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TREEWIDTH, CIRCLE GRAPHS, AND CIRCULAR DRAWINGS*

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Abstract. A circle graph is an intersection graph of a set of chords of a circle. We describe the unavoidable induced subgraphs of circle graphs with large treewidth. This includes examples that are far from the "usual suspects." Our results imply that treewidth and Hadwiger number are linearly tied on the class of circle graphs and that the unavoidable induced subgraphs of a vertex-minor-closed class with large treewidth are the usual suspects if and only if the class has bounded rank-width. Using the same tools, we also study the treewidth of graphs G that have a circular drawing whose crossing graph is well-behaved in some way. In this setting, we show that if the crossing graph is K_t-minor-free, then G has treewidth at most $12t-23$ and has no $K_{2,4t}$-topological minor. On the other hand, we show that there are graphs with arbitrarily large Hadwiger number that have circular drawings whose crossing graphs are 2-degenerate.

Key words. circle graphs, treewidth, circular drawings

MSC codes. 05C83, 05C10, 05C62

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1. Introduction.

This paper studies the treewidth of graphs that are defined by circular drawings. Treewidth is the standard measure of how similar a graph is to a tree and is of fundamental importance in structural and algorithmic graph theory; see [13, 40, 65] for surveys. The motivation for this study is twofold. See section 2 for definitions omitted from this introduction.

1.1. Theme #1: Circle graphs.

A *circle graph* is the intersection graph of a set of chords of a circle. Circle graphs form a widely studied graph class [19, 21, 23, 32, 36, 47, 50], and there have been several recent breakthroughs concerning them. In the study of graph colorings, Davies and McCarty [21] showed that circle graphs are quadratically χ-bounded, improving on a previous long-standing exponential upper bound. Davies [19] further improved this bound to $\chi(G) \in \mathcal{O}(\omega(G)\log\omega(G))$, which is best possible. Circle graphs are also fundamental to the study of vertex-minors and are conjectured to lie at the heart of a global structure theorem for vertex-minor-closed graph classes (see [54]). To this end, Geelen et al. [36] recently proved an analogous result to the excluded grid minor theorem for vertex-minors using circle graphs. In particular, they showed that a vertex-minor-closed graph class has bounded rankwidth if and only if it excludes a circle graph as a vertex-minor. For further motivation and background on circle graphs, see [20, 54].

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Our first contribution essentially determines when a circle graph has large treewidth.

THEOREM 1. *Let $t \in \mathbb{N}$, and let G be a circle graph with treewidth at least $12t + 2$. Then G contains an induced subgraph H that consists of t vertex-disjoint cycles $(C_1, \ldots, C_t)$ such that for all $i < j$, every vertex of C_i has at least two neighbors in C_j. Moreover, every vertex of G has at most four neighbors in any C_i $(1 \leqslant i \leqslant t)$.*

Observe that, in Theorem 1, the subgraph H has a K_t-minor obtained by contracting each of the cycles C_i to a single vertex, implying that H has treewidth at least $t - 1$. Moreover, since circle graphs are closed under taking induced subgraphs, H is also a circle graph. We now highlight several consequences of Theorem 1.

First, Theorem 1 describes the unavoidable induced subgraphs of circle graphs with large treewidth. Recently, there has been significant interest in understanding the induced subgraphs of graphs with large treewidth [2, 3, 4, 5, 6, 7, 8, 14, 51, 62, 72]. To date, most of the results in this area have focused on graph classes where the unavoidable induced subgraphs are the following graphs, the *usual suspects*: a complete graph K_t, a complete bipartite graph $K_{t,t}$, a subdivision of the $(t \times t)$-wall, or the line graph of a subdivision of the $(t \times t)$-wall (see [72] for definitions). Circle graphs do not contain subdivisions of large walls or the line graphs of subdivisions of large walls, and there are circle graphs of large treewidth that do not contain large complete graphs or large complete bipartite graphs (see Theorem 22). To the best of our knowledge, this is the first result to describe the unavoidable induced subgraphs of the large treewidth graphs in a natural hereditary class when they are not the usual suspects. Later we show that the unavoidable induced subgraphs of graphs with large treewidth in a vertex-minor-closed class $\mathcal{G}$ are the usual suspects if and only if $\mathcal{G}$ has bounded rankwidth (see Theorem 24).

Second, the subgraph H in Theorem 1 is an explicit witness to the large treewidth of G (with only a multiplicative loss). Circle graphs being χ-bounded says that circle graphs with large chromatic number must contain a large clique witnessing this. Theorem 1 can therefore be considered to be a treewidth analogue to the χ-boundedness of circle graphs. We also prove an analogous result for circle graphs with large pathwidth (see Theorem 23).

Third, since the subgraph H has a K_t-minor, it follows that every circle graph contains a complete minor whose order is at least one-twelfth of its treewidth. This is in stark contrast to the general setting where there are K_5-minor-free graphs with arbitrarily large treewidth (for example, grids). Theorem 1 also implies the following relationship between the treewidth, Hadwiger number, and Hajós number of circle graphs (see section 5).[1]

THEOREM 2. *For the class of circle graphs, the treewidth and Hadwiger number are linearly tied. Moreover, the Hajós number is quadratically tied to both of them. Both "linear" and "quadratic" are best possible.*

1.2. Theme #2: Graph drawing. The second thread of this paper aims to understand the relationship between circular drawings of graphs and their crossing graphs. A *circular drawing* (also called *convex* drawing) of a graph places the vertices on a circle with edges drawn as straight-line segments. Circular drawings are a well-studied topic; see [35, 48, 73], for example. The *crossing graph* of a drawing D of a graph G has vertex set $E(G)$, where two vertices are adjacent if the corresponding

[1]For a graph class $\mathcal{G}$, two graph parameters α and β are *tied on* $\mathcal{G}$ if there exists a function f such that $\alpha(G) \leqslant f(\beta(G))$ and $\beta(G) \leqslant f(\alpha(G))$ for every graph $G \in \mathcal{G}$. Moreover, α and β are *quadratically/linearly tied on* $\mathcal{G}$ if f may be taken to be quadratic/linear.

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edges cross. Circle graphs are precisely the crossing graphs of circular drawings. If a graph has a circular drawing with a well-behaved crossing graph, must the graph itself also have a well-behaved structure? Graphs that have a circular drawing with no crossings are exactly the outerplanar graphs, which have treewidth at most 2. Put another way, outerplanar graphs are those that have a circular drawing whose crossing graph is K_2-minor-free. Our next result extends this fact, relaxing "K_2-minor-free" to "K_t-minor-free."

THEOREM 3. *For every integer $t \geqslant 3$, if a graph G has a circular drawing where the crossing graph has no K_t-minor, then G has treewidth at most $12t-23$.*

Theorem 3 says that G having large treewidth is sufficient to force a complicated crossing graph in every circular drawing of G. A topological $K_{2,4t}$-minor also suffices.

THEOREM 4. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G contains no $K_{2,4t}$ as a topological minor.*

Outerplanar graphs are exactly those graphs that have treewidth at most 2 and exclude a topological $K_{2,3}$-minor. As such, Theorems 3 and 4 extend these structural properties of outerplanar graphs to graphs with circular drawings whose crossing graphs are K_t-minor-free. We also prove a product structure theorem for such graphs, showing that every graph that has a circular drawing whose crossing graph has no K_t-minor is isomorphic to a subgraph of $H \boxtimes K_{\mathcal{O}(t^3)}$, where $\mathrm{tw}(H) \leqslant 2$ (see Corollary 11).

In the other direction, we consider sufficient conditions for a graph G to have a circular drawing whose crossing graph has no K_t-minor. By Theorems 3 and 4, G must have bounded treewidth and no $K_{2,4t}$-topological minor. While these conditions are necessary, we show that they are not sufficient but that bounded treewidth with bounded maximum degree is; see Lemma 17 and Proposition 18 in subsection 4.2 for details.

In addition, we show that the assumption in Theorem 3 that the crossing graph has bounded Hadwiger number cannot be weakened to bounded degeneracy. In particular, we construct graphs with arbitrarily large complete graph minors that have a circular drawing whose crossing graph is 2-degenerate (Theorem 20). This result has applications to the study of general (noncircular) graph drawings and, in particular, leads to the solution of an open problem asked by Hickingbotham and Wood [42].

Our proofs of Theorems 1 to 3 are all based on the same core lemmas proved in section 3. The results about circle graphs are in section 5, while the results about graph drawings are in section 4.

2. Preliminaries.

2.1. Graph basics. We use standard graph-theoretic definitions and notation; see [24] for undefined terms, definitions, and notation.

For a tree T, a *T-decomposition* of a graph G is a collection $\mathcal{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in W_x$, and (ii) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a (connected) subtree of T. Each set W_x in $\mathcal{W}$ is called a *bag*. The *width* of $\mathcal{W}$ is $\max\{|W_x| : x \in V(T)\} - 1$. A *tree-decomposition* is a T-decomposition for any tree T. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G.

A *path-decomposition* of a graph G is a T-decomposition where T is a path. The *pathwidth* $\mathrm{pw}(G)$ of a graph G is the minimum width of a path-decomposition of G.

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Let $n \in \mathbb{N}$. The *$(n \times n)$-grid* is the graph with vertex set $\{(i,j) \colon i,j \in \{1,\dots,n\}\}$ and edge set

$$\{(i,j)(i+1,j) \colon i \in \{1,\dots,n-1\}, j \in \{1,\dots,n\}\}$$
$$\cup \{(i,j)(i,j+1) \colon i \in \{1,\dots,n\}, j \in \{1,\dots,n-1\}\}.$$

The *$(n \times n)$-wall* is the graph with vertex set $\{(i,j) \colon i,j \in \{1,\dots,n\}\}$ and edge set

$$\{(i,j)(i+1,j) \colon i \in \{1,\dots,n-1\}, j \in \{1,\dots,n\}\}$$
$$\cup \{(i,j)(i,j+1) \colon i \in \{1,\dots,n\}, j \in \{1,\dots,n-1\}, \, i+j \text{ even}\}.$$

Grids and walls are the canonical examples of graphs with large treewidth.

A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. The *Hadwiger number* $h(G)$ of a graph G is the maximum integer t such that K_t is a minor of G.

A graph $\tilde{G}$ is a *subdivision* of a graph G if $\tilde{G}$ can be obtained from G by replacing each edge vw by a path P_{vw} with endpoints v and w (internally disjoint from the rest of $\tilde{G}$). A graph H is a *topological minor* of G if a subgraph of G is isomorphic to a subdivision of H. The *Hajós number* $h'(G)$ of G is the maximum integer t such that K_t is a topological minor of G. A graph G is *H-topological minor-free* if H is not a topological minor of G.

It is well known that for every graph G,

$$h'(G) \leqslant h(G) \leqslant \operatorname{tw}(G) + 1.$$

A *graph class* is a collection of graphs closed under isomorphism. A graph class is *hereditary* if it is closed under induced subgraphs. A *graph parameter* is a real-valued function α defined on all graphs such that $\alpha(G_1) = \alpha(G_2)$ whenever G_1 and G_2 are isomorphic.

2.2. Drawings of graphs. A *drawing* of a graph G is a function ϕ that maps each vertex $v \in V(G)$ to a point $\phi(v) \in \mathbb{R}^2$ and maps each edge $e = vw \in E(G)$ to a non–self-intersecting curve $\phi(e)$ in $\mathbb{R}^2$ with endpoints $\phi(v)$ and $\phi(w)$, such that

- $\phi(v) \neq \phi(w)$ for all distinct vertices v and w;
- $\phi(x) \notin \phi(e)$ for each edge $e = vw$ and each vertex $x \in V(G) \setminus \{v,w\}$;
- each pair of edges intersect at a finite number of points: $\phi(e) \cap \phi(f)$ is finite for all distinct edge e, f;
- no three edges internally intersect at a common point: For distinct edges e, f, g, the only possible element of $\phi(e) \cap \phi(f) \cap \phi(g)$ is $\phi(v)$, where v is a vertex incident to all of e, f, g.

A *crossing* of distinct edges $e = uv$ and $f = xy$ is a point in $(\phi(e) \cap \phi(f)) \setminus \{\phi(u), \phi(v), \phi(x), \phi(y)\}$, that is, an internal intersection point. A graph is *planar* if it has a drawing with no crossings. A *plane graph* is a planar graph G equipped with a drawing of G with no crossings.

The *crossing graph* of a drawing D of a graph G is the graph X_D with vertex set $E(G)$, where for each crossing between edges e and f in D, there is an edge of X_D between the vertices corresponding to e and f. Note that X_D is actually a multigraph, where the multiplicity of ef equals the number of times e and f cross in D. In most drawings that we consider, each pair of edges cross at most once, in which case X_D has no parallel edges.

Numerous papers have studied graphs that have a drawing whose crossing graph is well-behaved in some way. Here we give some examples. The *crossing number*

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$\mathrm{cr}(G)$ of a graph G is the minimum number of crossings in a drawing of G; see the surveys [60, 68, 74] or the monograph [67]. Obviously, $\mathrm{cr}(G) \leqslant k$ if and only if G has a drawing D with $|E(X_D)| \leqslant k$. Tutte [75] defined the *thickness* of a graph G to be the minimum number of planar graphs whose union is G; see [44, 55] for surveys. Every planar graph can be drawn with its vertices at prespecified locations [39, 61]. It follows that a graph G has thickness at most k if and only if G has a drawing D such that $\chi(X_D) \leqslant k$. A graph is *k-planar* if G has a drawing D in which every edge is in at most k crossings; that is, X_D has maximum degree at most k; see [28, 29, 37, 59], for example. More generally, Eppstein and Gupta [34] defined a graph G to be *k-degenerate crossing* if G has a drawing D in which X_D is k-degenerate. Bae et al. [9] defined a graph G to be *k-gap-planar* if G has a drawing D in which each crossing can be assigned to one of the two involved edges and each edge is assigned at most k of its crossings. This is equivalent to saying that every subgraph of X_D has average degree at most $2k$. It follows that every k-degenerate crossing graph is k-gap-planar and that every k-gap-planar graph is a $2k$-degenerate crossing graph [45].

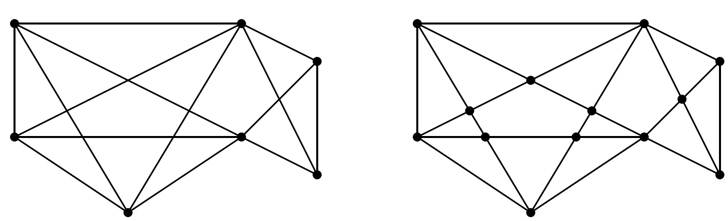
A drawing is *circular* if the vertices are positioned on a circle and the edges are straight-line segments. A theme of this paper is to study circular drawings D in which X_D is well-behaved in some way. Many papers have considered properties of X_D in this setting. The *convex crossing number* of a graph G is the minimum number of crossings in a circular drawing of G; see [68] for a detailed history of this topic. Obviously, G has convex crossing number at most k if and only if G has a circular drawing D with $|E(X_D)| \leqslant k$. The *book thickness* (also called *page-number* or *stack-number*) of a graph G can be defined as the minimum taken over all circular drawings D of G, of $\chi(X_D)$. This parameter is widely studied; see [10, 11, 27, 31, 78, 79], for example.

3. Tools. In this section, we introduce two auxiliary graphs that are useful tools for proving our main theorems.

For a drawing D of a graph G, the *planarization*, P_D, of D is the plane graph obtained by replacing each crossing with a dummy vertex of degree 4, as illustrated in Figure 1. Note that P_D depends on the drawing D (and not just on G).

For a drawing D of a graph G, the *map graph*, M_D, of D is obtained as follows. First, let P_D be the planarization of D. The vertices of M_D are the faces of P_D, where two vertices are adjacent in M_D if the corresponding faces share a vertex. If G is itself a plane graph, then it is already drawn in the plane, and so we may talk about the map graph, M_G, of G. Note that all map graphs are connected graphs. Figure 2 shows the map graph M_D for the drawing D in Figure 1.

The *radius* of a connected graph G, denoted $\mathrm{rad}(G)$, is the minimum nonnegative integer r such that for some vertex $v \in V(G)$ and for every vertex $w \in V(G)$, we have $\mathrm{dist}_G(v, w) \leqslant r$.



(a) Drawing D of a graph G. (b) Planarization P_D.

FIG. 1. *A drawing and its planarization.*

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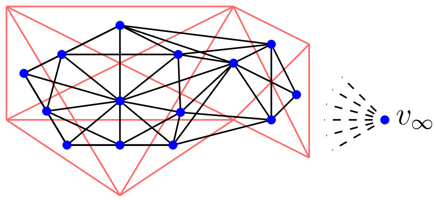

FIG. 2. *Map graph M_D. v_∞ is the vertex corresponding to the outer face: It is adjacent to all vertices except the unique vertex of degree* 10.

In subsection 3.1, we show that the radius of the map graph M_D acts as an upper bound for the treewidths of G and X_D. In subsection 3.2, we show that if D is a circular drawing and the map graph M_D has large radius, then X_D contains a useful substructure. Thus, the radius of M_D provides a useful bridge between the treewidth of G, the treewidth of X_D, and the subgraphs of X_D.

3.1. Map graphs with small radii. Here we prove that for any drawing D of a graph G, the radius of M_D acts as an upper bound for both the treewidth of G and the treewidth of X_D.

THEOREM 5. *For every drawing D of a graph G,*

$$\operatorname{tw}(G) \leqslant 6 \operatorname{rad}(M_D) + 7 \quad \textit{and} \quad \operatorname{tw}(X_D) \leqslant 6 \operatorname{rad}(M_D) + 7.$$

Wood and Telle [77, Prop. 8.5] proved that if a graph G has a circular drawing D such that whenever edges e and f cross e or f crosses at most d edges, then G has treewidth at most $3d + 11$. This assumption implies that $\operatorname{rad}(M_D) \leqslant \lfloor d/2 \rfloor + 1$, and so the first inequality of Theorem 5 generalizes this result.

It is not surprising that treewidth and radius are related for drawings. A classical result of Robertson and Seymour [66, eq. (2.7)] says that $\operatorname{tw}(G) \leqslant 3 \operatorname{rad}(G) + 1$ for every connected planar graph G. Several authors improved this bound as follows.

LEMMA 6 ([12, 29]). *For every connected planar graph G,*

$$\operatorname{tw}(G) \leqslant 3 \operatorname{rad}(G).$$

We now prove that if a planar graph G has large treewidth, then the map graph of any plane drawing of G has large radius. A *triangulation* of a plane graph G is a plane supergraph of G on the same vertex set and where each face is a triangle.

LEMMA 7. *Let G be a plane graph with map graph M_G. Then there is a plane triangulation H of G with $\operatorname{rad}(H) \leqslant \operatorname{rad}(M_G) + 1$. In particular,*

$$\operatorname{tw}(G) \leqslant 3 \operatorname{rad}(M_G) + 3.$$

Proof. Let F_0 be a face of G such that every vertex in M_G has distance at most $\operatorname{rad}(M_G)$ from F_0. For each face F of G, let $\operatorname{dist}_0(F)$ be the distance of F from F_0 in M_G.

Fix a vertex v_0 of G in the boundary of F_0, and set $\rho(v_0) := -1$. For every other vertex v of G, let

$$\rho(v) = \min\{\operatorname{dist}_0(F) \colon v \text{ is on the boundary of face } F\}.$$

Note that ρ takes values in $\{-1, 0, \ldots, \operatorname{rad}(M_G)\}$.

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We now construct a triangulation H of G such that every vertex $v \neq v_0$ is adjacent (in H) to a vertex u with $\rho(u) < \rho(v)$. In particular, the distance from v to v_0 in H is at most $\rho(v)+1 \leqslant \operatorname{rad}(M_G)+1$, and so H has the required radius. For each face F, let v_F be a vertex of F with the smallest ρ-value. Note that $v_{F_0} = v_0$. Triangulate G as follows. First, consider one by one each face F. For every vertex v of F that is not already adjacent to v_F, add the edge vv_F. Finally, let H be obtained by triangulating the resulting graph.

By Lemma 6, $\operatorname{tw}(G) \leqslant \operatorname{tw}(H) \leqslant 3\operatorname{rad}(H) \leqslant 3\operatorname{rad}(M_G)+3$. ∎

Note that a version of Lemma 7 with $\operatorname{rad}(M_G)$ replaced by the eccentricity of the outerface in M_G can be proved via outerplanarity.[2]

We use the following lemma about planarizations to extend Lemma 7 from plane drawings to arbitrary drawings.

LEMMA 8. *For every drawing D of a graph G, the planarization P_D of D satisfies*

$$\operatorname{tw}(G) \leqslant 2\operatorname{tw}(P_D)+1 \quad \text{and} \quad \operatorname{tw}(X_D) \leqslant 2\operatorname{tw}(P_D)+1.$$

Proof. Consider a tree-decomposition $(T, \mathcal{W})$ of P_D in which each bag has size at most $\operatorname{tw}(P_D)+1$. Now we prove the first inequality. Arbitrarily orient the edges of G. Each dummy vertex x of P_D corresponds to a crossing between two oriented edges ab and cd of G. For each dummy vertex x, replace each instance of x in the tree-decomposition $(T, \mathcal{W})$ by b and d. It is straightforward to verify that this gives a tree-decomposition $(T, \mathcal{W}')$ of G with bags of size at most $2\operatorname{tw}(P_D)+2$. Hence, $\operatorname{tw}(G) \leqslant 2\operatorname{tw}(P_D)+1$.

Now we prove the second inequality. Each dummy vertex x of P_D corresponds to a crossing between two edges e and f of G. For each dummy vertex x, replace each instance of x in $(T, \mathcal{W})$ by e and f. Also, for each vertex v of G, delete all instances of v from $(T, \mathcal{W})$. This gives a tree-decomposition $(T, \mathcal{W}'')$ of X_D with bags of size at most $2\operatorname{tw}(P_D)+2$. Hence, $\operatorname{tw}(X_D) \leqslant 2\operatorname{tw}(P_D)+1$. ∎

We are now ready to prove Theorem 5.

Proof of Theorem 5. Let P_D be the planarization of D. By definition, $M_D \cong M_{P_D}$. Lemma 7 implies that

$$2\operatorname{tw}(P_D)+1 \leqslant 2(3\operatorname{rad}(M_{P_D})+3)+1 = 6\operatorname{rad}(M_D)+7.$$

Lemma 8 now gives the required result. ∎

3.2. Map graphs with large radii.

The next lemma is a cornerstone of this paper. It shows that if the map graph of a circular drawing has large radius, then the crossing graph contains a useful substructure. For $a, b \in \mathbb{R}$, where $a < b$, let (a, b) denote the open interval $\{r \in \mathbb{R} \colon a < r < b\}$.

LEMMA 9. *Let D be a circular drawing of a graph G. If the map graph M_D has radius at least $2t$, then the crossing graph X_D contains t vertex-disjoint induced cycles $C_1, \ldots, C_t$ such that for all $i < j$, every vertex of C_i has at least two neighbors in C_j. Moreover, every vertex of X_D has at most four neighbors in any C_i $(1 \leqslant i \leqslant t)$.*

[2]Say that a plane graph G is *k-outerplane* if removing all the vertices on the boundary of the outerface leaves a $(k-1)$-outerplane subgraph, where a plane graph is 0-*outerplane* if it has no vertices. Consider a plane graph G, where v_∞ is the vertex of M_G corresponding to the outerface. Then one can show that if v_∞ has eccentricity k in M_G, then G is $(k+1)$-outerplane, and, conversely, if G is k-outerplane, then v_∞ has eccentricity at most k in M_G. Bodlaender [13] showed that every k-outerplanar graph has treewidth at most $3k-1$. The same proof shows that every k-outerplane graph has treewidth at most $3k-1$ (which also follows from [29]).

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Proof. Let $F \in V(M_D)$ be a face with distance at least $2t$ from the outer face of G. Let p be a point in the interior of F. Let R_0 be the infinite ray starting at p and pointing vertically upward. More generally, for $\theta \in \mathbb{R}$, let R_θ be the infinite ray with endpoint p that makes a clockwise angle of θ (radians) with R_0. In particular, R_π is the ray pointing vertically downward from p and $R_{\theta+2\pi} = R_\theta$ for all θ.

In the statement of the following claim and throughout the paper, "cross" means to internally intersect.

CLAIM. *Every R_θ crosses at least $2t-1$ edges of G.*

Proof. Consider moving along R_θ from p to the outer face. The distance in M_D only changes when crossing an edge or a vertex of G and changes by at most 1 when doing so. Since each R_θ contains at most one vertex of G, it must cross at least $2t-1$ edges. ◻

For each edge e of G, define $I_e := \{\theta \colon e \text{ crosses } R_\theta\}$. Since each edge is a line segment not passing through p, each I_e is of the form $(a,a')+2\pi\mathbb{Z}$, where $a < a' < a+\pi$. Also note that edges e and f cross exactly if $I_e \cap I_f \neq \varnothing$, $I_e \not\subseteq I_f$, and $I_f \not\subseteq I_e$.

For a set of edges $E' \subseteq E(G)$, define $I_{E'} = \bigcup\{I_e \colon e \in E'\}$. We say that E' is *dominant* if $I_{E'} = \mathbb{R}$ and is *minimally dominant* if no proper subset of E' is dominant. Note that if $e, f \in E'$ and E' is minimally dominant, then e and f cross exactly if $I_e \cap I_f \neq \varnothing$.

CLAIM. *If E' is minimally dominant, then*
(i) *every R_θ crosses at most two edges of E';*
(ii) *E' induces a cycle in X_D;*
(iii) *every edge of G crosses at most four edges of E'.*

Proof. We first prove (i). Suppose that there is some R_θ crossing distinct edges $e_1, e_2, e_3 \in E'$. Then $\theta \in I_{e_1} \cap I_{e_2} \cap I_{e_3}$ and $\theta + \pi \notin I_{e_1} \cup I_{e_2} \cup I_{e_3}$. Hence, we may write

$$I_{e_i} = (a_i, a_i') + 2\pi\mathbb{Z}, \qquad i = 1,2,3,$$

where $\theta - \pi < a_i < \theta < a_i' < \theta + \pi$. By relabeling, we may assume that $a_1 < a_2 < a_3 < \theta$. Now if a_3' is not the largest of a_1', a_2', a_3', then $(a_3, a_3') \subseteq (a_1, a_1') \cup (a_2, a_2')$, and so $I_{e_3} \subseteq I_{e_1} \cup I_{e_2}$, which contradicts the minimality of E'. Hence, $a_3' \geqslant a_1', a_2'$. But then $(a_2, a_2') \subseteq (a_1, a_1') \cup (a_3, a_3')$, and so $I_{e_2} \subseteq I_{e_1} \cup I_{e_3}$, which again contradicts minimality. This proves (i).

We next show that E' induces a connected subgraph of X_D. If E' does not, then there is a partition $E_1 \cup E_2$ of E' into nonempty sets such that no edge in E_1 crosses any edge in E_2. Since E' is minimally dominant, this means that $I_{E_1} \cap I_{E_2} = \varnothing$. Consider $\mathbb{R}$ with the topology induced by the Euclidean metric, which is a connected space. But I_{E_1} and I_{E_2} are nonempty open sets that partition $\mathbb{R}$. Hence, E' induces a connected subgraph.

We now show that E' induces a 2-regular graph in X_D, which together with connectedness establishes (ii). Let $e \in E'$, and write $I_e = (a, a') + 2\pi\mathbb{Z}$, where $a < a' < a + \pi$. Since E' is dominant, there are $f, f' \in E'$ with $a \in I_f$ and $a' \in I_{f'}$. If $f = f'$, then $I_e \subseteq I_f$, which contradicts minimality. Hence, f, f' are distinct, and so e has degree at least two in X_D. Suppose that e has some neighbor f'' in X_D distinct from f, f'. Since $I_{f''}$ is not a subset of I_e, it must contain at least one of a, a'. By symmetry, we may assume that $I_{f''}$ contains a. But then, for some sufficiently small $\varepsilon > 0$, all of $I_e, I_f, I_{f''}$ contain $a + \varepsilon$, and so $R_{a+\varepsilon}$ crosses three edges of E', which contradicts (i). Hence, e has exactly two neighbors in E', which establishes (ii).

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Finally, consider an arbitrary edge $e = uv$ of G. Let R_u be the infinite ray from p that contains u and R_v be the infinite ray from p that contains v. Observe that every edge of G that crosses e also crosses R_u or R_v. By (i), at most four edges in E' cross e, which proves (iii). ◻

For a set of edges $E' \subseteq E(G)$, say that an edge $e \in E'$ is *maximal in* E' if there is no $f \in E' \setminus \{e\}$ with $I_e \subseteq I_f$. Suppose that E' is dominant. Let $E'_{\max}$ be the set of maximal edges in E'. Clearly, $E'_{\max}$ is still dominant and so has a minimally dominant subset. In particular, every dominant set of edges E' has a subset E_1 that is minimally dominant and all of whose edges are maximal in E'.

CLAIM. *Let $E' \subseteq E(G)$ and $E_1, E_2 \subseteq E'$. Suppose that all the edges of E_1 are maximal in E' and that E_2 is dominant. Then every edge in E_1 crosses at least two edges in E_2.*

Proof. Let $e_1 \in E_1$, and write $I_{e_1} = (a, a') + 2\pi\mathbb{Z}$, where $a < a' < a + \pi$. Since E_2 is dominant, there are $e_2, e_3 \in E_2$ with $a \in I_{e_2}$ and $a' \in I_{e_3}$. If $e_2 = e_3$, then $I_{e_1} \subseteq I_{e_2}$, which contradicts the maximality of e_1 in E'.

By symmetry, it suffices to check that e_1 and e_2 cross. Note that for some sufficiently small $\varepsilon > 0$, $a + \varepsilon$ is in both I_{e_1} and I_{e_2}, and so $I_{e_1} \cap I_{e_2} \neq \varnothing$. As $a \in I_{e_2} \setminus I_{e_1}$, we have $I_{e_2} \not\subseteq I_{e_1}$. Finally, the maximality of e_1 in E' means that $I_{e_1} \not\subseteq I_{e_2}$. Hence, e_1 and e_2 do indeed cross. ◻

We are now ready to complete the proof. Note that a set of edges is dominant exactly if it crosses every R_θ. By the first claim, $E = E(G)$ is dominant. Let $E_1 \subseteq E$ be minimally dominant such that every edge of E_1 is maximal in E. By part (i) of the second claim, every R_θ crosses at most two edges of E_1 and so, by the first claim, crosses at least $2t - 3$ edges of $E \setminus E_1$. Hence, $E \setminus E_1$ is dominant. Let $E_2 \subseteq E \setminus E_1$ be minimally dominant such that every edge of E_2 is maximal in $E \setminus E_1$. Continuing in this way, we obtain pairwise disjoint $E_1, E_2, \ldots, E_t \subset E$ such that for all i,

- E_i is minimally dominant;
- every edge of E_i is maximal in $E \setminus (\bigcup_{i'<i} E_{i'})$;
- every R_θ crosses at most two edges of E_i;
- every R_θ crosses at least $2(t - i) - 1$ edges of $E \setminus (\bigcup_{i' \leqslant i} E_{i'})$.

By part (ii) of the second claim, every E_i induces a cycle C_i in X_D. Let $i < j$ and $E' := E \setminus (\bigcup_{i'<i} E_{i'})$. Then $E_i, E_j \subseteq E'$, and every edge of E_i is maximal in E'. Hence, by the third claim, every edge in E_i crosses at least two edges in E_j. In particular, every vertex of C_i has at least two neighbors in C_j.

Finally, by part (iii) of the second claim, every vertex of X_D has at most four neighbors in any C_i. ◻

4. Structural properties of circular drawings. Theorem 5 says that for any drawing D of a graph G, the radius of M_D provides an upper bound for $\operatorname{tw}(G)$ and $\operatorname{tw}(X_D)$. For a general drawing, it is impossible to relate $\operatorname{tw}(X_D)$ to $\operatorname{tw}(G)$. First, planar graphs can have arbitrarily large treewidth (for example, the $(n \times n)$-grid has treewidth n; see [40]) and admit drawings with no crossings. In the other direction, $K_{3,n}$ has treewidth 3 and crossing number $\Omega(n^2)$, as shown by Kleitman [46]. In particular, the crossing graph of any drawing of $K_{3,n}$ has average degree linear in n and thus has arbitrarily large complete minors [52, 53] and so arbitrarily large treewidth.

Happily, this is not true for circular drawings. Using the tools in section 3, we show that if a graph G has large treewidth, then the crossing graph of any circular

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drawing of G has large treewidth. In fact, the crossing graph must contain a large (topological) complete graph minor (see Theorems 3 and 10). In particular, if X_D is K_t-minor-free, then G has small treewidth. We further show that if X_D is K_t-minor-free, then G does not contain a subdivision of $K_{2,4t}$ (Theorem 4). Using these results, we deduce a product structure theorem for G (Corollary 11).

In the other direction, we ask what properties of a graph G guarantee that it has a circular drawing D where X_D has no K_t-minor. Certainly, G must have small treewidth. Adding the constraint that G does not contain a subdivision of $K_{2,f(t)}$ is not sufficient (see Lemma 17), but a bounded maximum degree constraint is: We show that if G has bounded maximum degree and bounded treewidth, then G has a circular drawing where the crossing graph has bounded treewidth (Proposition 18).

We also show that there are graphs with arbitrarily large complete graph minors that admit circular drawings whose crossing graphs are 2-degenerate (see Theorem 20).

4.1. Necessary conditions for K_t-minor-free crossing graphs. This subsection studies the structure of graphs that have circular drawings whose crossing graph is (topological) K_t-minor-free. Much of our understanding of the structure of these graphs is summarized by the next four results (Theorems 3, 4, and 10 and Corollary 11).

THEOREM 3. *For every integer $t \geqslant 3$, if a graph G has a circular drawing where the crossing graph has no K_t-minor, then G has treewidth at most $12t-23$.*

THEOREM 4. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G contains no $K_{2,4t}$ as a topological minor.*

THEOREM 10. *If a graph G has a circular drawing where the crossing graph has no topological K_t-minor, then G has treewidth at most $6t^2+6t+1$.*

From these, we may deduce a product structure theorem for graphs that have a circular drawing whose crossing graph is K_t-minor-free. For two graphs G and H, the *strong product* $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$ and with an edge between two vertices (v,w) and (v',w') if and only if $v=v'$ and $ww' \in E(H)$, $w=w'$ and $vv' \in E(G)$, or $vv' \in E(G)$ and $ww' \in E(H)$. Campbell et al. [17, Prop. 55] showed that if a graph G is $K_{2,t}$-topological minor-free and has treewidth at most k, then G is isomorphic to a subgraph of $H \boxtimes K_{\mathcal{O}(t^2k)}$, where $\operatorname{tw}(H) \leqslant 2$. Thus, Theorems 3 and 4 imply the following product structure result.

COROLLARY 11. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G is isomorphic to a subgraph of $H \boxtimes K_{\mathcal{O}(t^3)}$, where $\operatorname{tw}(H) \leqslant 2$.*

En route to proving these results, we use the cycle structure built by Lemma 9 to find (topological) complete minors in the crossing graph of circular drawings. We first show that the treewidth and Hadwiger number of X_D as well as the radius of M_D are all linearly tied.

LEMMA 12. *For every circular drawing D,*

$$\operatorname{tw}(X_D) \leqslant 6 \operatorname{rad}(M_D) + 7 \leqslant 12\, h(X_D) - 11 \leqslant 12 \operatorname{tw}(X_D) + 1.$$

Proof. The first inequality is exactly Theorem 5, while the final one is the well-known fact that $h(G) \leqslant \operatorname{tw}(G)+1$ for every graph G. To prove the middle inequality, we need to show that for any circular drawing D,

$$\operatorname{rad}(M_D) \leqslant 2\, h(X_D) - 3. \tag{4.1}$$

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Let $t := h(X_D)$, and suppose, for a contradiction, that $\mathrm{rad}(M_D) \geqslant 2t-2$. By Lemma 9, X_D contains $t-1$ vertex disjoint cycles $C_1, \dots, C_{t-1}$ such that for all $i<j$, every vertex of C_i has a neighbor in C_j. Contracting C_1 to a triangle and each C_i ($i \geqslant 2$) to a vertex gives a K_{t+1}-minor in X_D. This is the required contradiction. ∎

Clearly, the Hajós number of a graph is at most the Hadwiger number. Our next lemma implies that the Hajós number of X_D is quadratically tied to the radius of M_D and to the treewidth and Hadwiger number of X_D.

LEMMA 13. *For every circular drawing D,*

$$\mathrm{rad}(M_D) \leqslant h'(X_D)^2 + 3\,h'(X_D) + 1.$$

Proof. Let $t = h'(X_D)+1$, and suppose, for a contradiction, that $\mathrm{rad}(M_D) \geqslant t^2+t$. By Lemma 9, X_D contains $(t^2+t)/2$ vertex disjoint cycles $C_1, \dots, C_{(t^2+t)/2}$ such that for all $i<j$, every vertex of C_i has a neighbor in C_j. For each $i \in \{1,\dots,t\}$, let $v_i \in V(C_i)$. We assume that $V(K_t) = \{1,\dots,t\}$ and let $\phi\colon E(K_t) \to \{t+1,\dots,(t^2+t)/2\}$ be a bijection. Then for each $ij \in E(K_t)$, there is a (v_i,v_j)-path P_{ij} in X_D whose internal vertices are contained in $V(C_{\phi(ij)})$. Since ϕ is a bijection, it follows that $(P_{ij}\colon ij \in E(K_t))$ defines a topological K_t-minor in X_D, a contradiction. ∎

We are now ready to prove Theorems 3 and 10.

Proof of Theorem 3. Let D be a circular drawing of G with $h(X_D) \leqslant t-1$. By (4.1), $\mathrm{rad}(M_D) \leqslant 2t-5$. Finally, by Theorem 5, $\mathrm{tw}(G) \leqslant 12t-23$. ∎

Proof of Theorem 10. Let D be a circular drawing of G with $h'(X_D) \leqslant t-1$. By Lemma 13, $\mathrm{rad}(M_D) \leqslant t^2+t-1$. Finally, by Theorem 5, $\mathrm{tw}(G) \leqslant 6t^2+6t+1$. ∎

We now show that the bound on $\mathrm{tw}(G)$ in Theorem 3 is within a constant factor of being optimal. Let G_n be the $(n\times n)$-grid, which has treewidth n (see [40]). Theorem 3 says that in every circular drawing D of G_n, the crossing graph X_D has a K_t-minor, where $t = \Omega(n)$. On the other hand, let D be the circular drawing of G_n obtained by ordering the vertices $R_1, R_2, \dots, R_n$, where R_i is the set of vertices in the ith row of G_n (ordered arbitrarily). Let E_i be the set of edges in G_n incident to vertices in R_i; note that $|E_i| \leqslant 3n-1$. If two edges cross, then they have end-vertices in some E_i. Thus, $(E_1, \dots, E_n)$ is a path-decomposition of X_D of width at most $3n$. In particular, X_D has no K_{3n+2}-minor. Hence, the bound on $\mathrm{tw}(G)$ in Theorem 3 is within a constant factor of optimal. See [70, 71] for more on circular drawings of grid graphs.

Now we turn to subdivisions and the proof of Theorem 4. As a warm-up, we give a simple proof in the case of no division vertices.

PROPOSITION 14. *For every $k \in \mathbb{N}$, for every circular drawing D of $K_{2,4k-1}$, X_D contains $K_{k,k}$ as a subgraph.*

Proof. Let the vertex classes of $K_{2,4k-1}$ be X and Y, where $X = \{x,y\}$ and $|Y| = 4k-1$. Vertices x and y split the circle into two arcs, one of which must contain at least $2k$ vertices from Y. Label these vertices $x, v_1, \dots, v_s, y$, where $s \geqslant 2k$ in order around the circle. For every $i \in \{1,\dots,k\}$, define the edges $e_i = yv_i$ and $f_i = xv_{k+i}$. The e_i and f_i are vertices in X_D, and for all i and j, edges e_i and f_j cross, as required. ∎

We now work toward the proof of Theorem 4.

A *linear drawing* of a graph G places the vertices on the x-axis with edges drawn as semicircles above the x-axis. In such a drawing, we consider the vertices of G to be

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elements of $\mathbb{R}$ given by their x-coordinates. Such a drawing can be wrapped to give a circular drawing of G with an isomorphic crossing graph. For an edge $uv \in E(G)$, where $u < v$, define I_{uv} to be the open interval (u, v). For a set of edges $E' \subseteq E(G)$, define $I_{E'} := \bigcup\{I_e : e \in E'\}$. Two edges $uv, xy \in E(G)$, where $u < v$ and $x < y$, are *nested* if $u < x < y < v$ or $x < u < v < y$.

LEMMA 15. *Let $a, b \in \mathbb{R}$, where $a < b$, and let D be a linear drawing of a graph G, where G consists of two internally vertex-disjoint paths $P_1 = (v_1, \dots, v_n)$ and $P_2 = (u_1, \dots, u_m)$ such that $u_1, v_1 \leqslant a < b \leqslant u_m, v_n$. Then there exists $E' \subseteq E(G)$ such that $(a, b) \subseteq I_{E'}$ and E' induces a connected graph in X_D. Moreover, for $x \in \{a, b\}$, if $x \notin V(P_1) \cap V(P_2)$, then $x \in I_{E'}$.*

Proof. We first show the existence of E'. Observe that $(a, b) \subseteq I_{E(P_1)} \cup \{v_1, \dots, v_n\}$. If G contains an edge uv, where $u \leqslant a < b \leqslant v$, then we are done by setting $E' = \{uv\}$. So assume that G has no edge of that form. Then there is a vertex $v \in V(P_1)$ such that $a < v < b$. Each such vertex v is not in $V(P_2)$, implying that $v \in I_{E(P_2)}$. Therefore, $(a, b) \subseteq I_{E(G)}$. Let E' be a minimal set of edges of $E(G)$ such that $(a, b) \subseteq I_{E'}$. By minimality, no two edges in E' are nested. We claim that $X_D[E']$ is connected. If not, then there is a partition $E_1 \cup E_2$ of E' into nonempty sets such that no edge in E_1 crosses any edge in E_2. Since E' is minimal, this means that $I_{E_1} \cap I_{E_2} = \varnothing$. Consider (a, b) with the topology induced by the Euclidean metric, which is a connected space. But $I_{E_1} \cap (a, b)$ and $I_{E_2} \cap (a, b)$ are nonempty open sets that partition (a, b), a contradiction. Hence, $X_D[E']$ is connected.

Finally, let $x \in \{a, b\}$, and suppose that $x \notin V(P_1) \cap V(P_2)$. Then G has an edge uv such that $u < x < v$. If $x \in I_{E'}$, then we are done. Otherwise, E' contains an edge incident to x. Since $a < u < b$ or $a < v < b$, it follows that uv crosses an edge in E'. So adding uv to E' maintains the connectivity of $X_D[E']$ and now $x \in I_{E'}$. $\square$

LEMMA 16. *Let G be a subdivision of $K_{2,3}$, and let $x, y \in V(G)$ be the vertices with degree 3. For every circular drawing D of G, there exists a component Y in X_D that contains an edge incident to x and an edge incident to y.*

Proof. Let P_1, P_2, P_3 be the internally disjoint (x, y)-paths in G. Let $\mathcal{U} = (u_1, \dots, u_m)$ be the sequence of vertices on the clockwise arc from x to y (excluding x and y). Let $\mathcal{V} = (v_1, \dots, v_n)$ be the sequence of vertices on the anticlockwise arc from x to y (excluding x and y). Say that an edge $uv \in E(G)$ is *vertical* if $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Suppose that no edge of G is vertical. By the pigeonhole principle, we may assume that $V(P_1) \cup V(P_2) \subseteq \mathcal{U} \cup \{x, y\}$. The claim then follows by applying Lemma 15 along the clockwise arc from x to y.

Now assume that $E(G)$ contains at least one vertical edge. Let $e_1, \dots, e_k$ be an ordering of the vertical edges of G such that if e_i is incident to $u_{i'}$ and e_{i+1} is incident to $u_{j'}$, then $i' \leqslant j'$. In the case, when $u_{i'} = u_{j'}$, e_i and e_{i+1} are ordered by their endpoints in $\mathcal{V}$.

CLAIM. *For each $i \in \{1, \dots, k-1\}$, there exists $E_i \subseteq E(G)$ such that $E_i \cup \{e_i, e_{i+1}\}$ induces a connected subgraph of X_D.*

Proof. Clearly, the claim holds if e_i and e_{i+1} cross or if there is an edge in G that crosses both e_i and e_{i+1}. So assume that e_i and e_{i+1} do not cross and that no edge crosses both e_i and e_{i+1}. Assume that $e_i = u'v'$ and $e_{i+1} = u''v''$, where $u', u'' \in \mathcal{U}$ and $v', v'' \in \mathcal{V}$. Let $j \in \{1, 2, 3\}$. If P_j does not contain e_i, then P_j contains neither endpoint of e_i. Since e_i separates x from y in the drawing, P_j contains e_i or an edge that crosses e_i. Likewise, P_j contains e_{i+1} or an edge that crosses e_{i+1}. Let $P'_j = (p_1, \dots, p_m)$ be

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a vertex-minimal subpath of P_j such that p_1p_2 is e_i or crosses e_i and $p_{m-1}p_m$ is e_{i+1} or crosses e_{i+1}. By minimality, no edge in $E(P'_j)\setminus\{p_1p_2, p_{m-1}p_m\}$ crosses e_i or e_{i+1}. Therefore, by the ordering of the vertical edges, no edge in $E(P'_j)\setminus\{p_1p_2, p_{m-1}p_m\}$ is vertical. As such, either $\{p_2,\dots,p_{m-1}\}\subseteq\mathcal{U}$ or $\{p_2,\dots,p_{m-1}\}\subseteq\mathcal{V}$. By the pigeonhole principle, without loss of generality, $V(P'_1)\cup V(P'_2)\subseteq\mathcal{U}$. Since $V(P'_1)$ and $V(P'_2)$ have distinct endpoints, the claim then follows by applying Lemma 15 along the clockwise arc between u' and u''. ◻

It follows from the claim that all the vertical edges are contained in a single component Y of X_D. Now consider the three edges in G incident to x. By the pigeonhole principle, without loss of generality, two of these edges are of the form xu_i, xu_j, where $i<j$. Let u_a be the vertex in $\mathcal{U}$ incident to the vertical edge e_1. If $a<j$, then e_1 crosses xu_j. If $a=j$, then by the ordering of the vertical edges, the path P_i that contains the edge xu_i also contains an edge that crosses both e_1 and xu_j. Otherwise, $j<a$, and applying Lemma 15 to the clockwise arc between u_j and u_a, it follows that xu_j is also in Y. By symmetry, there is an edge incident to y that is in Y, as required. ◻

We are now ready to prove Theorem 4.

Proof of Theorem 4. Let G be a subdivision of $K_{2,4t}$, and let D be a circular drawing of G. We show that X_D contains a K_t-minor. Let x, y be the degree $4t$ vertices in G. Let $\mathcal{U}=(u_1,\dots,u_m)$ be the sequence of vertices on the clockwise arc from x to y (excluding x and y). Let $\mathcal{V}=(v_1,\dots,u_n)$ be the sequence of vertices on the anticlockwise arc from x to y (excluding x and y). Say that an edge $uv\in E(G)$ is *vertical* if $u\in\mathcal{U}$ and $v\in\mathcal{V}$.

Let ℓ be the number of vertical edges in G. Let $k:=\min\{\ell,t\}$, and let $d:=t-k$. Then G contains $4d$ paths $P_1,\dots,P_{4d}$ that contain no vertical edge. We say that P_i is a $\mathcal{U}$*-path* (resp., $\mathcal{V}$*-path*) if it contains an edge incident to a vertex in $\mathcal{U}$ ($\mathcal{V}$). By the pigeonhole principle, without loss of generality, $P_1,\dots,P_{2d}$ are $\mathcal{U}$-paths. By pairing the paths and then applying Lemma 15 to the clockwise arc from x to y, it follows that X_D contains d vertex-disjoint connected subgraphs $Y_1,\dots,Y_d$ in X_D, where each Y_i contains an edge (in G) incident to x and an edge incident to y. Consider distinct $i,j\in\{1,\dots,d\}$. Let $xu_{i'}\in V(Y_i)$ and $xu_{j'}\in V(Y_j)$, and assume that $i'<j'$. Since $xu_{j'}$ separates $u_{i'}$ from y in the drawing and $P_1,\dots,P_{2d}$ are internally disjoint, it follows that there is an edge in $V(Y_i)$ that crosses $xu_{j'}$. So $Y_1,\dots,Y_d$ are pairwise adjacent, which form a K_d-minor in X_D.

Let $\tilde{E}:=\{e_1,\dots,e_k\}$ be any set of k vertical edges in G. Since $t=d+k$, there are $4k$ internally disjoint (x,y)-paths distinct from $P_1,\dots,P_{4d}$, at least $3k$ of which avoid $\tilde{E}$. Grouping these paths into k sets each with three paths, it follows from Lemma 16 that there exists k vertex-disjoint connected subgraphs $Z_1,\dots,Z_k$ in X_D, where each Z_i contains an edge (in G) incident to x and an edge incident to y. Since each $e\in\tilde{E}$ separates x and y in the drawing, it follows that each $V(Y_i)$ and $V(Z_j)$ contains an edge (in G) that crosses e. Thus, by contracting each Y_i into a vertex and each $Z_j\cup\{e_j\}$ into a vertex and then deleting all other vertices in X_D, we obtain the desired K_t-minor in X_D. ◻

4.2. Sufficient conditions for K_t-minor-free crossing graphs. It is natural to consider whether the converse of Theorems 3 and 4 holds. That is, does there exist a function f such that if a $K_{2,t}$-topological minor-free graph G has treewidth at most k, then there is a circular drawing of G whose crossing graph is $K_{f(t,k)}$-minor-free? Our next result shows that this is false in general. A *t-rainbow* in a circular drawing

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of a graph is a noncrossing matching consisting of t edges between two disjoint arcs in the circle.

Lemma 17. *For every $t \in \mathbb{N}$, there exists a $K_{2,4}$-topological minor-free graph G with $\mathrm{tw}(G) = 2$ such that for every circular drawing D of G, the crossing graph X_D contains a K_t-minor.*

Proof. Let T be any tree with maximum degree 3 and sufficiently large pathwidth (as a function of t). Such a tree exists, as the complete binary tree of height $2h$ has pathwidth h. Let G be obtained from T by adding a vertex v complete to $V(T)$, so G has treewidth 2. Since $G - v$ has maximum degree 3, it follows that G is $K_{2,4}$-topological minor-free.

Let D be a circular drawing of G, and let D_T be the induced circular drawing of T. Since T has sufficiently large pathwidth, a result of Pupyrev [64, Thm. 2] implies that X_D has large chromatic number or a $4t$-rainbow.[3] Since the class of circle graphs is χ-bounded [38],[4] it follows that if X_D has large chromatic number, then it contains a large clique, and we are done. So we may assume that D_T contains a $4t$-rainbow. By the pigeonhole principle, there is a subset $\{a_1b_1, \ldots, a_{2t}b_{2t}\}$ of the rainbow edges such that a_ib_i topologically separates v from a_j and b_j whenever $i < j$. As such, a_ib_i crosses the edges va_j and vb_j in D whenever $i < j$. Therefore, X_D contains a $K_{t,2t}$ subgraph with bipartition $(\{a_1b_1, \ldots, a_tb_t\}, \{va_{t+1}, vb_{t+1}, \ldots, va_{2t}, vb_{2t}\})$, and this contains a K_t-minor. ∎

Lemma 17 is best possible in the sense that $K_{2,4}$ cannot be replaced by $K_{2,3}$. An easy exercise shows that every biconnected $K_{2,3}$-topological minor-free graph is either outerplanar or K_4. It follows (by considering the block-cut tree) that every $K_{2,3}$-minor-free graph has a circular 1-planar drawing, so the crossing graph consists of isolated edges and vertices.

While $K_{2,k}$-topological minor-free and bounded treewidth is not sufficient to imply that a graph has a circular drawing whose crossing graph is K_t-minor-free, we now show that bounded degree and bounded treewidth is sufficient.

Proposition 18. *For $k, \Delta \in \mathbb{N}$, every graph G with treewidth less than k and maximum degree at most Δ has a circular drawing in which the crossing graph X_D has treewidth at most $(6\Delta + 1)(18k\Delta)^2 - 1$.*

Proof. Refining a method from [25, 76], Distel and Wood [26] proved that any such G is isomorphic to a subgraph of $T \boxtimes K_m$, where T is a tree with maximum degree $\Delta_T := 6\Delta$ and $m := 18k\Delta$. Since the treewidth of the crossing graph does not increase when deleting edges and vertices from the drawing, it suffices to show that $T \boxtimes K_m$ admits a circular drawing in which the crossing graph X_D has treewidth at most $(\Delta_T + 1)m^2 - 1$. Without loss of generality, assume that $V(K_m) = \{1, \ldots, m\}$. Take a circular drawing of T such that no two edges cross (this can be done since T is outerplanar). For each vertex $v \in V(T)$, replace v by $((v, 1), \ldots, (v, m))$ to obtain a circular drawing D of $T \boxtimes K_m$. Observe that if two edges $(u, i)(v, j)$ and $(x, a)(y, b)$ cross in D, then $\{u, v\} \cap \{x, y\} \neq \varnothing$. For each vertex $v \in V(T)$, let W_v be the set of edges of $T \boxtimes K_m$ that are incident to some (v, i). We claim that $(W_v : v \in V(T))$ is a tree-decomposition of X_D with the desired width. Clearly, each vertex of X_D is in a bag, and for each vertex $e \in V(X_D)$, the set $\{x \in V(T) : e \in W_x\}$ induces a

[3] The result of Pupyrev [64] is in terms of stacks and queues but is equivalent to our statement.

[4] A class of graphs $\mathcal{G}$ is *χ-bounded* if there is a function f such that for every graph $G \in \mathcal{G}$, $\chi(G) \leqslant f(\omega(G))$.

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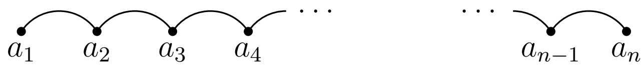

FIG. 3.

graph isomorphic to either K_2 or K_1 in T. Moreover, by the above observation, if $e_1e_2 \in E(X_D)$, then there exists some node $x \in V(T)$ such that $e_1, e_2 \in W_x$. Finally, since there are $\binom{m}{2}$ intra-K_m edges and $\Delta_T \cdot m^2$ cross-K_m edges, it follows that $|W_v| \leqslant (\Delta_T + 1)m^2$ for all $v \in V(T)$, as required. ∎

We conclude this subsection with the following open problem.

OPEN PROBLEM 19. *Does there exist a function f such that every $K_{2,k}$-minor-free graph G has a circular drawing D in which the crossing graph X_D is $K_{f(k)}$-minor-free?*

4.3. Circular drawings and degeneracy. Theorems 3 and 10 say that if a graph G has a circular drawing D, where the crossing graph X_D excludes a fixed (topological) minor, then G has bounded treewidth. Graphs excluding a fixed (topological) minor have bounded average degree and degeneracy [52, 53]. Despite this, we now show that X_D having bounded degeneracy is not sufficient to bound the treewidth of G. In fact, it is not even sufficient to bound the Hadwidger number of G.

THEOREM 20. *For every $t \in \mathbb{N}$, there is a graph G_t and a circular drawing D of G_t such that*

- *G_t contains a K_t-minor;*
- *G_t has maximum degree 3;*
- *X_D is 2-degenerate.*

Proof. We draw G_t with vertices placed on the x-axis (x-coordinate between 1 and t) and edges drawn on or above the x-axis. This can then be wrapped to give a circular drawing of G_t.

For real numbers $a_1 < a_2 < \cdots < a_n$, we say that a path P is drawn as a *monotone path* with vertices $a_1, \ldots, a_n$ if it is drawn as follows, where each vertex has x-coordinate equal to its label.

In all our monotone paths, $a_1, a_2, \ldots, a_n$ will be an arithmetic progression. We construct our drawing of G_t as follows (see Figure 4 for the construction with $t = 4$).

First, let P_0 be the monotone path with vertices $1, 2, \ldots, t$. For $s \in \{1, 2, \ldots, t-1\}$, let P_s be the monotone path with vertices

$$s + 2^{-s}, s + 3 \cdot 2^{-s}, s + 5 \cdot 2^{-s}, \ldots, t - 2^{-s}.$$

Observe that these paths are vertex-disjoint. For $0 \leqslant r < s \leqslant t - 1$, let $I_{r,s}$ be the interval

$$[s + 2^{-r} - 2^{-s}, s + 2^{-r}].$$

Note that the lower endpoint of $I_{r,s}$ is a vertex in P_s and that the upper endpoint is a vertex in P_r. Also note that no vertex of any P_i lies in the interior of $I_{r,s}$. Indeed, for $i > s$, the vertices of P_i have value at least $s + 2^{-r}$, and for $i \leqslant s$, the denominator of the vertices of P_i precludes them from being in the interior. Hence, for all $r < s$, we may draw a horizontal edge $e_{r,s}$ between the endpoints of $I_{r,s}$.

Graph G_t and the drawing D are obtained as a union of the P_s together with all the $e_{r,s}$. The paths P_s are vertex-disjoint, and edge $e_{r,s}$ joins P_r to P_s, so G_t contains a K_t-minor. We now show that the $I_{r,s}$ are pairwise disjoint. Note that $I_{r,s} \subset (s, s+1]$,

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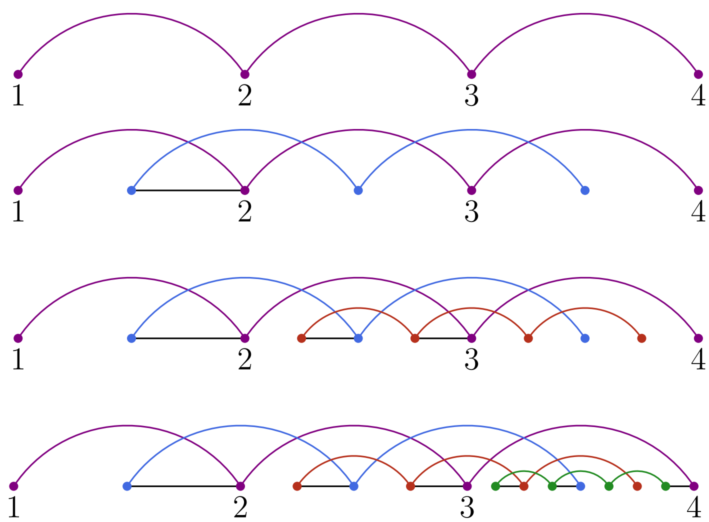

FIG. 4. *G_4 built up path by path, where P_0 is purple, P_1 is blue, P_2 is red, P_3 is green, and the $e_{r,s}$ are black.*

so two I with different s values are disjoint. Next, note that $I_{r,s} \subset (s+2^{-(r+1)}, s+2^{-r}]$ for $r \leqslant s-2$, while $I_{s-1,s} = [s+2^{-s}, s+2^{-(s-1)}]$, and so two I with the same s but different r values are disjoint. In particular, any vertex v is the endpoint of at most one $e_{r,s}$ and so has degree at most three. Hence, G_t has maximum degree three.

Each edge $e_{r,s}$ is horizontal and crosses no other edges and so has no neighbors in X_D. Next, consider an edge aa' of P_s. We have $a' = a + 2 \cdot 2^{-s}$. Exactly one vertex in $V(P_0) \cup V(P_1) \cup \cdots \cup V(P_s)$ lies between a and a': their midpoint, $m = a + 2^{-s}$. Vertex m has at most two nonhorizontal edges incident to it, and so in X_D, every $aa' \in E(P_s)$ has at most two neighbors in $E(P_0) \cup E(P_1) \cup \cdots \cup E(P_s)$. Thus, X_D is 2-degenerate, as required. $\square$

4.4. Applications to general drawings. This section studies the global structure of graphs admitting a general (not necessarily circular) drawing. In particular, consider the following question: If a graph G has a drawing D, then what graph-theoretic assumptions about X_D guarantee that G is well structured? Even 1-planar graphs contain arbitrarily large complete graph minors [28], so one cannot expect G to exclude a fixed minor.

The following definition works well in this setting. Eppstein [33] defined a graph class $\mathcal{G}$ to have the *treewidth-diameter property,* more recently called *bounded local treewidth,* if there is a function f such that for every graph $G \in \mathcal{G}$, for every vertex $v \in V(G)$ and for every integer $r \geqslant 0$, the subgraph of G induced by the vertices at distance at most r from v has treewidth at most $f(r)$. If f is linear (polynomial), then $\mathcal{G}$ has *linear* (*polynomial*) *local treewidth.*

Lemma 6 shows that planar graphs have linear local treewidth. More generally, Dujmović, Morin, and Wood [29] showed that k-planar graphs have linear local treewidth (in fact, k-planar graphs satisfy a stronger product structure theorem [30]). On the other hand, Hickingbotham and Wood [42] showed that 1-gap planar graphs do not have polynomial local treewidth. They also asked whether k-gap planar graphs have bounded local treewidth. We show that this is false in a stronger sense.

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Dawar, Grohe, and Kreutzer [22] defined a graph class $\mathcal{G}$ to *locally exclude a minor* if for each $r \in \mathbb{N}$, there is a graph H_r such that for every graph $G \in \mathcal{G}$, every subgraph of G with radius at most r contains no H_r-minor. Observe that if $\mathcal{G}$ has bounded local treewidth, then $\mathcal{G}$ locally excludes a minor.

By Theorem 20, for each $t \in \mathbb{N}$, there is a graph G_t that contains a K_t-minor and has a circular drawing D such that X_D is 2-degenerate. Let G'_t be the graph obtained from G_t by adding a vertex into the outer face of D that is complete to $V(G'_t)$. So the graph G'_t is a 2-degenerate crossing, has radius 1, and contains a K_t-minor. Thus, graphs that are 2-degenerate crossing do not locally exclude a minor, implying that they do not have bounded local treewidth, thus answering the above question of Hickingbotham and Wood [42]. Since every graph that is 2-degenerate crossing is 2-gap-planar, we conclude that 2-gap-planar graphs also do not locally exclude a minor (and do not have bounded local treewidth). This result highlights a substantial difference between k-planar graphs and k-gap-planar graphs (even for $k = 2$). We now prove the following stronger result. A *star-forest* is a forest where each component is a star.

PROPOSITION 21. *For every $t \in \mathbb{N}$, there is a graph G and a drawing D of G such that*

- *G has radius 1;*
- *G contains a K_{t+1}-minor;*
- *X_D is a star-forest.*

Thus, the graph G is a 1-degenerate crossing and 1-gap-planar.

Proof. Let $\phi\colon E(K_t) \to \{1, \dots, \binom{t}{2}\}$ be a bijection. As illustrated in Figure 5, for each $ij \in E(K_t)$, draw vertices at $(\phi(ij), i), (\phi(ij), j) \in \mathbb{R}^2$ together with a straight vertical edge between them (blue edges in Figure 5).

For each $i \in \{1, 2, \dots, t\}$, draw a straight horizontal edge between each pair of consecutive vertices along the $y = i$ line. Let G_0 be the graph obtained. Let P_i be the subgraph of G_0 induced by the vertices on the $y = i$ line. Then P_i is a path on $t - 1$ vertices (green edges in Figure 5).

For each vertex v in $P_1 \cup \cdots \cup P_t$, add a "vertical" edge from v to a new vertex v' drawn with y-coordinate $t + 1$ (brown edges in Figure 5).

For $i = 1, 2, \dots, t$, complete the following step. If two vertical edges e and f cross an edge g in P_i at points x and y, respectively, and no other vertical edge crosses g between x and y, then subdivide g between x and y, introducing a new vertex v, and

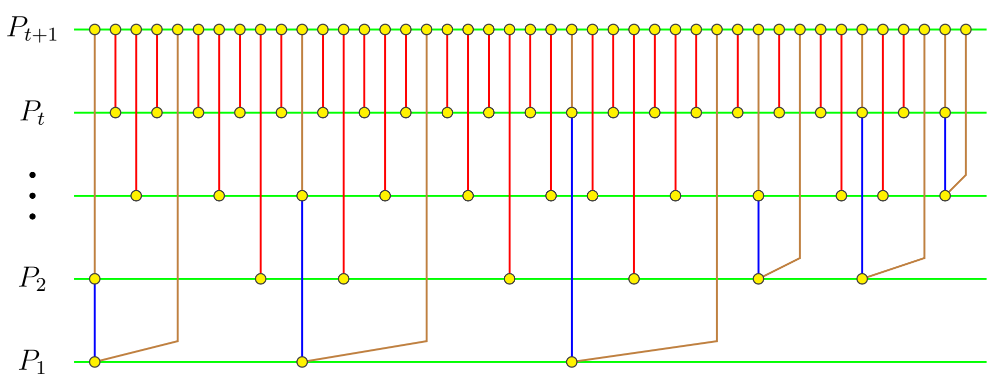

FIG. 5. *The graph G_1 in the proof of Proposition 21.*

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add a new vertical edge from v to a new vertex v' with y-coordinate $t+1$ (red edges in Figure 5).

Finally, add a path P_{t+1} through all the vertices with y-coordinate $t+1$. We obtain a graph G_1 and a drawing D_1 of G_1. Each crossing in D_1 is between a vertical and a horizontal edge, and each horizontal edge is crossed by at most one edge. Thus, X_{D_1} is a star-forest.

By construction, no edge in P_{t+1} is crossed in D_1, and every vertex has a neighbor in P_{t+1}. Thus, contracting P_{t+1} to a single vertex gives a graph G with radius 1 and a drawing D of G in which $X_D \cong X_{D_1}$. Thus, the graph G is 1-degenerate crossing and 1-gap-planar. Finally, G contains a K_{t+1}-minor, obtained by contracting each horizontal path P_i into a single vertex. ∎

5. Structural properties of circle graphs. Recall that a circle graph is the intersection graph of a set of chords of a circle. More formally, let C be a circle in $\mathbb{R}^2$. A *chord* of C is a closed line segment with distinct endpoints on C. Two chords of C either cross, are disjoint, or have a common endpoint. Let S be a set of chords of a circle C such that no three chords in S cross at a single point. Let G be the crossing graph of S. Then G is called a *circle graph*. Note that a graph G is a circle graph if and only if $G \cong X_D$ for some circular drawing D of a graph H, and in fact, one can take H to be a matching.

We are now ready to prove Theorems 1 and 2. While the treewidth of circle graphs has previously been studied from an algorithmic perspective [47], to the best of our knowledge, these theorems are the first structural results on the treewidth of circle graphs.

THEOREM 1. *Let $t \in \mathbb{N}$, and let G be a circle graph with treewidth at least $12t+2$. Then G contains an induced subgraph H that consists of t vertex-disjoint cycles $(C_1,\dots,C_t)$ such that for all $i<j$, every vertex of C_i has at least two neighbors in C_j. Moreover, every vertex of G has at most four neighbors in any C_i $(1 \leqslant i \leqslant t)$.*

Proof. Let D be a circular drawing of a graph such that $G \cong X_D$. Let M_D be the map graph of D. Since $\operatorname{tw}(X_D) = \operatorname{tw}(G) \geqslant 12t+2$, it follows by Theorem 5 that M_D has radius at least $2t$. The claim then follows from Lemma 9. ∎

THEOREM 2. *For the class of circle graphs, the treewidth and Hadwiger number are linearly tied. Moreover, the Hajós number is quadratically tied to both of them. Both "linear" and "quadratic" are best possible.*

Proof. Let G be a circle graph, and let D be a circular drawing with $G \cong X_D$. By Lemma 12,

$$\operatorname{tw}(G) \leqslant 6\operatorname{rad}(M_D) + 7 \leqslant 12\,h(G) - 11 \leqslant 12\operatorname{tw}(G) + 1.$$

So the Hadwiger number and treewidth are linearly tied for circle graphs. This inequality and Lemma 13 imply that

$$h'(G) - 1 \leqslant h(G) - 1 \leqslant \operatorname{tw}(G) \leqslant 6\operatorname{rad}(M_D) + 7 \leqslant 6h'(G)^2 + 18\,h'(G) + 13.$$

Hence, the Hajós number is quadratically tied to both the treewidth and the Hadwiger number for circle graphs. Finally, $K_{t,t}$ is a circle graph which has treewidth t, Hadwiger number $t+1$, and Hajós number $\Theta(\sqrt{t})$. Hence, "quadratic" is best possible. ∎

We now discuss several noteworthy consequences of Theorems 1 and 2. Recently, there has been significant interest in understanding the unavoidable induced

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subgraphs of graphs with large treewidth [2, 3, 4, 5, 6, 7, 8, 51, 62, 72]. Obvious candidates of unavoidable induced subgraphs include complete graphs, complete bipartite graphs, subdivision of large walls, and line graphs of subdivision of large walls. We say that a hereditary class of graphs $\mathcal{G}$ is *induced*-tw-*bounded* if there is a function f such that for every graph $G \in \mathcal{G}$ with $\text{tw}(G) \geqslant f(t)$, G contains K_t, $K_{t,t}$, a subdivision of the $(t \times t)$-wall, or a line graph of a subdivision of the $(t \times t)$-wall as an induced subgraph.[5] While the class of all graphs is not induced-tw-bounded [3, 14, 18, 63, 72], many natural graph classes are. For example, Aboulker et al. [1] showed that every proper minor-closed class is induced-tw-bounded, and Korhonen [49] recently showed that the class of graphs with bounded maximum degree is induced-tw-bounded. We now show the following.

THEOREM 22. *The class of circle graphs is not induced*-tw-*bounded.*

Proof. We first show that for all $t \geqslant 50$, no circle graph contains a subdivision of the $(t\times t)$-wall or a line graph of a subdivision of the $(t\times t)$-wall as an induced subgraph. As the class of circle graphs is hereditary, it suffices to show that for all $t \geqslant 50$, these two graphs are not circle graphs. These two graphs are planar (so K_5-minor-free) and have treewidth $t \geqslant 50$. However, Lemma 12 implies that every K_5-minor-free circle graph has treewidth at most 49, which is the required contradiction.

Now consider the family of couples of graphs $((G_t, X_t)\colon t \in \mathbb{N})$ given by Theorem 20, where X_t is the crossing graph of the drawing of G_t. Then $(X_t\colon t \in \mathbb{N})$ is a family of circle graphs. Since $(G_t\colon t \in \mathbb{N})$ has unbounded treewidth, Theorem 3 implies that $(X_t\colon t \in \mathbb{N})$ also has unbounded treewidth. Moreover, since X_t is 2-degenerate for all $t \in \mathbb{N}$, it excludes K_4 and $K_{3,3}$ as (induced) subgraphs, as required. ∎

While the class of circle graphs is not induced-tw-bounded, Theorem 1 describes the unavoidable induced subgraphs of circle graphs with large treewidth. To the best of our knowledge, this is the first theorem to describe the unavoidable induced subgraphs of a natural hereditary graph class that is not induced-tw-bounded. In fact, it does so with a linear lower bound on the treewidth of the unavoidable induced subgraphs.

Theorem 1 can also be used to describe the unavoidable induced subgraphs of circle graphs with large pathwidth.

THEOREM 23. *There exists a function f such that every circle graph G with $\text{pw}(G) \geqslant f(t)$ contains*

- *a subdivision of a complete binary tree with height t as an induced subgraph,*
- *the line graph of a subdivision of a complete binary tree with height t as an induced subgraph, or*
- *an induced subgraph H that consists of t vertex-disjoint cycles $(C_1, \ldots, C_t)$ such that for all $i < j$, every vertex of C_i has at least two neighbors in C_j. Moreover, every vertex of G has at most four neighbors in any C_i $(1 \leqslant i \leqslant t)$.*

Proof. If $\text{tw}(G) \geqslant 12t + 2$, then the claim follows from Theorem 1. Now assume that $\text{tw}(G) < 12t + 2$. Hickingbotham [41] showed that there is a function $g(k,t)$ such that every graph with treewidth less than k and pathwidth at least $g(k,t)$ contains a subdivision of a complete binary tree with height t as an induced subgraph or the line graph of a subdivision of a complete binary tree with height t as an induced subgraph. The result follows with $f(t) := \max\{g(12t+2, t), 12t+2\}$. ∎

[5]This definition is motivated by analogy to χ-boundedness; see [69]. Note that while the language of 'induced tw-bounded' is original to this paper, Abrishami et al. [6] previously used this definition under the guise of "special," and Abrishami et al. [2] used it under the guise of "clean."

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We now discuss applications of Theorem 1 to vertex-minor-closed classes. For a vertex v of a graph G, to *locally complement at* v means to replace the induced subgraph on the neighborhood of v by its complement. A graph H is a *vertex-minor* of a graph G if H can be obtained from G by a sequence of vertex deletions and local complementations. Vertex-minors were first studied by Bouchet [15, 16] under the guise of isotropic systems. The name "vertex-minor" is due to Oum [56]. Circle graphs are a key example of a vertex-minor-closed class.

We now show that a vertex-minor-closed graph class is induced-tw-bounded if and only if it has bounded rank-width. Rank-width is a graph parameter introduced by Oum and Seymour [58] that describes whether a graph can be decomposed into a tree-like structure by simple cuts. For a formal definition and surveys on this parameter, see [43, 57]. Oum [56] showed that rank-width is closed under vertex-minors.

THEOREM 24. *A vertex-minor-closed class $\mathcal{G}$ is induced-*tw*-bounded if and only if it has bounded rankwidth.*

Proof. Suppose that $\mathcal{G}$ has bounded rankwidth. By a result of Abrishami et al. [6], there is a function f such that every graph in $\mathcal{G}$ with treewidth at least $f(t)$ contains K_t or $K_{t,t}$ as an induced subgraph. Thus, $\mathcal{G}$ is induced-tw-bounded. Now suppose that $\mathcal{G}$ has unbounded rank-width. By a result of Geelen et al. [36], $\mathcal{G}$ contains all circle graphs. It therefore follows by Theorem 22 that $\mathcal{G}$ is not induced-tw-bounded. ∎

We conclude with the following question.

OPEN PROBLEM 25. *Let $\mathcal{G}$ be a vertex-minor-closed class with unbounded rank-width. What are the unavoidable induced subgraphs of graphs in $\mathcal{G}$ with large treewidth?*

The cycle structure (or variants thereof) in Theorem 1 must be included in the list of unavoidable induced subgraphs.

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