Classical-quantum dynamics with applications to gravity

By

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I, Zachary Weller-Davies, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signed .... Date ....

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Abstract

We develop autonomous classical-quantum dynamics by providing formalism and tools to study effective theories of interacting classical and quantum systems. We find the most general form of continuous classical-quantum master equation, which generalizes the Fokker-Planck equation of classical mechanics and the Lindblad equation of open quantum systems, allowing for coupling between the classical and quantum systems. We further show that this master equation can be unraveled by stochastic classical-quantum trajectories. The resulting dynamics is a natural generalization of the standard semi-classical equations of motion. However, because the dynamics are linear in the combined classical-quantum state, they are completely positive on all initial quantum states, providing a method to study semi-classical physics even in the presence of large quantum fluctuations.

We find a general path integral representation for classical-quantum dynamics, generalizing the Feynman-Vernon and classical stochastic path integrals, allowing for interaction between classical and quantum systems. Via path integral methods, we give the first examples of Lorentz invariant classical-quantum dynamics. We further find diffeomorphism invariant classical-quantum theories of gravity. We provide a methodology to derive the generalizations of the gravitational Hamiltonian and momentum constraints in such theories.

We prove that the consistency of classical-quantum coupling implies a general trade-off between the quantum decoherence rate and the degree of diffusion induced in the classical system, given by the back-reaction strength. Applying the trade-off relation to Newtonian gravity, we find an experimental signature of theories in which gravity is fundamentally classical. Bounds on decoherence rates arising from current interferometry experiments, combined with precision measurements of mass, place substantial restrictions on theories where Einstein's classical theory of gravity interacts with quantum matter, with part of the parameter space of such theories already squeezed out. We provide figures of merit that can be used in future experiments.

Impact Statement

Semi-classical descriptions of systems are ubiquitous in all areas of physics. In gravity, this is the regime we are interested in when studying black-hole evaporation or inflationary cosmology, where quantum fluctuations in an expanding universe give rise to the primordial seeds for all of the large-scale structures we see today. In measurement theory, a quantum system interacts with a macroscopic device, considered classical. In atomic physics and quantum chemistry, small molecules often interact with a classical environment. Furthermore, most quantum technologies utilize a mixture of classical and quantum methods. In quantum communication, one often considers the problem of speeding up classical communication using quantum mechanics – a protocol that can be viewed as a classical-quantum map. Similarly, future quantum computers will combine quantum and classical operations, using quantum devices as subroutines calculating classically intractable tasks.

Despite this, it is only recently that tools, largely borrowed from quantum information, have been used to understand classical-quantum dynamics in generality. This work uses these tools to develop the classical-quantum formalism. We derive the most general form of classical-quantum master equation continuous in the classical degrees of freedom. We find unraveling and path integral descriptions of classical-quantum dynamics, whose descriptions are often better suited to numerical simulation. The dynamics we find generalize previous methods of incorporating quantum back-reaction on classical degrees of freedom. However, unlike standard semi-classical approaches, they allow for correlations to be built up between classical and quantum degrees of freedom, being applicable in a wider regime and forming a more complete semi-classical description.

Recent experimental proposals [1, 2] have also re-ignited discussion on whether one can fundamentally treat space-time classically. Such experiments aim to rule out semi-classical theories of gravity by measuring gravitationally induced entanglement, which is not reproducible by a classical description. Though promising, these experiments are potentially decades away from being realized. In this work, we approach the problem from a different direction by studying classical-quantum theories of gravity and the generic predictions and experimental bounds that follow from a consistent treatment. This work finds generic predictions for the Newtonian limit of any theory which treats gravity classically. All classical-quantum dynamics must obey a trade-off between decoherence and diffusion, quantified in terms of the strength of the classical-quantum coupling. We use this to show that theories with a fundamentally classical gravitational field can be tested in the near term, while current experiments already substantially restrict the parameter space of such theories.

Beyond the trade-off, this work explores the conceptual and technical consequences of having a classical gravitational field, which, since we do not have a full theory of quantum gravity, is an important question in foundational physics. In particular, this work develops theories of classical-quantum gravity, finding the first examples of Lorentz invariant and diffeomorphism invariant classical-quantum dynamics via a path integral approach, which have been long sought after and could be used to propose a fundamental theory of classical gravity interacting with quantum matter.

List of Publications and Preprints

The work presented in this thesis contains material from the following publications and preprints:

1. Jonathan Oppenheim and Zachary Weller-Davies. The constraints of post-quantum classical gravity. arXiv:2011.15112. JHEP 02 (2022) 080 [3].

2. Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda and Zachary Weller-Davies. Gravitationally induced decoherence vs space-time diffusion: testing the quantum nature of gravity. arXiv:2203.01982. Nature Commun. 14 (2023) 1, 7910. [4]

3. Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda and Zachary Weller-Davies. The two classes of hybrid classical-quantum dynamics. arXiv:2203.01332. [5]

4. Isaac Layton, Jonathan Oppenheim and Zachary Weller-Davies. A healthier semi-classical dynamics. arXiv:2208.11722. [6]

5. Jonathan Oppenheim and Zachary Weller-Davies. Path integrals from completely positive classical-quantum dynamics. arXiv:2301.04677. [7]

6. Jonathan Oppenheim and Zachary Weller-Davies. Covariant path integrals for quantum fields back-reacting on classical space-time. arXiv:2302.07283. [8]

7. Isaac Layton, Jonathan Oppenheim, Andrea Russo and Zachary Weller-Davies. The weak field limit of quantum matter back-reacting on classical spacetime. arXiv:2307.02557. JHEP 08 (2023) 163. [9]

Other publications and preprints by the author are:

8. Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda and Zachary Weller-Davies. Objective trajectories in hybrid classical-quantum dynamics. arXiv:2011.06009. Quantum 7 (2023) 891. [10]

9. Juan F. Pedraza, Andrea Russo, Andrew Svesko and Zachary Weller-Davies. Lorentzian threads as 'gatelines' and holographic complexity. arXiv:2105.12735. Phys.Rev.Lett. 127 (2021) 27, 271602. [11]

10. Juan F. Pedraza, Andrea Russo, Andrew Svesko and Zachary Weller-Davies. Sewing spacetime with Lorentzian threads: complexity and the emergence of time in quantum gravity. arXiv:2106.12585. JHEP 02 (2022) 093. [12]

11. Juan F. Pedraza, Andrea Russo, Andrew Svesko and Zachary Weller-Davies. Computing spacetime. arXiv:2205.05705. Honorable mention for the 2022 Essay Competition of the Gravity Research Foundation. Int.J.Mod.Phys.D 31 (2022) 14, 2242010. [13]

12. Rafael Carrasco, Juan F. Pedraza, Andrew Svesko and Zachary Weller-Davies. Gravitation from optimized computation: Einstein and beyond. arXiv:2306.08503. JHEP 09 (2023) 167. [14]

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Contents

List of Tables

Chapter 1

Introduction

Effective theories are ubiquitous in physics: from particle physics to classical statistical mechanics, we often make approximations to an underlying physical theory to simplify the dynamical description. Broadly, two approaches exist to constructing effective theories [15]. In a Wilsonian approach [16, 15], one starts with a high energy theory and asks how the effective low energy description changes as high momentum modes are integrated out. Alternatively, one modifies the theory by hand to isolate the desired degrees of freedom while maintaining phenomenological accuracy. This approach usually involves changing the high-energy description to make the effective description simpler and easier to use.

Often, we are interested in the effective description of a system where one part behaves classically and the other quantum mechanically; the system is described by an effective theory of combined classical-quantum (CQ) dynamics. We do this when we model a quantum measurement via the Born rule, since we treat the measurement device as classical. Similarly, in atomic physics and quantum chemistry, small molecules often interact with an environment, or thermal reservoir, which can be treated classically. In gravity, we would like to study the back-reaction of thermal radiation emitted from black holes. While the matter fields can be described by quantum field theory, practically, we only know how to treat space-time classically. Likewise, in cosmology, vacuum fluctuations are a quantum effect that gives rise to the primordial seeds sourcing galaxy formation. However, the expanding space-time they live in can only be treated classically.

The history of defining a consistent coupling between classical and quantum systems has been controversial [17, 18]. When the quantum system does not back-react on the classical system, the situation is simple: the dynamics are described by unitary quantum mechanics, and the quantum state evolves according to a Hamiltonian $H(z)$ that depends on classical degrees of freedom z. However, defining consistent dynamics where the quantum system back-reacts on the classical system has been more problematic.

The most familiar example of CQ back-reaction is semi-classical gravity. The standard approach to define back-reaction is via the semi-classical Einstein equations, which source the Einstein tensor $G_{\mu\nu}$ of the gravitational metric $g_{\mu\nu}$ by the expectation value of the stress-energy tensor $T_{\mu\nu}$ [19, 20]

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle. \tag{1.1}$$

Though the scope and limitations of the semi-classical Einstein equations are not precisely understood [21, 22, 23], they are commonly understood to fail when fluctuations of the stress-energy tensor are large in comparison to its mean value [24, 25, 26, 21, 27]. The problem is that the standard semi-classical equations fail to properly account for correlations between the classical and quantum degrees of freedom. One often inputs this correlation by hand, considering situations when the quantum state is fully decohered and then evolving the classical system conditioned on the quantum state being in a particular eigenvalue. However, this does not give a dynamical description of the system before the quantum state has decohered, nor does it describe how the correlations between the classical and quantum systems are built up [28, 6]. The scenarios where quantum fluctuations are significant are often the regimes we hope to understand, such as in considering the gravitational field associated with Schrodinger cat states of massive bodies [29, 30], or vacuum fluctuations during inflation [31, 32, 33, 34]. For these regimes, background field methods are not appropriate, and an alternate effective theory of the back-reaction of quantum matter on classical gravity is required.

Moving away from effective theories, the lack of success in constructing a complete theory of quantum gravity valid beyond the Planck scale, combined with the lack of low energy signatures of quantum gravity, means the question of whether or not the gravitational field is quantum is still open for debate. The widespread belief is that gravity must be quantized, partly due to various no-go theorems surrounding consistent classical-quantum coupling. Feynman famously argued that a CQ coupling would prevent superpositions [17, 35], Epply and Hannah argued that it would lead to superluminal signalling [18], and various other arguments have been offered over the years [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. Meanwhile, it is well known that taking the semi-classical dynamics as fundamental is inconsistent and leads to violations of the standard principles of quantum theory, inducing a breakdown of either operational no-signaling, the Born rule, or composition of quantum systems under the tensor product [48, 18, 49, 50, 51].

Despite this, various attempts have been made to study a classical theory of gravity interacting with quantum systems, for example, by using a channel or measurement-based approach [52, 49, 53, 54, 55], which lead to linear dynamics on the quantum state. More generally, it is now known that these no-go theorems are circumvented by allowing for stochastic coupling between the classical and quantum degrees of freedom. By representing the combined CQ-state as a distribution over the classical degrees of freedom, and density matrix at each point in phase space, [56, 57] introduced dynamics which are linear, completely positive, trace preserving, and preserve the split between classical and quantum degrees of freedom. These are necessary and sufficient conditions for the CQ-state to give positive probability outcomes for measurements. The evolution laws have been studied in various contexts [58, 59], including gravity [60, 52, 49, 61]. Recently, it was shown that the dynamics are special cases of the master equation derived in [28], which is the most general map governing consistent classical-quantum dynamics. If the dynamics are autonomous (time-local and completely positive at all times), one can write down the most general form of master equation. The dynamics are related to the GKSL or Lindblad equation [62, 63], which for bounded dynamics, is the most general autonomous dynamics for an open quantum system.

If gravity were fundamentally classical, then the assumptions that go into autonomous CQ dynamics are reasonable: the assumptions of complete-positivity and linearity are necessary for sensible predictions for all initial classical and quantum states; the assumption of autonomy is reasonable for any theory viewed as fundamental. Viewed this way, one expects that CQ master equations provide a template to construct consistent CQ theories of gravity. Such dynamics can probe the quantum nature of gravity in an alternate direction to current experimental proposals that have been made to measure low-energy gravitational phenomena that cannot be reproduced classically. Currently, the most promising experiments include those which aim to detect gravitationally induced entanglement in table-top experiments via spin entanglement witnesses [64, 53, 1, 2, 65, 66, 67, 68]. There have also been proposals to measure intrinsically quantum features of gravity without studying entanglement directly [69, 70]. Though undoubtedly exciting, current estimates suggest that the technology required to perform the experiments is decades away. We use the classical-quantum formalism to consider the question from the opposite direction: can we construct a consistent fundamental classical-quantum theory of gravity? Can we find experimental signatures of such theories which can be used as indirect tests for the quantum nature of gravity? These are precisely the theories that experiments measuring gravitationally induced entanglement would rule out.

One might also expect classical-quantum dynamics to be useful as an effective theory. While the assumptions that go into autonomous CQ dynamics are reasonable for a fundamental theory, none of the assumptions need to hold, at least exactly. Nonetheless, exploring the autonomous CQ dynamics in the gravitational context is worthwhile as a starting point. It may be useful in certain regimes, but more importantly, it can be used to gain insight into the challenges that may arise when attempting to construct a more complete semi-classical description.

There are two main aims of this thesis. The first is to develop classical-quantum dynamics by providing formalism and tools to study effective theories of interacting classical and quantum systems. This thesis is primarily concerned with autonomous classical-quantum dynamics, and we do not attempt to understand when and how such a limit arises from quantum theory. Indeed in the general case, the effective classical-quantum description can be non-Markovian. Nonetheless, we hope that general lessons of classical-quantum dynamics can be used to gain insight into the effective regime. We leave a fuller understanding of taking the classical-quantum limit of quantum-quantum systems as an open question for future work.

Secondly, we develop classical-quantum theories of gravity. Though we expect such a theory could be applicable as an effective theory, we take a particular interest in developing theories of classical gravity that could be considered fundamental and studying the experimental signatures of such theories, which potentially lead to near-term indirect tests of quantum gravity. Moreover, classical-quantum dynamics provides a tractable arena to understand some of the problems which arise in theories of quantum gravity, such as diffeomorphism invariance and preservation of constraints [71, 72], which may not be solely due to treating the gravitational field as quantum; we find similar issues in classical-quantum gravity and even purely classical probabilistic theories of gravity.

We now outline some of the main contributions of the thesis:

A classical-quantum Pawula theorem

The most general form of autonomous quantum dynamics is the Lindblad, or GKLS equation [63, 62]. Meanwhile, the most general form of classical dynamics continuous in the classical degrees of freedom is the Fokker-Planck equation [73]. Respectively, they are often used to describe the physics of quantum or classical systems interacting with an environment.

A natural question is: what are the most general dynamics that allow for the interaction of classical and quantum systems which are continuous in the classical degrees of freedom? This question is of interest not only from a foundational point of view but of practical interest since we often consider interacting systems where one can be or needs to be treated classically.

In Chapter 4, we find the general form of consistent autonomous continuous dynamics between classical and quantum systems. Our work enables one to study arbitrary continuous back-reactions and reduces to previously known master equations [57, 74] for a specific choice of Lindblad couplings. The master equation we derive can be viewed as a generalization of [75] applicable to *any* continuous classical-quantum dynamics. It extends a famous theorem in stochastic dynamics – the Pawula theorem [76] – to include quantum degrees of freedom.

This work was done in collaboration with Jonathan Oppenheim, Carlo Sparaciari, and Barbara Šoda [5].

Continuous unraveling of classical-quantum dynamics

In Chapter 5, we explicitly show that the continuous classical-quantum master equation derived in Chapter 4 can be unraveled by coupled stochastic differential equations with continuous trajectories. The resulting equations of motion are natural generalizations of the standard semi-classical equations of motion. However, because the resulting dynamics are linear in the combined classical-quantum state, it does not lead to the same pathologies - it accounts for correlations between the classical and quantum systems. In addition, despite a breakdown of predictability in the classical degrees of freedom, the quantum state evolves deterministically conditioned on the classical trajectory, provided a trade-off between decoherence and diffusion is saturated. As a result, the quantum state remains pure when conditioned on the classical trajectory.

This work was done in collaboration with Isaac Layton and Jonathan Oppenheim [6].

A path integral approach to classical-quantum dynamics

In Chapter 6, we derive a general path integral representation for classical-quantum dynamics. The path integral we derive is a generalization of the Feynman path integral for quantum systems and the stochastic path integral used to study classical stochastic processes, allowing for interaction between the classical and quantum systems. When the classical-quantum Hamiltonian is at most quadratic in the momenta, we derive a configuration space path integral. In Chapter 8, we study configuration space path integrals in more detail without resorting to master equation methods. These path integrals allow one to readily impose space-time symmetries, including Lorentz invariance or diffeomorphism invariance.

This work was done in collaboration with Jonathan Oppenheim [7].

Towards a diffeomorphism invariant theory of classical gravity

We develop classical-quantum theories of gravity. In the gravitational setting, by taking the classical degrees of freedom to live in the phase space of general relativity, the CQ dynamics describe a probability distribution over 4-geometries, each associated with a quantum state $(g_{\mu\nu}, \rho(g_{\mu\nu}))$. In Chapter 7, we provide a methodology for arriving at the analogs of the Hamiltonian and momentum constraints[1] in classical-quantum theories of gravity.

In Chapter 8 we use the path integral formulation of CQ dynamics to find the first examples of classical-quantum dynamics which are Lorentz and diffeomoprhism invariant, providing proof of principle that a complete theory could exist. We introduce a diffeomorphism invariant

[1]Recall, in the ADM formulation of classical gravity [77, 78], the dynamics are generated by a Hamiltonian $H_{ADM} = \int d^3x N(x)\mathcal{H}(x) + N^a(x)\mathcal{H}_a(x)$ containing freely chosen "lapse" $N(x)$, and "shift" $N^a(x)$ functions. In order to ensure that the dynamics do not depend on this choice, one must impose $\mathcal{H} \approx 0$, $\mathcal{H}_a \approx 0$, called the Hamiltonian and momentum constraint, respectively. The notation $\approx$ indicates that the constraints are *weakly zero*, meaning they only vanish on a subset of the phase space - called the constraint surface.

theory based on the trace of Einstein's equations. We also introduce a path integral formulation of general relativity where the space-time metric is treated classically and we posit the generalizations of the Hamiltonian and momentum constraints which must be satisfied for the theory to be gauge invariant.

This work was done in collaboration with Jonathan Oppenheim [8, 3].

Generic predictions for Newtonian limit of classical-quantum gravity

In Chapter 9 we study the Newtonian limit of classical-quantum gravity and find a generic prediction of CQ theories: the Newtonian potential diffuses away from its classical solution by an amount that depends on the decoherence rate into mass eigenstates. In order for the dynamics to be completely positive, the amount of diffusion is necessarily lower bounded by the decoherence rate into mass eigenstates. This provides a way of testing CQ theories: one lower bounds the amount of diffusion the theory must have from coherence experiments, which can then be tested by measuring the noise in precision mass experiments. We explore the experimental consequences in detail in Chapter 10. We also show that the Newtonian limit we derived agrees with the Newtonian limit of the diffeomorphism invariant theory in Chapter 8, showing that it has a Newtonian limit which describes completely positive evolution on the subset of states satisfying the Newtonian gauge approximation, giving rise to the hope that the complete theory has constraints which can be preserved in time.

This work was done in collaboration with Jonathan Oppenheim and Andrea Russo [9].

A trade-off between decoherence and diffusion: indirect tests for the quantum nature of gravity

In Chapters 10, we prove that classical-quantum necessarily results in decoherence of the quantum system and a breakdown in predictability in the classical phase space. We further prove that a trade-off between the rate of this decoherence and the degree of diffusion induced in the classical system is a general feature of all classical quantum dynamics; long coherence times require strong diffusion in phase space relative to the strength of the coupling. Applying the trade-off relation to Newtonian gravity, we find a relationship between the strength of gravitationally-induced decoherence versus diffusion of the Newtonian potential and its conjugate momenta. The trade-off provides an experimental signature of theories in which gravity is fundamentally classical. Bounds on decoherence rates arising from current interferometry experiments, combined with precision measurements of mass, place significant restrictions on theories where Einstein's classical theory of gravity interacts with quantum matter. We find that part of the parameter space of such theories is already squeezed out. We provide figures of merit that can be used in future mass measurements and interference experiments.

This work was done in collaboration with Jonathan Oppenheim, Carlo Sparaciari, and Barbara Šoda [4].

Structure of the thesis

The thesis is split into a main body plus appendices. In the first part of the thesis (Chapters 2 and 3), we introduce the necessary background material on classical-quantum dynamics and gravity. The results presented in this section are known in the literature. The rest of the thesis is split into two parts and contains research work performed during my Ph.D. The first part – consisting of Chapters 4, 5, 6 – focuses on developing the CQ formalism, while the second part – consisting of the Chapters 7, 8, 9, and 10 – applies the CQ formalism to the gravitational setting. This work was done in collaboration with both internal and external researchers, all of whom contributed significantly to the ideas developed in this thesis. Other published work, completed during my Ph.D. with collaborators but not included in the thesis, includes [11, 12, 10].

In Chapter 2, we provide an introduction to stochastic classical dynamics, open quantum dynamics, and classical-quantum dynamics. This chapter aims to provide a reference point for concepts that are unfamiliar to the reader. We introduce the general form of autonomous CQ dynamics and its Kramers-Moyal expansion, which are central objects of study in the rest of the thesis. We also study Hamiltonian classical-quantum dynamics [28] and look at simple examples of CQ dynamics.

In Chapter 3, we provide the relevant gravity background required for the thesis. We review the initial value, or ADM formulation of General relativity, which is used in Chapter 7 to study the constraints in classical-quantum theories of gravity. We also discuss standard semi-classical approaches for incorporating quantum back-reaction on a classical gravitational field.

In Chapter 4, we study the consequences of the dynamics being completely positive. We use this to determine the general form of continuous CQ dynamics via a CQ Pawula theorem. This chapter is based on the paper [5], a collaboration with Jonathan Oppenheim, Carlo Sparaciari, and Barbara Šoda.

In Chapter 5, we show how one can unravel the continuous master equation in terms of coupled stochastic differential equations, which provides a concrete algorithm for simulating continuous classical quantum dynamics which go beyond the standard semi-classical equations. We also show mathematical equivalence between the general form of the continuous master equation and continuous measurement with feedback. This chapter is based on the paper [6], a collaboration with Isaac Layton and Jonathan Oppenheim.

In Chapter 6, we derive a path integral representation for classical-quantum dynamics, which includes conditions on the couplings necessary for complete positivity and trace preservation. The path integral generalizes the Feynman path integral for quantum systems and the stochastic path integral used in classical stochastic processes. When the classical-quantum action is at most quadratic in the momenta, we show we can arrive at a configuration space path integral, which we study in more detail in Chapter 8. This chapter is based on the paper [7], a collaboration with Jonathan Oppenheim.

In Chapter 7, we take steps towards constructing a complete theory of CQ gravity. We provide a methodology to derive the constraint equations of a classical-quantum theory of gravity by imposing invariance of the dynamics under time-reparametrization invariance in a geometrodynamic picture. We find generalizations of the Hamiltonian and momentum constraints for classical-quantum dynamics, and we compute their algebra for the case of a quantum scalar field interacting with gravity. This chapter is based on the paper [3], a collaboration with Jonathan Oppenheim.

In Chapter 8, we study configuration space classical-quantum path integrals in more detail. We show they can be used to construct Lorentz and diffeomorphism invariant theories, proving that such dynamics exist. We introduce a path integral formulation of general relativity where the space-time metric is treated classically. We introduce diffeomorphism invariant theory based on the trace of Einstein's equations, and another more complete theory which gives rise to all of the components of Einstein's equations. We posit a general form of constraints for this theory which look promising, leaving it as a question for future work whether they give rise to consistent dynamics. This chapter is based on the paper [8], a collaboration with Jonathan Oppenheim.

In Chapter 9, we investigate the non-relativistic limit of classical-quantum gravity, where a classical Newtonian potential interacts with quantum matter. The theory can be viewed as a gauge fixed version of the theory introduced in Chapter 8, where we only consider the scalar degrees of freedom to linear order. The theory generalizes previous discussions on Newtonian classical-quantum gravity [60, 49, 52]. We arrive at a general form dynamics, which gives rise to Poisson's equation on average but also acts to diffuse around the classical solution with associated decoherence on the quantum system. In Chapter 10, we use the Newtonian limit introduced in this chapter to study experimental tests of classical-quantum gravity in the non-relativistic, weak field regime. This chapter is based on upcoming work [9], a collaboration with Jonathan Oppenheim and Andrea Russo.

In Chapter 10, we further explore the conditions of complete positivity, and we use this to arrive at a trade-off between decoherence and diffusion. Applying the trade-off to a model of a classical, non-relativistic Newtonian potential interacting with quantum matter, we find experimental restrictions on theories where classical gravity interacts with quantum matter. We find that part of the parameter space of such theories is already squeezed out. We provide figures of merit that can be used in future mass measurements and interference experiments. This chapter is based on the paper [4], a collaboration with Jonathan Oppenheim, Carlo Sparaciari, and Barbara Šoda.

The last part of the thesis contains the appendices. We provide minor results used in the main chapters and longer proofs and derivations, which, if added to the main body, would complicate the exposition and the presentation of more important results.

Part I

Background

Chapter 2

Classical, quantum, and classical-quantum formalism

In this chapter, we introduce the background material on classical-quantum coupling required for the main body of the thesis. We first briefly overview both classical stochastic and quantum mechanics before discussing the classical-quantum formalism used in the rest of the thesis. When introducing the classical and quantum background material, we include snippet proofs of many concepts. We choose to do this because many of the classical-quantum formalism's introduced in the first half of the thesis will be extensions of their classical and quantum counterparts, often combining the classical and quantum concepts naturally; we hope this chapter can be a useful reference point. For example, in Chapter 4, we extend the classical Pawula theorem [76] to the CQ case, finding the most general form of continuous CQ dynamics, extending the Fokker-Plank and Lindblad equations. In Chapter 5, we find an unraveling for the general form of CQ dynamics; this extends and combines quantum unravelings of the Lindblad equations and unravelings of the Fokker-Plank equation by classical stochastic differential equations. While in Chapter 6, we find a path integral approach to CQ dynamics, which generalizes and combines the Feynman-Vernon path integral of open quantum systems [79] and the stochastic path integral of classical stochastic dynamics [80].

2.1 Stochastic classical dynamics

This section presents the basics of stochastic classical dynamics needed to understand the rest of the thesis. We do so in a simplified form by introducing the basic concepts required without going into the mathematical details of probability theory. We refer the reader to [73, 81, 82, 80] for a more detailed analysis.

2.1.1 The probability density and probability transition amplitude

We denote a generic classical degree of freedom by $z \in \mathcal{M}$, which could be a classical degree of freedom such as a position $z = q$ or a point in phase space $z = (q, p)$. When z is higher dimensional, we use d to refer to the dimension of the system and we denote the components as z_i. Because we are interested in stochastic classical dynamics, we let the classical degree of freedom be described by a random variable, denoted by a capital letter $Z(t)$ (or sometimes Z_t). At each time, the stochastic variable $Z(t) \in \mathcal{M}$ takes on the possible values of the classical variable.

In stochastic classical dynamics, the basic object of interest is the *probability density* $p(z, t)$, which is positive $p(z, t) \geq 0$ and normalized $\int dz\ p(z, t) = 1$. It is defined by

$$p(z, t) = \mathbb{E}[\delta(z - Z(t))], \tag{2.1}$$

where the expectation value $\mathbb{E}$ is an ensemble average over the stochastic variable. The probability of finding the stochastic variable $Z(t)$ in the interval $z \leq Z(t) \leq z + dz$ at time t is then given by $p(z, t)dz$.

The probability density in Equation (2.1) can be extended to joint distributions. The probability of finding $z_1 \leq Z(t_1) \leq z_1 + dz_1$ and $z_2 \leq Z(t_2) \leq z_2 + dz_2$ ... and $z_n \leq Z(t_n) \leq z_n + dz_n$ is given by

$$P_n(z_n, t_n; z_{n-1}, t_{n-1} \dots; z_1, t_1)dz_1 \dots dz_n, \tag{2.2}$$

where

$$P_n(z_n, t_n; z_{n-1}, t_{n-1} \dots; z_1, t_1) = \mathbb{E}[\delta(z_n - Z(t_n)) \dots \delta(z_1 - Z(t_1))]. \tag{2.3}$$

Knowing the hierarchy of joint probability distributions is equivalent to completely knowing the stochastic process $Z(t)$.

Given the joint probability distribution, we can define the stochastic process's conditional probability, or *probability transition amplitude*. The transition amplitude describes the probability of finding the random variable $Z(t)$ in a given interval $z_n \leq Z(t_n) \leq z_n + dz_n$ at time t_n, given it is in the interval $z_{n-1} \leq Z(t_{n-1}) \leq z_{n-1} + dz_{n-1}$ at t_{n-1}, in the interval $z_{n-2} \leq Z(t_{n-2}) \leq z_{n-2} + dz_{n-2}$ at t_{n-2} ..., and in the interval $z_1 \leq Z(t_1) \leq z_1 + dz_1$ at t_1. Mathematically it is defined by

$$\begin{aligned} P(z_n, t_n | z_{n-1}, t_{n-1}; \ldots; z_1, t_1) &= \frac{P_n(z_n; t_n; \ldots; z_1, t_1)}{P_{n-1}(z_{n-1}, t_{n-1}; \ldots z_1, t_1)} \\ &= \frac{P_n(z_n; t_n; \ldots; z_1, t_1)}{\int dz_n P_n(z_n, t_n; \ldots z_1, t_1)}. \end{aligned} \tag{2.4}$$

2.1.2 Markovian processes

We shall be interested in Markovian processes (i.e., stochastic processes that are time-local).

For a Markovian process

$$P(z_n, t_n | z_{n-1}, t_{n-1}; \ldots; z_1, t_1) = P(z_n, t_n | z_{n-1}, t_{n-1}), \tag{2.5}$$

so that the conditional probability for finding $Z(t)$ in the interval $z_n \leq Z(t_n) \leq z_n + dz$ depends only on the value of $Z(t)$ at the previous time, as opposed to its entire history. In particular, this allows us to write the *Chapman-Kolmogorov equation* for the conditional probability

$$P(z_3, t_3 | z_1, t_1) = \int dz_2 P(z_3, t_3 | z_2, t_2) P(z_2, t_2 | z_1, t_1), \tag{2.6}$$

and knowledge of the conditional probability $P(z, t+\delta t | z', t)$ is enough to completely understand the process. Note, by definition, the conditional probability is positive and

$$\int dz P(z, t + \delta t | z', t) = 1 \tag{2.7}$$

ensures probabilities are normalized.

2.1.3 Short time moment expansion and the Fokker-Plank equation

For Markovian dynamics, one can derive a master equation from the transition probability amplitude. In particular, we can introduce a *Kramers-Moyal* expansion [83, 84, 73] of the master equation, which is obtained via a short time moment expansion of the probability transition amplitude $P(z,t+\delta t|z',t)$

$$P(z,t+\delta t|z',t) = \delta(z,z') + \delta t W(z|z',t). \tag{2.8}$$

The moments of the transition amplitude are defined via

$$M_{n,i_1\ldots i_n}(z',t,\delta t) = \int dz P(z,t+\delta t|z',t)(z-z')_{i_1}\ldots(z-z')_{i_n}, \tag{2.9}$$

and their short time expansion can be calculated as

$$\begin{aligned} M_{n,i_1\ldots i_n}(z',t,\delta t) = \delta_n^0 + \delta t \int dz W(z|z',t)(z-z')_{i_1}\ldots(z-z')_{i_n} \\ := \delta_n^0 + \delta t n! D(z')_{n,i_1\ldots i_n}, \end{aligned} \tag{2.10}$$

where we have defined the short-time expansion of the moments

$$D(z',t)_{n,i_1\ldots i_n} = \frac{1}{n!}\int dz W(z|z',t)(z-z')_{i_1}\ldots(z-z')_{i_n}, \tag{2.11}$$

which define the *moments of the Kramers-Moyal expansion*. The subscripts $i_j \in \{1,\ldots d\}$ label the different components of the vectors $(z-z')$. For example, in the case where $d=2$ and the classical degrees of freedom are the position and momenta of a particle $z=(z_1,z_2)=(q,p)$, we have $(z-z')=(z_1-z_1',z_2-z_2')=(q-q',p-p')$. The components are then given by $(z-z')_1=(q-q')$ and $(z-z')_2=(p-p')$.

Sometimes in the literature, a distinction is made between processes that satisfy Equation (2.5) with a time-dependent transition amplitude and those which are time-independent. The former is often called time-dependent Markovianity [85], and we will also refer to it as an *autonomous process* [28]. For notational simplicity, we shall often suppress the potential explicit time dependence of the moments $D(z',t)$ and the short-time transition amplitude $W(z|z',t)$, which can always be added later. We also find it helpful to refer to the moments as $D_n(z')$, by which we mean the object with components $D(z')_{n,i_1\ldots i_n}$.

To derive the master equation for autonomous processes, we first define the characteristic function, which is the Fourier transform of the transition amplitude

$$C(u,z',\delta t) = \int dz e^{iu\cdot(z-z')}P(z,t+\delta t|z',t) = \sum_{n=0}^{\infty}\frac{(i^n)u_{i_1}\ldots u_{i_n}}{n!}M_{n,i_1\ldots i_n}(z',\delta t), \tag{2.12}$$

In Equation (2.12), and throughout the thesis, we use the summation convention, so that contracted indices are assumed to be over. Taking the inverse Fourier transform, we can relate the transition amplitude to its moments

$$\begin{aligned} P(z, t+\delta t|z', t) &= \int du\ e^{-iu(z-z')} C(u, z', \delta t) \\ &= \sum_{n=0}^{\infty} \frac{M_{n,i_1\ldots i_n}(z', \delta t)}{n!} \frac{1}{(2\pi)^d} \int du \quad e^{-iu(z-z')}(i^n) u_{i_1} \ldots u_{i_n}, \end{aligned} \tag{2.13}$$

which, using the definition of the delta function

$$\delta(z, z') = \frac{1}{(2\pi)^d} \int du e^{-iu(z-z')}, \tag{2.14}$$

we can write as

$$P(z, t+\delta t|z', t) = \sum_{n=0}^{\infty} \frac{1}{n!} M_{n,i_1\ldots i_n}(z', \delta t) \left(\frac{\partial^n}{\partial z'_{i_1} \ldots \partial z'_{i_n}} \right) \delta(z, z'). \tag{2.15}$$

Substituting the short time moment coefficients of Equation (2.11) back into (2.15), taking the limit $\delta t \to 0$, and using the probability preserving condition in (2.7), we can write the master equation in the form

$$\frac{\partial p(z,t)}{\partial t} = \sum_{n=1}^{\infty} (-1)^n \frac{\partial^n}{\partial z_{i_1} \ldots \partial z_{i_n}} \left(D_{n,i_1\ldots i_n}(z) p(z,t) \right). \tag{2.16}$$

Equation (2.16) is known as the*Kolmogorov forward equation.* The moments of the master equation can be related to useful physical quantities; for example, $D_{1,i}$ governs the evolution of $\frac{d\mathbb{E}[z_i]}{dt}$ and is associated with the *drift* in the system, while $D_{2,ij}$ characterizes the amount of *diffusion* in the system and computes the second moments $\frac{d\mathbb{E}[z_i z_j]}{dt}$.

2.1.4 Pauli rate equation

In the literature, it is common to find the master equation written in a slightly different form. Instead of defining the moments of the short-time probability amplitude, one considers

$$P(z, t+\delta t|z', t) = \delta(z, z')(1 - \delta t \tilde{W}(z, t)) \delta t \tilde{W}(z|z', t), \tag{2.17}$$

where the norm condition in Equation (2.7) defines $\tilde{W}(z) = \int dz \tilde{W}(z|z')$. Equation (2.17) redefines the $W(z|z', t)$ appearing in Equation (2.8) via $W(z|z') = \tilde{W}(z|z') - \delta(z, z')\tilde{W}(z)$.

When the expansion in Equation (2.17) is used, then by direct substitution of the short time moment expansion into (2.8), we arrive at the Pauli rate equation

$$\frac{\partial p}{\partial t} = \int dz' \tilde{W}(z|z')p(z') - \tilde{W}(z)p(z). \tag{2.18}$$

Note, the moments $D_n, n \geq 1$ for W are the same as for $\tilde{W}$ since they only differ by their zeroth moment, which does not contribute to the master equation. Hence, the positivity condition $\delta(z, z') + \tilde{W}(z|z', t) \geq 0$ is still the necessary and sufficient condition to preserve the positivity of probabilities. The form of the master equation in Equation (2.18) is useful since it is positive *and normalized* for any positive $\tilde{W}(z|z')$.

2.1.5 Pawula theorem

An important theorem of Pawula [76] says that in order for the dynamics to preserve the positivity of the probability distribution, the Kramers-Moyal expansion must terminate at second order or contain infinitely many terms; specifically, none of the even moments D_{2n} can vanish. If it terminates at second order, it gives the well-known *Fokker-Plank equation*

$$\frac{\partial p(z,t)}{\partial t} = -\frac{\partial}{\partial z_i}\left[D_{1,i}(z,t)p(z,t)\right] + \frac{\partial^2}{\partial z_i \partial z_j}\left[D_{2,ij}(z,t)p(z,t)\right], \tag{2.19}$$

otherwise, the system undergoes finite size jumps with non-zero probability [73]. More precisely,

Pawula Theorem. *The series of moments D_n^{00}, $n \geq 1$, appearing in the Kramers-Moyal expansion of* (2.16) *either contains infinitely many terms, or it truncates after second order, in which case we have a Fokker-Plank equation.*

We generalize the Pawula theorem to include classical-quantum coupling in Chapter 4, finding the general form of continuous classical quantum dynamics. Therefore we provide proof of the classical Pawula theorem, which can be used as a reference.

Proof. We start with the generalized Cauchy-Schwartz inequality

$$\left[\int f(\Delta)g(\Delta)P(\Delta)\mathrm{d}\Delta\right]^2 \leq \int f^2(\Delta)P(\Delta)\mathrm{d}\Delta \int g^2(\Delta)P(\Delta)\mathrm{d}\Delta, \tag{2.20}$$

which holds for any non-negative distribution $P(\Delta)$ and arbitrary real valued functions $f(\Delta), g(\Delta)$. Using Equation (2.20) with

$$P(\Delta) = P(z+\Delta, t+\delta t|z), \quad f(\Delta) = \Delta_{i_1} \dots \Delta_{i_n}, \quad g(\Delta) = \Delta_{i_{n+m}} \dots \Delta_{i_{2n+2m}} \tag{2.21}$$

gives the inequalities

$$(M_{2n+m, i_1 \dots i_{2n+m}})^2 \leq M_{2n, i_1 i_1 \dots i_n i_n} M_{2n+2m, i_{n+m} i_{n+m} \dots i_{2n+2m} i_{2n+2m}}, \tag{2.22}$$

where $M_{n,i_1\dots i_n}(z, \delta t)$ is defined in Equation (2.10). To prove the Pawula theorem, we first relate the coefficients $M_{n,i_1\dots i_n}(z, \delta t)$ to the short-time expansion coefficients which appear in the master equation. Recall, we have $M_{n,i_1\dots i_n}(z, \delta t) = \delta_n^0 + n! D_{n_{i_1,\dots i_n}}(z)\delta t + O(\delta t^2)$. We have to be a little careful since $M_n(z, \delta t) = O(\delta t)$ for $n \geq 1$ but $O(1)$ for $n = 0$.[1] For $n = m = 0$ the inequality in (2.22) is trivially satisfied, while for $n = 0, m \geq 1$, we have no constraints on the short time expansion coefficients since the right-hand side of equation (2.22) is $O(\delta t)$ while the left-hand side is $O(\delta t^2)$. For $n \geq 1, m \geq 0$ we find

$$\left[(2n+m)! D_{2n+m, i_1 \dots i_{2n+m}}\right]^2 \leq (2n)!(2n+2m)! D_{2n, i_1 i_1 \dots i_n i_n} D_{2n+2m, i_{n+m} i_{n+m} \dots i_{2n+2m} i_{2n+2m}}. \tag{2.23}$$

Equation (2.23) is enough to derive the Pawula theorem, which tells us that if any even moment vanishes, then all moments with $n \geq 3$ must also vanish. To see this explicitly, observe if any even moment vanishes so that $D_{2n} = 0$, then $D_{2n+m} = 0$ for all m. Hence, if any even moment is zero, all higher-order moments must also vanish. Furthermore, if $D_{2n+2m} = 0$ then it can be seen from (2.23) that $D_{2n+m} = 0$. Denoting $r = n + m$, then this says $D_{2r} = 0$ implies $D_{r+n} = 0$ for $n = 1 \dots r-1$. Hence if any even moment vanishes, $D_{2r} = 0$, we deduce all higher order moments D_{2r+n} must vanish, as well as the moments D_{r+n} for $n = 1 \dots r-1$. Except for the case $r = 1$, $r+n$ will always contain an even number, and so from repeated application of this property, we deduce D_n must vanish for $n \geq 3$. □

2.1.6 Stochastic differential equations

A very useful fact is that Fokker-Plank equations can be identified with stochastic differential equations [73]

$$dZ_i(t) = \mu_i(Z(t), t)dt + \sigma_{ij}(Z(t), t)dW_j(t), \tag{2.24}$$

[1]We use the simplifying notation $M_n(z, \delta t)$, which means the matrix with components $M_{n,i_1\dots i_n}(z, \delta t)$

where $W_i(t)$ is a standard d dimensional Wiener process satisfying the Ito rules

$$dW_i dW_j = \delta_{ij} dt, \; dW_i dt = 0. \tag{2.25}$$

Equation (2.24) is to be interpreted, formally, as an integral equation, and to get from Equation (2.24) to (2.19) we use the Ito definition of the stochastic integral [86]. Specifically, if a stochastic process $Z(t)$ satisfies the stochastic differential equation in (2.24) then the probability density for $p(z,t) = \mathbb{E}[\delta(z - Z(t))]$ satisfies the Fokker-Plank equation (2.19), where $D_2 = \frac{1}{2}\sigma\sigma^T$ and $D_{1,i} = \mu_i$. If we instead use the Strananovich definition of a stochastic integral, then the Fokker-Plank equation can be identified with Equation (2.24) but with a redefinition of the drift vector μ_i [73].

To make the identification between (2.24) and (2.19), we start by noting that the dynamics of Z_t induces the following evolution on the probability density $p(z,t) = \mathbb{E}[\delta(z - Z(t))]$,

$$dp(z,t) = \frac{\partial p(z,t)}{\partial t} dt = \mathbb{E}[d\delta(z - Z(t))]. \tag{2.26}$$

Because of Ito's lemma, we must go to second order in dW to find the master equation. With this in mind we find

$$\mathbb{E}[d\delta(z-Z(t))] = \mathbb{E}[\frac{\partial}{\partial Z_i}[\delta(z-Z(t))]\mu_i(Z(t),t)]dt + \mathbb{E}[\frac{1}{2}\frac{\partial^2}{\partial Z_i \partial Z_j}[\delta(z-Z(t))]\sigma_{ik}(Z_t,t)\sigma^T_{kj}(Z_t,t)]dt. \tag{2.27}$$

We can use some well-known facts about the delta functional to simplify Equation (2.27). Using the two identities $\partial_{Z_i}\delta(z-Z) = -\partial_{z_i}\delta(z-Z)$ and $f(Z)\delta(z-Z) = f(z)\delta(z-Z)$ for any function f, the right hand side of Equation (2.27) becomes

$$-\frac{\partial}{\partial z_i}\mathbb{E}[\delta(z - Z(t))\mu_i(z,t)]dt + \frac{\partial^2}{\partial z_i \partial z_j}\mathbb{E}[\delta(z - Z(t))D_{2,ij}(z)]dt. \tag{2.28}$$

Using the definition of the probability density $p(z,t) = \mathbb{E}[\delta(z - Z(t))]$ and dividing by dt, we arrive at the Fokker-Plank equation of Equation (2.19).

2.1.7 Path integrals for stochastic classical dynamics

Another representation of classical dynamics is through classical stochastic path integrals. We derive the path integral for classical-quantum dynamics in Chapter 6. As a warm-up, in this section, we sketch a derivation of the classical path integral associated with a 1 Dimensional Fokker-Plank equation

$$\frac{\partial p(z,t)}{\partial t} = -\frac{\partial}{\partial z}\left[D_1(z,t)p(z,t)\right] + \frac{\partial^2}{\partial z^2}\left[D_2(z,t)p(z,t)\right]. \tag{2.29}$$

The more general case can be found in [80, 87] and in Chapter 6.

To arrive at the classical path integral, one starts from the moment expansion of the transition amplitude in Equation (2.13). In particular, this tells us that the transition amplitude at time $t+\delta t$ relates to that at t via

$$\begin{aligned}\varrho(z', t+\delta t) &= \frac{1}{(2\pi)}\int dudz\; e^{-iu\cdot(z'-z)}\;(1+\delta t[iuD_1 p(z,t) - u^2 D_2])\varrho(z,t) \\ &= \frac{1}{(2\pi)}\int dudz \exp\left(-iu\frac{(z'-z)}{\delta t}\delta t + \delta t iuD_1 - \delta t u^2 D_2\right)\varrho(z,t).\end{aligned} \tag{2.30}$$

Equation (2.30) is enough to derive the path integral. In particular, we note that by considering many time steps and taking the limit $\delta t \to 0$ we arrive at the Fokker-Plank path integral with action

$$S_C[u,z] = \int dt[-iu\frac{dz}{dt} + iuD_1 - u^2 D_2]. \tag{2.31}$$

Because the action is quadratic in u, it is also possible to integrate out the *response variables* u by performing standard Gaussian integrals to arrive at a path integral for the z variables alone. In this case, one finds the action (see Chapter 6)

$$S_C[z] = \int dt\left[-\frac{1}{4}D_2^{-1}(\frac{dz}{dt} - D_1)^2\right]. \tag{2.32}$$

2.2 Quantum theory and open quantum systems

This section discusses the formalism of open quantum systems, which will be used throughout the thesis. We first review the standard rules of quantum theory for closed systems before discussing open quantum systems and their corresponding dynamics, including unraveling and path integral approaches to study open systems. This section takes much inspiration from [88], and we refer the reader to [88, 82] for a detailed analysis of open quantum systems.

2.2.1 Quantum theory for closed systems

In standard quantum theory for an isolated system, a quantum state is defined as a ray in a Hilbert space; a Hilbert space is a vector space over the complex numbers $\mathbb{C}$ which has an inner product $\langle\psi|\phi\rangle$ and is complete in the norm $||\psi|| = \langle\psi\rangle^{1/2}$. A ray is an equivalence class of vectors that differ by multiplication by a non-zero complex scalar $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$. For any non-zero ray, we choose a representative of the class $|\psi\rangle$ with unit norm. States evolve dynamically under unitary evolution $\psi(t)\rangle = U|\psi(0)\rangle$, with $UU^\dagger = U^\dagger U = \mathbb{I}$.

Observables are the properties of physical systems which can be measured. In classical mechanics, an observable is any functional over the classical phase space. In quantum mechanics, observables are self-adjoint operators $O = O^\dagger$ on the Hilbert space. Because observables are self-adjoint, they admit a spectral representation in terms of their eigenvectors $\{|i\rangle\}$, which form a complete orthonormal basis for the Hilbert space

$$O = \sum_i o_i |i\rangle\langle i| = \sum_i o_i E_i, \tag{2.33}$$

where $E_i = |i\rangle\langle i|$ denotes the projection operator onto the eigenvector $|i\rangle$.

When an observable O is measured in the quantum state $|\psi\rangle$, the outcome prepares an eigenstate of O. Specifically, the outcome o_i occurs with probability $p_i = \langle\psi|E_i|\psi\rangle$ and after the measurement the quantum state is given by $\frac{E_i|\psi\rangle}{\langle\psi|E_i|\psi\rangle}$. If many identical systems $|\psi\rangle$ are prepared, we can calculate the expectation value of a quantum observable $\langle O\rangle = \sum_i o_i p_i = \langle\psi|O|\psi\rangle$.

In the standard axioms of quantum theory, the puzzling distinction between deterministic unitary dynamics determining the dynamics of the quantum state and non-unitary probabilistic evolution of quantum systems when they are being measured is called the measurement problem in quantum theory. There are many equivalent descriptions and refinements of the measurement problem. In Chapter 5, we find another. We show that we can give an alternative, effective dynamical description of measurement using the classical-quantum formalism. Instead of using the Born rule directly, one can posit that in an effective theory of quantum measurement, we should specify a classical degree of freedom that we monitor (i.e., the measurement device), which couples to a quantum system to be measured. The combined CQ evolution gives rise to dynamics equivalent to the Born rule. The quantum state collapses dynamically when inferences can be made about it by measuring and conditioning the classical system. The measurement problem is then mapped to the problem of how and when some degrees of freedom classicalize.[2]

Quantum systems compose under the tensor product. If the Hilbert space of system A is $\mathcal{H}_A$ and the Hilbert space of system B is $\mathcal{H}_B$, then the Hilbert space of the composite systems AB is the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$. If system A is prepared in the state $|\psi\rangle_A$ and system B is prepared in the state $|\varphi\rangle_B$, then the composite system's state is the product $|\psi\rangle_A \otimes |\varphi\rangle_B$. When writing the tensor products of states, we will suppress the tensor product notation and write $|\psi\rangle_A|\varphi\rangle_B$.

Because composite systems are defined by the tensor product, there are some states in $\mathcal{H}_{AB}$ which do not decompose as a product of states defined on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. For example, for the composition of two, two-level Hilbert spaces spanned by $\{|0\rangle, |1\rangle\})_A$ and $\{|0\rangle, |1\rangle\})_B$ the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \tag{2.34}$$

does not decompose into a product state on A, B. Such states are called *entangled states.*

2.2.2 Open quantum systems

Because of entanglement, in open quantum systems - when we limit our attention to a part of a larger system - a different formalism is required to express the outcome of any measurement performed on a subsystem.

For example, consider a Hilbert space of a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$, and let $\{|i\rangle_A\}, \{|\mu\rangle_B\}$ denote an orthonormal basis for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, so that any quantum state can be expanded as

$$|\psi\rangle_{AB} = \sum_{i,\mu} a_{i\mu}|i\rangle_A|\mu\rangle_B. \tag{2.35}$$

The expectation value of any observable on the system A alone $O_A \otimes \mathbb{I}_B$ is found via

$$\begin{aligned}\langle O_A\rangle &= \langle\psi|O_A \otimes \mathbb{I}_B|\psi\rangle \\ &= \sum_{i,j,\mu}\langle j|O_A|i\rangle = \mathrm{Tr}_A\left[O_A\rho_A\right],\end{aligned} \tag{2.36}$$

[2]If one was to make the controversial statement that some degrees of freedom are fundamentally classical, for example, the gravitational field, then, in a similar way to collapse models, this provides a potential route to solve the measurement problem, since there would be no need to include the Born rule, just the dynamics of interacting classical and quantum systems.

where we have defined

$$\rho_A = \mathrm{Tr}_B\left[\rho_{AB}\right], \tag{2.37}$$

with $\mathrm{Tr}_B\left[\right]$ the partial trace operation over B and $\rho_{AB} = |\psi\rangle\langle\psi|$. The partial trace operation is a linear map that takes an operator O_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ to an operator on $\mathcal{H}_A$ alone, defined by

$$\mathrm{Tr}_B\left[O_{AB}\right] = \sum_{\mu} \langle\mu|_B O_{AB} |\mu\rangle_B. \tag{2.38}$$

Both ρ_{AB} and ρ_A define *density matricies* on $\mathcal{H}_{AB}$ and $\mathcal{H}_A$ respectively. These are positive $\langle v|\sigma|v\rangle \geq 0$ $v \in \mathcal{H}$, Hermitian $\sigma = \sigma^\dagger$ and normalized $\mathrm{Tr}_{\mathcal{H}}\left[\sigma\right] = 1$ operators on the Hilbert space. The density matrix for a *pure state* $|\psi\rangle$ is given by $\rho = |\psi\rangle\langle\psi|$, while density matrices on subsystems are obtained via the partial trace operation. We sometimes call density matrices for subsystems *reduced density matrices.*

Density matrices are used in open quantum systems to give an operational description for all outcomes on a subsystem A, even when considered part of a larger system. Because they are positive Hermitian operators, density matrices can always be diagonalized in an orthonormal basis, in which case they can always be written the form

$$\rho = \sum_i p_i |i\rangle\langle i|. \tag{2.39}$$

with $p_i \geq 0$.

2.2.3 Density matrices and ensembles

Not only do density matrices describe the outcomes of measurements on subsystems, but they also combine naturally when probabilistic mixtures of quantum states are involved.

First note that given any two density matrices ρ_1, ρ_2, any convex combination $\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2$ is also a density matrix. In particular,

$$\begin{aligned}\langle O\rangle &:= \lambda\langle O\rangle_1 + (1-\lambda)\langle O\rangle_2 \\ &= \lambda \mathrm{Tr}\left[O\rho_1\right] + (1-\lambda)\mathrm{Tr}\left[O\rho_2\right] \\ &= \mathrm{Tr}\left[O\rho(\lambda)\right].\end{aligned} \tag{2.40}$$

Consequently, density matrices are consistent with an ensemble interpretation when probabilistic mixtures of quantum states are considered. Suppose the quantum states $\rho_i = |\psi_i\rangle_{AB}\langle\psi_i|_{AB}$

are each prepared with probability p_i, then the expectation value of any quantum observable is given by

$$\langle O\rangle = \sum_i p_i \mathrm{Tr}\left[\rho_i\right] = \mathrm{Tr}\left[\rho O\right], \tag{2.41}$$

where $\rho = \sum_i p_i\rho_i$ defines the density matrix for the entire system, and we can interpret $\rho = \sum_i p_i\rho_i$ as describing the quantum state where each ρ_i is prepared with probability p_i. The probabilistic interpretation of density matrices follows because the trace operation, which computes the expectation values of quantum observables, is linear.

Similarly, the expectation value for an observable on a subsystem A alone is given by

$$\langle O_A\rangle = \sum_i p_i \mathrm{Tr}_A\left[O\rho_{i,A}\right] = \mathrm{Tr}_A\left[O_A\rho_A\right] \tag{2.42}$$

where $\rho_{i,A} = \mathrm{Tr}_B\left[\rho_i\right]$ is the reduced density matrix for each of the possible prepared states and $\rho_A = \sum_i p_i\rho_{i,A}$. In other words, the density matrix ρ_A gives an operational description where each of the quantum states ρ_i is prepared with the probability p_i. The probabilistic interpretation of reduced density matrices follows because the partial trace operation, used to compute the expectation values of quantum observables on subsystems, is also linear.

Because the density matrix provides a consistent formalism even when ensembles of quantum states are considered, we can write the density matrix for a quantum system after the measurement of an observable $O = \sum_i o_i E_i$ is performed. After the measurement of O, the quantum state will be given by $|\psi'\rangle_i = \frac{E_i|\psi\rangle}{\langle\psi|E_i|\psi\rangle}$ and occurs with probability $p_i = \langle\psi|E_i|\psi\rangle$. Expressing ρ as an ensemble of pure states, we see that the measurement modifies the state according to

$$\rho' = \sum_i p_i|\psi_i'\rangle\psi_i'| = \sum_i E_i\rho_i E_i. \tag{2.43}$$

Generically density matrices can be decomposed in terms of other density matrices in many different ways, and the ensemble they describe is ambiguous without information about the entire system. The exceptions are the pure states $\rho = |\psi\rangle\langle\psi|$, which admit a unique decomposition and cannot be written as a convex combination of other density matrices. When the state is not pure, we call it *mixed.* It can be shown that a reduced density matrix on a subsystem S is pure if and only if $\mathrm{Tr}_S\left[\rho_S^2\right] = 1$ and mixed if $\mathrm{Tr}_S\left[\rho_S^2\right] < 1$. This can be used to characterize entangled states since the reduced density matrix for one of its subsystems must necessarily be mixed. Though the density matrix provides a complete operational description of the statistics of any measurement on the subsystem A, the ambiguity of its decomposition means mixed states do not convey the same physical state as when they are accompanied by information about operations on the B system.

A simple example is the maximally mixed state $\rho_A = \frac{1}{2}\mathbb{I}$, which can be decomposed as $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ but also as $\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle$: the density matrix gives the same operational description of the system whether the states $|0\rangle, |1\rangle$ are prepared with probability 1/2, whether $|+\rangle, |-\rangle$ are prepared with probability 1/2, or whether the system $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is prepared, and the experimenter only has access to part of the system.

When a state is a mixed state decomposed in its diagonal form as $\rho_A = \sum_i p_i |i\rangle\langle i|$ we say that it is in an *incoherent mixture* of the states $\{|i\rangle\}$ because the relative phases of the $|i\rangle$ states are experimentally inaccessible, which appear as off-diagonal terms in the density matrix. We see that when systems become entangled, the reduced state for the subsystem is mixed, and the entanglement destroys the coherence of the subsystem; some of the phases in the superposition become inaccessible when looking at the subsystem alone. When modeling an open quantum system dynamically, as we do in Section 2.2.5, this is represented by the *decoherence* of the off-diagonal terms in the density matrix. Intuitively, as a system undergoes evolution, it becomes entangled with the environment, which destroys the available coherence to the subsystem, represented by an exponential decay in its off-diagonal elements.

One can see more explicitly that density matrices can be realized in many different ways by considering the purification of density matrices. One can always *purify* a density matrix $\rho_A = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$, realizing it as a partial trace over a larger quantum system

$$|\phi\rangle_{AB} = \sum_i \sqrt{p_i} |\varphi_i\rangle_A |\alpha_i\rangle_B, \tag{2.44}$$

$$\rho_A = \mathrm{Tr}_B \left[|\phi\rangle_{AB} \langle\phi|_{AB} \right]. \tag{2.45}$$

Given that a density matrix can always be purified, it is natural to ask if they can always be realized as a proper ensemble of quantum states by measurements on a purified system.

The fact that this can always be done is immediate from Equation (2.44). By measuring an observable which projects into the $|\alpha_i\rangle_B$ basis on system B, the post measurement state for $|\phi\rangle_{AB}$ will be given by $|\phi'\rangle_{AB} = |\varphi_i\rangle_A |\alpha_i\rangle_B$ with probability p_i, which prepares $\rho_A =$

$\sum_i p_i|\varphi_i\rangle\langle\varphi_i|$ as a proper probabilistic ensemble of these states.

Similarly, given another decomposition of the *same* density matrix into pure states $\rho_A = \sum_i p'_i|q_i\rangle\langle q_i|$, we can always purify it in the form

$$|\phi_2\rangle_{AB} = \sum_i \sqrt{p'_i}|q_i\rangle_A|\beta_i\rangle_B, \tag{2.46}$$

and realize ρ_A as a proper ensemble of $|q_i\rangle$ by orthonormal measurements in the basis $|\beta_i\rangle$ on the second system. Using the Schmidt decomposition, one can show that any two purifications of the density matrix are related by a unitary transformation on the enlarged system [89, 48]. That is

$$|\phi\rangle_{AB} = (\mathbb{I} \otimes U_B)|\phi_2\rangle_{AB}. \tag{2.47}$$

We see a complete operational description of the statistics of any measurement on the subsystem A, which is not the same physical state as ρ_A, accompanied with information about operations on the B system. In particular, though measurements on the B system do not appear in any change in the operational description of ρ_A, and hence cannot be used to superluminally signal, it is clear that measurements on the B system do alter the physical state of the system in a non-local way. The fact that the correlations implied by the quantum theory are not reproducible by a local physical model is the content of Bell's theorem [90].

2.2.4 Quantum channels

In open quantum systems, dynamics are no longer unitary but described by a map from density operators to density operators. These are called *quantum channels*, which are completely positive, trace preserving (CPTP), and linear map on the space on density matrices [82, 88].

Mathematically, a quantum channel is

- Linear: $\Phi(\lambda\rho_1 + (1-\lambda)\rho_2 = \lambda\Phi(\rho_1) + (1-\lambda)\Phi(\rho_2))$.
- Hermiticity preserving: $\rho = \rho^\dagger$ implies $\Phi(\rho) = \Phi(\rho)^\dagger$.
- Trace preserving: $\mathrm{Tr}\,[\rho] = \mathrm{Tr}\,[\Phi(\rho)]$.
- Completely positive: $\rho \geq 0$ implies $\Phi \otimes \mathbb{I}_n(\rho) \geq 0$ for all n.

Complete positivity (CP) $\rho \geq 0$ implies $\Phi(\rho) \geq 0$ and is required if we want the channel to be positive, even if we consider it to act on just part of a larger system. This is required if we want the channel to give positive probabilities even when acting on part of an entangled state.

Linearity is required to maintain the ensemble interpretation of density matrices. In particular, suppose we consider preparing an ensemble of states with the density operator $\rho = \sum_i p_i \rho_i$. Then the final state after applying the quantum channel is

$$\rho' = \Phi(\sum_i p_i \rho_i). \tag{2.48}$$

Conversely, the final state should be found by applying the quantum channel to each state ρ_i and then averaging $\sum_i p_i \Phi(\rho_i)$. Equating these two is equivalent to asking the quantum channel to be linear. The trace and Hermitian preserving properties are required for the system to describe density matrices with normalized probability outcomes.

Quantum channels are fully characterized by the Kraus theorem [91], which states that any quantum channel can always be written in the form

$$\rho' = \Phi(\rho) = \sum_{\mu\nu} \Lambda^{\mu\nu} K_\mu \rho K_\nu^\dagger, \tag{2.49}$$

where $\Lambda^{\mu\nu}$ is Hermitian, positive matrix $a_\mu^* \Lambda^{\mu\nu} a_\nu \geq 0$ for any vector a_μ. The K_μ are known as *Kraus operators* and can always be taken to describe an orthogonal set of operators on the Hilbert space. We call the matrix $\Lambda^{\mu\nu}$ the *Kraus matrix*. The preservation of trace enforces the normalization condition

$$\sum_{\mu\nu} \Lambda^{\mu\nu} K_\nu^\dagger K_\mu = \mathbb{I}. \tag{2.50}$$

2.2.5 The Lindblad (GKSL) equation

We will be interested in discussing the dynamics of time-local classical-quantum systems, so we review time-local quantum channels in this section. When the dynamics are time-local, the state at time $t + \delta t$ can be written in terms of the state at time t alone, and a master equation can be derived when the Kraus operators are taken to be bounded and trace-class. When considering the dynamics of quantum systems, we consider quantum channels $\rho(t) = \Phi_t(\rho(0))$ and allow t to vary $0 \leq t \leq T$ which defines a one-parameter family of dynamical CPTP maps Φ_t.

2.2.6 Time local dynamics

A family of dynamical maps Φ_t is said to be *time-local* if $\dot{\rho}_t = \mathcal{L}_t(\rho_t)$ for a linear map $\mathcal{L}_t$. An important feature of time-local dynamics is that they can always be written in Lindblad form [92]

$$\frac{\partial \rho_t}{\partial t} = -i[H, \rho_t] + \lambda^{\alpha\beta}(t)\left(L_\alpha \rho_t L_\beta^\dagger - \frac{1}{2}\{L_\beta^\dagger L_\alpha, \rho_t\}\right), \tag{2.51}$$

where the matrix $\lambda^{\alpha\beta}(t)$ is Hermitian, but in general, the conditions for complete positivity are not known [92, 93]. In Equation (2.51), we again use the summation convention, so that contracted $\alpha\beta$ are assumed to be over. The operators L_α appearing in Equation (2.51) are called *Lindblad operators* and the matrix $\lambda^{\alpha\beta}$ we call the *Lindblad coupling*. Though Φ_t is a completely positive map on initial states for all times t, the generator $\mathcal{L}_t$ need not generate CP dynamics for $0 < t \leq T$: one can consider scenarios where the initial quantum state decoheres at early times, but recoeheres at late times keeping the total dynamics Φ_t completely positive [93]. In other words, $\mathcal{L}_t$ needs only generate CP dynamics on the subset of states that are reachable at time t, $\{\sigma_t : \exists\ \rho_0 \text{ s.t } \sigma_t = \Phi_t(\rho_0)\}$. For example, suppose that for $0 \leq t \leq T/2$ the map Φ_t acts as the perfectly depolarizing channel, sending any density matrix to the identity. We can then construct a completely positive dynamics Φ_t for $0 \leq t \leq T$ by concatenating the depolarizing channel with any map (not necessarily CP) that preserves the identity (i.e, a unital map). The resulting dynamics will be CP on all initial states, but for $t > T/2$ the generator of the dynamics $\mathcal{L}_t$ doesn't need to be completely positive on all states.

Markovian and autonomous dynamics

When the coefficients $\lambda^{\alpha\beta}$ are time-independent positive matrices $a_\alpha^* \lambda^{\alpha\beta} a_\beta \geq 0$ for any a_α, then Equation (2.51) is the well-known Lindblad (or GKSL) equation familiar in open quantum systems [63, 62]. We call such dynamics *time-independent Markovian*. The Lindblad equation represents the most general form of allowed time-local dynamics when one also demands that the generator $\mathcal{L}_t = \mathcal{L}$ be time independent [63, 62].

When the generator is time-independent, $\mathcal{L}_t = \mathcal{L}$, it generates completely positive dynamics on *all* states. This leads to an alternative but equivalent definition of quantum Markovianity, which extends naturally to when $\lambda^{\alpha\beta}$ are time-dependent. In particular, we say dynamics are *autonomous* if $\mathcal{L}_t$ generates CPTP dynamics on all states. In this case, the general form of master Equation is given by Equation (2.51) where the coefficients $\lambda^{\alpha\beta}(t)$ are positive but can now be time-dependent. When the generator is time-independent, autonomy reduces to time-independent Markovianity, while when the generator is time-dependent but CPTP on all states, the dynamics have also been coined *time-dependent Markovianity* [85, 94, 92].

Deriving the master equation for autonomous dynamics

Because we are interested in autonomous CQ dynamics and the associated CQ master equation, it is helpful to include a derivation of the associated master equation, which is given by Equation (2.51) with time-dependent coefficients $\lambda^{\alpha\beta}(t)$ which are positive Hermitian matrices.

We start by performing a short-time expansion of the Kraus form in Equation (2.49). For autonomous dynamics, this always takes the form

$$\rho'(t+\delta t) = \sum_{\mu\nu} \Lambda^{\mu\nu}(t,\delta t) K_\mu \rho(t) K_\nu^\dagger \tag{2.52}$$

where $\Lambda^{\mu\nu}(t,\delta t)$ is a positive matrix for all times.

Since we can choose the basis of Kraus operators to be an arbitrary basis on the Hilbert space, for the derivation, we take them to be $K_\mu = (\mathbb{I}, L_\alpha)$ and assume they are bounded and trace-class. Since for $\delta t = 0$, we know the dynamical map reduces to the identity,

$$\Lambda^{\mu\nu}(t,\delta t) = \delta_0^\mu \delta_0^\nu + \delta t \lambda^{\mu\nu}(t), \tag{2.53}$$

we find

$$\begin{aligned}\rho(t+\delta t) &= \sum_{\mu\nu} \Lambda^{\mu\nu}(t) L_\mu \rho(t) L_\nu^\dagger = \rho + dt \sum_{\mu\nu} \lambda^{\mu\nu}(t) L_\mu \rho(t) L_\nu^\dagger \\ &= \rho(t) + dt[\lambda^{00}\rho + \lambda^{0\alpha}\rho(t)L_\alpha^\dagger + \lambda^{\alpha 0} L_\alpha \rho + \lambda^{\alpha\beta} L_\alpha \rho(t) L_\beta^\dagger],\end{aligned} \tag{2.54}$$

where positivity demands that the matrix

$$\Lambda^{\mu\nu}(t) = \begin{bmatrix} 1+\delta t\lambda^{00} & \delta t \lambda^{0\beta} \\ \delta t \lambda^{\alpha 0} & \delta t \lambda^{\alpha\beta} \end{bmatrix} + O(\delta t^2), \tag{2.55}$$

is positive. Explicitly, for any vector a_μ we have

$$a_\mu^* \Lambda^{\mu\nu} a_\mu = |a_0|^2 + \delta t[a_0^* \lambda^{0\alpha} a_\alpha + a_\alpha^* \lambda^{\alpha 0} a_0 + a_\beta^\dagger \lambda^{\alpha\beta} a_\alpha] \geq 0. \tag{2.56}$$

By choosing $a_0 = 0$, Equation (2.56) imposes that $\lambda^{\alpha\beta}(t)$ is a positive matrix in α, β. However, for any choice of a_μ, there are no additional restrictions on the off-diagonals $\lambda^{0\beta}$ since they are $O(\delta t)$ and in the limit $\delta t \to 0$ the dominant contribution arises from the Λ^{00} component which is of order $O(1)$, enforcing positivity.

To arrive at the Lindblad form of Equation (2.51), note that the norm condition imposes that

$$\begin{aligned}
&\sum_{\mu\nu} \Lambda^{\mu\nu} L_\nu^\dagger L_\mu = \mathbb{I}, \\
&\lambda^{00}\mathbb{I} + \lambda^{0\alpha} L_\alpha + \lambda^{\alpha 0} L_\alpha^\dagger + \lambda^{\alpha\beta} L_\beta L_\alpha = 0,
\end{aligned} \tag{2.57}$$

so substituting for λ^{00} and taking the $dt \to 0$ limit gives us the GKSL or Lindblad equation [63, 62]

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + \lambda^{\alpha\beta} L_\alpha \rho L_\beta^\dagger - \frac{1}{2}\lambda^{\alpha\beta}\{L_\beta^\dagger L_\alpha, \rho\}, \tag{2.58}$$

where we have defined the Hamiltonian $H = \frac{i}{2}(\lambda^{\alpha 0} L_\alpha - \lambda^{0\alpha} L_\alpha^\dagger)$. In Equation (2.58), the coefficients $\lambda^{\alpha\beta}(t)$ can be time-dependent but must be positive matrices in $\alpha\beta$. This concludes the proof that any autonomous dynamics can be brought to the form of Equation (2.58) for a specific choice of Lindblad operators $(\mathbb{I}, L_\alpha)$ and it is a well-known result [82, 95] that Equation (2.58) for arbitrary Lindblad operators L_α also defines autonomous dynamics. We will often write the anti-commutator $\{,\}$ as $\{,\}_+$, to distinguish it from the Poisson bracket appearing in Hamiltonian classical mechanics.

2.2.7 Decoherence

When considering an open quantum system according to the dynamics in Equation (2.58), the system generically undergoes decoherence due to its interaction with the environment. Intuitively, the system becomes entangled with its environment, destroying the subsystem's available coherence. This behavior is most easily seen when the Lindbladian operators are Hermitian. For example, consider the master equation

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + \lambda L \rho L^\dagger - \frac{1}{2}\lambda\{L^\dagger L, \rho\}, \tag{2.59}$$

for a Hermitian operator $L = L^\dagger$. Because L is Hermitian, it can be diagonalized, and we can choose to decompose ρ in terms of its eigenvectors $\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j|$. The action of the

Lindbladian term is to suppress the off-diagonal elements of the density matrix

$$\frac{\partial \rho_{ij}}{dt} \approx -\frac{\lambda}{2}(L_i - L_j)^2 \rho_{ij}. \tag{2.60}$$

2.2.8 Unraveling of Lindblad equations

In analogy with the unraveling of the Fokker-Plank equation by stochastic differential equations, the Lindblad equation can be unraveled by a set of stochastic differential equations. However, unlike the classical case, where the trajectories are objective, because of the ambiguities in the ensemble interpretation of the density matrix, the unravelings of the master equation are not unique, unless extra information about the state of the environment is provided. Nonetheless, they can be valuable tools in simulating open quantum systems [82].

One commonly found unraveling [96] is given by the stochastic differential equation for the pure quantum state

$$\begin{aligned} d|\psi\rangle_t = &-iH|\psi\rangle_t dt \\ &- \frac{1}{2}\lambda^{\alpha\beta}(L_\beta^\dagger - \langle L_\beta^\dagger\rangle)(L_\alpha - \langle L_\alpha\rangle)|\psi\rangle_t dt + \frac{1}{2}\lambda^{\alpha\beta}(\langle L_\beta^\dagger\rangle L_\alpha - \langle L_\alpha\rangle L_\beta^\dagger)|\psi\rangle_t dt \\ &+ (\lambda^{1/2})^{\alpha\beta}(L_\alpha - \langle L_\alpha\rangle)|\psi\rangle d\xi_\beta, \end{aligned} \tag{2.61}$$

where $\lambda^{1/2}\lambda^{1/2\dagger} = \lambda$ is the square route of the positive Hermitian matrix λ. In Equation (5.9), the noise process is a complex noise

$$d\xi_\alpha d\xi_\beta = 0, \; d\xi_\alpha d\xi_\beta^* = \delta_{\alpha\beta}, \tag{2.62}$$

and the density matrix is found by averaging over all possible realizations of the quantum state

$$\rho(t) = \mathbb{E}[|\psi(t)\rangle\langle\psi(t)|]. \tag{2.63}$$

To see the unraveling in Equation (2.61) gives the Lindblad equation when averaged over trajectories, first note that the evolution of the density matrix takes the form

$$d\rho = \mathbb{E}[|d\psi(t)\rangle\langle\psi| + |\psi(t)\rangle\langle d\psi(t)| + |d\psi(t)\rangle\langle d\psi(t)|] \tag{2.64}$$

where we have to go to second order because $d\xi d\xi^* = O(dt)$.

We can compute each term individually,

$$\mathbb{E}[|d\psi(t)\rangle\langle\psi|] = [-iH\rho - \frac{1}{2}\lambda^{\alpha\beta}L_\beta^\dagger L_\alpha\rho + \lambda^{\alpha\beta}\langle L_\beta^\dagger\rangle L_\alpha\rho - \frac{1}{2}\lambda^{\alpha\beta}\langle L_\beta^\dagger\rangle\langle L_\alpha\rangle\rho]dt \tag{2.65}$$

$$\mathbb{E}[|\psi(t)\rangle\langle d\psi|] = [i\rho H - \frac{1}{2}\lambda^{\alpha\beta}\rho L_\beta^\dagger L_\alpha + \lambda^{\alpha\beta}\rho L_\beta^\dagger\langle L_\alpha\rangle + \frac{1}{2}\lambda^{\alpha\beta}\rho\langle L_\beta^\dagger\rangle\langle L_\alpha\rangle]dt \tag{2.66}$$

$$\mathbb{E}[d|\psi(t)\rangle\langle d\psi|] = [\lambda^{\alpha\beta}L_\alpha\rho L_\beta^\dagger - \lambda^{\alpha\beta}\rho L_\beta^\dagger\langle L_\alpha\rangle - \lambda^{\alpha\beta}\langle L_\beta^\dagger\rangle L_\alpha\rho + \lambda^{\alpha\beta}\langle L_\beta^\dagger\rangle\langle L_\alpha\rangle\rho]dt, \tag{2.67}$$

and summing the contributions we find the Lindblad equation equation.

If one also includes information about the environment, one can arrive at unravelings that lead to pure state quantum evolution, which is objective, conditioned on the environment's history. Objective unravelings can be seen in [96], who interpret the Lindblad equation as a continuous measurement of an environment. We also see this more generally in Chapter 5, where we find the general form of unraveling for continuous classical-quantum dynamics; we find that, though the evolution of the quantum and classical states are stochastic, if one conditions on the entire classical trajectory the evolution of the pure quantum state is uniquely determined.

2.2.9 Path integrals for open quantum systems

In this section, we introduce the path integral formalism for open quantum systems (see [97] for an excellent review). We combine this with the path integral for stochastic classical mechanics in Chapter 6 to arrive at a combined classical-quantum path integral [8, 7].

It is well known in quantum mechanics that for a Hamiltonian system $H = \frac{p^2}{2m} + V(x)$ one can write a path integral governing the amplitude of the quantum state

$$\psi(x_f, t_f) = \int^{x(t_f)=x_f} \mathcal{D}x e^{iS[x]}\psi(x, t_i), \tag{2.68}$$

where $\psi(t,x) = \langle x|\psi(t)\rangle$ and $S[x] = \int dt[m\frac{\dot{x}^2}{2} - V(x)] = \int dt L[x]$, with $L[x]$ the Lagrangian of the system. Equation (2.68) can also be used to write a density matrix path integral

$$\psi(x_f^+, t_f)\psi^*(x_f^-, t_f) = \int^{x^+(t_f)=x_f^+, x^-(t_f)=x_f^-} \mathcal{D}x^-\mathcal{D}x^+ e^{iS[x^+]-iS[x^-]}\psi(x^+, t_i)\psi^*(x^-, t_i), \tag{2.69}$$

where x^+ are associated with the ket variables and x^- to the bra variables.

Similarly, we can derive a path integral representation for the Lindblad equation in Equation (2.51). To do so, we first expand the density matrix

$$\rho(t) = \int dx^+ dx^- \rho(x^+, x^-, t) |x^+\rangle\langle x^-|, \tag{2.70}$$

and to compute the path integral, we need to evaluate

$$\rho(t + \delta t) = \rho(t) + \delta t \mathcal{L}(\rho(t)). \tag{2.71}$$

The fundamental object of interest is given by relating the components $\rho(y^+, y^-, t+\delta t)$ to those at time t, $\rho(x^+, x^-, t)$

$$\rho(y^+, y^-, t + \delta t) = \rho(x^+, x^-, t) + \delta t \langle y^+ | \mathcal{L}(\rho) | y^- \rangle. \tag{2.72}$$

Because the Hamiltonian can depend on momentum, we will compute $\langle y^+ | \mathcal{L}(\mathbb{I}_+ \rho \mathbb{I}_+) | y^- \rangle$ and insert resolutions of the identity in the momentum basis $\mathbb{I}_\pm = \int dp^\pm |p^\pm\rangle\langle p^\pm|$. We find

$$\begin{aligned} &\rho(y^+, y^-, t + \delta t) = \int dx^+ dx^- dp^+ dp^- \langle y^+ | p^+ \rangle \langle p^+ | x^+ \rangle \langle x^- | p^- \rangle \langle p^- | y^- \rangle \rho(x^+, x^-, t) \times \\ &[1 + \delta t(-iH(p^+, x^+) + iH(p^-, x^-) \\ &+ \lambda^{\alpha\beta} L_\alpha(x^+) L_\beta(x^-) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^+) L_\alpha(x^+) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^-) L_\alpha(x^-))], \end{aligned} \tag{2.73}$$

which gives the path integral

$$\begin{aligned} \rho(y^+, y^-, t + \delta t) = \int dx^+ dx^- dp^+ dp^- \exp\Bigg(i\frac{y^+ - x^+}{\delta t}\delta t - i\frac{y^- - x^-}{\delta t}\delta t + -iH(p^+, x^+) + iH(p^-, x^-) \\ + \lambda^{\alpha\beta} L_\alpha(x^+) L_\beta(x^-) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^+) L_\alpha(x^+) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^-) L_\alpha(x^-)\Bigg) \rho(x^+, x^-, t). \end{aligned} \tag{2.74}$$

Taking the $\delta t \to 0$ limit, and considering many time intervals, we find (after skipping a few steps [97]) a path integral with action

$$\begin{aligned} iS[x^+, x^-, p^+, p^-] = \int dt \Bigg[i(\dot{x}^+ p^+ - H[p^+, x^+]) - i(\dot{x}^- p^- - H[p^-, x^-]) \\ \lambda^{\alpha\beta} L_\alpha(x^+) L_\beta(x^-) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^+) L_\alpha(x^+) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^-) L_\alpha(x^-) \Bigg]. \end{aligned} \tag{2.75}$$

Note, we can also integrate out the momentum variables, the result of which is to perform the same Legendre transformation as in standard quantum mechanics, in which case we arrive at a path integral with action

$$
\begin{aligned}
iS[x^+, x^-] = \int dt\ \Big[& iL[x^+] - iL[x^-] \\
& + \lambda^{\alpha\beta} L_\alpha(x^+) L_\beta(x^-) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^+) L_\alpha(x^+) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^-) L_\alpha(x^-) \Big].
\end{aligned}
\tag{2.76}
$$

The term

$$
iS_{FV} = \int dt \lambda^{\alpha\beta} L_\alpha(x^+) L_\beta(x^-) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^+) L_\alpha(x^+) - \frac{1}{2}\lambda^{\alpha\beta} L^*_\beta(x^-) L_\alpha(x^-) \Big] \tag{2.77}
$$

is known as the *Feynman-Vernon action* [79] and incorporates Lindbladian evolution into the path integral.

As a simple example, take $L(x) = x$, then the action of the Feynman-Vernon term is

$$
iS_{FV} = -\int dt\ \frac{1}{2}\lambda(x^+ - x^-)^2, \tag{2.78}
$$

and we see the path integral suppresses paths away from values where $x^+ \neq x^-$. i.,e, it causes the off-diagonal elements of the density matrix to be exponentially suppressed, causing decoherence of the quantum system.

2.3 Classical-quantum dynamics

We now introduce the general formalism used to describe a classical degree of freedom coupled to a quantum one, and we denote a generic classical degree of freedom by z. For example, it could be a classically treated position variable $z = q$ or a point in phase space $z = (q, p)$.

When considering a hybrid system, the natural set of states to consider are hybrid classical-quantum (CQ) states. Formally, a classical-quantum state associates to each classical variable an un-normalized density matrix $\varrho(z,t) = p(z,t)\sigma(z,t)$ such that $\mathrm{Tr}_{\mathcal{H}}[\varrho(z)] = p(z,t) \geq 0$ is a normalized probability distribution over the classical degrees of freedom and $\int dz \varrho(z,t)$ is a normalized density operator on a Hilbert space $\mathcal{H}$. Intuitively, $p(z,t)$ can be understood as the probability density of being in the phase space point z and $\sigma(z,t)$ as the *normalized* quantum state one would have given the classical state z occurs. This is consistent with the ensemble interpretation of density matrices given in Section 2.2.3.

An example of such a *CQ-state* is the CQ qubit, where we take a 2 dimensional Hilbert space and couple to classical position and momenta. The state then takes the form of a 2×2

matrix over phase space

$$\varrho(q,p,t) = \left(\begin{array}{cc} u_0(q,p,t) & c(q,p,t) \\ c^\star(q,p,t) & u_1(q,p,t) \end{array} \right). \tag{2.79}$$

We study the CQ qubit example in Section 2.3.3. Moving away from states, we can define any *CQ operator* $f(z)$ which associates a quantum operator to each point in phase space.

In analogy with the Kraus theorem for quantum mechanics and the probability transition equation for classical mechanics, it has been shown [28, 98] that any dynamics mapping CQ states onto themselves, if taken to be linear, will be completely positive if and only if it can be written in the form

$$\varrho(z,t_f) = \int dz' \Lambda(z,t_f|z',t_i)(\varrho(z',t)) = \int dz' \sum_{\mu\nu} \Lambda^{\mu\nu}(z,t_f|z',t_i) K_\mu \varrho(z',t_i) K_\nu^\dagger, \tag{2.80}$$

where $\Lambda^{\mu\nu}(z,t_f|z',t_i)$ is a completely positive Hermitian matrix kernel in $\mu\nu$ for each z,z', and the K_μ are an arbitrary orthogonal set of Kraus operators on the Hilbert space. Specifically, we ask that for any vector $a_\mu(z)$

$$\int dz dz' a_\mu^*(z) \Lambda^{\mu\nu}(z,t_f|z',t_i) a_\mu(z') \geq 0, \tag{2.81}$$

which also allows us to deal with the case where $\Lambda^{\mu\nu}(z,t_f|z',t_i)$ is only defined in a distributional sense, i.e, $\Lambda^{\mu\nu}(z,t_f|z',t_i) \sim \delta(z-z')$. The normalization of probabilities requires

$$\int dz \sum_{\mu\nu} \Lambda^{\mu\nu}(z,t_f|z',t_i) K_\nu^\dagger K_\mu = \mathbb{I}. \tag{2.82}$$

We compare the general dynamics for classical, quantum, and classical-quantum dynamics and their associated positivity and norm conditions in Table 2.1. Because we will everywhere deal with $\sum_{\mu\nu}$, we will hereon drop the summation and use the Einstein summation convention, implicitly assuming that contracted indices are to be summed over.

2.3.1 Moment expansion and the CQ master equation

We will be primarily interested in studying CQ master equations and will focus on time-local and autonomous dynamics. In the CQ case, we take autonomous to mean that the map in Equation (2.80) defines completely positive dynamics when applied to all states; in other words, CQ dynamics is autonomous if $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ is a positive matrix-kernel for all times.

	Classical
Dynamics	$p(z,t_f)=\int dz' P(z,t_f\|z',t_i)p(z',t_i)$
Positivity condition	$P(z,t_f\|z',t_i)\geq 0$ for all z,z'
Norm condition	$\int dz(z,t_f\|z',t_i)=1$
	Quantum
Dynamics	$\rho(t_f)=\sum_{\mu\nu}\Lambda^{\mu\nu}(t_f,t_i)K_\mu\rho(t_i)K_\nu^\dagger$
Positivity condition	$\Lambda^{\mu\nu}(t_f,t_i)$ a positive matrix in $\mu\nu$
Norm condition	$\sum_{\mu\nu}\Lambda^{\mu\nu}(t_f,t_i)K_\nu^\dagger K_\mu=\mathbb{I}$
	Classical-quantum
Dynamics	$\varrho(z,t+\delta t)=\int dz' \sum_{\mu\nu}\Lambda^{\mu\nu}(z,t_f\|z',t_i)K_\mu\varrho(z',t_i)K_\nu^\dagger$
Positivity condition	$\Lambda^{\mu\nu}(z,t_f\|z',t_i)$ a positive matrix in $\mu\nu$ for all z,z'
Norm condition	$\int dz \sum_{\mu\nu}\Lambda^{\mu\nu}(z,t_f\|z',t_i)K_\nu^\dagger K_\mu=\mathbb{I}$

Table 2.1: A table illustrating the general dynamics governing classical, quantum, and classical-quantum dynamics. We also show the positivity conditions required for dynamics to maintain positive probabilities, as well as the norm condition, which ensures probabilities sum to one. In this sense, Equation (2.80) is a natural classical-quantum generalization of the classical transition probability equation and the quantum Kraus decomposition theorem to the hybrid case.

Assuming autonomy, one can derive the CQ master equation in a similar fashion to the classical and quantum master equations by performing a short time expansion of Equation (2.80) in the case when the L_μ are trace-class [28]. When the dynamics are autonomous, we can write the transition equation of Equation (2.80) in the form

$$\varrho(z,t+\delta t) = \int dz' \Lambda(z,t+\delta t|z',t)(\varrho(z',t)) = \int dz' \ \Lambda^{\mu\nu}(z,t+\delta t|z',t) K_\mu \varrho(z',t) K_\nu^\dagger. \quad (2.83)$$

Moreover, we can perform a short-time expansion of the transition amplitude

$$\Lambda^{\mu\nu}(z,t+\delta t|z',t) = \delta_0^\mu \delta_0^\nu + \delta t W^{\mu\nu}(z|z',t). \quad (2.84)$$

We saw that in classical autonomous dynamics, the Kramers-Moyal expansion relates the master equation to the moments of the probability transition amplitude. We now perform the analogous calculation for the combined classical-quantum case. For the moment expansion, we work with the form of the dynamics in Equation (2.83), using a basis of Kraus operators on the Hilbert space which includes the identity, $K_\mu = \{\mathbb{I}, L_\alpha\} \equiv L_\mu$. We take the classical degrees of freedom to be d dimensional, $z = (z_1, \dots z_d)$, and we label the components as z_i, $i \in \{1, \dots d\}$.

We begin by introducing the moments of the CQ transition amplitude

$$M^{\mu\nu}_{n,i_1\dots i_n}(z',t,\delta t) = \int dz \ \Lambda^{\mu\nu}(z,t+\delta t|z',t)(z-z')_{i_1} \dots (z-z')_{i_n}, \quad (2.85)$$

where $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ are the components of the dynamics of the CP map in the basis $L_\mu = \{\mathbb{I}, L_\alpha\}$, as defined in Equation (2.99).

We define the characteristic function, which is the Fourier transform of the transition amplitude

$$C^{\mu\nu}(u,z',t,\delta t) = \int dz e^{iu\cdot(z-z')} \Lambda^{\mu\nu}(z,t+\delta t|z',t) = \sum_{n=0}^{\infty} \frac{(i^n) u_{i_1} \dots u_{i_n}}{n!} M^{\mu\nu}_{n,i_1\dots i_n}(z',t,\delta t). \quad (2.86)$$

Taking the inverse Fourier transform, we can relate the transition amplitude to its moments

$$\begin{aligned} \Lambda^{\mu\nu}(z,t+\delta t|z',t) &= \int du \ e^{-iu(z-z')} C^{\mu\nu}(u,z',t,\delta t) \\ &= \sum_{n=0}^{\infty} \frac{M^{\mu\nu}_{n,i_1\dots i_n}(z',t,\delta t)}{n!} \frac{1}{(2\pi)^d} \int du \ \ e^{-iu(z-z')}(i^n), u_{i_1} \dots u_{i_n}, \end{aligned} \quad (2.87)$$

which, using the definition of the delta distribution, we can write as

$$\Lambda^{\mu\nu}(z,t+\delta t|z',t) = \sum_{n=0}^{\infty} \frac{1}{n!} M^{\mu\nu}_{n,i_1\dots i_n}(z',t,\delta t) \left(\frac{\partial^n}{\partial z'_{i_1} \dots \partial z'_{i_n}} \right) \delta(z,z'). \quad (2.88)$$

Looking at the short time expansion coefficients of $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$, as defined in (2.84), we have

$$
\begin{aligned}
M^{\mu\nu}(z',t,\delta t)_{n,i_1\dots i_n} &= \delta_0^\mu \delta_0^\nu + \delta t \int dz W^{\mu\nu}(z|z',t)(z-z')_{i_1}\dots(z-z')_{i_n} \\
&\equiv \delta_0^\mu \delta_0^\nu + \delta t n! D^{\mu\nu}_{n,i_1\dots i_n}(z',t) + O(\delta t^2),
\end{aligned}
\tag{2.89}
$$

where we have implicitly defined the quantity $D^{\mu\nu}(z',t)_{n,i_1\dots i_n}$ via

$$
D^{\mu\nu}_{n,i_1\dots i_n}(z',t) := \frac{1}{n!}\int dz W^{\mu\nu}(z|z',t), t(z-z')_{i_1}\dots(z-z')_{i_n}. \tag{2.90}
$$

Just as for the classical case, we shall occasionally find it useful to refer to the moments as $D_n(z')$, by which we mean the object with components $D^{\mu\nu}(z')_{n,i_1\dots i_n}$. We shall also often suppress the time dependence of $D_n(z')$, which can be added later.

Substituting the short time moment coefficients back into Equation (2.88), taking the limit $\delta t \to 0$, and using the probability preserving condition in (2.82), we can write a classical-quantum master equation in the form

$$
\begin{aligned}
\frac{\partial \varrho(z,t)}{\partial t} = &\sum_{n=1}^{\infty}(-1)^n \left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}(z,\delta t)\varrho(z,t)\right) \\
&- i[H(z),\varrho(z)] + D_0^{\alpha\beta}(z)L_\alpha \varrho(z) L_\beta^\dagger - \frac{1}{2}D_0^{\alpha\beta}\{L_\beta^\dagger L_\alpha, \varrho(z)\}_+ \\
&+ \sum_{\mu\nu\neq 00}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{\mu\nu}_{n,i_1\dots i_n}(z)L_\mu \varrho(z,t) L_\nu^\dagger\right),
\end{aligned}
\tag{2.91}
$$

where we define the Hermitian operator $H(z) = \frac{i}{2}(D_0^{\mu 0}L_\mu - D_0^{0\mu}L_\mu^\dagger)$ (which is Hermitian since $D_0^{\mu 0} = D_0^{0\mu *}$). $H(z)$ defines a Hamiltonian which is dependent on the classical degrees of freedom.

We see the first line of Equation (2.91) describes purely classical dynamics and is fully described by the moments of the identity component of the dynamics $\Lambda^{00}(z,t+\delta t|z',t)$. The second line describes pure quantum Lindbladian evolution described by the zeroth moments of the components $\Lambda^{\alpha 0}(z,t+\delta t|z',t), \Lambda^{\alpha\beta}(z,t+\delta t|z',t)$; specifically the (block) off diagonals, $D_0^{\alpha 0}(z)$, describe the pure Hamiltonian evolution, while the components $D_0^{\alpha\beta}(z)$ describe the dissipative part of the pure quantum evolution. Note that the Hamiltonian and Lindblad couplings can depend on the classical degrees of freedom, so the second line describes the action of the classical system on the quantum one. The third line contains the non-trivial classical-quantum back-reaction, where changes in the distribution over phase space are induced and accompanied by changes in the quantum state.

Equation (2.91) is a natural generalization of the Lindblad equation and classical master equation in the case of classical-quantum coupling. The positivity conditions from (2.80) transfer to positivity conditions on the master equation via (2.84). We can write the positivity conditions in an illuminating form by writing the short time expansion of the transition amplitude $\Lambda^{\mu\nu}(z, t+\delta t|z', t)$, as defined by Equation (2.84), in block form

$$\Lambda^{\mu\nu}(z, t+\delta t|z', t) = \begin{bmatrix} \delta(z, z') + \delta t W^{00}(z|z', t) & \delta t W^{0\beta}(z|z', t) \\ \delta t W^{\alpha 0}(z|z', t) & \delta t W^{\alpha\beta}(z|z', t) \end{bmatrix} + O(\delta t^2), \tag{2.92}$$

and the dynamics will be completely positive if and only if $\Lambda^{\mu\nu}(z, t+\delta t|z', t)$ is a positive matrix. It is useful to note that from (2.92), we can immediately deduce that $\delta(z, z') + \delta t W^{00}(z|z', t)$ must be positive, which is the same positivity condition as for classical dynamics, as well as the matrix $W^{\alpha\beta}(z|z', t)$. Furthermore, if either of $W^{\alpha\beta}(z|z', t)$ or $W^{00}(z|z', t)$ vanish, then so must $W^{0\alpha}(z|z', t)$, except for its $\delta(z, z')$ component which generates pure Hamiltonian evolution. This tells us that to have non-trivial CQ coupling, we must have a non-zero $W^{\alpha\beta}(z|z', t)$. When the classical degrees of freedom are discrete, the Schur complement – assuming $W^{00}(z|z', t)$ is non-vanishing – informs us the matrix $\Lambda^{\mu\nu}(z, t+\delta t|z', t)$ will be positive if and only if $W^{00}(z|z', t)W^{\alpha\beta}(z|z', t) - W^{0\beta}(z|z', t)W^{\alpha 0}(z|z', t) \succeq 0$ is a positive matrix in $\alpha\beta$ for all $z \neq z'$. We must be more careful in the continuous case since the components $\Lambda^{\mu\nu}(z, t+\delta t|z', t)$ may only be defined in a distributional sense. We explore the positivity conditions in detail in Chapters 4 and 10, and we shall see that they have important consequences for CQ dynamics, such as a general trade-off between decoherence and diffusion, which can be used to constrain classical-quantum theories of gravity.

We have derived the master equation of Equation (2.91) for a specific set of Lindblad operators $L_\mu = (\mathbb{I}, L_\alpha)$. However, it can be shown that Equation (2.91), combined with the positivity condition (2.92), defines completely positive CQ dynamics for an arbitrary set of Lindblad operators $\{L_\mu\}$ [28, 95].

One of the contributions (Chapter 4) of this thesis is to prove that the most general form of continuous (in the classical degrees of freedom), autonomous classical-quantum dynamics takes the form

$$
\begin{aligned}
\frac{\partial \varrho(z,t)}{\partial t} &= \sum_{n=1}^{n=2} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \ldots \partial z_{i_n}} \right) \left(D^{00}_{n,i_1 \ldots i_n} \varrho(z,t) \right) \\
&- \frac{\partial}{\partial z_i} \left(D^{0\alpha}_{1,i} \varrho(z,t) L^\dagger_\alpha \right) - \frac{\partial}{\partial z_i} \left(D^{\alpha 0}_{1,i} L_\alpha \varrho(z,t) \right) \\
&- i[H(z), \varrho(z,t)] + D_0^{\alpha\beta}(z) L_\alpha \varrho(z) L^\dagger_\beta - \frac{1}{2} D_0^{\alpha\beta} \{ L^\dagger_\beta L_\alpha, \varrho(z) \}_+,
\end{aligned} \tag{2.93}
$$

where $2D^{00}_2 \succeq D_1 D_0^{-1} D_1^\dagger$ and $(\mathbb{I} - D_0 D_0^{-1}) D_1 = 0$. Here, and throughout, D_0^{-1} is the generalized inverse of the positive semi-definite Lindbladian coupling $D_0^{\alpha\beta}$, D_1 is a matrix in both α, i indices with entries $D^{0\alpha}_{1,i}$, which encodes the strength of the CQ back-reaction, and D^{00}_2 is a matrix in i, j with entries $D^{00}_{2,ij}$, which represents the necessity of diffusion in the classical phase space. The symbol $\succeq$ refers to matrix positivity, and $a \succeq b$ is equivalent to $a - b$ being a positive matrix. Equation (2.93) naturally generalizes the Fokker-Plank and Lindblad equations to the case of continuous classical-quantum coupling.

We refer to CQ dynamics undergoing dynamics according to Equation (2.93) as *continuous CQ dynamics* and thus undergoing the more general dynamics of Equation (2.91) as *jumping CQ dynamics* since it is accompanied by finite-sized jumps in the classical degrees of freedom with a finite probability.

Comments on non-Markovian CQ dynamics

When a CQ dynamics is autonomous, the generator can always be written in the form of Equation (2.91) where the moments D_n must satisfy the positivity conditions implied by the positivity conditions of $\Lambda^{\mu\nu}(z, t + \delta t | z', t)$.

If we instead asked only for the CQ dynamics to be time-local, but relaxed the assumption that the generator $\Lambda^{\mu\nu}(z, t + \delta t | z', t)$ be completely positive for all times, then - in analogy with the purely quantum case [92] - we still expect the dynamics to take the form of Equation (2.91), but where the positivity conditions are relaxed; indeed, the derivation of the master equation relied only on the time-local property of the dynamics. In particular, for general time-local dynamics, we expect that for intermediate times one can have CPTP classical-quantum dynamics where the quantum degrees of freedom recohere while simultaneously the classical degrees of freedom become less diffusive. It would be interesting to explore non-Markovian CQ dynamics further since it is currently not well understood but is crucial in understanding the role of hybrid dynamics in effective theories where dynamics is generally non-autonomous.

Rate equation form

Just as in the classical case, where the master equation can be written in either a rate-equation or expanded form, the same is true for the CQ case. To write it in a rate equation form, we define $W^{\mu\nu}(z|z',t) = \tilde{W}^{\mu\nu}(z|z',t) - \delta_0^\mu\delta_0^\nu\mathcal{N}(z,t)\delta(z,z')$, where the norm condition fixes $\mathcal{N}(z',t) = \int dz \sum_{\mu\nu}\tilde{W}^{\mu\nu}(z|z',t)L_\nu^\dagger L_\mu$.

By substituting the short-time expansion coefficients into (2.83) and taking the limit $\delta t \to 0$, we can write the master equation in the form

$$\frac{\partial\varrho(z,t)}{\partial t} = \int dz'\ \tilde{W}^{\mu\nu}(z|z',t)L_\mu\varrho(z',t)L_\nu^\dagger - \frac{1}{2}\tilde{W}^{\mu\nu}(z,t)\{L_\nu^\dagger L_\mu,\varrho\}_+, \tag{2.94}$$

where $\{,\}_+$ is the anti-commutator, and preservation of normalization under the trace and $\int dz$ defines

$$\tilde{W}^{\mu\nu}(z,t) = \int \mathrm{d}z'\tilde{W}^{\mu\nu}(z'|z,t). \tag{2.95}$$

Following standard convention, we refer to $\tilde{W}^{\mu\nu}(z|z')L_\mu\varrho(z')L_\nu^\dagger$ as the *jump term* and we call $\frac{1}{2}\tilde{W}^{\mu\nu}(z)\{L_\nu^\dagger L_\mu,\varrho\}_+$ the *no-event term.*[3].

Note, the redefinition of $W(z|z,t) \to \tilde{W}(z|z,t)$ does not change the moments appearing in the master equation in 2.91. In particular, except for D_0^{00}, which does not appear in the master equation, the moments D_n are found to be

$$\begin{aligned} D^{\mu\nu}_{n,i_1\ldots i_n}(z',t) &:= \frac{1}{n!}\int dz W^{\mu\nu}(z|z',t)(z-z')_{i_1}\ldots(z-z')_{i_n} \\ &= \frac{1}{n!}\int dz\tilde{W}^{\mu\nu}(z|z',t)(z-z')_{i_1}\ldots(z-z')_{i_n}, \end{aligned} \tag{2.96}$$

and to ensure complete positivity of the dynamics, it is sufficient to check the positivity of the matrix

$$\begin{bmatrix} \delta(z,z') + \delta t\tilde{W}^{00}(z|z') & \delta t\tilde{W}^{0\beta}(z|z') \\ \delta t\tilde{W}^{\alpha 0}(z|z') & \delta t\tilde{W}^{\alpha\beta}(z|z') \end{bmatrix}. \tag{2.97}$$

[3]These conventions come from studying the unravellings of Lindblad equations via stochastic pure state quantum trajectories. At each time step, the quantum state either undergoes continuous evolution via an effective Hamiltonian or with some probability jumps to a new state that depends on the Lindblad operators L_α appearing in the master equation [99, 100, 101]

	Classical
Master equation	$\frac{\partial p}{\partial t} = \int dz' W\left(z\|z'\right) p\left(z'\right) - \int dz' W\left(z'\|z\right) p(z)$
Positivity condition	$P(z, t+\delta t\|z', t) = \delta(z, z') + \delta t W(z\|z') + \mathcal{O}(\delta t^2) \geq 0 \ \forall z, z'$
	Quantum
Master equation	$\frac{\partial \sigma(t)}{\partial t} = -i[H, \sigma] + h^{\alpha\beta} L_\alpha \sigma L_\beta^\dagger - \frac{1}{2}\left\{h^{\alpha\beta} L_\beta^\dagger L_\alpha, \sigma\right\}_+$
Positivity condition	$h^{\alpha\beta}$ a positive matrix, $h \succeq 0$.
	Classical-quantum
Master equation	$\frac{\partial \varrho}{\partial t} = \int dz' W^{\mu\nu}\left(z\|z'\right) L_\mu \varrho\left(z'\right) L_\nu^\dagger - \frac{1}{2}\int dz' W^{\mu\nu}\left(z'\|z\right)\left\{L_\nu^\dagger L_\mu, \varrho(z)\right\}_+$
Positivity condition	$\Lambda^{\mu\nu}(z, t+\delta t\|z', t) = \begin{bmatrix} \delta\left(z, z'\right) + \delta t W^{00}\left(z\|z'\right) & \delta t W^{0\beta}\left(z\|z'\right) \\ \delta t W^{\alpha 0}\left(z\|z'\right) & \delta t W^{\alpha\beta}\left(z\|z'\right) \end{bmatrix} + \mathcal{O}(\delta t^2) \succeq 0 \ \forall z, z'$

Table 2.2: A table illustrating the Markovian master equations governing classical, quantum, and classical-quantum dynamics. We see that the CQ master equation is a natural generalization of the classical rate equation and the Lindblad equation. We have suppressed the explicit t dependence on the transition amplitudes, but this can be back added in.

Written in the form of Equation (2.94) is useful since it is automatically normalized, whereas Equation (2.83) is not. We compare the rate equation forms for the classical, quantum, and classical-quantum master equations in Table 2.2.

Vectorization of CQ dynamics

In Chapter 4, we shall find it useful to deal with superoperators and to double the quantum degrees of freedom using the vectorization map [102]. We do so by representing the CQ density operators $\varrho(z)$ as vectors by stacking the columns, i.e., sending $|i\rangle\langle j| \to |j\rangle \otimes |i\rangle$. We denote the vectorized form as $\vec{\varrho}(z)$. Then, superoperators are matrices acting on the stacked vector $\vec{\varrho}(z)$, for example

$$\vec{\varrho}(z, t+\delta t) = \int dz' \Lambda^{\mu\nu}(z, t+\delta t|z', t)(\bar{L}_\nu \otimes L_\mu)\vec{\varrho}(z, t) = \int dz' \Lambda^{vec}(z, t+\delta t|z', t)(\vec{\varrho}(z')), \quad (2.98)$$

where we write *vec* to remind us that we should view the superoperator as a matrix on the doubled Hilbert space. This is particularly useful since it allows us to identify the components of the superoperator in any orthogonal basis of operators $(\bar{L}_\nu \otimes L_\mu)$ via

$$\Lambda^{\mu\nu}(z,t+\delta t|z',t) = \mathrm{Tr}\left[(\bar{L}_\nu \otimes L_\mu)^\dagger \Lambda^{vec}(z,t+\delta t|z',t)\right]. \tag{2.99}$$

2.3.2 Physical interpretation of the moments

In classical Markovian dynamics, the moments of the short time expansion of the probability transition amplitude $P(z,t+\delta t|z',t)$ are useful since they are usually related to observable quantities. For example, the first moment characterizes the system's drift, while the second moment typically characterizes diffusion. In the CQ case, we have similar interpretations. We see from Equation (2.91) that the zeroth moments characterize the pure quantum evolution, with $D_0^{\alpha\beta}$ determining the rate of decoherence on the quantum system (and Lindbladian coupling more generally). As we shall see in Chapter 4, in order to have a non-trivial classical-quantum dynamics, positivity demands $D_0^{\alpha\beta}(z) \neq 0$ and so the classical system forces decoherence upon the quantum system. To give interpretation to the higher order moments, consider starting in a state of certainty in phase space $\varrho(z,t) = \delta(z,\bar{z})\sigma(z)$ and after some short time δt measuring the classical observable $(z-\bar{z})^n$, $n \geq 1$. In this case, using Equation (2.91), we find

$$\int dz (z-\bar{z})^n \mathrm{Tr}\left[\varrho(z,t+\delta t)\right] = \delta t n! D_n^{\mu\nu}(\bar{z}) \mathrm{Tr}\left[L_\nu^\dagger L_\mu \sigma\right], \tag{2.100}$$

hence we see the coefficients $D_n^{\mu\nu}(z)$ (for $\mu\nu \neq 00$) characterize the back-reaction of the quantum system on the classical system in the presence of non-trivial CQ coupling. In particular, the first moment $D_{1,i}^{\mu\nu}$, with $\mu,\nu \neq 0$, gives the force exerted by the quantum system on the classical system, and the second moment determines the diffusion induces on the classical degrees of freedom.

More generally, we can consider the expectation value of any CQ operator $O(z)$, $\langle O(z)\rangle := \int dz \mathrm{Tr}\left[O(z)\varrho\right]$ which, for simplicity, we assume does not have an explicit time dependence. Its evolution law can be determined via Equation (2.91)

$$
\begin{aligned}
\frac{d\langle O\rangle}{dt} &= \int dz \mathrm{Tr}\left[O(z)\frac{\partial\varrho}{\partial t}\right] \\
&= \int dz \mathrm{Tr}\varrho\left[-i[O(z),H(z)] + D_0^{\alpha\beta}(z)L_\beta^\dagger O(z)L_\alpha - \frac{1}{2}D_0^{\alpha\beta}\{L_\alpha L_\beta^\dagger, O(z)\}_+\right. \\
&\quad \left. + \sum_{n=1}^{\infty}\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D_{n,i_1\dots i_n}^{\alpha\beta}(z)L_\beta^\dagger L_\alpha O(z,t)\right)\right],
\end{aligned} \tag{2.101}
$$

where we have used cyclicity of trace and integration by parts to bring the equation of motion into a form that would enable us to write a CQ version of the *Heisenberg representation* [28] for a CQ operator. If we are interested in the expectation value of phase space variables, then $O(z) = z\mathbb{I}$ and Equation (2.101) gives

$$
\frac{d\langle z\rangle}{dt} = \int dz D_{1,i_1}^{\mu\nu}\,\mathrm{Tr}\left[L_\nu^\dagger L_\mu \varrho(z,t)\right], \tag{2.102}
$$

with all higher order terms vanishing, and we see that $\sum_{\mu\nu\neq 00} D_{1,i_1}^{\mu\nu}\langle L_\nu^\dagger L_\mu\rangle$ governs the rate at which the quantum system moves the classical system through phase space. The force of this back-reaction is especially apparent if the equations of motion are Hamiltonian in the classical limit as in [28].

Specifically, defining $H_I(z) := h^{\alpha\beta}L_\beta^\dagger L_\alpha$ and take $D_{1,i}^{\alpha\beta} = \omega_i^j d_j h^{\alpha\beta}$ with ω the symplectic form and d_j the exterior derivative. Then Equation (2.102) is analogous to Hamilton's equations, and the CQ evolution equation, after tracing out the quantum system, has the form of a Liouville's equation to first order

$$
\frac{\partial\rho(z,t)}{\partial t} = \{H_c, \rho(z,t)\} + \mathrm{tr}\left(\{H_I(z), \varrho(z)\}\right) + \dots, \tag{2.103}
$$

where $\rho(z) := \mathrm{Tr}\left[\varrho(z)\right]$ and the the $\dots$ represent the higher order terms in the moment expansion. We call any CQ master equation with Hamiltonian drift *Hamiltonian CQ-dynamics*.

The significance of the second moment is also seen via Equation (2.101) to be related to the variance of phase space variables $\sigma_{z_{i_1}z_{i_2}} := \langle z_{i_1}z_{i_2}\rangle - \langle z_{i_1}\rangle\langle z_{i_2}\rangle$

$$
\begin{aligned}
\frac{d\sigma^2_{z_{i_1},z_{i_2}}}{dt} &= 2\langle D_{2,i_1,i_2}^{\alpha\beta}L_\beta^\dagger L_\alpha\rangle + \langle z_2 D_{1,z_{i_1}}^{\alpha\beta}L_\beta^\dagger L_\alpha\rangle - \langle z_{i_2}\rangle\langle D_{1,z_{i_1}}^{\alpha\beta}L_\beta^\dagger L_\alpha\rangle \\
&\quad + \langle z_{i_1}D_{1,z_{i_2}}^{\alpha\beta}L_\beta^\dagger L_\alpha\rangle - \langle z_{i_1}\rangle\langle D_{1,z_{i_2}}^{\alpha\beta}L_\beta^\dagger L_\alpha\rangle.
\end{aligned} \tag{2.104}
$$

In the case when $D_{1,z_{i_1}}$ is uncorrelated with z_{i_2} and $D_{1,z_{i_2}}$ uncorrelated with z_{i_1}, then the growth of the variance only depends on the diffusion coefficient.

The zeroth moment $D_0^{\alpha\beta}$ is just the pure Lindbladian couplings. The simplest example is the case of a pure decoherence process with a single Hermitian Lindblad operator L and decoherence coupling D_0. Then we can define a basis $|a\rangle$ via the eigenvectors of L and

$$\langle a| \frac{\partial \varrho}{\partial t} |b\rangle = -i \langle a| [H(z), \varrho] |b\rangle - \frac{1}{2} D_0 (L(a) - L(b))^2 \langle a|\varrho|b\rangle, \tag{2.105}$$

and we see that the matrix elements of ϱ which quantify coherence between the states $|a\rangle$,$|b\rangle$ decay exponentially fast with a decay rate of $D_0(L(a) - L(b))^2$. For a damping/pumping process of a quantum harmonic oscillator with Hamiltonian $H = \omega a^\dagger a$, $L_\downarrow = a$, $L_\uparrow = a^\dagger$, a the creation operator, and $D_0^{\uparrow\uparrow}$, $D_0^{\downarrow\downarrow}$ the non-zero couplings, then standard calculations [82] show that an initial superposition $\frac{1}{\sqrt{2}}(|n\rangle + |m\rangle)$ with n,m large and $n \gg m$ will initially decohere at a rate of approximately $(D_0^{\uparrow\uparrow} + D_0^{\downarrow\downarrow})(m + n)/2$. The state will eventually thermalize to a temperature of $\omega / \log (D_0^{\downarrow\downarrow}/D_0^{\uparrow\uparrow})$. In this case, the Lindblad couplings determine not only the rate of decoherence but also the rate at which energy is pumped into the harmonic oscillator. In Chapter 10, we will derive a trade-off between Lindblad couplings and the diffusion coefficients. Although we sometimes refer to this as a trade-off between decoherence and diffusion, this terminology is only strictly appropriate for pure decoherence processes. More generally, it is a trade-off between Lindblad couplings and diffusion coefficients.

2.3.3 Master equation examples

In this section, we go through examples of CQ dynamics, illustrating the general properties of CQ theories discussed in this chapter.

A model with finite-sized jumps in the classical phase space

We shall now look at a model of a spin half particle interacting with a linear potential. This model was studied in detail in [10] and helps with intuition in understanding the consequences of the CQ coupling. We take the classical degrees of freedom to be position and momentum $z = (q, p)$ and a two-dimensional Hilbert space $\mathcal{H}$ to represent the quantum spin degrees of freedom. We take the pure classical evolution to be generated by a Hamiltonian $H_c = p^2/2m$ and an interaction Hamiltonian $H_I(q, p) = h^\alpha L_\alpha^\dagger L_\alpha = Bq\omega|0\rangle\langle 0| - Bq\omega|1\rangle\langle 1|$. We take the Lindblad operators to be $L_{\alpha=0} = |0\rangle\langle 0|, L_{\alpha=1} = |1\rangle\langle 1|$, which then defines $h^1(q, p) = \omega q B$, $h^2(q, p) =$

$-\omega q B$. We take the following dynamics for the system

$$\begin{aligned}\frac{\partial \varrho(z,t)}{\partial t} &= \{H_c(z), \varrho(z,t)\} - i\left[H_I(z), \varrho(z,t)\right] \\ &\quad + \frac{1}{\tau}\sum_\alpha \left(e^{\tau\{h^\alpha(z),\cdot\}} L_\alpha \varrho(z,t) L_\alpha^\dagger - \frac{1}{2}\left\{L_\alpha^\dagger L_\alpha, \varrho(z,t)\right\}_+\right),\end{aligned} \tag{2.106}$$

where the Poisson bracket in the exponential acts on the CQ state as a linear superoperator

$$\sum_\alpha \frac{1}{\tau} e^{\tau\,\{h^\alpha(z),\,\cdot\,\}}\, L_\alpha\, \varrho(z,t)\, L_\alpha^\dagger = \sum_\alpha \frac{1}{\tau}\left(1 + \tau\,\left\{h^\alpha(z), L_\alpha\, \varrho(z,t)\, L_\alpha^\dagger\right\} + ...\right). \tag{2.107}$$

This is seen to be completely positive due to Equation (2.94).

It is helpful to discuss the interpretation of the dynamics briefly. Equation (2.106) is modeled on a Poisson-like process with jump rate $\frac{1}{\tau}$ jump size $\delta = Bq\tau$.[4] In particular, a jump in phase space is accompanied by a Lindblad operator acting on the quantum state, which causes it to jump to being in the $|0\rangle$ or $|1\rangle$ state. The Lindblad operators, L_1, L_2, cause the off-diagonals of the quantum density matrix to decay exponentially with time. Specifically, a quantum state in superposition "collapses" to the $|0\rangle$ state with a rate given by $1/\tau$ – and when it does so, there is a jump in the classical momentum of the particle by an amount $B\omega\tau$ – or the state "collapses" to $|1\rangle\langle 1|$, and there is a jump of momentum $-B\omega\tau$. It was shown [10] that this leads to objective quantum state trajectories conditioned on the classical phase space since measuring the classical degree of freedom makes it possible to determine the quantum state exactly. The model, in some sense, mimics the features of the Stern-Gerlach experiment; the magnetic fields measure the quantum state in the $|0\rangle, |1\rangle$ basis and kicks the particle depending on the state the spin has collapsed to. As we evolve in time, classical degrees of freedom in the state naturally diffuse according to the distribution of momentum jumps the system undergoes.

Now, we can repeat the steps in Section 2.3.1 and perform a Kramers-Moyal expansion of the master equation to obtain

$$\begin{aligned}\frac{\partial \varrho}{\partial t} &= \{H_c, \varrho(z)\} - i[H_I(z), \varrho(z)] \\ &\quad + \sum_{n=0}(-1)^n \left(\frac{\partial^n}{\partial z_{i_1}\dots z_{i_n}}\right)\left[D^\alpha_{n,i_1\dots i_n}(z) L_\alpha \varrho(z) L_\alpha^\dagger\right] - \frac{1}{2} D^\alpha_0(z)\{L_\alpha^\dagger L_\alpha, \varrho(z)\}_+,\end{aligned} \tag{2.108}$$

[4] Indeed, it is easy to verify that the probability rate of a jump larger than δ, namely $R = \lim_{t\downarrow s} \frac{1}{t-s}\mathrm{Prob}[|z(t) - z'(s)| > \delta]$, is given by $R = \frac{1}{\tau}\mathrm{Tr}_{\mathcal{H}}\left[L_\alpha^\dagger L_\alpha \varrho\right]$ if $0 < \delta < \delta t \frac{\partial h^\alpha}{\partial q}$ and zero otherwise.

with the moments given by

$$D^1_{n,p\ldots p} = \frac{1}{\tau}\frac{1}{n!}(\omega B\tau)^n, \quad D^2_{n,p\ldots p} = \frac{1}{\tau}\frac{1}{n!}(-\omega B\tau)^n. \tag{2.109}$$

We can directly see that the zeroth moment, $D^\alpha_0 = \frac{1}{\tau}$, is characteristic of the decoherence rate of the quantum system. We write the CQ state in component form

$$\varrho(q,p,t) = \left(\begin{array}{cc} u_0(q,p,t) & c(q,p,t) \\ c^\star(q,p,t) & u_1(q,p,t) \end{array} \right). \tag{2.110}$$

Using this, we can write an equation of motion for the components

$$\frac{\partial u_i(q,p,t)}{\partial t} = -\frac{p}{m}\frac{\partial u_i(q,p,t)}{\partial q} + \frac{1}{\tau}\left(u_i\left(q, p + (-1)^i\omega B\tau, t\right) - u_i(q,p,t)\right), \quad i \in \{0,1\} \tag{2.111}$$

$$\frac{\partial c(q,p,t)}{\partial t} = -2\omega i q B\omega c(q,p,t) - \frac{p}{m}\frac{\partial c(q,p,t)}{\partial q} - \frac{1}{\tau}c(q,p,t) \tag{2.112}$$

and using the method of characteristics, one can analytically solve for the off-diagonals of the density operator to obtain

$$c(q,p,t) = \tilde{c}(q - \frac{p}{m}t)\, e^{-iB(\omega_0-\omega_1)\left(q - \frac{p}{2m}t\right)t - \frac{t}{\tau}}, \tag{2.113}$$

which illustrates that the off-diagonal terms of the quantum state vanish after a characteristic time $t_{dec} = \tau$.

Taking $\tau \to 0$ gives us a natural classical limit of CQ dynamics: the quantum state rapidly decoheres into its pointer basis. At the same time, higher order terms D_n, $n \geq 2$ arising from the exponentiated Poisson bracket are suppressed by powers of τ. The dominating contribution to the back-reaction is given by the term $\left\{h^\alpha(z), L_\alpha\, \varrho(z,t)\, L^\dagger_\alpha\right\}$ which, under trace, describes Hamiltonian evolution on the classical degrees of freedom as in Equation (2.103). Occasionally in the thesis, we will refer to the classical limit of CQ dynamics, and this is what we have in mind.

We can find the short-time variances in position and momentum by calculating

$$\int \mathrm{d}z \mathrm{Tr}\left[(z_i - \langle z_i\rangle)(z_j - \langle z_j\rangle)\varrho(z,\delta t)\right]. \tag{2.114}$$

If we take an initial state to be of the form $\varrho(z,0) = \delta(q-\bar{q}, p-\bar{q})\varrho$, then using integration by parts and taking the trace of the quantum system, the short time variance in momenta is easily seen to be

$$\sigma^2_{pp}(\delta t) = \delta t 2\frac{\tau(\omega_0 B)^2}{2}\mathrm{Tr}\left[(|0\rangle\langle 0| + |1\rangle\langle 1|)\varrho\right] = \delta t\tau(\omega B)^2 = \delta t(D^1_{2,pp} + D^2_{2,pp}), \tag{2.115}$$

and so the rate of diffusion in the classical phase space is characterized by the second moment in the Kramers- Moyal expansion. From equation (2.115), we see that changing the decoherence rate τ directly alters the amount of diffusion in the classical phase space. We can similarly calculate the drift

$$\langle p\rangle(\delta t) = \mathrm{Tr}\left[(D_p^1|0\rangle\langle 0| + D_p^2|1\rangle\langle 1|)\varrho\right]. \tag{2.116}$$

Finally, we can analyze the trade-off between diffusion and decoherence, seen in simulations of the model in [10] and discussed in full generality in Chapter 10. There we show that for CQ dynamics to be completely positive, one necessarily lower bounds the diffusion in the system $D_2^{\mu\nu}$ in terms of the Lindbladian coupling $D_0^{\alpha\beta}$ and the force exerted on the classical system, the latter quantified by the rectangular matrix $(D_1^{br})^{\alpha\mu} = D_1^{\alpha\mu}$. The bound on diffusion takes the form of a matrix inequality $2D_2D_0 - D_1^{br}D_1^{br\dagger} \succeq 0$, which in the present context tells us that $2D_2^{\alpha}D_0^{\alpha} \geq (D_{i,p}^{\alpha})^2$. Reading off the moments from (2.85), we see the inequality relating the decoherence and diffusion to the Hamiltonian is satisfied and in-fact saturated

$$\frac{1}{2D_0^{\alpha}}(D_{i,p}^{\alpha})^2 = \frac{\tau}{2}(\omega B)^2 = D_{2,pp}^{\alpha}. \tag{2.117}$$

An example of continuous dynamics

A simple example of a continuous master equation of Equation (2.93) is given by a classical oscillator coupled to a quantum one. The classical oscillator we describe by the classical Hamiltonian

$$H_c = \frac{1}{2}p^2 + \frac{1}{2}\omega_c^2 q^2, \tag{2.118}$$

and the quantum oscillator we describe by the quantum Hamiltonian

$$H_q = \frac{1}{2}P^2 + \frac{1}{2}\omega_q^2 Q^2. \tag{2.119}$$

We consider the coupling via the interaction Hamiltonian $H_{cq} = D_1 qQ$. From Equation (2.93), the deterministic part of the dynamics is given by

$$\begin{aligned} &\{H_c, \varrho\} - i[H_q, \varrho] - iD_1 q[Q, \varrho] + \frac{1}{2}D_1\{qQ, \varrho\} - \frac{1}{2}D_1\{\varrho, qQ\} \\ &= \{H_c, \varrho\} - i[H_q, \varrho] - iD_1 q[Q, \varrho] + \frac{1}{2}D_1\left(Q\frac{\partial\varrho}{\partial p} + \frac{\partial\varrho}{\partial p}Q\right). \end{aligned} \tag{2.120}$$

The back-reaction governed by the D_1 term and takes the form

$$\frac{1}{2}(\{H_{cq}, \varrho\} - \{\varrho, H_{cq}\}), \tag{2.121}$$

which is known as the *Alexandrov-Gerasimenko bracket* [103, 104]. However, Equation (2.120) is not completely positive, without adding decoherence and diffusion due to Equation (2.93). The full master equaiton reads

$$\begin{aligned}\frac{\partial \rho}{\partial t} =& \{H_c, \varrho\} - i[H_q, \varrho] - iD_1 q[Q, \varrho] + \frac{1}{2}D_1\left(Q\frac{\partial \varrho}{\partial p} + \frac{\partial \varrho}{\partial p}Q\right) \\ &+ \lambda\frac{1}{2}[Q, [\varrho, Q]] + D_2\frac{\partial^2 \rho}{\partial p^2},\end{aligned} \tag{2.122}$$

where complete positivity requires the decoherence-diffusion trade-off

$$D_2 \geq \frac{D_1^2}{\lambda}. \tag{2.123}$$

Roughly speaking, when the system is coherent, the diffusion has to mask the force that the quantum system exerts on the classical one. Equation (2.122) is of the form originally studied by Diosi [57], where one can also add friction term $\gamma\frac{\partial(p\rho)}{\partial p}$ to dampen the effect of the diffusion.

Chapter 3

Hamiltonian formulation of GR

In this chapter, we introduce background material for general relativity (GR) necessary for the second part of the thesis, where we discuss applications of the classical-quantum formalism to gravity. The language of CQ dynamics presented so far is given by a master equation formalism, where one specifies an initial classical-quantum state and evolves according to the dynamics of a CQ master equation. We, therefore, review the initial value, or ADM formulation [77], of GR (see [105] for a detailed overview), which is used in Chapter 7 to study the constraints in classical-quantum theories of gravity.[1] We also discuss standard semi-classical approaches for incorporating quantum back-reaction on a classical gravitational field, which gives rise to non-linear evolution on the quantum state but can be useful in certain regimes.

3.1 The Einstein equations

The setting of Einstein gravity is a Lorentzian manifold $(M, g_{\mu\nu})$, where $g_{\mu\nu}$ is a Lorentzian metric with signature $(-,+,+,+)$. The metric solves Einstein's equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{3.1}$$

which relate the curvature of space-time $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ to the stress energy tensor $T^{\mu\nu}$ of the matter degrees of freedom. The Einstein equations arise from variations of the gravitational

[1] We also consider a covariant approach via path integral methods in Chapter 8.

action $S = S_{EH} + S_m$, where

$$S_{EH} = \frac{c^4}{16\pi G} \int R\sqrt{-g}\ \mathrm{d}^4 x, \tag{3.2}$$

is the *Einstein-Hilbert action* and S_m is a matter action; for example, for a minimally coupled scalar field Φ the matter action S_m is given by

$$S_m = \int d^4x\sqrt{g}[-\frac{1}{2}g^{ab}\nabla_a\Phi\nabla_b\Phi - V(\Phi)]. \tag{3.3}$$

Einstein's equations are diffeomorphism invariant; a diffeomorphism $\theta : M \to M$ is a smooth map between manifolds which is one-to-one, onto, and has a smooth inverse. In particular, denoting the metric and matter degrees of freedom as (g, Φ_m), then if (g, Φ_m) solve the Einstein's equations, then the push-forward $((g)_*, (\Phi_m)_*)$, obtained by applying the diffeomorphism to the dynamical degrees of freedom, also solve the Einsteins equations. Because diffeomorphism invariance is a local symmetry, it is treated as a gauge symmetry of the theory. Because of this gauge invariance, there are 2 degrees of freedom per space-time point[2]; $G_{\mu\nu}$ is a symmetric tensor containing 10 equations, the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$ removes 4 degrees of freedom, and gauge symmetry removes another 4.

Written in the form of Equation (3.1), the symmetries of Einstein's equations are manifest. However, they are presented in a different format than is usual when considering the dynamics of physical systems; usually, one considers a physical problem by specifying initial data, with dynamics that evolve the initial data in time to find a solution. In closed systems, the dynamics are usually considered to be Hamiltonian. In open systems, one can consider more general master equations, such as the Fokker-Plank equation for open classical dynamics or the Lindblad equation for open quantum systems. In its covariant form (3.1), it is not apparent which initial conditions are sufficient to govern the evolution of Einstein's equations, nor how to ensure the initial value problem is well posed with a unique solution because the equations have a diffeomorphism gauge symmetry.

Because much of the thesis is devoted to studying classical-quantum master equations and their applications to gravity, we now introduce the Hamiltonian (initial value) formulation of GR.

[2]We define the number of physical degrees of freedom we define to be the number of generalized positions (here $g_{\mu\nu}$), whose evolution is given by a second order in time differential equation.

3.2 The 3+1 split

To obtain an initial value formulation of GR, one considers a foliation of the space-time M, which we assume is globally hyperbolic $M \cong \mathbb{R} \times \Sigma$. Practically, what this means is that there exists a scalar function $t(x)$ and a family of hypersurfaces Σ_t, such that $M = \cup_t \Sigma_t$ and each Σ_t is a level surface of the scalar field

$$\forall t \in \mathbb{R},\ \Sigma_t := \{p \in M, t(p) = t\}. \tag{3.4}$$

Given a foliation of space-time, one specifies initial data on a slice Σ_t which evolve according to the gravitational Hamiltonian equations of motion.

To arrive at the Hamiltonian formulation, we first need to decompose the metric into quantities that are intrinsic to the leaves of the foliation Σ_t. This is achieved by performing a $3+1$ split of the metric and writing it in a form adapted to the foliation.

To derive the 3+1 split, on each hypersurface Σ_t we define spatial co-co-ordinates x^i, and let $x^\mu = (t, x^i)$ define co-ordinates on M [105]. We call these coordinates the *coordinates adapted to the foliation*. We denote $n_\mu = -N\nabla_\mu t$, the unit normal to the surfaces Σ_t. In the adapted co-ordinates, $n_\mu = -N\delta^0_\mu$, and its normalization determines N via $g^{00} = -\frac{1}{N}$. The function N is called the *lapse function*.

We define the *normal evolution vector* $m^\mu = Nn^\mu$. This vector has a special role since the vector $\delta t m^\mu$ transports the hypersurface Σ_t to its neighboring one $\Sigma_{t+\delta t}$. To see this, note that the normal evolution vector satisfies $m^\mu \nabla_\mu t = 1$, so that along $\delta t m^\mu$ the scalar time function t changes according to $t' = t + \delta t m^\mu \nabla_\mu t = t + \delta t$.

This property is also satisfied by the vector ∂_t, which has components δ^0_μ in the adapted coordinates and corresponds to a simple translation of the scalar time variable. The difference between the vectors ∂_t and m is tangent to Σ_t and is called the *shift vector*, which we denote β^μ. In components, we have

$$\delta^\mu_0 = m^\mu + \beta^\mu = -N^2 g^{0\mu} + \beta^\mu. \tag{3.5}$$

By contracting Equation (3.5) with $n_\mu = -N\delta^0_\mu$, we see that $\beta^0 = 0$, which verifies that the shift vector lies tangent to Σ_t. We shall write $\beta^i = N^i$ and combine the lapse and shift vectors in the four vector $N^\mu = (N, N^i)$.

We can use (3.5) to find the components of the metric in the coordinates adapted to the foliation. We find

$$g_{00} = g_{\mu\nu}\delta^\mu_0\delta^\nu_0 = g_{\mu\nu}(-N^2 g^{0\mu} + N^\mu)(-N^2 g^{0\nu} + N^\nu) = -N^2 + g_{ij}N^iN^j, \tag{3.6}$$

where we have substituted in for (3.5) and used the fact that $g^{00} = -\frac{1}{N}$. Furthermore,

$$g_{0i} = g_{\mu\nu}\delta^\mu_0\delta^\nu_i = g_{\mu\nu}(-N^2 g^{0\mu} + N^\mu)\delta^\nu_i = N_i. \tag{3.7}$$

Writing $\gamma_{ij} = g_{ij}$, we arrive at the familiar *3+1 decomposition* of the metric

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.8}$$

which is also known as the *ADM decomposition* [105, 77].

3.2.1 Curvatures associated to the foliation

The 3-metric γ_{ij} lies tangent to the surface Σ_t, which defines a metric on Σ_t. From now onwards, we shall use abstract index notation with Roman letters a, b to represent tensors that are intrinsic to Σ_t; for example, we denote the 3 metric γ_{ab} to emphasize that it defines a tensor on Σ_t. This notation is consistent with Chapter 7, where we discuss constraints arising in Hamiltonian theories of CQ gravity.

Because γ_{ab} represents a metric on Σ_t, we can define a covariant derivative and associated curvature tensors. We denote D for the covariant derivative associated γ_{ab}, so that for any vector X^a tangent to Σ_t

$$D_bX^a = \partial_b X^a + \Gamma^a_{bc}X^c, \tag{3.9}$$

where Γ^a_{bc} denotes the Christoffel symbol of the covariant derivative D

$$\Gamma^a_{bc} := \frac{1}{2}\gamma^{ad}\left(\partial_b\gamma_{dc} + \partial_c\gamma_{db} - \partial_d\gamma_{bc}\right). \tag{3.10}$$

From D, we can also define an intrinsic Riemann tensor

$${}^{(3)}R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{cb} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{cb}, \tag{3.11}$$

as well as an *extrinsic curvature* tensor

$$K_{ab} = \frac{1}{2N}\left(\dot\gamma_{ab} - D_aN_b - D_bN_a\right), \tag{3.12}$$

which describes how the embedded surfaces Σ_t curve in M.

The intrinsic and extrinsic curvatures are not independent but are related by the Riemann tensor of the four geometry by the Gauss-Codazzi equations. In particular, one can use the Gauss-Codazzi equations [105] to find the G_{0a} and G_{00} projections of the Einstein tensor $G_{\mu\nu}$ in terms of K_{ab} and curvatures of γ_{ab}

$$\begin{aligned} G_{00} &= \frac{1}{2}({}^{(3)}R - K_{ab}K^{ab} + K^2) = \frac{8\pi G}{c^4}T_{00}, \\ G_{0a} &= D_b K^b_a - D_a K = \frac{8\pi G}{c^4}T_{0a}. \end{aligned} \tag{3.13}$$

The Equations in (3.13) are first order in the time derivatives of γ_{ab} and the matter degrees of freedom, at least for minimally coupled matter. Consequently, (3.13) are not dynamical equations but constraints on the initial data one can specify. To elaborate on the role of constraints in gravity, we now introduce the Hamiltonian formulation of GR, where the constraint structure is manifest.

3.3 Hamiltonian formulation of GR

In this section, we will arrive at the Hamiltonian formulation of GR, also known as the ADM formalism [77, 106]. We do so explicitly for vacuum GR, without matter, and quote the end result when matter is included. We assume the space-time is asymptotically flat so that boundary terms can be ignored.

To derive the Hamiltonian formulation of vacuum GR, we start from the Einstein-Hilbert action

$$S_{EH} = \frac{c^4}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x. \tag{3.14}$$

We first substitute for the ADM split in Equation (3.8). After integrating by parts and assuming an asymptotically flat space-time, we find the action

$$S_{EH} = \int dt d^3x \mathcal{L}[\gamma, N, N^a] = \frac{c^4}{16\pi G}\int dt d^3x \sqrt{\gamma} N \left({}^{(3)}R + K_{ab}K^{ab} - K^2\right), \tag{3.15}$$

where ${}^{(3)}R$ is the Ricci scalar of γ_{ab}, K_{ab} is the extrinsic curvature of a surface of constant t, as defined in Equation (3.12), and $K = K_{ab}\gamma^{ab}$.

The action S_{EH} is a functional of N, N^a and γ_{ab}. To introduce the Hamiltonian, we first need to determine the momenta conjugate to N, N^a and γ_{ab}. Since the action does not depend on time derivatives of N and N^a, their conjugate momenta are identically zero $P_N, P_{N^a} = 0$. The momentum conjugate to γ_{ab} is

$$\pi^{ab} \equiv \frac{\delta S}{\delta \dot{\gamma}_{ab}} = \frac{c^4\sqrt{\gamma}}{16\pi G}\left(K^{ab} - K\gamma^{ab}\right). \tag{3.16}$$

Note that the factor of $\sqrt{\gamma}$ means that π^{ab} is not a tensor but a tensor density of weight $\frac{1}{2}$.[3] The phase space is then generated by elements, (γ_{ab}, π^{ab}) which satisfy the canonical Poisson bracket relations[4]

$$\{\gamma_{ab}(x), \pi^{cd}(y)\} = \frac{1}{2}(\delta_a^c\delta_b^d + \delta_a^d\delta_b^c)\delta(x,y). \tag{3.17}$$

We define the Hamiltonian as the Legendre transform of the Lagrangian:

$$H = \int d^3x\left(\pi^{ab}\dot{\gamma}_{ab} - \mathcal{L}\right), \tag{3.18}$$

which, after integrating by parts and neglecting surface terms, takes the form of the *ADM Hamiltonian* [77, 78]

$$H_{ADM}[N, \vec{N}] = \int d^3x N^\mu \mathcal{H}_\mu = \int d^3x (N\mathcal{H} + N^a\mathcal{H}_a) =:= H[N] + H[\vec{N}], \tag{3.19}$$

where $\mathcal{H}_\mu = (\mathcal{H}, \mathcal{H}_a)$. In Equation (3.19)

$$\mathcal{H} = \frac{(16\pi G)}{c^4}\pi^{ab}G_{abcd}\pi^{cd} - \frac{c^4}{16\pi G}\gamma^{1/2}R, \quad \mathcal{H}_a = -2\gamma_{ac}D_b\pi^{cb}, \tag{3.20}$$

and G_{abcd} is the *deWitt metric* defined as $G_{abcd} = \frac{1}{2\sqrt{\gamma}}(\gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc} - \gamma_{ab}\gamma_{cd})$.[5] We can also define an extended phase space $(\gamma_{ab}, \pi^{ab}, N, P_N, N^a, P_{N^a})$, by including explicitly the constraints $P_\mu = (P_N, P_{N^a}) \approx 0$ in the *extended Hamiltonian*

$$H^e_{ADM} = \int d^3x (N^\mu \mathcal{H}_\mu + \lambda^\mu P_\mu). \tag{3.21}$$

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[3] A tensor density of weight p transforms under a coordinate transformation in the same way as γ^p, the determinant of γ, times a tensor.

[4] The convention here is that $\delta(x,y)$ is a scalar in x and a scalar density in y. It is defined by its action on scalar functions $f: \Sigma \to \mathbb{R}$ as $f(x) = \int_\Sigma dy \delta(x,y) f(y)$. It is useful to note that as a consequence, $\int_\Sigma dy \nabla_a^x \delta(x,y) f(y) = \nabla_a f(x)$ and $\int_\Sigma dy \nabla_a^y \delta(x,y) f(y) = -\int_\Sigma dy \delta(x,y) \nabla_a^x f(y) = -\nabla_a f(x)$ [107].

[5] We can extend the covariant derivative to act on tensor densities of weight W, we do this by subtracting $W\Gamma_c^{ca}$ to the usual covariant derivative D_a For example $D_a\pi^{bc} = \partial_a\pi^{bc} + \Gamma^b_{ad}\pi^{dc} + \Gamma^c_{ad}\pi^{bc} - \Gamma^d_{da}\pi^{bc}$ since π^{ab} is a tensor density of weight 1.

In the Hamiltonian formalism, γ_{ab} and π^{ab} are the dynamical variables. The lapse function N and shift vector N^a appearing in Equation (3.19) are arbitrary functions and play the role of Lagrange multipliers enforcing constraints. They are non-dynamical since $P_N, P_{N^a} \approx 0$, and as a result, we see that GR is a constrained theory. In terms of the extended Hamiltonian in Equation (3.21), one sees that the time derivatives of the lapse and shift vectors are functions of the arbitrary Lagrange multipliers λ^μ, $\dot{N}^\mu = \lambda^\mu$, The lapse and shift vectors arose when we performed the 3+1 split of the metric in Section 3.2 and represent the gauge degrees of freedom associated to picking a foliation of space-time; in particular, a different choice of time function t and coordinates x^i will yield different lapse and shift vectors.

Asking that the constraints $P_N, P_{N^a} \approx 0$ are preserved in time leads to the Hamiltonian and Momentum constraints, $\mathcal{H} = \mathcal{H}_a \approx 0$. Conservation of these constraints is ensured via the hypersurface deformation algebra [108]

$$\begin{aligned}
&\{H[N], H[M])\} = H[\vec{R}] \\
&\{H[\vec{M}], H[N]\} = H\left[L_{\vec{M}} N\right] \\
&\{H[\vec{N}], H[\vec{M}]\} = H[L_{\vec{N}} \vec{M}],
\end{aligned} \tag{3.22}$$

where $R^a := \gamma^{ab}\left(N D_b M - M D_b N\right)$ and L is the Lie derivative on Σ_t.

The dynamical equations of motion are found from Hamilton's equations

$$\dot{\gamma}_{ab} = \frac{\delta H}{\delta \pi^{ab}} \quad \dot{\pi}^{ab} = -\frac{\delta H}{\delta \gamma_{ab}}. \tag{3.23}$$

The first equation reproduces the definition of π^{ab}, while the second equation encodes the dynamics of Einstein's equations not associated with constraints, i.e., the spatial projections of Einstein's equations.

The Hamiltonian formulation is used to solve the system of variables (γ_{ab}, π^{ab}) once a lapse and shift vector (N, N^a) is specified. Since the N^μ are arbitrary functions of space and time, the interpretation is that $\{\mathcal{H}_\mu,\}$ generate gauge transformations. One way of understanding their role is that they generate gauge transformations on an initial value surface: they take initial data to other initial data, which, once the equations of motion have been solved, give rise to physically equivalent solutions [109]. To see this more explicitly, we can consider the evolution of any phase space functional $f(\gamma, \pi)$. Given it takes the value $f(0)$ at $t = 0$, after a small time ϵ, it can take the values

$$
\begin{aligned}
f(\epsilon) &= f(0) + \epsilon N^{\mu} \left\{ f(0), \mathcal{H}_{\mu} \right\} \\
f'(\epsilon) &= f(0) + \epsilon N^{\mu'} \left\{ f(0), \mathcal{H}_{\mu} \right\},
\end{aligned}
\tag{3.24}
$$

depending on the choice of lapse and shift. Consequently, given the same initial data set, we get two equally valid descriptions of the phase space functional. If we wish to retain any notion of predictability, we must identify that action of the constraints is to give new phase space variables which should be considered physically equivalent in the sense that they will yield equivalent solutions to the dynamics [106]. The difference between the function due to different choices of lapse and shift is given by

$$
\delta f(\epsilon) = \epsilon \left(N^{\mu\prime} - N^{\mu} \right) \left\{ f(0), \mathcal{H}_{\mu} \right\}, \tag{3.25}
$$

and hence one can deduce that $\{\mathcal{H}_{\mu}, \}$ generates equal time gauge transformations on the phase space, associated with the different possible choices of the lapse and shift-vector. It should be emphasized that the constraints $\mathcal{H}_{\mu}$ do not *directly* map solutions of initial data to other gauge equivalent solutions, nor do they generate diffeomorphisms on solutions; this is easily seen since they do not change the values of the lapse and shift vectors, which change under diffeomorphisms [109]. Note that the extended Hamiltonian in Equation (3.21) does act as a time diffeomorphism on the space of solutions in the extended phase pace. In the next section, we discuss the relationship between the constraint generators and diffeomorphism invariance in more detail.

Understanding that the constraints generate gauge transformations, we can check that the Hamiltonian system counts the same number of degrees of freedom as Einstein's equations. Because the lapse and shift are arbitrary functions corresponding to a choice of gauge, the physical degrees of freedom are (γ_{ab}, π^{ab}), which constitute 6 degrees of freedom, but satisfy 4 constraints, leading to two degrees of freedom per space-time point.[6]

Since the language of CQ dynamics is that of master equations, it is worth mentioning that – although it is not usually considered – we can write the dynamics of pure GR in a Liouville

[6]Note that (γ_{ab}, π^{ab}) consist of 12 phase space degrees of freedom, but since π^{ab} is defined in terms of $\dot{\gamma}^{ab}$ they lead to only 6 configuration space degrees of freedom undergoing second order time dynamics.

formulation. In particular, a phase space distribution $\rho(\gamma, \pi)$ will evolve under the dynamics as

$$\frac{\partial \rho}{\partial t} = \{H_{ADM}, \rho\}, \tag{3.26}$$

subject to the constraints $\mathcal{H}\rho = \mathcal{H}_a\rho \approx 0$; that is ρ must have support only on the constraint surface.

We can, of course, add matter to the discussion. For ease of calculation, we only consider coupling scalar fields to gravity in this thesis. A classical field minimally coupled to gravity will have a Hamiltonian of the form

$$H_T[N, \vec{N}] = H_{ADM}[N, \vec{N}] + H_m[N, \vec{N}] = \int d^3x N(\mathcal{H} + \mathcal{H}_m) + N^a(\mathcal{H}_a + \mathcal{H}_{m,a}), \tag{3.27}$$

where H_m is the Hamiltonian of the matter field. In the presence of matter, the constraint surface takes the form $\mathcal{H} + \mathcal{H}_m \approx 0, \mathcal{H} + \mathcal{H}_{m,a} \approx 0$.

For example, the Hamiltonian of the free scalar field reads

$$H_m[N, \vec{N}] = \int d^3x N(\frac{1}{2}\gamma^{-1/2}\pi^2 + \frac{1}{2}\gamma^{1/2}\gamma^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2}\gamma^{1/2}m^2\phi_\phi^2) + N^i\pi_\phi\partial_i\phi. \tag{3.28}$$

The Liouville equation for the phase space density $\rho(\gamma, \pi_\gamma, \phi, \pi_\phi, t)$ then takes the form

$$\frac{\partial \rho}{\partial t} = \{H[N, \vec{N}], \rho\} + \{H_m[N, \vec{N}], \rho\}, \tag{3.29}$$

where ρ must only have support on the constraint surface.

3.3.1 Symmetries in the ADM formalism

In this section, we briefly comment on the relationship between the constraint generators $\{\mathcal{H}_\mu, \}$ and that of diffeomorphism invariance in GR, which is a gauge symmetry on the space of *solutions* to the equations of motion.

Given a choice of lapse and shift vectors $(N(t), N^a(t))$ the equations of motion, generated by Equation (7.13), give us trajectories in phase-space $(\gamma_{ab}(t), \pi^{ab}(t))$, which naturally define a Lorentzian 4 metric on M via the ADM decomposition

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ab}(N^a dt + dx^a)(N^b dt + dx^b). \tag{3.30}$$

Note, Equation (3.30) is an isometric embedding so that $\gamma_{ab}(t)$ is the induced metric for surfaces of constant t, Σ_t, and we can invert $\pi_{ab}(t)$ to obtain $\dot{\gamma}_{ab} = F_{ab}[\gamma, \pi]$. We denote the set of phase-space solutions to the ADM equations by $\mathcal{S}$.

To discuss the relationship between diffeomorphism invariance and the hypersurface deformation algebra, consider applying a diffeomorphism θ to the solution (3.30), which acts to send $g_{\mu\nu} \to g'_{\mu\nu}$. Since GR is diffeomorphism invariant, $g'_{\mu\nu}$ will also be a solution to the Einstein equations. We can further find a 3+1 split of $g'_{\mu\nu}$

$$g'_{\mu\nu}dx^\mu dx^\nu = -N'^2 dt^2 + \gamma'_{ab}(t,x)(N'^a dt + dx^a)(N'^b dt + dx^b), \tag{3.31}$$

which defines a new set of variables $(\gamma'_{ab}(t), \pi^{ab\prime}(t), N'(t), N'^a(t))$ which will also lie in $\mathcal{S}$.

The fact that $g'_{\mu\nu}$ is also a solution to Einstein's equations leads naturally to a notion of diffeomorphism invariance on the phase space; more appropriately, this should be called a diffeomorphism *equivariance*. We say that the theory is diffeomorphism equivariant since $(\gamma_{ab}(t), \pi^{ab}(t), N(t), N^a(t))$ is a solution to the ADM equations of motion if and only if transformed solution $(\gamma'_{ab}(t), \pi^{ab\prime}(t), N'(t), N'^a(t))$ is also a solution to the ADM equations. We denote the mapping between the two solutions by ϕ.

In the case of the ADM equations of motion, the transformation ϕ is exactly that which arises from Noether's inverse theorem applied to gauge transformations generated by $\mathcal{H}, \mathcal{H}_a$ on the extended phase space [110]. Explicitly, on the extended phase space, ϕ is generated by [110]

$$G(t;\xi) = P_\mu \dot{\xi}^\mu + \xi^\mu \left(\mathcal{H}_\mu + N^\varrho C^\nu_{\mu\varrho} P_\nu\right), \tag{3.32}$$

where $G(t;\xi)$ acts on the space of solutions $\mathcal{S}$, taking one solution to a physically equivalent solution. In Equation (3.32), ξ^μ parameterizes the diffeomorphism, and $\{\mathcal{H}_\mu, \mathcal{H}_\nu\} = C^\sigma_{\mu\nu}\mathcal{H}_\sigma$. Specifically, ξ^0 parameterizes transformations normal to surfaces of constant time, while ξ^a parameterizes the spatial diffeomorphisms.

Importantly, we see that $\mathcal{H}_\mu$ by themselves do not generate diffeomorphisms on the space of solutions. Instead, they act as gauge transformations, taking initial data to other initial data, which, once the equations of motion have been solved, give rise to physically equivalent solutions; one can easily verify $\mathcal{H}_\mu$ cannot generate diffeomorphisms on the space of solutions, since they do not change the values of the lapse and shift vectors, which change under diffeomorphisms. They do, however, partially do the job, and one can check that they generate diffeomorphisms on the phase space variables (γ_{ab}, π^{ab}); the extra terms in Equation (3.32) then account for the changes in the lapse and shift vectors. One can also check that with the substitution $\xi^\mu = N^\mu$ Equation (3.32) reduces to the extended Hamiltonian of Equation (3.21), which can be understood as generating time diffeomorphisms along the vector ∂_t. Note that the dual role of the extended Hamiltonian does not imply that the dynamics are frozen or timeless since the roles of gauge transformations and dynamical evolution are quite distinct. In particular, as the generator of dynamics, the Hamiltonian takes initial data and generates a solution $p \in \mathcal{S}$ to the phase space equations of motion; in loose terms, the Hamiltonian acts within a single solution p. On the other hand, as a gauge transformation, it is to be understood as taking one solution $p \in \mathcal{S}$ to another, gauge equivalent, solution $p' \in \mathcal{S}$. We refer the reader to [109] for a more detailed discussion of these points.

3.4 Incorporating back-reaction: the semi-classical equations

We now discuss the case of incorporating back-reaction when the matter degrees of freedom are considered quantum. In the case of gravity, the standard approach to define back-reaction is via the semi-classical Einstein equations, which source the Einstein tensor $G_{\mu\nu}$ by the expectation value of the stress-energy tensor $T_{\mu\nu}$ [19, 20]

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle, \tag{3.33}$$

where the quantum state is understood to undergo unitary dynamics; the quantum state $|\psi\rangle$ at time t is determined by Hamiltonian evolution that depends on classical degrees of freedom (g, π).

$$\frac{d|\psi\rangle}{dt} = -iH(g,\pi)|\psi\rangle. \tag{3.34}$$

The semi-classical Einstein equations can be derived from effective low energy quantum gravity when there is a dominant background gravitational field, and fluctuations around it are small [111, 112, 113, 114]. Though the scope and limitations of the semi-classical Einstein equations are not precisely understood [21, 22, 23], they are commonly understood to fail when fluctuations of the stress-energy tensor are large in comparison to its mean value [24, 25, 26, 21, 27].

Though the semi-classical equations are valid when quantum fluctuations are small, and so the quantum state is essentially classical, the case where the fluctuations are significant are often precisely the regimes we most wish to understand; for example, when considering the gravitational field associated with Schrodinger cat states of massive bodies [29, 30], or vacuum fluctuations during inflation [31, 32, 33, 34]. For these regimes, background field methods are not appropriate, and an alternate effective theory of the back-reaction of quantum matter on classical gravity is required.

The problem with using the semi-classical equations in the presence of large quantum fluctuations is that they lead to non-linear evolution on the quantum state and, more importantly, the density matrix due to the expectation value sourcing the classical degree of freedom. Consequently, they also fail when probabilistic mixtures are considered since non-linear evolution on the density matrix is inconsistent with an ensemble interpretation [6, 28]. The non-linearity of the dynamics leads to violations of the standard principles of quantum theory; as an effective theory, small violations of quantum theory may be acceptable, which is the case for an essentially classical quantum state, but away from this, the theory should be deemed inapplicable. The reason for the failure of semi-classical gravity when large quantum fluctuations are considered is that the semi-classical equations fail to account for correlations between the classical and quantum degrees of freedom. In practice, the correlation is often put in by hand by considering situations when the quantum state is fully decohered and then evolving the classical system conditioned on the quantum state being in a particular eigenvalue, but this is quite distinct from a direct application of the semi-classical equations (see [6, 28] for a more detailed discussion of these points).

We can see the non-linearity of the semi-classical equations in more detail by considering the Newtonian limit of the semi-classical equations. The Newtonian limit of semi-classical gravity is described by the single Hamiltonian constraint

$$\nabla^2 \Phi = 4\pi G \langle \psi | m | \psi \rangle = 4\pi G m |\psi(x)|^2, \tag{3.35}$$

while the quantum state undergoes Hamiltonian evolution according to $H = H_m + H_I$,

$$\frac{d|\psi\rangle}{dt} = -iH_m|\psi\rangle - i\int d^3x \Phi(x) m(x) |\psi\rangle, \tag{3.36}$$

where H_I is the interaction Hamiltonian

$$H_I = \int d^3x \Phi(x) m(x), \tag{3.37}$$

and H_m denotes the matter Hamiltonian not associated with gravity. Inserting the solution for the gravitational potential

$$\Phi(x) = -G \int d^3x' \frac{\langle\psi|m(x)|\psi\rangle}{|x-x'|} \tag{3.38}$$

into Equation (3.36), we find the *Schrodinger-Newton equation* [115, 116, 117]

$$\frac{d|\psi\rangle}{dt} = -i(H_m)|\psi\rangle + iG \int d^3x \int d^3x' \frac{\langle\psi|m(x)|\psi\rangle}{|x-x'|} m(x)|\psi\rangle, \tag{3.39}$$

which describes the evolution of the quantum state in the Newtonian limit of semi-classical gravity.

It is well known that the dynamics of Equation (3.39) is not consistent when applied to quantum states with large fluctuations [29, 49, 51], and because it leads to non-linear evolution of the density matrix, it violates the standard principles of quantum theory, inducing a breakdown of either operational no-signaling, the Born rule, or composition of quantum systems under the tensor product [48, 18, 49, 50, 51]. Though one might be willing to expect some small violations of these properties as an effective theory, the same cannot be said if it is treated as a fundamental equation. Regardless, an improved semi-classical description should be sought after; in Chapter 5, we shall show that CQ dynamics can be used to give rise to a consistent semi-classical formalism, which upholds the standard principles of quantum theory, and leads to consistent dynamics when any quantum state is considered – even when quantum fluctuations are large.

We also mention another formalism for semi-classical gravity called *stochastic gravity* [21], an extension of semi-classical gravity aimed at incorporating higher-order corrections to Einstein's equations. The main object of study in stochastic gravity is the so-called *Einstein-Langevin equation*

$$G_{ab}[g+h] + \Lambda(g_{ab} + h_{ab}) = \frac{8\pi G}{c^4}(\langle T_{ab}[g+h]\rangle + \xi_{ab}[g]), \tag{3.40}$$

which describes the fluctuations h around a background metric g in terms of a stochastic noise source $\xi_{ab}[g]$, whose statistics are defined in terms of the two-point correlation function of the stress-energy tensor.[7] The Einstein-Langevin equation is solved to obtain a probability

[7] It is easy to check that Equation (3.40) is invariant under linear diffeomorphisms, so long as the background metric g is a solution to the semi-classical Einstein equation.

distribution over the perturbations h, from which one can attempt to learn about the back-reacting regime; for example, by studying correlation functions of metric degrees of freedom averaged over the noise $\xi_{ab}[g]$. Though useful as an effective theory in the presence of small quantum fluctuations, like the semi-classical Einstein equation, the Einstein-Langevin equation gives rise to non-linear evolution on the quantum state.

Part II

Developing classical-quantum dynamics

Chapter 4

A classical-quantum Pawula theorem

Having introduced the relevant background material, we now enter into the first part of the main thesis, which is concerned with developing the classical-quantum formalism.

In this chapter, we study the positivity conditions of CQ dynamics in detail. In classical dynamics, in Chapter 2, we saw that we could write the master equation in terms of the moments of the transition probability amplitude via the Kramers-Moyal expansion [83, 84, 73]. The complete positivity of the dynamics means the transition amplitude must be positive, which can be used to derive constraints on the moments appearing in the moment expansion. Of particular relevance is the Pawula theorem [76], which states that the moment expansion either stops after the first or second moments, or else it must contain an infinite number of terms; in the former case, this restricts continuous dynamics to the well-known Fokker-Planck equation [73].[1]

Here, we prove a classical-quantum version of the Pawula theorem, which follows from a combined CQ Cauchy-Schwarz inequality, Equation (4.9). We find that in order for a non-trivial classical-quantum interaction to be completely positive, the classical-quantum moment expansion must either contain an infinite number of terms or it must be of the form

[1] It is important to note that if one truncates the series after n terms with $n \geq 3$, the resulting equation, although not positive, can still be used as an approximation to the dynamics in an appropriate regime. Indeed, one might attain a better approximation of certain classical dynamics by using an approximation that is not positive; one just has to be careful about the validity of the approximation [73].

$$
\frac{\partial \varrho(z,t)}{\partial t} = \sum_{n=1}^{n=2}(-1)^n \left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}\varrho(z,t)\right) - \frac{\partial}{\partial z_i}\left(D^{0\alpha}_{1,i}\varrho(z,t)L^\dagger_\alpha\right) - \frac{\partial}{\partial z_i}\left(D^{\alpha 0}_{1,i}L_\alpha\varrho(z,t)\right)
$$
$$
- i[H(z),\varrho(z,t)] + D^{\alpha\beta}_0(z)L_\alpha\varrho(z)L^\dagger_\beta - \frac{1}{2}D^{\alpha\beta}_0\{L^\dagger_\beta L_\alpha,\varrho(z)\}_+, \tag{4.1}
$$

where $2D^{00}_2 \succeq D_1 D_0^{-1} D_1^\dagger$ and $(\mathbb{I} - D_0 D_0^{-1})D_1 = 0$ Here, D_0^{-1} is the generalized inverse of the matrix $D_0^{\alpha\beta}$, D_1 is a matrix in both α, i indices with entries $D^{0\alpha}_{1,i}$ and D^{00}_2 is a matrix in i,j with entries $D^{00}_{2,ij}$.

Previously in the literature, examples of continuous classical-quantum master equations have been given [57, 60], but the general form was unknown [74]; just like examples of the Fokker-Plank equation were known before Pawula proved that it was the most general form of continuous master equation. Infinite moments are indicative of a jump process, and so we find that Equation (4.13) is the most general form of CQ master equation that has continuous trajectories in the classical phase space.

As a further consequence of the CQ Pawula theorem, we show that for classical-quantum dynamics to be completely positive, one must have a term representing pure Lindbladian evolution on the quantum state. In other words, the nature of completely positive dynamics necessarily results in the classical degrees of freedom inducing decoherence on the quantum state. Classicality induces classicality.

This chapter is based on the paper [5], which is work done in collaboration with Carlo Sparaciari, Barbara Šoda, and Jonathan Oppenheim.

4.1 Positivity conditions for classical-quantum interactions

We start with the master equation for autonomous classical-quantum dynamics

$$
\begin{aligned}
\frac{\partial \varrho(z,t)}{\partial t} &= \sum_{n=1}^{\infty}(-1)^n \left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}(z)\varrho(z,t)\right) \\
&\quad - i[H(z),\varrho(z)] + D^{\alpha\beta}_0(z)L_\alpha\varrho(z)L^\dagger_\beta - \frac{1}{2}D^{\alpha\beta}_0\{L^\dagger_\beta L_\alpha,\varrho(z)\}_+ \\
&\quad + \sum_{\mu\nu\neq 00}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{\mu\nu}_{n,i_1\dots i_n}(z)L_\mu\varrho(z,t)L^\dagger_\nu\right),
\end{aligned} \tag{4.2}
$$

where we have suppressed the explicit t dependence on the moments $D_n(z,t)$. We know that the dynamics in Equation (4.2) will be positive so long as the transition amplitude $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ is a positive matrix and that positivity of $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ transfers naturally to positivity conditions on the short time moment expansions defined in Equation (2.84); for example, by considering the block form of (2.92)

$$\Lambda^{\mu\nu}(z,t+\delta t|z',t) = \begin{bmatrix} \delta(z,z') + \delta t W^{00}(z|z',t) & \delta t W^{0\beta}(z|z',t) \\ \delta t W^{\alpha 0}(z|z',t) & \delta t W^{\alpha\beta}(z|z',t) \end{bmatrix} + O(\delta t^2), \tag{4.3}$$

where we recall

$$D^{\mu\nu}_{n,i_1\dots i_n}(z',t) := \frac{1}{n!}\int dz W^{\mu\nu}(z|z',t), t(z-z')_{i_1}\dots(z-z')_{i_n}. \tag{4.4}$$

In this chapter, we also suppress the explicit time dependence of the transition amplitudes $W(z|z',t)$, which can always be added back in.

We first note that the pure classical positivity condition, given by the 00 component $\Lambda^{00}(z,t+\delta t|z',t) = \delta(z,z') + \delta t W^{00}(z|z',t)$, leads to the well known Pawula theorem of classical autonomous dynamics discussed in Chapter 2; if any even moment D^{00}_n vanishes, then all moments with $n \geq 3$ must also vanish.

For the classical-quantum interaction to be completely positive, Equation (4.3) tells us $W^{\alpha\beta}(z|z')$ must be a positive matrix in α, β. We shall now use this fact to derive a family of inequalities which their moments $D^{\alpha\beta}_n(z)$ must satisfy, enabling us to prove a strengthened version of the Pawula theorem to CQ dynamics. In particular, we use this to show that the most general form of an autonomous classical-quantum master equation continuous in the classical phase is given by (4.13); any other master equation must contain finite-sized jumps in the phase space with non-zero probability. We further show that we *must* have a non-zero pure decoherence term; the completely positivity of the CQ interaction necessarily causes the classical system to induce decoherence on the quantum system. This, in turn, requires diffusion in phase space above a certain threshold value which we quantify further in Chapter 10.

4.2 Inequalities on the moments from positivity conditions

In this section, we derive a Cauchy-Schwartz inequality, Equation (4.9), applicable to any CQ map which is completely positive. We use it to derive a set of inequalities relating the moments

$D_n^{\alpha\beta}(z)$ appearing in Equation (4.2). We first note that since $W^{\alpha\beta}(z|z')$ is a positive matrix, $W(z|z')(\varrho(z')) = W^{\alpha\beta}(z|z')L_\alpha\varrho(z')L_\beta^\dagger$ defines a completely positive operator. It will prove useful to use the vectorization map (2.99) to write the expansion coefficients $D_n^{\alpha\beta}(z)$ appearing in the dynamics of (4.2) in terms of the components of the completely positive operator $W(z|z')$. Explicitly,

$$D^{\alpha\beta}_{n,i_1\dots i_n}(z')L_\alpha\varrho(z')L_\beta^\dagger = \frac{1}{n!}\int dz\ W^{\alpha\beta}(z|z')L_\alpha\varrho(z')L_\beta^\dagger(z-z')_{i_1}\dots(z-z')_{i_n} \tag{4.5}$$

$$= \frac{1}{n!}\int dz \mathrm{Tr}\left[(\bar{L}_\beta\otimes L_\alpha)^\dagger W^{vec}(z|z')\right] L_\alpha\varrho(z')L_\beta^\dagger(z-z')_{i_1}\dots(z-z')_{i_n}. \tag{4.6}$$

We could equally well write the completely positive operator $W(z|z')$ in a different basis, which will prove useful. To that end, given an arbitrary basis on the underlying Hilbert space $\{|a\rangle\}$, we define the natural basis of operators on the Hilbert space E_{ab} via $E_{ab} = |a\rangle\langle b|$. In this basis

$$D^{\alpha\beta}_{n,i_1\dots i_n}(z)L_\alpha\varrho L_\beta^\dagger = D^{abcd}_{n,i_1\dots i_n}(z)E_{ca}\varrho E_{bd}, \tag{4.7}$$

where as in Equation (4.5)

$$D^{abcd}_{n,i_1\dots i_n}(z') := \int dz\frac{1}{n!}\mathrm{Tr}\left[(E_{db}\otimes E_{ca})^\dagger W^{vec}(z|z')\right](z-z')_{i_1}\dots(z-z')_{i_n}\ . \tag{4.8}$$

Now, let us prove a generalized form of the Cauchy-Schwartz inequality that we can use for the case of hybrid classical-quantum theories. It will take the form

$$\int d\Delta \mathrm{Tr}_{\mathcal{H}}\left[f(\Delta)^\dagger f(\Delta)T(\Delta)\right]\int d\Delta \mathrm{Tr}_{\mathcal{H}}\left[g(\Delta)^\dagger g(\Delta)T(\Delta)\right]$$
$$\geq \int d\Delta \mathrm{Tr}_{\mathcal{H}}\left[f^\dagger(\Delta)g(\Delta)T(\Delta)\right]\int d\Delta \mathrm{Tr}_{\mathcal{H}}\left[g^\dagger(\Delta)f(\Delta)T(\Delta)\right], \tag{4.9}$$

and it holds for any completely positive operator $T(\Delta)$ and arbitrary CQ operators $g(\Delta), f(\Delta)$. The above relation is easily derived by rearranging

$$\int d\Delta d\Delta' \mathrm{Tr}_{A,B}\left[(f_A(\Delta)g_B(\Delta') - g_A(\Delta)f_B(\Delta'))^\dagger(f_A(\Delta)g_B(\Delta') - g_A(\Delta)f_B(\Delta'))T_A(\Delta)T_B(\Delta')\right], \tag{4.10}$$

which is positive because each map $T_{A/B}(\Delta)$ acting on its share of a positive operator is a completely positive map. Using (4.9) with

$$T(z+\Delta, z) = W^{vec}(z+\Delta, z)$$
$$f(\Delta) = (E_{bb}\otimes E_{aa})\Delta_{i_1}\dots\Delta_{i_n},\quad g(\Delta) = (E_{bd}\otimes E_{ac})\Delta_{i_{n+m}}\dots\Delta_{i_{2n+2m}}, \tag{4.11}$$

and then integrating over z, we find the inequalities on the moments arising in the CQ equation

$$(2n!)(2n+2m)!D^{abab}_{2n,i_1i_1\dots i_ni_n}D^{cdcd}_{2n+2m,i_{n+m}i_{n+m}\dots i_{2n+2m}i_{2n+2m}} \geq |(2n+m)!D^{abcd}_{2n+m,i_1\dots i_{2n+m}}|^2, \tag{4.12}$$

where we have used $D^{abcd} = (D^{badc})^*$, which follows from the fact that $W^{\alpha\beta}(z|z')$ is Hermitian.

4.3 A classical-quantum Pawula theorem

The inequalities in Equation (4.12) possess essentially the same structure as those arising in the classical Pawula theorem [76]. However, crucially, they must hold for all $n, m \geq 0$. The difference between the CQ and classical case arises since the zeroth moment of the map $\Lambda^{\mu\nu}(z, t+\delta t|z', t)$ is of order $O(\delta t)$ for the classical-quantum interaction, while it is $O(1)$ for the classical case due to the consistency condition at $\delta t = 0$. More precisely, for $\delta t = 0$ the CQ map in Equation (2.80) takes the form $\Lambda^{\mu\nu}(z, t|z', t) = \delta^\mu_0\delta^\nu_0 + O(\delta t)$. As a result, the zeroth moment of the purely classical component of the CQ map is $O(1)$; there can be no inequalities relating the zeroth moment of the classical dynamics to any higher-order moments since the zeroth moment always dominates. However, for the classical-quantum interaction, the zeroth moment is $O(\delta t)$, so there exist inequalities relating the zeroth moment to the higher order moments, leading to a strengthened version of the Pawula theorem – which we now state and prove. We define *non-trivial CQ evolution* to be one where $W^{\alpha\beta}(z|z')$ is somewhere positive so that the quantum system back-reacts on the classical system.

CQ Pawula Theorem. *For non-trivial CQ evolution, we must have infinitely many moments defined in Equation (4.8), or else the master equation takes the form*

$$\frac{\partial \varrho(z,t)}{\partial t} = \sum_{n=1}^{n=2}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}\varrho(z,t)\right) - \frac{\partial}{\partial z_i}\left(D^{0\alpha}_{1,i}\varrho(z,t)L^\dagger_\alpha\right) - \frac{\partial}{\partial z_i}\left(D^{\alpha 0}_{1,i}L_\alpha\varrho(z,t)\right)$$
$$- i[H(z), \varrho(z,t)] + D^{\alpha\beta}_0(z)L_\alpha\varrho(z)L^\dagger_\beta - \frac{1}{2}D^{\alpha\beta}_0\{L^\dagger_\beta L_\alpha, \varrho(z)\}_+, \tag{4.13}$$

where $2D^{00}_2 \succeq D_1 D_0^{-1} D_1^\dagger$ and $(\mathbb{I} - D_0 D_0^{-1})D_1 = 0$ Here, D_0^{-1} is the generalized inverse of the matrix $D^{\alpha\beta}_0$, D_1 is a matrix in both α, i indices with entries $D^{0\alpha}_{1,i}$ and D^{00}_2 is a matrix in i, j with entries $D^{00}_{2,ij}$. Furthermore, the zeroth moment, $D^{\alpha\beta}_0(z)$ cannot vanish.

Proof. Consider the inequality in Equation (4.12) for $n, m \geq 1$. Suppose any even CQ moment vanishes, so that $D_{2n}^{abcd} = 0$ for all a, b, c, d, then so must $D_{2n+m}^{abcd} = 0$, meaning all higher order moments also vanish. Furthermore, if $D_{2n+2m}^{abcd} = 0$, for all a, b, c, d, then $D_{2n+m}^{abcd} = 0$. Denoting $r = n + m$ we see if $D_{2r}^{abcd} = 0$ then $D_{r+n}^{abcd} = 0$ for $n = 1 \dots r - 1$. To summarise: if any even moment vanishes $D_{2r}^{abcd} = 0$ we deduce that all higher order moments D_{2r+n}^{abcd} must vanish, as well as the moments D_{r+n}^{abcd} for $n = 1 \dots r - 1$. Except for the case $r = 1$, a moment expansion to order $r + n$ will always contain an even moment, and so from repeated application of these properties, if any even moment vanishes, then $D_n^{abcd} = 0$ for all $n \geq 3$. This is the usual Pawula theorem for the coefficients $D_n^{\alpha\beta}$. However, for the CQ case, we also know that the block diagonal matrix $W^{\alpha\beta}(z|z')$ in Equation (4.3) also defines a completely positive map. We can use this to strengthen the condition. In particular, we also have the inequality (4.12) for $n = 0, m \geq 1$ which tells us

$$(2n)! D_0^{abab} D_{2m,i_1 i_1 \dots i_m i_m}^{cdcd} \geq |(m)! D_{m,i_1 \dots i_m}^{abcd}|^2. \tag{4.14}$$

Taking any even moment to be zero, we deduce that $D_4^{abcd} = 0$. But then from (4.14) we must then have $D_2^{abcd} = 0$, which in turn implies $D_1^{abcd} = 0$. Hence we see that if any even moment vanishes, then all of the moments $D_n^{\alpha\beta}$, $n \geq 1$ vanish.

Hence, we conclude that the block $W^{\alpha\beta}(z|z')$ describes pure quantum evolution and is determined by the zeroth moment $D_0^{\alpha\beta}$. What remains is to prove that $D_n^{0\alpha} = 0$ for $n \geq 1$ and Equation (4.3) will be positive if and only if the remaining couplings satisfy the positivity conditions $2D_2 \succeq D_1 D_0^{-1} D_1^{\dagger}$ and $(\mathbb{I} - D_0 D_0^{-1}) D_1 = 0$ where D_0^{-1} is the generalized inverse of the matrix with elements $D_0^{\alpha\beta}$, D_1 is a matrix in both α, i indices with entries $D_{1,i}^{0\alpha}$, and D_2^{00} is a matrix in i, j with entries $D_{2,ij}^{00}$.

If any of the even moments of D_n^{00} and $D_n^{\alpha\beta}$ greater than two vanish, our findings show that we can write the transition amplitude in block form (4.3) as

$$\Lambda^{\mu\nu}(z, t{+}\delta t|z', t) = \begin{bmatrix} \delta(z,z') + \delta t \sum_{n=0}^{2} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \dots \partial z_{i_n}}\right) \left(D_{n,i_1 \dots i_n}^{00}(z) \delta(z,z')\right) & \delta t \sum_{n=0}^{\infty} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \dots \partial z_{i_n}}\right) \left(D_{n,i_1 \dots i_n}^{0\alpha}(z) \delta(z,z')\right) \\ \delta t \sum_{n=0}^{\infty} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \dots \partial z_{i_n}}\right) \left(D_{n,i_1 \dots i_n}^{\alpha 0}(z) \delta(z,z')\right) & \delta t D_0^{\alpha\beta} \delta(z,z') \end{bmatrix}, \tag{4.15}$$

where we must remember that

$$D_0^{00}(z,t)\mathbb{I} + D_0^{0\alpha} L_\alpha + D_0^{\alpha 0} L_\alpha^{\dagger} + D_0^{\alpha\beta} L_\beta^{\dagger} L_\alpha = \mathbb{I}, \tag{4.16}$$

from the normalization condition in Equation (2.82).

To finalize the proof, we consider the positivity condition of $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ directly, which states that

$$\int dz A^*_\mu(z,z')\Lambda^{\mu\nu}(z,t+\delta t|z',t)A_\nu(z,z') \geq 0. \tag{4.17}$$

Since the 00 component of (4.15) contains a delta function $\delta(z,z')$ which is order $O(1)$, whilst all other components are order $O(\delta t)$, Equation (4.17) will be always be positive unless we pick $A_0(z,z') = (z-z')^n f(z,z')$ where $f(z,z)$ is non-zero. Since we know the blocks $\Lambda^{00}(z,t+\delta t|z',t)$ and $\Lambda^{\alpha\beta}(z,t+\delta t|z',t)$ are positive, we must consider the case in which A_α is non-zero, or else the off-diagonal terms of the block matrix (4.15) do not contribute. The only choice of $A_\mu(z,z')$, which gets rid of the leading order $\delta(z,z')$, has well-defined distributional derivatives and keeps the off-diagonal terms is $A_0(z,z') \sim (z-z')f(z,z')^n$ and $A_\alpha = a_\alpha(z,z')$, where $a_\alpha(z,z)$ is non-zero.

In the case in which we have many classical degrees of freedom z_i, we pick $A_0(z,z') = b^{i_1\dots i_n}(z,z')(z-z')_{i_1}\dots(z-z')_{i_n}$ for some vector $b^{i_1\dots i_n}(z,z')$. Choosing $A_\mu(z,z') = (b^i(z,z')(z-z')_{i_1}\dots(z-z')_{i_n}, a_\alpha(z,z'))$, $n \geq 2$ gives the condition

$$\sum_{n=2}^{\infty} n![(b^{i_1\dots i_n *}(z,z)D^{0\alpha}_{n,i_1\dots i_n}(z)a_\alpha) + ((b^{i_1\dots i_n *}(z,z)D^{0\alpha}_{n,i_1\dots i_n}(z)a_\alpha)^*] \geq 0, \tag{4.18}$$

which can always be made negative for a suitable choice of b and a_α, hence we deduce $D^{0\alpha}_n = 0$ for $n \geq 2$.[2]

The remaining conditions arise from choosing $A_0(z,z') = b^i(z,z')(z-z)_i$. With this choice of $A_0(z,z')$ we find the condition for positivity of $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ is

$$2b^{i*}(z,z)D^{00}_{2,ij}b^j(z,z) + b^{i*}(z,z)D^{0\alpha}_{1,i}a_\alpha(z,z) + a^*_\alpha(z,z)D^{\alpha 0}_{1,i}b^i(z,z) + a^*_\alpha(z,z)D^{\alpha\beta}_0 a_\beta(z,z) \geq 0. \tag{4.19}$$

Defining D_2 to be the $n \times n$ matrix with elements $D^{00}_{2,ij}$, D_1 to be the $n \times p$ matrix in i, α with elements $D^{0,\alpha}_{1,i}$ and D_0 the $p \times p$ matrix in α, β with elements $D^{\alpha\beta}_0$ Equation (4.19) can be written in the form

$$[b^*, \alpha^*]\begin{bmatrix} 2D_2 & D_1 \\ D^*_1 & D_0 \end{bmatrix}\begin{bmatrix} b \\ \alpha \end{bmatrix} \geq 0, \tag{4.20}$$

[2] For example, suppose for a specific choice of α,b that Equation (4.18) was positive, then $b \to -b$ will be negative.

which is equivalent to the condition that the $(n+p)\times(n+p)$ matrix be positive semi-definite

$$D = \begin{bmatrix} 2D_2 & D_1 \\ D_1^* & D_0 \end{bmatrix} \succeq 0. \tag{4.21}$$

Since D_0 and D_2 must be positive semi-definite and since (4.21) is a block matrix, we know that the Schur complement of D_0 must be positive semi-definite, which implies

$$2D_2 \succeq D_1 D_0^{-1} D_1^\dagger,\ (\mathbb{I} - D_0 D_0^{-1}) D_1 = 0. \tag{4.22}$$

From Equation (4.22), we see that if D_0 vanishes, so does D_1, and so we must have decoherence for non-trivial CQ evolution. The master equation then takes the form of Equation (4.13). In the case of a single Lindblad operator, the requirements implied by Equation (4.22) reduce to that found in [57].

We further show in Appendix B that the Lindblad operators in (4.13) can be arbitrary rather than requiring them to be traceless and orthogonal. The map will still be completely positive – so long as the conditions on the moments in Equation (4.22) are satisfied.

□

We have therefore found a strengthened version of the Pawula theorem for the CQ couplings – we either have infinitely many terms in the moment expansion, or else the dynamics must take the form of Equation (4.13). It is useful to note that since Equation (4.21) is in block form, it follows the two following conditions are equivalent

$$\begin{aligned} D \succeq 0 \Leftrightarrow D_2 \succeq 0, 2D_0 - D_1^\dagger D_2^{-1} D_1 \succeq 0, \left(I - D_2 D_2^{-1}\right) D_1 = 0 \text{ and} \\ D \succeq 0 \Leftrightarrow D_0 \succeq 0, 2D_2 - D_1 D_0^{-1} D_1^\dagger \succeq 0, \left(I - D_0 D_0^{-1}\right) D_1^\dagger = 0. \end{aligned} \tag{4.23}$$

We shall refer to both conditions as the *decoherence-diffusion trade-off*, the consequences of which we explore in detail in Chapter 10. Although we refer to this as a trade-off between decoherence and diffusion, this terminology is only strictly appropriate for pure decoherence processes. More generally, it is a trade-off between Lindblad couplings and diffusion coefficients.

The decoherence-diffusion trade-off is precisely the reason that consistent classical-quantum dynamics exist. It necessitates stochastic unpredictability in the classical degree of freedom arising from D_2 and decoherence of quantum superpositions via D_0, both effects of which cannot be made small when the quantum system back-reacts on the classical one via D_1 due to

Equation (4.23). The trade-off between maintaining superpositions and classical uncertainty is what allows one to evade the no-go theorems of Feynman and others regarding the consistency of hybrid dynamics [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. The essence of arguments against quantum-classical interactions is that they would prohibit superpositions of quantum systems which source a classical field. Since different classical fields are perfectly distinguishable in principle, if the classical field is in a distinct state for each quantum state in the superposition, the classical field could always be used to determine the quantum system's state, causing it to decohere instantly. By satisfying the trade-off, the quantum system preserves coherence because diffusion of the classical degrees of freedom means that the state of the classical field does not determine the state of the quantum system.

In classical autonomous dynamics, the only time continuous autonomous process, in a sense that

$$\lim_{t\downarrow s}\frac{1}{t-s}\int_{|z-z'|>\delta}\mathrm{d}z\ p(t,z|s,z')=0,\quad\forall\delta>0,\tag{4.24}$$

is given by a diffusion process with dynamics described by the Fokker-Plank equation [73]. We can obtain a probability distribution for the classical degrees of freedom by taking the trace over the quantum system. To be precise, consider solving the CQ evolution; we can write $\varrho(z,t)=p(z,t)\bar{\varrho}(z,t)$ where $\sigma(z,t)$ is a normalized quantum state. The equation of motion for the classical degrees of freedom, obtained by tracing out the quantum system, is

$$\frac{\partial p(z,t)}{\partial t}=\sum_{n=1}^{\infty}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(\bar{D}_{n,i_1\dots i_n}(z,t)p(z,t)\right),\tag{4.25}$$

where we define $\bar{D}_{n,i_1\dots i_n}(z,t)=\sum_{\mu\nu}D_{n,i_1\dots i_n}(z)^{\mu\nu}(z)\mathrm{Tr}\left[L_\nu^\dagger L_\mu\sigma(z,t)\right]$. It is worth emphasizing, however, that the resulting dynamics, although arising from autonomous dynamics linear in the CQ state, are non-Markovian and so cannot be reduced to the models of [57]. Non-Markovianity is encoded by the fact that the $\bar{D}_n$ coefficients depend on the quantum state $\sigma(z,t)$, which can act as a memory for the full CQ dynamics. If we have an infinite number of terms in Equation (4.25), then the dynamics will contain finite-sized jumps with non-zero probability. More precisely, in the case with infinite moments in the classical-quantum dynamics, there will always exist states for which there are finite-sized jumps with finite probability in the classical phase space, take $\rho(z,t)=p(z,t)\mathbb{I}$ for example. In Chapter 5, we construct an explicit algorithm to study continuous classical-quantum dynamics in terms of continuous trajectories in phase space.

4.4 Discussion

In this chapter, we have introduced the most general form of autonomous classical-quantum master equations that are continuous in the classical phase space, given by Equation (4.13). Any other master equation necessarily causes discrete, finite-sized jumps in phase space, or else it violates the conservation of probabilities or fails to be completely positive on the quantum system. We achieved this by introducing a classical-quantum Cauchy-Schwarz inequality, which enabled us to derive various inequalities which the moments of the transition amplitude must satisfy.

The master equation we find generalizes the Fokker-Plank equation of open classical systems and the Lindblad equation of open quantum systems to the case of coupled classical-quantum systems, providing a reference for the study of hybrid classical-quantum dynamics in the future. Previously only examples of continuous classical coupling were given [57, 60], but its general form was unknown [74]. The continuous master equation will play a central role in many chapters of the thesis; in Chapter 5, we show that it can be unraveled by a set of coupled stochastic differential equations, in analogy with the unraveling of the Fokker-Plank and Lindblad equations by stochastic differential equations, which we show leads to an improved semi-classical formalism which is consistent even in the presence of large quantum fluctuations; in Chapters 6 and 8 we show that one can arrive at a path integral representation for the continuous master equation, which helps understand whether CQ dynamics can retain space-time symmetries such as diffeomorphism invariance.

Indeed, in the context of classical-quantum theories of gravity [60, 28, 49, 3], if the space-time metric undergoes continuous dynamics, then one expects a version of Equation (4.13) to generate it. To this end, the field-theoretic version of Equation (4.13) is given in [4, 3] and Appendix G. Constructing consistent theories of CQ general relativity then amounts to an appropriate choice of Lindblad operators and couplings D_0, D_1, D_2; realizations of the master equation for the case of gravity have been given in [60, 28, 8], which we study in the later parts of the thesis. The fact that the amount of classical diffusion D_2 is lower bounded by the decoherence coupling D_0 leads to potential experimental signatures of a classical gravitational field [4], which we discuss in detail in Chapter 10.

Chapter 5

Unraveling classical-quantum dynamics

In analogy with the unraveling of quantum Lindblad equations [99, 101, 100], and the unraveling of classical master equations [73, 82], hybrid master equations have been unraveled by stochastic differential equations. This is true for both the continuous master equations [74], and those which contain jumps [10]. In the literature, unravelings of continuous master equations have only been derived in specific examples, usually surrounding the context of continuous measurements [52, 74, 49]. In this chapter, we find the general form of continuous classical-quantum unraveling using the continuous master equation found in the previous chapter, which provides a concrete algorithm for simulating continuous classical quantum dynamics.

The resulting equations of motion are natural generalizations of the standard semi-classical equations of motion. However, since the resulting dynamics are linear in the combined classical-quantum state, it does not lead to the pathologies which usually follow evolution laws based on expectation values. In particular, the evolution laws we present account for correlations between the classical and quantum systems, which resolves issues associated with other semi-classical approaches. It is necessary to include stochasticity in the classical part of the dynamics and a Lindbladian coupling in the quantum part of the system in order to preserve the linearity of the dynamics and the positivity of the classical-quantum state. Nonetheless, despite a breakdown of predictability in the classical degrees of freedom, we find the quantum state evolves deterministically conditioned on the classical trajectory, the trade-off between the Lindbladian coupling and the diffusion coefficient found in Chapter 4 is saturated.

In the context of gravity, the solution to the dynamics is described by a probability distribution over continuous 4-geometries and an associated quantum state, which could potentially be used to simulate the *dynamics* of classical gravity interacting with quantum matter, going beyond the standard semi-classical regime. However, we do not study this in detail in this chapter.

We also prove that resulting dynamics completely parameterizes continuous measurement and (Markovian) feedback procedures, showing the equivalence of the continuous measurement [52, 74, 49] and hybrid [60, 28] approaches to continuous classical-quantum coupling, which are often treated as being mathematically distinct in discussions of classical-quantum gravity [118]. Because of the equivalence between continuous CQ dynamics, in Section 5.5.2, we propose that CQ dynamics gives rise to an effective theory of quantum measurement. One first identifies a classical system that acts as a measurement device and a quantum system to be measured and studies the effective CQ dynamics of the interacting classical and quantum systems. The details of the measurement apparatus and coupling are then encoded in a handful of phenomenological parameters, i.e., those appearing in Equation (4.13), which govern the strength of the back-reaction of the quantum system and the strength of the measurement. Within this prescription, there is no need to discuss a measurement outcome, nor when or how a measurement occurs; there are only effective dynamics of continuously interacting classical and quantum systems.

This chapter is based on [6], which is work done in collaboration with Isaac Layton and Jonathan Oppenheim.

5.1 The standard semi-classical equations

For the case of gravity, we reviewed the semi-classical approach to incorporate the back-reaction of a quantum system on a classical system in Chapter 3. There, the standard approach to define back-reaction is via the semi-classical Einstein equations, which source the Einstein tensor $G_{\mu\nu}$ by the expectation value of the stress-energy tensor $T_{\mu\nu}$ [19, 20]

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle, \tag{5.1}$$

with the quantum state $|\psi\rangle$ at time t determined by a Hamiltonian H that depends on classical degrees of freedom z (here taken to be the gravitational metric)

$$\frac{d|\psi\rangle}{dt} = -iH(z)|\psi\rangle. \tag{5.2}$$

The dynamics represented by Equations (5.2) and (5.1) may be understood as a special case of a more general approach taken to describe back-reaction, which we shall refer to as the *standard semi-classical approach*. We take the standard semi-classical equations to be the system of equations where the quantum evolution is governed by a phase-space dependent Hamiltonian, as in Equation (5.2), and the classical evolution undergoes a back-reaction force determined by an expectation value of the quantum state,

$$dz = \{H_c, z\} + \langle\{H_I(z), z\}\rangle_{|\psi\rangle}, \tag{5.3}$$

where H_c is a purely classical Hamiltonian and H_I is a quantum interaction Hamiltonian which determines the strength of the back-reaction.

The standard semi-classical equations are often studied in molecular dynamics and are called the mean-field equations [119, 120]. As with gravity, they can be useful in some regimes where the quantum fluctuations are small, and no correlation is generated between the classical and quantum systems.

As outlined in Chapter 3, the problem with the standard semi-classical equations, away from small fluctuations, is that they fail to properly account for correlations between the classical and quantum degrees of freedom. Consequently, they give rise to non-linear evolution on the density matrix, which leads to violations of the standard principles of quantum theory [48, 18, 49, 50, 51] when the quantum fluctuations are large and correlations are dynamically generated between the classical and quantum systems. In particular, when quantum fluctuations are significant, i.e., for the case of Schrodinger cat states of massive bodies [29, 30], or vacuum fluctuations during inflation [31, 32, 33, 34], the semi-classical equations do not yield consistent dynamics, and an alternate effective theory of back-reaction is required. The standard semi-classical equations share some similarities with the Born-Oppenheimer approximation in that they do not allow correlations to build up between the classical and quantum systems.

In this chapter, we will make progress towards this goal using the general form of autonomous classical-quantum dynamics found in the previous chapter. In particular, we show that CQ dynamics can be used to give rise to a consistent semi-classical formalism, which upholds the standard principles of quantum theory and leads to consistent dynamics when any quantum state is considered. Note, though the assumption of autonomy is reasonable for any theory viewed as fundamental, it may not hold in an effective theory, where the dynamics can generally be non-Markovian. Moreover, when the classical system arises as a limit of a quantum one, there may not always be a description of it in terms of continuous classical trajectories. A more complete semi-classical picture should be able to account for these issues in more detail. We discuss this in more detail in Section 5.7 when we conclude the chapter.

5.2 Classical-quantum trajectories

We start by defining the primary objects of our semi-classical description. The standard semi-classical equations describe deterministic classical-quantum trajectories. However, more generally, one can consider dynamics that generate probability distributions over classical trajectories in phase space and quantum trajectories in Hilbert space. We saw in Chapter 4 that this was necessary if the dynamics are autonomous due to the requirement of a non-zero diffusion coefficient D_2 in the master equation of Equation (5.5).

We assume that at all times, the semi-classical system is fully characterized by the pair (z, ρ). Here z denotes the classical degrees of freedom ρ denotes a quantum state. The entire evolution of the semi-classical system is thus characterized by a classical-quantum trajectory, which we denote by $\{(Z_t, \rho_t)\}_{t>0}$. Each of the trajectories occurs with some probability. Both Z_t and ρ_t can be understood as random variables, taking values in the classical phase space and the space of density operators, respectively. These are a natural combination of the trajectories considered in Chapter 2, where we saw the Fokker-Plank equation was unraveled by stochastic differential equations governing the evolution of the Z_t, and the Lindblad equation could be unraveled by stochastic differential equations governing the evolution of quantum trajectories $\rho_t = |\psi_t\rangle\langle\psi_t|$.

We can relate the classical-quantum trajectories to the CQ state by defining

$$\varrho(z,t) = \mathbb{E}[\delta(z - Z_t)\rho_t], \tag{5.4}$$

where the expectation value is over all possible values of the underlying noise process. Equation (5.4) is easily seen to be a normalized CQ state. In particular, taking the trace of the quantum system, we arrive at the probability distribution over classical variables $p(z,t)$ is described by $p(z,t) = \mathbb{E}[\delta(z - Z_t)]$ while integrating out the classical degrees of freedom z, we arrive at $\mathbb{E}[\rho_t]$, which is a normalized quantum state. Equation (5.4) can be understood as weighting the quantum state achieved for a particular value of the noise process by the probability it accompanies a specific classical value and then averaging over all possible values of the noise process.

5.3 Unraveling continuous classical-quantum dynamics

We now find the general classical-quantum unraveling for the continuous autonomous classical-quantum master equation found in the previous chapter

$$\begin{aligned}\frac{\partial \varrho(z,t)}{\partial t} &= \sum_{n=1}^{n=2} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \dots \partial z_{i_n}}\right) \left(D^{00}_{n,i_1\dots i_n} \varrho(z,t)\right) - \frac{\partial}{\partial z_i}\left(D^{0\alpha}_{1,i} \varrho(z,t) L^\dagger_\alpha\right) - \frac{\partial}{\partial z_i}\left(D^{\alpha 0}_{1,i} L_\alpha \varrho(z,t)\right) \\ &\quad - i[H(z), \varrho(z,t)] + D^{\alpha\beta}_0(z) L_\alpha \varrho(z) L^\dagger_\beta - \frac{1}{2} D^{\alpha\beta}_0 \{L^\dagger_\beta L_\alpha, \varrho(z)\}_+,\end{aligned} \tag{5.5}$$

recalling that the conditions for complete positivity are $2D^{00}_2 \succeq D_1 D_0^{-1} D_1^\dagger$ and $(\mathbb{I} - D_0 D_0^{-1}) D_1 = 0$, where, D_0^{-1} is the generalized inverse of the matrix $D^{\alpha\beta}_0$, D_1 is a matrix in both α, i indices with entries $D^{0\alpha}_{1,i}$ and D^{00}_2 is a matrix in i,j with entries $D^{00}_{2,ij}$.

Explicitly, the dynamics of Equation (5.5) is unraveled by the pair of stochastic differential equations

$$dZ_{t,i} = D_{1,i}(Z_t)dt + \langle D^{\alpha 0}_{1,i}(Z_t) L_\alpha + D^{0\alpha}_{1,i}(Z_t) L^\dagger_\alpha \rangle dt + \sigma_{ij}(Z_t) dW_j, \tag{5.6}$$

$$\begin{aligned}d\rho_t &= -i[H(Z_t), \rho_t]dt + D^{\alpha\beta}_0(Z_t) L_\alpha \rho L^\dagger_\beta dt - \frac{1}{2} D^{\alpha\beta}_0(Z_t)\{L^\dagger_\beta L_\alpha, \rho_t\}_+ dt \\ &\quad + D^{\alpha 0}_{1,j} \sigma^{-1}_{ji}(Z_t)(L_\alpha - \langle L_\alpha \rangle)\rho_t dW_i + D^{0\alpha}_{1,j} \sigma^{-1}_{ji}(Z_t)\rho_t (L^\dagger_\alpha - \langle L^\dagger_\alpha \rangle) dW_i,\end{aligned} \tag{5.7}$$

where σ_{ij} is a positive semi-definite matrix which is defined in terms of the diffusion matrix of Equation (5.5) by $D^{00}_{2,ij} = \frac{1}{2}\sigma_{ik}\sigma^T_{kj}$. In Equation's (5.6) and (5.7), σ^{-1} denotes the generalized inverse of σ, and W_i is the standard multivariate Wiener process satisfying the Ito rules

$$dW_i dW_j = \delta_{ij} dt, \ dW_i dt = 0. \tag{5.8}$$

Although the expectation values appear in the equations of motion for both Z_t and ρ_t when averaged over the noise process, they give rise to a linear master equation on the CQ state; the fact that the dynamics give rise to non-linear evolution on ρ_t is not inconsistent with quantum theory. Indeed, the same is true for unravelings of the GKSL equation [48] and also for quantum measurements. For example, the conditioning of the quantum state after a measurement outcome (which we label by the classical variable i) is obtained is a non-linear map, $\rho \to \frac{\rho_i}{\mathrm{Tr}[\rho_i]}$, but averaging over all possible outcomes is a linear map on the space of density matrices. In this case, the non-linearity arises purely because of the normalization of the quantum state. One can show the same is true for the expectation values appearing in the quantum part of the evolution. In [6], it is shown that one can write the quantum evolution in Equation (5.7) by considering un-normalized quantum states.

Conversely, the expectation value appearing in the classical equation of motion is not due to normalization. Instead, it represents that the drift of the classical system is generically unknown, even for pure quantum states, and the noise is required precisely so that one cannot determine the drift exactly while the quantum state has coherence. In particular, the equation for dZ_t should be interpreted as an equation that governs the *statistics* of dZ_t, not that the expectation value actually sources the drift. Again, this feature is also true for measurements on subsystems in standard quantum theory. Pure quantum states only encode the statistics of measurement outcomes; one can only say that the outcome i will occur with certainty if the quantum state is in an eigenvalue of the operator which is being measured. We shall show that the only case where the drift can be determined exactly in Equation (5.6), i.e., the $\sigma \to 0$ limit, is when the quantum states are in eigenstates of the drift operator.

Though the dynamics of Equation (5.7) is stochastic, an observer with knowledge of the entire classical trajectory up-to time t, $\{Z_t\}_{t\leq t}$ can deduce ρ_t from changes in Z_t. This property is shown explicitly in [6], where the quantum state evolution is presented without any noise terms, but instead in terms of changes to the classical degree of freedom dZ_i. This form of equation can be found by inverting Equation (5.6) to find dW_i in terms of dZ_i, and substituting into Equation (5.6). We also give an explicit example of this in Section 5.5.2.

On first inspection, the master Equation of (5.5), and the unraveling in Equation's (5.6) and (5.7) lead to a loss of quantum information due to the presence of the decoherence terms with coefficients D_0. However, for an initially pure quantum state, we find that when the decoherence-diffusion trade-off is saturated, which we take to mean that $D_0 = D_1(\sigma\sigma^T)^{-1}D_1^\dagger$, the quantum state ρ_t remains pure, i.e., $\mathrm{Tr}\left[(\rho_t + d\rho_t)^2\right] \to 1$ and there is no loss of quantum information [6].

We can see this more explicitly via the pure state unraveling

$$\begin{aligned} d|\psi\rangle_t &= -iH(Z_t)|\psi\rangle_t dt + D_{1,j}^{\alpha 0}\sigma_{ij}^{-1}(Z_t)(L_\alpha - \langle L_\alpha\rangle)|\psi\rangle_t dW_i \\ &- \frac{1}{2}D_0^{\alpha\beta}(Z_t)(L_\beta^\dagger - \langle L_\beta^\dagger\rangle)(L_\alpha - \langle L_\alpha\rangle)|\psi\rangle_t dt + \frac{1}{2}D_0^{\alpha\beta}(\langle L_\beta^\dagger\rangle L_\alpha - \langle L_\alpha\rangle L_\beta^\dagger)|\psi\rangle_t dt, \end{aligned} \tag{5.9}$$

which, using the standard Ito rules

$$d\rho = d|\psi\rangle\langle\psi| + |\psi\rangle d\langle\psi| + d|\psi\rangle d\langle\psi|, \tag{5.10}$$

is equivalent to Equation (5.7) when the decoherence diffusion trade-off is saturated. Thus, despite the loss of predictability in the classical degrees of freedom, classical-quantum theories saturating the trade-off can preserve the purity of the quantum states when conditioned on the classical trajectory; this is one important difference between classical-quantum unravelings and quantum unravelings of the GKSL equation.

In hindsight, the fact that one can preserve the purity of the quantum system conditioned on the classical trajectory is perhaps an expected feature of hybrid dynamics. In Section 5.5.1, we show that the dynamics of Equations (5.6) and (5.7) have an equivalent description in terms of a generalization of the procedure given in [96], where an auxiliary classical degree of freedom is sourced by the measurement signal of a continuous measurement, and we also allow for an auxiliary variable to have its own purely classical dynamics; in this sense, Equations (5.6) and (5.7) form a complete parameterization for continuous measurement-based classical-quantum control where one also allows for continuous control on the classical system, and are similar to measurement based feedback equations familiar in quantum control [121, 122].

The fact that the purity of quantum states can be maintained when conditioned on the classical trajectory is then an expected feature. In continuous quantum measurement, weak measurements of a pure quantum state are concatenated in time, with the dynamics of both the quantum system and the measurement signal forming a pair of stochastic differential equations similar to that of Equation (5.6) and (5.9). While, in general, the quantum state evolution will be random due to the probabilistic nature of quantum measurements, conditioned on the series of measurement outcomes, the state will remain pure. Similarly, in open quantum systems, pure states become mixed as they become entangled (correlated) with their environment, which is, practically speaking, inaccessible. However, with knowledge of both the system and the environment dynamics, the quantum state of the entire system will remain pure; here, the classical degree of freedom behaves as the environment in restoring the purity of the state.

5.3.1 Deriving the unraveling

To arrive at the unraveling of Equation's (5.6) and (5.7) we start by noting that the dynamics of Z_t and ρ_t induce the following evolution on the CQ state $\varrho(z,t) = \mathbb{E}[\delta(Z_t - z)\rho_t]$,

$$d\varrho(z,t) = \frac{\partial \varrho(z,t)}{\partial t} dt = \mathbb{E}[d(\delta(Z_t - z)\rho_t)]. \tag{5.11}$$

One must therefore calculate

$$\mathbb{E}[d(\delta(Z_t - z)\rho_t)] = \mathbb{E}[d\delta(Z_t - z)\rho_t + \delta(Z_t - z)d\rho_t + d\delta(Z_t - z)d\rho_t]. \tag{5.12}$$

For clarity, we shall go through each term individually. Combining Ito's lemma with the equation of motion for the classical variable in (5.6), the first term in Equation (5.12) reads

$$\begin{aligned}\mathbb{E}[d\delta(Z_t - z)\rho_t] =& \mathbb{E}[\frac{\partial}{\partial Z_i}[\delta(Z_t - z)]\rho_t(D^{00}_{1,i}(Z_t,t) + \langle D^{\alpha 0}_{1,i}(Z_t,t)L_\alpha + D^{0\alpha}_{1,i}(Z_t,t)L^\dagger_\alpha\rangle)]dt \\ &+ \mathbb{E}[\frac{1}{2}\frac{\partial^2}{\partial Z_i \partial Z_j}[\delta(Z_t - z)]\rho_t \sigma_{ik}(Z_t,t)\sigma^T_{kj}(Z_t,t)]dt.\end{aligned} \tag{5.13}$$

We can use some well-known facts about the delta functional to simplify Equation (5.13). Using the two identities $\partial_{Z_i}\delta(Z-z) = -\partial_{z_i}\delta(Z-z)$ and $f(Z)\delta(Z-z) = f(z)\delta(Z-z)$ for any function f, the right hand side of Equation (5.13) becomes

$$-\frac{\partial}{\partial z_i}\mathbb{E}[\delta(Z_t - z)\rho_t(D^{00}_{1,i}(z) + \langle D^{\alpha 0}_{1,i}(z)L_\alpha + D^{0\alpha}_{1,i}(z)L^\dagger_\alpha\rangle)]dt + \frac{\partial^2}{\partial z_i \partial z_j}\mathbb{E}[\delta(Z_t - z)\rho_t D^{00}_{2,ij}(z)]dt. \tag{5.14}$$

Using the definition of the CQ state in Equation (5.4), we arrive at

$$\mathbb{E}[d\delta(Z_t - z)\rho_t] = (-\frac{\partial}{\partial z_i}[\varrho(z)D^{00}_{1,i}(z)] - \frac{\partial}{\partial z_i}[\langle D^{\alpha 0}_{1,i}L_\alpha\rangle\varrho] - \frac{\partial}{\partial z_i}[\varrho(z)\langle D^{0\alpha}_{1,i}L^\dagger_\alpha\rangle] + \frac{\partial^2}{\partial z_i \partial z_j}[\varrho(z)D^{00}_{2,ij}(z)])dt \tag{5.15}$$

The second term in Equation (5.12) is simpler to calculate and gives the pure quantum evolution terms

$$\mathbb{E}[\delta(Z_t - z)d\rho_t] = i[H(Z_t), \rho_t]dt + D_0^{\alpha\beta}(Z_t)L_\alpha \rho L_\beta^\dagger dt - \frac{1}{2}D_0^{\alpha\beta}(Z_t)\{L_\beta^\dagger L_\alpha, \rho_t\}_+ dt \tag{5.16}$$

For the final term in Equation (5.12), only the second-order terms $dW^2 = dt$ are relevant. We find

$$\mathbb{E}[d\delta(Z_t - z)d\rho_t] = \mathbb{E}[\frac{\partial}{\partial Z_i}[\delta(Z_t - z)]\rho_t D_{1,i}^{\alpha 0}(Z_t, t)(L_\alpha - \langle L_\alpha \rangle) + D_{1,i}^{0\alpha}(Z_t, t)(L_\alpha^\dagger - \langle L_\alpha^\dagger \rangle))]dt, \tag{5.17}$$

and again using the standard properties of the delta function

$$\mathbb{E}[d\delta(Z_t - z)d\rho_t] = -\frac{\partial}{\partial z_i}[D_{1,i}^{\alpha 0}(z)(L_\alpha - \langle L_\alpha \rangle)\varrho(z) + D_{1,i}^{0\alpha}\varrho(z)(L_\alpha^\dagger - \langle L_\alpha^\dagger \rangle)]dt. \tag{5.18}$$

Summing the three contributions gives the equation of motion for $\partial_t \mathbb{E}[\delta(Z_t - z)\rho_t] = \partial_t \varrho(z)$ to be that of the continuous master equation in Equation (5.5).

5.4 Hamiltonian unravelings

Having presented the general dynamics for Z_t and ρ_t, we now turn to an important example of the dynamics where the back-reaction is generated by a Hamiltonian and the decoherence-diffusion trade-off is saturated. Specifically, we take the pure classical part of the drift determined by D_1^{00} to be generated by a classical Hamiltonian $H_C(z)$. For the interaction term, one can use the freedom in the choice of Lindblad operators to pick $L_\alpha = \{Z_\alpha, H_I\}$, where $H_I(z)$ is an interaction Hamiltonian, and then set $D_{1,i}^{0\alpha} = \frac{1}{2}\delta_i^\alpha$. In this case, we arrive at a set of equations that we called "the healed semi-classical equations" in [6]. The classical dynamics are given by

$$dZ_{t,i} = \{Z_{t,i}, H_C(Z_t)\}dt + \langle \{Z_{t,i}, H_I(Z_t)\}\rangle dt + \sigma_{ij}(Z_t)dW_j, \tag{5.19}$$

while the quantum evolution takes the form

$$\begin{aligned} d|\psi\rangle_t &= -i(H_0 + H_I(Z_t))|\psi\rangle_t dt \\ &+ \frac{1}{2}\sigma_{ij}^{-1}(\{Z_j, H_I\} - \langle \{Z_j, H_I\}\rangle)|\psi\rangle_t dW_i \\ &- \frac{1}{8}\sigma_{ij}^{-1}\sigma_{ik}^{-1}(\{Z_j, H_I\} - \langle \{Z_j, H_I\}\rangle)(\{Z_k, H_I\} - \langle \{Z_k, H_I\}\rangle)|\psi\rangle_t dt, \end{aligned} \tag{5.20}$$

where in the above σ is any real matrix satisfying $(\delta^i_j - (\sigma\sigma^{-1})^i_j)\delta^\alpha_i = 0$, which arises due to the complete positivity condition $(\mathbb{I} - D_2 D_2^{-1})D_1 = 0$. For a given initial quantum state $|\psi_i\rangle$ and classical state z_i, these coupled stochastic differential equations determine the probability of ending up in any final pair of states z_f and $|\psi_f\rangle$. An early example of this dynamics for the special case of linear coupling between two particles, one classical and one quantum, was described in [75]. Many examples and simulations can be found in our paper [6].

We now briefly discuss features of the healed semi-classical equations. The quantum state evolves under Equation (5.20), with the first term responsible for pure unitary evolution and the final two terms describing continuous stochastic evolution that tends to drive the quantum state towards a joint eigenstate of the operators $\{Z_i, H_I\}$ [123]; Equation (5.20) is mathematically equivalent to a continuous measurement of the operator $\{Z_i, H_I\}$ [121, 122], and exponentially suppress trajectories away from the minimum value of

$$\frac{1}{8}\sigma^{-1}_{ij}\sigma^{-1}_{ik}(\{Z_j, H_I\} - \langle\{Z_j, H_I\}\rangle)(\{Z_k, H_I\} - \langle\{Z_k, H_I\}\rangle)|\psi\rangle_t, \tag{5.21}$$

which is when the quantum state is in an eigenstate of $\{Z_i, H_I\}$ [123]. Note, this does not necessarily mean that Equation (5.20) has a fixed point since the pure quantum evolution does not always share the same eigenvectors as $\{Z_i, H_I\}$. In this case, there is a battle between the pure Hamiltonian evolution that creates coherence and the remaining evolution that destroys it.[1]

On the other hand, the classical evolution in Equation (5.19) consists of a term describing a purely classical drift, another which describes the quantum back-reaction on the classical system, and a diffusion term. Despite the appearance of an expectation value in the back-reaction term, the dynamics give statistics for Z_t as if the classical system were diffusing around a force given by a random eigenstate of the operators $\{Z_i, H_I\}$. In particular, since $\{Z_i, H_I\}$ is Hermitian, we can decompose the quantum state in terms of the eigenvectors of $\{Z_i, H_I\} = \sum_j h^j_i(Z)|h^j_i\rangle$ and write $|\psi\rangle = \sum_j \sqrt{p_j}|h^j_i\rangle$. We find that

$$\langle\{Z_i, H_I\}\rangle = \sum_j p_j h^j_i, \tag{5.22}$$

which can be interpreted as sources of the force by an eigenvector h^j_i with probability p_j.

[1] We thank Isaac Layton for pointing this out.

The only free parameters of the model are encoded in the matrix σ_{ij}, which governs both the rate at which the quantum state evolves to an eigenstate and the rate of diffusion of the classical system. The fact that the two rates are inversely related is a consequence of the decoherence-diffusion trade-off, which ensures that there can be no quantum back-reaction without associated diffusion or decoherence of the quantum system.

The differences between the healed semi-classical and standard semi-classical equations, given by Equations (5.2) and (5.3), are evident. The classical evolution of standard semi-classical Equation (5.3) takes the form of the healed semi-classical Equation (5.19) but without the noise term. Similarly, the quantum evolution of the standard semi-classical Equation (5.2) takes the form of the healed semi-classical Equation (5.20) but without both the stochastic diffusion term and the term which drives quantum states to an eigenvector of the drift operator $\{Z_i, H_I\}$. Taking the limit of deterministic classical evolution in Equation (5.19), $\sigma \to 0$ means that the quantum states are very quickly driven to eigenstates due to the appearance of σ^{-1} in Equation (5.20). Similarly, sending $\sigma^{-1} \to 0$ to make the quantum evolution unitary results in wildly stochastic classical dynamics. The decoherence diffusion trade-off prevents the recovery of both equations in any limit.

The standard semi-classical equations are thus inconsistent if applied to all states. Using Equations (5.19) and (5.20), we can find a prescription to test the validity of the standard semi-classical equations for any given quantum state $|\psi\rangle_t$. Firstly, we note that the low noise limit $\sigma \to 0$ must be taken in Equation (5.19) to recover the standard semi-classical equation for the classical degrees of freedom. For the quantum state $|\psi\rangle_t$ to then be also effectively described by unitary evolution, as in the standard semi-classical approach in Equation (5.2), the quantum state must then generally satisfy

$$\begin{aligned}&+\frac{1}{2}\sigma_{ij}^{-1}(\{Z_j, H_I\} - \langle\{Z_j, H_I\}\rangle)|\psi\rangle_t dW_i\\&-\frac{1}{8}\sigma_{ij}^{-1}\sigma_{ik}^{-1}(\{Z_j, H_I\} - \langle\{Z_j, H_I\}\rangle)(\{Z_k, H_I\} - \langle\{Z_k, H_I\}\rangle)|\psi\rangle_t dt \approx 0.\end{aligned} \tag{5.23}$$

More precisely, Equation (5.23) should be negligible in comparison to the pure Hamiltonian evolution of the quantum state, so the valid quantum states will be approximate eigenstates of $\{Z_i, H_I\}$, which minimize (5.23). It is worth mentioning that in the Newtonian limit of semi-classical gravity discussed in Chapter 3, the interaction is dominated by the mass density $\frac{\partial H_I}{\partial \Phi} = m(x)$ and we see that the standard semi-classical equations are valid only when the quantum state is in an approximate eigenstate of the mass density operator, which excludes macroscopic superpositions, as well as states which are spatially entangled: essentially the quantum state of matter must be approximately classical [24, 25, 31].

An alternative viewpoint is that when the low noise $\sigma \to 0$ limit is taken, the quantum dynamics rapidly cause the quantum state to evolve to the eigenstate $\{Z_i, H_I\}$, with the probabilities determined by the Born rule [123]. The classical evolution is then well approximated by conditioning on eigenstates of the quantum state and then evolving the classical degree of freedom using the eigenstate. The classical system is probabilistic in this limit, but only due to the probability distribution over the decohered quantum eigenstates. In practice, the semi-classical Einstein equations are often used this way; for example, when considering statistical mixtures or when describing vacuum fluctuations during inflation where quantum states decohere on super-horizon scales and give rise to a classical probability distribution over space-time perturbations. This use is justified as a limiting case of Equations (5.19) and (5.20) when the classical noise is small. The advantage of Equations (5.19) and (5.20) is that they can more generally be used to understand what happens to the dynamics of the classical system when the quantum state still has coherence, giving rise to a consistent semi-classical theory even in the presence of large quantum fluctuations.

5.5 Comparison to measurement and feedback

In this section, we compare our result to previous methods of generating consistent classical-quantum dynamics using continuous measurement and feedback approaches [75, 49, 52]. In these approaches, the classical degree of freedom is sourced by the outcomes of a continuous measurement. By construction, such approaches are completely positive and lead to consistent coupling between classical and quantum degrees of freedom. The stochasticity of the dynamics is due to the continuous measurement, and the non-linearity is due to the state update rule, meaning the dynamics of [75, 49, 52] take a similar form to Equations (5.19) and (5.20). However, it is worth noting some differences between the previous approaches based on continuous measurement and the one we have presented in this Chapter. Firstly, the dynamics we present allow for the classical degrees of freedom to be independent of the quantum degrees of freedom and have their own dynamics, described via the purely classical evolution term $D^{00}_{1,i}$. This allows us to apply the dynamics to more complex CQ scenarios, for example, when the Hamiltonian $H(z)$ is non-linear in z; this is necessary to consider semi-classical dynamics of gravity beyond the weak field limit. As a result, the dynamics, while autonomous on both the classical and quantum systems, can be non-Markovian on the classical and quantum systems alone. It, therefore, does not always reduce to pure Lindbladian evolution on the quantum system, such as in [49, 124, 52].

Secondly, we have taken the dynamics on the phase space degrees of freedom to be continuous. In the measurement and feedback approaches of [49, 52], the classical degrees of freedom evolve discontinuously because the classical coordinate is directly sourced by the outcome J_i of a continuous measurement which is a discontinuous stochastic random variable. To obtain continuous classical degrees of freedom, one can instead source the conjugate momenta of the canonical coordinates via the measurement signal $J_i dt$. This approach is taken in [75], which leads to a special case of our dynamics.

We now show that all continuous CQ dynamics have an equivalent description in terms of a generalization of the procedure given in [75], where an auxiliary classical degree of freedom is sourced by the measurement signal of a continuous measurement, and we also allow for an auxiliary variable to have its own purely classical dynamics. In other words, Equation's (5.6) and (5.7) form a complete parameterization for continuous measurement-based classical-quantum control where one also allows for continuous control on the classical system and are similar to measurement based feedback equations familiar in quantum control [121, 122]. It would be interesting to find a complete parameterization of the dynamics in the discontinuous case, which we do not consider here.

5.5.1 Continuous classical-quantum dynamics as continuous measurement

To show the equivalence between classical-quantum unravelings and continuous measurement, we will consider a continuous measurement described by a series of generalized measurements, also known as positive operator valued measures (POVMs). These will be described by the Kraus operators $\{\Omega_J\}$ performed in the interval $[t + dt)$. The outcome of the measurement we label by $J_{t,k}$, and we shall assume that this drives the auxiliary classical variable through a force term $dZ_{t,k} = D^{00}_{t,k} + J_{t,k}dt$, where we also allow for purely classical dynamics $D^{00}_{t,k}$. We could choose the outcome of the continuous measurement to drive the classical system differently, but this will be sufficient for our purposes.

We consider a generalized case of [96] and here let the measurement $\{\Omega_J\}$ at time t be explicitly Z_t dependent, i.e., we allow for the measurement to depend on the classical trajectory, which we write $\{\Omega_J(Z_t)\}$.

Specifically, we consider the measurement described via

$$\Omega_J(Z_t) = 1 - iH(Z_t)dt - \frac{1}{2}D_0^{\alpha\beta}(Z_t)L^\dagger_\beta L_\alpha dt + \frac{1}{2}L_\alpha D^{\alpha 0}_{1,i}(Z_t)(D_2^{-1})^{ij}(Z_z)J_{t,j}dt. \tag{5.24}$$

The normalization condition on the measurement

$$\int d\mu_0(J)\Omega^\dagger_J\Omega_J = 1, \tag{5.25}$$

is satisfied so long as we pick the measure $d\mu_0(J)$ to be such that

$$\int d\mu_0(J)(J_{t,i}dt) = 0, \int d\mu_0(J)(J_{t,i}dt)(J_{t,j}dt) = (\sigma\sigma^T)_{ij} = 2D_{2,ij}dt, \tag{5.26}$$

and we take $D_0^{\alpha\beta} = \frac{1}{2}D^{0\alpha}_{1,i}D_2^{-1ij}D^{\beta 0}_{1,j}$. Equation (5.26) has the same statistics as a multivariate Gaussian random variable, so we pick $\mu_0(J)$ to be a Gaussian measure with co-variance matrix $(\sigma\sigma^T)_{ij}$.

We can calculate the mean of $J_{t,i}$ in the quantum state ρ via

$$\int d\mu_0(J)\mathrm{Tr}\left[\rho\Omega^\dagger_J\Omega_J\right]J_{t,i} = \langle D^{0\alpha}_{1,i}(Z_t)L^\dagger_\alpha + D^{\alpha 0}_{1,i}(Z_t)L_\alpha\rangle + O(dt^2), \tag{5.27}$$

whilst we can similarly calculate the seconds moments $J_{t,i}J_{t,j}$. These turn out to be independent of the system ρ and hence equivalent to the statistics of a Gaussian random variable with variance $(\sigma\sigma^T)_{ij}$ [96]. As such, the statistics of the measurement outcomes can be described by the stochastic differential equation

$$J_{t,i}dt = dZ_t = \langle D^{0\alpha}_{1,i}(Z_t)L^\dagger_\alpha + D^{\alpha 0}_{1,i}(Z_t)L_\alpha\rangle dt + \sigma_{ij}(Z_t)dW_j. \tag{5.28}$$

Given the measurement outcome $J_{t,k}$, the conditioned density matrix takes the form

$$\rho' = \frac{\Omega_J\rho\Omega^\dagger_J}{Tr[\Omega_J\rho\Omega^\dagger_J]}. \tag{5.29}$$

Denoting $\tilde{L}^j = D_{1,i}^{\alpha 0}(\sigma^{-1})^{ij}(Z_t)$ we find

$$\begin{aligned}\rho' = \rho - i[H(Z_t),\rho]dt - \frac{1}{2}D_0^{\alpha\beta}(Z_t)\{L_\beta^\dagger L_\alpha,\rho\}dt + (\tilde{L}^i - \langle\tilde{L}^i\rangle)\rho J_{t,i}dt + \rho(\tilde{L}^{\dagger i} - \langle\tilde{L}^{\dagger i}\rangle)J_{t,i}dt \\ + \tilde{L}^i\rho\tilde{L}^{\dagger j}J_{t,i}dtJ_{t,j}dt + D_0^{\alpha\beta}(Z_t)\langle L_\beta^\dagger L_\alpha\rangle dt - \langle\tilde{L}^{\dagger j}\tilde{L}^i\rangle\ J_{t,j}dtJ_{t,i}dt \\ + \langle\tilde{L}^i + \tilde{L}^{\dagger i}\rangle\langle\tilde{L}^j + \tilde{L}^{\dagger j}\rangle J_{t,i}dtJ_{t,j}dt.\end{aligned} \tag{5.30}$$

Substituting for $J_{t,i}dt$ in Equation (5.28), one finds the continuous CQ unraveling equation which *saturates* the decoherence diffusion trade-off. To obtain the general unraveling form, one can simply include Lindbladian terms in the quantum state evolution, which we do not explicitly show. In this sense, any CQ master equation which does not saturate the trade-off can be interpreted as a continuous measurement process with inefficient quantum measurements [121].

Within this framework, the decoherence-diffusion trade-off is straightforwardly interpreted as a manifestation of the information-disturbance trade-off. In this case, the strength of the measurement is parameterized by D_0; the weaker the continuous measurement, the less decoherence on the quantum state. However, a weak measurement leads to less information being learned about the quantum system, so there is larger noise in the outcomes of the measurements. Since the measurement outcomes drive the classical system through D_1, this leads to greater diffusion on the classical state. The stronger the coupling is, the larger the coefficient pre-multiplying the noise is, thus explaining the appearance of the coupling strength D_1 in the trade-off.

5.5.2 CQ dynamics as an effective theory of quantum measurement

The fact that the classical-quantum unravelings form a parameterization for continuous measurement-based classical-quantum control means that they can be used to describe an effective theory of measurement dynamics. Though the unraveling equations are formally equivalent to those of continuous measurement, the equations can have a very different physical interpretation. In continuous measurement, the classical degrees of freedom are taken to be measurement outcomes, or signals, which result from applying the Born rule to the measured quantum system. Conversely, in CQ dynamics, the classical degrees of freedom can be considered any classical system. They can be dynamical; for example, they can be taken as a macroscopic system's classical position and momenta.

Hybrid formalisms to model measurement were originally introduced by Sherry and Sudarshan [125, 126, 127], though the resulting dynamics are generally not completely positive on the quantum system. Blanchard and Jadczyk [56, 128] later studied a consistent framework to study a class of completely positive CQ master equations that contain jumps in the classical phase space. Here we are able to provide a complete framework to study measurement by the interaction of continuous classical and quantum systems as an effective theory.

We can construct an effective theory of continuous quantum measurement - extending the ideas presented of [127, 125, 126, 56, 128] - as follows:

1. One identifies a classical system that acts as a measurement device.
2. One identifies the quantum system to be measured.
3. One takes the dynamics of the combined classical-quantum system to be generated by CQ dynamics of Equations (5.6) and (5.7), with the details of the measurement apparatus and coupling encoded in the *phenomenological* parameters D_0, D_1, D_2, which govern the strength of the back-reaction of the quantum system and the strength of the measurement respectively.

Notably, the CQ effective theory consists only of dynamics of interacting classical and quantum degrees of freedom, with all the fine-grained details of the measurement device encoded in a few parameters: D_1, which governs the strength of the back-reaction of the quantum system on the classical, while D_0 and D_2 are related to the strength of the measurement. When the trade-off is saturated, $2D_0 = D_1 D_0^{-1} D_1^{\dagger}$, which describes a noiseless measurement of a quantum system, while when the trade-off is not saturated, the dynamics describe a noisy quantum measurement. When the measurement is noiseless, we can describe the evolution of the quantum state by the pure state unraveling in Equation (5.9).

Within this prescription, there is no need to discuss what a measurement outcome is, nor when or how a measurement occurs; there are only effective dynamics of continuously interacting classical and quantum systems. Moreover, it removes the need to introduce auxiliary quantum probe systems. Instead, all of this information is encoded in the phenomenological parameters.

One way of interpreting this is that when measurements are included, for all practical purposes, we should be viewing quantum theory as an effective theory of interacting classical and quantum systems, with dynamics evolving according to Equations (5.6) and (5.7).

As a simple example, we can consider the case where we have a classical measurement device that we model via a classical position and momenta (x, p) of a pointer. We take the effective dynamics to be generated by

$$dx_t = \frac{p_t}{m_c}dt, \tag{5.31}$$

$$dp_t = D_1\langle L\rangle dt + \sigma d\xi, \tag{5.32}$$

$$d|\psi\rangle_t = -iH|\psi\rangle_t dt + \frac{D_1}{2\sigma}(L - \langle L\rangle)|\psi\rangle_t d\xi - \frac{D_1^2}{8\sigma^2}(L - \langle L\rangle)^2|\psi\rangle_t dt. \tag{5.33}$$

Given the classical trajectory (x_t, p_t) is observed, we can invert $d\xi = \frac{1}{\sigma}[dp_t - D_1\langle L\rangle dt]$ to uniquely determine the evolution of the quantum state, conditioned on the classical measurement apparatus

$$d|\psi\rangle_t = -iH|\psi\rangle_t dt + \frac{D_1}{2\sigma^2}(L - \langle L\rangle)|\psi\rangle_t[dp_c - D_1\langle L\rangle] - \frac{D_1^2}{8\sigma^2}(L - \langle L\rangle)^2|\psi\rangle_t dt. \tag{5.34}$$

The dynamics of the quantum state described by Equation (5.33) is precisely the quantum state dynamics that arise in continuous measurement theory [121, 96, 122]. Specifically, equation (5.33) represents the dynamics for a noiseless continuous measurement of the Hermitian operator L with measurement strength determined by the ratio of $\frac{D_1}{\sigma}$, recalling that $\sigma^2 = 2D_2$. The result is to continuously collapse the state vector into eigenstates of the L operator, where the final result of the measurement can be read off from the classical variables position from Equation (5.34). Taking the strong measurement limit $\frac{D_1}{\sigma} \to \infty$, one collapses the state instantaneously, and it can be shown this leads to the probabilities described by the Born rule [123].

Note that we do not expect the classical degree of freedom representing the measurement device to interact directly with the quantum system. On the contrary, we expect that the measurement device consists of many complex quantum systems which interact with the quantum system being measured. The advantage of the classical-quantum framework is that one arrives at a consistent, effective theory of the measurement procedure, which is phenomenologically accurate, by encoding this fine-grained information in the parameters D_1, σ which are to be experimentally determined: this is what we want in an effective theory of measurement.

It would be interesting to explore this effective theory of measurement in more detail. For example, one could use the path integral approach to classical-quantum dynamics which we discuss in Chapter 6 to better under the role of measurements in relativistic quantum dynamics [129, 130, 131, 132]. Furthermore, given that the classical-quantum unraveling is equivalent to the dynamics of quantum measurement, it would be interesting to see whether one could use it to give an alternative derivation of the Born rule based on the requirement of consistent coupling between classical and quantum systems; this is a subtle problem since the derivation of the classical-quantum dynamics relied on complete positivity, which is almost tantamount to assuming the Born rule to begin with [133].

5.6 Potential applications to gravity

Thus far, we have only considered continuous classical degrees of freedom. In Appendix C, we discuss how one can formally arrive at dynamics for fields – the result is the same but to replace quantities with their local counterparts and derivatives with functional derivatives. Effectively, the spatial coordinate x acts like an index of the Lindblad operators and the matrices D_n.

Our goal in this Chapter is not to reproduce a fully covariant semi-classical description of quantum gravity but rather a framework describing consistent semi-classical dynamics beyond the standard approach. We nonetheless conclude with a brief discussion of the full gravitational context. Classical-quantum dynamics in the full gravitational setting has previously been studied in [28, 3]. The idea, introduced in [28], was to take the classical degrees of freedom to be given by the Riemannian 3 metrics (on some 3 surface Σ) and their conjugate momenta $z = (\gamma_{ij}, \pi^{ij})$. One then considers completely positive dynamics, depending on some lapse N and shift N^i, which maps hybrid states $\varrho(\gamma, \pi, t)$ onto themselves, describing a geometrodynamic [108] picture of classical gravity interacting with quantum matter. One can also consider the lapse and shift and their conjugate momenta part of the phase space, in which case they enter into the Poisson bracket. While this changes nothing in the purely classical case, it offers some advantages in the CQ case. Here, by (formally) studying the unraveling of the dynamics for each realization of the noise process, we now have entire trajectories for each of the variables $(\gamma_{ij}, \pi^{ij}, N, N^i)$ each associated to a quantum state, $\rho(t|\gamma_{ij}, \pi^{ij}, N, N^i)$. This allows us to define a tuple $(g_{\mu\nu}, \rho_{\Sigma_t}(t))$ via the ADM embedding

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2(t,x)dt^2 + \gamma_{ij}(t,x)(N^i(t,x)dt + dx^i)(N^j(t,x)dt + dx^j). \tag{5.35}$$

Equation (5.35) associates a 4-metric and quantum state to each trajectory. The unraveling thus provides a method to study the dynamics of classical gravity interacting with quantum matter.

Taking the pure dynamics to be local and Hamiltonian, in the sense of Equations (5.19), (5.20), in Appendix C we find the dynamics

$$G_{ij} = 8\pi G\langle T_{ij}[g,\pi]\rangle + \sigma_{ij}^{kl}[g,\pi]d\xi_{kl}, \tag{5.36}$$

where $d\xi_{kl}$ is a white noise process. The evolution of the quantum state is given by Equation (5.20) where H_I is the matter Hamiltonian. Constructing a classical-quantum theory with the same degrees of freedom as GR amounts to constructing the hybrid versions of the gravitational constraints, which are the G_{00} and G_{0i} components of the Einstein tensor. In Chapter 7, we will provide the first study of hybrid classical-quantum constraints, while in Chapter 8, we introduce a diffeomorphism covariant and invariant theory of classical-quantum gravity. While we do not construct a complete theory and show it has constraints which are preserved, it serves as a proof of principle that classical-quantum theories may be made diffeomorphism invariant, which may lead to insight into the constraints. We leave whether these constraints can be preserved as a question for future research. Importantly, the decoherence-diffusion trade-off can be used to experimentally test for theories with a fundamentally classical gravitational field since they necessarily lead to diffusion in the gravitational potential and decoherence of masses in spatial superposition (see Chapter 10).

On a related note, the form of the classical evolution equation (5.36) looks similar to a Markovian version of the non-Markovian Einstein-Langevin equation [134, 135, 136, 137, 21]. The Einstein-Langevin equation is the central object of study in stochastic gravity [134, 135, 136, 137, 21] aimed at incorporating higher order corrections to Einstein's equations sourced by the quantum stress-energy tensor. Such corrections were initially motivated by studying the interaction of two linear quantum systems via a path integral approach, integrating out one of the quantum systems and looking at the $\hbar \to 0$ dynamics of the remaining system. The

Einstein-Langevin equation is believed to be valid whenever the dynamics can be approximated by correlation functions which are second order. The dynamics we introduce provide a semi-classical regime that goes beyond this since we have arrived at a consistent semi-classical picture that gives rise to consistent dynamics on any quantum state: this includes quantum states in macroscopic superposition, which are not approximated well by second-order correlation functions and for which the Einstein-Langevin equation fails to be a good approximation [21].

5.7 Discussion

The equations in this chapter parameterize the general form of completely-positive, linear, autonomous, and continuous classical-quantum dynamics in terms of combined classical-quantum trajectories. Given that the initial motivation was to arrive at a healthier theory of semi-classical gravity, it is worth considering when we expect these assumptions to hold. If gravity were to be fundamentally classical, then these assumptions are reasonable: the assumptions of complete-positivity and linearity are necessary for sensible predictions for all initial classical and quantum states; the assumption of autonomous dynamics is reasonable for any theory viewed as fundamental; and the assumption of continuous classical trajectories is necessary for the dynamics to describe probability distributions over space-times. Viewed this way, one expects the field-theoretic versions of Equation (5.6) and (5.9) to provide a template to construct consistent CQ theories of gravity.

The dynamics introduced here should also be useful as an effective theory. While we expect this to be true, it is crucial to note that none of the assumptions need necessarily hold, at least exactly, or for all times. For instance, if one allows for the non-Markovian evolution that generically arises in the study of open quantum systems, we necessarily violate the assumption of autonomous dynamics. In this case, a time-local non-Markovian theory takes the same form as Equation (5.5), but without the requirement for the decoherence-diffusion trade-off to hold for all times, [28]. Alternatively, one may construct dynamics that are not completely positive on all initial classical distributions but completely positive on those permissible by a quantum theory or on some more significant set subset of quantum states than those allowed in semi-classical gravity. While such theories may be useful as effective theories, they are likely incompatible with the assumption of classical trajectories. In this regard, the dynamics we present here are likely valid in a regime of the effective theory where classical trajectories are well-defined.

A more detailed treatment of classical-quantum dynamics may shed light on some of the open problems in semi-classical physics. Of potential interest would be in understanding the role of vacuum fluctuations in cosmology and structure formation. Since we wish to investigate the role that vacuum fluctuations play in density in-homogeneity, this is a regime in which the semi-classical Einstein Equation (5.1) cannot be used. In practice, researchers consider situations in which the density perturbations have decohered [34, 33, 138, 139, 140, 141, 142] so that they can condition on their value and feed this into the Friedman-Robertson-Walker equation governing the expansion of the local space-time patch [143, 144, 145]. As already discussed, this procedure can be understood as the low noise, $\sigma \to 0$, limit of the healed semi-classical equations (5.19) and (5.20). The semi-classical dynamics we have presented provide a framework where it is possible to ask what happens earlier when there are genuine quantum fluctuations. Exploring features such as these in more realistic models would be of great interest, especially since we can consistently evolve the system before the fluctuations have decohered.

Chapter 6

Path integrals for classical-quantum dynamics

In this chapter we make use of the developments in the understanding of classical-quantum master equations presented in previous chapters to write down a classical-quantum path integral, equivalent to dynamics which is CPTP. Specifically, using the most general form of autonomous classical-quantum dynamics of Equation's (2.91) and (4.13), we associate a path integral to *any* CQ master equation, from which the conditions on complete positivity can easily be read off. The general result is given by Equation (6.16). We study classical-quantum path integrals without resorting to master equation methods in Chapter 8.

In the master equation picture, the complete positivity, and general consistency, of classical-quantum dynamics is manifest. However, in a variety of contexts a path integral approach is perhaps more useful. For example, some numerical simulations are better suited to path integral methods [146, 147, 148, 149, 150, 151, 151, 152], especially when saddle point approximations are valid. For practical applications, we saw in the previous chapter that classical-quantum dynamics can be viewed as the natural framework to discuss quantum theory when measurements are involved, which is particularly relevant for quantum control procedures. Indeed, the most general operation one is allowed to perform in standard quantum theory is described by a series of CPTP maps which are performed conditioned on the outcomes of measurements. On the other hand, CQ dynamics is the framework to consider theories with a classical field, whether fundamental or effective, and a path integral approach allows one to impose space-time and gauge symmetries, as well as the possibility to enforce the modern principles used when studying effective field theories [15].

In quantum mechanics, it is well known that one can derive the path integral approach from the Schrödinger equation, and also from the more general Lindblad equation arising from open quantum systems [79, 97]. It is perhaps less well known that one can do the same for classical dynamics, arriving at an equivalence between general master equations and path integrals [153, 154, 80, 87]. For example, a Brownian particle whose conditional probability distribution $P(x, p|x', p')$ evolves according to the Fokker-Plank equation has an equivalent description in terms of a path integral which is (up to factors of $i, \hbar$) the same as the standard path integral of quantum mechanics. We showed explicitly how one can derive path integrals for open classical and quantum systems in Chapter 2, and the path integral combines them to the case of classical-quantum coupling. We compare the path integral for open quantum systems, the classical path integral for stochastic systems, and the classical-quantum hybrid path integrals we construct in this chapter in Table 2.2.

Classical-quantum path integrals have appeared previously [155, 156, 157, 158, 159]. These may be valid when applied to some initial probability densities, but generally lead to negative probabilities since the dynamics is not completely positive on all initial states. Here, we consider dynamics which is CPTP on all states at all times. Of particular relevance is the class of continuous master equations [57, 60], the most general form of which was introduced in Chapter 4. We find these path integrals have a natural decomposition into a pure classical part, representing the stochastic nature of the classical degrees of freedom, a pure quantum part, which includes a Feynman-Vernon term, and a classical-quantum part – which acts to exponentially suppress the paths which deviate from the averaged equations of motion – as summarized by Table 2.2. Under certain conditions, namely when the dynamics is at most quadratic in momenta, we can integrate out the momenta to arrive at a configuration space path integral. In the case where the dynamics is approximately Hamiltonian, as in [60, 28, 10], the configuration space path integral acts to exponentially suppress the paths which deviate from paths solving the averaged Euler-Lagrange equations.

The final form we find motivates a general form of configuration space path integrals, sum-

Table 6.1: A table representing the classical, quantum, and classical-quantum path integrals.

	Classical stochastic
Path integral	$p(q,p,t_f) = \int \mathcal{D}q\mathcal{D}p\ e^{iS_C[q,p]}\delta(\dot{q} - \frac{\partial H}{\partial p})p(q,p,t_i)$
Action	$iS_C = -\int_{t_i}^{t_f} dt \frac{1}{4}\ (\frac{\partial H}{\partial q} + \dot{p})D_2^{-1}(\frac{\partial H}{\partial q} + \dot{p})$
CP condition	D_2^{-1} a positive (semi-definite) matrix, $D_2^{-1} \succeq 0$

(a) The path integral for continuous, stochastic classical dynamics [153, 154, 80, 87] in phase space generated by a stochastic Hamiltonian. One sums over all classical configurations (q,p) with a weighting according to the difference between the classical path and its expected force $-\frac{\partial H}{\partial q}$, by an amount characterized by the diffusion matrix D_2. In the case where the force is determined by a Lagrangian L_C, the action S_C describes a suppression of paths away from the Euler-Lagrange equations $iS_C = -\int_{t_i}^{t_f} dt \frac{1}{4}(\frac{\delta L_C}{\delta q_i})(D_2^{-1})^{ij}(\frac{\delta L_C}{\delta q_j})$, by an amount determined by the diffusion coefficient D_2. The most general form of classical path integral is given by Equation (6.18).

	Quantum
Path integral	$\rho(\phi^{\pm}, t_f) = \int \mathcal{D}\phi^{\pm}\ e^{iS[\phi^+] - iS[\phi^-] + iS_{FV}[\phi^+,\phi^-]}\rho(\phi^{\pm}, t_i)$
Action	$S[\phi] = \displaystyle\int_{t_i}^{t_f} dt\big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\big),$ $iS_{FV} = \displaystyle\int_{t_i}^{t_f} dt\big(D_0^{\alpha\beta} L_\alpha^+ L_\beta^{*-} - \frac{1}{2} D_0^{\alpha\beta}(L_\beta^{*-} L_\alpha^- + L_\beta^{*+} L_\alpha^+)\big)$
CP condition	$D_0^{\alpha\beta}$ a positive (semi-definite) matrix, $D_0 \succeq 0$.

(b) The path integral for a general autonomous quantum system, here taken to be ϕ. The quantum path integral is doubled since it includes a path integral over both the bra and ket components of the density matrix, here represented using the $\pm$ notation. In the absence of the Feynman Vernon term S_{FV} [79], the path integral represents a quantum system evolving unitarily with an action $S[\phi]$. When the Feynman Vernon action S_{FV} is included, the path integral describes the path integral for dynamics undergoing Lindbladian evolution [62, 63] with Lindblad operators $L_\alpha(\phi)$.

	Classical-quantum
Path integral	$\rho(q,p,\phi^{\pm},t_f) =$ $\int \mathcal{D}q\mathcal{D}p\mathcal{D}\phi^{\pm}\ e^{iS_C[q,p]+iS[\phi^+]-iS[\phi^-]+iS_{FV}[\phi^{\pm}]+iS_{CQ}[q,p,\phi^{\pm}]}\delta(\dot{q}-\frac{p}{m})\rho(q,p,\phi^{\pm},t_i)$
Action	$iS_C[z]+iS_{CQ}[z,\phi^{\pm}] = -\frac{1}{4}\int_{t_i}^{t_f} dt\ D_2^{-1}\big(\frac{\partial H_C}{\partial q}+\frac{1}{2}\frac{\partial V_I[q,\phi^+]}{\partial q}+\frac{1}{2}\frac{\partial V_I[q,\phi^-]}{\partial q}+\dot{p}\big)^2.$
CP condition	$D_0 \succeq 0, D_2 \succeq 0$ and $8D_2 \succeq D_0^{-1}$

(c) The phase space path integral for continuous, autonomous classical-quantum dynamics. The path integral is a sum over all classical paths of the variables q, p, as well as a sum over the doubled quantum degrees of freedom $\phi^{\pm}$. The action contains the purely quantum term from the quantum path integral in Table 6.1b, but also includes the term $iS_C + iS_{CQ}$. This suppresses paths away from the drift, which is sourced by both purely classical terms described by the Hamiltonian H_C and the back-reaction of the quantum systems on the classical ones, described by a classical-quantum interaction potential V_I. The action acts to suppress paths which deviate from the $\pm$ averaged Hamilton's equations (see Equation (6.49)). The most general form of classical-quantum path integral is given by Equation (6.16). Under certain conditions, namely when the classical-quantum action (6.16) is quadratic in momenta, one can arrive at a configuration space path integral, where paths deviating from the $\pm$ averaged Euler-Lagrange equations as suppressed (see Equation (6.66)). In order for the dynamics to be completely positive, the decoherence-diffusion trade-off $8D_2 \succeq D_0^{-1}$ must be satisfied [5, 4], where D_0^{-1} is the generalized inverse of D_0, which must be positive semi-definite. As a consequence, there must be both a Feynman-Vernon term D_0, and deviation from paths away from their expected drift due to the diffusion coefficient D_2, and both effects cannot be made small. When the trade-off is saturated, the path integral preserves purity of the quantum state, conditioned on the classical degree of freedom [6] (see Section 6.3.3).

marized by Equation (6.72). In Chapter 8 we prove such path integrals are completely positive without resorting to master equation methods, meaning the general form is valid even when higher derivative terms are included. As a result, these path integrals provide a general framework to construct classical-quantum theories which respect space-time symmetries. In Chapter 8 we use them to construct CQ path integrals for gravity including a diffeomorphism invariant theory of CQ gravity. These CQ path integrals can be thought of as an effective theory where space-time is treated as classical. On the other hand, if taken as fundamental, the parameter space of the theory can be experimentally constrained via the decoherence diffusion trade-off [4], which has already been used to constrain theories with a fundamentally classical gravitational field. We will find that the trade-off plays a special role here. When it is saturated, the path integral takes on a particularly simple form. In this chapter we will primarily interested in deriving CQ path integrals from master equations, but for completeness we also include a brief discussion of their more general form.

This chapter is based on the paper [7], which is work done in collaboration with Jonathan Oppenheim.[1]

6.1 Moment expansion of the dynamics

In this chapter, we will assume classical degrees of freedom are described by a continuous measurable space, and we will generically denote elements of the space by z. We discuss the case of fields separately, and less rigorously, in Section 6.5 and also in Chapter 8.

Recall from the background section in Chapter 2 that we can write the transition amplitude $\Lambda^{\mu\nu}(z, t+\delta t|z', t)$ for an autonomous classical-quantum map as

$$\begin{aligned}\Lambda^{\mu\nu}(z, t+\delta t|z', t) &= \int du\ e^{-iu\cdot(z-z')} C^{\mu\nu}(u, z', \delta t) \\ &= \sum_{n=0}^{\infty} \frac{M^{\mu\nu}_{n,i_1\dots i_n}(z', \delta t)}{n!} \frac{1}{(2\pi)^d} \int du \quad e^{-iu\cdot(z-z')}(i^n) u_{i_1}\dots u_{i_n}.\end{aligned} \tag{6.1}$$

As a consequence, using the short-time expansion of the moments

$$M^{\mu\nu}(z', \delta t)_{n,i_1\dots i_n} = \delta^\mu_0\delta^\nu_0 + \delta t n! D^{\mu\nu}_{n,i_1\dots i_n}(z', t) + O(\delta t^2), \tag{6.2}$$

[1] The factor of 2 difference between this chapter and [7] is because in this thesis we have chosen to absorb the $\frac{1}{n!}$ appearing in the master equation of [7] into the definition of D_n. The same is true for Chapter 8.

we can write the state at $t+\delta t$ in terms of the coefficients $D^{\mu\nu}_{n,i_1\dots i_n}(z')$ and the state at t as

$$\varrho(z,t{+}\delta t) = \frac{1}{(2\pi)^d}\int dudz'\, e^{-iu(z-z')}\left(\varrho(z',t) + \sum_{n=0}^{\infty}\delta t(i^n)u_{i_1}\dots u_{i_n}D^{\mu\nu}_{n,i_1\dots i_n}(z',t)L_\mu\varrho(z',t)L^\dagger_\nu\right). \tag{6.3}$$

Equation (6.3) will be a key equation in deriving the CQ path integral. Recall, the moments appearing in Equation (6.3) can be related to physical quantities. For example, the first and second moments D_1, D_2 of the probability transition amplitude characterize the amount of drift and diffusion in the system, whilst the zeroth moment D_0 can be related to the amount of decoherence of the quantum system. One needs to remember that the moments are not independent of each other and using the conservation of probability (Equation 2.82) one can eliminate D^{00}_0 in favour of the other coefficients. In this section, we shall often write D_n to describe the object with entries $D^{\mu\nu}_{n,i_1\dots i_n}$ and we occasionally write the master equation in short-hand as

$$\frac{\partial\varrho}{\partial t} = \mathcal{L}(\rho), \tag{6.4}$$

where the *superoperator* $\mathcal{L}$ is defined via the right hand side of Equation (6.3). The formal solution to the dynamics can then be written as

$$\rho(t) = U(t,t_i) = \mathcal{T}\{e^{\int_{t_i}^{t}dt'\mathcal{L}(t')}\}(\rho(t_i)), \tag{6.5}$$

where $\mathcal{T}$ denotes the time ordering operator, which is required since in the time dependent case $\mathcal{L}(t)$ operations do not commute with each other at different times. Since the dynamics is autonomous, it forms a semi-group [63], so that $U(t,t_i) = U(t,t')U(t',t_i)$.

6.2 Derivation of the path integral formalism

In classical autonomous dynamics, the Kramers-Moyal expansion is used to obtain a path integral representation of the dynamics. For the reader unfamiliar with classical path integrals for open classical systems we recommend [80] (see also [73, 87]). The path integral for quantum systems is found after Trotterizing [160] the dynamics and inserting position and momentum resolutions of the identity – see [97] for a review of quantum path integrals for open quantum systems. In the hybrid case, we shall do both simultaneously to arrive at a CQ path integral, using the short time representation of the dynamics appearing in Equation (6.3).

We first derive a path integral for the most general CQ master equation to arrive at a phase space path integral, which includes an integral over response variables. The result is Equation (6.16). In its most general form, the path integral is a complicated object, however, in Section 6.3 we study the path integral for the class of continuous master Equations. In this case, we find one can always integrate out the response variables to arrive at a phase space path integral alone, given by Equation (6.41).[2] The resulting path integral has a natural interpretation in terms of suppressing paths away from there averaged equations of motion by an amount characterized by D_2^{-1}.[3] Simultaneously there is decoherence the quantum system, by an amount depending on D_0. The decoherence diffusion trade-off [4], necessary for complete positivity of the dynamics, tells us that one cannot simultaneously make the effects of decoherence and diffusion small if there is back-reaction on the classical system.

6.2.1 Derivation of phase space path integral for any CQ dynamics

Let us now derive the CQ path integral for the master equation defined by Equation (6.3). For ease of presentation, we shall take the Lindblad operators L_μ to be functions of two canonically conjugate operators ϕ, π, with $[\phi, \pi] = i$, $L_\mu(\phi, \pi)$, but the derivation also holds if they are functions of multiple operators and we can also write a coherent state path integral using similar methods to [97]. We use the convention that $\langle\phi|\phi'\rangle = \delta(\phi - \phi')$ and $\langle\pi|\pi'\rangle = 2\pi\delta(\pi - \pi')$ so that $\langle\phi|\pi\rangle = e^{i\pi\phi}$.

To derive the path integral, we first Trotterize the dynamics [160]. Defining $t_f = t_i + K\delta t$ and $t_k = k\delta t$, we use the identity

$$\begin{aligned} U(t_f, t_i) &= \lim_{K\to\infty} U(t_f, t_{k-1})U(t_{k-1}, t_{k-2})\dots U(t_2, t_i) \\ &= \lim_{K\to\infty} (1 + \delta t\mathcal{L}(t_{k-1}))(1 + \delta t\mathcal{L}(t_{k-2}))\dots(1 + \delta t\mathcal{L}(t_i)). \end{aligned} \tag{6.6}$$

In the time independent case, Equation (6.6) reduces to the familiar statement

$$U(t_f, t_i) = e^{\mathcal{L}(t_f - t_i)} = \lim_{K\to\infty} e^{\mathcal{L}\delta t} = \lim_{K\to\infty} (1 + \mathcal{L}\delta t)^K. \tag{6.7}$$

[2] In Section 6.4 we discuss the sufficient conditions to derive a configuration space path integral, namely that the classical-quantum action be at most quadratic in momenta.

[3] Here, and throughout, recall the $^{-1}$ denotes the generalized inverse of $D_2(D_0)$, since $D_2(D_0)$ are only required to be positive *semi-definite*

We can use Equation (6.6) to write the CQ state at time t_{k+1} in terms of the state at time t_k as

$$\varrho(z_{k+1}, t_{k+1}) = \int dz_k \delta(z_{k+1} - z_k)\varrho(z_k, t_k) + \delta t \mathcal{L}(z_{k+1}|z_k)(\varrho(z_k, t_k)). \tag{6.8}$$

Using the definition of the delta function, we can identify the right hand side of Equation (6.8) with that of Equation (6.3) to write

$$\begin{aligned}\varrho(z_{k+1}, t_{k+1}) = \frac{1}{(2\pi)^d} \int du_k dz_k \ e^{-iu_k\cdot(z_{k+1}-z_k)} \times \\ \left(\varrho(z_k, t_k) + \sum_{n=0}^{\infty} \delta t(i^n) u_{k,i_1} \dots u_{k,i_n} D^{\mu\nu}_{n,i_1\dots i_n}(z_k, t_k) L_\mu \varrho(z_k, t_k) L^\dagger_\nu\right)\end{aligned} \tag{6.9}$$

The next step is to map the Lindblad operators acting on the CQ state to c-numbers which can be exponentiated. Just as with the quantum path integral, we first write the state in terms of the ϕ basis

$$\varrho(z_k, t_k) = \int d\phi^-_k d\phi^+_k \ \varrho(z_k, \phi^+_k, \phi^-_k, t_k) \ |\phi^+_k\rangle\langle\phi^-_k|, \tag{6.10}$$

where $\varrho(z, \phi^+_k, \phi^-_k, t_k) = \langle\phi^+|\varrho(z, t_k)|\phi^-\rangle$. The $+,-$ terms arise because we expand the density matrix in terms of both bra and ket states. The convention is such that when we calculate the expectation value of operators $\mathrm{Tr}\,[O^+\varrho O^-]$, then, after using cyclicity of the trace $\mathrm{Tr}\,[\varrho O^- O^+] = \mathrm{Tr}\,[O^- O^+ \varrho]$, the operators O^- are always to the left of O^+.

Inserting Equation (6.9) into Equation (6.10), along with π^+, π^- resolutions of the identity $\mathbb{I} L_\mu \varrho L^\dagger_\nu \mathbb{I}$ gives the following expression for the transition amplitude

$$\begin{aligned}&\varrho(z_{k+1}, \phi^+_{k+1}, \phi^-_{k+1}, t_{k+1}) = \\ &\frac{1}{(2\pi)^d} \int d\phi^-_k d\phi^+_k d\pi^-_k d\pi^+_k du_k dz_k e^{\delta t \mathcal{I}(\phi^-_k, \phi^+_k, \pi^-_k, \pi^+_k, u_k, z_k, t_k)} \varrho(z_k, \phi^+_k, \phi^-_k, t_k),\end{aligned} \tag{6.11}$$

where we have implicitly defined the (time-discrete) classical-quantum action

$$\begin{aligned}\mathcal{I}(\phi^-_k, \phi^+_k, \pi^-_k, \pi^+_k, u_k, z_k, t_k) = -iu_k \cdot \frac{(z_{k+1} - z_k)}{\delta t} + i\frac{(\phi^+_{k+1} - \phi^+_k)}{\delta t}\pi^+ - i\frac{(\phi^-_{k+1} - \phi^-_k)}{\delta t}\pi^- \\ + \sum_{n=0}^{\infty} (i^n) u_{k,i_1} \dots u_{k,i_n} D^{\mu\nu}_{n,i_1\dots i_n}(z_k, t_k) L^+_\mu L^{-*}_\nu,\end{aligned} \tag{6.12}$$

using the shorthand $L^+_\mu := L_\mu(\phi^+, \pi^+) = \langle\pi^+|L_\mu(\phi, \pi)|\phi^+\rangle$, and similarly for L^-. Taking the $t \to 0, K \to \infty$ limit, with $\delta t K = t_f - t_i$, we arrive at the path integral representation of the transition amplitude

$$\varrho(z,\phi^+,\phi^-,t_f) = \lim_{K\to\infty}\int\left\{\prod_{k=1}^{K} d\phi_k^{\pm}\right\}\left\{\prod_{k=1}^{K}\frac{d\pi_k^{\pm}}{2\pi}\right\}\left\{\prod_{k=1}^{K}\frac{du_k}{(2\pi)^d}\right\}\left\{\prod_{k=1}^{K} dz_k\right\}\times$$
$$e^{\delta t\sum_{k=1}^{K}\mathcal{I}(\phi_k^-,\phi_k^+,\pi_k^-,\pi_k^+,u_k,z_k,t_k)}\varrho(z_i,\phi_i^+,\phi_i^-,t_i), \tag{6.13}$$

where it should always be understood that boundary conditions for the final state have been imposed. For ease of notation, we will write this formally as

$$\varrho(z,\phi^+,\phi^-,t_f) = \int\mathcal{D}\phi^{\pm}\mathcal{D}\pi^{\pm}\mathcal{D}u\mathcal{D}z\ e^{\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u,z,t_i,t_f)}\varrho(z_i,\phi_i^+,\phi_i^-,t_i), \tag{6.14}$$

where

$$\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u,z,t_i,t_f) =$$
$$\int_{t_i}^{t_f} dt\left[-iu\cdot\frac{dz}{dt} + i\dot{\phi}^+\pi^+ - i\dot{\phi}^-\pi^- + \sum_{n=0}^{\infty}(i^n)u_{i_1}\dots u_{i_n}D^{\mu\nu}_{n,i_1\dots i_n}(z,t)L^+_{\mu}L^{-*}_{\nu}\right], \tag{6.15}$$

and the $\pm$ denotes integration over both the $+,-$ variables. Finally, we can use the normalization condition to substitute in for the coefficient D_0^{00} to write the path integral in a way which reflects the structure of the master equation in (2.91), in which case we find our general expression for the *CQ action*

$$\begin{aligned}\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u,z,t_i,t_f) &= \int_{t_i}^{t_f} dt\Bigg[-iu\cdot\frac{dz}{dt} + \sum_{n=1}^{\infty}(i^n)u_{i_1}\dots u_{i_n}D^{00}_{n,i_1\dots i_n}(z,t)\\ &+ i\dot{\phi}^+\pi^+ - iH^+ - i\dot{\phi}^-\pi^- + iH^- + D_0^{\alpha\beta}(z,t)L^+_{\alpha}(\phi^+,\pi^+)L^{*-}_{\beta} - \frac{1}{2}D_0^{\alpha\beta}(z,t)\left(L^{*+}_{\beta}L^+_{\alpha} + L^{*+}_{\beta}L^+_{\alpha}\right)\\ &+ \sum_{\mu\nu\neq 00}\sum_{n=1}^{\infty}(i^n)u_{i_1}\dots u_{i_n}D^{\mu\nu}_{n,i_1\dots i_n}(z,t)L^+_{\mu}L^{*-}_{\nu}\Bigg].\end{aligned} \tag{6.16}$$

We can break down Equation (6.16) into its familiar parts, by writing

$$\begin{aligned}\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u,z,t_i,t_f) = iS_C[u,z] + iS[\phi^+,\pi^+] - iS[\phi^-,\pi^-] + iS_{FV}[z,\phi^{\pm},\pi^{\pm}]\\ + iS_{CQ}[u,z,\phi^{\pm},\pi^{\pm}].\end{aligned} \tag{6.17}$$

In Equation (6.17) $S_C[u,z]$ is the pure classical action [80]

$$iS_C[u,z] = \int_{t_i}^{t_f} dt\left[-iu\cdot\frac{dz}{dt} + \sum_{n=1}^{\infty}(i^n)u_{i_1}\dots u_{i_n}D^{00}_{n,i_1\dots i_n}(z,t)\right], \tag{6.18}$$

$S[\phi,\pi] = \dot{\phi}\pi - H$ is a pure quantum action (written in momentum variables), which appears in the combination $iS[\phi^+,\pi^+] - iS[\phi^+,\pi^+]$ due to the bra and ket components of the density matrix. $S_{FV}[z,\phi^\pm,\pi^\pm]$ is the Feynman-Vernon action familiar in the study of open quantum systems [79, 97], describing the pure Lindbladian part of the dynamics

$$S_{FV}[z,\phi^\pm,\pi^\pm] = -i\int_{t_i}^{t_f} dt\ D_0^{\alpha\beta}(z,t)L_\alpha^+ L_\beta^{*-} - \frac{1}{2}D_0^{\alpha\beta}(z,t)\left(L_\beta^{*+}L_\alpha^+ + L_\beta^{*+}L_\alpha^+\right), \tag{6.19}$$

and $S_{CQ}[u,z,\phi^\pm,\pi^\pm]$ is describes the novel non-trivial CQ interaction terms

$$iS_{CQ}[u,z,\phi^\pm,\pi^\pm] = \int_{t_i}^{t_f} dt \sum_{\mu\nu\neq 00}\sum_{n=1}^{\infty}(i^n)u_{i_1}\dots u_{i_n}D^{\mu\nu}_{n,i_1\dots i_n}(z,t)L_\mu^+ L_\nu^{*-}. \tag{6.20}$$

One sees that the quantum back-reaction on the classical system is encoded by the interaction of the Lindblad operators with the response variables u through the coupling $D^{\mu\nu}_{n,i_1\dots i_n}(z,t)$. Equation (6.13) is the most general path integral formulation for autonomous CQ dynamics. However, the fact that there are infinitely many terms appearing in the exponent make it potentially difficult to work with, at least exactly, and represents the fact that in general classical-quantum dynamics can involve finite sized jumps in the classical phase space.

Nonetheless, the path integral becomes much simpler for the class of continuous CQ dynamics, the most general form we introduced in Chapter 4. Recall, the general form of the continuous dynamics is given by

$$\begin{aligned}\frac{\partial\varrho(z,t)}{\partial t} &= \sum_{n=1}^{n=2}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}(z,t)\varrho(z,t)\right)\\ &\quad - \frac{\partial}{\partial z_i}\left(D^{0\alpha}_{1,i}(z,t)\varrho(z,t)L_\alpha^\dagger\right) - \frac{\partial}{\partial z_i}\left(D^{\alpha 0}_{1,i}(z,t)L_\alpha\varrho(z,t)\right) &(6.21)\\ &\quad - i[H(z),\varrho(z,t)] + +D_0^{\alpha\beta}(z,t)L_\alpha\varrho(z,t)L_\beta^\dagger - \frac{1}{2}D_0^{\alpha\beta}(z,t)\{L_\beta^\dagger L_\alpha,\varrho(z,t)\}_+, &(6.22)\end{aligned}$$

where completely positivity of the dynamics is equivalent to the condition that the matrix

$$D = \begin{bmatrix} 2D_2 & D_1 \\ D_1^\dagger & D_0 \end{bmatrix} \succeq 0 \tag{6.23}$$

is positive semi-definite. Recall that since Equation (6.23) is in block form it follows from the Schur complement that positive semi-definiteness of D is equivalent to the following conditions

$$\begin{aligned} &D \succeq 0 \Leftrightarrow D_2 \succeq 0, 2D_0 - D_1^\dagger D_2^{-1}D_1 \succeq 0, \left(I - D_2D_2^{-1}\right)D_1 = 0 \text{ and}\\ &D \succeq 0 \Leftrightarrow D_0 \succeq 0, 2D_2 - D_1 D_0^{-1}D_1^\dagger \succeq 0, \left(I - D_0D_0^{-1}\right)D_1^\dagger = 0.\end{aligned} \tag{6.24}$$

Here $^{-1}$ denotes the generalized inverse, since D_2, D_0 are only required to be positive *semi-definite*. For convenience, in this chapter we often write the generalized inverse of an object X as X^{-1} but we emphasise that it should always be understood as the generalized inverse if X is not invertible.

For the class of continuous master equations, the action of Equation (6.16) reads

$$\begin{aligned}\mathcal{I}(\phi^-, \phi^+, \pi^-, \pi^+, u, z, t_i, t_f) = \int_{t_i}^{t_f} dt \Big[&- iu \cdot \frac{dz}{dt} + iu_i D_{1,i}^{00} - u_i D_{2,ij}^{00} u_j \\ i\dot{\phi}^+ \pi^+ - iH^+ - i\dot{\phi}^- \pi^- + iH^- &+ D_0^{\alpha\beta} L_\alpha^+ L_\beta^{*-} - \frac{1}{2} D_0^{\alpha\beta} \left(L_\beta^{*-} L_\alpha^- + L_\beta^{*+} L_\alpha^+ \right) \\ + iu_i D_{1,i}^{0\alpha} L_\alpha^{*-} + iu_i D_{1,i}^{\alpha 0} L_\alpha^+ \Big]&,\end{aligned} \tag{6.25}$$

and we can identify the purely classical contribution with the *Fokker-Plank* action [80, 87]

$$iS_C = \int_{t_i}^{t_f} dt \left[-iu \cdot \frac{dz}{dt} + iu_i D_{1,i}^{00} - u_i D_{2,ij}^{00} u_j \right], \tag{6.26}$$

whilst the CQ interaction is determined via

$$S_{CQ} = \int_{t_i}^{t_f} dt \left[u_i D_{1,i}^{0\alpha} L_\alpha^{*-} + u_i D_{1,i}^{\alpha 0} L_\alpha^+ \right]. \tag{6.27}$$

Equation (6.25) gives the general form of path integrals for any continuous autonomous CQ master equation. In the next section we show that these path integrals can be simplified further since the action is quadratic in u. Therefore, we are able to integrate out the response variables u to obtain a path integral over $\phi^\pm, \pi^\pm, z$ alone. In this case, the coupling between quantum and classical degrees of freedom can be written down directly, as opposed to being mediated by response variables, and we show this has a natural interpretation as suppressing classical paths which deviate away from their averaged path, determined by both pure classical drift and drift sourced by quantum back-reaction on the classical degrees of freedom.

To summarize this section, we have seen that by introducing a recursive short time moment expansion of the classical-quantum state, Equation (6.9), we are able to write down the general form of the phase space classical-quantum path integral. The general expression is given by Equation (6.16), and for continuous dynamics we arrive at the considerably simpler path integral of Equation (6.25). In the general case, the action includes an integral over response variables u. However, for the continuous master equations the response variables couple quadratically in the action; in the next section we shall integrate these out to obtain a path integral formulation without response variables.

6.3 Path integral formulation for continuous CQ master Equations without response variables

In this section, we perform the integral over response variables in Equation (6.25) to obtain a path integral representation of the continuous master equation in terms of the variables $\phi^{\pm}, \pi^{\pm}, z$ alone. Since the pure quantum parts of the action are u independent, the relevant portion of the action, i.e, the u dependent part, is given by $S_C + S_{CQ}$ – as defined in Equations (6.26) and (6.27). We have to be careful since the diffusion matrix D_2 appearing in S_C is a symmetric, positive *semi-definite* matrix, as opposed to a positive definite matrix, and so the Gaussian integral over response variables is slightly less standard.

To deal with this issue, we first diagonalize the diffusion matrix by means of an orthogonal transformation $O_{ij}(z,t)$

$$D^{00}_{2,ij}(z,t) = \sum_k O_{ik}(z,t)^T \lambda_k(z,t) O_{kj}(z,t). \tag{6.28}$$

We expect the u path integral measure to be invariant under orthogonal transformations of u and so after applying the orthogonal transformation to Equation (6.25), or more properly to each u integral in Equation (6.13), we arrive at the diagonalized action for the continuous master equation

$$\begin{aligned}&\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u,z,t_i,t_f) = \int_{t_i}^{t_f} dt \Big[- i u_i O_{ij} \frac{dz_j}{dt} + i u_i O_{ij} D^{00}_{1,j} - u_i \lambda_i u_i \\ &i\dot{\phi}^+\pi^+ - iH^+ - i\dot{\phi}^-\pi^- + iH^- + D_0^{\alpha\beta} L^+_\alpha L^{*-}_\beta - \frac{1}{2} D_0^{\alpha\beta}\left(L^{*-}_\beta L^-_\alpha + L^{*+}_\beta L^+_\alpha \right) \\ &+ i u_i O_{ij} D^{0\alpha}_{1,j} L^{*-}_\alpha + i u_i O_{ij} D^{\alpha 0}_{1,j} L^+_\alpha \Big].\end{aligned} \tag{6.29}$$

Having diagonalized the diffusion matrix $D^{00}_{2,ij}$ the $u_i u_j$ cross terms appearing in Equation (6.25) decouple and we can perform each component u_i of the $\mathcal{D}u$ path integral separately. Since we know that $D^{00}_{2,ij}$ is a positive semi-definite matrix there are two cases to consider – when its eigenvalues vanish $\lambda_i = 0$ and when they are strictly positive $\lambda_i > 0$. We split the action into terms consisting of response variables which couple to a zero eigenvalue and those which couple to a strictly positive eigenvalue

$$\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u,z,t_i,t_f) = \mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u_{\lambda=0},z,t_i,t_f) + \mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,u_{\lambda>0},z,t_i,t_f). \tag{6.30}$$

Given this decomposition, we are then able to perform the integration over the response variables in turn, starting with those associated to a zero eigenvalue.

Integral over response variables u_i with $\lambda_i = 0$

When $\lambda_i = 0$ the action in Equation (6.29) is linear in u_i. We know that for the dynamics to be completely positive we must have (6.24) that

$$(I - D_2 D_2^{-1}) D_1 = 0, \tag{6.31}$$

and as a consequence

$$(I - \lambda\lambda^{-1}) O D_1 = 0, \tag{6.32}$$

where λ^{-1} is the generalized inverse of λ. We can write (6.32) in components as

$$\sum_j (1 - \delta(\lambda_i)) O_{ij} D_{1,j} = 0, \tag{6.33}$$

where $\delta(\lambda_i)$ is 0 if $\lambda_i = 0$ and one otherwise. In the case when $\lambda_i > 0$ this poses no extra restrictions. However, when $\lambda_i = 0$ the i, j component can only be satisfied if $\sum_j O_{ij} D_{1,j} = 0$. Hence, whenever $\lambda_i = 0$ the terms $u_i O_{1,j} D_{1,j}^{0\alpha}$ - and its complex conjugate term - will not contribute in Equation (6.29). This is expected: if we use a basis of classical variables such that the diffusion matrix is diagonal, then if any of the classical variables are deterministic we expect to have no quantum back-reaction on these degrees of freedom, since we know this must be of stochastic nature due to Equation (6.24) and the decoherence diffusion trade-off [4].

In fact, for vanishing λ_i, the only term involving the response variable u_i is given by the purely classical S_C action, and takes the form

$$\sum_{j,i|\lambda_i=0} \left(-i u_i O_{ij} \frac{dz_j}{dt} + i u_i O_{ij} D_{1,j}^{00} \right). \tag{6.34}$$

Performing the u_i path integral over (6.34) gives rise to a delta functional

$$\begin{aligned} \varrho(z, \phi^+, \phi^-, t_f) = \int \mathcal{D}\phi^- \mathcal{D}\phi^+ \mathcal{D}\pi^- \mathcal{D}\pi^+ \mathcal{D}u_{\lambda>0} \mathcal{D}z \prod_{k|\lambda_k=0} \delta(O_{kj} \frac{dz_j}{dt} - O_{kj} D_{1,j})) \times \\ e^{\mathcal{I}(\phi^-, \phi^+, \pi^-, \pi^+, u_{\lambda>0}, z, t_i, t_f)} \varrho(z_i, \phi_i^+, \phi_i^-, t_i), \end{aligned} \tag{6.35}$$

where we denote $\mathcal{D}u_{\lambda>0}$ as the remaining path integral over the variables with eigenvalue $\lambda_i > 0$. $\mathcal{D}u_{\lambda>0}$ should be understood as the $K \to \infty$ limit of $\prod_{k=1}^{K} \prod_{i=1}^{R} \frac{d\vec{u}_{k,i}}{2\pi}$, where $i = 1, \ldots R$ label the components of $u_{k,i}$ which have positive eigenvalue $\lambda_i > 0$.

Integral over response variables with $\lambda_i > 0$

Now let us consider the path integral for the response variables u_i which couple to a positive eigenvalue $\lambda_i > 0$. The terms in the CQ action (6.25) which involve these response variables are given by S_C and S_{CQ}

$$\sum_{i|\lambda_i \geq 0, j} \left[-iu_i O_{ij} \frac{dz_j}{dt} + iu_i O_{ij} D_{1,j}^{00} - u_i \lambda_i u_i + iu_i O_{ij} D_{1,j}^{0\alpha}(z) L_\alpha^{*-} + iu_i O_{ij} D_{1,j}^{\alpha 0}(z) L_\alpha^{+} \right]. \quad (6.36)$$

Equation (6.36) is quadratic in u_i and the quadratic terms couple to a now positive matrix $u_i \lambda_i \delta_{ij} u_i$. As such, at each timestep t_k we can perform a standard Gaussian integral

$$\int \exp d^R u \left(-\frac{1}{2} u \cdot A \cdot u + iJ \cdot u \right) = \sqrt{\frac{(2\pi)^R}{\det A}} \exp\left(-\frac{1}{2} J \cdot A^{-1} \cdot J \right), \quad (6.37)$$

and integrate out the remaining response variables. Explicitly, at each time-step we can perform the integration over the remaining u_i variables using Equation (6.37), identifying R as the rank of the positive definite block of $D_{2,ij}$, A as the $R \times R$ matrix with elements $A_{ij} = 2(\delta t)\delta_{ij}\lambda_j$ and finally $J_i = O_{ij} D_{1,diff}$ where $D_{1,j}^{diff}$ is the vector

$$D_{1,i}^{diff}(z, \phi^{\pm}, \pi^{\pm}, t) = D_{1,i}^{00}(z,t) + D_{1,i}^{0\alpha}(z,t) L_\alpha^*(\phi^-, \pi^-) + D_{1,i}^{\alpha 0}(z,t) L_\alpha(\phi^+, \pi^+) - \frac{dz_i}{dt}. \quad (6.38)$$

$D_{1,j}^{diff}$ describes the difference between the classical path $\frac{dz_i}{dt}$ and its expected drift, sourced by both quantum back-reaction $D_{1,i}^{0\alpha}(z) L_\alpha^*(\phi^-, \pi^-) + D_{1,i}^{\alpha 0}(z) L_\alpha(\phi^+, \pi^+)$, and purely classical drift $D_{1,i}^{00}$. Using Equation (6.37) we see that for each u_i integration we will pick up a contribution

$$\sqrt{\frac{(2\pi)^R}{(2\delta t)^R \det \lambda^+}} \exp\left(-\frac{1}{4} D_1^{diff} \cdot D_2^{-1} \cdot D_1^{diff} \delta t \right), \quad (6.39)$$

where $\det \lambda^+$ is defined as the product of all the positive eigenvalues of $D_{2,ij}$.

It is important to emphasise that the diffusion matrix $D_2(z,t)$ can be z dependent, as can its eigenvalues, and so the $\lambda^+(z,t)$ appearing in Equation (6.39) is also generically z dependent. As such, the prefactor $\frac{1}{\sqrt{\det \lambda^+}}$ is not simply a normalization constant which we would expect to cancel when computing correlation functions, instead it is important to keep the $\lambda^+(z)$ in the path integral. This term is similar in nature to the additional curvature term which arises when studying classical path integrals in curved space, as well as when studying Langevin equations with multiplicative noise [161, 162]. For notational convenience, we will absorb the prefactor into measure dz.

All together we can integrate out all the response variables u, to write the classical quantum path integral in terms of the variables $\phi^{\pm}, \pi^{\pm}, z$ alone

$$\begin{aligned}\varrho(z, \phi^+, \phi^-, t_f) = \int \mathcal{D}\phi^- \mathcal{D}\phi^+ \mathcal{D}\pi^- \mathcal{D}\pi^+ \mathcal{D}z \prod_{k|\lambda_k=0} \delta(O_{kj}\frac{dz_j}{dt} - O_{kj}D_{1,j})) \times \\ e^{\mathcal{I}(\phi^-, \phi^+, \pi^-, \pi^+, z, t_i, t_f)} \varrho(z_i, \phi_i^+, \phi_i^-, t_i),\end{aligned} \tag{6.40}$$

where the action takes the form

$$\begin{aligned}\mathcal{I}(\phi^-, \phi^+, \pi^-, \pi^+, z, t_i, t_f) = \int_{t_i}^{t_f} dt \Big[- \frac{1}{4} D_1^{diff} \cdot D_2^{-1} \cdot D_1^{diff} \\ + i\dot{\phi}^+ \pi^+ - iH^+ - i\dot{\phi}^- \pi^- + iH^- + D_0^{\alpha\beta} L_\alpha^+ L_\beta^{*-} - \frac{1}{2} D_0^{\alpha\beta} \left(L_\beta^{*-} L_\alpha^- + L_\beta^{*+} L_\alpha^+ \right) \Big],\end{aligned} \tag{6.41}$$

and we have redefined the $\mathcal{D}z$ integration measure, which due to Equation (6.39), now reads

$$\prod_{k=1}^{K} \left(\frac{\pi}{\delta t}\right)^{R/2} \frac{1}{\sqrt{\det \lambda^+}} dz_k. \tag{6.42}$$

Equation (6.41) gives the most general path integral for the entire class of continuous CQ master equations. The conditions for the underlying dynamics to be completely positive can be read off directly from Equation (6.24). In particular, the underlying dynamics will be CP if and only if the Lindbladian coefficient D_0 is positive semi-definite $D_0 \succeq 0$, $(\mathbb{I} - D_0 D_0^{-1}) D_1 = 0$ and $2D_2 \succeq D_1 D_0^{-1} D_1^\dagger$ where D_0^{-1} is the generalized inverse D_0.

The classical-quantum interaction of the action in Equation (6.41) is contained in the term

$$-\int_{t_i}^{t_f} dt \; \frac{1}{4} D_1^{diff}(z, \phi^{\pm}, \pi^{\pm}, t) \cdot D_2^{-1}(z, t) \cdot D_1^{diff}(z, \phi^{\pm}, \pi^{\pm}, t), \tag{6.43}$$

which, using the definition of D_1^{diff} in Equation (6.38), has a very natural interpretation as suppressing contributions to the path integral where paths $\frac{dz}{dt}$ differ from the expected drift, $D_{1,i}^{00} + D_{1,i}^{0\alpha} L_\alpha^{*-} + D_{1,i}^{\alpha 0} L_\alpha^+$, which is sourced by both the pure classical evolution and the quantum back-reaction on the system. We also see that the amount one is penalized for moving away from the expected classical trajectory depends on the inverse of the diffusion matrix; if there is a large amount of diffusion, then D_2^{-1} is expected to be small, and so classical paths which venture away from the expected value are penalized less. However, if there is very little diffusion then D_2^{-1} will be large, and classical trajectories are forced to stick nearby the most likely path. Since both D_2^{-1} and D_0 cannot simultaneously be made small by virtue of the complete positivity condition in Equation (6.24): there is a trade-off between the amount of diffusion in the classical system and the strength of the Lindbladian evolution of the quantum system which is characterized by the second line of Equation (6.41). To further gain intuition for the path integral, let us now discuss some cases in which the path integral in Equation (6.41) simplifies.

6.3.1 Hermitian Lindblad operators

A particularly nice interpretation of the path integral arises when the Lindblad operators L_α in Equation (6.41) are Hermitian. In this case we can write the pure Lindbladian term of the path integral S_{FV} in Equation (6.19) in terms of a negative semi-definite quadratic form

$$iS_{FV} = -\frac{1}{2}\int_{t_i}^{t_f} dt(L^-_\alpha - L^+_\alpha)D_0^{\alpha\beta}(L^-_\beta - L^+_\beta), \tag{6.44}$$

and so Equation (6.41) reads

$$\begin{aligned}\mathcal{I}(\phi^-, \phi^+, \pi^-, \pi^+, z, t_i, t_f) = \int_{t_i}^{t_f} dt\Big[& i\dot{\phi}^+\pi^+ - iH^+ - i\dot{\phi}^-\pi^- + iH^- + \\ - \frac{1}{2}(L^-_\alpha - L^+_\alpha)D_0^{\alpha\beta}(L^-_\beta - L^+_\beta) - \frac{1}{4}D_1^{diff} \cdot D_2^{-1} \cdot D_1^{diff}\Big]. & \end{aligned} \tag{6.45}$$

Reminding ourselves that the path integral can be used to compute the off-diagonal elements of the density operator $\varrho(z, \phi^+, \phi^-, t_f)$, we see that the Lindbladian part of the action suppresses the off-diagonal elements by an amount dependent on $L^-_\alpha - L^+_\alpha$ and the Lindbladian coupling matrix $D_0^{\alpha\beta}$. In particular when $D_0^{\alpha\beta}$ is large, we find that paths where $L^-_\alpha \sim L^+_\alpha$ are heavily preferred, causing the density matrix to decohere into the eigenbasis dictated by the Lindblad operators. On the other hand, if the magnitude of the decoherence is small, then one can maintain superpositions for a long time. However, we know from the CP conditions that long coherence times leads to necessarily large D_2 on the classical system, with the precise relationship determined by the strength of the quantum back-reaction D_1. Large diffusion means that D_2^{-1} will be small, and so paths can deviate away from there average without much suppression; there is a lot of classical uncertainty. In other words, if the decoherence rate is small, so that superpositions can be maintained, then measuring the classical degree of freedom necessarily gives you little information about the coherence of the quantum state.

6.3.2 Hamiltonian drift

Another simplification occurs when the drift D_1^{drift} is generated by a CQ Hamiltonian $H_C[q,p]+V_I[q,\phi]$, as in [28, 10, 60]. We assume the interaction is minimally coupled, so that the interaction potential $V_I[q,\phi]$ depends on classical and quantum position but not momenta, whilst $H_C[q,p]$ we take to be a purely classical Hamiltonian. When the drift is generated by a Hamiltonian, the back-reaction, described by the $D_1^{0\alpha}$ term in the master equation (6.21), is given by the *Alexandrov-Gerasimenko bracket* [163, 104]

$$\frac{1}{2}(\{V_I,\varrho\}-\{\varrho,V_I\}), \tag{6.46}$$

where for simplicity we consider only one classical degree of freedom. Furthermore, due to the trade-off in Equation (6.24), we know the full master equation must contain a term

$$\begin{aligned}\frac{\partial\varrho(q,p)}{\partial t} &= \{H_C,\varrho\}-i[H_Q,\varrho]\\ &+\frac{\partial^2}{\partial p^2}(D_2\varrho)+\frac{1}{2}\left(\{V_I(q,\phi),\varrho\}-\{\varrho,V_I(q,\phi)\}\right)-\frac{1}{2}D_0\{\frac{\partial V_I}{\partial q},\{\frac{\partial V_I^\dagger}{\partial q},\varrho\}\},\end{aligned} \tag{6.47}$$

where H_Q is a quantum Hamiltonian that can in general be phase space dependent and complete positivity demands $8D_2\geq D_0^{-1}$. For the master equation in Equation (6.47), the path integral action of Equation (6.41) is

$$\begin{aligned}\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,z,t_i,t_f) &= \int_{t_i}^{t_f}dt\Big[i\dot{\phi^+}\pi^+-iH_Q^+-i\dot{\phi^-}\pi^-+iH_Q^-+\\ &-\frac{D_0}{2}\left(\frac{\partial V_I}{\partial q}^--\frac{\partial V_I}{\partial q}^+\right)^2-\frac{1}{4}D_1^{diff}\cdot D_2^{-1}\cdot D_1^{diff}\Big],\end{aligned} \tag{6.48}$$

where the classical-quantum interaction takes the form

$$D_1^{diff}\cdot D_2^{-1}\cdot D_1^{diff}=D_2^{-1}\left(\{H_C,p\}+\frac{1}{2}\{V_I[q,\phi^+],p\}+\frac{1}{2}\{V_I[q,\phi^-],p\}-\dot{p}\right)^2. \tag{6.49}$$

Equation (6.49) thus acts to suppress paths which deviate from the $\pm$ averaged Hamiltonian equations of motion. In Section 6.4, we show that when the action is at most quadratic in the momenta of the classical and quantum systems, one can integrate out the momentum variables to arrive at a configuration space path integral, where paths deviating from their averaged Euler-Lagrange equations are suppressed.

6.3.3 When the trade-off is saturated

We now see that when one saturates the decoherence-diffusion trade-off, a remarkable set of cancellations occur, and the path integral takes on a very simple form. In particular, it factorises, so that there is no coupling between the ϕ^+ and ϕ^- fields. This reflects the fact that when the trade-off is saturated the classical-quantum dynamics keeps quantum states pure conditioned on the classical trajectories, as we saw in Chapter 5. When the trade-off is saturated, we can expand out the classical-quantum interaction term in the path integral of Equation (6.41) to find that all of the cross terms involving $\pm$ vanish. In particular those arising from D_0 cancel with those arising from D_2^{-1} and Equation (6.41) reduces to

$$\mathcal{I}(\phi^-, \phi^+, \pi^-, \pi^+, z, t_i, t_f) = \mathcal{I}^+_{CQ}(\phi^+, \pi^+, z, t_i, t_f) + \mathcal{I}^{-*}_{CQ}(\phi^-, \pi^-, z, t_i, t_f) - I_C(z, t_i, t_f), \quad (6.50)$$

where

$$\begin{aligned}\mathcal{I}_{CQ}(\phi, \pi, z, t_i, t_f) = \int_{t_i}^{t_f} \Big[& i\dot{\phi}\pi - iH - \frac{1}{2}(D^{00}_{1,i} - \frac{dz_i}{dt})(D_2^{-1})^{ij} D^{0\alpha}_{1,j} L^*_\alpha \\ & - \frac{1}{4}(L_\alpha D_0^{\alpha\beta} L^*_\beta + D^{0\alpha}_{1,i} L^*_\alpha (D_2^{-1})^{ij} D^{0\beta}_{1,j} L^*_\beta) \Big]\end{aligned} \quad (6.51)$$

$$I_C(z, t_i, t_f) = \int_{t_i}^{t_f} \left[\frac{1}{4}(D^{00}_{1,i} - \frac{dz_i}{dt})(D_2^{-1})^{ij}(D^{00}_{1,j} - \frac{dz_j}{dt}) \right]. \quad (6.52)$$

When the back-reaction is Hermitian, meaning that $D^{0\alpha}_{1,i} L_\alpha(\phi, \pi) = L_i(\phi, \pi, z)$ is Hermitian, Equation (6.51) simplifies further and after eliminating for $2D_0 = D_1 D_2^{-1} D_1$ its general form can be written in terms of $(D_2^{-1})^{ij}$ alone

$$\mathcal{I}_{CQ}(\phi, \pi, z, t_i, t_f) = \int_{t_i}^{t_f} \left[i\dot{\phi}\pi - iH - \frac{1}{2}(D^{00}_{1,i} - \frac{dz_i}{dt})(D_2^{-1})^{ij} L_j - \frac{1}{2}(L_i (D_2^{-1})^{ij} L_j) \right]. \quad (6.53)$$

An important example of dynamics where the trade-off is saturated is an ideal continuous measurement of a Hermitian operator $Z(z_t)$, where we can also allow the choice of continuous measurement, and its strength $k(z_t)$, to depend on the measurement outcome z_t at time t.

6.3.4 A path integral for continuous measurement and Markovian feedback

In this section we arrive at the general form of path integral representation for a continuous measurement procedure. To construct a continuous measurement, we divide time into a sequence of intervals of length δt, and consider a weak measurement in each interval. To obtain a continuous measurement, we make the strength of each measurement proportional to the time interval, and then take the limit in which the time intervals become infinitesimally short. We consider a continuous measurement of a Hermitian operator $Z(z_t)$, which can depend on the measurement signal z_t at time t, and which we take to be a functional of x, p. The measurement signal z_t is related to the measurement outcome α_t by $z_t = \alpha_t dt$ [121]. The measurement signal undergoes continuous evolution, whilst the measurement outcome α_t is wildly discontinuous, especially for weak measurements where little information is gained in each timestep. It is well known [122, 121] that for a continuous measurement the dynamics of the quantum state can be described by the set of coupled differential equations

$$d|\psi(t)\rangle = \left\{-k(z_t)(Z(z_t) - \langle Z(z_t)\rangle)^2 dt + \sqrt{2k(z_t)}(Z(z_t) - \langle Z(z_t)\rangle)d\xi\right\}|\psi(t)\rangle \tag{6.54}$$

$$dz_t = \langle Z(z_t)\rangle dt + \frac{d\xi}{\sqrt{8k(z_t)}}, \tag{6.55}$$

where z_t is the measurement signal, or record, and $k(z_t)$ parameterizes the measurement strength, which can in general also be z_t dependent, and for simplicity we are ignoring the pure quantum evolution in comparison to the measurement dynamics. The stochastic part of the evolution, described via $d\xi_i$, is the standard multivariate Wiener process satisfying the Ito rules $d\xi_i d\xi_j = \delta_{ij}dt, \;\; d\xi_i dt = 0$. The measurement outcome itself undergoes white noise dynamics $\alpha_t = \langle Z(z_t)\rangle + \frac{1}{8k(z_t)}\frac{d\xi}{dt}$ and is discontinuous in time which is why the signal z_t is preferred. From the measurement signal, one can obtain the measurement record by taking the time derivative.

From the set of non-linear stochastic differential equations given in (6.54), (6.55), it is possible using the methods outlined in Chapter 5 [6] to construct a *linear* classical-quantum master equation for the combined classical-quantum state given by

$$\frac{\partial \varrho(z)}{\partial t} = -k(z)[Z(z),[Z(z),\varrho(z)]] + \frac{Z(z)}{2}\frac{\partial \varrho(z)}{\partial z} + \frac{\partial \varrho(z)}{\partial z}\frac{Z(z)}{2} + \frac{1}{16}\frac{\partial^2}{\partial z^2}(k^{-1}(z)\varrho(z)), \tag{6.56}$$

from which we identify $D_1^{0,Z} = D_1^{Z,0} = \frac{1}{2}$, $D_0(z) = 2k(z)$ and $D_2(z) = \frac{1}{16k(z)}$. We can easily check that for perfect measurements, the decoherence diffusion trade-off in Equation (6.23) is satisfied and in fact saturated. Substituting into Equation (6.45), we find the corresponding action

$$
\begin{aligned}
\mathcal{I}(x^{\pm}, p^{\pm}, z, t_i, t_f) = \int_{t_i}^{t_f} dt \Big[& i\dot{x}^+ p^+ - i\dot{x}^- p^- - k(z)(Z^-(x,p) - Z^+(x,p))^2 \\
& - 4k(z) \left(\frac{1}{2}(Z^+(x,p) + Z^-(x,p)) - \frac{dz}{dt} \right)^2 \Big].
\end{aligned} \tag{6.57}
$$

To our knowledge, such a general form of Markovian continuous measurement path integral has not appeared in the literature, and is complementary to current approaches [148, 149, 150, 151, 151, 152]. One could also write down a coherent state path integral for continuous measurement using the methods introduced in [97], which could be of use in studying problems in optical quantum feedback [164]. One can also allow for noisy measurements by including an appropriate Feynman-Vernon term in Equation (6.57). We should also emphasize that we have derived this from slightly different considerations than other approaches, namely by starting from complete positivity of classical-quantum dynamics. Such path integrals have proved useful in optimizations of quantum control tasks, especially in the strong measurement regime where saddle-point approximations are valid [148, 149].

It is important to note that the path integral in Equation (6.41) is more general than the continuous measurement path integral of Equation (6.57). Firstly, it allows for the case where there are many measurement operators and outcomes. It also allows for noisy imperfect quantum measurements. More importantly, it allows for the case where both the classical and the quantum degrees of freedom have dynamics of their own, which is encoded in the fact that in Equation (6.41) D_1^{diff} can contain purely classical evolution determined by the drift D_1^{00}, and we can also include purely Hamiltonian, and more generally Lindbladian, quantum evolution.

To summarize this section, we have shown that one can integrate out the response variables for the class of continuous CQ master equations to arrive at the path integral in Equation (6.41), which is represented in terms of the quantum phase space variables ϕ, π and the classical variables z alone. In the next section, we look at examples where we can arrive at a configuration space path integral in both quantum and classical degrees of freedom.

6.4 Configuration space path integrals

In this section, we consider configuration space path integrals. A configuration space path integral is an important tool in understanding whether or not one can construct covariant theories (fundamental or effective) of interacting classical-quantum fields. The conditions for which we can arrive at a configuration space path integral are standard. In ordinary quantum mechanics we are able to integrate out the quantum momenta if the path integral is at most quadratic in the quantum momentum variables. In the CQ case, if the action is quadratic in both the classical and quantum momenta then we can arrive at a full configuration space path integral. For all such dynamics, the methodology of arriving at a configuration space path integral is the same: at each time step, one completes the square in the action and performs a Gaussian path integral. In this section we focus on a class of CQ dynamics which end up having interesting configuration space path integrals. In particular, as in [28], we take the classical degrees of freedom to live in a phase space and we take the CQ dynamics to be generated by an interaction potential V_I. We assume the dynamics are *minimally coupled*, which we take to mean that the CQ couplings depend only on position and not on momenta, and we diffuse only in momenta. This class of dynamics proves to be interesting since one finds a configuration space path integral which can be written in terms of a CQ proto-action W_{CQ}, as summarized by Equation (6.66).

6.4.1 An explicit derivation of a configuration space path integral

Consider a classical system $z = (q_i, p_j)$, coupled to a quantum system ϕ, π. In this subsection we consider continuous master equations of the form

$$
\begin{aligned}
\frac{\partial \varrho(q,p)}{\partial t} &= \{H_C, \varrho\} - i[H_Q, \varrho] + \frac{\partial^2}{\partial p_i \partial p_j}(D_{2,ij}(q_i)\varrho) + \frac{1}{2}\left(\{V_I(q_i,\phi), \varrho\} - \{\varrho, V_I(q_i,\phi)\}\right) \\
&\quad + D_0^{\alpha\beta}(q_i)\left(L_\alpha(\phi)\varrho L_\beta^\dagger(\phi) - \frac{1}{2}\{L_\beta^\dagger(\phi) L_\alpha(\phi), \varrho\}\right),
\end{aligned} \tag{6.58}
$$

where $H_C = \sum_i \frac{p_i^2}{2m_i} + V(q_i)$ is the purely classical Hamiltonian, $H_Q = \frac{\pi^2}{2m_Q} + V(\phi)$ is the purely quantum Hamiltonian and $V_I(q_i, \phi)$ is a classical-quantum interaction potential, a hybrid object which we take to only depend on the classical and quantum positions q_i, ϕ. We also assume that V_I is Hermitian, although in general this need not be the case. We take the Lindbladian coupling

$D_0^{\alpha\beta}(q_i, t)$ and the diffusion coefficient $D_{2,ij}(q_i, t)$ to depend on the classical positions only. We emphasize the conditions imposed on Equation (6.58) are not necessary to get a configuration space path integral; one only requires that the momentum dependence be at most quadratic. However, including q_i diffusion, or momentum dependence π in the Lindblad operators, alters the form of the momentum integral and complicates the final form of path integral we find.

Since we can make an arbitrary selection of Lindblad operators, we use this to fix $\frac{1}{2}D_{1,i}^{0\alpha} = \delta_\alpha^i$, and take a basis of Lindblad operators which includes $L_i = \frac{\partial V_I}{\partial q_i}$. With this choice, Equation (6.58) becomes

$$\frac{\partial \varrho(q,p)}{\partial t} = \{H_C, \varrho\} - i[H_Q + V_I, \varrho] + \frac{\partial^2}{\partial p_i \partial p_j}(D_{2,ij}(q_i)\varrho) + \frac{1}{2}\left(\{V_I(q_i,\phi), \varrho\} - \{\varrho, V_I(q_i,\phi)\}\right)$$
$$+ D_0^{ij}(q_i)\left(\frac{\partial V_I}{\partial q_i}\varrho\frac{\partial V_I^\dagger}{\partial q_j} - \frac{1}{2}\left\{\frac{\partial V_I^\dagger}{\partial q_j}\frac{\partial V_I}{\partial q_i}, \varrho\right\}\right) + \mathcal{L}_{Lindblad}(\varrho), \tag{6.59}$$

where $\mathcal{L}_{Lindblad}$ denotes the collection of pure Lindbladian terms one could include in Equation (6.59) that are not associated to $\frac{\partial V_I}{\partial q_i}$. Since these will not be accompanied by any back-reaction on the classical degrees of freedom we will ignore them, focusing on terms associated to back-reaction alone. The $\mathcal{L}_{Lindblad}$ terms could easily be added back to the final configuration space path integral by a suitable choice of Feynman-Vernon action. Since we have fixed $D_{1,i}^{0\alpha} = \frac{1}{2}\delta_\alpha^i$ the complete positivity condition of Equation (6.24) is now that $8D_2 \succeq D_0^{-1}$.

The transition amplitude for Equation (6.59) reads

$$\begin{aligned}&\varrho(q,p,\phi^+,\phi^-,t_f) = \\ &(\prod_i m_i)\int \mathcal{D}\phi^\pm \mathcal{D}\pi^\pm \mathcal{D}q\mathcal{D}p\delta(m_i\dot{q}_i - p_i))\ e^{\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,q,p,t_i,t_f)}\varrho(q,p,\phi_i^+,\phi_i^-,t_i),\end{aligned} \tag{6.60}$$

which can be read off from Equation (6.40). The corresponding action is given by

$$\begin{aligned}\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,q,p,t_i,t_f) = &\int_{t_i}^{t_f} dt\Big[i\dot{\phi}^+\pi^+ - iH^+ - iV_I^+ - i\dot{\phi}^-\pi^- + iH^- + iV_I^- \\ &- \frac{1}{2}\left(\frac{\partial V_I^-}{\partial q_i} - \frac{\partial V_I^+}{\partial q_i}\right)D_0^{ij}(q,t)\left(\frac{\partial V_I^-}{\partial q_j} - \frac{\partial V_I^+}{\partial q_j}\right) \\ &- \frac{1}{4}\left(\dot{p}_i + \frac{\partial H_C}{\partial q_i} + \frac{1}{2}\frac{\partial V_I^-}{\partial q_i} + \frac{1}{2}\frac{\partial V_I^+}{\partial q_i}\right)(D_2^{-1})^{ij}(q,t)\left(\dot{p}_j + \frac{\partial H_C}{\partial q_j} + \frac{1}{2}\frac{\partial V_I^-}{\partial q_j} + \frac{1}{2}\frac{\partial V_I^+}{\partial q_j}\right)\Big].\end{aligned} \tag{6.61}$$

Since there is no q_i diffusion, the pure Hamiltonian part of the classical evolution enforces the constraint $p_i = m_i\dot{q}_i$ via the delta functional in Equation (6.60). Combined with the form of back-reaction in Equation (6.61), we see the action suppresses paths which deviate from the $\pm$ averaged Hamilton's equations, just as in Section 6.3.2. We can now perform the classical momentum integration, including over the final momenta, to get a path integral over the classical configuration space. Doing this, Equation (6.61) becomes

$$
\begin{aligned}
&\mathcal{I}(\phi^-,\phi^+,\pi^-,\pi^+,q,t_i,t_f) = \int_{t_i}^{t_f} dt \Big[+ i\dot{\phi^+}\pi^+ - iH^+ - iV_I^+ - i\dot{\phi^-}\pi^- + iH^- + iV_I^- \\
&- \frac{1}{2}(\frac{\partial \Delta V_I}{\partial q_i}) D_0^{ij} (\frac{\partial \Delta V_I}{\partial q_j}) \\
&- \frac{1}{4}\left(m_i \ddot{q}_i + \frac{\partial H_C}{\partial q_i} + \frac{\partial \bar{V}_I}{\partial q_i}\right)(D_2^{-1})^{ij}(q)\left(m_j \ddot{q}_j + \frac{\partial H_C}{\partial q_j} + \frac{\partial \bar{V}_I}{\partial q_j}\right)\Big].
\end{aligned}
\tag{6.62}
$$

In Equation (6.62) we have introduced the notation $\bar{V}_I = \frac{1}{2}(V_I^- + V_I^+)$, the $\pm$ average potential and $\Delta V_I = V_I^- - V_I^+$ as the difference in the potential along the $\pm$ branches. If we further define the classical Lagrangian $L_C(q) = \sum_i p_i \dot{q}_i - H_C(q_i, p_i)$ then we recognise the CQ interaction term in Equation (6.62) as the Euler-Lagrange equations which result from varying $\int_{t_i}^{t_f} dt(L_C - \bar{V}_I)$.

We can write the action in a more compact form by defining the *CQ proto-action* W_{CQ} via

$$
W_{CQ}[q,\phi] = \int_{t_i}^{t_f} dt L_Q(\phi) + L_C(q) - V_I(q,\phi) := \int_{t_i}^{t_f} L_{CQ}(q,\phi), \tag{6.63}
$$

where $L_Q(\phi)$ is the quantum Lagrangian, $L_C(q)$ the classical Lagrangian, and $V_I(q,\phi)$ is the interaction potential. We can rewrite the CQ interaction in Equation (6.62) in terms of variations of the $\pm$ averaged CQ proto-action $\bar{W}_{CQ}$

$$
\begin{aligned}
&- \frac{1}{4}\left(m_i \ddot{q}_i + \frac{\partial H_C}{\partial q_i} + \frac{\partial \bar{V}_I}{\partial q_i}\right)(D_2^{-1})^{ij}(q)\left(m_j \ddot{q}_j + \frac{\partial H_C}{\partial q_j} + \frac{\partial \bar{V}_I}{\partial q_j}\right) \\
&= -\frac{1}{4}\frac{\delta}{\delta q_i}(\bar{W}_{CQ}[q,\phi^\pm])(D_2^{-1})^{ij}\frac{\delta}{\delta q_j}\bar{W}_{CQ}[q,\phi^\pm]).
\end{aligned}
\tag{6.64}
$$

In order to get a full configuration space path integral, all that remains is to do the integrals over the momentum of the quantum system. As with the standard quantum path integral, the technical requirement to be able to do this is that the action in Equation (6.62) is quadratic in π^+, π^- so that we can perform the π^+, π^- integral exactly by completing standard Gaussian integrals. Since we have taken the simplest case where the only momentum dependence π^-, π^+ comes from the Hamiltonian, the result of the momentum integration is to perform a Legendre transformation. We end up with the configuration space CQ path integral representation of the transition amplitude

$$\varrho(q,\phi^+,\phi^-,t_f) = \mathcal{N}\int \mathcal{D}\phi^-\mathcal{D}\phi^+\mathcal{D}q\ e^{\mathcal{I}(\phi^-,\phi^+,q,t_i,t_f)}\varrho(q,\phi_i^+,\phi_i^-,t_i), \tag{6.65}$$

where $\mathcal{N} = \prod_i m_i$ is a normalization constant arising from the classical momentum path integral, and we have absorbed the usual factors from the Gaussian integrals into the definition of $D\phi^\pm$ to obtain the standard path integral measures. The action in Equation (6.65) takes its final form

$$\mathcal{I}(\phi^-,\phi^+,q,t_i,t_f) = \int_{t_i}^{t_f} dt\Big[iL_{CQ}[q,\phi^+] - iL_{CQ}[q,\phi^-] - \frac{1}{2}(\frac{\delta\Delta W_{CQ}[q,\phi^\pm]}{\delta q_i})D_0^{ij}(\frac{\delta\Delta W_{CQ}[q,\phi^\pm]}{\delta q_j}) \\ - \frac{1}{4}\frac{\delta}{\delta q_i}(\bar{W}_{CQ}[q,\phi^\pm])(D_2^{-1})^{ij}\frac{\delta}{\delta q_j}(\bar{W}_{CQ}[q,\phi^\pm])\Big]. \tag{6.66}$$

Equation (6.66) is remarkably simple. All of the classical-quantum interaction is encoded in variations of a single classical-quantum proto-action W_{CQ} and, due to the choice of Lindblad operators, the complete positivity condition is that $8D_2 \succeq D_0^{-1}$. The classical trajectories are suppressed away from the $\pm$ averaged equations of motion which arise from varying the proto-action by an amount depending on D_2, whilst there is simultaneous decoherence by an amount which depends on the difference in the equations of motion between the $\pm$ branches. In Appendix D we discuss a simple toy example of the configuration space path integral and illustrate how one can use perturbative methods familiar in quantum theories to calculate CQ path integrals using CQ Feynman diagrams.

This direct derivation of the path integral was valid for the family of master equations given by Equation (6.58), which couple less than quadratically in the momenta. However, given the final and suggestive form of Equation (6.66) it is tempting to take it as a *definition* of classical-quantum dynamics and let W_{CQ} be an arbitrary functional of $q, \phi^\pm$ *and* their derivatives. Although the mapping from the path integral to the master equation may not be completed analytically, one might expect that the condition that $8D_2 \succeq D_0^{-1}$ is sufficient for the dynamics to be completely positive. One often does something similar in quantum theory by taking the path integral formulation to be the fundamental object of study, which often includes higher derivative terms in the action even though the mapping between master equation's and path integral can only be computed exactly when the master equation is at most quadratic in momenta. Inspired by the action in Equation (6.66), in Chapter 8 we prove directly from path integral methods that *any* configuration space path integral of the form

$$
\begin{aligned}
\mathcal{I}(\phi^-,\phi^+,q,t_i,t_f) = \mathcal{I}_{CQ}(q,\phi^+,t_i,t_f) + \mathcal{I}^*_{CQ}(q,\phi^-,t_i,t_f) - \mathcal{I}_C(q,t_i,t_f) \\
+ \int_{t_i}^{t_f} dt d\bar{x} \sum_\gamma c^\gamma(q,t,\bar{x})(L_\gamma[\phi^+]L^*_\gamma[\phi^-])
\end{aligned}
\tag{6.67}
$$

defines completely positive CQ dynamics (not necessarily normalized). In Equation (6.67) $c^\gamma \geq 0$, $L_\gamma[\phi^\pm]$ can be any functional of the bra and ket variables, $\mathcal{I}_{CQ}$ determines the CQ interaction on each of the ket and bra paths and $\mathcal{I}_C(q,t_i,t_f)$ is the purely classical action which takes real values. In order for the path integral to be convergent we impose that $\mathcal{I}_C$ is positive (semi) definite, as well as asking that the real part of $\mathcal{I}_{CQ}$ be negative (semi) definite. The term on the second line of Equation (6.67) can be thought of the path integral version of a Kraus operation, and allows one to incorporate a loss of quantum information into the path integral through Lindbladian terms when the decoherence-diffusion trade-off is not saturated. Equation (6.66) is a special case of Equation (6.67) when $8D_0 \succeq D_2^{-1}$ is satisfied, which can be seen by expanding out the CQ action and grouping terms by $8D_0 - D_2^{-1}$ [8]. This is true for an *arbitrary* CQ proto-action W_{CQ}.

This result of Equation (6.67) allows us to study consistent classical-quantum path integrals, even when we cannot perform the momentum integration exactly from the Hilbert space picture. Instead, using Equation (6.67) as a starting point, we can write down CP CQ dynamics by an appropriate choice of W_{CQ}. In doing so one must be careful to ensure that the action dynamics is also normalized. Normalization is ensured if one includes the appropriate classical and quantum kinetic terms, which we discuss in more detail in Chapter 8.

6.5 Path integrals for classical fields interacting with quantum fields

In this section, we comment on the path integral for classical fields interacting with quantum fields. This provides a natural arena to study the renormalization properties of classical-quantum dynamics, as well as covariant properties of classical-quantum field theories. We study this in more detail in Chapter 8. We treat the path integral as a formal object, making no attempt to prove anything rigorously, as is often the case with field theories.

The path integral remains largely unchanged for the case of fields. Starting with a classical-quantum master equation involving fields, one can (formally) insert various resolutions of the identity and arrive at analogous formulas for Equations (6.16), (6.41), (6.66). We do not reproduce these steps here, since they are identical to those in rest of the chapter. Instead we quote the final result for *ultra-local* classical-quantum dynamics, which is to send all the couplings appearing in the action $D^{\mu\nu}_{n,i_1\dots i_n}(z,t) \to D^{\mu\nu}_{n,i_1\dots i_n}(z,x)$, the classical variables appearing in the action $z_i \to z_i(x)$, the quantum variables to $(\phi,\pi) \to (\phi(x),\pi(x))$ and finally we one must integrate over all space in the action $\int dt \to \int dx$, where $x = (t,\vec{x})$. To be explicit in understanding the field theoretic case, in this section we consider a master Equation which has a Lorentz invariant path integral.[4]

For a quantum field $\phi(x)$ coupled to a classical field $q(x)$, the field theoretic version of the master Equation in (6.58) is

$$\begin{aligned}\frac{\partial \varrho(q,p)}{\partial t} &= \{H_C,\varrho\} - i[H_Q,\varrho] + \frac{1}{2}\left(\{V_I(q,\phi),\varrho\} - \{\varrho, V_I(q,\phi)\}\right)\\ &+ \int d\vec{x}\ \frac{\delta^2}{\delta p_i(\vec{x})\delta p_j(\vec{x})}(D_{2,ij}(q,t,\vec{x})\varrho)\\ &+ \int d\vec{x}\ D_0^{ij}(q,t,\vec{x})\left(\frac{\delta V_I}{\delta q_i(\vec{x})}\varrho\frac{\delta V_I^\dagger}{\delta q_j(\vec{x})} - \frac{1}{2}\Big\{\frac{\delta V_I^\dagger}{\delta q_j(\vec{x})}\frac{\delta V_I}{\delta q_i(\vec{x})},\varrho\Big\}\right),\end{aligned} \tag{6.69}$$

where $V_I[q,\phi] = \int d\vec{x}\mathcal{V}_I[q,\vec{x}]$ is an interaction potential and we take the purely classical part of the dynamics to be generated by the action $S_C(q) = \int dt \int d\vec{x}\mathcal{L}_C[q,x]$. It should be noted that Equation (6.69) needs regularizing, since there are multiple functional derivatives acting at the same point x. This corresponds to the fact that in the field theoretic case the path integral will require renormalization. For the choice of dynamics in Equation (6.69), the path integral

[4] In the purely quantum case, Lorentz invariant open systems, and their renormalization properties, have recently been studied in [165]. A simple example of Lorentz invariant quantum dynamics, which was shown to be renormalizable [165], is given by the Lorentz invariant Lindblad Equation which has the action

$$\mathcal{I}(\phi^-,\phi^+,t_i,t_f) = \int_{t_i}^{t_f} dt\Big[i\mathcal{L}_Q^+(x) - i\mathcal{L}_Q^-(x) - \frac{D_0}{2}\int dx\,\left(\phi^-(x)-\phi^+(x)\right)^2\Big]. \tag{6.68}$$

action is again found to be of the form in Equation (6.66)

$$\mathcal{I}(\phi^-,\phi^+,q,t_i,t_f)=\int_{t_i}^{t_f}dtd\vec{x}\Big[i\mathcal{W}^+_{CQ}(x)-i\mathcal{W}^-_{CQ}(x)$$
$$-\frac{1}{2}(\frac{\delta\Delta W_{CQ}}{\delta q_i})D_0^{ij}(z,t,\vec{x})(\frac{\delta\Delta W_{CQ}}{\delta q_j})-\frac{1}{4}\frac{\delta}{\delta q_i}(\bar{W}_{CQ}[q,\phi^{\pm}])(D_2^{-1})^{ij}(z,t,\vec{x})\frac{\delta}{\delta q_j}(\bar{W}_{CQ}[q,\phi^{\pm}])\Big],\tag{6.70}$$

where now

$$W_{CQ}[q,\phi]=\int dtd\vec{x}(\mathcal{L}_Q[\phi]+\mathcal{L}_C[q]-\mathcal{V}_I[q,\phi]):=\int dt\mathcal{W}_{CQ}[q,\phi]\tag{6.71}$$

is a space-time CQ proto-action.

The path integral enables us to construct CQ theories with space-time symmetries. For example, Equation (6.70) will describe Lorentz invariant CQ dynamics when W_{CQ} is chosen to be a Lorentz invariant scalar. We study field theoretic path integrals in more detail in Chapter 8 [8]. There, we start with Equation (6.67) and prove the resulting dynamics is CP. Given the form of Equation (6.70), it is natural to consider the class of dynamics

$$\begin{aligned}\mathcal{I}(\phi^-,\phi^+,q,t_i,t_f)=\int_{t_i}^{t_f}dtd\vec{x}\Big[i\mathcal{W}^+_{CQ}(x)-i\mathcal{W}^-_{CQ}(x)\\ -\frac{1}{2}\frac{\delta\Delta W_{CQ}}{\delta q_i}D_0^{ij}(q,t,\vec{x})\frac{\delta\Delta W_{CQ}}{\delta q_j}-\frac{1}{4}\frac{\delta\bar{W}_{CQ}}{\delta q_i}(D_2^{-1})^{ij}(q,t,\vec{x})\frac{\delta\bar{W}_{CQ}}{\delta q_j}\Big],\end{aligned}\tag{6.72}$$

where we impose the restriction $8D_0\succeq D_2^{-1}$ to ensure the action takes the form in Equation (6.67) and is therefore completely positive. Note, the action in Equation (6.72) takes the same form as Equation (6.70) but now we let W_{CQ} be an *arbitrary* CQ proto-action. In Chapter 8 we show one can use Equation (6.72) to construct examples of Lorentz and diffeomorphism invariant dynamics.

6.6 Discussion

In this chapter we have discussed the path integral for general classical-quantum master equations, emphasising the necessary and sufficient conditions for the dynamics to be consistent and completely positive on the quantum system. In the general case we find a path integral representation of the dynamics with response variables, given by Equation (6.16), whilst for the class of continuous master equations, we were able to integrate out the response variables to arrive at the phase space path integral of Equation (6.41). Under certain conditions, namely when the action of Equation (6.41) is at most quadratic in classical (quantum) momenta, we can integrate out the classical (quantum) momenta to arrive at a configuration space path integral. For the case of minimally coupled Hamiltonian theories, we end up with a simple path integral representation, Equation (6.66), where the dynamics is completely encoded via the proto-action W_{CQ}. Given its final form, we posited that the resulting CQ action should be completely positive for an arbitrary proto-action, a result we prove via path integral methods in Chapter 8. We then studied the classical-quantum path integral for fields.

Applications of the CQ path integral. It would be interesting to explore possible applications of the path integral to standard quantum mechanical scenarios. Generally, we expect CQ dynamics to be a good effective description of a quantum system when one part behaves effectively classical [166]. A particularly relevant scenario is perhaps measurement based quantum control [96, 167], or coherent quantum control with dissipative resources [121, 166]. In Appendix 6.3.4 we introduced the path integral for the most general Markovian continuous measurement procedure one can perform. In this context, these path integral can be understood as an extension of [148, 149] which has proved useful in simulating quantum control tasks, particularly in the strong measurement limit where saddle point approximations are valid. We also expect that the path integral could be useful for certain systems in quantum chemistry where hybrid classical-quantum coupling has previously been used to study systems beyond the mean field approximation [120, 119].

Lorentz invariant collapse models via a classical field and relativistic measurement. We saw that the CQ path integral corresponded to a Lorentz invariant path integral which causes decoherence of the quantum state via interaction with a classical field. Using Equation (6.67) we are able to write down families of covariant models with a fundamentally classical field which naturally give rise to a decoherence mechanism on the quantum state. It would be interesting to explore such theories further in the context of relativistic collapse models [168, 169, 170, 171, 123, 172, 173]. The main difference between the CQ dynamics considered here and standard collapse models is that here, the quantum system becomes classicalised through it's interaction with a dynamical physical field, rather than an unobservable auxiliary field.

Renormalization of classical-quantum field theories. The renormalization of open quantum field theories was recently studied in [165, 174, 61]. It was found that open ϕ^4 theory was perturbatively renormalizable, and the complete positivity condition for the Lindblad Equation was peturbatively preserved under renormalization citebaidya2017renormalization.[5] On the other hand, open Yukawa theory was found to be non-renormalizable [174]. It would be interesting to explore the renormalization of classical-quantum field theories. This is even more highly constrained than the renormalization of open QFT, since not only does the Lindbladian coupling need to remain positive semi-definite, but so does the diffusion coupling, and for complete positivity one also demands that these be inversely related. Having a renormalizable theory of interacting classical and quantum fields would by itself be an interesting result. On the contrary, if it was found that CQ field theories are not renormalizable, and only valid as effective theories when a physical cutoff is imposed, this would have important consequences for theories which treat the gravitational field as being fundamentally classical.

Implications for classical-quantum gravity. Finally, it would be interesting to explore further the consequences of the path integral in understanding theories where the gravitational field is treated classically, both as a fundamental theory, but also as an effective theory where non-Markovian effects could be incorporated. In Chapter 8 we show one can construct diffeomorphism invariant theories of CQ gravity and it is would be worthwhile to explore these models further to better understand this type of dynamics and constraints it imposes.

[5]Though it should be said that the most general form of dynamics introduced in [165] is not CP because of the negative definite Lindbladian momentum coupling $\partial_\mu \phi^+ \partial^\mu \phi^-$ which enters into the master equation. Specifically, when one goes to the master Equation picture the resulting Lindbladian is not CP since the Gaussian momentum integrals are altered due to additional momentum couplings in the Lindbladian. We show this explicitely in Appendix J

Part III

Applications to gravity

Chapter 7

Constraints in classical-quantum gravity

Thus far, we have studied and developed the general formalism of completely-positive, linear, autonomous, classical-quantum dynamics. In Chapter 4, we arrived at the general form of continuous classical-quantum dynamics, while in Chapter's 5 and 6, we found unraveling and path integral formalism for CQ coupling. Armed with this formalism, we now study applications of the classical-quantum dynamics to gravity. The motivations are two-fold. Firstly, the lack of success in constructing a complete theory of quantum gravity valid beyond the Planck scale, combined with the lack of low energy signatures of quantum gravity, means the question of whether or not the gravitational field is quantum is still open for debate.

Many recent proposals have been made to measure low-energy gravitational phenomena that cannot be reproduced classically. Currently, the most promising experiments include those which aim to detect gravitationally induced entanglement in table-top experiments via spin entanglement witnesses [64, 53, 1, 2, 65, 66, 67, 68]. There have also been proposals to measure intrinsically quantum features of gravity without studying entanglement directly [69, 70]. Though undoubtedly exciting, current estimates suggest that the technology required to perform the experiments is decades away. We use the classical-quantum formalism to consider the question from the opposite direction: can we construct a consistent fundamental classical-quantum theory of gravity? Can we find experimental signatures of such theories which can be used as indirect tests for the quantum nature of gravity? These are precisely the theories that experiments measuring gravitationally induced entanglement would rule out.

One might also expect classical-quantum dynamics to be useful as an effective theory. While the assumptions that go into autonomous CQ dynamics are reasonable for a fundamental theory, none of the assumptions need to hold exactly, as an effective theory. For instance, if one allows for the non-Markovian evolution that generically arises in the study of open quantum systems, we necessarily violate the assumption of autonomous dynamics. Nonetheless, exploring the autonomous CQ dynamics in the gravitational context is worthwhile as a starting point. It may be useful in certain regimes, but more importantly, it can be used to gain insight into the challenges that may arise when attempting to construct a more complete semi-classical description.

In this chapter, we take a first step towards studying the constraints which arise in a full theory of classical space-time coupled to a quantum field. In the later parts of the thesis, the models and assumptions entering this chapter will ultimately be refined to produce more appealing models. In particular, in Chapter 8, we study a path integral approach to coupling classical and quantum fields, which looks more promising than a master equation approach, and in Chapter 9, we consider the Newtonian limit of CQ theories, which leads to a constraint which violates the assumption of purely deterministic classical dynamics which enters into this section. Part of the reason for this is that in this chapter, we study constraints for the jumping master equations with infinite moments, which take a different form to the continuous master equation considered in Chapters 8 and 9; at the time this work was completed, we had not yet discovered the general form of the continuous master equation.

In this chapter, we take a geometrodynamic approach and study gauge symmetries of classical-quantum dynamics using master equations and the ADM formalism discussed in Chapter 3. It is here that other attempts to reconcile quantum theory with gravity have failed. In loop quantum gravity, the constraint algebra does not close off-shell [71, 72]. However, it is hoped that one can find an operator ordering such that it will. String theory is background dependent, although it is hoped that string field theory will resolve this.

To recap, in the ADM formulation of classical gravity [77, 78], the dynamics is generated by a Hamiltonian $H_{ADM} = \int d^3x N(x)\mathcal{H}(x) + N^a(x)\mathcal{H}_a(x)$ containing freely chosen lapse $N(x)$, and shift $N^a(x)$ functions. In order to ensure that the dynamics do not depend on this choice, one must impose $\mathcal{H} \approx 0$, $\mathcal{H}_a \approx 0$, called the Hamiltonian and momentum constraint, respectively. When classical matter is included, one finds the constraints $\mathcal{H} + \mathcal{H}_m \approx 0$, $\mathcal{H}_a + \mathcal{H}_{m,a} \approx 0$.

We wish to study the regime where the matter degrees are treated quantum, but back-react on a classical space-time undergoing the combined evolution according to an autonomous CQ master equation. To do so, we must understand the analogous constraints in the combined classical-quantum case, which we study in this chapter. In particular, we study a class of classical-quantum theories of gravity that retain some desirable properties, namely local time reparameterization invariance and spatial diffeomorphism invariance. In such models, by applying a classical-quantum version of the Dirac argument, we can derive a set of constraints that must be satisfied for the theory to be independent of the choice of lapse and shift. In a purely classical theory, one can restrict phase space variables to lie in the constraint surface. In a quantum theory, one can impose the constraints as projectors onto a subspace of the Hilbert space. For a theory that combines classical and quantum degrees of freedom, a different method is required, especially since it is not, as far as we know, easily derived from an action principle – though we discuss progress towards this in the Chapter 8. We find that independence of the choice of lapse and shift requires "commutation" (in a certain sense) of the spatial and temporal parts of the equations of motion. For this commutation to hold, one finds a condition on the CQ-state analogous to the momentum constraint. Conservation of this condition in time leads to the Hamiltonian constraint. This methodology is the central result of the chapter.

We study these constraints explicitly in a theory of a quantum scalar field coupled to classical gravity. Although we are unable to construct a realization of the theory which satisfies the constraint conditions, at least without a further restriction on the choice of lapse and shift, we expect that the general procedure for deriving the constraints will enable further study CQ theories of gravity in a concrete setting. This chapter studies constraints for the jumping master equations with infinite moments. However, we emphasize that the methodology here should apply to the case of continuous master equations as well.

This chapter is based on the paper [3], which is work done in collaboration with Jonathan Oppenheim.

7.1 Hybrid dynamics with a Hamiltonian limit

In this chapter, we will consider the set of autonomous jumping CQ master equations with a Hamiltonian limit. Recall, the general form of master equation takes the form of Equation (2.94)

$$\frac{\partial \varrho(z,t)}{\partial t} = \int dz' \; W^{\mu\nu}(z|z',t) L_\mu \varrho(z') L_\nu^\dagger - \frac{1}{2} W^{\mu\nu}(z,t)\{L_\nu^\dagger L_\mu, \varrho\}_+, \tag{7.1}$$

where the bracket $\{,\}_+$ is quantum anti-commutator, which should be distinguished from the classical Poisson bracket, which we write as $\{,\}$, and $W(z)$ is defined as

$$W^{\mu\nu}(z,t) = \int \mathrm{d}z' W^{\mu\nu}(z'|z,t). \tag{7.2}$$

In Equation (7.1), the Lindblad operators L_μ are arbitrary. One can easily verify that (7.1) is trace-preserving, and together with the condition that

$$\Lambda^{\mu\nu}(z, t+\delta t|z',t) = \begin{bmatrix} \delta(z,z') + \delta t W^{00}(z|z',t) & \delta t W^{0\beta}(z|z',t) \\ \delta t W^{\alpha 0}(z|z',t) & \delta t W^{\alpha\beta}(z|z',t) \end{bmatrix} + O(\delta t^2) \tag{7.3}$$

be a positive matrix in μ, ν for all z, z', guarantees that the dynamics is completely positive. To isolate the purely classical degrees of freedom, we choose to decompose Equation (7.1) in terms of the operators $L_\mu = (\mathbb{I}, L_\alpha)$, where L_α is an arbitrary Lindblad operator. In this case, the master equation can be written in the form

$$\begin{aligned} \frac{\partial \varrho(z,t)}{\partial t} &= \int dz' \; W^{00}(z|z',t)\varrho(z') - \frac{1}{2} W^{00}(z,t)\varrho(z), \\ &+ \int dz' \; W^{\alpha\beta}(z|z',t) L_\alpha \varrho(z') L_\beta^\dagger - \frac{1}{2} W^{\alpha\beta}(z,t)\{L_\beta^\dagger L_\alpha, \varrho\}_+. \end{aligned} \tag{7.4}$$

While the master equation (7.1) is completely general, we wish to restrict to dynamics which becomes approximately Hamiltonian in the classical limit. This form of dynamics was introduced in [28]. We shall review the formalism since we use it to construct CQ theories of gravity, particularly those which reproduce the Hamiltonian formulation of Einstein's gravity once we also take the classical limit of the quantum system.

We take the classical degrees of freedom to live in a phase space $\Gamma = (\mathcal{M}, \omega)$, where ω is the symplectic form. We further assume the pure classical evolution to be generated by a classical

Hamiltonian H_c. While we believe one may need to consider dynamics where the purely classical dynamics is also stochastic (see Chapters 8 and 9), we here consider the restrictive and simpler case where it is deterministic. In this case, we can perform a moment expansion of Equation (7.4) and write it as

$$\begin{aligned}\frac{\partial \varrho(z,t)}{\partial t} &= \{H_c, \varrho(z,t)\} - i[H(z), \varrho(z)] + D_0^{\alpha\beta}(z) L_\alpha \varrho(z) L_\beta^\dagger - \frac{1}{2} D_0^{\alpha\beta}\{L_\beta^\dagger L_\alpha, \varrho(z)\}_+ \\ &+ \sum_{\mu\nu\neq 00} \sum_{n=1}^{\infty} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \dots \partial z_{i_n}}\right) \left(D_{n,i_1\dots i_n}^{\alpha\beta}(z) L_\alpha \varrho(z,t) L_\beta^\dagger\right).\end{aligned} \tag{7.5}$$

Now, we can define a phase space Hamiltonian vector field via $X_h^{\alpha\beta,i} = (\omega^{-1})^{ij} d_j h^{\alpha\beta}$. Here, $h^{\alpha\beta}(z)$ is some phase space functional, and d_i is the exterior derivative on the phase space. Here, the form of $h^{\alpha\beta}(z)$ is motivated by wanting to reproduce an interacting Hamiltonian of the form

$$H_I(z) = h^{\alpha\beta} L_\beta^\dagger L_\alpha. \tag{7.6}$$

By picking the interaction term $W^{\alpha\beta}(z|z')$ in (7.4) to be such that the vector of first moments takes the form $D_{1i}^{\alpha\beta}(z) = X_h^{\alpha\beta,j}$, the interacting part of the dynamics in (7.5) becomes

$$\begin{aligned}&D_0^{\alpha\beta}(z) L_\alpha \varrho(z,t) L_\beta^\dagger - \frac{1}{2} D_0^{\alpha\beta}(z) \left\{L_\beta^\dagger L_\alpha, \varrho(z,t)\right\}_+ + \{h^{\alpha\beta}, L_\alpha \varrho(z,t) L_\beta^\dagger\} \\ &+ \frac{\partial^2}{\partial z_{i_1} \partial z_{i_2}} (D_{2,i_1 i_2}^{\alpha\beta} L_\alpha \varrho(z,t) L_\beta^\dagger) + \dots \ .\end{aligned} \tag{7.7}$$

We see that defining $H_I(z) = h^{\alpha\beta}(z) L_\beta^\dagger L_\alpha$ and taking the trace over the quantum system gives us an effective classical equation of motion

$$\frac{\partial p(z,t)}{\partial t} = \{H_c, \varrho(z,t)\} + \mathrm{Tr}\left[\{H_I(z), \varrho(z)\}\right] + \dots \ , \tag{7.8}$$

where the $\dots$ denote the higher order terms in the moment expansion. In order to reproduce Hamiltonian dynamics, we imagine the scenario in which the higher order moments in (7.8) are suppressed by some order parameter [28, 10], such as for the example given in Section 2.3.3. Recall, this example was generated by a diagonal Hamiltonian $H_I(z) = h^\alpha(z) L_\alpha^\dagger L_\alpha$

$$\begin{aligned}\frac{\partial \varrho(z,t)}{\partial t} &= \{H_c(z), \varrho(z,t)\} - i\left[H_I(z), \varrho(z,t)\right] \\ &+ \frac{1}{\tau} \sum_\alpha \left(e^{\tau\{h^\alpha(z),\cdot\}} L_\alpha \varrho(z,t) L_\alpha^\dagger - \frac{1}{2}\left\{L_\alpha^\dagger L_\alpha, \varrho(z,t)\right\}_+\right),\end{aligned} \tag{7.9}$$

with τ an order parameter suppressing higher-order contributions to the dynamics.

For clarity, we can compare the Hamiltonian limit of CQ dynamics to the case where we have two classical systems, (z_1, z_2), which interact via an interaction Hamiltonian H_I. If the dynamics of the total system are given by

$$\frac{\partial p(z_1, z_2, t)}{\partial t} = \{H_1, \rho\} + \{H_2, \rho\} + \{H_I, \rho\} \tag{7.10}$$

then integrating out the second system, and defining $\bar{\rho}(z_1) = \int dz_2 \rho(z_1, z_2)$, we get an effective equation of motion

$$\frac{\partial \bar{\rho}(z_1)}{\partial t} = \{H_1, \bar{\rho}(z_1)\} + \int \mathrm{d}z_2 \{H_I, \rho(z_1, z_2)\}, \tag{7.11}$$

which justifies (7.8) as the appropriate classical limit to reproduce Hamiltonian dynamics. It also ensures the classical degrees of freedom undergo Hamiltonian evolution on average – a type of Eherenfest theorem for CQ dynamics. There are some ambiguities in the construction; for example, there is no unique decomposition of the Hamiltonian into Lindblad operators L_α, but this shall not be relevant to the discussion in this chapter.

A natural question, then, is to ask whether or not this formalism can be used to construct a consistent theory of CQ gravity, where the gravitational field is considered classical, while the matter fields are considered quantum. In particular, the CQ theory should reproduce general relativity in the classical limit of the quantum system. The subtleties arise since gravity is a gauge theory; not only do the degrees of freedom undergo dynamics generated by the pure gravity Hamiltonian but they must also live on the constraint surface. Hence, although we can use the formalism introduced in this section to construct CQ dynamics which reproduces that of gravity, we must find a way of studying the constraints in CQ theories.

7.2 CQ theory which reproduces Einstein gravity

This section reviews how one can use the CQ formalism introduced to construct CQ models of gravity, which reproduce Einstein's gravity [28]. We here restrict ourselves to the case where the CQ equation has the purely classical evolution generated by the ADM Hamiltonian, the pure quantum evolution generated by the Klein-Gordon (KG) Hamiltonian, and the interaction term a CQ dynamics, whose first moment is such that it approximates the Hamiltonian formulation of gravity in the classical limit.

7.2.1 Liouville formulation of classical gravity

Recall from Chapter 3 that in classical gravity, the relevant configuration space is given by the space of Riemannian metrics on a surface Σ and their conjugate momenta. Elements of the phase space are denoted (γ_{ab}, π^{ab}) and are taken to satisfy the canonical Poisson bracket relations

$$\{\gamma_{ab}(x), \pi^{cd}(y)\} = \frac{1}{2}(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c)\delta(x, y) \tag{7.12}$$

The dynamics are generated by the ADM Hamiltonian [77, 78]

$$H_{ADM}[N, \vec{N}] = \int d^3x N^\mu \mathcal{H}_\mu = \int d^3x (N\mathcal{H} + N^a \mathcal{H}_a) = H[N] + H[\vec{N}] \tag{7.13}$$

where

$$\mathcal{H} = (16\pi G)\pi^{ab} G_{abcd} \pi^{cd} - \frac{1}{16\pi G}\gamma^{1/2} R, \quad \mathcal{H}_a = -2\gamma_{ac} D_b \pi^{cb} \tag{7.14}$$

G_{abcd} is the deWitt metric defined as $G_{abcd} = \frac{1}{2\sqrt{g}}(\gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc} - \gamma_{ab}\gamma_{cd})$ and D_a the covariant derivative with respect to the metric γ_{ab} on Σ. In this chapter, we use units in which $c = 1$ to simplify our formulas.

The lapse function N and shift vector N^a appearing in Equation (7.13) are arbitrary functions of (t, x). They arise when performing the $3+1$ split of space-time and represent the gauge degrees of freedom associated with picking a foliation of space-time. They are non-dynamical since $P_N, P_{N^a} = 0$, and as a result, the Hamiltonian formulation of GR is a constrained theory. Asking that the constraints $P_N, P_{N^a} = 0$ be preserved in time leads to the Hamiltonian and Momentum constraints, $\mathcal{H} = \mathcal{H}_a = 0$. Conservation of these constraints is ensured via the hypersurface deformation algebra [108]

$$\begin{aligned}
&\{H[N], H[M])\} = H[\vec{R}] \\
&\{H[\vec{M}], H[N]\} = H\left[L_{\vec{M}} N\right] \\
&\{H[\vec{N}], H[\vec{M}]\} = H[L_{\vec{N}} \vec{M}]
\end{aligned} \tag{7.15}$$

where $R^a := \gamma^{ab}\left(N D_b M - M D_b N\right)$ and L is the Lie derivative on Σ.

Since we study gravitational CQ master equations, writing the dynamics of pure GR in a Liouville formulation is useful. In particular, a phase space distribution $\rho(\gamma, \pi)$ will evolve under the dynamics as

$$\frac{\partial \rho}{\partial t} = \{H_{ADM}, \rho\} \tag{7.16}$$

subject to the constraints $\mathcal{H}\rho = \mathcal{H}_a\rho = 0$; that is ρ must have support only on the constraint surface. There are a few things to be wary of when using a Liouville formalism. Firstly, we must remember that ρ is a distribution, so its action is only defined once smeared over phase space test functions. Furthermore, in the Liouville picture, we solve for $\rho(\gamma, \pi, t)$ given a choice of lapse and shift vector N^μ. The solution can then be interpreted as describing a probability density over trajectories $(\gamma_{ab}(t), \pi^{cd}(t))$, which, using the ADM split.

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{7.17}$$

we can use to define a probability distribution over 4-geometries $g_{\mu\nu}$. Since the trajectories are deterministic, we can imagine starting in a state of certainty $\rho \sim \delta(\bar{\gamma}, \gamma)\delta(\bar{\pi}, \pi)$, we can imagine picking a different lapse and shift for each point in phase space $N(t, \gamma, \pi), N^a(t, \gamma, \pi)$. Here, we take the weakest thing we can ask for, which is that the lapse and shift only depend on x, t, and we do not consider the more general case where they can be phase-space dependent.

We now consider coupling matter to gravity; for ease of calculation, we will study scalar fields coupled to gravity. A classical field minimally coupled to gravity will have a Hamiltonian of the form

$$H_T[N, \vec{N}] = H_{ADM}[N, \vec{N}] + H_m[N, \vec{N}] = \int d^3x N(\mathcal{H} + \mathcal{H}_m) + N^a(\mathcal{H}_a + \mathcal{H}_{m,a}), \tag{7.18}$$

where H_m is the Hamiltonian of the matter field. In the presence of matter, the constraint surface takes the form $\mathcal{H} + \mathcal{H}_m = 0, \mathcal{H} + \mathcal{H}_{m,a} = 0$.

For example, the Hamiltonian of the free scalar field reads

$$H_m[N, \vec{N}] = \int d^3x N(\frac{1}{2}\gamma^{-1/2}\pi^2 + \frac{1}{2}\gamma^{1/2}\gamma^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2}\gamma^{1/2}m^2\phi_\phi^2) + N^i\pi_\phi\partial_i\phi. \tag{7.19}$$

The Liouville equation for the phase space density $\rho(\gamma, \pi_\gamma, \phi, \pi_\phi, t)$ then takes the form

$$\frac{\partial \rho}{\partial t} = \{H[N, \vec{N}], \rho\} + \{H_m[N, \vec{N}], \rho\}, \tag{7.20}$$

where ρ must only have support on the constraint surface.

In order to gain parallels to the CQ theory, in particular when looking at the gravitational degrees of freedom alone, it is insightful to integrate out the ϕ, π_ϕ degrees of freedom to get an effective equation for the evolution of the gravitational degrees of freedom. Defining $\bar{\rho}(\gamma, \pi_\gamma) = \int D\phi D_{\pi_\phi} \rho(\gamma, \pi_\gamma, \phi, \pi_\phi)$ then integrating (7.20) over the matter degrees of freedom gives an effective equation of motion

$$\frac{\partial \bar{\rho}(\gamma, \pi_\gamma, t)}{\partial t} = \{H, \bar{\rho}\} + \int D\phi D_{\pi_\phi} \{H_m, \rho\}, \tag{7.21}$$

which is the version of Equation (7.8) if the matter were to be treated classically.

7.2.2 CQ theories of gravity

We now construct CQ theories of gravity whose dynamics become approximately that of Einstein's gravity in the classical limit. To be slightly more precise: following [28] and the toy models in [10], the interaction between the classical and quantum degrees of freedom causes the quantum state to change while at the same time causing back-reaction on the classical degrees of in a way which approximates Hamiltonian evolution generated by the ADM Hamiltonian.

To that end, we consider the CQ dynamics of Equation (7.5), where we take the pure classical dynamics to be generated by H_{ADM}, while we take the pure quantum evolution to be generated by the Klein-Gordon Hamiltonian H_m

$$H_m[N, \vec{N}] = \int d^3x N h^{\alpha\beta} L_\beta^\dagger L_\alpha + N^a p_a^{\alpha\beta} L_\beta^\dagger L_\alpha = \int d^3x N(x) \mathcal{H}_m(x) + N^a(x) \mathcal{H}_{m,a}(x), \tag{7.22}$$

which we now take to be a quantum object. Here we have defined

$$\begin{gathered} h^{\pi\pi} := \frac{1}{2}\gamma^{-1/2}, \quad h^{\phi\phi} = \frac{1}{2}\gamma^{1/2}, \quad h^{ab} = \frac{1}{2}\gamma^{1/2}\gamma^{ab} \\ p^{a\pi} = 1/2, \quad p^{\pi a} = 1/2 \\ L_\pi(x) = \pi_\phi(x), \quad L_\phi(x) = \phi(x), \quad L_a(x) = \partial_a \phi(x), \end{gathered} \tag{7.23}$$

and the now quantum field operators satisfy the canonical commutation relations

$$[\phi(x), \phi(y)] = 0, \quad [\pi_\phi(x), \pi_\phi(y)] = 0, \quad [\phi(x), \pi_\phi(y)] = i\delta(x, y). \tag{7.24}$$

The $\delta(x, y)$ is defined as a scalar in x and a scalar density in y. We shall not construct the precise nature of the Hilbert space $\mathcal{H}$; ultimately, we only exploit the algebraic properties of the canonical commutation relations, so the details of the Hilbert space are not of primary importance. However, let us briefly comment on the problem of defining the Hilbert space, an important open problem in the study of classical-quantum gravity.

Even for a free field in a fixed background $g_{\mu\nu}$, infinitely many unitarily in-equivalent representations of the commutation relations on a Hilbert space exist. In general, there is no notion of a preferred state for which a Hilbert space representation can be defined [175, 176]. As such, the modern view takes an algebraic approach when studying quantum fields in curved space. One instead views the algebraic (commutation) relations satisfied by the field observables as fundamental, which are taken to belong to an algebra $\mathcal{A}$. States ω are then defined as positive linear functionals on $\mathcal{A}$. In this viewpoint, the algebraic structure is unique, but there are infinitely many *representations* of the algebra on a Hilbert space. In particular, the GNS construction [177, 178] shows that every state ω on the algebra defines a Hilbert space $\mathcal{H}$, a representation of the algebra on the Hilbert space, and a Hilbert space vector corresponding to ω. Thus, though technically equivalent, the algebraic approach allows one to formulate Quantum field theory (QFT) in curved space in a way that is independent of the representation of the algebra and does not require one to single out a preferred state in the theory.

The classical-quantum case has a similar problem in defining the Hilbert space. It is simple enough to define some preferred Hilbert space $\mathcal{H}$ and take the field operators to act on $\mathcal{H}$: as an example, we could take the standard Hilbert space of free field in Minkowski space, defined by the existence of a Poincare invariant vacuum state [176]. We could then consider classical-quantum dynamics in this Hilbert space, but this arbitrary choice of Hilbert space is unsatisfactory. As for quantum fields in curved space, we can view classical-quantum dynamics and the commutation relations (7.24) algebraically – for example, by using the Heisenberg representation of classical-quantum dynamics introduced in [28]. What needs to be added in the CQ case is a version of the GNS construction: given a classical-quantum state which enables us to calculate probabilities for observables, we need to determine how to construct a Hilbert space representation for the dynamics uniquely. Since this is potentially a challenging and technical problem, we view the operators acting on the Hilbert space as formal expressions. We leave an algebraic formulation of classical-quantum dynamics as an interesting open problem for future research.[1]

Now that we have specified the purely classical and quantum dynamics, all that remains is to specify the classical-quantum interaction term, which describes the back reaction of the quantum system on the classical degrees of freedom. We call a specific choice of the CQ coupling a *realization* of a CQ theory. Firstly, we demand that the first moments of the Kramers-Moyal expansion in (7.5) contain a term

$$\int d^3x N(x) \frac{\delta h^{\alpha\beta}}{\delta \gamma_{cd}} L_\alpha \varrho L^\dagger_\beta \tag{7.25}$$

in order to yield Einstein's gravity in the classical limit. Consequently, we view Equation (7.25) as a condition on CQ theories of gravity that a sensible realization of CQ dynamics must satisfy. Here the α, β indices run over $\phi, \pi_\phi, \nabla_a\phi$. In order to study CQ dynamics in a concrete setting, we shall make some further assumptions about the realizations. Firstly, we assume that the realizations are local so that the CQ interaction is fully specified by the set of transition amplitudes $W^{\alpha\beta}(z|z', x)$, $W^{\alpha\beta}_a(z|z', x)$, representing the generalizations of the quantum matter Hamiltonian density $\{\mathcal{H}_m, \}$ and momentum density $\{\mathcal{H}_{m,a}, \}$. Secondly, we shall focus on a natural class of dynamics, namely those with CQ couplings $W^{\alpha\beta}(z|z')$, which are linear in N and N^a.

There is a good reason to do this. From a physical standpoint, the free functions N and N^a represent the local time reparameterization invariance and space-like diffeomorphism invariance of the underlying theory. A natural question is whether we can have a CQ theory of gravity that upholds these symmetries. Furthermore, from a technical point of view, if we were to consider non-linear couplings in the lapse and shift – say we had an N^2 coupling – then the method we use to derive the constraints would lead to a constraint which itself depends on N. Preservation of such a constraint then leads to N becoming dynamical, which generically leads to a gauged fixed theory; this happens, for example, in Horava gravity [179, 180, 181]

[1] Note that the problem in defining a Hilbert space representation is compounded relative to the semi-classical case because the space-time metric is now dynamical and background independent: solutions are described by a probability distribution over 4-metrics, where each metric is associated with a quantum state. This is perhaps both a blessing and a curse; having a dynamical space-time makes it harder to find an algebraic formulation of classical-quantum dynamics, but it perhaps provides an interesting intermediate arena to study properties of QFT beyond the semi-classical regime.

(although there are exceptions, where additional secondary constraints fix N up to a global reparameterization invariance, for example in shape dynamics [182, 183, 184]). We leave gauge fixed CQ theories as a possible area for further study.

Assuming that realizations are linear in the lapse and shift, we are led to study dynamics of the form

$$\begin{aligned}\frac{\partial \varrho(z)}{\partial t} &= \{H_{ADM}, \varrho(z)\} - i[H_m, \varrho(z)] \\ &+ \int dz' \int d^3x \Big[(NW^{\alpha\beta}(z|z',x) + N^a W_a^{\alpha\beta}(z|z',x))L_\alpha(x)\varrho(z')L_\beta^\dagger(x)\Big] \\ &- \frac{1}{2}\int d^3x[(NW_0^{\alpha\beta}(z) + N^a W_{0a}^{\alpha\beta}(z))\{L_\beta^\dagger(x)L_\alpha(x), \varrho(z)\},\end{aligned} \tag{7.26}$$

where the first moments of the realizations satisfy Equation (7.25). Here, z labels points in the phase space of GR, $z = (\gamma_{ab}, \pi^{cd})$, and writing out equation (7.26) in full we obtain

$$\begin{aligned}&\frac{\partial \varrho(\gamma,\pi)}{\partial t} = \{H_{ADM}, \varrho(\gamma,\pi)\} - i[H_m, \varrho(\gamma,\pi)] \\ &+ \int \mathcal{D}\gamma'\mathcal{D}\pi' \int d^3x \Big[(NW^{\alpha\beta}(\gamma,\pi|\gamma',\pi',x) + N^a W_a^{\alpha\beta}(\gamma,\pi|\gamma',\pi',x))L_\alpha(x)\varrho(\gamma',\pi')L_\beta^\dagger(x)\Big] \\ &- \frac{1}{2}\int d^3x[(NW_0^{\alpha\beta}(\gamma,\pi) + N^a W_{0a}^{\alpha\beta}(\gamma,\pi))\{L_\beta^\dagger(x)L_\alpha(x), \varrho(\gamma,\pi)\}.\end{aligned} \tag{7.27}$$

In Equation (7.27), the integral over $\int \mathcal{D}\gamma\mathcal{D}\pi$ is to be treated as a formal integral over all configurations of Riemmanian 3 metrics γ_{ab} and their conjugate momenta π^{ab}, and we do not make any attempt to justify its existence rigorously. To simplify notation, we will often suppress the phase space integrals and write, for example, $\int dz' W^{\alpha\beta}(z|z',x)\varrho(z') = W^{\alpha\beta}(x)(\varrho)$ indicating that $W^{\alpha\beta}(x)$ is a differential (or in other cases, a CQ CP map) acting on ϱ. Furthermore, we will often write the master equation in a more compact form as

$$\frac{\partial \varrho(\gamma,\pi)}{\partial t} = \int d^3x N\mathcal{L}(\varrho) + N^a\mathcal{L}_a(\varrho), \tag{7.28}$$

implicitly defining

$$
\begin{aligned}
\mathcal{L}(x)(\varrho) = \{\mathcal{H}(x), \varrho\} - i[\mathcal{H}_m(x), \varrho] \\
+ W^{\alpha\beta}(x) L_\alpha(x) \varrho L^\dagger_\beta(x) - \frac{1}{2} W_0^{\alpha\beta}(z,x) \{L^\dagger_\beta(x) L_\alpha(x), \varrho\}
\end{aligned}
\tag{7.29}
$$

$$
\begin{aligned}
\mathcal{L}_a(x)(\varrho) = \{\mathcal{H}_a(x), \varrho\} - i[\mathcal{H}_{m,a}(x), \varrho] \\
+ W_a^{\alpha\beta}(x) L_\alpha(x) \varrho L^\dagger_\beta(x) - \frac{1}{2} W_{0a}^{\alpha\beta}(z,x) \{L^\dagger_\beta(x) L_\alpha(x), \varrho\}.
\end{aligned}
\tag{7.30}
$$

Equation (7.26) gives us a class of hybrid theories that give the dynamics of GR in their classical limit. Of course, things are more complicated than this since GR is a constrained system; not only do the degrees of freedom undergo dynamics generated by the ADM Hamiltonian, but they must also lie on the constraint surface. Hence, we expect constraints to enter any CQ theory of gravity that gives all the components of Einstein's equations.

In the usual picture, the constraints come directly from an action principle – which we do not have here, though we make progress towards this in Chapter 8. How, then, can we impose constraints on the CQ theory? It is known in classical GR that it is possible to derive the constraints by exploiting the algebroid nature of the hypersurface deformation algebra [108] (see also [106, 185, 186, 187, 188, 189, 107, 190], for discussion of the deformation algebra in other contexts). Similar consistency conditions can be found by considering the Dirac argument. If the equations of motion contain arbitrary functions of time, such as the lapse and shift, then to retain predictability, two solutions related by a different choice of the arbitrary function must be gauge equivalent [106]. The subsequent section shows how it leads to a constraint surface in GR. We then apply this to the CQ theory and use it to derive generalized Hamiltonian and momentum constraints in CQ theories of gravity.

7.3 Deriving constraints from gauge conditions

In this section, we briefly review the Dirac argument [106] for gauge theories. By applying a Dirac-like argument to GR, we then show that one can arrive at a set of consistency conditions that lead to constraints on the theory. The mathematics is similar to that used in HKT [108], which exploits the algebroid nature of GR to arrive at the constraint surface. We then extend this method to the case of CQ master equations which provides a methodology to derive a set of constraints for CQ theories that are linear in N, N^a, which is a central result of this chapter.

7.3.1 Dirac argument for Hamiltonian systems

Hamiltonian gauge theories generically have actions of the form [106, 191]

$$I[q^i, p_j, \lambda^a] = \int dt(p_i\dot{q}^i - H_0(p_i, q^j) - \lambda^a\phi_a(p_i, q_i)), \tag{7.31}$$

which define equations of motion and constraints

$$\dot{q}^i = \frac{\partial H_0}{\partial p_i} - \lambda^a\frac{\partial\phi_a}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_0}{\partial q_i} + \lambda^a\frac{\partial\phi_a}{\partial q_i}, \quad \phi_a = 0. \tag{7.32}$$

Preservation of the constraints requires that

$$\frac{d\phi_a}{dt} = [\phi_a, H_0] - [\phi_a, \phi_b]\lambda^b \approx 0. \tag{7.33}$$

where $\approx$ means the equation must be weakly zero, meaning it must vanish on the constraint surface [106]. Gauge theories are characterized by having $[\phi_a, \phi_b] \approx 0$, in which case the constraints are said to be *first class*. In this case, the $\lambda^a(t)$ remains undetermined, and the equations of motion contain arbitrary functions of time, as will their solutions. Gauge theories then generically have Hamiltonian's of the form

$$H_T = H_0 + \lambda^a(t)\phi_a, \tag{7.34}$$

where the $\lambda^a(t)$ are arbitrary and they multiply constraints. Usually, it is said that first-class constraints generate equal time gauge transformations. We can run the Dirac argument [106] as follows to see why this is. Suppose we have some initial data (q^i, p_j) at $t = 0$. Since the $\lambda(t)$ is undetermined, we can use either $\lambda(t)$ or $\lambda'(t)$ to solve the equations of motion. Let $f(q^i, p_j)$ be any functional over the phase space. Then at time $t = \epsilon$

$$f(\epsilon) = f(0) + \epsilon\{f(0), H_0\} + \epsilon\lambda^a\{f(0), \phi_a\} \tag{7.35}$$

$$f'(\epsilon) = f(0) + \epsilon\{f(0), H_0\} + \epsilon\lambda^{a\prime}\{f(0), \phi_a\}. \tag{7.36}$$

Consequently, given the same initial data set, we get two equally valid descriptions. We must identify these descriptions as physically equivalent to retain predictability. The difference between the two solutions is

$$\delta f(\epsilon) = \epsilon(\lambda' - \lambda)\{f(0), \phi_a\}, \tag{7.37}$$

and hence $\{\phi_a\}$ generates equal time gauge transformations. As discussed in Chapter 3, there is a subtle difference between first-class constraints generating gauge transformations on initial data - or data on a time slice - which, once the dynamics are solved, give classes of physically equivalent solutions, and gauge transformations which act on the space of solutions directly. The equal time gauge transformations are to be understood in the former sense – as gauge transformations between initial data configurations [109].

Given that constraints generate gauge transformations, it should then be true that the generators form an algebra (which follows from linearity and demanding that the gauge transformations be transitive). For first-class constraints, we have

$$\{\phi_a, \{\phi_b, f\}\} - \{\phi_b, \{\phi_a, f, \}\} = \{C^c_{ab}\phi_c, f\} = \{C^c_{ab}, f\}\phi_c + C^c_{ab}\{\phi_c, f\} \approx C^c_{ab}\{\phi_c, f\}, \quad (7.38)$$

so that the gauge generators close as an algebra on the constraint surface. Note that this holds even for the case of GR, where the C^a_{bc} are structure functions that depend on the phase space.

7.3.2 Deriving the constraint surface of GR from the Dirac argument

We now show how using the Dirac argument for classical gravity leads to constraints. Suppose we have a state $\rho(\gamma, \pi)$ which evolves according to the ADM equations of motion

$$\frac{\partial \rho}{\partial t} = \int d^3x \{N\mathcal{H} + N^a\mathcal{H}_a, \rho\}. \quad (7.39)$$

We will not take $N(t, x), N^a(t, x)$ to have a functional dependence on γ, π, the weakest condition we can ask for. We will have to be careful since the states $\rho(\gamma, \pi)$ are defined only in the distributional sense, so things like Poisson brackets are only defined by their action on test functions. We denote the smeared distributions as

$$\langle A, \rho\rangle = \int D\gamma D\pi \rho(\gamma, \pi, t) A(\gamma, \pi, t). \quad (7.40)$$

Since the lapse and shift functions are arbitrary $\{\mathcal{H}, \}$ and $\{\mathcal{H}_a, \}$ generate equal time gauge transformations. As a consequence

$$\langle A, \rho\rangle \sim \langle A, \rho\rangle + \epsilon\langle\{A, \mathcal{H}_a(x)\}, \rho\rangle. \quad (7.41)$$

Since all observables, $A(\gamma, \pi, t)$ must be independent of the choice of gauge, they must satisfy $\langle\{A(\gamma, \pi, t), \mathcal{H}_a(x)\}, \rho\rangle = 0$. In particular, picking $\rho = \delta(\bar{\gamma}, \gamma)\delta(\bar{\pi}, \pi)$, we see observables must satisfy

$$\{A, \mathcal{H}_a(x)\} \approx 0. \tag{7.42}$$

Note, we are not saying that time evolution is a gauge transformation and observables are frozen. The gauge generators take one initial configuration to another; both dynamically evolved in time. Furthermore, allowing for the case where the observables contain an explicit time dependence – which usually occurs once one has gauge fixed [192] – means they evolve in time even though the Poisson bracket with the Hamiltonian vanishes.

Demanding that the gauge transformations close as an algebra, which follows from asking that the equivalence relation be linear and transitive, we must have

$$\{H[N], \{H[M], \rho\}\} - \{H[M], \{H[N], \rho\}\} \approx \text{Gauge}(\rho). \tag{7.43}$$

When Equation (7.43) is smeared over observables, we see that the left-hand side must vanish since we know that observables Poisson commute with the constraints.

We now use the Jacobi identity and the hypersurface deformation algebra defined in Equation (7.15) to write Equation (7.43) as

$$\{\{H[N], H[M]\}, \rho\} = \int d^3x \{\gamma^{ab} \delta N_b \mathcal{H}_a(x), \rho\} \approx \text{Gauge}(\rho), \tag{7.44}$$

where we have defined $\delta N_b = N\partial_b M - M\partial_b N$. Smearing Equation (7.44) over a phase space smearing function A, one finds the left hand side of Equation (7.44) is

$$\int D\gamma D\pi \int d^3x \{\gamma^{ab}\delta N_b \mathcal{H}_a(x), \rho\} A = \int d^3x \langle \delta N_b \{A, \gamma^{ab}\}, \mathcal{H}_a \rho\rangle + \langle \{A, \mathcal{H}_a\} \delta N_b \gamma^{ab}, \rho\rangle. \tag{7.45}$$

When A is an observable, the final term in Equation (7.45) vanishes as it acts as a gauge transformation, and so we see that for the algebra of gauge transformations to close

$$\langle \{A, \gamma^{ab}\}, \mathcal{H}_a \rho\rangle \approx 0. \tag{7.46}$$

In other words, we require $\langle \{A, \gamma^{ab}\}, \mathcal{H}_a \rho\rangle = 0$ whenever A is a phase space observable. This condition is satisfied by the constraints of GR since $\langle A, \mathcal{H}_a \rho\rangle = 0$ for all phase space functions A when on the constraint surface. Preservation of this condition is guaranteed by the Hamiltonian constraint $\langle A, \mathcal{H}\rho\rangle = 0$, as can be seen by applying the evolution equation on it and using the Jacobi identity. However, the consistency condition in Equation (7.46) is weaker than asking the constraints hold. As a trivial example, we can consider a theory in which the only allowed observable is $A = I$, then Equation (7.46) holds for any ρ on the phase space.

In general, we shall ask: is there a set of sensible observables and states for which consistency conditions – such as that in equation (7.46) – hold and which are preserved in time? This is similar to the *on-shell closure* studied in loop quantum gravity [188]. For dynamics generated by the ADM Hamiltonian, it seems there is no non-trivial set of states/observables other than that of GR for which Equation (7.46) holds. The problem is that we can keep applying gauge transformations to generate more and more independent constraints. For example, suppose that Equation (7.46) is satisfied, then applying a spatial gauge transformation, we also require that

$$\langle \{\{\gamma^{ab}, A\}, \mathcal{H}_c\}, \mathcal{H}_b \rho \rangle \approx 0, \tag{7.47}$$

and similarly for the gauge transformation generated by the Hamiltonian. We can seemingly continue to do this indefinitely. We return to this subtlety for the CQ theory at the end of Section (7.5). For now, we will take the consistency conditions in a weak form asking whether or not there exists a sensible set of observables and states for which they hold.

7.3.3 Dirac argument for CQ master equations

The Dirac argument extends naturally to the CQ theory. In particular, suppose we are given a CQ master equation of the form

$$\frac{\partial \varrho}{\partial t} = \mathcal{L}_0(\varrho) + \lambda^a(t)\mathcal{L}_a(\varrho), \tag{7.48}$$

where the $\lambda^a(t)$ are undetermined functions of time. We can run the Dirac argument on ϱ. Without loss of generality, consider having an initial CQ state $\varrho(z, 0)$ at $t = 0$. The master equation depends on some arbitrary functions of time $\lambda^a(t)$. Consider picking two different functions $\lambda^a, \lambda^{a\prime}$. Then at time $t = \epsilon$ the solutions are

$$\varrho(\epsilon) = \varrho(0) + \epsilon\mathcal{L}_0(\varrho(0)) + \epsilon\lambda^a\mathcal{L}_a(\varrho) \tag{7.49}$$

$$\varrho'(\epsilon) = \varrho(0) + \epsilon\mathcal{L}_0(\varrho(0)) + \epsilon\lambda^{a\prime}\mathcal{L}_a(\varrho(0)). \tag{7.50}$$

The difference between the two solutions is

$$\delta\varrho(\epsilon) = \epsilon(\lambda^{a\prime} - \lambda^a)\mathcal{L}_a(\varrho(0)), \tag{7.51}$$

from which we conclude that $\mathcal{L}_a(\varrho)$ is generating equal time gauge transformations. We should then ask that the generators close as an algebra

$$\mathcal{L}_a(\mathcal{L}_b(\varrho)) - \mathcal{L}_b(\mathcal{L}_a(\varrho)) \approx C^c_{ab}\mathcal{L}_c(\varrho). \tag{7.52}$$

Since the classical part of the CQ state may only be defined in a distributional sense, we can re-state equation (7.52) as asking that

$$\langle A, \mathcal{L}_a(\mathcal{L}_b(\varrho)) - \mathcal{L}_b(\mathcal{L}_a(\varrho))\rangle \approx 0 \tag{7.53}$$

whenever A is an observable. Similar to the case of classical GR in the previous subsection, we will see that in CQ theories of gravity, the generators only close as an algebra if we are on a constraint surface. We will ask that the constraints are satisfied in the weakest sense. That is, we will look for a non-trivial set of states and observables for which the consistency condition is satisfied and preserved in time. Using this method, we can derive a generalized momentum constraint for the CQ theory; asking that this constraint is preserved will lead to a generalized Hamiltonian constraint.

7.4 Deriving constraints in post-quantum theories of Gravity

In this section, we will use the Dirac argument to derive consistency conditions for CQ theories of gravity. We expect the methods used here to extend to all CQ theories of the form introduced in Section 7.2, namely those linear in the lapse and shift. We briefly outline a general procedure for generating consistency conditions and constraints in such theories. We spend the remainder of the chapter performing the explicit calculations for a specific class of CQ theories of a quantum scalar field coupled to gravitational degrees of freedom.

7.4.1 A general method of arriving at constraints

We consider CQ dynamics linear in the lapse and shift, reproducing Hamiltonian evolution in the classical limit. In particular, we will consider we consider the simplified theory of Equation (7.26)

$$\frac{\partial \varrho(\gamma, \pi)}{\partial t} = \int d^3x N\mathcal{L}(\varrho) + N^a\mathcal{L}_a(\varrho) = \mathcal{L}[N] + \mathcal{L}[\vec{N}], \tag{7.54}$$

where

$$\begin{aligned}\mathcal{L}(x)(\varrho) &= \{\mathcal{H}(x), \varrho\} - i[\mathcal{H}_m(x), \varrho] \\ &\quad + W^{\alpha\beta}(x) L_\alpha(x) \varrho L_\beta^\dagger(x) - \frac{1}{2} W_0^{\alpha\beta}(x) \{L_\beta^\dagger(x) L_\alpha(x), \varrho\}\end{aligned} \tag{7.55}$$

$$\begin{aligned}\mathcal{L}_a(x)(\varrho) &= \{\mathcal{H}_a(x), \varrho\} - i[\mathcal{H}_{m,a}(x), \varrho] \\ &\quad + W_a^{\alpha\beta}(x) L_\alpha(x) \varrho L_\beta^\dagger(x) - \frac{1}{2} W_{0a}^{\alpha\beta}(x) \{L_\beta^\dagger(x) L_\alpha(x), \varrho\}.\end{aligned} \tag{7.56}$$

The Dirac argument tells us that data, or CQ states, related by different choices of N, N^a must lead to gauge equivalent solutions when the dynamics are solved, i.e., that $\mathcal{L}(x), \mathcal{L}_a(x)$ generate equal time gauge transformations.

As a consequence, denoting the dual operators (under $\int dz$) to $\mathcal{L}$ and $\mathcal{L}_a$ as $\mathcal{L}^*, \mathcal{L}_a^*$ respectively, this implies that

$$\int DgD\pi \mathcal{L}^*(A)\varrho \approx 0, \quad \int DgD\pi \mathcal{L}_a^*(A)\varrho \approx 0 \tag{7.57}$$

for all observables A.[2] In Section 7.5, we will only consider realizations that diffuse in the conjugate momenta π^{ab}. Hence, we expect that any constraints will be solved by functions of the form $\varrho(\gamma, \pi) = \delta(\gamma, \bar{\gamma})\varrho_c(\gamma, \pi)$ since the commutation relations will never lead to γ_{ab} diffusive terms.[3] As a consequence, we expect that

$$\int D\pi \mathcal{L}^*(A)(\bar{\gamma}, \pi)\varrho_c(\bar{\gamma}, \pi) \approx 0, \quad \int D\pi \mathcal{L}_a^*(A)(\bar{\gamma}, \pi)\varrho_c(\bar{\gamma}, \pi) \approx 0. \tag{7.58}$$

Multiplying Equation (7.58) by an arbitrary function of the metric $f(\bar{\gamma})$, putting back the delta $\delta(\gamma, \bar{\gamma})$, and integrating over γ, we expect more generally that

$$\int D\gamma D\pi \mathcal{L}^*(A) f(\gamma)\varrho \approx 0, \quad \int D\gamma D\pi \mathcal{L}_a^*(A) f(\gamma)\varrho \approx 0, \tag{7.59}$$

so that the gauge transformation can have a metric dependence on the lapse and shift. The requirement that the gauge transformations are transitive leads to consideration of the algebra of generators

$$[\mathcal{L}_a(x), \mathcal{L}_b(y)], \quad [\mathcal{L}_a(x), \mathcal{L}(y)], \quad [\mathcal{L}(x), \mathcal{L}(y)], \tag{7.60}$$

[2]Here, the dual operators are defined via integration by parts $\int DgD\pi A\mathcal{L}(\varrho) = \int DgD\pi \mathcal{L}^*(A)\varrho$.

[3]This assumption turns out to be violated by the improved class of theories we discuss in Chapters 8 and 9.

where once again we remind the reader that $\mathcal{L}, \mathcal{L}_a$ are classical-quantum operators acting on ϱ, in the sense of Equation (7.54), where the couplings $W^{\alpha\beta}(x)$ are to be interpreted as kernels $W^{\alpha\beta}(x)(\varrho) = \int dz' W^{\alpha\beta}(z|z', x)\varrho(z')$, and similarly for $W_a^{\alpha\beta}(x)$. For example, $[\mathcal{L}_a(x), \mathcal{L}_b(y)](\varrho) = \mathcal{L}_a(x)(\mathcal{L}_b(y)(\varrho)) - \mathcal{L}_b(x)(\mathcal{L}_a(y)(\varrho))$.

We require that the algebra closes; the commutator of two gauge transformations is weakly another gauge transformation, which vanishes when smeared over an observable $A \in O_{obs}$. Similarly to GR, we will see that demanding this leads to a notion of a constraint surface. We will often deal with the spatially smeared versions of these transformations and denote $\mathcal{L}[N] = \int d^3x N(x)\mathcal{L}(x)$ and $\mathcal{L}[\vec{N}] = \int d^3x N^a(x)\mathcal{L}_a(x)$.

7.5 A CQ theory of gravity coupled to a scalar field

Now that we have a general method of deriving constraints in CQ theories, we spend the remainder of the chapter exploring the consistency conditions for a quantum scalar field interacting with a classical gravitational field. We make several simplifying assumptions so that the theory considered here is a special case of the one derived in [28]. We outline all of our assumptions as follows

Assumption 1. *We take the evolution to be linear in the lapse and shift and consider the choice of N and $\vec{N}$ to be pure gauge.*

Assumption 2. *We will take the $W_a^{\alpha\beta}(x) = 0$ so that $\mathcal{L}_a(\varrho) = \{\mathcal{H}_a(x), \varrho\} - i[\mathcal{H}_{m,a}(x), \varrho]$ generates spatial diffeomorphisms and the theory will be spatially diffeomorphism invariant.*

Assumption 3. *We take the CQ couplings $W^{\alpha\beta}(z|z')$ to have the same Lindblad structure as the Hamiltonian so that the interaction terms can be written*

$$W^{\alpha\beta}(x)L_\alpha(x)\varrho L_\beta^\dagger(x) = W^{\phi\phi}(x)\phi(x)\varrho\phi(x) + W^{\pi\pi}(x)\pi_\phi(x)\varrho\pi_\phi(x)$$
$$+ W^{ab}(x)\partial_a\phi(x)\varrho\partial_b\phi(x). \tag{7.61}$$

In particular, $W^{\alpha\beta}(x)$ is taken to be local in x, while the more general case could be non-local and include a regulator $W^{\alpha\beta}(x-y)L_\alpha(x)\varrho L_\beta^\dagger(y)$.

Assumption 4. *We take the CQ coupling with the scalar field, in analogy with classical gravity, to have a functional dependence on the spatial metric only;* $W^{\alpha\beta}[\gamma](x)$*. We call such a coupling minimal coupling.*

Assumption 5. *We take the first moment of* $W^{\alpha\beta}(x)$ *to reduce to General Relativity in the classical limit so that* $\mathrm{Tr}\left[W^{\alpha\beta}(x)L_\alpha(x)\varrho L^\dagger_\beta(x)\right] = \mathrm{Tr}\left[\{H_m, \varrho\}\right]$*. Since one generally considers matter Hamiltonians which only depend on* γ_{ab} *and not* π^{ab}*, this motivates our previous assumption of minimal coupling.*

Assumption 6. *The CQ term couples states with different momenta* π^{ab} *only; we only jump in momentum. We call such theories* $\pi - dispersive$*, while in the more general case, one can have both dispersion in the* π^{ab} *and dispersion in* γ_{ab}*. If both terms are present, then the relationship between* π^{ab} *and* $\dot{\gamma}_{ab}$ *exists only on the level of expectation values.*

Assumption 7. *We take the pure gravity part of the master equation to be deterministic and given by general relativity* $\{H_{ADM}, \varrho\}$*. In the more general case, the pure gravity evolution can be stochastic.*

Assumption 1 is respected in Einstein's theory of General Relativity, but Einstein also considered what is now known as the unimodular theory of gravity [193, 194, 195, 196, 197, 198, 199] where N is chosen so that the cosmological constant becomes an integration constant. Assumption 1 also doesn't hold in Horava gravity [179, 180, 181] and shape dynamics [182, 183, 184], but here we explore the consequences of taking the full gauge symmetry. Assumption 3 seems reasonable when considering the jumping master equations. When computing the gauge transformations' commutators, it must be the case that $W^{\alpha\beta}$ term must transform like the Hamiltonian, or else the pure classical part and pure quantum part of the evolution will transform differently to the CQ coupling. This assumption is, however, violated for the continuous master equations since they have a different structure, which implements back-reaction via the off-diagonal elements such as $D_1 1, i^{\alpha 0} L_\alpha \varrho(z) + c.c.$ One could also imagine a theory with alternative, higher order, Lindblad operators that have the correct transformation properties – such as $W^{\phi\phi\phi\phi}(x)\phi\phi\varrho\phi\phi$ – this type of higher order coupling seems to be implied by the covariant path integral approach we discuss in Chapter 8. However, we do not discuss such theories in this chapter.

We can summarize Assumptions 4 and 6 as considering CQ couplings which take the form $W^{\alpha\beta}(z|z', x) = W^{\alpha\beta}(\gamma, \pi|\gamma', \pi', x)$, where the moments of $W^{\alpha\beta}(\gamma, \pi|\gamma, \pi', x)$ depend only on γ_{ab} and not π^{ab}. These assumptions are motivated twofold. Firstly, by analogy to pure classical gravity. There, when one considers minimally coupled scalar fields, the interaction term is of the form

$$W^{\alpha\beta}_{classical}(x)(\rho) = \int d^3y \frac{\delta \mathcal{H}_m(x)}{\delta\gamma^{ab}(y)} \frac{\delta\rho}{\delta\pi^{ab}(y)} = \int d^3y \frac{\delta h^{\alpha\beta}(x)}{\delta\gamma^{ab}(y)} L^\dagger_\beta L_\alpha \frac{\delta\rho}{\delta\pi^{ab}(y)}, \tag{7.62}$$

which only couples states with different momenta, and the coupling only has a functional dependence on the spatial metric through $\frac{\delta h^{\alpha\beta}}{\delta\gamma^{ab}}$. Secondly, since we end up calculating the commutation relations in (7.60), which includes Poisson brackets with the pure classical Hamiltonian and momentum, Assumptions 3, 4, 6 seem natural, though we shall find in Chapters 8 and 9 that more promising CQ theories violate the assumption of being only π diffusive.

Having made these simplifying assumptions, we are now ready to study their constraints. We have yet to specify the CQ coupling explicitly, except for some Lindbladian structure and functional dependence we would like it to have. As we shall see in the next section, there are various transformation properties that any realization must satisfy. We then study the constraints in such realizations, which arise from studying the algebra of equal time gauge generators in Equation (7.60). In particular, we find that the $[\mathcal{L}_a(x), \mathcal{L}_b(y)]$ closes, which is an expected artifact of the fact we are assuming (Assumption 2) $\mathcal{L}_a$ generates spatial diffeomorphisms. We find the $[\mathcal{L}_a(x), \mathcal{L}(y)]$ generator closes so long as the couplings $W^{\alpha\beta}(z|z', x)$ satisfy certain transformation rules; essentially telling us that $W^{\alpha\beta}(z|z', x)$ must transform correctly under spatial diffeomorphisms. Finally, we study the $[\mathcal{L}(x), \mathcal{L}(y)]$ commutator. which leads to a generalization of the momentum constraint to CQ dynamics. We find that the preservation of this constraint gives rise to a CQ analog of the Hamiltonian constraint.

$[\mathcal{L}_a(x), \mathcal{L}_b(y)]$ commutator for the scalar field

Due to Assumption 2, we are considering the case where $W^{\alpha\beta}_a = 0$, so that the total CQ momentum generator $\mathcal{L}_a$ reads

$$\mathcal{L}_a(x)(\varrho) = \{\mathcal{H}_a(x), \varrho\} - i[\mathcal{H}_{m,a}(x), \varrho]. \tag{7.63}$$

One can then easily verify that

$$[\mathcal{L}[\vec{N}], \mathcal{L}[\vec{M}]] = \mathcal{L}[\vec{N}, \vec{M}], \tag{7.64}$$

which vanishes when smeared over an observable since $\mathcal{L}_a$ is treated as gauge. Since Equation (7.64) represents the lie algebra of spatial diffeomorphisms, this verifies that Equation (7.63) is the generator of spatial diffeomorphisms.

$[\mathcal{L}_a(x), \mathcal{L}(y)]$ commutator for the scalar field

We now compute the $[\mathcal{L}_a(x), \mathcal{L}(y)]$ commutator. With the interpretation that $\mathcal{L}_a$ generates spatial diffeomorphisms, we find that the algebra closes so long as we pick the realizations to have the correct transformation properties under spatial diffeomorphisms. In total, we find the (smeared) commutation relation

$$\begin{aligned}
&[\mathcal{L}[\vec{N}], \mathcal{L}[N]](\rho) = \int d^3x \, [N^a D_a N \mathcal{L}(\rho)] \\
&+ \int d^3x N N^a \left[L_\alpha D_a(W^{\alpha\beta}\varrho) L_\beta^\dagger - \frac{1}{2}\{L_\beta^\dagger L_\alpha, D_a(W_0^{\alpha\beta}\varrho)\}\right] \\
&+ \int d^3x D_a N^a \left[W^{\phi\phi}\phi\varrho\phi - \frac{1}{2}\{W_0^{\phi\phi}\phi^2, \varrho\}\right] - D_a N^a \left[W^{\pi\pi}\pi\varrho\pi - \frac{1}{2}\{W_0^{\pi\pi}\pi^2, \varrho\}\right] \\
&+ \int d^3x \left[(D_c N^c W^{ab} - D_c N^b W^{ca} - D_c N^a W^{cb})(D_a\phi\varrho D_b\phi - \frac{1}{2}\{D_b\phi D_a\phi, \varrho\})\right] \\
&+ \int d^3x N(x) \left[\{H[\vec{N}], W^{\alpha\beta} L_\alpha \varrho L_\beta^\dagger - \frac{1}{2} W_0^{\alpha\beta} L_\beta^\dagger L_\alpha, \varrho\}\}\right. \\
&\left. - W^{\alpha\beta} L_\alpha \{H[\vec{N}], \varrho\} L_\beta^\dagger - \frac{1}{2} W_0^{\alpha\beta} L_\beta^\dagger L_\alpha, \{P[\vec{N}], \varrho\}\}\right].
\end{aligned} \tag{7.65}$$

Although somewhat daunting, we see that so long as $W^{\alpha\beta}$ satisfies certain transformation properties, specifically if

$$\begin{aligned}
&\{\mathcal{H}[\vec{N}], W^{\pi\pi}(\varrho)\} - W^{\pi\pi}(\{\mathcal{H}[\vec{N}], \varrho\}) = \int d^3x D_a N^a W^{\pi\pi}(\varrho) - N^a D_a W^{\pi\pi}(\varrho), \\
&\{\mathcal{H}[\vec{N}], W^{\phi\phi}(\varrho)\} - W^{\phi\phi}(\{\mathcal{H}[\vec{N}], \varrho\}) = \int d^3x - D_a N^a W^{\phi\phi}(\varrho) - N^a D_a W^{\phi\phi}(\varrho), \\
&\{\mathcal{H}[\vec{N}], W^{ab}(\varrho)\} - W^{ab}(\{\mathcal{H}[\vec{N}], \varrho\}) \\
&= \int d^3x [D_c N^b W^{ca}(\varrho) + D_c N^a W^{cb}(\varrho) - D_c N^c W^{ab}(\varrho)] - N^a D_a W^{ab}(\varrho),
\end{aligned} \tag{7.66}$$

then the algebra will close

$$[\mathcal{L}[\vec{N}], \mathcal{L}[N]](\rho) = \int d^3x N^a D_a N \mathcal{L}(\rho) = \mathcal{L}[L_{\vec{N}} M](\rho), \tag{7.67}$$

and the theory will be spatially diffeomorphism invariant. Here $L_{\vec{N}}N$ is the Lie derivative of N along N^a. The conditions in Equation (7.66) are demystified somewhat when one realizes they are analogous to terms arising in classical gravity. It is a general property of minimally coupled field theories that [108]

$$\{\mathcal{H}_{ma}(x), \mathcal{H}_m(y)\} = 2D_b\left(\frac{\delta \mathcal{H}_m(y)}{\delta \gamma_{ab}(x)}\right) + \mathcal{H}_m(x)\partial_a \delta(x,y) \tag{7.68}$$

$$\{\mathcal{H}_a(x), \mathcal{H}_m(y)\} = -2D_b\left(\frac{\delta \mathcal{H}_m(y)}{\delta \gamma_{ab}(x)}\right), \tag{7.69}$$

and these combine so that the matter Hamiltonian transforms as a scalar under spatial diffeomorphisms

$$\{\mathcal{H}_a + \mathcal{H}_{ma}(x), \mathcal{H}_m(y)\} = \mathcal{H}_m(x)\partial_a \delta(x,y). \tag{7.70}$$

Since we are assuming that the CQ Lindbladian has the same structure as the matter Hamiltonian, we often find terms that look similar to (7.68) and (7.69) – essentially the anomalous terms in (7.65). We expect these to cancel with terms arising from the Poisson bracket – the terms in the final line of Equation (7.65) – enforcing conditions on the allowed realizations. We interpret this as telling us that $W^{\alpha\beta}(z|z')L^\dagger_\beta L_\alpha$ must transform like the Hamiltonian under the action of the Poisson bracket. From now on, we will assume that these can be satisfied without reference to an explicit realization. Some realizations can be found in [28, 3]. We will now derive the constraints which arise from the final component of the algebra – the $[\mathcal{L}(x), \mathcal{L}(y)]$ commutator.

$[\mathcal{L}(x), \mathcal{L}(y)]$ commutator for the scalar field

We now move on to study the final commutator in algebra. So far, we have found restrictions on the realizations of the CQ theory. In this section, we will find that we will need to impose constraints in order for the theory to be consistent. We find

$$[\mathcal{L}[N], \mathcal{L}[M]](\varrho) = \int d^3x (N\partial_a M - M\partial_a N)\left[\{\gamma^{ab}\mathcal{H}_b, \varrho\} - i[\gamma^{ab}\mathcal{H}_{m,b}, \varrho] + \bar{\mathcal{C}}^a(\varrho)\right], \tag{7.71}$$

where

$$\begin{aligned}\bar{\mathcal{C}}^a(\varrho) = {} & C_J^{ab}(D_b\phi(x)\varrho\pi(x) + \pi(x)\varrho D_b\phi(x)) - \frac{1}{2}C_N^{ab}\{D_b\phi(x)\pi(x) + \pi(x)D_b\phi(x), \varrho\}_+ \\ & - iC_H^{ab}[\mathcal{H}_a, \varrho] + iC_{JN}^{ab}(D_a\phi(x)\varrho\pi(x) - \pi(x)\varrho D_a\phi(x)).\end{aligned} \tag{7.72}$$

In Equation (7.72), we have defined the couplings[4]

$$C^{ab}_J = 2(h^{ab}W^{\pi\pi} + h^{\pi\pi}W^{ab}), \quad C^{ab}_N = 2(W^{ab}_0 h^{\pi\pi} + W^{\pi\pi}_0 h^{ab})$$
$$C^{ab}_H = \frac{1}{4}(W^{\pi\pi}W^{ab} - W^{ab}_0 W^{\pi\pi}_0), \quad C^{ab}_{JN} = (W^{ab}W^{\pi\pi}_0 - W^{ab}_0 W^{\pi\pi}). \tag{7.73}$$

The first two terms of Equation (7.71) give rise to a component that is a gauge transformation. To see this, we smear over a phase space observable A to find

$$\begin{aligned}\langle A, \{\gamma^{ab}\mathcal{H}_b, \varrho\} - i[\gamma^{ab}\mathcal{H}_{m,b}, \varrho]\rangle &= \langle \{A, \gamma^{ab}\}, \mathcal{H}_b\varrho\rangle + \langle \{A, \mathcal{H}_b\}\gamma^{ab}, -i[\gamma^{ab}\mathcal{H}_{m,b}, \varrho]\rangle \\ &= \langle \{A, \gamma^{ab}\}, \mathcal{H}_b\varrho\rangle + \langle \mathcal{L}^*_b(A)\gamma^{ab}, \varrho\rangle,\end{aligned} \tag{7.74}$$

which, since the final term in (7.74) is a gauge transformation, is weakly equal to

$$\langle \{A, \gamma^{ab}\}, \mathcal{H}_b\varrho\rangle. \tag{7.75}$$

As a consequence, we find

$$[\mathcal{L}[N], \mathcal{L}[M]](\varrho) \approx \int d^3x (N\partial_a M - M\partial_a N)\left[\{\gamma^{ab}, \mathcal{H}_b\varrho\} + \bar{\mathcal{C}}^a(\varrho)\right], \tag{7.76}$$

and so, in order for the algebra of generators to close, we need to impose the CQ momentum constraint

$$\mathcal{C}^a = \{\gamma^{ab}, \mathcal{H}_b\varrho\} + \bar{\mathcal{C}}^a(\varrho) \approx 0, \tag{7.77}$$

which should hold when smeared over observables A. Using the definitions in Equation (7.73), we can write this out in full as

$$\begin{aligned}\mathcal{C}^a(\varrho) = \{\gamma^{ab}, \mathcal{H}_b\varrho\} &- iC^{ab}_H[\mathcal{H}_{m,a}, \varrho] + C^{ab}_J(D_b\phi(x)\varrho\pi(x) + \pi(x)\varrho D_b\phi(x)) \\ &- \frac{1}{2}C^{ab}_N\{\mathcal{H}_{m,a}, \varrho\}_+ + C^{ab}_{JN}(D_a\phi(x)\varrho\pi(x) - \pi(x)\varrho D_a\phi(x)).\end{aligned} \tag{7.78}$$

It is useful to perform a quick sanity check on Equation (7.78) to see if it gives the correct momentum constraint in the classical limit. First, taking the trace over the quantum system and defining $\mathrm{Tr}\,[\varrho(z)] = \rho(z)$, one sees

$$\mathrm{Tr}\,[C^a(\varrho)] = \{\gamma^{ab}, \mathcal{H}_b\rho\} + \mathrm{Tr}\left[(2C^{ab}_J - C^{ab}_N)\pi D_b\phi\varrho\right]. \tag{7.79}$$

[4]Here the labels J, N, H, JN stand for "Jump", "no-event", "Hamiltonian", "jump-no-event". Specifically, the C^{ab}_J term takes the form of a Jump term in a Lindblad equation, while C^{ab}_N takes the form of a no-event term in the Lindblad equation. The C^{ab}_H term takes the form of a Hamiltonian term. The final term C^{ab}_{JN} has no analog with the Lindblad equation but arises due to the commutation of the jump and no-event term in $\mathcal{L}$.

Performing a Kramers-Moyal expansion of the CQ couplings $W^{\alpha\beta}(z|z',x)$ to first order, one finds the zeroth order term cancels in Equation (7.79) and, remembering that the first moment is such that Einstein's equations hold, we are left with

$$\mathrm{Tr}\left[C^a(\varrho)\right] = \{\gamma^{ab}, \mathcal{H}_b\rho\} + \mathrm{Tr}\left[D_b\phi\pi\{\gamma^{ab},\varrho\}\right] + \dots \ . \tag{7.80}$$

Smearing over an observable A, Equation (7.80) becomes

$$\int DgD\pi A\mathrm{Tr}\left[C^a(\varrho)\right] = \int DgD\pi\{A,\gamma^{ab}\}(\mathcal{H}_b\rho + \pi\mathrm{Tr}\left[D_b\phi\varrho\right] + \dots) \ . \tag{7.81}$$

Comparing this CQ constraint to the standard momentum constraint of GR, we see that it gives a sensible constraint in the classical limit, i.e., one that is satisfied by classical gravity (recall that $\mathcal{H}_{m,a} = \pi D_b\phi$). We also see that we get a sensible constraint in the limit where the matter remains quantum. It appears it can be satisfiable even though it contains both functionals of the classical degrees of freedom and quantum operators. In particular, one could have been concerned that we would find that we had to satisfy the naive CQ constraint $\mathcal{H}_a + \pi D_a\phi\rho \approx 0$ when restricted to the constraint surface; this would have required setting a c-number equal to an operator equation. Instead, we get a CQ-equation equivalent to finding mixed fixed points of some dynamics, which is at least possible to hold in principle.

Conservation of the momentum constraint C^a for the scalar field

Now that we have a momentum constraint, we must check to see if it is preserved in time. In the classical case, preservation of the momentum constraint gives rise to the Hamiltonian constraint, and we expect something analogous for the CQ theory. Indeed, we find we get a constraint which is the standard Hamiltonian constraint in the classical limit – although it appears to require additional constraints or a restriction on the lapse and shift if one is to hope that it will be preserved in time. We discuss the possible implications for this when we conclude in Section 7.6.

Conservation of the momentum constraint requires calculating the quantity $C^a(\frac{\partial\varrho}{\partial t}) = C^a(\mathcal{L}[N,\vec{N}](\varrho))$. For calculation purposes, it is slightly simpler to consider the commutator $[C^a,\mathcal{L}[N,\vec{N}]](\varrho)$, noting that the difference between the commutator and the evolution of the constraint is given by the term $\int DgD\pi C^{a*}(\mathcal{L}^*(A))\varrho$; that is the momentum constraint but smeared over the phase space operator $\mathcal{L}^*(A)$ instead of A. When performing the calculation, we will smear the momentum constraint C^a with a lower index spatial smearing function $M_a(x)$ and write C^a $\mathcal{C}[\underline{\mathrm{M}}] = \int d^3x M_a \mathcal{C}^a$.

We first calculate the commutator with the spatial part of the evolution equation $[C[\underline{\mathrm{M}}], \mathcal{L}[\vec{N}]](\varrho)$; in other words, to check whether or not the momentum constraint transforms correctly under spatial diffeomorphisms. Assuming that the realization has the transformation properties defined in Equation (7.66), one finds[5]

$$[\mathcal{C}[\underline{\mathrm{M}}], [\mathcal{L}[\vec{N}]](\varrho) = -\int d^3x N^c D_c M_a \mathcal{C}^b(\varrho) + M_a D_c N^a \mathcal{C}^c(\varrho), \tag{7.82}$$

which vanishes on the constraint surface. We are then left to calculate the commutation with $\mathcal{L}[N]$, which, in analogy to the classical case, we expect to give a Hamiltonian constraint.

It shall be useful to split up the generators into the purely classical part involving the Poisson bracket and everything else. To that end, we write $\mathcal{L}(\varrho) = \{\mathcal{H}, \varrho\} + \bar{\mathcal{L}}(\varrho)$ and $C^a(\varrho) = \{\gamma^{ab}, \mathcal{H}_b \varrho\} + \bar{C}^a(\varrho)$. The evolution of the smeared constraint reads

$$\begin{aligned}
[C[\underline{\mathrm{M}}], \mathcal{L}[N]](\varrho) = \int d^3x d^3y M_a(y) N(x) \big[&\{\gamma^{ab}(y), \{\mathcal{H}_b(y), \mathcal{H}(x)\}\varrho\} \\
&+ \{\gamma^{ab}, \mathcal{H}_b \bar{\mathcal{L}}(\varrho)\} - \bar{\mathcal{L}}(\{\gamma^{ab}, \mathcal{H}_b \varrho\}) + [\bar{\mathcal{C}}^a, \bar{\mathcal{L}}](\varrho)) \\
&+ \{\{\gamma^{ab}(y), \mathcal{H}(x)\}, \mathcal{H}_b(y)\varrho\} + \bar{\mathcal{C}}^a(\{\mathcal{H}, \varrho\} - \{\mathcal{H}, \bar{\mathcal{C}}^a(\varrho)\}\big].
\end{aligned} \tag{7.83}$$

Much is going on in Equation (7.83), so we will break it into pieces and discuss what we expect to get back from each term before presenting our findings. We first comment on the third line of Equation (7.83), which consists of the term

$$\{\{\gamma^{ab}(y), \mathcal{H}(x)\}, \mathcal{H}_b(y)\varrho\} + \bar{\mathcal{C}}^a(\{\mathcal{H}, \varrho\}) - \{\mathcal{H}, \bar{\mathcal{C}}^a(\varrho)\}. \tag{7.84}$$

Firstly, we note that Equation (7.84) has the Lindblad structure of the momentum constraint.[6] To gain some intuition, it is useful to take the trace of (7.84) and look at the first order in the Kramers-Moyal expansion. Explicitly, Equation (7.84) becomes

$$\{\{\gamma^{ab}(y), \mathcal{H}(x)\}, \mathcal{H}_b(y)\rho\} + \mathrm{Tr}\left[D_b \phi \pi \{\{\gamma^{ab}(y), \mathcal{H}(x)\}, \varrho\}\right] + \dots \tag{7.85}$$

[5] This can be made to look similar to the $\mathcal{L}_a, \mathcal{L}_b$ commutator by using integration by parts on the second term of (7.82) where it differs slightly because $D_b C^a(\varrho)$ is not vanishing

[6] By this, we mean that the expression in (7.84) contains quantum operators with the same structure as the momentum constraint in Equation (7.78), i.e., those which come in the format $\sim \partial_a \phi \varrho \pi$.

and smearing this over an observable gives

$$\int DgD\pi\{A,\{\gamma^{ab}(y),\mathcal{H}(x)\}\}(\mathcal{H}_b\rho+\mathrm{Tr}\,[D_b\phi\pi\varrho]+\dots)\,. \tag{7.86}$$

We can compare (7.86) to the momentum constraint in (7.81), and we note that although almost identical, they are not quite the same. To be precise, Equation (7.86) is smeared over $\{A,\{\gamma^{ab},\mathcal{H}\}\}$ whilst Equation (7.81) is instead smeared over $\{A,\gamma^{ab}\}$. Of course, both the constraints are not independent; both vanish if we satisfy the effective classical constraint $\mathcal{H}_b\rho+\mathrm{Tr}\,[D_b\phi\pi\varrho]=0$. We, therefore, posit that in a sensible realization, Equation (7.84) should be weakly zero whenever the momentum constraint is satisfied. Otherwise, one is forced to view Equation (7.84) as a separate constraint to the momentum constraint, which must be preserved in time by itself. One then faces a similar issue looking at the time evolution of Equation (7.84) and must impose more constraints in a series that seems unlikely to terminate. We view the condition that Equation (7.84) should be weakly zero whenever the momentum constraint is satisfied as a transformation rule which any realization $W^{\alpha\beta}(z|z')$ must obey, telling us how they CQ couplings must transform under the action of the pure gravity Hamiltonian $\{\mathcal{H},\}$ – just as the transformation rules defined in Equation (7.66) place conditions on the realizations in order for the theory to be spatially diffeomorphism invariant.

We now study the remaining terms in Equation (7.83), given by the first two lines, which we expect to give rise to a Hamiltonian constraint. Before presenting the result, getting some intuition for each of the terms appearing is helpful. Using the Dirac algebra, defined in (7.15), the purely classical term, $\{\gamma^{ab}(y),\{\mathcal{H}_b(y),\mathcal{H}(x)\}\varrho\}$, can be written as $\partial^y_b\delta(y,x)\{\gamma^{ab}(y),\mathcal{H}(y)\varrho\}$, giving rise to the classical Hamiltonian part of the constraint. It is less obvious what we expect $[\bar{\mathcal{C}}^a,\bar{\mathcal{L}}](\varrho)$ to give back. Taking a step back and looking at the analogous term in classical gravity, one has instead of $[\bar{\mathcal{C}}^a,\bar{\mathcal{L}}](\varrho)$ the term

$$\{\mathcal{H}_{ma}(x),\mathcal{H}_m(y)\}=2D_b\left(\frac{\delta\mathcal{H}_m(y)}{\delta\gamma_{ab}(x)}\right)+\mathcal{H}_m(x)\partial_a\delta(x,y) \tag{7.87}$$

which gives rise to the Hamiltonian constraint and a term $2D_b\left(\frac{\delta\mathcal{H}_m(y)}{\delta\gamma_{ab}(x)}\right)$. In calculating the full Poisson bracket between the momentum constraint and the Hamiltonian constraint, this anomalous term cancels with a term arising from the Poisson bracket between the pure gravity momentum and the matter Hamiltonian $\{\mathcal{H}_a,\mathcal{H}_m\}$ so that in combination $\{\mathcal{H}_a+\mathcal{H}_{ma},\mathcal{H}_m\}\sim\mathcal{H}_m$ gives back the matter part of the Hamiltonian constraint.

By virtue of the Lindblad structure of the constraints, due to Assumption 3, we expect that under commutation $\bar{C}^a$ transforms like momentum and $\mathcal{L}$ like a Hamiltonian. As a consequence, we expect to find (morally)

$$[\bar{C}^a, \bar{\mathcal{L}}] \sim \mathcal{R} + \mathcal{L}_{constraint}(x)\partial_a \delta(x, y), \tag{7.88}$$

where $\mathcal{R}$ is the CQ version of $2D_b\left(\frac{\delta \mathcal{H}_m(y)}{\delta \gamma_{ab}(x)}\right)$ and $\mathcal{L}_{constraint}(x)$ is the CQ Hamiltonian constraint. Finally, in analogy with the classical case, we expect the $\mathcal{R}$ term appearing in Equation (7.88) to cancel with the Poisson bracket term arising in the second line of (7.83), namely $\{\gamma^{ab}, \mathcal{H}_b \bar{\mathcal{L}}(\varrho)\} - \bar{\mathcal{L}}(\{\gamma^{ab}, \mathcal{H}_b \varrho\})$, so that the first two lines of Equation (7.83) gives rise to the CQ generalization of the Hamiltonian constraint.

We now present the full Hamiltonian constraint – the first two lines of Equation (7.83). In doing so, it is first useful to introduce notation for a certain combination of terms that frequently arises. We define $\mathcal{R}_{AB}$ via

$$\begin{aligned}
&\mathcal{R}^{\alpha\beta}_{AB} L_\alpha \varrho L_\beta = \int d^3x N \big[W^{\phi\phi}_B \phi D_b (M_a C^{ab}_A \varrho)\phi - W^{\pi\pi}_B \pi D_b (M_a C^{ab}_A \varrho)\pi \\
&+ D_e \phi (D_b (M_a C^{af}_A \varrho) W^{eb}_B + D_b (M_a C^{ae}_A \varrho) W^{bf}_B - D_b (M_a C^{ab}_A \varrho) W^{ef}_B) D_f \phi \\
&+ M_a C^{ab}_A L_\alpha D_b (W^{\alpha\beta}_B \varrho) L_\beta \big].
\end{aligned} \tag{7.89}$$

Here the A sub-index denotes terms coming from the momentum constraint and is associated with C_J, C_H, C_{JN}, C_N. In contrast, the B sub-index denotes terms coming from the Hamiltonian constraint and is associated with the couplings $W^{\alpha\beta}_J = W^{\alpha\beta}(z|z'), W^{\alpha\beta}_N = W^{\alpha\beta}_0(z), W^{\alpha\beta}_H = h^{\alpha\beta}$.[7] These terms are the CQ analogy of the $2D_b\left(\frac{\delta \mathcal{H}_m(y)}{\delta \gamma_{ab}(x)}\right)$ terms, which arise from integration by parts due the transformation properties of $\bar{\mathcal{C}}^a, \bar{\mathcal{L}}$. A lengthy but straightforward calculation

[7]Again, the J, H, JN, N stand for jump, Hamiltonian, jump-no event and no-event and are an attempt to label sensibly the terms arising in the constraint. We remind the reader that the jump and no-event terms have the structure defined just before equation (7.4), while the jump-no event term appearing in the momentum constraint arises from a commutation of jump and no-event terms appearing in the $[\mathcal{L}(x), \mathcal{L}(y)]$ commutator.

then gives the total Hamiltonian constraint

$$
\begin{aligned}
&\mathcal{L}_{constraint} = \int d^3x M_a D_b N\big[(-2iC_H^{ab}h^{\alpha\beta} + \frac{i}{2}C_N^{ab}W_0^{\alpha\beta} - \frac{i}{2}C_J^{ab}W^{\alpha\beta})[L_\beta^\dagger L_\alpha, \varrho]\\
&+ \{\gamma^{ab}(y), \mathcal{H}(y)\varrho\} + 2(C_J^{ab}h^{\alpha\beta} + C_H^{ab}W^{\alpha\beta})L_\alpha\varrho L_\beta^\dagger - (C_N^{ab}h^{\alpha\beta} + C_H^{ab}W_0^{\alpha\beta})\{L_\beta^\dagger L_\alpha, \varrho\}_+\big]\\
&+ \int d^3x NM_a\mathcal{H}_b(\{\gamma^{ab}, \mathcal{H}_b\bar{\mathcal{L}}(\varrho)\} - \bar{\mathcal{L}}(\{\gamma^{ab}, \mathcal{H}_b\varrho\})\\
&+ (\frac{i}{2}\mathcal{R}_{NN}^{\alpha\beta} - \frac{i}{2}\mathcal{R}_{JJ}^{\alpha\beta} - 2i\mathcal{R}_{HH}^{\alpha\beta})[L_\beta^\dagger L_\alpha, \varrho] + (2\mathcal{R}_{HJ}^{\alpha\beta} + 2\mathcal{R}_{JH}^{\alpha\beta})L_\alpha\varrho L_\beta^\dagger\\
&- (\mathcal{R}_{NH}^{\alpha\beta} + \mathcal{R}_{HN}^{\alpha\beta})\{L_\beta^\dagger L_\alpha\}_+\\
&+ \int d^3x NM_a\big[(C_{JN}^{ab}(W_0^{\phi\phi} - 2ih^{\phi\phi}) - C_J^{ab}W_0^{\phi\phi} + C_N^{ab}W^{\phi\phi})D_b\phi\varrho\phi +\\
&+ (C_{JN}^{ab}(W_0^{\phi\phi} + 2ih^{\phi\phi}) + C_J^{ab}W_0^{\phi\phi} - C_N^{ab}W^{\phi\phi})\phi\varrho D_b\phi\\
&+ (C_{JN}^{ab}(-W_0^{\pi\pi} + 2ih^{\pi\pi}) - W_0^{\pi\pi}C_J^{ab} + C_N^{ab}W^{\pi\pi})D_b\pi\varrho\pi + (C_{JN}^{ab}(-W_0^{\pi\pi} - 2ih^{\pi\pi})\\
&+ (W_0^{\pi\pi}C_J^{ab} - C_N^{ab}W^{\pi\pi})\pi\varrho D_b\pi\\
&+ (C_{JN}^{ab}(-W_0^{\phi\phi} - 2ih^{\phi\phi}) + C_N^{ab}W^{ef} - W_0^{ef}h^{ef}C_J^{ab})(D_eD_f\phi)\varrho D_b\phi\\
&+ (C_{JN}^{ab}(-W_0^{\phi\phi} + 2ih^{\phi\phi}) - C_N^{ab}W^{ef} + W_0^{ef}C_J^{ab})D_b\phi\varrho(D_eD_f\phi)\big]\\
&\int d^3x\big[NW^{\pi\pi}\{\pi D_b(M_a\pi C_{JN}^{ab}, \varrho)\}_+ - NM_bC_{JN}^{ab}W^{\phi\phi}\{D_b\phi\phi, \varrho\}_+\\
&+ M_aC_{JN}^{ab}\{D_b\phi D_d(NW^{cd}D_c\phi, \varrho)\}_+\big]\\
&- 2\int d^3x\big[NW_0^{\pi\pi}\pi D_b(C_{JN}^{ab}M_a\varrho)\pi + M_bD_dNW_0^{cd}C_{JN}^{ab}D_c\phi\varrho D_b\phi\big] \approx 0,
\end{aligned}
\tag{7.90}
$$

which needs to be weakly zero for the theory to be gauge invariant. Again, a lot is going on in Equation (7.90), so we now summarize what each term in the constraint tells us.

- The first two lines of Equation (7.90) look like a potentially sensible Hamiltonian constraint. Indeed, if we take their quantum trace, we end up with

$$\{\gamma^{ab}(y), \mathcal{H}(y)\rho\} + 2(C_J^{ab}h^{\alpha\beta} + C_H^{ab}W^{\alpha\beta} - C_N^{ab}h^{\alpha\beta} - C_H^{ab}W_0^{\alpha\beta})\mathrm{Tr}\left[L_\beta^\dagger L_\alpha\varrho\right]. \tag{7.91}$$

  Performing the Kramer's-Moyal expansion to first order, we find this gives

$$\{\gamma^{ab}(y), \mathcal{H}(y)\rho\} + \mathrm{Tr}\left[h^{\alpha\beta}L_\beta^\dagger L_\alpha\{\gamma^{ab}, \varrho\}\right] + \dots \tag{7.92}$$

  and smeared over an observable A reads

$$\int DgD\pi\{A, \gamma^{ab}(y)\}(\mathcal{H}(y)\rho + \mathrm{Tr}\left[h^{\alpha\beta}L_\beta^\dagger L_\alpha\varrho\right] + \dots \tag{7.93}$$

and looks exactly like what we might expect as a Hamiltonian constraint in the classical limit.

- The third line is the part coming from the Poisson bracket of the constraint with $\bar{\mathcal{L}}$, which, as mentioned, we expect to cancel with the $\mathcal{R}$ terms (in the fourth and fifth lines) because $\bar{\mathcal{C}}^a$ transforms like momentum and $\bar{\mathcal{L}}$ like a Hamiltonian. This does not appear to happen here. Taking the trace over the quantum system, the third and fourth lines of Equation (7.90) combine to give

$$\begin{aligned}&\int d^3x N M_a \Big[\mathrm{Tr}\left[\bar{\mathcal{L}}(\{\gamma^{ab},\varrho\}) - \bar{\mathcal{L}}(\mathcal{H}_b\{\gamma^{ab},\varrho\}) + (2\mathcal{R}^{\alpha\beta}_{JH} - 2\mathcal{R}^{\alpha\beta}_{NH})(\varrho)L^\dagger_\beta L_\alpha) \right] \\ &+ \mathrm{Tr}\left[(2\mathcal{R}^{\alpha\beta}_{HJ} - 2\mathcal{R}^{\alpha\beta}_{HN})(\varrho)L^\dagger_\beta L_\alpha \right] \Big].\end{aligned} \tag{7.94}$$

If we perform a Kramers Moyal expansion to first order, we see that the first term in Equation (7.94) vanishes so that we are left with the final term, $\mathrm{Tr}\left[(2\mathcal{R}^{\alpha\beta}_{HJ} - 2\mathcal{R}^{\alpha\beta}_{HN})(\varrho)L^\dagger_\beta L_\alpha\right]$. Recalling the definition of R_{HJ}, R_{HN} in Equation (7.89), as well as the form of C_N in Equation (7.73), we see that the R_{HN} term cancels with the zeroth moment of the R_{HJ} term and we are left with the first moment of the $2\mathcal{R}^{\alpha\beta}_{HJ}$ term alone. Explicitly, $\mathcal{R}^{\alpha\beta}_{HJ}$ is written

$$\begin{aligned}\mathcal{R}^{\alpha\beta}_{HJ} = \int d^3x N \big[&W^{\phi\phi} D_b(M_a C^{ab}_H \varrho) - W^{\pi\pi} D_b(M_a C^{ab}_H \varrho) + (D_b(M_a C^{af}_H \varrho) W^{eb} \\ &+ D_b(M_a C^{ae}_H \varrho) W^{bf} - D_b(M_a C^{ab}_H \varrho) W^{ef}) + M_a C^{ab}_H L_\alpha D_b(W^{\alpha\beta}\varrho) L_\beta \big],\end{aligned} \tag{7.95}$$

and one can verify if the first moments of the Kramers-Moyal expansion are to give GR, this will not identically be zero. One might hope that one could include it in the Hamiltonian constraint. However, since it comes with different smearing functions to the (would be) Hamiltonian constraint in the first two lines of (7.90), we must either restrict the lapse and shift or Equation (7.95) must be imposed as a separate constraint, which would appear to be over-constraining the system. We also see offending terms of the form $D_b(C^{ab}_H \varrho)$; these are interesting since we do not get these in pure GR; the matter part of the momentum constraint contains no metric degrees of freedom. It is worth noting that we get these violating terms even in classical analogs of the CQ theory. To be precise, one can ask the question: can one have an autonomous master equation on the phase space of GR, which contains noise and is gauge invariant? For example, one can study a

Fokker-Plank type equation linear in the lapse and shift and apply the same arguments as outlined in this chapter to derive a momentum constraint, which must then be preserved in time. In doing so, we still find violating terms of the form $D_b(C^{ab}\varrho)$.

- The rest of the terms in Equation (7.90) are purely CQ terms with no classical analog. They come with different smearing functions to those in the (would be) Hamiltonian constraint in the first two lines of Equation (7.90). Consequently, they must be imposed as a separate constraint, itself preserved in time, or we could impose restrictions on the choice of lapse and shift; we do not check what this gives here. Given the form of the remaining terms, it seems they render the constraint non-satisfiable without restrictions on the lapse and shift.

  All of these terms come from the C_J, C_{JN} and C_N parts of the momentum constraint, which in turn come from the Jump, and no-event parts of the evolution equation when calculating the $[\mathcal{L}[N], \mathcal{L}[M]]$ commutation, i.e., from the consideration of the gauge algebra under two CQ jumps. This is where one might have expected the current demands for gauge invariance to break down. One might hope that if no no-event term was associated with CQ back-reaction, then the expression for the remaining terms in (7.90) is greatly simplified. This is the case for the class of continuous classical-quantum dynamics introduced in chapter 4. More generally, it would seem that the elimination of these terms will require us to weaken or change assumptions outlined in Section 7.5. We study improved theories in Chapters 8 and 9, which violate several of the assumptions considered here.

To summarize, although we have found a sensible-looking momentum constraint, when looking at its conservation in time we find Equation (7.90), which either requires a restriction on the choice of lapse and shift or additional constraints to be satisfied, which almost surely over-constrain the system.

We have broken down Equation (7.90) into multiple terms with different interpretations. In particular, we find several terms with different smearing functions, meaning we do not find a single constraint. We must either impose multiple constraints or restrict the lapse and shift so that the extra constraints vanish. Even though we impose constraints in a weak form, which can be viewed as a constraint on the moments of any allowed distribution ρ, imposing additional constraints, themselves preserved in time, is likely to over-constrain the system.

Take, for example, the pure gravity momentum constraint as derived in Equation (7.46), $\langle\{\gamma^{ab}, A\}, \mathcal{H}_b\rho\rangle \approx 0$. As mentioned, we can take this in its weakest form; it needs to vanish when smeared over observables instead of arbitrary smearing functions. However, we can keep applying gauge transformations to the weak form of the constraints to get more and more constraints on the moments of the distribution ρ. It would appear that one is forced eventually to impose a stronger version of the constraint. Instead of viewing Equation (7.46) as a constraint on the moments of the distribution, one is likely forced to impose a constraint on the state space, which must hold for any smearing function A. In this case, imposing multiple constraints will over-constrain the theory to no longer have 2 degrees of freedom per space-time point.

7.6 Discussion

In this chapter, we have presented a methodology to study the gauge invariance of a class of autonomous CQ theories of gravity linear in the lapse and shift N, N^a, and that reproduces the dynamics of GR in the classical limit. The theory could be regarded as fundamental or an effective theory of quantum gravity in the classical limit of the gravitational degrees of freedom. We have derived the constraints on the level of the equations of motion. The theory is invariant under spatial diffeomorphisms. We then demanded that the theory be invariant under an arbitrary choice of N and N^a, from which we can derive the theory's constraints, including an analog of the momentum and Hamiltonian constraint. The momentum constraint arises as a condition required for the theory to be invariant under the choice of lapse function. At the same time, the Hamiltonian constraint arises from demanding that the momentum constraint be preserved in time. Unlike classical GR, the constraints do not correspond to a constraint surface of the phase space but rather as an operator equation acting on the CQ state. This is reminiscent of approaches in quantum gravity where the super-Hamiltonian and super-momentum operators are applied to the state, and one constrains the wave function to be annihilated by these operators. However, here, we do not assume that the classical constraints are promoted to operators but derive them from the symmetry considerations.

Here, we have asked for the full invariance of the dynamics under the lapse and shift while still consistent with General Relativity. Full gauge invariance appears too strong a condition for the models considered in this chapter: it seems unlikely that the constraints can be solved, even in the weak sense. In Chapters 8 and 9, we study different models of CQ dynamics based on the form of the continuous master equation, which are more promising. These theories violate several assumptions presented in this chapter. The continuous master equations have a different Lindblad structure to the jumping models, sourcing back-reaction via the off-diagonal $D^{\alpha 0}_{1,i} L_\alpha \varrho + c.c$ components, violating Assumption 3. In the covariant path integral approach of Chapter 8, we also find that higher order Lindbladian terms are required for covariance, again violating Assumption 3. In both Chapter 8 and Chapter 9, we find that there is diffusion in the metric itself, not just the conjugate momenta, violating the Assumption 7 of deterministic classical dynamics; when studying the Newtonian limit of CQ theories in Chapter 9, we see that diffusion of the spatial metric is necessary in order to preserve the Newtonian constraint.

When studying the path integral approach in Chapter 8, we will show that it is possible to construct diffeomorphism invariant theories of CQ gravity, and we give an example where the trace of Einstein's equations are satisfied on average, showing that diffeomorphism invariance can be upheld in CQ theories. However, this theory is not constrained. We also attempt to construct a full theory that gives rise to the G_{0i}, G_{00} components of Einstein's equations in the classical limit. For this theory, we posit a set of CQ constraints, but we are unable to prove that they give rise to CPTP dynamics on the constraint surface.[8]

It could be that asking for both full gauge invariance and the classical limit of Einstein's equations is too strong a condition. Several theories, such as Horava gravity, shape dynamics, and unimodular gravity, fix the lapse and shift. The gauge group of the theory is then smaller or different than the gauge group of general relativity. One could, for example, be satisfied with foliation diffeomorphisms alone and impose that N must be spatially constant. In that case, the constraint algebroid would close since the smearing functions always appear with divergences acting on them. However, one hopes a weaker one can be found.

Another possibility is that if one thinks of this theory as the classical limit of quantum

[8]In Chapter 9, we study the Newtonian limit of CQ theories, which can be understood as a gauge fixed, weak field limit of a complete theory. We show that it is possible to have constraints that are preserved in time, leading to a CQ version of Poisson's equation.

gravity, we should lift the autonomous assumption since by taking the classical limit, we are throwing away quantum information, which could act as a memory for the evolution leading to non-Markovian dynamics. In this case, one hopes the methodology of studying CQ constraints could shed light on a possible theory of quantum gravity. Indeed the algebra shares some features of quantum algebra since the matter fields are quantized. At the same time, the classical nature of the gravity part allows for a much more tractable set of calculations, especially for the continuous master equation. Along the way, one sees that a number of conceptual issues that prove difficult to resolve in quantum gravity also occur in theories where one has a probability density over gravitational degrees of freedom.

Chapter 8

Covariant path integrals

So far, we have not studied classical-quantum dynamics in a manifestly covariant framework. The problem with studying gravity, or any field theory, in a master equation picture, is twofold. Firstly, from a practical point of view, field theories are generally better suited to path integral methods. Secondly, in a master equation picture, it is difficult to impose symmetries directly on the master equation. Indeed, writing down master equations for classical-quantum fields directly, without knowing whether or not they are covariant or uphold space-time symmetries, seems to go against much of the principles of modern physics, where one starts with actions based on symmetry principles. If one took the position that there is a fundamentally classical field, such as the gravitational field, it is also not obvious how one could couple it to the standard model while simultaneously ensuring symmetry principles and renormalizability are upheld.

This chapter studies configuration space path integrals for quantum fields interacting with classical fields. We show that this can be done consistently by proving that the dynamics are completely positive directly, without resorting to master equation methods. This is especially important since, in general, we saw in Chapter 6 that it was only possible to go from a master equation picture to a path integral picture when the master equation is less than quadratic in classical or quantum momenta. The path integrals allow one to readily impose space-time symmetries, including Lorentz invariance or diffeomorphism invariance.

Our results have consequences for any theory with a degree of freedom that behaves classically, whether effective or fundamental. With this in mind, we provide a possible template for studying CQ field theories. We introduce a class of classical-quantum actions motivated by

Chapter 6, which can be used to construct theories with a sensible classical limit. The corresponding path integral can be understood in terms of summing over all classical and quantum paths, where the classical paths deviating too much from their semi-classical configuration are suppressed by the coupling D_0, which also governs the strength of the quantum decoherence. Since we do not have a full theory of quantum gravity, of particular relevance is to construct an effective theory of quantum matter back-reacting on classical space-time, and we discuss the application of our work to the gravitational setting. We introduce a path integral formulation of general relativity where the space-time metric is treated classically and a diffeomorphism invariant theory based on the trace of Einstein's equations.

This chapter is based on the paper [8], which is work done in collaboration with Jonathan Oppenheim.

8.1 Completely positive path integrals

We saw in Chapter 6 that we could derive CQ path integrals from autonomous CQ master equations, and we could arrive at a configuration space path integral analytically when the action was at most quadratic in momenta. In this section, we shall show that one can prove the complete positivity of autonomous classical-quantum dynamics directly from the path integral. As a corollary, our result can also be to prove the complete positivity of the Feynman-Vernon path integral without resorting to the Lindblad equation. This is important, since often higher derivative path integrals are considered in the literature which turn out to not be completely positive (see Appendix J).

The path integral tells us how the components of the density matrix evolve. Including a classical variable z, the path integral should tell us how to evolve the components of a classical-quantum state

$$\varrho(z,t) = \int dz d\phi^+ d\phi^- \varrho(z,t,\phi^+,\phi^-) \; |\phi^+\rangle\langle\phi^-|, \tag{8.1}$$

where ϕ represents a continuous quantum degree of freedom and $\varrho(z,t,\phi^+,\phi^-) = \langle\phi^+|\rho(z,t)|\phi^-\rangle$ are the components of the CQ state. Writing Equation (8.1) out explicitly, generically, a path integral will take the form

$$\rho(z_f,\phi_f^+,\phi_f^-,t_f) = \int \mathcal{D}z\mathcal{D}\phi^+\mathcal{D}\phi^- e^{\mathcal{I}[\phi^+,\phi^-,z,t_i,t_f]}\rho(z_i,\phi_i^+,\phi_i^-,t_i). \tag{8.2}$$

In Equation (8.2), it is implicitly understood that boundary conditions are to be imposed at t_f. In the purely quantum case, one has $\mathcal{I}[\phi^+,\phi^-,t_i,t_f] = iS[\phi^+,t_i,t_f] - iS[\phi^-,t_i,t_f]$ and the path integral is doubled since we are considering density matrices so we must sum over all bra and ket paths.

When the action contains higher derivatives, we can also include additional initial conditions on the time derivatives of the fields in Equation (8.3) [200]. For example, if the action contains terms with second-time derivatives in the classical degrees of freedom, we can write down the action

$$\rho(z_f,\dot{z_f},\phi_f^+,\phi_f^-,t_f) = \int \mathcal{D}z\mathcal{D}\phi^+\mathcal{D}\phi^- e^{\mathcal{I}[\phi^+,\phi^-,z,t_i,t_f]}\rho(z_i,\dot{z}_i,\phi_i^+,\phi_i^-,t_i). \tag{8.3}$$

Having introduced the classical-quantum formalism, let us now state and prove our main result:

Any time-local classical-quantum path integral with action of the form

$$\begin{aligned}\mathcal{I}(\phi^-,\phi^+,z,t_i,t_f) &= \mathcal{I}_{CQ}(\phi^+,z,t_i,t_f) + \mathcal{I}^*_{CQ}(\phi^-,z,t_i,t_f) - \mathcal{I}_C(z,t_i,t_f) \\ &+ \int_{t_i}^{t_f} dt \sum_\gamma c^\gamma(z,t)(L_\gamma[\phi^+]L^*_\gamma[\phi^-])\end{aligned} \tag{8.4}$$

defines completely positive CQ dynamics when the terms in Equation (8.4) have the following properties: $L_\gamma[\phi^\pm]$ can be any functional of the bra and ket variables, $c^\gamma \geq 0$, $\mathcal{I}_C$ is positive (semi) definite, and the real part of $\mathcal{I}_{CQ}$ is negative (semi) definite. We implicitly assume that c^γ is chosen so that the path integral converges. The dynamics described by Equation (8.4) is CP but not necessarily norm preserving. However, via Table 2.2 time-local CP dynamics can always be normalized in a linear way, and including this we find normalized CP dynamics given by Equation (8.13).

In the field-theoretic case, the final line of Equation (8.4) is replaced by

$$\int_{t_i}^{t_f} dx \sum_\gamma c^\gamma(z,x)(L_\gamma[\phi^+](x)L^*_\gamma[\phi^-](x)), \tag{8.5}$$

and the resulting path integral in Equation (8.4) will be completely positive so long as $c^\gamma(z,x)$ is positive.

In Equation (8.4) $\mathcal{I}_{CQ}$ determines the CQ interaction on each of the ket and bra paths and $\mathcal{I}_C(z,t_i,t_f)$ is a purely classical action which takes real values. The above requirements on positive definiteness have been imposed for the path integral to converge. This condition also arose when studying path integrals associated with CQ master equations Chapter 6; for example, one can take the classical action $\mathcal{I}_C$ to be the action associated with the path integral of the Fokker-Planck equation (8.16) [153, 87, 80] which must be positive (semi) definite in order for the path integral to converge. The term on the final line of Equation (8.4) contains cross terms between the bra and ket branches ϕ^+, ϕ^-, which sends pure states to mixed states and corresponds to including additional noise in the dynamics. It takes the form of a Kraus map acting on the CQ state, which ensures complete positivity, and allows one to include classical-quantum Feynman-Vernon [79, 165] terms into the action.

If all the $c^\gamma = 0$, the ϕ^+ and ϕ^- integrals factorize in Equation (8.4), meaning the path integral preserves the purity of the quantum state conditioned on the classical trajectory. In this case, the absence of cross terms in the action, despite the requirement of Lindblad terms in the hybrid master equation, is a consequence of saturating the decoherence-diffusion trade-off. We shall primarily focus on this case; it can be shown that any CQ dynamics which does not preserve the purity of the quantum state conditioned on the classical degree of freedom can be embedded into a larger classical space where the quantum state remains pure, in a CQ version of purification [6].

It is useful to split $\mathcal{I}_{CQ}$ into its real and imaginary components $\mathcal{I}_{CQ} = \mathcal{R}_{CQ} + i\mathcal{S}_Q$. Then Equation (8.4) (with $c_n = 0$) reads

$$\mathcal{I}^{\pm} = \mathcal{R}^+_{CQ} + \mathcal{R}^-_{CQ} + i(\mathcal{S}^+_{CQ} - \mathcal{S}^-_{CQ}) - \mathcal{I}_C, \tag{8.6}$$

and we can get some intuition for each term. Heuristically expanding the actions, or more properly their Lagrangian's, in terms of their field dependence $\mathcal{S}_{CQ} \sim \sum_m a_m(z)s_m(\phi)$ and $\mathcal{R}_{CQ} \sim \sum_m b_m(z)r_m(\phi)$ we see that

$$\begin{aligned} \mathcal{S}^+_{CQ} - \mathcal{S}^-_{CQ} &\sim \sum_m a_m(z)(s_m(\phi^+) - s_m(\phi^-)) \\ \mathcal{R}^+_{CQ} + \mathcal{R}^-_{CQ} &\sim \sum_m b_m(z)(r_m(\phi^+) + r_m(\phi^-)). \end{aligned} \tag{8.7}$$

Hence, the imaginary part of the integral is associated with things like coherence, which depend on the difference between the ket and bra components of the density matrix. In contrast, the real part of the action depends on the sum of the left and right components of the density matrix, which are things like its expectation value. Moreover, conditioned on a classical trajectory $\bar{z}(t)$

- which can be represented by inserting a delta function $\delta(z(t)-\bar{z}(t))$ into the classical part of the path integral - we see that the evolution of the quantum state factorizes between the $\phi^{\pm}$ integrals and hence keeps pure quantum states pure.

The back-reaction of the quantum system on the classical one is contained in the real components of the CQ action $\mathcal{R}^{\pm}_{CQ}$. Indeed, when $\mathcal{R}^{\pm}_{CQ}=0$, the path integral in Equation (8.6) reduces to the standard quantum path integral for the density matrix but also includes a classical variable which can undergo its own autonomous dynamics due to the inclusion of the classical action $\mathcal{I}_C$. However, whenever there is back-reaction, Equation (8.4) *necessarily* describes non-unitary evolution: we saw this in Chapter 4 using master equation methods.

To prove that the dynamics described by Equation (8.4) gives rise to consistent CQ dynamics, we must show that it leads to completely positive dynamics preserving the positivity of the CQ state.

Recall that positivity of the CQ state means that for any Hilbert space vector $|v(z)\rangle$ we have $\mathrm{Tr}\left[|v(z)\rangle\langle v(z)|\varrho(z)\right]\geq 0$. In components, complete positivity is equivalent to asking that for any vector $|v(z)\rangle$ with components $v(\phi,z)=\langle\phi|v(z)\rangle$ we have

$$\int d\phi^+ d\phi^- v(\phi^+,z)^*\varrho(\phi^+,\phi^-,z)v(\phi^-,z)\geq 0. \tag{8.8}$$

A CQ dynamics Λ is positive if it preserves the positivity of CQ states and completely positive if $\mathbb{I}\otimes\Lambda$ is positive when we act with the identity on any larger system.

Since we assume the dynamics are time-local, we can perform a short-time expansion of the path integral. For the action in Equation (8.4) the path integral integrand always factorizes into the form

$$[e^{\mathcal{I}^+[\phi^+,z]}(e^{\mathcal{I}^-[\phi^-,z]})^*+\delta t\sum_{\gamma}c^{\gamma}(L^+_{\gamma}e^{\mathcal{I}^+[\phi^+,z]})(L^-_{\gamma}e^{\mathcal{I}^-[\phi^-,z]})^*]e^{-\mathcal{I}_C[z]}+\dots, \tag{8.9}$$

Because Equation (8.9) consists of a sum of terms that factorize between $\pm$ branches, even when higher order terms in the expansion are included, it is manifestly completely positive, which can be seen from the definition of complete positivity in Equation (8.8). Equation (8.9) is almost immediate from the form of path integral in Equation (8.4), and for completeness we show this in detail in Appendix E.1. It is important to note that because of the exponentials, Equation (8.9) is always strictly positive, meaning that we do not encounter zero norm states. Instead, the problem of negative norm states and ghosts is mapped to the problem of convergence of the path integral [200].

The path integral in Equation (8.4) is CP, but it is not always norm preserving. However, in the time-local case we can always normalize a CP map in a linear manner to arrive at a CP norm preserving dynamics.

To see this, first recall that in time-local classical dynamics a positive map

$$\frac{\partial p(z)}{\partial t} = \int dz' W(z|z') p(z') \tag{8.10}$$

is normalized by subtracting $W(z)p(z)$, where $W(z) = \int dz' W(z'|z)$. Similarly, in time-local quantum dynamics, the CP dynamics

$$\frac{\partial \sigma}{\partial t} = \lambda^{\mu\nu} L_\mu \sigma L_\nu^\dagger \tag{8.11}$$

is normalized by subtracting the no-event term $\frac{\lambda^{\mu\nu}}{2}\{L_\nu^\dagger L_\mu, \sigma\}$ appearing in the Lindblad equation. For combined CQ dynamics, we know from Table 2.2 that any positive CQ map

$$\frac{\partial \varrho(z)}{\partial t} = \int dz' W^{\mu\nu}\left(z|z'\right) L_\mu \varrho\left(z'\right) L_\nu^\dagger \tag{8.12}$$

can be normalized by subtracting $\frac{1}{2}\int dz' W^{\mu\nu}\left(z'|z\right)\{L_\nu^\dagger L_\mu, \varrho(z)\}_+$ to yield CP norm preserving dynamics.

With this in mind, for time-local dynamics, Equation (8.4) can always be normalized and taking this into account we can include a normalization factor $I_{\mathcal{N}}[\phi^+, \phi^-, z, t_i, t_f]$ in the CQ path integral

$$\rho(z_f, \phi_f^+, \phi_f^-, t_f) = \int \mathcal{D}z\mathcal{D}\phi^+\mathcal{D}\phi^- e^{\mathcal{I}[\phi^+,\phi^-,z,t_i,t_f] - I_{\mathcal{N}}[\phi^+,\phi^-,z,t_i,t_f]} \rho(z_i, \phi_i^+, \phi_i^-, t_i). \tag{8.13}$$

The path integrals studied in Chapter 6 where normalized since they correspond to CPTP master equations. In Section 8.3 we introduce a 'natural class' of path integrals that are based on a CQ proto action and are motivated by Chapter 6; these are the path integrals of interest in this thesis. In Section 8.3 we show that normalization can be guaranteed by including appropriate classical and quantum kinetic terms in the action. However, more generally, starting from Equation (8.4), finding a general closed form for the normalization function seems difficult. For this reason we leave an in-depth study of the normalization general CQ path integrals for future work. Note that we do not expect that this changes the discussion in this chapter since we know that the normalization factor can always be found. Rather, there may be mathematical tricks which can be utilized to write the normalization factor in a neat and more expressive form.

8.2 Comparison to classical path integrals

The path integral action we introduce in Equation (8.4) is general. Therefore, finding CQ actions that give rise to dynamics with a sensible physical interpretation is useful. We saw examples in Chapter 6, and we will here study a simple example of the path integral associated with the Fokker-Plank equation

$$\frac{\partial p(z)}{\partial t} = -\frac{\partial}{\partial z_i}[D_{1,i}(z)p(z)] + \frac{\partial^2}{\partial z_i \partial z_j}[D_{2,ij}(z)p(z)], \tag{8.14}$$

where $z = (q_1, p_1, \ldots, q_n, p_n)$ for an n dimensional system [73] and $p(z)$ is the classical probability density $p(z)$

In Equation (8.14), the coefficient $D_{1,i}$ characterizes the amount of drift in the system and is equal to the evolution of the expectation value of z, $\partial_t \langle z_i \rangle$. If D_{1,q_i} also depends on p_i, it contributes a friction term. The matrix $2D_{2,ij}$ characterizes the amount of diffusion in the system and characterizes $\partial_t \langle z_i z_j \rangle$. The corresponding path integral is given by [153, 154, 80, 87]

$$p(z, t_f) = \int \mathcal{D}z \; e^{-\mathcal{I}_C(z,t_i,t_f)} p(z_i, t_i), \tag{8.15}$$

where

$$\mathcal{I}_C(z, t_i, t_f) = \frac{1}{4} \int_{t_i}^{t_f} dt [\frac{dz_i}{dt} - D_{1,i}(z)] D^{-1}_{2,ij} [\frac{dz_j}{dt} - D_{1,j}(z)]. \tag{8.16}$$

The path integral has a natural interpretation in suppressing classical paths that deviate from their expected drift D_1 by an amount that depends on the inverse of the diffusion coefficient D_2^{-1}. If D_2 is z dependent, Equation (8.16) can also contain an anomalous contribution, arising from the z dependence of $D_2(z)$, as in Chapter 6, but we shall not include it here since (8.16) still defines positive classical dynamics.

The simplest non-trivial case is where one diffuses only in momenta. In this case, $\dot{q}_i = \frac{p_i}{m_i}$ and the momentum integral acts to enforce a delta function over $\delta(p_i - m_i \dot{q}_i)$. Integrating out the momentum variables, the result is a path integral over only the configuration space variables q_i with action

$$\mathcal{I}_C(q,t_i,t_f) = \frac{1}{4}\int_{t_i}^{t_f} dt[m_i\frac{d^2q_i}{dt^2} - D_{1,i}(q)]D^{-1}_{2,ij}[m_j\frac{d^2q_j}{dt^2} - D_{1,j}(q)], \tag{8.17}$$

from which we see that the path integral acts to suppress paths away from their expected equations of motion with the amount depending on D_2.

Taking the expected classical equation of motion to itself be generated by an action S_C, the action in (8.17) can be re-written as

$$\mathcal{I}(q,t_i,t_f) = \frac{1}{4}\int_{t_i}^{t_f} dt\frac{\delta S_C}{\delta q_i}D^{-1}_{2,ij}\frac{\delta S_C}{\delta q_j}. \tag{8.18}$$

Since S_C itself appears in the path integral action $\mathcal{I}(q,t_i,t_f)$, we shall henceforth refer to S_C as the classical proto-action, as in Chapter 6. It is important to note that, generally, one can and should include non-Lagrangian friction terms in the path integral, represented by a more general drift coefficient, as in Equation (8.17).

8.3 A natural class of path integrals

The classical action in Equation (8.18) generalizes to the combined classical-quantum case. A natural class of configuration space path integrals are those derivable from a classical-quantum proto-action that is the sum of a quantum Lagrangian, a classical Lagrangian, and a CQ interaction term

$$W_{CQ}[q,\phi] = \int d^4x\mathcal{W}_{CQ}[q,\phi] = \int d^4x\mathcal{L}_Q[\phi] + \mathcal{L}_C[q] - V_{CQ}[q,\phi], \tag{8.19}$$

$$\begin{aligned}\mathcal{I}(q,\phi^-,\phi^+,t_i,t_f) = \int_{t_i}^{t_f} dx\Big[&i\mathcal{W}^+_{CQ}(x) - i\mathcal{W}^-_{CQ}(x) - \frac{1}{2}\frac{\delta\Delta W_{CQ}}{\delta q_i(x)}D_{0,ij}(q,x)\frac{\delta\Delta W_{CQ}}{\delta q_j(x)}\\ &- \frac{1}{4}\frac{\delta\bar{W}_{CQ}}{\delta q_i(x)}D^{-1}_{2,ij}(q,x)\frac{\delta\bar{W}_{CQ}}{\delta q_j(x)}\Big],\end{aligned} \tag{8.20}$$

where we denote the configuration space classical variable by q, and we take $D_0(q,x), D_2(q,x)$ to be symmetric, positive semi-definite real matrices. We impose the matrix restriction $8D_0 \succeq D_2^{-1}$ to ensure the action takes the form of Equation (8.4) and hence is completely positive. We show this explicitly in Appendix E.2. When $8D_0 = D_2^{-1}$, the path integral preserves purity on the quantum system, as shown in Chapter 5 using unraveling methods. In Equation (8.20)

$W_{CQ}[q,\phi]$ is a real classical-quantum proto-action which generates the dynamics, and we have use the notation $\bar{W}_{CQ} = \frac{1}{2}(W_{CQ}[q,\phi^+] + W_{CQ}[q,\phi^-])$ for the $\pm$ averaged proto-action and $\Delta W_{CQ} = W_{CQ}[q,\phi^-] - W_{CQ}[q,\phi^+]$ for the difference in the proto-action along the $\pm$ branches. As in the classical case, one can add friction terms to Equation (8.20) though we shall not do this in the present work. For simplicity, we here deal with theories with local correlation kernels, but we also expect our results to extend to the case where D_0, D_2 are positive semi-definite matrix kernels $D_0(x,y), D_2(x,y)$ which have some range [4].

This form of action is motivated by the study of path integrals in Chapter 6 for CQ master Equations whose back-reaction is generated by a Hamiltonian [60, 28, 6], as well as the purely classical path integral in Equation (8.18).

Written in the form of Equation (8.20), we see that the action of D_2 is to suppress paths that deviate from the $\pm$ averaged Euler-Lagrange equations, which themselves follow from varying the bra-ket averaged proto-action $\bar{W}_{CQ}$, while the effect of the D_0 term is to decohere the quantum system. The decoherence diffusion trade-off $8D_0 \succeq D_2^{-1}$ [5, 4], required for the dynamics to be CP, means that if coherence is maintained for a long time, then there is necessarily lots of diffusion in the classical system away from its most likely path, with the amount depending on both D_0 and the strength of the coupling which enters in W_{CQ}.

One must further ensure that Equation (8.20) is normalized. When the CQ coupling does not involve higher derivative kinetic terms, it was shown in Chapter 6 that the dynamics generated by Equation (8.20) is normalized. In Appendix E.3, we extend this result and show that any CQ path integral of the form

$$\begin{aligned} I[q,\phi^+,\phi^-] = \int dt i\dot{\phi}_+^2 + iV(\phi^+) - i\dot{\phi}_+^2 - iV(\phi^-) \\ - \frac{D_0(q,\dot{q},\phi^+)}{2}(\ddot{q} + f(q,\dot{q},\phi^+))^2 - \frac{D_0(q,\dot{q},\phi^-)}{2}(\ddot{q} + f(q,\dot{q},\phi^-))^2, \end{aligned} \tag{8.21}$$

is normalized. In the case $D_0 > 0$ has a functional dependence on the fields, one must make sure to also include a factor of $\sqrt{\det(D_0(q,\dot{q},\phi)}$ in the path integral measure [153, 161]. We further show that any higher derivative CQ path integral of the form

$$\begin{aligned} I[q,\phi^+,\phi^-] = \int dt i\ddot{\phi}_+^2 + iV(\phi^+,\dot{\phi}^+) - i\ddot{\phi}_+^2 - iV(\phi^-,\dot{\phi}^-) \\ - \frac{D_0(q,\dot{q},\phi^+,\dot{\phi}^+)}{2}(\ddot{q} + f(q,\dot{q},\phi^+,\dot{\phi}^+))^2 - \frac{D_0(q,\dot{q},\phi^-,\dot{\phi}^-)}{2}(\ddot{q} + f(q,\dot{q},\phi^-,\dot{\phi}^-))^2, \end{aligned} \tag{8.22}$$

is also normalized up to constant factors. In the case $D_0 > 0$ has a functional dependence on the fields, one must again include a factor of $\sqrt{\det(D_0(q,\dot{q},\phi,\dot{\phi})}$ in the path integral measure [153, 161]. Note, if D_0 has a functional dependence on the fields, it is possible to re-exponentiate $\sqrt{\det(D_0)}$ through a Faddeev-Popov type action [201]

$$S[a,\bar{b},b] = \int dt D_0(a^2 + \bar{b}b), \tag{8.23}$$

where a is bosonic and $b, \bar{b}$ are anti-commuting Fermions. The integral over $b, \bar{b}$ yields $\det(D_0)$, whilst the integral over a yields $(\det(D_0))^{-1/2}$ [201]. We do not consider this explicitly here.

Equation's (8.21) and (8.22) are very generic type of action one gets through Equation's (8.20) when varying a CQ proto action that has second order equations of motion in the classical degree of freedom.

8.4 Lorentz invariant CQ dynamics

As a simple example, we can consider a classical field $q(x)$ coupled to a quantum field $\phi(x)$ with a manifestly Lorentz invariant proto-action

$$W_{CQ} = \int d^4x \big[\mathcal{L}_Q(\phi) - \frac{1}{2}\partial_\mu q \partial^\mu q - \frac{1}{2}m_q^2 q^2 - \frac{\lambda}{2}q^2\phi^2\big]. \tag{8.24}$$

In this case, assuming $8D_0 = D_2^{-1}$, we find the expressions for the CQ coupling terms

$$\frac{\delta \Delta W_{CQ}}{\delta q} D_0 \frac{\delta \Delta W_{CQ}}{\delta q} = \lambda^2 D_0 q^2((\phi^+)^2 - (\phi^-)^2)^2 \tag{8.25}$$

$$\frac{\delta \bar{W}_{CQ}}{\delta q} D_2^{-1} \frac{\delta \bar{W}_{CQ}}{\delta q} = 4D_0(\partial^\mu \partial_\mu q + m_q^2 q + \lambda q((\phi^+)^2 + (\phi^-)^2))^2. \tag{8.26}$$

We see that Equation (8.25) acts to decohere the quantum system into the $|\phi\rangle$ basis by suppressing configurations away from $\phi^+ = \phi^-$ by an amount proportional to $D_0\lambda^2$, where λ characterizes the back-reaction on the quantum system. On the other hand, Equation (8.26) acts to suppress configurations away from their semi-classical equations of motion - found from varying $\frac{\delta \bar{W}_{CQ}}{\delta q}$ - by an amount also proportional to D_0. Note that this does not depend on the coupling strength so that in the regime where the back-reaction is small, one can maintain coherence without deviating too much from the expected classical equations of motion. This can be used to evaluate CQ path integrals by working perturbatively in the back-reaction coupling (see Appendix D). Equation (8.25) takes the form of Equation (8.21) if one includes a quantum Lagrangian (such as the Klein-Gordon Lagrangian) and so defines normalized dynamics. Lorentz invariant or covariant pure Linbladians have been studied in [202, 203, 165].

As another example, in Equation (8.20), we could pick a proto-action based on the stress-energy tensor of the quantum matter $T_{\mu\nu}$. For example, by choosing

$$W_{CQ} = \int dx \mathcal{L}_Q(\phi) - \frac{1}{2}\partial_\mu q \partial^\mu q - \ \eta^{\mu\nu} T_{\mu\nu}(x) q(x), \tag{8.27}$$

where $T = \eta^{\mu\nu} T_{\mu\nu}$ is the trace of the stress energy tensor. In this case, we find Lorentz invariant dynamics that causes decoherence of the quantum state according to the stress energy tensor

$$\begin{aligned} \mathcal{I}(\phi^-, \phi^+, q, t_i, t_f) = \int_{t_i}^{t_f} dt d\vec{x} \bigg[& i\mathcal{L}_Q^+(x) - i\mathcal{L}_Q^-(x) - \frac{\lambda^2 D_0[q]}{2} (T^-(x) - T^+(x))^2 \\ & - \frac{1}{4D_2[q]} \left(-\partial_\mu \partial^\mu q(x) + \frac{\lambda}{2}(T^+(x) + T^-(x)) \right)^2 \bigg]. \end{aligned} \tag{8.28}$$

Such dynamics essentially amount to a Lorentz invariant collapse model, where in Equation (8.28), the collapse occurs dynamically due to interaction with a classical field. In particular, when the proto-action is based on the stress-energy tensor, there is an amplification mechanism by which states with small energy maintain coherence while macroscopic objects decohere; this is related to the amplification mechanism used in spontaneous collapse models [168, 169, 170, 171, 48, 172, 173]. Equation (8.28) contains higher derivative Lindblad operators due to the decoherence being according to the stress energy tensor. Hence, according to Equation (8.22), the dynamics will be normalized dynamics if a higher derivative quantum Lagrangian is included, which is indicative of the theory being an effective theory [15]. We leave the study of effective CQ theories to future work. We can further arrive at diffeomorphism invariant CQ dynamics by taking the CQ interaction potential W_{CQ} to be related to a gravitational action, which we now show.

8.5 Diffeomorphism invariant CQ gravity

Let us now comment on some of the consequences of classical-quantum theories of gravity. The goal is to attempt to construct a covariant classical-quantum dynamics theory that approximates Einstein's equations. Since in Equation (8.20) the paths away from $\frac{\delta}{\delta q_i}(\bar{W}_{CQ}[q, \phi^{\pm}])$ are exponentially suppressed by an amount depending on D_2^{-1}, the most likely path will be those for which

$$\frac{\delta}{\delta q_i}(\bar{W}_{CQ}[q,\phi^{\pm}]) \approx 0. \tag{8.29}$$

To get a theory that agrees with Einstein's gravity on average, we could therefore try to take $W_{CQ}[g,\phi]$ to be the sum of the Einstein Hilbert action $S_{EH}[g] = \frac{1}{16\pi G}\int \sqrt{g}R$ (in units where $c = 1$), and a matter action $S_m[g,\phi]$ including a cosmological constant. In the case where $W_{CQ} = S_{EH} + S_m$ we have

$$\frac{\delta}{\delta g_{\mu\nu}}(W_{CQ}[g,\phi]) = -\frac{\sqrt{-g}}{16\pi G_N}(G^{\mu\nu} - 8\pi G_N T^{\mu\nu}), \tag{8.30}$$

Thus, paths would be exponentially suppressed away from (a $\pm$ branch average of) Einstein's equations. Explicitly, taking the classical degree of freedom to be $g_{\mu\nu}$, the decoherence part of the CQ interaction in Equation (8.20) is given by

$$\frac{\delta\Delta W_{CQ}}{\delta g_{\mu\nu}} D_{0,\mu\nu\rho\sigma} \frac{\delta\Delta W_{CQ}}{\delta g_{\rho\sigma}} = \det(-g)\frac{1}{4}(T^{\mu\nu+} - T^{\mu\nu-})D_{0,\mu\nu\rho\sigma}(T^{\rho\sigma+} - T^{\rho\sigma-}), \tag{8.31}$$

whilst (assuming $8D_0 = D_2^{-1}$) the diffusion part takes the form

$$\frac{\delta\bar{W}_{CQ}}{\delta g_{\mu\nu}} D^{-1}_{2,\mu\nu\rho\sigma} \frac{\delta\bar{W}_{CQ}}{\delta g_{\rho\sigma}} = \frac{1}{64\pi^2 G_N^2}\det(-g)(G^{\mu\nu} - 8\pi G_N \bar{T}^{\mu\nu})D_{0,\mu\nu\rho\sigma}(G^{\rho\sigma} - 8\pi G_N \bar{T}^{\rho\sigma}). \tag{8.32}$$

The dynamics take the form of Equation (8.4); thus, the dynamics are completely positive, and the quantum state of the fields remains pure conditioned on the metric. The full action, without assuming the trade-off is saturated, takes the form

$$\begin{aligned}\mathcal{I}[\phi^-,\phi^+,g_{\mu\nu}] = \int dx\Big[& i\mathcal{L}_Q^+ - i\mathcal{L}_Q^- - \frac{\det(-g)}{8}(T^{\mu\nu+} - T^{\mu\nu-})D_{0,\mu\nu\rho\sigma}(T^{\rho\sigma+} - T^{\rho\sigma-}) \\ &- \frac{\det(-g)}{1024\pi^2}(G^{\mu\nu} - \frac{1}{2}(8\pi(T^{\mu\nu})^+ + 8\pi(T^{\mu\nu})^-)D^{-1}_{2,\mu\nu\rho\sigma}[g](G^{\rho\sigma} - \frac{1}{2}(8\pi(T^{\rho\sigma})^+ + 8\pi(T^{\rho\sigma})^-)\Big],\end{aligned} \tag{8.33}$$

where $\mathcal{L}_Q$ is the quantum Lagrangian density. Just as for the Lorentz invariant theory that decoheres according to the stress energy tensor, Equation (8.33) contains higher derivative Lindblad operators through $T^{\mu\nu}$. Hence, according to Equation (8.22), to normalize the dynamics means that a higher derivative quantum Lagrangian should be included – this implies that such as theory may only be valid as an effective theory to be considered up to some energy scale.

Because the path integral in Equation (8.33) contains both decoherence and diffusion, it does not suffer from the same pathologies as the standard semi-classical Einstein's equation's $G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle$ [28, 6]. In particular, it includes the correlation between the matter and gravitational degrees of freedom via the CQ interaction term on the second line. For example, consider starting in an initial state describing a planet in a superposition of left and right $|L\rangle, |R\rangle$ states. The action of the decoherence term will be to enforce (in the Newtonian limit) that the quantum state decoheres into mass eigenstates – meaning that after the decoherence time, the planet will be found on either the left or the right. Because of the CQ interaction, paths where the quantum state decoheres into being on the left are correlated with the classical paths in which the gravitational field is sourced by a planet on the left $T_L^{\mu\nu}$, and similarly for paths which decohere to the planet being found on the right.

When the trade-off is saturated $8D_2 = D_2^{-1}$, the action is fully characterized by the tensor density $D_{0,\mu\nu\rho\sigma}$. There are two possible demands one could make on this tensor. The first would be to require that it be a positive semi-definite matrix in the sense that $v^{\mu\nu}D_{0,\mu\nu\rho\sigma}v^{\rho\sigma} \geq 0$ for any matrix $v^{\rho\sigma}$. This condition would ensure that the dynamics are completely positive and normalizable on any initial state, and classical paths close to Einstein's equations are more probable. Constructing diffeomorphism invariant classical-quantum theories of gravity then amounts to trying to find a tensor $D_0^{\mu\nu\rho\sigma}$, which gives rise to a path integral which defines completely-positive dynamics.

To meet this demand, the simplest thing one can try is to take $D_{0,\mu\nu\rho\sigma} = D_0 g^{-1/2} g_{\mu\nu}g_{\rho\sigma}$, in which case one finds a diffeomorphism invariant CQ theory of gravity in which paths deviating from the *trace* of Einstein's equations are suppressed, moreover, according to Equation (8.31) the quantum state decoheres into eigenstates of the trace of the stress-energy tensor. In the Newtonian limit, where the trace of the stress-energy tensor is dominated by its mass term, it decoheres the quantum state into mass eigenstates. This decoherence is again related to the amplification mechanism used in spontaneous collapse models. However, here the decoherence mechanism arises as a consequence of treating the gravitational field classically and imposing diffeomorphism invariance on the CQ action. Furthermore, although the quantum state decoheres, it remains pure if we condition it on the classical trajectory.

The equations we find demonstrate that a diffeomorphism invariant CQ theory of gravity is possible, though it may only be valid as an effective theory. The challenge in constructing a complete theory is obtaining the transverse parts of the Einstein equation, which are the constraints, while ensuring the path integral over classical metrics remains negative definite so that the path integral converges. In Lorentzian signature, this does not appear possible within the current framework since it amounts to constructing a positive definite metric out of the metric tensor alone. One could instead choose a $D_{0,\mu\nu\rho\sigma}$, which is non-purely geometric, in which case it either introduces a preferred background or must be made dynamical. The former suggests an effective theory in which one obtains a classical metric by adding decoherence or tracing out degrees of freedom in some reference frame. In the latter case, one should add terms proportional to $D_{0,\mu\nu\rho\sigma}g^{\mu\nu}g^{\rho\sigma}$ and $D_{0,\mu\nu\rho\sigma}g^{\mu\sigma}g^{\nu\rho}$ into the classical part of the action. One then must ensure that such terms do not conflict with experimental bounds, which we explore in Chapter 10

Alternatively, we could relax the requirement that $D_{0,\mu\nu\rho\sigma}$ be positive semi-definite but merely require that the path integral be normalizable and preserve positivity on the physical degrees of freedom of a restricted class of initial states. It need not concern us if the negative eigenvalues of $D_{0,\mu\nu\rho\sigma}$ correspond to gauge degrees of freedom or if initial states violate the general constraints relativity evolve into distributions with negative probabilities. In this case, we could impose constraints conserved in the weak sense, similar to Chapter 7 and in loop quantum gravity [190].

As an example, we can consider a classical analogy via the time-independent Fokker-Plank equation in Equation (8.14). The stationary states $p_S(z)$ for the equation are given by

$$p_S(z) = \frac{C}{D_2(z)} \exp\left(\int^z dz' \frac{D_1(z')}{D_2(z')}\right). \tag{8.34}$$

Now we can consider the dynamics of Equation (8.14) with $D_2 \to -D_2$. This does not define positive dynamics on all classical states, which is represented by the fact that the Fokker-Plank path integral diverges. However, it does define positive dynamics on a subset of classical states. The most natural set of states to consider are the stationary states, which now read

$$p_S(z) = \frac{C}{D_2(z)} \exp\left(-\int^z dz' \frac{D_1(z')}{D_2(z')}\right). \tag{8.35}$$

If one starts off in a state of the form in Equation (8.35) the dynamics does not lead to inconsistent probabilities. In the combined classical-quantum case, we could therefore hope that imposing similar constraints on the state space, will lead to completely dynamics for on-shell configurations.

With this in mind, a general form of ultra-local diffusion matrix is then proportional to the generalized Wheeler-deWitt metric in $3+1$ dimensions

$$D_{0,\mu\nu\rho\sigma} = \frac{D_0}{2}(-g)^{-1/2}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2\beta g_{\mu\nu}g_{\rho\sigma}\right), \tag{8.36}$$

with D_0 a positive constant. Equation (8.36) is not a positive matrix, so does not define sensible dynamics acting on all CQ states. However, we would like to consider the case where we consider a constrained set of possible states. The situation is tenable in part because the different components in the CQ action represent both dynamical Fokker-Planck type terms and also components that act to enforce a diffusive, CQ version of the constraints of General Relativity. We posit that these have the form

$$\rho(C|\phi^+, \phi^-) = \frac{1}{\mathcal{N}}e^{-\int \det(-g)C(x)dx}, \tag{8.37}$$

with

$$C(x) := \frac{1}{2}\Delta C_\mu \sigma^{\mu\nu}\Delta C_\nu + \bar{C}_\mu \sigma^{\mu\nu}\bar{C}_\nu, \tag{8.38}$$

$$\bar{C}_\mu(g, \phi^+, \phi^-) := \bar{T}_{\mu 0} - \frac{G_{\mu 0}}{8\pi G_N}, \tag{8.39}$$

$$\Delta C_\mu(\phi^+, \phi^-) := \left(T^+_{0\mu} - T^-_{0\mu}\right), \tag{8.40}$$

with $\phi^\pm$ to be understood as a convenient notation for all fields over some initial Cauchy slice. We would like $\sigma^{\mu\nu}$ to be positive semi-definite so that classical states have probability distributions that are peaked around the constraints of general relativity. The dynamics would then suppress large violations of the constraints. The diffusion element $D_{0,}^{0000}$ associated with the Hamiltonian constraint is non-negative whenever $\beta \leq 1$.

If one chooses $\beta < 1/3$ [204, 205] then the sub-tensor $D_{0,ijkl}$ of spatial-indices is positive semi-definite. The elements of the diffusion matrix associated with the momentum constraint, as well as terms connecting the dynamical and constraint equations, depend on the choice of g^{0i}, which is usually associated with a gauge degree of freedom and the shift vector in the Hamiltonian formalism. This gives some plausibility to the conjecture that one can choose a diffusion tensor that is positive semi-definite on the true degrees of freedom.

The part of the path integral corresponding to the constraints can also be viewed as telling us that initial distributions must be of the form corresponding to Equation (8.37). The constraints then define allowed distribution in terms of the matrix elements ϕ^+, ϕ^-. The full initial state $\rho(g, \phi^+, \phi^-)$ is given by further specifying a quantum state of the field $\rho(\phi^+, \phi^-)$, and a distribution of the metric g conditional on C, ϕ^+, ϕ^- i.e.

$$\rho(g, \phi^+, \phi^-) = \rho(g, \phi^+, \phi^- | C, \phi^+, \phi^-)\rho(C|\phi^+, \phi^-)\rho(\phi^+, \phi^-). \tag{8.41}$$

This state can be pure conditioned on the metric. We then want to restrict ourselves to initial states of the above form, which remain positive and normalizable under the action of the path integral. We leave as an open question whether evolution preserves the constraint in general. In Chapter 9, we study the Newtonian limit of CQ theories, which can be understood as a non-relativistic gauge fixed version of the theories introduced in this chapter. We arrive at a CQ version of the Newtonian constraint, which takes a form similar to Equation (8.37), and show that it is possible to preserve this constraint in time, meaning the path integral in Equation (8.33) gives rise to sensible completely positive on the subset of CQ states which satisfy the Newtonian gauge.

Alternatively, we could impose complete positivity and normalizability of the evolution of these initial states by only summing over paths for which this holds. We regard this as unsatisfactory.

It is also possible to consider a $D_{0,\mu\nu\rho\sigma}(x, x')$, a positive-definite kernel in space-time coordinates x, x' in which case one has stochastic processes which are correlated in space-time. The CQ interaction terms then take the form

$$-\frac{1}{2}\int dxdx' \, \frac{\delta \Delta W_{CQ}}{\delta g_{\mu\nu}(x)} D_{0,\mu\nu\rho\sigma}(x, x') \frac{\delta \Delta W_{CQ}}{\delta g_{\mu\nu}(x')} - \frac{1}{4}\int dxdx' \, \frac{\delta \bar{W}_{CQ}}{\delta g_{\mu\nu}(x)} D^{-1}_{2,\mu\nu\rho\sigma}(x, x') \frac{\delta \bar{W}_{CQ}}{\delta g_{\mu\nu}(x')}. \tag{8.42}$$

In this case, one is breaking the autonomous property of the dynamics since it is not time-local, which is suggestive of an effective theory rather than a fundamental one.

8.6 Discussion

In this chapter, we have introduced a general path integral for classical-quantum dynamics, given by Equation (8.4), which opens up the way to study classical degrees of freedom coupled to quantum ones via path integral methods. The path integral provides an approach to study covariant theories of classical fields coupled to quantum ones. We have given an explicit example of a Lorentz invariant CQ theory and discussed potential applications to classical-quantum theories of gravity.

In particular, we have arrived at a diffeomorphism invariant theory of CQ gravity - summarized by Equations (8.31), (8.32) - which acts to suppress paths that deviate from the trace of Einstein's equations, while simultaneously decohering the quantum system according to the trace of the stress-energy tensor. This provides a first example of diffeomorphism invariant classical-quantum dynamics and, more generally, is a first example of diffeomorphism invariant collapse dynamics [168, 169, 170, 171, 48, 172, 173], where the loss of coherence is a derived consequence of the interaction of a quantum system with a classical dynamical variable. We have also proposed a theory reproducing all of Einstein's equations as a limiting case. The theory is diffeomorphism invariant, but we have not proven that the dynamics preserve the proposed constraints: this may necessitate the further study of the theory's constraint algebroid, a project we initiated in Chapter 7.

The theories introduced in this chapter violate several assumptions which went into the scalar field model considered in Chapter 7. Firstly, the dynamics in this chapter are motivated by the continuous master equation, which has a different Lindblad structure than the jumping models; the jumping models are generally associated with path integrals containing an infinite number of couplings. Secondly, Equations (8.31), (8.32) are higher order in the stress-energy tensor, they contain products $\sim T^2$, and these higher order terms were not considered in Chapter 7. Thirdly, the dynamics is diffusive in the configuration space variable $g_{\mu\nu}$ and hence in the 3-metric γ_{ij}, while the model considered in Chapter 7 was assumed to undergo deterministic classical dynamics. The danger with the model introduced in this chapter is that we have not shown that the constraints will be preserved in time. Furthermore, if the standard constraints of GR are violated too much, this will leave the vacuum state unstable [206]. The fact that large deviations from the standard constraints can be suppressed via the $D_0^{0\mu,0\nu}$ terms suggests that these effects can be made small, but this and other experimental checks need to be further explored.

Let us conclude by discussing some further possible extensions of the present work. Firstly, exploring the renormalization properties of classical-quantum dynamics would be interesting, which we have not touched upon here. Since the resulting classical-quantum action is essentially indistinguishable from a standard quantum field theory, we expect that similar methods could be used. Though effective theories can be non-renormalizable, the renormalizability of CQ dynamics has important foundational consequences for theories with a fundamentally classical field. For the classical-quantum theories generated by a proto-action (8.20), the classical part of the dynamics is generated by a higher derivative kinetic term. For example, in the Lorentz invariant dynamics of Equation (8.32), the propagator associated with the higher derivative term is given by $(\partial_\mu \partial^\mu q)^2$, which we expect to scale like $\sim \frac{1}{(p^2)^2}$ for large momentum; this appears to significantly help with the renormalization properties of any purely classical terms, as well as those associated with back-reaction. However, the gravitational action of Equation's (8.31) and (8.32) are not power counting renormalizable due to the terms which are quadratic in the stress-energy tensor, though, as noted in [165, 174], one must be careful with power counting renormalization when considering the density matrix path integral.

Secondly, we have approached CQ dynamics starting from the description of a system in terms of classical and variables and writing down dynamics which leads to consistent evolution. As an effective theory, it would be interesting to arrive at classical-quantum theories from a top-down approach. That is, starting from a quantum-quantum system, we should be able to arrive at an effective CQ description. We expect this occurs via some decoherence mechanism on one of the systems and is closely related to the quantum to classical transition [207, 208, 209, 210].

We have here given a general construction by which one can write down CQ path integrals that uphold space-time and gauge symmetries. It would be worthwhile to explore this further with concrete examples, and to study the mapping between covariant CQ path integrals and master equations in detail. One open question is to determine what the symmetries generators are, since they should necessarily be altered in a non-unitary theory [211]. In particular, for Lorentz invariant theories, we expect the symmetry generators should form a CQ generalization of the Lorentz algebra. In Appendix K, we make some progress towards this, by studying the symmetry generators in a simple example of a Lorentz invariant open quantum system. For classical-quantum gauge theories, which could be useful in an effective theory of light-matter interactions when there is classical back-reaction, the killing form provides a natural choice for D_0 since for a compact lie group, the killing form is positive semi-definite [212]. In the context of gravity, the theory presented here could be regarded as a theory with a fundamentally classical gravitational field. However, we expect it to be useful as an effective theory of semi-classical gravitational physics when back-reaction is involved. If such a theory were renormalizable, it would form a consistent way of coupling classical and quantum gravity with a sensible UV limit. Such a theory necessarily deviates from quantum mechanics; when there is back-reaction on the gravitational field, the dynamics are no longer unitary, and the coupling necessarily induces collapse of the wavefunction due to the decoherence term parameterized by D_0 in Equation (8.20), while there must also be diffusion in the gravitational field which leads to experimentally testable signatures of a classical gravitational field which we explore in Chapter 10 [4].

Chapter 9

The Newtonian limit of classical-quantum dynamics

This chapter studies the non-relativistic of a classical Newtonian potential interacting with quantum matter. This limit can be viewed as a gauge fixed version of both [28] and the theory introduced in Chapter 8, where we only consider the scalar degrees of freedom to linear order. Our results generalize previous discussions of Newtonian classical-quantum gravity, mainly studied using continuous measurement and feedback approaches [60, 49, 124, 52, 213].

Our goal will be to study the non-relativistic limit of general classical-quantum theories of gravity to provide a template for theorists and experimentalists to develop and test CQ theories. In particular, in Chapter 10, we generalize the decoherence-diffusion trade-off discussed so far in this thesis (Chapter 4) to obtain bounds relating the decoherence rate and diffusion of any CQ theory, given in terms of the strength of the back-reaction. By studying the non-relativistic limit, we can understand the predictions of CQ theories in the Newtonian regime. Indeed, we find a generic prediction of CQ theories: the Newtonian potential diffuses away from its classical solution by an amount that depends on the decoherence rate into mass eigenstates. The decoherence-diffusion trade-off, therefore, provides a way of testing CQ theories: one lower bounds the amount of diffusion the theory must have from coherence experiments, which can then be tested by measuring the noise in precision mass experiments. We explore this in detail in Chapter 10.

In this chapter, we consider the general case of CQ Newtonian gravity by reducing the degrees of freedom to scalar perturbations. In this "bottom-up" approach, we assume that the relevant dynamical degrees of freedom are scalar perturbations of the metric, i.e., the Newtonian potential. We shall also allow for vector perturbations of the shift vector at higher order in c, which we find are necessary to construct consistent dynamics. We impose phenomenological constraints on the dynamics and ask that Newton's equation for the gravitational field is satisfied on expectation. With these assumptions, we can construct the Newtonian limit of CQ theories via a reduction - even without a complete theory. In particular, by first identifying the relevant degree of freedom as the Newtonian potential, we use the master equation and unraveling formalism to construct CQ dynamics parameterized by the moments D_0, D_2 appearing in the dynamics. We verify our findings by showing that the Newtonian limit we derived agrees with the Newtonian limit of the theory introduced in Chapter 9.

This chapter is based on upcoming work [9], which is work done in collaboration with Jonathan Oppenheim and Andrea Russo.

9.1 Newtonian limit of classical GR

In this section, we study the Newtonian limit of classical general relativity (GR), which motivates our study of the Newtonian limit of classical-quantum theories of gravity. By the Newtonian limit, we mean the linearised expansion of the metric around a flat Minkowski background, where the $c \to \infty$ limit is taken, discarding terms higher order in c.

The Newtonian limit of GR is represented by a non-dynamical scalar perturbation of flat Minkowski spacetime expressed through the metric:

$$ds^2 = -c^2 \left(1 + \frac{2\Phi}{c^2}\right) dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j, \tag{9.1}$$

where Φ satisfies the gravitational Poisson equation. The usual derivation of this limit arises from a gauge fixing of the full Einstein theory. There, one starts with a generalized scalar-vector tensor perturbation of the metric in the form [214] of

$$ds^2 = -c^2 \left(1 + \frac{2\Phi}{c^2}\right) dt^2 + \frac{w_i}{c}(dt dx^i + dx^i dt) + \left[\left(1 - \frac{2\psi}{c^2}\right) \delta_{ij} + \frac{2 s_{ij}}{c^2}\right] dx^i dx^j, \tag{9.2}$$

where $\partial_i w^i = \partial_i s^{ij} = 0$ and takes the infinite c limit of Einstein's equations. When the stress-energy tensor is chosen to represent a pressureless dust distribution, or a point particle, only one non-dynamical scalar perturbation Φ remains at the end, and it is constrained to obey the gravitational Poisson's equation.

We instead present a derivation of the Newtonian limit by directly reducing the degrees of freedom to scalar perturbations. In the reduced degrees of freedom approach, we *first* assume that the relevant physical degrees of freedom are scalar perturbations. We shall also allow for vector perturbations of the shift vector at higher order in c, which we find are necessary to construct a consistent CQ theory. In the classical case, we see we can set these to zero. This allows us to construct the Newtonian $c \to \infty$ limit of CQ theories by considering only scalar perturbations of the gravitational degrees of freedom - even without a complete theory, though with assumptions on its completion.

9.1.1 Newtonian limit via a reduced action

We now arrive at the Newtonian limit of GR via a reduced Hamiltonian. We take as a starting point the linearised Einstein Hilbert Lagrangian density, which is equivalent to the Fierz-Pauli action [215] for the metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$S[h_{\mu\nu}] = \frac{c^4}{16\pi G}\int d^4x\, \mathcal{L}(h_{\mu\nu}), \tag{9.3}$$

$$\mathcal{L}(h_{\mu\nu}) = -\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h^{\rho\sigma}\partial_\rho h^\mu_\sigma - \frac{1}{4}\eta^{\mu\nu}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} + \frac{1}{4}\eta^{\mu\nu}\partial_\mu h \partial_\nu h. \tag{9.4}$$

We are interested in constructing CQ dynamics for a Newtonian theory, so we further make a Newtonian approximation of the metric. We take the ADM decomposition of the metric

$$\mathrm{d}s^2 = -(Nc\ \mathrm{d}t)^2 + g_{ij}\left(\mathrm{d}x^i + N^i c\ \mathrm{d}t\right)\left(\mathrm{d}x^j + N^j c\ \mathrm{d}t\right), \tag{9.5}$$

and we assume that

$$N = \left(1 + \frac{\Phi}{c^2}\right),\ N^i = \left(0 + \frac{n^i}{c^3}\right),\ g_{ij} = \left(1 - \frac{2\psi}{c^2}\right)\delta_{ij}. \tag{9.6}$$

The extra factor of c in the choice of shift-vector is related to the fact that, classically, the h_{0i} component occurs at a higher order than the h_{00}, h_{ij} components [214]. Since in the ADM formalism, the physical degrees of freedom are the spatial metric g_{ij} and its conjugate momenta, assuming that $g_{ij} = \left(1 - \frac{2\psi}{c^2}\right)\delta_{ij}$ amounts to assuming the physical degrees of freedom are scalar perturbations of the metric. We assume that all fields vanish at infinity. In the purely classical case, we find that when the stress-energy tensor $T_{0i} = 0$, then $n^i = 0$, but we will show that in the combined CQ case $n^i \neq 0$ even in the absence of the stress-energy tensor. Instead, this correction is required to preserve the theory's Hamiltonian constraint.

With the gauge fixing of Equation (9.6), the linearized action in Equation (9.3) is

$$\begin{aligned} S = \frac{1}{8\pi G}\int d^4x \Big[&- \frac{3\dot{\psi}^2}{c^2} + \frac{\partial_i n^i}{2c^2}(\dot{\Phi} - \dot{\psi}) \\ &- \frac{\dot{n}^i}{2c^2}(\partial_i \Phi + 3\partial_i \psi) - \frac{1}{4c^2}\partial_i n^j \partial_j n^i + \partial_i \psi \partial^i \psi - 2\partial_i \Phi \partial^i \psi + \frac{1}{4c^2}\partial_i n^j \partial_j n^i \Big]. \end{aligned} \tag{9.7}$$

To go to the Hamiltonian picture, we first calculate the functional derivatives with respect to $\dot{\psi}$, $\dot{\phi}$ and $\dot{n}^i$ to find the conjugate momenta

$$\pi_\psi = -\frac{1}{16c^2 G\pi}(12\dot{\psi} + \partial_i n^i), \;\; \pi_\Phi = \frac{\partial_i n^i}{16\pi c^2 G}, \;\; \pi_i = -\frac{1}{16\pi c^2 G}(\partial_i \Phi + 3\partial_i \psi). \tag{9.8}$$

Equation (9.8) defines two primary constraints given by

$$\Pi_\Phi = \pi_\Phi - \frac{\partial_i n^i}{16\pi c^2 G} \approx 0, \tag{9.9}$$

$$\Pi_i = \pi_i + \frac{1}{16\pi c^2 G}(\partial_i \Phi + 3\partial_i \psi) \approx 0. \tag{9.10}$$

In the $c \to \infty$ limit, Equations (9.9) and (9.10) become the constraints $\pi_\Phi, \pi_i \approx 0$, which enforce the Hamiltonian and momentum constraints.

One might worry that the constraints appearing in Equations (9.9) and (9.10) appear to be different concerning the constraints one obtains by first considering the full ADM formalism constraints and then linearising them. This difference has been studied in [216], where it was shown that the two forms are related by a canonical transformation. Alternatively, one could follow the approach of [106] and add a specific non-covariant term to the linearised action of Equation (9.3). The additional term vanishes on-shell and simplifies the primary constraints to match them with those derived from the ADM formalism. Since we are interested in the $c \to \infty$ limit, these distinctions do not matter, as we end up imposing the constraints $\pi_\Phi, \pi_i \approx 0$, which are equivalent to the primary constraints $\pi_N, \pi_{N_i} \approx 0$, where N, N^i are the lapse and shift vectors.

Using the definitions of conjugate momenta in Equation (9.8), and working to leading order in c, we arrive at the Newtonian Hamiltonian

$$H = \int d^3x \left[-\frac{2\pi G c^2}{3}\pi_\psi^2 - \frac{1}{12}\pi_\psi \partial_i n^i + \frac{\partial_i \psi \partial_i \Phi}{4\pi G} - \frac{\partial_i \psi \partial_i \psi}{8\pi G} + \lambda_\Phi \pi_\Phi + \lambda^i \pi_i \right]. \tag{9.11}$$

To find the Newtonian interaction Hamiltonian, we need to couple gravity to matter. We shall consider the matter distribution to be that of a particle with mass density $m(x)$, in which case to leading order in c only T_{00} contributes to the gravitational equations. The corresponding interaction Hamiltonian can then be written as

$$H_I = \int d^3x \Phi(x) m(x). \tag{9.12}$$

The total Hamiltonian is given by $H_{tot} = H + H_I$:

$$H_{tot} = H_m \int d^3x \left[-\frac{2\pi G c^2}{3}\pi_\psi^2 - \frac{1}{12}\pi_\psi \partial_i n^i + \frac{\partial_i \psi \partial_i \Phi}{4\pi G} - \frac{\partial_i \psi \partial_i \psi}{8\pi G} + \lambda_\Phi \pi_\Phi + \lambda^i \pi_i + \Phi(x) m(x) \right], \tag{9.13}$$

where the dynamics associated to H_{tot} is given by:

$$\dot{\psi} = -\frac{4G\pi c^2 \pi_\psi}{3} - \frac{1}{12}\partial_i n^i,\ \dot{\pi}_\psi = \frac{\nabla^2(\Phi - \psi)}{4\pi G},\ \dot{\Phi} = \lambda_\Phi,\ \dot{\pi}_\Phi = \frac{\nabla^2 \Phi}{4\pi G} - m,\ \dot{n}^i = \lambda^i,\ \dot{\pi}_i = -\frac{1}{12}\partial_i \pi_\psi. \tag{9.14}$$

We arrive at the Newtonian limit by imposing the constraints $\pi_i, \pi_\Phi \approx 0$ and solving Equation (9.14). Note the constraint $\pi_\Phi \approx 0$ imposes

$$\frac{\nabla^2 \Phi}{4\pi G} - m \approx 0 \quad \Rightarrow \quad \Phi(t, x) = -G \int d^3x' \, \frac{m(x)}{|x - x'|}, \tag{9.15}$$

on the potential Φ i.e, Φ must solve Poisson's equation. On the other hand, the constraint $\pi_i \approx 0$ imposes $\pi_\psi \approx 0$, where we have used the fact that π_ψ vanishes at infinity. Preservation of the $\pi_\psi \approx 0$ constraint imposes that $\Phi = \psi$. Moreover, the time derivative of the Newtonian potential directly dictates the Lagrange multiplier via $\lambda_\Phi = \dot{\Phi}$ and the divergence part of the shift vector via $\dot{\Phi} = -\frac{1}{12}\partial_i n^i$.

Since we assume a stationary source, where only T_{00} contributes, $\partial_i n^i = 0$. This equation only partially fixes the shift n^i since related by different choices of the shift vector will be gauge equivalent. In the classical theory, it is common to assume the gauge $n^i = 0$, in which case we arrive at the Newtonian metric of Equation (9.1), where Φ satisfies Poisson's equation.

We have arrived at the Newtonian limit of general relativity by making the Newtonian approximation on the metric in Equation (9.6) and then deriving the dynamics in the $c \to \infty$ limit. While deriving the Newtonian limit from a full GR approach requires a complete diffeomorphism invariant theory, we have seen that we can construct a consistent reduced theory by first identifying the correct degrees of freedom (in this case, scalar perturbations of the metric) and then writing down their dynamics according to a reduced Hamiltonian.

9.1.2 A stochastic classical analog of the CQ theory

In the CQ case, we will construct the Newtonian limit by assuming the relevant degrees of freedom are scalar perturbations of the metric of the form in Equation (9.6) and then considering a reduced CQ master equation governing the dynamics of the perturbations. Since we will be interested in describing the non-relativistic limit of a quantum mass interacting with classical gravity, the back-reaction on the gravitational field from the quantum matter is dominated by the T_{00} component. Any classical-quantum momentum constraint should be unchanged since it does not involve matter. In particular, the back-reaction of the quantum system on the classical system enters through $\dot{\pi_\Phi}$ in Equation (9.14), through the action of the interaction Hamiltonian $H_I = \int d^3x \Phi(x) m(x)$. Because quantum back-reaction must necessarily involve diffusion, in the CQ case, the equation of motion for $\dot{\pi}_\Phi$ will be modified to include a stochastic term.

To gain some intuition, we can consider the classical analog of the CQ theory by considering a Langevin equation for $\dot{\pi}_\Phi$

$$\dot{\pi}_\Phi = \frac{\nabla^2 \Phi}{4\pi G} - m - \sigma\xi, \tag{9.16}$$

where $\sigma(x)$ is a coefficient and $\xi(t, x)$ is a white noise process

$$\mathbb{E}[d\xi(x)] = 0, \quad \mathbb{E}[\xi(t, x)\xi(t', y)] = \delta(t, t')\delta(x, y)dt. \tag{9.17}$$

Note, multiplying Equation (9.16) by dt, the equation takes the same form as the stochastic unravelings studied in Chapter 5, but with $\langle m \rangle \to m$ since we consider the classical analog of the back-reaction. We have also chosen to define $\sigma \to -\sigma$ so that the noise term in Equation (9.16) can be interpreted as a random contribution to the mass term.

With the modified dynamics for π_Φ, we find the constraint $\pi_\Phi \approx 0$ imposes the stochastic

Newtonian constraint

$$\nabla^2\Phi = 4\pi G(m + \sigma\xi) \tag{9.18}$$

on the potential Φ.

The full set of dynamics then takes the form

$$\dot{\psi} = -\frac{4G\pi c^2\pi_\psi}{3} - \frac{1}{12}\partial_i n^i,\ \dot{\pi}_\psi = \frac{\nabla^2(\Phi-\psi)}{4\pi G},\ \dot{\Phi} = \lambda_\Phi,\ \dot{\pi}_\Phi = \frac{\nabla^2\Phi}{4\pi G} - m - \sigma\xi, \tag{9.19}$$

$$\dot{n}^i = \lambda^i,\ \dot{\pi}_i = -\frac{1}{12}\partial_i\pi_\psi. \tag{9.20}$$

Since the back-reaction is in T^{00}, the momentum constraint π_i remains unchanged from the deterministic case. Its preservation imposes the constraint $\pi_i \approx 0$, which further imposes the constraint $\Phi = \psi$. The constraint $\pi_\Phi \approx 0$ gives the Newtonian constraint. However, with the addition of the noise, the Newtonian potential is no longer stationary but instead solves the randomly sourced Poisson equation

$$\Phi = -G\int dx'\ \frac{m(x') + \sigma(x',t)\xi(x',t)}{|x-x'|}. \tag{9.21}$$

Equation (9.21) then determines λ_Φ, and using that $\Phi = \psi$, also determines $\partial_i n^i$ and λ^i through the Equation

$$\dot{\psi} = -\frac{1}{12}\partial_i n^i. \tag{9.22}$$

In particular, with the gauge choice given by Equation (9.6), we see that n^i is *required* for the theory to be consistent. The presence of diffusion in the equation of motion for π_Φ makes the Newtonian potential $\psi = \Phi$ fluctuate, and we see the shift vector $\partial_i n^i$ is required to account for this fluctuation. Had we not included it, we would have found an inconsistent set of equations since $\dot{\psi}$ would have been vanishing through Equation (9.22). However, the stochastic constraint requires it to fluctuate.

We point out once again that this does not fix the shift n^i uniquely since we are free to add a divergenceless term and get the same solution to the equation of motion. Moreover, in a complete calculation, we expect that contributions from T_{0i} will also determine the components n^i without affecting the Newtonian contribution, given by the h_{00} component. Regardless, the set of Equation's (9.19) are together consistent, and performing higher-order calculations is beyond the scope of the current work.

9.1.3 A simplified starting point

In the stochastic case, we still find that the dynamics set $\Phi = \psi$. Hence, one can instead start with the metric perturbation:

$$N = \left(1 + \frac{\Phi}{c^2}\right),\ N^i = \left(0 + \frac{n^i}{c^3}\right),\ g_{ij} = \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}. \tag{9.23}$$

and consider the dynamics obtained by setting $\Phi = \psi$ in Equation (9.11). One can also remove the kinetic term $-\frac{2\pi G c^2}{3}\pi_\psi^2$, which does not contribute to the equations of motion on the constraint surface, and the Lagrange multiplier involving π_Φ, which also vanishes on the constraint surface. In this case, the Hamiltonian reads

$$H + H_I = \int d^3x \left[\frac{(\nabla\Phi)^2}{8\pi G} + m\Phi - \frac{1}{12}\pi_\Phi \partial_i n^i\right], \tag{9.24}$$

subject to the constraint $\pi_\Phi \approx 0$. Because Equation's (9.23) and (9.24) are considerably simpler than Equation's (9.6) and (9.11) but result in the same dynamics, we will use the Hamiltonian in Equation (9.24) to describe the Newtonian limit of CQ theories.

Since the noise process is white noise, technically, the metric perturbation will describe a probability measure, and we should use it to compute averaged quantities; this is true of both Φ and the shift vector n^i, which are now both stochastic quantities. In particular, averaging over a timescale ΔT and length scale ΔL, $\Delta L/\Delta T \ll c$, we have that $\frac{\Delta\Phi}{\Delta T} \sim \frac{n^i}{\Delta L}$, so that $\frac{\Delta L \Delta\Phi}{\Delta T} \sim n^i \ll c^3$ which verifies our initial assumption to include the perturbation h_{0i} as $\frac{n^i}{c^3}$ in Equation (9.6).

With this in mind, we now study the Newtonian limit of the full CQ theory. In the $c \to \infty$ limit, we arrive at Poisson's equation on average. However, because of the CQ interaction, the Newtonian limit also predicts diffusion around this solution according to Equation (9.21), with simultaneous decoherence on the quantum system.

Before discussing how a quantum system's back-reaction on the classical Newtonian field is implemented through diffusion processes, we would like to comment on the choice of gauge. The end goal of this chapter is to formulate the Newtonian limit of gravity for CQ-hybrid theories; we are still determining if a complete CQ theory can be made fully diffeomorphism invariant in a way that also accounts for the constraints of the theory. Regardless, our choice of gauge is motivated by the need to preserve the gravitational constraints. By choosing the gauge as in Equation (9.6), we know that we have a way of consistently selecting trajectories that stay on the constraint surface, where the conjugate momenta vanish as described in this section. We verify our findings by showing that with the gauge fixing described by Equation (9.6), the theory introduced in Chapter 8 agrees with the Newtonian limit we find using a bottom-up approach.

9.2 Hamiltonian CQ dynamics reproducing the Newtonian limit

Having discussed in detail the Newtonian limit of GR, we are now in a position to discuss how to write down classical-quantum theories that give rise to the Newtonian interaction on average. Before discussing the specifics of continuous and discrete master equations, we shall outline the general procedure and assumptions.

Assumption 1. *We assume that the evolution of the combined classical-quantum system undergoes autonomous CQ dynamics.*

We expect this assumption to hold if CQ is treated as a fundamental theory. However, as an effective theory, this assumption may break down since the dynamics can be non-Markovian. We comment on the differences between a fundamental and effective theory in Section 9.8 (see also [6]).

Assumption 2. *We take a bottom-up approach and assume that in the weak field $c \to \infty$, the appropriate gravitational degrees of freedom are the perturbations of the metric in the form of Equation (9.23).*

In particular, the leading order contribution which governs the geodesics of test particles is described by $g_{00} = (1 + \frac{\Phi}{c^2})$.

Assumption 3. *We take the variables $(\Phi(t), \pi_\Phi(t), n^i(t))$ to be classical stochastic variables coupled to a stochastic quantum state $\rho(t)$. We assume that the purely classical part of the evolution is generated by the reduced Hamiltonian (9.24), that the interaction between classical and quantum degrees of freedom is Hamiltonian and that it is governed by the reduced interaction Hamiltonian in Equation (9.12), where the constraints $\pi_\Phi(t) \approx 0$ should also be imposed.*

Specifically, we require that the first moment D_{1,π_Φ} is picked to reproduce the Newtonian back-reaction on average:

$$\mathrm{Tr}\left[\{H_I,\varrho\}\right] = \int d^3x\ \mathrm{Tr}\left[m(x)\frac{\delta\rho}{\delta\pi_\Phi(x)}\right], \tag{9.25}$$

so that the dynamics are approximately Hamiltonian on average.

Since the back-reaction of the quantum system on the classical system is associated with T_{00}, we expect the CQ momentum constraint to be unchanged from its classical counterpart, as it is not associated with any back-reaction. One caveat, however, to keep in mind is that relativistic corrections at high energy may affect the low-energy behavior of the theory. In particular, general relativity has yet to be tested at distances shorter than the millimeter scale. Here we assume it holds to arbitrarily short distances.

As a consequence of Assumption 3, we know from the decoherence-diffusion trade-off that there must be diffusion in the classical variable and also Lindbladian evolution on the quantum state. In particular, in the next chapter we generalize the decoherence-diffusion trade-off found for the continuous master Equations in Chapter 4 to show that for any completely positive autonomous CQ master equation, the moments $D_n^{\mu\nu}$ appearing in the master equation of Equation (2.91) must satisfy $2D_{2,\pi_\Phi\pi_\Phi} \succeq D^{br}_{1,\pi_\Phi}D_0^{-1}(D^{br}_{1,\pi_\Phi})^\dagger$, where $D_{2,\pi_\Phi\pi_\Phi}$ is the full matrix of diffusion coefficients $D^{\mu\nu}_{2,\phi_\Phi\pi_\Phi}$, $D^{br}_{1,\pi_\Phi} = D^{\mu\alpha}_{1,\pi_\Phi}$ is the matrix describing the drift of the back-reaction of the quantum system and $(D_0^{-1})_{\alpha\beta}$ is the generalized inverse of the Lindbladian coefficient $D_0^{\alpha\beta}$.

Assumption 4. *In this chapter, we will take the coefficients D_n entering the master equation to be minimally coupled, by which we mean they depend only on the Newtonian potential Φ, $D_n(\Phi)$ and not their conjugate momenta π_Φ.*

This assumption is motivated by the fact that in Einstein's gravity, the mass density couples to the Newtonian potential, not its conjugate momenta, and we are imposing the constraint that $\pi_\Phi \approx 0$. Nonetheless, one could generalize the master equations to the non-minimally coupled case by considering couplings $D_n(\Phi) \to D_n(\Phi,\pi_\Phi)$ in all of the equations.

Note, Assumptions 1-4 violate those given in Chapter 7 and enable us to find constraints preserved in time. In particular, Assumption 2 amounts to a choice of gauge, while the theories in Chapter 7 were assumed to be gauge independent. It may be that no consistent gauge invariant theory exists, which gives rise to the full Einstein's equations, but we can find gauge fixed dynamics that give rise to its Newtonian limit. More importantly, via Assumption 3, we let the Newtonian potential be stochastic, which amounts to letting γ_{ij} be a stochastic variable. Conversely, in Chapter 7, we assumed deterministic classical dynamics for the configuration space variable γ_{ij}. Here, we find that stochasticity in the Newtonian potential is required to find consistent constraints that close, perhaps shedding light on why the theory considered in Chapter 7 fell short.

We now consider the dynamics consistent with our assumptions. We discuss the case of continuous back-reaction in detail since, in this case, our assumptions fully determine the dynamics up to a choice of Lindbladian strength. We then discuss how one can use the decoherence-diffusion trade-off to reason about the dynamics in the case of jumping master equations.

9.3 Continuous gravitational back-reaction

In Chapter 4 [5], it was shown that there are two classes of CQ master equations, and the general form of the continuous master equation was found to be Equation (4.13). We also found the unraveling representation of such master equations in Chapter 5, characterized only by the couplings D_0, σ, D_1, with $\sigma\sigma^T = 2D_2$. In this case, our assumptions on Hamiltonian back-reaction are enough to specify the combined classical-quantum system's dynamics fully. They take the form of the Hamiltonian unraveling of Chapter 5 (see also Appendix C for a discussion of unravelings with fields)

$$
\begin{aligned}
d\Phi &= -\frac{1}{12}\partial_i n^i dt, \\
d\pi_\Phi &= \frac{\nabla^2\Phi}{4\pi G}dt - \langle m(x)\rangle dt - \int d^3y\,\sigma(\Phi;x,y)dW(y), \\
d\rho(t) &= -i[H_m + H_I, \rho]dt + \frac{1}{2}\int d^3y D_0(\Phi;x,y)\left([m(x),[\rho, m(y)]]\right)dt \\
&\quad + \frac{1}{2}\int d^3y\,\sigma^{-1}(\Phi;x,y)\big(m(x)\rho + \rho\, m(x) - 2\rho\langle m(x)\rangle\big)dW(y).
\end{aligned}
\tag{9.26}
$$

In Equation (9.26) H_m is a purely quantum Hamiltonian, $m(x)$ is the quantum mass density operator, $\langle O\rangle$ is the expectation value of the normalized quantum state ρ, $\mathrm{Tr}\,[\rho O]$ and $W_i(x)$ is a Wiener process in space-time satisfying

$$
\mathbb{E}[dW(x)] = 0, \quad \mathbb{E}[dW(x)dW(y)] = \delta(x,y)dt. \tag{9.27}
$$

For notational simplicity, we will suppress the dependence of the couplings D_0, D_2 on the Newtonian potential and write $D_0(x, y), D_2(x, y)$. The Lindbladian term, characterized by D_0, and the diffusion term D_2 are required for the back-reaction to be completely positive, which can be seen from the decoherence diffusion trade-off [4] apparent in Equation (4.13).

In Equation (9.26), σ is related to the diffusion coefficient D_2 appearing in the master equation via

$$2D_2(\Phi; x, y) = \int dw\ \sigma(\Phi; x, w)\sigma(\Phi; w, y). \tag{9.28}$$

and σ^{-1} the generalized inverse of σ.

The choice of the possible master equation is therefore fully constrained up to the functional choices of the couplings $D_0(\Phi; x, y), D_2(\Phi; x, y)$. Both $D_2(\Phi; x, y), D_2(\Phi; x, y)$ are required to be positive kernels, where a positive kernel $f(x, y)$ is a kernel such that $\int dxdya^*(x)f(x, y)a(y) \geq 0$ for any function $a(x)$. They are also constrained to satisfy the decoherence diffusion trade-off:

$$8D_2 \succeq D_0^{-1}, \tag{9.29}$$

where Equation (9.29) is to be understood as a matrix kernel equation:

$$\int dxdy\ a(x)^*\big[8D_2(\Phi; x, y) - D_0^{-1}(\Phi; x, y)\big]a(y) \geq 0, \tag{9.30}$$

which must hold for an arbitrary function $a(x)$. We give example kernels that satisfy the decoherence-diffusion trade-off in Table 9.3.

9.3.1 Imposing the Newtonian constraint

To arrive at the classical-quantum version of Poisson's equation, we must impose the constraint $\pi_\Phi \approx 0$ according to the Hamiltonian in Equation (9.24). We impose the constraint on the level of trajectories.[1] In practice, what we are really doing is choosing n^i stochastically such

[1]This is conceptually very similar to what is done in quantum theory when constraints are imposed via a path integral approach, where one associates to each path a measure given by the action, then selects only the paths which satisfy the constraint. Take for example, a Hamiltonian with $H(q, p) = H_0(q, p) + \lambda C(q, p)$. The phase space partition function for the theory is represented by $\mathcal{Z} = \int \mathcal{D}q\mathcal{D}p\mathcal{D}\lambda e^{\frac{i}{\hbar}\int dt[\dot{q}p - H(q,p) - \lambda C(q,p)]}$. Since the Hamiltonian is linear in λ, the path integral over the Lagrange multiplier in λ enforces a delta function over

Master Equation	Decoherence $D_0^{\alpha\beta}(x,y)$	Diffusion $D_{2,\alpha\beta}(x,y)$
Continuous (ultra-local)	$D_0^{\alpha\beta}(x,y) = D_0^{\alpha\beta}\delta(x,y)$	$D_{2,\alpha\beta}(x,y) = \frac{1}{8}(D_0^{-1})_{\alpha\beta}\delta(x,y)$
Continuous (Gaussian)	$D_0^{\alpha\beta}(x,y) = D_0^{\alpha\beta}g_{\mathcal{N}(x,y)}$	$D_{2,\alpha,\beta}(x,y) = \frac{(D_0^{-1})_{\alpha\beta}}{8}F(x,y)g_{\mathcal{N}(x,y)}$
Continuous (D.P)	$D_0^{\alpha\beta}(x,y) = \frac{D_0^{\alpha\beta}}{\lvert x-y\rvert}$	$D_{2,\alpha\beta}(x,y) = \frac{1}{8}\frac{(D_0^{-1})_{\alpha\beta}}{4\pi}\nabla_y^2(\delta(x,y))$

Table 9.1: Possible choices of kernels for the continuous master equations and the resulting diffusion/decoherence coefficients, assumed to saturate the trade-off in Equation (9.29). In the table, D.P stands for the Diosi-Penrose kernel, $g_{\mathcal{N}(x,y)}$ is a normalized Gaussian distribution, and $F(x,y) = \prod_{i=1}^{d}\sum_{n=0}^{\infty} c_n(r_0)H_{2n}(\frac{x_i-y_i}{r_0})$ where $c_n(r_0) = \frac{(-1)^n(r_0)^{2n}n!}{2^n}$, d is the spatial dimension and H_{2n} denote the Hermite polynomials [217]. In particular $g_{\mathcal{N}}^{-1}(x,y) = F(x,y)g_{\mathcal{N}}(x,y)$.

that $\dot{\pi}_\Phi \approx 0$. Doing so turns Φ_t into a white noise variable with values given by the solution of Equation (9.31). However, naively replacing all occurrences of Φ_t with its solution in terms of an Itô white noise variable, in particular, that which appears in H^I, does not lead one to a CPTP dynamics. Before the constraints are imposed, the dynamics of Φ_t are continuous, and thus any back reaction from the quantum matter on Φ_t only returns to affect the quantum matter degrees of freedom at later times. To ensure that this time-ordering is maintained even in the limit that Φ_t no longer evolves continuously, one must be careful to ensure the action of H^I occurs after the other stochastic terms [49]. One possible way to ensure this is to first write the unravelling of the density matrix in the Stratonovich form and then impose the constraint that turns Φ_t, and hence H^I, into white noise. This allows us to correctly rewrite the unravelling such that when converting back into the Itô formalism, we pick up an extra decoherence term given by the backreaction of the stochastic gravitational field and allows us to get rid of the non-linear evolution term picked up from the solution of the noise Poisson equation.[2] This then gives the final form of unravelling in the Newtonian limit:

$$\frac{\nabla^2\Phi_t}{4\pi G} = \langle m(x)\rangle + \int d^3y\,\sigma(\Phi_t;x,y)\xi_t(y), \tag{9.31}$$

$$\begin{aligned}
d\rho_t = {} & -i\,[H_m + V_m, \rho_t]\,dt - i\int d^3x\,d^3y\,d^3y'\left[-G\,\frac{m(x)\sigma(\Phi_t,y,y')}{|x-y|},\rho_t\right] dW_t(y') \\
& + \frac{1}{2}\int d^3x\,d^3y\,D_0(\Phi_t;x,y)\big([m(x),[\rho_t,m(y)]]\big)\,dt \\
& + \frac{1}{2}\int d^3x\,d^3y\,d^3y'\,[\sigma(\Phi_t;x,y),[\rho_t,\sigma(\Phi_t;x,y')]]\,dt \\
& + \frac{1}{2}\int d^3x\,d^3y\,\sigma^{-1}(\Phi_t;x,y)\big(m(x)\rho_t + \rho_t\,m(x) - 2\rho_t\langle m(x)\rangle\big)\,dW_t(y),
\end{aligned} \tag{9.32}$$

where $\xi_t(x) = \frac{dW_t(x)}{dt}$ is the formal definition of white noise, and

$$\begin{aligned}
V_m &= -\frac{G}{2}\int d^3x\,d^3y\frac{m(x)m(y)}{|x-y|}, \\
\sigma(\Phi_t;x,y) &= -G\int d^3\,y''\frac{m(x)\sigma(\Phi_t,y,y'')}{|x-y|}.
\end{aligned} \tag{9.33}$$

$\delta(C(q,p))$ so that the partition function reads $\mathcal{Z} = \int \mathcal{D}q\mathcal{D}p\delta(C(q,p))e^{\frac{i}{\hbar}\int dt[\dot{q}p - H(q,p)]}$ which can be interpreted as summing over all paths with weight $e^{\frac{i}{\hbar}\int dt[\dot{q}p-H(q,p)]}$ and then selecting only those that satisfy the constraint $C(q(t),p(t)) = 0$.

[2] In the present context, this extra decoherence term was discovered by Isaac Layton

These equations were first written down by Tilloy and Diósi in [49] and their derivation from a fundamental theory is a central result of this chapter. The Newtonian limit of CQ theories is described by a Newtonian potential diffusing around Poisson's equation by an amount defined by D_2, while simultaneously, the density matrix decoheres into the mass eigenbasis by an amount determined by the Lindbladian coefficient D_0.

The details of the functional dependence of σ and D_0 on Φ_t have been left unspecified. When they are independent of Φ_t the equations coincide with those of a continuous measurement of a quantum mass, where the measurement outcome is used to source the Newtonian potential, as given by Equation (24) of [49]. In this case, the additional decoherence term appearing in Equation (9.26) proportional to $\sigma^2 \sim D_0^{-1}$ was studied in [124], where the kernels σ, D_0 leading to the minimal amount of decoherence where found. On the other hand, one can consider the couplings to be a general Markovian functional of Φ_t. This will generically lead to additional terms, as was observed with H^I above, which may not preserve the CPTP property of the dynamics. Exploring the details of these functional dependencies is an interesting question which we leave open for future work.

The spatial dependence of the couplings have also been left unspecified. The simplest example of Equation (9.31) is found when the couplings D_0, D_2 are taken to be *ultra-local*, which we take to mean that $D_0, D_2 \sim \delta(x, y)$, and also saturate Equation (9.29). For the case of ultra-local couplings, Equation (9.31) has an equivalent path integral description and we can arrive at the Newtonian limit by considering a gauge fixed Newtonian limit of the covariant theory introduced in Chapter 8. We show this in Section 9.4.

9.3.2 Linearity of the dynamics and white noise

Though the dynamics of Equation (9.31) gives rise to completely positive dynamics, this is surprising at first glance since the expectation value of the quantum state directly sources the Newtonian potential. This is in contrast to the unravellings in Chapter 5, where the change in the classical variable dz was sourced by the expectation value of a quantum operator multiplied dt.

For example, suppose that Alice entangles the state of a mass in the left and right with the state of a massless qubit, preparing the entangled state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle|L\rangle + |1\rangle|R\rangle). \tag{9.34}$$

Then naively, it appears this could be used to violate the linearity of quantum theory.

In particular, in the state $|\psi\rangle_{AB}$, the Newtonian potential has an average around $\mathbb{E}[\nabla^2\Phi] = \frac{1}{2}(m_L + m_R)$, with a probability distribution $u_0(\Phi)$. Now consider if Alice measures the $|0\rangle, |1\rangle$ basis. After the measurement, with probability half, the state collapses to $|0\rangle|L\rangle$, and with probability half, it collapses to $|1\rangle|L\rangle$. Thus after the measurement, the Newtonian potential will be sourced either by m_L, with a probability distribution $u_1(\Phi)$ with $\mathbb{E}[\nabla^2\Phi] = m_L$, or by m_R, with a probability distribution $u_2(\Phi)$ with $\mathbb{E}[\nabla^2\Phi] = m_R$. At first glance, this appears to be in contradiction with the linearity of the dynamics; the initial reduced density matrix $u_0(\Phi)\mathbb{I}$ is mapped to $u_0(\Phi)\mathbb{I} \to \frac{1}{2}u_1(\Phi)|L\rangle\langle L| + \frac{1}{2}u_2(\Phi)|R\rangle\langle R|$.

The resolution to this puzzle has to do with the properties of white noise, which is the time-derivative of a Wiener process. Recall that Wiener increments δW are Gaussian distributed with variance δt

$$P(\delta W) \sim \exp\left(-\frac{(\delta W)^2}{2\delta t}\right). \tag{9.35}$$

Since the Newtonian potential undergoes a white noise process, the signal $\alpha = \delta t \nabla^2\Phi$, $\alpha = \langle m\rangle\delta t + \delta W$ is a Gaussian random variable with mean $\langle m\rangle\delta t$. Hence α is distributed according to

$$P(\alpha) \sim \exp\left(-\frac{(\alpha - \langle m\rangle\delta t)^2}{2\delta t}\right). \tag{9.36}$$

Consequently, the distribution for Φ at any given time is given by

$$P(\Phi) \sim \exp\left(-\frac{\delta t(\nabla^2\Phi - \langle m\rangle)^2}{2}\right). \tag{9.37}$$

Equation (9.37) describes a wildly stochastic process with variance $\frac{1}{\delta t}$. In particular, in the $\delta t \to 0$ limit, one learns nothing about the distribution of the Newtonian potential, and the distributions $p_0(\Phi)$, $u_1(\Phi), u_2(\Phi)$ all coincide. The timescale to learn about the distribution depends on the strength of the ratio between the diffusion coefficient σ and the back-reaction, which determines the strength of the collapse of the continuous measurement. In this sense, linearity is restored because by the time one could - in theory - learn about Alice's entangled measurement, the reduced density matrix on the local quantum system will have decohered due to the appearance of the Lindbladian evolution in the mass operators, which act to continuously measure the mass density.

9.4 Deriving the Newtonian limit as a gauge fixing of a complete theory

In this section, we show that we can arrive at the Newtonian limit of (9.31) by taking the Newtonian limit of the theory introduced in Equation (8.33) of Chapter 8. This acts as a sanity check, both for the Newtonian limit of Equation (9.31) and that the theory introduced in the previous Chapter has constraints with a sensible Newtonian limit. In the Newtonian limit, keeping only the highest order terms in c, we find that the problematic off-diagonal terms appearing in Equation (8.33) disappear. In other words, we show the dynamics of Equation (8.33) defines CP dynamics on the subset of states defined by the Newtonian limit. We leave it as a question for further work whether the CQ constraints would be preserved in the more general case, and in particular, if the dynamics of Equation (8.33) lead to stable dynamics which preserves the Newtonian limit once higher order terms are c are considered.

We start by noting that for ultra-local couplings, Equation (9.31) has an equivalent description in terms of the classical-quantum path integral

$$\rho(t_f, \Phi_f, m_f^+, m_f^-) = \int \mathcal{D}\Phi \mathcal{D}m^{\pm} \exp\Bigg[\int_{t_i}^{t_f} dt d^3x \big(i\mathcal{L}_Q[m(x)^+, \Phi] - i\mathcal{L}_Q[m(x)^-, \Phi]\big) \\ -\frac{D_0[\Phi]}{2}\big(m(x)^+ - m(x)^-\big)^2 - \frac{1}{4D_2[\Phi]}\left(\frac{\nabla^2\Phi}{4\pi G} - \frac{1}{2}\big(m(x)^+ + m(x)^-\big)\right)^2\Bigg]\rho(t_i, \Phi_i, m_i^+, m_i^-), \tag{9.38}$$

where $\rho(t, \Phi, m^+, m^-) = \langle m^+|\varrho(t, \Phi)|m^-\rangle$ denote the components of the CQ state in the mass eigenstate basis, and a boundary condition is imposed on $\Phi(t_f) = \Phi_f$. We show this in detail in [9] but this formula can also be derived by performing a weak continuous measurement on the mass and using the measurement outcome to source Φ [49]. Equation (9.38) describes an integral over paths of the classical Newtonian potentials and a doubled path integral over the quantum mass eigenstates $m^{\pm}$. The Newtonian constraint is encoded via a combination of non-dynamical terms in the path integral (i.e., they do not contain time derivatives). This path integral contains precisely the constraints we posited in Chapter 8.

We now show that Equation (9.38) can be understood as a gauge fixed version of the full theory introduced in Chapter 8. For simplicity, we assume that the decoherence trade-off has been saturated. Consider the full theory of Equation (8.33), which, when coupled to a quantum mass density, has an action of the form

$$\begin{aligned}\mathcal{I}[m^+, m^-, g_{\mu\nu}] = \int dx \Big[& i\mathcal{L}_Q^+ - i\mathcal{L}_Q^- - \frac{\det(-g)}{8}(T^{\mu\nu+} - T^{\mu\nu-})D_{0,\mu\nu\rho\sigma}(T^{\rho\sigma+} - T^{\rho\sigma-}) \\ & - \frac{\det(-g)c^8}{128G^2\pi^2}(G^{\mu\nu} - \frac{4\pi G}{c^4}((T^{\mu\nu+} + T^{\mu\nu-})D_{0,\mu\nu\rho\sigma}[g](G^{\rho\sigma} - \frac{4\pi G}{c^4}(T^{\rho\sigma+} + 8\pi T^{\rho\sigma-})\Big],\end{aligned} \tag{9.39}$$

where $\mathcal{L}_Q$ is the quantum Lagrangian density.

For simplicity, we take D_0, D_2 to saturate the trade-off. We take the couplings to be ultra-local

$$D_{0,\mu\nu\rho\sigma} = \frac{D_0}{2c^5}(-g)^{-1/2}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2\beta g_{\mu\nu}g_{\rho\sigma}\right), \tag{9.40}$$

where the factor of $\frac{1}{c^5}$ is to ensure that D_0 has the same units as that appearing in Equation (9.38). To obtain the Newtonian limit, we write the path integral in an ADM form, described by summing over all paths N, N^i, γ and considering the action as a functional of the variables appearing in the ADM decomposition $\mathcal{I}[m^+, m^-, N, \vec{N}, \gamma]$. The Newtonian limit can then be understood as a Gauge fixing of the complete theory, computing the transition amplitudes between the CQ states defined on surfaces Σ_t

$$\begin{aligned}\rho(t_f, m_f^+, m_f^-, \gamma_f) = \int & \mathcal{D}\gamma dN d\vec{N}\mathcal{D}m^+\mathcal{D}m^-\delta(\gamma_{ij} - (1 - \frac{2\Phi}{c^2}\delta_{ij})\delta(N - (1 + \frac{\Phi}{c^2}))\delta(N^i - \frac{n^i}{c^3})) \\ \times \; & e^{\mathcal{I}[m^+, m^-, N, \vec{N}, \gamma]}\rho(t_i, m_i^+, m_i^-, \gamma_i).\end{aligned} \tag{9.41}$$

We now substitute for the Newtonian gauge by performing the delta functional integrals. In particular, we have $g_{00} = c^2 N = (c^2 + \Phi)$, whilst $g_{ij} \sim (1 - \frac{2\Phi}{c^2})\delta_{ij}$ and $g_{0i} = \frac{n^i}{c^3}$. The components of the Einstein tensor are calculated as

$$G^{00} = \frac{2\nabla^2\Phi}{c^4}, \tag{9.42}$$

$$G_{0i} = -\frac{2}{c^5}\partial_0\partial_i\Phi + \frac{1}{2c^5}\nabla^2 n_i, \tag{9.43}$$

$$G_{ij} = -\frac{2}{c^4}\partial_t\partial_t\Phi. \tag{9.44}$$

Similarly, noting that $\det(-g) = c^2$, we see that due to the powers of c, the de-Witt metric is dominated by its 0000 component, which to leading order is given by

$$D_{0,0000} = D_0 c^3 (1 - \beta). \tag{9.45}$$

Keeping only terms leading order in c, the path integral action in Equation (9.39) is dominated by terms only involving D_{0000} and reduces to

$$\mathcal{I}[m^+, m^-, \Phi] = \int dx \Big[i\mathcal{L}_Q^+ - i\mathcal{L}_Q^- - \frac{D_0(1-\beta)}{8}(m^+ - m^-)^2 - \frac{D_0(1-\beta)}{2}(\frac{\nabla^2\Phi}{4\pi G} - \frac{1}{2}(m^+ + m^-))^2 \Big]. \tag{9.46}$$

Redefining $D_0' = \frac{D_0(1-\beta)}{4}$ gives the same path integral as we found for the general Newtonian limit derived from Equation (9.38).

Hence, provided $\beta \leq 1$, we find the dynamics $c \to \infty$ limit of the full theory described by Equation (9.39) gives rise to complete positive dynamics, which describe a randomly sourced Poisson's equation, with associated decoherence on the quantum state. This provides a sanity check for the Newtonian limit derived in Equation (9.31) and gives rise to the hope that the theory in Chapter 8 has constraints that are preserved in time. However, we note that here we have arrived at the Newtonian limit by a gauge fixing of the full theory after neglecting all the terms higher order in c. Because we neglected terms higher order in c, we have eliminated the potential violating terms in the full path integral (9.39), which are not positive semi-definite. Specifically, the violating terms involving D_{000i}, D_{0i0j} still arise, but they are higher order in c; what we have shown the dynamics of Equation (9.39) defines CP dynamics on the subset of states defined by the Newtonian limit. We have not shown that the dynamics are consistent away from this limit, for example they could be unstable for finite c, and we leave this as a question for future research. A more general consideration is essential since we have not shown that the complete theory preserves the form of the Newtonian limit. It could be that by including higher-order terms in the calculation, one causes it to deviate from it in some way; for example, in deriving the Newtonian limit, we have assumed that Poisson's equation holds, on average, at any scale. By including higher-order terms, one might find that it is altered in the low-energy regime.

9.5 Comparison to previous classical-quantum theories

This section compares the Newtonian limit derived in the previous section to previously considered classical-quantum theories.

The Equations described via (9.26) have previously been studied in [49]: when the couplings σ and D_2 are independent of Φ they describe the same equations as a continuous measurement of a quantum mass, where the measurement outcome is used to source the Newtonian potential. Here we have shown that such dynamics arise on general grounds for all continuous classical-quantum theories of gravity, which are completely positive and agree with the Newtonian limit on expectation. As described in [49], because the Newtonian potential is sourced by a white noise process, one must be careful when implementing the quantum state evolution in Equation (9.31) since the couplings can depend on the Newtonian potential. In implementing the evolution when the couplings σ and D_2 are dependent on Φ, one can then be led to extra decoherence on the quantum state ρ due to Itô calculus involving multiplicative terms in Φ, potentially leading to faster decoherence rates than is implied by D_0 alone. However, studying this in detail is technically challenging and beyond the scope of the current work.

A variation of Equation (9.26) was also studied in [55], in a theory of strongly incoherent gravity. This model can be understood similarly to [49]. However, instead of performing a series of frequent weak measurements described by a finite coupling σ, one considers a series of strong measurements that happen infrequently and probabilistically. When the measurement dynamics are not occurring, the evolution of the quantum state is unitary, and there is no back-reaction on the gravitational field: it is stationary. Conversely, when the strong measurement occurs, this can be understood as taking the parameter $\sigma \to 0$, in which case $\nabla^2\Phi = 4\pi G\langle m\rangle$, and the quantum state instantaneously collapses into its mass eigenstates.

In Equation's (9.26) and (9.31), we have taken the drift to be local in x. At the same time, we allow for the decoherence and diffusion terms to have some range. The interaction law between the classical and quantum systems is still local in this case, but non-local correlations can be created [218]. Importantly, if the Lindbladian coupling D_0 has some range, then even though the CQ interaction is local, the master equation can, in principle, generate entanglement between two spatially separated quantum systems via the Lindbladian coefficient. However, this effect is likely to be small.

If the Lindbladian coupling in Equation (9.26) D_0 is ultra-local, the dynamics do not generate entanglement between spatially separated regions, meaning that the models with local couplings parameterize the general form of the continuous master equation which would be ruled out by gravity induced entanglement (GIE) experiments. We give three examples of kernels D_0, D_2 for continuous master equations in Table 9.3. It should be noted that with an appropriate choice of D_0, the pure Lindbladian evolution appearing in (9.26) can be taken to resemble the Lindbladian part of spontaneous collapse models [219, 168, 169, 170, 171] except that, in our model, there is no need to think about any ad-hoc field, nor think of the collapse as a physical process. Rather, one necessarily gets decoherence of the wave function for free via gravitationally induced decoherence [220, 49, 124, 28].

Models with ultra-local couplings form perhaps the most natural class of CQ dynamics. However, in Chapter 10, we show that considerations of the decoherence diffusion trade-off have already ruled out non-relativistic versions of these models. In other words, (in line with assumptions 1-4) classical-quantum theories, which have continuous gravitational degrees of freedom with local interactions and correlations, are already ruled out by experiment. Note, this does not necessarily rule out the theory of Equation (8.33) described in Chapter 8 since we have not shown that higher order corrections can change the Newtonian limit at small distance scales. Rather, we have shown that assuming 1-4 hold on the CQ state, the dynamics of Equation (8.33) give rise to a consistent Newtonian limit with the same dynamics of Equation (9.31).

Specifically, one can constrain CQ theories via Equation (9.31) and the decoherence diffusion trade-off in Equation (9.29). The Newtonian limit of CQ gravity predicts diffusion of the Newtonian potential by an amount depending on D_2. This can be upper bounded by precision mass experiments, which precisely measure the acceleration of particles. Conversely, coherence and heating experiments can be used to upper bound the inverse Lindbladian coefficient D_0^{-1}, which gives a lower bound on D_2 via the decoherence diffusion trade-off. Hence, when combined, it is possible to get an experimental squeeze on CQ theories [4].

9.6 Decoherence rates

For completeness, we briefly discuss the decoherence rates which follow from Equation (9.31), since they will be used in Chapter 10. We shall compute the decoherence rate for a quantum mass initially in a partially decohered superposition of state $|L\rangle$ and $|R\rangle$. We describe the quantum state using creation and annihilation operators $\psi(x), \psi^\dagger(x)$ on a Fock space, related to the usual momentum-based Fock operators as $\psi(x) = \int dp e^{i\vec{p}\cdot\vec{x}} a_{\vec{p}}$. The mass density operator is defined via $m(x) = m\psi^\dagger(x)\psi(x)$, where m is the mass of the particle. We assume that the state remains well approximated by a state with a fixed particle number. The superposition can be taken to be distributions centered around $x = x_L$ and $x = x_R$ with total mass m, i.e., for a one-particle state we could take $|L/R\rangle = \int d^3x f_{L/R}(x)\psi^\dagger(x)|0\rangle$. We will take them to be well separated so that $f_L(x)f_R(x) \approx 0$. With this orthogonality condition, the result of the equation's quantum part is to decohere the quantum state by an amount D_0. We can compute the off-diagonal elements of Equation (9.31) to see this. We shall assume that the range of $D_0(x, y)$ is smaller than the length scale entering the superposition, so that $D_0(x, y)m_L(x)m_R(y) \approx 0$, and ignore the effects of the pure Hamiltonian evolution – though these can lead to *additional* decoherence [49]. The result is that $\rho_{LR} = \langle L|\rho|R\rangle$ undergoes dynamics according to

$$\frac{d\rho_{LR}}{dt} \approx -\frac{1}{2}\int d^3x d^3y D_0(\Phi; x, y)(m_L(x)m_L(y) + m_R(x)m_R(y))\rho_{LR}. \tag{9.47}$$

In particular, we see leading order off-diagonals decay exponentially at a rate determined by

$$\lambda = \frac{1}{2}\int d^3x d^3y D_0(\Phi; x, y)(m_L(x)m_L(y) + m_R(x)m_R(y)). \tag{9.48}$$

In practice, one does not have access to the entire history of the Newtonian potential. So one should also integrate out the Newtonian potential to find the true decoherence of the quantum state. This calculation is generally complicated since the couplings $D_0(\Phi), D_2(\Phi)$ can depend on the Newtonian potential. Nonetheless, Equation (9.47) gives a lower bound for the decoherence of the quantum state if one has full knowledge of the trajectories in the unraveling. Alternatively, in the presence of a background Newtonian potential, such as that of the Earth's, and assuming polynomial dependence of the D_0 on the Newtonian potential, then we to leading order we can approximate D_0 by its background value, which gives Equation (9.47) as the decoherence rate. We use Equation (9.47) in Chapter 10 to constrain CQ theories of gravity experimentally.

As an example of a decoherence kernel, we can take $D_0(x, x')$ to be the *Diosi-Penrose kernel* (D.P kernel) defined via $D_0(x, y) = \frac{D_0}{|x-y|}$. In this case, the off-diagonals decay exponentially with a rate proportional to the Diosi-Penrose decoherence rate

$$\lambda = \frac{1}{2} \int d^3x d^3y \frac{D_0}{|x-y|}(m_L(x)m_L(y) + m_R(x)m_R(y)). \tag{9.49}$$

For identical spherical distributions of radius R and total mass M, and mass density ρ, Equation (9.49) can be re-written as

$$\lambda = -\frac{1}{G} \int d^3x D_0 \Phi(x)\rho(x), \tag{9.50}$$

which is proportional to the average gravitational self-energy of the mass distribution $\lambda = \frac{3D_0M^2}{5R}$.

For a composite particle of mass M, made up of N constituents each of radius R, the mass density will be represented by a sum over all of the particles $m(x) = \sum_i m_i(x)$. The decoherence rate, in this case, is given by

$$\lambda = \frac{1}{2} \int d^3x d^3y \frac{D_0}{|x-y|}(\sum_{i,j} m_{L,i}(x)m_{L,j}(y) + \sum_{i,j} m_{R,i}(x)m_{R,j}(y)). \tag{9.51}$$

Since the cross terms involving i, j are suppressed by a factor of inter-atomic scales to leading order, the contribution to the decoherence rate is lower bounded by the $i = j$ component of the sums in Equation (9.51), which gives an extra factor of N relative to the single-particle case $\lambda = \frac{3D_0NM^2}{5R}$.

Similarly, we can calculate the decoherence rate for ultra-local couplings $D_0(\Phi; x, y) = D_0(\Phi)(x)\delta(x, y)$. We find that the decoherence rate is calculated as

$$\lambda = \frac{D_0M^2}{V}, \tag{9.52}$$

where V is the volume of the particle. For N composite particles, the decoherence rate is additive in N and found to be $\lambda = \frac{ND_0M^2}{V}$.

9.7 Newtonian limit for general master equations

So far in this chapter, we have only considered the Newtonian limit of theories with continuous back-reaction. This section considers the more general case, which includes when the master equation and back-reaction are associated with jumps in the Newtonian potential.

As a consequence of Assumption 3, we know from the decoherence-diffusion trade-off that there must be diffusion in the classical variable and also Lindbladian evolution on the quantum state. As mentioned, in the next chapter, we generalize the decoherence-diffusion trade-off found for the continuous master Equations in Chapter 4 to show that for any completely positive autonomous CQ master equation, the moments $D_n^{\mu\nu}$ appearing in the master equation (2.91) must satisfy

$$2D_{2,\pi_\Phi\pi_\Phi} \succeq D^{br}_{1,\pi_\Phi} D_0^{-1} (D^{br}_{1,\pi_\Phi})^\dagger, \tag{9.53}$$

where $D_{2,\pi_\Phi\pi_\Phi}$ is the full matrix of diffusion coefficients $D^{\mu\nu}_{2,\phi_\Phi\pi_\Phi}$, $D^{br}_{1,\pi_\Phi} = D^{\mu\alpha}_{1,\pi_\Phi}$ is the matrix describing the drift of the back-reaction of the quantum system and $(D_0^{-1})_{\alpha\beta}$ is the generalized inverse of the Lindbladian coefficient $D_0^{\alpha\beta}$.

We can use Equation (9.53) to understand general predictions about the second moments of the dynamics generated by general CQ master equations. In particular, from the general form of the CQ master equation in Equation (2.91) (see Appendix G for a discussion on field-theoretic master equations) that the first moment of $\dot{\pi_\Phi}$ is described by

$$\partial_t \langle \pi_\Phi \rangle = \mathrm{Tr}\left[D_1^{\alpha\mu} L_\alpha \varrho(z) L_\nu^\dagger \right] + \mathrm{Tr}\left[D_1^{\mu\alpha} L_\mu \varrho(z) L_\alpha^\dagger \right] = \mathrm{Tr}\left[\{\pi_\Phi, H_I\} \right] = \langle m \rangle, \tag{9.54}$$

while the second moment is described via

$$\partial_t \langle \pi_\Phi(x) \pi_\Phi(y) \rangle = 2\mathrm{Tr}\left[D_2^{\mu\nu}(x,y) L_\mu(x) \varrho L_\nu^\dagger(y) \right]. \tag{9.55}$$

As a consequence, the statistics of the first two moments of π_Φ can be described by adding a stochastic random process to the equation of motion $\dot{\pi_\Phi}$

$$\dot{\pi}_\Phi = \frac{\nabla^2 \Phi}{4\pi G} - \langle m(x) \rangle - u(\Phi, m)\xi(t,x), \tag{9.56}$$

where, at each time, the noise process satisfies

$$\mathbb{E}_{m,\Phi}[u\xi(t,x)] = 0, \quad \mathbb{E}_{m,\Phi}[u\xi(t,x) u\xi(y,t')] = 2\langle D_2(x,y,\Phi)\rangle \delta(t,t'), \tag{9.57}$$

where we have defined $\langle D_2(x,y,\Phi)\rangle = \mathrm{Tr}\left[D_2^{\mu\nu}(x,y,\Phi) L_\mu(x) \rho L_\nu^\dagger(y) \right]$, and ρ is the quantum state at time t. In Equation 9.57, the m, Φ subscripts of $\mathbb{E}_{m,\Phi}$ allow for the possibility that the statistics of the noise process can be dependent on the Newtonian potential and mass distribution of the particle.

In the non-relativistic limit, where $c \to \infty$, we impose the momentum constraint $\pi_\Phi \approx 0$, and we again recover Poisson's equation for gravity, but with a stochastic contribution to the mass

$$\Phi(t,x) \approx -G \int d^3x' \frac{[\langle m(x',t)\rangle - u(\Phi,m)J(x',t)]}{|x-x'|}. \tag{9.58}$$

If $u\xi(x,t)$ is Gaussian, Equation (10.28) completely determines the noise process, but in general, higher-order correlations are possible, but we can use Equation (9.57) to bound the effects due the second moments of Φ, which are fully described by $D_2^{\mu\nu}(x,y,\Phi)$.

In particular, from the trade-off, we know that there must be an accompanying decoherence on the quantum state given by the coupling $D_0^{\alpha\beta}$, which lower bounds the amount of diffusion through (9.53). In Chapter 10, we see that this can be used to constrain CQ dynamics which contain jumps experimentally.

9.7.1 Jumping master equation

An example of a jumping master equation satisfying Assumptions 1-4 is

$$\frac{\partial \varrho}{\partial t} \approx \{H_c(\Phi),\varrho\} - i[H_m + +H_I,\varrho] + \int d^3x \left[D_0(\Phi;x)e^{D_0^{-1}(\Phi;x)\frac{\delta}{\delta\pi_\Phi(x)}} m\psi(x)\varrho\psi^\dagger(x) \right.$$
$$\left. -\frac{1}{2}\{m(x),\varrho\}_+\right].$$

Equation (9.59) needs regularizing, which we assume is performed by computing smeared observables. I.e., instead of calculating quantities such as the variation in force $\partial_t\langle\nabla\Phi_\Phi(x)\nabla\Phi_\Phi(x)\rangle$ directly, we assume that one calculates smeared quantities, such the variations of the time and spatial averaged forces $\bar{F} = -\int d^3x dt L(x,t)\nabla\Phi(x,t)$ where $L(t,x)$ is a regulator; we do this explicitly in Chapter 10.

Expanding Equation (9.59) to second order, we see that

$$\begin{aligned}\frac{\partial \varrho}{\partial t} &\approx \{H_c(\Phi),\varrho\} - i[H_m + H_I,\varrho] \\ &+ \int d^3x D_0(\Phi;x)[m\psi(x)\varrho\psi(x)^\dagger - \frac{1}{2}\{m(x),\varrho\}] \\ &+ \int d^3x m\psi(x)\frac{\delta\varrho}{\delta\pi_\Phi(x)}\psi^\dagger(x) + \frac{1}{2}\int d^3x D_0(\Phi;x)m\psi(x)\frac{\delta^2\varrho}{\delta\pi_\Phi(x)\delta\pi_\Phi(x)}\psi^\dagger(x).\end{aligned} \tag{9.59}$$

The back-reaction is Hamiltonian, since we obtain Equation (9.25) under trace. The Lindblad operators are ψ, and in this basis we see that the back-reaction matrix $D_1^{\alpha\mu}$ has components

$D_1^{\psi\psi} = 1$, whilst the diffusion coupling $D_2^{\mu\nu}$ is found to be $D_2^{\psi\psi} = \frac{1}{2}D_0$, which saturates the trade-off in Equation (9.53). The decoherence rate for a mass in superposition, as in Section 9.6, is found from the no-event term $-\frac{1}{2}\{m(x), \varrho\}$ and is given by

$$\lambda = D_0 M, \tag{9.60}$$

while for a system of N composite particles, the decoherence rate is additive and found to be $\lambda = D_0 N M$.

9.8 Discussion

In this chapter, we have considered, on general grounds, the weak field limit of classical-quantum theories of gravity, which give rise to linear, completely positive dynamics. This provides a template from which classical-quantum theories of gravity can be experimentally tested and theoretically developed.

The Newtonian limit we find generalizes previous approaches to coupling classical and quantum systems in the Newtonian limit. In [49, 52, 55], the Newtonian potential is sourced by the outcome of certain measurements of the mass operator. The behavior in these models is qualitatively the same as those presented here: the Newtonian potential diffuses by an amount that depends on the inverse of the strength of the measurement. At the same time, the quantum system decoheres into its mass eigenbasis because it is being continuously measured. In this chapter, we have arrived at this behavior in complete generality for CQ theories which agree with the Newtonian limit on expectation, with the diffusion of the Newtonian potential and decoherence on the quantum system described by the parameters $D_2(\Phi; x, y), D_0(\Phi; x, y)$ satisfying the decoherence-diffusion trade-off. Previously discussed models of Newtonian classical-quantum gravity [49, 52, 55] then arise as limits of the more general theory considered in this chapter.

The weak field CQ theories we studied gave a generic prediction: the Newtonian potential diffuses away from its average solution, and for the dynamics to be completely positive, the amount of the diffusion is lower bounded by the coherence time for masses in superposition. This feature is perhaps most elegantly described via Equation (9.31). This behavior has also been derived from the path integral formulation of CQ, ensuring the positivity of the state is preserved.

Several proposals to test the quantum nature of gravity via gravitationally induced entanglement are expected to be realizable within the next few decades with technological advancements [64, 53, 1, 2, 65, 66, 67, 68]. If the underlying theory is local, then witnessing entanglement would imply that gravity is not a classical field. Within the framework of consistent classical-quantum coupling, we can inquire from the other direction, asking about the general experimental signatures of treating the gravitational field as being classical.

The combination of imposing the Newtonian limit of GR together with the decoherence-diffusion trade-off, which follows from complete positivity, means that CQ theories allow for very concrete predictions which can be experimentally tested with current experimental technologies, which we explore in detail in Chapter 10.

As an effective theory of Newtonian gravity, we still expect the path integrals and the unravellings derived in this paper to be valid to all dynamics with a time-local description [28]. However, in general, they can be non-Markovian, which means that the couplings D_0, D_2 need not be positive semi-definite for all times, [92, 93], nor satisfy the decoherence-diffusion trade-off for all times since this is a consequence of the Markovian assumption.

An open theoretical problem is to construct a complete diffeomorphism invariant theory of classical-quantum gravity and to study the low energy effects of such a theory in a top-down approach. We have made some progress towards this goal by considering the Newtonian limit of the theory introduced in Chapter 8. We have seen that the Newtonian limit of the theory gives rise to qualitatively and quantitatively similar behavior to the general approach considered in this chapter. A more detailed calculation would involve checking whether the constraints involved give rise to consistent dynamics away from the Newtonian limit and whether higher order corrections necessitate changing any of the assumptions outlined in this chapter; for example, if the Newtonian description should be altered even at low energies.

Chapter 10

The trade-off between decoherence and diffusion

This chapter discusses the experimental signatures that follow from consistent CQ dynamics. We prove a generic trade-off between the rate of decoherence and the amount of diffusion in the classical phase space, extending that found in the CQ Pawula theorem in Chapter 4. The stronger the interaction between the quantum and classical systems, the greater the trade-off. One cannot have quantum systems with long-coherence times without inducing much diffusion in the classical system. One can also generalize this result to a trade-off between the rate of diffusion and the strength of more general couplings to Lindblad operators, with decoherence being a special case. This is expressed as Equations (10.14) and (10.12), which bounds the product of diffusion coefficients and Lindblad coupling constants in terms of the strength of the CQ-interaction. It is precisely this trade-off which allows the theories considered here to evade the no-go arguments of Feynmann [17, 35], Aharonov [40], Eppley and Hannah [18] and others [17, 39, 221, 59, 48, 222, 37, 38, 41, 42, 43, 44, 46, 47]. The essence of arguments against quantum-classical interactions is that they would prohibit superpositions of quantum systems that source a classical field. Since different classical fields are perfectly distinguishable in principle, if the classical field is in a distinct state for each quantum state in the superposition, the classical field could always be used to determine the quantum system's state, causing it to decohere instantly. By satisfying the trade-off, the quantum system preserves coherence because diffusion of the classical degrees of freedom means that the state of the classical field does not determine the state of the quantum system [61, 28]. Equation (10.14) and other variants we derive quantify the amount of diffusion required to preserve any coherence. If space-time curvature is treated classically, then complete positivity of the dynamics means its interaction with quantum fields necessarily results in unpredictability and gravitationally induced decoherence.

This trade-off between the decoherence rate and diffusion provides an experimental signature, not only of models of hybrid Newtonian dynamics such as [60] or post-quantum theories of General Relativity such as [28] but of any theory which treats gravity as being fundamentally classical. The metric and their conjugate momenta necessarily diffuse away from what Einstein's General Relativity predicts, and this experimental signature squeezes classical-quantum theories of gravity from both sides. If one has shorter decoherence times for superpositions of different mass distributions, one necessarily has more diffusion of the metric and conjugate momenta. The latter effect causes imprecision in measurements of mass such as those undertaken in the Cavendish experiment [223, 224, 225] or in measurements of Newton's constant "Big G" [226, 227, 228]. The precision at which a mass can be measured in a short time thus provides an upper bound on the amount of gravitational diffusion, as quantified by Equation (10.30). At the same time, decoherence experiments place a lower bound on the diffusion. Our estimates suggest that experimental lower bounds on the coherence time of large molecules [229, 230, 231, 232, 233, 170], combined with gravitational experiments measuring the acceleration of small masses [234, 235, 236], already place substantial restrictions on theories where space-time is not quantized. In Section 10.3, we show that several realizations of CQ gravity are already ruled out. In contrast, other realizations produce enough diffusion away from General Relativity to be detectable by future table-top experiments. Although the absence of such deviations from General Relativity would not be a direct confirmation of the quantum nature of gravity, such as experiments proposed in [64, 53, 1, 2, 65, 66, 67, 68] to exhibit entanglement generated by gravitons, it would effectively rule out any sensible theory which treats space-time classically. In comparison, confirmation of gravitational diffusion would suggest that space-time is fundamentally classical.

This chapter is based on the paper [4], which is work done in collaboration with Carlo

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10.1 A general Trade off between decoherence and diffusion

In this section, we use positivity conditions to prove that the trade-off between decoherence and diffusion seen in models such as those of [57, 28, 10] are, in fact, a general feature of *all* classical-quantum interactions. We shall also generalize this and derive a trade-off between diffusion and arbitrary Lindbladian coupling strengths. The trade-off is in relation to the strength of the dynamics and is captured by Equations (10.9), (10.12) and (10.14). In Section 10.2, we extend the trade-off to the case where the classical and quantum degrees of freedom can be fields and use this to show that treating the metric as being classical necessarily results in diffusion of the gravitational field. We consider the general form of autonomous CQ master equation. The master equation is given by Equation (2.91), and we reproduce it here for convenience

$$\begin{aligned}\frac{\partial \varrho(z,t)}{\partial t} &= \sum_{n=1}^{\infty}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}(z,\delta t)\varrho(z,t)\right)\\ &- i[H(z),\varrho(z)] + D_0^{\alpha\beta}(z)L_\alpha\varrho(z)L_\beta^\dagger - \frac{1}{2}D_0^{\alpha\beta}\{L_\beta^\dagger L_\alpha,\varrho(z)\}_+\\ &+ \sum_{\mu\nu\neq 00}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{\mu\nu}_{n,i_1\dots i_n}(z)L_\mu\varrho(z,t)L_\nu^\dagger\right).\end{aligned} \tag{10.1}$$

Where the moments D_n are related to the short-time expansion of the transition amplitude

$$\Lambda^{\mu\nu}(z,t+\delta t|z',t) = \delta_0^\mu\delta_0^\nu + \delta t W^{\mu\nu}(z|z',t), \tag{10.2}$$

through the equation

$$D^{\mu\nu}_{n,i_1\dots i_n}(z',t) := \frac{1}{n!}\int dz W^{\mu\nu}(z|z',t), t(z-z')_{i_1}\dots(z-z')_{i_n}. \tag{10.3}$$

There are two separate possible sources for the force (or drift) of the back-reaction of the quantum system on phase space – it can be sourced by either the $D^{0\alpha}_{1,i}$ components or the Lindbladian components $D^{\alpha\beta}_{1,i}$. We shall deal with both sources simultaneously by considering a CQ Cauchy-Schwartz inequality which arises from the positivity of

$$\mathrm{Tr}\left[\int dz dz' \Lambda^{\mu\nu}(z,t+\delta t|z',t)O_\mu(z,z')\rho(z')O_\nu^\dagger(z,z')\right] \geq 0, \tag{10.4}$$

which must be positive for any vector of CQ operators O_μ. One can verify that this must be positive directly from the positivity conditions on $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ and we go through the details in Appendix F. A common choice for O_μ would be the set of operators $L_\mu = \{\mathbb{I}, L_\alpha\}$ appearing in the master equation.

The inequality in Equation (10.4) turns out to be especially useful since it can be used to define a (pseudo) inner product on a vector of operators with components O_μ via

$$\langle \bar{O}_1, \bar{O}_2\rangle = \int dz dz' \mathrm{Tr}\left[\Lambda^{\mu\nu}(z,t+\delta t|z',t)(z|z')O_{1\mu}\varrho(z')O_{2\nu}^\dagger\right] \tag{10.5}$$

where $||\bar{O}|| = \sqrt{\langle \bar{O}, \bar{O}\rangle} \geq 0$ due to (10.4). Technically this is not positive definite, but this shall not be important for our purpose. Taking the combination $O_\mu = ||\bar{O}_2||^2 O_{1\mu} - \langle \bar{O}_1, \bar{O}_2\rangle O_{2\mu}$ for vectors $O_{1\mu}, O_{2\mu}$, positivity of the norm gives

$$||\bar{O}||^2 = \ ||\bar{O}_2||^2\bar{O}_1 - \langle \bar{O}_1, \bar{O}_2\rangle \bar{O}_2||^2 = ||\bar{O}_2||^2\left(||\bar{O}_1||^2||\bar{O}_2||^2 - |\langle \bar{O}_1, \bar{O}_2\rangle|^2\right) \geq 0, \tag{10.6}$$

and as long as $||\bar{O}_2|| \neq 0$ we have a Cauchy- Schwartz inequality

$$||\bar{O}_1||^2||\bar{O}_2||^2 - |\langle \bar{O}_1, \bar{O}_2\rangle|^2 \geq 0. \tag{10.7}$$

We can use (10.7) to get a trade-off between the observed diffusion and decoherence by picking $O_{2\mu} = \delta_\mu^\alpha L_\alpha$ and $O_{1\mu} = b^i(z-z')_i L_\mu$, where $L_\mu = \{\mathbb{I}, L_\alpha\}$ are the Lindblad operators appearing in the master equation. In this case, $||\bar{O}_2|| = \int dz \mathrm{Tr}\left[D_0^{\alpha\beta} L_\alpha \varrho L_\beta^\dagger\right]$ and one can verify using CQ Pawula theorem [5][1] that in order to have non-trivial back-reaction on the quantum system, complete positivity demands that $||\bar{O}_2|| > 0$, meaning the Cauchy-Schwartz inequality in Equation (10.7) must hold. By using the short time moment expansion of $\Lambda^{\mu\nu}(z,t+\delta t|z',t)$ defined in Equation (10.2) and using integration by parts, we then arrive at the observational trade-off between decoherence and diffusion

$$\int dz \mathrm{Tr}\left[2b^{i*}D_{2,ij}^{\mu\nu}b^j L_\mu \varrho(z) L_\nu^\dagger\right]\int dz \mathrm{Tr}\left[D_0^{\alpha\beta}L_\alpha\varrho(z)L_\beta^\dagger\right] \geq \left|\int dz \mathrm{Tr}\left[b^i D_{1,i}^{\mu\alpha}L_\mu\varrho(z)L_\alpha^\dagger\right]\right|^2, \tag{10.8}$$

which must hold for any positive CQ state $\varrho(z)$. Stripping out the b^i vectors, (10.8) is equivalent to the matrix positivity condition

$$0 \preceq 2\langle D_2\rangle\langle D_0\rangle - \langle D_1^{br}\rangle\langle D_1^{br}\rangle^\dagger, \quad \forall \varrho(z), \tag{10.9}$$

[1]In particular, to reach this conclusion one can insert the CQ state into the CQ Cauchy-Schwartz inequality and repeat the proof of the Pawula theorem [5], which must now hold once averaged over the state.

where we define

$$\langle D_0 \rangle = \int dz \mathrm{Tr}\left[D_0^{\alpha\beta} L_\alpha \varrho(z) L_\beta^\dagger\right], \ \langle D_1^{br} \rangle_i = \int dz \mathrm{Tr}\left[D_{1,i}^{\mu\alpha} L_\mu \varrho(z) L_\alpha^\dagger\right]$$
$$\langle D_2 \rangle_{ij} = \int dz \mathrm{Tr}\left[D_{2,ij}^{\mu\nu} L_\mu \varrho(z) L_\nu^\dagger\right] \tag{10.10}$$

and $D_1^{br} = D_{1,i}^{\mu\alpha}$ defines the *back-reaction matrix,* describing the back-reaction of the quantum system on the classical one. Since (10.9) holds for all states, the tightest bound is provided by the infimum over all states

$$0 \preceq \inf_{\varrho(z)} \{2\langle D_2 \rangle \langle D_0 \rangle - \langle D_1^{br} \rangle \langle D_1^{br} \rangle^\dagger\}. \tag{10.11}$$

The quantities $\langle D_2 \rangle$ and $\langle D_0 \rangle$ appearing in Equation (10.9) are related to observational quantities. In particular, $\langle D_2 \rangle$ is the expectation value of the classical diffusion observed, and $\langle D_0 \rangle$ is related to the amount of decoherence on the quantum system. The expectation value of the back-reaction matrix $\langle D_1^{br} \rangle$ quantifies the back-reaction on the classical system. In the trivial case $D_1^{br} = 0$, Equation (10.9) places little restriction on the diffusion and Lindbladian rates appearing on the left-hand side. We already knew from [62, 63] that the $D_0^{\alpha\beta}$ must be a positive semi-definite matrix, and we also know that diffusion coefficients must be positive semi-definite. However, in the non-trivial case, the larger the back-reaction exerted by the quantum system, the stronger the trade-off between the diffusion coefficients and Lindbladian coupling. Equation (10.9) gives a general trade-off between observed diffusion and Lindbladian rates. We call the trade-offs involving expectation values over the state *observational trade-offs.*

We can also find a trade-off regarding a theory's coupling coefficients alone. We show in Appendix F that the general matrix trade-off

$$D_1^{br} D_0^{-1} D_1^{br\dagger} \preceq 2D_2 \tag{10.12}$$

holds for the matrix whose elements are the couplings $D_{2,ij}^{\mu\nu}, D_{1,i}^{\alpha\mu}, D_0^{\alpha\beta}$ for any CQ dynamics. Moreover, $(\mathbb{I} - D_0 D_0^{-1}) D_1^{br} = 0$, which tells us that D_0 cannot vanish if there is non-zero back-reaction. Equation (10.12) quantifies the required amount of decoherence and diffusion for the dynamics to be completely positive. In Equation (10.12), and throughout, D_0^{-1} is the generalized inverse of $D_0^{\alpha\beta}$, since $D_0^{\alpha\beta}$ is only required to be positive semi-definite. In the special case of a single Lindblad operator $\alpha = 1$ and classical degree of freedom, and when the only non-zero couplings are $D_0^{11} := D_0$, $D_{2,pp}^{00} := 2D_2$ and $D_{1,q}^{0} = 1$ this trade-off reduces to the condition $D_2 D_0 \geq 1$ used in [57].

It is also useful to try to obtain an observational trade-off in terms of the total drift due to back-reaction as calculated in Equation (2.102)

$$\langle D_1^T \rangle_i = \sum_{\mu\nu \neq 00} \int dz \mathrm{Tr}\left[D_{1,i}^{\mu\nu} L_\mu \varrho(z) L_\nu^\dagger \right]. \tag{10.13}$$

It follows directly from Equation (10.9) that when the back-reaction is sourced by either $D_{1,i}^{0\mu}$ or $D_{1,i}^{\alpha\beta}$ we can arrive at the observational trade-off in terms of the total drift[2]

$$0 \preceq 8\langle D_2 \rangle \langle D_0 \rangle - \langle D_1^T \rangle \langle D_1^T \rangle^\dagger, \quad \forall \varrho(z), \tag{10.14}$$

where the quantities appearing in Equation (10.14) are now all observational quantities related to drift, decoherence, and diffusion.

In the case where the back-reaction is Hamiltonian at first order in the sense of Equation (2.103), then (10.14) can be written as

$$\langle \omega \cdot \frac{\partial H_I}{\partial \vec{z}} \rangle \langle \omega \cdot \frac{\partial H_I}{\partial \vec{z}} \rangle^\dagger \preceq 8 \langle D_2 \rangle \langle D_0 \rangle, \quad \forall \varrho(z). \tag{10.15}$$

As a result, we can derive a trade-off between diffusion and decoherence for any theory that reproduces this classical limit and treats one of the systems classically.

To summarize, whenever the back-reaction of the quantum system on the classical system induces a force on the phase space, then we have a trade-off between the amount of diffusion on the classical system and the strength of decoherence on the quantum system (or, more precisely, the strength of the Lindbladian couplings $D_0^{\alpha\beta}$). This can be expressed both as a condition on the matrix of coupling coefficients in the master equation, via Equation (10.12) or in terms of observable quantities using Equation's (10.9) and (10.14). When the back-reaction is Hamiltonian, we further have Equation (10.15). We want to apply this trade-off to the case of gravity in the non-relativistic, Newtonian limit, which we considered in Chapter 9. In order to do so, we will need to generalize the trade-off to the case of quantum fields interacting with classical ones, which we do in Section 10.2. The goal will be to understand the implications of treating the metric (or Newtonian potential) as being classical by using the trade-off when the

[2] We believe that (10.14) should hold more generally, though we do not have a general proof.

quantum back-reaction induces a force on the gravitational field, which, on expectation, is the same as the weak field limit of general relativity.

10.2 Trade off in the presence of fields

We want to explore the trade-off in the gravitational setting and the consequences of treating the gravitational field as classical and matter quantum. Since gravity is a field theory, we must first discuss classical-quantum master equations in the presence of fields. In the field-theoretic case, both the Lindblad operators and the phase space degrees of freedom can have spatial dependence, $z(x), L_\mu(x)$ and a general bounded CP map which preserves the classicality of the two systems can be written [28]

$$\rho(z,t) = \int dz' dx dy \Lambda^{\mu\nu}(z, t+\delta t|z', t; x, y) L_\mu(x, z, z') \varrho(z', 0) L_\nu^\dagger(y, z, z'), \tag{10.16}$$

where, as is usually the case with fields, in Equation (10.16), it should be implicitly understood that a smearing procedure has been implemented. We elaborate on some details when fields are introduced in Appendix G. The condition for (10.16) to be completely positive on *all* CQ states is

$$\int dz dx dy A_\mu^*(x, z, z') \Lambda^{\mu\nu}(z, t+\delta t|z', t; x, y) A_\nu(y, z, z') \geq 0 \tag{10.17}$$

meaning that $\Lambda^{\mu\nu}(z, t+\delta t|z', t; x, y)$ can be viewed as a positive matrix in $\mu\nu$ and a positive kernel in x, y. In the field-theoretic case, one can still perform a Kramers-Moyal expansion and find a trade-off between the coefficients $D_0(x,y), D_1(x,y), D_2(x,y)$ appearing in the master equation. Due to the Lindblad operators' spatial dependence, the coefficients now have an x, y dependence. The coefficients $D_1(x,y), D_2(x,y)$ still have a natural interpretation as measuring the amount of force (drift) and diffusion, while $D_0(x,y)$ describes the purely quantum evolution on the system and can be related to decoherence.

Using the positivity condition in Equation (10.17) we find the same trade of between coupling constants in Equation (10.12) but where now $D_2(x,y)$ is the $(p+1)n \times (p+1)n$ matrix-kernel with elements $D_{2,ij}^{\mu\nu}(x,y)$, $D_1^{br}(x,y)$ is the $(p+1)n \times p$ matrix-kernel with rows labeled by μi, columns labelled by β, and elements $D_{1,i}^{\mu\beta}(x,y)$, and $D_0(x,y)$ is the $p \times p$ decoherence matrix-kernel with elements $D_0^{\alpha\beta}(x,y)$.[3] In the field theoretic trade off we are treating

[3] Here $i \in \{1, \dots, n\}$ $\alpha \in \{1, \dots, p\}$ and $\mu \in \{1, \dots, p+1\}$,

the objects in Equation (10.12) as matrix-kernels, so that for any position dependent vector $b^i_\mu(x)$, $(D_2 b)^\mu_i(x) = \int dy D^{\mu\nu}_{2,ij}(x,y) b^j_\nu(y)$, whilst for any position dependent vector $a_\alpha(x)$, $(D_0\alpha)^\alpha(x) = \int dy D_0^{\alpha\beta}(x,y)\alpha_\beta(y)$. Explicitly, we find that the complete positivity of the dynamics is equivalent to the matrix condition

$$\int dxdy[b^*(x), \alpha^*(x)] \begin{bmatrix} 2D_2(x,y) & D_1^{br}(x,y) \\ D_1^{br}(x,y) & D_0(x,y) \end{bmatrix} \begin{bmatrix} b(y) \\ \alpha(y) \end{bmatrix} \geq 0 \tag{10.18}$$

which should be positive for any position dependent vectors $b^i_\mu(x)$ and $a_\alpha(x)$. This is equivalent to a trade-off between coupling constants in Equation (10.12) if we view (10.12) as a matrix-kernel equation.

Though we make no assumption on the locality of the Lindbladian and diffusion couplings, we hereby assume that the drift back-reaction is local, so that $D_1^{br}(x,y) = \delta(x,y) D_1^{br}(x)$. As we shall see in the next section, this is a natural assumption if we want back-reaction given by a local Hamiltonian. However, one might not want to assume that the form of the Hamiltonian remains unchanged to arbitrarily small distances. With this locality assumption, Equation (10.18) gives rise to the same trade-off of Equation (10.12), where the trade-off is to be interpreted as a matrix kernel inequality. Writing this out explicitly, we have

$$\int dxdy \alpha^{i*}_\nu(x) D^{\mu\alpha}_{1,i}(x)(D_0^{-1})_{\alpha\beta}(x,y) D^{\beta\nu}_{1,j}(x')\alpha^i_\nu(x') \leq \int dxdy 2\alpha^{i*}_\mu(x) D^{\mu\nu}_{2,ij}(x,y)\alpha^j_\nu(y), \tag{10.19}$$

where asking that this inequality holds for all vectors $\alpha^i_\mu(x)$ is equivalent to the matrix-kernel trade-off condition of Equation (10.12). We saw examples of gravitational master Equations satisfying the coupling constant trade-off in Chapter 9. The decoherence-diffusion trade-off tells us how much diffusion and stochasticity is required to maintain coherence when the quantum system back-reacts on the classical one.

In the field-theoretic case, we can similarly find an observational trade-off, relating the expected value of the diffusion matrix $\langle D_2(x,y)\rangle$ to the expected value of the drift in a physical state ϱ as we did in Section 10.1. This is done explicitly in Appendix G, using a field-theoretic version of the Cauchy-Schwartz inequality given by Equation (G.16), we find

$$2\langle D_2(x,x)\rangle \int dx'dy' \langle D_0(x',y')\rangle \succeq \langle D_1^{br}(x)\rangle \langle D_1^{br}(x)\rangle^\dagger, \tag{10.20}$$

where equation (10.22) is to be understood as a matrix inequality with entries

$$\begin{aligned}
\langle D_0(x,y)\rangle &= \int dz \mathrm{Tr}\left[D_0^{\alpha\beta} L_\alpha(x)\varrho L_\beta^\dagger(y)\right],\\
\langle D_1^{br}(x,y)\rangle_i &= \int dz \mathrm{Tr}\left[D_{1,i}^{\mu\alpha} L_\mu(x)\varrho L_\alpha^\dagger(x)\right],\\
\langle D_2(x,y)\rangle_{ij} &= \int dz \mathrm{Tr}\left[D_{2,ij}^{\mu\nu} L_\alpha(x)\varrho L_\beta^\dagger(y)\right].
\end{aligned} \tag{10.21}$$

Similarly, when the back-reaction is sourced by either $D_{1,i}^{0\mu}$ or $D_{1,i}^{\alpha\beta}$ it follows from Equation (10.20) we can arrive at the observational trade-off in terms of the total drift due to back-reaction

$$8\langle D_2(x,x)\rangle \int dx'dy' \langle D_0(x',y')\rangle \succeq \langle D_1^T(x)\rangle\langle D_1^T(x)\rangle^\dagger, \tag{10.22}$$

where

$$\langle D_1^T(x)\rangle_i = \int dz \mathrm{Tr}\left[D_{1,i}^{0\alpha}(x)\varrho L_\alpha^\dagger(x) + D_{1,i}^{\alpha 0}(x) L_\alpha \varrho(x) + D_{1,i}^{\alpha\beta}(x) L_\alpha(x)\varrho L_\beta^\dagger(x)\right]. \tag{10.23}$$

We shall now use the trade-off to study the consequences of treating the gravitational field classically. We will consider the back-reaction of the mass on the gravitational field to be governed by the Newtonian interaction. We shall then find that experimental bounds on coherence lifetimes for particles in superposition require significant diffusion in the gravitational field to be maintained. This can be upper bounded by gravitational experiments.

To summarise this section, we have derived the trade-off between decoherence and diffusion for classical-quantum field theories, both in terms of coupling constants of the theory and in terms of observational quantities. This trade-off puts tight observational constraints on classical theories of gravity, which we now discuss.

10.3 Physical constraints on the classicality of gravity

In this section, we apply the trade-off of Equation (10.18) to the case of gravity. Since the trade-offs derived in the previous section depend only on the back-reaction or drift term, they are insensitive to the particulars of the theory. We shall consider the Newtonian, non-relativistic limit of a classical gravitational field which we studied in Chapter 9, and for completeness we outline all of our assumptions.

Assumption 1. *We assume that the evolution of the combined classical-quantum system undergoes autonomous CQ dynamics. In particular, the theory is a completely positive norm-preserving autonomous map, and we can perform a short-time Kramers-Moyal expansion of the dynamics.*

Assumption 2. *We apply the theory to the weak field limit of General Relativity, where, as recalled in Chapter 9, the Newtonian potential interacts with matter through its mass density* $m(x)$. *We assume that the purely classical part of the evolution is generated by the reduced Hamiltonian of Equation (9.24), that the interaction between classical and quantum degrees of freedom is Hamiltonian, and that it is governed by the reduced interaction Hamiltonian*

$$H_I(\Phi) = \int d^3x \Phi(x) m(x), \tag{10.24}$$

where the constraints $\pi_\Phi(t)$ *should also be imposed.*

Assumption 3. *In this work, we will take the coefficients* D_n *entering the master equation to be minimally coupled, by which we mean they depend only on the Newtonian potential* Φ, $D_n(\Phi)$ *and not their conjugate momenta* π_Φ.

Assumption 4. *In relating* D_0 *to the decoherence rate of a particle in superposition, we shall assume that the state of interest is well approximated by a state living in a Hilbert space of fixed particle number. We believe this is a mild assumption: ordinary non-relativistic quantum mechanics is described via a single particle Hilbert space. We frequently place massive composite particles in superposition, and they do not typically decay into multiple particles.*

These assumptions can be understood as describing the non-relativistic Newtonian limit of classical-quantum theories of gravity. In particular, we call such theories *non-relativistic* to emphasize that they assume the Newtonian approximation holds to short distances, i.e., sub millimeter scales where coherence experiments are performed, but general relativity has yet to be tested at distances shorter than the millimeter scale. It may be that higher order, or relativistic corrections, affect the low-energy behavior of the theory but we do not discuss such *relativistic* theories here.

With these assumptions and treating the matter density as a quantum operator, this tells us that in order for the back-reaction term to reproduce the Newtonian interaction on average

$$\mathrm{Tr}\left[\{H_I,\varrho\}\right]=\mathrm{Tr}\left[\int d^3x\ \hat{m}(x)\frac{\delta\varrho}{\delta\pi_\Phi}\right]=-\sum_{\mu\nu\neq 00}\mathrm{Tr}\left[\int d^3xD^{\mu\nu}_{1,\pi_\Phi}(\Phi,\pi_\Phi,x)L_\mu(x)\frac{\delta\varrho}{\delta\pi_\Phi}L^\dagger_\nu(x)\right],\tag{10.25}$$

then we must pick

$$\langle D^T_{1,\pi_\phi}(\Phi,\pi_\Phi,x)\rangle=-\langle\hat{m}(x)\rangle,\tag{10.26}$$

meaning that the back-reaction matrix $D^{\mu\alpha}_{1,\pi_\Phi}$ is non vanishing. In the previous chapter (see also the appendices of [4]), we saw examples of master equations for which Equation (10.26) is satisfied. However, their details are irrelevant since we only require the expectation of the back-reaction force to be the mass – a necessary condition for the theory to reproduce Newtonian gravity.

As a consequence of the coupling constant and observational trade-offs derived in Equations (10.19) and (10.20), a non-zero D_{1,π_Φ} implies that there must be diffusion in the momenta conjugate to π_Φ. For decohered mass, we saw in Chapter 9 that this diffusion is equivalent to adding a stochastic random process $\xi(x,t)$ to the equation of motion for $\dot{\pi}_\Phi$ to give

$$\dot{\pi}_\Phi=\frac{\nabla^2\Phi}{4\pi G}-m(x)+u(\Phi,\hat{m})\xi(t,x),\tag{10.27}$$

where we allow some *colouring* to the noise via a function $u(\Phi,\hat{m})$ which can depend on Φ, and the matter distribution $\hat{m}$. The noise process satisfies

$$\mathbb{E}_{m,\Phi}[u\xi(x,t)]=0,\quad \mathbb{E}_{m,\Phi}[u\xi(x,t)u\xi(y,t')]=2\langle D_2(x,y,\Phi)\rangle\delta(t,t'),\tag{10.28}$$

where we have defined $\langle D_2(x,y,\Phi)\rangle=\mathrm{Tr}\left[D^{\mu\nu}_2(x,y,\Phi)L_\mu(x)\rho L^\dagger_\nu(y)\right]$, and ρ is the quantum state for the decohered mass density. Here the m,Φ subscripts of $\mathbb{E}_{m,\Phi}$ allow for the possibility that the statistics of the noise process can be dependent on the Newtonian potential and mass distribution of the particle. If $u\xi(x,t)$ is Gaussian, Equation (10.28) completely determines the noise process, but in general, higher-order correlations are possible, although they need not concern us here, since we are only interested in bounding the effects due to $D_2(x,y,\Phi)$. In this chapter, we often suppress the explicit dependence of the couplings D_n on the Newtonian potential for notational convenience.

In the non-relativistic limit, where $c\to\infty$, we impose the momentum constraint $\pi_\Phi\approx 0$ and recover Poisson's equation for gravity, but with a stochastic contribution to the mass. This is precisely as expected on purely physical grounds: in order to maintain the coherence of any mass in superposition, there must be noise in the Newtonian potential, and this must be such that we cannot tell which element of the superposition the particle will be in, meaning the Newtonian potential should look like it is being sourced in part by a random mass distribution. In other words, the trade-off requires that the stochastic component of the coupling obscures the amount of mass m at any point.

The solution to Equation (10.27) is given by

$$\Phi(t,x) \approx -G \int d^3x' \frac{[m(x',t) - u(\Phi,\hat{m})\xi(x',t)]}{|x-x'|}, \tag{10.29}$$

and a formal treatment of solutions to non-linear stochastic integrals of the form of Equation (10.27) can be found in [237]. A higher precision calculation would involve a full simulation of CQ dynamics, for example, using unraveling methods [6, 28]. It may also be that relativistic corrections may constrain the degree of diffusion even at low energy. One should consider this when drawing conclusions from the models presented here.

We have seen that there are two classes of CQ dynamics, in the sense that there are those with continuous trajectories in phase space and those with discrete jumps. For the class of continuous CQ models, we know that $\xi(x,t)$ should be described by a white noise process in time, and its statistics should be independent of the mass density of the particle. For the discrete class, $\xi(x,t)$ can involve higher order moments and will generally be described by a jump process [5, 10]. Its statistics can also depend on the mass density since, generally, the diffusion matrix $D^{\mu\nu}_{2,ij}$ couples to Lindblad operators. It is worth noting that the discrete CQ theories considered in [28, 3, 10] generically suppress higher order moments, and often we expect that we can approximate the dynamics by a Gaussian process, but this need not be the case in general.

This variation in Newtonian potential leads to observational consequences, which can be used to experimentally test and constrain CQ theories of gravity for various choices of kernels appearing in the CQ master equation. One immediate consequence is that the variation in Newtonian potential leads to a variation of force experienced by a particle or composite mass via $\vec{F}_{tot} = -\int d^3x m(x)\nabla\Phi(x)$. We can also estimate the time-averaged force via $\frac{1}{\Delta T}\int_0^{\Delta T} \vec{F}_{tot}$ where ΔT is the time resolution over which the force is measured and is the useful quantity when comparing with experiments. Using Equation (10.29), in Appendix I.1, we find that the

Master Equation	Diffusion Kernel	Experimental squeeze
Continuous (ultra-local)	$D_2(\Phi;x,y) = D_2(\Phi)\delta(x,y)$ $D_2(\Phi) = \sum_n c^n \Phi^n$	$10^{-41} \geq D_2 \geq 10^{-9}\ kg^2 sm^{-3}$ (Eqn (10.32))
Continuous (D.P)	$D_2(\Phi;x,y) = -l_p^2 D_2(\Phi)\nabla^2\delta(x,y)$ $D_2(\Phi) = \sum_n c^n \Phi^n$	$10^{-9} \geq l_P^2 D_2 \geq 10^{-35}\ kg^2 sm^{-1}$ (Eqn (10.35))
Discrete (ultra-local)	$D_2(\Phi;x,y) = \frac{l_P^2}{m_P} D_2(\Phi)\delta(x,y)$ $D_2(\Phi) = \sum_n c^n \Phi^n$	$10^{-1} \geq \frac{l_P^2 D_2}{m_P} \geq 10^{-25}\ kgs$ (Eqn (10.34))

Table 10.1: Current experimental bounds on non-relativistic classical-quantum theories for different master equations and functional dependence on the diffusion coefficient. The diffusion coefficient is bounded from above by observed acceleration variations σ_a^2 seen in precision mass experiments via Equation (10.30). In all cases, the master equation is assumed to saturate the bound, which is used to find the lower bound the amount of diffusion on the quantum system by bounding D_0 from coherence rates via Equation (10.31). It is seen that minimally coupled continuous models, which are non-relativistic and do not create spatial correlations (we call these *ultra-local*) and have polynomial dependence on the Newtonian potential, are ruled out. In contrast, continuous models with non-local correlations, such as the Diosi-Penrose (D.P.) kernel, and ultra-local discrete models are less constrained. Here l_P, m_P denote the Planck length and Planck mass, respectively, which are required for the dimensions of $D_2(\Phi)$ to be the same in all cases.

variance of the magnitude of the time-averaged force experienced by a particle in a Newtonian potential is given by

$$\sigma_F^2 = \frac{2G^2}{\Delta T}\int d^3xd^3yd^3x'd^3y'm(x)m(y)\frac{(\vec{x}-\vec{x}')\cdot(\vec{y}-\vec{y}')}{|x-x'|^3|y-y'|^3}\langle D_2(x',y',\Phi)\rangle, \tag{10.30}$$

where the variation is averaged by the time resolution ΔT. We will use this to estimate the variation in precision measurements of mass, such as modern versions of the Cavendish experiment for various choices of $\langle D_2(x',y',\Phi)\rangle$.

On the other hand, experimentally measured decoherence rates can be related to D_0, which we calculated for both the continuous master equation and the jumping master equation in Chapter 9. In Appendix H, we show that for a mass whose quantum state is a superposition of two states $|L\rangle$ and $|R\rangle$ of approximately orthogonal mass densities $m_L(x), m_R(x)$, and whose separation we take to be larger than the correlation range of $D_0(x,y)$, the decoherence rate is given by

$$\lambda = \frac{1}{2}\int dxdyD_0^{\alpha\beta}(x,y)(\langle L|L_\beta^\dagger(y)L_\alpha(x)|L\rangle + \langle R|L_\beta^\dagger(y)L_\alpha(x)|R\rangle), \tag{10.31}$$

which generalizes the calculations performed in Chapter 9. Via the coupling constant trade-off, Equations (10.30) and (10.31) give rise to a double-sided squeeze on the coupling D_2. Equation (10.30) upper bounds D_2 in terms of the uncertainty of acceleration measurements seen in gravitational torsion experiments, while the coupling constant trade-off Equation (10.31) lower bounds D_2 in terms of experimentally measured decoherence rates arising from interferometry experiments.

We now show this for various choices of diffusion kernel for the models considered in Chapter 9. We have already calculated the decoherence rates, and we calculate the associated force variations in Appendix I.1. The bounds are summarized in Table 10.1. The diffusion coupling strength will be characterized by the coupling constant D_2, which we consider a dimension-full quantity with units kg^2sm^{-3}, and is related to the rate of diffusion for the conjugate momenta of the Newtonian potential. We upper bound D_2 by considering the variation of the time-averaged acceleration $\sigma_a = \frac{\sigma_F}{M}$ for a composite mass M which contains N atoms which we treat as spheres of constant density ρ with radius r_N and mass m_N. We lower bound D_2 via the coupling constant trade-off of Equation (10.18) and then by considering bounds on the coherence time for composite particles with total mass M_λ and which are made up of N_λ

constituents, each with typical length scale when in superposition R_λ and volume V_λ.

For continuous dynamics $\langle D_2(x,y,\Phi)\rangle = D_2(x,y,\Phi)$ since the diffusion is not associated with any Lindblad operators. The full dynamics for these models were considered in Section 9.3 of Chapter 9. Let us now consider a very natural kernel, namely $D_2(x,y;\Phi) = D_2(\Phi)\delta(x,y)$, which is both translation invariant and does not create any correlations over space-like separated regions. We call dynamics which do not create correlations over space-like separated regions *ultra-local* since theories that are not of this form can still be non-signaling. The squeeze will generally depend on the functional choice of $D_2(\Phi)$ on the Newtonian potential. However, in the presence of a large background potential Φ_b, such as that of the Earth's, we will often be able to approximate $D_2(\Phi) = D_2(\Phi_b)$. This is true for kernels that depend on Φ and $\nabla\Phi$, though the approximation does not hold for all kernels, for example, $D_2 \sim -\nabla^2\Phi$ which creates diffusion only where there is mass density. For diffusion kernels $D_2(\Phi_b)$ where the background potential is dominant, we find the promised squeeze on $D_2(\Phi_b)$

$$\frac{\sigma_a^2 N r_N^4 \Delta T}{V_b G^2} \geq D_2 \geq \frac{N_\lambda M_\lambda^2}{V_\lambda \lambda}, \tag{10.32}$$

where V_b is the volume of space over which the background Newtonian potential is significant. V_b enters since the variation in acceleration is found to be

$$\sigma_a^2 \sim \frac{D_2 G^2}{r_N^4 N \Delta T} \int d^3x' D_2(\Phi_b), \tag{10.33}$$

where the d^3x' integral is over all space. This immediately rules out continuous theories with noise everywhere, i.e., with a diffusion coefficient independent of the Newtonian potential since the integral will diverge.

Standard Cavendish type classical torsion balance experiments [223] measure accelerations of the order $10^{-7}ms^{-2}$ over minutes $\Delta T \sim 10^2$, so a very conservative bound is $\sigma_a \sim 10^{-7}ms^{-2}$, whilst for a kg mass $N \sim 10^{26}$ and $r_N \sim 10^{-15}m$. Conservatively taking $V_b \sim r_E^2 h\ m^3$ where r_E is the radius of the Earth and h is the atmospheric height gives $D_2 \leq 10^{-41}kg^2sm^{-3}$. The decoherence rate λ is bounded by various experiments [238]. Typically, such experiments aim to witness interference patterns of molecules that are as massive as possible. Taking a conservative bound on λ, for example, that arising from the interferometry experiment of [233] which saw coherence in large organic fullerene molecules with total mass $M_\lambda = 10^{-24}kg$ over a timescale of $0.1s$, gives an upper bound on the decoherence rate $\lambda < 10^1 s^{-1}$. Fullerene molecules comprise

$N_\lambda \sim 10^3$ particles with typical atomic size $10^{-15}m$. After passing through the slits, the molecule becomes delocalized in the transverse direction on the order of $10^{-7}m$ before being detected. Since the interference effects are due to the superposition in the transverse x direction, which is the direction of alignment of the gratings, it seems like a reasonable assumption to take the size of the wave-packet in the remaining y, z direction to be the size of the fullerene, since we could imagine measuring the y, z directions without effecting the coherence. We, therefore, take the volume $V_\lambda \sim 10^{-15}10^{-15}10^{-7}m^3 = 10^{-37}m^3$, which gives $D_2 \geq 10^{-9}kg^2sm^{-3}$, and suggests that classical-quantum theories of gravity with ultra-local continuous noise are ruled out by experiment.

On the other hand, the discrete models appear less constrained due to the suppression of the noise away from the mass density. For example, consider the ultra-local discrete jumping models, such as the one given in Section 9.7 of Chapter 9, which have $\langle D_2(x, y, \Phi_b)\rangle = \frac{l_P^3 D_2(\Phi_b)}{m_P} m(x)$, where $m_P = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass and $l_P = \sqrt{\frac{\hbar G}{c^3}}$ is the Planck length, required to ensure D_2 has the units of kg^2sm^{-3}.

We find the squeeze on D_2

$$\frac{\sigma_a^2 N r_N^4 \Delta T}{m_N G^2} \geq \frac{l_P^3}{m_P} D_2 \geq \frac{M_\lambda}{\lambda}, \tag{10.34}$$

and plugging in the numbers tells us that discrete theories of classical gravity are not ruled out by experiment, and we find $10^{-1}kgs \geq \frac{l_P^3}{m_P} D_2 \geq 10^{-25}kgs$.

We can also consider other noise kernels which are not ultra-local. A natural kernel is the Diosi-Penrose kernel for the class of continuous models studied in Section 9.3 of Chapter 9 with $D_2(x, y, \Phi_b) = -l_P^2 D(\Phi_b)\nabla^2\delta(x, y)$. The inverse Lindbladian kernel satisfying the coupling constants trade-off is to zeroeth order in $\Phi(x)$, the Diosi-Penrose kernel $D_0(x, y, \Phi_b) = \frac{D_0(\Phi_b)}{|x-y|}$.

For this choice of dynamics, we find the squeeze for D_2 in terms of the variation in acceleration

$$\frac{\sigma_a^2 N r_N^3 \Delta T}{G^2} \geq l_P^2 D_2 \geq \frac{N_\lambda M_\lambda^2}{R_\lambda \lambda}. \tag{10.35}$$

Using the same numbers as for the ultra-local continuous model, with $R_\lambda \sim V_\lambda^{1/3} \sim 10^{-12}m$, we find that classical torsion experiments upper bound D_2 by $10^{-9}kg^2sm^{-1} \geq l_P^2 D_2$, while interferometry experiments bound D_2 from below via $l_P^2 D_2 \geq 10^{-35}kg^2sm^{-1}$.

Equations (10.32), (10.34), and (10.35) show that experiments squeeze classical theories of gravity from both ways. We have been highly conservative here and anticipate that further analysis and near-term experiments can tighten the bounds by orders of magnitude. Several proposals for table-top experiments to measure gravity precisely have recently been performed and could give rise to tighter upper bounds on D_2. Some of these experiments involve millimeter-sized masses whose gravitational coupling is measured via torsional pendula [234, 235], or rotating attractors [236]. With such devices, the gravitational coupling between small masses can be measured while limiting the amount of other noise sources. There are proposals for further mitigating the noise due to the environment, including the inertial noise, gas particles collisions, photon scattering on the masses, and curvature fluctuations due to other sources [239, 240, 241]. Other experiments are based on interference between masses; for example, atomic interferometers allow for measuring the curvature of space-time over a macroscopic superposition [242, 243, 244].

We can get stronger lower bounds via improved coherence experiments. Typically, such experiments aim to witness interference patterns of molecules that are as massive as possible. At the same time, we see that the experimental bound on CQ theories is generically obtained by maximizing the coherence time for massive particles with a small wave-packet size V_λ.

Thus far, we have considered local effects on particles due to diffusion. While this enables us to rule out some theories, the bounds are generally weak if one wants to rule out all of them. However, it may be possible to do so via cosmological considerations. In attempting to place experimental constraints on this diffusion, it is also worth considering other regimes, such as longer-range effects, which might be detected by gravitational wave detectors such as LIGO. We leave a detailed study of the effect of gravitational diffusion on cosmological scales and LIGO to future work. The effect will again depend on the form of the kernel $D_2(x, x')$. Our estimates [245] suggest that local effects from table-top experiments place a stronger bound on gravitational theories than LIGO. In particular, unlike gravitational wave measurements, which are reasonably high-frequency events requiring extraordinarily high precision in the relative displacement of the arm length from its average, it is preferential to have a lower precision measurement, which occurs over a longer period to allow for the diffusion in path length to build up, and with a smaller uncertainty in the average length of the arm itself. Furthermore, since the LIGO arm is kept in a vacuum, we do not expect strong bounds on discrete models where the diffusion is associated with an energy density.

10.4 Discussion

Several direct proposals to test the quantum nature of gravity are expected to come online in the next decade or two. These are based on detecting entanglement between mesoscopic masses inside matter-wave interferometers [64, 53, 1, 2, 65, 66, 67, 68]. For these experiments, some theoretical assumptions are needed: one requires that it is only gravitons that travel between the two masses and mediate the creation of entanglement. If this is the case, then the onset of entanglement implies that gravity is not a classical field. These can be thought of as experiments that, if successful, would confirm the quantum nature of gravity (although other alternatives to quantum theory are possible [51]).

Here, we come from the other direction by supposing that gravity is instead classical and then exploring the consequences. Theories in which gravity is fundamentally classical were thought to have been ruled out by various no-go theorems and conceptual difficulties. However, these no-go theorems are avoided if one allows for non-deterministic coupling in the dynamics.

We have here proven that this feature is necessary and made it quantitative by exploring the consequences of complete positivity on any dynamics that couples quantum and classical degrees of freedom. Complete positivity is required to ensure the probabilities of measurement outcomes remain positive throughout the dynamics. We have shown that any theory that preserves probabilities and treats one system classically must have fundamental decoherence of the quantum system and diffusion in phase space, which are signatures of information loss. Using a CQ version of the Kramers-Moyal expansion, we have derived a trade-off between decoherence on the quantum system and the system's diffusion in phase space. The trade-off is expressed in terms of the strength of the back-reaction of the quantum system on the classical one. We have derived the trade-off in terms of the theory's coupling constants and observational quantities that can be measured experimentally.

In the case of gravity, the trade-off places a lower bound on the rate of diffusion of the gravitational degrees of freedom in terms of the decoherence rate of particles in superposition. Current experiments do not rule out theories that treat gravity as fundamentally classical.

However, we have been able to rule out a broad parameter space of such theories. This is done partly through table-top observations via Equations (10.32), (10.34), and (10.35). Given any diffusion kernel, we can compute the inaccuracy of mass measurements due to fluctuations in the gravitational field, and we can derive a bound on the associated decoherence rate using the trade-off. This allows us to rule out broad classes of theories in terms of their diffusion kernel. For example, we can rule out several ultra-local non-relativistic theories which back-react continuously in phase space.

Any theory which treats gravity classically has fairly limited freedom to evade the effects of the trade-off. There is the freedom to choose the diffusion or decoherence kernels $D_2(x, x')$ and $D_0(x, x')$, but the trade-off restricts one in terms of the other. Then, because of the results proven in [5], one can consider two classes of theory, those which are continuous realizations and whose diffusion can only depend on the gravitational degrees of freedom, and discrete theories whose diffusion can also depend directly on the matter fields. Chapter 9 gave examples of both theory classes.

Finally, one could consider theories that do not reproduce the weak field limit of General Relativity to all distances. We could imagine that the interaction Hamiltonian of Equation (10.24) does not hold to arbitrarily short distances or arbitrarily high mass densities. This would correspond to modifying $D_1(x, x')$ in some way, either by making it slightly non-local, by disallowing arbitrarily high mass densities, or by including an additional contribution such as a friction term. All of these modifications would seem to violate Lorentz invariance in some way. One caveat, however, to keep in mind is that relativistic corrections may affect the low-energy behavior of the theory, which we have not considered here.

We have only given an order of magnitude estimate of when gravitational diffusion will lead to appreciable deviations from Newtonian gravity or General Relativity. The most promising experiments bounding the diffusion appear to be table-top experiments that precisely measure the mass of an object. This area is important from the perspective of weight standards, for example, those undertaken by NIST on the 1kg mass standard K20 and K4 [246]. The increased precision and measuring time of Kibble Balances [247] and atomic interferometers [242, 243, 248, 249] would make such measurements an ideal testing ground, both to further constrain the diffusion kernel and to look for diffusion effects, whose dependence on the test mass is outlined in Appendix I. Here, we have found that the resolution time ΔT over which variations of acceleration are estimated affects the strength of the bound. It would be helpful if future experiments reported this value. Since we have found that CQ theories predict uncertainty in mass measurements, it is perhaps intriguing that different experiments to measure Newton's constant G yield results whose relative uncertainty differs by as much as $5 \cdot 10^{-4}$ $\mathrm{m^3kg^{-1}s^{-2}}$, which is more than an order of magnitude larger than the average reported uncertainty [226, 227, 228]. If one were to try and explain the discrepancy in G measurements via gravitational diffusion, then for all the kernels we studied in Section 10.3, we find that the variation in acceleration depends on $\frac{1}{\sqrt{N}}$ the number of nucleons in the test mass so that masses with smaller volume should yield more considerable uncertainty and this would be the effect to look for in measurement discrepancies. The relatively large uncertainty in such measurements makes it challenging for table-top experiments to place strong upper bounds on gravitational diffusion.

Now turning to the other side of the trade-off. Improved decoherence times would further squeeze theories in which gravity remains classical. While a current experimental challenge is to demonstrate interference patterns using larger and larger mass particles, we here find the bounds in Equation's (10.32) (10.34) depend on the expectation of the particle's mass density in ways which depend on the particular kernel. Thus interference experiments with particles of high *mass density* rather than mass can be preferable. There are also kernels, for which the relevant quantity is the expectation of the mass density, which will depend on the size of the wave-packet used in the interference experiment. This quantity is rarely obtainable from most papers that report on such experiments. While this dependence might initially appear counter-intuitive, it follows from the fact that in order to relate the trade-off in terms of coupling constants to observational quantities, and in particular, the decoherence rate, we took expectation values of the relevant quantities to get a trade-off in terms of only averages. And indeed, the decoherence rate, which is an expectation value, can easily depend on the wave-packet density, as we see from examples in Chapter 9

Since we here show that all theories which treat gravity classically necessarily decohere the quantum system, another constraint on theories that treat gravity classically is given by constraints from anomalous heating of the quantum system [206, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262]. In standard collapse models, these constraints are not in themselves very strong, since fundamental decoherence effects can be made arbitrarily weak by appropriate scaling of the decoherence parameter. Here, however, we see less freedom than one might imagine. If the Lindbladian coupling constants are made small to reduce heating, the gravitational diffusion must be large. Thus, heating constraints that place bounds on $D_0(x, x')$ place additional constraints on $D_2(x, x')$.

While the absence of diffusion could rule out theories where gravity is fundamentally classical, the presence of such deviations, at least on short time scales, might not confirm gravity's classical nature. Such effects could instead be caused by quantum theories of gravity, whose classical limit is effectively described by a CQ theory. In other words, one might expect some gravitational diffusion because, from an effective theory point of view, one is in a regime where space-time behaves classically. However, the trade-off we have derived directly results from treating the background space-time as fundamentally classical. In a fully quantum theory of gravity, the interaction of the gravitational field with particles in a superposition of two trajectories will cause decoherence. However, coherence can then be restored when the two trajectories converge. This happens when electrons interact with the electromagnetic field while passing through a diffraction grating yet still form an interference pattern at the screen. This is a non-Markovian effect, and the trade-off we derived is a direct consequence of the positivity condition, which is a direct consequence of the Markovian assumption. In the time-local non-Markovian theory where General Relativity is treated classically, one still expects the master equation to take the form found in [28], but without the matrix whose elements are $D_n^{\mu\nu}$ needing to be positive semi-definite [92, 93].

Appendix A

CQ states with continuous classical degrees of freedom

In this appendix, we formulate CQ dynamics in terms of continuous degrees of freedom more rigorously by defining a CQ state as an operator-valued measure, which formalizes the notion "to each z we associate a sub-normalized density matrix such that its trace defines a probability distribution over phase space". We then use this to argue why any completely positive CQ dynamics can be written in the form of Equation (2.80), even in the case where the classical degrees of freedom are continuous.

Let Ω be a set and $\mathcal{A}$ be a σ algebra. A map $\varrho : \mathcal{A} \to S_{\leq\infty}(\mathcal{H})$, where $S_{\leq\infty}(\mathcal{H})$ denotes the space of un-normalized density matrices, is called a *CQ state* if

1. For each $A \in \mathcal{A}$, $\varrho(A)$ is an un-normalised, density operator on $\mathcal{H}$.
2. $\varrho(\emptyset) = 0$ and $\varrho(\Omega)$ is a normalised density matrix.
3. If E_i are disjoint then $\varrho\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \varrho\left(A_j\right)$
4. $\mu_\varrho(A) = \mathrm{Tr}\left[\varrho(A)\right]$ defines a probability measure on $\mathcal{A}$

Now from any CQ state ϱ we can form a real-valued measure by setting $\varrho_v(A) = \langle v, \varrho(A)v\rangle$ for any $v \in \mathcal{H}$. Then, there exists a unique linear map, denoted $f \to \int_\Omega f\varrho(d\omega)$ with the property that

$$\left\langle v, \left(\int_\Omega f\varrho(d\omega)\right) v\right\rangle = \int_\Omega f\varrho_v(d\omega) \tag{A.1}$$

for all bounded measurable complex functions $f : \Omega \to \mathbb{C}$ and all $v \in \mathcal{H}$, where the right-hand side of (A.1) is the ordinary Lebesgue integral; this follows in the same way as the proof of unique integration for projector-valued operators [263], after all a density matrix can be written as a sum of projectors. We can compute the operator-valued integral of an arbitrary bounded measurable function as follows. Take a sequence s_n of simple functions converging uniformly to f, then the integral of f is the limit, in the operator norm topology, of the integral of the s_n [263].

CQ evolution is a map taking CQ states to CQ states. In order to make sense of this, we need to define a notion of measurable super-operators. We first define the space of completely positive super-operators, which maps $S_{\leq\infty}(\mathcal{H})$ to itself as P. We want to write ultimately

$$\varrho'(A) = \int_{\Omega} \Lambda(\omega, A)(\varrho(d\omega)), \tag{A.2}$$

where $\Lambda : \Omega \times \mathcal{A} \to P$ is such that $\Lambda(\omega, A)$ is a completely positive super-operator for $\omega \in \Omega$ and $A \in \mathcal{A}$. In the rest of the section, and occasionally in the next section, we write the integral in Equation (A.2) as

$$\varrho(z) = \int dz' \Lambda(z|z')\varrho(z'). \tag{A.3}$$

This section aims to give a slightly more precise definition of the integral so that we can prove a CQ version of Kraus' theorem in the case where the classical degrees of freedom are continuous.

For CQ dynamics, we further ask that $\varrho'(A)$ define a CQ state. We give meaning to the integral $\varrho'(A) = \int_{\Omega} \Lambda(\omega, A)(\varrho(d\omega))$ by taking the inner product with a Hilbert space vector. In particular, if we let $\{|a\rangle\}$ denote an arbitrary basis of vectors in $\mathcal{H}$, and we write the super-operator as $\Lambda(\omega, A)(\varrho) = \sum_{abcd} \Lambda^{abcd}(\omega, A)|a\rangle\langle b|\varrho|c\rangle\langle d|$, then we can make sense of the integral as follows

$$\langle a|\varrho'(A)|d\rangle = \sum_{bc} \langle b| \left[\int_{\Omega} \Lambda^{abcd}(\omega, A)\varrho(d\omega) \right] |c\rangle, \tag{A.4}$$

and we loosely define measurable CQ dynamics as dynamics for which Equation (A.4) is well defined. For simplicity, we assume here that the Hilbert space dimension is finite. However, by analogy with Kraus' theorem for quantum operations, we expect that this can be extended to any bounded trace-class operation.

A.1 Proof of Kraus theorem for CQ dynamics

In this section we sketch a proof of a CQ Kraus theorem when the classical degrees of freedom are allowed to be continuous, and the Hilbert space is finite-dimensional; that is, we give an outline of a proof that every completely positive CQ map can be written in the form of Equation (2.80), with normalization conditions in Equation (2.82).

We assume we have a completely positive linear CQ map Λ. By linearity, the most general form of the dynamics can be written in the form $\varrho'(A) = \int_\Omega \Lambda(\omega|A)(\varrho(d\omega))$ – we take it given that $\Lambda(\omega|A)$ is measurable in the sense that (A.4) is well defined.

If it is completely positive, then it is certainly n positive. To that end, consider the Choi matrix

$$\varrho'_{\bar{z}}(A) = \int_\Omega \sum_{ab} (I \otimes \Lambda(\omega, A))(E_{ab} \otimes E_{ab}\delta_{\bar{z}}(d\omega)) = \sum_{ab} E_{ab} \otimes \Lambda(\bar{z}|A)(E_{ab}), \tag{A.5}$$

where $\delta_{\bar{z}}$ is the delta measure ($\delta_{\bar{z}}(A) = 1$ if $\bar{z} \in A$ and 0 otherwise), E_{ab} is the natural basis of operators on $\mathcal{H}$, $E_{ab} = |a\rangle\langle b|$ and $\bar{z} \in \Omega$. Since $\sum_{ab} E_{ab} \otimes E_{ab}\delta_{\bar{z}}$ is positive, and Λ is assumed to be a completely positive CQ evolution map, $\varrho'_{\bar{z}}(A)$ defines a CQ state on $\mathcal{H}_R \otimes \mathcal{H}$, where $\mathcal{H}_R$ is a reference Hilbert space. Hence, for each $A \in \mathcal{A}$, $\varrho_{\bar{z}}(A)$ can be diagonalized

$$\varrho'_{\bar{z}}(A) = \sum_\mu \lambda^\mu(\bar{z}, A)|\phi_\mu(\bar{z}, A)\rangle\langle\phi_\mu(\bar{z}, A)|, \tag{A.6}$$

where the eigenvalues $\lambda^\mu(\bar{z}, A)$ are positive for each $A, \bar{z}$. This is equivalent to the statement that $\int dz dz' \lambda(z, z')P_\mu(z, z') \geq 0$ for any positive $P_\mu(z, z')$. Here $|\phi_\mu(\bar{z}, A)\rangle$ is an element of the product Hilbert space $\mathcal{H}_R \otimes \mathcal{H}$. Now we can find the map $\Lambda(\bar{z}|A)(E_{ab})$ by projection of the Choi matrix on the reference system

$$\mathrm{Tr}_R\left[(E_{ba} \otimes I)\varrho'_{\bar{z}}(A)\right] = \langle a_R|\varrho'_{\bar{z}}(A)|b_R\rangle = \Lambda(\bar{z}|A)(E_{ab}). \tag{A.7}$$

If we define the operator $V_\mu : \mathcal{H} \to \mathcal{H}$ via its action on the basis of $\mathcal{H}$, $\{|a\rangle\}$, via $V_\mu(\bar{z}, A)|a\rangle = \langle a_R|\phi_\mu(\bar{z}, A)\rangle$ then

$$\Lambda(\bar{z}|A)(E_{ab}) = \sum_\mu \lambda^\mu(\bar{z}, A)V_\mu(\bar{z}, A)E_{ab}V^\dagger_\mu(\bar{z}, A) \tag{A.8}$$

Since $\bar{z}$ is arbitrary and E_{ab} is a basis of operators on $\mathcal{H}$ we conclude

$$\Lambda(\omega|A) = \sum_\mu \lambda^\mu(\omega, A)V_\mu(\omega, A) \odot V^\dagger_\mu(\omega, A) \tag{A.9}$$

for all $\omega \in \Omega$. Hence, we can write any complete positive, measurable CQ dynamics in the form

$$\varrho'(A) = \int_\Omega \Lambda(\omega|A)(\varrho(d\omega)) = \sum_\mu \int_\Omega \lambda^\mu(\omega, A) V_\mu(\omega, A)\varrho(d\omega) V_\mu^\dagger(\omega, A). \tag{A.10}$$

By changing basis to an arbitrary basis of operators $'V_\mu(\omega, A) = U_{\mu\nu}(\omega, A) V_\nu^\dagger(\omega, A)$ the CQ map can be written in the form of (2.80).

We can recover the normalization conditions in (2.82) as follows. We first note, since the $|\phi_\mu(\bar{z}, A)\rangle$ are orthogonal, so are the matrices $V_\mu(\bar{z}, A)$

$$\sum_{a,b} \langle \phi_\mu(\bar{z}, A)|a_R, b\rangle\langle a_R, b|\phi_\nu(\bar{z}, A)\rangle = \mathrm{Tr}_\mathcal{H}\left[V_\mu^\dagger(\bar{z}, A) V_\nu(\bar{z}, A)\right] = \delta_{\mu\nu} \tag{A.11}$$

Finally, we note that

$$\begin{aligned}\sum_\mu \lambda^\mu(\bar{z}, A) V_\mu^\dagger(\bar{z}, A) V_\mu(\bar{z}, A) &= \sum_{\mu ab} \lambda^\mu(\bar{z}, A) E_{ab} \langle \phi_\mu(\bar{z}, A)|(E_{ab}\otimes I)|\phi_\mu(\bar{z}, A)\rangle \\ &= \sum_{ab} \mathrm{Tr}_{\mathcal{H}_R\otimes\mathcal{H}}\left[(E_{ab}\otimes I)\varrho_{\bar{z}}(A)\right] E_{ab}\end{aligned} \tag{A.12}$$

which using equation (A.5) gives

$$\sum_\mu \lambda^\mu(\bar{z}, A) V_\mu^\dagger(\bar{z}, A) V_\mu(\bar{z}, A) = \sum_\mu \lambda^\mu(\bar{z}, A) I \tag{A.13}$$

Since, $\mathrm{Tr}_{\mathcal{H}_\mathcal{R}\otimes\mathcal{H}}\left[\varrho_{\bar{z}}(A)\right] = \sum_\mu \lambda^\mu(\bar{z}, A)$ defines a probability measure on $\mathcal{A}$ we deduce that

$$\int_\Omega \sum_\mu \lambda^\mu(\bar{z}, d\omega) V_\mu^\dagger(\bar{z}, \omega) V_\mu(\bar{z}, \omega) = I \tag{A.14}$$

which is the normalization condition in (K.3).

The main lesson here is that once we treat the CQ state as an operator valued measure, then the intuition of $\Lambda(z|z')$ as describing a quantum operator for each z, z', holds, even when the degrees of freedom are continuous.

Appendix B

Continuous CP evolution with arbitrary Lindblad operators

We have shown that any continuous CP CQ map can be written as Equation (4.13) where the Lindblad operators are traceless. This appendix shows that one can pick arbitrary Lindblad operators, L_α, in (4.13), and the map will still be completely positive. In other words, we show that Equation (4.13) where the Lindblad operators are arbitrary is also completely positive, so long as the moments satisfy the positivity conditions.

We first write the arbitrary Lindblad operators, L_α, in terms of a set of traceless matrices $L_\alpha = \bar{L}_\alpha + b_\alpha \mathbb{I}$. The equation then takes the same form

$$
\begin{aligned}
\frac{\partial \varrho(z,t)}{\partial t} = &\sum_{n=1}^{n=2} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \dots \partial z_{i_n}}\right) \left(D^{00}_{n,i_1\dots i_n} \varrho(z,t)\right) + \frac{\partial}{\partial z_i}\left(D^{0\alpha}_{1,i} \varrho(z,t) \bar{L}^\dagger_\alpha\right) + \frac{\partial}{\partial z_i}\left(D^{\alpha 0}_{1,i} \bar{L}_\alpha \varrho(z,t)\right) \\
&- i[H(z), \varrho(z,t)] + +D^{\alpha\beta}_0(z) \bar{L}_\alpha \varrho(z) \bar{L}^\dagger_\beta - \frac{1}{2} D^{\alpha\beta}_0 \{\bar{L}^\dagger_\beta \bar{L}_\alpha, \varrho(z)\}_+,
\end{aligned} \tag{B.1}
$$

but with a re-scaled Hamiltonian

$$
H(z) \to H(z) + \frac{1}{2i}(D^{\alpha\beta}_0 b^*_\beta \bar{L}_\alpha - D^{\alpha\beta}_0 b_\alpha \bar{L}^\dagger_\beta), \tag{B.2}
$$

and a re-scaled *classical* drift coefficient

$$
D^{00}_{1,i} \to D^{00}_{1,i} + D^{0\alpha}_{1,i} b^*_\alpha + D^{\alpha 0}_{1,i} b_\alpha. \tag{B.3}
$$

We can then write the traceless Lindblad operators in terms of a basis of traceless Lindblad operators $\bar{L}_\alpha = V_\alpha^\beta \tilde{L}_\beta$ which span the same vector space. We can always choose V_α^β is invertible since $\tilde{L}_\beta$ form a basis for the traceless operators. Defining $\tilde{D}^{00}_{n,i_1\ldots i_n} = D^{00}_{n,i_1\ldots i_n}$, $\tilde{D}^{\beta 0}_{1,i} = D^{\alpha 0}_{1,i} V_\alpha^\beta$ and $\tilde{D}_0^{\gamma\sigma} = V_\gamma^\alpha D_0^{\gamma\sigma} (V^\dagger)^\beta_\sigma$ we find the master equation takes the form

$$\begin{aligned}\frac{\partial \varrho(z,t)}{\partial t} = &\sum_{n=1}^{n=2} (-1)^n \left(\frac{\partial^n}{\partial z_{i_1} \ldots \partial z_{i_n}}\right) \left(\tilde{D}^{00}_{n,i_1\ldots i_n} \varrho(z,t)\right) + \frac{\partial}{\partial z_i} \left(\tilde{D}^{0\alpha}_{1,i} \varrho(z,t) \tilde{L}^\dagger_\alpha\right) + \frac{\partial}{\partial z_i} \left(\tilde{D}^{\alpha 0}_{1,i} \tilde{L}_\alpha \varrho(z,t)\right) \\ &- i[H(z), \varrho(z,t)] + +\tilde{D}_0^{\alpha\beta}(z) \tilde{L}_\alpha \varrho(z) \tilde{L}^\dagger_\beta - \frac{1}{2} \tilde{D}_0^{\alpha\beta} \{\tilde{L}^\dagger_\beta \tilde{L}_\alpha, \varrho(z)\}_+ \end{aligned} \tag{B.4}$$

which is now of the form in Equation (4.13). Furthermore,

$$2\tilde{D}_2 = 2D_2 \succeq D_1^\dagger \tilde{D}_0^{-1} \tilde{D}_1 = D_1^\dagger D_0^{-1} D_1, \tag{B.5}$$

where we have used the invertibility of V_β^α. Hence any equation of the form (4.13) with arbitrary Lindblad operators and coefficients satisfying the positivity conditions will be completely positive.

Appendix C

Unraveling of classical-quantum field theory

Since gravity is a field theory, we discuss unravelings in the context of fields in this appendix. The field theoretic version of the continuous master equation in (4.13) of Chapter 4 is [4]

$$\begin{aligned}\frac{\partial \varrho(z,t)}{\partial t} &= -\int dx \frac{\delta}{\delta z_i(x)} \left(D_{1,i}^{00}(z;x)\varrho(z,t)\right) - \int dxdy \frac{\delta^2}{\delta z_i(x)\delta z_j(y)} \left(D_{2,ij}^{00}(z;x,y)\varrho(z,t)\right) \\ &\quad - i[H(z),\varrho(z,t)] + \int dxdy \left[D_0^{\alpha\beta}(z;x,y)L_\alpha(x)\varrho(z)L_\beta^\dagger(y) - \frac{1}{2}D_0^{\alpha\beta}(z;x,y)\{L_\beta^\dagger(y)L_\alpha(x),\varrho(z)\}_+\right] \\ &\quad \int dx \frac{\delta}{\delta z_i(x)} \left(D_{1,i}^{0\alpha}(z;x)\varrho(z,t)L_\alpha^\dagger(x)\right) + \frac{\delta}{\delta z_i(x)} \left(D_{1,i}^{\alpha 0}(z;x)L_\alpha(x)\varrho(z,t)\right).\end{aligned} \tag{C.1}$$

The complete positivity of the dynamics is enforced by the positivity of the matrix

$$\int dxdy[b^*(x),\alpha^*(x)] \begin{bmatrix} 2D_2(x,y) & D_1(x,y) \\ D_1(x,y) & D_0(x,y) \end{bmatrix} \begin{bmatrix} b(y) \\ \alpha(y) \end{bmatrix} \geq 0, \tag{C.2}$$

which should be positive for any position dependent vectors $b_\mu^i(x)$ and $a_\alpha(x)$ [4]. Using the same methods as for the derivation of the unraveling in the case of continuous classical-quantum dynamics but replacing derivatives by functional derivatives, Equation (C.1) can be unraveled by the coupled stochastic differential equations

$$
\begin{aligned}
&d\rho_t = \mathcal{L}(Z_t)(\rho_t)dt + \int dxdy \big[D^{\alpha 0}_{1,j}(Z_t;x)\sigma^{-1}_{ij}(Z_t;x,y)(L_\alpha(x) - \langle L_\alpha(x)\rangle)\rho_t dW_i(y) \\
&+ D^{0\alpha}_{1,j}(Z_t;x)\sigma^{-1}_{ij}(Z_t;x,y)\rho_t(L^\dagger_\alpha(x) - \langle L^\dagger_\alpha(x)\rangle)dW_i(y)\big] \\
&dZ_{t,i}(x) = (D^{00}_{1,i}(Z_t;x) + \langle D^{\alpha 0}_{1,i}(Z_t;x)L_\alpha(x) + D^{0\alpha}_{1,i}(Z_t;x)L^\dagger_\alpha(x)\rangle)dt + \int dy\sigma_{ij}(Z_t;x,y)dW_j(y).
\end{aligned}
\tag{C.3}
$$

Where now $W_i(x)$ is a spatially dependent Wiener process satisfying

$$
\mathbb{E}[W_i(x)] = 0, \quad \mathbb{E}[dW_i(x)dW_j(y)] = \delta_{ij}\delta(x,y)dt, \tag{C.4}
$$

and we have used the notation $\mathcal{L}(Z)(\rho)$ as shorthand for the pure Lindbladian term appearing in Equation (C.1). In (C.3) $\sigma^{-1}(x,y)$ denotes the generalized kernel inverse, so that $\int dydz\ \sigma(x,y)\sigma^{-1}(y,z)\sigma(z,w) = \sigma(z,w)$. The equations will be local if one picks $\sigma(x,y) \sim \delta(x,y)$, but we can also allow for the more general case. In Equation (C.4) $\frac{dW_i(x)}{dt}$ is a white noise process, and as a result, the solutions to the dynamics will, in general, require regularization. Studying this is beyond the scope of the current work. However, a promising approach would be to study the renormalization properties of classical-quantum path integrals in Chapters 6 and 8.

C.1 A gravitational CQ theory example

As an example, we can use the theory of [28] to formally write down dynamics that agree with the ADM equations of motion on expectation for minimally coupled matter (we consider the Newtonian limit of this theory elsewhere [9, 4]). The idea of [28] was to assume that the dynamics are approximately Einstein's gravity in the classical limit. Here we see that this fixes the drift terms so that the Hamiltonian form of the semi-classical Einstein's equations (1.1) are obeyed on average, and we arrive at a general form for the evolution of the 3-metric γ_{ij} and its conjugate momenta π^{ij}

$$\begin{aligned}
&d\gamma_{ij} = \{\gamma_{ij}, H_{ADM}[N,\vec{N}]\}dt,\\
&d\pi^{ij} = \{\pi^{ij}, H_{ADM}[N,\vec{N}]\}dt - \langle\frac{\delta \hat{H}^{(m)}[N,\vec{N}]}{\delta\gamma_{ij}}\rangle dt + \int dy \sigma^{ij}_{kl}[\gamma,\pi;x,y]dW_{kl}(x)\\
&d\rho_t = -i[\hat{H}^{(m)}[N,\vec{N}],\rho_t] - \frac{1}{2}\int dxdy D_0^{ij;kl}[\pi;x,y][\frac{\partial \hat{H}^{(m)}[N,\vec{N}]}{\delta\gamma_{ij}(x)},[\frac{\partial H_m}{\delta\gamma_{kl}(y)},\rho_t]]dt,\\
&+\frac{1}{2}\int dy(\sigma^{-1})^{kl}_{ij}[\gamma,\pi;x,y](\frac{\delta \hat{H}^{(m)}[N,\vec{N}]}{\delta\gamma_{ij}(y)} - \langle\frac{\delta \hat{H}^{(m)}[N,\vec{N}]}{\delta\gamma_{ij}(y)}\rangle)\rho_t dW_{kl}(y)\\
&+\frac{1}{2}\int dy(\sigma^{-1})^{kl}_{ij}[\gamma,\pi;x,y]\rho_t(\frac{\delta \hat{H}^{(m)}[N,\vec{N}]}{\delta\gamma_{ij}(y)} - \langle\frac{\delta \hat{H}^{(m)}[N,\vec{N}]}{\delta\gamma_{ij}(y)}\rangle)dW_{kl}(y).
\end{aligned} \tag{C.5}$$

We obtain the semi-classical equation (5.36) when the dynamics are taken to be ultra-local, $\sigma \sim \delta(x,y)$, where we use equations of motion to invert $\pi_{ij}[\dot{\gamma}]$ and obtain the expression for G_{ij}. Equation (5.36) is sourced by a white noise term, since $G_{ij} \sim \ddot{\gamma}_{ij} \sim \frac{dW_{kl}}{dt}$ which is a white noise process. In Equation (C.5), we can also consider the case where the lapse and shift N, N^i and their conjugate momenta are included as phase space degrees of freedom. While adding them does nothing in the purely classical case when the constraints are satisfied, it does have some advantages concerning the weak field limit [9] and the constraint algebra [3] of the CQ theory. In this case, one has additional diffusion and Lindbladian terms. We could also add a term that describes any information loss, classical or quantum, not due to the decoherence diffusion trade-off, but we have omitted such terms.

The dynamics will generally depend on the lapse and shift functions N, N^i as in the Hamiltonian formulation of general relativity. On each realization of the noise process, we now have entire trajectories for each of the variables $(\gamma_{ij}, \pi^{ij}, N, N^i)$ each associated with a quantum state, $\rho(t|\gamma_{ij}, \pi^{ij}, N, N^i)$. This allows us to define a tuple $(g_{\mu\nu}, \rho_{\Sigma_t}(t))$ via the ADM embedding

$$\begin{aligned}
&g_{\mu\nu}dx^\mu dx^\nu = -N^2(t,x)dt^2\\
&+\gamma_{ij}(t,x)(N^i(t,x)dt + dx^i)(N^j(t,x)dt + dx^j).
\end{aligned} \tag{C.6}$$

This associates to each trajectory a *4-metric* and quantum state on a 1-parameter family of hypersurfaces Σ. The unraveling provides a method for studying the dynamics of classical gravity interacting with quantum matter.

In the gravitational context, one also expects that one should consider theories that retain diffeomorphism symmetry. Treated as an effective theory, we expect that the coefficients

D_0, D_1, D_2 entering into the dynamics result from integrating out high energy modes of dynamical fields. As such, we should demand that an effective theory be diffeomorphism *covariant*, meaning that a solution to the dynamics in one frame should be a solution to the dynamics in any other frame, where we should transform the parameters entering into the master equation by hand since they arise from hidden dynamical degrees of freedom.

On the other hand, if there are no degrees of freedom that have been integrated out, as would be the case in a fundamental theory of classical-quantum gravity, it is natural to impose diffeomorphism invariance on the dynamics and the coefficients entering the master equation should be constructed out of the gravitational degrees of freedom alone. It remains to be seen if such dynamics can be made diffeomorphism invariant and give rise to full general relativity with preserved constraints. However, we progressed towards this in Chapter 8. Such dynamics could, in principle, be taken as fundamental, and we leave it as an open question of whether or not such dynamics can be made diffeomorphism invariant with a full classical-quantum constraint surface.

Appendix D

Perturbative methods for CQ path integrals

In this appendix, we study a simple model of CQ interaction to illustrate how one can use standard perturbative methods to calculate classical-quantum correlation functions via CQ Feynman diagrams.

In Chapters 6 and 8, we considered the path integral, which constructs a CQ state at a time t_f from a CQ state at time t_i. In computing correlation functions of classical-quantum observables, the final state is not important, so we can perform a t_f integral over the classical-quantum fields to arrive at the partition function

$$Z = \int dq_f \mathrm{Tr}\left[\varrho(q_f, t_f)\right] \tag{D.1}$$

which for the configuration space path integral takes the form

$$Z_0 = \mathcal{N} \int \mathcal{D}\phi^- \mathcal{D}\phi^+ \mathcal{D}q \; e^{\mathcal{I}(\phi^-, \phi^+, q, t_i, t_f)} \varrho(q_i, \phi_i^+, \phi_i^-, t_i), \tag{D.2}$$

where now there are no final boundary conditions imposed on the path integral.

Formally, we can calculate correlation functions by inserting sources J^+, J^-, J_q into the path integral and taking functional derivatives with respect to the sources. The partition function of interest is, therefore

$$Z[J^+, J^-, J_q] = \mathcal{N} \int \mathcal{D}\phi^- \mathcal{D}\phi^+ \mathcal{D}q \; e^{\mathcal{I}(\phi^-, \phi^+, q, t_i, t_f) - iJ_+\phi^+ + iJ^-\phi^- - J_q q} \varrho(q_i, \phi_i^+, \phi_i^-, t_i). \tag{D.3}$$

In general, the form of the path integral depends on the initial CQ state $\varrho(q_i, \phi_i^+, \phi_i^-, t_i)$ and any calculation of correlation function must be performed on a case-by-case basis depending on the initial state.

However, often we are interested in stationary states, and we would like to obtain information on correlation functions over arbitrary long times by taking the limit $t_i \to -\infty, t_f \to \infty$. In open systems, as well as when calculating scattering amplitudes, it is often assumed that the initial state in the infinite past does not affect the stationary state of the system so that there is a complete loss of memory of the initial state [97]. Under this assumption, it is possible to ignore the boundary term containing the initial CQ state $\varrho(q_i, \phi_i^+, \phi_i^-, t_i)$ and we arrive at the partition function

$$Z[J^+, J^-, J_q] = \mathcal{N} \int \mathcal{D}\phi^- \mathcal{D}\phi^+ \mathcal{D}q \; e^{\mathcal{I}(\phi^-, \phi^+, q, -\infty, \infty) - iJ_+\phi^+ + iJ^-\phi^- - J_q q}. \tag{D.4}$$

Using equation (D.4), we can then use standard perturbation methods for computing correlation functions in CQ theories.

As a simple example, consider the zero-dimensional CQ theory with CQ proto-action

$$W_{CQ} = S_Q - \frac{m_q^2 q^2}{2} - \frac{\lambda q^2 \phi^2}{2}, \tag{D.5}$$

and a pure quantum action given by $S_Q = -\frac{m_\phi^2 \phi^2}{2}$. Assuming the decoherence diffusion trade-off is saturated, we arrive at the total action

$$\mathcal{I}[\phi^\pm, q] = -\frac{i}{\hbar}\frac{m_\phi^2(\phi^+)^2}{2} + \frac{i}{\hbar}\frac{m_\phi^2(\phi^-)^2}{2} - \frac{1}{4D_2}\left(q^2 m_q^4 + \frac{1}{2}\lambda^2 q^2((\phi^+)^4 + (\phi^-)^4) + \frac{1}{2}\lambda q m_q^2((\phi^+)^2 + (\phi^-)^2)\right). \tag{D.6}$$

We see from Equation (D.6) that D_2 in an interacting CQ theory plays the same role as $\hbar$ in an interacting quantum theory. To compute correlation functions, we can therefore work perturbatively in D_2. Note, the double limit $D_2 \to 0, D_2^{-1}\lambda \to 0$ defines a deterministic quantum theory with no classical back-reaction.

We define the free theory as the action independent of any CQ back-reaction

$$I_{free} = iS^+ - iS^- - I_C = -i\frac{m_\phi^2(\phi^+)^2}{2\hbar} + i\frac{m_\phi^2(\phi^-)^2}{2\hbar} - \frac{1}{4D_2}q^2 m_q^4. \tag{D.7}$$

Inserting sources, we find the partition function

$$Z_{free}[J_+, J_-, J_q] = \int d\phi^\pm dq e^{I_{free} - iJ_+\phi^+ + iJ_-\phi^- - J_q q}, \tag{D.8}$$

which can be performed exactly by performing each Gaussian integral individually

$$Z_{free}[J_+,J_-,J_q] = \left(\int d\phi^+ e^{-i\frac{m_\phi^2(\phi^+)^2}{2\hbar} - iJ_+\phi^+}\right)\left(\int d\phi^- e^{+i\frac{m_\phi^2(\phi^-)^2}{2\hbar} + iJ_-\phi^-}\right)\left(\int dq e^{-\frac{1}{4D_2}q^2 m_q^4 - J_q q}\right). \tag{D.9}$$

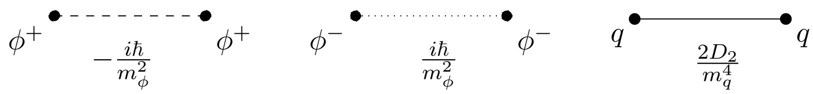
Equation (D.9) is evaluated as

$$Z_{free}[J_+,J_-,J_q] = \left(\frac{-2\pi i\hbar}{m_\phi^2}\right)e^{\frac{iJ_+^2}{2\hbar m_\phi^2}}\left(\frac{2\pi i\hbar}{m_\phi^2}\right)e^{\frac{-iJ_-^2}{2\hbar m_\phi^2}}\left(\frac{\pi 2D_2}{m_q^4}\right)e^{\frac{J_q^2}{4D_2 m_q^4}} = Z_0 e^{\frac{iJ_+^2}{2\hbar m_\phi^2}} e^{\frac{-iJ_-^2}{2\hbar m_\phi^2}} e^{\frac{J_q^2}{4D_2 m_q^4}}. \tag{D.10}$$

From Equation (D.10), we can then define the propagators for the free theory

$$\langle \phi^+\phi^+\rangle = -\frac{i\hbar}{m_\phi^2},\ \langle \phi^-\phi^-\rangle = \frac{i\hbar}{m_\phi^2},\ \langle qq\rangle = \frac{2D_2}{m_q^4}, \tag{D.11}$$

and we can represent each of the propagators by the following Feynman diagrams



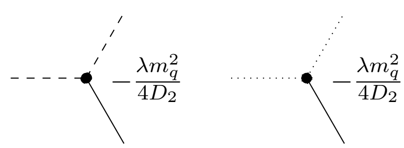
(D.12)

The full partition function with the CQ interaction turned on then takes the form

$$Z[J_+,J_-,J_q] = \langle e^{\mathcal{I}_{CQ}}\rangle = \langle e^{-\frac{1}{4D_2}\left(\frac{1}{2}\lambda^2 q^2((\phi^+)^4+(\phi^-)^4)+\frac{1}{2}\lambda q m_q^2((\phi^+)^2+(\phi^-)^2)\right)}\rangle \tag{D.13}$$

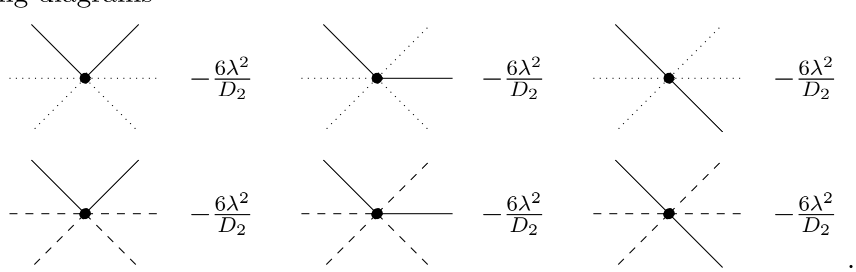
and we can perform an asymptotic expansion of the CQ interaction in terms of D_2 to arrive at the usual Feynman rules for computing correlation functions. Specifically, for terms in the action like $\lambda_{nml}\phi_+^n\phi_-^m q^l$, we assign the vertex with value $\lambda_{nml}n!m!l!$ to each topologically distinct diagram.

As an example, the CQ interaction term $q(\phi^\pm)^2$ in Equation (D.6) has two tri-vertices with strength $-\frac{2!}{8D_2}\lambda m_q^2$ and can be represented by the diagrams



(D.14)

We also have the sextic $q^2(\phi^\pm)^4$ interaction with vertex value $-\frac{\lambda^2 4!2!}{8D_2}$ which is assigned to each of the following diagrams



(D.15)

Appendix E

Complete postivity of classical-quantum path integrals

In this appendix, we prove the statement made in Chapter 8 that Equation (8.4) defines completely positive CQ dynamics and that the dynamics defined by Equation (8.20) takes the form of Equation (8.4) and is hence completely positive. We also show that normalization of the path integral occurs via the inclusion of appropriate classical and quantum kinetic terms in Section E.3.

E.1 Proof of complete positivty

The proof of complete positivity of Equation (8.9) is almost immediate from an expansion of the path integral. To see this in detail, we can perform a short-time expansion of the full path integral, which we can always do since we assume the dynamics are time-local.

Let us first consider the case where the quantum state remains pure, so that $c^{\gamma} = 0$ in Equation (8.9). Defining $t_f = t_0 + K\delta t$, $t_i = t_0 + i\delta t$, and discretizing the path integral into steps of size δt we have that

$$\varrho_{i+1} = \int d\phi_i^+ d\phi_i^- dz_i (e^{\mathcal{I}^+_{i+1,i}})(e^{\mathcal{I}^-_{i+1,i}})^* e^{-\mathcal{I}_{C,i+1,i}} \varrho_i, \tag{E.1}$$

where we use the shorthand $\varrho_i = \varrho(\phi_i^+, \phi_i^-, z_i, t_i)$, $\mathcal{I}_{i+1,i} = \mathcal{I}[\phi_{i+1}, \phi_i, z_{i+1}, z_i]$ and $\mathcal{I}_{C,i+1,i} = \mathcal{I}_C[z_{i+1}, z_i]$.

More generally, we can allow for the case where the action contains higher time derivatives, in which case we have $\mathcal{I}[\phi_{i+k_q},\dots,\phi_i,z_{i+k_c},\dots,z_i]$ and $\mathcal{I}_C[z_{i+k_c},z_i]$ with $k_c,k_q\geq 2$. In order to retain the usual composition law for the path integral, we must also let increasingly higher derivative terms describe the state $\varrho(\phi_i^{\pm},\dots,\frac{d^{k_q-1}\phi_i^{\pm}}{dt^{k_q-1}},z_i,\frac{d^{k_c-1}z_i}{dt^{k_c-1}})$. [200]. The final state then imposes boundary conditions on the components of the action, which contain higher derivative terms so that Equation (E.1) is still well defined.

With this in mind, we can take the trace with respect to an arbitrary vector $|v(q)\rangle$, and for complete positivity, we need to show

$$\int d\phi_{i+1}^{+}d\phi_{i+1}^{-}v_{i+1}^{+*}\varrho_{i+1}v_{i+1}^{-}\geq 0. \tag{E.2}$$

Denoting $\tilde{v}_{i+1,i}^{+}=e^{\mathcal{I}_{i+1,i}^{+}}v_{i+1}^{+*}$, then inserting Equation (E.1) into Equation (E.2) we have

$$\int d\phi_{i+1}^{+}d\phi_{i+1}^{-}d\phi_i^{+}d\phi_i^{-}dz_i\tilde{v}_{i+1,i}^{+}\tilde{v}_{i+1,i}^{-*}e^{-\mathcal{I}_{C,i+1,i}}\varrho_i. \tag{E.3}$$

Because the integral factorizes into $\pm$ conjugates, Equation (E.3) will always be positive. To see this explicitly, we first perform the $\phi_i^{\pm}$ integrals to obtain

$$c_{i+1}=\int d\phi_i^{+}d\phi_i^{-}\tilde{v}_{i+1,i}^{+}\tilde{v}_{i+1,i}^{-*}\varrho.\geq 0, \tag{E.4}$$

where we have used the positivity of the state CQ $\varrho(\phi_i^{+},\phi_i^{-},q_i,t_i)$. What remains is the integral

$$\int d\phi_{i+1}^{+}d\phi_{i+1}^{-}dz_ie^{-\mathcal{I}_{C,i+1,i}}c_{i+1}\geq 0, \tag{E.5}$$

which is positive since both c_{i+1} and the exponential are both positive. In Equation (E.5), there is still a free z_{i+1} variable, which corresponds to the fact that positivity of the CQ state demands that the CQ dynamics keep quantum states positive conditioned on the classical degrees of freedom. We thus see that the state after applying the time-evolved state will also be positive. Hence the dynamics are positive. When we consider the dynamics as part of a larger system, we apply the identity map to the larger system. The dynamics still factorize this way - we perform a delta function path integral on the auxiliary system. Hence, Equation (6.67) defines completely positive dynamics.

In the more general case, we can have non-zero $c^{\gamma}[z,x]$, and there is information loss since the dynamics can send pure states to mixed states. In this case, the only thing which changes is the definition of $\tilde{v}^{+}_{i+1,i}$ in Equation (E.2). In particular, in the general case, we must also expand out the terms involving c^{γ}_{n} in the action of Equation (8.4)

$$e^{\delta t\sum_{\gamma}c^{\gamma}_{i+1,i}(L^{+}_{\gamma})_{i+1,i}(L^{-}_{\gamma})^{*}_{i+1,i}} = 1+\delta t\sum_{\gamma}c^{\gamma}_{i+1,i}(L^{+}_{\gamma})_{i+1,i}(L^{-}_{\gamma})^{*}_{i+1,i}+\dots, \tag{E.6}$$

where $c^{\gamma}_{i+1,i}=c^{\gamma}[q_{i+1},q_i]$, $L^{+}_{i+1,i}=L^{+}[\phi^{+}_{i+1},\phi^{+}_{i}]$ and similarly for the $^{-}$ branch.

With this in mind, the integrand of the path integral in Equation (6.67) factorizes according to Equation (E.3), and the steps to prove complete positivity are the same but with

$$\tilde{v}^{\gamma+}_{i+1,i}=(\sqrt{\delta t c^{\gamma}_{i+1,i}}(L^{+}_{\gamma})_{i+1,i})e^{\mathcal{I}^{+}_{i+1,i}}v^{+*}_{i+1}, \tag{E.7}$$

from which the complete positivity of the dynamics follows from the same arguments outlined in Equation's (E.3) and (E.4), where we now also sum over γ. Note, though we need only work to first order in δt, had we included them, the higher order δt terms also factorize similarly.

In the field-theoretic case, the total CQ action is

$$\begin{aligned}\mathcal{I}(\phi^{-},\phi^{+},q,t_i,t_f) &= \mathcal{I}_{CQ}(q,\phi^{+},t_i,t_f)+\mathcal{I}^{*}_{CQ}(q,\phi^{-},t_i,t_f)-\mathcal{I}_{C}(q,t_i,t_f)\\ &\quad+\int_{t_i}^{t_f}dx\sum_{\gamma}c^{\gamma}(q,t,x)(L_{\gamma}[\phi^{+}](x)L^{*}_{\gamma}[\phi^{-}](x))\end{aligned} \tag{E.8}$$

and we can repeat the argument for complete positivity, which again follows from the factorization of the path integral integrand. In this case, complete positivity follows from the fact that

$$\begin{aligned}&\int\mathcal{D}\phi^{+}_{i+1}\mathcal{D}\phi^{-}_{i+1}\mathcal{D}\phi^{+}_{i}\mathcal{D}\phi^{-}_{i}\mathcal{D}z_i\,\times\\ &\sum_{\gamma}\int d\vec{x}c^{\gamma}_{i+1,i}(x)\big(v^{+}_{i+1,i}(L^{+}_{\gamma})_{i+1,i}(x)e^{\mathcal{I}^{+}_{i+1,i}}\big)\big(L^{-}_{\gamma})_{i+1,i}(x)v^{-}_{i+1,i}e^{\mathcal{I}^{-}_{i+1,i}}\big)^{*}e^{-\mathcal{I}_{C,i+1,i}}\varrho_i,\end{aligned} \tag{E.9}$$

is positive when $c^{\gamma}\geq 0$ and ρ_i is a positive density matrix.

E.2 Showing the natural class of CQ dynamics is CP

In this section, we show that the dynamics defined by Equation (8.20) takes the form of Equation (8.4) and is hence completely positive. Since the purely quantum Lagrangian terms appearing in Equation (8.4) are manifestly CP, we shall focus on the CQ interaction term

$$-\frac{1}{2}\int dx\Delta X^{i}(q,x)D_{0,ij}(q,x)\Delta X^{j}(x)-\frac{1}{4}\int dx\bar{X}^{i}(q,x)D^{-1}_{2,ij}(q,x)\bar{X}^{j}(q,x), \tag{E.10}$$

where we use the shorthand notation $X^i(q,x) = \frac{\delta W_{CQ}}{\delta q_i(x)}$, which we assume is Hermitian since it is generated by a real proto-action W_{CQ}. For ease of presentation, we will here suppress any potential q, x dependence from X^i, D_0, D_2, but these can be added back in.

Expanding Equation (E.10), we can group terms according to D_0, D_2^{-1} as

$$\begin{aligned}&-\frac{1}{16}\int dxd\vec{y}(8D_{0,ij}+D_{2,ij}^{-1})((X^+)^i(X^+)^j+(X^-)^i(X^-)^j)\\&+\frac{1}{16}\int dx(8D_{0,ij}-D_{2,ij}^{-1})((X^+)^i(X^-)^j+(X^-)^i(X^+)^j).\end{aligned} \tag{E.11}$$

We see that the first line in Equation (E.11) is of the form $I^+_{CQ} + (I^-_{CQ})^*$ and so adheres to the form in Equation (8.4). If the trade-off is saturated, this completes the proof that (8.20) takes the form of Equation (8.4). When not saturated, we can write $8D_0 - D_2 = c \succeq 0$, where $c_{ij}(q,x)$ is a real, symmetric positive semi-definite matrix. We can then expand the second line of Equation (E.11) as

$$\frac{1}{16}\int dx\; c_{ij}(q,x)((X^+)^i(X^-)^j+(X^-)^i(X^+)^j), \tag{E.12}$$

which, after diagonalizing c_{ij}, takes the form of Equation (E.8), and hence defines CP dynamics whenever the condition $8D_0 - D_2 = c \succeq 0$ is satisfied.

When the trade-off $8D_0 - D_2 = c \succeq 0$ is saturated, i.e., when $c = 0$, we can reproduce the full action for the gravity theory of Equation (8.33)

$$\begin{aligned}\mathcal{I}[\phi^-,\phi^+,g_{\mu\nu}] = \int dx\Big[&i\mathcal{L}^+_{KG} - i\mathcal{L}^-_{KG} - \frac{\det(-g)}{8}(T^{\mu\nu+}-T^{\mu\nu-})D_{0,\mu\nu\rho\sigma}(T^{\rho\sigma+}-T^{\rho\sigma-})\\&-\frac{\det(-g)}{128\pi^2}(G^{\mu\nu}-\frac{1}{2}(8\pi(T^{\mu\nu})^+ + 8\pi(T^{\mu\nu})^-)D_{0,\mu\nu\rho\sigma}[g](G^{\rho\sigma}-\frac{1}{2}(8\pi(T^{\rho\sigma})^+ + 8\pi(T^{\rho\sigma})^-)\Big],\end{aligned} \tag{E.13}$$

which we have now verified takes the form of Equation (8.4).

E.3 Ensuring the CQ path integral is normalized

In this section, we show that the CQ action defined by Equation's (8.22) and (8.21) are normalized.

To see the problem of normalization of higher derivative path integrals in more detail, we will review how the normalization of quantum states occurs in Lindbladian path integrals with a Feynman-Vernon action [79], and how probabilities are conserved in higher-order classical path integrals. Let us first consider higher-order classical path integrals. We refer the reader to Chapter 6 for a complete derivation of normalized CQ path integrals from master equations.

E.3.1 Normalization of higher derivative classical path integrals

When considering a classical path integral that contains higher derivatives, we should treat $q, \dot{q}$ as independent variables. This is outlined in detail in [200]. To that end, we will show how the normalization of the path integral

$$p(q_f, \dot{q}_f, t_f) = \int^B \mathcal{D}q e^{-\int_{t_i}^{t_f} dt[\ddot{q}-f(\dot{q},q)]^2} p(q_i, t_i) \tag{E.14}$$

occurs. In Equation (E.14), note that the boundary conditions are given by $B = \{q(t_f) = q_f, \dot{q}(t_f) = \dot{q}_f\}$, which involve both q and $\dot{q}$.

To check normalization, we consider Equation (E.14) for small δt, with $t_n = \delta t + t_{n-1}$

$$p(q_{n+1}, q_{n+2}, t_{n+1}) = \int dq_n e^{-\delta t[\frac{q_{n+2}-2q_n+q_{n+1}}{\delta t} - f(q_{n+1}, q_n)]^2} p(q_n, q_{n+1}, t_n). \tag{E.15}$$

The norm of the probability distribution is found by performing the integral over the final variables q_{n+1}, q_{n+2}

$$\int dq_n dq_{n+1} dq_{n+1} e^{-\delta t[\frac{q_{n+2}-2q_n+q_{n+1}}{\delta t} - f(q_{n+1}, q_n)]^2} \times p(q_n, q_{n+1}, t_n). \tag{E.16}$$

Equation (E.16) defines a standard Gaussian integral over the q_{n+2} coordinate. Hence, the q_{n+2} integral eats the action up to a Gaussian normalization factor that we can calculate exactly, and we are left with

$$1 = \int dq_n dq_{n+1} N p(q_n, q_{n+1}, t_n), \tag{E.17}$$

so we can simply absorb N into the measure, and the path integral will be normalized. If we were to include a q dependent diffusion coefficient $D_2(q, \dot{q})$ in Equation (E.14), then the Gaussian integral will be q dependent, and this will need to be included in the measure for $\mathcal{D}q$. One can also re-exponentiate this term via a Faddeev-Popov action, as in Equation (8.23). The message is that the higher derivative terms in the classical path integral are standard Gaussian integrals if we consider q and $\dot{q}$ as independent variables. Hence, the classical contribution to the path integral defined by Equation (8.20), which also involves the $\pm$ bra-ket branch average, will be normalized in a similar way. We show this explicitly in Section E.3.3

E.3.2 Normalization of higher derivative Feynman-Vernon path integrals

Let us now consider a Feynman-Vernon quantum path integral with a decoherence term. Consider first the path integral for a quantum state σ

$$\sigma(\phi_f^+, \phi_f^-, t_f) = \int^B \mathcal{D}\phi^+ \mathcal{D}\phi^- e^{\int_{t_i}^{t_f} dt i[\dot{\phi_+}^2 + V(\phi_+)] - i[\dot{\phi_-}^2 + V(\phi_-)] - \frac{D_0}{2}(L(\phi_+) - L(\phi_-))^2} \sigma(\phi_i^+, \phi_i^-, t_i), \tag{E.18}$$

where B imposes the final state boundary conditions on the bra and ket fields, and $L(\phi)$ is an arbitrary operator of ϕ but not of its derivatives.

For Equation (E.18), it will prove insightful to show how the kinetic term enforces the normalization of the quantum state. To that end, consider the short time version of Equation (E.19)

$$\begin{aligned}\sigma(\phi_{n+1}^+, \phi_{n+1}^-, t_f) = \int d\phi_n^+ d\phi_n^- e^{\delta t[i(\frac{\phi_{n+1}^+ - \phi_n^+}{\delta t})^2 + iV(\phi_n^+) - i(\frac{\phi_{n+1}^- - \phi_n^-}{\delta t})^2 - iV(\phi_n^-)]} \\ \times e^{-\delta t \frac{D_0}{2}(L(\phi_n^+) - L(\phi_n^-))^2} \sigma(\phi_n^+, \phi_n^-, t_n).\end{aligned} \tag{E.19}$$

The trace of the quantum state is found by matching the $\phi_{n+1}^+ = \phi_{n+1}^- = \phi$ fields and integrating over ϕ

$$\int d\phi_n^+ d\phi_n^- d\phi e^{\frac{i}{\delta t}\phi(\phi_n^+ - \phi_n^-)} e^{i\delta t[V(\phi_n^+) - iV(\phi_n^-)]} e^{-\delta t \frac{D_0}{2}(L(\phi_n^+) - L(\phi_n^-))^2} \sigma(\phi_n^+, \phi_n^-, t_n). \tag{E.20}$$

Performing the integration over ϕ gives rise to a delta function $\delta(\phi_n^+ - \phi_n^-)$. Hence, the quantum state is normalized to constant factors that can be absorbed.

However, had we included higher-order kinetic terms in the decoherence sector, we would not have found this normalization. In particular, if the decoherence term was instead

$$\int_{t_i}^{t_f} dt \frac{1}{2} D_0 (\dot{\phi}_+^2 - \dot{\phi}_-^2)^2, \tag{E.21}$$

then the delta function integral is not imposed, and the state is no longer normalized to constant factors.

As such, it seems inevitable that for the path integral to be normalized with higher derivative decoherence terms, one needs to also add higher derivative kinetic terms in the action. In this case, the action

$$S = \int_{t_i}^{t_f} dt i[\dot{\phi_+}^2 + \ddot{\phi_+}^2 - V(\phi_-)] - i[\dot{\phi_-}^2 + \ddot{\phi_-}^2 - V(\phi_-)] - \frac{1}{2} D_0 (\dot{\phi}_+^2 - \dot{\phi}_-^2)^2 \tag{E.22}$$

is normalized up to constant factors by the same argument, so long as we treat ϕ and $\dot{\phi}$ as independent variables to be specified in the quantum state; this is also argued for independent reasons in [200]. To see this, one does the short time expansion, treating ϕ and $\dot{\phi}$ as independent variables as in the higher derivative classical path integral. Computing the trace then sets $\phi^{\pm}_{n+2}$ equal to each other, as well as the setting the $\phi^{\pm}_{n+1}$ fields equal to be the same. The ϕ_{n+2} integral then enforces a delta function over $\delta(\phi^+_n - \phi^-_n)$, which kills the decoherence term and means that the path integral is normalized up to constant factors.

E.3.3 Normalization of CQ path integrals

In this section, we show that Equation's (8.21) give rise to normalized CQ dynamics (8.22). The proofs will follow in the same way as the discussion of normalization of classical and quantum path integrals.

Let us start with the higher derivative case. We show that any CQ path integral with action

$$
\begin{aligned}
I[q,\phi^+,\phi^-] = \int dt i\ddot{\phi}_+^2 + iV(\phi^+,\dot{\phi}^+) - i\ddot{\phi}_-^2 - iV(\phi^-,\dot{\phi}^-) \\
- \frac{D_0(q,\dot{q},\phi^+,\dot{\phi^+})}{2}(\ddot{q} + f(q,\dot{q},\phi^+,\dot{\phi}^+))^2 - \frac{D_0(q,\dot{q},\phi^-,\dot{\phi^-})}{2}(\ddot{q} + f(q,\dot{q},\phi^-,\dot{\phi}^-))^2
\end{aligned}
\tag{E.23}
$$

is normalized up-to constant factors when $D_0 > 0$. In the case where D_0 has a functional dependence on the fields one must make sure to also include a factor of $\sqrt{\det(D_0(q,\dot{q},\phi)}$ in the path integral measure. Equation (E.23) is a generic type of action one gets from varying Equation (8.20) with a CQ proto action that has second order equations of motion for the classical degree of freedom.

The steps in showing Equation (E.23) follow in the same way as the discussions of classical and quantum path integrals. Firstly, because the action is higher derivative, the CQ state is specified through $\varrho(q,\dot{q},\phi^{\pm},\dot{\phi}^{\pm})$.

Taking the trace at the $t_{n+1} = t_n + \delta t$ time-step therefore involves identifying $\phi^+_{n+2} = \phi^-_{n+2} = \phi_{n+2}$ and $\phi^+_{n+1} = \phi^-_{n+1} = \phi_{n+1}$. We then integrate over the ϕ_{n+2} and ϕ_{n+1} variables, as well as over the q_{n+2}, q_{n+1} classical degrees of freedom.

Let us first look at the higher derivative quantum kinetic term. This can be expanded as

$$\begin{aligned}\ddot{\phi}_+^2 - \ddot{\phi}_-^2 &\sim (\phi_{n+2} - 2\phi_n^+ + \phi_{n+1})^2 - (\phi_{n+2} - 2\phi_n^- + \phi_{n+1})^2 \\ &= 4\big[(\phi_n^+)^2 - (\phi_n^-)^2 + \phi_{n+2}(\phi_n^- - \phi_n^+) + \phi_{n+1}(\phi_n^- - \phi_n^+)\big].\end{aligned} \tag{E.24}$$

Hence, integrating over ϕ_{n+2} gives a delta function in $\delta(\phi_n^- - \phi_n^+)$. As a consequence of this, all the bra and ket fields in the path integral are identified. We are therefore left with the action

$$I'[q,\phi] = -\int dt D_0(q,\dot{q},\phi,\dot{\phi})(\ddot{q} + f(q,\dot{q},\phi,\dot{\phi}))^2. \tag{E.25}$$

Since all the bra and ket quantum fields are identified, normalization of Equation (E.23) is equivalent to ensuring that Equation (E.25) is normalized.

As we saw for the classical path integrals, integrating Equation (E.25) over the $\ddot{q}$ at second time step implements a standard Gaussian integral. If D_0 is dependent on the fields, we therefore pick up a term $(\sqrt{\det(D_0(q,\dot{q},\phi)})^{-1/2}$, which we must cancel in the measure by including a $\sqrt{\det(D_0(q,\dot{q},\phi)}$ term. It can also be exponentiated into the action by introducing Bosonic and Fermionic Faddeev-Poppov fields [201]. This term commonly arises in the study of Fokker-Plank type equations when the noise is multiplicative [153, 161, 201]. With this in mind, once we have integrated over $\ddot{q}$, the action vanishes and we are left with the normalization of the initial CQ state. Hence the path integral preserves the normalization of CQ states.

In a similar manner, we can also show that the path integral of Equation (8.21) is also normalized

$$\begin{aligned}I[q,\phi^+,\phi^-] = \int dt i\dot{\phi}_+^2 + iV(\phi^+) - i\dot{\phi}_-^2 - iV(\phi^-) - \frac{D_0(q,\dot{q},\phi^+)}{2}(\ddot{q} + f(q,\dot{q},\phi^+))^2 \\ - \frac{D_0(q,\dot{q},\phi^-)}{2}(\ddot{q} + f(q,\dot{q},\phi^-))^2,\end{aligned} \tag{E.26}$$

where $D_0 > 0$. To see this, we first take the trace of the system, setting $\phi_{n+1}^+ = \phi_{n+1}^- = \phi$. Integrating over ϕ then enforces a delta function $\delta(\phi^+ - \phi^-)$. We are then left with the action

$$I'[q,\phi] = -\int dt D_0(q,\dot{q},\phi)(\ddot{q} + f(q,\dot{q},\phi))^2, \tag{E.27}$$

and we can again perform the Gaussian integral over $\ddot{q}$ to arrive at a normalized path integral if $\sqrt{\det(D_0(q,\dot{q},\phi)}$ is included in the measure.

Appendix F

The trade-off between decoherence and diffusion coupling constants

This appendix will introduce the positivity conditions used to prove the decoherence diffusion trade-off of chapter 10.

Recall from the definition of positivity of the transition amplitude that $\Lambda^{\mu\nu}(z+\delta t|z't)$ must satisfy

$$\int dz A^*_\mu(z,z')\Lambda^{\mu\nu}(z+\delta t|z',t)A_\nu(z,z') \geq 0, \tag{F.1}$$

for any $A_\mu(z,z')$ for which (F.1) is well defined: i.e. so that the distributional derivatives in (F.1) are well defined.

As a consequence of Equation (F.1) being positive, we also know that

$$\mathrm{Tr}\left[\int dz \Lambda^{\mu\nu}(z+\delta t|z',t)O_\mu(z,z')\rho(z')O^\dagger_\nu(z,z')\right] \geq 0 \tag{F.2}$$

will be positive for *any* vector of operators (potentially phase space dependent) $O_\mu(z,z')$. This follows from the cyclicity of the trace and the fact that $\Lambda^{\mu\nu}(z+\delta t|z',t)O^\dagger_\nu(z,z')O_\mu(z,z')$ will be a positive operator so long as (F.1) holds. A common choice of O_μ would be the Lindblad operators L_μ appearing in the master equation.

The inequality in Equation (F.1) proves useful to derive positivity conditions on the coupling constants appearing in the master equation. In contrast, Equation (F.2) is useful in deriving the observational trade-off for the continuous master equations.

We can get a general trade-off between the decoherence and diffusion coefficients which appear in the master equation, arriving at a trade-off between the decoherence and diffusion coefficients in terms of the back-reaction drift coefficient $D^{\mu\alpha}_{1,i}$.

Consider Equation (F.1), with $A_\mu = \delta^\alpha_\mu a_\alpha + b^i_\mu (z-z')_i$. By integrating by parts over the phase space degrees of freedom, we find

$$2b^{i*}_\mu D^{\mu\nu}_{2,ij} b^j_\nu + b^{i*}_\mu D^{\mu\beta}_{1,i} a_\beta + a^*_\alpha D^{\alpha\mu}_{1,i} b^i_\mu + a^*_\alpha D^{\alpha\beta}_0 a_\beta \geq 0 \tag{F.3}$$

Taking $i \in \{1,\ldots,n\}$ $\alpha \in \{1,\ldots,p\}$ and $\mu \in \{1,\ldots,p+1\}$, we can write this as a matrix positivity condition

$$[b^*, \alpha^*] \begin{bmatrix} 2D_2 & D^{br}_1 \\ D^{br}_1 & D_0 \end{bmatrix} \begin{bmatrix} b \\ \alpha \end{bmatrix} \geq 0 \tag{F.4}$$

where D_2 is the $(p+1)n \times (p+1)n$ matrix with elements $D^{\mu\nu}_{2,ij}$, D^{br}_1 is the $(p+1)n \times p$ matrix with rows labeled by μi and columns labelled by β with elements $D^{\mu\beta}_{1,i}$ and D_0 is the $p \times p$ decoherence matrix with elements $D^{\alpha\beta}_0$. $D^{br}_{1,i}$ describes the quantum back-reacting components of the drift. Equation (F.4) is equivalent to the condition that the $((p+1)n+p) \times ((p+1)n+p)$ matrix

$$\begin{bmatrix} 2D_2 & D^{br}_1 \\ D^{br}_1 & D_0 \end{bmatrix} \succeq 0 \tag{F.5}$$

Since we know D_2 and D_0 must be positive semi-definite, we know from Schur decomposition that

$$2D_2 \succeq D^{br}_1 D^{-1}_0 D^{br\dagger}_1 \tag{F.6}$$

where D_0 is the generalized inverse of D_0. Furthermore, if D_0 vanishes, then clearly D^{br}_1 must vanish for (F.5) to be positive semi-definite.

Appendix G

Classical-quantum dynamics with fields

In this appendix, we describe CQ dynamics in the case where the Lindblad operators and the phase-space degrees of freedom can have spatial dependence $z(x), L_\mu(x)$.

For the case of fields, operators $O(x)$ constructed out of local fields $\phi(x)$ will generally be unbounded; hence, the Stinespring dilation theorem does not hold. This problem is common in the study of algebraic quantum field theory. We can get around it by considering the case in which operators of interest are obtained by smearing the local fields over bounded functionals F. For example, we can first smear the local fields over a smearing function f, $\phi_f = \int dx \phi(x) f(x)$ and then consider bounded functions of ϕ_f such as $F(\phi_f) = e^{i\phi_f}$. In doing this, we can write a CQ version of the Stinespring dilation theorem exactly and proceed along the lines of [28] to show that any completely positive CQ map can be written in the form

$$\rho'(z) = \int dz dx dy \Lambda^{\mu\nu}(z|z'; x, y) L_\mu(x, z, z') \varrho(z') L_\nu^\dagger(y, z, z') \tag{G.1}$$

where the positivity condition states

$$\int dz dx dy A_\mu^*(x, z, z') \Lambda^{\mu\nu}(z|z'; x, y) A_\nu(y, z, z') \geq 0. \tag{G.2}$$

We shall assume that we deal with dynamics which can be written in Lindblad form, as is usually assumed in the unbounded case [264].

G.1 CQ Kramers-Moyal expansion for fields

Just as in Chapter 2, we can formally introduce the moments of the transition amplitude

$$M^{\mu\nu}_{n,i_1\dots i_n}(w_1,\dots w_n;x,y,\delta t)=\int Dz\Lambda^{\mu\nu}(z|z';x,y,\delta t)(z-z')_{i_1}(w_1)\dots(z-z')_{i_n}(w_n) \tag{G.3}$$

which we assume to exist, which might involve the smearing of the operators $z(x)$. Defining $L_0(x)=\delta(x)\mathbb{I}$, we can define the coefficients $D^{\mu\nu}_{n,i_1\dots i_n}$ implicitly via

$$M^{\mu\nu}_{n,i_1\dots i_n}(z',w_1,\dots w_n;x,y,\delta t)=\delta^\mu_0\delta^\nu_0+\delta t n! D^{\mu\nu}_{n,i_1\dots i_n}(w_1,\dots w_n;x,y,\delta t). \tag{G.4}$$

The characteristic function then takes the form

$$C^{\mu\nu}(u,z';x,y)=\int Dz e^{i\int dwu(w)\cdot(z(w)-z'(w))}\Lambda^{\mu\nu}(z|z';x,y) \tag{G.5}$$

and expanding out the exponential, this takes the form

$$C^{\mu\nu}(u,z';x,y)=\sum_{n=0}^{\infty}\int dw_1\dots dw_n\frac{u_i(w_1)\dots u_{i_n}(w_n)}{n!}M^{\mu\nu}_{n,i_1\dots i_n}(z',w_1,\dots w_n;x,y,\delta t) \tag{G.6}$$

performing the inverse Fourier transform allows us to write the transition amplitude in terms of functional derivatives of the delta function

$$\Lambda^{\mu\nu}(z|z';x,y,\delta t)=\sum_{n=0}^{\infty}\int dw_1\dots dw_n\frac{M^{\mu\nu}_{n,i_1\dots i_n}(z',w_1,\dots w_n;x,y,\delta t)}{n!}\frac{\delta^n}{\delta z'_{i_1}(w_1)\dots z'_{i_n}(w_n)}\delta(z,z') \tag{G.7}$$

and we can use this to write a CQ master equation in the form

$$\begin{aligned}\frac{\partial\varrho(z,\delta t)}{\partial t}&=\sum_{n=1}^{\infty}\int dw_1\dots dw_n(-1)^n\frac{\delta^n}{\delta z_{i_1}(w_1)\dots z_{i_n}(w_n)}\left(D^{00}_{n,i_1\dots i_n}(z,w_1,\dots w_n)\varrho(z)\right)\\&-i[H,\varrho(z)]+\int dxdy D^{\alpha\beta}_0(z;x,y)L_\alpha(x)\varrho(z)L_\beta(y)-\frac{1}{2}D^{\alpha\beta}_0(z;x,y)\{L^\dagger_\beta(y)L_\alpha(x),\varrho\}\\&+\sum_{n=0}^{\infty}\sum_{\mu\nu\neq 00}\int dxdydw_1\dots dw_n(-1)^n\frac{\delta^n}{\delta z_{i_1}(w_1)\dots z_{i_n}(w_n)}\left(D^{\mu\nu}_{n,i_1\dots i_n}(z,w_1,\dots w_n;x,y)L_\mu(x)\varrho(z)L^\dagger_\nu(y)\right).\end{aligned} \tag{G.8}$$

Since we are interested in studying dynamics with local back-reaction, we shall take $D^{\mu\nu}_1(z,w;x,y)=D^{\mu\nu}_1(x)\delta(x,y)\delta(x,w)$. By the decoherence diffusion trade-off, which we derive in the next subsection[1], this also means that the diffusion matrix $D^{\mu\nu}_{2,ij}(z,w_1,w_2,x,y)$ is lower bounded by the

[1]More precisely, take Equation (G.2) with $A_\mu(x)=\delta^\alpha_\mu\alpha_\alpha(x)+\int dwb^i_\mu(x,w)(z-z')(x,w)$ and apply the same methods as in subsection G.2.

matrix $D_1^{\mu\alpha}(x)(D_0^{-1})_{\alpha\beta}(x,y)D_1^{\beta\nu*}(y)\delta(w_1,x)\delta(w_2,y)$. Without loss of generality we thus take $D_2(z,w_1,w_2,x,y) = D_2(z,x,y)\delta(x,w_1)\delta(y,w_2)$

G.2 Trade-off between diffusion and decoherence couplings for fields

In the field-theoretic case, the positivity condition is given by Equation (G.2), and we can find a trade-off between decoherence and diffusion by considering $A_\mu(x) = \delta_\mu^\alpha \alpha_\alpha(x) + \int dx b_\mu^i(x)(z-z')(x)$. So that

$$\int dxdy\big[2b_\mu^{i*}(x)D_{2,ij}^{\mu\nu}(x,y)b_\nu^j(y) + b_\mu^{i*}(x)D_{1,i}^{\mu\beta}(x,y)a_\beta(y) \\ +a_\alpha^*(x)D_{1,i}^{\alpha\mu}(x,y)b_\mu^i(y) + a_\alpha^*(x)D_0^{\alpha\beta}(x,y)a_\beta(y)\big] \geq 0 \tag{G.9}$$

where we use the shorthand notation $D_{2,ij}^{\mu\nu}(z,x,y) := D_{2,ij}^{\mu\nu}(x,y)$ and similarly $D_{1,i}^{\alpha\mu}(z;x,y) := D_{1,i}^{\alpha\mu}(x,y)$.

Taking $i \in \{1,\dots,n\}$ $\alpha \in \{1,\dots,p\}$ and $\mu \in \{1,\dots,p+1\}$, we can write this as a matrix positivity condition

$$\int dxdy[b^*(x),\alpha^*(x)]\begin{bmatrix} 2D_2(x,y) & D_1^{br}(x,y) \\ D_1^{br}(x,y) & D_0(x,y)\end{bmatrix}\begin{bmatrix} b(y) \\ \alpha(y)\end{bmatrix} \geq 0 \tag{G.10}$$

where $D_2(x,y)$ is the $(p+1)n \times (p+1)n$ matrix-kernel with elements $D_{2,ij}^{\mu\nu}(x,y)$, $D_1^{br}(x,y)$ is the $(p+1)n \times p$ matrix-kernel with rows labeled by μi and columns labelled by β with elements $D_{1,i}^{\mu\beta}(x,y)$ and $D_0(x,y)$ is the $p \times p$ decoherence matrix-kernel with elements $D_0^{\alpha\beta}(x,y)$. $D_{1,i}^{br}$ describes the quantum back-reacting components of the drift.

Equation (G.10) is equivalent to the condition that the $((p+1)n+p)\times((p+1)n+p)$ matrix of operators

$$\begin{bmatrix} 2D_2 & D_1^{br} \\ D_1^{br} & D_0 \end{bmatrix} \succeq 0 \tag{G.11}$$

be positive semi-definite. Here we are viewing the objects of (G.11) as matrix-kernels, so that for any position dependent vector $b_\mu^i(x)$, $(D_2 b)_i^\mu(x) = \int dy D_{2,ij}^{\mu\nu}(x,y)b_\nu^j(y)$.

Since we know D_2 and D_0 must be positive semi-definite, we know from Schur decomposition that

$$2D_2 \succeq D_1^{br} D_0^{-1} D_1^{br\dagger} \tag{G.12}$$

and

$$(\mathbb{I} - D_0 D_0^{-1}) D_1^{br} = 0, \tag{G.13}$$

where D_0^{-1} is the generalized inverse of D_0. Furthermore, from Equation (G.13), we see if D_0 vanishes, then clearly D_1^{br} must also vanish in order for (G.11) to be positive semi-definite.

G.3 Observational trade-off for fields

We can use the same methods to arrive at an observational trade-off using the field-theoretic version of the Cauchy-Schwartz inequality in (10.7). This arises from the positivity of

$$\mathrm{Tr}\left[\int dz dz' dx dy \Lambda^{\mu\nu}(z|z', x, y) O_\mu(z, z', x)\rho(z') O_\nu^\dagger(z, z', y)\right] \geq 0 \tag{G.14}$$

for any local vector of CQ operators $O_\mu(z, z', x)$. We have to be careful since (G.14) is not in general well defined since O_μ may not be trace-class. We hence assume that we consider states $\rho(z)$ and operators $O_\mu(z, z', x)$ for which (G.14) is well defined. Since we are interested in getting an observational trade-off, we expect this always to be the case for physical classical-quantum states $\rho(z)$.

We shall use Equation (G.14) to arrive at a (pseudo) inner product on a vector of operators O_μ via

$$\langle \bar{O}_1, \bar{O}_2\rangle = \int dz dz' dx dy \mathrm{Tr}\left[\Lambda^{\mu\nu}(z|z' x, y) O_{1\mu}(x) \varrho(z') O_{2\nu}^\dagger(y)\right] \tag{G.15}$$

where $||\bar{O}|| = \sqrt{\langle \bar{O}, \bar{O}\rangle} \geq 0$ due to (G.14). Technically this is not positive definite, but again, this will not worry us. Hence, so long as $||\bar{O}_2|| \neq 0$ we again have a Cauchy- Schwartz inequality

$$||\bar{O}_1||^2 ||\bar{O}_2||^2 - |\langle \bar{O}_1, \bar{O}_2\rangle|^2 \geq 0. \tag{G.16}$$

Choosing $O_{1,\mu}(x) = \delta^\alpha_\mu L_\alpha(x)$ and $O_{2,\mu}(x) = \int dx' b^i(x)(z-z')_i(x)L_\mu(x)$, one finds

$$
\begin{aligned}
&||\bar{O}_1||^2 = \int dzdxdy \mathrm{Tr}\left[D_0^{\alpha\beta}(z;x,y)L_\alpha \varrho(z)L_\beta^\dagger\right] := \langle D_0\rangle \\
&||\bar{O}_2||^2 = 2\int dzdxdy \mathrm{Tr}\left[b^{j*}(x)D^{\mu\nu}_{2,ij}(z;x,y)L_\mu(x)\varrho(z)L_\nu^\dagger(y)b^i(y)\right] \\
&|\langle \bar{O}_1, \bar{O}_2\rangle|^2 = |\int dzdx \mathrm{Tr}\left[b^{i*}(x,x)D^{\alpha\nu}_{1,i}(z;x)L_\alpha(x)\varrho(z)L_\nu^\dagger(x)\right]|^2 := |\langle\int dx b^{i*}(x)D^{br}_{1,i}(x)\rangle|^2
\end{aligned}
\tag{G.17}
$$

Taking the limit $b^i(x) \to \delta(x,\bar{x})b^i(\bar{x})$, we arrive at a local trade-off between diffusion, drift, and the total decoherence. In particular, using G.17 and the fact that for back-reaction, the expectation value of D_0 cannot vanish, we arrive at the observational trade-off of Equation (10.22)

$$
b^i(\bar{x})\left[2\langle D_{2,ij}(\bar{x},\bar{x})\rangle\langle D_0\rangle - |\langle D^{br}_{1,i}(\bar{x})\rangle|^2\right]b^j(\bar{x}) \geq 0 \tag{G.18}
$$

which we write in matrix form as

$$
2\langle D_2(\bar{x},\bar{x})\rangle\langle D_0\rangle \succeq \langle D_1^{br}(\bar{x})\rangle\langle D_1^{br}(\bar{x})\rangle^\dagger \tag{G.19}
$$

We obtain the trade-off between decoherence, diffusion, and total drift

$$
8\langle D_2(\bar{x},\bar{x})\rangle\langle D_0\rangle \succeq \langle D_1^{T}(\bar{x})\rangle\langle D_1^{T}(\bar{x})\rangle^\dagger \tag{G.20}
$$

using the same methods as in Chapter 10 for the continuous case, which we do not reproduce here.

Appendix H

Relating decoherence rates to the observational trade-off

This appendix studies the decoherence rates for general CQ master equations. We consider the case of a quantum mass initially in a partially decohered superposition of state $|L\rangle$ and $|R\rangle$. We describe the quantum state using creation and annihilation operators $\psi(x), \psi^\dagger(x)$ on a Fock space, related to the usual momentum-based Fock operators as $\psi(x) = \int dp e^{i\vec{p}\cdot\vec{x}} a_{\vec{p}}$. We assume that the state remains well approximated by a state with a fixed particle number. The superposition can be taken to be distributions centered around $x = x_L$ and $x = x_R$ with total mass m, i.e., for a one-particle state we could take $|L/R\rangle = \int d^3x f_{L/R}(x)\psi^\dagger(x)|0\rangle$. We will take them to be well separated so that $f_L(x)f_R(x) \approx 0$, and we take the separation distance to be larger than the scale of the non-locality in $D_0(x, y)$. Mathematically this means that $\langle L|D_0^{\alpha\beta}(z; x, y)L_\beta^\dagger(y)L_\alpha(x)|R\rangle \approx 0$ for *any* local operators $L_\alpha(x)$ and $L_\beta(y)$.

With this orthogonality condition, we can then (at least initially) consider the joint quantum classical state restricted to the 2 dimensional Hilbert space of these two states so that the total quantum-classical system can be written as

$$\varrho(\Phi, \pi_\Phi, t) = \begin{pmatrix} u_L(\Phi, \pi_\Phi, t) & \alpha(\Phi, \pi_\Phi, t) \\ \alpha^\star(\Phi, \pi_\Phi, t) & u_R(\Phi, \pi_\Phi, t) \end{pmatrix}, \tag{H.1}$$

where $u_L(\Phi, \pi_\Phi, t)$ and $u_R(\Phi, \pi_\Phi, t)$ corresponds to some sub-normalized probability distribution over the classical states of the gravitational field.

We define the total quantum state ρ_Q by integrating over the classical degrees of freedom

$$\rho_Q = \int D\Phi D\pi_\Phi \varrho(\Phi, \pi_\Phi, t) \tag{H.2}$$

and we shall relate $\langle D_0 \rangle$ appearing in the trade-off to the decoherence rate of the off diagonals of ρ_Q. Integrating over the classical phase space in Equation (10.1), one finds the follows expression for the evolution of ρ_Q

$$\begin{aligned}\frac{\partial \rho_Q}{\partial t} &= \int D\phi D\pi_\Phi - i[H(\Phi, \pi_\Phi), \varrho(\Phi, \pi_\Phi)] \\ &+ \int D\phi D\pi_\Phi \int dxdy \Big[D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y) L_\alpha(x) \varrho(\Phi, \pi_\Phi, t) L_\beta^\dagger(y) \\ &- \frac{1}{2} D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y) \{L_\beta^\dagger(y) L_\alpha(x), \varrho(\Phi, \pi_\Phi, t)\} \Big].\end{aligned} \tag{H.3}$$

In particular, one finds that the off-diagonals $\langle L| \frac{\partial \rho_Q}{\partial t} |R\rangle$ evolve in part according to standard unitary evolution, and in part due to the Lindbladian term

$$\begin{aligned}&\int D\phi D\pi_\Phi \int dxdy \Big[\langle L| D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y) L_\alpha(x) \varrho(\Phi, \pi_\Phi, t) L_\beta^\dagger(y) |R\rangle \\ &- \frac{1}{2} D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y) \langle L| \{L_\beta^\dagger(y) L_\alpha(x), \varrho(\Phi, \pi_\Phi, t)\} |R\rangle \Big].\end{aligned} \tag{H.4}$$

We shall now study the two terms appearing in Equation (H.4) separately, starting with the first term. Since we assume that the state is well approximated by a state with a fixed particle number, then the contributions to the first term in Equation (H.4) only come from terms where $L_\alpha(x)$ and $L_\beta(y)$ have the same number of creation and annihilation operators. To compute the expression, one commutes through the creation operators to act on the $\langle L|$ bra and picks up a term $f_L(x)$. Similarly, one commutes the annihilation operators to the act on the $|R\rangle$ ket and picks up a term $f_R(y)$. As a consequence

$$\langle L| D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y) L_\alpha(x) \varrho(\Phi, \pi_\Phi, t) L_\beta^\dagger(y) |R\rangle \sim D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y) f_L(x) f_R(y) \approx 0 \tag{H.5}$$

where the last equality follows from the fact that we are taking the masses to be well separated, and the range of D_0 is assumed to be much less than the separation between the masses.

Hence, the evolution of the off-diagonals comes from the unitary evolution and the second term in Equation (H.4), the so-called *no-event* term

$$-\frac{1}{2} \int D\phi D\pi_\Phi \int dxdy D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y) \langle L| \{L_\beta^\dagger(y) L_\alpha(x), \varrho(\Phi, \pi_\Phi, t)\} |R\rangle, \tag{H.6}$$

which is negative definite and acts to suppress the off-diagonals exponentially. To see this, note that expanding out $\varrho(\Phi, \pi_\Phi, t)$ in terms of the approximate 2 dimensional Hilbert space

$$\varrho(\Phi, \pi_\Phi, t) = u_L(\Phi, \pi_\Phi, t)|L\rangle\langle L| + u_R(\Phi, \pi_\Phi, t)|R\rangle\langle R| + \alpha(\Phi, \pi_\Phi, t)|L\rangle\langle R| + \alpha^*(\Phi, \pi_\Phi, t)|R\rangle\langle L|, \tag{H.7}$$

and using the fact that the range of D_0 is much less than the separation between the left and right masses, we can write the no-event term as

$$-\frac{1}{2}\int D\Phi D\pi D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y)\left(\langle L|L^\dagger_\beta(y)L_\alpha(x)|L\rangle + \langle R|L^\dagger_\beta(y)L_\alpha(x)|R\rangle\right)\langle L|\varrho(\Phi, \pi_\Phi)|R\rangle. \tag{H.8}$$

Equation (H.8) already expresses that the off-diagonal terms will decay, and the particle will decohere at a rate determined by the integrand of Equation (H.8). We can go slightly further in the presence of a background Newtonian potential such as the Earth's Φ_b. The Earth's background potential dominates over small fluctuations in Φ due to the particles, and we can approximate Equation (H.8) by

$$-\frac{1}{2}D_0^{\alpha\beta}(\Phi_b, \pi_{\Phi_b}; x, y)(\langle L|L^\dagger_\beta(y)L^\dagger_\beta(y)L_\alpha(x)|L\rangle + \langle R|L^\dagger_\beta(y)L^\dagger_\beta(y)L_\alpha(x)|R\rangle)\langle L|\rho_Q|R\rangle. \tag{H.9}$$

The result is to exponentially decrease the coherence $\langle L|\rho_Q|R\rangle$ with a rate λ_{LR} determined by

$$\lambda_{LR} = \frac{1}{2}\int dxdy D_0^{\alpha\beta}(\Phi_b, \pi_{\Phi_b}; x, y)(\langle L|L^\dagger_\beta(y)L_\alpha(x)|L\rangle + \langle R|L^\dagger_\beta(y)L_\alpha(x)|R\rangle). \tag{H.10}$$

Note Equation (H.10) is the same decoherence rate as if we choose not to integrate out the Newtonian potential, which gives rise to the same bound for the decoherence rate and is what we use for the experimental bounds considered in Chapter 10.

We can also show that the $\langle D_0\rangle$ term appearing in the observational trade-off of Equation (10.22) is always less than (twice) this decoherence rate, though we do not use it in Chapter 10.

Specifically, we show that

$$\int D\Phi D\pi_\Phi \int dxdy \mathrm{Tr}\left[D_0^{\alpha\beta}(\Phi, \pi_\Phi; x, y)L^\dagger_\beta(y)L_\alpha(x)\varrho(\Phi, \pi_\Phi)\right] \leq 2\lambda_{LR} \tag{H.11}$$

To see this, we first expand out the CQ state in terms of Equation (H.7) and use the fact that D_0 has a range less than the separation of the masses. We then arrive at the following expression for the left-hand side of Equation (H.11)

$$\int D\Phi D\pi_\Phi \int dxdy D_0^{\alpha\beta}(\Phi,\pi_\Phi;x,y)(\langle L|L_\beta^\dagger(y)L_\alpha(x)|L\rangle u_L(\Phi,\pi_\Phi,t)+\langle R|L_\beta^\dagger(y)L_\alpha(x)|R\rangle u_R(\Phi,\pi_\Phi,t)). \tag{H.12}$$

Due to the positivity of the CQ density matrix, u_L and u_R must both be positive. Furthermore, $u_L + u_R \leq 1$. Hence this must be less than Equation (H.8), from which (H.10) directly follows.

It is also important to note that though λ_{LR} is the decoherence rate of a particle in a superposition of L/R states, the bound of Equation (H.11) holds even for fully decohered masses in any mixture of $|L\rangle\langle L|, |R\rangle\langle R|$ states. This can be seen directly from Equation (H.12), which depends only on u_L, u_R.

Appendix I

Detecting gravitational diffusion

This appendix shows how the diffusion induced on the Newtonian potential can be measured experimentally, giving rise to the bounds in Chapter 10.

As shown in Chapter 9, in the non-relativistic limit, $c \to \infty$, the CQ dynamics can be approximated by sourcing the Newtonian potential by a random mass term, and in order to maintain the coherence of any mass in superposition, there must be noise in the Newtonian potential such that we cannot tell which element of the superposition the particle will be in

$$\nabla^2 \Phi = 4\pi G[m(x,t) + u(\Phi, \hat{m})\xi(x,t)], \tag{I.1}$$

with

$$\mathbb{E}_m[\xi(x,t)] = 0,\ \mathbb{E}_m[u\xi(x,t)u\xi(y,t')] = 2\langle D_2(x,y,\Phi)\rangle \delta(t,t'), \tag{I.2}$$

where $\langle D_2(x,y,\Phi)\rangle := \mathrm{Tr}\left[D_2^{\mu\nu}(x,y,\Phi_b)L_\mu(x)\varrho L_\nu^\dagger(y)\right]$ and ϱ is the quantum state for the decohered mass density.

The solution to Equation (I.1) is given by

$$\Phi(t,x) = -G\int d^3x' \frac{[m(x',t) - u(\Phi,\hat{m})\xi(x',t)]}{|x-x'|}, \tag{I.3}$$

where the statistics of ξ are described by Equation (I.2). A formal treatment of solutions to non-linear stochastic integrals of the form Equation (I.1) can be found in [237].

I.1 Table-top experiments

In this section, we estimate the variation in force that would be seen in table-top experiments. This bounds the diffusion of classical theories of gravity from above, giving a squeezed bound on D_2 due to lower bounds on diffusion arising from coherence experiments. We do this for dynamics in Equation (I.1), but the methodology is general and could also be used in a full simulation of CQ dynamics.

The variation in force induced on a composite mass is found via

$$\vec{F}_{tot} = -\int d^3 x m(x) \nabla \Phi. \tag{I.4}$$

Using the solution in Equation (I.3), the total force can be written

$$\vec{F}_{tot} = -G \int d^3 x d^3 x' m(x) \frac{(\vec{x} - \vec{x}')}{|x - x'|^3} [m(x', t) - J(x', t)]. \tag{I.5}$$

In reality, we measure time-averaged force by measuring time averaged accelerations over the time resolution of the experiment ΔT $\frac{1}{\Delta T} \int_0^{\Delta T} dt F_{tot}$. The total variation in the forces time averaged magnitude[1] $\sigma_F^2 := \vec{F}_{tot} \cdot \vec{F}_{tot}$ can be written as

$$\sigma_F^2 = \frac{1}{\Delta T} 2G^2 \int d^3 x d^3 y d^3 x' d^3 y' m(x) m(y) \frac{(\vec{x} - \vec{x}') \cdot (\vec{y} - \vec{y}')}{|x - x'|^3 |y - y'|^3} \langle D_2(x', y', \Phi) \rangle. \tag{I.6}$$

We shall use Equation (I.6) to provide an upper bound on coupling constants of CQ theories for different choices of kernels $D_2(x', y', \Phi)$. Given a choice of functional form of the kernel, all that remains is the strength of the diffusion coupling, which for the translation invariant kernels we consider here takes the form of a single coupling constant D_2. We take D_2 as a dimension-full quantity with units $kg^2 s m^{-3}$, which characterizes the diffusion rate for the conjugate momenta of the Newtonian potential.

For a composite mass, we can approximate the mass density by summing over N individual atoms of mass density $m_i(x)$, $m(x) = \sum_i m_i(x)$. The total force is the given by $\vec{F}_{tot} = \sum_i \vec{F}_i$, where $\vec{F}_i$ is the force on each individual atom $\vec{F}_i = -\int_V dx m_i(x) \nabla \Phi(x)$, and the total variation of force is then $\sigma_F^2 = \mathbb{E}[\sum_{ij} F_i F_j] - \mathbb{E}[\sum_i F_i]^2$.

The squeeze will generally depend on the functional choice of $D_2(x, y, \Phi)$ on the Newtonian potential. As mentioned in the main body, in the presence of a large background potential Φ_b,

[1] The full covariance matrix for various kernels is given in [265]

such as that of the Earth's, we will often be able to approximate $D_2(x,y,\Phi) = D_2(x,y,\Phi_b)$. This is true for the kernels with functional dependence of the form $D_2 \sim \Phi^n, D_2 \sim \nabla\Phi$, though the approximation does not hold for all kernels, for example, $D_2 \sim \nabla^2\Phi$ which creates diffusion only where there is mass density. We shall only consider diffusion kernels $D_2(x,y,\Phi_b)$ where the background potential is dominant, leaving more general considerations for future work.

I.1.1 Ultra-local continuous models

For local translation invariant dynamics for which the background Newtonian potential is dominant, for example, $D_2 \sim \Phi^n$, we have $\langle D_2(x,y,\Phi_b)\rangle = \langle D_2(\Phi_b)\rangle\delta(x,y)$, and we arrive at the expression for the total variation in time-averaged force

$$\sigma_F^2 = \frac{2G^2}{\Delta T}\sum_{ij}\int d^3x d^3y d^3x' m_i(x) m_j(y) \frac{(\vec{x}-\vec{x}')\cdot(\vec{y}-\vec{x}')}{|x-x'|^3|y-x'|^3}\langle D_2(x',\Phi_b)\rangle. \tag{I.7}$$

To leading order, the integral in Equation (I.7) is dominated by the self variation term where $i = j$, since nuclear scales $10^{-15}m$ dominate over inter-atomic scales $10^{-9}m$, so that $\mathbb{E}[\sum_{ij} F_i F_j] \sim \sum_i \mathbb{E}[F_i^2]$. Approximating the mass density of the atoms as coming from their nucleus and taking them to be spheres of constant density ρ with radius r_N and mass m_N, we find that the integral in Equation (I.7) is approximately

$$\sigma_F^2 \sim \frac{NG^2\rho^2 r_N^2}{\Delta T}\int d^3x'\langle D_2(\Phi_b)\rangle. \tag{I.8}$$

For the class of ultra-local continuous dynamics studied in Section 9.3 of Chapter 9, we have $\langle D_2(\Phi_b)\rangle = D_2(\Phi_b)$ since the diffusion is not associated to any Lindblad operators. If noise is everywhere throughout space, then the integral in Equation (I.8) diverges and gives evidence that continuous CQ theories with noise everywhere should be ruled out.

As such, we expect that continuous CQ theory must contain non-linear terms proportional to the Newtonian potential appearing in Equation (I.1), in which case we can approximate $\int dx' D_2$ by $V_b D_2$ where V_b is the volume of the region over which the background Newtonian potential is significant. In total, we find for continuous local CQ dynamics

$$\sigma_F^2 \sim \frac{D_2 NG^2\rho^2 r_N^2 V_b}{\Delta T}. \tag{I.9}$$

From this, we can calculate D_2 in terms of the total variance of the acceleration $\sigma_a^2 = \frac{\sigma_F^2}{m_{tot}^2}$ to get a lower bound

$$D_2 \leq \frac{\sigma_a^2 N r_N^4 \Delta T}{V_b G^2}. \tag{I.10}$$

Standard Cavendish-type classical torsion experiments measure accelerations of the order $10^{-7} ms^{-2}$, and we can take the time over which the acceleration is averaged to be that of minutes $\Delta T \sim 10^2 s$, so a very conservative bound is $\sigma_a \sim 10^{-7} ms^{-2}$, while N will be $N \sim 10^{26}$ and $r_N \sim 10^{-15} m$. We take the background Newtonian potential to be that of the Earth's, and we (conservatively) take V_b to be $V_b \sim r_E^2 h \sim 10^{15}\ m^3$ where r_E is the Earths radius, and h is the atmospheric height. We see that this bounds D_2 from above by $D_2 \leq 10^{-41} kg^2 s m^{-3}$.

On the other hand, D_2 is bounded from below from interferometry experiments which bound the decoherence rate. We calculated the decoherence rate in Section 9.6 of Chapter 9. Using the coupling constant trade-off, for the kernel $D_2(x,y) = D_2 \delta(x,y)$, we see (ignoring constant factors) that the decoherence rate is found to be

$$\lambda \sim \frac{N_\lambda M_\lambda^2}{V_\lambda D_2}, \tag{I.11}$$

where M_λ is the mass of a composite particle in the interferometry experiment, which is made up of N_λ particles, each with volume V_λ. This gives rise to the squeeze

$$\frac{\sigma_a^2 N r_N^4 \Delta T}{V_b G^2} \geq D_2 \geq \frac{N_\lambda M_\lambda^2}{V_\lambda \lambda}. \tag{I.12}$$

Using the numbers from [233], with $M_\lambda \sim 10^{-24} kg$, $N_\lambda \sim 10^3$, and $V_\lambda \sim 10^{-15} 10^{-15} 10^{-7} m^3 = 10^{-37} m^3$, $\lambda \sim 10^1 s^{-1}$ we find that $D_2 \geq 10^{-9} kg^2 s m^{-3}$. This suggests that the $D_2(x,y) = D_2 \delta(x,y)$ kernel for classical gravity is already ruled out by experiment.

I.1.2 Ultra-local jumping models

For the local discrete models, such as that of Equation (9.59) studied in Section 9.7 of Chapter 9, the theory is less constrained due to the dependence of the diffusion on the mass density. In this case, $\langle D_2(\Phi_b) \rangle = \frac{l_P^3}{m_P} D_2(\Phi_b) m(x)$, where the factors of Planck length and Planck mass are to ensure that $D_2(\Phi_b)$ has the required units. We arrive at the upper bound for D_2

$$\frac{\sigma_a^2 N r_N^4 \Delta T m_P}{m_N G^2 l_P^3} \geq D_2. \tag{I.13}$$

Meanwhile, we calculated the decoherence rate in Section 9.7 of Chapter 9, which, using the coupling constant trade-off (C.2) goes as $\lambda \sim \frac{M_\lambda m_P}{l_P^3 D_2}$, which lower bounds D_2. From this, we arrive at the squeeze

$$\frac{\sigma_a^2 N r_N^4 \Delta T}{m_N G^2} \geq \frac{l_P^3 D_2}{m_P} \geq \frac{M_\lambda}{\lambda}, \tag{I.14}$$

and plugging in the numbers, we find the bound given by Equation (10.34), which gives rise to the squeeze for local discrete models $10^{-1} kgs \geq \frac{l_P^3}{m_P} D_2 \geq 10^{-25} kgs$.

I.1.3 Continous Diosi-Penrose model

We can also consider other diffusion kernels; for example, we can consider the continuous dynamics of Section 9.3 of Chapter 9 with D_0 the Diosi-Penrose kernel. In this case we have that $D_2(x,y) = -l_P^2 D_2(\Phi_b)\nabla^2\delta(x,y)$. The Lindbladian kernel saturating the coupling constants trade-off at zeroeth order in $\Phi(x)$ is the Diosi-Penrose kernel $D_0(x,y,\Phi_b) = \frac{D_0(\Phi_b)}{|x-y|}$, as we saw in Table 9.3. Approximating the masses as spheres of constant density, we find from a substitution of the kernel into Equation (I.6) that the variation in time-averaged force is given by

$$\sigma_F^2 \sim \frac{l_P^2 G^2 m_N^2 N D_2}{\Delta T r_N^3}. \tag{I.15}$$

We, therefore, find a lower bound for D_2 in terms of the variation in acceleration

$$D_2 \leq \frac{\Delta T l_P^2 \sigma_a^2 N r_N^3}{G^2}, \tag{I.16}$$

which for classical torsion experiments $\sigma_a \sim 10^{-7} ms^{-2}$, $T \sim 10^2 s$, $N \sim 10^{26}$ and $r_N \sim 10^{-15} m$ gives $D_2 l_p^2 \leq 10^{-9} kgsm^{-1}$. On the other hand, for this kernel, the decoherence rate was calculated in Section 9.6 of Chapter 9, which via the trade-off reads

$$\lambda \sim \frac{N M_\lambda^2}{l_P^2 D_2 R_\lambda}, \tag{I.17}$$

which gives the squeeze on D_2

$$\frac{\Delta T \sigma_a^2 N r_N^3}{G^2} \geq l_P^2 D_2 \geq \frac{N_\lambda M_\lambda^2}{R_\lambda \lambda}. \tag{I.18}$$

For the numbers used in the main body of the text [233] $M_\lambda \sim 10^{-24} kg$, $N_\lambda \sim 10^3$, $R_\lambda \sim V^{1/3} = 10^{-12} m$, $\lambda \sim 10^1 s$, this yields $D_2 l_P^2 \geq 10^{-35} kgsm^{-1}$ and so this model is not ruled out by experiment.

In general, we can squeeze D_2 from above and below by simulating full CQ dynamics satisfying the decoherence-diffusion trade-off. We bound D_2 from above by studying the effects of diffusion on gravitational experiments. We bound D_2 from below using the coupling constant trade-off and coherence experiments lower bounding the decoherence rate. As we have seen in this section, it appears that classes of continuous CQ hybrid theories of gravity obeying the assumptions outlined in Chapter 10, including models without spatial correlations, are already experimentally ruled out, while others, such as the ultra-local jumping models, require stronger bounds from both gravitational and coherence experiments. We have been very conservative in our estimates, so we expect a more thorough analysis to tighten the bounds by orders of magnitude.

Appendix J

Positivity constraints in open quantum field theory

In the path integral approach, the space-time Lorentz invariance of the open quantum field theory manifests. However, in the open quantum theory literature, path integrals that are not completely positive, but are thought to be, are often considered. This chapter shows that the path integral introduced in [165] is not completely positive.

The path integral

$$\begin{aligned} S_\phi = &-\int d^d x \left[\frac{1}{2} z\left(\partial \phi_{\mathrm{R}}\right)^2+\frac{1}{2} m^2 \phi_{\mathrm{R}}^2+\frac{\lambda_3}{3!} \phi_{\mathrm{R}}^3+\frac{\lambda_4}{4!} \phi_{\mathrm{R}}^4+\frac{\sigma_3}{2!} \phi_{\mathrm{R}}^2 \phi_{\mathrm{L}}+\frac{\sigma_4}{3!} \phi_{\mathrm{R}}^3 \phi_{\mathrm{L}}\right] \\ &+\int d^d x\left[\frac{1}{2} z^*\left(\partial \phi_{\mathrm{L}}\right)^2+\frac{1}{2} m^{2 *} \phi_{\mathrm{L}}^2+\frac{\lambda_3^*}{3!} \phi_{\mathrm{L}}^3+\frac{\lambda_4^*}{4!} \phi_{\mathrm{L}}^4+\frac{\sigma_3^*}{2!} \phi_{\mathrm{L}}^2 \phi_{\mathrm{R}}+\frac{\sigma_4^*}{3!} \phi_{\mathrm{L}}^3 \phi_{\mathrm{R}}\right] \\ &+i \int d^d x\left[z_{\Delta}\left(\partial \phi_{\mathrm{R}}\right) \cdot\left(\partial \phi_{\mathrm{L}}\right)+m_{\Delta}^2 \phi_{\mathrm{R}} \phi_{\mathrm{L}}+\frac{\lambda_{\Delta}}{2!2!} \phi_{\mathrm{R}}^2 \phi_{\mathrm{L}}^2\right], \end{aligned} \tag{J.1}$$

is Lorentz invariant and was shown to be renormalizable [165]. However, we now show that the $\partial_\mu \phi_L \partial^\mu \phi_R$ terms give rise to non-positive dynamics, severely restricting the allowed dissipation to be of the form $\phi_L^n \phi_R^m$.

As an example, we can consider the Lindblad equation

$$\frac{\partial \rho}{\partial t}=-i[H, \rho]-\frac{1}{2} \pi \rho \pi+\frac{1}{2} \partial_i \phi \rho \partial_i \rho-\frac{1}{2}\{-\frac{1}{2} \pi^2+\frac{1}{2} \partial_i \phi \partial_i \phi, \rho\} \equiv \mathcal{L}(\rho), \tag{J.2}$$

which corresponds to the Lorentz invariant path integral

$$i S=\int d^4 x[\frac{i}{2}(\partial_\mu \phi^L \partial^\mu \phi^L-\partial_\mu \phi^R \partial^\mu \phi^R)+\frac{1}{2}(\partial_\mu \phi^R \partial^\mu \phi^L)-\frac{1}{4}(\partial_\mu \phi^R \partial^\mu \phi^R+\partial_\mu \phi^L \partial^\mu \phi^L)]. \tag{J.3}$$

However, Equation (J.2) is not positive. To be precise, the density operator ρ needs to be positive for all vectors $|v\rangle$ in the Hilbert space $\langle v|\rho|v\rangle \geq 0$. For the dynamics to be positive means that positive operators are mapped to positive operators. To show that the dynamics in Equation (J.2) is not positive, we need to find a positive operator mapped to a non-positive operator under the dynamics.

The dynamics of equation (J.2) can be written in terms of creation and annihilation operators

$$\frac{\partial\rho}{\partial t} = -i[H,\rho] + \int \frac{1}{2}\frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left[a_{\vec{p}}\rho a_{-\vec{p}} + a^\dagger_{\vec{p}}\rho a^\dagger_{-\vec{p}} - \frac{1}{2}\{a_{-\vec{p}}a_{\vec{p}} + a^\dagger_{-\vec{p}}a^\dagger_{\vec{p}}, \rho\}\right] \equiv \mathcal{L}(\rho) \tag{J.4}$$

Under short time evolution, we have $\rho(t+\delta t) = \rho(t) + \delta t\mathcal{L}(\rho)$. To show non-positivity, we shall construct explicitly a positive operator σ for which $\sigma + \delta t\mathcal{L}(\sigma)$ is not positive. Note, for the proof of non-positivity, we do not have to normalize, and for notational convenience, we deal with un-normalized states and vectors.

Now, consider the positive operator $a^\dagger_{\vec{q}}|0\rangle\langle 0|a_{\vec{q}} \equiv |\vec{q}\rangle\langle\vec{q}|$. We can compute $\mathcal{L}(|\vec{q}\rangle\langle\vec{q}|)$, which gives

$$\begin{aligned}\mathcal{L}(|\vec{q}\rangle\langle\vec{q}|) = \frac{1}{2}\frac{1}{(2\pi)^3}\omega_{\vec{q}}\left[|0\rangle\langle 0|a_{\vec{q}}a_{-\vec{q}} + a^\dagger_{\vec{q}}a^\dagger_{-\vec{q}}|0\rangle\langle 0|\right] - \\ \frac{1}{2}\int \frac{1}{2}\frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left[a^\dagger_{\vec{q}}|0\rangle\langle 0|a_{\vec{q}}a_{\vec{p}}a_{\vec{p}} + a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle\langle 0|a_{\vec{q}}\right].\end{aligned} \tag{J.5}$$

We note that this takes a diagonal operator to a non-diagonal operator, which can now be used to show the non-positivity of the dynamics. To that end consider $\langle v|\mathcal{L}(|\vec{q}\rangle\langle\vec{q}|)|v\rangle$ where the (un-normalised) vector $|v\rangle$ is defined via

$$|v\rangle = |0\rangle - \int d^3k d^3l \; c_{\vec{k}}c_{\vec{l}}a^\dagger_{\vec{k}}a^\dagger_{\vec{l}}|0\rangle, \tag{J.6}$$

with $c_{\vec{k}}, c_{\vec{l}} > 0$. We then compute $\langle v|(|\vec{q}\rangle\langle\vec{q}| + \delta t\mathcal{L}(|\vec{q}\rangle\langle\vec{q}|)|v\rangle$ which gives

$$\langle v|(|\vec{q}\rangle\langle\vec{q}| + \delta t\mathcal{L}(|\vec{q}\rangle\langle\vec{q}|)|v\rangle = -\delta t\frac{\omega_{\vec{q}}c_{\vec{q}}c_{-\vec{q}}}{(2\pi)^3}|\langle 0|0\rangle|^2, \tag{J.7}$$

which is negative. To summarize, the operator defined via the dynamics has taken a positive operator to a non-positive operator. So the dynamics defined via Equation (J.2) is not positive.

In general, if one includes a $-\pi\rho\pi$ term in the Lindblad equation as a jump operator, which corresponds to a $\partial_\mu\phi^L\partial_\mu\phi^R$ terms in the path integral, while the dynamics are Lorentz invariant then it will not be completely positive.

Appendix K

Symmetry generators with information loss

In this appendix, we study a simple model of a Lorentz invariant theory with information loss, originally studied in [211]. Our goal will be to understand the role of the Lorentz symmetry generators in theories with information loss, which are currently not well understood [211]. It would be interesting to further explore the consequences for the generators of symmetries in the covariant path integrals found in Chapter 8, and this appendix helps us gain intuition for some of the subtleties that arise in transforming states when information is lost.

We start by recalling the definition of Lorentz invariance in unitary quantum systems. For unitary quantum evolution, one can consider the equations of motion for the fields $\Phi : M \to \mathbb{R}$ in the Heisenberg representation, in which case the definition of Lorentz invariance and covariance is unchanged relative to the classical case: Φ solves the equations of motion if and only if $\Phi' = \Phi(\Lambda^{-1}x)$ solves the same equations of motion, where Λ is a Lorentz boost. Consequently, Hermitian operators Q_a represent the Lorentz algebra (and, more generally, the Poincare algebra). The Q^a are such that $\Phi(t,x)$ solves the equations of motion if and only if $\Phi - i\epsilon_a[Q^a,\Phi]$ also solve the equation of motion.

When there is information loss in the quantum system, there are two cases we can consider. When there is an underlying classical noise, we can easily extend the notion of space-time symmetry of the dynamical fields. We denote a *realization* of a noise process as $\xi(t,x)$. We denote a particular trajectory of the dynamical fields obeying the stochastic equations of motion as Φ_ξ. We say that a stochastic theory is Lorentz invariant if the trajectory Φ_ξ occurs with probability $P[\xi(t,x)]$ if and only if the trajectory $\Phi'_{\xi'} = \Phi_\xi(\lambda^{-1}x)$ is a solution to the same equations of motion, with a realization of the noise ξ' which occurs with the same probability as ξ. With this definition, two Lorentz-related observers use the same equations of motion. Although they both see different realizations of the noise process, the statistics of the dynamics remain the same, so they cannot decipher which frame they are in without reference to some external system.

On the other hand, we could consider the case where the noise is intrinsically quantum. The dynamics are then generated by Lindbladian $\mathcal{L}$ on the quantum system, which does not have a unique unraveling in terms of a classical noise process. We shall be interested in the case where the dynamics are considered fundamental. Our goal will be to understand the role of the generators of Lorentz symmetry in this case.

We first study a concrete example of a Lorentz invariant theory with a classical noise process and use this to gain intuition into the case of fundamental Lindbladian evolution.

K.1 Transformations with a classical noise process

As an example, we consider a theory whose dynamics are generated by the random Hamiltonian [206, 211]

$$H_\xi = H + \int d^3x \xi(t,x)\phi(x), \tag{K.1}$$

with $H = \int d^3x \mathcal{H}(x) = \frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\phi\partial_i\phi$. We take $\xi(t,x)$ to be a white noise process

$$\mathbb{E}[\xi(t,x)] = 0,\ \mathbb{E}[\xi(t,x)\xi(t',x')] = \delta(t-t')\delta^3(x-x'). \tag{K.2}$$

The model is not physical [206, 266, 211] since it gives rise to infinite particle production, however, it provides a simple model where we are able to explore the consequences of Lorentz invariance in detail.

For white noise, it is useful to note that the probability of a particular realization of ξ is given by

$$\text{Prob}[\xi(t,x)] = \mathcal{N}\exp\left(-\int d^4x \frac{\xi(t,x)\cdot\xi(t,x)}{2}\right), \tag{K.3}$$

where $\mathcal{N}$ is to ensure that probabilities are normalised $\int D\xi(t,x)\mathrm{Prob}[\xi(t,x)] = 1$. It is clear from Equation (K.3) that the realization $\xi(t,x)$ occurs with the same probability as $\xi(\Lambda^{-1}(t,x))$[1]. In the Heisenberg representation, the dynamics generated by Equation (K.1) give rise to a Langevin type of equation for ϕ, π

$$\dot{\phi} = \pi,\ \dot{\pi} = \partial_i \partial_i \phi + \phi \xi(t,x), \tag{K.4}$$

which can be written in a manifestly covariant manner as

$$-\partial^\mu \partial^\mu \phi = \phi \xi(t,x). \tag{K.5}$$

We denote solutions to the equations of motion in the Heisenberg representation as $\phi_\xi(t,x)$; we shall not construct them explicitly, and we refer the reader to [237] for an in-depth discussion of solutions to non-linear Equations of the form in Equation (K.5).

By averaging over the noise ξ, Equation (K.4) corresponds to the Lindblad equation

$$\frac{\partial \rho}{\partial t} = -i[H,\rho] + \frac{1}{2}\int d^3x \phi(x)\rho\phi(x) - \frac{1}{2}\{\phi^2(x),\rho\}_+. \tag{K.6}$$

Let us now discuss the case of symmetry generators for the dynamics generated by Equation (K.5). On a solution to the dynamics, the active space-time Lorentz transformation relating two observers is given by

$$\phi'_\xi(x) = \phi_\xi(\Lambda^{-1}x), \tag{K.7}$$

which depends on the realization of the noise process $\xi(t,x)$ since it involves time derivatives of the fields: to transform between frames, we perform a space-time transformation conditioned on the realization of the classical variable ξ. In particular, for the case of a classical noise process, we find the algebra of Lorentz boosts are the same as for the deterministic quantum case, but with $H \to H_\xi$ and $Q^i \to Q^i_\xi$. In particular, the Lorentz boosts Q^i_ξ now contain an explicit dependence on the noise process

$$Q^{ti}_\xi = \int d^3x\ tP^{0i} - x^i(\frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi\nabla\phi + \xi(t,x)\phi(x)) \tag{K.8}$$

where $P^{0i} = \int d^3x \pi \partial^i \phi$ is standard the momentum generator.

[1] We have to be careful here since the measure is not well defined on its own, but $D\xi(t,x)\mathrm{Prob}[\xi(t,x)]$ is well defined.

An important point is that these Lorentz transformations act on the solution to the dynamics, not on a time slice. They do not project to the phase space since they involve the time coordinate; this can be seen explicitly from the time dependence in Equation (K.8). As a consequence, an observer (call them Alice) who has initial data on a space-time slice Σ, $(\phi(0, x), \pi(0, x))$, at $t = 0$, cannot transform their state to a Lorentz boosted observer (Bob) in their own $t = 0$ frame without knowing the realization of the noise process ξ, since Bob has initial data at space-time points which lie in the future and past of Alice.

K.1.1 Best guess for transformed states: prediction vs. retrodiction

Since the Lorentz transformation depends on the realization which has occurred, one can ask what we should do if we do not know the realization of the noise process ξ everywhere: what is Alice's best guess for transforming to Bob's state? Since averaging over the noise process for the theory gives rise to a Lindblad equation, this "best guess" Lorentz transformation gives intuition for the generator of the Lorentz boost when fundamental Lindbladian evolution is considered – we now study it in more detail.

To be concrete, let us consider the case where Alice has a known state at some time $\bar{t}$, Alice cannot see into the future, and she has forgotten the past, but she is persistent and so tries to come up with an estimate for the state Bob sees at his own time. The best thing Alice can do is run the dynamics forward and backward in time to find $\phi_\xi(t, x)$ everywhere. She simulates the state $\phi_\xi(t, x)$ to the future of $\bar{t}$ using her state as an initial condition, and for $t < \bar{t}$, she uses her known state as a final state. Bob is related to Alice by a Lorentz transformation. So for each realization of the noise process, she performs a Lorentz boost, obtaining the field $\phi_\xi(\Lambda^{-1}x)$, which represents the state Bob would see if the realization ξ occurs. Alice then averages over all possible realizations, arriving at a best estimate for the boosted state. As we shall see, we can explicitly construct the generator taking Alice's state to Bob's, represented by a CP map in Equation (K.9). To understand this, it will first prove helpful to comment on some points pertaining to prediction vs. retrodiction in stochastic theories.

Given initial data at some time t, we use the Lindbladian dynamics $\mathcal{L}$ to predict the system's state in the future. Conversely, given initial data at t, we can use adjoint $\mathcal{L}^\dagger$ to retrodict what the state was in the past. To understand this, let us first consider the simple case of time-translation symmetry. Alice has an initial state $\rho(t)$ and wishes to guess what Bob, who is to Alice's future or past, sees. First, we consider the case where Bob lives in the future of Alice. In this case, Alice simulates over all classical trajectories ξ for time ϵ and averages over the result; this is found via the Lindbladian evolution in Equation (K.6). However, if Bob lives to Alice's past, she must retrodict her state. Alice uses her data as a final condition and evolves backward for all possible realizations of the noise ξ in the past to time $t-\epsilon$. Averaging over the noise process, she evolves her state with the backward equation to find the best guess for Bob's state. Both maps define CP evolution on initial quantum states.

Note, this is *not* what one would do if the Lindblad equation were treated as fundamental; this is to do with the omelet of inference and causation arising from information loss. If the Lindblad equation is taken to be fundamental, then to predict the past state $\rho(t-\epsilon)$, one uses the inverse generator $-\mathcal{L}$: it is known that $\rho(t) = e^{\mathcal{L}t}\rho(0)$ is a solution to the Lindblad equation with some earlier initial state $\rho(\bar{t})$, $\bar{t} < t$. In this case, the back-wards generators $-\mathcal{L}$ is not a CP map, but it is CP for all $t > 0$ on states which are solutions to the dynamics $\rho(t) = e^{\mathcal{L}t}\rho(0)$. We shall consider this point in detail in section K.2.

Moving back to the best guess, Lorentz boost generator. We consider an infinitesimal Lorentz boost $(t, x) \to (t + \epsilon x, x + \epsilon t)$ and a scalar field undergoing the dynamics in Equation (K.4). We take the space-time representation of the Lorentz group on the trajectories $\phi'_\xi(t, x) = \phi_\xi(t - \epsilon x, x - \epsilon t)$, which implicitly defines $\pi'_\xi(t, x)$ as its time derivative from Equation (K.4). Alice has initial data at some time $\bar{t}$, we take this to be given by $(\phi(\bar{t}, x), \pi(\bar{t}, x))$. We evolve forwards and backward in time to get $\phi_\xi(t, x)$ everywhere.

The Lorentz transformation sends $\phi_\xi(t, x) \to \phi'_\xi(t, x) = \phi_\xi(t - \epsilon x, x - \epsilon t)$. We want to use this to get the best estimate for Bob's state at his time $\bar{t}$. Importantly, for $x > 0$, $\phi'_\xi(\bar{t}, x) = \phi_\xi(\bar{t} - \epsilon x, x - \epsilon\bar{t})$ is to the past of $\bar{t}$, and so is obtained by retrodiction on Alices's initial state, while for $x < 0$ $\phi'_\xi(\bar{t}, x) = \phi_\xi(\bar{t} - \epsilon x, x - \epsilon\bar{t})$ is obtained from the state which Alice has predicted from his initial state. Taking this into account, the resulting best guess boosts at time t is found to be

$$\frac{\partial \rho(t)}{\partial \epsilon} = -i[Q^{ti}, \rho(t)] + \frac{1}{2}\int d^3x |x^i| \left[\phi(x)\rho\phi(x) - 1/2\{\phi^2(x), \rho\}_+\right] \equiv \mathcal{Q}(\rho(t)) \tag{K.9}$$

Where $Q^{ti} = \int d^3x \left[t\phi\partial^i\pi - x^i\left(\pi^2 + \partial_i\phi\partial_i\phi\right)\right]$ is the free Lorentz charge. We should note that after boosting $\rho'(t) = \rho(t) + \epsilon\mathcal{Q}(\rho(t))$ does not satisfy the same equation of motion as $\rho(t)$, but this is to be expected since for $x>0$ we are retrodicting on the state.

Equation (K.9) takes the same form as one would expect if including in the boost the component due to its Lindbladian evolution

$$\frac{\partial \rho(t)}{\partial \epsilon}=-i[Q^{ti},\rho(t)]-\frac{1}{2}\int d^3x x^i\left[\phi(x)\rho\phi(x)-1/2\{\phi^2(x),\rho\}_+\right]\equiv \mathcal{Q}(\rho(t)). \tag{K.10}$$

The difference is that for $x>0$, the best guess state is obtained by retrodiction, so the operator $\mathcal{L}^\dagger$ should be used, which leads to the modulus $|x^i|$ appearing in Equation (K.9) as opposed to x^i alone, which does not define a completely positive operator on all states $\rho(t)$.

However, the best guess generator for the state is not what one would do if the Lindblad equation

$$\frac{\partial \rho}{\partial t}=-i[H,\rho]+\frac{1}{2}\int d^3x\phi(x)\rho\phi(x)-\frac{1}{2}\{\phi^2(x),\rho\}_+ \tag{K.11}$$

was treated to be fundamental. As discussed, we can consider the simple example of time-translation invariance: if the Lindblad equation is taken to be fundamental, then to predict the past state $\rho(t-\epsilon)$ one uses the inverse generator $-\mathcal{L}$. In this case, the backward-in-time generators $-\mathcal{L}$ is not a CP map. However, it is clearly CP for all $t>0$ on the subset of states which are solutions to the dynamics $\rho(t)=e^{\mathcal{L}t}\rho(0)$. We might then expect that Equation (K.10) can define Lorentz boosts for a theory with fundamental information loss. We now show this is the case.

K.2 Lorentz invariance for Lindblad equations

To discuss the Lorentz invariance of Lindblad equations, we note that Equation (K.11) has an equivalent path integral description. The path integral tells us how to relate states at two different times via

$$\langle\tilde{\phi}_+|\rho(t_f)|\tilde{\phi}_-\rangle=\int_{\phi^\pm(t_i)=\bar{\phi}^\pm}^{\phi^\pm(t_f)=\tilde{\phi}^\pm} D\phi_+D\phi_-e^{iS_{t_i}^{t_f}[\phi_+\phi_-]}\langle\bar{\phi}_+|\rho(t_i)|\bar{\phi}_-\rangle, \tag{K.12}$$

where ϕ_L is the ket field, and ϕ_R is the bra field arising due to the density matrix's two-sided evolution. The action is given by

$$iS_{t_i}^{t_f}[\phi_L,\phi_R]=\int_{t_i}^{t_f}dt\int d^3x\ [i\mathcal{L}_{KG}^+-i\mathcal{L}_{KG}^--\frac{1}{2}(\phi^+-\phi^-)^2)]. \tag{K.13}$$

and can be found using the methods introduced in Chapter 2. We can find the generator of the Lorentz boost from the charge associated with the Lorentz invariance of the action in the path integral. We find

$$Q^{ti} = \int d^3x t T^{0i} - x^i T^{00}, \tag{K.14}$$

with

$$\begin{aligned} -iT^{\mu\nu}[\phi^+, \phi^-] &= \partial^\mu \phi^- \partial^\nu \phi^- - \partial^\mu \phi^+ \partial^\nu \phi^+ \\ &\quad - \frac{1}{2} n^{\mu\nu} (\partial_\sigma \phi^- \partial^\sigma \phi^- - \frac{i}{2} \phi^- \phi^- - \partial_\sigma \phi^+ \partial^\sigma \phi^+ - \frac{i}{2} \phi^+ \phi^+ + i \phi^+ \phi^-). \end{aligned} \tag{K.15}$$

We can write the Hilbert space operator corresponding to the Lorentz boost as

$$Q^{ti}(\rho) = -i[Q_H^{ti}, \rho] - \frac{1}{2} \int d^3x x^i \left[\phi(x) \rho \phi(x) - \frac{1}{2} \{\phi^2(x), \rho\}_+ \right] \tag{K.16}$$

Where Q_H^{tx} is the unitary Lorentz charge, $Q^{tx} = \int d^3x \left[t\phi \partial_i \pi - x \left(\pi^2 + \partial_i \phi \partial_i \phi \right) \right]$, which takes exactly the form of Equation (K.10).

One can easily verify that it is a symmetry of the equations of motion. That is, $\rho(t)$ solves $\frac{\partial \rho}{\partial t} = \mathcal{L}(\rho)$ if and only if $\frac{\partial Q^{tx}(\rho)}{\partial t} = \mathcal{L}(Q^{tx}(\rho))$.

More generally, in combination with the time evolution operator $\mathcal{L}$, and the standard momentum P_i and rotation generators R_i, we see that the Lorentz boosts defined via Equation (K.14), (K.15) satisfy the same algebra as the Poincare algebra

$$\begin{aligned} &[Q_i, P_k] = i\eta_{ik} \mathcal{L}, \ [Q_i, \mathcal{L}] = -iP_i, \\ &[R_m, Q_n] = i\epsilon_{mnk} Q_k, \ [Q_m, Q_n] = -i\epsilon_{mnk} R_k, \ [\mathcal{L}, P_i] = 0, [\mathcal{L}, R_i] = 0, \end{aligned} \tag{K.17}$$

with all of the other commutation relations unchanged.

Consequently, we see that the dynamics with these generators are Lorentz invariant - as is to be expected via the manifestly invariant path integral. In particular, ρ is a solution to the Lindblad equation of motion given by Equation (K.11) if and only if $\rho(t) + \epsilon_a Q^a(\rho)$ is a solution to the equation of motion, where Q^a is a generator of the Lorentz boosts (or more generally Poincare boosts) satisfying Equation (K.17). The fact that the boosts are Lindbladian represents the facts that the dynamics are not unitary, and since the boosts involve mixing space and time, they are not unitary either.

The generators defined in Equation (K.17) define a generalization of the Poincare algebra to the case of information loss. Notably, the generator defined by Equation (K.16) is *not* CP, which initially looks worrying. However, if the Lindblad equation is fundamental, then states are found by solving $\rho(t) = e^{\mathcal{L}(t-t_i)}\rho(t_i)$ and the generator will be CP on states which are solutions to the dynamics. In particular, consider applying an infinitesimal tx Lorentz boost, parameterized by ϵ. In the classical case, we have $\phi'(t,x) = \phi(t-\epsilon x, x-\epsilon t)$. For $x > 0$, the state comes in from the past, and to describe the boosted state on a sub-region $-X \leq x \leq X$ we will therefore have to know the original state at time $t - \epsilon X$. For example, suppose we have a quantum state which only has support on a subregion X, which we denote ρ_X. We then expect that (K.17) describes CP evolution on $\rho_X(t) = \rho_X e^{(t-t_i)\mathcal{L}}\rho_X(t_i)$, so long as $(t-t_i) \geq \epsilon X$. Taking the $t_i \to -\infty$ limit, we see that the evolution is CP on all time evolved states. If we want to further want to demand consistency with initial states in the infinite past $\rho(-\infty)$, we should consider the subset of initial states for which the action of Q^{tx} is completely positive which is not completely. For the dynamics of Equation (K.11), this is satisfied by the states which are initially decohered in the ϕ basis.

To summarize, we have seen that for simply the theory defined by Equation (K.6), we find a Lorentz invariant path integral, which has led us to Equation (K.17), which defines a generalization of the Poincare algebra to the case where there is fundamental information loss. Notably, we see that because the Lindbladian dynamics are not unitary, the generators of the symmetry group are not unitary. Exploring the consequence of this in more detail, as well as studying the analogous algebra for the covariant path integrals studied in Chapter 8, would be interesting, which we leave for future work.

Commonly used notation

This section summarizes notation commonly used in the thesis and where the notation is introduced.

Classical stochastic mechanics

$D_{1,i}$ Drift coefficient, page 32

$D_{2,ij}$ Diffusion coefficient, page 32

z Classical degree of freedom, page 29

z_i d dimensional classical degree of freedom, page 29

$\approx$ Weakly vanishing, page 156

$\delta(z, z'), \delta(z - z')$ Dirac delta functional, page 32

δ_{ij} Kronecker delta, page 35

$\mathbb{E}$ Expectation value, page 29

ω Phase space symplectic form, page 60

$\xi(t, x)$ White noise process, page 202

$\{,\}$ Poisson bracket, page 60

$C(u, z, \delta t)$ Characteristic function, page 32

d Dimension of the classical system, page 29

$D(z,t)_{n,i_1\dots i_n}$ Moments of the Kramers-Moyal expansion, page 31

d_i Phase space exterior derivative, page 60

$M_{n,i_1\dots i_n}(z,t,\delta t)$ Moments of the probability transition amplitude, page 31

$P(z,t+\delta t|z',t)$ Probability transition amplitude, page 31

$p(z,t)$ Probability density, page 29

S_C Classical path integral action, page 36

u Response variables, page 36

$W(z|z',t)$ Short time expansion of the probability transition amplitude, page 31

$W_i(t)$ Wiener process, page 35

$Z(t), Z_t$ Classical trajectory, page 29

Fokker-Plank equation Equation (2.19), page 33

Quantum theory

$\lambda^{\alpha\beta}$ Lindblad coupling, page 44

$\Lambda^{\mu\nu}$ Kraus matrix, page 43

$\langle\psi|\phi\rangle$ Inner product, page 37

$\langle O\rangle$ Expectation value of a quantum observable, page 37

$\mathbb{I}$ Identity operator, page 43

$\mathcal{H}$ Hilbert space, page 37

$\mathcal{L}_t$ Lindblad generator, page 44

$\otimes$ Tensor product, page 38

Φ_t Dynamical quantum map, page 43

ρ Density matrix, page 39

$\mathrm{Tr}_{\mathcal{H}}\,[]$ Trace over over Hilbert space $\mathcal{H}$, page 39

$\mathrm{Tr}_B\,[]$ Partial trace over system B, page 39

$|\psi\rangle$ Quantum state, page 37

ξ_α Complex Wiener process, page 47

$\{,\}_+$ Anti-commutator, page 46

E_i Projection operator onto the eigenvector $|i\rangle$, page 37

H Quantum Hamiltonian, page 46

K_μ Kraus operators, page 43

L_α Lindblad operator, page 44

Q_a Generator of the Lorentz boosts, page 287

S Path integral action, page 50

S_{FV} Feynman-Vernon action, page 50

U Unitary operator, page 37

x^+, x^- Ket and bra variables, page 49

GKSL equation Equation (2.58), page 46

Classical-quantum theory

$\frac{1}{2}(\{H(z),\varrho\} - \{\varrho, H(z)\})$ Alexandrov-Gerasimenko bracket, page 65

$\Lambda^{\mu\nu}(z, t+\delta t|z', t)$ Classical-quantum transition amplitude, page 53

$\mathcal{I}$ Full classical-quantum path integral action, page 122

$D^{\mu\nu}_{n,i_1\ldots i_n}(z,t)$ Moments of the classical-quantum Kramers-Moyal expansion, page 54

$\bar{O} = \frac{1}{2}(O^- + O^+)$ The bra-ket average of a classical-quantum function, page 135

ϱ Classical-quantum state, page 50

$\Delta O = (O^- - O^+)$ The bra-ket difference of a classical-quantum function, page 135

λ Decoherence rate, page 218

Λ^{vec} Vectorization of a superoperator, page 59

$\langle A, \rho \rangle$ Smeared classical-quantum state, page 158

$\langle O(z) \rangle := \int dz \mathrm{Tr}\left[O(z)\varrho\right]$ Expectation value of a classical-quantum operator $O(z)$, page 59

$\mathcal{L}(x)$ Local generator of CQ dynamics $\partial_t \varrho = \int d^3x \mathcal{L}(x)(\varrho)$, page 155

$\det \lambda^+$ Determinant of the positive semi-definite block of $D^{00}_{2,ij}$, page 126

$\sigma, D^{00}_{2,ij} = \frac{1}{2}(\sigma\sigma^T)_{ij}$ Decomposition of the classical diffusion coefficient, page 97

$C^{\mu\nu}(u, z, t, \delta t)$ Classical-quantum characteristic function, page 53

$D^{\alpha\beta}_0(z)$ Lindbladian coefficient, page 55

$D^{\mu\nu}_{1,ij}(z)$ Diffusion coefficient, page 55

$D^{diff}_{1,i}$ Difference vector between the classical path $\frac{dz_i}{dt}$ and its expected drift, page 126

$D^{\mu\nu}_{1,i}(z)$ Drift coefficient, page 55

D^{br}_1 Back-reaction matrix, page 228

$f(z; x, y)$ Classical-quantum matrix kernel, page 208

$H(z)$ Classical-quantum Hamiltonian, page 54

L_μ Lindblad operators, page 53

$M^{\mu\nu}_{n,i_1\ldots i_n}(z, t, \delta t)$ Moments of the classical-quantum transition amplitude, page 53

S_{CQ} Classical-quantum path integral coupling, page 122

$W^{\mu\nu}(z|z', t)$ Short time expansion of the classical-quantum transition amplitude, page 53

W_{CQ} Classical-quantum proto-action, page 135

$X \succeq 0$ Positive semi-definite matrix, page 82

$X^{-1}, X \succeq 0$ Generalized inverse of a positive semi-definite matrix, page 82

General form of classical-quantum master equation Equation (2.91), page 54

General form of continuous classical-quantum master equation Equation (2.93), page 56

Gravity

Γ^a_{bc} Christoffel symbol of the covariant derivative D, page 69

(M, g) Lorentzian manifold, page 66

γ_{ab} Intrinsic metric on Σ_t, page 69

$\mathcal{C}[\underline{\mathrm{M}}]$ Smeared classical-quantum momentum constraint, page 169

$\mathcal{C}^a$ Classical-quantum momentum constraint, page 167

$\mathcal{H}$ Hamiltonian constraint, page 71

$\mathcal{H}_a$ Momentum constraint, page 71

$\mathcal{L}_{constraint}$ Classical-quantum Hamiltonian constraint, page 172

∇_μ Levi-Civita covariant derivative of g, page 67

$\mathcal{R}_{AB}$ Gravitational classical-quantum transformation couplings, page 171

Φ Newtonian potential, page 78

π^{ab} Momentum conjugate to γ_{ab}, page 71

π_Φ Momentum conjugate to Φ, page 200

Σ_t Hypersurface of a foliation, page 68

C_J, C_N, C_H, C_{JN} Jump, no-event, Hamiltonian, and jump-no-event components of the CQ momentum constraint, page 167

D Covariant derivative of γ, page 69

$D_{0,\mu\nu\rho\sigma}$ Generalized de-Witt metric, page 192

$G_{\mu\nu}$ Einstein tensor, page 18

$g_{\mu\nu}$ Metric tensor, page 18

G_{abcd} de-Witt metric, page 71

$H[\vec{N}] = \int d^3x N^a(x)\mathcal{H}_a(x)$ Smeared momentum constraint, page 71

$H[N] = \int d^3x N(x)\mathcal{H}(x)$ Smeared Hamiltonian constraint, page 71

$h^{\alpha\beta}$ Decomposition of a matter super-Hamiltonian into Lindblad operators, page 151

$H_m(x)$ Matter super-Hamiltonian, page 151

$H_m[N, \vec{N}]$ Matter Hamiltonian, page 151

$h_{\mu\nu}$ Perturbation of the metric tensor, page 199

$H_{ADM}[N, \vec{N}]$ ADM Hamiltonian, page 71

$H_{m,a}(x)$ Matter super-momentum, page 151

K_{ab} Extrinsic curvature of the foliation Σ_t, page 70

$m(x)$ Mass density, page 78

N Lapse function, page 68

N^i Shift vector, page 68

$p_a^{\alpha\beta}$ Decomposition of a matter super-momenta into Lindblad operators, page 151

P_N, P_{N^a} Momentum conjugate to the lapse and shift vectors, page 71

R Ricci scalar, page 66

$R_{\mu\nu}$ Ricci tensor, page 66

S_{EH} Einstein-Hilbert action, page 67

$T_{\mu\nu}$ Stress-energy tensor, page 18

${}^{(3)}R^a_{bcd}$ Riemann tensor of γ, page 69

Constants

$\hbar = 6.626 \times 10^{-34}\ m^2 kg/s$ Planck's constant, page 239

$c = 2.997 \times 10^8\ m/s$ Speed of light, page 18

$G = 6.674 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$ Newtons constant, page 18

$l_P = 1.616 \times 10^{-35}\ m$ Planck's length, page 239

$m_P = 2.176 \times 10^{-8}\ kg$ Planck's mass, page 239

Experiment

M_λ Mass of a composite particle, page 238

V_λ Volume of a composite particle, page 238

ΔT Time resolution of an experiment, page 237

λ Decoherence rate, page 237

ρ Density of a single particle, page 238

σ_a Variation of the time-averaged acceleration, page 238

σ_F Variation of the time-averaged force, page 237

$\vec{F}_{tot}$ Time-averaged force, page 235

$D_0(x,y) = \frac{D_0}{|x-y|}$ Diosi-Penrose (D.P) kernel, page 219

$D_0(x,y) = D_0\delta(x,y)$ Ultra-local kernel, page 219

m_N Mass of a single particle, page 238

N Number of particles in a composite particle, page 238

r_N Radius of a single particle, page 238

R_λ Radius of a composite particle, page 238

V_b Volume of space with significant background Newtonian potential, page 238

Abbreviations

CP Completely positive, page 43

$CPTP$ Completely positive and trace preserving, page 42

CQ Classical-quantum, page 17

$D.P$ Diosi-Penrose, page 219

GIE Gravity induced entanglement, page 217

GR General relativity, page 66

$POVM$ Positive operator valued measure, page 104

QFT Quantum field theory, page 152

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