Optimal stopping of the stable process with state-dependent killing

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We describe the solution of an optimal stopping problem for a stable Lévy process killed at state-dependent rate. The killing rate is chosen in such a way that the killed process remains self-similar, and the solution to the optimal stopping problem is obtained by characterising a self-similar Markov process associated with the stable process. The optimal stopping strategy is to stop upon first passage into an interval, found explicitly in terms of the parameters of the model.

Keywords: Lamperti transformation; Lévy process; Markov additive process; omega-clock; optimal stopping; self-similar Markov process; stable process

1. Introduction

Consider a stable Lévy process $X = (X_t)_{t\geq 0}$ with index $\alpha \in (0,2)$, to which we introduce a state-dependent killing, occuring at rate $\omega(X_t)$ at time t, where

$$\omega(x) = \begin{cases} k(-x)^{-\alpha}, & x < 0 \\ 0, & x \geq 0, \end{cases}$$

for some parameter $k > 0$. Denote the random killing time by T, and the killed process by

$$X_t^\dagger = X_t \mathbb{1}_{\{t<T\}}.$$

We consider the gain function

$$g(x) = \begin{cases} (x^r - K)^+, & x > 0 \\ 0, & x \leq 0, \end{cases}$$

for some $r \in \mathbb{R} \setminus \{0\}$ and $K > 0$, and wish to solve the optimal stopping problem

$$v(x) = \sup_\tau \mathbb{E}_x \left[g(X_\tau^\dagger) \right], \tag{1}$$

where the supremum is over all (possibly infinite) stopping times τ for the natural enlargement of the filtration of $X^\dagger$, and the probability measure $\mathbb{P}_x$ with expectation $\mathbb{E}_x$ indicates that $X_0 = x$.

Optimal stopping problems for Lévy processes have been studied for many years, but the introduction of state-dependent killing brings new complexity to the problem. Rodosthenous and Zhang [26] have considered a similar question for general Lévy processes without upward jumps (and with a different gain function g and killing rate ω). They applied classical techniques in novel ways to characterise a stopping region which has quite a complex form. We take a very different approach: the particular form of ω means that the process $X^\dagger$ inherits from X the self-similarity property characteristic of stable processes, and we show that $X^\dagger$ has an intimate relationship with a member of a newly-discovered class of Lévy processes, the double hypergeometric processes.

This relationship makes it possible to find a surprisingly explicit solution to the optimal stopping problem: it is optimal to stop when $X^\dagger$ enters an interval, the bounds of which can be given explicitly in terms of the parameters of the model. This is outlined in the theorem below, which is the main result of this work and which we initially present only in its broad strokes.

Theorem 1. *There exists $\delta > \max(0, \alpha - 1)$, uniquely characterised in terms of the parameters of the stable process and the killing coefficient k, such that the following holds for the process X started in $x \neq 0$.*

(i) *When $0 < r < \delta$, the solution of the optimal stopping problem* (1) *is given by the upwards first passage time*

$$\tau^* := \inf\{t \geq 0 : X_t \geq b^*\},$$

where b^* can be found explicitly.

(ii) *When $-(\delta - \alpha + 1) < r < 0$, the solution of the optimal stopping problem* (1) *is given by the first entrance time*

$$\tau^* := \inf\{t \geq 0 : 0 < X_t \leq 1/b^*\}$$

where b^* can again be found explicitly.

(iii) *When $r < -(\delta - \alpha + 1)$ or $r > \delta$, $v(x) = \infty$ for all $x \neq 0$.*

In both instances, the quantity b^* can be found in terms of the parameters of the stable process, the killing parameter k and the parameters r, K of the gain function, and there is an explicit expression for δ. The full version of this result appears as Theorem 15, once we have introduced the necessary notation, and an integral expression for v is given in Remark 16.

This work has two purposes: firstly, to describe the solution of an optimal stopping problem for a stable Lévy process with an omega-clock; and secondly, to explicitly characterise a new self-similar Markov process, the killed path-censored stable process.

In the context of stochastic optimal control, the omega-clock was introduced by Albrecher, Gerber and Shiu [1], who studied the dividend problem. Since then, the model has enjoyed some popularity as a way to account for bankruptcy: if X represents the capital of some company, then since the killing rate is only positive when the process itself is negative, the killing time can be seen as a non-immediate ruin event. One could also consider $1/X$ to be the capital process (in which case ω is decreasing in the capital level), or indeed look at a time-change of one of these processes by some additive functional. The study of the ruin behaviour of risk processes with omega-clock has been very successful [2,10–12,23]. Optimal stopping problems have been considered for this process in the case of Brownian motion [13] and of general spectrally negative Lévy processes [26], which is the situation closest to ours. In [26], Rodosthenous and Zhang consider an American call option with the underlying equal to the exponential of a spectrally negative Lévy processs, and with an omega-clock function $\omega(x) = k\mathbb{1}_{(-\infty,0)}(x)$ for some $k \geq 0$. Without the omega-clock, this problem was solved by Mordecki [24], but with the clock, it becomes substantially harder, and the authors show that it is optimal to stop upon first passage into the union of a half-line and a (possibly empty) compact interval. The main differences in the present work are that the underlying is a Lévy process (not its exponential), that the omega-clock function is different in form, and that the process has both upward and downward jumps.

There is a long tradition of studying stable processes using a transformation into a positive self-similar Markov process (pssMp); as examples, we mention the stable process killed on going below zero and its conditionings [7], censored [5] and path-censored stable processes [19] and the radial part of an isotropic stable process [8]. Our first step in dealing with this process with omega-clock is to cut out the excursions below zero, and this gives rise to a new pssMp, the killed path-censored stable process, which we describe in explicit detail by relating it to a member of the double hypergeometric class of Lévy processes. This allows us to obtain the solution to the optimal stopping problem. The relation with a pssMp also explains the form of the omega-clock function ω, which ensures that the self-similarity property of the stable process is maintained after killing.

The remainder of the paper is structured as follows. In Section 2, we construct a pssMp by transforming the path of the killed stable process, and we apply the Lamperti transform to obtain a new Lévy process from this. In Section 3, we show that the Lévy process defined in the preceding section belongs to the double hypergeometric class and solve an auxiliary optimal stopping problem based on it. Finally, Section 4 provides the proof of the main results of the paper. A short concluding Section 5 offers comments on extensions of this work.

2. The killed path-censored stable process and its Lamperti transform

Since the gain function g in the optimal stopping problem (1) is zero on $(-\infty, 0)$, it is natural to consider removing the path sections of X where it is negative. In this section, we will show that this gives rise to a positive self-similar Markov process, and identify its distribution using the Lamperti transform.

We begin with a formal presentation of Lévy processes, stable processes, the omega-clock and the theory of self-similarity. This leads to a description of a new positive self-similar Markov process Y, the killed path-censored stable process, which we then characterise and which will be vital to our solution, in Section 3, of an auxiliary optimal stopping problem. The process Y was described briefly in the case of a symmetric stable process in [20], citing communication with the second author of this article, and here we offer a full characterisation in the general case.

A process $\xi = (\xi_t)_{t\geq 0}$ is called a (killed) Lévy process if it has state space $\mathbb{R} \cup \{\partial\}$, càdlàg paths and stationary, independent increments. Such a process is characterised by the *Lévy-Khintchine formula*, which states that for all $\theta \in \mathbb{R}$, the characteristic exponent given by $e^{-t\Psi(\theta)} = \mathbb{E}\left[e^{i\theta\xi_t}\right]$ satisfies

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|<1\}}\right) \Pi(\mathrm{d}x) + q, \tag{2}$$

where $q \geq 0$ is the killing rate, $a \in \mathbb{R}$ is the linear coefficient, $\sigma \geq 0$ is the Gaussian coefficient and Π is the Lévy measure, concentrated on $\mathbb{R}\backslash\{0\}$ and such that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. When $q > 0$, the process ξ is sent to the cemetery state ∂ at an exponential random time with rate q, which is otherwise independent of the path of ξ, and remains at ∂ forever. We write $\mathbb{P}_x$ for the law of the process started from x, and we will retain this notation for other stochastic processes wherever this is unambiguous.

A Lévy process $X = (X_t)_{t\geq 0}$ is called a *stable process* if it enjoys the *scaling property*, namely, that when started from $X_0 = 0$, the process $(cX_{tc^{-\alpha}})_{t\geq 0}$ has the same law as X for any $c > 0$. The parameter $\alpha \in (0, 2]$ is called the *index* of X. Stable processes can be described in terms of their Lévy-Khintchine formula as follows [16, §1.2.6 and §6.5.3]. The Lévy measure Π is absolutely continuous with density given by

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{\{x>0\}} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{\{x<0\}}, \quad x \in \mathbb{R},$$

where

$$c_+ = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \text{ and } c_- = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

The parameter ρ is called the *positivity parameter* of X and satisfies $\rho = \mathbb{P}_0(X_t \geq 0)$ for all $t > 0$. For convenience, we write $\hat{\rho} = 1 - \rho$. We restrict ourselves to the following set of admissible parameters:

$$\mathscr{A}_{st} = \{(\alpha,\rho) : \alpha \in (0,1), \rho \in (0,1)\} \cup \{(\alpha,\rho) : \alpha \in (1,2), \rho \in (1 - \frac{1}{\alpha}, \frac{1}{\alpha})\} \cup \{(\alpha,\rho) = (1, \frac{1}{2})\},$$

which encompasses (up to a multiplicative factor in their spatial scale) all stable processes with the exception of Brownian motion, processes jumping only in one direction and symmetric Cauchy processes with non-zero drift.

If E is either $\mathbb{R}$ or $[0,\infty)$, then an E-valued standard Markov process Y (in the sense of [4]) is said to be an *E-self-similar Markov process* if, for some fixed $\alpha > 0$ and all $y \in E$ and $c > 0$,

$$\text{the law of } (cY_{c^{-\alpha}t})_{t\geq 0} \text{ under } \mathbb{P}_y \text{ is } \mathbb{P}_{cy}. \tag{3}$$

When $E = [0,\infty)$, such a process is called a *positive self-similar Markov process (pssMp)*, and when $E = \mathbb{R}$, it is called a *real self-similar Markov process (rssMp)*. The stable process X is an rssMp.

We are now ready to describe the omega-clock killing of the stable process, and our procedure for removing its path sections lying in $(-\infty, 0)$.

Let X be a stable process. Due to self-similarity, we can regard the measures $\mathbb{P}_x$ as being defined by $\mathbb{E}_x[F(X_t, t \geq 0)] = \mathbb{E}_{\operatorname{sgn} x}[F(|x| X_{t|x|^{-\alpha}}, t \geq 0)]$ for measurable functionals F and $x \in \mathbb{R} \setminus \{0\}$; this will be convenient at times. Denote by $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ the natural enlargement of the filtration $(\sigma(X_s, s \leq t))_{t\geq 0}$ in the sense of Bichteler [3, Definition 1.3.38]. Note in particular that the enlargement does not depend on the initial value of X. Define the positive continuous additive functional

$$A_t = \int_0^t \omega(X_s)\,\mathrm{d}s, \quad t \geq 0,$$

where ω is as defined in the introduction. Let e_1 be an exponential random variable of rate 1, independent of X, and define the omega-clock killing time by

$$T = \inf\{t \geq 0 : A_t > \mathrm{e}_1\}.$$

The *omega-killed stable process* is given by

$$X_t^\dagger = X_t \mathbb{1}_{\{t<T\}}, \quad t \geq 0.$$

We note that here, 0 functions as a cemetery state for $X^\dagger$, which is a common convention with ssMps.

Lemma 2. *The process $X^\dagger$ is a standard Markov process in the filtration $\mathbb{F}^\dagger$, the natural enlargement of $(\sigma(X_s^\dagger, s \leq t))_{t\geq 0}$.*

Proof. We give a sketch of the proof, since the argument follows familiar lines. Recall that $T < T_0$ almost surely, which implies that, for a random time τ, we have $\{T \leq \tau\} = \{X_\tau^\dagger = 0\}$. The strong Markov property of $X^\dagger$ at the stopping time τ can be proved by partitioning on the event $\{X_\tau^\dagger = 0\}$ and using the corresponding property of X and the definition of T. Next, the bounded convergence theorem implies that, for any measurable, bounded f, $\lim_{t\to 0} \mathbb{E}_x[f(X_t^\dagger)] = f(x)$. This and the strong Markov property are the only facts required to prove the quasi-left-continuity of $X^\dagger$, by reproducing the proof of [16, Lemma 3.2]. This completes the proof. ☐

Let $C = (C_t)_{t \geq 0}$ be given by

$$C_t = \int_0^t \mathbb{1}_{\{X_s^\dagger \geq 0\}} \, \mathrm{d}s, \quad t \geq 0,$$

and call its right-continuous inverse C^{-1}. C counts the time that $X^\dagger$ spends above 0. The *killed path-censored stable process* Y is the stochastic process

$$Y_t = X^\dagger_{C_t^{-1}}, \quad t \geq 0.$$

The effect of the Markov time-change C^{-1} is to erase the negative components of $X^\dagger$ and glue the non-negative parts together at the endpoints, up until the time that $X^\dagger$ is killed during one of these negative components.

The role of 0 deserves special attention. Write $T_0 = \inf\{t \geq 0 : X_t = 0\}$ for its hitting time. When $\alpha \leq 1$, the process X cannot hit zero, whereas when $\alpha > 1$, T_0 is finite almost surely. Even in the latter case, $A_{T_0} = \infty$ a.s., and this can be seen by noting that $k^{-1} A_{T_0}$ is the lifetime of the Lamperti representation of the path-censored stable process formed from $-X$, which by [18, Lemma 13.3] is infinite. Therefore, $X^\dagger$ and Y are always killed before reaching zero.

Moreover, regardless of the value of α, when the process X is started from zero, it is killed immediately. We can see this is follows. Fix t; due to scaling, the distribution of $k^{-1} A_t$ under $\mathbb{P}_x$ converges, as $x \to 0$, to that of $k^{-1} A_{T_0}$ under $\mathbb{P}_1$ (recalling that $T_0 < \infty$ iff $\alpha > 1$, and the discussion above). Applying [18, Lemma 13.3] again, we see that $A_t = \infty$ a.s. for any $t > 0$ when $X_0 = 0$. In conclusion, we can regard 0 as an absorbing state for $X^\dagger$ and Y, which is consistent with the convention for ssMps mentioned above.

Proposition 3. *The process Y is a positive self-similar Markov process with respect to the filtration $\mathbb{F}^\dagger \circ C^{-1} = (\mathscr{F}^\dagger_{C_t^{-1}})_{t \geq 0}$.*

Proof. To prove the fact that Y is a standard Markov process, we use [4], specifically the remark following Chapter V, Proposition 4.11. This treats Y as a time-change of $X^\dagger$, and one ingredient we require is that $X^\dagger$ has a reference measure. This is a measure μ such that a set $B \subset \mathbb{R}$ is null for μ if and only if it has zero potential (meaning in [4, §II.3] that, for some $q \geq 0$ and every $x \in \mathbb{R}$, $U_q^\dagger(x, B) \coloneqq \int_0^\infty e^{-qt} \mathbb{P}_x(X_t^\dagger \in B) \, \mathrm{d}t = 0$). We will prove that $\mu(\mathrm{d}x) = \delta_0(\mathrm{d}x) + \mathrm{d}x$, the sum of a Dirac mass at zero and Lebesgue measure, is a reference measure for $X^\dagger$. Fix $q > 0$. A short calculation shows that

$$U_q^\dagger(x, B) = \mathbb{E}_x \left[\int_0^\infty e^{-qt - A_t} \mathbb{1}_{\{X_t \in B\}} \, \mathrm{d}t + (1 - e^{-A_t}) \delta_0(B) \right]. \tag{4}$$

From this we see that there exists $a > 0$ such that for all x and B, $U_q^\dagger(x, B) \leq a(U_q(x, B) + \delta_0(B))$, where U_q is the q-potential of X. Since for every $t > 0$, X_t is absolutely continuous with respect to Lebesgue measure and has support $\mathbb{R}$, it follows that U_q is equivalent to Lebesgue measure, and thence that $U_q^\dagger$ is absolutely continuous with respect to some scalar multiple of μ.

On the other hand, take a measurable set $B \subset \mathbb{R}$ such that $\mu(B) > 0$, and fix $x \neq 0$. If $\delta_0(B) > 0$, then by (4), $U_q^\dagger(x, B) > 0$ as well. If not, then B has positive Lebesgue measure, and then $U_q(x, B) > 0$ by the discussion above. But this implies that X has positive probability of reaching B before an exponential time of rate q; and $X^\dagger$ has positive probability of not being killed before reaching B. Therefore, $U_q^\dagger(x, B) > 0$. This completes the proof that μ is a reference measure for $X^\dagger$.

We now return to the result of [4]. Y is the time-change of $X^\dagger$ by the continuous additive functional C. $X^\dagger$ is a standard process with a reference measure and C has support $[0,\infty)$, which is closed in $\mathbb{R}$. This is enough to conclude that Y is a standard process.

We prove the scaling property in two steps: first, we show that $X^\dagger$ is self-similar, and then that Y inherits this property.

1. $X^\dagger$ is self-similar; that is, the scaling property (3) applies to it. Fix $c > 0$. Let $\widetilde{X}_t = cX_{tc^{-\alpha}}$ and define the rescaled process $\widetilde{A}$ as

$$\widetilde{A}_t = \int_0^t \omega(\widetilde{X}_s)\,\mathrm{d}s = \int_0^t kc^{-\alpha}\left(-(X_{c^{-\alpha}s})\right)^{-\alpha}\mathbb{1}_{\{X_s<0\}}\mathrm{d}s$$
$$= \int_0^{c^{-\alpha}t} k(-X_u)^{-\alpha}\mathbb{1}_{\{X_u<0\}}\mathrm{d}u = A_{c^{-\alpha}t}$$

Let $\widetilde{T} = \inf\{t : \widetilde{A}_t > \mathrm{e}_1\} = c^\alpha T$. The scaling property of $X^\dagger$ now follows, using in the third line the scaling of X:

$$\text{under } \mathbb{P}_x,\ (cX^\dagger_{c^{-\alpha}t})_{t\ge 0} = \left(cX_{c^{-\alpha}t}\mathbb{1}_{\{c^{-\alpha}t<T\}}\right)_{t\ge 0} = \left(\widetilde{X}_t\mathbb{1}_{\{t<\widetilde{T}\}}\right)_{t\ge 0}$$
$$\overset{d}{=} \left(X_t\mathbb{1}_{\{t<T\}}\right)_{t\ge 0} = X^\dagger \text{ under } \mathbb{P}_{cx}.$$

2. Y is self-similar.

   Let $\tilde{C}$ be the functional C applied to the process $(cX^\dagger_{tc^{-\alpha}})_{t\ge 0}$; a calculation similar to the one above yields that $\tilde{C}_t^{-1} = c^\alpha C^{-1}_{c^{-\alpha}t}$. We deduce the scaling property of Y:

$$\text{under } \mathbb{P}_x,\ (cY_{c^{-\alpha}t})_{t\ge 0} = \left(cX^\dagger_{C^{-1}_{c^{-\alpha}t}}\right)_{t\ge 0} = \left(cX^\dagger_{c^{-\alpha}\tilde{C}^{-1}_t}\right)_{t\ge 0}$$
$$\overset{d}{=} \left(X^\dagger_{C_t^{-1}}\right)_{t\ge 0} = Y \text{ under } \mathbb{P}_{cx},$$

   where we used step 1 in the third equality.

Since Y evidently has state space $[0,\infty)$, this completes the proof. □

The work of Lamperti [22] provides a bijection between the class of Lévy processes killed at an independent and exponentially distributed time and the class of positive self-similar Markov processes, which can be expressed through a straightforward space-time transformation; [16, §13] offers a textbook treatment. Let $T(s) = \left(\int_0^\cdot Y_u^{-\alpha}\,\mathrm{d}u\right)^{-1}(s)$, and

$$\xi_s = \log Y_{T(s)}, \quad s \ge 0. \tag{5}$$

Then, ξ is a Lévy process killed at positive rate.

Our next aim is to obtain the characteristic function of ξ, using the structure of Y in terms of gluing path sections of $X^\dagger$.

Define a stopping time

$$\tau_0^- = \inf\{t \ge 0 : X_t < 0\},$$

at which the stable process X passes below zero for the first time. We will denote the first 'gluing time' of Y by $\sigma_0 = \inf\{t \geq 0 : C_t^{-1} > C_{t-}^{-1}\}$; in fact, $\sigma_0 = \tau_0^-$, but the latter notation would be misleading for Y, since it actually stays positive at this time. R1:(vii) R2:6:1

Lemma 4. *For any $x > 0$, the joint law of $(Y_{\sigma_0}, Y_{\sigma_0 -})$ under $\mathbb{P}_x$ is equal to that of $(xY_{\sigma_0}, xY_{\sigma_0 -})$ under $\mathbb{P}_1$.*

Proof. The proof is very similar to [28, Lemma 3.5], but as the situation is slightly different, we include it for completeness. Fix $c > 0$ and define the two rescaled processes $(\widetilde{X}_t)_{t\geq 0}$ by $\widetilde{X}_t = cX_{c^{-\alpha}t}$ and $(\widetilde{Y}_t)_{t\geq 0}$ by $\widetilde{Y}_t = cY_{c^{-\alpha}t}$. Let $\tilde{\tau}_0^- = \inf\{t \geq 0 : \tilde{X}_t < 0\}$, and analogously define $\tilde{\sigma}_0$, the first gluing time of $\tilde{Y}$, which is equal to $\tilde{\tau}_0^-$. Then,

$$c^\alpha \tau_0^- = \inf\{c^\alpha t : t \geq 0, X_t < 0\} = \inf\{t \geq 0 : cX_{c^{-\alpha}t} < 0\} = \tilde{\tau}_0^- .$$

Therefore, for every $c, x > 0$ and $y, z \in \mathbb{R}$, $\mathbb{P}_x(Y_{\sigma_0 -} \in \mathrm{d}y, Y_{\sigma_0} \in \mathrm{d}z) = \mathbb{P}_{cx}(c^{-1}Y_{\sigma_0 -} \in \mathrm{d}y, c^{-1}Y_{\sigma_0} \in \mathrm{d}z)$. The lemma follows by setting $c = 1/x$. □

Denote by p the *killing probability* of Y at the first gluing event, namely

$$p = \mathbb{P}_x\left(Y_{\sigma_0} = 0\right),$$

which we assert is independent of x. The following lemma gives the explicit expression for p.

Lemma 5 (Killing probability). *The killing probability p is independent of x and is given by*

$$p = \frac{k}{c_+/\alpha + k}.$$

As a consequence, $T < \infty$.

Proof. Recall that T is the time at which $X^\dagger$ is killed, and so C_T is the killing time of Y. Let $R = \inf\{t > \tau_0^- : X_t \geq 0\}$, the first return time of X above zero. In terms of these quantities, $p = \mathbb{P}(T \leq R)$.

Consider the dual process $\hat{X}$ with distribution $-X$, which is still a stable process (with different parameters). Let $\hat{X}^*$ denote the process $\hat{X}$ sent to zero at the first time it passes below zero. It is well-known [7, Theorem 2] that the Lamperti transform of the pssMp $\hat{X}^*$ is killed at exponential time of rate c_+/α, regardless of the value of $\hat{X}_0^*$, and said killing time corresponds (through the Lamperti time-change) to the first time that $\hat{X}$ passes below zero.

Let $\hat{\tau}_0^- = \inf\{t \geq 0 : \hat{X}_t < 0\}$. The discussion above amounts to the statement that, whatever the initial value of $\hat{X}$, the distribution of $\int_0^{\hat{\tau}_0^-} (\hat{X}_u)^{-\alpha}\, \mathrm{d}u$ is exponential with parameter c_+/α. This leads us to the following calculation, in which e_1 is a random variable with exponential distribution of rate 1, independent of $\hat{X}$. R1:(xii) R2:6:-1

$$\begin{aligned}
\mathbb{P}_x(R < T) &= \mathbb{E}_x\left[\mathbb{P}_{-X_{\tau_0^-}}\left(\int_0^{\hat{\tau}_0^-} k(\hat{X}_u)^{-\alpha}\, \mathrm{d}u < e_1\right)\right] \\
&= \mathbb{E}_x\left[\mathbb{P}_{-X_{\tau_0^-}}\left(\int_0^{\hat{\tau}_0^-} (\hat{X}_u)^{-\alpha}\, \mathrm{d}u < e_1/k\right)\right] \\
&= \frac{\frac{c_+}{\alpha}}{\frac{c_+}{\alpha} + k}.
\end{aligned}$$

The killing probability is $p = 1 - \mathbb{P}(R < T)$, and this completes the proof.

Since p is indeed independent of X, and since $X^\dagger$ has fixed, positive probability of being killed in every excursion it makes below the level 0, it must eventually be killed; that is, $T < \infty$. □

Write X^* for the stable process killed upon exiting $[0, \infty)$, and ξ^* for the Lévy process appearing in its Lamperti representation. The law of the latter was characterised in [7, Theorem 2], and its characteristic exponent computed in [14, Theorem 1]. Making use of the preparatory work above, we are now able to describe the path structure of ξ and compute its characteristic exponent.

Proposition 6 (Structure of ξ). *The Lévy process ξ is the sum of two independent Lévy processes ξ^1 and ξ^2, which are characterised as follows:*

(i) *The Lévy process ξ^1 has characteristic exponent*

$$\Psi^1(\theta) = \Psi^*(\theta) - \frac{c_-}{\alpha}, \quad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of the process ξ^*.

(ii) *The process ξ^2 has has characteristic exponent*

$$\Psi^2(\theta) = (1-p)\Psi^{\mathrm{cpp}}(\theta) + p\frac{c_-}{\alpha},$$

where $p = \frac{k}{c_+/\alpha + k}$ is the killing probability and Ψ^{cpp} is the characteristic exponent of ξ^c, a compound Poisson process with jump rate c_-/α, which is expressed as

$$\Psi^{\mathrm{cpp}}(\theta) = \frac{c_-}{\alpha}\left(1 - \frac{\Gamma(1-\alpha\rho+i\theta)\Gamma(\alpha\rho-i\theta)\Gamma(1+i\theta)\Gamma(\alpha-i\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha)}\right),$$

for $\theta \in \mathbb{R}$.

Proof. The proof is almost identical to that of Proposition 3.4 in [19], which operates by considering the path up to the gluing time σ_0. The only difference is that at time σ_0, the process $X^\dagger$ may be sent to zero, which corresponds to killing of the process ξ^2 in place of a jump. Accounting for this in the obvious way, making use of Lemmas 4 and 5 in the appropriate place, yields the result on the path structure. The expression for Ψ^{cpp} is obtained from [19, Proposition 4.2]. □

This structure will allow us to compute the characteristic exponent of ξ. To give the explicit expression, we first require some more notation.

Lemma 7. *Let*

$$\delta = \frac{1}{2}\Big(\alpha - \frac{1}{\pi}\arccos\big(p\cos\pi(\alpha\rho - \alpha\hat{\rho}) + (1-p)\cos\pi\alpha\big)\Big).$$

Then, δ uniquely satisfies the conditions

$$\max(0, \alpha - 1) < \delta < \min(\alpha\rho, \alpha\hat{\rho}) \tag{6}$$

and

$$(1-p)\sin\pi\alpha\rho\sin\pi\alpha\hat{\rho} = \sin\pi(\alpha\rho - \delta)\sin\pi(\alpha\hat{\rho} - \delta). \tag{7}$$

Proof. Using product-to-sum identities, condition (7) can be rewritten as follows:

$$(1-p)\big(\cos\pi(\alpha\rho-\alpha\hat\rho)-\cos\pi\alpha\big)=\cos\pi(\alpha\rho-\alpha\hat\rho)-\cos\pi s$$
$$\cos(\pi s)=p\cos\pi(\alpha\rho-\alpha\hat\rho)+(1-p)\cos\pi\alpha, \tag{8}$$

where $s=2(\alpha/2-\delta)$, and the inequalities (6) are equivalent to $\max(\alpha\rho-\alpha\hat\rho,\alpha\hat\rho-\alpha\rho)<s<\min(\alpha,2-\alpha)$.

We divide our analysis into two cases depending on the value of α. When $\alpha\in(0,1]$, we have that $-\alpha<\alpha\rho-\alpha\hat\rho<\alpha$. If $\rho\geq 1/2$, then taking

$$s=\frac{1}{\pi}\arccos\big(p\cos\pi(\alpha\rho-\alpha\hat\rho)+(1-p)\cos\pi\alpha\big)$$

yields $0<\alpha\rho-\alpha\hat\rho<s<\alpha\leq 1$. Moreover, this is the unique value of s in the interval specified which satisfies (8). The analysis is similar when $\rho<1/2$.

On the other hand, when $\alpha\in(1,2)$, we have instead $\alpha-2<\alpha\rho-\alpha\hat\rho<2-\alpha$. If $\rho\geq 1/2$, then taking s as above gives $0<\alpha\rho-\alpha\hat\rho<s<2-\alpha<1$; again, the uniqueness argument and the case $\rho<1/2$ are similar. □

We briefly note a few extensions and special cases. When $p=1$, which is not part of our parameter set but corresponds formally to $k=\infty$, that is, immediate killing when X goes below zero, we have $\delta=\min(\alpha\rho,\alpha\hat\rho)$. When $\rho=1/2$, the symmetric case, we have $\delta=\frac{1}{2}\big(\alpha-\frac{1}{\pi}\arccos\big(p+(1-p)\cos\pi\alpha\big)\big)=\frac{\alpha}{2}-\frac{1}{\pi}\arcsin\big(\sqrt{1-p}\sin(\pi\alpha/2)\big)$; this calculation corresponds to the one cited in [20] for the process denoted there by $Y^{\natural}$.

The structure of ξ allows us to deduce explicitly the characteristic exponent of the process.

Corollary 8. *The characteristic exponent of ξ is expressed as*

$$\Psi(\theta)=\frac{\Gamma(\alpha-i\theta)\Gamma(\alpha\rho-i\theta)\Gamma(1+i\theta)\Gamma(1-\alpha\rho+i\theta)}{\Gamma(\alpha-\delta-i\theta)\Gamma(\delta-i\theta)\Gamma(\delta+1-\alpha+i\theta)\Gamma(1-\delta+i\theta)}. \tag{9}$$

Proof. The beginning of the proof resembles that of [19, Theorem 5.3], but it then diverges due to the killing. Using the structure described in Proposition 6 and substituting the expression for Ψ^* found in [15, Theorem 1], we obtain:

$$\begin{aligned}\Psi(\theta)&=\Psi^*(\theta)+(1-p)\Psi^{\mathrm{cpp}}(\theta)-(1-p)\frac{c_-}{\alpha}\\&=\frac{\Gamma(\alpha-i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\hat\rho-i\theta)\Gamma(1-\alpha\hat\rho+i\theta)}\\&\quad+(1-p)\frac{c_-}{\alpha}-(1-p)\frac{c_-}{\alpha}\frac{\Gamma(1-\alpha\rho+i\theta)\Gamma(\alpha\rho-i\theta)\Gamma(\alpha-i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha)}\\&\quad-(1-p)\frac{c_-}{\alpha}.\end{aligned}$$

Rearranging this and substituting the expression for c_-, yields

$$\begin{aligned}\Psi(\theta)&=\Gamma(\alpha-i\theta)\Gamma(1+i\theta)\\&\quad\times\left[\frac{1}{\Gamma(\alpha\hat\rho-i\theta)\Gamma(1-\alpha\hat\rho+i\theta)}-(1-p)\frac{\Gamma(\alpha\rho-i\theta)\Gamma(1-\alpha\rho+i\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)}\right]\end{aligned}$$

$$= \frac{1}{\pi^2}\Gamma(\alpha - i\theta)\Gamma(1+i\theta)\Gamma(\alpha\rho - i\theta)\Gamma(1-\alpha\rho+i\theta)$$
$$\times \left[\sin\pi(\alpha\rho - i\theta)\sin\pi(\alpha\hat{\rho} - i\theta) - (1-p)\sin\pi\alpha\rho\sin\pi\alpha\hat{\rho}\right], \quad (10)$$

where we apply the reflection formula to the last equality. Applying (7) gives that

$$\sin\pi(\alpha\rho - i\theta)\sin\pi(\alpha\hat{\rho} - i\theta) - (1-p)\sin\pi\alpha\rho\sin\pi\alpha\hat{\rho}$$
$$= \frac{1}{2}\left[\cos\pi(\alpha\rho - \alpha\hat{\rho}) - \cos\pi(\alpha - 2i\theta) - \cos\pi(\alpha\rho - \alpha\hat{\rho}) + \cos\pi(\alpha - 2\delta)\right]$$
$$= \sin\pi(\alpha - \delta - i\theta)\sin\pi(\delta - i\theta)$$
$$= \frac{\pi^2}{\Gamma(\alpha-\delta-i\theta)\Gamma(1-\alpha+\delta+i\theta)\Gamma(\delta-i\theta)\Gamma(1-\delta+i\theta)}, \quad (11)$$

using product-to-sum and sum-to-product identities followed by the reflection formula. Finally, substituting (11) into (10) yields the expresssion in the statement. □

Remark 9. We close the section by describing the situation where $k = 0$. This is not part of our assumptions, but there is a link to the existing literature. In this case, noting $T = \infty$, one should replace $X^\dagger$ with the definition $X^\dagger_t = X_t \mathbb{1}_{\{t<T_0\}}$ to ensure that the state 0 is absorbing.

The Lévy process ξ is unkilled in this case, and in Lemma 7, the lower bound $\delta = \max(0, \alpha - 1)$ is attained. Here we have $p = 0$, which represents the fact that Y is simply the path-censored stable process defined in [19]. Many of the arguments that appear in this section are analogous to ones in that work, though the presence of killing when $k > 0$ introduces some interesting novel features.

3. An optimal stopping problem for the Lamperti transform ξ

Having identified the pssMp Y via its Lamperti transform ξ, we are in a position to solve a related optimal stopping problem for the latter process, which we will later translate into a solution of the original problem.

The solution to the optimal stopping problem for ξ will rely on the *Wiener-Hopf factorisation* of Lévy processes. This can briefly be described as follows in terms of characteristic and Laplace exponents, with the meaning tha a function ϕ is the Laplace exponent of a subordinator H if $\mathbb{E}e^{-\lambda H_t} = e^{-t\phi(\lambda)}$. If Ψ is the characteristic exponent of a Lévy process ξ which is killed at rate $q \geq 0$, then there exists a factorisation of Ψ of the form

$$\Psi(\theta) = \kappa(q, -i\theta)\hat{\kappa}(q, i\theta), \quad \theta \in \mathbb{R}, \quad (12)$$

where $\kappa(q,\cdot)$ and $\hat{\kappa}(q,\cdot)$ are Laplace exponents of two (possibly killed) subordinators, known as the ascending and descending ladder height processes. The functions κ and $\hat{\kappa}$ are called the *Wiener-Hopf factors* of Ψ (or of ξ), and this factorisation is unique up to multiplication of each factor by a positive constant.

These subordinators describe the way that ξ makes new maxima and minima, which goes some way to explaining their utility in the context of this problem. A full treatment of the theory of Wiener-Hopf factorisation can be found in [16, §6].

Our first goal in this section is to characterise ξ by identifying it as a double hypergeometric Lévy process. This is a recently defined class of processes with explicit Wiener-Hopf factorisation. The process ξ is the second known example of a double hypergeometric process found ‘in the wild’, the other being the ricocheted stable process described by Budd [6].

3.1. Identification of the Lamperti transform ξ

Double hypergeometric processes, introduced in [20], are a family of Lévy processes with known Wiener-Hopf factorisation. The class can be characterised as follows. Let $\mathscr{O}$ be the set of all $(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}) \in [0,\infty)^4$ satisfying one of

(i) for some $n \in \mathbb{N} \cup \{0\}$, $\mathtt{c}+n \le \mathtt{a}+n \le \mathtt{d} \le \mathtt{b} \le \mathtt{c}+n+1$, or
(ii) for some $n \in \mathbb{N}$, $\mathtt{a}+n-1 \le \mathtt{c}+n \le \mathtt{b} \le \mathtt{d} \le \mathtt{a}+n$.

When $(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}) \in \mathscr{O}$, the function

$$\mathrm{B}(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d};\lambda) := \frac{\Gamma(\lambda+\mathtt{a})\Gamma(\lambda+\mathtt{b})}{\Gamma(\lambda+\mathtt{c})\Gamma(\lambda+\mathtt{d})}, \quad \mathrm{Re}\,\lambda \ge 0,$$

is the Laplace exponent of a subordinator.

Moreover, it is shown in [20, Corollary 2.1] that, when $(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}), (\hat{\mathtt{a}},\hat{\mathtt{b}},\hat{\mathtt{c}},\hat{\mathtt{d}}) \in \mathscr{O}$, the function

$$\Psi(\theta) = \mathrm{B}(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d};-i\theta)\mathrm{B}(\hat{\mathtt{a}},\hat{\mathtt{b}},\hat{\mathtt{c}},\hat{\mathtt{d}};i\theta), \quad \theta \in \mathbb{R},$$

is the characteristic exponent of a Lévy process in the *double hypergeometric class*.

Lemma 10. *The process ξ is a double hypergeometric Lévy process with parameters*

$$\mathtt{a} = \alpha\rho, \mathtt{b} = \alpha, \mathtt{c} = \delta, \mathtt{d} = \alpha-\delta,$$

and

$$\hat{\mathtt{a}} = 1-\alpha\rho, \hat{\mathtt{b}} = 1, \hat{\mathtt{c}} = \delta+1-\alpha, \hat{\mathtt{d}} = 1-\delta.$$

Accordingly, the Wiener-Hopf factors of ξ are

$$\kappa(q,z) = \frac{\Gamma(\alpha\rho+z)\Gamma(\alpha+z)}{\Gamma(\delta+z)\Gamma(\alpha-\delta+z)}, \tag{13}$$

and

$$\hat{\kappa}(q,z) = \frac{\Gamma(1-\alpha\rho+z)\Gamma(1+z)}{\Gamma(\delta+1-\alpha+z)\Gamma(1-\delta+z)}, \tag{14}$$

where

$$q = \frac{\Gamma(\alpha\rho)\Gamma(\alpha)\Gamma(1-\alpha\rho)}{\Gamma(\delta)\Gamma(\alpha-\delta)\Gamma(\delta+1-\alpha)\Gamma(1-\delta)} = \frac{c_-}{\alpha}\frac{k}{k+\frac{c_+}{\alpha}}.$$

Proof. The interval for δ given in (6) implies that both $(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d})$ and $(\hat{\mathtt{a}},\hat{\mathtt{b}},\hat{\mathtt{c}},\hat{\mathtt{d}})$ satisfy condition (i) for membership of $\mathscr{O}$, with $n=0$.

It follows from Corollary 2.1 in [20] that the Lévy process ξ with characteristic exponent

$$\begin{aligned}\Psi(\theta) &= \mathrm{B}(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d};-i\theta)\mathrm{B}(\hat{\mathtt{a}},\hat{\mathtt{b}},\hat{\mathtt{c}},\hat{\mathtt{d}};i\theta),\\ &= \frac{\Gamma(\alpha\rho-i\theta)\Gamma(\alpha-i\theta)\Gamma(1-\alpha\rho+i\theta)\Gamma(1+i\theta)}{\Gamma(\delta-i\theta)\Gamma(\alpha-\delta-i\theta)\Gamma(\delta+1-\alpha+i\theta)\Gamma(1-\delta+i\theta)}\end{aligned}$$

exists as a member of the double hypergeometric class. Comparing with (9) shows that this identifies our process ξ, and the result of [20] yields the stated Wiener-Hopf factorisation. □

When $k = 0$, the process ξ is a hypergeometric Lévy process in the simple ($\alpha \le 1$) or extended ($\alpha > 1$) class, as described in [19, §5]. The expressions given in the above result for its Wiener-Hopf factors remain valid in this case, though they can be simplified further.

3.2. The optimal stopping problem as a perpetual call option

In this part, we derive the solutions for the optimal stopping problem

$$w(y) = \sup_{\sigma \in \mathscr{S}_{\mathbb{G}}} \mathbb{E}_y \left[\left(e^{r\xi_\sigma} - K \right)^+ \right], \tag{15}$$

where $r \in \mathbb{R} \setminus \{0\}$, and $\mathscr{S}_{\mathbb{G}}$ indicates the set of all stopping times with respect to $\mathbb{G} = (\mathscr{G}_t)_{t\ge 0}$, the natural enlargement of $(\sigma(\xi_s, s \le t))_{t \ge 0}$, defined as in Section 2. This is a perpetual American call option in which the underlying is the process $e^{r\xi}$, and has been addressed in [24].

The Wiener-Hopf factors κ and $\hat{\kappa}$ of ξ appear as components in the solution. We note that $\kappa(q,z)$ is a well-defined holomorphic function for $\operatorname{Re} z > -\alpha\rho$, and the same applies to $\hat{\kappa}(q,z)$ for $\operatorname{Re} z > \alpha\rho - 1$.

Theorem 11. *Consider the optimal stopping problem* (15).

(i) *When* $0 < r < \delta$, *the solution is given by*

$$w(y) = \frac{1}{\mathbb{E}[e^{r\bar{\xi}_{\zeta-}}]} \mathbb{E}\left[\left(e^{r(y+\bar{\xi}_{\zeta-})} - K\mathbb{E}\left[e^{r\bar{\xi}_{\zeta-}} \right] \right)^+ \right], \tag{16}$$

where ζ is the killing time of the process ξ, which has exponential distribution with rate q, and $\bar{\xi}_t = \sup_{0\le s\le t} \xi_s$. We remark that $\mathbb{E}\left[e^{r\bar{\xi}_{\zeta-}}\right] = \frac{\kappa(q,0)}{\kappa(q,-r)}$. The optimal stopping time is given by

$$\sigma^* = \inf\{t \ge 0 : \xi_t \ge c^*\}$$

where

$$c^* = \frac{1}{r} \log K \frac{\kappa(q,0)}{\kappa(q,-r)}.$$

(ii) *When* $-(\delta + 1 - \alpha) < r < 0$, *the solution is given by*

$$w(y) = \frac{1}{\mathbb{E}[e^{r\underline{\xi}_{\zeta-}}]} \mathbb{E}\left[\left(e^{r(y+\underline{\xi}_{\zeta-})} - K\mathbb{E}\left[e^{r\underline{\xi}_{\zeta-}} \right] \right)^+ \right] \tag{17}$$

where $\underline{\xi}_t = \inf_{0\le s\le t} \xi_s$, and $\mathbb{E}\left[e^{r\underline{\xi}_{\zeta-}}\right] = \frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)}$. The optimal stopping time is given by

$$\sigma^* = \inf\{t \ge 0 : \xi_t \le -c^*\}$$

where

$$c^* = \frac{1}{|r|} \log K \frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)}.$$

(iii) *When $r > \delta$ or $r < -(\delta + 1 - \alpha)$, $w(y) = \infty$ for all y.*

Proof. Theorem 1 in [24] provides a solution for the valuation of the perpetual American call option for a general Lévy process, expressed in terms of the moment generating function of its supremum. The proof follows from an application of this result, the main prerequisite for which is to check the inequality $\mathbb{E}e^{r\xi_1} < 1$.

(i) Take $r > 0$. Condition $\mathbb{E}e^{r\xi_1} < 1$ is equivalent to $\Psi(-ir) < 0$, where Ψ is the characteristic exponent of the killed Lévy process ξ.
The first zero of $\Psi(-ir)$ occurs at δ, and so in this case $\mathbb{E}e^{r\xi_1} < 1$ if and only if $0 < r < \delta$.
The value function w is now expressed in terms of the moment generating function of the overall supremum of $r\xi$, which is given by [16, Theorem 6.15] in terms of the Wiener-Hopf factor:

$$\mathbb{E}\left[e^{r\bar{\xi}_{\zeta-}}\right] = \frac{\kappa(q,0)}{\kappa(q,-r)}.$$

The result follows from [24].

(ii) Take now $r < 0$. In this case, $\mathbb{E}e^{r\xi_1} < 1$ if and only if $0 < -r < \delta + 1 - \alpha$. Since $r < 0$, the value function is expressed in terms of

$$\mathbb{E}[e^{r\underline{\xi}_{\zeta-}}] = \frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)},$$

and the result again follows from [24].

(iii) In this case, it holds that $\mathbb{E}[e^{r\xi_1}] > 1$, and so the result follows from [24, Theorem 1(c) and (d)], where Mordecki shows that arbitrarily large values can be obtained by stopping at deterministic times. □

Corollary 12. *When* $0 < r < \delta$*, we can express*

$$w(y) = \kappa(q,-r)\int_0^\infty \left(e^{r(y+z)} - K\frac{\kappa(q,0)}{\kappa(q,-r)}\right)^+ u(z)\,\mathrm{d}z, \quad y \in \mathbb{R},$$

where

$$u(x) = \frac{1}{\Gamma(\alpha\rho)} e^{-\delta x}(1-e^{-x})^{\alpha\rho-1}{}_2F_1(\delta-\alpha\hat{\rho},\delta,\alpha\rho,1-e^{-x}), \quad x \geq 0,$$

and ${}_2F_1$ *is the hypergeometric function. When* $-(\delta+1-\alpha) < r < 0$*, we have*

$$w(y) = \hat{\kappa}(q,r)\int_0^\infty \left(e^{r(y-z)} - K\frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)}\right)^+ \hat{u}(z)\,\mathrm{d}z, \quad y \in \mathbb{R},$$

where

$$\hat{u}(x) = \frac{1}{\Gamma(\alpha\hat{\rho})} e^{-(1-\delta)x}(1-e^{-x})^{\alpha\hat{\rho}-1}{}_2F_1(\alpha-\delta,\alpha\hat{\rho}-\delta,\alpha\hat{\rho},1-e^{-x}), \quad x \geq 0.$$

Proof. The random variable $\bar{\xi}_{\zeta-}$ has Laplace transform given by

$$\lambda \mapsto \frac{\kappa(q,0)}{\kappa(q,\lambda)}, \quad \mathrm{Re}\,\lambda > \max(-\delta,\delta-\alpha),$$

and the double beta subordinator with Laplace exponent $\kappa(q,\cdot)$ has a renewal density u whose Laplace transform is given by $\lambda \mapsto \frac{1}{\kappa(q,\lambda)}$. The function u is the convolution of the two functions

$$u_1(x) = \frac{1}{\Gamma(\delta)} e^{-(\alpha-\delta)x}(1-e^{-x})^{\delta-1}, \quad x \geq 0, \text{ and}$$

$$u_2(x) = \frac{1}{\Gamma(\alpha\rho-\delta)} e^{-\delta x}(1-e^{-x})^{\alpha\rho-\delta-1}, \quad x \geq 0,$$

which are the renewal densities of Lamperti-stable subordinators corresponding, respectively, to the Laplace exponents $\lambda \mapsto \frac{\Gamma(\alpha\rho+\lambda)}{\Gamma(\delta+\lambda)}$ and $\lambda \mapsto \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha-\delta+\lambda)}$. Computing the convolution of u_1 and u_2 yields the expression in the statement.

The proof in the case $r < 0$ is very similar, but we instead need to compute the convolution of functions

$$\hat{u}_1(x) = \frac{1}{\Gamma(\delta)}(e^x - 1)^{\delta-1}, \quad x \geq 0, \text{ and}$$

$$\hat{u}_2(x) = \frac{1}{\Gamma(\alpha\hat{\rho}-\delta)} e^{-(1-\alpha+\delta)x}(1-e^{-x})^{\alpha\hat{\rho}-\delta-1}, \quad x \geq 0,$$

in order to compute the density of $\underline{\xi}_{\zeta-}$. □

Expressions for u and $\hat{u}$ in general (as potential densities of a double beta subordinator) appear in [20, Theorem 2.1], but for our particular parameter set, the ones above are simpler.

4. Solution of the optimal stopping problem

In this section, we return to our original problem (1), which we can rewrite as

$$v(x) = \sup_{\tau \in \mathscr{S}_{\mathbb{F}^\dagger}} [g(X_t^\dagger)], \tag{18}$$

where $\mathscr{S}_{\mathbb{H}}$ represents the set of stopping times associated with some filtration $\mathbb{H}$, and, as defined earlier, $\mathbb{F}^\dagger$ is the natural enlargement of the filtration of $X^\dagger$. We also recall the gain function g defined in the introduction:

$$g(x) = \begin{cases} (x^r - K)^+, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

The reason for our introduction of the notation $\mathscr{S}_{\mathbb{H}}$ is that we will have cause to work with several filtrations as we connect (18) to the problem we considered in Section 3.

The following lemma will be important in understanding the structure of the solution of the optimal stopping problem.

Lemma 13. *When $-(\delta+1-\alpha) < r < \delta$, $\mathbb{E}_x \left[\sup_{t\geq 0} g\left(X_t^\dagger\right)\right] < \infty$ for all $x \in \mathbb{R}$.*

Proof. Assume initially that $x > 0$. First, observe that $\sup_{t\geq 0} g(X_t^\dagger)$ can only be attained at a time when X is positive. The time-change by C^{-1}, which yields Y, does not remove any such times. Applying this and an elementary bound for g, we obtain:

$$\mathbb{E}_x\left[\sup_{t\geq 0} g(X_t^\dagger)\right] = \mathbb{E}_x\left[\sup_{t\geq 0} g(Y_t)\right] \leq \mathbb{E}_x\left[\sup_{t\geq 0} Y_t^r \mathbb{1}_{\{Y_t\neq 0\}}\right]. \tag{19}$$

Given a path of Y starting at x, the corresponding path of ξ starts at $\log x$ and is a deterministic space and time transformation, which implies that

$$\mathbb{E}_x\left[\sup_{t\geq 0} Y_t^r \mathbb{1}_{\{Y_t\neq 0\}}\right] = \mathbb{E}_{\log x}\left[\sup_{s\geq 0} e^{r\xi_s}\right]. \tag{20}$$

We next need to show that the right hand side of (20) is finite. For this, we recall that by Theorem 1(a) in [24], $\mathbb{E}_0\left[\sup_{s\geq 0} e^{r\xi_s}\right] < \infty$ is true if $\mathbb{E}_0\left[e^{r\xi_1}\right] < 1$ holds, and as outlined in the proof of Theorem 11, this can be reduced to the condition that $-(\delta + 1 - \alpha) < r < \delta$. This completes the proof in the case $x > 0$.

We now consider the possibility that $x < 0$. In this case, the supremum in question can only be positive if it occurs after the time τ_0^+ at which X returns above zero. This together with the scaling property and a bound on g implies that

$$\mathbb{E}_x\left[\sup_{t\geq 0} g(X_t^\dagger)\right] \leq \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^+}}\left[\sup_{t\geq 0}(X_t^\dagger)^r \mathbb{1}_{\{X_t>0\}}\right]\right] = \mathbb{E}_x\left[X_{\tau_0^+}^r\right]\mathbb{E}_1\left[\sup_{t\geq 0}(X_t^\dagger)^r \mathbb{1}_{\{X_t>0\}}\right]$$

The second expectation is equal to the left-hand side of (20) at $x = 1$, and hence is finite. For the first expectation, a result of Rogozin [27, Lemma 3] is useful; note that in applying it, we correct for the error in that lemma (q, which is denoted ρ here, should be $1 - q$). This gives:

$$\mathbb{E}_x\left[X_{\tau_0^+}^r\right] = \frac{\sin \pi\alpha\rho}{\pi}(-x)^{\alpha\rho}\int_0^\infty y^{r-\alpha\rho}(y-x)^{-1}\,\mathrm{d}y,$$

and thanks to the conditions on r given in the statement, this is finite for all x.

If $x = 0$, then $X_t^\dagger = 0$ and the result follows. ☐

This lemma leads to a key theorem about the stopping problem.

Theorem 14. *When* $0 < r < \delta$ *or* $-(\delta + 1 - \alpha) < r < 0$, *there exists a set* $D \subset (0, \infty)$ *such that*

$$\tau_D \coloneqq \inf\{t \geq 0 : X_t^\dagger \in D\}$$

is an optimal stopping time in (18).

Proof. We begin by considering the case $r > 0$, and we will apply [25, Corollary 2.9]. That result is formulated for finite-horizon stopping problems, but since $\lim_{t\to\infty} X_t^\dagger = 0$, Remark 2.10 in the same work notes that the application remains valid in this case. We remark that what we call a 'stopping time' is known as a 'Markov time' in [25].

Let

$$D = \{x \in \mathbb{R} : v(x) = g(x)\}.$$

The result of [25] states that the first entrance time τ_D of $X^\dagger$ into D is optimal for (18) if we can verify the following points:

(i) $\mathbb{E}_x[\sup_{t\geq 0} g(X_t^\dagger)] < \infty$. This is Lemma 13.

(ii) g is upper semicontinuous. This holds because g is continuous when $r > 0$.
(iii) v is lower semicontinuous.

As mentioned in Section 2, we can regard X under $\mathbb{P}_x$ as being defined by $X_t = |x|\tilde{X}_{t|x|^{-\alpha}}$, where $\tilde{X}$ has the law of X started from $\operatorname{sgn} x$. This allows us to reduce the treatment of all the measures $(\mathbb{P}_x)_{x\in\mathbb{R}\setminus\{0\}}$ to just that of $\mathbb{P}_1$ and $\mathbb{P}_{-1}$, and the identity extends also to $X^\dagger$, since it too is self-similar. In particular, it means that we can write

$$\mathbb{E}_x[g(X^\dagger_\tau)] = \mathbb{E}_{\operatorname{sgn} x}[g(|x|X^\dagger_{\tau|x|^{-\alpha}})],$$

for any $x \neq 0$ and random time τ. This will be quite useful.

Fix $x \neq 0$. Take $\epsilon > 0$, and let τ_ϵ be a stopping time satisfying

$$\mathbb{E}_x[g(X^\dagger_{\tau_\epsilon})] \geq v(x) - \epsilon/2.$$

Now, the right-continuity of $X^\dagger$, the dominated convergence theorem and Lemma 13 together imply that

$$\lim_{\eta\downarrow 0}\mathbb{E}_x[g(X^\dagger_{\tau_\epsilon+\eta})] = \mathbb{E}_x[g(X^\dagger_{\tau_\epsilon})],$$

which means that, if we set $\tau'_\epsilon = \tau_\epsilon + \eta$ for $\eta > 0$ small enough,

$$\mathbb{E}_x[g(X^\dagger_{\tau'_\epsilon})] \geq v(x) - \epsilon.$$

Since $\tau'_\epsilon = \lim_{\eta'\uparrow\eta}(\tau_\epsilon + \eta')$ gives an expression in terms of an increasing limit of stopping times, and $X^\dagger$ is quasi-left-continuous by Lemma 2, it follows that $X^\dagger$ is continuous at τ'_ϵ almost surely.

Take $(x_n)_{n\geq 0}$ to be a sequence converging to x and, from now on, take n sufficiently large that $\operatorname{sgn} x_n = \operatorname{sgn} x$. We first observe that

$$v(x_n) \geq \mathbb{E}_{x_n}[g(X^\dagger_{\tau'_\epsilon})] = \mathbb{E}_{\operatorname{sgn} x}[g(|x_n|X^\dagger_{|x_n|^{-\alpha}\tau'_\epsilon})].$$

Applying Fatou's lemma and using the continuity of $X^\dagger$ at τ'_ϵ, we obtain:

$$\begin{aligned}
\liminf_{n\to\infty} v(x_n) &\geq \mathbb{E}_{\operatorname{sgn} x}[\liminf_{n\to\infty} g(|x_n|X^\dagger_{|x_n|^{-\alpha}\tau'_\epsilon})] \\
&\geq \mathbb{E}_{\operatorname{sgn} x}[\min\{g(|x|X^\dagger_{|x|^{-\alpha}\tau'_\epsilon}), g(|x|X^\dagger_{|x|^{-\alpha}\tau'_\epsilon -})\}] \\
&= \mathbb{E}_x[\min\{g(X^\dagger_{\tau'_\epsilon}), g(X^\dagger_{\tau'_\epsilon -})\}] \\
&= \mathbb{E}_x[g(X^\dagger_{\tau'_\epsilon})] \geq v(x) - \epsilon.
\end{aligned}$$

Finally, letting $\epsilon \to 0$, we have

$$\liminf_{n\to\infty} v(x_n) \geq v(x),$$

which is the lower semicontinuity we require for $x \neq 0$.

Take $x = 0$, a stopping time τ and a sequence $(x_n)_{n\geq 0}$ convenging to 0. Then,

$$\liminf_{n\to\infty} v(x_n) = \liminf_{n\to\infty}\mathbb{E}_{x_n}[g(X^\dagger_\tau)] \geq 0 = v(0).$$

Therefore, v is lower semicontinuous also at 0, which completes the proof of this part.

Now turn to the case $r < 0$, in which g is no longer continuous. Our approach will be to make a space-time transformation of X, known as the Riesz–Bogdan–Żak transformation [17, Theorem 3.1]. Let $\theta(t) = \left(\int_0^\cdot |X_u|^{-2\alpha}\,\mathrm{d}u\right)^{-1}(t)$, and define $X_t^\circ = 1/X_{\theta(t)}$. The associated probability measures are $\mathbb{P}_x^\circ = \mathbb{P}_{1/x}$ for $x \neq 0$.

The process X° is a Markov process which is self-similar of index α, in the sense that it satisfies (3) for $y \in \mathbb{R} \setminus \{0\}$. Note that $\theta^{-1}(T_0) = \infty$ a.s., so the paths of X and X° are in correspondence up to T_0, the first time X hits zero, if at all.

It is not hard to show that $A_t^\circ = \int_0^t \omega(X_s^\circ)\,\mathrm{d}s = A_{\theta(t)}$, and we can define $T^\circ = \inf\{t \geq 0 : A_t^\circ > \mathrm{e}_1\}$, where e_1 is the independent random variable of rate 1 which was used in the definition of T. Note that $T = \theta(T^\circ)$.

Let $X_t^{\circ\dagger} = \frac{1}{X_{\theta(t)}} \mathbb{1}_{\{\theta(t)<T\}} = X_t^\circ \mathbb{1}_{\{t<T^\circ\}}$. For this process we can also define the probability measure $\mathbb{P}_0^\circ$ under which $X^{\circ\dagger}$ remains at 0 for all time. In this way, we obtain a real self-similar Markov process of index α, following essentially the same argument as in the proofs of Lemma 2 and Proposition 3.

Every stopping time τ for $\mathbb{F}^\dagger$ can be replaced by a stopping time $\tau' = \tau \wedge T_0$ without reducing the payoff, and $\theta^{-1}(\tau')$ is a stopping time for the filtration $\mathbb{F}^{\circ\dagger} = \left(\mathscr{F}^\dagger_{\theta(t)}\right)_{t\geq 0}$, which is also the natural enlargement of the filtration of $X^{\circ\dagger}$. Defining

$$g^\circ(x) = \begin{cases} (x^{-r} - K)_+, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

this discussion gives us that

$$v^\circ(x) \coloneqq \sup_{\tau \in \mathscr{S}_{\mathbb{F}^{\circ\dagger}}} \mathbb{E}_x^\circ[g^\circ(X_\tau)] = v(1/x), \tag{21}$$

for $x \neq 0$, and $v^\circ(0) = 0$. From this, we can use the result of [25] to conclude that the the stopping region for this new, equivalent optimal stopping problem is

$$D^\circ = \{x \in \mathbb{R} : v^\circ(x) = g^\circ(x)\},$$

provided we can verify the analogues of points (i–iii) above.

Point (i) follows immediately, and point (ii) holds because g° is continuous when $r < 0$. Point (iii) must be verified, but the proof above works with essentially no change, since it uses only the fact that the process in question is self-similar, strong Markov and has no fixed jumps, and this is true for $X^{\circ\dagger}$ just as for $X^\dagger$.

Defining $D = \{1/x : x \in D^\circ \setminus \{0\}\} \cup \{0\}$, it follows from (21) and the definition of $X^{\circ\dagger}$ that the entrance time of D is optimal for (18).

Finally, it is simple to see that 0 belongs to the sets D defined above, since $v(0) = 0 = g(0)$ in all cases. However, this point, and more importantly any points in $(-\infty, 0]$, can be removed from D without changing the value function, as we now show. Set $D' = D \cap (0, \infty)$, and then consider the following calculation:

$$\begin{aligned}
\mathbb{E}_x[g(X_{\tau_D})\mathbb{1}_{\{\tau_D<T\}}] &= \mathbb{E}_x[g(X_{\tau_D})\mathbb{1}_{\{\tau_D<T\}}\mathbb{1}_{\{X_{\tau_D}>0\}}] + \mathbb{E}_x[g(X_{\tau_D})\mathbb{1}_{\{\tau_D<T\}}\mathbb{1}_{\{X_{\tau_D}\leq 0\}}] \\
&= \mathbb{E}_x[g(X_{\tau_{D'}})\mathbb{1}_{\{\tau_{D'}<T\}}\mathbb{1}_{\{\tau_D=\tau_{D'}\}}] + 0 \\
&\leq \mathbb{E}_x[g(X_{\tau_{D'}})\mathbb{1}_{\{\tau_{D'}<T\}}].
\end{aligned}$$

From this it follows that, if $D \not\subset (0, \infty)$, we can replace it with D' and obtain at least as good a value; indeed, since τ_D is optimal, the values obtained from τ_D and $\tau_{D'}$ will actually be equal. □

Our main result, which implies Theorem 1 appearing in the introduction, is as follows. Recall that T is the killing time of the process $X^\dagger$.

Theorem 15. *Let $x \in \mathbb{R} \setminus \{0\}$. The solution of the optimal stopping problem* (18) *is given as follows.*

(i) *If $0 < r < \delta$, then the optimal stopping time is given by*

$$\tau^* = \inf\{t \geq 0 : X_t \geq b^*\}$$

where

$$b^* = \left(K \frac{\kappa(q,0)}{\kappa(q,-r)}\right)^{1/r}.$$

Moreover $\mathbb{E}_1\left[\bar{X}_T^r\right] < \infty$ and the optimal value is given for $x > 0$ by

$$v(x) = \frac{1}{\mathbb{E}_1\left[\bar{X}_T^r\right]}\mathbb{E}_1\left[\left(\left(x\bar{X}_T\right)^r - K\mathbb{E}_1\left[\bar{X}_T^r\right]\right)^+\right],$$

where $\bar{X}_t = \sup_{s \leq t} X_s$.

(ii) *If $-(\delta + 1 - \alpha) < r < 0$, then the optimal stopping time is given by*

$$\tau^* = \inf\{t \geq 0 : 0 < X_t \leq 1/b^*\}$$

where

$$b^* = \left(K \frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)}\right)^{1/|r|}.$$

Moreover, $\mathbb{E}_1\left[J(X)_T^r\right] < \infty$ and the optimal value is given for $x > 0$ by

$$v(x) = \frac{1}{\mathbb{E}_1\left[J(X)_T^r\right]}\mathbb{E}_1\left[\left((xJ(X)_T)^r - K\mathbb{E}_1\left[J(X)_T^r\right]\right)^+\right],$$

where $J(X)_t = \inf\{X_s : s \leq t, X_s > 0\}$.

(iii) *If $r > \delta$ or $r < -(\delta + 1 - \alpha)$, then $v(x) = \infty$.*

Proof. Since $g(x) = 0$ for $x \leq 0$, it is never optimal to stop when X is negative. The processes X and Y have the same range when restricted to $(0, \infty)$, so the optimal stopping problem (18) is equivalent to

$$v(x) = \sup_{\tau' \in \mathscr{S}_{\mathbb{F}^\dagger \circ C^{-1}}} \mathbb{E}_x\left[g\left(Y_{\tau'}\right)\right],$$

where we recall $\mathbb{F}^\dagger \circ C^{-1} = \left(\mathscr{F}^\dagger_{C_t^{-1}}\right)_{t \geq 0}$. Moreover, as already outlined, the pssMp Y corresponds to ξ under a time and space change, which implies that

$$v(y) = \sup_{\tau'' \in \mathscr{S}_{\mathbb{F}^\dagger \circ C^{-1} \circ T}} \mathbb{E}_y\left[g\left(e^{\xi_{\tau''}}\right)\right], \tag{22}$$

where $y = \log x$ and here again we have $\mathbb{F}^\dagger \circ C^{-1} \circ T = \left(\mathscr{F}^\dagger_{C^{-1}_{T(t)}}\right)_{t \geq 0}$.

We know from Theorem 14 that the hitting time τ_D of some set $D \subset (0,\infty)$ is optimal for (18), and hence that the first passage time σ_H of ξ into set $H = \{\log x : x \in D\}$ is optimal for (22).

The solution in Theorem 11 showed that the solution of the problem (15) for ξ is given by a hitting time σ^*. In (22), we optimise over $\mathscr{S}_{\mathbb{F}^\dagger \circ C^{-1} \circ T}$, and in (15) we optimise over $\mathscr{S}_{\mathbb{G}}$. The former set of stopping times is larger, since the filtration contains information about the times that $X^\dagger$ spends below zero. Since $\sigma_H \in \mathscr{S}_{\mathbb{G}}$, it follows that it is also optimal for (15). Comparing the form of σ^* with σ_H and hence with the original stopping region D, we obtain that

$$D = \left[\left(K \frac{\kappa(q,0)}{\kappa(q,-r)} \right)^{1/r}, \infty \right) \text{ if } 0 < r < \delta,$$

and

$$D = \left(0, \left(\frac{\hat{\kappa}(q,-r)}{K\hat{\kappa}(q,0)} \right)^{1/|r|} \right] \text{ if } -(\delta + 1 - \alpha) < r < 0.$$

For part (iii), we turn to the corresponding part of Theorem 11, where it is shown that arbitrarily large values of w can be obtained by stopping at a deterministic time, say t_0. Since time t_0 for ξ corresponds to time $C^{-1}_{T(t_0)}$ for X, this is a viable time at which to stop X. It follows that $v(x) = \infty$ for $x > 0$. When $x < 0$, one can first wait for X to pass above zero, which happens without being killed with positive probability, and then act as above, again attaining unbounded values. □

Remark 16. When $x > 0$, the value function can be expressed as an integral by noting that $v(x) = w(\log x)$, where w is the value function computed in Corollary 12.

When $x < 0$, it is still possible to obtain a (double) integral expression for v, though it is less direct. In this case, $X^\dagger$ is started below zero and we should wait for it to either be killed (with probability p independent of x) or jump back above zero (with probability $1-p$). Let $\tau_0^+ = \inf\{t \geq 0 : X_t^\dagger \geq 0\}$. Then, for all $x < 0$,

$$\begin{aligned} v(x) &= \mathbb{E}_x[v(X^\dagger_{\tau_0^+})] \\ &= (1-p) \int_0^\infty v(y) \frac{\sin \pi\alpha\rho}{\pi} (-x)^{\alpha\rho} y^{-\alpha\rho} (y-x)^{-1} \, \mathrm{d}y, \end{aligned}$$

where the integral expression comes from the same result of [27] which we used in Lemma 13. In conjunction with the expression for w given in Corollary 12, this can be used to write v as a double integral suitable for numerical computation.

5. Remark on a variant stopping problem

In this section we briefly describe how a gain function akin to that of a put option requires a different analysis, despite the superficial similarities. We consider the optimal stopping problem

$$v(x) = \sup_\tau \mathbb{E}_x \left[g(X_\tau) \mathbb{1}_{\{\tau < T\}} \right], \qquad g(x) = (K-x)^+, \tag{23}$$

where $K \in \mathbb{R}$. Since $g(x)$ may be positive for $x < 0$, it no longer makes sense to erase the sojourns of X in $(-\infty, 0)$. Instead, we may describe the problem using the so-called Lamperti-Kiu transform. This gives $X^\dagger$ in terms of a Markov additive process (MAP), which is a process (ξ, J) on $\mathbb{R} \times \{\pm 1\}$ obtained by

$$\xi_t = \log|X^\dagger_{T(t)}| \text{ and } J_t = \operatorname{sgn} X^\dagger_{T(t)}, \quad t \geq 0,$$

where T is a time-change.

The process (ξ, J) corresponding to $X^\dagger$ will be killed at a rate $\omega(\xi_t, J_t)$, where

$$\omega(y, j) = \begin{cases} 0, & j = 1, \\ k, & j = -1, \end{cases}$$

and the problem (23) will correspond to

$$v(x) = v(y, j) = \sup_\sigma \mathbb{E}_{y,j} \left[g(\xi_\sigma, J_\sigma) \mathbb{1}_{\{t<\zeta\}} \right],$$

where $(y, j) = (\log|x|, \operatorname{sgn} x)$, ζ is the killing time of (ξ, J) and

$$g(y, j) = \begin{cases} (K - e^y)^+, & j = 1, \\ (K + e^y)^+, & j = -1. \end{cases}$$

The MAP (ξ, J) can be described explicitly in terms of its matrix exponent. Unfortunately, though this translation to a MAP problem is relatively simple to describe, two new issues arise. The first is that the presence of J-dependent killing means that the matrix Wiener-Hopf factorisation of (ξ, J), which is known when $k = 0$ [17,21], is no longer evident. The second is that, even if the factorisation were known, the theory of optimal stopping is much less developed for these processes, outside of the spectrally negative case [9].

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