Typicality in random quantum scattering

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We consider scattering processes where a quantum system comprises an inner subsystem and a boundary and is subject to Haar-averaged random unitaries acting on the boundary-environment Hilbert space only. We show that, regardless of the initial state, a single scattering event will disentangle the unconditional state (i.e., the scattered state when no information about the applied unitary is available) across the inner subsystem-boundary partition. Also, we apply Lévy's lemma to constrain the trace norm fluctuations around the unconditional state. Finally, we derive analytical formulas for the mean scattered purity for initial globally pure states and provide one with numerical evidence of the reduction of fluctuations around such mean values with increasing environmental dimension.

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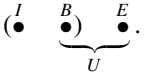
I. TYPICAL STATES UNDER RESTRICTED INTERACTIONS

Typicality arguments, based on Haar averages in the high-dimensional limit, have been repeatedly advocated as a plausible avenue to justify the second law and the ubiquity of the thermal state [1–4]. Such arguments rely on averages over Haar-distributed unitaries over a composite Hilbert space $\mathcal{H}_{\sigma\epsilon} = \mathcal{H}_\sigma \otimes \mathcal{H}_\epsilon$ pertaining to a physical system σ interacting with an environment ϵ (see the next section for a detailed definition of "Haar-distributed"). The typicality approach, closely related to the general study of typical partial entropies [5,6], does allow one to shed light on the general thermodynamic behavior of quantum systems without getting bogged down in unnecessary detail, and has been more recently extended to other quantifiers as well, such as quantum coherence [7]. Besides, typical entanglement and states play an important role in quantum information science, both directly and through the construction of related t designs, with applications to quantum cryptography, teleportation thresholds, state estimation, channel capacities, and randomized benchmarking [8–14]. Random unitaries, possibly under specific restrictions, have also recently been applied to explore thermal behaviours in many-body systems [15], with further applications to emergent hydrodynamics [16,17], quantum chaos [18], and driven systems [19–21].

However, it must be noted that actual physical interactions hardly ever comply with the notion of typicality set out above, as they do not extend to the whole system-plus-environment Hilbert space. It is therefore worthwhile to reconsider the study of typical states and entropies by generalizing the system or environment framework to one where the system comprises an inner part with Hilbert space $\mathcal{H}_I$ and a boundary with Hilbert $\mathcal{H}_B$, and where only the latter interacts with the environment, with Hilbert space $\mathcal{H}_E$, through a Haar-distributed unitary over the space $\mathcal{H}_B \otimes \mathcal{H}_E$, as per this sketch:



Somewhat loosely, we refer to this framework of restricted, "partial" mixing as "random scattering." This framework will allow us to inquire into how equilibration [22] and thermalization play out for many-body systems with only a partial interface interacting with the environment.

Notably, a setup similar to the above one plays a central role in the seminal Hayden-Preskill model for information retrieval from a black hole's Hawking radiation [23]. Therein, the internal dynamics of a black hole is modeled through a random, Haar-distributed unitary acting only on two specific subsystems (which are maximally entangled with two other reference systems). Building on previous work [10,11], Hayden and Preskill make use of an upper bound constraining the trace distance between the averaged system reference state and a product state, given in terms of dimensions and inner state purity. Such structured black hole models, based on the selective application of "pseudorandom" interactions, are currently drawing substantial attention [24–27]. Strictly related to these models is the study of "information scrambling" [29–31], which would be carried out by black holes through random interactions. It is also worthwhile to mention

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that, in a similar vein, pseudorandom interactions are finding application in other fundamental investigations, such as the study of wormhole growth in the AdS/CFT correspondence [28].

On a less extraordinary, and arguably more concrete, level, our model of "buffered" random interactions would be relevant at the mesoscopic scale, for nanowires [32] or quantum dots in 2D structures [33,34], when the bulk's and boundary's sizes become comparable and interactions with the environment are mediated by the latter (and, indeed, the relation between quantum dots' and black holes' thermalization has already been spotted [35]).

In this work, we determine the average state resulting from a scattering event and show that it is always completely uncorrelated across the inner subsystem-boundary partition; also, by applying a standard argument based on Lévy's lemma, we exponentially bound the fluctuations around such a mean state for initial states that are separable across the inner subsystem-boundary partition and for any initial state of the environment. Next, we determine the average purity of the system (inner plus boundary) state under Haar-averaged scattering interactions for an initial global pure state, as well as for an initial factorized state of the environment as a function of the initial inner subsystem-boundary Schmidt coefficients and subsystems' dimensions alone.

The plan of the paper is as follows: in Sec. II we define the random scattering process, set out the basic terminology, and introduce some of the quantities adopted; in Sec. III we determine the average unconditional state and study its fluctuations for initially separable states; in Sec. IV we determine the mean local purity of initially pure scattered states and, by addressing numerically GHZ and W states, present evidence that the fluctuations around such mean values shrink with increasing environmental dimension; in Sec. V we draw some conclusions; derivations and preliminary technical material concerning Haar averages, Lévy's lemma, and concentration of measure are deferred to the Appendixes.

II. THE SCATTERING MAP

Let us consider the scattering map Φ sending the global initial state ρ, defined on the Hilbert space $\mathcal{H}_I \otimes \mathcal{H}_B \otimes \mathcal{H}_E$, into the state of subsystem IB after a random scattering interaction acting on subsystem BE alone:

$$\Phi(\rho) = \mathrm{Tr}_E\left[\int_U dU_{BE}(\mathbb{I}_I \otimes U_{BE})\rho(\mathbb{I}_I \otimes U_{BE})^\dagger\right], \quad (1)$$

where the Haar average over the unitary group, denoted by $\int_U dU_{BE}$, takes into account the lack of information about the boundary-environment interaction in a random scattering process. To emphasize the Haar averaging involved, we shall also refer to the scattered system state as $\langle\rho_{IB}\rangle = \Phi(\rho)$ in what follows.

Let us recall that the Haar measure is the only left-invariant measure on the unitary group (and, more generally, and up to the more technical requirements of being finite on any compact subset as well as inner and outer regular, on any locally compact topological group [36]). In our integral notation, this amounts to stating that, for any measurable subset $\mathcal{S}$ of the unitary group U and any $V \in U$, one has

$$\int_{\mathcal{S}} dU = \int_{V\mathcal{S}} dU. \quad (2)$$

Basic techniques for the evaluation of Haar averages are sketched in Appendix A. The mathematical privilege of such a measure is thus clear and has motivated its adoption in physics to describe sets of random states since the late 1970s [1]. Yet the physical and operational relevance of such an adoption is less clear, and the remit of its applicability in practice is to some degree questionable, which is precisely one of the main motivations behind this paper.

As customary in this context, we shall adopt the purity $\mu = \mathrm{Tr}(\rho^2)$ as an entropic measure. The purity is related to the so-called linear entropy $S_L = (1-\mu)$, which can in turn be derived from the von Neumann entropy $S_V = -\mathrm{Tr}(\rho\ln\rho)$ through a Taylor expansion at lowest order in the quantum state. At variance with the von Neumann entropy, the purity is not endowed with a direct operational interpretation but is much more expedient to evaluate and is still very effective in characterizing the mixedness and thermal character of a state, especially around its extremal values (1 for pure states and $1/d$ for maximally mixed states, d being the system dimension). It is easy to see directly from its definition that the purity is a convex function on the set of quantum states.

In the following we will consider both the average scattered state $\langle\rho_{IB}\rangle = \Phi(\rho)$ as well as the average of the purity under the Haar measure

$$\langle\mu(\rho_{IB})\rangle = \mathrm{Tr}_{IB}\left[\int_U dU(\mathrm{Tr}_E(\mathbb{I}_I \otimes U)\rho(\mathbb{I}_I \otimes U)^\dagger)^2\right]. \quad (3)$$

Both $\langle\rho_{IB}\rangle$ and $\langle\mu(\rho_{IB})\rangle$ can be evaluated exactly, and, as we will see in the next sections, the fluctuations around them can be bound by means of Lévy's lemma.

Borrowing from well-established terminology in quantum control theory, we will refer henceforth to the state $\langle\rho_{IB}\rangle$ as to the "unconditional" state, which is prepared after a scattering event when the interaction between boundary and environment is completely unknown, and thus Haar-distributed. The unconditional state is an average of conditional states ρ_{IB}, which would be prepared if information about the unitary interaction were somehow retrieved. The purity distribution of conditional states may be of interest too and will therefore be considered in what follows.

III. THE UNCONDITIONAL STATE

Let us first analyze some remarkable properties of the unconditional state, resulting from the Haar average, and consider their thermodynamical implications. Throughout the paper, we will use a convention such that the indices of all matrices and coefficients will always follow the order inner system-boundary-environment. Also, we shall set $d_X := \dim(\mathcal{H}_X)$.

A. A decoupling theorem

Quite remarkably, the scattering map of Eq. (1) has the effect of totally suppressing any initial correlation between the boundary and the inner part of the system.

Proposition 1. For any given initial state ρ, the expectation value over the Haar measure of the reduced IB state after the scattering interaction defined in Eq. (1) is a factorized state:

$$\langle \rho_{IB} \rangle = \rho_I \otimes \frac{\mathbb{I}_B}{d_B}, \tag{4}$$

with

$$\rho_I = \mathrm{Tr}_{BE}(\rho). \tag{5}$$

The proof of this proposition is deferred to Appendix B.

Notice that the local parts of this unconditional state might have been predicted by inspection, since the maximally mixed state in the boundary is a result of the Haar-averaged mixing with the environment, while the inner system is untouched by the interaction, which thus will not alter its local state. However, the complete destruction of correlations, regardless of the dimensions involved, is not trivial. This proposition complies with what is known from the study of Hawking radiation [23], in a form and to a degree that will be made more explicit in the next subsection.

Before moving on to quantifying the statistical deviation from the unconditional state, it is worthwhile to say a few words about its thermodynamic implications. If one takes the stance that thermalization can be justified by localized, yet unknown, surface interactions with an environment, this statement shows that environmental thermalization must be entirely mediated by the surface, i.e., it cannot act on the bulk of a system directly through previous correlations, regardless of the subsystem sizes at play.

B. Fluctuations around the unconditional state

Lévy's lemma (see Appendix E for a discussion of its derivation) is a powerful standard tool to characterize the Haar-generated distribution of conditional states. Let us first recall the lemma.

Lemma 1. (Lévy's lemma) Given a Lipschitz function $f : S^d \to \mathbb{R}$ defined on the d-dimensional hypersphere $\mathbb{S}^d$ and a point $\phi \in \mathbb{S}^d$ chosen at random, the probability P for f to deviate from its mean value is given by

$$P[|f(\phi) - \langle f \rangle| \geqslant \epsilon] \leqslant 2\exp\left(-\frac{(d+1)\epsilon^2}{9\pi^3\eta^2}\right), \tag{6}$$

where ϵ is an arbitrarily small positive constant and η is the Lipschitz constant of f, i.e., $\eta : |f(\phi_1) - f(\phi_2)| \leqslant \eta|\phi_1 - \phi_2|, \forall(\phi_1, \phi_2) \in \mathbb{S}^d$.

The above lemma can be applied anytime one deals with pure global quantum states which, for a Hilbert space of dimension d, live on the surface of a $(2d-1)$-dimensional hypersphere, and is in fact key to thermodynamic typicality arguments, as in [4]. In our case, though, the action of the random, Haar-averaged unitaries is constrained to the BE subspace, so we cannot apply the lemma directly.

Yet let us consider an initial state that is separable across the inner-boundary sector, i.e., a state of the form

$$\rho = \sum_j p_j \rho_{I,j} \otimes \rho_{BE,j} \tag{7}$$

$$= \sum_{j,k} p_j p_{BE,j,k} \rho_{I,j} \otimes |\psi_{BE,j,k}\rangle\langle\psi_{BE,j,k}|, \tag{8}$$

where p_j and $p_{BE,j,k}$ are positive probabilities adding up to one when summed over j and k respectively, the $\rho'_{I,j}$s are quantum states of the inner sector, and $|\psi_{BE,j,k}\rangle$ are unit vectors of the Hilbert space $\mathcal{H}_B \otimes \mathcal{H}_E$.

Observe now that, in general, the action of global, Haar-averaged unitaries on any given pure state results in a probability distribution of pure states that is independent from the original pure state. Hence, the probability distribution resulting from the action of Haar-averaged unitaries acting on $|\psi_{BE,j,k}\rangle \in \mathcal{H}_B \otimes \mathcal{H}_E$ in each element of the sum above does not depend on j and k. Such a probability distribution may be parametrized, for all j and k, on the $(2d_Bd_E - 1)$-dimensional hypersphere of pure states on $\mathcal{H}_B \otimes \mathcal{H}_E$. Let ϕ denote the variables that parametrize such a hypersphere, and $|\phi\rangle$ the corresponding pure state of the BE subsystem. Then note that the conditional state resulting from the BE Haar average on the state of Eq. (8) is characterized by a specific choice of the ϕ variables, while j and k are just summed upon. Hence, the initial state ρ is mapped into the following conditional state:

$$\sum_{j,k} p_j p_{BE,j,k} \rho_{I,j} \otimes |\phi\rangle\langle\phi| = \rho_I \otimes |\phi\rangle\langle\phi|, \tag{9}$$

where $\rho_I = \sum_j p_j \rho_{I,j} = \mathrm{Tr}_{BE}(\rho)$. Then, as proved in detail in Appendix F, Lévy's lemma may be applied to obtain the following characterization of the distribution of trace distances of the scattered states:

$$P\left[\|\rho_{IB,\phi} - \Phi(\rho)\|_1 \geqslant \epsilon + \sqrt{\frac{d_B^2 - 1}{d_E d_B + 1}}\right] \leqslant 2\mathrm{e}^{-\frac{d_B d_E \epsilon^2}{18\pi^3}}. \tag{10}$$

Let us also recall that, by virtue of Helstrom's theorem [37], the minimum error probability in discriminating between two quantum states ρ and σ, optimized over all POVMs, is given by $\frac{1}{2} - \frac{1}{4}\|\rho - \sigma\|_1$. In this precise sense, states at vanishing trace distance become operationally indistinguishable, an argument that was also applied to justify perceived thermodynamic regularities in the appropriate limit [4].

Inequality (10) applies only to separable inner-boundary initial states (which also include cases where the environment is initially completely uncorrelated, although this requirement was not explicitly needed), as Lévy's lemma does not allow one to make any direct inference for initial entangled states, where the effect of random unitaries on off-diagonal elements must be taken into account. Under such a separability assumption, the dimension of the inner Hilbert space does not play any role in bounding the typical fluctuations. We will explore more general initial states and address situations where the inner Hilbert space does play a role by investigating the purity distribution of the scattered states in the next section.

Notice also that, as typical in such cases [4], our argument does not demonstrate an exponential shrinking of the deviation from the average when the trace distance is arbitrarily small, although this is recovered in the limit $d_E \gg d_B$: As one should expect, concentration of measure occurs, even under partial averaging, as the environment dimension grows, regardless of the other dimensions involved.

IV. MEAN SCATTERED PURITY OF INITIAL PURE STATES

A scenario which is amenable to an insightful evaluation is that of initial pure states. Denoting with $\{|i\rangle\}$, $\{|b\rangle\}$, and $\{|e\rangle\}$ the bases of choice in the spaces $\mathcal{H}_I$, $\mathcal{H}_B$, and $\mathcal{H}_E$, and with ψ_{ibe} arbitrary state vector coefficients, one would have for pure states:

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{ii'bb'ee'} \psi_{ibe}\psi^*_{i'b'e'}|ibe\rangle\langle i'b'e'|. \tag{11}$$

Through a rather lengthy calculation, reported in Appendix C, one then obtains the following mean purity:

$$\langle\mu(\rho_{IB})\rangle = \frac{d_B + d_E}{d_B d_E + 1} - \frac{d_B\left(d_E^2 - 1\right)}{(d_B d_E)^2 - 1}(\Delta - \Gamma), \tag{12}$$

where we have defined

$$\Delta := \sum_{\substack{i,i',(be),(be)' \\ i\neq i',(be)\neq(be)'}} |\psi_{i(be)}|^2|\psi_{i'(be)'}|^2, \tag{13}$$

$$\Gamma := \sum_{\substack{i,i',(be),(be)' \\ i\neq i',(be)\neq(be)'}} \psi_{i(be)}\psi^*_{i'(be)}\psi_{i'(be)'}\psi^*_{i(be)'}. \tag{14}$$

Notice that, in general, $(\Delta - \Gamma) \geqslant 0$.

When the system-environment state is factorized, a global initial pure state may be written as $(\sum_i^M c_i|I_i\rangle \otimes |B_i\rangle) \otimes |E\rangle$, such that the only additional parameters at play, other than the Hilbert spaces' dimensions, are the Schmidt coefficients c_i's of the IB initial state, on which the mean purity must necessarily depend. As shown in Appendix D, the latter then reads

$$\langle\mu(\rho_{IB})\rangle = \frac{d_B + d_E}{d_B d_E + 1} + \frac{d_B\left(1 - d_E^2\right)}{(d_B d_E)^2 - 1}\sum_{\substack{jl \\ j\neq l}} |c_j|^2|c_l|^2, \tag{15}$$

where $\{c_j\}$ are the Schmidt coefficients of the initial IB state.

Clearly, Eq. (15) implies that, when the initial state of the IB system is also separable, i.e., when $|\Psi\rangle = |I\rangle \otimes |B\rangle \otimes |E\rangle$, one recovers the well-known result in the absence of inner system-boundary separation [6,44]:

$$\langle\mu(\rho_{IB})\rangle = \frac{d_B + d_E}{d_B d_E + 1}. \tag{16}$$

This is actually true regardless of the initial correlations between boundary and environment. Indeed, when $|\Psi\rangle = |I\rangle \otimes |BE\rangle = \sum_{i=1}^{d_I}\sum_{j=1}^{d_B d_E} \gamma_i\tau_j|\Gamma_i\rangle|T_j\rangle$:

$$\begin{aligned}\Delta - \Gamma &= \sum_{ii'jj'}[|\gamma_i\tau_j|^2|\gamma_{i'}\tau_{j'}|^2 - \gamma_i\tau_j\gamma^*_{i'}\tau^*_j\gamma_{i'}\tau_{j'}\gamma^*_i\tau^*_{j'}] \\ &= \sum_{ii'jj'}[|\gamma_i\tau_j|^2|\gamma_{i'}\tau_{j'}|^2 - |\gamma_i\tau_j|^2|\gamma_{i'}\tau_{j'}|^2] \\ &= 0,\end{aligned} \tag{17}$$

so that Eq. (12) reduces to the standard bipartite system-environment case.

If, on the other hand, the initial inner system-boundary state is maximally entangled, assuming, as is reasonable, $d_B \leqslant d_I$, one can insert $c_j = 1/\sqrt{d_B}\ \forall\ j$ to obtain, from Eq. (15), the mean purity

$$\langle\mu(\rho_{IB})\rangle = \frac{\left(d_B^2 - 1\right)d_E + d_E^2 - 1}{\left(d_B^2 d_E^2 - 1\right)}, \tag{18}$$

which, in the limit $d_E \to \infty$, yields $\langle\mu(\rho_{IB})\rangle = 1/d_B^2$: the boundary's dimension constrains the equilibrium mean purity to a value higher than the minimum value $1/(d_I d_B)$.

This corresponds to the fact that full mixing (and, in settings where the total energy is set, full thermalization) cannot be achieved via interaction with a boundary, unless further boundary-inner subsystem interactions are also taken into account.

The fundamental difference between our scattering model and the conventional scenario in which the whole system interacts with an environment lies in the fact that in the latter case the expectation value of the purity is independent of the initial state, while in the former case, in general, this is clearly not true. This is the case even in the limit of infinite dimension of the environment:

$$\lim_{d_E\to\infty}\langle\mu(\rho_{IB})\rangle = \frac{1}{d_B}(1 - \Delta + \Gamma). \tag{19}$$

Here Lévy's lemma cannot be applied directly to bound the fluctuations around these mean values, as the random unitary transformations do not in general result in a state distribution that can be parametrized on a hypersphere. At variance with the case considered in Sec. III B, in fact, the initial state is in this case not separable: nonvanishing off-diagonal entries in an entangled basis prevent the simple decomposition and parametrization that was possible as a consequence of applying the Haar average on the state of Eq. (8). However, we have checked numerically the validity of Eqs. (12) for initial GHZ and W states. For N qubits, these two classes of states, which are important for the study of multipartite entanglement [38], are defined as (note that these states describe both system and environment)

$$|\mathrm{GHZ}\rangle_N = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}), \tag{20}$$

$$|W\rangle_N = \frac{1}{\sqrt{N}}(|10\ldots0\rangle + |01\ldots0\rangle + \cdots + |00\ldots1\rangle). \tag{21}$$

Defining $\rho^{\mathrm{GHZ}} = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|_N$ and $\rho^W = |W\rangle\langle W|_N$, a direct calculation leads to the following mean scattered purities:

$$\begin{aligned}\left\langle\mu\left(\rho_{IB}^{\mathrm{GHZ}}\right)\right\rangle &= \frac{d_B + d_E}{d_B d_E + 1} - \frac{1}{2}\frac{d_B\left(d_E^2 - 1\right)}{(d_B d_E)^2 - 1}, \\ \left\langle\mu\left(\rho_{IB}^{W}\right)\right\rangle &= \frac{d_B + d_E}{d_B d_E + 1} - \frac{2N_I(N - N_I)}{N^2}\frac{d_B\left(d_E^2 - 1\right)}{(d_B d_E)^2 - 1},\end{aligned} \tag{22}$$

where $D = d_I d_B d_E = 2^N$ [here the X subsystem comprises $N_X = \log_2(d_X)$ qubits]. Interestingly, the average scattered purity of W states is affected by the inner subsystem dimension. Also, it can be seen that the average purity of initial W states is larger than the GHZ one (for given d_B and d_E) for all N_I other than for $N_I = N/2$ (in which case they are equal), since $2N_I(N - N_I)/N^2 \geqslant 1/2$.

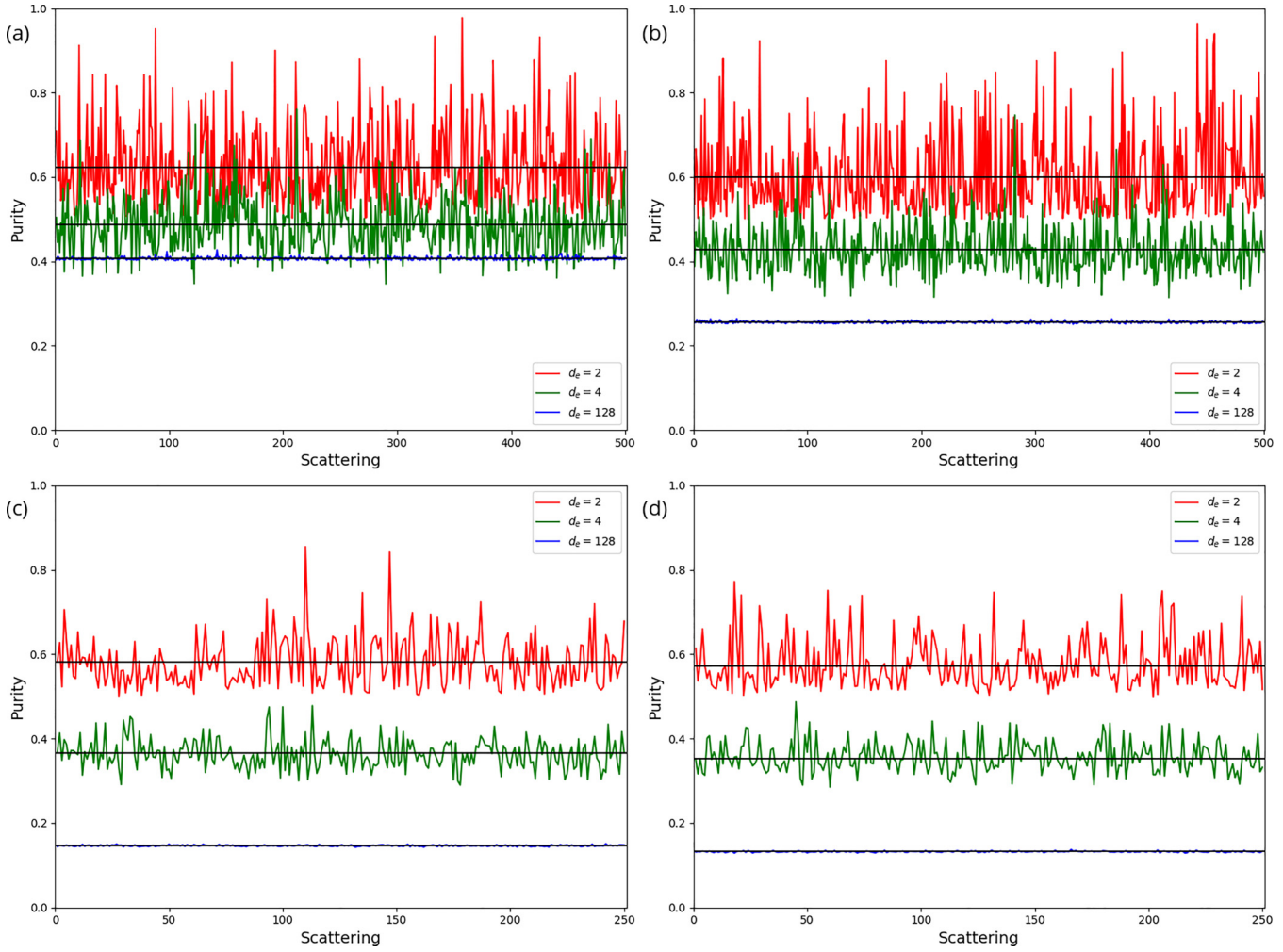

FIG. 1. Purity of the state resulting from a random scattering boundary-environment interaction for [see Eqs. (22) and (23)] (top left) ρ^W, (top right) ρ^{GHZ}, (bottom left) $\tilde{\rho}^W$, and (bottom right) $\tilde{\rho}^{\mathrm{GHZ}}$ initial states. Each value on the x axis labels a different U_{BE} random extraction. The straight horizontal lines in the plots represent the expected values of the purity, computed with Eqs. (22) and 23, where in (a) and (b) $d_I = d_B = 2$, whereas in (c) and (d) $d_I = 2$, $d_B = 4$.

We have also considered the case where three-qubit GHZ and W states interact with an $N_E = N - 3$ qubit environment prepared in a fiducial state $\tilde{\rho}^{\mathrm{GHZ}} := |GHZ\rangle\langle \mathrm{GHZ}|_3 \otimes |0\rangle\langle 0|_E^{\otimes(N-3)}$, $\tilde{\rho}^W := |W\rangle\langle W|_3 \otimes |0\rangle\langle 0|_E^{\otimes(N-3)}$, finding

$$\langle \mu(\tilde{\rho}_{IB}^{\mathrm{GHZ}})\rangle = \langle \mu(\rho_{IB}^{\mathrm{GHZ}})\rangle,$$
$$\langle \mu(\tilde{\rho}_{IB}^{W})\rangle = \frac{d_B + d_E}{d_B d_E + 1} - \frac{4}{9}\frac{d_B(d_E^2 - 1)}{(d_B d_E)^2 - 1}. \tag{23}$$

Note that these formulas apply to both $N_I = 1$, $N_B = 2$ and to $N_I = 2$, $N_B = 1$, since the Schmidt coefficients across the inner boundary partition are the same in both setups.

The results of the numerical analyses [39] are shown in Fig. 1. In the top row, we simulated 500 scattering processes of two-qubit GHZ and W states [Eqs. (20)]. For each scattering, a unitary matrix of dimension $2^{(N_B+N_E)}$ is drawn at random according to the Haar measure, tensored with the identity on the inner qubit, and applied to the global (I + B + E) state. The resulting state is then reduced by tracing out the environment, and the purity of the reduced (I + B) state is computed. This is repeated for different dimensions d_E of the environment. In the bottom row, the same process is applied to a global state composed of three-qubit GHZ and W states tensored with an environment of the form $|0\rangle\langle 0|_E^{\otimes N_E}$. As can be appreciated in the figure, the fluctuations around the expected value of μ are quite large for an environment composed of $N_E = 1$, $N_E = 2$ qubits, but are almost completely damped for $N_E = 7$.

V. CONCLUSIONS

Summing up, we have considered a scattering scenario where random unitary transformations act jointly on an environment and a boundary subsystem rather than on the whole system, which comprises an inner subsystem too, and derived the following:

(1) A proof that any single, Haar-distributed scattering event is able to disentangle completely boundary and inner subsystems.

(2) A proof that, for initially separable inner-boundary states and large enough environments, the conditional scattered states concentrate exponentially in trace norm around the average conditional state, by virtue of Lévy's lemma.

(3) An analytical formula for the mean purity of initially pure scattered states in terms of the subsystems' dimensions and initial state's coefficients.

(4) An analytical formula for the mean scattered purity of initially globally pure states uncorrelated with the environment, in terms of the Schmidt coefficients of the initial state.

As well, by addressing initial GHZ and W states, we have provided numerical evidence for the concentration of measure of scattered purities with increasing environmental dimension.

Two main traits of random scattering interactions may be inferred from our analysis and are worth emphasizing: (1) concentration of measure—and, with it, the typicality approach it entails—does not require interactions involving the whole system, as can be seen by noting that Eq. (10) implies exponential concentration in trace distance for any d_B in the limit $d_E \to \infty$; (2) full thermalization (in the sense of attaining the maximally mixed state), however, cannot be achieved by partial interactions, as indicated by the $d_E \to \infty$ limit of Eq. (18), where the mean purity is lower bounded by the boundary dimension.

Our approach and findings are a step towards a generalized study of thermalization and equilibration for structured systems and restricted interactions and may be of interest to quantum thermodynamics approaches based on typicality, to their application to mesoscopic conductors, as well as to other fundamental investigations, such as the study of black hole entropies. Some of these generalizations will require the extension of our results for typical purities to initial mixed states, allowing one to take into account more general situations, such as uncorrelated environments in thermal Gibbs states.

APPENDIX A: INTEGRALS OVER THE UNITARY GROUP

Here we just summarize the results of [42], which are relevant for the purposes of our model.

The maps (1) and (3) involve the calculation of integrals of the form

$$\int_U dU\,[U^*_{i,j}U_{k,l}], \quad \int_U dU\,\big[U^*_{i_1,j_1}U^*_{i_2,j_2}U_{k_1,l_1}U_{k_2,l_2}\big], \tag{A1}$$

where $\int_U$ means integrating over the Haar measure. Integrals of this kind can be tackled using Schur's lemma (see, e.g., [43,44]), but here we follow the, somewhat easier, approach described in [42].

In general, for some degrees of the polynomials p, q in U^* and U, one wants to compute

$$\int_U dU\big[U^*_{i_1,j_1}\dots U^*_{i_p,j_p}U_{k_1,l_1}\dots U_{k_q,l_q}\big] = \int_U dU\left[\prod_{a=1}^{p}U^*_{i_a,j_a}\prod_{b=1}^{q}U_{k_b,l_b}\right] \equiv \int_U dU\big[U^*_{I_pJ_p}U_{K_qL_q}\big] \equiv \langle I_p, J_p|K_q, L_q\rangle, \tag{A2}$$

where we have defined $X_p = (x_1, x_2, \dots, x_p)$. It is shown in [42] that the only nonzero integrals are the ones in which the degrees are such that $p = q$ (thus we will drop this index), $K = I$ and $L = J_Q$, where J_Q represents any permutation Q of the p indices in the set $J = (j_1 j_2 \dots j_p)$ [i.e., $J_Q = (j_{Q(1)} j_{Q(2)} \dots j_{Q(p)})$]:

$$\langle I_p, J_p|K_q, L_q\rangle = \langle I, J|I, J_Q\rangle. \tag{A3}$$

When $p = 1$

$$\int_U dU[U^*_{i,j}U_{k,l}] = \int_U dU[U^*_{i,j}U_{i,j}] = \langle i, j|i, j\rangle = \frac{1}{d}, \tag{A4}$$

where d is the dimension of U.

When $p = 2$, the nonzero integrals are the following:

$$(i_1 \neq i_2, j_1 \neq j_2): \int_U dU\big[U^*_{i_1,j_1}U^*_{i_2,j_2}U_{i_1,j_1}U_{i_2,j_2}\big] = \frac{1}{d^2-1}; \tag{A5}$$

$$(i_1 \neq i_2, j_1 \neq j_2): \int_U dU\big[U^*_{i_1,j_1}U^*_{i_2,j_2}U_{i_1,j_2}U_{i_2,j_1}\big] = -\frac{1}{d(d^2-1)}; \tag{A6}$$

$$(i_1 = i_2, j_1 \neq j_2): \int_U dU\big[U^*_{i_1,j_1}U^*_{i_1,j_2}U_{i_1,j_1}U_{i_1,j_2}\big] = \frac{1}{d(d+1)},$$

$$\int_U dU\big[U^*_{i_1,j_1}U^*_{i_1,j_2}U_{i_1,j_2}U_{i_1,j_1}\big] = \frac{1}{d(d+1)}; \tag{A7}$$

$$(i_1 \neq i_2, j_1 = j_2): \int_U dU\big[U^*_{i_1,j_1}U^*_{i_2,j_1}U_{i_1,j_1}U_{i_2,j_1}\big] = \frac{1}{d(d+1)}; \tag{A8}$$

$$(i_1 = i_2, j_1 = j_2): \int_U dU\big[U^*_{i_1,j_1}U^*_{i_1,j_1}U_{i_1,j_1}U_{i_1,j_1}\big] = \frac{2}{d(d+1)}. \tag{A9}$$

Knowing how to deal with this kind of integrals allows one to compute relevant quantities such as the mean reduced state $\overline{\rho_S}$ of a bipartite system-environment (SE) density matrix:

$$\begin{aligned}
\overline{\rho_S} &= \mathrm{Tr}_E(\langle\rho_{SE}\rangle_H) = \mathrm{Tr}_E\left\{\int_U dU\,[U\,\rho_{SE}U^\dagger]\right\}\\
&= \mathrm{Tr}_E\left\{\int_U dU \sum_{acge=1}^{d_S}\sum_{ik=1}^{d_S}\sum_{bdhf=1}^{d_E}\sum_{jl=1}^{d_E} U_{(ab)(cd)}U^*_{(ef)(gh)}\psi_{ij}\psi^*_{kl}|ab\rangle\langle cd|\cdot|ij\rangle\langle kl|\cdot|gh\rangle\langle ef|\right\}\\
&= \mathrm{Tr}_E\left\{\sum_{ikae=1}^{d_S}\sum_{jlbf=1}^{d_E}\psi_{ij}\psi^*_{kl}\underbrace{\int_U dU\,[U_{(ab)(ij)}U^*_{(ef)(kl)}]}_{=\frac{1}{d_Sd_E}\delta^{ab}_{ef}\delta^{ij}_{kl}}\right\}\\
&= \mathrm{Tr}_E\left\{\frac{1}{d_Sd_E}\sum_{ij}|\psi_{ij}|^2\sum_{a=1}^{d_S}\sum_{b=1}^{d_E}|ab\rangle\langle ab|\right\} = \frac{1}{d_Sd_E}\sum_{\xi=1}^{d_E}\langle\xi|\sum_{a=1}^{d_S}\sum_{b=1}^{d_E}|ab\rangle\langle ab|\cdot|\xi\rangle = \frac{1}{d_S}\sum_{a=1}^{d_S}|a\rangle\langle a|.
\end{aligned} \tag{A10}$$

APPENDIX B: PROOF OF PROPOSITION 1

Proof. Due to the linearity of averaging and partial tracing, it will suffice to prove this statement for initially pure states. So, substituting Eq. (11) into Eq. (1) and expressing the scattering unitaries in terms of their components in the BE basis, $U_{BE} = \sum_{i,k=1}^{d_B}\sum_{j,l=1}^{d_E} U_{(ij)(kl)}|ij\rangle\langle kl|$, yields

$$\begin{aligned}
\Phi(\rho_{IBE}) &= \mathrm{Tr}_E\left[\int_U dU\,(\mathbb{I}_I\otimes U_{BE})\rho_{IBE}(\mathbb{I}_I\otimes U_{BE})^\dagger\right]\\
&= \sum_{\xi=1}^{d_E}\langle\xi|\int_U dU\sum_{\substack{jklm\\mnop}}\sum_{\substack{sbe\\s'b'e'}} U_{(ij)(kl)}\psi_{sbe}|ij\rangle\langle kl|sbe\rangle\langle s'b'e'|mn\rangle\langle op|\psi^*_{s'b'e'}U^*_{(op)(mn)}|\xi\rangle\\
&= \sum_{\xi=1}^{d_E}\sum_{\substack{jklm\\mnop}}\sum_{\substack{sbe\\s'b'e'}}\psi_{sbe}\psi^*_{s'b'e'}\underbrace{\langle\xi|ij\rangle\langle kl|sbe\rangle\langle s'b'e'|mn\rangle\langle op|\xi\rangle}_{\delta^\xi_j\delta^k_b\delta^l_e\delta^m_{b'}\delta^n_{e'}\delta^\xi_o}\int_U dU\,[U^*_{(op)(mn)}U_{(ij)(kl)}]\\
&= \sum_{\xi=1}^{d_E}\sum_{\substack{sbe\\s'b'e'}}\sum_{io}\psi_{sbe}\psi^*_{s'b'e'}|si\rangle\langle s'o|\int_U dU\,[U^*_{(o\xi)(b'e')}U_{(i\xi)(be)}] = \sum_{\xi=1}^{d_E}\sum_{\substack{sbe\\s'b'e'}}\sum_{io}\psi_{sbe}\psi^*_{s'b'e'}|si\rangle\langle s'o|\frac{1}{d_Bd_E}\delta^o_i\delta^{be}_{b'e'}\\
&= \frac{1}{d_Bd_E}\sum_{\xi ss'bei}\psi_{sbe}\psi^*_{s'be}|si\rangle\langle s'i| = \sum_{ss'be}\psi_{sbe}\psi^*_{s'be}|s\rangle\langle s'|\otimes\frac{1}{d_B}\sum_{i=1}^{d_B}|i\rangle\langle i| = \rho_I\otimes\overline{\rho}_B,
\end{aligned} \tag{B1}$$

where for the fourth equality we have used the results of Appendix A. ■

APPENDIX C: EXPECTATION VALUE OF THE PURITY

Here we explicitly show how to get to Eq. (12). Substituting Eq. (11) into Eq. (3) and expressing the scattering unitaries in terms of their components in the BE basis, $U_{BE} = \sum_{i,k=1}^{d_B}\sum_{j,l=1}^{d_E} U_{(ij)(kl)}|ij\rangle\langle kl|$:

$$\begin{aligned}
\langle\mu(\rho_{IB})\rangle &= \mathrm{Tr}_{IB}\left\{\int_U dU\,[\mathrm{Tr}_E((\mathbb{I}_I\otimes U_{BE})\rho_{IBE}(\mathbb{I}_I\otimes U_{BE})^\dagger)]^2\right\}\\
&= \int_U dU\sum_{\gamma\gamma'\beta\beta'\xi\xi'}\langle(\gamma\beta)\xi|\sum_{\substack{ibe\\i'b'e'}}\sum_{\substack{jklm\\nopq}}\psi_{ibe}\psi^*_{i'b'e'}U_{(jk)(lm)}U^*_{(pq)(no)}|jk\rangle\langle lm|ibe\rangle\langle i'b'e'|no\rangle\langle pq|(\gamma'\beta')\xi\rangle\langle(\gamma'\beta')\xi'|\\
&\quad\times\sum_{\substack{i^2b^2e^2\\i^3b^3e^3}}\sum_{\substack{rstu\\vwxy}}\psi_{i^2b^2e^2}\psi^*_{i^3b^3e^3}U_{(rs)(tu)}U^*_{(xy)(vw)}|rs\rangle\langle tu|i^2b^2e^2\rangle\langle i^3b^3e^3|vw\rangle\langle xy|(\gamma'\beta')\xi'\rangle
\end{aligned}$$

TABLE I. Decomposition of I into a sum of the nonzero integrals (A5)–(A9). In the second to fourth columns, each of these factors is multiplied by their multiplicity

$\sum_{n_0 n_1 n_2 n_3}$	**I** ($\xi \neq \xi'$)	**II** ($\beta \neq \beta'$)	**III** ($\beta\xi = \beta'\xi'$)
$n_0 = n_1 \neq n_2 = n_3$	$d_B d_E (d_E - 1)$ (A5)	$d_B d_E (d_B - 1)$ (A6)	$d_B d_E$ (A7)
$n_0 = n_3 \neq n_1 = n_2$	$d_B d_E (d_E - 1)$ (A6)	$d_B d_E (d_B - 1)$ (A5)	$d_B d_E$ (A7)
$n_0 = n_1 = n_2 = n_3$	$d_B d_E (d_E - 1)$ (A8)	$d_B d_E (d_B - 1)$ (A8)	$d_B d_E$ (A9)

$$
\begin{aligned}
&= \sum_{\gamma\gamma'\beta\beta'\xi\xi'} \sum_{\substack{beb'e' \\ b^2e^2b^3e^3}} \psi_{\gamma be}\psi^*_{\gamma' b'e'}\psi_{\gamma' b^2e^2}\psi^*_{\gamma b^3e^3} \int_U dU [U^*_{(\beta'\xi)(b'e')} U^*_{(\beta\xi')(b^3e^3)} U_{(\beta\xi)(be)} U_{(\beta'\xi')(b^2e^2)}] \\
&\equiv I + J,
\end{aligned} \tag{C1}
$$

where we have split the sum into two components:

$$
\begin{aligned}
I &:= \sum_{\gamma\beta\beta'\xi\xi'} \sum_{n_0 n_1 n_2 n_3} \psi_{\gamma n_0}\psi^*_{\gamma n_1}\psi_{\gamma n_2}\psi^*_{\gamma n_3} \int_U dU \left[U^*_{(\beta'\xi)n_1} U^*_{(\beta\xi')n_3} U_{(\beta\xi)n_0} U_{(\beta'\xi')n_2}\right], \\
J &:= \sum_{\substack{\gamma\gamma'\beta\beta'\xi\xi' \\ (\gamma\neq\gamma')}} \sum_{n_0 n_1 n_2 n_3} \psi_{\gamma n_0}\psi^*_{\gamma' n_1}\psi_{\gamma' n_2}\psi^*_{\gamma n_3} \int_U dU \left[U^*_{(\beta'\xi)n_1} U^*_{(\beta\xi')n_3} U_{(\beta\xi)n_0} U_{(\beta'\xi')n_2}\right].
\end{aligned} \tag{C2}
$$

In the above equalities we have also shortened the notation, merging the boundary and environment indices pertaining to both the coefficients of $|\Psi\rangle$ and U into a single one. Let us take care of I first:

$$
\begin{aligned}
I = &\underbrace{\sum_{\substack{\gamma\beta\xi\xi' \\ \xi\neq\xi'}} \sum_{n_0 n_1 n_2 n_3} \psi_{\gamma n_0}\psi^*_{\gamma n_1}\psi_{\gamma n_2}\psi^*_{\gamma n_3} \int_U dU \left[U^*_{(\beta\xi)n_1} U^*_{(\beta\xi')n_3} U_{(\beta\xi)n_0} U_{(\beta\xi')n_2}\right]}_{=:\mathbf{I}(\xi\neq\xi')} \\
&+ \underbrace{\sum_{\substack{\gamma\beta\beta'\xi \\ \beta\neq\beta'}} \sum_{n_0 n_1 n_2 n_3} \psi_{\gamma n_0}\psi^*_{\gamma n_1}\psi_{\gamma n_2}\psi^*_{\gamma n_3} \int_U dU \left[U^*_{(\beta'\xi)n_1} U^*_{(\beta\xi)n_3} U_{(\beta\xi)n_0} U_{(\beta'\xi)n_2}\right]}_{=:\mathbf{II}(\beta\neq\beta')} \\
&+ \underbrace{\sum_{\gamma\beta\xi} \sum_{n_0 n_1 n_2 n_3} \psi_{\gamma n_0}\psi^*_{\gamma n_1}\psi_{\gamma n_2}\psi^*_{\gamma n_3} \int_U dU \left[U^*_{(\beta\xi)n_1} U^*_{(\beta\xi)n_3} U_{(\beta\xi)n_0} U_{(\beta\xi)n_2}\right]}_{=:\mathbf{III}(\beta\xi=\beta'\xi')}.
\end{aligned} \tag{C3}
$$

We should now further decompose the sums above to get to a sum of integrals like Eqs. (A5)–(A9). Writing it explicitly would be rather unmanageable, though. Arguably the best way to work it out is to group the decomposition into a table. Using Table I, after a little bookkeeping we obtain

$$
\begin{aligned}
I = &(d(d_E - 1) + d(d_B - 1)) \left(\sum_{\substack{n_0 n_2 \gamma \\ n_0\neq n_2}} \frac{|\psi_{\gamma n_0}|^2 |\psi_{\gamma n_2}|^2}{(d^2 - 1)} + \sum_{n_0\gamma} \frac{|\psi_{\gamma n_0}|^2 |\psi_{\gamma n_0}|^2}{d(d+1)} - \sum_{\substack{n_0 n_2 \gamma \\ n_0\neq n_2}} \frac{|\psi_{\gamma n_0}|^2 |\psi_{\gamma n_2}|^2}{d(d^2-1)} \right) \\
&+ 2d \left(\sum_{\substack{n_0 n_2 \gamma \\ n_0\neq n_2}} \frac{|\psi_{\gamma n_0}|^2 |\psi_{\gamma n_2}|^2}{d(d+1)} + \sum_{n_0\gamma} \frac{|\psi_{\gamma n_0}|^2 |\psi_{\gamma n_0}|^2}{d(d+1)} \right)
\end{aligned} \tag{C4}
$$

with $d = d_E d_B$.

We can do exactly the same thing with J. First, decompose the sum as in Eq. (C3):

$$\begin{aligned} J = &\underbrace{\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{\beta\xi\xi'\\ \xi\neq\xi'}}\sum_{n_0n_1n_2n_3}\psi_{\gamma n_0}\psi^*_{\gamma' n_1}\psi_{\gamma' n_2}\psi^*_{\gamma n_3}\int_U dU\left[U^*_{(\beta\xi)n_1}U^*_{(\beta\xi')n_3}U_{(\beta\xi)n_0}U_{(\beta\xi')n_2}\right]}_{\mathbf{I}_{(\xi\neq\xi')}}\\ &+\underbrace{\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{\beta\beta'\xi\\ \beta\neq\beta'}}\sum_{n_0n_1n_2n_3}\psi_{\gamma n_0}\psi^*_{\gamma' n_1}\psi_{\gamma' n_2}\psi^*_{\gamma n_3}\int_U dU\left[U^*_{(\beta'\xi)n_1}U^*_{(\beta\xi)n_3}U_{(\beta\xi)n_0}U_{(\beta'\xi)n_2}\right]}_{\mathbf{II}_{(\beta\neq\beta')}}\\ &+\underbrace{\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\beta\xi}\sum_{n_0n_1n_2n_3}\psi_{\gamma n_0}\psi^*_{\gamma' n_1}\psi_{\gamma' n_2}\psi^*_{\gamma n_3}\int_U dU\left[U^*_{(\beta\xi)n_1}U^*_{(\beta\xi)n_3}U_{(\beta\xi)n_0}U_{(\beta\xi)n_2}\right]}_{\mathbf{III}_{(\beta\xi=\beta'\xi')}}, \end{aligned} \tag{C5}$$

and then use Table I [the integrals are the same of Eq. (C3)] to obtain, after some algebra,

$$\begin{aligned} J = &\, d(d_E-1)\left(\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\frac{\psi_{\gamma n_0}\psi^*_{\gamma' n_0}\psi_{\gamma' n_2}\psi^*_{\gamma n_2}}{(d^2-1)}-\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\frac{\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_2}\right|^2}{d(d^2-1)}+\sum_{\substack{n_0\gamma\gamma'\\ \gamma\neq\gamma'}}\frac{\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_0}\right|^2}{d(d+1)}\right)\\ &+d(d_B-1)\left(\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\frac{\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_2}\right|^2}{(d^2-1)}-\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\frac{\psi_{\gamma n_0}\psi^*_{\gamma' n_0}\psi_{\gamma' n_2}\psi^*_{\gamma n_2}}{d(d^2-1)}+\sum_{\substack{n_0\gamma\gamma'\\ \gamma\neq\gamma'}}\frac{\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_0}\right|^2}{d(d+1)}\right)\\ &+\frac{1}{(d+1)}\left(\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\psi_{\gamma n_0}\psi^*_{\gamma' n_0}\psi_{\gamma' n_2}\psi^*_{\gamma n_2}\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_2}\right|^2+2\sum_{\substack{n_0\gamma\gamma'\\ \gamma\neq\gamma'}}\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_0}\right|^2\right). \end{aligned} \tag{C6}$$

Putting it all together and rearranging a bit:

$$\begin{aligned} I+J &= \frac{d_E+d_B}{d+1}\left(\sum_{\substack{\gamma n_0n_2\\ n_0\neq n_2}}\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma n_2}\right|^2+\sum_{\gamma n_0}\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma n_0}\right|^2+\sum_{\substack{\gamma\gamma' n_0\\ \gamma\neq\gamma'}}\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_0}\right|^2\right)\\ &= \left[\frac{dd_B-d_E}{(d-1)(d+1)}\right]\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_2}\right|^2+\left[\frac{dd_E-d_B}{(d-1)(d+1)}\right]\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\psi_{\gamma n_0}\psi^*_{\gamma' n_0}\psi_{\gamma' n_2}\psi^*_{\gamma n_2}. \end{aligned} \tag{C7}$$

The calculation to get to the final expression of the average purity is rather tedious but quite trivial. It only involves some algebra and some care in grouping the right terms to form the trace of the global state. Ultimately, this leads to

$$\langle\mu(\rho_{IB})\rangle = \frac{d_B+d_E}{d_Ed_B+1}+\frac{d_B\left(1-d_E^2\right)}{(d_Ed_B)^2-1}\sum_{\substack{\gamma\gamma'\\ \gamma\neq\gamma'}}\sum_{\substack{n_0n_2\\ n_0\neq n_2}}\left(\left|\psi_{\gamma n_0}\right|^2\left|\psi_{\gamma' n_2}\right|^2-\psi_{\gamma n_0}\psi^*_{\gamma' n_0}\psi_{\gamma' n_2}\psi^*_{\gamma n_2}\right). \tag{C8}$$

■

APPENDIX D: MEAN PURITY OF INITIALLY UNCORRELATED PURE STATES

Proof. The initial state $\rho_{IBE} = |\Psi\rangle\langle\Psi|$ is such that

$$|\Psi\rangle = |\Psi_{IB}\rangle\otimes|\Psi_E\rangle = \left(\sum_i^M c_i|I_i\rangle\otimes|B_i\rangle\right)\otimes\left(\sum_j\xi_j|E_j\rangle\right), \tag{D1}$$

where $M = \min((d_B, d_I)$. By plugging the coefficients of the above equation into Eq. (12), we have that the last term in the sum must (Γ) be null (this is due to the fact that $\psi_{i(be)} \propto \delta_{ib}$, because of the Schmidt decomposition). So we have

$$\Delta - \Gamma = \sum_{\substack{i,i',(be),(be)' \\ i\neq i',(be)\neq(be)'}} [|\psi_{i(be)}|^2|\psi_{i'(be)'}|^2 - \psi_{i(be)}\psi^*_{i'(be)}\psi_{i'(be)'}\psi^*_{i(be)'}] = \sum_{\substack{jklm \\ j\neq l}} |c_j\xi_k|^2|c_l\xi_m|^2$$
$$= \sum_{\substack{jl \\ j\neq l}} |c_j|^2|c_l|^2 \sum_k |\xi_k|^2 \sum_m |\xi_m|^2 = \sum_{\substack{jl \\ j\neq l}} |c_j|^2|c_l|^2, \tag{D2}$$

where the last equality follows from the fact that the reduced state of the environment is trace-one. Hence when the initial state is separable such as is Eq. (D1), the resulting purity is

$$\langle\mu(\rho_{IB})\rangle = \frac{d_B + d_E}{d_B d_E + 1} + \frac{d_B\left(1 - d_E^2\right)}{(d_B d_E)^2 - 1} \sum_{\substack{jl \\ j\neq l}} |c_j|^2|c_l|^2. \tag{D3}$$

■

APPENDIX E: MEASURE CONCENTRATION AND LÉVY'S LEMMA

d-dimensional pure quantum states can be described as points on the surface of a $(2d-1)$-dimensional unit sphere. This can be realized by expressing a generic state in complex coordinates $|\psi\rangle = (z_1, z_2, \ldots, z_d)$, where $z_j \in \mathbb{C}$, for $j = 1, 2, \ldots, d$, with $\sum_j^d |z_j|^2 = 1$ and writing the coordinates in real components $z_j = x_j + iy_j$, so that $\sum_j^d x_j^2 + \sum_j^d y_j^2 = 1$.

Heuristically, the phenomenon of measure concentration on a unit sphere $S^{(2d-1)}$ in $\mathbb{R}^{2d}$ translates to the fact that almost all surface measure of the sphere is concentrated around the equator, for *any* equator. That is, for any random choice of a coordinate x_j, consider an equator of width ϵ,

$$E_\epsilon := \left\{x_j \in S^{(2d-1)} \,|\, d(x_j, 0) \leqslant \frac{\epsilon}{2}\right\}, \tag{E1}$$

where $d(x, y) = \arccos\langle x, y\rangle \;\; \forall x, y \in S^{(2d-1)}$ is the angular distance. Provided a normalized surface measure $\xi(S^{(2d-1)}) = 1$, it can be shown that

$$\xi(E_\epsilon) \geqslant 1 - e^{(-kd\epsilon^2)}, \tag{E2}$$

where $k > 0$ is some constant.

Measure concentration is at the basis of Lévy's lemma (Sec. IV), as we show in the following. The kind of Lévy's lemma we will sketch the derivation of here is slightly different from the one applied in the next section, but the two formulations are strictly related. For space reasons, not all the details of the calculations will be shown; the interested reader can find them in [45].

To proceed, we first need to define two quantities:

Definition 1. (Median): Let X be a metric space and $f : X \to \mathbb{R}$ a continuous function. A median M_f is defined by

$$\xi\{x \in X \,|\, f(x) \leqslant M_f\} = \tfrac{1}{2}. \tag{E3}$$

Definition 2. (Concentration function): Let X be a metric space and S a subset of it, with $\xi(S) = \frac{1}{2}$. For any $\epsilon > 0$, the concentration function is defined as

$$\alpha_X(\epsilon) := sup\{\xi(X \setminus N_\epsilon(S))\}, \tag{E4}$$

where $N_\epsilon(S)$ is the ϵ neighborhood of S:

$$N_\epsilon(S) := \{x \in X \,|\, \exists s \in S \,:\, d(s, x) < \epsilon\}. \tag{E5}$$

These definitions allow one to formulate the following lemma:

Lemma 2. Let X be a metric space and $f : X \to \mathbb{R}$ a Lipschitz-continuous function with constant 1, then

$$\xi\{x \in X \,|\, f(x) \geqslant M_f + \epsilon\} \leqslant \alpha_X(\epsilon). \tag{E6}$$

Proof. Take $S : \{x \,|\, f(x) \leqslant M_f\}$ so that $\xi(S) = 1/2$ and consider a subset $B \subseteq X$ such that $f(b) \geqslant M_f + \epsilon$, $\forall b \in B$. Because f is Lipschitz continuous, all points $x \in N_\epsilon(S)$ satisfy $f(x) < M_f + \epsilon$, so it must be $b \notin N_\epsilon(S)$, $\forall b \in B$. That means B is a subset of X: $\{b \in X \,|\, f(b) \geqslant M_f + \epsilon\} \subseteq X \setminus N_\epsilon(S)$ and thus $\xi\{x \in X \,|\, f(x) \geqslant M_f + \epsilon\} \leqslant \xi(X \setminus N_\epsilon(S)) \leqslant \alpha_X(\epsilon)$. ■

In terms of probabilities, and by rescaling of the ϵ to $\epsilon \to \epsilon' = \eta\epsilon$ for Lipschitz functions such that $|f(x) - f(y)| \leqslant \eta\|x - y\| \leqslant \eta\epsilon$, the above lemma reads

$$\text{Prob}(f(x) \geqslant M_f + \epsilon') \leqslant \alpha_X\left(\frac{\epsilon'}{\eta}\right). \tag{E7}$$

In order to calculate the value of the concentration function $\alpha_{S^{(2d-1)}}$, one needs to invoke the isoperimetric inequality for the sphere (see, e.g., [45]):

Lemma 3. (Isoperimetric inequality for the sphere): Let $A \subseteq S^{(2d-1)}$ be a closed subset of the sphere and let $C(a, r) := \{x \in X \,|\, d(a, x) \leqslant r\} \subset S^{(2d-1)}$ a spherical cap around any point $a \in S^{(2d-1)}$, with the radius r chosen such that $\xi(C(a, r)) = \xi(A)$. Then

$$\xi\left(N_\epsilon(A)\right) \geqslant \xi\left(N_\epsilon(C(a, r))\right). \tag{E8}$$

Therefore we have

$$\alpha_{S^{(2d-1)}}(\epsilon) = \sup\{\xi\left(S^{(2d-1)} \setminus N_\epsilon(S)\right)\}$$
$$= \xi(S^{(2d-1)}) - \inf\{\xi\left(N_\epsilon(S)\right)\} = 1 - \inf\{\xi\left(N_\epsilon(S)\right)\}$$
$$= 1 - \xi\left(C(a, \frac{\pi}{2} + \epsilon)\right) \leqslant e^{-d\epsilon^2}, \tag{E9}$$

where the details of the calculation leading to the last inequality can be found in [46].

So far, then, for functions f with Lipschitz constant $\eta \leqslant 1$:

$$\xi\{f(x) \geqslant M_f + \epsilon\} \leqslant \alpha_{S^{(2d-1)}}(\epsilon) \leqslant e^{-d\epsilon^2}. \tag{E10}$$

Applying Lemma 2 to the function $g(x) = -f(x)$, one gets $\xi\{g(x) \geqslant M_f - \epsilon\} \leqslant \alpha_{S^{(2d-1)}}(\epsilon)$, thus

$$\xi\{|f(x) - M_f| \geqslant \epsilon\} \leqslant 2\alpha_{S^{(2d-1)}}(\epsilon). \tag{E11}$$

By rescaling $\epsilon \to \epsilon\eta$ for functions with $\eta \geqslant 1$ and interpreting the relative measure above as a probability, we get

$$\text{Prob}\{|f(x) - M_f| \geqslant \epsilon\} \leqslant 2\text{e}^{-d\frac{\epsilon^2}{\eta^2}}. \tag{E12}$$

Finally, an inequality can be shown which relates median and expectation value of f, bringing the missing factors in the exponential which appear in the version of the Lévy's lemma we made use of in Eq. (6).

APPENDIX F: BOUND TO STATE FLUCTUATIONS

Let us now make use of Lévy's lemma to bound the fluctuations around the average, unconditional state $\langle\rho_{IB}\rangle = \rho_I \otimes \mathbb{I}/d_B$. To this aim, we intend to apply the lemma to the trace distance between states, which quantifies their operational distinguishability [37,47], in particular between the conditional state $\rho_I \otimes |\phi\rangle\langle\phi|$ under the assumption of separability between the initial inner and boundary systems [see Eq. (9)], and the unconditional, average state: $f(\phi) = \|\rho_I \otimes \text{Tr}_E(|\phi\rangle\langle\phi|) - \rho_I \otimes \mathbb{I}/d_B\|_1 = \|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_1$. Lévy's lemma applied to $f(\phi)$ reads

$$\text{P}[|\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_1 - \langle\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_1\rangle| \geqslant \epsilon] \leqslant 2\text{exp}\left(-\frac{2d_Bd_E\epsilon^2}{9\pi^3\eta^2}\right). \tag{F1}$$

It is convenient to rearrange Eq. (F1) such that we get to an expression of the form

$$\text{P}[\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_1 \geqslant \gamma] \leqslant \gamma', \tag{F2}$$

where

$$\gamma = \epsilon + \langle\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_1\rangle, \quad \gamma' = 2\text{exp}\left(-\frac{2d_Bd_E\epsilon^2}{9\pi^3\eta^2}\right). \tag{F3}$$

So in order to estimate the fluctuations around the mean state we need to bound $\langle\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_1\rangle$. Following an argument presented in [4], it is now convenient to turn to the more accessible Hilbert-Schmidt norm $\|M\|_2 = \sqrt{\text{Tr}(M^\dagger M)}$ by exploiting its relationship with the trace norm $\|M\|_1 = \text{Tr}\sqrt{M^\dagger M}$, which satisfies, for any $n \times n$ matrix M, the relation $\|M\|_1^2 \leqslant n\|M\|_2^2$. We thus have

$$\begin{aligned}\langle\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_1\rangle &\leqslant \sqrt{d_B}\langle\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_2\rangle \leqslant \sqrt{d_B\langle\|\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B\|_2^2\rangle} \\ &= \sqrt{d_B\langle\text{Tr}[(\text{Tr}_E(|\phi\rangle\langle\phi|) - \mathbb{I}/d_B)^2]\rangle} = \sqrt{d_B\langle\text{Tr}_B[(\text{Tr}_E(|\phi\rangle\langle\phi|))^2]\rangle - 1} \\ &= \sqrt{\frac{d_B^2 - 1}{d_Bd_E + 1}},\end{aligned} \tag{F4}$$

where we inserted the average local purity $(d_B + d_E)/(d_Bd_E + 1)$ in the absence of initial IB correlations [see Eq. (C8)].

To complete the Lévy's bound we are looking for, we make use of the following lemma [4].

Lemma 4. The Lipschitz constant η of the function $f(\phi) = \|\text{Tr}_E[|\phi\rangle\langle\phi|-] - \mathbb{I}/d_B\|_1$ satisfies $\eta \leqslant 2$.

Proof. One has

$$\begin{aligned}|f(\phi_1) - f(\phi_2)|^2 &= |\|\text{Tr}_E[|\phi_1\rangle\langle\phi_1|] - \mathbb{I}/d_B\|_1 - \|\text{Tr}_E[|\phi_2\rangle\langle\phi_2|] - \mathbb{I}/d_B\|_1|^2 \leqslant \|\text{Tr}_E[|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|]\|_1^2 \\ &\leqslant \||\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|\|_1 = 4(1 - |\langle\phi_1|\phi_2\rangle|^2) \leqslant 4||\phi_1\rangle - |\phi_2\rangle|^2,\end{aligned} \tag{F5}$$

where we used the nonincrease of the trace norm under partial tracing as well as the reverse triangle inequality. The last inequality is equivalent to $\eta \leqslant 2$. ∎

The lemma above allows one to upper bound the quantity γ' in Eq. (F2) and thus obtain Inequality (10) (where $\rho_{IB,\phi} \equiv \rho_I \otimes \text{Tr}[|\phi\rangle\langle\phi|]$):

$$\text{P}\left[\|\rho_I \otimes \text{Tr}[|\phi\rangle\langle\phi|] - \Phi(\rho_\phi)\|_1 \geqslant \epsilon + \sqrt{\frac{d_B^2 - 1}{d_Bd_E + 1}}\right] \leqslant 2\text{exp}\left(-\frac{d_Bd_E\epsilon^2}{18\pi^3}\right), \tag{F6}$$

which bounds the fluctuations around the average state of Eq. (4).

[1] E. Lubkin, J. Math. Phys. **19**, 1028 (1978).
[2] J. Gemmer, A. Otte, and G. Mahler, Phys. Rev. Lett. **86**, 1927 (2001).
[3] J. Gemmer, M. Michel, and G. Mahler, *Quantum Thermodynamics* (Springer, Dordrecht, 2009).
[4] S. Popescu, A. J. Short, and A. Winter, Nat. Phys. **2**, 754 (2006).
[5] D. N. Page, Phys. Rev. Lett. **71**, 1291 (1993).
[6] O. C. O. Dahlsten, C. Lupo, S. Mancini, and A. Serafini, J. Phys. A: Math. Theor. **47**, 363001 (2014).
[7] U. Singh, L. Zhang, and A. K. Pati, Phys. Rev. A **93**, 032125 (2016).
[8] A. Hayashi, T. Hashimoto, and M. Horibe, Phys. Rev. A **72**, 032325 (2005).
[9] S. Iblisdir and J. Roland, Phys. Lett. A **358**, 368 (2006).
[10] P. Hayden, M. Horodecki, A. Winter, and J. Yard, Open Syst. Inf. Dyn. **15**, 7 (2008).
[11] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, Proc. R. Soc. A **465**, 2537 (2009).
[12] J. Emerson, R. Alicki, and K. Życzkowski, J. Opt. B: Quantum Semiclass. Opt. **7**, S347 (2005).
[13] E. Knill, D. Leibfried, R. Reichle, J. Britton, R. B. Blakestad, J. D. Jost, C. Langer, R. Ozeri, S. Seidelin, and D. J. Wineland, Phys. Rev. A **77**, 012307 (2008).
[14] E. Magesan, J. M. Gambetta, and J. Emerson, Phys. Rev. A **85**, 042311 (2012).
[15] M. P. A. Fisher, V. Khemani, A. Nahum, and S. Vijay, Annu. Rev. Condens. Matter Phys. **14**, 335 (2023).
[16] V. Khemani, A. Vishwanath, and D. A. Huse, Phys. Rev. X **8**, 031057 (2018).
[17] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Phys. Rev. X **8**, 031058 (2018).
[18] A. Chan, A. De Luca, and J. T. Chalker, Phys. Rev. X **8**, 041019 (2018).
[19] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. E **90**, 012110 (2014).
[20] L. D'Alessio and M. Rigol, Phys. Rev. X **4**, 041048 (2014).
[21] P. Ponte, A. Chandran, Z. Papić, and D. A. Abanin, Ann. Phys. (NY) **353**, 196 (2015).
[22] N. Linden, S. Popescu, A. J. Short, and A. Winter, Phys. Rev. E **79**, 061103 (2009).
[23] P. Hayden and J. Preskill, J. High Energy Phys. 09 (2007) 120.
[24] B. Yoshida, Phys. Rev. D **100**, 086001 (2019).
[25] L. Piroli, C. Sünderhauf, and X.-L. Qi, J. High Energy Phys. 04 (2020) 063.
[26] C. Akers, N. Engelhardt, D. Harlow, G. Penington, and S. Vardhan, arXiv:2207.06536.
[27] I. H. Kim and J. Preskill, J. High Energy Phys. 02 (2023) 233.
[28] A. Bouland, B. Fefferman, and U. Vazirani, arXiv:1910.14646.
[29] Y. Sekino and L. Susskind, J. High Energy Phys. 10 (2008) 065.
[30] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, J. High Energy Phys. 04 (2013) 022.
[31] S. H. Shenker and D. Stanford, J. High Energy Phys. 05 (2015) 132.
[32] T. Jalabert, E. F. C. Driessen, F. Gustavo, J. L. Thomassin, F. Levy-Bertrand, and C. Chapelier, Nat. Phys. **19**, 956 (2023).
[33] M. Gurioli, A. Vinattieri, M. Zamfirescu, M. Colocci, S. Sanguinetti, and R. Nötzel, Phys. Rev. B **73**, 085302 (2006).
[34] W. Cao, Z. Zhang, R. Patterson, Y. Lin, X. Wen, B. P. Veetil, P. Zhang, Q. Zhang, S. Shrestha, G. Conibeera, and S. Huang, RSC Adv. **6**, 90846 (2016).
[35] A. R. Kolovsky and D. L. Shepelyansky, Europhys. Lett. **117**, 10003 (2017).
[36] J. Diestel and A. Spalsbury, *The Joys of Haar Measure*, Graduate Studies in Mathematics (American Mathematical Society, 2014).
[37] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[38] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[39] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.109.034114 for the numerics' scripts, which includes Refs. [40,41].
[40] G. Toth, Comput. Phys. Commun. **179**, 430 (2008).
[41] T. Cubitt, https://www.dr-qubit.org/matlab.html.
[42] S. Aubert and C. S. Lam, J. Math. Phys. **44**, 6112 (2003).
[43] W. Fulton and J. Harris, *Representation Theory* (Springer, Berlin and Heidelberg, 2004).
[44] M. Müller, *Lecture Notes on Selected Advanced Topics in Quantum Information* (University of Waterloo, 2012).
[45] M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs Vol. 89 (American Mathematical Society, 2001).
[46] M. Gerken, *Selected Topics in Mathematical Physics: Quantum Information* (2013).
[47] A. S. Kholevo, Theory Probab. Appl. **23**, 411 (1979).