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ERRATUM: ANALYSIS AND NUMERICAL APPROXIMATION OF STATIONARY SECOND-ORDER MEAN FIELD GAME PARTIAL DIFFERENTIAL INCLUSIONS*

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Abstract. We correct the proofs of Theorems 3.3 and 5.2 in [Y. A. P. Osborne and I. Smears, *SIAM J. Numer. Anal.*, 62 (2024), pp. 138–166]. With the corrected proofs, Theorems 3.3 and 5.2 are shown to be valid without change to their hypotheses or conclusions.

Key words. mean field games, Hamilton–Jacobi–Bellman equations, nondifferentiable Hamiltonians, partial differential inclusions, monotone finite element method, convergence analysis

MSC codes. 65N15, 65N30

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1. Introduction. In [3, Proof of Theorem 3.3], the passage to the limit from [3, eq. (4.12)] to [3, eq. (4.15)] is not generally valid when considering a convergent subsequence of $\{m_j\}_{j\in\mathbb{N}}$, since [3, eq. (4.12)] holds for m_{j+1} instead of m_j. The consequence is that the argument given there is not sufficient to prove the existence of a weak solution of the MFG PDI system that is being considered. Likewise, the argument of the proof of [3, Theorem 5.2], which is entirely analogous in the discrete setting, is affected in the same way.

We now give correct proofs of Theorems 3.3 and 5.2 of [3]. There is no change to the statement of the theorems or their hypotheses. The correct argument is similar to the one in [2, Theorem 3.3], and is based on an application of Kakutani's fixed-point theorem [5, Chap. 9, Theorem 9.B], which we recall below for completeness.

THEOREM 1.1 (Kakutani's fixed point theorem). *Suppose that*
1. *$\mathcal{B}$ is a nonempty, compact, convex set in a locally convex space X;*
2. *$\mathcal{V}:\mathcal{B}\rightrightarrows\mathcal{B}$ is a set-valued map such that $\mathcal{V}(\tilde{b})$ is nonempty, closed, and convex for all $\tilde{b}\in\mathcal{B}$; and*
3. *$\mathcal{V}$ is upper semicontinuous.*

Then $\mathcal{V}$ has a fixed point: there exists a $\tilde{b}_*\in\mathcal{B}$ such that $\tilde{b}_*\in\mathcal{V}(\tilde{b}_*)$.

2. Correction.

2.1. Proof of [3, Theorem 3.3]. Recall that $c_4=\|b\|_{C(\overline{\Omega}\times\mathcal{A};\mathbb{R}^n)}$ is the Lipschitz constant of the Hamiltonian given in [3, eq. (2.3b)]. We equip the space $L^\infty(\Omega;\mathbb{R}^n)$ with its weak-$*$ topology, noting that it is then a locally convex topological vector space. Let $\mathcal{B}$ denote the ball

$$\mathcal{B}:=\left\{\tilde{b}\in L^\infty(\Omega;\mathbb{R}^n):\|\tilde{b}\|_{L^\infty(\Omega;\mathbb{R}^n)}\le c_4\right\}. \tag{2.1}$$

We note that $\mathcal{B}$ is nonempty, convex, and is also closed in the weak-$*$ topology. Since $L^1(\Omega;\mathbb{R}^n)$ is separable, the weak-$*$ topology on $\mathcal{B}$ is metrizable [4, Chap. 15]. Moreover, Helly's theorem implies that $\mathcal{B}$ is compact.

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Let $M:\mathcal{B}\to H_0^1(\Omega)$ be the map defined as follows: for each $\tilde{b}\in\mathcal{B}$, let $M[\tilde{b}]$ in $H_0^1(\Omega)$ be the unique solution of

$$\int_\Omega \big(\nu\nabla M[\tilde{b}]\cdot\nabla\phi + M[\tilde{b}]\tilde{b}\cdot\nabla\phi+\kappa M[\tilde{b}]\phi\big)\mathrm{d}x = \langle G,\phi\rangle_{H^{-1}\times H_0^1} \quad \forall\phi\in H_0^1(\Omega). \tag{2.2}$$

The map M is well-defined thanks to [3, Lemma 4.5]. Next, let $U:L^2(\Omega)\to H_0^1(\Omega)$ be the map defined as follows: for each $m\in L^2(\Omega)$, let $U[m]\in H_0^1(\Omega)$ denote the unique solution of

$$\int_\Omega \big(\nu\nabla U[m]\cdot\nabla\psi + H(x,\nabla U[m])\psi+\kappa U[m]\psi\big)\mathrm{d}x = \langle F[m],\psi\rangle_{H^{-1}\times H_0^1} \quad \forall\psi\in H_0^1(\Omega). \tag{2.3}$$

The map U is well-defined as a result of [3, Lemma 4.6].

We define the set-valued map $\mathcal{V}:\mathcal{B}\rightrightarrows L^\infty(\Omega;\mathbb{R}^n)$ as follows: for each $\tilde{b}\in\mathcal{B}$, let

$$\mathcal{V}[\tilde{b}] := D_pH\big[U\big[M\big[\tilde{b}\big]\big]\big]. \tag{2.4}$$

In [3, Lemma 4.3] it is implied that $\mathcal{V}[\tilde{b}]\subset\mathcal{B}$ for each $\tilde{b}\in\mathcal{B}$, so $\mathcal{V}:\mathcal{B}\rightrightarrows\mathcal{B}$. Moreover, for every $\tilde{b}\in\mathcal{B}$, the set $\mathcal{V}[\tilde{b}]$ is nonempty and convex. Indeed, for each $\tilde{b}\in\mathcal{B}$ the set $\mathcal{V}[\tilde{b}]$ is nonempty by [3, Lemma 4.3]. Also, $\mathcal{V}[\tilde{b}]$ is convex since ∂_pH has convex images. Furthermore, $\mathcal{V}[\tilde{b}]$ is closed for all $\tilde{b}\in\mathcal{B}$ thanks to [3, Lemma 4.4].

The existence of a weak solution of the MFG PDI in the sense of [3, Definition 3.1] is equivalent to showing the existence of a fixed point of $\mathcal{V}$, i.e., that there exists a $\tilde{b}_*\in\mathcal{B}$ such that $\tilde{b}_*\in\mathcal{V}[\tilde{b}_*]$. Indeed, if $\tilde{b}_*\in\mathcal{B}$ satisfies $\tilde{b}_*\in\mathcal{V}[\tilde{b}_*]$ then a solution pair (u,m) of [3, eq. (3.1)] is given by $m:=M[\tilde{b}_*]$ and $u:=U[m]$ with $\tilde{b}_*\in D_pH[u]$, while the converse is obvious.

We now verify that $\mathcal{V}$ is upper semicontinuous. To this end, it suffices to prove that the graph of $\mathcal{V}$ is closed; cf. [1, Chap. 1, Corollary 1, p. 42]. Let $\mathcal{W}$ denote the graph of $\mathcal{V}$, which is defined by

$$\mathcal{W} := \left\{(\tilde{b},\bar{b})\in\mathcal{B}\times\mathcal{B} : \bar{b}\in\mathcal{V}[\tilde{b}]\right\}. \tag{2.5}$$

Since $\mathcal{B}$ is metrizable, to show that the graph $\mathcal{W}$ is a closed it is enough to show that whenever a sequence $\{(\tilde{b}_i,\bar{b}_i)\}_{i\in\mathbb{N}}\subset\mathcal{W}$ converges weakly-$*$ in $\mathcal{B}\times\mathcal{B}$ to a point $(\tilde{b},\bar{b})$ as $i\to\infty$, then $(\tilde{b},\bar{b})\in\mathcal{W}$, which is equivalent to $\bar{b}\in\mathcal{V}[\tilde{b}]$. Let us then suppose that we are given a sequence $\{(\tilde{b}_i,\bar{b}_i)\}_{i\in\mathbb{N}}\subset\mathcal{W}$ that converges weakly-$*$ in $\mathcal{B}\times\mathcal{B}$ to a point $(\tilde{b},\bar{b})$ as $i\to\infty$. To begin, we claim that $M[\tilde{b}_i]\to M[\tilde{b}]$ in $L^2(\Omega)$ as $i\to\infty$. Indeed, since $\{\tilde{b}_i\}_{i\in\mathbb{N}}\subset\mathcal{B}$, for each $i\in\mathbb{N}$ we apply [3, Lemma 4.5] to obtain the uniform bound

$$\sup_{i\in\mathbb{N}}\|M[\tilde{b}_i]\|_{H^1(\Omega)}\le C_1\|G\|_{H^{-1}(\Omega)}. \tag{2.6}$$

We deduce from this that any given subsequence $\{M[\tilde{b}_{i_j}]\}_{j\in\mathbb{N}}$ is uniformly bounded in $H_0^1(\Omega)$. The Rellich–Kondrachov compactness theorem then implies that there exists a further subsequence $\{M[\tilde{b}_{i_{j_s}}]\}_{s\in\mathbb{N}}$ and some $m\in H_0^1(\Omega)$ such that $M[\tilde{b}_{i_{j_s}}]\rightharpoonup m$ in $H_0^1(\Omega)$ and $M[\tilde{b}_{i_{j_s}}]\to m$ in $L^2(\Omega)$ as $s\to\infty$. By L^∞-weak-$*$ $\times$ L^2-strong convergence, we also have that $M[\tilde{b}_{i_{j_s}}]\tilde{b}_{i_{j_s}}\rightharpoonup m\tilde{b}$ in $L^2(\Omega;\mathbb{R}^n)$ as $s\to\infty$. Passing to the limit in the KFP equation (2.2) satisfied by $M[\tilde{b}_{i_{j_s}}]$ for $s\in\mathbb{N}$, we deduce that m satisfies

$$\int_\Omega\big(\nu\nabla m\cdot\nabla\phi + m\tilde{b}\cdot\nabla\phi+\kappa m\phi\big)\mathrm{d}x = \langle G,\phi\rangle_{H^{-1}\times H_0^1}\quad\forall\phi\in H_0^1(\Omega). \tag{2.7}$$

But by definition of $M[\tilde{b}]$ in (2.2), we see that $m = M[\tilde{b}]$ in $H^1_0(\Omega)$. In other words, every subsequence of $\{M[\tilde{b}_i]\}_{i\in\mathbb{N}}$ has a further subsequence that converges to $M[\tilde{b}]$ as $i \to \infty$. It follows that the entire sequence $\{M[\tilde{b}_i]\}_{i\in\mathbb{N}}$ converges to $M[\tilde{b}]$ in $L^2(\Omega)$ as $i \to \infty$. Then, [3, Lemma 4.6] implies that $U[M[\tilde{b}_i]] \to U[M[\tilde{b}]]$ in $H^1_0(\Omega)$ as $i \to \infty$. By hypothesis, $\overline{b}_i \in \mathcal{V}[\tilde{b}_i] = D_pH[U[M[\tilde{b}_i]]]$ for all $i \in \mathbb{N}$ and $\overline{b}_i \rightharpoonup^* \overline{b}$ in $L^\infty(\Omega;\mathbb{R}^n)$ as $i \to \infty$. We conclude from [3, Lemma 4.4] that $\overline{b} \in D_pH[U[M[\tilde{b}]]]$, i.e., $\overline{b} \in \mathcal{V}[\tilde{b}]$. We have, therefore, shown that $\mathcal{W}$ is closed, so $\mathcal{V}$ is upper semicontinuous.

We have thus shown that the map $\mathcal{V} : \mathcal{B} \rightrightarrows \mathcal{B}$ satisfies all of the conditions of Kakutani's fixed-point theorem, so $\mathcal{V}$ admits a fixed point and there exists a weak solution of the MFG PDI in the sense of [3, Definition 3.1]. The bounds [3, eqs. (3.5)–(3.6)] then follow directly from [3, Lemmas 4.5 and 4.6].

2.2. Proof of [3, Theorem 5.2]. The proof of [3, Theorem 5.2] on the existence of solutions of the discrete problems [3, eq. (5.4)] for each $k \in \mathbb{N}$ is analogous to the argument above, where the operators M and U are replaced by their discrete counterparts.

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