# THE $\chi$-RAMSEY PROBLEM FOR TRIANGLE-FREE GRAPHS*

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**Abstract.** In 1967, Erdős asked for the greatest chromatic number, $f(n)$, amongst all $n$-vertex, triangle-free graphs. An observation of Erdős and Hajnal together with Shearer's classical upper bound for the off-diagonal Ramsey number $R(3,t)$ shows that $f(n)$ is at most $(2\sqrt{2}+o(1))\sqrt{n/\log n}$. We improve this bound by a factor $\sqrt{2}$, as well as obtaining an analogous bound on the list chromatic number which is tight up to a constant factor. A bound in terms of the number of edges that is similarly tight follows, and these results confirm a conjecture of Cames van Batenburg et al. [*Electron. J. Combin.*, 27 (2020), P2.34].

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**1. Introduction.** The classical Ramsey question for triangle-free graphs asks for the value of $R(3,t)$[1] or, equivalently, the smallest independence number amongst $n$-vertex triangle-free graphs. Every graph $G$ satisfies $\alpha(G) \cdot \chi(G) \geqslant |G|$, which suggests a natural "$\chi$-Ramsey" question first asked by Erdős [17] in 1967. He asked for the greatest chromatic number of a triangle-free graph in terms of its number of vertices, $n$, or number of edges, $m$ (see also Appendix B in the first edition of [4]). Denoting the maximum for the vertex problem by $f(n)$, an early suggestion of the correct growth rate of $f(n)$ was given by Ajtai, Komlós, and Szemerédi's [2] upper bound $R(3,t) = \mathcal{O}(t^2/\log t)$, which shows that every $n$-vertex triangle-free graph has independence number at least $\Omega(\sqrt{n \log n})$ and so *Hall ratio*

$$\rho(G) = \max_{\varnothing \neq H \subseteq G} \frac{|H|}{\alpha(H)} = \mathcal{O}(\sqrt{n/\log n}).$$

The Hall ratio is a natural lower bound for the chromatic number. Erdős and Hajnal [16] (see [21, pp. 124–125] for details) noted that iteratively pulling out the large independent sets guaranteed by Ajtai, Komlós, and Szemerédi's result and giving each one a different color matches this bound. That is, $f(n) = \mathcal{O}(\sqrt{n/\log n})$. A well-known result of Shearer [30] gives the best-known constant in the theorem of Ajtai, Komlós, and Szemerédi, which can be used to sharpen the bound to

$$f(n) \leqslant (2\sqrt{2} + o(1))\sqrt{n/\log n}.$$

The classical Ramsey and $\chi$-Ramsey problems for triangle-free graphs were considered by Kim [24], who constructed $n$-vertex, triangle-free graphs with independence number $\mathcal{O}(\sqrt{n \log n})$. Specifically, for the $\chi$-Ramsey problem in triangle-free graphs, Kim

[1]$R(3,t)$ is defined to be the least $n$ such that every $n$-vertex graph contains either a triangle or an independent set of size $t$.

proved that $f(n) \geqslant (1/9 - o(1))\sqrt{n/\log n}$, and brought attention to the upper bound that follows from Shearer's result. Improving Kim's lower bound, Fiz Pontiveros, Griffiths, and Morris [18] and Bohman and Keevash [9] followed the triangle-free process to its asymptotic end, showing that there are $n$-vertex triangle-free graphs with Hall ratio (and hence chromatic number) at least $(1/\sqrt{2} - o(1))\sqrt{n/\log n}$, so there is a factor of four between these upper and lower bounds for $f(n)$.

A bound in terms of the number of edges was given by Poljak and Tuza [29] (see also [19, 28]), who showed that every $m$-edge triangle-free graph $G$ has chromatic number $\chi(G) = \mathcal{O}(m^{1/3}/(\log m)^{2/3})$. This is tight up to a constant factor due to Kim's construction or the triangle-free process. An improved constant follows from a more refined method due to Gimbel and Thomassen [19], together with a theorem of Molloy that we discuss below. These improvements were observed in [12, Prop. 4.6] which states the bound $\chi(G) \leqslant (2^{1/3}3^{5/3} + o(1))m^{1/3}/(\log m)^{2/3}$. Gimbel and Thomassen also showed that any triangle-free graph $G$ that can be embedded on an orientable or nonorientable surface of genus $g$ has $\chi(G) = \mathcal{O}(g^{1/3}/(\log g)^{2/3})$, which is tight up to a constant factor.

Cames van Batenburg et al. [12] recently highlighted the problems of tightening the asymptotic constants above, and of giving bounds for more refined graph coloring parameters including the fractional and list chromatic numbers.

**1.1. Definitions.** Our terminology is standard, but we briefly introduce the required notions of graph coloring here. Recall that a *k-coloring* of a graph is a partition of its vertex set into $k$ independent sets (color classes). The *chromatic number* of a graph $G$, denoted $\chi(G)$, is the least $k$ for which $G$ has a $k$-coloring.

A *fractional coloring of weight at most $k$* is a probability distribution on the independent sets of a graph such that every vertex has probability at least $1/k$ of being in the random independent set. The *fractional chromatic number*, $\chi_f(G)$, of a graph $G$ is the least $k$ for which $G$ has a fractional coloring of weight at most $k$. Equivalently, $\chi_f(G)$ is the solution to a fractional relaxation of the natural integer program that gives $\chi(G)$.

Given a graph $G$, a *list assignment $L$* is an assignment of a list $L(v) \subseteq \mathbb{N}$ to each vertex $v$ of $G$. An *L-coloring of $G$* is a coloring $c \colon V(G) \to \mathbb{N}$ that is consistent with $L$ (each vertex $v$ has $c(v) \in L(v)$) and proper ($c(u) \neq c(v)$ whenever $uv$ is an edge). The *list chromatic number*, $\chi_\ell(G)$, of a graph $G$ is the least $k$ such that $G$ is $L$-colorable for any list assignment $L$ whose lists have size at least $k$.

It is straightforward to show that every $G$ has $\rho(G) \leqslant \chi_f(G) \leqslant \chi(G) \leqslant \chi_\ell(G)$ by the definition of $\rho$, considering the uniform probability distribution over the color classes in a $\chi(G)$-coloring of $G$, and by considering list assignments where every vertex is given the same list.

**1.2. Results.** Table 1 summarizes the best known upper bounds in terms of the number of vertices. Given the difficulty of improving Shearer's Ramsey number bound, one should compare upper bounds for $\chi_f$, $\chi$, $\chi_\ell$ to his bound for $\rho$.

Our first result improves the upper bound for chromatic number by a factor of $\sqrt{2}$, thus matching the bound for the fractional chromatic number established in [12]. We apply this new bound and a tactic of Gimbel and Thomassen [19] to improve the previous best bound in terms of the number of edges [12, Prop. 4.6].

THEOREM 1.1. *As $n \to \infty$, any triangle-free graph on $n$ vertices has chromatic number at most $(2 + o(1))\sqrt{n/\log n}$.*

*As $m \to \infty$, any triangle-free graph with at most $m$ edges has chromatic number*

at most $(3^{5/3} + o(1))m^{1/3}/(\log m)^{2/3}$.

TABLE 1
*Upper bounds for n-vertex triangle-free graphs G.*

| Parameter | Previous bound | | This work |
|---|---|---|---|
| $\rho(G)$ | $(\sqrt{2} + o(1))\sqrt{n/\log n}$ | [30] | |
| $\chi_f(G)$ | $(2 + o(1))\sqrt{n/\log n}$ | [12] | |
| $\chi(G)$ | $(2\sqrt{2} + o(1))\sqrt{n/\log n}$ | [30, 24] | $(2 + o(1))\sqrt{n/\log n}$ |
| $\chi_\ell(G)$ | $(2\sqrt{2} + o(1))\sqrt{n}$ | [12] | $(4\sqrt{2} + o(1))\sqrt{n/\log n}$ |

A particularly noticeable feature of the previous bounds in Table 1 is the upper bound for the list chromatic number, which does not have the same growth rate as the lower bound provided by the triangle-free process. We give a short argument to rectify this. As with the case of chromatic number, we use this new result to obtain a bound in terms of the number of edges that is tight up to a constant factor.

THEOREM 1.2. *As $n \to \infty$, any triangle-free graph on $n$ vertices has list chromatic number at most $(4\sqrt{2} + o(1))\sqrt{n/\log n}$.*

*As $m \to \infty$, any triangle-free graph with at most $m$ edges has list chromatic number at most $(12 \cdot 3^{2/3} + o(1))m^{1/3}/(\log m)^{2/3}$.*

This result confirms Conjecture 6.1 from [12], though we do not believe the given constants are tight. In particular, they are worse than what we prove for the usual chromatic number.

Turning to a bound in terms of genus as studied by Gimbel and Thomassen [19], using Theorem 1.2 we obtain the correct growth rate for the list chromatic number as well as sharpening the constant for chromatic number with an application of Theorem 1.1. The key observation underlying the proof is that the genus and number of edges are at most a constant factor apart in the critical range.

COROLLARY 1.3. *As $g \to \infty$, any triangle-free graph of genus at most $g$ has chromatic number at most $(3 \cdot 6^{2/3} + o(1))g^{1/3}/(\log g)^{2/3}$.*

*As $g \to \infty$, any triangle-free graph of genus at most $g$ has list chromatic number at most $(12 \cdot 6^{2/3} + o(1))g^{1/3}/(\log g)^{2/3}$.*

*The same bounds hold for graphs that can be embedded on a closed nonorientable surface of genus at most $2g$.*

Due to our Theorem 1.1, there is now an asymptotic factor $\sqrt{2}$ between the best known upper bounds for chromatic number and Hall ratio. Cames van Batenburg et al. [12] conjectured that it is possible to remove this $\sqrt{2}$ for the fractional chromatic number, and suggested this may be possible for the chromatic number, too. Our results establish bounds on the list chromatic number of triangle-free graphs that are tight up to the constant factor, but one might ask whether bounds of the same order hold for an even more general notion of graph colouring known as *correspondence coloring* or *DP-coloring*. In section 4 we discuss these constant factors, related work, some nice conjectures, and the correspondence coloring version of the $\chi$-Ramsey question.

**2. Proof ideas and tools.** We first outline the proof of the bound in terms of the number of vertices in Theorem 1.1. The key idea underpinning our improvement is induction, splitting into cases depending on the maximum degree of the graph. If some vertex of the graph has large degree, then we give its neighborhood (which

is an independent set by triangle-freeness) one color and color the remainder of the graph by induction. Otherwise, the graph has small maximum degree and so we may apply a result bounding the chromatic number of a triangle-free graph in terms of its maximum degree. We will use a recent groundbreaking result of Molloy [27], which is a chromatic strengthening of Shearer's lower bound for the independence number.

THEOREM 2.1 (Molloy). *As $\Delta \to \infty$, any triangle-free graph of maximum degree $\Delta$ has (list) chromatic number at most $(1 + o(1))\Delta/\log \Delta$.*

We are now in position to sketch the proof of Theorem 1.1. We will ignore all $o(1)$ terms. Let $G$ be an $n$-vertex triangle-free graph—we are trying to prove that $\chi(G) \leqslant 2\sqrt{n/\log n}$ and will assume the result holds for all smaller $n$. First, if $G$ has maximum degree at most $d(n) := \sqrt{n \log n}$, then Molloy's theorem immediately gives the result. Otherwise, some vertex $v$ of $G$ has degree greater than $d(n)$. Let $G'$ be $G$ with all neighbors of $v$ deleted. As $G$ is triangle-free, the neighborhood of $v$ is an independent set, so $\chi(G) \leqslant \chi(G') + 1$. Also, $n' = |G'| < n - d(n)$ and induction gives

$$\chi(G') \leqslant 2\sqrt{n'/\log n'}.$$

For this sketch, consider $\log n'$ and $\log n$ as identical and so

$$\chi(G') \leqslant 2\sqrt{\frac{n - d(n)}{\log n}} \leqslant 2\sqrt{\frac{n}{\log n}} - 1,$$

where the final inequality follows by squaring both sides and cancelling terms. As $\chi(G) \leqslant \chi(G') + 1$, we are done. The full proof has no new ideas, we merely have to overcome the technical challenge of the $o(1)$ terms. We do this in section 3, as well deriving the bound in terms of the number of edges.

Consider trying the same proof strategy for the list chromatic number, in pursuit of Theorem 1.2. If there is a vertex of large degree, then we cannot necessarily color its neighborhood with one color as there may be no color appearing on the lists of all its neighbors. In place of degree we use the notion of *color-degree*.

DEFINITION 2.2. *Let $G$ be a graph with list-assignment $L$. For a vertex $v$ and a color $c \in L(v)$, the* color-degree *of $c$ at $v$ is*

$$\deg_L(v, c) = |\{u \colon uv \in E(G), \ c \in L(u)\}|.$$

Suppose that we have a graph $G$ and list assignment $L$ which assigns lists of size $k$ to the vertices of $G$. If some color-degree, say $\deg_L(v, c)$, is large, then we color the neighbors of $v$ whose lists contain $c$ with color $c$, remove $c$ from all other lists and delete the colored vertices. What remains is a graph $G'$ of order $n - \deg_L(v, c)$, and crucially the graph $G$ admits an $L$-coloring provided $\chi_\ell(G') \leqslant k - 1$. Indeed, set $L'(u) = L(u) \setminus \{c\}$ for every vertex $u$ and note that if $\chi_\ell(G') \leqslant k - 1$, then $G'$ is $L'$-colorable and the vertices in $V(G) \setminus V(G')$ can all be colored with $c$. We bound $\chi_\ell(G')$ from above by induction. In order to carry out the same argument as above, we need an analogue of Molloy's theorem with $\Delta$ replaced by the maximum color-degree, and such results (with larger leading constants) have been proved by Amini and Reed [5] and Alon and Assadi [3].

THEOREM 2.3 (Alon–Assadi). *The following holds for all sufficiently large $d$. Let $G$ be a triangle-free graph with lists $L(v)$ for every vertex $v$. If, for every vertex $v$*

*and color $c \in L(v)$,*

$$|L(v)| \geqslant 8d/\log d,$$
$$\deg_L(v, c) \leqslant d,$$

*then $G$ admits an $L$-coloring.*

If the factor of 8 could be replaced with $1 + o(1)$, then our bound for the list chromatic number would match those for the fractional and usual chromatic number, that is, be a factor $\sqrt{2}$ away from the bound for the Hall ratio. We discuss this further in section 4.

The bound in terms of the number of edges given in Theorem 1.2 follows from the bound in terms of the number of vertices, via a simple argument that resembles its counterpart for the chromatic number and uses a list-partitioning idea present in [12]. We give this argument in the next section.

**3. The proofs.** In this section we give the proofs of Theorems 1.1 and 1.2. We stress that for the bounds in terms of $n$ the main ideas appeared in section 2, and what remains is a technical exercise.

*Proof of Theorem* 1.1. Let $f(x) = (2 + A(x))\sqrt{x/\log x}$, where $A = o(1)$ is smooth, nonnegative and nonincreasing (specified more precisely later). For the first stated bound it suffices to show that any $n$-vertex triangle-free graph $G$ has chromatic number at most $f(n)$. We will induct upon $n$, and we may choose $A$ so that the theorem holds for all $n \leqslant 20$. Assume from now on that $n \geqslant 20$.

First, suppose that every vertex of $G$ has degree at most $d(n) = \sqrt{n \log n}$. Then, by Theorem 2.1 there is an $\varepsilon(x) = o(1)$ such that

$$\chi(G) \leqslant (1 + \varepsilon(n)) \frac{d(n)}{\log d(n)} \leqslant (1 + \varepsilon(n)) \frac{d(n)}{\log(n^{1/2})}$$
$$= 2(1 + \varepsilon(n)) \sqrt{\frac{n}{\log n}}.$$

Thus, we are done in this case provided

$$(3.1) \qquad\qquad A(x) \geqslant 2\varepsilon(x).$$

In the second case there is some vertex $v$ with degree greater than $d(n)$. Let $G'$ be the graph obtained from $G$ by deleting all the neighbors of $v$. Then $G'$ has fewer than $n - d(n)$ vertices and

$$\chi(G) \leqslant \chi(G') + 1 \leqslant f(n - d(n)) + 1,$$

where the second inequality follows by induction. Hence, to complete the proof we need

$$(3.2) \qquad\qquad f(n) - f(n - d(n)) \geqslant 1$$

for all $n \geqslant 20$. It remains to check that it is possible to choose an $A$ such that (3.1) and (3.2) hold. We will assume that $A$ decays sufficiently slowly so that (3.1) holds.

The function $\sqrt{x/\log x}$ is concave for $x \geqslant 6$. If we choose the nonnegative function $A$ decaying sufficiently slowly, then $f$ ought to be concave, too. Indeed, if we choose

$A$ so that $|A''(x)| \leqslant 1/(10x^2)$, then a quick calculation[2] shows that for all $x \geqslant 10$, $f''(x) < 0$ (note that $A' \leqslant 0 \leqslant A$). By concavity, for all $n \geqslant 20$,

$$f(n) - f(n - d(n)) \geqslant f'(n)d(n) = \left(1 + \tfrac{A(n)}{2}\right)\left(1 - \tfrac{1}{\log n}\right) + nA'(n).$$

Choosing $A$ so that $|A'(x)| \leqslant 1/(x \log x)$ and $A(x) \geqslant 8/\log x$ gives (3.2). One should worry that the conditions we have placed on the derivatives of $A$ might preclude it from tending to zero. Happily, integrating these shows that this is not the case.

Now for the bound in terms of the number of edges. We apply the method of Gimbel and Thomassen [19]. For $d = ((m \log m)/3)^{1/3}$, let $V_1 \subseteq V(G)$ be those vertices of degree at most $d$, and let $V_2 = V(G) \setminus V_1$.

The subgraph of $G$ induced by $V_1$ can be properly colored with at most $(1 + o(1))d/\log d$ colors by Theorem 2.1. On the other hand, $2m \geqslant \sum_{v \in V_2} \deg(v) \geqslant |V_2|d$ and so $|V_2| \leqslant 2m/d = 2 \cdot 3^{1/3} \cdot m^{2/3}/(\log m)^{1/3}$. Applying the first part of this theorem we obtain

$$\chi(G) \leqslant (1 + o(1))\frac{d}{\log d} + (2 + o(1))\sqrt{\frac{|V_2|}{\log|V_2|}} \leqslant (3^{3/5} + o(1))\frac{m^{1/3}}{(\log m)^{2/3}},$$

which concludes the proof. $\qquad\square$

Before we prove Theorem 1.2, we recall the Chernoff bound.

THEOREM 3.1 (Chernoff bound, [20, Cor. 2.3]). *Fix $p \in (0, 1)$, and let the random variable $X \sim \mathrm{Bin}(k, p)$ be binomially distributed. Then for $\varepsilon \in (0, 1)$,*

$$\Pr(X \leqslant (1 - \varepsilon)kp) \leqslant 2e^{-\varepsilon^2 kp/3}.$$

*Proof of Theorem* 1.2. Let $g(x) = (4\sqrt{2} + B(x))\sqrt{x/\log x}$, where $B = o(1)$ is smooth, nonnegative, and nonincreasing. We may choose $B$ so that Theorem 1.2 holds for all small $n$. Assume from now on that $n \geqslant n_0$ for some fixed $n_0$.

Associate with each vertex $v$ a list $L(v)$ of colors such that $|L(v)| \geqslant g(n)$. It suffices to show that there is a proper coloring of $G$ from these lists. First, suppose all color-degrees are at most $d(n) = \sqrt{2}/4 \cdot \sqrt{n \log n}$. Now

$$\frac{8d(n)}{\log d(n)} \leqslant \frac{8d(n)}{\log(n^{1/2})} \leqslant g(n),$$

where the first inequality holds provided $n_0 \geqslant e^8$. Provided $n_0$ is large enough, Theorem 2.3 guarantees that there is an $L$-coloring of $G$.

Otherwise, there is some vertex $v$ and color $c \in L(v)$ with $\deg_L(v, c) > d(n)$. Let $G'$ be the graph obtained from $G$ by deleting all the neighbors of $v$ with color $c$ on their list. Then $G'$ has fewer than $n - d(n)$ vertices, and $G$ admits an $L$-coloring provided $\chi_\ell(G') \leqslant g(n) - 1$. By induction, we have $\chi_\ell(G') \leqslant g(n - d(n))$ and hence to show $\chi_\ell(G) \leqslant g(n)$ it suffices to prove that $g(n) - g(n - d(n)) \geqslant 1$. We finish as in the proof of Theorem 1.1: choosing the nonnegative function $B$ so that $|B''(x)| \leqslant 1/(10x^2)$ guarantees that $g$ is concave for $x \geqslant 10$. Thus, for $n \geqslant 20$

$$g(n) - g(n - d(n)) \geqslant g'(n)d(n)$$
$$= \left(1 + B(n) \cdot \tfrac{\sqrt{2}}{8}\right)\left(1 - \tfrac{1}{\log n}\right) + nB'(n) \cdot \tfrac{\sqrt{2}}{4}.$$

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[2] With $g(x) = \sqrt{x/\log x}$ we have $f'(x) = (2 + A(x))g'(x) + A'(x)g(x)$ and $g'(x) = \frac{\log x - 1}{2\sqrt{x(\log x)^3}}$.

Choosing $B$ so that $|B'(x)| \leqslant 1/(x \log x)$ and $B(x) \geqslant 32/\log x$ gives $\chi_\ell(G) \leqslant g(n)$, as required.

For the second part of Theorem 1.2, we partition $V(G)$ according to the largest color-degree at each vertex. Let $\varepsilon > 0$ be arbitrary, and without loss of generality suppose that $\varepsilon < 1/2$ and $m$ is sufficiently large. Let $G$ be a triangle-free graph on $n$ vertices with $m$ edges, and consider a list assignment $L$ giving each vertex a list of size at least

$$k = (1 + \varepsilon)^2 \cdot 12 \cdot 3^{2/3} \frac{m^{1/3}}{(\log m)^{2/3}}.$$

For $d = ((m \log m)/24)^{1/3}$, consider the partition of $V(G)$ given by

$$V_1 = \left\{ v \in V(G) \colon \max_{c \in L(v)} \deg_L(v, c) \leqslant d \right\},$$

$$V_2 = V(G) \setminus V_1.$$

Every vertex in $V_2$ has some color-degree at least $d$ and so has degree at least $d$. Hence, degree-counting gives $|V_2| \leqslant 2m/d$. We partition the colors in $\bigcup_{v \in V(G)} L(v)$ into two parts $L_1$ and $L_2$, and let $L_i(v) = L(v) \cap L_i$. We insist on using the colors in $L_1$ to color $G[V_1]$ and those in $L_2$ to color $G[V_2]$. Assume for now that every vertex $v$ satisfies

(3.3)
$$|L_1(v)| \geqslant (4 \cdot 3^{2/3} + \varepsilon) \frac{m^{1/3}}{(\log m)^{2/3}},$$

$$|L_2(v)| \geqslant (8 \cdot 3^{2/3} + \varepsilon) \frac{m^{1/3}}{(\log m)^{2/3}}.$$

For every vertex $v \in V_1$ and $c \in L_1(v)$: $\deg_{L_1}(v, c) \leqslant \deg_L(v, c) \leqslant d$ and

$$\frac{8d}{\log d} \leqslant (4 \cdot 3^{2/3} + o(1)) \frac{m^{1/3}}{(\log m)^{2/3}} \leqslant |L_1(v)|.$$

By Theorem 2.3, $G[V_1]$ is $L_1$-colorable (provided $m$ is large enough). Next, by the first part of this theorem,

$$\chi_\ell(G[V_2]) \leqslant (4 + o(1)) \sqrt{\frac{2|V_2|}{\log|V_2|}} \leqslant (8 \cdot 3^{2/3} + o(1)) \frac{m^{1/3}}{(\log m)^{2/3}},$$

so $G[V_2]$ is $L_2$-colorable. As $L_1$ and $L_2$ are disjoint sets of colors, these colorings can be combined to give an $L$-coloring of $G$.

We finally check that there is a partition $L_1 \cup L_2$ of the colors for which (3.3) holds. Each color $c \in \bigcup_{v \in V(G)} L(v)$ is independently placed in $L_1$ with probability $1/3$ and is otherwise placed in $L_2$. By the Chernoff bound, each vertex $v \in V(G)$ has

$$\Pr\left( |L_1(v)| \leqslant (4 \cdot 3^{2/3} + \varepsilon) \frac{m^{1/3}}{(\log m)^{2/3}} \right) \leqslant \Pr(|L_1(v)| \leqslant (1 - \varepsilon)k/3),$$

$$\leqslant 2e^{-\varepsilon^2 k/9}.$$

Similarly, each vertex $v$ has

$$\Pr\left( |L_2(v)| \leqslant (8 \cdot 3^{2/3} + \varepsilon) \frac{m^{1/3}}{(\log m)^{2/3}} \right) \leqslant 2e^{-2\varepsilon^2 k/9}.$$

We may assume that $G$ has no vertices of degree 0 or 1, as any such vertices can be removed before coloring $G$ and compatible colors found when they are readded. Then $G$ has at most $m$ vertices, and so $k > n^{1/3}/(\log n)^{2/3}$ and $2e^{-\varepsilon^2 k/9} = o(1/n)$. A union bound over the $n$ vertices of $G$ now gives that there is a partition of the lists of colors that satisfies (3.3). $\square$

We conclude this section with the short proof of Corollary 1.3, which is simply a slightly more precise version of the argument given in [19].

*Proof of Corollary* 1.3. Let $G$ be a graph on $n$ vertices, with $m$ edges and genus at most $g$. We may assume that $G$ is connected. By removing vertices of degree at most $d = g^{1/3}(\log g)^{-2/3}$, which can then be colored as they are readded, we may assume that $G$ has minimum degree at least $d$. Then the number $m$ of edges of $G$ satisfies $2m \geqslant dn$, and hence as $g \to \infty$ we have $n = o(m)$.

Now consider an embedding of $G$ on an orientable surface of genus $g$ such that the drawing of $G$ has $r$ regions. Since $G$ is triangle-free, each region is surrounded by at least four edges. Also, each edge bounds at most two regions and so $r \leqslant m/2$. Euler's formula gives $n - m/2 \geqslant n - m + r \geqslant 2 - 2g$, and hence $m \leqslant (4+o(1))g$. The corollary now follows by applications of Theorems 1.1 and 1.2. For nonorientable genus the argument is the same, but we must use Euler's formula in the form $n - m + r \geqslant 2 - k$, where $k$ is the nonorientable genus. $\square$

## 4. Related research and open problems.

**4.1. Fractional coloring.** Cames van Batenburg et al. [12, Conjs. 4.3 and 4.4] conjectured that upper bounds for the fractional chromatic number in terms of $n$ and $m$ should match Shearer's bound on the Hall ratio in triangle-free graphs. Recall that a fractional coloring of weight at most $k$ is a probability distribution on independent sets such that for every vertex $v$ we have $\Pr(v \in I) \geqslant 1/k$ and so it can be particularly useful to study distributions with a "local" lower bound on $\Pr(v \in I)$ that depends on parameters such as $\deg(v)$.

In a triangle-free graph on $n$ vertices, taking a uniform random neighborhood gives $\Pr(v \in I) = \deg(v)/n$ and combining this distribution with one derived from a suitable local version of Molloy's theorem should perform well. A local fractional coloring result from [13] associates to each vertex $v$ of a triangle-free graph a subset $w(v)$ of the positive reals of measure 1 such that $w(u)$ and $w(v)$ are disjoint for edges $uv$, and $w(v) \subset [0, m_v)$ for some $m_v = (1 + o(1))\deg(v)/\log\deg(v)$. Choosing a positive real number at random with a nonincreasing density function such as

$$p(r) = \max\left\{0, \sqrt{\frac{n}{\log n}} - \frac{r}{2}\frac{n}{\log n}\right\}$$

gives a random independent set $\{v\colon r \in w(v)\}$ which is more likely to contain vertices of lower degree. Combining this with the distribution obtained by taking the neighborhood of a uniformly random vertex, one recovers the fractional case of Theorem 1.1 (originally proved in [12] by a slightly more involved argument).

Kelly and Postle [23] conjectured an improved (and more natural) version of a distribution which favors independent sets containing low degree vertices, and specifically noted that combining this with the "random neighborhood" distribution matches Shearer's bound. Their conjecture is that every triangle-free graph admits a probability distribution on independent sets such that for every vertex $v$, $\Pr(v \in I) \geqslant (1 - o(1))\log\deg(v)/\deg(v)$.

**4.2. List and correspondence coloring.** A generalization of list coloring known as *correspondence coloring* or *DP-coloring* permits an arbitrary matching of colors from $L(u)$ to $L(v)$ to be "forbidden" at the edge $uv$. Formally, given a list assignment $L$ for a graph $G$, we consider the sets $\hat{L}(v) = \{(v, c) : c \in L(v)\}$ and a graph $H$ on vertex set $\bigcup_{v \in V(G)} \hat{L}(v)$. To form $H$ we put a clique on each $\hat{L}(v)$ and for all edges $uv \in E(G)$ and colors $c \in L(u) \cap L(v)$ we connect each $(u, c)$ to $(v, c)$. This gives a *cover graph* such that independent sets in $H$ of size $|V(G)|$ in $H$ are in 1-to-1 correspondence with the $L$-colorings of $G$. Correspondence coloring arises when we relax the requirements on the edges of $H$ between each $\hat{L}(u)$ and $\hat{L}(v)$, instead allowing an arbitrary matching between $\hat{L}(u)$ and $\hat{L}(v)$ for each edge $uv$ of $G$. The *correspondence chromatic number* of $G$, denoted $\chi_c(G)$, is then the least $k$ such that whenever we start with a list assignment with lists of size $k$ and construct a cover $H$ in this fashion with arbitrary matchings, the resulting $H$ contains an independent set of size $|V(G)|$. This definition and comparison to list coloring shows that we have $\chi_\ell(G) \leqslant \chi_c(G)$ for any graph $G$. Correspondence coloring is well studied in our setting, for example, Bernshteyn [7] showed that Theorem 2.1 holds for the correspondence chromatic number. Going further, Cambie and Kang [11] conjectured a version of Theorem 2.3 for correspondence coloring and with eight replaced by $1 + o(1)$, having proved this in the special case of bipartite graphs.

Local versions of the results we rely on are natural in the settings of list and correspondence coloring, where the necessary lower bound on $|L(v)|$ is a function of $\deg(v)$. See [10, 13, 15] for bounds in terms of degrees, and [22, Conj. 9.3.2] for a conjectured bound in terms of color-degrees for list coloring. It is natural to combine [22, Conj. 9.3.2] and [11, Conj. 4] and propose a local color-degree version of Theorem 2.1 in the case of correspondence coloring (though some care with a minimum list size is necessary; see [13]).

It is interesting to note that the solution to the "$\chi_c$-Ramsey" problem for triangle-free graphs follows easily (up to a constant factor) from results of Bernshteyn [6, 7] and Král, Pangrác, and Voss [25]. The independent works [6, 25] contain the lower bound $\chi_c(G) \geqslant (1/2 - o(1))d/\log d$ for arbitrary graphs $G$ of average degree $d$, so a balanced $n$-vertex complete bipartite graph has $\chi_c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \geqslant (1/4 - o(1))n/\log n$. Then in [7] Bernshteyn gave the strengthening of Theorem 2.1 to correspondence coloring and, since the maximum degree is at most the number of vertices, showed that when $G$ is an $n$-vertex triangle-free graph we have $\chi_c(G) \leqslant (1 + o(1))n/\log n$. This is therefore tight up to a factor of at most 4. This correspondence coloring problem behaves very differently from the list coloring problem: balanced complete bipartite graphs are perhaps the extremal example and the extremal value is $\Theta(n/\log n)$ rather than its square root.

**4.3. Relaxing the triangle-free condition.** Coloring graphs with sparse neighborhoods is natural generalization of the problems we mention here. A triangle-free graph has independent neighborhoods, but this can be relaxed to neighborhoods inducing a bounded number of edges, or bounded (fractional) chromatic number, Hall ratio, or clique number.[3] A classic problem in Ramsey theory is to upper bound the Hall ratio of $K_4$-free graphs in terms of the number of vertices, where the best-known bound is due to Li, Rousseau, and Zang [26]. We do not dwell on the $\chi$-Ramsey version of this problem as there is still a polynomial gap between this upper bound

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[3]Consider a sequence of assumptions inspired by the inequalities $\omega \leqslant \rho \leqslant \chi_f \leqslant \chi$ in subgraphs induced by neighborhoods.

and the lower bound due to Bohman [8] proved by analyzing the $K_4$-free process, so improving the constant factor with our methods is an unedifying prospect.

We focus on the $\chi$-Ramsey question for triangle-free graphs because known upper and lower bounds on the Hall ratio differ by only a constant factor, and we now match the growth rate for the list chromatic number, too. Much less is known for more general sparse neighborhood conditions and there are usually large gaps between known upper and lower bounds on Hall ratio. Notably, the case of graphs in which neighborhoods induce a bounded number of edges was settled up to a small constant factor in [14]. One can always transfer upper bounds on the Hall ratio to ones on chromatic number by iteratively pulling out large independent sets as color classes (see [21, pp. 124–125] or [12, Lem. 4.1]), and this raises the question of whether our methods can improve upon such arguments. Broadly, the answer is yes as generalizations of Theorem 2.1 for these settings with good leading asymptotic constants have been given in [1, 15] which means that one can prove analogues of our Theorem 1.1 with the same method. Rather than chasing constants in the upper bounds on chromatic number, we suggest that it would be interesting to focus on improving upper bounds for the Hall ratio, and to find good lower bounds. Both of these are deep and challenging problems.

The situation for list chromatic number is rather different, however, as there is no obvious, generic way to transfer bounds on the Hall ratio to list chromatic number without losing some factor of $(\log n)^c$ (see, e.g., [12, Thm. 6.4] for an argument that can be generalized). The "$\chi_\ell$-Ramsey" question for graphs in which neighborhoods induce a bounded number of edges was, in fact, already asked in [14]. Our proof of Theorem 1.2 shows how one might match the growth rate of Hall ratio bounds in this setting and others, but we do not have the necessary analogues of Theorem 2.3 (which gives a bound in terms of color-degree) in the more general sparse neighborhood settings. The $\chi_\ell$-Ramsey problem motivates the pursuit of such results, though they are certainly worth investigating in their own rights, too.

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