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The chromatic profile of locally bipartite graphs

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ABSTRACT

In 1973, Erdős and Simonovits asked whether every n-vertex triangle-free graph with minimum degree greater than $1/3 \cdot n$ is 3-colourable. This question initiated the study of the chromatic profile of triangle-free graphs: for each k, what minimum degree guarantees that a triangle-free graph is k-colourable. This problem has a rich history which culminated in its complete solution by Brandt and Thomassé. Much less is known about the chromatic profile of H-free graphs for general H.

Triangle-free graphs are exactly those in which each neighbourhood is one-colourable. Locally bipartite graphs, first mentioned by Łuczak and Thomassé, are the natural variant of triangle-free graphs in which each neighbourhood is bipartite. Here we study the chromatic profile of locally bipartite graphs. We show that every n-vertex locally bipartite graph with minimum degree greater than $4/7 \cdot n$ is 3-colourable ($4/7$ is tight) and with minimum degree greater than $6/11 \cdot n$ is 4-colourable. Although the chromatic profiles of locally bipartite and triangle-free graphs bear some similarities, we will see there are striking differences.

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1. Introduction

The Andrásfai-Erdős-Sós theorem is a classical result which considers the chromatic number of dense graphs (those with large minimum degree) that do not contain the clique K_{r+1} (for a short proof see Brandt [5]). It can be viewed as a minimum degree analogue of Erdős and Simonovits's stability theorem [9,10,20] for the structure of K_{r+1}-free graphs with close to the maximum number of edges.

Theorem 1 *(Andrásfai-Erdős-Sós, [3]). Let $r \geqslant 2$ and G be a K_{r+1}-free graph with n vertices and minimum degree greater than*

$$\left(1 - \frac{1}{r - 1/3}\right) \cdot n.$$

Then G is r-colourable. Furthermore $1 - 1/(r - 1/3)$ is tight.

At around the same time, Erdős and Simonovits [11] implicitly posed a very general problem: for a fixed graph H and positive integer k, compute

$$\delta_\chi(H, k) = \inf\{d \colon \text{if } \delta(G) \geqslant d|G| \text{ and } G \text{ is } H\text{-free, then } \chi(G) \leqslant k\}.$$

Theorem 1 says exactly that $\delta_\chi(K_{r+1}, r) = 1 - 1/(r - 1/3)$. The values of $\delta_\chi(H, k)$ as k varies form the *chromatic profile* of the family of H-free graphs. Erdős and Simonovits stated that determining the chromatic profile for general H was 'too complicated' and this sentiment was reiterated recently by Allen, Böttcher, Griffiths, Kohayakawa and Morris [1] – indeed, for many H, even the chromatic number of the H-free graph with the most edges is unknown. The chromatic profile of triangle-free graphs has been extensively studied [3,4,7,8,13,16,17,21] and is now known (and has been extended to K_{r+1}-free graphs [12,19]). Table 1 summarises this concisely.

Table 1
Chromatic profile of a triangle-free graph G.

$\delta(G)\lvert G\rvert^{-1} >$	$2/5$	$10/29$	$1/3$	$1/3 - \varepsilon$
$\chi(G) \leqslant$	2	3	4	∞

Clearly the sequence $\delta_\chi(H, k)$ is decreasing and so must tend to a limit, called the *chromatic threshold* of H-free graphs. Table 1 exhibits an interesting phenomenon: $\delta_\chi(K_3, k) = 1/3$ for all $k \geqslant 4$ and so the chromatic threshold of triangle-free graphs is $1/3$. More generally, the chromatic threshold of H-free graphs is defined as

$$\begin{aligned}\delta_\chi(H) &= \inf_k \delta_\chi(H, k)\\ &= \inf\{d \colon \exists C = C(H, d) \text{ such that if } \delta(G) \geqslant d|G| \text{ and } G \text{ is } H\text{-free, then } \chi(G) \leqslant C\}.\end{aligned}$$

Much more is known about the chromatic threshold than the chromatic profile. Indeed, in [1], Allen et al. determined the chromatic threshold of H-free graphs for every non-bipartite graph H. They noted that there are other natural families of graphs which are beyond the reach of their techniques. For a family of graphs, $\mathcal{F}$, one defines the chromatic profile and threshold analogously.

$$\begin{aligned}\delta_\chi(\mathcal{F},k) &= \inf\{d\colon \text{if } \delta(G)\geqslant d|G| \text{ and } G\in\mathcal{F}, \text{then } \chi(G)\leqslant k\},\\ \delta_\chi(\mathcal{F}) &= \inf_k \delta_\chi(\mathcal{F},k)\\ &= \inf\{d\colon \exists C=C(\mathcal{F},d) \text{ such that if } \delta(G)\geqslant d|G| \text{ and } G\in\mathcal{F}, \text{then } \chi(G)\leqslant C\}.\end{aligned}$$

In [15] we extend the Andrásfai-Erdős-Sós theorem to non-complete graphs. A certain family of graphs plays a central role. This family is a natural extension of triangle-free graphs and has previously appeared in the literature. Triangle-free graphs are exactly those in which each neighbourhood is independent (i.e. 1-colourable). Graphs in which each neighbourhood is 2-colourable (or 3-colourable ...) are termed *locally bipartite* (*locally tripartite* ...). Locally bipartite graphs were first mentioned a decade ago by Łuczak and Thomassé [18] who asked for their chromatic threshold, conjecturing it was 1/2. This was confirmed by Allen et al. [1]. However, in contrast to the well-studied case of triangle-free graphs, the chromatic profile of locally bipartite graphs, and more generally that of locally b-partite graphs ($b\geqslant 3$), has not previously been examined. In this paper we focus on the chromatic profile of locally bipartite graphs, deferring the chromatic threshold and profile of locally b-partite graphs to [14].

Locally bipartite graphs, just like triangle-free ones, exhibit a spectrum of thresholds. There are, however, some interesting differences which we explore in Section 1.2. Our understanding of their profile is summarised in our main result, Theorem 2. The graphs $\overline{C}_7$ and H_2^+ can be seen in Fig. 1 where they are discussed more thoroughly – for now it suffices to note that they are both small 4-chromatic locally bipartite graphs which are edge-maximal with respect to local bipartiteness.

Theorem 2 *(Locally bipartite graphs). Let G be a locally bipartite graph.*

- *If $\delta(G) > 4/7\cdot|G|$, then G is 3-colourable.*
- *If $\delta(G) > 5/9\cdot|G|$, then there is a homomorphism $G\to\overline{C}_7$.*
- *There is an absolute constant $\varepsilon>0$ such that if $\delta(G) > (5/9-\varepsilon)\cdot|G|$, then there is either a homomorphism $G\to\overline{C}_7$ or $G\to H_2^+$.*
- *If $\delta(G) > 6/11\cdot|G|$, then G is 4-colourable.*

Furthermore 4/7 and 5/9 are tight, as demonstrated by balanced blow-ups of $\overline{C}_7$ and suitable blow-ups of H_2^+ (see Fig. 2 on page 351), respectively.

The first bullet point corresponds to the $r=2$ case of Theorem 1: every n-vertex triangle-free graph with minimum degree greater than $2/5\cdot n$ is bipartite. That result has a very short proof, while the first bullet point requires a more substantial argument whose sketch precedes Section 2.1. Theorem 2 is a summary of our understanding, see the start of Section 2 for further details.

Theorem 2 gives the following information about the profile of the family of locally bipartite graphs, which we denote by $\mathcal{F}_{1,2}$:

$$\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7, \quad \delta_\chi(\mathcal{F}_{1,2}, 4) \leqslant 6/11.$$

As mentioned above, Allen et al. [1] and Łuczak and Thomassé [18] showed that $\delta_\chi(\mathcal{F}_{1,2}) = 1/2$. In [14], we will extend Theorem 2 below 6/11 to give more structural (but not colourability) results for locally bipartite graphs (essential for our analysis of locally b-partite graphs and extension of Theorem 1).

1.1. The graphs

Throughout the paper the following graphs will appear frequently and here we note a few of their properties to acquaint the reader.

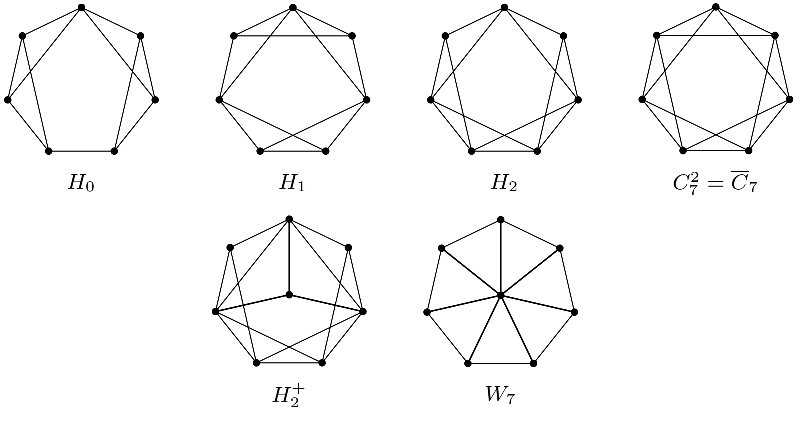


Fig. 1. The graphs appearing in Theorem 2 and its proof.

- All graphs shown are 4-chromatic and all bar W_7 are locally bipartite.
- The graph H_0 is isomorphic to the *Moser Spindle* – the smallest 4-chromatic unit distance graph. H_0 is also the smallest 4-chromatic locally bipartite graph and so it is natural that it should play such an integral part in many of our results. The graph $\overline{C}_7$ is the complement (and also the square) of the 7-cycle.
- Adding a single edge to H_0 while maintaining local bipartiteness can give rise to two non-isomorphic graphs, one of which is H_1. The other will appear fleetingly in Section 2.1. Adding a single edge to H_1 while maintaining local bipartiteness gives rise to a unique (up to isomorphism) graph – H_2. There is only one way to add a single edge to H_2 and maintain local bipartiteness – this gives $\overline{C}_7$. H_2^+ is H_2 with a degree 3 vertex added.

- $\overline{C}_7$ and H_2^+ are both edge-maximal locally bipartite graphs.
- W_7 is called the *7-wheel*. More generally, a single vertex joined to all the vertices of a k-cycle is called a *k-wheel* and is denoted by W_k. We term any edge from the central vertex to the cycle a *spoke* of the wheel and any edge of the cycle a *rim* of the wheel. Note that a graph is locally bipartite exactly if it does not contain any odd wheel (there is no such nice characterisation for a graph being locally tripartite, locally 4-partite, . . .).

The following observation gives a useful link between local bipartiteness and some of these graphs. We will use it frequently when copies of H_0, H_1, H_2 or $\overline{C}_7$ appear.

Remark 3. Any five vertices of H_0 contain a triangle or a 5-cycle. In particular, if G is a locally bipartite graph, then any vertex has at most four neighbours in any copy of H_0 appearing in G.

It will sometimes be useful to check whether there is a homomorphism from one locally bipartite graph to another. This is done with the following lemma. Two vertices of a graph are called *twins* if they have the same neighbourhood (in particular, twins cannot be adjacent) and we say a graph is *twin-free* if no two vertices have the same neighbourhood.

Lemma 4. *Let F be a twin-free, edge-maximal locally bipartite graph and let G be a locally bipartite graph. If there is a homomorphism $F \to G$, then F is an induced subgraph of G.*

Proof. Let φ: $F \to G$ be a homomorphism. If φ is injective, then F is a subgraph of G. But G is locally bipartite and F is edge-maximal locally bipartite, so any copy of F appearing in G must be induced. If φ is not injective, then there are distinct vertices u, v of F with $\varphi(u) = \varphi(v)$. The neighbourhoods of u and v are not the same, so we may assume there is a vertex w in F with w adjacent to v but not u. But then φ is a homomorphism from $F + uw$ to G: $\varphi(u) = \varphi(v)$ and $\varphi(v)$ is adjacent to $\varphi(w)$.

However, F is edge-maximal locally bipartite and uw is not an edge of F, so $F + uw$ is not locally bipartite. Hence φ is a homomorphism from a graph which is not locally bipartite, $F + uw$, to a locally bipartite graph, G, which is absurd. □

1.2. Comparison with triangle-free graphs and open questions

Here we explain the structural results behind the complete determination of the chromatic profile of triangle-free graphs and compare them to those in Theorem 2.

To properly discuss the chromatic profile of triangle-free and locally bipartite graphs it is useful to define blow-ups and weighted graphs. Given a graph G, a *blow-up* of G is a graph obtained by replacing each vertex v of G by a non-empty independent set I_v and each edge uv by a complete bipartite graph between classes I_u and I_v. We note in passing that a graph has the same chromatic and clique numbers as any of its blow-ups and furthermore if H is blow-up of G, then G is locally bipartite if and only if H is. We say a vertex v has been blown-up by n if $|I_v| = n$. It is often helpful to think of this as *weighting* vertex v by n. To be precise, a *weighted graph* (G, ω) is a graph G together with a weighting $\omega\colon V(G) \to \mathbb{R}^+$ and so (G, ω) can be viewed as the blow-up of G in which each vertex v has been blown-up by $\omega(v)$. An unweighted graph can be viewed as a weighted graph where each vertex has weight 1. If a weighted graph has a pair of twins (see Section 1.1), then merging those vertices and giving the new vertex the sum of their weights produces an equivalent graph with the same total weight.

Suppose we start with a triangle-free graph G. We can repeatedly add edges to G and merge twins to obtain a twin-free, edge-maximal triangle-free weighted graph (H, ω) whose total weight, $\omega(H)$, equals $|G|$ and whose minimum degree, $\delta(H, \omega)$, is at least $\delta(G)$ (the degree of a vertex in (H, ω) is the total weight of its neighbours). Note that there is a homomorphism $G \to H$. In particular, to understand the chromatic profile of triangle-free graphs, one only needs to understand the twin-free, edge-maximal triangle-free graphs H which have a weighting ω with $\delta(H, \omega) > 1/3 \cdot \omega(H)$ (we will refer to this last property as "H beating 1/3"). The above reasoning holds if we replace "triangle-free" by "locally bipartite" and replace 1/3 by 1/2 (the corresponding chromatic threshold). Hence, we are particularly interested in twin-free, edge-maximal locally bipartite graphs which beat 1/2. Lemma 4 applies and so for two such graph G and H, there is a homomorphism $G \to H$ if and only if G is an induced subgraph of H.

In the triangle-free case, the endeavour of finding all such graphs was implicitly started by Haggkvist [13], continued by Chen, Jin and Koh [8], and finished by Brandt and Thomassé [7]: there are two important sequences of triangle-free graphs, the 3-colourable Andrásfai graphs [2] ($\Gamma_1 = K_2, \Gamma_2 = C_5, \Gamma_3, \dots$) and the 4-chromatic Vega graphs [6] (which we denote by Υ_j). Brandt and Thomassé showed that the twin-free, edge-maximal triangle-free graphs beating 1/3 are exactly the Andrásfai and Vega graphs and so every triangle-free graph with $\delta(G) > 1/3 \cdot |G|$ has a homomorphism to one of these (and hence is 4-colourable). Furthermore, for any $c > 1/3$, only finitely many graphs of each sequence beat c. In particular, if a triangle-free G has $\delta(G)/|G| \geqslant c$ for $c > 1/3$, then there is a homomorphism from G to some early Γ_i or to some early Υ_j.

Theorem 2 effectively shows that $K_3, \overline{C}_7, H_2^+$ play the same role for locally bipartite graphs as the first three Andrásfai graphs do for triangle-free graphs. Furthermore, Theorem 2, together with Lemma 4, shows that they are the only twin-free, edge-maximal locally bipartite graphs which beat $5/9 - \varepsilon$ (in fact, we believe this is true down to 6/11). These results display similarities with the triangle-free case but also give a couple of striking differences. Firstly, the Andrásfai graphs are nested, while H_2^+ does not contain $\overline{C}_7$ and so, by Lemma 4, there is not even a homomorphism from one to the other. Secondly, the Andrásfai and Vega graphs have weightings in which all vertices have the same degree as is expected for extremal examples. However, H_2^+ has no such weighting and, in fact, its n-vertex weighting with greatest minimum degree $(5/9 \cdot n)$ has $1/9 \cdot n$ vertices with degree $2/3 \cdot n$ (this is shown in Fig. 2 on page 351).

It is natural to ask what graphs come after H_2^+. There is an infinite nested sequence of twin-free, edge-maximal locally bipartite graphs all beating 1/2: define Δ_ℓ as the complement of $C_{4\ell-1}^{\ell-1}$ (this is very natural as Γ_i is the complement of C_{3i-1}^{i-1}). Then Δ_ℓ has $4\ell - 1$ vertices, is (2ℓ)-regular, is 4-chromatic (its independence number is ℓ), and is edge-maximal locally bipartite (the addition of any edge gives a 4-clique). Note that $\Delta_2 = \overline{C}_7$. In fact, Δ_3 satisfies $\delta(\Delta_3)/|\Delta_3| = 6/11$ suggesting it is the next key graph when extending Theorem 2 below 6/11. Unlike the triangle-free case, the Δ_ℓ are not the only 4-chromatic twin-free, edge-maximal locally bipartite graphs beating 1/2. Indeed, H_2^+ is not a Δ_ℓ and nor is the graph shown in Fig. 4 on page 370. Intriguingly, neither of these graphs is contained in (nor, by Lemma 4, has a homomorphism to) any Δ_ℓ, since no Δ_ℓ contains an induced H_2 (no neighbourhood in Δ_ℓ contains two edges with no edges between). It would be interesting to have an infinite sequence of such non-Δ_ℓ graphs. Also, for each $c > 1/2$, are there only finitely many twin-free, edge-maximal locally bipartite graphs beating c (for triangle-free graphs this was first shown by Łuczak [17])?

A final question is whether there are any locally bipartite graphs beating 1/2 that are not 4-colourable – such graphs would be the analogue of Vega graphs in the triangle-free case. If there were none, then $\delta_\chi(\mathcal{F}_{1,2}, 5) = 1/2 = \delta_\chi(\mathcal{F}_{1,2})$ and so the chromatic profile of locally bipartite graphs would have only two thresholds (4/7 and 1/2) compared to three (2/5, 10/29 and 1/3) for triangle-free graphs.

1.3. Notation

Let G be a graph and $X \subset V(G)$. We write $\Gamma(X)$ for $\cap_{v \in X}\Gamma(v)$ (the common neighbourhood of the vertices of X) and $\deg(X)$ or $d(X)$ for $|\Gamma(X)|$. We often omit set parentheses so $\Gamma(u, v) = \Gamma(u) \cap \Gamma(v)$ and $d(u, v) = |\Gamma(u, v)|$. We write G_X for $G[\Gamma(X)]$ so, for example, $G_{u,v}$ is the induced graph on the common neighbourhood of vertices u and v. We make frequent use of the fact that for two vertices u and v of G

$$d(u, v) = d(u) + d(v) - |\Gamma(u) \cup \Gamma(v)| \geqslant d(u) + d(v) - |G| \geqslant 2\delta(G) - |G|.$$

Given a set of vertices $X \subset V(G)$, we write $e(X, G)$ for the number of ordered pairs of vertices (x, v) with $x \in X$, $v \in G$ and xv an edge in G. In particular, $e(X, G)$ counts each edge in $G[X]$ twice and each edge from X to $G - X$ once and satisfies

$$e(X, G) = \sum_{x \in X} \deg(x) = \sum_{v \in G} |\Gamma(v) \cap X|.$$

We generalise this notation to vertex weightings which will appear in many of our arguments. We will take a set of vertices $X \subset V(G)$ and assign weights $\omega\colon X \to \mathbb{Z}_{\geqslant 0}$ to the vertices of X. Then we define

$$\omega(X,G) = \sum_{x\in X} \omega(x)\deg(x) = \sum_{v\in G} \text{Total weight of the neighbours of } v \text{ in } X.$$

We will often use the word *circuit* (as opposed to cycle) in our arguments. A circuit is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_\ell$ with $\ell > 1$, v_i adjacent to v_{i+1} (for $i = 1, 2, \ldots, \ell - 1$) and v_ℓ adjacent to v_1. Note that in a locally bipartite graph the neighbourhood of any vertex does not contain an odd circuit (and, of course, does not contain an odd cycle). We use circuit to avoid considering whether some pairs of vertices are distinct when it is unnecessary to do so.

For two graphs G and H, we say there is a *homomorphism* $G \to H$ if there is a map $\varphi\colon V(G) \to V(H)$ such that for every edge uv of G, $\varphi(u)\varphi(v)$ is an edge of H. Note that there is a homomorphism $G \to H$ if and only if G is a subgraph of some blow-up of H. In particular, if there is a homomorphism $G \to H$, then $\chi(G) \leqslant \chi(H)$ and moreover if H is locally bipartite, then G is also.

2. Theorem 2 without homomorphisms

Theorem 2 follows from Theorems 5 to 9 which we state here. Theorem 5 establishes containing H_0 as an obstruction to a locally bipartite graph being 3-colourable. Theorem 6 leverages H_0 up to H_2^+ and $\overline{C}_7$. We prove these two results in this section: they give the required starting structure for the proofs of the homomorphism results, Theorems 7 to 9, which we carry out in Section 3.

Theorem 5. *Let G be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$, then G is either 3-colourable or contains H_0.*

Theorem 6. *Let G be a locally bipartite graph which contains H_0.*

- *Firstly, it must be the case that $\delta(G) \leqslant 4/7 \cdot |G|$.*
- *Secondly, if $\delta(G) > 5/9 \cdot |G|$, then G contains $\overline{C}_7$.*
- *Thirdly, if $\delta(G) > 6/11 \cdot |G|$, then G contains H_2^+ or $\overline{C}_7$.*

The graph $\overline{C}_7$ (and any of its balanced blow-ups) shows that $4/7$ is tight. Below are weightings (blow-ups) of H_2^+ and H_2 – in the former, 0^+ represents some tiny positive weight (we have not deleted the vertices entirely just given them a small weight relative to the rest). These blow-ups show that $5/9$ and $6/11$ are tight respectively (it follows from Lemma 4 that there is no homomorphism from H_2^+ to $\overline{C}_7$, and no homomorphism from either of H_2^+ or $\overline{C}_7$ to H_2).

Theorem 7. *Let G be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$ and G contains $\overline{C}_7$, then there is a homomorphism $G \to \overline{C}_7$.*

Theorem 8. *Let G be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$, then G is 4-colourable.*

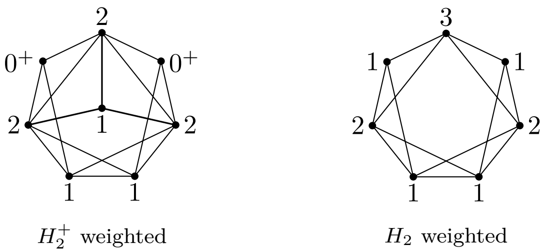


Fig. 2. Weightings of H_2^+ and H_2.

Theorem 9. *There is an $\varepsilon > 0$ such that if G is a locally bipartite graph with $\delta(G) > (5/9 - \varepsilon) \cdot |G|$ and G does not contain $\overline{C}_7$, then there is a homomorphism $G \to H_2^+$.*

Remark 10. We make no attempt to optimise the proof to obtain the 'best value' of ε as we believe that it is in fact possible to replace $5/9 - \varepsilon$ by $6/11$ (but our arguments do not achieve this).

Proof of Theorem 2. Let G be a locally bipartite graph. Theorems 5 and 6 together show that

- If $\delta(G) > 4/7 \cdot |G|$, then G is 3-colourable.
- If $\delta(G) > 5/9 \cdot |G|$, then G is either 3-colourable or contains $\overline{C}_7$.
- If $\delta(G) > 6/11 \cdot |G|$, then G is either 3-colourable or contains H_2^+ or contains $\overline{C}_7$.

The first bullet point of Theorem 2 is immediate and the second follows from Theorem 7.

Let $\varepsilon > 0$ as in Theorem 9. Suppose $\delta(G) > (5/9 - \varepsilon) \cdot |G|$. If G contains $\overline{C}_7$, then there is a homomorphism $G \to \overline{C}_7$, by Theorem 7. Otherwise, by Theorem 9, there is a homomorphism $G \to H_2^+$. This gives the third bullet point of Theorem 2. The final bullet point of Theorem 2 follows immediately from Theorem 8.

The balanced blow-up of $\overline{C}_7$ on n vertices has minimum degree at least $4\lfloor n/7 \rfloor$ and is 4-chromatic showing that $4/7$ is tight. Finally the blow-up of H_2^+ displayed in Fig. 2 shows that $5/9$ is tight (by Lemma 4, there is no homomorphism from $\overline{C}_7$ to H_2^+). $\square$

In Section 2.1 we carry out a careful edge-counting/vertex weighting argument which proves Theorem 6. The proofs of Theorems 7 to 9 are deferred to Section 3. We now introduce a key definition that will be crucial for our proofs and particularly for the proof of Theorem 5. To motivate this, consider a locally bipartite, H_0-free graph G with $\delta(G) > 6/11 \cdot |G|$. To prove Theorem 5 we need to show that G is 3-colourable. We may as well assume that G is edge-maximal: that is, the addition of any edge to G introduces either a copy of H_0 or creates a vertex with a non-bipartite neighbourhood. Thus, any non-edge of G is either a missing edge of a K_4, a missing rim of an odd wheel, a missing spoke of an odd wheel, or a missing edge of an H_0. This motivates a key definition.

Definition 11 *(Dense and sparse).* A pair of *non-adjacent, distinct* vertices u, v in a graph G is *dense* if $G_{u,v}$ contains an edge and *sparse* if $G_{u,v}$ does not contain an edge.

First note that every pair of distinct vertices in any graph is exactly one of ‘adjacent’, ‘dense’ or ‘sparse’. Another way to view being dense is as being the missing edge of a K_4. Locally bipartite graphs are K_4-free so any pair of distinct vertices with an edge in their common neighbourhood must be non-adjacent and so dense. Our initial observations above show that a sparse pair in the edge-maximal G is either a missing edge of an H_0, a missing rim of an odd wheel or a missing spoke of an odd wheel. In Section 2.2 we will rule out the possibility that a sparse pair is the missing spoke of an odd wheel and in Section 2.3 we will complete the proof of Theorem 5.

We finish the introduction to this section by collecting four simple but very effective lemmata about dense pairs of vertices. The second of these exhibits an edge-counting method that we will use frequently. These give us some control over dense pairs and much of the proof of Theorem 5 involves understanding in what configurations sparse pairs appear.

Lemma 12. *Let G be a graph with $\delta(G) > 1/2 \cdot |G|$ and let I be any largest independent set in G. Then, for every distinct $u, v \in I$, the pair u, v is dense.*

Proof. Fix distinct $u, v \in I$. Note that $\Gamma(u), \Gamma(v) \subset V(G) \setminus I$ so $|\Gamma(u) \cup \Gamma(v)| \leqslant |G| - |I|$. Hence

$$
\begin{aligned}
|\Gamma(u) \cap \Gamma(v)| &= d(u) + d(v) - |\Gamma(u) \cup \Gamma(v)| \geqslant 2\delta(G) - (|G| - |I|) \\
&= |I| + 2\delta(G) - |G| > |I|.
\end{aligned}
$$

But I is a largest independent set in G and so $\Gamma(u) \cap \Gamma(v)$ is not independent: $G_{u,v}$ contains an edge so u, v is dense. □

Lemma 13. *Let G be a graph with $\delta(G) > 1/2 \cdot |G|$ and suppose C is an induced 4-cycle in G. Then at least one of the non-edges of C is a dense pair.*

Proof. Suppose the result does not hold. We have an induced 4-cycle $C = v_1v_2v_3v_4$ in G with edges v_1v_2, v_2v_3, v_3v_4, v_4v_1 where the pairs v_1, v_3 and v_2, v_4 are both sparse. Note that any vertex has at most two neighbours in C. Indeed if u is adjacent to both v_1 and v_3, then u cannot be adjacent to either v_2 or v_4, as the pair v_1, v_3 is sparse; similarly, if u is adjacent to both v_2 and v_4, then u cannot be adjacent to either v_1 or v_3. Counting the edges between C and G from both sides gives

$$
4\delta(G) \leqslant d(v_1) + d(v_2) + d(v_3) + d(v_4) = e(C, G) \leqslant 2 \cdot |G|,
$$

which contradicts $\delta(G) > 1/2 \cdot |G|$. □

Lemma 14. *Let G be a locally bipartite graph which does not contain H_0. For any vertex v of G,*

$$D_v := \{u\colon \text{the pair } u, v \text{ is dense}\}$$

is an independent set of vertices.

Proof. Suppose that in fact there are distinct vertices v, u_1 and u_2 with the pairs v, u_1 and v, u_2 both dense and with u_1 adjacent to u_2. Let x_1x_2 be an edge in the common neighbourhood of v and u_1 and x_3x_4 be an edge in the common neighbourhood of v and u_2.

If $\{x_1, x_2\} = \{x_3, x_4\}$, then $u_1u_2x_1x_2$ is a K_4 in G. If $\{x_1, x_2\}$ and $\{x_3, x_4\}$ have one element in common, say $x_1 = x_3$, then G_{x_1} contains the 5-cycle $vx_2u_1u_2x_4$. Finally, if $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are disjoint, then $G[\{v, x_1, x_2, u_1, u_2, x_3, x_4\}]$ contains a copy of H_0. □

We can combine Lemmata 12 and 14 to give the following which will play a crucial role in finishing the proof of Theorem 5.

Lemma 15. *Let G be an H_0-free, locally bipartite graph with $\delta(G) > 1/2 \cdot |G|$. Let I be any largest independent set in G. For any distinct vertices u, v with $u \in I$: $v \in I$ if and only if the pair u, v is dense.*

Proof. Consider the set $\{u\} \cup D_u$ consisting of u and all the vertices which form a dense pair with u. It suffices to show that $I = \{u\} \cup D_u$. By Lemma 12, $I \subset \{u\} \cup D_u$. However, by the definition of dense and Lemma 14, $\{u\} \cup D_u$ is an independent set. Hence, by the maximality of I, we have $I = \{u\} \cup D_u$. □

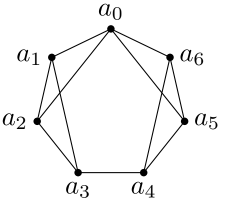
2.1. From H_0 to $\overline{C}_7$

In this subsection we prove Theorem 6. The strategy is to start with a copy of H_0 and consider the edges between it and the rest of G. Using the high minimum degree we are able to find a vertex with the correct neighbours in the copy of H_0 so that a copy of H_1 is present. We then play the same game to get a copy of H_2 and a copy of H_2^+ or $\overline{C}_7$.

For ease of reading we split the proof of Theorem 6 into a sequence of claims from 6/11 up to 4/7. Each claim corresponds to a bullet point of Theorem 6 and we are addressing the bullet points in reverse order. The proof of the first claim is by far the longest.

Claim 16. *Let G be a locally bipartite graph containing H_0. If $\delta(G) > 6/11 \cdot |G|$, then G contains H_2^+ or $\overline{C}_7$.*

Proof. This proof has the following structure. We will first show that G contains H_1, then that it contains H_2 and finally that it contains one of H_2^+ or $\overline{C}_7$. We label a copy of H_0 in G as below and let $X = \{a_0, a_1, \ldots, a_6\}$. Our first aim is to show that G contains H_1.



Let U_4 be the set of vertices with exactly four neighbours in X. Remark 3 says that no vertex has five neighbours in a copy of H_0 so all other vertices have at most three neighbours in X, and hence

$$42/11 \cdot |G| < 7\delta(G) \leqslant e(X, G) \leqslant 4|U_4| + 3(|G| - |U_4|) = 3|G| + |U_4|,$$

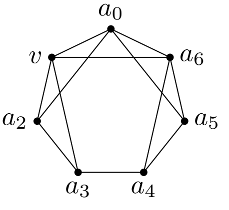
and so

$$|U_4| > 9/11 \cdot |G|.$$

Now $|U_4| + d(a_0) > |G|$ and so some vertex v is adjacent to a_0 and has four neighbours in X. Note that v cannot be adjacent to both a_1, a_2 as otherwise $va_0a_1a_2$ is a K_4, so by symmetry we may assume that v is not adjacent to a_1. Similarly we may assume that v is not adjacent to a_5. But v has four neighbours in X so must be adjacent to at least one of a_2, a_6 – by symmetry we may assume v is adjacent to a_2.

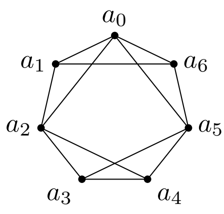
There are two possibilities: v is adjacent to a_0, a_2, a_3, a_4, or v is adjacent to a_0, a_2, a_6 and one of a_3, a_4. In the latter case we may assume by symmetry that v is adjacent to a_3. Hence there are two possibilities for $\Gamma(v) \cap X$: $\{a_0, a_2, a_3, a_4\}$ and $\{a_0, a_2, a_3, a_6\}$. In both cases v cannot be any a_i except for possibly a_1.

If $\Gamma(v) \cap X = \{a_0, a_2, a_3, a_4\}$, then G contains H_1 (replace a_1 by v). If $\Gamma(v) \cap X = \{a_0, a_2, a_3, a_6\}$, then G contains the following graph where, in particular, v is not adjacent to a_4.



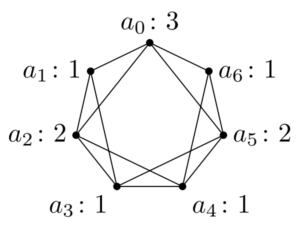
Vertex a_6 is not adjacent to a_3 else G_{a_6} contains the 5-cycle $a_0va_3a_4a_5$. Hence $va_6a_4a_3$ is an induced 4-cycle in G. By Lemma 13, at least one of the pairs v, a_4 and a_3, a_6 is dense. By symmetry we may assume that v, a_4 is dense: let $a_2'a_3'$ be an edge in G_{v,a_4}. Vertex v is not adjacent to a_5, so a_5 is neither a_2' nor a_3'. Note that a_0 is not adjacent to a_4 (else $a_0a_4a_5a_6$ is a K_4) and so a_0 is neither a_2' nor a_3'. If $a_6 = a_2'$, then G_{a_6} contains the 5-cycle $a_3'va_0a_5a_4$, which is impossible. Similarly $a_6 \neq a_3'$. Hence, a_2', a_3' are distinct from a_4, a_5, a_6, a_0, v and so $G[\{a_6, a_0, v, a_2', a_3', a_4, a_5\}]$ contains a copy of H_1 (with apex a_6). Hence in all cases G contains a copy of H_1.

We now show that G contains a copy of H_2. Consider a copy of H_1 with vertices $X = \{a_0, a_1, \ldots, a_6\}$ as shown below and again let U_4 be the set of vertices with 4 neighbours in X. As before, $|U_4| > 9/11 \cdot |G|$.



No vertex is adjacent to all of a_0, a_2, a_5, else G_{a_0} contains an odd circuit. In particular, $\Gamma(a_0, a_2), \Gamma(a_0, a_5)$ are disjoint. Each of these sets has size at least $2\delta(G) - |G| > 1/11 \cdot |G|$. But then $|U_4| + d(a_0, a_2) + d(a_0, a_5) > |G|$ and so there is some vertex $v \in \Gamma(a_0, a_2) \cup \Gamma(a_0, a_5)$ with four neighbours in X. By symmetry, we may assume v is adjacent to both a_0 and a_2 and not to a_5. Also v cannot be adjacent to a_1 otherwise $va_0a_1a_2$ is a K_4. Similarly v cannot be adjacent to both a_3 and a_4. Hence, v is adjacent to a_0, a_2, a_6 and one of a_3, a_4. By symmetry, we may assume v is adjacent to a_3. If $v = a_5$, then a_2a_5 is an edge, so G contains a K_4. If $v = a_4$, then a_0 has five neighbours in X, a contradiction. If v is neither a_4 nor a_5, then G contains H_2 (replace a_1 by v).

We finally show that G contains a copy of H_2^+ or $\overline{C}_7$. Consider a copy of H_2 in G with vertices $X = \{a_0, a_1, \ldots, a_6\}$. We assign weights $\omega \colon X \to \mathbb{Z}_{\geqslant 0}$ as shown in the diagram below, so, for example, $\omega(a_0) = 3$ and $\omega(a_1) = 1$ (recall this notation from Section 1.3). For each vertex $v \in G$, let $f(v)$ be the total weight of the neighbours of v in X.



Now

$$6|G| < 11\delta(G) \leqslant \omega(X,G) = \sum_{v \in G} f(v),$$

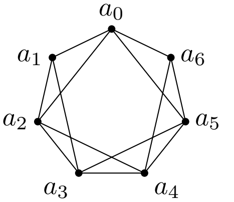
so some vertex v has $f(v) \geqslant 7$. But all vertices have at most four neighbours in a copy of H_2 so either v is adjacent to all of a_0, a_2, a_5 or v is adjacent to a_0, to exactly one of a_2 and a_5, and to exactly two of a_1, a_3, a_4, a_6.

First suppose that v is adjacent to all of a_0, a_2 and a_5. Note that v cannot be in X. Indeed, if $v = a_1$, then G_{a_5} contains the 5-cycle $a_0 v a_3 a_4 a_6$ and similarly if $v = a_6$. On the other hand, if $v = a_3$ then $a_0 a_1 a_2 v$ is a K_4 and similarly if $v = a_4$. Hence v together with H_2 gives a copy of H_2^+ in G.

Second suppose that v is adjacent to a_0, to one of a_2 and a_5, and to two of a_1, a_3, a_4, a_6. By symmetry we may assume that v is adjacent to a_2 and not to a_5. Then v is not adjacent to a_1 else $v a_0 a_1 a_2$ is a K_4 and v is not adjacent to a_4 else $v a_0 a_1 a_3 a_4$ is an odd circuit in G_{a_2}. Thus v is adjacent to a_0, a_2, a_3 and a_6. Note v is neither a_4 nor a_5 as v does not have five neighbours in X. Thus $G[X \setminus \{a_1\} \cup \{v\}]$ contains a copy of $\overline{C}_7$. □

Claim 17. *Let G be a locally bipartite graph containing H_0. If $\delta(G) > 5/9 \cdot |G|$, then G contains $\overline{C}_7$.*

Proof. From Claim 16, G contains H_2. Let $X = \{a_0, a_1, \ldots, a_6\}$ be a copy of H_2 in G.



Let U_4 be the set of vertices with exactly four neighbours in X. All other vertices have at most three neighbours in X so $|U_4| \geqslant 7\delta(G) - 3|G|$. Thus

$$|U_4| + |\Gamma(a_0, a_2)| \geqslant 7\delta(G) - 3|G| + 2\delta(G) - |G| = 9\delta(G) - 4|G| > |G|,$$

so U_4 and $\Gamma(a_0, a_2)$ are not disjoint: there is a vertex v which is adjacent to both a_0 and a_2 and has four neighbours in X. Vertex v is not adjacent to a_1 otherwise $v a_0 a_1 a_2$ is a K_4. Also v is not adjacent to a_4 otherwise G_{a_2} contains the odd circuit $v a_0 a_1 a_3 a_4$. Note v is not adjacent to both a_5 and a_6 otherwise $v a_0 a_5 a_6$ is a K_4. Hence v is adjacent to a_3. But then v is not adjacent to a_5 else G_{a_5} contains the odd circuit $v a_0 a_6 a_4 a_3$.

Therefore $\Gamma(v) \cap X = \{a_0, a_2, a_3, a_6\}$. In particular, v is neither a_4 nor a_5. Thus $G[X \setminus \{a_1\} \cup \{v\}]$ contains $\overline{C}_7$. □

Claim 18. *Let G be a locally bipartite graph containing H_0. Then $\delta(G) \leqslant 4/7 \cdot |G|$.*

Proof. Let X be a set of seven vertices in G with $G[X]$ containing H_0. By Remark 3, every vertex has at most four neighbours in X so

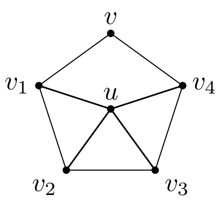
$$7\delta(G) \leqslant e(X, G) \leqslant 4|G|. \quad \square$$

2.2. Ruling out sparse pairs being spokes of odd wheels

In this subsection we make a start on the proof of Theorem 5, by ruling out the possibility that G contains a sparse pair of vertices which is the spoke of an odd wheel.

Lemma 19. *Let G be a locally bipartite graph with $\delta(G) > 1/2 \cdot |G|$ and which does not contain H_0. Then G does not contain a sparse pair u, v with uv being the missing spoke of a 5-wheel.*

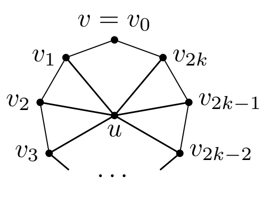
Proof. Suppose the conclusion does not hold: label the configuration as follows where u, v is a sparse pair.



As u, v is sparse, v_1 is not adjacent to v_4 and so uv_1vv_4 is an induced 4-cycle in G. By Lemma 13, the pair v_1, v_4 must be dense. But the pair v_1, v_3 is also dense and so $D_{v_1} = \{x \colon v_1, x \text{ is dense}\}$ contains the edge v_3v_4. This, however, contradicts Lemma 14. $\square$

Lemma 20. *Let G be a locally bipartite graph with $\delta(G) > 6/11 \cdot |G|$ which does not contain H_0. Then G does not contain a sparse pair u, v with uv being the missing spoke of an odd wheel.*

Proof. Suppose that u, v is a sparse pair which is the missing spoke of a $(2k + 1)$-wheel, where we assume that k is minimal. By Lemma 19, k is at least 3. Label the configuration as follows and write v_0 for v (we consider indices modulo $2k+1$). Let $C = \{v, v_1, \ldots, v_{2k}\}$.



Consider a vertex x that is adjacent to u. Suppose that x is adjacent to two vertices in C which are not two apart: x is adjacent to v_i and v_{i+r} where $r \in \{1, 3, 4, \ldots, k\}$. Firstly if $r = 1$, then either G contains the K_4 uxv_iv_{i+1} or $G_{u,v}$ contains an edge (if one of v_i or v_{i+1} is v) which contradicts the sparsity of u, v. Secondly if $r > 1$ is odd, then $C' = xv_iv_{i+1} \cdots v_{i+r}$ is an odd cycle which is shorter than C. Either C' is in G_u (if $v \notin C'$) contradicting the local bipartness of G or we have found a shorter odd cycle than C which satisfies the properties of C (if $v \in C'$). Finally if $r > 2$ is even, then $C' = xv_{i+r}v_{i+r+1} \cdots v_{i-1}v_i$ is an odd cycle which is shorter than C. Again we either obtain an odd cycle in G_u or contradict the minimality of C. Hence every neighbour of u has at most two neighbours in C.

All vertices have at most $2k$ neighbours in C as otherwise G contains a $(2k+1)$-wheel. Hence,

$$\begin{aligned}(2k+1)\delta(G) &\leqslant e(G, C) \leqslant 2d(u) + 2k(|G| - d(u)) \\ &= 2k|G| - (2k-2)d(u) \leqslant 2k|G| - (2k-2)\delta(G),\end{aligned}$$

so

$$\frac{6}{11} < \frac{\delta(G)}{|G|} \leqslant \frac{2k}{4k-1},$$

which implies that $k < 3$, a contradiction. □

2.3. The proof of Theorem 5

Here we will prove Theorem 5 which is restated below for convenience. The argument proceeds as follows. We start with an H_0-free, locally bipartite graph G with $\delta(G) > 6/11 \cdot |G|$ and wish to show that G is 3-colourable. We may assume that G is edge-maximal (so if it is 3-colourable, then it will in fact be complete tripartite). Edge-maximality and the previous two sections will allow us to classify in what configurations sparse pairs arise.

We then take I to be a largest independent set in G. It is enough to show that all edges between I and $G \setminus I$ are present as then $G \setminus I$ is bipartite (since G is locally bipartite) and so G is 3-colourable. Recall Lemma 15 which provides information on how I interacts with the rest of G. It says that if $u \in I$ and $v \notin I$ are not adjacent, then the pair u, v is sparse. In fact, we can extract more. The following definition will be helpful.

Definition 21 *(Quasidense).* A pair of vertices u, v is *quasidense* if there is a sequence of vertices $u = d_1, d_2, \ldots, d_k = v$ such that all pairs d_i, d_{i+1} are dense ($i = 1, 2, \ldots, k-1$).

Lemma 15 immediately implies that if the pair u, v is quasidense and $u \in I$, then $v \in I$ also. So if there are vertices $u \in I$, $v \notin I$ with u not adjacent to v, then the pair u, v is sparse and, furthermore, not quasidense. This will contradict our classification of the configurations in which sparse pairs appear.

Theorem 5. *Let G be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$, then G is either 3-colourable or contains H_0.*

Proof. Let G be an H_0-free, locally bipartite graph that satisfies $\delta(G) > 6/11 \cdot |G|$. We are required to show that G is 3-colourable. We may assume that G is edge-maximal: for any sparse pair u, v of G, the addition of edge uv to G introduces an odd wheel or a copy of H_0. By Theorem 6, the addition of uv to G introduces an odd wheel, a copy of H_2^+, or a copy of $\overline{C}_7$ (note that G itself does not contain these).

Firstly, if the addition of uv introduces an odd wheel, then, by Lemma 20, uv must be a rim of that wheel – this case is depicted in Fig. 3a. Secondly, if the addition of uv introduces a copy of $\overline{C}_7$, then that copy of $\overline{C}_7$ less the edge uv must not contain H_0 – this case is depicted in Fig. 3b. Finally, if the addition of uv introduces a copy of H_2^+, then that copy of H_2^+ less the edge uv must not contain H_0 – this case is depicted in Figs. 3c to 3g.

Thus, in G, any sparse pair u, v must appear in one of the following configurations (with the labels of u and v possibly swapped).

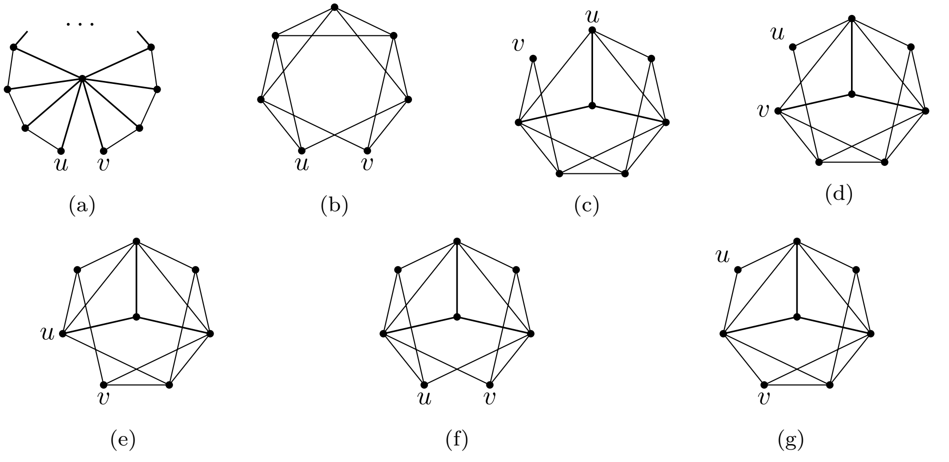
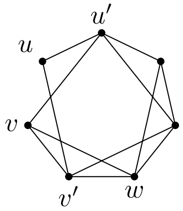
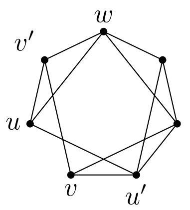


Fig. 3. Configurations in which a sparse pair u, v may appear (labels u and v possibly swapped).

Consider a largest independent set in G: an independent set I of size $\alpha(G)$. We will now show that all edges between I and $G \setminus I$ are present. Fix a vertex $u \in I$ and let v be any other vertex which is not adjacent to u. It suffices to show that $v \in I$. If the pair u, v is (quasi)dense, then $v \in I$ so we may assume that u, v is sparse (and not quasidense). Thus u, v appears in one of the configurations given in Fig. 3 (with labels u

and v possibly swapped). However, in each of Figs. 3a, 3b, 3c, 3f and 3g the pair u, v is quasidense. Hence we may assume that u, v appear in one of Figs. 3d and 3e. We consider these two configurations together (ignoring the central vertex). For ease we label some more of the vertices as follows.





In both cases, the pair u', w is dense and so, by Lemma 14, the pair u', v' is not dense. However, $u'v'$ is not an edge, as the pair u, v is sparse, and so u', v' is a sparse pair. But then $uu'vv'$ is an induced 4-cycle in which both non-edges are sparse which contradicts Lemma 13.

Thus all edges between I and $G \setminus I$ are present. Let $u \in I$, so $G_u = G[V(G) \setminus I]$. But G is locally bipartite, so $G[V(G) \setminus I]$ is bipartite. Using a third colour for the independent set I gives a 3-colouring of G. □

3. Homomorphism results

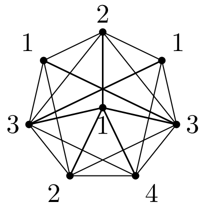
In this section, we will prove Theorems 7 to 9, which we restate here for convenience.

Theorem 7. *Let G be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$ and G contains $\overline{C}_7$, then there is a homomorphism $G \to \overline{C}_7$.*

Theorem 8. *Let G be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$, then G is 4-colourable.*

Theorem 9. *There is an $\varepsilon > 0$ such that if G is a locally bipartite graph with $\delta(G) > (5/9 - \varepsilon) \cdot |G|$ and G does not contain $\overline{C}_7$, then there is a homomorphism $G \to H_2^+$.*

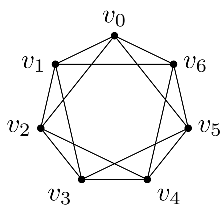
The proof of Theorem 7 (which appears in Section 3.1) takes a copy of $\overline{C}_7$ in G and builds structure around it, focussing initially on those vertices with four neighbours in the copy of $\overline{C}_7$ (of which there are many – more than $9/11 \cdot |G|$ in fact) and then tacking the rest onto these. The proof of Theorem 8 (which appears in Section 3.2) is eminently similar but longer. In place of a copy of $\overline{C}_7$ we take a copy of H_2^+ (if one of these is not present, then, by Theorems 5 and 6, G is either 3-colourable or contains $\overline{C}_7$ and so we are done by Theorem 7 – note that $\overline{C}_7$ is 4-colourable). Around this copy of H_2^+, structure is built in an analogous way to the proof of Theorem 7 with the aim of showing that there is a homomorphism $G \to H_2^+$ (as we believe it is). We will not fully complete this endeavour but will show that there is a homomorphism from G to the following graph which is H_2^+ with four extra edges.



Thankfully, this graph is 4-colourable (colouring shown in the diagram) and so we have Theorem 8. Finally, the proof of Theorem 9 (which appears in Section 3.3) uses all the machinery developed in the proof of Theorem 8 and makes use of taking ε sufficiently small so that the overall structure of G is very similar to that of the weighted H_2^+ shown in Fig. 2 (which has minimum degree 5/9).

3.1. Proof of Theorem 7

In this subsection we prove Theorem 7. Fix a locally bipartite graph G with $\delta(G) > 6/11 \cdot |G|$ that contains a copy of $\overline{C}_7$ which we label as follows. We will always consider indices modulo seven.



Let

$$D = \{x \in V(G) \colon x \text{ is adjacent to four of } v_0, v_1, \ldots, v_6\},$$
$$R = \{x \in V(G) \colon x \text{ is adjacent to at most three of } v_0, v_1, \ldots, v_6\}.$$

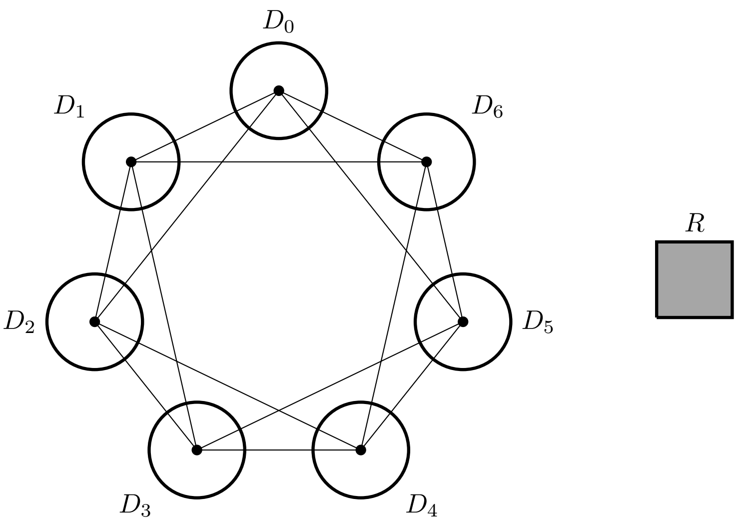
By Remark 3, no vertex is adjacent to five of the v_i so $D \cup R$ partitions $V(G)$. More precisely, note that no vertex is adjacent to three consecutive v_i (otherwise there is a K_4) nor to all of v_{i-2}, v_i, v_{i+2} (otherwise there is a 5-wheel centred at v_i). In particular, if we let

$$D_i = \Gamma(v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}),$$

then $D_0 \cup D_1 \cup \cdots \cup D_6$ partition D. We have a simple upper bound on the size of R.

$$\begin{aligned} &7\delta(G) \leqslant e(\{v_0, v_1, \ldots, v_6\}, G) \leqslant 4|D| + 3|R| = 4|G| - |R| \\ &\Rightarrow |R| \leqslant 4|G| - 7\delta(G). \end{aligned} \tag{1}$$

Also note that $D_i \cup D_{i+3}$ is independent for all i: if there is an edge dd' inside $D_i \cup D_{i+3}$, then $dv_{i+1}v_{i+2}d'$ is a K_4. In particular, there is a homomorphism $G[D] \to \overline{C}_7$. Our aim is to get a handle on R.



We will use the following lemma frequently.

Lemma 22. *Let $X \subset V(G)$ be a set of four vertices. Either there is $x \in R$ adjacent to all of X or there is $x \in D$ with at least three neighbours in X.*

Proof. Since $\delta(G) > 6/11 \cdot |G|$, we have $4\delta(G) > 6|G| - 7\delta(G)$. We will use this inequality multiple times in this section. This together with inequality (1) gives

$$e(X, G) \geqslant 4\delta(G) > 6|G| - 7\delta(G) \geqslant 2|G| + |R| = 2|D| + 3|R|.$$

But $D \cup R$ partition $V(G)$ so either some vertex in D has more than two neighbours in X or some vertex in R has more than three neighbours in X. $\square$

Our first two claims show that the collections of v_i to which vertices can be adjacent are similar to the collections of the D_i in which vertices can have neighbours.

Claim 23. *For all i, no vertex has a neighbour in each of D_{i-1}, D_i, D_{i+1}.*

Proof. If not, without loss of generality we may choose d_6, d_0, d_1 in D_6, D_0, D_1 respectively with common neighbour u such that $e(\{d_6, d_0\}) + e(\{d_0, d_1\})$ is maximal. We now apply Lemma 22 to $\{u, d_6, d_0, d_1\}$.

Suppose some x is adjacent to all of u, d_6, d_0, d_1. Now apply Lemma 22 to $X = \{u, x, d_6, d_1\}$: as uxd_6 and uxd_1 are triangles, no vertex is adjacent to all of X and furthermore, any vertex with three neighbours in X must be adjacent to both d_6 and d_1. In particular, some $d' \in D$ is adjacent to d_6, d_1 and to one of u, x. But then $d' \in D_0$, so, in our choice of d_6, d_0, d_1, u at the start, we could swap d' for d_0 and u for whichever of u and x is adjacent to d'. This contradicts the maximality unless d_0 is adjacent to both d_6 and d_1. But then $d_0 d_1 u x$ is a K_4.

Hence, in fact, there is some $d \in D$ adjacent to three of u, d_6, d_0, d_1. No vertex in D is adjacent to all of d_6, d_0, d_1, so d is adjacent to u. By symmetry we may assume d is adjacent to d_6. If d is adjacent to d_0 as well, then d is adjacent to both v_6 and v_0. But then $u d_6 v_0 v_6 d_0$ is an odd circuit in G_d.

Thus d is adjacent to u, d_6 and d_1. But then, $d \in D_0$ and so $u d_6 v_1 v_6 d_1$ is an odd circuit in G_d. □

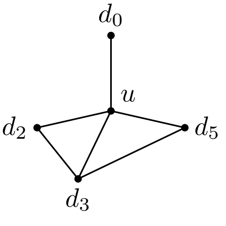
Claim 24. *For all i, no vertex has a neighbour in each of D_{i-2}, D_i, D_{i+2}.*

Proof. If not, without loss of generality we may choose d_5, d_0, d_2 in D_5, D_0, D_2 respectively with common neighbour u such that $e(\{d_5, d_0\}) + e(\{d_0, d_2\})$ is maximal. No vertex in D has a neighbour in each of D_5, D_0 and D_2, so $u \in R$. Apply Lemma 22 to $\{u, d_5, d_0, d_2\}$.

Suppose some x is adjacent to all of u, d_5, d_0, d_2. Now apply Lemma 22 to $\{u, x, d_0, d_2\}$: as $u x d_0$ and $u x d_2$ are triangles, there must be some $d' \in D$ is adjacent to d_0, d_2 and one of u, x. But then $d' \in D_1$ and so one of u, x has a neighbour in each of D_0, D_1 and D_2 which contradicts Claim 23.

Hence, there is some $d \in D$ adjacent to three of u, d_5, d_0, d_2. No vertex in D is adjacent to all of d_5, d_0, d_2 so d is adjacent to u. By symmetry, we may assume d is adjacent to d_2. If d is adjacent to d_0 as well, then $d \in D_1$, so u has a neighbour in each of D_0, D_1, D_2 contradicting Claim 23.

Thus d is adjacent to u, d_5 and d_2, so $d \in D_0 \cup D_3 \cup D_4$. If $d \in D_0$, then $u d_5 v_6 v_1 d_2$ is an odd circuit in G_d. Hence, we may assume by symmetry that $d \in D_3$. Write d_3 for d.



We now show there is some $d_0' \in D_0$ adjacent to both u and d_5. Apply Lemma 22 to $\{u, d_5, v_6, d_0\}$: by Claim 23, no vertex is adjacent to all of d_5, v_6, d_0 so there is $d'' \in D$ adjacent to u and to two of d_5, v_6, d_0.

- If d'' is adjacent to d_5 and d_0, then $d'' \in D_6$, so u has a neighbour in each of D_5, D_6, D_0, contrary to Claim 23.

- If d'' is adjacent to d_5 and v_6, then $d'' \in D_4 \cup D_0$. But if $d'' \in D_4$, then u has a neighbour in each of D_2, D_3, D_4, contrary to Claim 23, so $d'' \in D_0$. We may take $d'_0 = d''$.
- If d'' is adjacent to v_6 and d_0, then $d'' \in D_5 \cup D_1$. But if $d'' \in D_1$, then u has a neighbour in each of D_0, D_1, D_2, contrary to Claim 23, so $d'' \in D_5$. By maximality at the start we must have d_5 adjacent to d_0. We may take $d'_0 = d_0$.

Thus there is some $d'_0 \in D_0$ adjacent to both u and d_5. But then $d'_0 u d_3 v_4 v_6$ is an odd circuit in G_{d_5}. □

From the previous two claims it follows that for every vertex v there is an i such that

$$\Gamma(v) \cap D \subset \Gamma(v_i) \cap D = D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}.$$

For $i = 0, 1, \ldots, 6$ choose

$$R_i \subset \{v \in R \colon \Gamma(v) \cap D \subset \Gamma(v_i) \cap D\},$$

so that $R_0 \cup R_1 \cup \cdots \cup R_6$ is a partition of R. There may be some flexibility in the choice of the R_i (e.g. if $\Gamma(v) \cap D \subset D_0 \cup D_3 \cup D_4$, then we could take v in R_2 or R_5) – we will make use of this later. For now we just take any arbitrary choice. Note, by definition, that

$$e(R_i, D_i \cup D_{i-3} \cup D_{i+3}) = 0.$$

For each i, let $T_i = D_i \cup R_i$ – note that these partition $V(G)$. We can give a lower bound for the size of T_i. For each i,

$$\begin{aligned} d(v_{i-1}) + d(v_{i+1}) &= |D_i| + |D| + |R \cap \Gamma(v_{i-1})| + |R \cap \Gamma(v_{i+1})| \\ &= |D_i| + |D| + |R \cap (\Gamma(v_{i-1}) \cup \Gamma(v_{i+1}))| + |R \cap \Gamma(v_{i-1}, v_{i+1})| \\ &\leqslant |D_i| + |D| + |R| + |R_i| = |G| + |T_i|, \end{aligned}$$

so

$$|T_i| \geqslant 2\delta(G) - |G|. \tag{2}$$

We will eventually show that $T_i \cup T_{i+3}$ is independent for all i and so the map sending all vertices in T_i to v_i is a homomorphism from G to $\overline{C}_7$.

Claim 25. *Every $d \in D_i$ and $u \in T_{i+1}$ have a common neighbour in D. Similarly, every $d \in D_i$ and $u \in T_{i-1}$ have a common neighbour in D.*

Proof. By symmetry it suffices to prove this for $d_0 \in D_0$ and $u \in T_1$. As $e(D_0 \cup T_1, D_4) = 0$, we have $\Gamma(u) \cup \Gamma(d_0) \subset V(G) \setminus D_4$ so

$$\begin{aligned} |\Gamma(d_0) \cap \Gamma(u)| &= d(d_0) + d(u) - |\Gamma(d_0) \cup \Gamma(u)| \geqslant 2\delta(G) + |D_4| - |G| \\ &\geqslant 4\delta(G) - 2|G| - |R_4| > 4|G| - 7\delta(G) - |R_4| \geqslant |R| - |R_4| \\ &\geqslant |\Gamma(d_0) \cap R|, \end{aligned}$$

where we have used inequality (2), $\delta(G) > 6/11 \cdot |G|$, inequality (1) and $e(d_0, R_4) = 0$ respectively for the final four inequalities. In particular, d_0 and u have a common neighbour $d_u \in \Gamma(d_0) \cap D$. □

Claim 26. *For all i and $d \in D_i$, the sets $\Gamma(d) \cap T_{i-1}$ and $\Gamma(d) \cap T_{i+1}$ are independent.*

Proof. By symmetry it suffices to prove this for $\Gamma(d_0) \cap T_1$ where $d_0 \in D_0$. Suppose that $\Gamma(d_0) \cap T_1$ contains an edge uv. By Claim 25, d_0 and u have a common neighbour $d_u \in D$. As d_u is adjacent to $u \in T_1$, it must also be adjacent to v_1. Similarly there is $d_v \in D$ adjacent to d_0, v, v_1. But then $v_1 d_u u v d_v$ is an odd circuit in G_{d_0}. □

Claim 27. *For all i, T_i is independent.*

Proof. By symmetry it suffices to prove this for $i = 0$. Suppose that uv is an edge in T_0. We already have that D_0 is independent and $e(D_0, R_0) = 0$ so $u, v \in R_0$. We may choose the edge uv in R_0 so that $e(\{u, v, v_1, v_6\})$ is maximal. Apply Lemma 22 to $\{u, v, v_1, v_6\}$.

Suppose some $x \in R$ is adjacent to all of u, v, v_1, v_6. Then $x \in R_0$. By maximality of $e(\{u, v, v_1, v_6\})$, we must have had u and v adjacent to both v_1, v_6 and so uvv_1v_6 is a K_4.

Hence, there is some $d \in D$ with three neighbours amongst u, v, v_1, v_6. Either d is adjacent to both u, v or to both v_1, v_6. If d is adjacent to both v_1, v_6, then $d \in D_0$. But then d is adjacent to neither u nor v, as $e(D_0, R_0) = 0$. Hence d is adjacent to both u, v. By Claim 26, $d \notin D_1 \cup D_6$ so $d \in D_2 \cup D_5$. By symmetry, we may assume that $d = d_2 \in D_2$.

Apply Lemma 22 to $\{u, v, d_2, v_5\}$: d_2uv is a triangle so there is $d' \in D$ adjacent to v_5 and to two of u, v, d_2. As d' is adjacent to v_5 and at least one of $u, v \in R_0$, d' is in D_6. But then d' is not adjacent to d_2 and so is to both u and v. However, edge uv lies in $\Gamma(d') \cap R_0$, contrary to Claim 26. □

This shows that there is a homomorphism $G \to K_7$ and so G is 7-colourable. Before proceeding it will help to give structure to G_d for each $d \in D$.

Claim 28. *For any $i \in \{0, 1, 2, \dots, 6\}$ and any $d \in D_i$, G_d is connected bipartite. Furthermore, there is a bipartition of G_d into two vertex classes A_d, B_d which satisfy $(T_{i-1} \cup D_{i+2}) \cap \Gamma(d) \subset A_d$, $(T_{i+1} \cup D_{i-2}) \cap \Gamma(d) \subset B_d$ and at least one of $R_{i+2} \cap \Gamma(d) \subset A_d$, $R_{i-2} \cap \Gamma(d) \subset B_d$ occurs.*

Proof. We may assume $i = 0$. Fix $d \in D_0$ and define for $j = 5, 6, 1, 2$,

$$D_j^d = D_j \cap \Gamma(d),$$
$$R_j^d = R_j \cap \Gamma(d),$$
$$T_j^d = T_j \cap \Gamma(d) = D_j^d \cup R_j^d,$$

and note that the T_j^d partition $V(G_d)$. Since $v_6 \in G_d$, we can define

$$A_d = \{x \in G_d\colon\ \mathrm{dist}_{G_d}(x, v_6) \text{ is even}\},$$
$$B_d = \{x \in G_d\colon\ \mathrm{dist}_{G_d}(x, v_6) \text{ is odd}\}.$$

G is locally bipartite, so G_d is bipartite and so A_d and B_d are independent sets. Now

- $v_6 \in A_d$, $v_1 \in B_d$.
- v_6 is adjacent to all of $D_5^d \cup D_1^d$, so $D_5^d \cup D_1^d \subset B_d$.
- v_1 is adjacent to all of $D_6^d \cup D_2^d$, so $D_6^d \cup D_2^d \subset A_d$.

We next show that $R_6^d \subset A_d$. Let $x \in R_6^d$: by Claim 25, d and x have a common neighbour $d' \in D$. As $d \in D_0$ and $x \in R_6$, d' must be in $D_1 \cup D_5$. Hence, $d' \in B_d$ and so $x \in A_d$. Similarly $R_1^d \subset B_d$.

We now show that at least one of $R_2^d \subset A_d$, $R_5^d \subset B_d$ occurs. If not, then there is $u \in R_2^d \backslash A_d$ and $v \in R_5^d \backslash B_d$. Focus on u: $u \notin A_d$ so $\Gamma_{G_d}(u) \subset V(G_d) - B_d \subset T_6^d \cup T_2^d \cup R_5^d$. But $u \in R_2$, the set T_2 is independent and $e(R_2, D_6) = 0$, so, in fact,

$$\Gamma_{G_d}(u) \subset R_5^d \cup R_6^d.$$

Similarly,

$$\Gamma_{G_d}(v) \subset R_1^d \cup R_2^d.$$

In particular,

$$|\Gamma_{G_d}(u)| + |\Gamma_{G_d}(v)| \leqslant |R_1^d| + |R_2^d| + |R_5^d| + |R_6^d| \leqslant |R|.$$

But then, by inequality (1),

$$4|G| - 7\delta(G) \geqslant |R| \geqslant d(d, u) + d(d, v) \geqslant 4\delta(G) - 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$.

Finally we need to show that G_d is connected. We will do this by showing $A_d \cup B_d = V(G_d)$. We already have $T_1^d \cup T_6^d \cup D_2^d \cup D_5^d \subset A_d \cup B_d$ and at least one of R_2^d, R_5^d is a subset of $A_d \cup B_d$ – we need only show that the other one is too. By symmetry, we may assume $R_2^d \subset A_d \cup B_d$. Fix $x \in R_5^d$. Now $\deg_{G_d}(x) = d(x,d) \geqslant 2\delta(G) - |G| > 0$, so x has some neighbour in G_d. But R_5 is an independent set, so x has a neighbour in $A_d \cup B_d$ and so $x \in A_d \cup B_d$. □

We are finally in a position to show that there is a homomorphism $G \to \overline{C}_7$. It is here that we will make use of the flexibility in the choice of the R_i.

Claim 29. *It is possible to choose the R_j so that the sets $T_i \cup T_{i+3}$ are all independent.*

Proof. Note that $T_i, T_{i+3}, D_i \cup D_{i+3}$ are all independent and $e(D_i, R_{i+3}) = e(D_{i+3}, R_i) = 0$ so it suffices to show that it is possible to ensure $e(R_i, R_{i+3}) = 0$ for all i. We choose the R_i so that

$$S = \sum_{i=0}^{6} e(R_i, R_{i+3})$$

is minimal. Suppose that S is not zero: by symmetry, we may assume there is some $u \in R_2$, $v \in R_6$ with u adjacent to v. Apply Lemma 22 to $\{u, v, v_0, v_1\}$. Note that any common neighbour of v_0, v_1 is in $T_2 \cup T_6$ so is adjacent to at most one of u, v. Moreover, any common neighbour of v_0, v_1 which lies in D is in $D_2 \cup D_6$ so is adjacent to neither u nor v. Hence there is $d \in D$ which is adjacent to both u, v and to one of v_0, v_1. By symmetry, we may assume d is adjacent to v_1 and so $d \in D_0$. That is, there is at least one $d \in D_0$ adjacent to both u and v.

For any $d \in D_0 \cap \Gamma(u, v)$, consider the bipartition of G_d given by the previous claim:

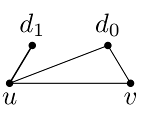
- $(T_6 \cup D_2) \cap \Gamma(d) \subset A_d$.
- $(T_1 \cup D_5) \cap \Gamma(d) \subset B_d$.
- At least one of $R_2 \cap \Gamma(d) \subset A_d$, $R_5 \cap \Gamma(d) \subset B_d$ occurs.

Now $v \in R_6$ is adjacent to d, so $v \in A_d$. Also u is adjacent to v and d, so $u \in B_d$. But $u \in R_2 \cap \Gamma(d)$ so $R_5 \cap \Gamma(d) \subset B_d$ occurs. Now $\Gamma_{G_d}(u) \subset A_d \subset (T_6 \cup T_2) \cap \Gamma(d)$. But $u \in R_2$, the set T_2 is independent and $e(R_2, D_6) = 0$, so

$$\Gamma_{G_d}(u) \subset R_6 \cap \Gamma(d). \tag{3}$$

Note that this holds for any choice of $d \in D_0 \cap \Gamma(u, v)$.

We first deal with the case where u has some neighbour in D_1. Pick any $d_0 \in \Gamma(u, v) \cap D_0$, $d_1 \in \Gamma(u) \cap D_1$ and apply Lemma 22 to $\{u, v, d_0, d_1\}$.



Vertices d_0uv form a triangle so some $d \in D$ is adjacent to d_1 and to two of d_0, u, v. If d is adjacent to d_0, then $d \in D_2 \cup D_6$, so d is adjacent to neither u nor v. Hence $d \in \Gamma(u, v, d_1) \cap D$, so $d \in D_0$. Thus $d \in \Gamma(u, v) \cap D_0$, $d_1 \in \Gamma(u) \cap D_1$ and d is adjacent to d_1. But then $\Gamma_{G_d}(u)$ contains $d_1 \notin R_6$ contradicting (3).

We are finally left with the case where u has no neighbours in D_1. This means we could have put u in R_5 rather than R_2 when we chose the R_i. In particular, by the minimality of S,

$$e(u, R_5) + e(u, R_6) \leqslant e(u, R_1) + e(u, R_2),$$

hence,

$$2\big(e(u, R_5) + e(u, R_6)\big) \leqslant e(u, R_1 \cup R_2 \cup R_5 \cup R_6) \leqslant |R|.$$

Pick any $d \in \Gamma(u, v) \cap D_0$: as $\Gamma_{G_d}(u) \subset R_6 \cap \Gamma(d)$, we have

$$\deg_{G_d}(u) \leqslant e(u, R_6) \leqslant 1/2 \cdot |R|.$$

Thus

$$|R| \geqslant 2d(d, u) \geqslant 4\delta(G) - 2|G| > 4|G| - 7\delta(G) \geqslant |R|,$$

where we used $\delta(G) > 6/11 \cdot |G|$ and inequality (1) for the final two inequalities. This is a contradiction and so $S = 0$, as required. □

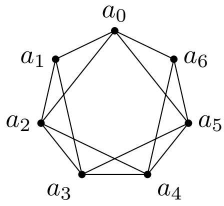
Proof of Theorem 7. For $i \in \{1, \dots, 7\}$, let R_i be given by Claim 29 and $T_i = D_i \cup R_i$. Then the T_i partition $V(G)$ and there are no edges within $T_i \cup T_{i+3}$. Thus the map which sends all vertices in T_i to a vertex v_i is a homomorphism from G to a copy of $\overline{C}_7$. □

3.2. Proof of Theorem 8

In this subsection we prove Theorem 8. This has many similarities with the proof of Theorem 7, although not having the full symmetry of $\overline{C}_7$ available adds some technicalities.

Fix a locally bipartite graph G with $\delta(G) > 6/11 \cdot |G|$. By Theorems 5 and 6, G is either 3-colourable, contains $\overline{C}_7$ or contains H_2^+. In the first two cases we are done (using Theorem 7), so we assume that G does not contain a copy of $\overline{C}_7$ but does contain a copy of H_2^+ (and so also a copy of H_2).

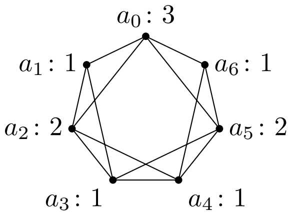
We say "$a_0a_1 \dots a_6$ is a copy of H_2 in G" to mean that the following configuration appears in G. We will continue to use Remark 3: no vertex is adjacent to five of the vertices which form a copy of H_2. We will always consider indices modulo seven.



Our first few claims will nail down to which a_i other vertices may be adjacent. This will eventually allow us to define the sets D_i in a similar way to the proof of Theorem 7.

Claim 30. *Let $a_0a_1 \dots a_6$ be a copy of H_2 in G. Then there is a vertex $u \notin \{a_0, a_1, \dots, a_6\}$ adjacent to a_5, a_0 and a_2 (i.e. any copy of H_2 'extends' to a copy of H_2^+). Furthermore, $|\Gamma(a_5, a_0, a_2)| \geqslant 11\delta(G) - 6|G|$.*

Proof. Consider a copy of H_2 in G with vertices $X = \{a_0, a_1, \dots, a_6\}$. We assign weight $\omega \colon X \to \mathbb{Z}_{\geqslant 0}$ as shown in the diagram below (recall this notation from Section 1.3). For each vertex $v \in G$, let $f(v)$ be the total weight of the neighbours of v in X.



The final part of the proof of Claim 16 showed that any vertex with $f(v) \geqslant 7$ is either adjacent to all of a_5, a_0, a_2 or is in a copy of $\overline{C}_7$. As G does not contain $\overline{C}_7$, $\Gamma(a_5, a_0, a_2)$ is exactly the set of vertices with $f(v) \geqslant 7$. However, any vertex in $\Gamma(a_5, a_0, a_2)$ is adjacent to no other vertex of X as G is locally bipartite. Hence, $\Gamma(a_5, a_0, a_2)$ is exactly the set of vertices with $f(v) = 7$ and all other vertices have $f(v) \leqslant 6$. In particular,

$$11\delta(G) \leqslant \sum_{v \in G} f(v) \leqslant 7|\Gamma(a_5, a_0, a_2)| + 6[|G| - |\Gamma(a_5, a_0, a_2)|] = |\Gamma(a_5, a_0, a_2)| + 6|G|.$$

Thus $|\Gamma(a_5, a_0, a_2)| \geqslant 11\delta(G) - 6|G|$ and so $\Gamma(a_5, a_0, a_2)$ is non-empty. As G is locally bipartite, no vertex in X is adjacent to all of a_5, a_0, a_2, so the copy of H_2 extends to a copy of H_2^+. □

Claim 31. *Let $a_0a_1 \dots a_6$ be a copy of H_2 in G. Then no vertex is adjacent to all of a_6, a_0, a_1, a_3.*

Proof. Suppose some vertex a is adjacent to all of a_6, a_0, a_1, a_3. All vertices are adjacent to at most four of the a_i so a cannot be one of the a_i. Let $A = \{a, a_0, a_1, \dots, a_6\}$ and gives weights, ω, to the vertices of A as shown.

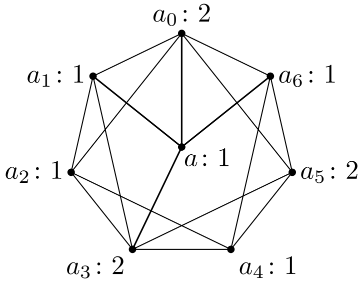


Fig. 4. The configuration of Claim 31.

For a vertex v, let $f(v)$ be the total weight of the neighbours of v in A. Now,

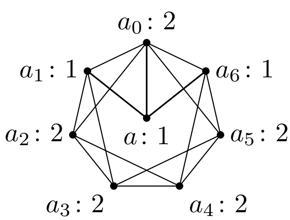
$$\sum_{v \in V(G)} f(v) = \omega(A, G) \geqslant 11\delta(G) > 6|G|,$$

so some vertex v has $f(v) \geqslant 7$. Vertex v is not adjacent to all of a_0, a_3, a_5 else $va_0a_6a_4a_3$ is an odd circuit in G_{a_5} and v is not adjacent to all of a, a_1, a_2, a_4, a_6 as these form a 5-cycle. Thus v is adjacent to exactly two of a_0, a_3, a_5 and at least three of a, a_1, a_2, a_4, a_6. As v is adjacent to at most four of the a_i, v must be adjacent to a.

- If v is adjacent to a_0, then v is not adjacent to a_1 (else vaa_0a_1 is a K_4), and v is not adjacent to a_6 (else vaa_6a_0 is a K_4). Thus v is adjacent to a_2 and a_4. But then $va_0a_1a_3a_4$ is an odd circuit in G_{a_2}.
- If v is adjacent to both a_3, a_5, then v is not adjacent to a_4 (else $va_3a_4a_5$ is a K_4), and v is not adjacent to a_1 (else vaa_1a_3 is a K_4). Thus v is adjacent to a_2 and a_6. But then G_a contains the odd circuit $va_6a_0a_1a_3$. □

Claim 32. *Let $a_0a_1 \ldots a_6$ be a copy of H_2 in G. Then no vertex is adjacent to all of a_6, a_0, a_1.*

Proof. Suppose some vertex a is adjacent to all of a_6, a_0, a_1. All vertices are adjacent to at most four of the a_i so a cannot be one of the a_i. Let $A = \{a, a_0, a_1, \ldots, a_6\}$ and gives weights, ω, to the vertices of A as shown.



For a vertex v, let $f(v)$ be the total weight of the neighbours of v in A. Now,

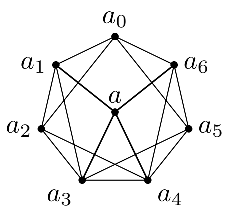
$$\sum_{v \in V(G)} f(v) = \omega(A, G) \geqslant 13\delta(G) > 7|G|,$$

so some vertex v has $f(v) \geqslant 8$. Vertex v must be adjacent to at least three of a_0, a_2, a_3, a_4, a_5. First suppose that v is adjacent to at least four of a_0, a_2, a_3, a_4, a_5. Vertex v cannot be adjacent to all of a_2, a_3, a_4 (else $va_2a_3a_4$ is a K_4) so v is adjacent to a_0 and a_5. Similarly v is adjacent to a_2. By symmetry, we may assume that v is adjacent to a_3. But then $va_0a_1a_4a_3$ is an odd circuit in G_{a_5}. Thus v is adjacent to exactly three of a_0, a_2, a_3, a_4, a_5 and so at least two of a, a_1, a_6. As v is adjacent to at most four of the a_i, v must be adjacent to a.

- If v is adjacent to a_0, then v cannot be adjacent to a_1 (else vaa_0a_1 is a K_4) and v cannot be adjacent to a_6 (else vaa_6a_0 is a K_4). Thus v is adjacent to only one of a, a_1, a_6 – a contradiction.
- Otherwise v is not adjacent to a_0. Certainly v is not adjacent to all of a_2, a_3, a_4 (else $va_2a_3a_4$ is a K_4) so v must be adjacent to a_5. Similarly v must be adjacent to a_2. By symmetry, we may assume that v is adjacent to a_4. Then v is not adjacent to a_6 (else $va_4a_5a_6$ is a K_4) so v is adjacent to a and a_1. Thus v is adjacent to a_1, a_2, a_4, a_5 and a. In particular, v is not a nor any of the a_i except possibly a_3. But then $a_0a_1a_2va_4a_5a_6$ is a copy of H_2 in G and a is adjacent to all of a_6, a_0, a_1, v which contradicts Claim 31. □

Claim 33. *Let $a_0a_1 \ldots a_6$ be a copy of H_2 in G. Then no vertex is adjacent to all of a_1, a_3, a_4, a_6.*

Proof. Suppose vertex a is adjacent to all of a_1, a_3, a_4, a_6. All vertices are adjacent to at most four of the a_i, so a cannot be an a_i. Let $Z = \{a, a_6, a_0, a_1\}$.



We claim that each vertex has at most two neighbours in Z. By Claim 32, no vertex is adjacent to all of a_6, a_0, a_1. If a vertex v is adjacent to all of a, a_0, a_1, then $va_0a_2a_3a$ is an odd circuit in G_{a_1} while if v is adjacent to all of a, a_6, a_0, then $va_0a_5a_4a$ is an odd circuit in G_{a_6}. Finally if v is adjacent to all of a, a_6, a_1, then $va_1a_3a_4a_6$ is an odd circuit in G_a. Thus

$$4\delta(G) \leqslant e(Z, G) \leqslant 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$. $\square$

We can now show that any vertex with four neighbours in a copy of H_2 'looks like' one of the vertices of the H_2.

Claim 34. *Let $a_0a_1 \dots a_6$ be a copy of H_2 in G. Suppose a vertex v is adjacent to at least four of the a_i. Then there is $i \in \{0, 2, 3, 4, 5\}$ such that*

$$\Gamma(v) \cap \{a_0, a_1, \dots, a_6\} = \Gamma(a_i) \cap \{a_0, a_1, \dots, a_6\}.$$

Proof. Fix a vertex v which is adjacent to at least four of the a_i. Firstly there is no i with v adjacent to all of a_{i-1}, a_i, a_{i+1} else there is a K_4, or v is adjacent to all of a_6, a_0, a_1 contradicting Claim 32. Now suppose there is an i with v adjacent to all of a_{i-2}, a_i, a_{i+2}. If $i = 2, 3, 4, 5$, then $va_{i-2}a_{i-1}a_{i+1}a_{i+2}$ is an odd circuit in G_{a_i}. If i is 1 or 6, then, by symmetry, we may assume $i = 1$: vertex v is adjacent to all of a_6, a_1, a_3. Now v is adjacent to none of a_0 (by Claim 32), a_2 (else there is a K_4) or a_4 (by Claim 33). Thus v is adjacent to a_5. But then v is adjacent to a_1, a_3, a_5 which is the already discounted case of $i = 3$. Finally if $i = 0$, then v is adjacent to all of a_5, a_0, a_2 so v is adjacent to neither a_1 nor a_6 (else there is a K_4). By symmetry, we may assume that v is adjacent to a_3. But then v is adjacent to all of a_3, a_5, a_0 which is the already discounted case of $i = 5$.

Hence there is an i with v adjacent to all of $a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}$ and no other a_j. Now i is not 1 as otherwise $a_0va_2a_3a_4a_5a_6$ is a copy of $\overline{C}_7$. Similarly i is not 6. For all other i, $\Gamma(v) \cap \{a_0, a_1, \dots, a_6\} = \Gamma(a_i) \cap \{a_0, a_1, \dots, a_6\}$. $\square$

Fix some copy, $v_0v_1 \dots v_6$, of H_2 in G. We are ready to build some structure around this copy of H_2. Let

$$\begin{aligned} D_i &= \Gamma(v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}) \quad \text{for } i = 0, 2, 3, 4, 5, \\ D_1 &= \Gamma(v_0, v_2, v_3), \qquad\qquad D_6 = \Gamma(v_4, v_5, v_0), \\ D &= \cup_{i=0}^{6} D_i, \qquad\qquad R = V(G) \setminus D. \end{aligned}$$

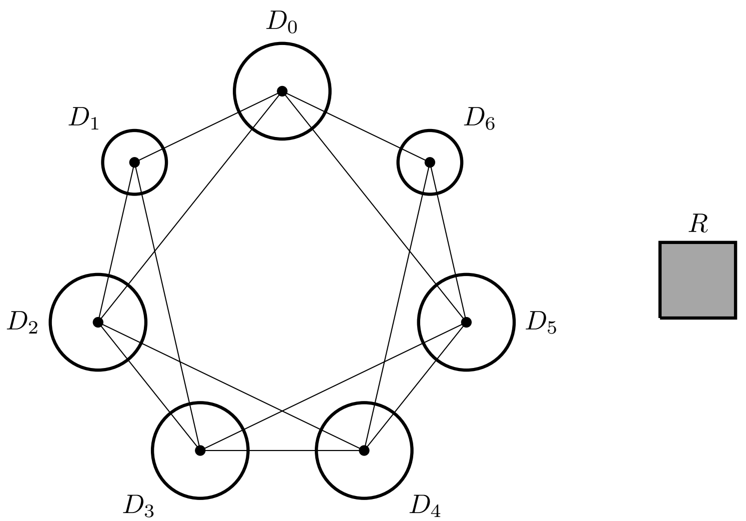
By Claim 34, no vertex is adjacent to five of the v_i, and so the D_i are pairwise disjoint. Also from Claim 34, the vertices adjacent to exactly four of the v_i are those in

$$D^* := D_0 \cup D_2 \cup D_3 \cup D_4 \cup D_5,$$

and all other vertices are adjacent to at most three of the v_i. Thus we can give a simple upper bound on the size of $R \cup D_1 \cup D_6 = V(G) \setminus D^*$.

$$7\delta(G) \leqslant e(\{v_0, v_1, \ldots, v_6\}, G) \leqslant 4|D^*| + 3|R \cup D_1 \cup D_6| = 4|G| - |R \cup D_1 \cup D_6|$$
$$\Rightarrow |R \cup D_1 \cup D_6| \leqslant 4|G| - 7\delta(G). \tag{4}$$

Each D_i is independent (if dd' is an edge in D_i, then at least one of $dd'v_{i-2}v_{i-1}$, $dd'v_{i+1}v_{i+2}$ is a K_4). Suppose there is an edge dd' between D_i and D_{i+3}. If $i \neq 5, 6$, then $dv_{i+1}v_{i+2}d'$ is a K_4. If $i = 5$, then $d'v_3v_4v_6v_0$ is a 5-cycle in G_d and if $i = 6$, then $dv_0v_1v_3v_4$ is a 5-cycle in $G_{d'}$. Hence $D_i \cup D_{i+3}$ is an independent set for all i. Finally, if $d_1 \in D_1$ and $d_6 \in D_6$ are adjacent, then $v_0d_1v_2v_3v_4v_5d_6$ is a copy of $\overline{C}_7$. Thus there is a homomorphism $G[D] \to H_2$. Our aim is to get a handle on R.



The following lemma corresponds to Lemma 22 and is just as useful. Its proof is identical with inequality (4) in place of (1), D^* in place of D and $R \cup D_1 \cup D_6$ in place of R.

Lemma 35. *Let $X \subset V(G)$ be a set of four vertices. Either there is $x \in R \cup D_1 \cup D_6$ adjacent to all of X or there is $x \in D^*$ with at least three neighbours in X.*

Our first three claims show that the collections of v_i to which vertices can be adjacent are similar to the collections of the D_i in which vertices can have neighbours. This will eventually allow us to define R_i in a similar way to the proof of Theorem 7.

Claim 36. *For all i, no vertex has a neighbour in each of D_{i-1}, D_i, D_{i+1}.*

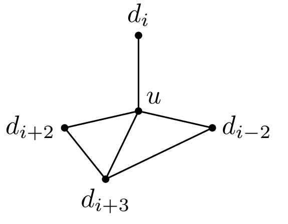
Proof. If not, choose d_{i-1}, d_i, d_{i+1} in D_{i-1}, D_i, D_{i+1} respectively with common neighbour u such that $e(\{d_{i-1}, d_i\}) + e(\{d_i, d_{i+1}\})$ is maximal. Proceeding exactly as in the proof of Claim 23 (with Lemma 35 in place of Lemma 22) shows there is some $d \in D^*$ adjacent to u, d_{i-1} and d_{i+1}. Then $d \in D_i$.

- If $i = 0$, then $dd_1v_2v_3v_4v_5d_6$ form a copy of H_2, while u is adjacent to d_6, d and d_1, contrary to Claim 32.
- If $i \neq 0$, then v_{i-1} is adjacent to d_{i+1}, v_{i+1} and v_{i+1} is adjacent to d_{i-1}, v_{i-1} so $ud_{i-1}v_{i+1}v_{i-1}d_{i+1}$ is an odd circuit in G_d. □

Claim 37. *For all $i \neq 0$, no vertex has a neighbour in each of D_{i-2}, D_i, D_{i+2}.*

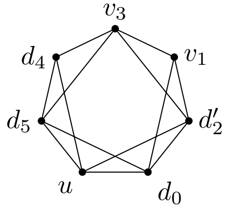
Proof. If not, let $i \neq 0$ and choose d_{i-2}, d_i, d_{i+2} in D_{i-2}, D_i, D_{i+2} respectively with common neighbour u such that $e(\{d_{i-2}, d_i\}) + e(\{d_i, d_{i+2}\})$ is maximal.

Proceeding exactly as in the proof of Claim 24 (with Lemma 35 for Lemma 22 and Claim 36 for Claim 23) shows that $u \in R$ and there is some $d \in D^*$ adjacent to u, d_{i-2} and d_{i+2}. In particular, $d \in D_i \cup D_{i+3} \cup D_{i-3}$. Suppose $d \in D_i$: $i \neq 0$ so v_{i-1}, v_{i+1} are adjacent so $ud_{i-2}v_{i-1}v_{i+1}d_{i+2}$ is an odd circuit in G_d. Thus $d \in D_{i-3} \cup D_{i+3}$. By symmetry we may assume that $d \in D_{i+3}$. Write d_{i+3} for d.



Continuing as in the proof of Claim 24 shows that there is $d_i' \in D_i$ adjacent to both u and d_{i-2}.

If $i \neq 2$, then v_{i-1} and v_{i-3} are adjacent, so $d_i'ud_{i+3}v_{i-3}v_{i-1}$ is an odd circuit in $G_{d_{i-2}}$. Finally if $i = 2$, then $v_3d_4d_5ud_0d_2'v_1$ is a copy of H_2 in G (note that all the vertices are distinct: vertex $u \in R$ and the others are in distinct D_i). However, v_2 is adjacent to all of v_1, v_3, d_4 contrary to Claim 32.



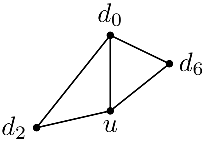
□

Claim 38. *No vertex has a neighbour in each of D_6, D_0, D_2 and no vertex has a neighbour in each of D_5, D_0, D_1.*

Proof. Suppose not – by symmetry we may assume that some vertex has a neighbour in each of D_6, D_0, D_2. Choose $d_6 \in D_6, d_0 \in D_0$ and $d_2 \in D_2$ with common neighbour u such that $e(\{d_6, d_0\}) + e(\{d_0, d_2\})$ is maximal. We first show that d_0 is adjacent to both d_2 and d_6. Apply Lemma 35 to $\{u, d_6, d_0, d_2\}$.

Suppose some x is adjacent to all of u, d_6, d_0, d_2. Now apply Lemma 35 to $\{u, x, d_0, d_2\}$: uxd_0 and uxd_2 are triangles so some vertex in D^* is adjacent to d_0 and d_2 and one of u, x. But no vertex in D^* has a neighbour in each of D_0, D_2.

Hence, there is some $d \in D^*$ adjacent to three of u, d_6, d_0, d_2. No vertex in D^* has a neighbour in each of D_0, D_2 so d is adjacent to u, d_6 and one of d_0, d_2. If d is adjacent to d_0, then $d \in D_5$. But then u has a neighbour in each of D_5, D_6, D_0 contrary to Claim 36. Thus d is adjacent to d_2 and so $d \in D_0 \cup D_4$. If $d \in D_4$, then u has a neighbour in each of D_2, D_4, D_6, contradicting Claim 37. Hence $d \in D_0$. By maximality at the start we must have d_0 adjacent to both d_2 and d_6.



Now $d_0v_1d_2v_3v_4v_5d_6$ is a copy of H_2 so can be extended, by Claim 30, to a copy of H_2^+. That is, there is some other vertex, v, adjacent to v_5, d_0, d_2. But then G_{d_0} contains the odd circuit $ud_2vv_5d_6$. □

Corollary 39. *Every v in G satisfies one of the following properties.*

- *v has a neighbour in each of D_5, D_0, D_2 and $\Gamma(v) \cap D \subset D_5 \cup D_0 \cup D_2$.*
- *There is an i such that $\Gamma(v) \cap D \subset \Gamma(v_i) \cap D$.*

Proof. Fix a vertex v. First suppose that v has a neighbour in each of D_5, D_0, D_2. By Claim 36, v has no neighbours in $D_1 \cup D_6$. By Claim 37, v has no neighbours in $D_3 \cup D_4$. Thus $\Gamma(v) \cap D \subset D_5 \cup D_0 \cup D_2$.

Otherwise v does not have a neighbour in each of D_5, D_0, D_2. By Claims 36 and 37, there is an i with $\Gamma(v) \cap D \subset D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}$. If $i \neq 1, 6$, then $D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2} = \Gamma(v_i) \cap D$ and so we are done. Otherwise we may assume, by symmetry, that $i = 1$: $\Gamma(v) \cap D \subset D_6 \cup D_0 \cup D_2 \cup D_3$. By Claim 38, we have one of the following.

- $\Gamma(v) \cap D \subset D_0 \cup D_2 \cup D_3 = \Gamma(v_1) \cap D$,
- $\Gamma(v) \cap D \subset D_6 \cup D_2 \cup D_3 \subset \Gamma(v_4) \cap D$,
- $\Gamma(v) \cap D \subset D_6 \cup D_0 \cup D_3 \subset \Gamma(v_5) \cap D$. □

This corollary gives structure to R. Firstly let

$$R_{502} = \{v \in R\colon v \text{ has a neighbour in each of } D_5, D_0, D_2\}.$$

Then, for $i = 0, 1, \ldots, 6$ choose

$$R_i \subset \{v \in R\colon \Gamma(v) \cap D \subset \Gamma(v_i) \cup D\},$$

so that $R_{502} \cup R_0 \cup R_1 \cup \cdots \cup R_6$ is a partition of R. There may be some flexibility in the choice of the R_i – we will make use of this later. For now we just take any arbitrary choice. Note, for each i, that

$$e(R_i, D_i \cup D_{i-3} \cup D_{i+3}) = 0,$$

and also that

$$e(R_1, D_6) = e(R_6, D_1) = e(R_{502}, D_1 \cup D_3 \cup D_4 \cup D_6) = 0.$$

For $i = 0, 1, \ldots, 6$, let $T_i = D_i \cup R_i$. We may give a lower bound for the size of T_i. Firstly,

$$\begin{aligned} d(v_1) + d(v_6) &= |D_0| + |D| - |D_1| - |D_6| + |R \cap (\Gamma(v_1) \cup \Gamma(v_6))| + |R \cap \Gamma(v_1, v_6)| \\ &\leqslant |D_0| + |D| - |D_1| - |D_6| + |R| - |R_1| - |R_6| - |R_{502}| + |R_0|, \end{aligned}$$

so

$$|T_0| \geqslant 2\delta(G) - |G| + |T_1| + |T_6| + |R_{502}|. \tag{5}$$

Next,

$$\begin{aligned} d(v_0) + d(v_2) &= |D_1| + |D| + |R \cap (\Gamma(v_0) \cup \Gamma(v_2))| + |R \cap \Gamma(v_0, v_2)| \\ &\leqslant |D_1| + |D| + |R| + |R_1| + |R_{502}|, \end{aligned}$$

and so (using symmetry for the second inequality)

$$\begin{aligned} |T_1| + |R_{502}| &\geqslant 2\delta(G) - |G|, \\ |T_6| + |R_{502}| &\geqslant 2\delta(G) - |G|. \end{aligned} \tag{6}$$

A similar argument applied to $d(v_1) + d(v_3)$ gives

$$\begin{aligned} |T_2| &\geqslant 2\delta(G) - |G| + |T_6| + |R_{502}|, \\ |T_5| &\geqslant 2\delta(G) - |G| + |T_1| + |R_{502}|, \end{aligned} \tag{7}$$

and one applied to $d(v_2) + d(v_4)$ gives

$$\begin{aligned} |T_3| &\geqslant 2\delta(G) - |G|, \\ |T_4| &\geqslant 2\delta(G) - |G|. \end{aligned} \tag{8}$$

In particular, for all $i \neq 1, 6$,

$$|T_i| \geqslant 2\delta(G) - |G|, \tag{9}$$

and, using inequality (6) combined with (5) and (7), we have for $i = 0, 2, 5$,

$$|T_i| \geqslant 4\delta(G) - 2|G|. \tag{10}$$

The next three claims are technical in nature but will speed up what follows.

Claim 40. *Every $d \in D_i$ and every $u \in R_{i+1}$ have a common neighbour in D^* unless $i = 2$, in which case they have a common neighbour in $D^* \cup R_{502}$. Similarly, every $d \in D_i$ and every $u \in R_{i-1}$ have a common neighbour in D^* unless $i = 5$, in which case they have a common neighbour in $D^* \cup R_{502}$.*

Proof. It is enough to prove the assertion when $d \in D_i$ and $u \in R_{i+1}$, the other assertion following symmetrically. Now $\Gamma(d) \cup \Gamma(u) \subset V(G) \setminus D_{i-3}$, so

$$|\Gamma(d) \cap \Gamma(u)| = d(d) + d(u) - |\Gamma(d) \cup \Gamma(u)| \geqslant 2\delta(G) - |G| + |D_{i-3}|.$$

Now, for $i = 0, 1, 3, 5, 6$, inequality (9) gives

$$\begin{aligned}|\Gamma(d) \cap \Gamma(u)| &\geqslant 2\delta(G) - |G| + |D_{i-3}| \geqslant 4\delta(G) - 2|G| - |R_{i-3}| \\ &> 4|G| - 7\delta(G) - |R_{i-3}| \geqslant |R \cup D_1 \cup D_6| - |R_{i-3}| \\ &\geqslant |\Gamma(d) \cap (R \cup D_1 \cup D_6)|,\end{aligned}$$

where we used $\delta(G) > 6/11 \cdot |G|$, inequality (4) and $e(D_i, R_{i-3}) = 0$ in the third, fourth and fifth inequalities respectively. Hence there is a common neighbour of d and u which is not in $R \cup D_1 \cup D_6$, so is in D^*.

For $i = 4$, inequality (6) gives

$$\begin{aligned}|\Gamma(d) \cap \Gamma(u)| &\geqslant 2\delta(G) - |G| + |D_1| \geqslant 4\delta(G) - 2|G| - |R_1| - |R_{502}| \\ &> 4|G| - 7\delta(G) - |R_1| - |R_{502}| \geqslant |R \cup D_1 \cup D_6| - |R_1| - |R_{502}| \\ &\geqslant |\Gamma(d) \cap (R \cup D_1 \cup D_6)|,\end{aligned}$$

where we used $\delta(G) > 6/11 \cdot |G|$, inequality (4) and $e(D_4, R_1 \cup R_{502}) = 0$ in the third, fourth and fifth inequalities respectively.

For $i = 2$, inequality (6) gives

$$\begin{aligned}|\Gamma(d) \cap \Gamma(u)| &\geqslant 2\delta(G) - |G| + |D_6| \geqslant 4\delta(G) - 2|G| - |R_6| - |R_{502}| \\ &> |R \cup D_1 \cup D_6| - |R_6| - |R_{502}| \geqslant |\Gamma(d) \cap ((R \setminus R_{502}) \cup D_1 \cup D_6)|. \quad \square\end{aligned}$$

Claim 41. *Every $d \in D_0$ and every $u \in R_{502}$ have a common neighbour in $D_2 \cup D_5$. Every $d \in D_2 \cup D_5$ and every $u \in R_{502}$ have a common neighbour in D_0.*

Proof. Fix $d \in D_0$ and $u \in R_{502}$. Now $\Gamma(d) \cup \Gamma(u) \subset V(G) \setminus D_3$, so, as in the previous claim,

$$|\Gamma(d) \cap \Gamma(u)| \geqslant 4\delta(G) - 2|G| - |R_3| > |R \cup D_1 \cup D_6| - |R_3| \\ \geqslant |\Gamma(d) \cap (R \cup D_1 \cup D_6)|,$$

so d, u have a common neighbour in D^*. But $d \in D_0$ so this common neighbour must be in $D_2 \cup D_5$.

Suppose, for contradiction, $d \in D_2$, $u \in R_{502}$ have no common neighbour in D_0: then d, u have no common neighbour in D. Now $u \in R_{502}$ so u has a neighbour $d_0 \in D_0$, $d_5 \in D_5$. Apply Lemma 35 to $\{u, d, d_0, d_5\}$: no vertex in D^* is adjacent to d, d_0 or to d_0, d_5 or to d, u so there is some v adjacent to all of u, d, d_0, d_5.

Now apply Lemma 35 to $\{u, v, d_0, d_5\}$: as uvd_0 and uvd_5 are triangles, some vertex in D^* is adjacent to both d_0, d_5 (and one of u, v). But no vertex in D^* has a neighbour in each of D_0, D_5. □

Claim 42. *For each $i \in \{0, 1, \ldots, 6, 502\}$, every two vertices in R_i have a common neighbour in D^*.*

Proof. Before addressing the claim, we show that

$$|R_{502}| + |T_j| + |T_{j+3}| + |T_{j-3}| \geqslant 3(2\delta(G) - |G|) \tag{11}$$

for all $j \in \{0, 1, \ldots, 6\}$. Note that at most one of $j, j-3, j+3$ can be 1 or 6. If none of $j, j-3, j+3$ is 1 or 6, then (11) follows from inequality (9). If exactly one of them is 1 or 6, then (11) follows from inequalities (6) and (9).

We first deal with $i \in \{0, 1, \ldots, 6\}$. If $u, v \in R_i$ have no common neighbour in D^*, then $\Gamma(u) \cap D^*$ and $\Gamma(v) \cap D^*$ are disjoint subsets of $D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}$. Now, by inequality (4),

$$|\Gamma(u) \cap D^*| \geqslant d(u) - |R \cup D_1 \cup D_6| \geqslant 8\delta(G) - 4|G|,$$

so $|D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}| \geqslant 16\delta(G) - 8|G|$. This, together with inequality (11), gives

$$|G| \geqslant |D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}| + |R_{502}| + |T_i| + |T_{i+3}| + |T_{i-3}| \\ \geqslant (16\delta(G) - 8|G|) + 3(2\delta(G) - |G|) = 22\delta(G) - 11|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$.

Now we deal with $i = 502$. If $u, v \in R_{502}$ have no common neighbour in D^*, then, as above,

$$|D_5 \cup D_0 \cup D_2| \geqslant 16\delta(G) - 8|G|.$$

But then combining this with inequality (11) for $j = 1$ gives

$$\begin{aligned}|G| &\geqslant |D_5 \cup D_0 \cup D_2| + |R_{502}| + |T_1| + |T_4| + |T_3| \\ &\geqslant 16\delta(G) - 8|G| + 3(2\delta(G) - |G|) = 22\delta(G) - 11|G|,\end{aligned}$$

which contradicts $\delta(G) > 6/11 \cdot |G|$. □

We now aim to show that $T_i = D_i \cup R_i$ is independent for all i.

Claim 43. *For all i and $d \in D_i$: $\Gamma(d) \cap T_{i-1}$ and $\Gamma(d) \cap T_{i+1}$ are independent.*

Proof. Suppose there is $d_i \in D_i$ such that $\Gamma(d_i) \cap T_{i+1}$ contains the edge uv. As D_{i+1} is independent and $e(D_{i+1}, R_{i+1}) = 0$, both $u, v \in R_{i+1}$.

We first deal with the case when i is not 2. Then, by Claim 40, there is $d_u \in D^*$ adjacent to both u, d_i. As d_u is adjacent to $u \in R_{i+1}$, d_u is adjacent to v_{i+1}. Similarly there is $d_v \in D^*$ adjacent to v, d_i, v_{i+1}. But then $v_{i+1}d_u uvd_v$ is an odd circuit in G_{d_i}.

Now suppose $i = 2$ and write $d = d_2$. By Claim 40, d_2 and u have a common neighbour $x_u \in D^* \cup R_{502}$. If $x_u \in D^*$, then as x_u is adjacent to $u \in R_3$, x_u is adjacent to v_3, while if $x_u \in R_{502}$, then, by Claim 41, x_u and d_2 have a common neighbour $d' \in D_0$. Taking $d_u = v_3$ in the former case and $d_u = d'$ in the latter, we see that x_u and d_2 have a common neighbour $d_u \in D$ which is adjacent to v_1. Similarly d_2 and v have a common neighbour x_v such that x_v and d_2 have a common neighbour d_v which is adjacent to v_1. But then $v_1 d_u x_u uvx_v d_v$ is an odd circuit in G_{d_2}. □

Claim 44. *For all i, T_i is independent.*

Proof. Suppose uv is an edge in T_i – as D_i is independent and $e(D_i, R_i) = 0$ we have $u, v \in R_i$. By Claim 42, u, v have a common neighbour in $d \in D^*$. By Claim 43, $d \notin D_{i-1} \cup D_{i+1}$ so $d \in D_{i-2} \cup D_{i+2}$. We complete the argument assuming that $d \in D_{i-2}$. The other case is analogous. Write $d = d_{i-2}$.

Apply Lemma 35 to $\{u, v, d_{i-2}, v_{i+2}\}$: as uvd_{i-2} is a triangle, there is $d' \in D^*$ adjacent to v_{i+2} and to two of u, v, d_{i-2}. In particular, d' is adjacent to v_{i+2} and to at least one of $u, v \in R_i$, so $d' \in D_{i+1}$. But then d' cannot be adjacent to d_{i-2} and so is adjacent to both u and v. However, edge uv lies in $\Gamma(d') \cap R_i$, contradicting Claim 43. □

Claim 45. *R_{502} is independent.*

Proof. Suppose uv is an edge in R_{502}. By Claim 42, u, v have a common neighbour $d \in D_5 \cup D_0 \cup D_2$.

First suppose that $d \in D_0$. By Claim 41, d, u have a common neighbour $d_u \in D_2 \cup D_5$ and d, v have a common neighbour $d_v \in D_2 \cup D_5$. We may assume that $d_u \in D_2$.

- If $d_v \in D_2$, then $v_1 d_u uvd_v$ is an odd circuit in G_d.

- If $d_v \in D_5$, then $dv_1d_uv_3v_4d_vv_6$ is a copy of H_2 so, by Claim 30, can be extended to a copy of H_2^+: there is x adjacent to all of d, d_u, d_v. But then xd_uuvd_v is an odd circuit in G_d.

Next suppose that $d \in D_2 \cup D_5$. By symmetry, we may assume that $d \in D_2$. By Claim 41, d, u have a common neighbour $d_u \in D_0$ and d, v have a common neighbour $d_v \in D_0$. But then $v_1d_uuvd_v$ is an odd circuit in G_d. □

We have made good progress: we now know that there is a homomorphism $G \to K_8$ and so G is 8-colourable.

Claim 46. $e(R_{502}, T_1 \cup T_6) = 0$.

Proof. If not, then we may assume there is an edge uv with $u \in R_{502}$ and $v \in T_1$. As $e(R_{502}, D_1) = 0$, we have $v \in R_1$. We first show that u, v have a common neighbour in $D_0 \cup D_2$. Apply Lemma 35 to $\{u, v, v_0, v_2\}$: any common neighbour of v_0, v_2 is in $T_1 \cup R_{502}$ so is adjacent to at most one of u, v. Hence there is $d \in D^*$ adjacent to three of u, v, v_0, v_2. No vertex of D^* is adjacent to both v_0, v_2 so d is a common neighbour of u, v. As d is adjacent to u and v, $d \in D_0 \cup D_2$.

First suppose $d \in D_0$. By Claim 40, d, v have a common neighbour $d_v \in D^*$: d_v is adjacent to both d, v so $d_v \in D_2$. By Claim 41, d, u have a common neighbour $d_u \in D_2 \cup D_5$.

- If $d_u \in D_2$, then $v_1d_uuvd_v$ is an odd circuit in G_d.
- If $d_u \in D_5$, then $dv_1d_vv_3v_4d_uv_6$ is a copy of H_2 so, by Claim 30, there is a vertex x adjacent to all of d, d_u, d_v. But then xd_uuvd_v is an odd circuit in G_d.

Now suppose $d \in D_2$. By Claim 41, d, u have a common neighbour $d_u \in D_0$. By Claim 40, d, v have a common neighbour $d_v \in D^*$: d_v is adjacent to both d, v so $d_v \in D_0 \cup D_3$. But then $v_1d_uuvd_v$ is an odd circuit in G_d. □

Claim 47. $e(T_1, T_6) = 0$.

Proof. If not, then there is an edge uv with $u \in T_1$, $v \in T_6$. As $e(T_1, D_6) = e(T_6, D_1)$, we have $u \in R_1$ and $v \in R_6$. We first show that u, v have a common neighbour $d \in D_0$. Apply Lemma 35 to $\{v_0, v_2, u, v\}$: any common neighbour of v_0, v_2 is in $T_1 \cup R_{502}$ so is not adjacent to u. Hence there is $d \in D^*$ adjacent to three of v_0, v_2, u, v. No vertex in D^* is adjacent to v_0, v_2 so d is adjacent to u, v. But $u \in R_1, v \in R_6$ so $d \in D_0$.

By Claim 40, d, u have a common neighbour $d_u \in D^*$ and d, v have a common neighbour $d_v \in D^*$. As $d \in D_0$ and $u \in R_1$, we have $d_u \in D_2$. Similarly, $d_v \in D_5$. Now $dv_1d_uv_3v_4d_vv_6$ is a copy of H_2, so, by Claim 30, there is a vertex x adjacent to all of d, d_u, d_v. But then xd_uuvd_v is an odd circuit in G_d. □

Before proceeding it will help to give structure to G_d for each $d \in D_1 \cup D_2 \cup \cdots \cup D_6$. This corresponds to Claim 28 in the proof of Theorem 7.

Claim 48. *For each $i \in \{1, 2, \ldots, 6\}$ and every $d \in D_i$, G_d is connected bipartite. Furthermore, there is a bipartition of G_d into two vertex classes A_d, B_d which satisfy $(T_{i-1} \cup D_{i+2}) \cap \Gamma(d) \subset A_d$, $(T_{i+1} \cup D_{i-2}) \cap \Gamma(d) \subset B_d$ and at least one of $R_{i+2} \cap \Gamma(d) \subset A_d$, $R_{i-2} \cap \Gamma(d) \subset B_d$ occurs. If $i = 2$, then $R_{502} \cap \Gamma(d) \subset A_d$ and if $i = 5$, then $R_{502} \cap \Gamma(d) \subset B_d$.*

Proof. Fix $d \in D_i$ and define for $j = i-2, i-1, i+1, i+2$,

$$D_j^d = D_j \cap \Gamma(d),$$
$$R_j^d = R_j \cap \Gamma(d),$$
$$R_{502}^d = R_{502} \cap \Gamma(d),$$

and note that these partition $V(G_d)$ (and some of them can be empty). Also let $T_j^d = T_j \cap \Gamma(d)$. Vertex $v_{i-1} \in G_d$. We let

$$A_d = \{x \in G_d \colon \operatorname{dist}_{G_d}(x, v_{i-1}) \text{ is even}\},$$
$$B_d = \{x \in G_d \colon \operatorname{dist}_{G_d}(x, v_{i-1}) \text{ is odd}\}.$$

G is locally bipartite so G_d is bipartite and so A_d and B_d are independent sets. Now, as i is not 0,

- $v_{i-1} \in A_d$, $v_{i+1} \in B_d$.
- v_{i-1} is adjacent to all of $D_{i-2}^d \cup D_{i+1}^d$, so $D_{i-2}^d \cup D_{i+1}^d \subset B_d$.
- v_{i+1} is adjacent to all of $D_{i-1}^d \cup D_{i+2}^d$, so $D_{i-1}^d \cup D_{i+2}^d \subset A_d$.

If $i = 2$, then, by Claim 41, any $x \in R_{502}^d$ has a neighbour in $D_0^d \subset B_d$, so $R_{502}^d \subset A_d$. If $i = 5$, then, by Claim 41, any $x \in R_{502}^d$ has a neighbour in $D_0^d \subset A_d$ so $R_{502}^d \subset B_d$. For other i, R_{502}^d is empty.

We next show that $R_{i-1}^d \subset A_d$. Fix $x \in R_{i-1}^d$ – it suffices to show $x \in A_d$. Suppose x, d have a common neighbour in $d' \in D$. As $x \in R_{i-1}$ and $d \in D_i$, d' must be in $D_{i-2} \cup D_{i+1}$. Hence $d' \in B_d$ and so $x \in A_d$. On the other hand if x, d do not have a common neighbour in D, then Claim 40 guarantees that $i = 5$ and x has a neighbour in $R_{502}^d \subset B_d$, so $x \in A_d$. Similarly $R_{i+1}^d \subset B_d$.

We now show that at least one of $R_{i+2}^d \subset A_d$, $R_{i-2}^d \subset B_d$ occurs. If not, then there is $u \in R_{i+2}^d \setminus A_d$ and $v \in R_{i-2}^d \setminus B_d$. Focus on u: $u \notin A_d$ so $\Gamma_{G_d}(u) \subset V(G_d) - B_d \subset T_{i-1}^d \cup T_{i+2}^d \cup R_{i-2}^d \cup R_{502}^d$. But $u \in R_{i+2}$, the set T_{i+2} is independent and $e(R_{i+2}, D_{i-1}) = 0$, so, in fact,

$$\Gamma_{G_d}(u) \subset R_{i-1}^d \cup R_{i-2}^d \cup R_{502}^d.$$

Similarly

$$\Gamma_{G_d}(v) \subset R^d_{i+1} \cup R^d_{i+2} \cup R^d_{502}.$$

If u and v both have a neighbour in R^d_{502}, then R^d_{502} would be non-empty, so $i = 2, 5$ and either $R^d_{502} \subset A_d$ or $R^d_{502} \subset B_d$. The former contradicts $v \notin B_d$ and the latter contradicts $u \notin A_d$. Thus, at most one of u, v has a neighbour in R^d_{502}. In particular,

$$|\Gamma_{G_d}(u)| + |\Gamma_{G_d}(v)| \leqslant |R^d_{i-2}| + |R^d_{i-1}| + |R^d_{i+1}| + |R^d_{i+2}| + |R^d_{502}| \leqslant |R|.$$

But then inequality (4) gives

$$4|G| - 7\delta(G) \geqslant |R| \geqslant d(d, u) + d(d, v) \geqslant 4\delta(G) - 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$.

Finally the proof that G_d is connected is identical to that part of the proof of Claim 28. □

To prove that there is a homomorphism $G \to H_2^+$ we would need to show that $e(R_i, R_{i+3}) = 0$ for all i and $e(R_3 \cup R_4, R_{502}) = 0$. We make a start.

Claim 49. $e(T_i, T_{i+3}) = 0$ *for* $i = 0, 2, 4$.

Proof. Suppose not: there is an edge uv with $u \in T_{i+3}$, $v \in T_i$. As $e(T_i, D_i \cup D_{i-3} \cup D_{i+3}) = 0$, we must have $u \in R_{i+3}$ and $v \in R_i$. We first show that u, v have a common neighbour $d \in D^* \cap (D_{i+1} \cup D_{i+2})$. Apply Lemma 35 to $\{u, v, v_{i+1}, v_{i+2}\}$: any common neighbour of v_{i+1}, v_{i+2} is in $T_i \cup T_{i+3}$ so is adjacent to at most one of u, v. Moreover, any common neighbour of v_{i+1}, v_{i+2} in D^* is in $D_i \cup D_{i+3}$ and so is adjacent to neither u nor v. Hence there is $d \in D^*$ adjacent to both u, v and one of v_{i+1}, v_{i+2} – in particular, $d \in D_{i+1} \cup D_{i+2}$.

If $i = 2$, we may take $d \in D_3$, by symmetry. If $i = 0, 4$ we may assume, by symmetry that $i = 4$. Since $d \in D^*$, we have $d \in D_5$. In conclusion, we have adjacent vertices $u \in R_{i+3}$, $v \in R_i$ with common neighbour $d \in D_{i+1}$ where i is 2 or 4. Consider the bipartition of G_d given by Claim 48:

- $(T_i \cup D_{i+3}) \cap \Gamma(d) \subset A_d$.
- $(T_{i+2} \cup D_{i-1} \cup R_{502}) \cap \Gamma(d) \subset B_d$.
- At least one of $R_{i+3} \cap \Gamma(d) \subset A_d$ or $R_{i-1} \cap \Gamma(d) \subset B_d$ occurs.

As $v \in R_i$, we have $v \in A_d$ and so $u \in B_d$. But $u \in R_{i+3} \cap \Gamma(d)$, so $R_{i-1} \cap \Gamma(d) \subset B_d$ occurs. Now $\Gamma_{G_d}(u) \subset A_d \subset (T_i \cup T_{i+3}) \cap \Gamma(d)$. But $u \in R_{i+3}$, the set T_{i+3} is independent and $e(R_{i+3}, D_i) = 0$, so

$$\Gamma_{G_d}(u) \subset R_i \cap \Gamma(d).$$

Thus, $|R_i| \geqslant d(d,u) \geqslant 2\delta(G) - |G|$. But then, using inequalities (4) and (6),

$$4|G| - 7\delta(G) \geqslant |R \cup D_1 \cup D_6| \geqslant |R_i| + |T_1 \cup R_{502}| \geqslant 4\delta(G) - 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$. □

We have been flexible about the R_i and so all of our results thus far hold for any R_i satisfying their definition. Now is the time to make a further choice. We choose the R_i so that

$$S = \sum_{i=0}^{6} e(R_i, R_{i+3}) + e(R_3 \cup R_4, R_{502}) \tag{12}$$

is minimal.

Claim 50. $e(T_i, T_{i+3}) = 0$ *for* $i = 1, 3$.

Proof. By symmetry it suffices to prove this for $i = 3$. Suppose we have $u \in T_6$, $v \in T_3$ with u adjacent to v. As $e(D_3, T_6) = e(D_6, T_3) = 0$, we must have $u \in R_6$ and $v \in R_3$. Just as in the proof of Claim 49, u and v have a common neighbour $d \in D_4 \cup D_5$. When $d \in D_4$, the argument of Claim 49 works again (with $i = 3$). We deal with the more difficult $d \in D_5$ case. For any $d \in D_5 \cap \Gamma(u, v)$, consider the bipartition of G_d given by Claim 48:

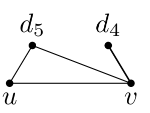
- $(T_4 \cup D_0) \cap \Gamma(d) \subset A_d$.
- $(T_6 \cup D_3 \cup R_{502}) \cap \Gamma(d) \subset B_d$.
- At least one of $R_0 \cap \Gamma(d) \subset A_d$ or $R_3 \cap \Gamma(d) \subset B_d$ occurs.

As $u \in R_6$, $u \in B_d$ and so $v \in A_d$. But $v \in R_3 \cap \Gamma(d)$, so $R_0 \cap \Gamma(d) \subset A_d$ occurs. Now $\Gamma_{G_d}(v) \subset B_d \subset T_6 \cup T_3 \cup R_{502}$. But $v \in R_3$, the set T_3 is independent and $e(R_3, D_6) = 0$, so

$$\Gamma_{G_d}(v) \subset (R_6 \cup R_{502}) \cap \Gamma(d).$$

Note that this holds for any choice of $d \in D_5 \cap \Gamma(u, v)$.

We first deal with the case where v has at least one neighbour in D_4. Pick any $d_5 \in \Gamma(u, v) \cap D_5$, $d_4 \in \Gamma(v) \cap D_4$.



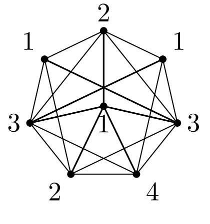
Apply Lemma 35 to $\{u, v, d_4, d_5\}$: the vertices d_5, u, v form a triangle so some $d \in D^*$ is adjacent to d_4 and to two of d_5, u, v. If d is adjacent to d_5, then $d \in D_3 \cup D_6$, so d is adjacent to neither u nor v. Hence $d \in \Gamma(u, v, d_4) \cap D^*$, so $d \in D_5$. But then $d \in \Gamma(u, v) \cap D_5$ and $\Gamma_{G_d}(v)$ contains $d_4 \notin (R_6 \cup R_{502}) \cap \Gamma(d)$, a contradiction.

We are finally left with the case where v has no neighbours in D_4: when we were choosing the R_i we could have put u in R_0. In particular, Claim 49 gives $e(u, R_4) = 0$. Also R_3 is independent, so $e(u, R_3) = 0$. Thus if we put $u \in R_0$, then u would contribute 0 to S while currently it contributes at least 1 (the edge uv contributes to $e(R_6, R_3)$). This contradicts the minimality of S. □

Proof of Theorem 8. Let G be a locally bipartite graph with $\delta(G) > 6/11 \cdot |G|$. By Theorems 5 and 6, G is either 3-colourable, contains $\overline{C}_7$ or contains H_2^+. In the first two cases we are done (using Theorem 7 – note that $\overline{C}_7$ is 4-colourable). Hence, we may assume that G does not contain a copy of $\overline{C}_7$ but does contain a copy of H_2^+. We can thus follow the argument of this subsection, defining the D_i and R_i and establishing all the claims concerning them.

Note that the T_i and R_{502} together partition $V(G)$. By Claim 44, each T_i is independent and, by Claim 45, R_{502} is independent. By Claims 49 and 50, $e(T_i, T_{i+3}) = 0$ for $i = 0, 1, \ldots, 4$ and, by Claim 47, $e(T_1, T_6) = 0$. Finally, $e(R_{502}, T_1 \cup T_6) = 0$ by Claim 46.

Hence, identifying T_i with v_i and R_{502} with a single vertex gives a homomorphism from G to the following graph which is H_2^+ with four extra edges. This graph is 4-colourable (colouring shown in the diagram) and so $\chi(G) \leqslant 4$.



□

3.3. Proof of Theorem 9

In this subsection we prove Theorem 9. We remind the reader that there is now a stronger minimum degree condition on G: $\delta(G) \geqslant (5/9 - \varepsilon) \cdot |G|$ for some small positive ε. We are also given that G does not contain $\overline{C}_7$. We are in the same position as at the start of the proof of Theorem 8 but with a stronger minimum degree condition that we will leverage to give G greater structure. Hence, we may use all of the machinery from our proof of Theorem 8 and, in particular, we only need to show that $e(R_3 \cup R_4, R_{502}) = e(R_1, R_5) = e(R_2, R_6) = 0$. As before, we choose the R_i so that S, as given in equation (12), is minimal. Using inequalities (8) and (10), we have

$$|T_3|, |T_4| \geqslant 2\delta(G) - |G| > (1/9 - 2\varepsilon)|G|,$$

$$|T_0|, |T_2|, |T_5| \geqslant 4\delta(G) - 2|G| > (2/9 - 4\varepsilon)|G|.$$

Also, by Claim 30, we have

$$|R_{502}| \geqslant |\Gamma(v_5, v_0, v_2)| \geqslant 11\delta(G) - 6|G| > (1/9 - 11\varepsilon)|G|.$$

Now $2(1/9 - 2\varepsilon) + 3(2/9 - 4\varepsilon) + (1/9 - 11\varepsilon) = 1 - 27\varepsilon$, so in fact we have

$$\begin{aligned} |T_3|, |T_4|, |R_{502}| &= (1/9 - O(\varepsilon))|G|, \\ |T_0|, |T_2|, |T_5| &= (2/9 - O(\varepsilon))|G|. \end{aligned}$$

Throughout we will use $O(\varepsilon)$ to denote a quantity for which there is an absolute positive constant C (in particular, independent of G and ε) such that the quantity lies between $-C\varepsilon$ and $C\varepsilon$. By inequality (4),

$$|D_1 \cup D_6 \cup R| \leqslant 4|G| - 7\delta(G) < (1/9 + 7\varepsilon)|G|.$$

Putting all this together (and noting that $R_{502} \subset R$) we have

$$\begin{aligned} |R \setminus R_{502}|, |D_1|, |D_6| &= O(\varepsilon)|G|, \\ |D_3|, |D_4|, |R_{502}| &= (1/9 + O(\varepsilon))|G|, \\ |D_0|, |D_2|, |D_5| &= (2/9 + O(\varepsilon))|G|. \end{aligned} \tag{13}$$

Note that these numbers match the weighting of H_2^+ given in Fig. 2. That was a weighting of H_2^+ with minimum degree attaining 5/9.

Claim 51. *Provided $\varepsilon > 0$ is sufficiently small, $e(R_1, R_5) = e(R_2, R_6) = 0$.*

Proof. Suppose this is false. By symmetry we may take $r_2 \in R_2$ and $r_6 \in R_6$ where r_2r_6 is an edge. We first claim that r_2 has a neighbour $t_1 \in T_1$. Indeed, if r_2 does not, then, when we chose the R_i, we could have put r_2 in R_5. Thus, by the minimality of S,

$$e(r_2, R_5) + e(r_2, R_6) \leqslant e(r_2, R_1) + e(r_2, R_2).$$

However, the left-hand is positive (the edge r_2r_6 contributes to it), while the right-hand side is zero (R_2 is independent and r_2 has no neighbours in R_1 by assumption). Thus r_2 does indeed have a neighbour in $t_1 \in T_1$.

Now, $\Gamma(r_2) \subset T_0 \cup T_1 \cup T_3 \cup T_4 \cup R_{502} \cup R_6$ and this union has size $(5/9+O(\varepsilon))|G|$, so r_2 has at most $O(\varepsilon)|G|$ non-neighbours in $D_0 \cup D_3 \cup D_4$. Also, $\Gamma(t_1) \subset T_0 \cup T_2 \cup T_3 \cup R_5$ and this union has size $(5/9+O(\varepsilon))|G|$, so t_1 has at most $O(\varepsilon)|G|$ non-neighbours in $D_0 \cup D_3$. Similarly, r_6 has at most $O(\varepsilon)|G|$ non-neighbours in $D_0 \cup D_4$. But D_0, D_3 both have size at least $(1/9 + O(\varepsilon))|G|$, so, provided ε is small enough, there is $d_0 \in D_0$ adjacent to all of r_2, t_1, r_6 and there is $d_3 \in D_3$ adjacent to both t_1, r_2.

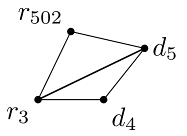


Fig. 5. The configuration of Claim 52.

Finally, $\Gamma(d_3) \subset T_1 \cup T_2 \cup T_4 \cup T_5$ and this union has size $(5/9 + O(\varepsilon))|G|$, so d_3 has at most $O(\varepsilon)|G|$ non-neighbours in D_4. But D_4 has size $(1/9 + O(\varepsilon))|G|$, so, provided ε is small enough, there is $d_4 \in D_4$ adjacent to all of r_2, d_3, r_6. But then $d_0 t_1 d_3 d_4 r_6$ is a 5-cycle in G_{r_2}. □

Claim 52. *Provided $\varepsilon > 0$ is sufficiently small, $e(R_3 \cup R_4, R_{502}) = 0$.*

Proof. Suppose this is false. By symmetry we may take some $r_3 \in R_3$ that has at least one neighbour in R_{502}. We first claim that r_3 has at least one neighbour in D_4. Indeed, if r_3 does not, then, when we chose the R_i, we could have put r_3 in R_0. Thus, by the minimality of S,

$$e(r_3, R_0) + e(r_3, R_6) + e(r_3, R_{502}) \leqslant e(r_3, R_3) + e(r_3, R_4).$$

But we showed in Claim 49 that $e(R_0, R_4) = 0$ and we did this before we made the choice to minimise S. In particular, as we could have put r_3 in R_0 we know that $e(r_3, R_4) = 0$. Also R_3 is independent so $e(r_3, R_3) = 0$. But then, the right-hand side of the inequality is zero, while the left-hand side is positive (r_3 has at least one neighbour in R_{502}).

Thus r_3 has at least one neighbour in R_{502} and at least one neighbour in D_4. We may write,

$$|\Gamma(r_3) \cap D_4| = c_4|G|, \quad |\Gamma(r_3) \cap R_{502}| = c_{502}|G|,$$

where $0 < c_4, c_{502} \leqslant 1/9 + O(\varepsilon)$ (using our knowledge of $|D_4|$, $|R_{502}|$). Also,

$$\begin{aligned}|\Gamma(r_3) \cap D_5| &\geqslant d(r_3) - |\Gamma(r_3) \cap R_{502}| - |\Gamma(r_3) \cap D_4| - |T_2| - |T_1| - |R \setminus R_{502}| \\ &\geqslant (1/3 - c_4 - c_{502} + O(\varepsilon))|G|.\end{aligned}$$

But $1/3 - c_4 - c_{502} \geqslant 1/9 - O(\varepsilon)$, so, provided ε is sufficiently small, r_3 has at least one neighbour in D_5.

We next claim that the configuration in Fig. 5 appears with $d_5 \in D_5$, $d_4 \in D_4$, $r_{502} \in R_{502}$.

First suppose that $c_4, c_{502} \geqslant 1/27$. Pick a neighbour $d_5 \in D_5$ of r_3. Now $\Gamma(d_5) \subset T_3 \cup T_4 \cup T_6 \cup T_0 \cup R_{502}$ and this union has size $(5/9 + O(\varepsilon))|G|$, so d_5 has at most $O(\varepsilon)|G|$ non-neighbours in $D_4 \cup R_{502}$. But $\Gamma(r_3) \cap D_4$ and $\Gamma(r_3) \cap R_{502}$ both have size at least $1/27 \cdot |G|$, so, provided ε is small enough, d_5 has a neighbour in each of $\Gamma(r_3) \cap D_4$, $\Gamma(r_3) \cap R_{502}$ giving the configuration in Fig. 5.

Otherwise $\min\{c_4, c_{502}\} < 1/27$ and so

$$|\Gamma(r_3) \cap D_5| \geqslant (1/3 - c_4 - c_{502} + O(\varepsilon))|G| \geqslant (1/3 - 1/9 - 1/27 + O(\varepsilon))|G| = (5/27 + O(\varepsilon))|G|.$$

Pick $r_{502} \in R_{502}$ and $d_4 \in D_4$ both adjacent to r_3. Now, $\Gamma(d_4) \subset T_2 \cup T_3 \cup T_5 \cup T_6$ and this union has size $(5/9 + O(\varepsilon))|G|$, so d_4 has at most $O(\varepsilon)|G|$ non-neighbours in D_5. Also, $\Gamma(r_{502}) \subset T_0 \cup T_2 \cup T_5 \cup R_3 \cup R_4$ and this union has size $(2/3 + O(\varepsilon))|G|$, so r_{502} has at most $(1/9 + O(\varepsilon))|G|$ non-neighbours in D_5. But $5/27 > 1/9$, so, provided ε is sufficiently small, there is some $d_5 \in \Gamma(r_3) \cap D_5$ adjacent to both d_4 and r_{502}.

Hence, in all cases, the configuration in Fig. 5 appears with $d_5 \in D_5, d_4 \in D_4$ and $r_{502} \in R_{502}$. Consider the bipartition of G_{d_5} given by Claim 48:

- $(T_4 \cup D_0) \cap \Gamma(d_5) \subset A_{d_5}$.
- $(T_6 \cup D_3 \cup R_{502}) \cap \Gamma(d_5) \subset B_{d_5}$.

In particular, $d_4 \in A_{d_5}$ and $r_{502} \in B_{d_5}$. But then $r_3 \in G_{d_5}$ has a neighbour in both A_{d_5} and B_{d_5}, so can be in neither, which is a contradiction. □

Proof of Theorem 9. Take $\varepsilon > 0$ sufficiently small so that Claims 51 and 52 hold. Note that the T_i and R_{502} together partition $V(G)$. By Claim 44, each T_i is independent and, by Claim 45, R_{502} is independent. By Claims 49 to 51, $e(T_i, T_{i+3}) = 0$ for all i and by Claim 47, $e(T_1, T_6) = 0$. Finally, by Claims 46 and 52, $e(R_{502}, T_1 \cup T_3 \cup T_4 \cup T_6) = 0$.

Hence, identifying T_i with v_i and R_{502} with a single vertex gives a homomorphism from G to H_2^+. □

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