OPEN ACCESS
International Journal of Theoretical and Applied Finance
(2023) 2450001 (55 pages)

DOI: 10.1142/S0219024924500018

SYSTEMIC PERSPECTIVE OF TERM RISK IN BANK FUNDING MARKETS

ANDREA MACRINA*

Department of Mathematics
University College London (UCL)
Gower Street, London WC1E 6BT, UK

African Institute for Financial Markets
and Risk Management (AIFMRM)
University of Cape Town (UCT)
Private Bag X3, Rondebosch 7701, South Africa
**a.macrina@ucl.ac.uk*

OBEID MAHOMED

African Institute for Financial Markets
and Risk Management (AIFMRM)
University of Cape Town (UCT)
Private Bag X3, Rondebosch 7701, South Africa

Received 17 February 2023
Revised 15 December 2023
Accepted 21 December 2023
Published 15 February 2024

The transition from term-based reference rates to overnight reference rates has created a dislocation in the market-making processes between the interbank and non-interbank funding, and their respective derivatives markets. This dislocation can be attributed to differences in funding and corresponding interest rate swap transactions, a thesis we explain and characterize in detail. It is then shown how this dislocation may be resolved. Based on a systemic perspective of a stylized financial system, an aggregated banking system is constructed that is void of idiosyncratic credit risks but still vulnerable to liquidity risks. Within this setup, a mathematical modeling framework for term-cognizant interest rate systems is derived that enables the pricing and valuation of bank term funding and associated derivatives transactions with varying liquidity characteristics. Other outcomes include: (i) a detailed analysis of the incomplete market paradigm that encapsulates bank term funding rates and the risk management processes involved therein; and

*Corresponding author.

(ii) a recovery of consistency in the pricing and valuation between funding and related interest rate swap transactions, along with a mechanism to exchange term risk.

Keywords: Term rates; interbank market; money market; interest rate derivatives; pricing kernels; liquidity risk; SONIA and SOFR; LIBOR transition; benchmark reform.

1. Introduction

The global transition from *risky* term-based reference rates (TBRRs) to *near risk-free* overnight reference rates (ONRRs) is generally considered as an innovation that restores the *single-curve framework* within bank funding and associated derivatives markets. In this paper, it is shown that while this may hold true within the interbank funding and derivatives markets, it does not within the non-interbank counterparts of these markets. In particular, it is conjectured that the ONRR regime creates a *dislocation* between the *market-making* processes that is required within the interbank and non-interbank funding markets — one that did not exist before the 2007/2008 global financial crisis (GFC), as described briefly below.

Before the GFC, commercial banks were able to source term funding at *near risk-free term rates*. This may be evidenced by the following:

(a) short-term interbank reference rates[a] moved in lockstep with central bank (CB) policy rates, with subdued spreads and the three-month term being the key funding rate,[b] in general;
(b) medium- to long-term deposit rates coincided with those implied from forward rate agreements (FRAs) and interest rate swaps (IRSs), modulo insignificant spreads; and
(c) spreads between IRSs that referenced interbank reference rates with different tenors, or Basis Swap (BS) spreads, were negligible or nonmaterial.

These stylized features of the bank funding and linear derivatives markets are described and corroborated in, for instance, the research undertaken by Beau *et al.* (2014), Mercurio (2009), Bianchetti & Carlicchi (2011), Morini (2009) and Collin-Dufresne & Solnik (2001). For an insightful book on systemic risk in bank systems, and more generally in financial networks, we refer to Hurd (2016) and references cited therein. Two important implications of these features, within these markets, were the following:

(i) *Cross-sectional term agnosticism*, i.e. nonmaterial credit and liquidity risks implied that all term funding-related forward rates were tradable and replicable, at least synthetically[c] via the linear derivatives market, i.e. using FRAs and/or IRSs; and

[a]For example, the set of London Interbank Offered Rates (LIBORs).
[b]Thereby also the main derivative reference rate and transmission tool for the CB's monetary policy.
[c]It would have also been possible to trade and replicate term funding-related forward rates via the classic textbook *cash-and-carry* type of strategy. However, the respective market-maker would have to consider the sourcing and placement of funding carefully, for this to be practically viable.

(ii) *Funding–swap duality*, i.e. consistency and coherence between the *market-making* of banks' term funding and linear interest rate derivative transactions.

Both of these implications manifested a *single-curve framework* for the *primary market* pricing and *secondary market* valuation of funding and interest rate derivative transactions.

Post the GFC, cognizance of significant credit and liquidity risks has persisted within bank funding markets. As a result, only feature (a) from above remains, albeit reduced in form, with the bulk of interbank funding transactions transitioning from term-based to overnight rates — see, for instance, Schrimpf & Sushko (2019) for a review of this change.[d] Interbank markets have all transitioned, or are in the process of transitioning, from indicative TBRRs to transacted ONRRs, with the latter considered as being *near risk-free* while the former (in particular, the three-month term) now incorporates material elements of credit and liquidity risk.

Therefore, the pre-GFC interbank market microstructure constructed upon key three-month funding and reference rates has now, post-GFC, been *replaced* by a similar one based on overnight rates, such that the interbank funding and its associated linear derivatives[e] market remains the source of *near risk-free term rates*. General bank funding that originates within the non-interbank funding market will not transition into the overnight rate regime in general, primarily due to the fact that nonbank entities will seek to earn funding yields in excess of risk-free rates. In other words, these entities would want to be exposed to *term risk*, which naturally manifests due to credit and liquidity risks.

From the perspective of implication (ii), the discussion above reveals that the funding–swap duality should be maintained within the interbank market, conditional on funding transactions that reference overnight rates. However, there is now a dislocation between the interbank funding and derivatives market and the non-interbank funding market, conditional on funding transactions that reference term rates and yield term risk premia. This dislocation arises as a result of implication (i), cross-sectional term agnosticism, no longer holding in the presence of term risk. While it is impossible to recover implication (i) within the post-GFC market microstructure, it is possible to recover implication (ii) from a systemic perspective, which is the primary objective of this research.

In order to achieve this objective, it is assumed that only the interbank cash[f] and non-interbank funding markets exist, along with various reference rates. This setup creates the potential for *term-dependent floating interest rate risk*.[g] Since each

[d]Also, see Guggenheim (2020), Klingler & Syrstad (2021), Nelson (2020), Schrimpf & Sushko (2019) and Skov & Skovmand (2021) for further discussion and specific details regarding the potential creation and replacement of term rates.

[e]This market consists of futures and IRSs that reference the respective ONRR, with the latter IRS referred to as an Overnight Indexed Swap (OIS).

[f]This market includes overnight and term funding and borrowing activity amongst major banks within a given economy, and is therefore classified as a *primitive* market.

term rate embeds credit and liquidity risk, floating interest rate risk only manifests contingent on: (1) available initial and subsequent funding at each *roll-over* time; and (2) survival of the deposit-taking or borrowing bank, over each roll-over term that constitutes the total funding tenor under consideration. In order to simplify the exposition, a systemic perspective is postulated and assumed which preserves liquidity risk, from a *funding*, *market* and *systemic* perspective as defined by Acerbi & Scandolo (2008), as well as systemic bank credit risk, but precludes the need to consider idiosyncratic bank credit risks. The market-making process which enables the exchange of *floating-for-fixed interest rate risk* in the derivatives market is then developed, and is shown to be the dual process that would be required to do the same within the funding market, thereby achieving the primary objective.

The analysis within this systemic perspective yields a pricing kernel (PK) framework for the pricing and valuation of FRAs that is based on replication, which in turn provides the following results and outcomes:

(i) modeled fair FRA rates are systematically lower than corresponding market forward rates implied from the systemic term funding curve, which may be attributed to potential liquidity risks within the postulated systemic setting;
(ii) multiple interbank swap curves[h] distinguished by the term of the underlying interbank reference rates, i.e. the underlying FRA or IRS instruments associated with each curve reference the same interbank reference rate;[i]
(iii) a mechanism to exchange fixed interbank interest rate risk across the systemic term funding and interbank swap curves; and
(iv) a framework for consistent pricing and valuation of interbank funding instruments across curves, which also enables fair early liquidation or settlement values.

This paper should be considered as the prequel to the framework developed in Macrina & Mahomed (2018). In particular, the *curve-conversion factor process* and the resultant *across-curve pricing formula* developed in Macrina & Mahomed (2018) is rigorously justified here via the careful construction of the mechanism that enables the exchanges of risk — this is shown in detail in Sec. 7. The research by Filipović & Trolle (2013), Gallitschke *et al.* (2017) and Gefang *et al.* (2011) is also related — in their work tractable models are developed that capture multiple term structures of interbank risk post the GFC, and then they attempt to untangle credit and liquidity risks by calibrating these models to market data.[j] Also related

[g]This is the general interest rate risk that will emanate from a funding transaction that *rolls-over* multiple times at the frequency of the term of the funding rate under consideration.

[h]For a thorough account of multi-curve approaches, see Grbac & Runggaldier (2015).

[i]While not a primary objective of the paper, the results in Secs. 4 and 5 advocate for term-dependent market prices of risk in interest rate markets, which is related to the work of Campbell (1986).

[j]The untangling of credit and liquidity risk is also considered by Schwarz (2019) using an econometric approach.

is the research undertaken in Alfeus *et al.* (2020) and Backwell *et al.* (2023), where a reduced-form modeling framework is developed to model the dynamics of ONRRs and TBRRs, with the use of a novel modeling quantity that captures liquidity risk in the guise of *refinancing* or *roll-over risk*. However, these avenues of research are focused on pricing and valuation within multi-curve interest rate derivatives markets, while the work here focuses on developing a systemic framework for pricing and valuation within bank funding markets. Section 2 also discusses the subtle difference between *replicated* and *traded* FRAs, with the former playing a fundamental role in the theory that is developed. Unfortunately, a market for such replicated FRAs does not exist, which hinders the possibility of specific model calibration.

The rest of this paper is structured as follows. Section 2 contextualizes the funding–swap duality problem, the systemic financial system and PK mathematical framework that are relevant to this research. Section 3 considers the modeling of a systemic financial system with perfect liquidity and a single term rate. Section 4 builds upon Sec. 3 by considering the effect of the availability of multiple term rates. Section 5 then considers the same market considered in Sec. 4, but now in the presence of liquidity risk. Section 6 shows how the findings from Sec. 5 may be adapted for practical use, which takes the form of reduced-form PK-based models, and Sec. 7 concludes with an exposition of how exchanges of risk enable pricing and valuation. Some of the proofs and further supporting material are found in the appendix to this paper.

2. Funding–Swap Duality and Dislocation

In this section, a systemic version of the financial system that is under consideration is postulated. This setting enables the description of the notion of funding–swap duality and the subsequent dislocation thereof. This section then concludes with an outline of how funding–swap duality may be recovered at the systemic level, along with the mathematical framework that enables the required analysis and the derivation of the associated results.

2.1. *A stylized systemic financial system*

In any modern economy, idiosyncratic credit and liquidity risks must be carefully modeled over and above the risk-free interest rate dynamics that underpin the financial system. Figure B.1 shows a stylized financial system, along with an understanding of how various interest rates are determined using an axiomatic approach. This reveals idiosyncratic credit and liquidity risks faced by important agents, bank *treasuries* (TRs), who are tasked with the market-making of term funding. Another important set of agents are bank *sales and trading units* (STs), who are tasked with the market-making of financial derivatives. Being an internal client to their respective TR, each ST is *indirectly* exposed to the same risks. These idiosyncratic exposures are difficult to model in a concise and consistent manner, so that one may make rigorous systemic inferences. Therefore, we make two simplifying adjustments,

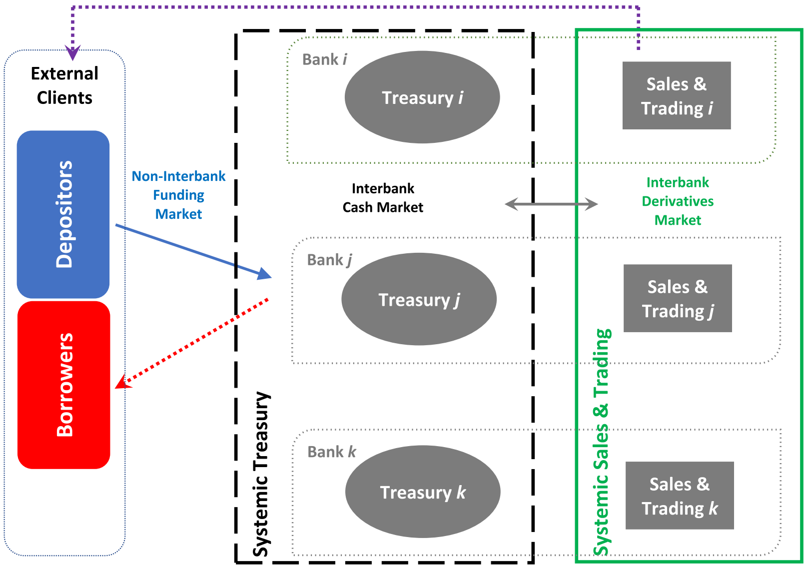

Fig. 1. Stylized systemic version of the financial system under consideration.

and consider the financial system depicted in Fig. 1, instead of the more involved system presented in Appendix B.

In this setting, the banking system has a systemic representation and consists of a *systemic treasury* (STR) along with a *systemic sales and trading unit* (SST). The STR and SST are collections of the respective individual entities, and by definition and construction subsume the interbank cash and derivatives markets, respectively. This representation enables a systematic analysis of all risks that affects the market-making processes undertaken by the STR and the SST. This includes general interest rate risk, systemic credit and liquidity risk. Idiosyncratic credit risk only remains in loans offered by the STR to *external borrowers* (EBs), which is not the focus here. Before moving on, the following description of shorthand notation simplifies the technical exposition in the rest of the paper.

Notation 1 (Simple rates). Let $A(u, u + n\delta)$ denote an arbitrary simple rate with accrual period $[u, u + n\delta]$, where $\delta > 0$, $n \in \mathbb{N}$ and $u \in \mathbb{R}_{\geq 0}$ denotes the start of the accrual period. The shorthand notation A_u^n will be used for such rates, such that an investment of one unit of currency at this rate will yield $(1 + n\delta A_u^n)$ at maturity time $u + n\delta$.

Within this new systemic context, Axiom B.4 which describes term funding and interbank lending for an individual bank, may now be replaced by the following systemic version.

Axiom 2.1 (Systemic term funding and liquidity). The STR will quote to internal (SST) and external clients (ECs) a funded, credit-risky and liquidity-cost-adjusted simple rate

$$R_u^{n,N} = x_u^n + d_u^{n,N} + \ell_u^{n,N}, \tag{2.1}$$

for term deposits with a nominal value of N and a tenor $n\delta$ at time u, where x_u^n is the STR's expectation of the risk-free component; and $d_u^{n,N}$ and $\ell_u^{n,N}$ are systemic term- and nominal-dependent debt and liquidity spreads/premiums. A quote (or liquidity) is not guaranteed for all nominal values and terms.

Remark 1 (Systemic debt premium and default paradox). Since the STR is an aggregation of individual credit-risky TRs, the systemic debt premium is also a suitable aggregation of each TR's idiosyncratic debt premiums, all of which are positive in general, indicating a chance for systemic default. However, in reality, the CB will preclude the possibility of a systemic default, and hence the systemic banking entity (STR and SST), which is also what is assumed within this theoretical context. Hence the paradox, which is a natural artefact of the systemic perspective.

Reference term rates are now a fair aggregated representation of the STR's term funding rates, i.e. an aggregate of those articulated in Axiom 2.1, and as shown in Axiom 2.2. This is by assumption and construction since the STR subsumes the interbank cash market.

Axiom 2.2 (Reference rates). Benchmark rates R_u^n are constructed by specific aggregate functions.[k] Given the structure of such rates, as postulated in Axiom 2.1, the aggregated benchmark is conjectured to be

$$R_u^n := x_u^n + d_u^n + \ell_u^n, \tag{2.2}$$

where d_u^n and ℓ_u^n are now an aggregated expectation of the debt premium and liquidity spread, respectively. Accordingly, R_u^n is also referred to as an interbank reference or market-based term rate. The result here is identical to Axiom B.5, as expected and required.

In addition, this enables a direct representation of the STR's term funding curve, provided next.

Definition 1 (Systemic term funding curve). The fair aggregated representation of the STR's term funding curve at time u is then

$$\{R_u^n\,; n \in \{1, 2, \ldots, m\}\}, \tag{2.3}$$

with the $m\delta$-term assumed to be the longest available funding tenor. Analogous in form to Definition B.1, the fair aggregated representation of the STR's $n\delta$-

[k]For example, a trimmed median or average, which is utilized for rates such as JIBAR and LIBOR, or a volume-weighted average for reference rates such as the SAFEX overnight rate and SONIA (also trimmed).

term funding rate at time u is given by the $n\delta$-term reference rate, Eq. (2.2) from Axiom 2.2.

The systemic term funding curve is therefore a collection of nonhomogeneous reference rates, which, in the absence of other interest rate financial markets and information, forms the basis for fair valuation at the systemic level. The SST is assumed to be an internal client of the STR, therefore both entities exhibit the same level of credit risk. As a result, for the purpose of pricing in their market-making process, the SST need only consider: (i) the arbitrage-free dynamics of the set of reference rates or, equivalently, the systemic term funding curve; along with (ii) the ad hoc liquidity risk that emanates from Axiom 2.1.

2.2. *Funding–swap duality*

At the current time t, consider an external depositor (ED) that offers funding to the STR over $[t, t+2\delta]$ at the δ-term rate.[1] Moreover, the ED would like to fix the δ-term rate that is applicable over $[t+\delta, t+2\delta]$. The STR therefore has one of two funding transaction options:

(i) **Actual Fixed Deposit:** For one unit worth of funding, the STR will pay the ED

$$(1+\delta R_t^1)(1+\delta K_t^{\mathrm{STR}}), \tag{2.4}$$

at time $t+2\delta$, with the STR assuming the responsibility of market-making the fixed rate K_t^{STR} at the current time t, which has an accrual period equal to $[t+\delta, t+2\delta]$.

(ii) **Synthetic Fixed Deposit:** For one unit worth of funding, the STR can passively pay

$$(1+\delta R_t^1)(1+\delta R_{t+\delta}^1). \tag{2.5}$$

If the STR enters a long position in a fair $\delta \times 2\delta$ FRA at time t with nominal equal to $(1+\delta R_t^1)$, offered by the SST, its net terminal payoff will be

$$\begin{aligned} &-(1+\delta R_t^1)(1+\delta R_{t+\delta}^1) + (1+\delta R_t^1)[R_{t+\delta}^1 - K_t^{\mathrm{SST}}]\delta \\ &\quad = -(1+\delta R_t^1)(1+\delta K_t^{\mathrm{SST}}), \end{aligned} \tag{2.6}$$

at time $t+2\delta$, where K_t^{SST} is the fair $\delta \times 2\delta$ FRA rate, which must be market-made by the SST. This is a negative cash flow for the STR, since it must be paid to the ED.

Options (i) and (ii) have the same payoff apart from K_t^{STR} and K_t^{SST}. However, take note that option (ii) eliminates all floating interest rate risk for the STR, at the

[1]The choice of δ and 2δ as the terms under consideration is arbitrary. The same implications and intuitions will apply for single- versus multiple-term funding, in general.

cost of a residual short $\delta \times 2\delta$ FRA position for the SST. Therefore, by comparison, this implies that option (i) must hide a residual synthetic short $\delta \times 2\delta$ FRA position for the STR. As is standard in mathematical finance, devising hedging strategies for these residual risks will provide an objective path to pricing the fixed rates. Both of these options require the replication of a long position in a fair $\delta \times 2\delta$ fair FRA rate in order to eliminate all of the residual risk.

From the above, it is conjectured that the pricing of such a funding transaction is equivalent to the dual process of pricing a long position in the associated FRA, which may also be thought of as a single-period IRS, hence the moniker *funding–swap duality*.

Remark 2 (STR lending). By similar logic, it is possible to show that the pricing of a similarly structured lending (or offering of funding) transaction is equivalent to the dual process of pricing a short position in the associated FRA. However, a lending transaction would expose the STR to the idiosyncratic credit risk of the borrowing entity. Therefore, to preclude such risk one would have to postulate and assume a lending transaction to another systemic banking entity.

2.3. *Replicated versus traded FRAs and funding–swap dislocation*

In order to eliminate risk in options (i) and (ii) above, a long position in a replicated FRA is advocated. This is achieved by depositing one unit of currency at the δ-term rate over the interval $[t, t+2\delta]$, which yields $(1+\delta R_t^1)(1+\delta R_{t+\delta}^1)$, i.e. the floating leg of the long $\delta \times 2\delta$ FRA payoff. Then, the fixed leg of the FRA, which is the source of the funding for the aforementioned deposit, is created as follows: In option (ii), the deposit is funded by the SST borrowing one unit at the 2δ-term rate. In option (i), the STR has already acquired funding of one unit for the period $[t, t+2\delta]$. Considering the risks associated with these transactions in combination with the short FRA exposure (i.e. the actual or implied exposure from options (i) and (ii)) will ultimately enable the pricing of K_t^{STR} and K_t^{SST}, respectively, and the hedging of the residual risk. It is important to note that the replication approach is theoretically appealing since it is instructive of the fundamental nature of the funding risks at play, and therefore a novel tool for modeling and analysis, as will be shown in sections that follow.

Remark 3 (Replicated FRAs and exchanges of term risk). Observe that replicating a long FRA position, from the perspectives of both the STR and SST, embeds a natural *transformation of term risk*, whereby funding acquired over the 2δ-term is actually exchanged for a deposit over the same tenor but at the δ-term frequency. Such an exchange is considered to be fundamental in nature, since the risk characteristics of the underlying funding and deposit transactions are maintained. Within the systemic perspective constructed in this paper, the residual risk is completely due to floating interest rate and liquidity risks.

In practice, the replication approach is not viable for the market-making of a FRA due to its punitive use of funding, and the associated credit and liquidity risks. Traded FRAs are *quasi-primitive* instruments with the FRA rate treated as the key pricing variable along with *collateral* and *central clearing* market microstructure to mitigate *counterparty credit risk*. The FRA rate is practically determined via demand and supply, but may be theoretically justified via classical risk-neutral valuation, in addition to *valuation adjustments* (most notably, the margin value adjustment (MVA), for centrally cleared traded FRAs).

Remark 4 (Traded FRAs and exchanges of floating-for-fixed interest rate risk). Traded FRAs treat the FRA rate as a tradable variable in and of itself, and may be liquidated at any time in the secondary market. Therefore, apart from enabling the exchange of floating-for-fixed interest rate risk, none of the other fundamental features of related funding transactions are captured. Moreover, this is further exacerbated by the fact that the respective market-maker must also consider valuation adjustments, such as MVA, which are idiosyncratic and only necessary for the fair valuation of derivatives within a risky market context.

Conjecture 2.1 (Funding–swap dislocation). *The reason for the dislocation between the funding and its associated linear derivatives market is three-fold*:

(a) *traded FRAs are not directly related to underlying funding transactions*;
(b) *the market-making process for traded FRAs requires idiosyncratic valuation adjustments thereby exacerbating the issue from* (*a*); *and*
(c) *the transition from TBRRs to ONRRs implies that associated modern linear derivatives only enable the exchange of floating-for-fixed interest rate risk related to the overnight term, and are thereby only applicable to interbank funding transactions.*[m]

In this paper, it is shown how funding–swap duality may be recovered in the modern risky market context. This is shown to be achievable with replicated FRAs within the constructed systemic setting, and therefore in the presence of liquidity risk only — the main ideas are articulated in the next section.

2.4. *Systemic funding–swap duality*

In the constructed systemic setting along with perfect liquidity, the following will apply: (1) The STR will be indifferent between sourcing funding at the δ- or the 2δ-term rate over $[t, t+2\delta]$, since there are no liquidity risks. (2) The STR will be indifferent between replicated and traded FRAs, since the traded FRA is replicable, at least theoretically, which also means that $K_t^{\mathrm{STR}} = K_t^{\mathrm{SST}}$. This setting, which is presented in Secs. 3 and 4, actually recovers the characteristics of the pre-GFC bank

[m]The assumption here, which is commensurate with current market activity, is that the majority of interbank funding will reference overnight rates, in order to limit exposure to credit risk.

funding and associated derivatives market. Considering the same systemic setting but now with illiquidity, the following applies: (1) The STR will rationally prefer 2δ- over δ-term funding over $[t, t+2\delta]$, since the latter implies that the funder is only willing to bear liquidity risk over the δ-term, or put differently, the funder requires the availability of liquidity at the δ-term frequency over the interval $[t, t+2\delta]$. (2) The STR will rationally prefer the replicated over the traded FRA, based on the previous section. However, within this context, since the STR will be the counterparty to the SST, this precludes the SST from any default risk and therefore the need for any collateralization or valuation adjustments. In addition, the STR will not demand the liquidity characteristics of a traded FRA, therefore the SST may once again consider replication in its market-making process. Section 5 models this particular illiquid setting along with details regarding the process undertaken by the SST to market-make a replicated FRA.

Remark 5 (SST market-making replicated FRAs). Implication (2), combined with Remark 2, is the reason that the market-making of replicated FRAs is assumed to be the natural role of the SST.

Remark 6 (Funding–swap duality $\Rightarrow$ Liquidation–liquidity duality). A key quantity in the replicated FRA market-making process is the *systemic liquidity indicator*, described in Definition 7. When market-making the $\delta \times 2\delta$ replicated FRA, one specific version of this abstract quantity, viz. $L^1_{t+\delta}$, defines the availability of symmetric[n] systemic liquidity in the δ-term rate at time $t+\delta$ from the SST's perspective.

For the market-making of a long position in a replicated FRA, the lack of systemic liquidity for a δ-term deposit may seem counter-intuitive, however viewed from the perspective of the STR's dual process of market-making the corresponding funding transaction, i.e. the actual fixed deposit, or option (i), this event coincides with the funder choosing to liquidate their funding position at $t+\delta$ and thereby forcing the STR to renege on rolling-over the δ-term deposit at this time. Similar logic may be applied to the market-making process for a short position in a replicated FRA; however, the lack of funds to borrow at the δ-term rate at time $t+\delta$ is an intuitive event in the realization of systemic illiquidity. Therefore, an implication of funding–swap duality is liquidation–liquidity duality.

2.5. *Mathematical framework for systemic interest rate modeling*

The financial system depicted in Fig. 1 forms the backdrop for the modeling framework, with the addition of the following assumptions:

(i) The *non-interbank funding* and interbank cash markets are the *primitive* financial markets, with the only market-maker being the STR.

[n]Symmetric here refers to the SST's ability to deposit and borrow at the δ-term rate at time $t+\delta$.

(ii) The *interbank derivatives market* is the considered *derivative* financial market, with the SST and its constituents being market-makers.
(iii) There are *no transaction costs*, *profit margins* or *taxes*.

It is assumed that this market system is *incomplete*, *arbitrage-free* and supported by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\geq 0}, \mathbb{P})$ satisfying the *usual conditions*, where $\mathbb{P}$ denotes the real-world probability measure, and where

$$\mathcal{F}_u := \mathcal{G}_u \vee \overline{\mathcal{L}}_u, \tag{2.7}$$

for $u \geq 0$. The filtration $(\mathcal{G}_u)_{u\geq 0}$ models information about all *tradable* variables, $(\overline{\mathcal{L}}_u)_{u\geq 0}$ models information about all *liquidity* variables associated with the funding markets (i.e. items (i) and (ii)). The filtration $(\mathcal{L}_u)_{u\geq 0}$, where $\mathcal{L}_u \subset \overline{\mathcal{L}}_u$, models information about *liquidity* variables associated with the funding markets only (i.e. item (i) only). Therefore, $(\mathcal{F}_u)_{u\geq 0}$ models information about all tradable variables and their liquidity characteristics.

It is assumed that the SST's starting point for modeling the systemic term funding curve or, equivalently, its set of nonhomogeneous constituent rates is the specification of a corresponding set of statistically estimated stochastic discount factors (SDFs) — for more information on SDFs, see Cochrane (2005), Constantinides (1992), Cox *et al.* (1981), Duffie (2001) and Hunt & Kennedy (2000). The next definition describes one such SDF for an arbitrary $n\delta$-term along with the calculation of estimated $n\delta$-term rates, where $n \in \{1, 2, \ldots, m\}$.

Definition 2 (Estimated $n\delta$-term SDF and rates). The estimated $n\delta$-term SDF, $(\widehat{D}^n_u)_{u\geq 0}$, is assumed to be a $\{(\mathcal{G}_u), \mathbb{P}\}$-continuous semimartingale. At any time u, it is possible to calculate an estimate for the $n\delta$-term zero coupon bond (ZCB) price by writing

$$\widehat{P}^n_{u,u+n\delta} := \frac{1}{\widehat{D}^n_u}\mathbb{E}^{\mathbb{P}}[\widehat{D}^n_{u+n\delta} \,|\, \mathcal{G}_u], \tag{2.8}$$

making use of the estimated $n\delta$-term SDF. Then

$$\widehat{R}^n_u := \frac{1}{n\delta}\left(\frac{1}{\widehat{P}^n_{u,u+n\delta}} - 1\right), \tag{2.9}$$

is the estimated $n\delta$-term rate at time u, using the definition of a simple rate. If the current time is t, then it is assumed that the parameters associated with the SDF are optimally estimated, with respect to $\mathbb{P}$, using the historical time series data for the $n\delta$-term reference rate: $\{R^n_u\,; u \in \{t_0, t_1, \ldots, t_k\}\} \subset \mathcal{G}_t$, where $\{t_0, t_1, \ldots, t_k\}$ denote the set of trading days that lie within the interval $[0, t]$. For further context, we refer to Appendix C for a specific example of a set of estimated $n\delta$-term SDFs.

Having this setup, the objective now is to model the aforementioned interest rate markets in a systematic manner. The following scenarios are analyzed for the non-interbank funding and interbank cash markets: (i) a single term rate with perfect liquidity; (ii) multiple term rates with perfect liquidity; and (iii) multiple term rates with illiquidity. Each of these are considered in the next three sections.

3. Single Term Rate with Perfect Liquidity

The first system that we consider is constituted by a non-interbank funding and interbank cash market with only a single reference term rate. This rate is assumed to be tradable, i.e. any of the entities within the systemic structure may deposit or borrow at this rate. Perfect liquidity also prevails, i.e. unlimited funding is available for all tenors via the use of this single rate. Without any loss of generality, the δ-term rate is chosen as this rate and is formally defined next.

Definition 3 (Perfectly liquid δ-term rate). The δ-term rate is defined by Axiom 2.2 (Axiom B.5) when $n = 1$, with the STR being the market-maker. Being tradable, R_u^1 is assumed to be $\mathcal{G}_u$-measurable, and it has the form

$$R_u^1 = x_u^1 + d_u^1, \tag{3.1}$$

since perfect liquidity implies that $\ell_u^1 = 0$.

Now, we assume that the SST's objective is to market-make interest rate derivatives based on the δ-term rate. With an estimated δ-term SDF, from Definition 2, the SST may now estimate a set of ZCB prices, for various tenors but using the δ-term rate only. This procedure is described next.

Definition 4 (Estimated δ-term ZCB prices). Assuming that the current time is t, then for $i, j \in \mathbb{N}_0$ with $i \leq j$, the expression

$$\widehat{P}^1_{t+i\delta,t+j\delta} := \frac{1}{\widehat{D}^1_{t+i\delta}} \mathbb{E}^{\mathbb{P}}[\widehat{D}^1_{t+j\delta} \,|\, \mathcal{G}_{t+i\delta}], \tag{3.2}$$

is the estimated price at time $t + i\delta$ for a δ-term ZCB, with unit nominal, that matures at time $t + j\delta$, i.e. a ZCB with $(j - i)\delta$-tenor that accrues interest via compounding $(j - i)$ fixed δ-term rates that are implied by the estimated δ-term SDF and information available at time $t+i\delta$. Shorthand notation: $\widehat{P}^1_{t,i,j} := \widehat{P}^1_{t+i\delta,t+j\delta}$ for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $t \in \mathbb{R}_{\geq 0}$.

The SST's estimated δ-term ZCB term structure at time t is then given by $\{\widehat{P}^1_{t,0,j}\,; j \in \mathbb{N}_0\}$. These ZCB prices are completely model-dependent and are therefore not tradable, in general. Rather they may be utilized by the SST in the market-making process for such products. These ZCBs are equivalent to synthetic term rates with $j\delta$-tenors, which accrue compounded interest at a δ-term frequency. Later it will be shown how such ZCBs may be structured with FRAs or, equivalently, IRSs. There is however one ZCB that is linked to the δ-term rate, and therefore tradable. This ZCB is introduced next.

Definition 5 (Tradable δ-term ZCB). Assuming that the current time is t then, from Definition 3, the tradable δ-term rate is R_t^1. The price at time t of a tradable

δ-term ZCB, with unit nominal and δ-tenor, is

$$P^1_{t,t+\delta} := \frac{1}{1+\delta R^1_t}. \tag{3.3}$$

Shorthand notation: $P^1_{t,i,1} := P^1_{t+i\delta,t+\delta}$ for each $i \in \{0,1\}$ and $t \in \mathbb{R}_{\geq 0}$.

The SST's estimated price $\widehat{P}^1_{t,0,1}$ for this ZCB will not be equal to $P^1_{t,0,1}$, in general. This discrepancy would expose the SST to potential arbitrage if their estimated model were used for pricing and valuation. Therefore, their estimated model must be adjusted to recover the price of the tradable ZCB. In the following lemma, we introduce the *δ-term systemic pricing measure*,[o] denoted here by $\mathbb{P}_1$, the *calibrated δ-term SDF*, and the *PK*.

Lemma 1 (Calibrated δ-term SDF and PK). *At time $t+\delta$, the $\mathcal{G}_t$-measurable SDF associated with the δ-term systemic pricing measure $\mathbb{P}_1$ is given by*

$$D^1_{t+\delta} := \frac{1}{\Lambda^1_t\,\widehat{D}^1_t}\mathbb{E}^{\mathbb{P}}[\Lambda^1_{t+\delta}\,\widehat{D}^1_{t+\delta}\,|\,\mathcal{G}_t] = \frac{1}{\widehat{D}^1_t}\mathbb{E}^{\mathbb{P}_1}[\widehat{D}^1_{t+\delta}\,|\,\mathcal{G}_t]. \tag{3.4}$$

The $\{(\mathcal{G}_u)_{t\leq u\leq t+\delta},\mathbb{P}\}$-density martingale $(\Lambda^1_u)_{t\leq u\leq t+\delta}$, with $\Lambda^1_t = 1$, induces a measure change $\mathbb{P}\to\mathbb{P}_1$ via the Radon–Nikodym derivative $\Lambda^1_u = (\mathrm{d}\mathbb{P}_1/\mathrm{d}\mathbb{P})|_{\mathcal{G}_u}$. Moreover, $D^1_{t+\delta} = P^1_{t,0,1}$. For $j\in\{0,1\}$, the $\mathcal{G}_{t+j\delta}$-measurable δ-term PK is thus given by

$$\pi^1_{t+j\delta} := \Lambda^1_{t+j\delta}D^1_{t+j\delta}, \tag{3.5}$$

where $D^1_t = 1$, and where the PK is calibrated to the tradable δ-term ZCB.

Proof. The estimated δ-term SDF $\{\widehat{D}^1_{t+j\delta}\,; j\in\{0,1\}\}$ is considered as an initial candidate for the calibrated δ-term SDF, which must be defined in discrete-time on the set $\{t, t+\delta\}$. Since $\widehat{D}^1_t\widehat{P}^1_{t,0,1} = \mathbb{E}^{\mathbb{P}}[\widehat{D}^1_{t+\delta}\,\big|\,\mathcal{G}_t]$ and $\widehat{P}^1_{t,0,1} \neq P^1_{t,0,1}$ in general, the estimated δ-term SDF and $\mathbb{P}$ are not viable candidates for the calibrated SDF and pricing measure, respectively. Constructing and calibrating the change-of-measure $\{(\mathcal{G}_u),\mathbb{P}\}$-martingale $(\Lambda^1_{t+v\delta})_{0\leq v\leq 1}$ such that Eq. (3.4) holds, with $D^1_{t+\delta} = P^1_{t,0,1}$, yields the correct calibrated SDF specification. The correct and calibrated SDF model is obtained by introducing the $\{(\mathcal{G}_u,\mathbb{P})\}$-measure-change martingale $(\Lambda^1_u)_{t\leq u\leq t+\delta}$, with $D^1_{t+\delta} = P^1_{t,0,1}$, which leads to Eq. (3.4). For $i\in\{0,1\}$, the δ-term PK specification, Eq. (3.5), follows trivially, from where it may be verified that $\frac{1}{\pi^1_{t+i\delta}}\mathbb{E}^{\mathbb{P}}[\pi^1_{t+\delta}\,\big|\,\mathcal{G}_{t+i\delta}] = \frac{1}{D^1_{t+i\delta}}\mathbb{E}^{\mathbb{P}_1}[D^1_{t+\delta}\,\big|\,\mathcal{G}_{t+i\delta}] = P^1_{t,i,1}$, which concludes the proof. □

[o] If the δ-term rate is a perfectly liquid ONRR, then d^1_u is approximately zero, R^1_u is risk-free for all practical purposes, and $\mathbb{P}_1$ is an approximation of the classical risk-neutral measure.

For a more practical perspective on Lemma 1, we refer to Appendix D for a specific example of a calibrated δ-term SDF.

Remark 7 (Single-period arbitrage-free model). The availability of only one tradable term rate, and its associated ZCB, enables the SST to construct a single-period arbitrage-free model only, over $[t, t+\delta]$, with volatility estimated statistically since no derivative market exists.

Remark 8 (Market price of systemic risk). Under the real-world measure $\mathbb{P}$ and with respect to the traded information filtration $(\mathcal{G}_u)_{u\geq t}$, the martingale $\{\Lambda^1_{t+j\delta}\,; j \in \{0,1\}\}$ adjusts the real-world estimated δ-term ZCB price to the arbitrage-free tradable price, and therefore encodes the market price of δ-term interest rate systemic risk over $[t, t+\delta]$.

4. Multiple Term Rates with Perfect Liquidity

In this section, a second perfectly liquid reference term rate,, the 2δ-term rate R^2_u, is introduced. Its definition is analogous to that of the δ-term counterpart. As indicated later in Remark 10, the extension to a $n\delta$-term rate system is produced by postulating R^n_u term rates as we do next for the 2δ-term rate R^2_u. Using the estimated 2δ-term SDF, $(\widehat{D}^2_u)_{u\geq 0}$, the SST may estimate the 2δ-term ZCB-system $\{\widehat{P}^2_{t+2i\delta,t+2j\delta}; i,j \in \mathbb{N}_0, i \leq j\}$ at the current time t by directly replicating all of the results from the previous section. In what follows, we adopt the shorthand notation $\widehat{P}^2_{t,2i,2j} := \widehat{P}^2_{t+2i\delta,t+2j\delta}$ for all $i,j \in \mathbb{N}_0$, where $i \leq j$ and $t \in \mathbb{R}_{\geq 0}$. From Lemma 1, we obtain the $\mathcal{G}_{t+2j\delta}$-measurable 2δ-term PK

$$\pi^2_{t+2j\delta} := \Lambda^2_{t+2j\delta} D^2_{t+2j\delta}, \tag{4.1}$$

where $j \in \{0,1\}$. The $\{(\mathcal{G}_u), \mathbb{P}\}$-martingale $(\Lambda^2_{t+2v\delta})_{0\leq v\leq 1}$, with $\Lambda^2_t := 1$, enables the change-of-measure from $\mathbb{P}$ to $\mathbb{P}_2$, the 2δ-term systemic pricing measure on $\mathcal{G}_{t+2\delta}$. This PK is calibrated to

$$P^2_{t,t+2\delta} := \frac{1}{1+2\delta R^2_t} = \frac{1}{\pi^2_t}\mathbb{E}^{\mathbb{P}}[\pi^2_{t+2\delta}\,|\,\mathcal{G}_t], \tag{4.2}$$

which is the price of the tradable 2δ-term ZCB at the current time t. We bear in mind the notation $P^2_{t,2i,2} := P^2_{t+2i\delta,t+2\delta}$ for each $i \in \{0,1\}$ and $t \in \mathbb{R}_{\geq 0}$.

In general, the estimated δ- and 2δ-term SDFs are not equal, almost surely. Even if they are specified as the same model, their statistical estimation relies upon the historical time series of two nonhomogeneous reference term rates. Each of these SDFs encodes different sources of floating interest rate risk. Accordingly, the estimated prices of δ- and 2δ-term ZCBs with tenor equal to 2δ will also not be equal in general, i.e.

$$\widehat{P}^2_{t,0,2} = \frac{1}{\widehat{D}^2_t}\mathbb{E}^{\mathbb{P}}[\widehat{D}^2_{t+2\delta}\,|\,\mathcal{F}_t] \neq \frac{1}{\widehat{D}^1_t}\mathbb{E}^{\mathbb{P}}[\widehat{D}^1_{t+2\delta}\,|\,\mathcal{F}_t] = \widehat{P}^1_{t,0,2}. \tag{4.3}$$

However, there are two relations between the tradable δ- and 2δ-term ZCBs that must hold to preclude arbitrage from the perspective of the SST. These relations, described in the lemmas below, allow the definition of the δ-term PK to be extended from $t+\delta$ to $t+2\delta$.

Lemma 2 (Early liquidation enforced by replacement). *At time $t+\delta$, the fair early liquidation value of the tradable 2δ-term ZCB, issued at time t, is equal to*

$$P^1_{t,1,2} := \frac{1}{1+\delta R^1_{t+\delta}} = \frac{1}{\pi^1_{t+\delta}}\mathbb{E}^{\mathbb{P}}[\pi^1_{t+2\delta}\,|\,\mathcal{G}_{t+\delta}] = \frac{1}{D^1_{t+\delta}}\mathbb{E}^{\mathbb{P}_1}[D^1_{t+2\delta}\,|\,\mathcal{G}_{t+\delta}], \tag{4.4}$$

which is the initial value of the tradable δ-term ZCB. Moreover, the calibrated δ-term SDF, defined in Lemma 1, may be specified at time $t+2\delta$ as

$$D^1_{t+2\delta} := \frac{D^1_{t+\delta}}{\Lambda^1_{t+\delta}\widehat{D}^1_{t+\delta}}\mathbb{E}^{\mathbb{P}}[\Lambda^1_{t+2\delta}\,\widehat{D}^1_{t+2\delta}\,|\,\mathcal{G}_{t+\delta}] = \frac{D^1_{t+\delta}}{\widehat{D}^1_{t+\delta}}\mathbb{E}^{\mathbb{P}_1}[\widehat{D}^1_{t+2\delta}\,|\,\mathcal{G}_{t+\delta}], \tag{4.5}$$

with the definition of the $\{(\mathcal{G}_u),\mathbb{P}\}$-martingale $(\Lambda^1_{t+v\delta})_{0\le v\le 2}$ extended to $t+2\delta$ with time-inhomogeneous parameters, such that $\Lambda^1_{t+2\delta}/\Lambda^1_{t+\delta} = \mathrm{d}\mathbb{P}_1/\mathrm{d}\mathbb{P}\,|_{\mathcal{G}_{t+2\delta}}$ and $D^1_{t+2\delta} = D^1_{t+\delta}P^1_{t,1,2}$.

See Appendix E.1 for the proof. The previous lemma enabled the specification and calibration of the δ-term PK up to time $t+2\delta$, from the vantage point of time $t+\delta$. The next result will provide information that will enable calibration up to the same time, but using information at the current time t.

Lemma 3 (Synthetic δ-term ZCB with 2δ-tenor). *At the current time t, the tradable δ- and 2δ-term ZCBs allow the SST to create a fair FRA, i.e. at zero cost, with payoff*

$$V_{t+2\delta} = \alpha N\left[(1+\delta R^1_{t+\delta}) - \frac{P^1_{t,0,1}}{P^2_{t,0,2}}\right], \tag{4.6}$$

at time $t+2\delta$, where α is equal to 1 (-1) for a long (short) position and N denotes the nominal amount. This in turn enables the creation of $P^1_{t,t+2\delta}$, a tradable synthetic δ-term ZCB at time t with maturity time equal to $t+2\delta$, with $P^1_{t,t+2\delta} = P^2_{t,0,2}$. Shorthand notation: $P^1_{t,i,2} := P^1_{t+i\delta,t+2\delta}$ for $i\in\{0,1,2\}$ and $t\in\mathbb{R}_{\ge 0}$.

See Appendix E.2 for the proof. With the result of Lemma 3 at hand, it is now possible to consider the calibration of the δ-term PK up to time $t+2\delta$, using market information that is available at time t.

Theorem 1 (Initial calibration of the δ-term PK to $t+2\delta$). *At the current time t, the tradable synthetic δ-term ZCB with 2δ-tenor is modeled by*

$$P^1_{t,0,2} := D^1_{t+\delta}\mathbb{E}^{\mathbb{P}_1}\left[\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda^1_{t+2\delta}\widehat{D}^1_{t+2\delta}}{\Lambda^1_{t+\delta}\widehat{D}^1_{t+\delta}}\,\middle|\,\mathcal{G}_{t+\delta}\right]\,\middle|\,\mathcal{G}_t\right] = D^1_{t+\delta}\,\mathbb{E}^{\mathbb{P}_1}\left[\frac{\widehat{D}^1_{t+2\delta}}{\widehat{D}^1_{t+\delta}}\,\middle|\,\mathcal{G}_t\right], \tag{4.7}$$

where the time-inhomogeneous $\{(\mathcal{G}_u),\mathbb{P}\}$-martingale $(\Lambda^1_u)_{t\le u\le t+2\delta}$ induces the change-of-measure from $\mathbb{P}\to\mathbb{P}_1$ on $(\mathcal{G}_u)_{t\le u\le t+2\delta}$, via the Radon–Nikodym derivative $\Lambda^1_{t+2\delta}/\Lambda^1_t = (\mathrm{d}\mathbb{P}_1/\mathrm{d}\mathbb{P})\big|_{\mathcal{G}_{t+2\delta}}$, and the parameters associated with $(\Lambda^1_u)_{t+\delta<u\le t+2\delta}$ is calibrated such that $\mathbb{E}^{\mathbb{P}_1}[\widehat{D}^1_{t+2\delta}/\,\widehat{D}^1_{t+\delta}\,|\,\mathcal{G}_t] = P^2_{t,0,2}/P^1_{t,0,1}$.

Proof. The definitions of the calibrated δ-term SDF and PK are specified for maturity time $t+2\delta$ with information at time $t+\delta$ in Lemma 2, viz. $\pi^1_{t+2\delta} := \Lambda^1_{t+2\delta}D^1_{t+2\delta}$. That structure is maintained here along with the observation that the no-arbitrage initial value of the tradable synthetic δ-term ZCB with tenor equal to 2δ must be

$$P^1_{t,0,2} := \frac{1}{\pi^1_t}\mathbb{E}^{\mathbb{P}}[\,\pi^1_{t+2\delta}\,|\,\mathcal{G}_t] = \frac{1}{D^1_t}\mathbb{E}^{\mathbb{P}_1}[D^1_{t+2\delta}\,|\,\mathcal{G}_t], \tag{4.8}$$

where $D^1_t := 1$ and $\Lambda^1_t := 1 \Rightarrow \pi^1_t = 1$. Substituting expression (4.5) into the right-hand side of the above equation yields

$$\mathbb{E}^{\mathbb{P}_1}\left[D^1_{t+2\delta}\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}_1}\left[\frac{D^1_{t+\delta}}{\widehat{D}^1_{t+\delta}}\,\mathbb{E}^{\mathbb{P}_1}\left[\widehat{D}^1_{t+2\delta}\,\middle|\,\mathcal{G}_{t+\delta}\right]\,\middle|\,\mathcal{G}_t\right] = D^1_{t+\delta}\,\mathbb{E}^{\mathbb{P}_1}\left[\frac{\widehat{D}^1_{t+2\delta}}{\widehat{D}^1_{t+\delta}}\,\middle|\,\mathcal{G}_t\right], \tag{4.9}$$

where the second equality follows by the tower property of conditional expectations, and since $D^1_{t+\delta}$, defined in Eq. (3.4), is $\mathcal{G}_t$-measurable. Now, working with the expression after the second equality, observe that

$$\mathbb{E}^{\mathbb{P}_1}\left[\frac{\widehat{D}^1_{t+2\delta}}{\widehat{D}^1_{t+\delta}}\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}_1}\left[\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda^1_{t+2\delta}\widehat{D}^1_{t+2\delta}}{\Lambda^1_{t+\delta}\widehat{D}^1_{t+\delta}}\,\middle|\,\mathcal{G}_{t+\delta}\right]\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}_1}\left[P^1_{t,1,2}\,\middle|\,\mathcal{G}_t\right], \tag{4.10}$$

i.e. the inner expectation computed with information up to $\mathcal{G}_{t+\delta}$ must equal $P^1_{t,1,2}$, such that

$$\mathbb{E}^{\mathbb{P}_1}[D^1_{t+2\delta}\,|\,\mathcal{G}_t] = D^1_{t+\delta}\mathbb{E}^{\mathbb{P}_1}[P^1_{t,1,2}\,|\,\mathcal{G}_t], \tag{4.11}$$

which follows after substituting Eq. (4.10) back into Eq. (4.9). Lemma 3 states that $P^1_{t,0,2}$ must equal $P^2_{t,0,2}$, and therefore it follows from Eqs. (4.8) and (4.11) that

$$\mathbb{E}^{\mathbb{P}_1}[P^1_{t,1,2}\,|\,\mathcal{G}_t] = \frac{P^2_{t,0,2}}{D^1_{t+\delta}} = \frac{P^2_{t,0,2}}{P^1_{t,0,1}}, \tag{4.12}$$

i.e. this enforces a numerical value for the expectation from Eq. (4.10). Since the time-dependent parameter associated with $(\Lambda^1_u)_{t+\delta<u\leq t+2\delta}$ is free to specify at time t, it is possible to calibrate this quantity such that Eq. (4.11) holds, which concludes the proof. □

Remark 9 (Term-dependent market price of systemic risk). Since the estimated δ-term SDF is used to model the tradable 2δ-term ZCB, it is conjectured that a time-inhomogeneous market price of risk structure is necessary. There are two notions of time in this framework: (i) universal calendar time defined by the variable u and a current time denoted by t; and (ii) term and tenor times determined by natural number multiples of δ. Therefore, the time-inhomogeneous process $\{\Lambda^1_{t+j\delta}\,;j\in\{0,1,2\}\}$ actually implies that the framework advocates for term-dependent parameters, or term-dependent market prices of systemic risk.

Remark 10 (Multiple term rates). Theorem 1 can be iteratively repeated out to $t+j\delta$, for $j\in\{3,4,5,\ldots,m\}$ if the corresponding reference term rates, R^j_t, exist and are perfectly liquid. The calibrated δ-term SDF may then be extended as

$$D^1_{t+j\delta} := \frac{D^1_{t+(j-1)\delta}}{\Lambda^1_{t+(j-1)\delta}\widehat{D}^1_{t+(j-1)\delta}}\mathbb{E}^{\mathbb{P}}[\Lambda^1_{t+j\delta}\,\widehat{D}^1_{t+j\delta}\,|\,\mathcal{G}_{t+(j-1)\delta}]$$

$$= \frac{D^1_{t+(j-1)\delta}}{\widehat{D}^1_{t+(j-1)\delta}}\mathbb{E}^{\mathbb{P}_1}[\widehat{D}^1_{t+j\delta}\,|\,\mathcal{G}_{t+(j-1)\delta}].$$

If one or a subset of these reference rates are not quoted, the result of the theorem still applies but there will now be a range of viable values for the missing rates for arbitrage-free calibration. For a specific example of a calibrated δ-term SDF, we refer to Appendix D.

Remark 11 (Term-specific SDFs but a unique term structure). Each term has a distinct estimated SDF, which encodes floating interest rate risk, along with a unique single-period PK. The ability to replicate all tradable ZCBs via a system of FRAs leads to a single-curve interest rate term structure, and enables multi-period calibration for all PKs. Since the δ-term bears the least credit risk relative to all other terms, its pricing measure is the closest to the classical risk-neutral measure.

Theorem 1 relied on the SST's ability to create a FRA. The resultant δ-term PK therefore encodes the arbitrage-free mechanics to price and value such a product. Pricing this FRA, which is formalized in the next corollary, will reveal the fair FRA rate to be the simple forward rate that is constructed from the δ- and 2δ-term rates, which is defined next.

Definition 6 ($\delta\times 2\delta$ forward rate). At $u\in\mathbb{R}_{\geq 0}$, the $\delta\times 2\delta$ forward rate is a simple rate, denoted by $F(u;u+\delta,u+2\delta)$, at which one may deposit (borrow) money over the future δ-term $[u+\delta, u+2\delta]$. The net capital plus interest yield (cost) at time $u+2\delta$ is

$$1 + \delta F(u; u+\delta, u+2\delta) := (1 + 2\delta R_u^2)\big/(1 + \delta R_u^1), \tag{4.13}$$

with $F(u; u+\delta, u+2\delta)$ being $\mathcal{G}_u$-measurable. Shorthand notation: In general the $j\delta \times (j+1)\delta$ forward rate will be denoted as

$$F_{u,i,j}^1 := F(u + i\delta; u + j\delta, u + (j+1)\delta), \tag{4.14}$$

for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Corollary 1 ($\delta \times 2\delta$ FRA pricing). *The fair strike rate process for the general version of the $\delta \times 2\delta$ FRA defined in Lemma 3 is given by the $\mathcal{G}_{t+i\delta}$-measurable process*

$$F_{t,i,1}^1 = \frac{1}{\delta}\left(\frac{P_{t,i,1}^1}{P_{t,i,2}^1} - 1\right), \tag{4.15}$$

for $i \in \{0, 1\}$, with $F_{t,1,1}^1 = R_{t+\delta}^1$, the δ-term rate at time $t+\delta$.

See Appendix E.3 for the proof.

Remark 12 (Multi-period arbitrage-free models). The availability of multiple tradable term rates and ZCBs enables the SST to construct a multi-period arbitrage-free model, over $[t, t+m\delta]$ for $m \geq 2$. Equation (4.15) emphasizes the fact that volatility is still statistically estimated, even with the existence of FRAs. Within this context, the SST may use the multi-period δ-term PK for general pricing and valuation, however to use the PK to risk manage derivatives written on rates other than the δ-term rate would be inconsistent.

5. Multiple Term Rates with Illiquidity

The multiple term rate system of the previous section is again considered here, except that the assumption of perfect liquidity is revoked. The definition of the respective term rates revert back to the general form in Axiom 2.2 (Axiom B.5). From the perspective of the SST, the added illiquidity features proposed by Axiom 2.1 has to be modeled into the framework.

In Appendix F, one such model for a general $n\delta$-term quote rate is provided, in Definition F.1, that incorporates term, nominal size, asymmetric liquidity spreads (due to loans and deposits), as well as specific SST and systemic illiquidity. Through fair valuation at the systemic level and suitable aggregation, Proposition F.1 and Corollary F.1 provide the necessary justification for a simpler symmetric model specification based on a *systemic liquidity indicator*. This construction is formalized in the next definition, followed by the definition of a potentially illiquid $n\delta$-term rate, from the perspective of the SST. A more general three-state costly systemic liquidity indicator is also presented in Appendix F, in Definition F.2, along with comparable results, in Lemma F.1 that are derived later in this section. In order to ease the exposition, the simpler two-state specification is considered here. All of the results derived here still hold within the more general setting, modulo minor adjustments and assumptions.

Definition 7 (Systemic liquidity indicators). At time $u \in \mathbb{R}_{\geq 0}$, the binary random variable L_u^n assumes a value of 1 if perfect systemic liquidity exists for the $n\delta$-term rate, or 0 otherwise. If the current time is t, then the natural filtration associated with liquidity is

$$\mathcal{L}_t := \sigma(\{\{L_u^1, L_u^2, \ldots, L_u^m\}\,; u \in \{t_0, t_1, \ldots, t_k\}\}), \tag{5.1}$$

where $\{t_0, t_1, \ldots, t_k\}$ denotes the set of trading days that lie within the interval $[0, t]$. The systemic liquidity indicators are assumed to exhibit both serial and cross-sectional independence, or more formally

$$\mathbb{E}^{\mathbb{P}}[L_u^n \,|\, \mathcal{L}_t \cap \sigma(\{L_u^n \notin \{0, 1\}\})] = \mathbb{E}^{\mathbb{P}}[L_u^n] = \mathbb{P}[L_u^n = 1] := q_u^n, \tag{5.2}$$

for all $t \leq u$, with $q_u^n := q(u, u + n\delta)$ being a deterministic function for the probability of perfect systemic $n\delta$-term liquidity at time u.

Definition 8 (Potentially illiquid $n\delta$-term rate). At some arbitrary time $u < t$, the $n\delta$-term rate

$$\widetilde{R}_t^n := R_t^n L_t^n, \tag{5.3}$$

is potentially illiquid from the vantage point of u if $L_t^n = 0$, i.e. it will not be possible for the SST to borrow from (or deposit with) the STR, at time t, for a tenor equal to $n\delta$.

These liquidity indicators enable the definition of various liquidity regimes, which in turn enables the definition of a set of term- and liquidity-dependent PKs (LDPKs), all from the perspective of the SST. First, the various regimes of liquidity are defined.

Definition 9 (Liquidity regimes). Let $i, j \in \mathbb{N}_0$, with $i < j$, and define the counting sets

$$\begin{aligned} \mathbb{N}_{i,j} &:= \{i, i+1, \ldots, j-1, j\}, \quad \text{and} \\ \mathbb{N}_{i,j}^n &:= \{i, i+n, \ldots, i+(k-1)n, i+kn\}, \end{aligned} \tag{5.4}$$

where $k := \lfloor (j-i)/n \rfloor - 1$, $\mathbb{N}_{i,i} := \emptyset$ and $\mathbb{N}_{i,i}^n := \emptyset$. At time u, the following liquidity regimes are possible over the interval $[u, u + m\delta]$:

(i) **NPFL** — No present nor future liquidity exists on the set

$$\mathcal{L}^{\mathrm{NPFL}}_{u,u+m\delta} := \sigma(\{L^n_{u+i\delta} = 0\,; n \in \mathbb{N}_{1,m}, i \in \mathbb{N}^n_{0,m}\}). \tag{5.5}$$

(ii) **NPL** — No present liquidity only exists on the set

$$\mathcal{L}^{\mathrm{NPL}}_{u,u+m\delta} := \sigma(\{L^n_u = 0\,; n \in \mathbb{N}_{1,m}\}). \tag{5.6}$$

(iii) **PPL** — Only partial present liquidity exists on the set

$$\mathcal{L}^{\mathrm{PPL}}_{u,u+m\delta} := \sigma(\{L^n_u = 1\,; n \subset \mathbb{N}_{1,m}\}). \tag{5.7}$$

(iv) **CPL** — Complete present liquidity only exists on the set

$$\mathcal{L}^{\mathrm{CPL}}_{u,u+m\delta} := \sigma(\{L^n_u = 1\,; n \in \mathbb{N}_{1,m}\}). \tag{5.8}$$

(v) **CPFL** — Complete present and future liquidity exists on the set

$$\begin{aligned}\mathcal{L}^{\mathrm{CPFL}}_{u,u+m\delta} &:= \mathcal{L}^1_{u,u+m\delta} \vee \mathcal{L}^2_{u,u+m\delta} \vee \cdots \vee \mathcal{L}^m_{u,u+m\delta} \\ &= \mathcal{L}^{\mathrm{CPL}}_{u,u+m\delta} \vee \mathcal{L}^1_{u+\delta,u+m\delta} \vee \mathcal{L}^2_{u+2\delta,u+m\delta} \vee \cdots \vee \mathcal{L}^m_{u+m\delta,u+m\delta},\end{aligned} \tag{5.9}$$

where $\mathcal{L}^n_{u+i\delta,u+m\delta} := \sigma\big(\{L^n_{u+j\delta} = 1\,; j \in \mathbb{N}^n_{i,m}\}\big)$ models liquidity in the $n\delta$-term rate over $[u+i\delta, u+m\delta]$ if $(m-i) \bmod n = 0$, or $[u+i\delta, u+m\delta)$ otherwise, for $i \in \mathbb{N}_{0,m}$.

Regimes (i) and (ii) are complements of (iv) and (v), respectively. Regimes (ii), (iii) and (iv) pose uncertain future liquidity, with (iii) also posing uncertain present liquidity for some terms. Shorthand notation: $\mathcal{L}^{\mathrm{X}}_{u,i,j} := \mathcal{L}^{\mathrm{X}}_{u+i\delta,u+j\delta}$ for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Remark 13 (Implications of the CPFL regime). Under the CPFL regime all of the results from Secs. 3 and 4 may be recovered, i.e. multi-period arbitrage-free term-dependent PKs with associated term-dependent systemic pricing measures.

In Appendix G, the definitions of an $n\delta$-term LDPK is provided over a single period $[t, t+n\delta]$, and then over multiple periods $[t, t+i\delta]$, for $i \in \mathbb{N}^n_{0,m+n}$. Both Definitions G.1 and G.2 clearly reveal that the regime of liquidity has a significant impact on the form of the PK associated with each tradable term. The impact of present liquidity or illiquidity is fundamental, with the latter requiring the subjective process of market-making. Modeling the term rate market-making process of the STR is not an objective of this research, therefore the prevalence of the CPL regime will be a minimal assumption in all that follows. Then, from a practical perspective, in order to deal with potential future illiquidity, presently available liquidity must be fully exploited. This is achieved through the definition of the following *hybrid-term LDPK*.

Definition 10 (Hybrid-term LDPK). At the current time t, the PK defined by

$$\widetilde{\pi}_{t+i\delta} = \begin{cases} \pi^1_{t+i\delta}, & \text{conditional on } \mathcal{L}^{\text{CPL}}_{t,0,i} \vee \mathcal{L}^1_{t,1,i}, \text{ for } i \in \mathbb{N}^1_{1,m+1}, \\ \pi^2_{t+i\delta}, & \text{conditional on } \mathcal{L}^{\text{CPL}}_{t,0,i} \vee \mathcal{L}^2_{t,2,i}, \text{ for } i \in \mathbb{N}^2_{2,m+2}, \\ \quad\vdots\ , & \qquad\vdots \qquad\qquad\qquad \vdots\ , \qquad \vdots \qquad\quad , \\ \pi^m_{t+i\delta}, & \text{conditional on } \mathcal{L}^{\text{CPL}}_{t,0,i} \vee \mathcal{L}^m_{t,m,i}, \text{ for } i \in \mathbb{N}^m_{m,2m}, \end{cases} \tag{5.10}$$

with $\widetilde{\pi}_t := 1$, is called the hybrid-term LDPK.

The hybrid-term LDPK is only well defined at time t, and may be used to present value future cash flows due at time $t + i\delta$ back to time t only, for $i \in \mathbb{N}_{1,m}$. For the intertemporal valuation of the same cash flows back to a future time $t + h\delta$, for $h \in \mathbb{N}_{1,i-1}$, but from the vantage point of time t, one would require the prevalence of the CPFL regime at time t in order to have a well-defined PK over $[t + h\delta, t + i\delta]$. However, invoking the CPFL regime would recover the setup of the previous section, the PK would coincide with one of the term-dependent PKs defined in that section, and there'd be no need for the hybrid-term LDPK.

With the hybrid-term LDPK it is now possible to consider again the pricing (or market-making) of a $\delta \times 2\delta$ FRA, from the perspective of the SST, under potential future illiquidity. Assuming that the CPL regime prevails at the time of pricing, the SST will have to subjectively specify the probability of future liquidity, articulated here via the liquidity indicators.

Lemma 4 ($\delta \times 2\delta$ FRA pricing under potential future illiquidity). *The fair strike rate process for the general version of the $\delta \times 2\delta$ FRA defined in Lemma 3 is*

$$\overline{F}(t + i\delta; t + \delta, t + 2\delta) = \begin{cases} q^1_{t+\delta} F^1_{t,0,1}, & i = 0 \quad \text{and conditional on } \ \mathcal{L}^{\text{CPL}}_{t,0,2}, \\ F^1_{t,1,1}, & i = 1 \ \text{ and conditional on } \ \mathcal{L}^{\text{CPL}}_{t,1,2}, \end{cases} \tag{5.11}$$

which is also $\mathcal{G}_{t+i\delta}$-measurable. Shorthand notation: In general, the $j\delta \times (j + 1)\delta$ FRA rate is denoted by

$$\overline{F}^1_{u,i,j} := \overline{F}(u + i\delta; u + j\delta, u + (j + 1)\delta), \tag{5.12}$$

for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Proof. Assuming that $\mathcal{L}_t = \mathcal{L}_{t-} \vee \mathcal{L}^{\text{CPL}}_{t,0,2}$, the standard FRA replication strategy yields

$$\begin{aligned} \widetilde{V}_{t+2\delta} &= \alpha N[(1 + \delta \widetilde{R}^1_{t+\delta}) - (1 + \delta F^1_{t,0,1})] \\ &= V_{t+2\delta} - \alpha N \delta (1 - L^1_{t+\delta}) R^1_{t+\delta}, \end{aligned} \tag{5.13}$$

where $F^1_{t,0,1}$ is defined in Corollary 1 and $V_{t+2\delta}$ is the payoff of a fair $\delta \times 2\delta$ FRA that is not exposed to liquidity risk. Let $\mathcal{M}_t := \mathcal{G}_t \vee \mathcal{L}_t$, then using the hybrid-term LDPK from Definition 10, the current value of the above payoff is

$$\begin{aligned}
\widetilde{\pi}_t \widetilde{V}_t &= \mathbb{E}^{\mathbb{P}}[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,|\, \mathcal{M}_t] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,|\, \mathcal{M}_t\,, L^1_{t+\delta}] \,|\, \mathcal{M}_t] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,|\, \mathcal{M}_t\,, L^1_{t+\delta} = 1]\mathbb{P}(L^1_{t+\delta} = 1) \,|\, \mathcal{M}_t] \\
&\quad + \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,|\, \mathcal{M}_t\,, L^1_{t+\delta} = 0]\mathbb{P}(L^1_{t+\delta} = 0) \,|\, \mathcal{M}_t]\,, \qquad (5.14)
\end{aligned}$$

which follows by the tower property of conditional expectations. Since $\widetilde{\pi}_t := 1$ and observing that $\mathcal{L}^1_{t,0,2} = \sigma(\{L^1_{t+\delta} = 1\})$ and $\mathcal{L}^2_{t,0,2} = \emptyset$, it follows that

$$\begin{aligned}
\widetilde{V}_t &= \mathbb{E}^{\mathbb{P}}[\pi^1_{t+2\delta}V_{t+2\delta} \,|\, \mathcal{M}_t]q^1_{t+\delta} - \mathbb{E}^{\mathbb{P}}[\pi^2_{t+2\delta}\alpha N\delta F^1_{t,0,1} \,|\, \mathcal{M}_t](1 - q^1_{t+\delta}) \\
&= q^1_{t+\delta}V_t - \alpha N(1 - q^1_{t+\delta})\delta F^1_{t,0,1}P^2_{t,0,2}, \qquad (5.15)
\end{aligned}$$

using the definition of the hybrid-term LDPK. V_t is the fair value of the FRA under perfect liquidity, and therefore equal to 0. Trading this FRA with the strike rate equal to the fair FRA rate defined in the perfect liquidity setting therefore leads to an initial loss (gain) if the market-maker is long (short). The assumption here is that $V_{t+2\delta}$ will still be the FRA payoff even when $L^1_{t+\delta} = 0$ and the strong case of no systemic liquidity is in effect. In reality, there will still be a reference δ-term rate that is contractually specified for such a case by the relevant FRA ISDA documentation. Setting the FRA strike rate to an arbitrary value, $\overline{F}^1_{t,0,1}$, and pricing via the same process gives

$$\widetilde{V}_t = \alpha q^1_{t+\delta}N[P^1_{t,0,1} - (1 + \delta\overline{F}^1_{t,0,1})P^1_{t,0,2}] - \alpha N(1 - q^1_{t+\delta})\delta\overline{F}^1_{t,0,1}P^2_{t,0,2}, \quad (5.16)$$

while setting $\widetilde{V}_t = 0$, recalling that $P^2_{t,0,2} = P^1_{t,0,2}$, and solving for the fair FRA strike rate yields $\overline{F}^1_{t,0,1} = q^1_{t+\delta}F^1_{t,0,1}$, as required. Repeating this pricing process at time $t + \delta$, for exactly the same contract and assuming that the CPL liquidity regime, $\mathcal{L}^{\text{CPL}}_{t,1,2}$, prevails at this time, it is trivial to show that $\overline{F}^1_{t,1,1} = F^1_{t,1,1}$, which completes the proof. □

Contingent on present liquidity, the pricing of a FRA still requires a subjective view on future liquidity. Therefore, the SST must be afforded some level of *risk appetite* in order to market-make such derivatives. This is in stark contrast with the perfect liquidity setting where the SST could replicate FRAs perfectly, and thereby required (nor deserved) any risk appetite. In general then, the SST will have the capacity for exposure to residual risk. This, combined with the *zero net* *supply*[p] and *unfunded*[q] nature of linear derivatives, such as FRAs, allows the SST substantial flexibility in their market-making process.

Remark 14 (FRA liquidity is not completely contingent on the CPL regime). Even if the NPL regime were to prevail, the SST's capacity to carry residual risk and potentially hedge in the future, through offsetting positions, will still enable the pricing of FRAs. Practically, this decouples the theoretical contingency of the SST on the STR, or equivalently, interest rate derivatives on the set of term rates. However, Lemma 4 reveals structure to the decoupling with forward rates required to dominate corresponding FRA rates.

Lemma 3 enforced the early liquidation value of a 2δ-term rate (or ZCB) by replacement. This result was used in conjunction with the replication of a $\delta \times 2\delta$ FRA to create a synthetic δ-term ZCB with 2δ-tenor. An analogous result is possible here, however it is contingent on $\delta \times 2\delta$ FRA and δ-term rate liquidity. Therefore, based on the discussion leading up to and including Remark 14, it is now assumed that a FRA market has been established within the interbank derivatives market. While the individual STs that constitute the SST would be responsible for the establishment of the FRA market through active market-making via model creation; here the SST is considered to be a separate entity that is observing this market at a systemic level and considering the problem of passive market-making via model calibration. Only the stylized problem from Lemma 3 and Theorem 1 is considered again here — the general version of this problem is considered in the next section.

Assumption 1 ($\delta \times 2\delta$ FRA market-making). The individual STs have sufficient risk appetite to market-make and enable liquidity of the $\delta \times 2\delta$ FRA at time t. At this time, the fair or mid-market FRA rate, denoted here by $\widehat{F}^1_{t,0,1}$, is used by the SST together with the result from Lemma 4 for the purpose of calibration.

In particular, setting $\overline{F}^1_{t,0,1} = \widehat{F}^1_{t,0,1}$ and assuming that $\mathcal{L}_t = \mathcal{L}_{t^-} \vee \mathcal{L}^{\text{CPL}}_{t,0,2}$ enables the SST to compute

$$q^1_{t+\delta} = \frac{\overline{F}^1_{t,0,1}}{F^1_{t,0,1}}, \tag{5.17}$$

using Eq. (5.11), which is now the market-implied probability of perfect δ-term liquidity at time $t+\delta$ using information available at the current time t.

Definition 11 (Systemic $\delta \times 2\delta$ FRA liquidity indicator). At time $t + i\delta$, for each $i \in \{0, 1\}$, the binary random variable $\overline{L}^1_{t,i,1}$ assumes a value of 1 if perfect

[p] A derivative transaction only exists once there is a willing buyer and seller — the market-maker may be either.

[q] In general, linear financial derivatives are exchange-traded and margined or over-the-counter and collateralized, subject to a zero-threshold credit support annex, and require no initial capital outlay.

systemic liquidity exists for the $\delta \times 2\delta$ FRA, or 0 otherwise. When $i = 0$, perfect systemic liquidity means that Assumption 1 holds, and it is assumed that

$$\sigma(\{\overline{L}^1_{t,0,1} = 1\}) \supset \mathcal{L}^{\mathrm{CPL}}_{t,0,2}. \tag{5.18}$$

When $i = 1$, perfect systemic liquidity is equivalent to $L^1_{t+\delta} = 1$, or

$$\sigma(\{\overline{L}^1_{t,1,1} = 1\}) = \mathcal{L}^1_{t,1,2}. \tag{5.19}$$

If the current time is t, then the natural filtration associated with liquidity is now

$$\overline{\mathcal{L}}_t := \mathcal{L}_t \vee \sigma(\{\overline{L}^1_{u,0,1}\,;\, u \in \{t_0, t_1, \ldots, t_k\}\}), \tag{5.20}$$

where $\{t_0, t_1, \ldots, t_k\}$ denotes the set of trading days that lie within the interval $[0, t]$ and $\mathcal{L}_t$ is defined in Definition 7, Eq. (5.1). Since the systemic FRA liquidity indicators will only be used to indicate regimes of liquidity and will not be used for pricing, the probabilistic structure of these are left unspecified.

Using the $\delta \times 2\delta$ FRA along with the δ-term rate, it is now possible to formulate the analog to Lemma 3 within this setting of potential illiquidity. The synthetic δ-term ZCB that is constructed here is referred to as a liquidity-contingent ZCB (LCZCB), since its definition relies on the availability of liquidity in the aforementioned instruments.

Lemma 5 (Synthetic δ-term LCZCB with 2δ-tenor). *Assuming that $\overline{L}^1_{t,0,1} = 1$, and setting $\overline{F}^1_{t,0,1} = \widehat{F}^1_{t,0,1}$, it is possible to replicate the following ZCB*:

$$\overline{P}^1_{t,i,2} := \begin{cases} D^1_{t+\delta}\big/(1 + \delta \overline{F}^1_{t,0,1}), & i = 0, \\ P^1_{t,1,2}, & i = 1, \\ 1, & i = 2, \end{cases} \tag{5.21}$$

provided that $L^1_t = L^1_{t+\delta} = 1$, or equivalently that $\mathcal{L}^1_{t,0,2}$ holds.

Proof. Assuming that $L^1_t = L^1_{t+\delta} = 1$, it is possible to borrow (deposit) M units of currency at the δ-term rate at time t and refinance (redeposit) the total cost (proceeds) thereof at time $t + \delta$, such that the cumulative cost (yield) is $M\big/D^1_{t+2\delta}$ at time $t + 2\delta$. Combining this loan (deposit) with a long (short) position in a fair $\delta \times 2\delta$ market FRA with strike rate $\overline{F}_{t,0,1}$ and $N = M\big/D^1_{t+\delta}$ will enable the conversion of the floating cost (yield) to a fixed cost (yield) equal to $M(1 + \delta\overline{F}^1_{t,0,1})\big/D^1_{t+\delta}$ at time $t + 2\delta$. Setting $M = D^1_{t+\delta}\big/(1 + \delta\overline{F}^1_{t,0,1})$, enables the creation of the synthetic δ-term LCZCB, with 2δ-tenor, given by Eq. (5.21). Since it is assumed that $L^1_{t+\delta} = 1$, it is clear that $\overline{P}^1_{t,1,2} = P^1_{t,1,2}$, as required. ◻

Lemma 4 and 5 provides the basis for the construction of a δ-term LCZCB-system, one that is created by exchanging δ-term floating-for-fixed interest rate risk. It is possible to model this system via the definition of a liquidity-contingent PK (LCPK). The δ-term LCPK is defined over the interval $[t, t+2\delta]$ in the next theorem.

Theorem 2 (δ-term LCPK). *Contingent on* $\overline{L}^1_{t,0,1} = L^1_t = L^1_{t+\delta} = 1$, *or equivalently*

$$\overline{\mathcal{L}}_t := \mathcal{L}_{t-} \vee \sigma(\{\overline{L}^1_{t,0,1} = 1\}) \vee \mathcal{L}^1_{t,0,2}, \tag{5.22}$$

using the result from Lemma 5 *and recalling that* $\mathcal{F}_t := \mathcal{G}_t \vee \overline{\mathcal{L}}_t$, *the* δ*-term LCPK may be defined as*

$$\overline{\pi}^1_{t+j\delta} := \pi^1_{t+j\delta}\Theta^1_{t+j\delta}, \tag{5.23}$$

for $j \in \{0,1,2\}$, *where the time-inhomogeneous* $\{(\mathcal{G}_u), \mathbb{P}_1\}$*-martingale* $(\Theta^1_{t+v\delta})_{0\le v\le 2}$, *with* $\Theta^1_t := 1$, *enables the change-of-measure from* $\mathbb{P}_1$ *to* $\mathbb{Q}_1$ *on* $\mathcal{G}_{t+j\delta}$, *i.e.* $\frac{\Theta^1_{t+j\delta}}{\Theta^1_t} = \frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}_1}\big|_{\mathcal{G}_{t+j\delta}}$, *such that the* $\mathbb{E}^{\mathbb{P}}[\overline{\pi}^1_{t+\delta} \,|\, \mathcal{F}_t] = P^1_{t,0,1}$ *and the* $\mathbb{E}^{\mathbb{P}}[\overline{\pi}^1_{t+2\delta} \,|\, \mathcal{F}_t] = \overline{P}^1_{t,0,2}$.

Proof. Since $\mathcal{L}^{\mathrm{CPL}}_{t,0,2} \subset \sigma(\{\overline{L}^1_{t,0,1} = 1\})$, by Definition 11, it follows that $\mathcal{L}^{\mathrm{CPFL}}_{t,0,2} \subset \overline{\mathcal{L}}_t$ and that the δ-term PK is well defined over $[t, t+2\delta]$. Therefore, $\{\pi^1_{t+j\delta}, j \in \{0,1,2\}\}$ is a good initial candidate for the LCPK, however it does not recover the initial price of the synthetic δ-term LCZCB with 2δ-tenor. The definition of the $\{(\mathcal{G}_u), \mathbb{P}_1\}$-martingale $(\Theta^1_{t+v\delta})_{0\le v\le 2}$ enables a change-of-measure such that

$$\mathbb{E}^{\mathbb{P}}[\Lambda^1_{t+2\delta}\Theta^1_{t+2\delta}D^1_{t+2\delta} \,|\, \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}_1}[\Theta^1_{t+2\delta}D^1_{t+2\delta} \,|\, \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}_1}[D^1_{t+2\delta} \,|\, \mathcal{F}_t] := \overline{P}^1_{t,0,2}, \tag{5.24}$$

as required, recalling that $\Lambda^1_t = \Theta^1_t = 1$, i.e. the free time-dependent parameters associated with $(\Theta^1_{t+v\delta})_{1<v\le 2}$ is free to specify at time t such that the expectation equals $\overline{P}^1_{t,0,2}$. Also, at the future time $t+\delta$, since $\overline{\mathcal{L}}_{t+\delta} \supset \overline{\mathcal{L}}_t \supset \mathcal{L}^1_{t,0,2}$ it follows $L^1_{t+\delta} = 1$ and recalling $D^1_{t+2\delta}$ is $\mathcal{G}_{t+\delta}$-measurable, then

$$\begin{aligned}\frac{1}{\overline{\pi}^1_{t+\delta}}\mathbb{E}^{\mathbb{P}}[\Lambda^1_{t+2\delta}\Theta^1_{t+2\delta}D^1_{t+2\delta} \,|\, \mathcal{F}_{t+\delta}] &= \frac{D^1_{t+2\delta}}{D^1_{t+\delta}\Theta^1_{t+\delta}}\mathbb{E}^{\mathbb{P}_1}[\Theta^1_{t+2\delta} \,|\, \mathcal{F}_{t+\delta}] \\ &= \frac{D^1_{t+2\delta}}{D^1_{t+\delta}} = P^1_{t,1,2},\end{aligned} \tag{5.25}$$

which shows that the value of the synthetic δ-term LCZCB, given by Eq. (5.21), is recovered by the δ-term LCPK. Since $D^1_{t+\delta} = P^1_{t,0,1}$ is $\mathcal{G}_t$-measurable, it follows straightforwardly that

$$\mathbb{E}^{\mathbb{P}}[\Lambda^1_{t+\delta}\Theta^1_{t+\delta}D^1_{t+\delta}\,|\,\mathcal{F}_t] = P^1_{t,0,1}\mathbb{E}^{\mathbb{P}_1}[\Theta^1_{t+\delta}\,|\,\mathcal{F}_t] = P^1_{t,0,1}, \tag{5.26}$$

which completes the proof, showing that: (i) the free time-dependent parameters associated with $(\Theta^1_{t+v\delta})_{0<v\leq 1}$ may be specified freely at time t; and (ii) the δ-term LCPK is calibrated to the δ-term rate and the synthetic δ-term LCZCB with 2δ-tenor. □

It may not be apparent but the definitions of the synthetic δ-term LCZCB and its associated LCPK, from Lemma 5 and Theorem 2, had two steps and associated contingencies:

(i) the interval $[t, t+2\delta]$, or more specifically the set $\{t, t+\delta, t+2\delta\}$, requires that $\mathcal{L}^1_{t,0,2}$ holds, or equivalently that $L^1_t = L^1_{t+\delta} = 1$, and that $\overline{L}^1_{t,0,1} = 1$; and
(ii) the future interval $[t+\delta, t+2\delta]$, or more specifically the set $\{t+\delta, t+2\delta\}$, requires that $\mathcal{L}^1_{t,1,2}$ holds, or equivalently that $L^1_{t+\delta} = 1$.

In general, to extend these definitions over the interval $[t, t+m\delta]$, would require m steps:

(i) at the current time t and over the set $\{t, t+\delta, \ldots, t+m\delta\}$, one would require that $\mathcal{L}^1_{t,0,m}$ holds and that $\overline{L}^1_{t,0,j} = 1$ for $j \in \mathbb{N}_{1,m-1}$;
(ii) at each future time $t+i\delta$ and over the set $\{t+i\delta, t+(i+1)\delta, \ldots, t+m\delta\}$, for $i \in \mathbb{N}_{1,m-2}$, one would require $\mathcal{L}^1_{t,i,m}$ and that $\overline{L}^1_{t,i,j} = 1$ for $j \in \mathbb{N}_{i+1,m-1}$; and
(iii) at the future time $t+(m-1)\delta$ and over the set $\{t+(m-1)\delta, t+m\delta\}$, one would require that $\mathcal{L}^1_{t,m-1,m}$ holds.

This will form the basis for the reduced-form modeling approach that is developed in the next section. This section is concluded with a few remarks that aim to assist the reader to build intuition in relation to all of the theory that has been presented thus far.

Remark 15 (LCZCBs are not tradable). The δ-term ZCB-system that was introduced in Sec. 4, assuming perfect liquidity, viz.

$$\{P^1_{t,i,j}\,;\, i \in \mathbb{N}_{0,j}\,,\, j \in \mathbb{N}_{0,m}\}, \tag{5.27}$$

denotes a set of tradable ZCBs whose tenors span the interval $[t, t+m\delta]$. Moreover, recall that under perfect liquidity all term ZCBs are replicated via the δ-term system, i.e. $P^1_{t,i,j} = P^{j-i}_{t,i,j}$. Under potential illiquidity, the definition and tradability of the above set of ZCBs requires $\mathcal{L}^{\mathrm{CPFL}}_{t,0,m}$ to hold. When $\mathcal{L}^{\mathrm{CPL}}_{t,0,m}$ holds, this set reduces to

$$\{P^n_{t,0,n}\,;\, n \in \mathbb{N}_{0,m}\}, \tag{5.28}$$

i.e. the set of ZCBs derived from the systemic term funding curve, defined in Definition 1. Then, the δ-term LCZCB-system, that was introduced in this section, is

$$\{\overline{P}^{1}_{t,i,j}\,;i\in\mathbb{N}_{0,j}\,,j\in\mathbb{N}_{0,m}\},\tag{5.29}$$

and is contingent upon $\overline{L}^{1}_{t,i,j}=1$ for all $j\in\mathbb{N}_{i+1,m-1}$ and $\mathcal{L}^{1}_{t,i,m}$ holding for each $i\in\mathbb{N}_{0,m}$. Apart from $\overline{P}^{1}_{t,i,i+1}=P^{1}_{t,i,i+1}$ which requires present liquidity at $t+i\delta$, i.e. $L^{1}_{t+i\delta}=1$, for each $i\in\mathbb{N}_{0,m-1}$, the remainder of the LCZCBs require the availability of future liquidity and are therefore not tradable, in general.[r]

Remark 16 (Liquidity-contingent term-dependent market price of systemic risk). Under potential illiquidity, the δ-term market price of systemic risk modeled by $\{\Lambda^{1}_{t+j\delta}\,;j\in\mathbb{N}_{0,m}\}$ is effectively adjusted for the potential cost of illiquidity incurred when market-making FRAs through the process $\{\Theta^{1}_{t+j\delta}\,;j\in\mathbb{N}_{0,m}\}$. However, this is all strictly contingent on the availability of δ-term and FRA liquidity, as described above. Therefore, the product of the above processes $\{\Lambda^{1}_{t+j\delta}\Theta^{1}_{t+j\delta}\,;j\in\mathbb{N}_{0,m}\}$ models the liquidity-contingent term-dependent market price of systemic risk associated with the δ-term LCPK.

Remark 17 (Multiple LCPKs and liquidity-contingent term structures). Each term, $n\delta$, will have a distinct LCPK, $\{\overline{\pi}^{n}_{t+j\delta}\,;j\in\mathbb{N}^{n}_{0,m}\}$, with an associated liquidity-contingent systemic pricing measure, $\mathbb{Q}_n$, modeled upon its perfect liquidity counterparts, $\{\pi^{n}_{t+j\delta}\,;j\in\mathbb{N}^{n}_{0,m}\}$ and $\mathbb{P}_n$. This will be further illustrated in the next section. The inability to replicate all tradable ZCBs via the system of FRAs, along with the contingency on future term rate liquidity, leads to liquidity-contingent multi-period calibration for each tradable term. This in turn leads to multiple liquidity-contingent term structures.

Remark 18 (Classical risk-neutral measure). Within this context, $\mathbb{Q}_1$ is the best proxy for the classical risk-neutral measure, however it is not clear that this measure produces consistent and coherent expectations of risk-free term rates considering the idiosyncratic and subjective market-making processes of the STR and the constituents of the SST, and the interactions thereof.

6. Reduced-Form Model Development

In order to formalize the construction of an arbitrary $n\delta$-term LCZCB-system and LCPK over an arbitrary horizon $[t,t+pn\delta]$, for $p\in\mathbb{N}$, it is useful to provide a

[r]From the perspective of an EC, long positions in LCZCBs may be enabled by the SST offering FRA liquidity and the STR issuing floating rate notes (FRNs) that reference the δ-term rate. Short positions in LCZCBs would require the EC to secure bespoke variable rate loan agreements from the STR that reference the δ-term rate with zero additional spread for credit risk. This would not be possible however, unless the credit risk of the EC was comparable to that of constituent banking entities.

definition for FRA liquidity regimes akin to the term loan and deposit regimes from Definition 9.

Definition 12 (FRA liquidity regimes). At an arbitrary time $u+i\delta$, complete $n\delta$-term FRA liquidity over the interval $[u+i\delta, u+m\delta]$ exists on the set

$$\overline{\mathcal{L}}^n_{u+i\delta,u+m\delta} := \sigma\big(\big\{\overline{L}^n_{u,i,i+j} = 1\,;\, j \in \mathbb{N}^n_{n,m-i}\big\}\big), \tag{6.1}$$

where $i, m \in \mathbb{N}_0$ with $i \le m$ and, as in Definition 11, the binary random variable $\overline{L}^n_{u,i,i+j}$ is equal to 1 if perfect systemic liquidity exists for the $j\delta \times (j+n)\delta$ FRA, or is equal to 0 otherwise. Also, it is assumed that

$$\sigma\left(\left\{\overline{L}^n_{u,i,i+j} = 1\right\}\right) \supset \sigma\left(\left\{L^j_{u+i\delta} = 1, L^{j+n}_{u+i\delta} = 1\right\}\right), \tag{6.2}$$

which is the analogous assumption to Eq. (5.18) from Definition 11. Shorthand notation: $\overline{\mathcal{L}}^n_{u,i,j} := \overline{\mathcal{L}}^n_{u+i\delta,u+j\delta}$ for all $i, j \in \mathbb{N}_0$, with $i \le j$ and $u \in \mathbb{R}_{\ge 0}$.

Analogous to the construction of the δ-term LCZCB-system and its associated LCPK that was described in the previous section, the construction of the comparable $n\delta$-term quantities, viz. $\{\overline{P}^n_{t,in,jn}; i, j \in \mathbb{N}_{0,p}\,, i \le j\}$ and $\{\overline{\pi}^n_{t+jn\delta}; j \in \mathbb{N}_{0,p}\}$, over the interval $[t, t+pn\delta]$, would require p steps: (1) At the current time t and over the set $\{t, t+n\delta, t+2n\delta, \ldots, t+pn\delta\}$, one would require that $\mathcal{L}^n_{t,0,pn}$ and $\overline{\mathcal{L}}^n_{t,0,pn}$ holds; (2) at each future time $t+in\delta$ and over the set $\{t+in\delta, t+(i+1)n\delta, \ldots, t+pn\delta\}$, for $i \in \mathbb{N}_{1,p-2}$, one would require that $\mathcal{L}^n_{t,in,pn}$ and $\overline{\mathcal{L}}^n_{t,in,pn}$ holds; and (3) at the future time $t+(p-1)n\delta$ and over the set $\{t+(p-1)n\delta, t+pn\delta\}$, one would require that $\mathcal{L}^n_{t,(p-1)n,pn}$ holds.

While the construction of such term-dependent and -consistent quantities is theoretically appealing, it is far too rigid for real-world pricing, valuation and risk management. This is practically demonstrated by the inability of the $n\delta$-term LCPK to model the natural tenor transformation associated with fixed maturity financial instruments through the passage of time, as well as the asynchronicity between calendar (t), term ($n\delta$) and tenor ($pn\delta$) time. The objective of this section is to adapt the framework to cater for the aforementioned considerations. This is achieved through a *reduced-form modeling approach*.

To construct a reduced-form $n\delta$-term LCZCB-system and LCPK over an arbitrary time interval $[t+i\delta, t+m\delta]$, where $n \le m$ and $i \in \mathbb{N}_{0,m}$, the following assumptions are required.

Assumption 2 (CPL). At time $t+i\delta$, the STR enables the CPL regime $\mathcal{L}^{\mathrm{CPL}}_{t,i,m}$.

Assumption 3 (FRA market-making). At time $t+i\delta$, the individual STs have sufficient risk appetite to market-make and enable the liquidity of each $k\delta \times (k+n)\delta$

FRA with fair or mid-market FRA rate $\widehat{F}^n_{t,i,i+k}$, for $k \in \mathbb{N}_{1,m-n-i}$. This FRA liquidity regime exists on the set

$$\overline{\mathcal{L}}^{(n)}_{u+i\delta,u+m\delta} := \sigma\big(\big\{\overline{L}^n_{u,i,i+k} = 1\,;\,, k \in \mathbb{N}_{1,m-n-i}\big\}\big) \supset \overline{\mathcal{L}}^n_{u+i\delta,u+m\delta}, \tag{6.3}$$

and is therefore a richer set than that defined in Definition 12. Using Lemma 4 within this context, the model fair $k\delta \times (k+n)\delta$ FRA rate is

$$\overline{F}^n_{t,i,i+k} := q^n_{t+(i+k)\delta} F^n_{t,i,i+k}, \tag{6.4}$$

which the SST may set equal to $\widehat{F}^n_{t,i,i+k}$ in order to calibrate $q^n_{t+(i+k)\delta}$, the probability of $n\delta$-term liquidity at time $t+(i+k)\delta$. Assumption 2 enables the computation of $F^n_{t,i,i+k}$.

Assumption 4 (Future $n\delta$-term liquidity). At time $t+i\delta$, future $n\delta$-term liquidity according to the set

$$\mathcal{L}^{(n)}_{t,i,m} := \sigma(\{L^n_{t+(i+k)\delta} = 1\,; k \in \mathbb{N}_{1,m-n-i}\}) \supset \mathcal{L}^n_{t,i,m}, \tag{6.5}$$

is assumed to exist, with the definitions of the reduced-form $n\delta$ LCZCB-system and LCPK being contingent upon thus assumption.

Assumption 5 (Reduced-form $n\delta$-term PK). Using the estimated $n\delta$-term SDF from Definition 2 and contingent on the CPFL regime, the calibrated reduced-form $n\delta$-term SDF is

$$D^{(n)}_{t+j\delta} := \mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda^{(n)}_{t+j\delta}}{\Lambda^{(n)}_{t+(j-1)\delta}} \widehat{D}^n_{t+j\delta} \,\middle|\, \mathcal{G}_{t+(j-1)\delta}\right], \tag{6.6}$$

where the time-inhomogeneous process $\big(\Lambda^{(n)}_{t+v\delta}\big)_{0\le v\le m}$ is a $\{(\mathcal{G}_u),\mathbb{P}\}$-martingale, with $\Lambda^{(n)}_t := 1$, that enables a change-of-measure from $\mathbb{P}$ to $\mathbb{P}_{(n)}$, the reduced-form $n\delta$-term systemic pricing measure. Then, commensurate with the δ-term, the calibrated reduced-form $n\delta$-term PK is defined by $\pi^{(n)}_{t+j\delta} := \Lambda^{(n)}_{t+j\delta} D^{(n)}_{t+j\delta}$, with the time-inhomogeneous parameters associated with $\Lambda^{(n)}_{t+j\delta}$ chosen such that

$$P^{(n)}_{t,i,j} := \frac{1}{\pi^{(n)}_{t+i\delta}} \mathbb{E}^{\mathbb{P}}[\pi^{(n)}_{t+j\delta} \mid \mathcal{G}_{t+i\delta}] = \frac{1}{D^{(n)}_{t+i\delta}} \mathbb{E}^{\mathbb{P}_{(n)}}[D^{(n)}_{t+j\delta} \mid \mathcal{G}_{t+i\delta}] = P^{j-i}_{t,i,j}, \tag{6.7}$$

which defines the reduced-form $n\delta$-term ZCB-system, for $i,j \in \mathbb{N}_{0,m}$ with $i \le j$. Finally, the reduced-form $n\delta$-term rate is defined by

$$R^{(n)}_{t+i\delta} := \frac{1}{n\delta}\left(\frac{1}{P^{(n)}_{t,i,j}} - 1\right), \tag{6.8}$$

when $j-i=n$. For $n=1$, the reduced-form δ-term PK is identical to its counterpart.

It is now possible to define the reduced-form synthetic $n\delta$-term LCZCB-system, the intertemporal values of which will be used in the definition of the reduced-form $n\delta$-term LCPK.

Lemma 6 (Reduced-form synthetic $n\delta$-term LCZCB-system). *Given Assumptions* 2–5, *the reduced-form synthetic $n\delta$-term LCZCB system is defined by*

$$\overline{P}^{(n)}_{t,i,j} := \begin{cases} \frac{D^1_{t+(i+1)\delta}}{D^1_{t+i\delta}} \prod_{k=0}^{(j-i-n-1)/n} (1+n\delta\overline{F}^n_{t,i,i+nk+1})^{-1}, & \text{mod}(j-i,n)=1, \\ \frac{D^2_{t+(i+2)\delta}}{D^2_{t+i\delta}} \prod_{k=0}^{(j-i-n-2)/n} (1+n\delta\overline{F}^n_{t,i,i+nk+2})^{-1}, & \text{mod}(j-i,n)=2, \\ \vdots \qquad , & \vdots \qquad , \\ \frac{D^{n-1}_{t+(i+n-1)\delta}}{D^{n-1}_{t+i\delta}} \prod_{k=0}^{(j-i-2n+1)/n} (1+n\delta\overline{F}^n_{t,i,i+nk+n-1})^{-1}, & \text{mod}(j-i,n)=n-1, \\ \frac{D^n_{t+(i+n)\delta}}{D^n_{t+i\delta}} \prod_{k=0}^{(j-i-2n)/n} (1+n\delta\overline{F}^n_{t,i,i+n(k+1)})^{-1}, & \text{mod}(j-i,n)=0, \end{cases} \tag{6.9}$$

for $n < (j-i) \leq m$, while for $0 \leq (j-i) \leq n$ the definition resolves to

$$\overline{P}^{(n)}_{t,i,j} := \begin{cases} P^n_{t,j-n,j}, & i = j-n, \\ P^{n-1}_{t,j-n+1,j}, & i = j-(n-1), \\ \vdots, & \vdots, \\ P^1_{t,j-1,j}, & i = j-1, \\ 1, & i = j, \end{cases} \tag{6.10}$$

with $i,j \in \mathbb{N}_{0,m}$ and $i \leq j$.

See Appendix H.1 for the proof.

Theorem 3 (Reduced-form $n\delta$-term LCPK). *Maintaining the setup of Lemma* 6, *as well as the result thereof, a reduced-form $n\delta$-term LCPK may be defined as*

$$\overline{\pi}^{(n)}_{t+j\delta} := \pi^{(n)}_{t+j\delta}\Theta^{(n)}_{t+j\delta}, \tag{6.11}$$

for $j \in \mathbb{N}_{0,m}$, where $\Theta^{(n)}_t := 1$ and

$$\frac{\Theta^{(n)}_{t+j\delta}}{\Theta^{(n)}_{t+i\delta}} = \begin{cases} 1, & 0 \leq j-i \leq n, \\ \frac{X^n_{t+j\delta}}{X^n_{t+i\delta}}, & n < j-i \leq m. \end{cases} \tag{6.12}$$

The process $\left(X^n_{t+v\delta}\right)_{0\le v\le m}$ *is chosen to be a time-inhomogeneous* $\{(\mathcal{G}_u),\mathbb{P}_{(n)}\}$*-martingale, with* $X^n_t := 1$*, that enables a change-of-measure from* $\mathbb{P}_{(n)}$ *to* $\mathbb{Q}_{(n)}$ *on* $\mathcal{G}_{t+j\delta}$*, i.e.* $\frac{X^n_{t+j\delta}}{X^n_t} = \frac{\mathrm{d}\mathbb{Q}_{(n)}}{\mathrm{d}\mathbb{P}_{(n)}}\big|_{\mathcal{G}_{t+j\delta}}$*, such that*

$$\overline{\pi}^{(n)}_{t+i\delta} P^{(n)}_{t,i,j} = \mathbb{E}^{\mathbb{P}}[\overline{\pi}^{(n)}_{t+j\delta} \,|\, \mathcal{G}_{t+i\delta}], \tag{6.13}$$

for all $i,j\in\mathbb{N}_{0,m}$ *with* $i\le j$*.*

Proof. By Assumption 5 and construction, the reduced-form $n\delta$-term PK recovers the reduced-form synthetic $n\delta$-term LCZCB value for $0\le(j-i)\le n$, i.e. $\overline{P}^{(n)}_{t,i,j} = P^{j-i}_{t,i,j} = P^{(n)}_{t,i,j}$. Therefore, the reduced-form $n\delta$-term PK $\{\pi^{(n)}_{t+j\delta}\,;j\in\mathbb{N}_{0,m}\}$ is a good initial candidate for the reduced-form $n\delta$-term LCPK. However, when $n<(j-i)\le m$ then $P^{(n)}_{t,i,j} = P^{j-i}_{t,i,j} \ne \overline{P}^n_{t,i,j}$. The definition of the $\{(\mathcal{G}_u),\mathbb{P}_{(n)}\}$-martingale $\left(X^n_{t+v\delta}\right)_{0\le v\le m}$ enables a change-of-measure such that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda^{(n)}_{t+j\delta}}{\Lambda^{(n)}_{t+i\delta}}\frac{\Theta^n_{t+j\delta}}{\Theta^n_{t+i\delta}}\frac{D^{(n)}_{t+j\delta}}{D^{(n)}_{t+i\delta}}\,\middle|\,\mathcal{G}_{t+i\delta}\right] &= \mathbb{E}^{\mathbb{P}_{(n)}}\left[\frac{X^n_{t+j\delta}}{X^n_{t+i\delta}}\frac{D^{(n)}_{t+j\delta}}{D^{(n)}_{t+i\delta}}\,\middle|\,\mathcal{G}_{t+i\delta}\right]\\ &= \mathbb{E}^{\mathbb{Q}_{(n)}}\left[\frac{D^{(n)}_{t+j\delta}}{D^{(n)}_{t+i\delta}}\,\middle|\,\mathcal{G}_{t+i\delta}\right],\end{aligned} \tag{6.14}$$

may be set to the value of $\overline{P}^n_{t,i,j}$ by calibrating the free time-dependent parameters associated with $X^n_{t+j\delta}$. Finally, observe that

$$\frac{1}{\overline{\pi}^{(n)}_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^{(n)}_{t+j\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \begin{cases}1, & j-i=0,\\ P^{j-i}_{t,i,j}, & j-i\le n,\\ \dfrac{1}{D^{(n)}_{t+i\delta}}\mathbb{E}^{\mathbb{Q}_{(n)}}[D^{(n)}_{t+j\delta}\,|\,\mathcal{G}_{t+i\delta}], & j-i>n,\end{cases} \tag{6.15}$$

for all $i,j\in\mathbb{N}_{0,m}$ with $i\le j$, which is the required dynamics and completes the proof. □

7. Exchanges of Risk, Pricing and Valuation

Consider an EC, with negligible credit risk, that has initiated an agreement, at some past time u, for a forward-starting term loan/deposit with the STR, should there be liquidity at this future time. At the current time $t\ge u$, the remaining tenor until initiation and maturity equals $i\delta$ and $(n+i)\delta$, respectively. Importantly, this agreement does not guarantee the EC liquidity in the respective term loan/deposit, nor does it guarantee a fixed rate. Let the cash flow that the EC will pay/receive at maturity time $t+(n+i)\delta$ be denoted by N. Then, at initiation time $t+i\delta$, the value of this cash flow will be

$$NP^{n}_{t,i,n+i} = N\frac{1}{\overline{\pi}^{n}_{t+i\delta}}\mathbb{E}^{\mathbb{P}}[\overline{\pi}^{n}_{t+(n+i)\delta}\,|\,\mathcal{G}_{t+i\delta}], \tag{7.1}$$

which is the effective nominal or principal value of the underlying transaction. At the current time, the theoretical value of the overall agreement is

$$N\overline{P}^{(n)}_{t,0,n+i} = N\frac{1}{\overline{\pi}^{(n)}_{t}}\mathbb{E}^{\mathbb{P}}[\overline{\pi}^{(n)}_{t+i\delta}P^{n}_{t,i,n+i}\,|\,\mathcal{G}_{t}], \tag{7.2}$$

since $P^{n}_{t,i,n+i} = \overline{P}^{(n)}_{t,i,n+i}$, by construction. Since there is no guarantee in this agreement, the value above is theoretical and need not be exchanged between the EC and STR. If the future transaction is a deposit (loan), then the second equation above is the amount that the EC must pay to (receive from) the STR. At any time prior to initiation, the EC or the STR may wish to change the nature of the future transaction. They have one of the following options: (i) early terminate the agreement prior to initiation; (ii) restructure interest and capital cash flows while preserving the effective term rate; (iii) restructure the remaining tenor until maturity, from the initiation time; (iv) perform (iii) and then (ii); or (v) perform (iii) and also reduce the term of the underlying interest rate. Option (i) bears no current risk, nor any exchange of cash flows, but the STR potentially loses: (a) term funding in the case of a deposit; or (b) interest income in the case of a loan. In the case of a loan, option (ii) is beneficial to the STR since it may fund the loan via a corresponding term deposit and then demand for periodic loan repayments (capital plus interest) at an *internal rate of return* that matches the effective cost of the aforementioned term funding, and hence the corresponding term rate as well. In reality, such a mechanism reduces credit risk exposure for the STR. In the case of a deposit, this option enables the STR to offer periodic interest payments to the EC that matches the effective term rate. The EC will therefore have improved liquidity, while effectively securing a term rate. It should be noted however, that restructured interest rate and periodic capital payments in an amortizing capital structure, will affect the effective duration of the term loan/deposit, which will in turn impact the effective term rate offered by the STR. Such a restructure will therefore overlap with option (iii). The restructured cash flow for option (ii) at time $t+(n+i)\delta$ may be generally represented as

$$N\frac{P^{n}_{t,i,n+i}}{P^{(n)}_{t,i,n+i}} = N, \tag{7.3}$$

since $P^{(n)}_{t,i,n+i}$, which is equivalent in value to $P^{n}_{t,i,n+i}$ by construction, is the reduced-form version of the $n\delta$-term ZCB from Assumption 5, and thereby encodes potential interest rate structures that yield the same effective $n\delta$-term rate. The $n\delta$-term LCPK is still valid here, hence the current and initiation time valuation remains unchanged. Apart from the restructured interest cash flows, this option is therefore equivalent to the original agreement. Option (iii) leaves the effective nominal of the transaction unchanged, at initiation time, but reduces the tenor, say for example to $j\delta$, with $j \leq n$. From the vantage point of the initiation time, the restructured cash flow at time $t+(j+i)\delta$ then becomes

$$N\frac{P^n_{t,i,n+i}}{P^j_{t,i,j+i}}, \tag{7.4}$$

and effects (a) and (b) on the STR are diminished in accordance with how much smaller j is in comparison to n. The value of this cash flow at initiation time is then given by

$$NP^n_{t,i,n+i} = \frac{1}{\overline{\pi}^j_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^j_{t+(j+i)\delta}\,N\frac{P^n_{t,i,n+i}}{P^j_{t,i,j+i}}\,\middle|\,\mathcal{G}_{t+i\delta}\right], \tag{7.5}$$

and is therefore unchanged from the original agreement. The choice to change the tenor of the future transaction at the time of initiation implies a change of the underlying numeraire and PK from the $n\delta$- to the $j\delta$-term, when viewed from the vantage point of the current time. This is enabled by an exchange of units between the respective numeraires at time $t+i\delta$, when viewed from time t, which yields

$$\frac{\overline{\pi}^{(n)}_{t+i\delta}}{\overline{\pi}^{(j)}_{t+i\delta}}NP^{(n)}_{t,i,n+i}, \tag{7.6}$$

with the reduced-form quantities ensuring time synchronicity between the $j\delta$- and $n\delta$-term. The theoretical value of the new agreement at the current time is then

$$N\overline{P}^{(n)}_{t,0,n+i} = \frac{1}{\overline{\pi}^{(j)}_t}\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^{(j)}_{t+i\delta}\,\frac{\overline{\pi}^{(n)}_{t+i\delta}}{\overline{\pi}^{(j)}_{t+i\delta}}NP^{(n)}_{t,i,n+i}\,\middle|\,\mathcal{G}_t\right], \tag{7.7}$$

since $\overline{\pi}^{(n)}_t = \overline{\pi}^{(j)}_t := 1$. Again, this initial theoretical value is unchanged from the original agreement, as required. Option (iv) is identical to (iii), but $P^j_{t,i,j+i}$ is replaced with $P^{(j)}_{t,i,j+i}$. Option (v) builds upon (iii) and (iv), with the EC/STR wanting to pay/receive fixed interest on the effective nominal at a frequency less than $j\delta$, say for example δ. Assuming reinvestment of interim interest, the cash flow at time $t+(n+j)\delta$ now becomes

$$N\frac{P^n_{t,i,n+i}}{\overline{P}^1_{t,i,j+i}}, \tag{7.8}$$

from the vantage point at $t+i\delta$. This cash flow depends on the creation of the synthetic δ-term LCZCB with $j\delta$-tenor, and therefore exposes the EC to liquidity risk in the δ-term rate over the period $[t+i\delta, t+(j+i)\delta]$. The SST will separately enable the exchange of δ-term floating-for-fixed interest rate risk for the EC. Effect (a) and (b) on the STR is therefore exacerbated as well, with term funding reduced to the shortest available term and interest income being generally lower, while being contingent on the availability of δ-term liquidity. The value of this cash flow at initiation time is given by

$$NP^n_{t,i,n+i} = \frac{1}{\overline{\pi}^1_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^1_{t+(j+i)\delta}\, N\frac{P^n_{t,i,n+i}}{\overline{P}^1_{t,i,j+i}}\,\middle|\,\mathcal{G}_{t+i\delta}\right], \tag{7.9}$$

and is therefore unchanged from the original agreement. As with the previous case, an exchange of units is required between the respective numeraires at time $t+i\delta$

$$\frac{\overline{\pi}^{(n)}_{t+i\delta}}{\overline{\pi}^1_{t+i\delta}}NP^{(n)}_{t,i,n+i}, \tag{7.10}$$

when viewed from the vantage point at time t. The theoretical value at this time is then

$$N\overline{P}^{(n)}_{t,0,n+i} = \frac{1}{\overline{\pi}^1_t}\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^1_{t+i\delta}\,\frac{\overline{\pi}^{(n)}_{t+i\delta}}{\overline{\pi}^1_{t+i\delta}}NP^{(n)}_{t,i,n+i}\,\middle|\,\mathcal{G}_t\right], \tag{7.11}$$

since $\overline{\pi}^{(n)}_t = \overline{\pi}^1_t := 1$. The next lemma generalizes this concept, using the reduced-form quantities defined in Lemma 6 and Theorem 3.

Lemma 7 (Exchange of risk). *Assuming that the current time is* t, *a fixed cash flow* N *that is due at a future time* $t+(i+j_1)\delta$ *with a fixed accrual rate based on the* $n_1\delta$*-term that is determined at time* $t+i\delta$ *may be exchanged for an equivalent cash flow*

$$N\frac{\overline{\pi}^{(n_1)}_{t+i\delta}\,\overline{P}^{(n_1)}_{t,i,i+j_1}}{\overline{\pi}^{(n_2)}_{t+i\delta}\,\overline{P}^{(n_2)}_{t,i,i+j_2}}, \tag{7.12}$$

which is payable at time $t+(i+j_2)\delta$, *accrues interest at a fixed rate that is also determined at time* $t+i\delta$ *but based on the* $n_2\delta$*-term, where* $i, j_1, j_2 \in \mathbb{N}_0$ *an* $n_1, n_2 \in \mathbb{N}$.

Proof. Using the $n_1\delta$-term LCPK for valuation, the original cash flow has a value of $N\overline{P}^{(n_1)}_{t,0,i+j_1}$ and $N\overline{P}^{(n_1)}_{t,i,i+j_1}$ at times t and $t+i\delta$, respectively. Exchanging the underlying term rate requires an exchange of units across the respective numeraires at the determination time of the respective term-dependent accrual rates. This is enabled by Eq. (7.12). Then, valuing this new cash flow using the $n_2\delta$-term LCPK yields the same values as the original cash flow at times t and $t+i\delta$, respectively, as required. □

The next result shows how the stochastic cash flow given by Eq. (7.12) may be exchanged for an equivalent fixed cash flow through the creation of a forward contract.

Corollary 2 (Fair forward pricing). *The equivalent exchanged cash flow, given by Eq.* (7.12), *may be considered as a fixed cash flow* H *that is due at time* $t+(i+$

$j_2)\delta$ with a fixed accrual rate derived from the $n_2\delta$-term LCPK over the interval $[t, t+(i+j_2)\delta]$.

Proof. Consider a forward contract with the following terminal payoff:

$$V_{t+(i+j_2)\delta} := N\frac{\overline{\pi}^{(n_1)}_{t+i\delta}\overline{P}^{(n_1)}_{t,i,i+j_1}}{\overline{\pi}^{(n_2)}_{t+i\delta}\overline{P}^{(n_2)}_{t,i,i+j_2}} - H, \tag{7.13}$$

where H is a fixed strike price. Such a contract enables the exchange of the floating/uncertain cash flow (7.12), measurable at time $t+i\delta$ and payable at time $t+(i+j_2)\delta$, for a fixed cash flow H at time $t+(i+j_2)\delta$. Since the underlying cash flow naturally accrues interest that is defined by the $n_2\delta$-term LCPK, the term- and liquidity-consistent current value of such a forward is given by

$$\overline{\pi}^{(n_2)}_t V_t = \mathbb{E}^{\mathbb{P}}[\overline{\pi}^{(n_2)}_{t+(i+j_2)\delta} V_{t+(i+j_2)\delta} \,|\, \mathcal{G}_t], \tag{7.14}$$

$$V_t = \frac{N}{\overline{\pi}^{(n_2)}_t}\mathbb{E}^{\mathbb{P}}[\overline{\pi}^{(n_1)}_{t+i\delta}\overline{P}^{(n_1)}_{t,i,i+j_1} \,|\, \mathcal{G}_t] - \frac{H}{\overline{\pi}^{(n_2)}_t}\mathbb{E}^{\mathbb{P}}[\overline{\pi}^{(n_2)}_{t+(i+j_2)\delta} \,|\, \mathcal{G}_t] \tag{7.15}$$

$$= N\frac{\overline{\pi}^{(n_1)}_t}{\overline{\pi}^{(n_2)}_t}\overline{P}^{(n_1)}_{t,0,i+j_1} - H\overline{P}^{(n_2)}_{t,0,i+j_1}, \tag{7.16}$$

where the first term on the right-hand side of the second line follows by the tower property of conditional expectation, since $V_{t+(i+j_2)\delta}$ is $\mathcal{G}_{t+i\delta}$-measurable. Setting $V_t = 0$, noting that $\overline{\pi}^{(n_1)}_t = \overline{\pi}^{(n_2)}_t = 1$, and solving for H reveals the fair forward price to be

$$H = N\frac{\overline{\pi}^{(n_1)}_t\overline{P}^{(n_1)}_{t,0,i+j_1}}{\overline{\pi}^{(n_2)}_t\overline{P}^{(n_2)}_{t,0,i+j_2}} = N\frac{\overline{P}^{(n_1)}_{t,0,i+j_1}}{\overline{P}^{(n_2)}_{t,0,i+j_2}}. \tag{7.17}$$

Lastly, observe that the fair forward price is $\mathcal{G}_t$-measurable, due at time $t+(i+j_2)\delta$ and accrues interest commensurate with the $n_2\delta$-term LCZCB over $[t, t+(i+j_2)\delta]$, as required. □

Up to now, only *replicated* FRAs, as defined in Sec. 2.3, have been considered. For a general $i\delta \times (i+n)\delta$ replicated FRA, the perfect liquidity setting yielded a fair FRA rate equal to the corresponding forward rate $F^n_{t,0,i}$ at the current time t, with $i, m \in \mathbb{N}$. The potentially illiquid setting yields $\overline{F}^n_{t,0,i} = q^n_{t+i\delta}F^n_{t,0,i}$ as the fair FRA rate. The exchange of risk mechanism developed in Lemma 7 and Corollary 2 may now be utilized to model the pricing and recover the risk-free value of a *traded* FRA.

Lemma 8 (Traded FRA). *At time $t < t+i\delta$, the traded version of the generic $i\delta \times (i+n)\delta$ FRA has a fair strike rate given by*

$$\overline{F}^{1,n}_{t,0,i} := \frac{\overline{P}^{(n)}_{t,0,i+n}}{\overline{P}^{1}_{t,0,i+m}}\,\overline{F}^n_{t,0,i} = \frac{\overline{P}^{(n)}_{t,0,i+n}}{\overline{P}^{1}_{t,0,i+n}}\,q^n_{t+i\delta}F^n_{t,0,i}, \tag{7.18}$$

so that the fair value of such a FRA with arbitrary strike rate H is

$$V_t = \alpha N \overline{P}^{1}_{t,0,i+n}[\overline{F}^{1,n}_{t,0,i} - H]n\delta, \tag{7.19}$$

where the δ-term rate has been assumed to be an ONRR.

Proof. First, an exchange of risk is required to convert the replicated FRA payoff

$$\alpha N \left[R^{(n)}_{t+i\delta} - K\right] n\delta, \tag{7.20}$$

into an equivalent traded version that may be liquidated, and therefore must be valued, at the highest frequency, the δ-term. This is achieved by applying the result from Lemma 1, with $n_1 = n$, $n_2 = 1$ and $j_1 = j_2 = i + n$, which yields the equivalent payoff

$$V_{t+(i+n)\delta} = \alpha N \frac{\overline{\pi}^{(n)}_{t+i\delta}\overline{P}^{(n)}_{t,i,i+n}}{\overline{\pi}^{1}_{t+i\delta}\overline{P}^{1}_{t,i,i+n}}[R^{(n)}_{t+i\delta} - K]n\delta, \tag{7.21}$$

while an application of Corollary 2 enables the equivalent representation of the exchanged fixed strike rate as a new fixed strike rate $H := K\,\overline{P}^{(n)}_{t,0,i+n}/\overline{P}^{1}_{t,0,i+n}$. The current value of the FRA is then given by

$$V_t = \frac{1}{\overline{\pi}^1_t}\mathbb{E}^{\mathbb{P}}[\overline{\pi}^1_{t+(i+n)\delta} V_{t+(i+n)\delta} \,|\, \mathcal{G}_t] \tag{7.22}$$

$$= \alpha N \left(\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^{(n)}_{t+i\delta}\overline{P}^{(n)}_{t,i,i+n} R^{(n)}_{t+i\delta} \,\middle|\, \mathcal{G}_t\right] - H\overline{P}^{1}_{t,0,i+n}\right) n\delta \tag{7.23}$$

$$= \alpha N \overline{P}^{1}_{t,0,i+n}\left[\frac{\overline{P}^{(n)}_{t,0,i+n}}{\overline{P}^{1}_{t,0,i+n}}\,\overline{F}^{n}_{t,0,i} - H\right] n\delta. \tag{7.24}$$

The second line follows by the tower property and from the fact that $\mathbb{E}^{\mathbb{P}}[\overline{\pi}^1_{t+(i+n)\delta} \,|\, \mathcal{G}_{t+i\delta}] = \overline{\pi}^1_{t+i\delta}\overline{P}^1_{t,i,i+n}$ and $\mathbb{E}^{\mathbb{P}}[\overline{\pi}^{(n)}_{t+i\delta}\overline{P}^{(n)}_{t,i,i+n}R^{(n)}_{t+i\delta} \,|\, \mathcal{G}_t] = \overline{\pi}^{(n)}_t \overline{P}^{(n)}_{t,0,i+n}\overline{F}^{n}_{t,0,i}$. Setting this to 0 and solving for H yields the fair FRA strike rate, given by Eq. (7.18). Then, for any traded FRA that references the $n\delta$-term rate with arbitrary strike rate H, the value of such a FRA at time t is given by Eq. (7.19). □

From Eq. (7.18), note that the fair traded FRA rate is an exchanged version of the corresponding fair replicated FRA rate, and is lower in magnitude in general. However at the reset times of the FRAs, while the replicated version resets exactly at the respective underlying term rate, the traded version will reset at the exchanged version of the aforementioned term rate. This is explained further below.

Remark 19 (Traded FRA reference rates). At any interim time between reset and expiry, the traded FRA's fair value is

$$\overline{P}^{1}_{t,j,i+n}\alpha N\left[\frac{P^{(n)}_{t,i,i+n}}{\overline{P}^{1}_{t,i,i+n}}R^{(n)}_{t+i\delta}-H\right]n\delta, \tag{7.25}$$

where $j \in \{i, i+1, \ldots, i+n\}$. Therefore, the theoretically correct reference rate for a generic traded FRA is the exchanged version of the $n\delta$-term rate, i.e. the first term within the squared brackets above. The difference between this reference rate and the $n\delta$-term rate is that the latter includes the premium for liquidity risk over the $n\delta$-term, while the latter does not. In practice, traded FRAs reference key bank funding reference rates, and since banks are forced, in general, to offer secondary market liquidity on funding transactions, it is logical that these reference rates should not offer any liquidity risk premium. This also highlights the fixed term contractual characteristics of the reference rates defined in Axioms 2.1 and 2.2, i.e. these term rates do not guarantee the existence of secondary market liquidity.

8. Conclusions

This research paper analyzes liquidity risk implications for bank funding markets before the GFC, and after the reference rate reform that has followed the GFC in recent years. Theoretically, this is equivalent to analyzing a *near risk-free* versus a *risky* market context. Using a systemic financial system, postulated in Sec. 2, we show how one can focus the analysis on liquidity risk, only. Sections 3 and 4 present results for bank funding and linear derivatives markets when there are no systemic-liquidity risks, revealing the fungibility of term rates and thereby the recovery of the funding–swap duality — all enabled via the exchange of floating-for-fixed interest rate risk across terms. Section 5 introduces liquidity risk which alters the fundamental nature of the set of term rates, with each offering a liquidity risk premium to the provider of funding in return for foregoing their liquidity for the respective term under consideration. Since liquidity risk cannot be hedged, and is not fungible, a term-based multi-curve framework is derived that consistently fragments and encodes the nature of term-based liquidity risks in the bank funding market. This is achieved using FRA replication from first principles which recovers the funding–swap duality within each term. Sections 6 and 7 reveal how the term-based multi-curve framework may be modeled in a reduced-form setting. This in turn enables a method for an exchange of risk across terms, and the consistent pricing and valuation of funding transactions, and traded standard derivatives.

Appendix A. Characterization of Liquidity Risk

The systemic perspective of the stylized financial system constructed in Sec. 2 enables the modeling undertaken in this paper to focus on liquidity risk. The chosen characterization of liquidity risk is consistent with that provided in Acerbi & Scandolo (2008), Bianchetti & Carlicchi (2011) and Morini (2009). Within the context of funding, this translates into the following three categories:

(i) *funding-liquidity risk*,[s] which refers to the uncertainty associated with the general availability of funding at the initiation or roll-over time of a funding transaction;
(ii) *market-liquidity risk*,[t] which refers to the uncertainty related to the difficulty and cost of liquidating an existing funding transaction; and
(iii) *systemic-liquidity risk*,[u] which is the realization of risks (i) or (ii) as a result of systemic instability.

It is hopefully clear that categories (i)–(iii) are all related and are terms that are used synonymously in practice, as reported in Acerbi & Scandolo (2008). For the purpose of defining term rates in Sec. 2, the current existence and the expectation for the existence of any subset, or all, of the categories above, over the duration of the term of the rate under consideration, leads to the existence of a term-dependent *liquidity premium*. This premium is captured in the definition of various term rates via the specification of a term-dependent *liquidity spread*. For modeling future liquidity and its uncertainty, this characterization is used again in the definition of the set of systemic liquidity indicators, i.e. in Definition F.1.

Appendix B. Stylized Financial and Axiomatic Interest Rate Systems

This section provides a description of the financial system under consideration and an axiomatic construction of the interest rates therein.

B.1. *Stylized financial system*

Figure B.1 provides a depiction of a financial system within a generic economy, along with interactions between the constituent entities.

We draw attention to the TR and ST of each banking entity, highlighted in Fig. B.1. The TR is solely responsible for the sourcing of funding (or deposits) and lending, while the ST is responsible for engineering financial products. Interactions

[s]The work by Eisenschmidt & Tapking (2009) supports the existence of this type of liquidity risk within money markets.
[t]For an in-depth analysis on funding and market liquidity, albeit with a slightly different interpretation, see Brunnermeier & Pedersen (2009).
[u]For a more practical macroeconomic, perspective on systemic-liquidity risk within the banking context, see Acharya *et al.* (2011) and Acharya & Skeie (2011).

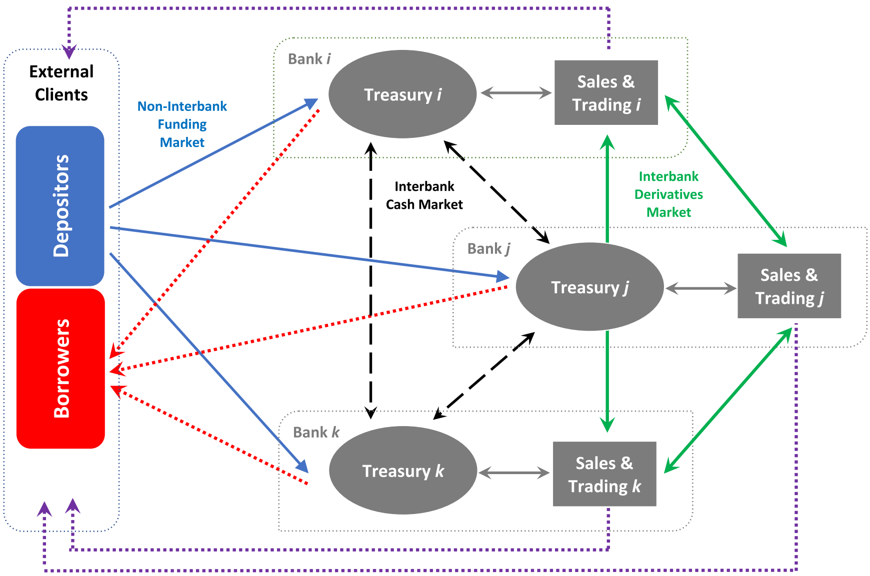

Fig. B.1. Stylized version of the financial system under consideration.

between TRs define the *interbank cash market*, which enables the transfer of surplus funds among banks. Each ST will borrow (or deposit) funds required for (generated through) creation of products, and for trading and hedging processes with their respective TR. Interactions among STs define the *interbank derivatives market*, which enables hedging, arbitrage and speculative strategies within the interbank system. Individuals, corporations and government entities constitute ECs of the banking system. Functionally, this group may be categorized further as *depositors* (EDs), *borrowers* (EBs) and *end-product users* (EPUs). In this setup, EDs and EBs interact with TRs while EPUs interact with STs. These categories are not mutually exclusive, in general. The interactions between the EDs and the TRs define the *non-interbank funding market*, which is a subset of the *money market* and an important source of term funding for banks. An important entity that is not explicitly shown in Fig. B.1 is the CB, which is the regulator and central entity for all banking agents and activities. Banks may engage in repurchase (repo) transactions with the CB to settle reserve account deficits. The CB is thus considered as the lender of last resort. This facility is used in Remark 1 to argue against the possibility of a banking system default.

B.2. *Interest rate system: An axiomatic construction*

Axiom B.1 (CB policy rate). CBs enable local banks to earn interest on account surpluses or to settle account deficits via a short-term (one week or less) repo facility, offering government bonds as collateral. The respective rate for these facilities is the policy rate set by the monetary policy committee (MPC) periodically in response to changes in inflation and economic growth expectations. Assuming that δ is representative of the tenor of these transactions, it is assumed that r_u^1 denotes the MPC's repo rate and a pure risk-free rate. This rate is only accessible by local banks.

Axiom B.2 (Government bond repos). The secondary government bond market enables all participants to engage in repo, or buy-sell back, transactions. These are short-term (one year or less) collateralized loans, where the borrower offers government bonds as collateral. Suitably aggregating many such transactions, the effective simple rate for an $n\delta$-term transaction is conjectured to be

$$S_u^n := x_{u,r}^n + \ell_{u,r}^n + c_{u,r}^n, \tag{B.1}$$

and is also referred to as a secured funding rate,[v] where

(i) $x_{u,r}^n := \frac{1}{n\delta}(\mathbb{E}_u\big[\prod_{i=0}^{n-1}\big(1+\delta r_{u+i\delta}^1\big)\big]-1)$ is an $n\delta$-term risk-free rate based on lenders' expectations for the evolution of the MPC's policy rate over this period given information available at time u, expressed here via the operator $\mathbb{E}_u[\cdot]$;

(ii) $\ell_{u,r}^n$ is a liquidity spread, which is term-dependent and may be less than or equal to zero when lenders have significant surplus funds available but is generally positive, and increases with term, since funding liquidity and market liquidity is generally limited;

(iii) $c_{u,r}^n$ is a spread for potential collateral-liability mismatch, which is nonnegative and dependent on term, $x_{u,r}^n$ and $\ell_{u,r}^n$, government bond volatility and the initial loan-to-collateral value ratio.

For an understanding of liquidity risk as it is applied here, refer to Appendix A.

Axiom B.3 (Government bonds). The local currency government bond market provides a benchmark for funded default-free term rates.[w] While cash flow structures and quoted yield conventions may be nonhomogeneous, a set of consistent effective simple term rates

$$G_u^n := x_{u,g}^n + \ell_{u,g}^n, \tag{B.2}$$

may be recovered from traded government bonds. This effective rate is conjectured to constitute a risk-free component, $x_{u,g}^n$, as defined in Axiom B.2 but now based on primary market borrowers' and secondary market agents' expectations of the MPC policy rate. It also constitutes a liquidity spread, $\ell_{u,g}^n$, akin to $\ell_{u,r}^n$ from Axiom B.2 is now based on the supply of funding for local government debt in the primary market and the market liquidity of the respective government bond under consideration in the secondary market.

[v]The Secured Overnight Financing Rate (SOFR), the benchmark rate for USD-denominated derivatives and loans is derived from the US Treasury repo market.
[w]Based on the assumption that a government will not default on debt issued in its local currency.

Axiom B.4 (Term funding and interbank lending). The TR that forms part of the arbitrary banking entity i will quote a funded, credit-risky simple term rate

$$R_{u,i}^{n,N} := x_{u,i}^{n} + d_{u,i}^{n,N} + \ell_{u,i}^{n,N}, \tag{B.3}$$

to internal (i.e. ST of bank i) and ECs (other TRs and EDs) for term deposits with a nominal size equal to N and a tenor of $n\delta$ at time u. The component $x_{u,i}^{n}$ is the risk-free component according to the ith TR's policy rate expectations.[x] The components $d_{u,i}^{n,N}$ and $\ell_{u,i}^{n,N}$ are term- and nominal-dependent debt and liquidity premiums offered as compensation for bearing the credit and liquidity risk of bank i over the fixed term. The credit spread will generally be positive; however, the liquidity spread can be negative, zero or positive depending on current funding-liquidity risk and expected market- and systemic-liquidity risks.

The pricing/quoting mechanism for interbank loans are also consistent with Axiom B.4, since such a loan may be considered as one bank's TR (the borrower) quoting a deposit rate to another bank's TR (the lender), subject to the usual processes of negotiation involved in the general market-making process. With Axiom B.4, it is now possible to define an individual bank's *term funding curve*. This is a term structure of nonhomogeneous rates that is created, or market-made, by the respective bank's TR.

Definition B.1 (Idiosyncratic term funding curve). Consider an arbitrary time u, funding term $n\delta$, and banking entity i. Let $\{N_{u,i,1}^{n}, N_{u,i,2}^{n}, \ldots, N_{u,i,a}^{n}\}$ and $\{w_{u,i,1}^{n}, w_{u,i,2}^{n}, \ldots, w_{u,i,a}^{n}\}$, where $a \in \mathbb{N}$, denote a set of nominals and corresponding weights that reflect the respective likelihoods of the TR, associated with bank i, securing funding at the respective nominal values for the $n\delta$-term at time u, with $\sum_{j=1}^{a} w_{u,i,j}^{n} = 1$. Then, a fair aggregated representation of the TR's $n\delta$-term funding rate at time u is

$$\overline{R}_{u,i}^{n} := x_{u,i}^{n} + \overline{d}_{u,i}^{n} + \overline{\ell}_{u,i}^{n}, \tag{B.4}$$

where $\overline{R}_{u,i}^{n} := \sum_{j=1}^{m} w_{u,i,j}^{n} R_{u,i}^{n,N_{u,i,j}}$, with its components defined similarly. Moreover, a fair aggregated representation of the TR's term funding curve at time u is then $\{\overline{R}_{u,i}^{n}; n \in \{1, 2, \ldots, m\}\}$, with the $m\delta$-term assumed to be the longest available funding tenor.

Axiom B.5 (Reference rates). Benchmark term rates R_u^n are constructed by specific aggregate functions,[y] applied to collated indicative quotes or retrospective traded bank deposits or interbank lending data. Given the structure of such rates,

[x] Being a risk-free term rate, $x_{u,i}^{n}$ is not contingent on nominal value, by definition.

[y] For example, a trimmed median or average, which is utilized for rates such as JIBAR and LIBOR, or a volume-weighted average for reference rates such as the SAFEX overnight rate and SONIA (also trimmed).

as postulated in Axiom B.4, the aggregated benchmark is conjectured to be

$$R_u^n := x_u^n + d_u^n + \ell_u^n, \tag{B.5}$$

where x_u^n, d_u^n and ℓ_u^n are now an aggregated expectation of the risk-free rate, debt premium and liquidity spread, respectively. Accordingly, R_u^n is also referred to as an interbank reference or market-based term rate.

Appendix C. Example of an Estimated $n\delta$-Term SDF

Assume that $\{(Z_u^i)_{u\geq 0}\,; i \in \{1, 2, \ldots, \ell\}\}$ is a set of real-valued continuous-time homogeneous $\{(\mathcal{G}_u), \mathbb{P}\}$-Markov processes, which represent latent factors underpinning all tradable market variables. Then, the classical notion of a short rate process $(r_u^n)_{u\geq 0}$ and estimated SDF $(\widehat{D}_u^n)_{u\geq 0}$, associated with the $n\delta$-term rate, may be defined by

$$r_u^n := f_n(Z_u) \quad \text{and} \quad \widehat{D}_u^n := \exp(-\int_0^u r_v^n \,\mathrm{d}v),$$

respectively, for $n \in \{1, 2, \ldots, m\}$. Both quantities above are $\mathcal{G}_u$-measurable, and $f_n : \mathbb{R}^p \to \mathbb{R}$ is some suitably behaved and measurable function, with $Z_u := [Z_u^1, Z_u^2, \ldots, Z_u^p]^\intercal$, for some $p \in \{1, 2, \ldots, \ell\}$. As a practical example, assume that the set of time-homogeneous Markov processes are independent Ornstein–Uhlenbeck processes of the form

$$\mathrm{d}Z_u^i = a_i(b_i - Z_u^i)\,\mathrm{d}u + c_i\,\mathrm{d}W_u^i,$$

for $u \geq 0$, with $(W_u^i)_{u\geq 0}$ being a standard $\{(\mathcal{G}_u), \mathbb{P}\}$-Wiener process, for $i \in \{1, 2, \ldots, \ell\}$. Next, assume that the set of short rate processes are defined by $r_u^n := \sum_{i=1}^h Z_u^i$, for $n \in \{1, 2, \ldots, m\}$ and $h \leq \ell$. Then, the estimated $n\delta$-term SDF is given by $\widehat{D}_u^n = \prod_{i=1}^h \exp\left(-\int_0^u Z_v^i\,\mathrm{d}v\right)$, and

$$\begin{aligned}\widehat{P}_{u,u+n\delta}^n := \mathbb{E}^{\mathbb{P}}\left[\frac{\widehat{D}_{u+n\delta}^n}{\widehat{D}_u^n}\,\middle|\,\mathcal{G}_u\right] &= \mathbb{E}^{\mathbb{P}}\left[\prod_{i=1}^h \exp\left(-\int_0^{n\delta} Z_{u+v}^i\,\mathrm{d}v\right)\,\middle|\,\mathcal{G}_u\right] \\ &= \prod_{i=1}^h \exp\left[A_i(n\delta) - B_i(n\delta)Z_u^i\right],\end{aligned}$$

where

$$B_i(x) := \frac{1}{a_i}(1 - e^{-a_i x}) \quad \text{and} \quad A_i(x) := \left(b_i - \frac{c_i^2}{2a_i^2}\right)[B_i(x) - x] - \frac{c_i^2}{4a_i}B_i(x)^2,$$

and $x \geq 0$. It follows that if the current time is t, the set of parameters

$$\left\{\{Z_0^i, a_i, b_i, c_i\}\,; i \in \{1, 2, \ldots, h\}\right\}, \tag{C.1}$$

may be estimated using the historical set of $n\delta$-term rates $\{R^n_{t_0}, R^n_{t_1}, \ldots, R^n_{t_k}\}$, where $\{t_0, t_1, \ldots, t_k\}$ denotes the set of trading days in the interval $[0, t]$.

Appendix D. Example of a Calibrated δ-Term PK

Consider an estimated δ-term SDF as given in Appendix C, but only driven by a single Ornstein–Uhlenbeck process. Then, $\widehat{P}^1_{u,u+\delta} = \exp[A_1(\delta) - B_1(\delta)Z^1_u]$. Assuming a multiple term rate in a perfect liquidity market context, the initial δ-term ZCB term structure is given by $\{P^1_{t,t+\delta}, P^1_{t,t+2\delta}, \ldots, P^1_{t,t+m\delta}\}$, for the current time t. A calibrated δ-term PK may then be defined via a time-inhomogeneous change-of-measure from $\mathbb{P}$ to $\mathbb{P}_1$ through the density martingale

$$\mathrm{d}\Lambda^1_u = \phi_1(u)\Lambda^1_u\,\mathrm{d}W^1_u,$$

for $u \in [t, t + m\delta]$. Here, $\Lambda^1_t := 1$, and $\phi_1(\cdot)$ is a real-valued deterministic function of time, so that

$$\mathrm{d}W^1_u = \phi_1(u)\,\mathrm{d}u + \mathrm{d}W^{\mathbb{P}_1}_u,$$

where $(W^{\mathbb{P}_1}_u)_{u\geq t}$ is a standard $\{(\mathcal{G}_u), \mathbb{P}_1\}$-Wiener process. Setting $\phi_1(u) := \frac{a_1}{c_1} d_1(u)$, the factor process under the $\mathbb{P}_1$-measure becomes

$$\mathrm{d}Z^1_u = a_1(b_1 + d_1(u) - Z^1_u)\,\mathrm{d}u + c_1\,\mathrm{d}W^{\mathbb{P}_1}_u.$$

The deterministic function $d_1(u)$ may be defined by

$$d_1(u) = \begin{cases} d^1_1, & u \in [t, t+\delta], \\ d^1_2, & u \in (t+\delta, t+2\delta], \\ \vdots, & \quad\vdots\quad, \\ d^1_m, & u \in (t+(m-1)\delta, t+m\delta], \end{cases}$$

where $d^1_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, m\}$. The following result is then required:

$$\mathbb{E}^{\mathbb{P}_1}\left[\exp\left(-\int_{i\delta}^{j\delta} Z^1_{t+v}\mathrm{d}v\right)\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \exp[A^*_1(t+i\delta, t+j\delta) - B_1((j-i)\delta)Z^1_{t+i\delta}], \tag{D.1}$$

where $A^*_1(t+i\delta, t+j\delta) := A_1((j-i)\delta) + C_1(t+i\delta, t+j\delta)$ and

$$C_1(t+i\delta, t+j\delta) := \sum_{k=i+1}^{j} d^1_k[B_1((j-k+1)\delta) - B_1((j-k)\delta) - \delta],$$

for $i < j \in \{1, 2, \ldots, m\}$. Then, using this result and Lemma 1, it follows that

$$D^1_{t+\delta} = \mathbb{E}^{\mathbb{P}_1}\left[\exp\left(-\int_0^\delta Z^1_{t+v}\mathrm{d}v\right)\,\middle|\,\mathcal{G}_t\right] = \exp[A^*_1(t, t+\delta) - B_1(\delta)Z^1_t],$$

where $A^*_1(t, t+\delta) = A_1(\delta) + C_1(t, t+\delta)$ and $C_1(t, t+\delta) = d^1_1[B_1(\delta) - \delta]$, and d^1_1 may be set such that $D^1_{t+\delta} = P^1_{t,t+\delta} = P^1_{t,0,1}$. Next, using the above result and Lemma 2, it follows that

$$\begin{aligned}\frac{D^1_{t+2\delta}}{D^1_{t+\delta}} &= \mathbb{E}^{\mathbb{P}_1}\left[\exp\left(-\int_\delta^{2\delta} Z^1_{t+v}\mathrm{d}v\right)\,\middle|\,\mathcal{G}_{t+\delta}\right]\\ &= \exp\left[A^*_1(t+\delta, t+2\delta) - B_1(\delta)Z^1_{t+\delta}\right],\end{aligned}$$

where $A^*_1(t+\delta, t+2\delta) = A_1(\delta) + C_1(t+\delta, t+2\delta)$ and $C_1(t+\delta, t+2\delta) = d^1_2[B_1(\delta) - \delta]$. Thus

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_1}\left[D^1_{t+2\delta}\,\middle|\,\mathcal{G}_t\right] &= \mathbb{E}^{\mathbb{P}_1}\left[D^1_{t+\delta}\exp\left[A_1(\delta) + d^1_2[B_1(\delta) - \delta] - B_1(\delta)Z^1_{t+\delta}\right]\,\middle|\,\mathcal{G}_t\right]\\ &= D^1_{t+\delta}\exp\left[A_1(\delta) + d^1_2[B_1(\delta) - \delta]\right]\mathbb{E}^{\mathbb{P}_1}\left[\exp\left[-B_1(\delta)Z^1_{t+\delta}\right]\,\middle|\,\mathcal{G}_t\right]\\ &= P^1_{t,0,1}\exp\left[A_1(\delta) + d^1_2[B_1(\delta) - \delta]\right]\mathbb{E}^{\mathbb{P}_1}\left[\exp\left[-B_1(\delta)Z^1_{t+\delta}\right]\,\middle|\,\mathcal{G}_t\right],\end{aligned}$$

which reveals that d^1_2 is a free parameter that may be set to ensure that $\mathbb{E}^{\mathbb{P}_1}[D^1_{t+2\delta}\,|\,\mathcal{G}_t] = P^1_{t,0,2}$. In general, this construction may be repeated iteratively and it can be shown that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}_1}[D^1_{t+j\delta}\,|\,\mathcal{G}_t]\\ &\quad= P^1_{t,0,j-1}\exp[A_1(\delta) + d^1_j[B_1(\delta) - \delta]]\mathbb{E}^{\mathbb{P}_1}[\exp[-B_1(\delta)Z^1_{t+(j-1)\delta}]\,|\,\mathcal{G}_t],\end{aligned}$$

such that d^1_j is a free parameter that enables $\mathbb{E}^{\mathbb{P}_1}[D^1_{t+j\delta}\,|\,\mathcal{G}_t] = P^1_{t,0,j}$, for $j \in \{3, 4, \ldots, m\}$.

Appendix E. Multiple Term Rates With Perfect Liquidity

E.1. *Proof of Lemma* 2

Proof. While the 2δ-term rate and its associated ZCB is a fixed-term product, the prevailing assumptions along with perfect liquidity enables early liquidation via replacement. The SST, as a market-taker, could easily terminate (redeem) the loan (deposit) at time $t + \delta$, by taking an opposite position using the tradable δ-term rate. Therefore, to preclude arbitrage, Eq. (4.4) must be the fair liquidation value of the tradable 2δ-term ZCB at time $t+\delta$. At time $t+2\delta$, the calibrated δ-term SDF must have the representation

$$D^1_{t+2\delta} := \frac{1}{(1+\delta R^1_t)}\frac{1}{\left(1+\delta R^1_{t+\delta}\right)}, \tag{E.1}$$

to preclude arbitrage. Since R^1_u is $\mathcal{G}_u$-measurable, respectively, it should be clear that $D^1_{t+2\delta}$ is $\mathcal{G}_{t+\delta}$-measurable. From Eq. (4.4), it then follows that $D^1_{t+2\delta} = D^1_{t+\delta}P^1_{t,1,2}$. Finally, Eq. (4.5) follows in a similar manner to Lemma 1 with the free time-dependent parameter associated with $\left(\Lambda^1_{t+v\delta}\right)_{1<v\leq 2}$ enabling the calibration, which concludes the proof. □

E.2. *Proof of Lemma* 3

Proof. The long (short) FRA payoff, Eq. (4.6), may be replicated, at zero cost (i.e. $V_t = 0$), by borrowing (depositing) $NP^1_{t,0,1}$ at the 2δ-term rate and simultaneously depositing (borrowing) $NP^1_{t,0,1}$ at the δ-term rate at time t, and depositing (borrowing) the proceeds thereof, which is the nominal amount N, again at the δ-term rate at time $t+\delta$.[z] Using Lemma 2, it is straightforward to show that the fair value of the FRA at time $t+\delta$ is

$$V_{t+\delta} = \alpha N\left[1 - \frac{P^1_{t,0,1}}{P^2_{t,0,2}}\,P^1_{t,1,2}\right]. \tag{E.2}$$

Borrowing (depositing) $P^2_{t,0,2}$ units of currency at the δ-term rate at time t and refinancing (redepositing) the total cost (proceeds) thereof at time $t+\delta$ would cumulatively cost (yield) $P^2_{t,0,2}/D^1_{t+2\delta}$ at time $t+2\delta$. Combining this position with a long (short) FRA, with $N = P^2_{t,0,2}/D^1_{t+\delta} = P^2_{t,0,2}/P^1_{t,0,1}$, would result in a net cost (yield) of one unit of currency at time $t+2\delta$, and a net cost (yield) of $P^1_{t,1,2}$ at time $t+\delta$, using Eq. (E.2). The combined strategy therefore has a value of $P^2_{t,0,2}$ at time t and replicates the value of the 2δ-term ZCB at times $t+\delta$ and $t+2\delta$. Having the δ-term SDF as the key building block, this strategy creates a synthetic δ-term ZCB, $\left\{P^1_{t,i,2}\,; i\in\{0,1,2\}\right\}$, at time t with maturity time $t+2\delta$, such that $P^1_{t,0,2} = P^2_{t,0,2}$, $P^1_{t,2,2} = P^2_{t,2,2} = 1$ and $P^1_{t,1,2}$ is the interim fair value at time $t+\delta$, which concludes the proof. □

E.3. *Proof of Corollary* 1

Proof. The general version of the $\delta\times 2\delta$ FRA has the following terminal payoff

$$V_{t+2\delta} = \alpha N[(1+\delta R^1_{t+\delta}) - (1+\delta K)] = \alpha N\left[\frac{1}{P^1_{t,1,2}} - (1+\delta K)\right], \tag{E.3}$$

[z]Depositing (borrowing) at one of the term rates is equivalent to buying (selling) the associated ZCB.

where K is an arbitrary fixed rate specified at initiation of the contract, at time t. Using the calibrated δ-term PK from Theorem 1, the initial arbitrage-free value of the FRA is

$$\begin{aligned} V_t &= \frac{1}{\pi_t^1}\,\mathbb{E}^{\mathbb{P}}[\pi_{t+2\delta}^1 V_{t+2\delta} \,|\, \mathcal{G}_t] = \frac{1}{D_t^1}\,\mathbb{E}^{\mathbb{P}_1}[D_{t+2\delta}^1 V_{t+2\delta} \,|\, \mathcal{G}_t] \\ &= \alpha\mathbb{E}^{\mathbb{P}_1}[D_{t+\delta}^1 \,|\, \mathcal{G}_t] - \alpha(1+\delta K)\mathbb{E}^{\mathbb{P}_1}[D_{t+2\delta}^1 \,|\, \mathcal{G}_t] \\ &= \alpha P_{t,0,1}^1 - \alpha(1+\delta K) P_{t,0,2}^1, \end{aligned} \tag{E.4}$$

where the third line follows from Lemma 2, viz. $D_{t+2\delta}^1 = D_{t+\delta}^1 P_{t,1,2}^1$, and the last line by definition of the δ-term PK. Setting $V_t = 0$ and solving for the fair strike rate yields $K = (P_{t,0,1}^1 / P_{t,0,2}^1 - 1)/\delta$. Repeating this process at time $t+\delta$, for exactly the same contract, would then yield $K = (P_{t,1,1}^1 / P_{t,1,2}^1 - 1)/\delta$, as required, which concludes the proof. □

Appendix F. Systemic Liquidity Indicators

Definition F.1 (General $n\delta$-term quoted rate). Assume that u and t are quoting and trading times, respectively, with $u < t$. For a nominal amount N, the SST may model a future STR $n\delta$-term deposit/loan quote rate as

$$R_{t,\alpha,\beta}^{n,N} := R_t^n L_{t,\alpha,\beta}^{n,N}, \tag{F.1}$$

where α is equal to sgn(1) for a deposit, sgn(−1) for a loan; β is a state variable equal to 3 if the SST can source perfect liquidity, 2 if the SST can source costly liquidity, 1 if only the SST can't source liquidity and 0 if there is no systemic liquidity; and the $\mathcal{L}_t$-measurable liquidity indicator

$$L_{t,\alpha,\beta}^{n,N} := \begin{cases} 1, & \text{if } \beta = 3, \text{ with probability } q_{t,\alpha,3}^{n,N}, \\ 1 - \alpha\Delta_{t,\alpha}^{n,N} / (n\delta R_t^n P_{t,0,n}^n), & \text{if } \beta = 2, \text{ with probability } q_{t,\alpha,2}^{n,N}, \\ 0, & \text{if } \beta = 1, \text{ with probability } q_{t,\alpha,1}^{n,N}, \\ 0, & \text{if } \beta = 0, \text{ with probability } q_{t,\alpha,0}^{n,N}, \end{cases} \tag{F.2}$$

with $\Delta_{t,\alpha}^{n,N} := \Delta(t, t+n\delta, N, \alpha)$, a positive real-valued function which models the absolute future cost per unit nominal, and the probability $q_{t,\alpha,\beta}^{n,N} := q(t, t+n\delta, N, \alpha, \beta)$ both assumed to be deterministic functions. By the law of total probability $\sum_{\beta=0}^{3} q_{t,\alpha,\beta}^{n,N} = 1$, while it must also hold that $L_{t,+,0}^{n,N}(\omega) = L_{t,-,0}^{n,N}(\omega)$ a.s., so that the likelihood of systemic illiquidity is equal for both loans and deposits, i.e. $q_{t,+,0}^{n,N} = q_{t,-,0}^{n,N}$.

Proposition F.1 (Expected future value and cost of $n\delta$-term liquidity). *Consider* $\{N_{t,\alpha,1}^n, N_{t,\alpha,2}^n, \ldots, N_{t,\alpha,b}^n\}$, *a set of nominals with associated weights* $\{w_{t,\alpha,1}^n, w_{t,\alpha,2}^n, \ldots, w_{t,\alpha,b}^n\}$ *that reflect the respective likelihood of the SST engaging* *in deposit and loan transactions at such nominals at time t for a term of $n\delta$, with $\sum_{i=1}^{b} w^n_{t,\alpha,i} = 1$, where $b \in \mathbb{N}$. To ease notation here, N_i and w_i are used to denote $N^n_{t,\alpha,i}$ and $w^n_{t,\alpha,i}$, respectively, for $i \in \{1, 2, \ldots, b\}$. From the vantage point of the SST at time u, the weighted average future value at time t of an $n\delta$-term deposit/loan with unit nominal is*

$$V^n_{t,\alpha} = \alpha \left[q^n_{t,\alpha,0} \widehat{P}^n_{t,0,n} + q^n_{t,\alpha,1} P^n_{t,0,n} + q^n_{t,\alpha,2}(1 - \alpha\Delta^n_{t,\alpha}) + q^n_{t,\alpha,3} \right], \tag{F.3}$$

where the aggregated probabilities and cost function are, respectively, defined by

$$q^n_{t,\alpha,\beta} := \left(\sum_{i=1}^{b} w_i N_i q^{n,N_i}_{t,\alpha,\beta} \right) \Big/ \left(\sum_{i=1}^{b} w_i N_i \right), \text{and} \tag{F.4}$$

$$\Delta^n_{t,\alpha} := \left(\sum_{i=1}^{b} w_i N_i q^{n,N_i}_{t,\alpha,2} \Delta^{n,N_i}_{t,\alpha} \right) \Big/ \left(q^n_{t,\alpha,2} \sum_{i=1}^{b} w_i N_i \right). \tag{F.5}$$

The expected future cost of $n\delta$-term liquidity per unit nominal at time t is given by $(\alpha - V^n_{t,\alpha})$.

Proof. At time t if $\beta = 3$ then perfect liquidity prevails, the reference $n\delta$-term rate will exist and the fair value of the SST's deposit/loan will be

$$\frac{1}{\pi^n_t} \mathbb{E}^{\mathbb{P}} \left[\pi^n_{t+n\delta} \alpha N_i \left(1 + n\delta R^{n,N_i}_{t,\alpha,3} \right) \,\middle|\, \mathcal{G}_t \right] = \alpha N_i P^n_{t,0,n} \left(1 + n\delta R^{n,N_i}_{t,\alpha,3} \right) = \alpha N_i. \tag{F.6}$$

If $\beta = 2$, costly liquidity prevails, the reference $n\delta$-term rate will exist and the fair value of the SST's deposit/loan will be

$$\begin{aligned} \frac{1}{\pi^n_t} \mathbb{E}^{\mathbb{P}} \left[\pi^n_{t+n\delta} \alpha N_i \left(1 + n\delta R^{n,N_i}_{t,\alpha,2} \right) \,\middle|\, \mathcal{G}_t \right] &= \alpha N_i P^n_{t,0,n} \left(1 + n\delta R^{n,N_i}_{t,\alpha,2} \right) \\ &= \alpha N_i - N_i \Delta^{n,N_i}_{t,\alpha}. \end{aligned} \tag{F.7}$$

If $\beta = 1$, only the SST can't access liquidity, the reference $n\delta$-term rate will still exist and the fair value of the SST's position will now be

$$\frac{1}{\pi^n_t} \mathbb{E}^{\mathbb{P}} \left[\pi^n_{t+n\delta} \alpha N_i \left(1 + n\delta R^{n,N_i}_{t,\alpha,1} \right) \,\middle|\, \mathcal{G}_t \right] = \alpha N_i P^n_{t,0,n}. \tag{F.8}$$

In the case of a deposit, this represents the value foregone by not being able to access the $n\delta$-term rate. For a loan, this represents the value gained by having to settle a liability early as opposed to deferring payment by accessing funding through the $n\delta$-term rate. When $\beta = 0$, there is no systemic liquidity and therefore no reference $n\delta$-term rate. In this scenario, the SST may estimate the fair value of their position as

$$\frac{1}{\widehat{D}_t^n}\mathbb{E}^{\mathbb{P}}\left[\widehat{D}_{t+n\delta}^n \alpha N_i(1+n\delta R_{t,\alpha,0}^{n,N_i}) \,\middle|\, \mathcal{G}_t\right] = \alpha N_i \widehat{P}_{t,0,n}^n, \tag{F.9}$$

by making use of the estimated $n\delta$-term SDF. Then, the estimated value at time t is

$$V_{t,\alpha}^{n,N_i} = \alpha N_i \left[q_{t,\alpha,0}^{n,N_i}\widehat{P}_{t,0,n}^n + q_{t,\alpha,1}^{n,N_i}P_{t,0,n}^n + q_{t,\alpha,2}^{n,N_i}(1-\alpha\Delta_{t,\alpha}^{n,N_i}) + q_{t,\alpha,3}^{n,N_i}\right], \tag{F.10}$$

and the weighted average future value equation (F.3) is recovered by setting

$$V_{t,\alpha}^n := \left(\sum_{i=1}^b w_i V_{t,\alpha}^{n,N_i}\right) \Big/ \left(\sum_{i=1}^b w_i N_i\right), \tag{F.11}$$

which holds for both $n\delta$-term loans or deposits, and concludes the proof. □

Corollary F.1 (Mid-expected future value and cost of $n\delta$-term liquidity). *From Eq.* (F.3), *it follows that the mid-weighted average future value is*

$$V_t^n = \alpha\left[q_{t,0}^n\widehat{P}_{t,0,n}^n + q_{t,1}^n P_{t,0,n}^n + q_{t,2}^n(1+\epsilon_t^n) + q_{t,3}^n\right], \tag{F.12}$$

where the mid-probabilities and cost function are, respectively, defined by

$$q_{t,\beta}^n := \frac{1}{2}(q_{t,+,\beta}^n + q_{t,-,\beta}^n), \quad \text{and} \tag{F.13}$$

$$\epsilon_t^n := \frac{1}{2q_{t,2}^n}(q_{t,-,2}^n\Delta_{t,-}^n - q_{t,+,2}^n\Delta_{t,+}^n), \tag{F.14}$$

with the expected future cost of $n\delta$-term liquidity per unit nominal now given by $(\alpha - V_t^n)$.

Proof. Setting $V_t^n := \alpha(V_{t,+}^n - V_{t,-}^n)/2$ yields the mid-future value equation (F.12), with the expected future cost of liquidity result then following trivially. □

Remark F.1 (Spread quantity ϵ_t^n). Having constructed a mid-value in Corollary F.1, the quantity ϵ_t^n may be interpreted as the mid-value of the bid-offer spread associated with $n\delta$-term deposit and loan liquidity at time t, suitably weighted by the probability of each transaction at specific nominals, from the perspective of the SST. The magnitude and sign of this mid-spread depends on the funding market climate. In particular, one would expect

(1) $\epsilon_t^n > 0$, in a *stressed market* where the STR has difficulty sourcing term funding;
(2) $\epsilon_t^n \approx 0$, in a *normal market* where $\Delta_{t,+}^n$ and $\Delta_{t,-}^n$ may be attributed to profit margins;

(3) $\epsilon_t^n < 0$, in a *stressed market* where the STR has excess access to term funding, a scenario that is most likely to realize for near or shorter terms-to-maturity.

Remark F.2 (Systemic liquidity indicators). Proposition F.1 and Corollary F.1 have enabled the aggregation of the nominal effect in the general $n\delta$-term quoted rates, as well as the averaging of the spread asymmetry due to loans and deposits. The structure of the mid-expected future value, Eq. (F.12), indicates that a simpler symmetric and systemic specification for the liquidity indicator will suffice, especially under the assumption of a normal market, or $\epsilon_t^n \approx 0$. Therefore, in order to ease the exposition, the $\mathcal{L}_t$-measurable random variable

$$L_t^n := \begin{cases} 1, & \text{perfect systemic liquidity with probability } q_t^n, \\ 0, & \text{no systemic liquidity with probability } 1 - q_t^n, \end{cases} \tag{F.15}$$

models systemic $n\delta$-term liquidity at time t, with $q_t^n := q(t, t + n\delta)$ being a deterministic function that determines the probability of perfect systemic liquidity. With this indicator, states $\beta \in \{2, 3\}$ and $\beta \in \{0, 1\}$ of the general indicator are essentially combined, with $V_t^n = \alpha \left[(1 - q_t^n)\widehat{P}_{t,0,n}^n + q_t^n\right]$ now being the mid-expected future value of $n\delta$-term liquidity per unit nominal at time t.

A more general version of the systemic liquidity indicator, defined in Remark F.2, which incorporates the state of costly liquidity is considered next. Using the definition of this new liquidity indicator, the lemma below reveals the impact of costly liquidity on the fair $\delta \times 2\delta$ FRA rate derived in Lemma 4.

Definition F.2 (Costly systemic liquidity indicators). At time $u \in \mathbb{R}_{\geq 0}$, the random variable

$$C_u^n := \begin{cases} 0, & \text{no systemic liquidity with probability } q_{u,0}^n, \\ 1, & \text{perfect systemic liquidity with probability } q_{u,1}^n, \\ 1 + \epsilon_u^n, & \text{costly systemic liquidity with probability } q_{u,2}^n, \end{cases} \tag{F.16}$$

models $n\delta$-term systemic liquidity, where $\epsilon_u^n \in \mathbb{R}$ is the deterministic spread quantity defined in Corollary F.1 and Remark F.1. If the current time is t, then the natural filtration associated with liquidity is

$$\mathcal{L}_t := \sigma(\{\{C_u^1, C_u^2, \ldots, C_u^m\}\,;\, u \in \{t_0, t_1, \ldots, t_k\}\}), \tag{F.17}$$

where $\{t_0, t_1, \ldots, t_k\}$ denotes the set of trading days that lie within the interval $[0, t]$. These costly systemic liquidity indicators are assumed to exhibit both serial and cross-sectional independence, or more formally

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[C_u^n \,\middle|\, \mathcal{L}_t \cap \sigma\left(\{C_u^n \notin \{0,1,1+\epsilon_u^n\}\}\right)\right] &= \mathbb{E}^{\mathbb{P}}\left[C_u^n\right] \\ &= \mathbb{P}\left[C_u^n = 1\right] + (1+\epsilon_u^n)\,\mathbb{P}\left[C_u^n = 1+\epsilon_u^n\right] \\ &= q_{u,1}^n + q_{u,2}^n\left(1+\epsilon_u^n\right), \qquad \text{(F.18)}\end{aligned}$$

for all $t \leq u$, with $q_{u,i}^n := q_i(u, u+n\delta)$ being a deterministic function for $i \in \{0,1,2\}$.

Lemma F.1 ($\delta \times 2\delta$ FRA pricing under potentially costly liquidity). *The fair strike rate process for the general version of the $\delta \times 2\delta$ FRA defined in Lemma 3 is*

$$\overline{F}_{t,i,1}^1 = \begin{cases} F_{t,0,1}^1\left[q_{t+\delta,1}^1 + q_{t+\delta,2}^1(1+\epsilon_{t+\delta}^1)\right], & i=0 \text{ and conditional on } C_t^1 = C_t^2 = 1, \\ F_{t,1,1}^1, & i=1 \text{ and conditional on } C_{t+\delta}^1 = 1, \end{cases} \qquad \text{(F.19)}$$

which is also $\mathcal{G}_{t+i\delta}$-measurable.

Proof. Assuming that $\mathcal{L}_t = \mathcal{L}_{t^-} \vee \sigma(\{C_t^1 = 1, C_t^2 = 1\})$, the FRA replication strategy yields $\widetilde{V}_{t+2\delta} = V_{t+2\delta} - \alpha N\delta(1 - C_{t+\delta}^1)R_{t+\delta}^1$, as was the case in Lemma 4, with the proof following similarly. □

Appendix G. Liquidity-Dependent Pricing Kernels

Definition G.1 ($n\delta$-term LDPK over a single period). At the current time t, under the CPL or CPFL liquidity regime the $n\delta$-term LDPK is defined by

$$\widetilde{\pi}_{t+in\delta}^n := \pi_{t+in\delta}^n, \qquad \text{(G.1)}$$

for $i \in \{0,1\}$, i.e. perfect liquidity enables the definition of the $n\delta$-term PK. This scenario therefore allows a market-taker, such as the SST, to access $n\delta$-term liquidity. If the NPFL or NPL liquidity regime prevails then the $n\delta$-term LDPK is given by

$$\widetilde{\pi}_{t+in\delta}^n := \widehat{D}_{t+in\delta}^n, \qquad \text{(G.2)}$$

for $i \in \{0,1\}$, i.e. no liquidity requires the estimation of an $n\delta$-term rate using the respective estimated SDF. This scenario therefore requires market-making to create liquidity. Under the PPL liquidity regime with the $n\delta$-term being illiquid but the $i\delta$- and $j\delta$-terms being liquid such that $j < n < k$, the SST may define the $n\delta$-term LDPK by

$$\widetilde{\pi}_{t+in\delta}^n := \Lambda_{t+in\delta}^n D_{t+in\delta}^n, \qquad \text{(G.3)}$$

where the $\{(\mathcal{G}_u), \mathbb{P}\}$-martingale $(\Lambda^n_{t+vn\delta})_{0\leq v\leq 1}$ must be chosen so that

$$\mathbb{E}^{\mathbb{P}_n}[\widehat{D}^n_{t+n\delta}\,|\,\mathcal{G}_t] = D^n_{t+n\delta} \in (D^k_{t+k\delta}\,,\, D^j_{t+j\delta}), \tag{G.4}$$

in order to ensure positive forward rates over $[t+j\delta, t+k\delta]$. Therefore, this scenario also requires market-making to create liquidity, however, liquidity in the other adjacent terms provides information to create an arbitrage-free range for the calibrated SDF.

Definition G.2 ($n\delta$-term LDPK over multiple periods). At the current time t, under the CPFL liquidity regime the $n\delta$-term LDPK is defined by

$$\widetilde{\pi}^n_{t+i\delta} := \pi^n_{t+i\delta}, \tag{G.5}$$

for $i \in \mathbb{N}^n_{0,m+n}$, i.e. perfect liquidity enables the definition of the $n\delta$-term PK over the interval $[t, t+m\delta]$ if m mod $n = 0$, or $[t, t+m\delta)$ otherwise. If the CPL or PPL (as defined in Definition G.1) liquidity regime prevails then

$$\widetilde{\pi}^n_{t+i\delta} = \begin{cases} \pi^n_{t+i\delta}, & i \in \mathbb{N}^n_{0,2n}\ , \\ \widehat{D}^n_{t+i\delta}, & i \in \mathbb{N}^n_{2n,m+n}, \end{cases} \tag{G.6}$$

i.e. potential future illiquidity requires market-making beyond the first period. If the NPL or NPFL liquidity regime prevails then the $n\delta$-term LDPK is defined by

$$\widetilde{\pi}^n_{t+i\delta} := \widehat{D}^n_{t+i\delta}, \tag{G.7}$$

for $i \in \mathbb{N}^n_{0,m+n}$, i.e. no present and no/uncertain future liquidity requires market-making to create liquidity.

Appendix H. Reduced-Form Model Development

H.1. *Proof of Lemma* 6

Proof. At time $t+i\delta$, the present value of one unit of currency due at time $t+j\delta$ is equal to

$$P^{j-i}_{t,i,j} = \frac{1}{1+(j-i)\delta R^{j-i}_{t+i\delta}}, \tag{H.1}$$

provided that $i \leq j \leq m$, according to Assumption 2. Therefore, considering a synthetic $n\delta$-term LCZCB with tenor less than or equal to $n\delta$, i.e. $(j-i) \leq n$, it follows that $\overline{P}^{(n)}_{t,i,j} = P^{j-i}_{t,i,j}$, which is the result shown in Eq. (6.10).

For the case of $n < (j-i) \leq m$ and $\mathrm{mod}(j-i, n) = h$, $h \in \mathbb{N}_{0,n-1}$, a synthetic $n\delta$-term LCZCB with $(j-i)\delta$-tenor may be constructed using Assumptions 2–4 as follows. At $t+i\delta$, if $h > 0$

(i) Borrow (deposit) M units of currency at the $h\delta$-term rate.

(0) Long (short) the $h\delta \times (h+n)\delta$ fair FRA with nominal equal to $M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}$.

(1) Long (short) the $(h+n)\delta \times (h+2n)\delta$ fair FRA with nominal equal to $M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\left(1+n\delta\overline{F}^n_{t,i,i+h}\right)$.

$\vdots$ $\qquad\qquad$ $\vdots$

(N) Long (short) the $(j-n)\delta \times j\delta$ fair FRA with nominal equal to

$$M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}(1+n\delta\overline{F}^n_{t,i,i+h})(1+n\delta\overline{F}^n_{t,i,i+h+n})\ldots(1+n\delta\overline{F}^n_{t,i,j-2n}). \tag{H.2}$$

At time $t+(i+h)\delta$, using Assumption 4

(i) The loan (deposit) matures which costs (yields): $M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}$.

(ii) Refinance (redeposit) the costs (proceeds) from (i) at the $n\delta$-term rate.

At time $t+(i+h+n)\delta$, using Assumption 4

(ii) The loan (deposit) matures which costs (yields): $M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\frac{D^n_{t+(i+h)\delta}}{D^n_{t+(i+h+n)\delta}}$.

(0) The long (short) FRA payoff: $(-)M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\left[\frac{D^n_{t+(i+h)\delta}}{D^n_{t+(i+h+n)\delta}}-(1+n\delta\overline{F}^n_{t,i,i+h})\right]$.

(iii) Add (ii) and (0), and refinance (redeposit) the costs (proceeds) at the $n\delta$-term rate.

Repeating this process at each time $t+(i+h+nk)\delta$, for $k=2,3,\ldots,(j-i-n-h)/n$, will eventually result in a total cost (yield) equal to

$$M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}(1+n\delta\overline{F}^n_{t,i,i+h})(1+n\delta\overline{F}^n_{t,i,i+h+n})\ldots(1+n\delta\overline{F}^n_{t,i,j-n}), \tag{H.3}$$

at time $t+j\delta$, which is measurable at time $t+i\delta$. This strategy may then be used to create the synthetic $n\delta$-term LCZCB with $(j-i)\delta$-tenor by setting

$$M := \frac{D^h_{t+(i+h)\delta}}{D^h_{t+i\delta}}(1+n\delta\overline{F}^n_{t,i,i+h})^{-1}(1+n\delta\overline{F}^n_{t,i,i+h+n})^{-1}\ldots(1+n\delta\overline{F}^n_{t,i,j-n})^{-1}, \tag{H.4}$$

and therefore $\overline{P}^{(n)}_{t,i,j}=M$, which is consistent with Eq. (6.9) for $h>0$. If $h=0$ then the relevant contracts are the $(k+n)\delta \times (k+2n)\delta$ FRA contracts for $k=0,n,2n,\ldots,(j-i-2n)$. The same strategy may be employed for $h=0$, which completes the proof. □

Acknowledgments

We are grateful to A. Backwell, E. Schlögl and D. Skovmand for discussions on term structures of interest rates. We also thank seminar and conference participants at the SFRA Colloquium and Workshop, ICMS, Edinburgh, UK (February 2019), the Research in Options 2019 Conference, IMPA, Rio de Janeiro, Brazil (December 2019), at the Union Bank of Switzerland (UBS), London, UK (February 2020), and the second and fourth Interest Rate Reform Conferences, World Business Strategies (WBS), London, UK (March 2020 and October 2021) for comments and suggestions. We thank two anonymous reviewers for a detailed reading of our paper and their constructive comments. This work is dedicated to the memory of Tom Hurd, dear friend and much valued member of the international Financial Mathematics community.

ORCID

Andrea Macrina https://orcid.org/0000-0001-6957-270X

Obeid Mahomed https://orcid.org/0000-0001-8509-199X

References

C. Acerbi & G. Scandolo (2008) Liquidity risk theory and coherent measures of risk, *Quantitative Finance* **8** (7), 681–692.

V. V. Acharya, D. Gale & T. Yorulmazer (2011) Rollover risk and market freezes, *Journal of Finance* **66** (4), 1177–1209.

V. V. Acharya & D. Skeie (2011) A model of liquidity hoarding and term premia in interbank markets, *Journal of Monetary Economics* **58**, 436–447.

M. Alfeus, M. Grasselli & E. Schlögl (2020) A consistent stochastic model of the term structure of interest rates for multiple tenors, *Journal of Economic Dynamics and Control* **114**, 103861.

A. Backwell, A. Macrina, E. Schlögl & D. Skovmand (2023) Term rates, multicurve term structures and overnight rate benchmarks: A roll-over risk approach, *Frontiers of Mathematical Finance* **2** (3), 340–384.

E. Beau, J. Hill, T. Hussain & D. Nixon (2014) Bank funding costs: What are they, what determines them and why do they matter? Bank of England Quarterly Bulletin Q4.

M. Bianchetti & M. Carlicchi (2011) Interest rates after the credit crunch: Multiple-curve vanilla derivatives and SABR. arXiv:1103.2567.

M. K. Brunnermeier & L. H. Pedersen (2009) Market liquidity and funding liquidity, *Review of Financial Studies* **22** (6), 2201–2238.

J. Y. Campbell (1986) A defense of traditional hypotheses about the term structure of interest rates, *Journal of Finance* **41** (1), 183–193.

J. H. Cochrane (2005) *Asset Pricing*. Princeton: Princeton University Press.

P. Collin-Dufresne & B. Solnik (2001) On the term structure of default premia in the swap and LIBOR markets, *Journal of Finance* **56** (3), 1095–1115.

G. M. Constantinides (1992) A theory of the nominal term structures of interest rates, *The Review of Financial Studies* **5**, 531–552.

J. C. Cox, J. E. Ingersoll & S. A. Ross (1981) A re-examination of traditional hypotheses about the term structure of interest rates, *Journal of Finance* **36** (4), 769–799.

D. Duffie (2001) *Dynamic Asset Pricing Theory*. Princeton: Princeton University Press.

J. Eisenschmidt & J. Tapking (2009) Liquidity risk premia in unsecured interbank money markets. Technical Report 1025, European Central Bank.

D. Filipović & A. B. Trolle (2013) The term structure of interbank risk, *Journal of Financial Economics* **109**, 707–733.

J. Gallitschke, S. Seifried & F. T. Seifried (2017) Interbank interest rates: Funding liquidity risk and XIBOR basis spreads, *Journal of Banking and Finance* **78**, 142–152.

D. Gefang, G. Koop & S. M. Potter (2011) Understanding liquidity and credit risks in the financial crisis, *Journal of Empirical Finance* **18**, 903–914.

Z. Grbac & W. J. Runggaldier (2015) *Interest Rate Modeling: Post-Crisis Challenges and Approaches*, SpringerBriefs in Quantitative Finance. Cham: Springer.

B. Guggenheim (2020) At the crossroads in the transition awa from LIBOR: From overnight to term rates. BIS Working Papers No. 819. https://www.bis.org/publ/work891.pdf.

P. J. Hunt & J. E. Kennedy (2000) *Financial Derivatives in Theory and Practice*. Hoboken: Wiley.

T. R. Hurd (2016) *Contagion! Systemic Risk in Financial Networks*, SpringerBriefs in Quantitative Finance. Cham: Springer.

S. Klingler & O. Syrstad (2021) Life after LIBOR, *Journal of Financial Economics* **141** (2), 783–801.

A. Macrina & O. Mahomed (2018) Consistent valuation across curves using pricing kernels, *Risks* **6** (18), 1–32.

F. Mercurio (2009) Interest rates and the credit crunch: New formulas and market models. Bloomberg Portfolio Research Paper No. 2010-01-FRONTIERS.

M. Morini (2009) Solving the puzzle in the interest rate market. SSRN: 1506046.

B. Nelson (2020) A replacement for LIBOR as a loan benchmark would ideally be sensitive to liquidity risk as well as credit risk. Technical Report, Bank Policy Institute.

A. Schrimpf & V. Sushko (2019) Beyond LIBOR: A primer on the new benchmark rates. BIS Quarterly Review March.

K. Schwarz (2019) Mind the gap: Disentangling credit and liquidity in risk spreads, *Review of Finance* **23** (3), 557–597.

J. B. Skov & D. Skovmand (2021) Dynamic term structure models for SOFR futures, *Journal of Futures Markets* **41**, 1520–1544.