Z$ be a separable fibration between normal varieties with a pair $(X/Z, B)$. Let $W \subset X$ be a subvariety such that $(W_{\overline{\eta}})_{\mathrm{red}}$ is a log canonical centre of $(X_{\overline{\eta}}, B_{\overline{\eta}})$, where $\overline{\eta}$ is the geometric generic point of $Z$. Even if $f|_W$ is separable, it may be that $W$ is not a log canonical centre of $(X, B)$. This is because, in order to extract a log canonical place over $(W_{\overline{\eta}})_{\mathrm{red}}$, we may have to blow-up centres $\overline{V}$ defined over $X_{\overline{\eta}}$ such that the variety $V$ over $X$ "corresponding" to $\overline{V}$ is not geometrically reduced over $Z$.

**Lemma 4.2.7.** *Let $f\colon X \to Z$ be a separable fibration between normal varieties such that the geometric generic fibre $X_{\overline{\eta}}$ is normal. Let $B$ be a $\mathbb{Q}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $\sigma\colon Z' \to Z$ be a generically finite separable map and consider the diagram:*

$$\begin{tikzcd} X' \arrow[r, "s"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "\sigma"] & Z, \end{tikzcd}$$

*where $X'$ is the normalisation of the main component of the fibre product $X \times_Z Z'$. Define a $\mathbb{Q}$-divisor $B'$ on $X'$ by log pullback, so that $K_{X'} + B' = s^*(K_X + B)$. Let $\eta$ be the generic point of $Z$. Then a subset $W \subseteq X$ such that $W_\eta$ is non-empty, is a log canonical centre (resp. non-log canonical centre) of $(X, B)$ if and only if there exists $W' \subseteq X'$, irreducible component of $s^{-1}(W)$, which is a log canonical centre (resp. non-log canonical centre) of $(X', B')$.*

*Proof.* By Lemma 2.2.8, the conductor of $X'$ is vertical, so, up to shrinking $Z$, we can assume the fibre product is already normal. Moreover, up to shrinking $Z$ further, we can assume both $\sigma$ and its Galois closure are étale, so $s$ and its Galois closure are étale as well. In particular, they do not have any (wild) ramification. Thus, the discrepancies over the two pairs are the same ([KM98, Proposition 5.20]). qed

**Definition 4.2.8.** Let $f\colon X \to Z$ be a separable flat fibration between normal varieties and $B$ a $\mathbb{Q}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Assume that the geometric generic fibre $X_{\overline{\eta}}$ is normal. A **geometric non-klt centre** (resp. **geometric log canonical centre/ geometric non-log canonical centre**) of $(X, B)$ is a subvariety $W \subset X$ such that, if $\overline{\eta}$ is the geometric generic point of $Z$, $(W_{\overline{\eta}})_{\mathrm{red}}$ is a non-klt (resp. log canonical/ non-log canonical) centre of $(X_{\overline{\eta}}, B_{\overline{\eta}})$.

**Proposition 4.2.9.** *Let $f\colon X \to Z$ be a separable fibration between normal varieties such that the geometric generic fibre $X_{\overline{\eta}}$ is normal. Let $B$ be a $\mathbb{Q}$-divisor on $X$ such that $K_X + B$, $B^v$ and all the vertical divisors $D$ whose multiplicity is $\ell_D > 1$ are $\mathbb{Q}$-Cartier. Let $\overline{W}$ be a log canonical (resp. non-log canonical) centre of $(X_{\overline{\eta}}, B_{\overline{\eta}})$ and let $W$ be a subvariety of $X$ such that an irreducible component of $(W_{\overline{\eta}})_{\mathrm{red}}$ is exactly $\overline{W}$. Then, there exists $e \gg 0$ such that $W^{(e)}$ is a log canonical (resp. non-log canonical) centre of $(X^{(e)}, B_e)$, where $B_e := \beta_e^* B^h + \beta_e^{-1} B^v$.*

*Proof.* Since $\overline{W}$ is a log canonical (resp. non-log canonical) centre of $(X_{\overline{\eta}}, B_{\overline{\eta}})$, there exists $\overline{Y} \to X_{\overline{\eta}}$ proper birational map that extracts an exceptional divisor $\overline{E}$ over $\overline{W}$ with discrepancy $-1$ (resp. $< -1$).

By Lemma 2.2.9, there is a generically finite map $\varphi\colon Z' \to Z$ such that, if $X'$ is the normalisation of the main component of $X \times_Z Z'$, there exists $\sigma\colon Y \to X'$ birational with $Y_{\overline{\eta}} = \overline{Y}$. Up to possibly taking a further purely inseparable base change, the map $\varphi$ can be decomposed as $F^e \circ \psi$, with $\psi$ separable and $e \in \mathbb{N}$, so that we have the following diagram:

$$\begin{tikzcd} Y \arrow[r, "\sigma"] & X' \arrow[r, "\gamma"] \arrow[d] & X^{(e)} \arrow[r, "\beta_e"] \arrow[d] & X \arrow[d, "f"] \\ & Z' \arrow[r, "\psi"] & Z \arrow[r, "F^e"] & Z. \end{tikzcd}$$

Let $B_e := \beta_e^* B^h + \beta_e^{-1}(B^v)$ and define $B'$ by log pullback from $X^{(e)}$, so $K_{X'} + B' = \gamma^*(K_{X^{(e)}} + B_e)$. Note that, by Theorem 2.3.27 and the $\mathbb{Q}$-Cartier assumptions in the statement, $K_{X^{(e)}} + B_e$ is $\mathbb{Q}$-Cartier. Since $X_{\overline{\eta}}$ is normal, by Lemma 2.2.8, $X'_{\overline{\eta}} = Y_{\overline{\eta}} = X^{(e)}_{\overline{\eta}}$, and $\beta_e$ and $\gamma$ are isomorphisms on the geometric generic fibre, therefore, $B_{\overline{\eta}} = B_{e,\overline{\eta}} = B'_{\overline{\eta}}$.

Let $E \subset Y$ be an exceptional divisor on $Y$ such that $E_{\overline{\eta}} = \overline{E}$. The discrepancy of $E$ with respect to the pair $(X', B')$ can be computed at $\overline{\eta}$, therefore it is 1 (resp. $< -1$). Hence, $\sigma(E)$ is a log canonical (resp. non-log canonical) centre of $(X', B')$ and $(\beta_e \circ \gamma \circ \sigma)(E) = W$. We conclude by Lemma 4.2.7. qed

*Remark* 4.2.10. The above Proposition 4.2.9 says that, if $W$ is a geometric non-klt (resp. log canonical/ non-log canonical) centre of $(X/Z, B)$, it is not necessarily a non-klt (resp. log canonical/ non-log canonical) centre of $(X, B)$. However, for $e \gg 0$, its base change $W^{(e)}$ inside $X^{(e)}$ is a non-klt (resp. log canonical/ non-log canonical) centre of $(X^{(e)}, B_e)$, where $B_e := \beta_e^* B^h + \beta_e^{-1} B^v$.

Moreover, since the non-klt centres of the geometric generic fibre $(X_{\overline{\eta}}, B_{\overline{\eta}})$ are finitely many, for $e \gg 0$, they all appear as (geometric) non-klt centres of $(X^{(e)}, B_e)$. Assuming the existence of log resolutions, we can construct a birational model of $(X^{(e)}, B_e)$ which extracts all these geometric non-klt centres. More precisely, for $e \gg 0$, there exists a birational map $Y \to X^{(e)}$ such that, if $\overline{W}$ is a non-klt centre of $(X_{\overline{\eta}}, B_{\overline{\eta}})$, there is a place $E \subset Y$ that extracts $\overline{W}$.

**Theorem 4.2.11** (Existence of geometric $(*)$-modifications)**.** *Assume the LMMP and the existence of log resolutions in dimension $n$. Let $f\colon X \to Z$ be a fibration from a normal variety $X$ of dimension $n$ onto a normal curve $Z$, such that the geometric generic fibre $X_{\bar\eta}$ is normal. Let $B \geq 0$ be a $\mathbb{Q}$-divisor on $X$ such that $K_X+B$, $B^v$ and all the vertical divisors $D$ whose multiplicity is $\ell_D > 1$ are $\mathbb{Q}$-Cartier. Assume that the coefficients of $B$ and $B_{\bar\eta}$ are $\leq 1$. Let $\overline{W}$ be a log canonical centre of $(X_{\bar\eta}, B_{\bar\eta})$ and let $W$ be a subvariety of $X$ such that an irreducible component of $(W_{\bar\eta})_{\mathrm{red}}$ is exactly $\overline{W}$. For $e \geq 0$, let $B_e := \beta_e^* B^h + \beta_e^{-1} B^v$. Then, there exists $e_0$ such that, for all $e \geq e_0$, there exists a $(*)$-modification of $(X^{(e)}, B_e)$ which extracts an exceptional divisor $E$ over $W^{(e)}$ with discrepancy $-1$. More precisely, there exist a dlt GGLC pair $(Y/Z, C+E)$ with $Y$ $\mathbb{Q}$-factorial, $C \geq 0$, and a diagram:*

$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[dr, "g"'] & X^{(e)} \arrow[d, "f_e"] \\ & Z, \end{tikzcd}$$

*where $\mu$ is a proper birational map which extracts $E$ over $W^{(e)}$ and $g$ is a fibration. Moreover, there exist an effective exceptional $\mathbb{Q}$-divisor $R$ on $Y$, whose image in $X^{(e)}$ is supported in the non-log canonical locus of $(X^{(e)}, B_e)$, and a vertical effective $\mathbb{Q}$-divisor $G$ on $Y$, such that*

$$K_Y + C + E + R = \mu^*(K_{X^{(e)}} + B_e) + G.$$

*Additionally, $(K_Y + C + E)$ is $\mu$-nef.*

*Proof.* **Step 1.** First of all note that $(X_{\bar\eta}, B_{\bar\eta})$ is not necessarily log canonical. In this first step we find a model over $(X^{(e)}, B_e)$ that satisfies the GGLC property.

Choose $e \in \mathbb{N}$ big enough, so that, for all $\overline{V}$ non-log canonical centre of $(X_{\bar\eta}, B_{\bar\eta})$, there exists $V_e \subseteq X^{(e)}$ (geometric) non-log canonical centre of $(X^{(e)}, B_e)$ such that $V_{e,\bar\eta}$ is reduced and one of its components is $\overline{V}$. Such $e$ exists by Proposition 4.2.9. Note that, by the assumptions on $B$, the coefficients of $B_e$ are all $\leq 1$.

Let $\sigma\colon X_1 \to X^{(e)}$ be a log resolution of $(X^{(e)}, B_e)$ which extracts all the geometric non-log canonical centres.

Write $K_{X_1} + D_1 = \sigma^*(K_{X^{(e)}} + B_e) + T_1$, where $T_1$ is effective and $\sigma$-exceptional, and $D_1$ is effective and it does not have any components in common with $T_1$. If $D_1 = \sum_i a_i D_i$, with $D_i$ prime divisors, let $B_1 := \sum_i \min\{a_i, 1\} D_i$ and $R_1 = D_1 - B_1$, so that

$$K_{X_1} + B_1 + R_1 = \sigma^*(K_{X^{(e)}} + B_e) + T_1.$$

Note that $R_1$ is $\sigma$-exceptional and supported on the non-log canonical places of $(X^{(e)}, B_e)$. Moreover, $(X_1, B_1)$ is log smooth and $(X_{1,\bar\eta}, B_{1,\bar\eta})$ is log canonical since we have extracted all the non-log canonical places.

Now, we run a $(K_{X_1}+B_1)$-MMP over $X^{(e)}$. Let $\varphi\colon X_1 \dashrightarrow X_1'$ and $\tau\colon X_1' \to X^{(e)}$ be the resulting morphisms, $B_1' := \varphi_* B_1$ and $R_1' := \varphi_* R_1$. By the Negativity lemma [KM98, Lemma 3.39], $\varphi_* T_1 = 0$. In fact, $-R_1' + \varphi_* T_1$ is $\tau$-nef and $\tau_*(R_1' - \varphi_* T_1) \geq 0$, whence $-\varphi_* T_1 \geq 0$. All in all, we have:

$$K_{X_1'} + B_1' + R_1' = \tau^*(K_{X^{(e)}} + B_e),$$

where $R_1'$ is $\tau$-exceptional and supported on the non-log canonical places of $(X^{(e)}, B_e)$. Moreover, both $(X_1', B_1')$ and $(X_{1,\bar\eta}', B_{1,\bar\eta}')$ are log canonical and $X_1'$ is $\mathbb{Q}$-factorial.

**Step 2.** In this step, we extract a log canonical place over $W^{(e)}$.

Let $s\colon X_2 \to X_1'$ be a birational morphism extracting a log canonical place $E_2$ over $W^{(e)}$. Define $B_2$ and $T_2$ effective $\mathbb{Q}$-divisors with no common components so that

$$K_{X_2} + B_2 + E_2 = s^*(K_{X_1'} + B_1') + T_2.$$

Note that $X_2$ is $\mathbb{Q}$-factorial. Run a $(K_{X_2} + B_2 + E_2)$-MMP over $X^{(e)}$. Define $\psi\colon X_2 \dashrightarrow X_2'$ and $t\colon X_2' \to X^{(e)}$ to be the resulting morphisms, $B_2' := \psi_* B_2$ and $E_2' := \psi_* E_2$. Note that, since $(K_{X_2}+B_2+E_2) \sim_{\mathbb{Q},s} T_2$, all the contracted (or flipped) curves are in the support of $T_2$; in particular, $E_2$ is not contracted. Let $s' : X_2' \to X_1'$ be the induced map, then, $\psi_* T_2$ is $s'$-nef and $-s'_*\psi_* T_2 = 0$, therefore, $-\psi_* T_2 \geq 0$ by the Negativity lemma [KM98, Lemma 3.39], whence $\psi_* T_2 = 0$. All in all, we have:

$$K_{X_2'} + B_2' + E_2' + R_2' = t^*(K_{X^{(e)}} + B_e),$$

where $R_2' := \psi_* s^* R_1'$ is $t$-exceptional and supported on the non-log canonical places of $(X^{(e)}, B_e)$. Moreover, both $(X_2', B_2' + E_2')$ and $(X_{2,\bar\eta}', B_{2,\bar\eta}' + E_{2,\bar\eta}')$ are log canonical, $X_2'$ is $\mathbb{Q}$-factorial and $K_{X_2'} + B_2' + E_2'$ is $t$-nef.

**Step 3.** In this step, we find a $(*)$-modification of $(X_2', B_2' + E_2')$.

We apply Theorem 4.2.3 to $(X_2', B_2'+E_2')$ to find a $(*)$-modification $(Y'/Z, C' + E')$ with $\lambda\colon Y' \to X_2'$ birational, where $E'$ is the strict transform of $E_2'$. Let $g'\colon Y' \to Z$ be the induced fibration. In particular,

(a) $(Y'/Z, C' + E')$ is GGLC and satisfies Property $(*)$;

(b) $Y'$ is $\mathbb{Q}$-factorial and $(Y', C' + E')$ is dlt;

(c) there exists an effective $\mathbb{Q}$-divisor $G'$, such that we have $K_{Y'} + C' + E' = \lambda^*(K_{X_2'} + B_2' + E_2') + G'$ and $G'$ is $g'$-vertical;

(d) $K_{Y'} + C' + E'$ is $\lambda$-nef and, if $U := Y' \setminus \operatorname{Supp}(G')$, $(K_{Y'} + C' + E')|_U$ is $\mu'$-nef, where $\mu'\colon Y \to X^{(e)}$;

(e) let $R_{Y'} := \lambda^* R_2'$, then $K_{Y'} + C' + E' + R_{Y'} = \mu'^*(K_{X^{(e)}} + B_e) + G'$.

**Step 4.** As last step, we run a $(K_{Y'} + C' + E')$-MMP over $X^{(e)}$.

By point (d) in Step 3, this MMP is an isomorphism on $U$, so it does not contract $E'$. Let $Y$ be the resulting variety, $\mu\colon Y \to X^{(e)}$ the induced morphism, and $C, E, R$ and $G$ the push-forward on $Y$ of $C', E', R_{Y'}$ and $G'$, respectively. Let $g\colon Y \to Z$ be the induced morphism. Then:

(a) $(Y/Z, C+E)$ is GGLC and satisfies Property $(*)$ by Proposition 3.3.3;

(b) $Y$ is $\mathbb{Q}$-factorial and $(Y, C+E)$ is dlt since these properties are preserved under the MMP;

(c) $K_Y + C + E + R = \mu^*(K_{X^{(e)}} + B_e) + G$ by point (e) in Step 3;

(d) $K_Y + C + E$ is $\mu$-nef.

qed

### 4.2.4. Adjunction of the moduli part

Now, we study the restriction of the moduli part to geometric log canonical centres for fibrations onto curves.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Lemma 4.2.12.** *Let $(X, B+S)$ be a dlt pair, where $S$ is a prime divisor and $X$ is a normal $\mathbb{Q}$-factorial variety. Then, the normalisation morphism $S^\nu \to S$ is an isomorphism in codimension 1. Moreover, if either*

*(a) $X$ has dimension $\leq 3$ and is defined over a perfect field of characteristic $p > 5$, or*

*(b) $S$ satisfies the $S_2$ property;*

*then, $S$ is normal.*

*Proof.* The pair $(S^\nu, B_{S^\nu})$ induced on $S^\nu$ by adjunction is log canonical. Moreover, by [HW23, Lemma 2.1], $S^\nu \to S$ is a universal homeomorphism, therefore the singularities in codimension 1 are worse than nodal. Thus, if $S^\nu \to S$ was not an isomorphism in codimension 1, the conductor would have coefficients $> 1$, leading to a contradiction.

If $X$ is a threefold over an algebraically closed field of characteristic $p > 5$, the claim is proven in [Bir16, Lemma 5.2]. If $X$ is defined over a perfect field $k$ of characteristic $p > 5$, by the same argument, we conclude that $\overline{S} := S \times_k \overline{k}$ is normal, which means that $S$ is geometrically normal, whence normal. On the other hand, if $S$ satisfies the $S_2$ property, the pair $(S, B_S)$ induced on $S$ by adjunction is slc and, since $S^\nu \to S$ is a universal homeomorphism, there cannot be nodal singularities in codimension 1, thus $S$ is normal. qed

*Remark* 4.2.13. If $X$ has dimension $> 3$, or the characteristic of the base field is $p \leq 5$, then it is no longer true in general that plt centres are normal (see [Ber19] and [CT19]).

We now set some notation that we will use in the following results.

($\diamondsuit$) Set-up

(i) Let $f\colon X \to Z$ be an equidimensional fibration between normal varieties, where $Z$ is a curve. Let $(X/Z, B+S)$ be a GGLC pair associated with it, where $S$ is a prime horizontal divisor.

(ii) Assume that $X$ is $\mathbb{Q}$-factorial and $(X, B+S)$ is dlt. Then, by Lemma 4.2.12, $S$ is normal in codimension 1. For the sake of the computations we are going to do, we can suppose $S = S^\nu$. In fact, we will only be interested in working around codimension 1 points.

(iii) Let $(S^\nu, B_{S^\nu})$ be the pair induced on $S^\nu$ by adjunction.

(iv) Assume the LMMP and the existence of log resolutions in dimension up to $n := \dim(X)$. Suppose that $(X/Z, B+S)$ satisfies Property $(*)$.

(v) Let $f|_{S^\nu} = g \circ \varphi$, be the Stein factorisation of $f|_{S^\nu}$; in particular $g$ is a fibration and $\varphi\colon Z' \to Z$ is a finite morphism.

*Remark* 4.2.14. In the above Set-up ($\diamondsuit$), since $(X/Z, B+S)$ is GGLC, if $\overline{\eta}$ is the geometric generic point of $Z$, $S_{\overline{\eta}}$ is reduced, therefore, by Proposition 2.1.2, $\varphi$ is separable.

Moreover, we have some control over the ramification of $\varphi$. More precisely, if $\delta \subseteq Z$ is a prime divisor and $\varphi^*\delta = \sum_i m_i \delta'_i$, then $m_i$ divides the multiplicities with respect to $f$ of all the vertical divisors over $\delta$ intersecting $S$ non-trivially. Indeed, without loss of generality, we can suppose $S = S^\nu$. Since $(X/Z, B+S)$ has Property $(*)$, $(S, f^{-1}(\delta)|_S)$ is log canonical around $f^{-1}(\delta)|_S$, whence $f^{-1}(\delta)|_S$ is reduced. Then

$$\begin{aligned} f^*\delta|_S &= \sum_j \ell_j D_j|_S = \sum_j \ell_j \left( \sum_l D^S_l \right) \\ &= g^*\varphi^*\delta = \sum_i m_i g^* \delta'_i. \end{aligned}$$

The same computation shows that, if $D$ is a prime vertical divisor in $X$ and $D_S$ is a prime component of $D|_S$,the multiplicity of $D_S$ with respect to $g$ divides the multiplicity of $D$ with respect to $f$. All in all, if we work around a general point of $D_S$ and we let $\delta' := g(D_S)$ and $\delta := f(D)$, we can write $\varphi^*\delta = m\delta'$, $g^*\delta' = nD_S$ and $f^*\delta = \ell D$. Then, $\ell = mn$.

*Remark* 4.2.15. Recall that, if $\delta' \subseteq Z'$ is a prime divisor such that its multiplicity with respect to $\varphi$ is divisible by $p$, $\delta'$ is called a divisor of **wild ramification**. If the multiplicity is $\geq 2$ and coprime with $p$, $\delta'$ is a divisor of **tame ramification**. If $D$ is a wild fibre for $f$, then $\delta' := g(D)$ may be a divisor of wild ramification for $\varphi$. On the other hand, if $D$ is tame, then $\delta'$ is either unramified or of tame ramification.

**Proposition 4.2.16.** *In the above Set-up (✧), the pair $(S^\nu/Z', B_{S^\nu})$ associated to the fibration $g\colon S^\nu \to Z'$ is GGLC and satisfies Property $(*)$. Moreover, $(B_{S^\nu})^v = B^v|_{S^\nu}$.*

*Proof.* First of all, note that the geometric generic points of $Z$ and $Z'$ coincide. Since $S_{\bar\eta}$ is a log canonical centre of $(X_{\bar\eta}, B_{\bar\eta})$, the pair $(S^\nu_{\bar\eta}, B_{S^\nu_{\bar\eta}})$ is log canonical. By the universal properties of the normalisation, $(S^\nu)^\nu_{\bar\eta} = S^\nu_{\bar\eta}$. Thus, $(S^\nu/Z', B_{S^\nu})$ is GGLC.

Since $S^\nu \to S$ is an isomorphism in codimension 1, for the purpose of the proof, we can assume $S$ is normal. Let us verify that $(S/Z', B_S)$ satisfies Property $(*)$. Let $\Sigma_Z$ be the discriminant part of $(X/Z, B+S)$. Since $X$ is $\mathbb{Q}$-factorial, we define $B^h_S$ by adjunction, so that $(K_X + B^h + S)|_S = K_S + B^h_S$ and $K_S + B_S = K_S + B^h_S + B^v|_S$.

(a) We claim that $B^h_S$ does not contain any vertical component. In particular, $(B_S)^v = B^v|_S$ and $(B_S)^h = B^h_S$. Indeed, if this was not the case, let $D_S$ be a vertical divisor contained in $\mathrm{Supp}(B^h_S)$ and $z := f|_S(D_S)$. Since $(X, B^h + S + f^{-1}(z))$ is log canonical, $(S, B^h_S + D_S)$ is log canonical as well. Therefore, the coefficient of $D_S$ in $B^h_S$ must be 0, contradiction. To conclude, note that $(B^v)|_S \leq (B_S)^v$ since $S$ is horizontal.

(b) Let $\Sigma_{Z'} := \varphi^{-1}(\Sigma_Z)$. Then $B^v_S = f|_S^{-1}(\Sigma_Z) = g^{-1}(\Sigma_{Z'})$ and, since $Z'$ is a normal curve, $(Z', \Sigma_{Z'})$ is log smooth.

(c) Let $z' \in Z' \setminus \mathrm{Supp}(\Sigma_{Z'})$ be a closed point and $z := \varphi(z') \in Z \setminus \mathrm{Supp}(\Sigma_Z)$. In particular, $f$ is unramified at $z$, therefore, by the computations in Remark 4.2.15, both $\varphi$ and $g$ are unramified at $z'$. Since $(X, B + S + f^*z)$ is log canonical around $f^{-1}(z)$, $(S, B_S + f|_S^* z) = (S, B_S + g^*z' + D)$ is log canonical around $g^{-1}(z')$ by adjunction, where $f|_S^* z = g^*z' + D$.

The "moreover" part follows from point (a). qed

**Proposition 4.2.17.** *In Set-up (✧), let $M_X$ and $M_{S^\nu}$ be the moduli parts of $(X/Z, B+S)$ and $(S^\nu/Z', B_{S^\nu})$, respectively. Then,*

$$M_{S^\nu} = M_X|_{S^\nu} - \sum_{D_S} v_{D_S} D_S,$$

*where the sum is on those divisors $D_S$ that lie over points of wild ramification of $\varphi$ and the coefficients $v_{D_S}$ are positive integers.*

*Proof.* For the sake of the proof, we can assume $S = S^\nu$. By Proposition 4.2.16, $(S/Z', B_S)$ satisfies Property $(*)$ and $(B_S)^v = B^v|_S$, therefore, by Proposition 4.2.1,

$$M_X = K_X + B^h - f^* K_Z - R(f) \quad \text{and} \quad M_S = K_S + B^h_S - g^* K_{Z'} - R(g),$$

where $B^h_S$ satisfies $(K_X + B^h)|_S = K_S + B^h_S$. By the Hurwitz formula [Har77, Proposition 2.3, Chapter IV],

$$K_{Z'} = \varphi^* K_Z + \sum_i (m_i - 1) z'_i + \sum_j (a_j + 1) z'_j,$$

where the first sum is taken over all ramified point of $\varphi$ (wild and tame), with $m_i$ being their multiplicity, and the second sum only over the wildly ramified points of $\varphi$, with $a_j$ being a non-negative integer. Now, let us do the computations locally around a vertical prime divisor $D_S \subseteq S$. Let $D \subseteq X$ be a vertical prime divisor such that $D_S \leq D|_S$, $z := f(D)$ and $z' := g(D_S)$. We can suppose $f^* z = \ell D$, $\varphi^* z = m z'$, $g^* z' = n D_S$ and $\ell = mn$. If $\varphi$ is tamely ramified at $z'$, locally we have:

$$\begin{aligned} M_S &= K_S + B^h_S - g^* K_{Z'} - (n-1) D_S \\ &= K_S + B^h_S - f|_S^* K_Z - (m-1) g^* z' - (n-1) D_S \\ &= (K_X + B^h - f^* K_Z - (mn-1) D)|_S = M_X|_S. \end{aligned}$$

If $\varphi$ is wildly ramified at $z'$, locally we have:

$$\begin{aligned} M_S &= K_S + B^h_S - g^* K_{Z'} - (n-1) D_S \\ &= K_S + B^h_S - f|_S^* K_Z - g^*(m+a) z' - (n-1) D_S \\ &= (K_X - f^* K_Z - (mn-1) D)|_S - (an+n) D_S = M_X|_S - (an+n) D_S, \end{aligned}$$

for some $a \geq 0$. Set $v_{D_S} := an + n$ to conclude. qed

*Remark* 4.2.18. In Set-up (✧), let $\Sigma_Z$ be the discriminant part of $(X/Z, B + S)$. Define $\Sigma_{Z'} := \varphi^{-1}(\Sigma_Z)$. If $Z$ is not a curve, then $(Z', \Sigma_{Z'})$ may be even not log canonical, due to the possible presence of wild ramification in $\varphi$. Therefore, we may not be able to describe the moduli part $M_{S^\nu}$ as $K_{S^\nu} + B^h_{S^\nu} - g^* K_{Z'} - R(g')$ and we cannot conclude a formula similar to the one in Proposition 4.2.17 for adjunction of the moduli part.

## 4.3. Bend and Break theorem for the moduli divisor

A crucial step in the proof of positivity of the moduli part in characteristic 0 is the Cone theorem for foliations [ACSS21, Theorem 3.9]. If the canonical bundle of a foliation is not nef, we can find rational curves that are *tangent to the foliation*. The idea is to find them by applying Miyaoka–Mori's Bend and Break theorem [MM86] to the variety reduced modulo a big enough prime (see [SB92]). If our setting is in positive characteristic to start with, we do not have the possibility of changing the prime. However, for fibrations under Property $(*)$ conditions, we can relate the moduli part to the canonical divisor of the variety obtained after a Frobenius base change using Theorem 2.3.27. This is a key ingredient for our strategy and it constitutes one of the main differences between the situation in characteristic 0 and over fields of positive characteristic.

In this section we consider varieties defined over an algebraically closed field, unless otherwise stated. We will specify the characteristic in each result. We use the notation of (❋) in Section 2.3.2.

We will use the following version of the Bend and Break theorem.

**Theorem 4.3.1** (Bend and Break theorem, [Kol96, Theorem II.5.8])**.** *Let $X$ be a normal projective variety of dimension $n$ over an algebraically closed field of any characteristic and let $\xi$ be a smooth curve such that $X$ is smooth around $\xi$. Let $x \in \xi$ be a general point and $G$ a nef divisor on $X$. If $K_X \cdot \xi < 0$, there exists a rational curve $\zeta_x$ such that $x \in \zeta_x$ and*

$$G \cdot \zeta_x \leq 2n \frac{G \cdot \xi}{-K_X \cdot \xi}.$$

The next result is a direct consequence of the Bend and Break theorem; the proof is inspired by [KMM94, Theorem 6.1] and [Spi20, Corollary 2.28].

**Corollary 4.3.2.** *Let $X$ be a normal projective variety of dimension $n$ over an algebraically closed field of any characteristic and let $D_1, ..., D_n, G$ be nef divisors on $X$. Assume that*

*(a)* $D_1 \cdot ... \cdot D_n = 0$*;*

*(b)* $K_X \cdot D_2 \cdot ... \cdot D_n < 0$.

*Then, for $x \in X$ general point, there is a rational curve $\zeta_x \subseteq X$ containing $x$ such that*

*(i)* $G \cdot \zeta_x \leq 2n \frac{G \cdot D_2 \cdot ... \cdot D_n}{-K_X \cdot D_2 \cdot ... \cdot D_n}$*;*

*(ii)* $D_1 \cdot \zeta_x = 0$.

*Proof.* Let $H$ be an ample divisor and $0 \leq \varepsilon \ll 1$ small enough such that, if $H_i := D_i + \varepsilon H$ for $i = 2, ..., n$, we have that $K_X \cdot H_2 \cdot ... \cdot H_n < 0$. Now, choose $m_i \in \mathbb{N}$ such that $m_i H_i$ are very ample for $i = 2, ..., n$ and let $\xi$ be a curve in the intersection of the linear systems $|m_i H_i|$, passing through a general point of $X$ and such that $\xi$

is smooth and contained in the regular locus of $X$. Let $G_{\varepsilon,k} := kD_1 + G + \varepsilon H$. By Theorem 4.3.1, there exists $\zeta_{\varepsilon,k}$ through a general point of $\xi$ such that

$$(\ast) \qquad G_{\varepsilon,k} \cdot \zeta_{\varepsilon,k} \leq 2n \frac{G_{\varepsilon,k} \cdot \xi}{-K_X \cdot \xi}.$$

Since $D_1 \cdot D_2 \cdot ... \cdot D_n = 0$, there exists a constant $c \geq 0$ with the following property. For every $k > 0$, there exists $\varepsilon_k > 0$ such that the RHS of the equation $(\ast)$ is $\leq c$ for every $\varepsilon \leq \varepsilon_k$. Therefore the family $\{\zeta_{\varepsilon_k,k}\}_k$ is bounded. Since bounded integral points in the cone of curves are finitely many, up to passing to a sub-sequence, we can assume $\zeta_{\varepsilon_k,k} = \zeta$ is constant.

Now, letting $k$ approach $+\infty$, we get that $(kD_1 + G + \varepsilon_k H) \cdot \zeta$ is bounded, whence $D_1 \cdot \zeta = 0$. Therefore, $(\ast)$ becomes

$$(G + \varepsilon H) \cdot \zeta \leq 2n \frac{(kD_1 + G + \varepsilon H) \cdot H_2 \cdot ... \cdot H_n}{-K_X \cdot H_2 \cdot ... \cdot H_n}.$$

Finally, letting $\varepsilon$ go to 0, we get the conclusion. qed

The next proposition is one of the key steps for the proof of the main result of this chapter, Theorem 4.4.6. Roughly it says that, when the fibration satisfies Property $(*)$ and the boundary divisor is vertical, if the moduli part is $f$-nef, it is non-negative on curves that are general enough.

**Proposition 4.3.3.** *Assume the LMMP and the existence of log resolutions in dimension $n$. Let $f\colon X \to Z$ be a fibration between normal projective varieties and $(X/Z, B)$ a GGLC pair of dimension $n$ associated with it over an algebraically closed field of characteristic $p > 0$. Assume that $(X/Z, B)$ satisfies Property $(*)$ with $X$ $\mathbb{Q}$-factorial and let $M_X$ be the moduli part of $(X/Z, B)$. Assume there exist $D_2, ..., D_n$ nef $\mathbb{Q}$-divisors, $k \in \mathbb{Q}_{>0}$ and $A$ ample $\mathbb{Q}$-divisor such that:*

*(a)* $(M_X + kA) \cdot D_2 \cdot ... \cdot D_n = 0$*;*

*(b)* $M_X \cdot D_2 \cdot ... \cdot D_n < 0$*;*

*(c)* $(M_X + kA)$ *is nef.*

*Then, through a general point $x \in X$, there exists a vertical rational curve $\zeta$ such that $M_X \cdot \zeta < 0$.*

*Proof.* By Corollary 4.2.2,

$$\alpha_e^* K_{X^{(e)}} \cdot D_2 \cdot ... \cdot D_n = (p^e - 1)(M_X - B^h) \cdot D_2 \cdot ... \cdot D_n + K_X \cdot D_2 \cdot ... \cdot D_n - W_e \cdot D_2 \cdot ... \cdot D_n,$$

where $W_e := \sum_{D \text{ wild}} w_{D,e} D$, for some $w_{D,e} \geq 0$.

We claim that for each effective $\mathbb{Q}$-divisor $E$, $E \cdot D_2 \cdot ... \cdot D_n \geq 0$. Indeed, let $H$ be an ample divisor, $\varepsilon > 0$ and define $H_i := D_i + \varepsilon H$ for $i = 2, ..., n$. Let $\xi_\varepsilon$ be a curve not contained in $\mathrm{Supp}(D)$, taken in the intersection of the linear systems $|m_iH_i|$, where $m_i > 0$ is chosen so that $m_iH_i$ are very ample. Then, $E \cdot \xi_\varepsilon > 0$, and, letting $\varepsilon$ go to 0, we get the claim.

Therefore, $B^h \cdot D_2 \cdot ... \cdot D_n \geq 0$ and $D \cdot D_2 \cdot ... \cdot D_n \geq 0$ for every wild fibre $D$. All in all, for $e \gg 0$, we have $\alpha_e^* K_{X^{(e)}} \cdot D_2 \cdot ... \cdot D_n < 0$.

Let $D_1 := M_X + kA$ and $D_{i,e} := \beta_e^* D_i$ for $i = 1, ..., n$. Note that $\alpha_e^* D_{i,e} = p^e D_i$. Then,

$$\alpha_e^* K_{X^{(e)}} \cdot p^e D_2 \cdot ... \cdot p^e D_n = \deg(\alpha_e) K_{X^{(e)}} \cdot D_{2,e} \cdot ... \cdot D_{n,e} < 0.$$

Therefore, for $e \gg 0$,

(a) $D_{1,e} \cdot D_{2,e} \cdot ... \cdot D_{n,e} = \deg(\beta_e) D_1 \cdot ... \cdot D_n = 0$;

(b) $K_{X^{(e)}} \cdot D_{2,e} \cdot ... \cdot D_{n,e} < 0$.

Let $G_e := \beta_e^* G$, for some ample Cartier divisor $G$ on $X$. We apply Corollary 4.3.2 on $X^{(e)}, D_{1,e}, ..., D_{n,e}, G_e$ to find, through a general point of $X^{(e)}$, a rational curve $\zeta_e$ such that:

(i) $G_e \cdot \zeta_e \leq 2n \frac{G_e \cdot D_{2,e} \cdot ... \cdot D_{n,e}}{-K_{X^{(e)}} \cdot D_{2,e} \cdot ... \cdot D_{n,e}}$;

(ii) $D_{1,e} \cdot \zeta_e = 0$.

Let $\zeta_e^X$ be the image of $\zeta_e$ in $X$. By point (ii) and since $A$ is ample,

$$M_X \cdot \zeta_e^X < 0.$$

To conclude, we show that there exists $e$ such that $\zeta_e^X$ is vertical. Assume, for the sake of a contradiction, that this was not the case. Let $p^{\varphi(e)}$ be the minimum between the purely inseparable degree of $f|_{\zeta_e^X}$ and $p^e$. Remark that, by Lemma 2.3.15, the degree of $\beta_e|_{\zeta_e}$ is $p^{e-\varphi(e)}$. We claim that $\varphi(e) = e - O(1)$, where $O(1) \geq 0$ is a bounded function. Indeed,

$$\begin{aligned} p^e G \cdot p^e D_2 \cdot ... \cdot p^e D_n &= \alpha_e^* G_e \cdot \alpha_e^* D_{2,e} \cdot ... \cdot \alpha_e^* D_{n,e} \\ &= \deg(\alpha_e) G_e \cdot D_{2,e} \cdot ... \cdot D_{n,e}. \end{aligned}$$

Therefore:

$$\begin{aligned} p^{e-\varphi(e)} &\leq p^{e-\varphi(e)} G \cdot \zeta_e^X = G_e \cdot \zeta_e \leq 2n \frac{G_e \cdot D_{2,e} \cdot ... \cdot D_{n,e}}{-K_{X^{(e)}} \cdot D_{2,e} \cdot ... \cdot D_{n,e}} \\ &= 2n \frac{p^e G \cdot D_2 \cdot ... \cdot D_n}{-(p^e - 1)(M_X - B^h) \cdot D_2 \cdot ... \cdot D_n - K_X \cdot D_2 \cdot ... \cdot D_n + W_e \cdot D_2 \cdot ... \cdot D_n} \\ &\leq 2n \frac{p^e G \cdot D_2 \cdot ... \cdot D_n}{-(p^e - 1)(M_X - B^h) \cdot D_2 \cdot ... \cdot D_n - K_X \cdot D_2 \cdot ... \cdot D_n}, \end{aligned}$$

where the equality on the second line is given by Corollary 4.2.2. The last line is bounded as $e$ grows, whence the claim.

From the above chain of inequalities, we also infer that $G \cdot \zeta_e^X \leq p^{e-\varphi(e)} G \cdot \zeta_e^X$ is bounded. Therefore, $\{\zeta_e^X\}_e$ is a bounded family of curves. Since the integral points in the cone of curves that are bounded are finitely many, up to considering a sub-sequence of indices, $\zeta_e^X$ is eventually constant. However, this contradicts the fact that $\varphi(e) = e - O(1)$. qed

*Remark* 4.3.4. The Bend and Break theorem needs the base field to be algebraically closed. If the base field $k$ is only perfect, the rational curves we find may be defined over an extension of $k$. However, if $f\colon X \to Z$ is defined only over a perfect field $k$ and we find a rational curve over an extension of $k$, then its Galois orbit descends to a curve over $k$ which satisfies similar inequalities.

*Remark* 4.3.5. If $f\colon X \to Z$ is a separable equidimensional tame fibration between normal projective varieties, a similar proof shows a "generic" Bend and Break theorem for $K_{\mathcal{F}}$, the canonical divisor of the foliation induced by $f$. However, if $f$ is not tame, the proof does not go through for $K_{\mathcal{F}}$ due to the correction term given by the wild fibres.

## 4.4. The canonical bundle formula

In this section, we prove positivity of the moduli part. First, we prove that, for pairs satisfying Property $(*)$, the moduli part is nef. Then, we use $(*)$-modifications to recover this situation.

### 4.4.1. Property $(*)$ case

In this section we consider varieties defined over a perfect field $k$ of characteristic $p > 0$, unless otherwise stated. We use the notation of (❋) in Section 2.3.2.

Before stating the theorem, we prove that we can assume the field $k$ to be algebraically closed.

**Definition 4.4.1.** Let $X$ be a variety defined over a perfect field $k$ and $\overline{k}$ the algebraic closure of $k$. Define $\overline{X} := X \times_k \overline{k}$. If $W \subseteq \overline{X}$ is a subvariety, let $k \subseteq k'$ be a normal finite extension of $k$ over which $W$ is defined. Let $G := \mathrm{Gal}(\overline{k}/k)$ and $G' := \mathrm{Gal}(k'/k)$. Then $\sum_{g \in G'} g(W)$ descends to a cycle $\mathcal{C}$ on $X$ defined over $k$. Let $\mathcal{C} := \sum a_i C_i$ be the decomposition of $\mathcal{C}$ into irreducible components defined over $k$, then we define $W^G := \sum C_i$. Note that an integral component of $W^G \times_k \overline{k}$ is exactly $W$.

**Lemma 4.4.2.** *Let $f\colon X \to Z$ be a fibration between normal projective varieties over a perfect field $k$ and $D$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on it. Let $\overline{k}$ be the algebraic closure of $k$, $\overline{f}\colon \overline{X} \to \overline{Z}$ the base change of $f$ with $\overline{k}$, and $\overline{D} := D \times_k \overline{k}$. Then, $D$ is $f$-nef if and only if $\overline{D}$ is $\overline{f}$-nef.*

*Proof.* Let $G := \mathrm{Gal}(\overline{k}/k)$ and $\xi \subset \overline{X}$ a curve. Note that $\overline{f}(\xi)$ has dimension 0 if and only if $f(\xi^G)$ has dimension 0. Since $D \cdot \xi^G$ is a positive multiple of $\overline{D} \cdot \xi$, if $D$ is $f$-nef, then $\overline{D}$ is $\overline{f}$-nef as well. The converse is trivial. qed

**Lemma 4.4.3.** *Let $(X, B)$ be a pair over a perfect field $k$ and $(\overline{X}, \overline{B})$ its base change to the algebraic closure $\overline{k}$. Then $(X, B)$ is log canonical if and only if $(\overline{X}, \overline{B})$ is.*

*Proof.* Note that, since $k$ is perfect, $\overline{X}$ is normal by [Sta22, Tag 0C3M]. Moreover, by [Sta22, Tag 01V0], $K_{\overline{X}} = K_X \times_k \overline{k}$.

If $(\overline{X}, \overline{B})$ is log canonical, then $(X, B)$ is log canonical. Indeed, let $Y \to X$ be a birational map over $X$ and $E$ an exceptional divisor. Consider the base change with $\overline{k}$, $\overline{Y} \to \overline{X}$, $\overline{E} \subseteq \overline{Y}$. Since $k$ is perfect, $\overline{E}$ is reduced, thus the discrepancy of $E$ over $X$ coincides with the discrepancy of $\overline{E}$ over $\overline{X}$.

Conversely, assume for the sake of a contradiction that there exists a non-log canonical place $E' \subseteq Y' \xrightarrow{\sigma} \overline{X}$ over $\overline{X}$ with discrepancy $a' < -1$. Without loss of generality, we can assume $E', Y'$ and $\sigma$ are defined over $k'$, a finite Galois extension of $k$. Let $R := \mathcal{O}_{Y', \eta_{E'}}$, where $\eta_{E'}$ is the generic point of $E'$, let $\mathfrak{m}_R := (t)$, where $t \in k'(Y')$ is a local equation for $E'$, and let $v'$ be the valuation induced by $E'$ on $k'(Y')$. Let $t_0 = t, t_1, ..., t_d$ be the Galois conjugates of $t$. Note that, since $k'(Y')$ is a separable extension of $k(X)$, $t_i \neq t_j$ for every $i \neq j$. Then $v := v'|_{k(X)}$ is a valuation on $k(X)$ whose associated DVR is $(S := R \cap k(X), \mathfrak{m}_S := (\prod_{i=1,\dots,d} t_i))$. By [KM98, Lemma 2.45], there exists a proper birational map $\widetilde{X} \to X$ extracting a divisor $\widetilde{E}$ with induced valuation $v$. Let $a$ be its discrepancy and $\widetilde{X}' \to X'$, $\widetilde{E}' \subseteq \widetilde{X}'$ be the base change to $k'$. Then, $\widetilde{E}'$ is locally the zero locus of $\prod_{i=1,\dots,d} t_i$ and it is reduced. Moreover, its discrepancy $\widetilde{a}$ coincides with $a$. Let $E_0$ be the zero locus of $t_0$. Since the local ring around $E_0$ is exactly $(R, \mathfrak{m}_R)$, by [KM98, Remark 2.23], its discrepancy coincides with $a'$. Therefore, $a = \widetilde{a} = a' < -1$, contradiction. qed

**Corollary 4.4.4.** *Assume the LMMP and the existence of log resolutions in dimension $n$. Let $f \colon X \to Z$ be a fibration between normal varieties, where $\dim(X) = n$, and $(X/Z, B)$ a GGLC pair associated with it. Suppose that $(X/Z, B)$ satisfies Property $(*)$. Let $\overline{k}$ be the algebraic closure of $k$, $\overline{f} \colon \overline{X} \to \overline{Z}$ the base change of $f$ with $\overline{k}$, $\overline{B} := B \times_k \overline{k}$. Then $(\overline{X}/\overline{Z}, \overline{B})$ is GGLC, it satisfies Property $(*)$, $B_{\overline{Z}} = B_Z \times_k \overline{k}$ and $M_{\overline{X}} = M_X \times_k \overline{k}$.*

*Proof.* If $Y$ is a variety over $k$, we denote by $\overline{Y} := Y \times_k \overline{k}$. Similarly, if $D$ is a divisor on $Y$, we denote by $\overline{D} := D \times_k \overline{k}$. Let $G := \mathrm{Gal}(\overline{k}/k)$.

By Lemma 4.4.3, given $t \in \mathbb{R}_{\geq 0}$ and $\delta \subseteq Z$ prime divisor, if $(X, B + tf^*\delta)$ is log canonical around $\delta$, then $(\overline{X}, \overline{B} + t\overline{f}^*\overline{\delta})$ is log canonical around $\overline{\delta}$.

Conversely, given $t \in \mathbb{R}_{\geq 0}$, $\delta \subseteq \overline{Z}$ prime divisor, if $(\overline{X}, \overline{B} + t\overline{f}^*\delta)$ is log canonical around $\delta$, then $(X, B + tf^*\delta^G)$ is log canonical around $\delta^G$. Indeed, let $Y \to X$ be a birational map over $X$ and $E$ a place over $f^*\delta^G$. Consider the base change with $\overline{k}$,

$\overline{Y} \to \overline{X}$, $\overline{E} \subseteq \overline{Y}$. Since $k$ is perfect, $\overline{E}$ is reduced, thus the discrepancy of $E$ over $X$ coincides with the discrepancy of $\overline{E}$ over $\overline{X}$.

To conclude the proof, note that it is enough to show that $B_{\overline{Z}} = B_Z \times_k \overline{k}$, which follows from the above discussion. qed

We are now ready to prove that the moduli part is nef for fibrations satisfying Property $(*)$.

**Theorem 4.4.5.** *Assume the LMMP and the existence of log resolutions in dimension up to $n$. Let $f\colon X \to Z$ be a fibration from a normal projective variety $X$ of dimension $n$ onto a normal projective curve $Z$ and $(X/Z, B)$ a GGLC pair associated with it. Suppose that $(X/Z, B)$ satisfies Property $(*)$ and $X$ is $\mathbb{Q}$-factorial. Assume that $K_X + B$ is $f$-nef, then the moduli part $M_X$ is nef.*

*Proof.* The strategy to prove Theorem 4.4.5 is inspired by the proof of [ACSS21, Lemma 3.12]. First of all, if $k$ is not algebraically closed, let $\overline{k}$ be its algebraic closure, $\overline{f}\colon \overline{X} \to \overline{Z}$ the base change of $f$ with $\overline{k}$ and $\overline{B} := B \times_k \overline{k}$. By Corollary 4.4.4 and Lemma 4.4.2, $(\overline{X}/\overline{Z}, \overline{B})$ is GGLC, it satisfies Property $(*)$ and $M_{\overline{X}} = M_X \times_k \overline{k}$ is $\overline{f}$-nef. If we prove nefness of $M_{\overline{X}}$, by Lemma 4.4.2, this implies that $M_X$ is nef as well. Therefore, we can assume $k$ is algebraically closed.

If $M_X$ was not nef, there would exist $\rho$, extremal ray whose support dominates $Z$, such that $M_X \cdot \rho < 0$. Let $A$ be an ample divisor on $X$ such that $H_\rho := M_X + A$ is a supporting hyperplane for $\rho$. In particular, $H_\rho$ is nef.

**Outline of the proof**

Non big case. The idea when $H_\rho$ is not big is that we can find a negative curve that is general enough on which $M_X$ is negative. Then we apply Proposition 4.3.3.

Big case. If $H_\rho$ is big, we produce a geometric log canonical centre containing $\rho$. Thanks to Theorem 4.2.11, we extract this centre over $X^{(e)}$, i.e. after a base change of $f$ with a high enough power of the Frobenius morphism. The aim is to prove nefness of the moduli part by induction on the dimension, thus we perform adjunction using Proposition 4.2.17. Note that the case $\dim(X) = 1$ is trivial.

**Non big case.**

Let $d$ be the numerical dimension of $H_\rho$. Since $H_\rho$ is not big, $d < n$. Define $D_i := H_\rho$ for $2 \le i \le d+1$ and $D_i := A$ for $d+1 < i \le n$. Since $H_\rho^{d+1} \cdot A^{n-d-1} = 0$, $M_X \cdot H_\rho^d \cdot A^{n-d-1} = (H_\rho - A) \cdot H_\rho^d \cdot A^{n-d-1} < 0$. Then,

(a) $(M_X + A) \cdot D_2 \cdot \ldots \cdot D_n = 0$;

(b) $M_X \cdot D_2 \cdot \ldots \cdot D_n < 0$;

(c) $M_X + A$ is nef.

Therefore, by Proposition 4.3.3, there exists a vertical curve $\zeta$ such that $M_X \cdot \zeta < 0$, contradiction.

**Big case.**

**Step 1.** In this step, we produce a log canonical centre on the geometric generic fibre of $f$, containing a curve that is negative on the moduli part. More precisely, we find a horizontal curve $\xi \subseteq X$ and an effective $\mathbb{Q}$-divisor $\Gamma$ on $X$ such that:

(a) $M_X \cdot \xi < 0$;

(b) $\Gamma \cdot \xi \leq 0$;

(c) $\xi|_{X_{\bar{\eta}}}$ is a log canonical centre of $(X_{\bar{\eta}}, B_{\bar{\eta}} + \Gamma_{\bar{\eta}})$;

(d) all non-log canonical centres of $(X_{\bar{\eta}}, B_{\bar{\eta}} + \Gamma_{\bar{\eta}})$ are disjoint from $\xi|_{X_{\bar{\eta}}}$.

By [CMM14, Theorem 1.1], if $\varepsilon > 0$ is small enough and $m \gg 0$ is divisible enough, then $\mathrm{Bs}(m(H_\rho - \varepsilon A))$ is supported on those subvarieties $W \subseteq X$ such that $H_\rho|_W$ is not big. Let $\sum_{i=1}^r a_i \zeta_i$ be a representative of the ray $\rho$, where $a_i > 0$ and $\zeta_i$ are irreducible curves on $X$. Since $H_\rho$ is nef, $H_\rho \cdot \zeta_i = 0$, therefore $\mathrm{Bs}(m(H_\rho - \varepsilon A)) = \bigcup_{i=1}^r \zeta_i$. Consider the restriction maps

$$r_m \colon H^0(X, m(H_\rho - \varepsilon A)) \to H^0(X_{\bar{\eta}}, m(H_\rho - \varepsilon A)|_{X_{\bar{\eta}}}).$$

Define the sub-linear system $|V_\bullet| := (|m(H_\rho - \varepsilon A)|_{\bar{\eta}})_{m \in \mathbb{N}}$ on $X_{\bar{\eta}}$, then, for some $\overline{m} > 0$ sufficiently divisible, $\mathrm{Bs}(V_{\overline{m}}) = \bigcup_{i=1}^r \zeta_i|_{X_{\bar{\eta}}}$. Let $V_{\overline{m}}(k)$ be the image of $r_{\overline{m}}$ and let $\{s_1, ..., s_\ell\}$ be a choice of basis of $V_{\overline{m}}(k)$, considered as a $k$-vector space. Since $M_X$ is $f$-nef and $H_\rho - \varepsilon A \sim_{\mathbb{Q}} M_X + (1-\varepsilon)A$, there exists a horizontal curve $\xi \subseteq \bigcup_{i=1}^r \zeta_i$ such that $(H_\rho - \varepsilon A) \cdot \xi < 0$ and $M_X \cdot \xi < 0$. Let $\rho_{\bar{\eta}}$ be the ray corresponding to $\rho$ in $X_{\bar{\eta}}$ and $\xi_{\bar{\eta}} := \xi|_{X_{\bar{\eta}}}$. By Corollary 1.2.10, there exists an effective $\mathbb{Q}$-divisor $\overline{\Gamma} \in |V_{\overline{m}m}|$ such that $(X_{\bar{\eta}}, B_{\bar{\eta}} + \frac{1}{m}\overline{\Gamma})$ is log canonical outside $\bigcup_{i=1}^r \zeta_i|_{X_{\bar{\eta}}}$ and has a non-klt centre at $\xi_{\bar{\eta}}$. Moreover, $\overline{\Gamma}$ has a decomposition as $\sum_{i=1}^m D_i$ with $D_i \in |V_{\overline{m}}(k)|$; in particular, $\overline{\Gamma}$ belongs to the image of $r_{\overline{m}m}$. Assume there is $0 < \lambda \leq \frac{1}{m}$ such that $(X_{\bar{\eta}}, B_{\bar{\eta}} + \lambda\overline{\Gamma})$ has a log canonical centre at $\xi_{\bar{\eta}}$ and is log canonical outside $\bigcup_{i=1}^r \zeta_i|_{X_{\bar{\eta}}}$. Let $\Gamma' \in |\overline{m}m(H_\rho - \varepsilon A)|$ be a lift of $\overline{\Gamma}$ to $X$ and let $\Gamma := \lambda \Gamma'^h$, we claim that it satisfies the required properties. Indeed, points (a), (c) and (d) follow by construction and point (b) because $\Gamma \cdot \xi = \lambda \overline{m}m(H_\rho - \varepsilon A) \cdot \xi - \lambda \Gamma'^v \cdot \xi < 0$.

If such $\lambda$ does not exist, it means that $(X_{\bar{\eta}}, B_{\bar{\eta}})$ has already a log canonical centre at $\xi_{\bar{\eta}}$ and we define $\Gamma := 0$.

**Step 2.** In this step, we perform a Frobenius base change on the base of the fibration in order to find a curve $\xi_e \subseteq X^{(e)}$, that is a log canonical centre of $(X^{(e)}, B_e + \Gamma_e)$, where $B_e := \beta_e^* B^h + \beta_e^{-1} B^v$ and $\Gamma_e := \beta_e^* \Gamma$.

Let $\Delta := B^h + \Gamma$ and $\Delta_e := \beta_e^* \Delta$. By choosing $e \gg 0$, by Corollary 2.3.23, we can suppose $f_e$ is tame. Now, we want to compare the moduli part $M_X$ of $(X/Z, B)$

to $K_{X^{(e)}}$. By Theorem 2.3.27, we have:

$$\alpha_e^* K_{X^{(e)}} = (p^e - 1)(K_X - f^*K_Z - R(f)) + K_X - \sum_{D \text{ wild}} w_{D,e} D,$$

where $w_{D,e} \geq 0$ for every $D$ wild fibre. Since $f_e$ is tame, by Theorem 2.3.6:

$$\begin{aligned} &\alpha_e^*(K_{\mathcal{F}_e} + \Delta_e) = \alpha_e^*(K_{X^{(e)}} - f_e^* K_Z - R(f_e) + \Delta_e) \\ &= (p^e - 1)(K_X - f^*K_Z - R(f)) + K_X - \sum_{D \text{ wild}} w_{D,e} D - f^*K_Z - \alpha_e^* R(f_e) + p^e \Delta. \end{aligned}$$

Now, if $D$ is a vertical prime divisor in $X$, let $\ell_D = n_D p^{e_D}$ be its multiplicity with respect to $f$, where $e_D \geq 0$ and $n_D$ is coprime with $p$. By Lemma 2.3.25:

$$\alpha_e^* R(f_e) = R(f) - \sum_{D \text{ wild}} (p^{e_D} - 1) D.$$

Bringing everything together:

$$\begin{aligned} &\alpha_e^*(K_{\mathcal{F}_e} + \Delta_e) = \\ &(p^e - 1)(K_X - f^*K_Z - R(f)) + K_X - f^*K_Z - R(f) + p^e \Delta \\ &- \sum_{D \text{ wild}} w_{D,e} D + \sum_{D \text{ wild}} (p^{e_D} - 1) D = \\ &p^e(M_X + \Gamma) - \sum_{D \text{ wild}} w_{D,e} D + \sum_{D \text{ wild}} (p^{e_D} - 1) D. \end{aligned}$$

Let $\xi_e$ be a curve corresponding to $\xi$ in $X^{(e)}$. Note that $\sum_{D \text{ wild}} (p^{e_D} - 1) D \cdot \xi$ is independent of $e$ for $e \gg 0$. Therefore, up to choosing a bigger $e \geq 0$:

$$(K_{\mathcal{F}_e} + \Delta_e) \cdot \xi_e < 0.$$

**Step 3**. In this step, we construct a geometric $(*)$-modification that extracts a log canonical place over $\xi$ and we compute its moduli part.

Up to possibly choosing an even bigger $e \geq 0$, by Theorem 4.2.11, there exists a dlt GGLC pair $(Y/Z, C + E)$ with $Y$ $\mathbb{Q}$-factorial, and a diagram:

$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[dr, "g"'] & X^{(e)} \arrow[d, "f_e"] \\ & Z, \end{tikzcd}$$

where $\mu$ is a birational map and the centre of $E$ is $\xi_e$. The induced fibration $E \to Z$ is separable. Moreover, there exist an effective exceptional $\mathbb{Q}$-divisor $R$, whose image in $X^{(e)}$ is supported on the non-log canonical locus of $(X^{(e)}, \Delta_e + B_e^v)$, and a vertical effective $\mathbb{Q}$-divisor $G$, such that

$$K_Y + C + E + R = \mu^*(K_{X^{(e)}} + \Delta_e + B_e^v) + G,$$

where $K_Y + C + E$ is $\mu$-nef.

Now, we want to compare the moduli part $M_X$ of $(X/Z, B)$ to the moduli part $M_Y$ of $(Y/Z, C + E)$. Recall that $f_e$ is tame and all the vertical divisors with multiplicity $> 1$ lie over the discriminant part $B_Z$ of $(X/Z, B)$, which is reduced by Property $(*)$. Thus, by Theorem 2.3.6:

$$K_{\mathcal{F}_e} + \Delta_e = K_{X^{(e)}} + \Delta_e + B_e^v - f_e^*(K_Z + B_Z).$$

Therefore:

$$\begin{aligned}
&\mu^*(K_{\mathcal{F}_e} + \Delta_e) = \\
&K_Y + C + E - g^*(K_Z + C_Z) + R + g^*(C_Z - B_Z) - G = \\
&M_Y + R + g^*(C_Z - B_Z) - G,
\end{aligned}$$

where $C_Z$ is the discriminant part of $(Y/Z, C + E)$.

Now, we study more closely the $\mathbb{Q}$-divisor $g^*(C_Z - B_Z) - G$. Let $z \in Z$ and $b_z$ and $c_z$ be its coefficients in $B_Z$ and $C_Z$, respectively. If $b_z = 1$, then the fibre over $z$ is contained in $B^v$, so $c_z = 1$ as well by construction. If $b_z = 0$, it may happen that $c_z = 1$. Let $D$ be a non $\mu$-exceptional vertical prime divisor contained in $g^{-1}(z)$. By abuse of notation, call $D$ also $\beta_e(\mu(D))$. Then, by construction, the coefficient of $D$ in $G$ is 1 and, since $(X/Z, B)$ has Property $(*)$ and $b_z = 0$, the coefficient of $D$ in $f^*(z)$ must be 1, hence its coefficient in $g^*(z)$ is 1 as well ($\mu$ is an isomorphism at the generic point of $D$). All in all, we get that $g^*(C_Z - B_Z) - G$ is $\mu$-exceptional. Then, applying the Negativity lemma ([KM98, Lemma 3.39]), we conclude that $R + g^*(C_Z - B_Z) - G$ is effective. By abuse of notation, let $\xi \subseteq \operatorname{Supp}(E)$ be a curve mapping to $\xi_e$. Then, since $\xi$ is horizontal, $(R^v + g^*(C_Z - B_Z) - G) \cdot \xi \geq 0$. Moreover, $R^h \cdot \xi \geq 0$ by point (d) in Step 1. Thus,

$$M_Y \cdot \xi < 0.$$

**Step 4.** In this step we do adjunction on $E$ and we conclude by induction on the dimension.

Since $(Y, C+E)$ is $\mathbb{Q}$-factorial and dlt, the normalisation morphism $\nu\colon E^\nu \to E$ is an isomorphism in codimension 1 by Lemma 4.2.12. Define $C_{E^\nu}$ on $E^\nu$ by adjunction, so that we have $(K_Y + C + E)|_{E^\nu} = K_{E^\nu} + C_{E^\nu}$, and let $E^\nu \xrightarrow{g_E} Z' \xrightarrow{\varphi} Z$ be the Stein factorisation of $g|_{E^\nu}$. By Proposition 4.2.16, $(E^\nu/Z', C_{E^\nu})$ is GGLC and satisfies Property $(*)$. Moreover, by Proposition 4.2.17, there exists a vertical effective divisor $V$ on $E^\nu$ such that $M_Y|_{E^\nu} - V = M_{E^\nu}$, the moduli part of $(E^\nu/Z', C_{E^\nu})$. By abuse of notation, let $\xi$ be a curve in $E^\nu$ mapping onto $\xi$. Then, $M_{E^\nu} \cdot \xi < 0$.

If $\dim(X) = 2$, $E^\nu \to Z'$ is the identity, whence $M_{E^\nu} = 0$, giving a contradiction. Assume then that $\dim(X) \geq 3$. Note that, if $\zeta \subseteq \mathrm{Supp}(E^\nu)$ is vertical over $Z'$, $\nu(\mu(\zeta))$ is a point because $\mu(E)$ is supported on a horizontal curve. Therefore, $(K_Y + C + E)|_{E^\nu} \cdot \zeta \geq 0$. If $E^\nu$ is not $\mathbb{Q}$-factorial, substitute it with a $(*)$-modification $(E'/Z', C')$ constructed with Theorem 4.2.3. Let $\mu' \colon E' \to E^\nu$ be the resulting map, $g' \colon E' \to Z'$ the induced fibration, and let $C_{E^\nu,Z'}, C'_{Z'}$ be the discriminant parts of $(E^\nu/Z', C_{E^\nu})$ and $(E'/Z', C')$, respectively. By construction, there is a vertical effective $\mathbb{Q}$-divisor $D'$ such that $K_{E'} + C' = \mu'^*(K_{E^\nu} + C_{E^\nu}) + D'$ and $K_{E'} + C'$ is $\mu'$-nef. Note also that $K_{E'} + C'$ is $g'$-nef over $U := Z' \setminus g'(\mathrm{Supp}(D'))$. With computations similar to the ones in Step 3, we find that

$$\mu'^* M_{E^\nu} = M_{E'} + g'^*(C'_{Z'} - C_{E^\nu,Z'}) - D',$$

where $g'^*(C'_{Z'} - C_{E^\nu,Z'}) - D'$ is effective and $M_{E'}$ is the moduli part of $(E'/Z', C')$. Let $\xi'$ be a curve mapping to $\xi$, then $M_{E'} \cdot \xi' < 0$.

If $M_{E'}$ is not $g'$-nef, run a $(K_{E'} + C_{E'})$-MMP over $Z'$ and let $(E''/Z', C'')$ be the resulting pair with associated fibration $g'' \colon E'' \to Z'$. Since $K_{E'} + C'$ is $g'$-nef over $U$, $E'$ and $E''$ are isomorphic on $g'^{-1}(U)$. In particular, $\xi'$ is not contracted and the MMP does not terminate with a Mori fibre space. Let $\xi''$ be the image of $\xi'$ in $E''$. Let $p_1 \colon \widetilde{E} \to E'$ and $p_2 \colon \widetilde{E} \to E''$ be the projections from a resolution of $E' \dashrightarrow E''$. Then, $p_2^*(K_{E''} + C'') = p_1^*(K_{E'} + C') - D''$, where $D''$ is an effective $p_2$-exceptional $\mathbb{Q}$-divisor. We can choose $p_2$ to be an isomorphism on $g'^{-1}(U)$, therefore $D''$ is vertical over $Z'$. By Proposition 3.3.3, $(E''/Z', C'')$ satisfies Property $(*)$ and its discriminant part coincides with the discriminant part of $(E'/Z', C')$, whence:

$$M_{E''} \cdot \xi'' = M_{E'} \cdot \xi' - D'' \cdot \widetilde{\xi} < 0,$$

where $M_{E''}$ is the moduli part of $(E''/Z', C'')$ and $\widetilde{\xi}$ is a lift of $\xi'$ to $\widetilde{E}$.

To sum up, $(E''/Z', C'')$ satisfies Property $(*)$, $E''$ is $\mathbb{Q}$-factorial, $M_{E''}$ is $g''$-nef and there is a horizontal curve $\xi'' \subseteq E''$ such that $M_{E''} \cdot \xi'' < 0$. By the inductive assumption $M_{E''}$ is nef, contradiction. qed

### 4.4.2. General case

In the general case, we may need to go to a higher model of $X$ to achieve positivity of the moduli part.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Theorem 4.4.6.** *Assume the LMMP and the existence of log resolutions in dimension up to $n$. Let $f \colon X \to Z$ be a fibration from a normal projective variety $X$ of* *dimension $n$ onto a normal projective curve $Z$ and $(X/Z, B)$ a GGLC pair associated with it such that $B \geq 0$ and $(X, B)$ is log canonical. Suppose that $K_X + B$ is $f$-nef. Then, there exist a projective pair $(Y, C)$ satisfying Property $(*)$, with $C \geq 0$, and a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, dashed, "b"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Z, \end{tikzcd}$$

*with $b$ birational and such that:*

*(i) the pairs $(X/Z, B)$ and $(Y/Z, C)$ are crepant over the generic point of $Z$ (as in Definition 3.3.9);*

*(ii) the moduli part $M_Y$ of $(Y/Z, C)$ is nef.*

*Proof.* By Theorem 4.2.3, there exists $(X', B')$ a $\mathbb{Q}$-factorial dlt pair satisfying Property $(*)$ with $B' \geq 0$ and a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\mu"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & Z, \end{tikzcd}$$

where $\mu$ is projective birational and $K_{X'} + B' = \mu^*(K_X + B) + G$ with $G$ a vertical effective $\mathbb{Q}$-divisor. The two pairs are crepant over $\eta$, the generic point of $Z$, thus $(X'_{\overline{\eta}}, B'_{\overline{\eta}})$ is log canonical. The divisor $K_{X'} + B'$ may be not $f'$-nef anymore. In this case, we run a $(K_{X'} + B')$-MMP over $Z$. Let $(Y, C)$ be the resulting pair, $b\colon Y \dashrightarrow X$ the induced birational map and $g\colon Y \to Z$ the induced fibration. Note that the MMP only contracts (or flips) curves inside the support of $G$ and it is an isomorphism elsewhere, so the MMP cannot end with a Mori fibre space and $(K_{X'} + B')|_{X'_\eta} = (K_Y + C)|_{X'_\eta}$, whence $(Y/Z, C)$ and $(X/Z, B)$ are crepant over the generic point of $Z$. Moreover, $(Y/Z, C)$ satisfies Property $(*)$ by Proposition 3.3.3 and $K_Y + C$ is $g$-nef. Thus, we conclude by applying Theorem 4.4.5. qed

### 4.4.3. The $K$-trivial case

As a corollary of the previous results we get the canonical bundle formula in the classical setting, i.e. when the fibration is $K$-trivial. The proof is very similar to the one in the characteristic 0 case ([ACSS21, Theorem 1.3]).

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Theorem 4.4.7.** *Assume the LMMP and the existence of log resolutions in dimension up to $n$. Let $f\colon X \to Z$ be a fibration from a normal projective variety $X$ of dimension $n$ onto a normal projective curve $Z$ and $(X/Z, B)$ a GGLC pair associated* *with it such that $B \geq 0$ and $(X, B)$ is log canonical. Assume that $K_X + B \sim_{\mathbb{Q}} f^*L_Z$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L_Z$ on $Z$. Then, $M_X = f^*M_Z$ is nef.*

*Proof.* First, let $\mu\colon X' \to X$ be a $(*)$-modification as in Theorem 4.2.3. Let $f'\colon X' \to Z$ be the induced morphism and $B' \geq 0$ defined so that $K_{X'} + B' = \mu^*(K_X + B)$. Let $B'_Z$ and $M'$ be respectively the discriminant and the moduli parts of $(X'/Z, B')$. Then $M' = \mu^*M_X \sim_{\mathbb{Q}} f'^*L'_Z$, for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L'_Z$ on $Z$, so it is enough to show that $M'$ is nef. There exists an effective vertical $\mathbb{Q}$-divisor $G$ such that, if $B^* := B' + G$, $(X'/Z, B^*)$ satisfies Property $(*)$ and $B^* \geq 0$. Let $\Sigma_Z$ be the discriminant part of $(X'/Z, B^*)$ and $M^*$ its moduli part. Let $\gamma^*_z$ be the log canonical threshold of $f'^*(z)$ with respect to $(X'/Z, B^*)$ and $\gamma'_z$ the one with respect to $(X'/Z, B')$, for $z \in Z$. Note that $\gamma^*_z \leq \gamma'_z$. In particular, $\Sigma_Z \geq B'_Z$.

Now, define $B'' := B' + f'^*(\Sigma_Z - B'_Z)$, so that $K_{X'} + B'' \sim_{\mathbb{Q},Z} 0$, the discriminant part of $(X'/Z, B'')$ is $\Sigma_Z$ and its moduli part is $M'$. Note that $B'' \leq B^*$ and the difference is vertical.

It is possible that $K_{X'} + B^*$ is not $f'$-nef, in which case we perform a $(K_{X'} + B^*)$-MMP over $Z$. Let $\varphi\colon X' \dashrightarrow Y$ be the result of this MMP and $(Y, C)$ the resulting pair, where $C := \varphi_*B^*$. Note that this MMP cannot end with a Mori fibre space since $K_{X'} + B'$ is $f'$-nef. Call $g\colon Y \to Z$ the induced fibration and $M_Y$ the moduli part of $(Y/Z, C)$. Since $(X, B^*)$ is log canonical, so is $(Y, C)$ and, by Proposition 3.3.3, $(Y/Z, C)$ satisfies Property $(*)$ and its discriminant part is $\Sigma_Z$. Let $C'' := \varphi_*B''$. Since $K_{X'} + B'' \sim_{\mathbb{Q},Z} 0$, the divisor $(K_Y + C) - (K_Y + C'')$ is $g$-nef. Moreover, by construction, it is effective and supported on a vertical $\mathbb{Q}$-divisor $\Phi$ which does not contain any fibre. Let $\Psi$ be an effective vertical $\mathbb{Q}$-divisor such that $\Phi + \Psi \sim_{\mathbb{Q},Z} 0$. Then, $(K_Y + C) - (K_Y + C'') \sim_{\mathbb{Q},Z} -\Psi$ is $g$-nef, whence $\Psi = 0 = \Phi$ and $C = C''$. The pair $(Y/Z, C)$ satisfies Property $(*)$ and $M_Y$ is $g$-nef by construction. Therefore, $M_Y$ is nef by Theorem 4.4.5.

Now, we want to compare $M_Y$ and $M'$. Let $\widetilde{X}$ be a common resolution of $X' \dashrightarrow Y$ with $p\colon \widetilde{X} \to X'$, $q\colon \widetilde{X} \to Y$ the induced projections. We have that both $p^*M' - q^*M_Y$ and $q^*M_Y - p^*M'$ are $q$-nef since $M'$ is $f'$-trivial. Moreover, $q_*(p^*M' - q^*M_Y) = \varphi_*B'' - C = 0 = q_*(q^*M_Y - p^*M')$. Therefore, by the Negativity lemma ([KM98, Lemma 3.39]), $p^*M' = q^*M_Y$, and it is nef, whence the conclusion. qed

### 4.4.4. The canonical bundle formula in dimension 3

For threefolds over perfect fields of characteristic $p > 5$, the LMMP and existence of log resolutions are known to hold, so our results hold unconditionally.

**Corollary 4.4.8.** *Let $f\colon X \to Z$ be a fibration from a normal projective variety $X$ of dimension 3 onto a normal projective curve $Z$ over a perfect field of characteristic $p > 5$, and $(X/Z, B)$ a GGLC pair associated with it such that $B \geq 0$ and $(X, B)$ is log canonical. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is a log* *canonical pair. Suppose that $K_X + B$ is $f$-nef. Then, there exist a pair $(Y, C)$ satisfying Property $(*)$, with $C \geq 0$, and a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, dashed, "b"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Z, \end{tikzcd}$$

*with $b$ birational and*

*(i) $(X/Z, B)$ and $(Y/Z, C)$ are crepant over the generic point of $Z$;*

*(ii) the moduli part $M_Y$ of $(Y/Z, C)$ is nef.*

*Moreover, if $K_X + B \sim_{\mathbb{Q}} f^*L_Z$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L_Z$ on $Z$, $M_X = f^*M_Z$ is nef.*

*Proof.* By Remark 4.0.1, we can conclude that Theorem 4.4.5, Theorem 4.4.6 and Theorem 4.4.7 hold unconditionally for threefolds over a perfect field of characteristic $p > 5$. qed

# Chapter 5

# Iitaka conjectures

One of the most important invariants in the birational classification of varieties is the Kodaira dimension $\kappa(X, K_X)$; it measures the growth of pluricanonical sections. A fundamental problem is to relate Kodaira dimensions in fibrations: the Iitaka conjecture. In this chapter, we recall the definition of Iitaka dimension, alongside with some of its properties. Then, we present a variant of the Iitaka conjecture for the anticanonical Iitaka dimension.

## 5.1. Iitaka dimension

In this section we consider varieties defined over any field.

**Definition 5.1.1** ([Laz04a, 2.1.A]). Let $X$ be a normal projective variety and $L$ a $\mathbb{Q}$-divisor on it. For every $m > 0$ such that $mL$ is integral and the linear system $|mL|$ is not empty, $|mL|$ defines a rational map $\phi_{|mL|}\colon X \dashrightarrow \mathbb{P}^{N_m} \simeq \mathbb{P}(H^0(X, mL)^*)$. For $m \gg 0$ and sufficiently divisible, $\dim(\phi_{|mL|}(X))$ stabilises. The **Iitaka dimension of** $L$ is defined as:

$$\kappa(X, L) := \begin{cases} -\infty & \text{if } |mL| = \emptyset \text{ for all } m \geq 0; \\ \max_{m\geq 1} \dim(\phi_{|mL|}(X)) & \text{otherwise.} \end{cases}$$

*Remark* 5.1.2. Sometimes it is useful to work with different characterisations of the Iitaka dimension. We recall a couple of them here. Fix $X$ a normal projective variety over a field $k$ and $L$ a $\mathbb{Q}$-divisor on it.

Define the **section ring of** $L$ as $R(X, L) := \bigoplus_{m=0}^{\infty} H^0(X, mL)$, if $R(X, L) \neq 0$, it is an integral domain. If $|L|_{\mathbb{Q}} \neq 0$, then

$$\kappa(X, L) = \mathrm{tr.deg}_k R(X, L) - 1,$$

the transcendence degree of $R(X, L)$ over $k$. For more details, we refer to [Mor87, 1.3].

When $|L|_{\mathbb{Q}} \neq \emptyset$ the Iitaka dimension can be characterised also as

$$\kappa(X, L) = \sup \left\{ d \geq 0 \text{ s.t. } \limsup_{m \to \infty} \frac{\dim(H^0(X, mL))}{m^d} \neq 0 \right\}.$$

For more details, we refer to [Laz04a, Chapter 2].

*Remark* 5.1.3. Let $X$ be a normal projective variety over a field $k$, $L$ a $\mathbb{Q}$-divisor, and $k' \supseteq k$ a field extension. Denote by $X'$ and $L'$ the base change of $X$ and $L$, and suppose that $X'$ is normal. Then $\kappa(X, L) = \kappa(X', L')$ by flat base change. Note that, if $k$ is perfect, $X$ is geometrically normal by [Sta22, Tag 038N], therefore $X'$ is always normal.

**Theorem 5.1.4** ([Laz04a, Theorem 2.1.3.3]). *Let $X$ be a normal projective variety and let $L$ be a Cartier $\mathbb{Q}$-divisor such that $|L|_{\mathbb{Q}} \neq \emptyset$. Then there exists a fibration of normal projective varieties $\phi_\infty \colon X_\infty \to Z_\infty$ fitting in the following commutative diagram for all sufficiently divisible $m$:*

$$\begin{tikzcd} X_\infty \arrow[r, "u"] \arrow[d, "\phi_\infty"'] & X \arrow[d, dashed, "\phi_{|mL|}"] \\ Z_\infty \arrow[r, dashed, "v"] & Z_m, \end{tikzcd}$$

*where the maps $u, v$ are birational, $Z_m$ is the closure of the image of $\phi_{|mL|}$, and, for $z \in Z_\infty$ very general, $\kappa(X_{\infty,z}, (u^*L)|_{X_{\infty,z}}) = 0$. In particular, $\dim(Z_\infty) = \kappa(X, L)$.*

*Remark* 5.1.5. Let $X$ be a normal projective variety over a field $k$, and let $L$ be a semiample Cartier divisor; i.e. $\phi_{|mL|}$ is a morphism for some $m \geq 1$.

Moreover, suppose that $|V| \subseteq |mL|$ is a base point free linear subsystem so that, in particular, $L$ is semiample. Consider the Stein factorisation

$$\phi_{|V|} \colon X \xrightarrow{g} Y \xrightarrow{h} Z.$$

Then $g = \phi_{|mL|}$ for all $m \geq 1$ sufficiently divisible. In other words, the Stein factorization of $\phi_{|V|}$ recovers the Iitaka fibration of $L$. Indeed, if this was not the case, let $h_1 \circ h_2$ be the Stein factorisation of $h$. Since $V$ is base point free, there are ample divisors $H$ and $H'$ such that $H^0(X, mL) = H^0(X, g^*H)$ and $V \supseteq H^0(X, (h_2 \circ g)^*H')$. In particular, the inclusion map $V \subseteq H^0(X, mL)$ is given by multiplication by an element in $H^0(X, (h_2 \circ g)^*H' - g^*H)$. Since $V$ is base point free, this element is a constant and $h_2$ is an isomorphism. We will make use of this fact repeatedly. Such $\phi_{|mL|}$ will also be called the **semiample contraction of** $L$.

**Lemma 5.1.6.** *Let $\varphi \colon X' \to X$ be a surjective morphism between normal projective varieties and let $L$ be a Cartier divisor on $X$. Then $\kappa(X', \varphi^*L) = \kappa(X, L)$. Moreover, if $\varphi$ is equidimensional and $L$ is Weil, then the equality still holds.*

*Proof.* Since $\varphi$ is surjective we have $\kappa(X, L) \leq \kappa(X', \varphi^* L)$. If $\varphi$ is a fibration then the result follows by the projection formula. By considering the Stein factorisation of $\varphi$ we can thus reduce to the case where $\varphi$ is finite.

If $\varphi$ is purely inseparable, there exists $\psi \colon X \to X'$ such that $\varphi \circ \psi = F^e$, for some $e \geq 0$. Then:

$$\kappa(X, L) \leq \kappa(X', \varphi^* L) \leq \kappa(X, \psi^* \phi^* L) = \kappa(X, L).$$

If $\varphi$ is a Galois cover, the result is proven in [Mor87, Proposition 1.5][1]. If $\varphi$ is separable, there exists $\psi \colon X'' \to X'$ such that $\varphi \circ \psi$ is Galois. Thus:

$$\kappa(X, L) \leq \kappa(X', \varphi^* L) \leq \kappa(X'', \psi^* \phi^* L) = \kappa(X, L).$$

As for the general case, we can decompose $\phi = \psi_1 \circ \psi_2$, where $\psi_1 \colon X'' \to X$ is separable and $\psi_2 \colon X' \to X''$ is purely inseparable. Then, by the discussion above applied to $\psi_1$ and $\psi_2$, we have:

$$\kappa(X, L) = \kappa(X'', \psi_1^* L) = \kappa(X', \psi_2^* \psi_1^* L).$$

The "moreover" part follows from the same argument, after observing that $f^* L$ is well-defined by Remark 2.0.3, and replacing the projection formula by Lemma 2.0.6.

qed

## 5.2. Easy Additivity theorems

Let $f \colon X \to Z$ be a fibration between normal projective varieties. Some inequalities between the Kodaira dimensions of $X$, $Z$ and the general fibre $X_z$ have been known for a long time in any characteristic. They are the so-called "Easy Additivity" theorems.

In this section we consider varieties defined over a perfect field of any characteristic. However, by Remark 5.1.3, for the proofs we can assume the field of definition is algebraically closed.

**Theorem 5.2.1** (Easy Additivity, [Uen75, Theorem 6.12], [Fuj20, Lemma 2.3.31])**.** *Let $f \colon X \to Z$ be a separable fibration between normal projective varieties. Let $L$ be a $\mathbb{Q}$-divisor on $X$, then:*

$$\kappa(X, L) \leq \kappa(X_z^\nu, L|_{X_z^\nu}) + \dim(Z),$$

*where $X_z$ is a general fibre of $f$ and $X_z^\nu$ is its normalisation.*

[1] In *loc.cit.* this is proven over fields of characteristic 0, but the same proof works in positive characteristic.

*Proof.* We follow the proof in [Fuj20, Lemma 2.3.31]. By possibly substituting $L$ with $mL$ for $m \gg 0$, we may suppose that $L$ is integral. The inequality is obvious if $\kappa(X,L) = -\infty$. If $\kappa(X,L) = 0$, then $\kappa(X_z^\nu, L|_{X_z^\nu}) \geq 0$, proving the inequality.

If $\kappa(X,L) > 0$, consider the rational map $\phi_{|mL|} \colon X \dashrightarrow \mathbb{P}^N = \mathbb{P}(H^0(X,mL)^*)$ induced by the linear system $|mL|$, for some $m \gg 0$ computing $\kappa(X,L)$. Define a morphism $\varphi := \phi_{|mL|} \times f$, so that we have the following diagram:

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] \arrow[d, "f"'] & \mathbb{P}^N \times Z \arrow[r, "p_1"] \arrow[dl, "p_2"] & \mathbb{P}^N \\ Z, & & \end{tikzcd}$$

where $p_1$ and $p_2$ are the projections onto the two factors. Let $Y$ be the image of $\varphi$ and note that, for a general point $z$ of $Z$, $Y_z := Y \cap p_2^{-1}(z) = \varphi(f^{-1}(z))$. Hence, we have:

$$\kappa(X,L) = \dim(p_1(Y)) \leq \dim(Y) = \dim(Y_z) + \dim(Z) \leq \kappa(X_z^\nu, L|_{X_z^\nu}) + \dim(Z).$$

The last inequality follows from $H^0(X_z^\nu, mL|_{X_z^\nu}) \supseteq H^0(X,mL)|_{X_z^\nu}$. qed

**Theorem 5.2.2** (Easy Additivity 2, [Fuj77, Proposition 1])**.** *Let $f \colon X \to Z$ be a fibration between normal projective varieties. Assume the general fibre $X_z$ is normal. Let $L$ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ and $H$ a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Z$. Then,*

$$\kappa(X, L + f^*H) \geq \kappa(X_z, L|_{X_z}) + \dim(Z).$$

*Remark* 5.2.3. With similar arguments, it is possible to prove the inequalities in Theorem 5.2.1 and Theorem 5.2.2 for $X_\eta$, the *generic* fibre of $f$, instead of the general one. For the proof, see [Fuj20, Remark 2.3.32] and [BCZ18, Lemma 2.20].

**Lemma 5.2.4.** *Let $f \colon X \to Z$ be a fibration between normal projective varieties. Assume that a very general fibre $X_z$ is reduced and normal. Let $\eta$ be the generic point of $Z$ and $L$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Then,*

$$\kappa(X_{\bar{\eta}}, L_{\bar{\eta}}) = \kappa(X_\eta, L_\eta) = \kappa(X_z, L_z).$$

*Moreover, if $\kappa(X_\eta, L_\eta) \geq 0$, the above equalities hold also for a general fibre $X_z$.*

*Proof.* The first equality is a consequence of the flat base change theorem. As for the second, we can assume that $f$ is flat without loss of generality, hence we conclude by [Har13, Theorem 12.8, Chapter III].

Now, suppose $\kappa(X_\eta, L_\eta) \geq 0$. Let $H$ be an ample enough Cartier divisor on $Z$ such that $L + f^*H$ is $\mathbb{Q}$-effective. By the Easy Additivity theorems (Theorem 5.2.1, Theorem 5.2.2 and Remark 5.2.3), for a general fibre $X_z$ we have:

$$\kappa(X, L + f^*H) = \kappa(X_z, L_z) + \dim(Z) \quad \text{and} \quad \kappa(X, L + f^*H) = \kappa(X_\eta, L_\eta) + \dim(Z).$$

Thus, $\kappa(X_z, L_z) = \kappa(X_\eta, L_\eta)$. qed

*Remark* 5.2.5. With notation as in Lemma 5.2.4, when $\kappa(X_\eta, L_\eta) = -\infty$ and $z \in Z$ is general, the equality $\kappa(X_z, L_z) = -\infty$ may fail. Let $A$ be an Abelian variety, let $Z$ denote its dual, and consider the second projection $f\colon X := A \times Z \to Z$. Then the Poincaré line bundle $L$ on $X$ gives a counterexample, as $\kappa(X_z, L_z) = 0$ for all torsion points $z \in Z$.

**Proposition 5.2.6.** *Let $f\colon X \to Z$ be a fibration of normal projective varieties with general fibre $X_z$. Let $L$ be a $\mathbb{Q}$-divisor such that there exists an integer $m \geq 1$ for which* $\mathrm{Bs}(mL)$ *does not dominate $Z$. Then*

$$\kappa(X, L) \geq \kappa(X_z, L_z).$$

*Proof.* For all $n \geq 1$ let $V_n := H^0(X, nmL)$ and $r_n$ be the dimension of $V_{n,z}$ for a general point $z \in Z$. Define $\mathcal{F}_n := f_*\mathcal{O}_X(nmL)$. A choice of basis for $V_n$ yields a map

$$\mathcal{O}_Z^{\oplus r_n} \hookrightarrow \mathcal{F}_n.$$

By taking global sections we obtain $\dim(H^0(X, nmL)) \geq r_n$. By letting $n$ go to infinity and considering the respective rates of growth, we conclude $\kappa(X, L) \geq \kappa(X_z, L_z)$. qed

## 5.3. The Iitaka conjecture

The Iitaka conjecture complements the Easy Additivity theorems. It is also called $C_{n,m}$ conjecture, where the $C$ stands for "contractions", another name for fibrations, and $n, m$ are the dimensions of the source and the target of the fibration, respectively.

Although still open in general, over fields of characteristic 0 this conjecture is proven for many important classes of fibrations ([Vie77, Vie82, Vie83, Kaw82, Kaw81, Kaw85, Fuj03, HPS18, Bir09, Cao18, CP17, CH11]). In particular, $C_{n,m}$ holds when $\dim(Z) \leq 2$, and when $X_z$ admits a good minimal model.

It is then natural to ask whether the same inequality holds over fields of positive characteristic. Given the substantial difference between the generic fibre and the geometric generic fibre, in positive characteristic, we have two versions of the conjecture.

*Questions* 5.3.1 ([Zha19, Conjecture 1.2, Conjecture 1.4]). Let $f\colon X \to Z$ be a fibration between normal projective varieties over a perfect field of positive characteristic and let $\eta$ be the generic point of $Z$. Assume that $X_{\overline{\eta}}$ admits a smooth birational model $\overline{Y}$.

(1) (Strong form) Is it true that $\kappa(X, K_X) \geq \kappa(X_\eta, K_{X_\eta}) + \kappa(Z, K_Z)$?

(2) (Weak form) Is it true that $\kappa(X, K_X) \geq \kappa(\overline{Y}, K_{\overline{Y}}) + \kappa(Z, K_Z)$?

In [CZ15, Theorem 1.2] the conjecture is proven when the general fibre of $f$ is a curve and $f$ is separable. The papers [Eji17], [BCZ18], [EZ18] show that the conjecture does hold when $\dim(X) = 3$, the characteristic of the base field is $p > 5$ and the geometric generic fibre is smooth. In [Zha19], the author proves $C_{3,m}$ when the base of the fibration is of general type and the relative canonical divisor is relatively big, but they relax the assumptions on the smoothness of the generic fibre. However, it is known that over fields of positive characteristic the Iitaka conjecture does not hold in general. The paper [CEKZ21] gives some counterexamples using the construction of Tango–Raynaud surfaces.

## 5.4. The anticanonical Iitaka conjecture

Recently, an Iitaka-type statement for the anticanonical divisor has been proven in characteristic 0 by Chang in [Cha23]. We call $C^-_{n,m}$ this version of the Iitaka conjecture. In this section, first we recall the theorem and then we explain a heuristic on why we should expect such an inequality to hold. We conclude with some counterexamples to the inequality over fields of positive characteristic, where the geometric generic fibre is not normal. In the next Chapter 6, we discuss some positive results, assuming that the general fibres have "nice" $F$-singularities.

### 5.4.1. Results in characteristic 0

In this section we consider varieties defined over a perfect field, we specify the characteristic if needed.

**Theorem 5.4.1** ([Cha23, Theorem 1.1])**.** *Let $f \colon X \to Z$ be a fibration between normal projective $\mathbb{Q}$-Gorenstein varieties over a field*[2] *of characteristic 0 and let $X_z$ be a general fibre of $f$. Suppose $X$ has at worst klt singularities, $-K_X$ is effective and there exists an integer $m \geq 1$ such that $\mathrm{Bs}(-mK_X)$ does not dominate $Z$, then:*

$$\kappa(X, -K_X) \leq \kappa(X_z, -K_{X_z}) + \kappa(Z, -K_Z).$$

The result is generalised also to pairs $(X, B)$ with similar assumptions (see [Cha23, Theorem 4.1]).

Counterexamples for $C^-_{n,m}$, in any characteristic, removing the hypothesis on the anticanonical base locus, are easy to obtain already for ruled surfaces.

[2] In the original paper, the result is stated over an algebraically closed field, but, by Remark 5.1.3, the same proof gives the inequality over perfect fields.

*Example* 5.4.2 ([Cha23, Example 1.7]). Let $Z$ be a smooth projective curve of genus $g \geq 2$ and let $D$ be a divisor of degree $d > 2g - 2$. Consider the ruled surface

$$f\colon X := \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(-K_Z - D)) \to Z.$$

Let $Z_0$ be the fixed section of $f$. Then, $-K_X = 2Z_0 + f^*D \geq Z_0 + f^*D$. We claim that the divisor $Z_0 + f^*D$ is big. Using the Riemann–Roch Theorem, we get:

$$h^0(X, n(Z_0 + f^*D)) \geq \frac{1}{2}n(f^*D + Z_0) \cdot (n(f^*D + Z_0) + 2Z_0 + f^*D) + O(n).$$

Given our choice of $d$, it is immediate to see that the coefficient of $n^2$ on the RHS is strictly positive, whence the claim.
Note that $-K_X \cdot Z_0 = 2 - 2g < 0$. This tells us that $Z_0$ is contained in the base locus of $-nK_X$ for every integer $n \geq 1$, which, therefore, surjects onto $Z$. In conclusion, we have $\kappa(X, -K_X) = 2$, $\kappa(Z, -K_Z) = -\infty$, thus $C^-_{n,m}$ does not hold for this example.

We expect a more general version of Theorem 5.4.1 to hold: more precisely we expect one should be able to relax the condition on $\mathrm{Bs}(-K_X)$ being $f$-vertical.

**Definition 5.4.3.** Let $(X, B)$ be a projective sub-pair over a field of characteristic 0 and $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ a graded linear subsystem. Let $\mathcal{J}(X, B; V_\bullet)$ be the *asymptotic multiplier ideal* of $V_\bullet$ (for the definition see [Laz04b, §11.1]). We define the **non-klt locus of** $(X, B, V_\bullet)$ as

$$\mathrm{Nklt}(X, B, V_\bullet) := \{x \in X \text{ s.t. } \mathcal{J}(X, B; V_\bullet)_x \subseteq \mathfrak{m}_x\}.$$

When $V_\bullet$ is the full graded linear series associated to some $\mathbb{Q}$-divisor $L$, we write $\mathrm{Nklt}(X, B, ||L||) := \mathrm{Nklt}(X, B, V_\bullet)$

*Remark* 5.4.4. If $Z \not\subseteq \mathrm{Nklt}(X, B, V_\bullet)$, for $m \gg 0$ and $D_1, ..., D_k$ general elements in $V_m$, then $Z \not\subseteq \mathrm{Nklt}\left(X, B + \frac{1}{mk}(D_1 + ... + D_k)\right)$, by [Laz04b, Proposition 9.2.26, Definition 9.3.9].

*Remark* 5.4.5. We state a generalisation of [Cha23, Theorem 4.1] (see Theorem 5.4.1).

Let $f\colon X \to Z$ be a fibration between normal projective varieties over a field of characteristic 0, where $Z$ is $\mathbb{Q}$-Gorenstein. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$ and let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Z$. Let $L := -K_X - B - f^*D$, and assume that $L$ is $\mathbb{Q}$-effective and $\mathrm{Nklt}(X, B, ||L||)$ does not dominate $Z$. Then

$$\kappa(X, L) \leq \kappa(X_z, L_z) + \kappa(Z, -K_Z - D).$$

Furthermore, if $\kappa(Z, -K_Z - D) = 0$, equality holds.

Instead of asking $\mathrm{Bs}(mL)$ to not dominate $Z$ for some integer $m \geq 1$, it is enough to ask for $\mathrm{Nklt}(X, B, ||L||)$ to not dominate $Z$. Indeed, the way the hypothesis on $\mathrm{Bs}(mL)$ is used is to find a $\mathbb{Q}$-divisor $\Lambda \in |L|_{\mathbb{Q}}$ such that $(X_z, B_z + \Lambda_z)$ is still klt.

This is clearly true when $\mathrm{Bs}(mL)$ is vertical, since in this case $|mL|_z$ will be base point free. However, Remark 5.4.4 shows that the same is true under the weaker assumption on $\mathrm{Nklt}(X, B, ||L||)$. Using this, it is possible to prove the above version of [Cha23, Theorem 4.1].

We give an example showing that $C^-_{n,m}$ can fail when $\mathrm{Nklt}(X, B, L)$ is horizontal.

*Example* 5.4.6. Let $Z$ be an elliptic curve and $D$ a divisor on $Z$ of degree $d > 0$. Define $X := \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(-D))$ and let $f\colon X \to Z$ be the structure map. Let $B := Z_0$, where $Z_0$ is the section of $f$ corresponding to the surjection $\mathcal{O}_Z \oplus \mathcal{O}_Z(-D) \to \mathcal{O}_Z(-D)$, so that $\mathcal{O}_X(Z_0) = \mathcal{O}_X(1)$. Then, $-K_X - B \sim Z_0 + dX_z$, where $z$ is a general point of $Z$. In this case, $(X, B)$ is strictly log canonical and the non-klt locus dominates the base. We have

$$2 = \kappa(X, -K_X - B) > \kappa(X_z, -K_{X_z} - B_z) + \kappa(Z, -K_Z) = 1.$$

In this case we even have $\mathrm{Bs}(-K_X - B) = \emptyset$.

### 5.4.2. Heuristic

Let $f\colon X \to Y$ be a fibration between normal projective varieties over an algebraically closed field. The first step to prove $C^-_{n,m}$ is a positivity descent result, showing that, if $-K_X$ is $\mathbb{Q}$-effective, then so is $-K_Y$. In characteristic zero this is done by picking a sufficiently general $\Gamma \in |-K_X|_{\mathbb{Q}}$, so that $f\colon X \to Y$ with the pair structure $(X, \Gamma)$ is a Calabi–Yau fibration. By the canonical bundle formula we then write

$$K_X + \Gamma \sim_{\mathbb{Q}} f^*(K_Y + M_Y + B_Y) \sim_{\mathbb{Q}} 0,$$

where $M_Y + B_Y$ is $\mathbb{Q}$-effective, from which we conclude. In positive characteristic we do not have a canonical bundle formula in such generality, but we do have it in some cases (see Section 3.4). Hence, we conclude by the same argument as above, once we show that there exists $\Gamma \in |-K_X|_{\mathbb{Q}}$ such that $(X_y, \Gamma_y)$ has “controlled” singularities.

Suppose now that we have a fibration $f\colon X \to Y$ where both $-K_X$ and $-K_Y$ have sections; for simplicity, let us assume that they are both semiample. Then we can think of $X$ as being “built from” Calabi–Yau and Fano varieties: more precisely, letting $X \to X_{\mathrm{Fano}} := \mathrm{Proj}\left(\bigoplus_{m\geq 0} H^0(X, -K_X)\right)$ be the semiample contraction of $-K_X$ (see Remark 5.1.5), by the canonical bundle formula we have that $X_{\mathrm{Fano}}$ is a Fano-type variety with $\kappa(X, -K_X) = \dim(X_{\mathrm{Fano}})$, and a general fibre $X_{\mathrm{CY}}$ is a Calabi–Yau variety. Analogous descriptions hold for $Y$ and $X_y$. Then we have the following commutative diagram

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] \arrow[dd, bend right, "h"'] & V \arrow[r] \arrow[ddl, "h_{\mathrm{Fano}}"] & X_{\mathrm{Fano}} \\
Y \arrow[d, "g"] & & \\
Y_{\mathrm{Fano}}, & &
\end{tikzcd}
$$

where $V := \mathrm{Proj}_{Y_{\mathrm{Fano}}}\left(\bigoplus_{m\geq 0} h_*\mathcal{O}_X(-mK_X)\right)$. In particular, for $z \in Y_{\mathrm{Fano}}$ general, we have $V_z = (X_z)_{\mathrm{Fano}}$. Consider the induced fibration $f_z \colon X_z \to Y_z$: the base is a Calabi–Yau variety.

For such fibrations we prove an injectivity theorem which implies $C^-_{n,m}$ for $f_z$. The idea of this result can be easily explained when $\dim(Y_z) = 1$ and $X_z$ is normal. We want to prove that the restriction map $H^0(X_z, -mK_{X_z}) \to H^0(X_y, -mK_{X_y})$ is injective for $y \in Y_z$ general and $m \gg 0$. If this was not the case, there would exist an effective $\mathbb{Q}$-divisor $N \sim_{\mathbb{Q}} -K_{X_z}$ containing $X_y$ in its support. This implies that, for a positive constant $c \in \mathbb{Q}$, $-K_{X_z} - cX_y$ is $\mathbb{Q}$-effective. With the canonical bundle formula we can descend this effectivity to $Y_z$, proving that $-K_{Y_z} - \varepsilon y$ is effective for some $\varepsilon > 0$ small enough, contradicting the fact that $Y_z$ is Calabi–Yau.

We then conclude:

$$
\begin{aligned}
\kappa(X, -K_X) &= \dim(X_{\mathrm{Fano}}) \\
&\leq \dim(V) \\
&= \dim(X_z)_{\mathrm{Fano}} + \dim(Y_{\mathrm{Fano}}) \\
&= \kappa(X_z, -K_{X_z}) + \kappa(Y, -K_Y) \\
&\leq \kappa(X_y, -K_{X_y}) + \kappa(Y, -K_Y).
\end{aligned}
$$

### 5.4.3. Counterexamples in positive characteristic

The next examples, studied in [Ben22, §5], show that in positive characteristic we cannot expect the result in Remark 5.4.5 to hold in such generality. They are constructed from Tango–Raynaud surfaces. Analysing them, in the paper [CEKZ21], the authors found counterexamples to $C_{n,m}$ in characteristic $p$, for any $p > 0$. A similar construction gives counterexamples to $C^-_{7,6}$ in characteristics 2 and 3. Roughly speaking, they are based on the failure of generic smoothness.

In this section, we consider varieties defined over an algebraically closed field of characteristic $p > 0$.

**Definition 5.4.7.** Let $C$ be a smooth projective curve of genus $g_C \geq 2$. Define the

**Tango invariant**:

$$n(C) := \max\left\{\deg\left(\left\lfloor\frac{(df)}{p}\right\rfloor\right)\ \text{s.t.}\ f \in k(C)\right\},$$

where $(df)$ denotes the divisor of zeroes and poles of the differential $df$. Note that $pn(C) \leq 2g_C - 2$. We say that $C$ is a **Tango curve** if $n(C) > 0$ and that it is a **Tango–Raynaud curve** if, moreover, $pn(C) = 2g_C - 2$.

*Example* 5.4.8 ([Muk13, Example 1.3]). Let $e \in \mathbb{N}$ and let $C$ be the plane curve defined by the equation $Y^{pe} - YX^{pe-1} = Z^{pe-1}X$ in $\mathbb{P}^2_{\mathbb{F}_p}$ with coordinates $[X : Y : Z]$. It is smooth and, by adjunction, if $g_C$ is its genus, $2g_C - 2 = pe(pe-3)$. Consider the differential form $d\left(\frac{Z}{X}\right)$ and denote by $\infty$ the point $[0 : 0 : 1] \in C$. Then, $\left(d\left(\frac{Z}{X}\right)\right) = pe(pe-3)(\infty) = (2g_C - 2)(\infty)$, showing that $C$ is a Tango–Raynaud curve.

Let $C$ be a normal projective curve and consider the Frobenius map $F\colon C \to C$. Denote by $\mathcal{B}$ the cokernel of the induced map, $\mathcal{O}_C \to F_*\mathcal{O}_C$. Thus, for any Cartier divisor $D$, we have the exact sequence:

$$0 \to \mathcal{O}_C(-D) \to F_*(\mathcal{O}_C(-pD)) \to \mathcal{B}(-D) \to 0.$$

**Lemma 5.4.9** ([Xie10, Lemma 2.5]). *With the same notation as above,*

$$H^0(C, \mathcal{B}(-D)) = \{df\ s.t.\ f \in k(C),\ (df) \geq pD\}.$$

Note that $H^0(C, \mathcal{B}(-D))$ is the kernel of $F^*\colon H^1(C, \mathcal{O}_C(-D)) \to H^1(C, \mathcal{O}_C(-pD))$ when $D$ is effective.

Now, let $C$ be a Tango–Raynaud curve equipped with an effective divisor $D$ and a non-zero element $df$ in $H^0(C, \mathcal{B}(-D))$ such that $(df) = pD$. This determines a non-zero element of $H^1(C, \mathcal{O}_C(-D))$ which is mapped to zero by the Frobenius morphism. Notice that $df$ determines a (non-split) short exact sequence:

$$0 \to \mathcal{O}_C(-D) \to \mathcal{E} \to \mathcal{O}_C \to 0, \tag{❀}$$

where $\mathcal{E}$ is a rank two vector bundle on $C$. After applying the Frobenius morphism, the above exact sequence becomes split, thus we also get:

$$0 \to \mathcal{O}_C \to F^*\mathcal{E} \to \mathcal{O}_C(-pD) \to 0. \tag{✻}$$

Let $g\colon P := \mathbb{P}(\mathcal{E}) \to C$ and $g_1\colon P_1 := \mathbb{P}(F^*\mathcal{E}) \to C$. Thus, we have a commutative diagram:

$$\begin{tikzcd}
P \arrow[r, "F_{P/C}"] \arrow[rr, bend left, "F_P"] \arrow[dr, "g"'] & P_1 \arrow[r, "\varphi"] \arrow[d, "g_1"] & P \arrow[d, "g"] \\
& C \arrow[r, "F_C"] & C,
\end{tikzcd}$$

where the rightmost square is a fibre product diagram and $F_{P/C}$ is the relative Frobenius.
Let $T$ be a divisor on $P$ such that $\mathcal{O}_P(T) \simeq \mathcal{O}_P(1)$. The short exact sequence (❀) defines a section of $g$, $A \sim T + g^*D$. On the other hand, the short exact sequence (✻), defines a section of $g_1$, $B_1 \sim \varphi^*A - pg_1^*D$. Let $B := F_{P/C}^*B_1$, then $B \sim pA - pg^*D \sim pT$. Computing the arithmetic genus of $A$ and $B$ using the adjunction formula and the fact that $\deg(D) = \frac{2g_C - 2}{p}$, we see that the curves $A$ and $B$ are smooth of genus $g_C$ and they are disjoint.

If there exists $l > 0$ such that $l$ divides $p+1$ and $D = lD'$ for an effective divisor $D'$, then:

$$A + B \sim (p+1)T + g^*D = l(rT + g^*D') = lM,$$

where $r = \frac{p+1}{l}$ and $M := rT + g^*D'$. Explicit cases where we can find such $l$ are the curves of Example 5.4.8, by choosing an appropriate $e$. Since the support of the divisor $A + B$ is smooth, the $l$-cyclic cover defined by the above equivalence yields a smooth surface $S$. Call $\pi \colon S \to P$ the cover and $f := g \circ \pi \colon S \to C$.
The last step in this construction consists in taking $m$-times the fibre product of $S$ over $C$: $X_m := S \times_C S \cdots \times_C S$. Let $p_i \colon X_m \to S$ be the projection to the $i^{\text{th}}$ factor and $f_m \colon X_m \to C$ the composition of $f$ with any of these projections.

**Theorem 5.4.10.** *Let $\{p, l\} = \{2, 3\}$ and consider the fibration*

$$X_6 \to X_5.$$

*The stable base locus of the anticanonical divisor of $X_6$ is empty and its non-klt locus does not dominate $X_5$. Moreover, $\kappa(X_6, -K_{X_6}) = 0$, while $\kappa(X_5, -K_{X_5}) = -\infty$. Therefore, this fibration gives counterexamples in characteristics 2 and 3 to $C^-_{7,6}$ as stated in Remark 5.4.5.*

*Proof.* The relative anticanonical divisors of $g$ and $f$, for our choices of $p$ and $l$ are:

$$K_{P/C} = -2T - pg^*D;$$
$$K_{S/C} = \pi^*(K_{P/C} + (l-1)M) = -f^*D'.$$

Then, the anticanonical sheaf of $X_m$ is:

$$\omega_{X_m}^{-1} = \left(\prod_{i=1}^{m} p_i^*\omega_{S/C}^{-1}\right) \otimes \omega_C^{-1}.$$

Therefore, for any positive integer $n$ we have $-nK_{X_m} = f_m^*((nm-6n)D')$, whence:

$$\kappa(X_m, -K_{X_m}) = \begin{cases} 1 & \text{for } m > 6 \\ 0 & \text{for } m = 6 \\ -\infty & \text{for } m < 6 \end{cases} .$$

Note that the base locus is empty for $m \geq 6$.

Studying local equations, it is easy to see that the varieties $X_m$ are normal and their singular locus is the union of $\mathrm{Supp}(T_i) \cap \mathrm{Supp}(T_j)$ for $i \neq j$, where $T_i := (\pi \circ p_i)^* T$. In particular, for the chosen fibration $X_6 \to X_5$ the non-smooth locus does not dominate the base. Indeed, let $s$ be a local parameter on the fibres of $g$ such that locally $A = \{s = \infty\}$ and $B = \{s^p = 0\}$. First we look at what happens away from $A$. Let $(x_1, ..., x_r) = \underline{x}$ be local affine coordinates on $C$, then, as showed in [Muk13, §2] there is $f(\underline{x})$ such that $S$ affine locally has equation $t^l = s^p - f(\underline{x})$ inside $\mathbb{A}^1 \times \mathbb{A}^1 \times C$. In a similar way, affine locally, we can see $X_m \subseteq \mathbb{A}^m \times \mathbb{A}^m \times C$ with coordinates $t_1, ...., t_m, s_1, ..., s_m, \underline{x}$ and equations $t_i^l = s_i^p - f(\underline{x})$ for $i = 1, ..., m$. The derivatives along the variables $s_i$ are all 0, therefore, applying the Jacobi criterion we get that $X_m$ is singular where the rank of the matrix

$$\begin{pmatrix} t_1^{l-1} & 0 & ... & 0 \\ 0 & t_2^{l-1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & t_m^{l-1} \end{pmatrix}$$

is strictly less than $m-1$, whence the claim. On the other hand, if $\sigma = 1/s$ is a local parameter defining $A$ on $P$, an affine local equation for $S$ away from $B$ is $\tau^l = \sigma$. Therefore, $X_m$ is smooth in this open subset. qed

*Remark* 5.4.11. The above computations also show that the fibres of $X_6 \to X_5$ are not normal.

**Corollary 5.4.12.** *Let $p = 2$ or $p = 3$ and assume the existence of resolutions of singularities and that we can run the birational Minimal Model Program for smooth varieties of dimension 7 in characteristic $p$. Let $\{p, l\} = \{2, 3\}$, then, there exists a projective klt variety $Y$ of dimension 7, with a fibration $Y \to X_5$ for which $C^-_{7,6}$ does not hold. In particular, the base locus of $-nK_Y$ does not dominate $X_5$ for any $n \in \mathbb{N}$.*

*Proof.* For simplicity of notation, let $X := X_6$. Let $U \subseteq X$ be the regular locus of $X$ and $\mu\colon X' \to X$ a resolution which is an isomorphism over $U$. Write $K_{X'} = \mu^*(K_X) + E$, where $E$ is an exceptional $\mathbb{Q}$-divisor. Run a $K_{X'}$-MMP over $X$ and let $\varphi\colon X' \dashrightarrow Y$ and $\sigma\colon Y \to X$ be the resulting birational maps. Note that $Y$ is klt. Then, by the Negativity lemma ([KM98, Lemma 3.39]), $-\varphi_* E \geq 0$. Thus

$-K_Y = -\sigma^* K_X - \varphi_* E \geq -\sigma^* K_X$. Since $\mu(\mathrm{Supp}(E))$ is disjoint from $U$, the base locus of $-nK_Y$ does not dominate $X_5$ for any $n \in \mathbb{N}$. Furthermore, $\kappa(Y, -K_Y) \geq \kappa(X, -K_X) = 0$, while $\kappa(X_5, -K_{X_5}) = -\infty$. qed

# Chapter 6

# Iitaka conjecture for anticanonical divisors

In this chapter we present the main results in [Ben22] and [BBC23]. In particular, we prove the $C^-_{n,m}$ inequality over perfect fields of positive characteristic, assuming that the general fibres have "nice" $F$-singularities. In low dimension, it is enough to make local assumptions on the singularities (strongly $F$-regular), whereas to prove the result in any dimension, we assume global conditions ($K$*-globally* $F$*-regular*).

## 6.1. $C^-_{n,m}$ in low dimension

In this section we present the results proven in [Ben22]. In particular, we conclude that $C^-_{n,m}$ holds when the source of the fibration is a threefold or when the target is a curve, the general fibre is regular and the pair induced on it from the ambient space is strongly $F$-regular.

### 6.1.1. Weakly positive sheaves

The main property studied in [EG19] is weak positivity of sheaves. We recall here what we need to use in the following.

In this section we consider varieties defined over a perfect field of any characteristic, unless otherwise stated.

**Definition 6.1.1.** Let $X$ be a normal quasi-projective variety and $\mathcal{G}$ a coherent sheaf on it. We say that $\mathcal{G}$ is **generically globally generated** if the map

$$H^0(X, \mathcal{G}) \otimes \mathcal{O}_X \to \mathcal{G}$$

is surjective over the generic point of $X$.
The sheaf $\mathcal{G}$ is called **weakly positive** if, given any ample divisor $A$ on $X$ and any natural number $\alpha$, there exists an integer $\beta > 0$ such that $(\mathrm{Sym}^{\alpha\beta}(\mathcal{G}))^{**} \otimes \mathcal{O}_X(\beta A)$ is generically globally generated, where the double star indicates the double dual.

**Lemma 6.1.2.** *Let $X$ be a normal quasi-projective variety and $\mathcal{G}$ a coherent sheaf on it. Fix an integer $n \geq 2$. If there exists a generically globally generated invertible sheaf $H$ with the property that for all $\alpha > 0$, $\alpha \in \mathbb{Z} \setminus n\mathbb{Z}$, there is an integer $\beta > 0$ such that $(\mathrm{Sym}^{\alpha\beta}(\mathcal{G}))^{**} \otimes \mathcal{O}_X(\beta H)$ is generically globally generated, then $\mathcal{G}$ is weakly positive.*

*Proof.* By [Vie83, Remark 1.3(ii)], it is enough to check the condition of Definition 6.1.1 for one invertible sheaf, not necessarily ample, and all $\alpha \in \mathbb{Z}_{>0}$. Fix $H$ invertible sheaf that is generically globally generated. Assume that for all $\alpha > 0$, $\alpha \in \mathbb{Z} \setminus n\mathbb{Z}$, there is an integer $\beta > 0$ such that $(\mathrm{Sym}^{\alpha\beta}(\mathcal{G}))^{**} \otimes \mathcal{O}_X(\beta H)$ is generically globally generated. Let $\alpha' \in n\mathbb{Z}_{>0}$. By the above assumption, there is an integer $\beta > 0$ such that $(\mathrm{Sym}^{(\alpha'+1)\beta}(\mathcal{G}))^{**} \otimes \mathcal{O}_X(\beta H)$ is generically globally generated. Let $\beta' := (\alpha' + 1)\beta$. Then,

$$(\mathrm{Sym}^{\alpha'\beta'}(\mathcal{G}))^{**} \otimes \mathcal{O}_X(\beta' H) = (\mathrm{Sym}^{\alpha'}((\mathrm{Sym}^{(\alpha'+1)\beta}\mathcal{G})^{**} \otimes \mathcal{O}_X(\beta H)))^{**} \otimes \mathcal{O}_X(\beta H)$$

is generically globally generated, whence the conclusion. qed

**Lemma 6.1.3.** *Let $X$ be a normal quasi-projective variety and $\mathcal{G}$ a coherent sheaf on it. Assume $\mathcal{G} = \mathcal{O}_X(D)$ is an invertible sheaf, then it is generically globally generated if and only if $D$ is linearly equivalent to an effective divisor. Moreover, it is weakly positive if and only if $D$ is pseudoeffective.*

*Proof.* The first claim follows directly from the definition. Indeed in this case $\mathcal{G}$ is generically globally generated if and only if $H^0(X, \mathcal{G}) \neq 0$. Let us prove the second claim. The condition of being weakly positive translates to the fact that, for any $A$ ample and any $n \in \mathbb{Z}_{>0}$, $D + \frac{1}{n}A$ is $\mathbb{Q}$-effective. But then, $D = \lim_{n\to\infty} \left(D + \frac{1}{n}A\right)$ is a limit of $\mathbb{Q}$-effective divisors, thus it is pseudoeffective. On the other hand, if $D$ is pseudoeffective, then it is in the closure of the $\mathbb{Q}$-effective cone. Consider the line defined by $D + tA$. For $t \in \mathbb{Q}_{>0}$, each of these divisors is $\mathbb{Q}$-effective, thus $D$ is weakly positive. qed

We state the next result in the assumptions in which we need to use it later. In the original paper it is proven in greater generality.

**Theorem 6.1.4** ([Eji17, Theorem 5.1, Example 3.11])**.** *Let $f\colon X \to Z$ be a surjective morphism between normal projective varieties over an algebraically closed field of characteristic $p > 0$. Let $B$ be an effective Weil divisor on $X$ such that $aB$ is integral for some $a \in \mathbb{Z}_{>0}$, not divisible by $p$. Let $\overline{\eta}$ be the geometric generic point of $Z$. Suppose that:*

*(a) the geometric generic fibre $X_{\overline{\eta}}$ is normal;*

*(b) $K_{X_{\overline{\eta}}} + B_{\overline{\eta}}$ is $\mathbb{Z}_{(p)}$-Cartier and ample;*

*(c)* $(X_{\bar{\eta}}, B_{\bar{\eta}})$ *is sharply F-pure.*

*Then,*

$$(f_*\mathcal{O}_X(am(K_X+B)))\otimes\omega_Z^{\otimes -am}$$

*is weakly positive for every* $m \gg 0$*.*

### 6.1.2. Proof of $C^-_{n,1}$

In this section, we use a variation of the results in [EG19, §3, §4] to prove $C^-_{n,1}$ in positive characteristic. The authors of [EG19] assume that the relative anticanonical divisor is nef. We weaken a bit the assumption with (iii) in Set-up (☆) below.

(☆)Set-up

Let $f\colon X\to Z$ be a fibration between normal projective varieties over an algebraically closed field of characteristic $p>0$. Consider $\mathbb{Q}$-divisors $B$ and $D$ on $X$ and $Z$ respectively such that $K_X+B$ is $\mathbb{Q}$-Cartier and $D$ is $\mathbb{Z}_{(p)}$-Cartier. Let $B=B^+-B^-$ and $L:=-(K_X+B)-f^*D$. Suppose that:

(i) $f$ is equidimensional and $K_Z$ is $\mathbb{Z}_{(p)}$-Cartier;

(ii) the general fibre $X_z$ is regular and $(X_z, B_z^+)$ is strongly F-regular, where $B_z^+$ is defined by $(K_X+B^+)|_{X_z}=K_{X_z}+B_z^+$ (this restriction is well-defined since $X_z$ is normal);

(iii) there exists $\overline{m}\in\mathbb{Z}_{>0}$ not divisible by $p$ such that $\mathrm{Bs}(\overline{m}L)$ does not dominate $Z$ (in particular, $L$ is $\mathbb{Z}_{(p)}$-effective);

(iv) $\mathrm{Supp}(B^-)$ does not dominate $Z$;

(v) $B$ is a $\mathbb{Z}_{(p)}$-divisor.

**Theorem 6.1.5.** *In Set-up (☆), for all* $l\gg 0$ *and sufficiently divisible such that* $l$ *is not divisible by* $p$*,*

$$\mathcal{O}_X(l(-f^*(K_Z+D)+B^-))$$

*is weakly positive.*

*Proof.* First of all, note that we can assume that $K_X+B$ is $\mathbb{Z}_{(p)}$-Cartier. In fact, if $K_X+B$ is not $\mathbb{Z}_{(p)}$-Cartier, let $\mu, e$ be the minimal positive integers, with $\mu$ not divisible by $p$, such that $\mu p^e(K_X+B)$ is Cartier and let $\Gamma\geq 0$ be a $\mathbb{Z}_{(p)}$-divisor such that $\Gamma\sim_{\mathbb{Z}_{(p)}} L$. Define $B':=B+\frac{1}{p^{ee'}+1}\Gamma$, with $e'\gg 0$. Since $D$ is $\mathbb{Z}_{(p)}$-Cartier, $K_X+B'$ is $\mathbb{Z}_{(p)}$-Cartier and all the other assumptions in Set-up (☆) are satisfied if we replace $B$ with $B'$. In particular, $(ii)$ holds by Remark 1.3.25 if we choose $e'$ big enough.

Let $l$ be any positive integer not divisible by $p$ such that $l(K_X+B)$, $l(K_Z+D)$ are Cartier and $lB$ is integral.

We follow the proof of [EG19, Theorem 3.1], highlighting the small differences. First, we prove the statement when $f$ is equidimensional. Set $\mathcal{F} := \mathcal{O}_X(l(-f^*(K_Z+D)+B^-))$ and let $A$ be a very ample Cartier divisor on $X$. We need to show weak positivity of $\mathcal{F}$, in particular, by Lemma 6.1.2, it is enough to see that for any $\alpha \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}$, there is some $\beta \in \mathbb{Z}_{>0}$ such that $\mathcal{F}^{[\alpha\beta]}(\beta l A)$ is weakly positive.
The $\mathbb{Q}$-divisor $L+\alpha^{-1}A$ is $\mathbb{Z}_{(p)}$-effective since both $L$ and $A$ are. In particular, there exists $n \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}$ such that $|n(L+\alpha^{-1}A)| \neq \emptyset$. Since $\mathrm{Bs}(\overline{m}L)$ does not dominate $Z$ and $\overline{m}$ is not divisible by $p$, the graded linear system $\{|V_m| := |mn(L+\alpha^{-1}A)|_z\}_{m\in\mathbb{N}}$ is $\mathbb{Z}_{(p)}$-semiample for a general fibre $X_z$. By Corollary 1.3.28, there exists an effective $\mathbb{Z}_{(p)}$-divisor $\Gamma \sim_{\mathbb{Z}_{(p)}} (L+\alpha^{-1}A)$, such that $(X_z, B_z^+ + \Gamma_z)$ is still sharply F-pure. By Theorem 2.2.13, this implies that $(X_{\overline{\eta}}, (B^+ + \Gamma)_{\overline{\eta}})$ is sharply F-pure. Note that $(K_X+B^++\Gamma)_{\overline{\eta}} \sim_{\mathbb{Z}_{(p)}} \alpha^{-1}A_{\overline{\eta}}$ is ample. Thus, Theorem 6.1.4 proves weak positivity of the sheaf

$$(f_*\mathcal{O}_X(ll'm(K_X+B^++\Gamma))) \otimes \omega_Z^{\otimes -ll'm},$$

for any $m \gg 0$, where $l'$ is a positive integer not divisible by $p$ such that $ll'(B^++\Gamma)$ is integral. Then, as shown in [EG19, Theorem 3.1], for $\beta \gg 0$ sufficiently divisible, there is a generically surjective morphism

$$f^*((f_*\mathcal{O}_X(\alpha\beta l(K_X+B^++\Gamma))) \otimes \omega_Z^{\otimes -\alpha\beta l}) \to \mathcal{F}^{[\alpha\beta]} \otimes \mathcal{O}_X(\beta l A),$$

therefore the latter sheaf is weakly positive as well. qed

(✮)<u>Set-up</u>

Let $f\colon X \to Z$ be a fibration between normal projective varieties over an algebraically closed field of characteristic $p>0$, let $B$ be an effective $\mathbb{Z}_{(p)}$-divisor on $X$ such that $K_X+B$ is $\mathbb{Q}$-Cartier and $D$ a $\mathbb{Z}_{(p)}$-Cartier divisor on $Z$. Let $L := -(K_X+B) - f^*D$. Suppose that:

(i) $f$ is equidimensional and $K_Z$ is $\mathbb{Z}_{(p)}$-Cartier;

(ii) the general fibre $X_z$ is regular and the pair induced on it $(X_z, B_z)$ is strongly $F$-regular, where $B_z$ is defined by $(K_X+B)|_{X_z} = K_{X_z} + B_z$;

(iii) there exists $\overline{m} \in \mathbb{Z}_{>0}$ not divisible by $p$ such that $\mathrm{Bs}(\overline{m}L)$ does not dominate $Z$ (in particular, $L$ is $\mathbb{Z}_{(p)}$-effective).

All results in [EG19, §4] hold with almost the same proofs in this Set-up (✮) using Theorem 6.1.5 instead of [EG19, Theorem 3.1].

**Proposition 6.1.6.** *In Set-up (✭), let $E$ be a $\mathbb{Z}_{(p)}$-Cartier divisor on $Z$. Assume there exists a $\mathbb{Q}$-Cartier $\mathbb{Z}_{(p)}$-divisor $\Gamma \geq 0$ on $X$ such that $\Gamma \sim_{\mathbb{Z}_{(p)}} L - f^*E$. Then, for $\varepsilon \in \mathbb{Z}_{(p)>0}$ small enough, $\mathcal{O}_X(-mf^*(K_Z + D + \varepsilon E))$ is weakly positive for any sufficiently divisible $m \in \mathbb{Z}_{>0}$ not divisible by $p$.*

*Proof.* This proof follows [EG19, Proposition 4.4]. For any $\varepsilon \in [0,1) \cap \mathbb{Z}_{(p)}$, fix:

$$B^\varepsilon := B + \varepsilon\Gamma; \quad D^\varepsilon := D + \varepsilon E; \quad L^\varepsilon := -(K_X + B^\varepsilon) - f^*D^\varepsilon.$$

Note that $B^\varepsilon$ is a $\mathbb{Z}_{(p)}$-divisor, $K_X+B^\varepsilon$ is $\mathbb{Q}$-Cartier and $D^\varepsilon$ is $\mathbb{Z}_{(p)}$-Cartier. Moreover, there exists $\bar{n} \in \mathbb{Z}_{>0}$ not divisible by $p$ such that $\bar{n}(1-\varepsilon) \in \mathbb{Z}$, $\bar{n}L^\varepsilon \sim (1-\varepsilon)L$ and $\mathrm{Bs}(\bar{n}\bar{m}L^\varepsilon)$ does not dominate $Z$.

Take a general fibre $X_z$ of $f$; since $(X_z, B_z)$ is SFR, so is $(X_z, B_z^\varepsilon)$ for $\varepsilon$ small enough by Remark 1.3.25, where $B_z^\varepsilon$ is defined by $(K_X + B^\varepsilon)|_{X_z} = K_{X_z} + B_z^\varepsilon$. We apply Theorem 6.1.5 to $f\colon X \to Z, B^\varepsilon, D^\varepsilon$ to get that $\mathcal{O}_X(-mf^*(K_Y + D^\varepsilon))$ is weakly positive for any sufficiently divisible $m \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}$. qed

**Corollary 6.1.7.** *In Set-up (✭), assume there is an effective $\mathbb{Z}_{(p)}$-Cartier divisor $E$ on $Z$ such that $L - f^*E$ is $\mathbb{Z}_{(p)}$-equivalent to an effective $\mathbb{Q}$-Cartier divisor on $X$. Then $E = 0$.*

*Proof.* Fix an ample Cartier divisor $A$ on $X$. Applying Proposition 6.1.6 with $D = -K_Z$, we see that $\mathcal{O}_X(-mf^*E)$ is weakly positive, for some $m \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}$. We want to show $mf^*E = 0$. This is proven in the same way as in [EG19, Corollary 4.5]. qed

**Theorem 6.1.8.** *In Set-up (✭), let $D = -K_Z$. Let $X_z$ be a closed fibre of $f$ over a regular point $z \in Z$. Suppose the following conditions hold:*

*(i) $f$ is flat at every point of $X_z$;*

*(ii) $\mathrm{Supp}(B)$ does not contain $X_z$;*

*(iii) $X_z$ is normal.*

*Then the support of every effective $\mathbb{Z}_{(p)}$-divisor $\Gamma$ such that there exists an integer $m_0 \geq 1$ not divisible by $p$ with $m_0\Gamma \sim m_0 L$, does not contain $X_z$.*

*Proof.* This proof follows [EG19, Theorem 4.2]. First of all, note that, by the same considerations as in the proof of Theorem 6.1.5, we can assume $K_X + B$ is $\mathbb{Z}_{(p)}$-Cartier. Assume there exists $\Gamma$ such that $\Gamma \sim_{\mathbb{Z}_{(p)}} L$ and $\mathrm{Supp}(\Gamma)$ contains $X_z$. Consider the diagram:

$$\begin{array}{ccc} H \subseteq X' & \xrightarrow{\pi} & X \supseteq X_z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ E \subseteq Z' & \xrightarrow{\mu} & Z \ni z \end{array}$$

where

- $\mu\colon Z' \to Z$ is the blow-up at $z$;
- $\pi\colon X' \to X$ is the blow-up at $X_z := f^{-1}(z)$, note that, by flatness of $f$ near $z$ and normality of the fibres, $X'$ is normal and it coincides with $X \times_Z Z'$;
- $H := \mathrm{Exc}(\pi) \cong X_z \times E = f'^*E$, with $E := \mathrm{Exc}(\mu)$.

Let $L' := -\pi^*(K_X + B) + f^*K_{Z'} = \pi^*L$, then $L'$ is $\mathbb{Z}_{(p)}$-effective and $\mathrm{Bs}(\overline{m}L')$ does not dominate $Z'$. Let $c \in \mathbb{Z}_{(p)}$ be the coefficient of $H$ in $\pi^*\Gamma$, by assumption it is $> 0$. Then $L' - f'^*(cE) \sim_{\mathbb{Z}_{(p)}} \pi^*\Gamma - cH \geq 0$. By Corollary 6.1.7, $cE = 0$, contradiction. qed

**Corollary 6.1.9.** *In Set-up (☆), let $D = -K_Z$, then the restriction map*

$$\alpha\colon \bigoplus_{m \in \mathbb{N}\setminus p\mathbb{N}} H^0(X, mlL) \to \bigoplus_{m \in \mathbb{N}\setminus p\mathbb{N}} H^0(X_z, mlL_z)$$

*is injective, where $l$ is a positive integer not divisible by $p$ such that $H^0(X, lL) \neq 0$. In particular,*

$$\kappa(X, L) = \kappa(X_z, L_z).$$

*Proof.* This proof follows [EG19, Corollary 4.7]. If $\alpha$ was not injective, there would be a section of $\bigoplus_{m\in\mathbb{N}\setminus p\mathbb{N}} H^0(X, mlL)$ whose zero locus contains the fibre $X_z$ in its support. So, it suffices to show that for every effective divisor $\Gamma \sim_{\mathbb{Z}_{(p)}} L$, $\mathrm{Supp}(\Gamma)$ does not contain $X_z$. Therefore, we conclude by Theorem 6.1.8.

Now, let $m \in \mathbb{Z}_{>0}$ such that $mL$ computes the Iitaka dimension. If $p|m$, since $L$ is $\mathbb{Z}_{(p)}$-effective, there is $n$ coprime with $p$ such that $H^0(X, nL) \neq 0$. Thus $(m+n)L$ computes the Iitaka dimension as well and $m + n$ is not divisible by $p$. Therefore, we conclude by the first part and Proposition 5.2.6. qed

With this last result it is straightforward to prove $C^-_{n,1}$.

**Theorem 6.1.10** ($C^-_{n,1}$)**.** *Let $f\colon X \to Z$ be a fibration from a normal projective variety onto a smooth projective curve over a perfect field of characteristic $p > 0$. Let $B$ be an effective $\mathbb{Z}_{(p)}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Suppose that the general fibre $X_z$ is regular and the pair $(X_z, B_z)$ is strongly $F$-regular, where $B_z$ is defined by $(K_X + B)|_{X_z} = K_{X_z} + B_z$. Assume moreover that there exists an integer $\overline{m} \geq 1$ not divisible by $p$ such that $\mathrm{Bs}(-\overline{m}(K_X + B))$ does not dominate $Z$. Then:*

$$\kappa(X, -(K_X + B)) \leq \kappa(X_z, -(K_{X_z} + B_z)) + \kappa(Z, -K_Z).$$

*Moreover, if $\kappa(Z, -K_Z) = 0$, equality holds.*

*Proof.* First of all, by Remark 5.1.3, we can assume that the base field $k$ is algebraically closed.

Then, note that we can assume $\kappa(X, -(K_X + B)) \geq 0$. We distinguish three cases according to the genus of $Z$.

(1) Let $Z$ be a curve of genus 0.
The result follows immediately from the Easy Additivity theorem (see Theorem 5.2.1).

(2) Let $Z$ be a curve of genus 1.
Then, $K_Z \sim 0$, so Corollary 6.1.9 gives the desired inequality.

(3) Let $Z$ be a curve of genus $\geq 2$.
By the Riemann-Roch Theorem on curves, since $K_Z$ is ample, it is also $\mathbb{Z}_{(p)}$-effective, thus $-(K_X+B)+f^*K_Z$ is $\mathbb{Z}_{(p)}$-effective. This also implies that there exists $\overline{m}' \in \mathbb{Z}_{>0}$, not divisible by $p$ such that $\mathrm{Bs}(-\overline{m}'(K_X+B)+\overline{m}'f^*K_Z) \subseteq \mathrm{Bs}(-\overline{m}'(K_X+B))$ does not dominate $Z$. Thus, we apply Theorem 5.2.2 and Corollary 6.1.9 to get:

$$\kappa(X_z, -(K_{X_z}+B_z))+\dim(Z) \leq \kappa(X, -(K_X+B)+f^*K_Z) \leq \kappa(X_z, -(K_{X_z}+B_z)).$$

Contradiction. Such case never happens.

For the "moreover" part, apply Proposition 5.2.6 with $L := -(K_X+B)$ qed

### 6.1.3. Partial results on $C^-_{n,n-1}$

The goal of this section is the proof of $C^-_{n,n-1}$ in positive characteristic when the relative dimension of the fibration is one and the target has zero anticanonical Iitaka dimension. In particular, the main result is an injectivity theorem similar to Corollary 6.1.9 for fibrations of relative dimension 1 and it is an analogue of [Cha23, Theorem 3.8]. In order to prove it in characteristic 0, the techniques involve the use of some canonical bundle formula results as in [Amb05]. Here, we use the canonical bundle formula Theorem 3.4.1, instead.

In this section, we work in the following setting.

(❊)Set-up

Let $f\colon X \to Z$ be a fibration between normal projective varieties over an algebraically closed field of characteristic $p>0$. Suppose that $f$ is of relative dimension one. Let $B$ be an effective $\mathbb{Z}_{(p)}$-divisor on $X$ such that $K_X+B$ is $\mathbb{Q}$-Cartier and let $D$ be a $\mathbb{Q}$-Cartier divisor on $Z$. Let $L := -(K_X+B)-f^*D$ and assume that:

(i) $Z$ is $\mathbb{Q}$-Gorenstein;

(ii) the general fibre $X_z$ is regular and the pair $(X_z, B_z)$ is strongly F-regular, where $B_z$ is defined by $(K_X+B)|_{X_z} = K_{X_z}+B_z$;

(iii) there exists $\overline{m} \in \mathbb{Z}_{>0}$ not divisible by $p$ such that $\mathrm{Bs}(\overline{m}L)$ does not dominate $Z$ (in particular, $L$ is $\mathbb{Z}_{(p)}$-effective).

**Proposition 6.1.11.** *In Set-up (❋), $-K_Z - D$ is $\mathbb{Q}$-effective.*

*Proof.* Pick a general fibre $X_z$. By condition $(iii)$ in Set-up (❋), the graded linear system $(V_m)_{m\in\mathbb{N}}$, with $|V_m| := |mL|_z \subseteq |mL_z|$ is $\mathbb{Z}_{(p)}$-semiample. By Corollary 1.3.28, there exists an effective $\mathbb{Z}_{(p)}$-divisor $\Gamma \sim_{\mathbb{Z}_{(p)}} L$ on $X$ such that $(X_z, B_z + \Gamma_z)$ is sharply F-pure. Note that, since $X_z$ is a regular curve, $(X_z, B_z + \Gamma_z)$ is log smooth. Moreover, $K_X + B + \Gamma \sim_{\mathbb{Z}_{(p)}} -f^*D$. Let $X_{\bar\eta}$ be the geometric generic fibre of $f$. By Theorem 2.2.13, $(X_{\bar\eta}, B_{\bar\eta} + \Gamma_{\bar\eta})$ is sharply F-pure as well. Since sharply $F$-pure singularities are in particular log canonical, we can apply Theorem 3.4.1 to find an effective $\mathbb{Q}$-divisor $\Delta_Z$ on $Z$, such that $-K_Z - D \sim_{\mathbb{Q}} \Delta_Z$. qed

*Remark* 6.1.12. Applying the above Proposition 6.1.11 to the case $D = 0$, we conclude that, in the assumptions of Set-up (❋), $-K_Z$ is $\mathbb{Q}$-effective whenever $-(K_X + B)$ is.

Following the proofs of [Cha23, Proposition 4.2, Proposition 4.3], we get the same results in positive characteristic in the more restrictive Set-up (❋).

**Proposition 6.1.13.** *In Set-up (❋), let $E$ be a $\mathbb{Q}$-Cartier divisor on $Z$. Assume there exists $\Gamma$, effective $\mathbb{Q}$-divisor on $X$, such that $L - f^*E \sim_{\mathbb{Z}_{(p)}} \Gamma$. Then, for $0 < \varepsilon \ll 1$, $-K_Z - D - \varepsilon E$ is $\mathbb{Q}$-effective.*

*Proof.* The proof follows the one of [Cha23, Proposition 4.2]. By Remark 1.3.25, we can choose $0 < \varepsilon \ll 1$ such that $\varepsilon\Gamma$ is a $\mathbb{Z}_{(p)}$-divisor and $(X_z, B_z + \varepsilon\Gamma_z)$ is SFR. Choose $\varepsilon \in \mathbb{Q}$, with denominator not divisible by $p$ and define

$$B_\varepsilon := B + \varepsilon\Gamma, \quad D_\varepsilon := D + \varepsilon E, \quad L_\varepsilon := -(K_X + B_\varepsilon) - f^*D_\varepsilon.$$

Note that $B_\varepsilon$ is an effective $\mathbb{Z}_{(p)}$-divisor and there exists $\bar{n} \in \mathbb{Z}_{>0}$ not divisible by $p$ such that $\mathrm{Bs}(\bar{n}\bar{m}L^\varepsilon)$ does not dominate $Z$. Now, apply Proposition 6.1.11 to $(X, B_\varepsilon)$, $D_\varepsilon$, $L_\varepsilon$. qed

**Theorem 6.1.14.** *In Set-up (❋), assume that $\kappa(Z, -K_Z - D) = 0$. Then, the map defined by restriction on a general fibre $X_z$,*

$$\alpha \colon \bigoplus_{m\in\mathbb{N}\setminus p\mathbb{N}} H^0(X, mlL) \to \bigoplus_{m\in\mathbb{N}\setminus p\mathbb{N}} H^0(X_z, mlL_z)$$

*is injective, where $l$ is a positive integer not divisible by $p$ such that $H^0(X, lL) \neq 0$.*

*Proof.* If the theorem did not hold, there would exist $s$, non-zero section, in the kernel of $\alpha$. Then, $s$ defines an effective $\mathbb{Z}_{(p)}$-divisor $N \sim_{\mathbb{Z}_{(p)}} -(K_X + B) - f^*D$ containing $X_z$ in its support. Since $\kappa(Z, -K_Z - D) = 0$, there exists a unique effective $\mathbb{Q}$-divisor $M \sim_{\mathbb{Q}} -K_Z - D$. Suppose that $f(X_z) = z \in Z$ is such that:

(i) $f$ is flat in a neighbourhood of $z$;

(ii) $z$ is a regular point of $Z$;

(iii) $z \notin \operatorname{Supp}(M)$;

(iv) $X_z \not\subseteq \operatorname{Supp}(B)$.

Note that all the conditions above are open on $Z$, therefore a general point $z \in Z$ does satisfy them. Consider the diagram:

$$\begin{tikzcd} H \subseteq X' \arrow[r, "\pi"] \arrow[d, "f'"'] & X \supseteq X_z \arrow[d, "f"] \\ E \subseteq Z' \arrow[r, "\mu"] & Z \ni z, \end{tikzcd}$$

where

- $\mu \colon Z' \to Z$ is the blow-up at $z$;
- $\pi \colon X' \to X$ is the blow-up at $X_z := f^{-1}(z)$, note that, by flatness of $f$ near $z$ and normality of the fibres, $X'$ is normal and it coincides with $X \times_Z Z'$;
- $H := \operatorname{Exc}(\pi) \cong X_z \times E = f'^*E$, with $E := \operatorname{Exc}(\mu)$.

Let $D' := \mu^*D$ and let $B'$ be the strict transform of $B$, then:

$$\begin{aligned} -K_{Z'} &= \mu^*(-K_Z) - aE, && a = \dim(Z) - 1; \\ -(K_{X'} + B') &= \pi^*(-(K_X + B)) - bH, && b = \operatorname{codim}(X_z) - 1 = a. \end{aligned}$$

By assumption, $\operatorname{Bs}(\overline{m}\pi^*L)$ does not dominate $Z'$ and

$$\pi^*L = \pi^*(-(K_X + B) - f^*D) = -(K_{X'} + B') + bf'^*E - f'^*D'.$$

Let $c$ be the coefficient of $H = f'^*E$ in $\pi^*N$. Since $X_z$ is in the support of $N$, $c > 0$ and since $N$ is a $\mathbb{Z}_{(p)}$-divisor, $c \in \mathbb{Z}_{(p)}$. The $\mathbb{Z}_{(p)}$-divisor $\pi^*N - cf'^*E \geq 0$ is effective, thus, by Proposition 6.1.13, there exists an effective $\mathbb{Q}$-divisor $\Gamma_{Z'}$ which is $\mathbb{Q}$-linearly equivalent to $-K_{Z'} + bE - D' - \varepsilon cE$, for some $0 < \varepsilon \ll 1$. But then we would have:

$$\Gamma_{Z'} + \varepsilon cE \sim_{\mathbb{Q}} \mu^*(-K_Z - D) \sim_{\mathbb{Q}} \mu^*M.$$

Both sides of the above equation are effective, the LHS has $E$ in its support, while the RHS does not. However, $\kappa(Z', \mu^*M) = 0$, contradiction. qed

**Corollary 6.1.15** ($C^-_{n,n-1}$). *In the above Set-up (✻), assume that the varieties are defined over a perfect field of characteristic $p > 0$ and that $\kappa(Z, -K_Z) = 0$. Then:*

$$\kappa(X, -(K_X + B)) = \kappa(X_z, -(K_{X_z} + B_z)).$$

*Proof.* First of all note that, by Remark 5.1.3, we can assume the varieties are defined over an algebraically closed field of characteristic $p > 0$.

Let $m \in \mathbb{Z}_{>0}$ such that $mL$ computes the Iitaka dimension. If $p|m$, since $L$ is $\mathbb{Z}_{(p)}$-effective, there is $n$ coprime with $p$ such that $H^0(X, nL) \neq 0$. Thus $(m+n)L$ computes the Iitaka dimension as well and $m+n$ is not divisible by $p$. Therefore, we conclude by Theorem 6.1.14 and Proposition 5.2.6. qed

### 6.1.4. Proof of $C^-_{3,m}$

Here we use the results of the previous sections to prove $C^-_{3,m}$ over fields of positive characteristic.

One of the main technical difficulties in extending the proof of [Cha23, Theorem 4.1] in positive characteristic is that it is hard to control the singularities of the fibres of a fibration. The next lemma is needed to tackle this problem.

In this section we consider varieties defined over a perfect field, where we specify the characteristic for each result.

**Lemma 6.1.16.** *Let $S$ be a normal projective $\mathbb{Q}$-factorial surface over a perfect field of any characteristic, such that $\kappa(S, -K_S) = 1$ and let $g\colon S \dashrightarrow C$ be the rational map induced by the linear system $|-mK_S|$ for $m \gg 0$. Write $|-mK_S| = |M| + B$, where $|M|$ is the movable part (i.e. its base locus has codimension $\geq 2$) and $B$ is the fixed divisor.*
*Then $g$ is a morphism well-defined everywhere, it coincides with the Iitaka fibration of $-K_S$ and it is a (quasi-)elliptic fibration. Moreover, $M \sim_{\mathbb{Q}} g^*A$ for an ample $\mathbb{Q}$-divisor $A$ on $C$ and the support of $B$ does not dominate $C$.*

*Proof.* Since the base locus of $M$ has dimension 0, $M$ is semiample by Zariski–Fujita's Theorem [Fuj80, Theorem 2.8]. By possibly substituting it with a multiple, it is then linearly equivalent to $g^*A$, for an ample divisor $A$ on $C$. Let $B^h$ and $B^v$ be effective divisors decomposing $B$ into its horizontal and vertical components, respectively. We need to show that $B^h = 0$. Suppose this was not true. Then, the divisor $B^h$ would be relatively big, thus there exist $\mathbb{Q}$-divisors $H$ and $E$ such that $H$ is effective and relatively ample, $E$ is effective and $B^h \sim_{\mathbb{Q}} H + E$. Indeed, for $A'$ effective and relatively ample, we have the exact sequence

$$0 \to g_*\mathcal{O}_S(mB^h - A') \to g_*\mathcal{O}_S(mB^h) \to g_*\mathcal{O}_S(mB^h)|_{A'}.$$

For $m \to \infty$, if $B^h$ was relatively big, the rank of $g_*\mathcal{O}_S(mB^h)$ would grow as $m$, while the rank of $g_*\mathcal{O}_S(mB^h)|_{A'}$ is bounded, therefore the map $g_*\mathcal{O}_S(mB^h) \to g_*\mathcal{O}_S(mB^h)|_{A'}$ cannot be injective for $m \gg 0$, whence $g_*\mathcal{O}_S(mB^h - A') \neq 0$. Now write, for $0 < \varepsilon \ll 1$,

$$-mK_S \sim_{\mathbb{Q}} (g^*A + \varepsilon H) + (1-\varepsilon)H + E + B^v.$$

By the Nakai–Moishezon criterion, $g^*A + \varepsilon H$ is ample and $(1-\varepsilon)H + E + B^v$ is effective, showing that $-K_S$ is big. Contradiction.
Therefore, $g$ can be extended everywhere and it coincides with the Iitaka fibration of $-K_S$. Since $B$ is vertical, the general fibre of $g$ has arithmetic genus 1. qed

**Theorem 6.1.17** $(C^-_{3,m})$**.** *Let $f\colon X \to Z$ be a fibration from a normal projective threefold $X$ to a normal projective $\mathbb{Q}$-factorial variety $Z$, over a perfect field of characteristic $p \geq 5$. Let $B$ be an effective $\mathbb{Z}_{(p)}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Suppose that the general fibre $X_z$ is regular and the pair $(X_z, B_z)$ is strongly $F$-regular, where $B_z$ is defined by $(K_X + B)|_{X_z} = K_{X_z} + B_z$. Assume moreover that there exists an integer $\overline{m} \geq 1$ not divisible by $p$ such that $\mathrm{Bs}(-\overline{m}(K_X + B))$ does not dominate $Z$. Then,*

$$\kappa(X, -(K_X + B)) \leq \kappa(X_z, -(K_{X_z} + B_z)) + \kappa(Z, -K_Z).$$

*Moreover, if $\kappa(Z, -K_Z) = 0$, equality holds.*

*Proof.* First of all, by Remark 5.1.3 we can assume that the base field $k$ is algebraically closed.

Then, note that we can also assume $\kappa(X, -(K_X + B)) \geq 0$. If $Z$ is a curve, the result holds by Theorem 6.1.10. Thus, we only need to consider what happens when $Z$ is a surface. By Proposition 6.1.11, $-K_Z$ is $\mathbb{Q}$-effective. If $\kappa(Z, -K_Z) = 0$, we conclude by Corollary 6.1.15. If $\kappa(Z, -K_Z) = 2$, the Easy Additivity theorem gives the conclusion. We are therefore only left with the case $\kappa(Z, -K_Z) = 1$. We want to reduce to a situation where we can apply Corollary 6.1.15. To do this, we consider the Iitaka fibration of $Z$ following the ideas of [Cha23, Theorem 4.1].

Fix an $m \gg 0$ and let $g\colon Z \to C$ be the Iitaka fibration induced by $|-mK_Z|$. By Lemma 6.1.16, this is well-defined everywhere. Write $-mK_Z \sim g^*A + B$, where $B$ is the fixed vertical divisor and $A$ is ample on $C$. Call $X_c$ the general fibre of $h := g \circ f$, it is reduced by Proposition 2.1.2 since $h$ is a fibration to a curve, hence separable by Theorem 2.1.4. Let $\nu\colon X_c^\nu \to X_c \subseteq X$ be its normalisation and $\varphi\colon X_c^\nu \to Z_c$ the induced morphism obtained via restriction from $f$, where $Z_c$ is a general fibre of $g$. By the Easy Additivity theorem:

$$\begin{aligned}\kappa(X, -(K_X + B)) &\leq \kappa(X_c^\nu, -\nu^*(K_X + B)) + \dim(C) \\ &= \kappa(X_c^\nu, -\nu^*(K_X + B)) + \kappa(Z, -K_Z).\end{aligned}$$

To conclude, it is enough to prove that $\kappa(X_c^\nu, -\nu^*(K_X + B)) \leq \kappa(X_z, -(K_{X_z} + B_z))$ for a general fibre $X_z$ of $f$. By the above Lemma 6.1.16, the arithmetic genus of a general fibre of $g$ is 1, so, since the characteristic of the base field is $\geq 5$ by assumption, it is smooth because quasi-elliptic fibrations exist only in characteristics 2 and 3 by [Tat52, Corollary 1]. Moreover, since the general fibre of $f$ is smooth, there exists $U \subseteq Z_c$ such that $f^{-1}(U) \subseteq X_c$ is smooth. In particular, $X_c^\nu \to X_c$ is an isomorphism over $U$. Therefore, if we define a $\mathbb{Q}$-divisor $B_{X_c^\nu}$ by $K_{X_c^\nu} + B_{X_c^\nu} = (K_X + B)|_{X_c^\nu}$, then

$$(K_{X_c^\nu} + B_{X_c^\nu})|_{X_z} = K_{X_z} + B_z,$$

where $X_z$ is a fibre of $\varphi$ over a point in $U$. By [PW22, Corollary 1.3], there exists an effective Weil divisor $\mathcal{C}$ such that

$$K_{X_c^\nu} + \mathcal{C} = K_X|_{X_c^\nu},$$

thus $B_{X_c^\nu} = \nu^* B + \mathcal{C}$ is a $\mathbb{Z}_{(p)}$-divisor. Note that, since $X_c$ is $S_2$, the restriction of a $\mathbb{Q}$-Weil divisor on $X_c$ is well-defined. Indeed, if $X^{\mathrm{sm}}$ is the regular locus of $X$, $X^{\mathrm{sm}} \cap X_c$ has still complement of codimension $\geq 2$ for $X_c$ general. Let us study the base locus of $-\overline{m}(K_{X_c^\nu} + B_{X_c^\nu})$. Note that a general fibre of $g$ is not contained in $f(\mathrm{Bs}(-\overline{m}(K_X + B)))$. Indeed, fix one of these fibres, say $Z_c$, and let $X_c$ be the fibre of $h$ over $Z_c$. There exist effective divisors $D_1, \dots, D_\ell \sim -\overline{m}(K_X + B)$, such that $Z_c \not\subseteq f(\mathrm{Supp}(D_1) \cap \dots \cap \mathrm{Supp}(D_\ell))$. Since $0 \leq \nu^* D_i \sim -\overline{m}(K_{X_c^\nu} + B_{X_c^\nu})$, we conclude that the base locus of $-\overline{m}(K_{X_c^\nu} + B_{X_c^\nu})$ does not dominate $Z_c$. Thus, the fibration $\varphi$ satisfies:

- $B_{X_c^\nu}$ is an effective $\mathbb{Z}_{(p)}$-divisor;
- the general fibre $X_z$ of $\varphi$ is regular and $(X_z, B_z)$ is SFR;
- there exists an integer $\overline{m} \geq 1$ not divisible by $p$ such that $\mathrm{Bs}(-\overline{m}(K_{X_c^\nu} + B_{X_c^\nu}))$ does not dominate $Z_c$;
- $Z_c$ is smooth and $\kappa(Z_c, -K_{Z_c}) = 0$.

We are in the right setting to apply Corollary 6.1.15, whence:

$$\kappa(X_c^\nu, -\nu^*(K_X + B)) \leq \kappa(X_z, -(K_{X_z} + B_z)).$$

As for the "moreover" part, apply Proposition 5.2.6 to $L := -(K_X + B)$. qed

## 6.2. $C^-_{n,m}$ for $F$-split fibrations

### 6.2.1. $K$-globally $F$-regular varieties and a Weak Ordinarity conjecture

In [BBC23], we introduce a new class of $F$-singularities, namely *$K$-globally $F$-regular* varieties, that interpolates between globally $F$-split and globally $F$-regular varieties. The advantage of these singularities is that they are stable under small perturbations of the boundary by elements of the anticanonical $\mathbb{Q}$-linear system (Proposition 6.2.2) and that, for fibrations with $K$-globally $F$-regular fibres, a canonical bundle formula applies. We conjecture that $K$-globally $F$-regular varieties satisfy a *Weak Ordinarity* statement, namely, that klt pairs with semiample anticanonical bundle are $K$-globally $F$-regular when reduced modulo a dense set of primes.

In this section we consider varieties defined over a perfect field of characteristic $p > 0$, unless otherwise stated.

**Definition 6.2.1.** A projective $\mathbb{Z}_{(p)}$-pair $(X, B)$ is said to be $K$**-globally F-regular** or **KGFR** if

(a) $-K_X - B$ is semiample, with induced fibration $f\colon X \to Z$;

(b) $(X_{\bar{\eta}}, B_{\bar{\eta}})$ is globally $F$-split, where $\eta \in Z$ is the generic point;

(c) $K_X + B \sim_{\mathbb{Z}_{(p)},Z} 0$;

(d) $(Z, B^Z)$ is globally $F$-regular, where $B^Z$ is the $F$-discriminant, as in Definition 3.4.5.

Note that $(X, B)$ is globally $F$-split *a posteriori* thanks to Proposition 3.4.3(iii).

**Proposition 6.2.2.** *Let $(X, B)$ be a projective $K$-globally $F$-regular $\mathbb{Z}_{(p)}$-pair and let $\Lambda$ be an element of $|-K_X - B|_{\mathbb{Z}_{(p)}}$. Then $(X, B + \varepsilon\Lambda)$ is $K$-globally $F$-regular for all sufficiently small positive $\varepsilon \in \mathbb{Z}_{(p)}$.*

*Proof.* Let $f\colon X \to Z$ be the semiample contraction of $-K_X - B$. Let $B^Z$ be the divisor induced on $Z$ by Proposition 3.4.3. Note that $\Lambda = f^*\Lambda'$ for some $\mathbb{Z}_{(p)}$-divisor $\Lambda' \geq 0$ on $Z$ and $(B+\varepsilon\Lambda)^Z = B^Z+\varepsilon\Lambda'$ by Proposition 3.4.3(iv). We conclude because $-(K_X + B + \varepsilon\Lambda) \sim_{\mathbb{Z}_{(p)}} -(1-\varepsilon)(K_X + B)$ is semiample with induced fibration $f$, the pair $(X_{\bar{\eta}}, (B+\varepsilon\Lambda)_{\bar{\eta}} = B_{\bar{\eta}})$ is GFS and, for all sufficiently small positive $\varepsilon \in \mathbb{Z}_{(p)}$, $(Z, B^Z + \varepsilon\Lambda')$ is GFR by Lemma 1.3.12 qed

This is an analogue of Proposition 3.4.10 for $K$-globally $F$-regular singularities.

**Proposition 6.2.3.** *Let $f\colon X \to Y$ be a surjective projective morphism of normal varieties, such that* $\mathrm{St.deg}(f)$ *is not divisible by $p$ (see Definition 2.1.5), and let $B \geq 0$ be a $\mathbb{Z}_{(p)}$-divisor on $X$. Assume $(X, B)$ is globally $F$-split and $(1-p^e)(K_X + B) \sim_Y 0$. Then there exists a canonically determined effective $\mathbb{Z}_{(p)}$-divisor $B^Y$ on $Y$ such that, if $(X, B)$ is $K$-globally $F$-regular, then $(Y, B^Y)$ is globally $F$-regular.*

*Proof.* It is enough to apply Proposition 3.4.3 and Proposition 3.4.7 to the Stein factorisation $f\colon X \xrightarrow{g} Z \xrightarrow{h} Y$. qed

Now, we give examples of $K$-globally $F$-regular pairs.

*Example* 6.2.4. Let $C$ be a smooth projective curve. Then $C$ is KGFR if and only if either $C = \mathbb{P}^1$ or $C$ is an ordinary elliptic curve.

*Example* 6.2.5. Let $S$ be a normal projective surface such that $-K_S$ is semiample.

(1) If $\kappa(S, -K_S) = 0$, then $S$ is KGFR if and only if it is GFS.

(2) If $\kappa(S, -K_S) = 1$, let $f\colon S \to C$ be the fibration induced by $-K_S$. Then $S$ is KGFR if and only if the following conditions hold.

- The general fibre is an ordinary elliptic curve; this is always the case if $f$ is an elliptic non-isotrivial fibration.
- The base $C = \mathbb{P}^1$; this is automatically satisfied since $-K_C$ is big by the canonical bundle formula Proposition 3.4.3.
- The pair $(C, B^C)$, induced with Proposition 3.4.3 for $B = 0$, is GFR; in general this condition is hard to control. However, below we will see two examples (Proposition 6.2.7 and Example 6.2.9) where we check it explicitly.

(3) If $\kappa(S, -K_S) = 2$, then $S$ is KGFR if and only if it is the crepant blow-up of some GFR surface.

The following is an analogue of Conjecture 1.3.20 for $K$-globally $F$-regular singularities.

**Conjecture 6.2.6** (Relative Weak Ordinarity)**.** *Let $(X, B)$ be a projective klt pair over $\mathbb{C}$ such that $-(K_X + B)$ is semiample. Let $A$ be a finitely generated $\mathbb{Z}$-algebra. Then, for every model $(\mathcal{X}, \mathcal{B}) \to \operatorname{Spec}(A)$ of $(X, B)$, the set of primes $\mathfrak{p} \subseteq A$ such that $(X_{\bar{\mathfrak{p}}}, B_{\bar{\mathfrak{p}}})$ is $K$-globally $F$-regular, is dense in* $\operatorname{Spec}(A)$*.*

The heuristic behind this conjecture is that, given a fibration $f\colon X \to Z$ such that $K_X + B \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-Cartier divisor $L$, we can apply the canonical bundle formula in characteristic 0 to write $L = K_Z + B_Z + M_Z$, where $B_Z$ is the discriminant part, $M_Z$ is the moduli part, and $(Z, B_Z)$ is log Fano [Amb04, Theorem 3.1]. If $f_p\colon X_p \to Z_p$ is the reduction of $f$ modulo $p$, we compute the $F$-discriminant $B^{Z_p}$ with Proposition 3.4.3. Now, we compare $B^{Z_p}$ with the reduction of $B_Z$ modulo $p$, call it $B_{Z,p}$. The difference between $B_{Z,p}$ and $B^{Z_p}$ is controlled by the reduction modulo $p$ of the moduli part $M_Z$. Assuming Conjecture 1.3.20, by [DS17, Theorem 6.2], for every prime divisor $\delta \subseteq Z$, there exist infinitely many primes $p > 0$ such that $B_{Z,p}$ coincides with $B^{Z_p}$ around $\delta$. This is saying that the moduli part is somehow semiample "for a general prime".

Now, we give some evidence for the conjecture. First, we prove that, if $-(K_X+B)$ induces an isotrivial fibration onto $\mathbb{P}^1$, Conjecture 6.2.6 holds assuming Conjecture 1.3.20.

**Proposition 6.2.7.** *Assume Conjecture 1.3.20. Let $(X, B)$ be a projective klt pair over $\mathbb{C}$ such that $-(K_X + B)$ is semiample. Let $f\colon X \to Z$ be the induced fibration and write $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$ where $B_Z$ is the discriminant part and*

*$M_Z$ the moduli part in the canonical bundle formula as in Definition 3.2.6. Assume $Z = \mathbb{P}^1$ and $M_Z = 0$. Then, for every model $(\mathcal{X}, \mathcal{B}) \to \operatorname{Spec}(A)$ of $(X, B)$, the set of primes $\mathfrak{p} \subseteq A$ such that $(X_{\bar{\mathfrak{p}}}, B_{\bar{\mathfrak{p}}})$ is $K$-globally $F$-regular is dense in $\operatorname{Spec}(A)$.*

*Proof.* By construction, there exists $H$ ample divisor on $Z$ such that $K_X + B \sim_{\mathbb{Q}} -f^*H$. By [Amb04, Theorem 3.1], $(Z, B_Z)$ is klt and log Fano. Let $\varphi \colon \mathcal{X} \to \mathcal{Z}$ be a model of $f$ over $\operatorname{Spec}(A)$ and denote by $f_{\bar{\mathfrak{p}}} \colon X_{\bar{\mathfrak{p}}} \to Z_{\bar{\mathfrak{p}}}$ its reduction modulo a prime $\mathfrak{p}$ of $\operatorname{Spec}(A)$. If $D$ is a divisor on $X$ or on $Z$, we can assume we can lift it to the corresponding model and we denote by $D_{\bar{\mathfrak{p}}}$ its reduction modulo $\mathfrak{p}$. For a dense open set of primes $\mathfrak{p}$, $f_{\bar{\mathfrak{p}}}$ is a separable fibration between normal projective varieties and there is $e_{\mathfrak{p}}$ such that $(1-p^{e_{\mathfrak{p}}})(K_{X_{\bar{\mathfrak{p}}}} + B_{\bar{\mathfrak{p}}}) \sim_{Z_{\bar{\mathfrak{p}}}} 0$, where $p$ is the characteristic of the residue field at $\mathfrak{p}$. Up to possibly replacing $A$ with $A'$, a finite extension of the localisation of $A$ at finitely many primes, we can assume there exists a section $\operatorname{Spec}(A') \to \mathcal{Z}$. Let $\mathcal{X}_{A'} \to \operatorname{Spec}(A')$ be the induced model of a fibre, where $\mathcal{X}_{A'} := \mathcal{X} \times_{\mathcal{Z}} \operatorname{Spec}(A')$. By Conjecture 1.3.20, $X_{A',\bar{\mathfrak{p}}}$ is GFS for a dense set of primes of $\operatorname{Spec}(A')$. Applying Lemma 2.2.12, we conclude that the geometric generic fibre of $f_{\bar{\mathfrak{p}}}$ is GFS for a dense set of primes $\mathcal{P}$ of $\operatorname{Spec}(A)$.

Moreover, by Theorem 1.3.19 there exists $p_0 \in \mathbb{N}$ such that $(Z_{\bar{\mathfrak{p}}}, B_{Z,\bar{\mathfrak{p}}})$ is GFR for all $\mathfrak{p}$ with residue field of characteristic $p \geq p_0$. Thus, we can apply Proposition 3.4.3 to get an effective $\mathbb{Q}$-divisor $B_{\bar{\mathfrak{p}}}^{Z_{\bar{\mathfrak{p}}}}$ on $Z_{\bar{\mathfrak{p}}}$ such that $K_{X_{\bar{\mathfrak{p}}}} + B_{\bar{\mathfrak{p}}} \sim_{\mathbb{Q}} f_{\bar{\mathfrak{p}}}^*(K_{Z_{\bar{\mathfrak{p}}}} + B_{\bar{\mathfrak{p}}}^{Z_{\bar{\mathfrak{p}}}})$. By [DS17, Proposition 5.7] we have $B_{\bar{\mathfrak{p}}}^{Z_{\bar{\mathfrak{p}}}} \geq B_{Z,\bar{\mathfrak{p}}}$. As $B_{\bar{\mathfrak{p}}}^{Z_{\bar{\mathfrak{p}}}} \sim_{\mathbb{Q}} B_{Z,\bar{\mathfrak{p}}}$, we conclude $B_{\bar{\mathfrak{p}}}^{Z_{\bar{\mathfrak{p}}}} = B_{Z,\bar{\mathfrak{p}}}$. This implies that $(X_{\bar{\mathfrak{p}}}, B_{\bar{\mathfrak{p}}})$ is KGFR for all $\mathfrak{p} \in \mathcal{P}$ such that the characteristic of the residue field is $p \geq p_0$. qed

*Remark* 6.2.8. If $B = 0$ and the induced fibration $f \colon X \to Z$ has elliptic fibres, the above Proposition 6.2.7 holds unconditionally. In fact, Conjecture 1.3.20 is known to hold for elliptic curves ([Har77, Remark 4.23.4, Chapter IV]).

*Example* 6.2.9. In this example we check that Conjecture 6.2.6 holds for the Legendre family.

Let $S$ be the projective closure of $V(y^2 - x(x-1)(x-\lambda))$ inside $\mathbb{P}^2_{\mathbb{Z},[x:y:z]} \times \mathbb{P}^1_{\mathbb{Z},[\lambda:\mu]}$ over $\operatorname{Spec}(\mathbb{Z})$ and let $f \colon S \to C = \mathbb{P}^1_{\mathbb{Z}}$ be the induced projection. We will denote by $f_0 \colon S_0 \to C_0$ the fibration over $\operatorname{Spec}(\mathbb{C})$ and by $f_p \colon S_p \to C_p$ the fibration over $\operatorname{Spec}(\overline{\mathbb{F}}_p)$. The canonical bundle of $S_0$ is $\mathcal{O}_{\mathbb{P}^2_0 \times \mathbb{P}^1_0}(0,-1)|_{S_0}$, and the same formula holds for the canonical bundle of $S_p$. In particular, $f_0$ and $f_p$ are the fibrations induced by the anticanonical divisors.

By Example 3.1.6, the discriminant part of $f_0$ is $\frac{1}{2}(\infty)$. Now, let us consider what happens over $\operatorname{Spec}(\overline{\mathbb{F}}_p)$ for $p \geq 3$. By Example 3.4.6, if $B_p^{C_p}$ is the $F$-discriminant computed with the canonical bundle formula as in Proposition 3.4.3,

$$B_p^{C_p} = \frac{1}{2}(\infty) + \frac{1}{p-1} \sum_{\lambda \in \Lambda_p} (\lambda),$$

where $\Lambda_p$ is the set of those $\lambda$'s corresponding to supersingular elliptic curves.

By [HW02, Lemma 3.1], the pairs $(\mathbb{P}^1_p, B_p^{C_p})$ are GFR, whence $S_p$ is KGFR.

With the help of [GS], we can actually compute explicitly which $\lambda$'s belong to $\Lambda_p$ by looking at the $j$-invariant of the corresponding elliptic curve since there is a classification of the $j$-invariants corresponding to supersingular elliptic curves in characteristic $p$ (see [BK06, Table 6]). Note that, if $j = 0$, we will have two corresponding $\lambda$'s, if $j = 1728$, there are three of them and if $j \neq 0, 1728$, there are six of them. Recall that a pair $(X, B)$ is GFR if and only if the affine cone over it with the corresponding $\mathbb{Q}$-divisor $(C_X, C_B)$ is GFR around the vertex [SS10, Proposition 5.3]. In our case this amounts to checking whether $(\mathbb{A}^2_p, L)$ is GFR around the origin, where $L$ is the $\mathbb{Q}$-divisor of lines passing through the origin corresponding to $B_p^{C_p}$. Let $N := \frac{p-1}{2}$ and define the ideal

$$I_p := \left( y^N \cdot \prod_{\lambda \in \Lambda_p} (x - \lambda y) \right).$$

At this point, we can check with [GS] whether $(k[x,y], I^{\frac{1}{2N}})$ is GFR, where $k$ is an appropriate extension of $\mathbb{F}_p$ where all $\lambda \in \Lambda_p$ are defined. Here the script for $p = 5$ and a table with the results for $3 \leq p \leq 23$.

```
i1: needsPackage "RationalPoints2"; needsPackage "TestIdeals";
i3: k=GF 25; R=k[t];
i5: k[x]; rationalPoints(ideal(x^2-x+1))
o6: {{-2a-1}, {2a+2}}
i7: f= t^4*((t-2*a-2)^3)*((t+2*a+1)^3);
i8: isFRegular(1/12,f)
o8: true
```

| p | Supersingular $j$-invariants | $N$ | isFRegular |
|---|---|---|---|
| 3 | 1728 | 1 | true |
| 5 | 0 | 2 | true |
| 7 | 1728 | 3 | true |
| 11 | 0, 1728 | 5 | true |
| 13 | 5 | 6 | true |
| 17 | 0,8 | 8 | true |
| 19 | 7, 1728 | 9 | true |
| 23 | 0, 19, 1728 | 11 | true |

### 6.2.2. $F$-complements

We introduce a positive characteristic analogue of the notion of *complement* (see [Sho93, §5]). When $X$ is a normal projective variety, a complement is an effective $\mathbb{Q}$-divisor $\Lambda \in |-K_X|_{\mathbb{Q}}$ such that $(X, \Lambda)$ is log canonical.

In this section we consider varieties defined over an algebraically closed field $k$ of characteristic $p > 0$.

**Definition 6.2.10.** Let $f\colon X \to Z$ be a fibration of normal projective varieties, and let $B$ be a $\mathbb{Z}_{(p)}$-divisor such that $\mathrm{Supp}(B^-)$ does not dominate $Z$ and the geometric generic fibre $(X_{\bar{\eta}}, B_{\bar{\eta}})$ is globally $F$-split. Let $L$ be a $\mathbb{Z}_{(p)}$-divisor on $X$ such that $K_X + B + L \sim_{\mathbb{Z}_{(p)},Z} 0$. We say $L$ **admits an $F$-complement for** $(X/Z, B)$ if there exists $\Lambda \in |L|_{\mathbb{Z}_{(p)}}$ such that $(X_{\bar{\eta}}, B_{\bar{\eta}} + \Lambda_{\bar{\eta}})$ is globally $F$-split. We then say $\Lambda$ is an **F-complement for** $(X/Z, B)$. When $Z = \mathrm{Spec}(k)$ we refer to $\Lambda$ as an $F$-complement for $(X, B)$.

By results of Schwede and Smith we have that globally $F$-split couples admit $F$-complements.

**Theorem 6.2.11** ([SS10, Theorem 4.3(ii)])**.** *Let $(X, B)$ be a projective $\mathbb{Z}_{(p)}$-couple. If $(X, B)$ is globally $F$-split, then it admits an $F$-complement.*

We give a sufficient condition for the existence of $F$-complements on fibrations with $K$-globally $F$-regular fibres.

**Theorem 6.2.12.** *Let $f\colon X \to Z$ be a fibration of normal projective varieties with general fibre $X_z$, and let $B$ be a $\mathbb{Z}_{(p)}$-divisor on $X$ such that $\mathrm{Supp}(B^-)$ does not dominate $Z$. Let $D$ be a $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor on $Z$, set $L := -K_X - B - f^*D$. Assume*

*(a) $(X_z, B_z)$ is $K$-globally $F$-regular;*

*(b) there exists an integer $m \geq 1$ not divisible by $p$ such that $mL$ is integral and $mL_z$ is Cartier;*

*(c) there exists $|V| \subseteq |mL|$ such that $\phi_{|V|_z}$ is a morphism;*

*(d) $p$ does not divide $\mathrm{St.deg}(\phi_{|V|_z})$ (see Definition 2.1.5).*

*Then $L$ admits an $F$-complement for $(X/Z, B)$.*

Before we give a proof, we need the following result.

**Proposition 6.2.13.** *Let $G$ be a non-normal projective variety, let $A$ be an ample $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor on $G$, let $\nu\colon G^\nu \to G$ be the normalisation morphism, and let $\Delta^\nu \geq 0$ be a $\mathbb{Z}_{(p)}$-divisor on $G^\nu$ such that*

*(a) $(G^\nu, \Delta^\nu)$ is a globally $F$-regular pair;*

*(b) $-K_{G^\nu} - \Delta^\nu \sim_{\mathbb{Z}_{(p)}} \nu^*A$.*

*Then there exists an $F$-complement $\Gamma^\nu$ for $(G^\nu, \Delta^\nu)$ such that $\Gamma^\nu = \nu^*\Gamma$ for some $\Gamma \in |A|_{\mathbb{Z}_{(p)}}$.*

*Proof.* Let $m = p^e - 1$ for $e \geq 1$ divisible enough, so that $L := mA$ and $L^\nu := m(-K_{G^\nu} - \Delta^\nu) \sim \nu^* L$ are both Cartier. Let $\mathcal{C} := \operatorname{Ann}_{\mathcal{O}_G}(\nu_* \mathcal{O}_{G^\nu}/\mathcal{O}_G) \subseteq \mathcal{O}_G$ be the conductor ideal, and let $\mathcal{C}^\nu := \mathcal{O}_{G^\nu} \cdot \mathcal{C} \subseteq \mathcal{O}_{G^\nu}$ so that, for all $l \geq 0$, we have isomorphisms induced by pullback

(✣) $$\nu^\sharp \otimes \mathcal{O}_G(lL) \colon \mathcal{O}_G(lL) \otimes \mathcal{C} \to \nu_*(\mathcal{O}_{G^\nu}(lL^\nu) \otimes \mathcal{C}^\nu).$$

Let now $R$ be an effective Cartier divisor on $G^\nu$ such that $\mathcal{O}_{G^\nu}(-R) \subseteq \mathcal{C}^\nu$. For all $l = p^d - 1$ with $d \geq 1$ sufficiently divisible, we have that $(G^\nu, \Delta^\nu + R/l)$ is still GFR by Lemma 1.3.12, thus we have an $F$-complement $\Xi^\nu$ for $(G^\nu, \Delta^\nu + R/l)$ by Theorem 6.2.11. In particular, $\Gamma^\nu := \Xi^\nu + R/l$ is an $F$-complement for $(G^\nu, \Delta^\nu)$. For some $n \geq 1$ not divisible by $p$ we have that $lmn\Gamma^\nu$ is the divisor of a section $\gamma^\nu \in H^0(G^\nu, \mathcal{O}_{G^\nu}(lnL^\nu) \otimes \mathcal{C}^\nu)$, and (✣) shows that $\gamma^\nu = \nu^*\gamma$ for some $\gamma \in H^0(G, \mathcal{O}_G(lnL) \otimes \mathcal{C})$. We conclude by setting $\Gamma := (\gamma)/lmn$, where $(\gamma)$ is the divisor defined by $\gamma$. qed

*Proof of Theorem 6.2.12.* Up to replacing $m$ with a multiple we may assume $m = p^e - 1$ for some $e \geq 1$. Throughout the rest of the proof we will freely implicitly replace $m$ with $md$, where $d$ is not divisible by $p$ (equivalently, replace $e$ with a multiple). Consider the following diagram

(✶) $$\begin{tikzcd} & G^\nu \arrow[dr, "\nu"] & & \\ X_z \arrow[ur, "\varphi"] \arrow[r, "\phi_{|V|_z}"] \arrow[d, hook] & G \arrow[d, hook] \arrow[dr, hook] & \\ & & \mathbb{P}V^* \\ X \arrow[r, dashed, "\phi_{|V|}"] & W \arrow[ur, hook] & \end{tikzcd}$$

where $\nu$ is the normalisation and $\varphi$ is the natural morphism. Since $(X_z, B_z)$ is KGFR, by Proposition 6.2.3 there is an effective $\mathbb{Z}_{(p)}$-divisor $B_z^{G^\nu}$ on $G^\nu$, such that $(G^\nu, B_z^{G^\nu})$ is GFR and $-m(K_{X_z} + B_z) \sim \varphi^*(-m(K_{G^\nu} + B_z^{G^\nu}))$. By Proposition 6.2.13 there is an $F$-complement $\Lambda_{G^\nu} \in |-(K_{G^\nu} + B_z^{G^\nu})|_{\mathbb{Z}_{(p)}}$ for $(G^\nu, B_z^{G^\nu})$ such that $\Lambda_{G^\nu} = \nu^*\Lambda_G$ for some $\Lambda_G \in |\mathcal{O}_G(1)/m|_{\mathbb{Z}_{(p)}}$. Letting $\Lambda_z := (\nu \circ \varphi)^*\Lambda_G$ we then have that $(X_z, B_z + \Lambda_z)$ is GFS and $K_{X_z} + B_z + \Lambda_z \sim_{\mathbb{Z}_{(p)}} 0$, by Proposition 3.4.10. Then (✶) induces the following diagram on global sections for all $l \geq 0$:

$$
\begin{tikzcd}
H^0(X_z, \mathcal{O}_{X_z}(mlL_z)) & H^0(G, \mathcal{O}_G(l)) \arrow[l, "\phi^*_{|V|_z}"'] & \\
& & H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(l)) \arrow[ul] \arrow[dl] \\
H^0(X, \mathcal{O}_X(mlL)) \arrow[uu] & H^0(W, \mathcal{O}_W(l)). \arrow[l, "\phi^*_{|V|}"'] \arrow[uu] &
\end{tikzcd}
$$

As the two rightmost maps are surjective for all $l \gg 0$ by Serre Vanishing, we conclude we can lift $\Lambda_z$ to $\Lambda \in |L|_{\mathbb{Z}_{(p)}}$. As $(X_z, B_z + \Lambda_z)$ is GFS, we have that $(X_{\overline{\eta}}, B_{\overline{\eta}} + \Lambda_{\overline{\eta}})$ is also GFS by Lemma 2.2.12. qed

Thanks to the $K$-globally $F$-regular condition on the fibres, we are able to find $F$-complements even after a small perturbation of the boundary.

**Corollary 6.2.14.** *Let $f\colon X \to Z$ be an equidimensional fibration of normal projective varieties with general fibre $X_z$. Let $B$ be a $\mathbb{Z}_{(p)}$-divisor on $X$ such that* $\operatorname{Supp}(B^-)$ *does not dominate $Z$. Let $D$ be a $\mathbb{Z}_{(p)}$-divisor on $Z$, set $L := -K_X - B - f^*D$. Assume*

*(a) $(X_z, B_z)$ is $K$-globally $F$-regular,*

*(b) there exists an integer $m \geq 1$ not divisible by $p$ such that $mL$ is integral and $mL_z$ is Cartier,*

*(c) there exists $|V| \subseteq |mL|$ such that $\phi_{|V|_z}$ is a morphism,*

*(d) $p$ does not divide* $\mathrm{St.deg}(\phi_{|V|_z})$.

*Let $E$ be a $\mathbb{Z}_{(p)}$-divisor on $Z$ and suppose there exists a $\mathbb{Z}_{(p)}$-divisor $0 \leq \Gamma \sim_{\mathbb{Z}_{(p)}} L - f^*E$. Then $(1-\varepsilon)L$ admits an $F$-complement for $(X/Z, B+\varepsilon\Gamma)$ for all sufficiently small positive $\varepsilon \in \mathbb{Z}_{(p)}$.*

*Proof.* Let

$$B_\varepsilon := B + \varepsilon\Gamma, \quad D_\varepsilon := D + \varepsilon E, \quad L_\varepsilon := -K_X - B_\varepsilon - f^*D_\varepsilon$$

so that $L_\varepsilon \sim_{\mathbb{Z}_{(p)}} (1-\varepsilon)L$. The corollary will follow from Theorem 6.2.12 as soon as we verify that

(A) $(X_z, B_{\varepsilon,z})$ is KGFR,

(B) there is $n \geq 1$ not divisible by $p$ such that $nL_\varepsilon$ is integral and $nL_{\varepsilon,z}$ is Cartier,

(C) there exists $|V_\varepsilon| \subseteq |nL_\varepsilon|$ such that $\phi_{|V_\varepsilon|_z}$ is a morphism,

(D) $p$ does not divide $\mathrm{St.deg}(\phi_{|V_\varepsilon|_z})$.

Let $\psi\colon X_z \to H$ be the semiample fibration of $-K_{X_z} - B_z$. Then Proposition 3.4.3 yields an effective $\mathbb{Z}_{(p)}$-divisor $B_z^H$ such that $(H, B_z^H)$ is GFR. Let $\Gamma_H$ be an element of $|-K_H - B_z^H|_{\mathbb{Z}_{(p)}}$ such that $\Gamma_z = \psi^*\Gamma_H$. Since $(H, B_z^H)$ is GFR, by Lemma 1.3.12 we have $(H, B_z^H + \varepsilon\Gamma_H)$ is GFR for all sufficiently small positive $\varepsilon \in \mathbb{Z}_{(p)}$ such that $B_z^H + \varepsilon\Gamma_H$ is a $\mathbb{Z}_{(p)}$-divisor, and $K_H + B_z^H + \varepsilon\Gamma_H$ is $\mathbb{Z}_{(p)}$-Cartier, hence $(X_z, B_z + \varepsilon\Gamma_z)$ is also KGFR by Proposition 3.4.3, proving (A). To show (B), let $l$ be a positive integer not divisible by $p$, and such that $(1-\varepsilon)lm$ is an integer, and let $n := lm$. As for (C), let $V_\varepsilon := \operatorname{Im}\left(V^{\otimes(1-\varepsilon)l} \to H^0(X, nL_\varepsilon)\right) \subseteq H^0(X, nL_\varepsilon)$. Lastly, (D) follows from the fact that $\phi_{|V_\varepsilon|_z}$ and $\phi_{|V|_z}$ agree up to a Veronese embedding. qed

### 6.2.3. Proof of $C^-_{n,m}$

Now we have all the ingredients to prove the inequality $C^-_{n,m}$ in all dimensions for fibrations with $K$-globally $F$-regular fibres.

#### Calabi–Yau base

The following theorem is an analogue of [Cha23, Theorem 4.3] in positive characteristic.

In this section we consider varieties defined over an algebraically closed field of characteristic $p > 0$.

**Theorem 6.2.15** (Injectivity Theorem)**.** *Let $f\colon X \to Z$ be an equidimensional fibration between normal projective varieties, with normal general fibre $X_z$. Let $B$ be a $\mathbb{Z}_{(p)}$-divisor on $X$ such that* $\operatorname{Supp}(B^-)$ *does not dominate $Z$, let $D$ be a $\mathbb{Z}_{(p)}$-divisor on $Z$, and set $L := -K_X - B - f^*D$. Assume*

*(a) $(X_z, B_z)$ is $K$-globally $F$-regular,*

*(b) there exists an integer $m \geq 1$ not divisible by $p$ such that $mL$ is integral and $mL_z$ is Cartier,*

*(c) there is $|V| \subseteq |mL|$ such that $\phi_{|V|_z}$ is a morphism,*

*(d) $p$ does not divide the Stein degree of $\phi_{|V|_z}$,*

*(e) $\kappa(X, f^*(-K_Z - D) + P) = 0$, for some $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor $P \geq B^-$.*

*Then the restriction map $H^0(X, nL) \to H^0(X_z, nL_z)$ is injective for all $n \geq 0$. In particular, the inequality $\kappa(X, L) \leq \kappa(X_z, L_z)$ holds.*

Provided that $L$ admits $F$-complements for $(X/Z, B)$, we can follow the same proof as in [Cha23, Theorem 3.8, Proposition 4.2].

**Proposition 6.2.16.** *Let $f\colon X \to Z$ be a fibration of normal projective varieties, and let $B$ be a $\mathbb{Z}_{(p)}$-divisor such that* $\operatorname{Supp}(B^-)$ *does not dominate $Z$. Let $D$ be a $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor on $Z$, and let $L := -K_X - B - f^*D$. Assume*

(a) *$L$ admits an $F$-complement for $(X/Z, B)$, and either*

(b) *$f$ is equidimensional, or*

(b') *$B \geq 0$ and $Z$ is $\mathbb{Z}_{(p)}$-Gorenstein.*

*Then, $f^*(-K_Z - D) + B^-$ is $\mathbb{Z}_{(p)}$-effective.*

*Proof.* Let $\Lambda \in |L|_{\mathbb{Z}_{(p)}}$ be an $F$-complement for $(X/Z, B)$ and consider $\Delta := B + \Lambda$, so that $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is globally $F$-split and $K_X + \Delta \sim_{\mathbb{Z}_{(p)},Z} 0$. By Proposition 3.4.3, there is a canonically defined $\mathbb{Z}_{(p)}$-divisor $\Delta^Z$ such that

$$K_X + \Delta \sim_{\mathbb{Z}_{(p)}} f^*(K_Z + \Delta^Z) \sim_{\mathbb{Z}_{(p)}} f^*(-D).$$

Hence it is enough to show that $f^*\Delta^Z + B^-$ is an effective $\mathbb{Z}_{(p)}$-divisor. If $B \geq 0$ then $\Delta^Z$ is effective by Proposition 3.4.3(ii). Suppose now $f$ is equidimensional: then every component $P$ of $\mathrm{Supp}(\Delta^v)$ is mapped to a prime divisor $\delta$ of $Z$, hence $f^*\Delta^Z \geq \Delta^v$. Indeed, Remark 3.4.4 yields $\mathrm{coeff}_\delta(\Delta_Z) = 1 - d_\delta$, where

$d_\delta := \sup\{t \text{ s.t. } (X, \Delta + f^*(t\delta)) \text{ is globally sub } F\text{-split over the generic point of } \delta\}$.

As globally sub-$F$-split sub-couples are sub-log canonical in codimension one ([DS17, Lemma 2.14]), we have:

$$\begin{aligned} \mathrm{coeff}_P(\Delta^v) &\leq 1 - d_\delta \,\mathrm{coeff}_P(f^*(\delta)) \\ &\leq \mathrm{coeff}_P(f^*(\delta))(1 - d_\delta) \\ &= \mathrm{coeff}_P(f^*(\Delta_Z)). \end{aligned}$$

Since $\Delta^v + B^- \geq 0$, we conclude that $f^*\Delta_Z + B^- \geq 0$. qed

**Corollary 6.2.17.** *Let $f\colon X \to Z$ be a fibration of normal projective varieties, and let $B$ be a $\mathbb{Z}_{(p)}$-divisor on $X$ such that $\mathrm{Supp}(B^-)$ does not dominate $Z$. Let $D, E$ be $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisors on $Z$, set $L := -K_X - B - f^*D$, and assume that*

(a) *there exists $0 \leq \Gamma \sim_{\mathbb{Z}_{(p)}} L - f^*E$;*

(b) *$(1 - \varepsilon)L$ admits an $F$-complement for $(X, B + \varepsilon\Gamma)$, for $\varepsilon \in [0, 1) \cap \mathbb{Z}_{(p)}$, and either*

(c) *$f$ is equidimensional, or*

(c') *$B \geq 0$ and $Z$ is $\mathbb{Z}_{(p)}$-Gorenstein.*

*Then, $f^*(-K_Z - D - \varepsilon E) + B^-$ is $\mathbb{Z}_{(p)}$-effective.*

*Proof.* Let

$$B_\varepsilon := B + \varepsilon\Gamma, \quad D_\varepsilon := D + \varepsilon E, \quad L_\varepsilon = -K_X - B_\varepsilon - f^*D_\varepsilon$$

so that $L_\varepsilon \sim_{\mathbb{Z}_{(p)}} (1-\varepsilon)L$ and all the hypotheses of Proposition 6.2.16 are satisfied with respect to $f\colon (X, B_\varepsilon) \to Z$, $L_\varepsilon$, and $D_\varepsilon$. Then Proposition 6.2.16 yields that $f^*(-K_Z - D_\varepsilon) + B_\varepsilon^- = f^*(-K_Z - D - \varepsilon E) + B^-$ is $\mathbb{Z}_{(p)}$-effective. qed

We are now ready to prove our injectivity theorem.

*Proof of Theorem 6.2.15.* Let $X_z := f^{-1}(z)$ for a general $z \in Z$. We may assume that around $z$ the variety $Z$ is smooth, the morphism $f$ is flat, and $\mathrm{Supp}(B^-)$ does not contain $X_z$. Let $n > 0$, by contradiction suppose the map $H^0(X, nL) \to H^0(X_z, nL_z)$ is not injective. Then, there exists a divisor $0 \le \Delta \sim nL$ such that $X_z \subseteq \mathrm{Supp}(\Delta)$. As $L$ is $\mathbb{Z}_{(p)}$-effective, after possibly replacing $\Delta$ with $\Delta + \Delta'$ for some $\Delta' \sim mL$, we may assume $n$ is not divisible by $p$. Set $N := \frac{1}{n}\Delta$ so that $0 \le N \sim_{\mathbb{Z}_{(p)}} L$. Note that, by hypothesis, there exists a unique effective $\mathbb{Q}$-divisor $M \sim_{\mathbb{Q}} f^*(-K_Z - D) + P$, hence we may assume $X_z \not\subseteq \mathrm{Supp}(M)$. Consider now the diagram

$$\begin{tikzcd} (X', B') \arrow[r, "\pi"] \arrow[d, "f'"'] & (X, B) \arrow[d, "f"] \\ Z' \arrow[r, "\mu"] & Z, \end{tikzcd}$$

where notation is as follows:

- $Z'$ is the blowup of $Z$ at $z$ with exceptional divisor $E$, so that $K_{Z'} = \mu^*K_Z + aE$, with $a = \dim(Z) - 1$;
- $X'$ is the fibre product (hence it is also the blowup of $X$ at $X_z$ with exceptional divisor $G$, since blowup and flat base change commute);
- $f'$ is the induced morphism (since $f$ is equidimensional, so is $f'$);
- $B'$ is the strict transform of $B$, so that $K_{X'} + B' = \pi^*(K_X + B) + aG$.

Let also $D' := \mu^*D - aE$ and $L' := -K_{X'} - B' - f'^*D' \sim_{\mathbb{Z}_{(p)}} \pi^*L$. Note that $\mu^*\Theta$ is well-defined for all $\mathbb{Q}$-divisors $\Theta$, since they are all $\mathbb{Q}$-Cartier in a neighbourhood of $z$. Similarly, $\pi^*(K_X+B)$ is well-defined since $K_X+B$ is $\mathbb{Q}$-Cartier in a neighbourhood of the fibre $X_z$. As $\mathrm{Supp}(N) \supseteq X_z$ we have $\mathrm{Supp}(\pi^*N) \supseteq G$. Thus, letting $N' := \pi^*N$, we have $\mathrm{Supp}(N') \supseteq G$ too. In particular, for sufficiently small positive $\delta \in \mathbb{Z}_{(p)}$, we have an effective divisor

$$0 \le \Gamma' := N' - \delta G \sim_{\mathbb{Z}_{(p)}} L' - f'^*E', \quad E' := \delta E.$$

The data $f'\colon X' \to Z', B', D', L', E', \Gamma'$ satisfy the hypotheses of Corollary 6.2.14, so $L'$ admits an $F$-complement for $(X'/Z', B' + \varepsilon\Gamma')$. By Corollary 6.2.17 there exists a $\mathbb{Z}_{(p)}$-divisor $\overline{\Gamma}$ such that

$$\begin{aligned} 0 \le \overline{\Gamma} &\sim_{\mathbb{Z}_{(p)}} f'^*(-K_{Z'} - D' - \varepsilon E') + (B')^- \\ &= f'^*(\mu^*(-K_Z - D) - \varepsilon E') + (B')^- \\ &\le f'^*(\mu^*(-K_Z - D) - \varepsilon E') + \pi^* P \\ &= \pi^*(f^*(-K_Z - D) + P) - \varepsilon\delta G \\ &\sim_{\mathbb{Q}} \pi^* M - \varepsilon\delta G, \end{aligned}$$

contradicting the assumption $X_z \not\subseteq \operatorname{Supp}(M)$. qed

*Remark* 6.2.18. Under the assumptions of Theorem 6.2.15 with $B \geq 0$, by combining Theorem 6.2.15 and Proposition 5.2.6, we get that

$$\kappa(X, L) = \kappa(X_z, L_z).$$

Note that we cannot expect an equality when $B$ is not effective. Indeed, let $E$ be an ordinary elliptic curve, and $f\colon X := E \times E \to Z := E$ be the second projection. Let $z \in E$ be a closed point and $B := -f^*z$. Then, $f\colon X \to Z$ satisfies the assumptions of Theorem 6.2.15, whereas $f\colon X \to Z$ with the pair structure $(X, B)$ does not. We can check that:

$$1 = \kappa(X, -K_X - B) > \kappa(X_z, -K_{X_z}) = 0.$$

### General case

We are now ready to prove the main result of [BBC23].

In this section we consider varieties defined over a perfect field of characteristic $p > 0$.

**Theorem 6.2.19** (Tame $C^-_{n,m}$). *Let $f\colon X \to Y$ be a fibration of normal projective varieties with general fibre $X_y$. Let $B$ be an effective $\mathbb{Z}_{(p)}$-divisor on $X$, let $D$ be a $\mathbb{Z}_{(p)}$-Cartier $\mathbb{Z}_{(p)}$-divisor on $Y$, and set $L := -K_X - B - f^*D$. Assume*

*(a) $Y$ is $\mathbb{Z}_{(p)}$-Gorenstein,*

*(b) $(X_y, B_y)$ is $K$-globally $F$-regular,*

*(c) there exists an integer $m \geq 1$ not divisible $p$ such that $mL$ is Cartier,*

*(d) there is $|V| \subseteq |mL|$ such that $\phi_{|V|_y}$ is a morphism,*

*(e) $p$ does not divide the Stein degree of $\phi_{|V|_y}$.*

*Then*

$$\kappa(X, L) \leq \kappa(X_y, L_y) + \kappa(Y, -K_Y - D).$$

*Furthermore, if* $\kappa(Y, -K_Y - D) = 0$, *equality holds.*

*Proof.* By Remark 5.1.3 we can assume that $k$ is uncountable, and by Theorem 6.2.12 and Proposition 6.2.16 that $-K_Y - D$ is $\mathbb{Z}_{(p)}$-effective. Let $d \geq 1$ be sufficiently divisible so that $\phi_{|d(-K_Y-D)|}$ is (birational to) the Iitaka fibration of $-K_Y - D$. Consider now the following commutative diagram:

(✺)
$$\begin{tikzcd}
X^{(e)} \arrow[r, "b"] \arrow[d, "f_e"] & X' \arrow[r] \arrow[d, "f'"] \arrow[rr, bend left, "\pi"] & X_\infty \arrow[r] \arrow[d, "f_\infty"] & X \arrow[d, "f"] \\
Y^{(e)} \arrow[r, "a"] \arrow[d, "g_e"] & Y' \arrow[r, "u'"] \arrow[d, "g'"] \arrow[rr, bend left, "\mu"] & Y_\infty \arrow[r, "u"] \arrow[d, "\phi_\infty"] & Y \arrow[d, dashed, "\phi_{|d(-K_Y-D)|}"] \\
Z^e \arrow[r, "F^e"] & Z \arrow[r, "="] & Z \arrow[r, dashed] & Z_d,
\end{tikzcd}$$

where the notation is as follows.

(I) The lower rightmost diagram is constructed using Theorem 5.1.4 and $\phi_\infty$ is the Iitaka fibration of $-K_Y - D$.

(II) The variety $X_\infty$ is the normalisation of the main component of $X \times_Y Y_\infty$, and $f_\infty$ is the induced morphism.

(III) We construct $u'$ as a flattening of $f_\infty$ and $f'$ as the normalisation of the proper transform of $f_\infty$ as in Lemma 2.0.4. In particular $f'$ is equidimensional. After possibly replacing $u'$ with its pre-composition with some blowups along non-Cartier Weil divisors on $Y'$, and $X'$ by the corresponding normalised base change, we may further assume that there exists an effective Cartier divisor $\Theta'$ on $Y'$ such that $\mathrm{Exc}(\mu) = \mathrm{Supp}(\Theta')$ and $\mathrm{Exc}(\pi) \subseteq \mathrm{Supp}(f'^*\Theta')$, where $\mathrm{Exc}(\pi)$ and $\mathrm{Exc}(\mu)$ are the exceptional divisors of $\pi$ and $\mu$, respectively.

(IV) We define $Y^{(e)}$ and $X^{(e)}$ to be respectively the normalisations of the varieties $(Y' \times_Z Z^e)_{\mathrm{red}}$ and $(X' \times_Z Z^e)_{\mathrm{red}}$, and $a, b, g_e, f_e$ the naturally induced morphisms.

(V) We will denote by $h_e$ and $h'$ the compositions $g_e \circ f_e$ and $g' \circ f'$, respectively.

By the projection formula and Theorem 5.1.4, we have

$$\kappa(Y'_z, (\mu^*(-K_Y - D))|_{Y'_z}) = \kappa(Y_{\infty,z}, (u^*(-K_Y - D))|_{Y_{\infty,z}}) = 0.$$

By Lemma 2.3.30 we can pick $e$ so that, for general $z \in Z^e$, the fibres $X_z^{(e)}, Y_z^{(e)}$ are reduced and normal. Since $f'$ is equidimensional, we apply Corollary 2.3.32: there exist Weil divisors $D_X$, $D_Y$ on $X^{(e)}$ and $Y^{(e)}$ respectively, and an effective Weil divisor $C$ on $X^{(e)}$, such that

(1) $K_{X^{(e)}/X'} \sim D_X$ and $K_{Y^{(e)}/Y'} \sim D_Y$;

(2) $K_{X^{(e)}/X'} - f_e^* K_{Y^{(e)}/Y'} \sim -C$;

(3) $(f_e^* D_Y - D_X)|_{X_z^{(e)}} \sim C|_{X_z^{(e)}} \geq 0$.

Note that, by construction of diagram (✺), the general fibres of $f_e, f'$ and $f$ are isomorphic to each other via the restrictions of the morphisms $b$ and $\pi$, respectively. In particular, they are all normal. Combining this with point (1) we obtain:

(4) $D_X|_{X_y^{(e)}} \sim 0$, whence $C|_{X_y^{(e)}} = 0$.

By the Easy Additivity Theorem 5.2.1 applied to $h_e$, we obtain

$$\begin{aligned}\kappa(X^{(e)}, (\pi \circ b)^* L) &\leq \kappa(X_z^{(e)}, ((\pi \circ b)^* L)|_{X_z^{(e)}}) + \dim(Z) \\ &= \kappa(X_z^{(e)}, ((\pi \circ b)^* L)|_{X_z^{(e)}}) + \kappa(Y, -K_Y - D).\end{aligned}$$

Since Lemma 5.1.6 implies $\kappa(X^{(e)}, (\pi \circ b)^* L) = \kappa(X, L)$, it is then enough to show

$$\kappa(X_z^{(e)}, ((\pi \circ b)^* L)|_{X_z^{(e)}}) \leq \kappa(X_y, L_y).$$

The goal now is to apply Theorem 6.2.15 to the fibration $f_{e,z}$ given by the Cartesian diagrams

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X_z^{(e)} \arrow[r] \arrow[d, "f_{e,z}"] & X^{(e)} \arrow[d, "f_e"] \\ \{y\} \arrow[r] & Y_z^{(e)} \arrow[r] \arrow[d] & Y^{(e)} \arrow[d, "g_e"] \\ & \{z\} \arrow[r] & Z^e, \end{tikzcd}$$

where $z$ is a very general point of $Z^e$ and $y$ is a general point of $Y_z^{(e)}$. We now introduce appropriate divisors to which we apply Theorem 6.2.15.

(A) $K_{X'} + B' := \pi^*(K_X + B)$; note that $B'$ is a $\mathbb{Z}_{(p)}$-divisor, and $\mathrm{Supp}(B'^{,-})$ is $\pi$-exceptional.

(B) $K_{Y'} + \Xi' := \mu^* K_Y$; note that $\Xi'$ is a $\mu$-exceptional $\mathbb{Z}_{(p)}$-divisor.

(C) $K_{Y'} + D' := \mu^*(K_Y + D)$; note that $D'$ is still a $\mathbb{Z}_{(p)}$-divisor, possibly non $\mathbb{Q}$-Cartier.

(D) $L' := \pi^*L$ and $\overline{B}' := B' - f'^*\Xi'$, so that $L' = -K_{X'} - \overline{B}' - f'^*D'$. Note that, as $\pi$ is an isomorphism over the generic point of $Y$, conditions (b), (d), and (e) of Theorem 6.2.19 hold on $X'$ with respect to $L'$ and $\overline{B}'$. Moreover, $mL'$ is integral and $mL'_y$ is Cartier.

(E) $B_e := b^*\overline{B}' + C$ and $D_e := a^*D' - D_Y$; note that $C_z$ is $f_{e,z}$-vertical by point (4), and $a^*\mu^*(K_Y + D) \sim_{\mathbb{Z}_{(p)}} a^*(K_{Y'} + D') \sim_{\mathbb{Z}_{(p)}} K_{Y^{(e)}} + D_e$.

(F) For any $\mu$-exceptional prime divisor $E$, we write $E = E_{\mathrm{b}}$ in the case that $\mathrm{Supp}(E) \subseteq u^{-1}(\mathrm{Supp}(G))$, for all $G \in |-K_{Y'} - D'|_{\mathbb{Q}} = \mu^*|-K_Y - D|_{\mathbb{Q}}$, and $E = E_{\mathrm{f}}$ otherwise. Thus, when $E'$ is any $\mu$-exceptional $\mathbb{Q}$-divisor, we obtain an induced decomposition $E' = E'_{\mathrm{b}} + E'_{\mathrm{f}}$, where $E'_{\mathrm{b}}$ and $E'_{\mathrm{f}}$ have no common components.

(G) $P := f_e^*(a^*E)$, where $E \geq 0$ is a $\mu$-exceptional Cartier divisor, such that $P \geq B_e^-$; note that such an $E$ exists, since $\mathrm{Supp}(B_e^-) \subseteq f_e^{-1}(a^{-1}(\mathrm{Exc}(\mu)))$ by points (III), (A), (B), and (D). Note also that $\kappa(X_z^{(e)}, f_e^*(-K_{Y_z^{(e)}} - D_{e,z}) + P_z) = 0$. Indeed

$$\begin{aligned} a_z^*((\mu^*(-K_Y - D))|_{Y'_z}) &\sim_{\mathbb{Q}} -K_{Y_z^{(e)}} - D_{e,z} \\ &\leq_{\mathbb{Q}} -K_{Y_z^{(e)}} - D_{e,z} + P_z \\ &= a^*((\mu^*(-K_Y - D))|_{Y'_z}) + a_z^*(E_{\mathrm{b}}|_{Y'_z}) + a_z^*(E_f|_{Y'_z}) \\ &\leq_{\mathbb{Q}} a^*((\mu^*((1+\alpha)(-K_Y - D)))|_{Y'_z}) + a_z^*(E_f|_{Y'_z}), \end{aligned}$$

where $\alpha$ is some positive rational number. As $a_z^*(E_{\mathrm{f}})_z$ is $(\mu_z \circ a_z)$-exceptional, we conclude by taking global sections in the above chain of inequalities, and applying the projection formula combined with Theorem 5.1.4.

(H) $L_e := b^*L'$, so that $L_e \sim_{\mathbb{Z}_{(p)}} -K_{X^{(e)}} - B_e - f_e^*D_e$. Note that, by point (D), $mL_e$ is integral and $mL_{e,y}$ is Cartier.

Consider now the fibration $f_{e,z} \colon X_z^{(e)} \to Y_z^{(e)}$, with pair structure $(X^{(e)}, B_{e,z})$ and the $\mathbb{Z}_{(p)}$-divisors $D_{e,z}$, $L_{e,z}$, $P_z$. Conditions (a), (c), and (d) of Theorem 6.2.15 hold since, by points (D) and (H) above, we have injective pullback maps

$$H^0(X, mL) \xrightarrow{(\pi\circ b)^*} H^0(X^{(e)}, mL_e)$$

for all sufficiently divisible $m \geq 0$, which restrict to isomorphisms on general fibres of $f_e$ and $f$. Condition (b) of Theorem 6.2.15 holds by point (H), and condition (e) by point (G). Then we conclude, since we have

$$\begin{aligned} \kappa(X_z^{(e)}, ((\pi \circ b)^* L)|_{X_z^{(e)}}) &= \kappa(X_z^{(e)}, L_{e,z}) && \text{by points (e),(g) and Lemma 5.1.6} \\ &\leq \kappa(X_y, L_{e,z}|_{X_y}) && \text{by Theorem 6.2.15} \\ &= \kappa(X_y, L_y) && \text{as } \pi \circ b \text{ is an isomorphism over } X_y. \end{aligned}$$

To conclude the first part of the proof, note that, by Lemma 5.2.4, we can take $y \in Y$ to be general, rather than very general.

As for the "furthermore" part, apply Proposition 5.2.6 to get the opposite inequality. qed

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