Commun. Math. Phys. (2024) 405:46
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-024-04934-7

Communications in
Mathematical Physics

Spectral Asymptotic Properties of Semiregular Non-commutative Harmonic Oscillators

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Received: 5 January 2023 / Accepted: 4 January 2024

Abstract: We study here the spectral Weyl asymptotics for a semiregular system, extending to the vector-valued case results of Helffer and Robert, and more recently of Doll, Gannot and Wunsch. The class of systems considered here contains the important example of the Jaynes–Cummings system that describes light-matter interaction.

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Published online: 15 February 2024

1. Introduction

In this paper we will establish a result about the asymptotics of the Weyl spectral counting function $\mathsf{N}(\lambda)$ of a class of semiregular (see Sect. 2) globally elliptic pseudodifferential $N \times N$ systems (of order 2) that contains the important model of the Jaynes–Cummings system (see Sect. 3.1) that describes the interaction of light and matter (see [21]). The class we consider extends to a semiregular case (with scalar principal symbol) that of non-commutative harmonic oscillators (NCHOs) introduced by Parmeggiani and Wakayama in [16–18] (see also [14,15]). Namely, while the pseudodifferential class considered in [14] had a step $-2j$ in the homogeneity of the jth-term in the asymptotic expansion of the symbol, we consider here a step $-j$, an example of such a scaling in homogeneity being in fact the symbol of the Jaynes–Cummings Hamiltonian (we call *semiregular* this kind of classical symbols).

In the scalar case, this kind of asymptotics for global operators was initially established by Chazarain [3] (in a semiclassical setting) and then generalized by Shubin [22] (see also Hörmander [8]), and Helffer and Robert [6] (see also [19] and Helffer's book [7], and references therein), and more recently made more precise by Doll, Gannot and Wunsch in [4] (see also Doll and Zelditch [5] for a precise study of the singularities of the trace of the Schrödinger propagator).

In general, the importance of having asymptotics of $\mathsf{N}(\lambda)$ lies in the fact that passing to the inverse gives asymptotics of large eigenvalues. Hence, the more precise is the asymptotics of $\mathsf{N}(\lambda)$, the more precise is the asymptotics of λ_{N}. On the strict physical level, the importance of $\mathsf{N}(\lambda)$ is due to the fact that it determines the number of states of the system per unit energy as a function of the energy itself.

The asymptotics of the Weyl spectral counting function will be given in terms of the symbol of the system, and more precisely in terms of the principal part, the semiprincipal part and the subprincipal part (respectively the terms of order 2, 1 and 0 in the asymptotic expansion of the symbol). We will show that one can blockwise diagonalize (through a decoupling theorem) the system so as to be able to implement the scalar results mentioned above. This is, however, not straightforward, since we have to compose certain Fourier integral operators and ψdos in the Weyl calculus keeping track of the (matrix) symbols.

We will be retaining the notation relative to the Hörmander-Weyl pseudodifferential calculus (also in the semiclassical case) as in Parmeggiani [14] (see also [15] and [13]).

The plan of the paper is the following. In the next section we briefly recall the class of semiregular symbols and its main properties that we will be using in this paper. We then define the class of systems we will be concerned with here. In Sect. 3, we recall the Jaynes–Cummings model and its variations to encompass also atoms with several energy levels. We show that it is possible to associate with such systems coming from physics, geometrical models related to complexes of vector-valued differential forms. This is interesting in our opinion, for it shows that very likely higher Lie groups of symmetries are allowed in the theory. In Sect. 5, we state and prove the decoupling theorem, which shows that for the class we consider here it is possible to obtain a pseudodifferential block-reduction of the system. This is fundamental in the study of a parametrix of the Schrödinger flow associated with our system, which in turn is the basic object to study for obtaining the Weyl asymptotics we are interested in. The decoupling theorem will be stated both in the semiclassical case and in the semiregular case, and the proof given in the semiclassical setting (in fact, it will be useful, for further projects, to have also the semiclassical version). Since the subprincipal part enters the picture, we discuss in Sect. 6 the transformation properties of the subprincipal symbol of the system, along with its transformation law when changing a "gauge" (that is, when changing the unitary symbol which diagonalizes the semiprincipal part). In Sect. 7, we will state and prove the basic Weyl asymptotic results, the first one extending to our class of systems the asymptotics due to Helffer and Robert [6], and the second presenting a better error term when the zero-set of the determinant of the semiprincipal part has small dimension (see Theorems 7.8 and 7.9 below). The results are based on the construction of a reduced propagator, following the approach of Doll, Gannot and Wunsch [4], and it is here that the diagonalization theorem plays a fundamental role. The extension to systems, however, is not for free, for we have to control the conjugation of the Fourier integral operators (FIOs) with quadratic phases by the pseudodifferential diagonalizers, without losing the symbol-calculus properties. This point is very delicate and we follow here the approach of Doll and Zelditch [5], having, however, to adapt it to our case.

In the closing Sect. 8, we shall show the resulting asymptotics in the 2×2 Jaynes–Cummings system, and of a perturbation of the 3×3 Jaynes–Cummings system and its 6×6 geometric counterpart.

2. Semiregular Symbols and Our Class

We give in this section the definition of semiregular symbols that we will be considering throughout this paper, recall their basic properties and then introduce the class of systems we consider here.

In order to prepare the ground also to the study of extensions of this kind of systems to more general classes of systems, we will be using the following notation for the Hörmander metric and admissible weight (see Hörmander [9]): with $X = (x, \xi)$, $Y = (y, \eta)$ etc. belonging to the phase-space $\mathbb{R}^n \times \mathbb{R}^n$, and $m(X) := \langle X \rangle = (1 + |X|^2)^{1/2}$ the usual "Japanese bracket", we consider the Hörmander metric $g_X = |dX|^2/m(X)^2$. Then m is an admissible function (and so is m^μ for any given $\mu \in \mathbb{R}$), and we may exploit the full power of the Weyl-Hörmander pseudodifferential calculus.

We will write $\dot{\mathbb{R}}^{2n}$ for $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$.

Definition 2.1. Let M_N denote the algebra of $N \times N$ complex matrices. A symbol $a \in S(m^\mu, g; \mathsf{M}_N)$ is said to be classical (see Remark 3.2.4 of [14]) if it possesses an asymptotic expansion $\sum_{j\geq 0} a_{\mu-2j}$ in isotropic (i.e. positively homogenous and smooth outside the origin) terms $a_{\mu-2j}$ positively homogeneous of degree $\mu - 2j$. We write $a \in S_{\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$.
We say that $a \in S(m^\mu, g; \mathsf{M}_N)$ is semiregular if $a = a^{(0)} + a^{(1)}$, where $a^{(0)} \in S_{\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ and $a^{(1)} \in S_{\mathrm{cl}}(m^{\mu-1}, g; \mathsf{M}_N)$. We write $a \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$.
In other words, a symbol a is semiregular if it possesses an asymptotic expansion $\sum_{j\geq 0} a_{\mu-j}$ in isotropic terms $a_{\mu-j}$ positively homogeneous of degree $\mu - j$.
The terms a_μ, $a_{\mu-1}$ and $a_{\mu-2}$ are called the principal symbol, the semiprincipal symbol and the subprincipal symbol, respectively, of the operator $a^{\mathrm{w}}(x, D)$.

Hence, $a \in S_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$ if there exists a sequence $(a_{\mu-j})_{j\geq 0} \subset C^\infty(\dot{\mathbb{R}}^{2n}; \mathsf{M}_N)$ where $a_{\mu-j}$ is positively homogeneous of degree $\mu - j$ (in X) and, for an excision function χ,

$$a - \chi \sum_{j=0}^{N} a_{\mu-j} \in S(m^{\mu-(N+1)}, g; \mathsf{M}_N), \quad \forall N \in \mathbb{Z}_+.$$

As usual, we write

$$a \sim \sum_{j\geq 0} a_{\mu-j}.$$

Comment on the notation. Helffer in [7] and the authors of [4] and of [5] use $\Gamma_{\rm cl}$ for the set of semiregular symbols. We decided to adopt our notation $S_{\rm sreg}$ because, as the natural homogeneity of the Poisson bracket of homogeneous symbols is the sum of the orders minus 2, it is natural in the global calculus to call "regular" those symbols whose asymptotic expansion is made of homogeneous symbols for which the j-th term has order $\mu - 2j$ where μ is the order of the principal term. This is indeed parallel to the use of "semiregular" appearing in the paper by Boutet de Monvel [2] on the hypoellipticity of the $\bar{\partial}$ operator.

Remark 2.2. It is clear that composition of semiregular symbols yields a semiregular symbol.

Of course, when the symbol $a \in S(m^\mu, g; \mathsf{M}_N)$ is Hermitian, then the corresponding pseudodifferential operator $a^{\rm w}(x, D)$, obtained by Weyl-quantizing $a(X)$, is formally self-adjoint. We write $\Psi(m^\mu, g; \mathsf{M}_N)$, resp. $\Psi_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$, for the ψdos obtained by Weyl-quantization of symbols in $S(m^\mu, g; \mathsf{M}_N)$, resp. $S_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$.

Definition 2.3. A symbol $a \in S_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$ is said to be globally elliptic when its principal part a_μ satisfies

$$|\det(a_\mu(X))| \approx |X|^{\mu N}, \quad \forall X \in \dot{\mathbb{R}}^{2n}.$$

When a is globally elliptic, we will say that the corresponding ψdo $a^{\rm w}(x, D)$ is globally elliptic.

As usual, for $A, B > 0$, we write $A \lesssim B$ when there is $C > 0$ such that $A \leq CB$, and write $A \approx B$ when there are $C, C' > 0$ such that $CA \leq B \leq C'A$.

When $\mu > 0$ and $a = a^* \in S_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$ is globally elliptic (hence, $a^*_{\mu-j} = a_{\mu-j}$ for all $j \geq 0$ and a_μ is globally elliptic), the existence of a (semiregular) two-sided parametrix yields that $a^{\rm w}(x, D)$, realized as an unbounded operator on $L^2(\mathbb{R}^n; \mathbb{C}^N)$ with maximal domain the Shubin Sobolev space $B^\mu(\mathbb{R}^n; \mathbb{C}^N)$ (see [22], or [7] or [12]), is *self-adjoint* with a discrete spectrum. When furthermore $a_\mu > 0$ (as a Hermitian matrix), then $a^{\rm w}(x, D)$ is semibounded from below, and hence it has a spectrum bounded from below.

We are now in a position to introduce the class of systems we are interested in.

Definition 2.4. We say that an $N \times N$ symbol $a \in S_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$ is a **semiregular metric globally elliptic system** (SMGES for short) of order μ, when

$$a(X) = a(X)^* = p_\mu(X) I_N + a_{\mu-1}(X) + a_{\mu-2}(X) + S_{\rm sreg}(m^{\mu-3}, g; \mathsf{M}_N), \quad X \neq 0,$$

where:

- $p_\mu \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$ is positively homogeneous of degree μ and such that $|X|^\mu \approx p_\mu(X)$ for all $X \neq 0$ (below, when $\mu = 2$ we will always take p_2 to be the standard harmonic oscillator $p_2(X) = |X|^2/2$);
- $a_{\mu-1} = a_{\mu-1}^*$ is such that there exists $r \geq 1$ and $e_0 \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathsf{M}_N)$ **unitary** and positively homogeneous of degree 0 such that

$$e_0(X)^* a_{\mu-1}(X) e_0(X) = \mathsf{diag}(\lambda_{\mu-1,j}(X) I_{N_j}; 1 \leq j \leq r), \quad X \neq 0,$$

where $N = N_1 + N_2 + \ldots + N_r$ and $\lambda_{\mu-1,j} \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$ are positively homogeneous of degree $\mu - 1$ and such that

$$j < k \Longrightarrow \lambda_{\mu-1,j}(X) < \lambda_{\mu-1,k}(X), \quad \forall X \neq 0.$$

3. Some Models

We give here a few examples of semiregular NCHOs in the class SMGES, relevant to Quantum Optics (see [21]), that serve as a model of the class we consider in this paper. We will then show that they can indeed be set into a geometric framework, giving rise to connections on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$ which are in general non-flat.

It turns then out that the JC models in the various configurations (see below) are actually covariant Laplacians (the operators $\Box_k^{(N)}$ that will be introduced below).

The existence of such geometric generalizations suggests, as pointed out in the Introduction, that one may consider higher groups of symmetries (e.g. $\mathrm{SU}(N)$, $N \geq 3$, instead of $\mathrm{SU}(2)$ as in the JC model).

It will be convenient to use the following notation. We denote by $\boldsymbol{\sigma}_j$, $j = 0, \ldots, 3$, the Pauli-matrices, i.e.

$$\boldsymbol{\sigma}_0 = I_2, \quad \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$\boldsymbol{\sigma}_\pm = \frac{1}{2}(\boldsymbol{\sigma}_1 \pm i\boldsymbol{\sigma}_2).$$

Let $\langle \cdot, \cdot \rangle$ be the canonical Hermitian product in $\mathbb{C}^N$, and $e_1, \ldots, e_N$ be the canonical basis of $\mathbb{C}^N$. Let

$$E_{jk} := e_k^* \otimes e_j, \quad 1 \leq j, k \leq N,$$

be the basis of $\mathsf{M}_N(\mathbb{C}) = \mathfrak{gl}(N, \mathbb{C})$, where E_{jk} acts on $\mathbb{C}^N$ as

$$E_{jk} w = \langle w, e_k \rangle e_j, \quad w \in \mathbb{C}^N.$$

Hence we have the relation

$$E_{jk} E_{hm} = (e_k^* \otimes e_j)(e_m^* \otimes e_h) = e_k^*(e_h)(e_m^* \otimes e_j) = \langle e_h, e_k \rangle (e_m^* \otimes e_j) = \delta_{hk} E_{jm}.$$

We also let, for $X = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$,

$$\psi_j(X) := \frac{x_j + i\xi_j}{\sqrt{2}}, \quad 1 \leq j \leq n,$$

so that $\psi_j^{\mathrm{w}}(x, D)$ is the annihilation operator and $\psi_j^{\mathrm{w}}(x, D)^* = (\bar{\psi}_j)^{\mathrm{w}}(x, D)$ is the creation operator, with respect to the j-th variable. Hence, with $p_2(X) = |X|^2/2$ being the (standard) harmonic oscillator,

$$\sum_{j=1}^{n} \psi_j^{\mathrm{w}}(x, D)^* \psi_j^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) - \frac{n}{2}.$$

We will also have to consider $2N \times 2N$ matrices of the form $\boldsymbol{\sigma}_j \otimes E_{jk}$, in which case the product is given by

$$(\boldsymbol{\sigma}_j \otimes E_{hk})(\boldsymbol{\sigma}_{j'} \otimes E_{h'k'}) = \boldsymbol{\sigma}_j \boldsymbol{\sigma}_{j'} \otimes E_{hk} E_{h'k'},$$

and the action on a vector $w \in \mathbb{C}^{2N}$, written as

$$w = \sum_{j=1}^{N} \begin{bmatrix} w_{2j-1} \\ w_{2j} \end{bmatrix} \otimes e_j,$$

given by

$$(\boldsymbol{\sigma}_m \otimes E_{hk})w = \sum_{j=1}^{N} (\boldsymbol{\sigma}_m \begin{bmatrix} w_{2j-1} \\ w_{2j} \end{bmatrix}) \otimes (E_{hk} e_j).$$

We next list a few important models due to Jaynes and Cummings, recalling at the same time the physics that leads to the mathematical expression of the first of them (the 2×2 JC-model).

3.1. The Jaynes–Cummings (JC) model. This is the model of a two-level atom in one cavity, given by the 2×2 system in one real variable $x \in \mathbb{R}$

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_2 + \alpha \Big(\boldsymbol{\sigma}_+ \psi^{\mathrm{w}}(x, D)^* + \boldsymbol{\sigma}_- \psi^{\mathrm{w}}(x, D) \Big) + \gamma \boldsymbol{\sigma}_3, \quad \alpha, \gamma \in \mathbb{R}. \tag{3.1}$$

The JC-model is a solvable, fully quantum mechanical model of an atom in a field, introduced by E. Jaynes and F. Cummings [1] in 1963. It has served as a theoretical description of the light-matter interaction and has continued to fulfil in unanticipated ways the objectives of its discoverers, making it possible to examine the basic properties of quantum electrodynamics. The relative simplicity of the JC-model and the ease with which it can be extended through analytic expressions or numerical computations continue to motivate attention.

More precisely the JC-model was first introduced to study the classical aspects of spontaneous emission (SE) and to reveal the existence of *Rabi oscillations* in atomic excitation probabilities for fields with sharply defined energy (or photon number). In fact, Jaynes and Cummings considered a ***single two-state atom (molecule) interacting with a single near-resonant quantized cavity mode of the electromagnetic field.*** In case of fields with a statistical distribution of photon numbers, the oscillations collapse to an expected steady value. Thus:

The Jaynes–Cummings model consists of a single two-level atom coupled to a quantized single-mode field, represented as a harmonic oscillator (HO): the coupling between atom and field is characterized by a Rabi frequency. Loss of excitation in the atom appears as a gain in excitation of the oscillator.

For more on the history of the JC-model, see Shore and Knight [21].

We next give some physical insight of the Hamiltonian (3.1), starting from the physical meaning of its terms and the description of the light-matter interaction phenomenon.

- Representing mathematically the *photon creation* and *annihilation operators* $\hat{a}^\dagger$ and $\hat{a}$, respectively, by $\psi^{\mathrm{w}}(x, D)^*$ and $\psi^{\mathrm{w}}(x, D)$, the term
$$p_2^{\mathrm{w}}(x, D)I_2 - \frac{1}{2}I_2 = \psi^{\mathrm{w}}(x, D)^*\psi^{\mathrm{w}}(x, D)(=: \hat{a}^\dagger\hat{a})$$
is the *photon number operator* which acts on photon number states $|n\rangle$,
$$\hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle\,,$$
since
$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,;$$
- α denotes one half of the *atom-field coupling constant* normalized by the *frequency ω of the mono-modal field*;
- $\frac{1}{2} \pm \gamma$ denote the *energies of the two atomic states* $|g\rangle$ and $|e\rangle$ (respectively, *ground* and *excited*, that we represent mathematically by the vectors e_j for $j = 1, 2$, respectively) normalized by $\hbar\omega$, $\hbar$ being the Plank constant;
- $\boldsymbol{\sigma}_\pm$ denote the *transition operators acting on the atomic states*, defined as
$$\boldsymbol{\sigma}_+\,|s\rangle = \begin{cases} |g\rangle\,, & \text{if } s = e \\ 0, & \text{if } s = g \end{cases} \quad \text{and} \quad \boldsymbol{\sigma}_-\,|s\rangle = \begin{cases} |e\rangle\,, & \text{if } s = g \\ 0, & \text{if } s = e \end{cases}\,; \tag{3.2}$$
- $\boldsymbol{\sigma}_3$ denotes the *commutator of the transition operators acting on the atomic states*, that is $\sigma_3 := [\sigma_+, \sigma_-]$.

The phenomenon described is the following. In the full Hamiltonian (3.1) we distinguish three parts:

$$\begin{aligned} \hat{H}_{\text{atom}} &= \gamma\boldsymbol{\sigma}_3 + \frac{1}{2}I_2, \\ \hat{H}_{\text{field}} &= p_2^{\mathrm{w}}(x, D)I_2 - \frac{1}{2}I_2, \\ \hat{H}_{\text{int}} &= \alpha\Big(\boldsymbol{\sigma}_+\psi^{\mathrm{w}}(x, D)^* + \boldsymbol{\sigma}_-\psi^{\mathrm{w}}(x, D)\Big). \end{aligned}$$

The terms $\hat{H}_{\text{atom}}$ and $\hat{H}_{\text{field}}$ describe, respectively, the variation of the energy contained in the atom and in the electromagnetic field, while $\hat{H}_{\text{int}}$ is the most relevant part of the full Hamiltonian since it describes the interaction, that is the energy transfer between the two physical objects. Namely, $\boldsymbol{\sigma}_+\psi^{\mathrm{w}}(x, D)^*$ describes the photon emission by the atom with loss of excitation of the atom itself, while $\boldsymbol{\sigma}_-\psi^{\mathrm{w}}(x, D)$ describes the photon absorption by the atom which becomes excited.

3.2. The JC-model for an N-level atom and $n = N - 1$ cavity-modes in the Ξ-configuration. In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N - 1$, given by

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_N + \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x, D)^* E_{k,k+1} + \psi_k^{\mathrm{w}}(x, D) E_{k+1,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}.$$

In this case, the levels of the atom are given by 0 and the γ_k.

3.3. The JC-model for an N-level atom and $n = N - 1$ cavity-modes in the $\bigwedge$-configuration. In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N - 1$, given by

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_N + \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x, D)^* E_{k,N} + \psi_k^{\mathrm{w}}(x, D) E_{N,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}.$$

In this case, the levels of the atom are given by 0 and the γ_k.

3.4. The JC-model for an N-level atom and $n = N - 1$ cavity-modes in the so-called $\bigvee$-configuration. In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N - 1$, given by

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_N + \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x, D)^* E_{1,k+1} + \psi_k^{\mathrm{w}}(x, D) E_{k+1,1} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}.$$

In this case, the levels of the atom are given by 0 and the γ_k.

3.5. The diagonalizability of the first-order part in the above JC-models. We next show that the first-order part of the above JC-models may be diagonalized, so that the Jaynes–Cummings models all belong to the class of systems we consider in this paper. The result of the 2×2 system is straightforward. We consider therefore only the 3×3 and the $N \times N$ cases.

Lemma 3.1. *The JC-model for a 3-level atom and 2 cavity-modes in the Ξ-configuration, for a N-level atom and $N - 1$ cavity-modes in the $\bigwedge$-configuration and in the $\bigvee$-configuration may all be smoothly diagonalized.*

Proof. Let $A_1(X)$ for the first order part of the system. We compute the characteristic polynomial $p(\lambda; X) = \det(\lambda - A_1(X))$ for each of the models in the statement, that we call for short JC-3-Ξ, JC-$\bigwedge$ and JC-$\bigvee$ respectively.

- As for JC-3-Ξ we have

$$p(\lambda; X) = \lambda\Big(\lambda^2 - (\alpha_1^2|\psi_1(X)|^2 + \alpha_2^2|\psi_2(X))|^2)\Big), \quad X \in \mathbb{R}^4.$$

Hence, there are three eigenvalues

$$\lambda_0(X) \equiv 0, \quad \lambda_\pm(X) = \pm\sqrt{\alpha_1^2|\psi_1(X)|^2 + \alpha_2^2|\psi_2(X)|^2},$$

that may be ordered as

$$\lambda_-(X) < \lambda_0(X) < \lambda_-(X), \quad X \in \dot{\mathbb{R}}^4.$$

Since their pairwise differences in absolute value are bounded from below by $|X|$, the diagonalization Theorem 5.1 below can be applied.

- As for JC-$\bigwedge$ we have

$$p(\lambda; X) = \lambda^{N-2}\left(\lambda^2 - \sum_{j=1}^{N-1} \alpha_j^2|\psi_j(X)|^2\right), \quad X \in \mathbb{R}^{2n}.$$

In fact, the expression above for $p(\lambda; X)$ can be obtained by induction on (the number of atomic levels) N. Hence, there are N eigenvalues

$$\lambda_0(X) \equiv 0, \quad \lambda_\pm(X) = \pm\Big(\sum_{j=1}^{N-1} \alpha_j^2|\psi_j(X)|^2\Big)^{1/2},$$

that we may order as

$$\lambda_-(X) < \lambda_1(X) = \ldots = \lambda_{N-2}(X) =: \lambda_0(X) < \lambda_+(X), \quad X \in \dot{\mathbb{R}}^{2n}.$$

Thus, Theorem 5.3 can be applied with respect to the blockwise diagonalization with blocks

$$\lambda_{1,1} = 0_{N-2} \quad \text{and} \quad \lambda_{2,1} = \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix}.$$

- As for JC-$\bigvee$, Theorem 5.3 can be applied because the blockwise diagonalization is the same as in the previous case, since the characteristic polynomial of this model is the same as that of JC-$\bigwedge$ above, and have the same structure.

□

3.6. Possible extensions. In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots, \gamma_N \in \mathbb{R}$ with $\gamma_1 \le \gamma_2 \le \dots \le \gamma_N$, we consider the following $2N \times 2N$ systems in $\mathbb{R}^n$, with $n = N-1$, given by

$$A^{\mathrm{w}}(x,D) = p_2^{\mathrm{w}}(x,D) I_{2N} + \sum_{k=1}^{N}\sum_{j=1}^{n} \alpha_k \Big(\psi_j^{\mathrm{w}}(x,D)^* \boldsymbol{\sigma}_- \otimes E_{kk} + \psi_j^{\mathrm{w}}(x,D) \boldsymbol{\sigma}_+ \otimes E_{kk} \Big) + \sum_{k=1}^{N} \gamma_k \boldsymbol{\sigma}_3 \otimes E_{kk},$$

and given by

$$A^{\mathrm{w}}(x,D) = p_2^{\mathrm{w}}(x,D) I_{2N} + \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x,D)^* \boldsymbol{\sigma}_- \otimes E_{k,k+1} + \psi_k^{\mathrm{w}}(x,D) \boldsymbol{\sigma}_+ \otimes E_{k+1,k} \Big) + \sum_{k=1}^{N} \gamma_k \boldsymbol{\sigma}_3 \otimes E_{kk}.$$

4. Geometric Examples Generalizing the JC-Model for an N-Level Atom and $n = N - 1$ Cavity-Modes

Let $\Omega^k(\mathbb{R}^n)$ be the space of smooth (C^∞) k-differential forms over $\mathbb{R}^n$. We will denote by $\Omega^k(\mathbb{R}^n; \mathbb{C}^N) = \Omega^k(\mathbb{R}^n) \otimes \mathbb{C}^N$. Consider the exterior derivative operator d_k acting on k-forms, and its adjoint d_k^* acting on $k+1$-forms which has the expression $d_k^* = (-1)^{nk+1} \star d \star$, where $\star$ is the Hodge-$\star$ operator induced by the Euclidean metric. We may hence define the operators

$$D = D_k := \frac{1}{\sqrt{2}} \Big(d_k + \sum_{j=1}^{n} x_j dx_j \wedge \Big) \colon \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n),$$

and its $\star$-adjoint

$$D^* = D_k^* := \frac{1}{\sqrt{2}} \Big(d_k^* + \sum_{j=1}^{n} x_j i_{\partial/\partial x_j} \Big) \colon \Omega^{k+1}(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n).$$

One has

$$\Box_k := D_k^* D_k + D_{k-1} D_{k-1}^* = (p_2^{\mathrm{w}}(x,D) + k - \frac{n}{2}) \mathbf{1}_k \colon \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n),$$

where $\mathbf{1}_k$ stands for the identity operator on $\bigwedge^k(\mathbb{R}^n)$. We consider ordered multiindices of length k, $I = (i_1, i_2, \dots, i_k)$ where $1 \le i_1 < i_2 < \dots < i_k \le n$. The set of all such multiindices is denoted by $\mathsf{I}(n,k)$. We say that $j \in I$ if j appears as one of the entries of I. We also put $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$, so that the dx_I, for $I \in \mathsf{I}(n,k)$, form a basis of $\bigwedge^k(\mathbb{R}^n)$. We have the following set of formulae.

Proposition 4.1. *Let* $\omega = \omega_I dx_I \in \Omega^k(\mathbb{R}^n)$, $I \in \mathsf{I}(n,k)$. *We have:*

(i) For $1 \leq j \leq n$,

$$D_{k-1}(i_{\partial/\partial x_j}\omega) = \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)\omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I);$$

(ii) For $1 \leq j \leq n$,

$$dx_j \wedge D_{k-1}^{*}\omega = \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)^{*}\omega_I dx_j \wedge i_{\partial/\partial x_j}(dx_I);$$

(iii) For $1 \leq j \leq n$,

$$i_{\partial/\partial x_j}(D_k\omega) = \psi_j^{\mathrm{w}}(x, D)\omega_I dx_I - \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)\omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I);$$

(iv) For $1 \leq j \leq n$,

$$D_k^{*}(dx_j \wedge \omega) = \psi_j^{\mathrm{w}}(x, D)^{*}\omega_I dx_I - \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)^{*}\omega_I dx_j \wedge i_{\partial/\partial x_h}(dx_I).$$

Proof. The proof is based on the following elementary formula

$$\star\,(dx_h \wedge \star(dx_j \wedge dx_I)) = (-1)^{nk} i_{\partial/\partial x_h}(dx_j \wedge dx_I). \tag{4.1}$$

Since $d(dx_J) = 0$ we have

$$\begin{aligned} D_{k-1}(i_{\partial/\partial x_j}\omega) &= \frac{1}{\sqrt{2}} \sum_{h=1}^{n} \Big(\frac{\partial \omega_I}{\partial x_h} dx_h \wedge i_{\partial/\partial x_j}(dx_I) + x_h \omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I) \Big) \\ & \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)\omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I), \end{aligned}$$

and this proves (i).
Next, using once more the fact that dx_J is closed and using (4.1) gives

$$\begin{aligned} dx_j \wedge D_{k-1}^{*}\omega &= \frac{1}{\sqrt{2}} \sum_{h=1}^{n} \Big((-1)^{n(k-1)+1} \frac{\partial \omega_I}{\partial x_h} dx_j \wedge \star(dx_h \wedge \star dx_I) + x_h \omega_I dx_j \wedge i_{\partial/\partial x_h}(dx_I) \Big) \\ &= \frac{1}{\sqrt{2}} \sum_{h=1}^{n} \Big(-\frac{\partial \omega_I}{\partial x_h} dx_j \wedge i_{\partial/\partial x_h}(dx_I) + x_h \omega_I dx_j \wedge i_{\partial/\partial x_h}(dx_I) \Big) \\ &= \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)^{*}\omega_I dx_j \wedge i_{\partial/\partial x_j}(dx_I), \end{aligned}$$

which proves (ii).
To prove (iii), we just note that

$$i_{\partial/\partial x_j}(D_k\omega) = \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)\omega_I i_{\partial/\partial x_j}\Big(dx_h \wedge dx_I\Big)$$
$$= \psi_j^{\mathrm{w}}(x, D)\omega_I dx_I - \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)\omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I).$$

Finally, to prove (iv), we compute

$$D_k^*(dx_j \wedge \omega) = \frac{1}{\sqrt{2}} \sum_{h=1}^{n} \Big((-1)^{nk+1}$$
$$\frac{\partial \omega_I}{\partial x_h} \star (dx_h \wedge \star(dx_j \wedge dx_I)) + x_h \omega_I i_{\partial/\partial x_h}(dx_j \wedge dx_I)\Big)$$

(by (4.1))

$$= \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)^* \omega_I \Big(\delta_{jh} dx_I - dx_j \wedge i_{\partial/\partial x_h}(dx_I)\Big),$$

which completes the proof of the proposition. □

Remark 4.2. By convention, if ω is a 0-form then $i_{\partial/\partial x_j}\omega = 0$, for every j.

4.1. The geometric N-level atom in the Ξ*-configuration.* Next, let $N \geq 2$ be a fixed positive integer and let $n = N - 1$. We define, for $\alpha_1, \ldots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, the following connection D on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$

$$\mathsf{D} := D \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,j+1}.$$

The connection D extends to the following covariant exterior operator and adjoint covariant exterior operator

$$\mathsf{D}_k := D_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,j+1} \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N),$$
$$\mathsf{D}_k^* := D_k^* \otimes I_N + \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} \otimes E_{j+1,j} \colon \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

The connection D is non-flat, as the following lemma shows.

Lemma 4.3. *For the curvature* $F_{\mathsf{D}} = \mathsf{D}^2 \in \Omega^2(\mathbb{R}^n; \mathsf{M}_N)$ *of the covariant exterior operator* D *we have*

$$F_{\mathsf{D}} = \sum_{j=1}^{N-2} \alpha_j \alpha_{j+1} (dx_j \wedge dx_{j+1} \wedge\ ) \otimes E_{j,j+2}.$$

(We put by definition $E_{j,N+1} = 0$*, for every* j*.)*

Proof. We have

$$\mathsf{D}^2 = D^2 \otimes I_N + \sum_{h=1}^{N-1} \alpha_h \Big(D_{k+1}(dx_h \wedge\ ) + dx_h \wedge D \Big) \otimes E_{h,h+1}$$
$$+ \sum_{j,h=1}^{N-1} \alpha_j \alpha_h (dx_j \wedge dx_h \wedge\ ) \otimes \underbrace{E_{j,j+1} E_{h,h+1}}_{=\delta_{j+1,h} E_{j,h+1}} = F_{\mathsf{D}},$$

for the first and second term vanish. In fact, $D^2 = 0$ and for $\omega = \omega_I dx_I$,

$$D(dx_h \wedge \omega) + dx_h \wedge D\omega = \sum_{j=1}^{N-1} \psi_j^{\mathrm{w}}(x, D) \omega_I \Big(dx_h \wedge dx_j \wedge dx_I + dx_j \wedge dx_h \wedge dx_I \Big) = 0.$$

This concludes the proof. □

We next consider the associated Laplacian

$$\square_k^{(N)} = \mathsf{D}_k^* \mathsf{D}_k + \mathsf{D}_{k-1} \mathsf{D}_{k-1}^* \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

Lemma 4.4. *We have*

$$\square_k^{(N)} = (p_2^{\mathrm{w}}(x, D) + k - \frac{n}{2}) \mathbf{1}_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j \Big(\psi_j^{\mathrm{w}}(x, D)^* \mathbf{1}_k$$
$$\otimes E_{j,j+1} + \psi_j^{\mathrm{w}}(x, D) \mathbf{1}_k \otimes E_{j+1,j} \Big)$$
$$+ \sum_{j=1}^{N-1} \alpha_j^2 \mathbf{1}_k \otimes E_{j+1,j+1}$$
$$+ \sum_{j=1}^{N-1} \alpha_j^2 dx_j \wedge i_{\partial/\partial x_j} \mathbf{1}_k \otimes (E_{j,j} - E_{j+1,j+1}).$$

Proof. One has

$$\square_k^{(N)} = (D_k^* D_k + D_{k-1} D_{k-1}^*) \otimes I_N$$
$$+ \sum_{j=1}^{N-1} \alpha_j D_k^* (dx_j \wedge\ ) \otimes E_{j,j+1}$$
$$+ \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} D_k \otimes E_{j+1,j}$$
$$+ \sum_{j=1}^{N-1} \alpha_j D_{k-1} i_{\partial/\partial x_j} \otimes E_{j+1,j}$$
$$+ \sum_{j=1}^{N-1} \alpha_j dx_j \wedge D_{k-1}^* \otimes E_{j,j+1}$$

$$+\sum_{h,j=1}^{N-1}\alpha_j\alpha_h\Big(i_{\partial/\partial x_h}(dx_j\wedge\ )$$
$$\otimes E_{h+1,h}E_{j,j+1}+dx_j\wedge i_{\partial/\partial x_h}\otimes E_{j,j+1}E_{h+1,h}\Big),$$

from which the lemma follows by virtue of the formulae of Proposition 4.1. □

Corollary 4.5. *When $k=0$ we have*

$$\begin{aligned}\Box_0^{(N)} &= (p_2^{\mathrm{w}}(x,D)-\frac{n}{2})\otimes I_N\\ &\quad+\sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\otimes E_{j,j+1}+\psi_j^{\mathrm{w}}(x,D)\\ &\quad\otimes E_{j+1,j}\Big)+\sum_{j=1}^{N-1}\alpha_j^2\,\mathbf{1}_0\otimes E_{j+1,j+1}.\end{aligned}$$

Hence the JC-Ξ model is related to the Laplacian $\Box_0^{(N)}$.

Lemma 4.6. *The term of order 1 of $\Box_1^{(3)}$ can be blockwise-diagonalized with three blocks.*

Proof. Fix the basis $\{dx_i\otimes e_j\}_{i=1,2;\,j=1,2,3}$. We have that the semiprincipal symbol $A_1(X)$ of $\Box_1^{(3)}$,

$$A_1(X)=\sum_{j=1}^{2}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\,\mathbf{1}_k\otimes E_{j,j+1}+\psi_j^{\mathrm{w}}(x,D)\mathbf{1}_k\otimes E_{j+1,j}\Big),$$

can be rewritten in the above basis as

$$A_1(X):=\left[\begin{array}{c|c|c}0_2 & \overline{A_{11}(X)} & 0_2\\ \hline A_{11}(X) & 0_2 & \overline{A_{12}(X)}\\ \hline 0_2 & A_{12}(X) & 0_2\end{array}\right],$$

where $A_{1j}(X)=\alpha_j\begin{bmatrix}\psi_j(X) & 0\\ 0 & \psi_j(X)\end{bmatrix}$, $j=1,2$, and where 0_2 is the 2×2 zero-matrix. A computation gives that the characteristic polynomial $p(\lambda;X)=\det(\lambda-A_1(X))$ of $A_1(X)$ is

$$p(\lambda;X)=\lambda^2\Big(\lambda^2-(\alpha_1^2|\psi_1(X)|^2+\alpha_2^2|\psi_2(X)|^2)\Big)^2.$$

Hence, the zeros of $p(\lambda;X)$ are

$$0,\quad \lambda_\pm(X)=\pm\sqrt{\alpha_1^2|\psi_1(X)|^2+\alpha_2^2|\psi_2(X)|^2},$$

with multiplicity 2 each (recall that $\alpha_1,\alpha_2\in\mathbb{R}\setminus\{0\}$).

Thus, Theorem 5.3 can be applied with respect to the blockwise diagonalization with three blocks

$$\lambda_{1,1}=0_2\quad,\quad \lambda_{2,1}=\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_+\end{bmatrix}\quad\text{and}\quad \lambda_{3,1}=\begin{bmatrix}\lambda_- & 0\\ 0 & \lambda_-\end{bmatrix}.$$

□

4.2. The geometric N-level atom in the $\bigwedge$-configuration. Next, let $N \geq 2$ be a fixed positive integer and let $n = N-1$. We define, for $\alpha_1, \ldots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, the following connection D on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$

$$\mathsf{D} := D \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,N}.$$

The connection D extends to the following covariant exterior operator and adjoint covariant exterior operator

$$\mathsf{D}_k := D_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,N} \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N),$$

$$\mathsf{D}_k^* := D_k^* \otimes I_N + \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} \otimes E_{N,j} \colon \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

The connection D is flat as the following lemma shows.

Lemma 4.7. *The operators D_k form a complex. Hence the curvature of D vanishes.*

Proof. We have to to prove that $\mathsf{D}_k \mathsf{D}_{k+1} = 0$. We have

$$\begin{aligned} \mathsf{D}_{k+1}\mathsf{D}_k &= D_{k+1} D_k \otimes I_N + \sum_{h=1}^{N-1} \alpha_h \Big(D_{k+1}(dx_h \wedge\ ) + dx_h \wedge D_k \Big) \otimes E_{h,N} \\ &\quad + \sum_{j,h=1}^{N-1} \alpha_j \alpha_h (dx_j \wedge dx_h \wedge\ ) \otimes \underbrace{E_{j,N} E_{h,N}}_{=\delta_{h,N} E_{j,N} = 0} = 0, \end{aligned}$$

for the first and second term, as before, vanish. This concludes the proof. ◻

We next consider the associated Laplacian

$$\square_k^{(N)} = \mathsf{D}_k^* \mathsf{D}_k + \mathsf{D}_{k-1} \mathsf{D}_{k-1}^* \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

Lemma 4.8. *We have*

$$\begin{aligned} \square_k^{(N)} &= (p_2^{\mathrm{w}}(x, D) + k - \frac{n}{2}) \mathbf{1}_k \otimes I_N \\ &\quad + \sum_{j=1}^{N-1} \alpha_j \Big(\psi_j^{\mathrm{w}}(x, D)^* \, \mathbf{1}_k \otimes E_{j,N} + \psi_j^{\mathrm{w}}(x, D) \mathbf{1}_k \otimes E_{N,j} \Big) \\ &\quad + \sum_{j=1}^{N-1} \alpha_j^2 \mathbf{1}_k \otimes E_{N,N} - \sum_{j=1}^{N-1} \alpha_j^2 dx_j \wedge i_{\partial/\partial x_j} \mathbf{1}_k \otimes E_{N,N} \\ &\quad + \sum_{j,h=1}^{N-1} \alpha_j \alpha_h dx_j \wedge i_{\partial/\partial x_h} \mathbf{1}_k \otimes E_{j,h}. \end{aligned}$$

Proof. One has

$$\begin{aligned}\square_k^{(N)} &= (D_k^* D_k + D_{k-1} D_{k-1}^*) \otimes I_N + \sum_{j=1}^{N-1} \alpha_j D_k^* (dx_j \wedge\ ) \otimes E_{j,N} + \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} D_k \otimes E_{N,j} \\ &+ \sum_{j=1}^{N-1} \alpha_j D_{k-1} i_{\partial/\partial x_j} \otimes E_{N,j} + \sum_{j=1}^{N-1} \alpha_j dx_j \wedge D_{k-1}^* \otimes E_{j,N} \\ &+ \sum_{h,j=1}^{N-1} \alpha_j \alpha_h \Big(i_{\partial/\partial x_h} (dx_j \wedge\ ) \otimes E_{N,h} E_{j,N} + dx_j \wedge i_{\partial/\partial x_h} \otimes E_{j,N} E_{N,h} \Big),\end{aligned}$$

from which the lemma follows by virtue of the formulae of Proposition 4.1. □

Corollary 4.9. *When $k = 0$ we have*

$$\begin{aligned}\square_0^{(N)} &= (p_2^{\mathrm{w}}(x, D) - \frac{n}{2}) \otimes I_N \\ &+ \sum_{j=1}^{N-1} \alpha_j \Big(\psi_j^{\mathrm{w}}(x, D)^* \otimes E_{j,N} + \psi_j^{\mathrm{w}}(x, D) \otimes E_{N,j} \Big) + \Big(\sum_{j=1}^{N-1} \alpha_j^2 \Big) \mathbf{1}_0 \otimes E_{N,N}.\end{aligned}$$

Hence the JC-$\bigwedge$ model is related to the Laplacian $\square_0^{(N)}$.

4.3. The geometric N-level atom in the $\bigvee$-configuration. Next, let $N \geq 2$ be a fixed positive integer and let $n = N - 1$. We define, for $\alpha_1, \ldots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, the following connection D on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$

$$\mathsf{D} := D \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{1,j+1}.$$

The connection D extends to the following covariant exterior operator and adjoint covariant exterior operator

$$\mathsf{D}_k := D_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{1,j+1} \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N),$$

$$\mathsf{D}_k^* := D_k^* \otimes I_N + \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} \otimes E_{j+1,1} \colon \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

The connection D is flat, by the following lemma.

Lemma 4.10. *The operators D_k form a complex. Hence the curvature of D vanishes.*

Proof. We have to to prove that $\mathsf{D}_k \mathsf{D}_{k+1} = 0$. We have

$$\mathsf{D}_{k+1} \mathsf{D}_k = D_{k+1} D_k \otimes I_N + \sum_{h=1}^{N-1} \alpha_h \Big(D_{k+1} (dx_h \wedge\ ) + dx_h \wedge D_k \Big) \otimes E_{1,j+1}$$

$$+\sum_{j,h=1}^{N-1}\alpha_j\alpha_h(dx_j\wedge dx_h\wedge\ )\otimes\underbrace{E_{1,j+1}E_{1,h+1}}_{=\delta_{j+1,1}E_{1,h+1}=0}=0,$$

for the first and second term, once more, vanish. This concludes the proof. □

We next consider the associated Laplacian

$$\Box_k^{(N)}=\mathsf{D}_k^*\mathsf{D}_k+\mathsf{D}_{k-1}\mathsf{D}_{k-1}^*\colon\Omega^k(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow\Omega^k(\mathbb{R}^n;\mathbb{C}^N).$$

Lemma 4.11. *We have*

$$\begin{aligned}\Box_k^{(N)}&=(p_2^{\mathrm{w}}(x,D)+k-\frac{n}{2})\mathbf{1}_k\otimes I_N\\&+\sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\,\mathbf{1}_k\otimes E_{1,j+1}+\psi_j^{\mathrm{w}}(x,D)\mathbf{1}_k\otimes E_{j+1,1}\Big)\\&+\sum_{j=1}^{N-1}\alpha_j^2\mathbf{1}_k\otimes E_{j+1,j+1}+\Big(\sum_{j=1}^{N-1}\alpha_j^2dx_j\wedge i_{\partial/\partial x_j}\mathbf{1}_k\Big)\otimes E_{1,1}\\&-\sum_{j,h=1}^{N-1}\alpha_j\alpha_h dx_j\wedge i_{\partial/\partial x_h}\otimes E_{h+1,h+1}.\end{aligned}$$

Proof. In fact,

$$\begin{aligned}\Box_k^{(N)}&=(D_k^*D_k+D_{k-1}D_{k-1}^*)\otimes I_N+\sum_{j=1}^{N-1}\alpha_jD_k^*(dx_j\wedge\ )\otimes E_{1,j+1}\\&+\sum_{j=1}^{N-1}\alpha_ji_{\partial/\partial x_j}D_k\otimes E_{j+1,1}\\&+\sum_{j=1}^{N-1}\alpha_jD_{k-1}i_{\partial/\partial x_j}\otimes E_{j+1,1}+\sum_{j=1}^{N-1}\alpha_jdx_j\wedge D_{k-1}^*\otimes E_{1,j+1}\\&+\sum_{h,j=1}^{N-1}\alpha_j\alpha_h\Big(i_{\partial/\partial x_h}(dx_j\wedge\ )\otimes E_{h+1,1}E_{1,j+1}+dx_j\wedge i_{\partial/\partial x_h}\otimes E_{1,j+1}E_{h+1,1}\Big),\end{aligned}$$

from which the lemma follows once more by virtue of the formulae of Proposition 4.1. □

Corollary 4.12. *When $k=0$ we have*

$$\begin{aligned}\Box_0^{(N)}&=(p_2^{\mathrm{w}}(x,D)-\frac{n}{2})\otimes I_N\\&+\sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\otimes E_{1,j+1}+\psi_j^{\mathrm{w}}(x,D)\otimes E_{j+1,1}\Big)+\sum_{j=1}^{N-1}\alpha_j^2\,\mathbf{1}_0\otimes E_{j+1,j+1}.\end{aligned}$$

Hence the JC-$\bigvee$ model is related to the Laplacian $\Box_0^{(N)}$.

Remark 4.13. Note that, therefore, the JC models possess extensions to states that are vector-valued k-forms. Loosely speaking, one may think of this mathematical generalization as a transposition to a fermionic, or more generally supersymmetric, picture.

Lemma 4.14. *The semiprincipal term of* $\square_1^{(3)}$ *can be blockwise-diagonalized with three blocks.*

Proof. Fix the basis $\{dx_i \otimes e_j\}_{i=1,2;\, j=1,2,3}$. We have for the semiprincipal symbol $A_1(X)$ of $\square_1^{(3)}$,

$$A_1(X) = \sum_{j=1}^{2} \alpha_j \Big(\overline{\psi_j(X)} \mathbf{1}_k \otimes E_{1,j+1} + \psi_j(X) \mathbf{1}_k \otimes E_{j+1,1} \Big),$$

that it may be rewritten in the above basis as

$$A_1(X) = \left[\begin{array}{c|c|c} 0_2 & \overline{A_{11}(X)} & \overline{A_{12}(X)} \\ \hline A_{11}(X) & 0_2 & 0_2 \\ \hline A_{12}(X) & 0_2 & 0_2 \end{array} \right],$$

where $A_{1j}(X) = \begin{bmatrix} \alpha_j \psi_j(X) & 0 \\ 0 & \alpha_j \psi_j(X) \end{bmatrix}$, $j = 1, 2$, and where 0_2 is the 2×2 zero-matrix. A computations gives that the characteristic polynomial $p(\lambda; X) = \det(\lambda - A_1(X))$ of $A_1(X)$ is

$$p(\lambda; X) = \lambda^2 \Big(\lambda^4 - 2\big(\alpha_1^2 |\psi_1(X)|^2 + \alpha_2^2 |\psi_2(X)|^2\big)\lambda^2 + \big(\alpha_1^2 |\psi_1(X)|^2 + \alpha_2^2 |\psi_2(X)|^2\big)^2 \Big).$$

Hence, the zeros of $p(\lambda; X)$ are given by

$$0, \quad \lambda_\pm(X) = \pm\sqrt{\alpha_1^2 |\psi_1(X)|^2 + \alpha_2^2 |\psi_2(X)|^2},$$

each with constant multiplicity 2 (for $X \neq 0$).

Thus, the diagonalization Theorem 5.3 (see the next section) can be applied to obtain a blockwise diagonalization with three blocks

$$\lambda_{1,1} = 0_2 \quad , \quad \lambda_{2,1} = \lambda_+ I_2, \quad \text{and} \quad \lambda_{3,1} = \lambda_- I_2.$$

□

5. The Decoupling Theorem

In this section we prove a decoupling theorem for classes of semiregular global pseudodifferential systems from our class SMGES (that is, of the Jaynes–Cummings kind). For future purposes, we prove the theorem in the semiclassical case (hence $\sharp_h$ will denote the semiclassical composition of symbols in the semiclassical setting, see [14]) and then state the corresponding version valid for the semiregular case. The proof follows the lines of the decoupling theorem in [14], but it has a main twist due to the fact that the terms a_μ and $a_{\mu-1}$ may interact in the composition formula due to the conjugation of the symbol of the diagonalizer, but can be simultaneously blockwise diagonalized. Recall that $S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ stands for the set of *regular semiclassical symbols* (see Point 2 of Definition 9.1.9 of [14]), that is, they are h-dependent symbols that admit an asymptotic expansion in half-integer powers of h, with the h^j-coefficient which is an h-independent symbol of order the order of a decreased by $2j$. A semiclassical symbol A then belongs to $S^0_{0,\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ if it can be written in the form $A_\mu + h^{1/2}A_{\mu-1}$, where $A_\mu \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ and $A_{\mu-1} \in S^0_{0,\mathrm{cl}}(m^{\mu-1}, g; \mathsf{M}_N)$.

Theorem 5.1. *Let $\mu > 0$ and let $A = A^* = A_\mu + h^{1/2}A_{\mu-1} \in S^0_{0,\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ where*

$$A_\mu \sim \sum_{j\geq 0} h^j a_{\mu-2j} \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N), \quad A_{\mu-1} \sim \sum_{j\geq 0} h^j a_{\mu-1-2j} \in S^0_{0,\mathrm{cl}}(m^{\mu-1}, g; \mathsf{M}_N),$$

with $a_{-k} = a^*_{-k} \in S(m^{-k}, g, \mathsf{M}_N)$. Moreover, suppose $a_\mu = p_\mu I_N$ with $p_\mu \in S(m^\mu, g)$, and that $a_{\mu-1}$, for some $e_0 \in S(1, g; \mathsf{M}_N)$ such that $e_0 e_0^* = e_0^* e_0 = I_N$, can be written as

$$a_{\mu-1} = e_0 b_{\mu-1} e_0^*, \quad \text{where } b_{\mu-1} = b^*_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right],$$

where the $\lambda_{j,\mu-1} \in S(m^{\mu-1}, g; \mathsf{M}_{N_j})$, $j = 1, 2$ and $N = N_1 + N_2$, are such that

$$d_{\lambda_1,\lambda_2}(X) \gtrsim m(X)^{\mu-1}, \quad \forall X \in \mathbb{R}^{2n}, \tag{5.1}$$

with

$$d_{\lambda_1,\lambda_2}(X) = \inf\{|\zeta_1 - \zeta_2|;\ \ \zeta_j \in \mathrm{Spec}(\lambda_{j,\mu-1}(X)),\ \ j = 1, 2\}.$$

Then there exists $E \in S^0_{0,\mathrm{sreg}}(1, g; \mathsf{M}_N)$ with $E \sim \sum_{j\geq 0} h^{j/2} e_{-j}$ (with $e_{-k} \in S(m^{-k}, g, \mathsf{M}_N)$) and principal symbol e_0 such that

$$E^w(x, hD)^* E^w(x, hD) - I, \quad E^w(x, hD) E^w(x, hD)^* - I \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N), \tag{5.2}$$

and

$$E^w(x, hD)^* A^w(x, hD) E^w(x, hD) - B^w(x, hD) \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N), \tag{5.3}$$

where the symbol $B \sim \sum_{j\geq 0} h^{j/2} b_{\mu-j} \in S^0_{0,\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ is blockwise diagonal, with

$$b_{\mu-j}(X) = \left[\begin{array}{c|c} b_{\mu-j,1}(X) & 0 \\ \hline 0 & b_{\mu-j,2}(X) \end{array}\right], \quad \forall X \in \mathbb{R}^{2n}, \forall j \geq 0,$$

the blocks $b_{\mu-j,k}$ being of sizes $N_k \times N_k$, $k = 1, 2$, with

$$b_\mu = a_\mu = p_\mu I_N, \quad b_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right].$$

Remark 5.2. We shall call B an h^∞-(blockwise)diagonalization of A. Notice that B depends on A and e_0.

Proof. We immediately observe that once $E^{\mathrm{w}}(x, hD)$ has been constructed with the property that

$$E^{\mathrm{w}}(x, hD)^* E^{\mathrm{w}}(x, hD) = I + r^{\mathrm{w}}(x, hD), \text{ with } r \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N),$$

then by the ellipticity of $E^{\mathrm{w}}(x, hD)^*$ (namely, the existence of a parametrix) we also get

$$E^{\mathrm{w}}(x, hD) E^{\mathrm{w}}(x, hD)^* = I + s^{\mathrm{w}}(x, hD), \text{ with } s \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N).$$

Hence it suffices to prove the existence of E and B with the required properties. We show that for every integer $N_0 \in \mathbb{Z}_+$ there exist

$$e_{-k} \in S(m^{-k}, g; \mathsf{M}_N), \quad 0 \le k \le N_0,$$

and

$$b_{\mu-k,j} \in S(m^{\mu-k}, g; \mathsf{M}_j), \quad j = 1, 2, \quad 0 \le k \le N_0,$$

such that, with $E_{N_0}(X) := \sum_{k=0}^{N_0} h^{k/2} e_{-k}(X))$,

$$E^*_{N_0} \#_h E_{N_0} = I + h^{(N_0+1)/2} S^0_0(m^{-(N_0+1)}, g; \mathsf{M}_N),$$

and

$$E^*_{N_0} \#_h A \#_h E_{N_0} = \sum_{k=0}^{N_0} h^{k/2} b_{\mu-k} + h^{(N_0+1)/2} S^0_0(m^{\mu-(N_0+1)}, g; \mathsf{M}_N),$$

where the $b_{\mu-k} = \left[\begin{array}{c|c} b_{\mu-k,1} & 0 \\ \hline 0 & b_{\mu-k,2} \end{array}\right]$. We shall then take $E \sim \sum_{k\ge 0} h^{k/2} e_{-k}$.

First of all, we have that $e_0 \in S(1, g; \mathsf{M}_N)$ is such that $e_0^* a_\mu e_0$ and $e_0^* a_{\mu-1} e_0$ are diagonal matrices and e_0 satisfies the unitarity condition (note that a_μ and $a_{\mu-1}$ commute since a_μ is a scalar matrix).

We proceed by induction on N_0, and start by proving that the assertion is true for $N_0 = 1$ and for $N_0 = 2$. (We will omit the dependence on (x, hD) and write e_k^{w} in place of $e_k^{\mathrm{w}}(x, hD)$.) Hence, we look for $e_{-1} \in S(m^{-1}, g; \mathsf{M}_N)$ such that

$$(e_0 + h^{1/2} e_{-1})^* \#_h (e_0 + h^{1/2} e_{-1}) - I \in h S^0_0(m^{-2}, g; \mathsf{M}_N). \tag{5.4}$$

Now, the coefficient s_{-1} of $h^{1/2}$ in $e_0^* \#_h e_0$ is zero (because of the step-decrease of the global calculus), whence for the coefficient of $h^{1/2}$ in (5.4) we have

$$s_{-1} + e_0^* e_{-1} + e_{-1}^* e_0 = e_0^* e_{-1} + e_{-1}^* e_0 = 0. \tag{5.5}$$

Equation (5.5) has a general solution

$$e_{-1} = e_0 \alpha_{-1}, \tag{5.6}$$

where $\alpha_{-1} \in S(m^{-1}, g; \mathsf{M}_N)$ and

$$\alpha^*_{-1} + \alpha_{-1} = 0. \tag{5.7}$$

We next look for α_{-1} in such a way that $b_{\mu-1}$ is blockwise diagonal. Hence, we write

$$(e_0^{\mathrm{w}}+h^{1/2}e_{-1}^{\mathrm{w}})^*A^{\mathrm{w}}(e_0^{\mathrm{w}}+h^{1/2}e_{-1}^{\mathrm{w}}) = (e_0^{\mathrm{w}})^*A^{\mathrm{w}}e_0^{\mathrm{w}} + h^{1/2}\Big((e_{-1}^{\mathrm{w}})^*A^{\mathrm{w}}e_0^{\mathrm{w}} + (e_0^{\mathrm{w}})^*A^{\mathrm{w}}e_{-1}^{\mathrm{w}}\Big) + hr_{\mu-2}^{\mathrm{w}},$$

where $r_{\mu-2}\in S_0^0(m^{\mu-2},g,\mathsf{M}_N)$.
Now, recalling the definition of A, we have

$$(e_0+h^{1/2}e_{-1})^*\#_h A\#_h(e_0+h^{1/2}e_{-1}) = e_0^*\#_h a_\mu\#_h e_0 + h^{1/2}\Big(e_{-1}^*\#_h a_\mu\#_h e_0 + e_0^*\#_h a_\mu\#_h e_{-1} + e_0^*\#_h a_{\mu-1}\#_h e_0\Big) + hr_{\mu-2},$$

with $r_{\mu-2}\in S_0^0(m^{\mu-2},g,\mathsf{M}_N)$.
It follows that, since a_μ is a scalar matrix and hence it commutes with every other matrix, we look for e_{-1} such that

$$q_{\mu-1}+a_\mu(e_{-1}^*e_0+e_0^*e_{-1})+e_0^*a_{\mu-1}e_0 \tag{5.8}$$

is diagonal, where $q_{\mu-1}$ is the coefficient of $h^{1/2}$ in $e_0^*\#_h A_\mu\#_h e_0$ and in this case $q_{\mu-1}=0$. We have that $e_{-1}^*e_0+e_0^*e_{-1}=0$ and that $e_0^*a_{\mu-1}e_0$ is already diagonal by the hypothesis on e_0.

Hence (5.8) is blockwise diagonal *without* any further conditions on α_{-1}, which is therefore only required to be skew-hermitian. However, further constraints on it will arise in the next step.

This completes the case $N_0=1$.

Next we look for $e_{-2}\in S(m^{-2},g;\mathsf{M}_N)$ such that

$$(e_0+h^{1/2}e_{-1}+he_{-2})^*\#_h(e_0+h^{1/2}e_{-1}+he_{-2})-I\in h^{3/2}S_0^0(m^{-3},g;\mathsf{M}_N).$$

Hence, since

$$e_0^*\#_h e_0 - I = hs_{-2}+h^{3/2}s',\quad s_{-2}=s_{-2}^*\in S(m^{-2},g;\mathsf{M}_N),\quad s'\in S_0^0(m^{-3},g;\mathsf{M}_N),$$

we require that e_{-2} be a solution of

$$s_{-2}+e_0^*e_{-2}+e_{-2}^*e_0+e_{-1}^*e_{-1}=0. \tag{5.9}$$

Equation (5.9) has as a general solution

$$e_{-2}=-\frac{1}{2}e_0\big(s_{-2}+\underbrace{e_{-1}^*e_{-1}}_{=\alpha_{-1}^*\alpha_{-1}}\big)+e_0\alpha_{-2},$$

where $\alpha_{-2}\in S(m^{-2},g;\mathsf{M}_N)$ and

$$\alpha_{-2}^*+\alpha_{-2}=0. \tag{5.10}$$

We next determine α_{-2} so as to have b_μ in blockwise diagonal form with the diagonal blocks $b_{j,\mu},\ j=1,2$. Write

$$
\begin{aligned}
&(e_0+h^{1/2}e_{-1}+he_{-2})^*\#_h A\#_h(e_0+h^{1/2}e_{-1}+he_{-2})\\
&\quad= e_0^*\#_h A\#_h e_0 + h^{1/2}\Big(e_{-1}^*\#_h A\#_h e_0 + e_0^*\#_h A\#_h e_{-1}\Big) + h\Big(e_0^*\#_h A_\mu\#_h e_{-2}\\
&\qquad+e_{-2}^*\#_h A_\mu\#_h e_0 + e_0^*\#_h A_{\mu-1}\#_h e_{-1} + e_{-1}^*\#_h A_{\mu-1}\#_h e_0 + e_{-1}^*\#_h A_\mu\#_h e_{-1}\Big)\\
&\qquad+h^{3/2}S_0^0(m^{\mu-3},g;\mathsf{M}_N).
\end{aligned}
$$

Because of the form of A, we have

$$
\begin{aligned}
&(e_0+h^{1/2}e_{-1}+he_{-2})^*\#_h A\#_h(e_0+h^{1/2}e_{-1}+he_{-2})\\
&\quad= e_0^*\#_h a_\mu\#_h e_0 + h^{1/2}\Big(e_{-1}^*\#_h a_\mu\#_h e_0 + e_0^*\#_h a_\mu\#_h e_{-1} + e_0^*\#_h a_{\mu-1}\#_h e_0\Big)\\
&\qquad+h\Big(e_0^*\#_h a_\mu\#_h e_{-2} + e_0^*\#_h a_{\mu-1}\#_h e_{-1} + e_0^*\#_h a_{\mu-2}\#_h e_0 + e_{-1}^*\#_h a_\mu\#_h e_{-1}\\
&\qquad+e_{-1}^*\#_h a_{\mu-1}\#_h e_0 + e_{-2}^*\#_h a_\mu\#_h e_0\Big) + h^{3/2}r_{\mu-3},\quad r_{\mu-3}\in S_0^0(m^{\mu-3},g,\mathsf{M}_N).
\end{aligned}
$$

Hence, we look for e_{-2} such that the coefficient of h in

$$
(e_0+h^{1/2}e_{-1}+he_{-2})^*\#_h A\#_h(e_0+h^{1/2}e_{-1}+he_{-2}),
$$

given by

$$
q_{\mu-2}+e_0^*a_\mu e_{-2}+e_0^*a_{\mu-1}e_{-1}+e_{-1}^*a_\mu e_{-1}+e_{-1}^*a_{\mu-1}e_0+e_{-2}^*a_\mu e_0, \tag{5.11}
$$

is diagonal. Now, (5.11) can be rewritten, by (5.5), as

$$
q_{\mu-2}+a_\mu(e_0^*e_{-2}+e_{-2}^*e_0)+(e_0^*a_{\mu-1}e_0)\alpha_{-1}+a_\mu\alpha_{-1}^*\alpha_{-1}+\alpha_{-1}^*(e_0^*a_{\mu-1}e_0). \tag{5.12}
$$

Since by (5.9) and (5.6)

$$
e_0^*e_{-2}+e_{-2}^*e_0=-s_{-2}-\alpha_{-1}^*\alpha_{-1},
$$

we obtain that (5.12) becomes

$$
q_{\mu-2}-a_\mu s_{-2}+(e_0^*a_{\mu-1}e_0)\alpha_{-1}+\alpha_{-1}^*(e_0^*a_{\mu-1}e_0). \tag{5.13}
$$

We now split (5.13) into two (hermitian) parts (note that $q_{\mu-2}=q_{\mu-2}^*$ and $s_{-2}=s_{-2}^*$). The first part is given by

$$
q_{\mu-2}-a_\mu s_{-2}=\left[\begin{array}{c|c} u_1 & \gamma \\ \hline \gamma^* & u_2\end{array}\right], \tag{5.14}
$$

where $u_j=u_j^*$ are blocks of sizes $N_j\times N_j$. The second part is given by

$$
(e_0^*a_{\mu-1}e_0)\alpha_{-1}+\alpha_{-1}^*(e_0^*a_{\mu-1}e_0)=\big[e_0^*a_{\mu-1}e_0,\alpha_{-1}\big]
$$

(recall that α_{-1} is skew-hermitian by (5.7)). We therefore look for α_{-1} in the form

$$
\alpha_{-1}=\left[\begin{array}{c|c} 0 & \delta \\ \hline -\delta^* & 0\end{array}\right]. \tag{5.15}
$$

Using the fact that $e_0^* a_{\mu-1} e_0$ is blockwise diagonal with blocks $\lambda_{\mu-1,1}$ and $\lambda_{\mu-1,2}$, in order to make (5.13) blockwise diagonal, we are led to the equation

$$\lambda_{\mu-1,1}\delta - \delta\lambda_{\mu-1,2} = -\gamma, \tag{5.16}$$

which imposes a condition on α_{-1}.
By Lemma 9.2.2 in [14], the equation has a solution and this completes the case $N_0 = 2$.

It is important to note at this point that the only condition that α_{-2} must satisfy so far is that it be skew-hermitian, that is $\alpha_{-2} + \alpha_{-2}^* = 0$.

Now, we proceed by induction. So, suppose we have already constructed the symbols $e_0, e_{-1}, \dots, e_{-N_0}$, and $b_\mu, b_{\mu-1}, \dots, b_{\mu-N_0}$, independent of h, with the required properties. Moreover, suppose that we have constructed

$$e_{-N_0} = -\frac{1}{2} e_0 \Big(s_{-N_0} + e_{-1}^* e_{-(N_0-1)} + e_{-(N_0-1)}^* e_{-1} \Big) + e_0 \alpha_{-N_0}, \tag{5.17}$$

where $s_{-N_0} = s_{-N_0}^* \in S(m^{-N_0}, g, \mathsf{M}_N)$ is the coefficient of $h^{N_0/2}$ in $E_{N_0-2} \#_h E_{N_0-2}$, and the only condition that α_{-N_0} must satisfy is

$$\alpha_{-N_0} + \alpha_{-N_0}^* = 0. \tag{5.18}$$

Proceeding as in the case $N_0 = 2$, we look for $e_{-(N_0+1)}$ such that

$$\Big(E_{N_0} + h^{\frac{N_0+1}{2}} e_{-(N_0+1)} \Big)^* \#_h \Big(E_{N_0} + h^{\frac{N_0+1}{2}} e_{-(N_0+1)} \Big) = I + h^{\frac{N_0+2}{2}} S_0^0(m^{-(N_0+2)}, g; \mathsf{M}_N).$$

Thus, using the symbol-composition formula $\#_h$ and (part of) the inductive hypothesis, that is, $E_{N_0}^* \#_h E_{N_0} = I + h^{(N_0+1)/2} S_0^0(m^{-(N_0+1)}, g; \mathsf{M}_N)$ (the other part of the inductive hypothesis being relative to the diagonal form of the conjugated operator), we are led to the equation

$$s_{-(N_0+1)} + e_0^* e_{-(N_0+1)} + e_{-(N_0+1)}^* e_0 + e_{-1}^* e_{-N_0} + e_{-N_0}^* e_{-1} = 0, \tag{5.19}$$

where $s_{-(N_0+1)} = s_{-(N_0+1)}^*$ is the coefficient of $h^{(N_0+1)/2}$ in $E_{N_0-1}^* \#_h E_{N_0-1}$. Since

$$\begin{aligned} &\Big(E_{N_0} + h^{(N_0+1)/2} e_{-(N_0+1)} \Big)^* \#_h A \#_h \Big(E_{N_0} + h^{(N_0+1)/2} e_{-(N_0+1)} \Big) \\ &\qquad = \sum_{k=0}^{N_0} h^{k/2} b_{\mu-k} + h^{(N_0+1)/2} S_0^0(m^{\mu-(N_0+1)}, g; \mathsf{M}_N), \end{aligned}$$

with $b_{\mu-k}$ diagonal, we next look for $e_{-(N_0+1)}$ such that the $h^{(N_0+1)/2}$-coefficient in the symbol composition $E_{N_0+1}^* \#_h A \#_h E_{N_0+1}$, given by

$$\begin{aligned} &q_{\mu-(N_0+1)} + e_0^* a_\mu e_{-(N_0+1)} + e_{-(N_0+1)}^* a_\mu e_0 \\ &\quad + e_0^* a_{\mu-1} e_{-N_0} + e_{-N_0}^* a_{\mu-1} e_0 + e_{-1}^* a_\mu e_{-N_0} + e_{-N_0}^* a_\mu e_{-1}, \end{aligned} \tag{5.20}$$

is diagonal. Now, we rewrite (5.20) as

$$\begin{aligned} &q_{\mu-(N_0+1)} + a_\mu e_0^* e_{-(N_0+1)} + a_\mu e_{-(N_0+1)}^* e_0 + (e_0^* a_{\mu-1} e_0) e_0^* e_{-N_0} \\ &\quad + e_{-N_0}^* e_0 (e_0^* a_{\mu-1} e_0) + a_\mu e_{-1}^* e_{-N_0} + a_\mu e_{-N_0}^* e_{-1}, \end{aligned} \tag{5.21}$$

so that (5.21) becomes (using the fact that a_μ is a scalar)

$$q_{\mu-(N_0+1)} + a_\mu \underbrace{(e_0^* e_{-(N_0+1)} + e^*_{-(N_0+1)} e_0 + e^*_{-1} e_{-N_0} + e^*_{-N_0} e_{-1})}_{=-s_{-(N_0+1)} \text{ by (5.19)}} + (e_0^* a_{\mu-1} e_0) e_0^* e_{-N_0} + e^*_{-N_0} e_0 (e_0^* a_{\mu-1} e_0). \tag{5.22}$$

Next, using (5.17), we write

$$\begin{cases} e_0^* e_{-N_0} = -\frac{1}{2} s_{-N_0} + \alpha_{-N_0} =: \tau + \alpha_{-N_0}, \\ e^*_{-N_0} e_0 = \tau - \alpha_{-N_0}, \end{cases} \tag{5.23}$$

where

$$\tau = -\frac{1}{2} s_{-N_0} = \tau^*, \quad \alpha_{-N_0} = -\alpha^*_{-N_0}.$$

Hence (5.22) can be rewritten as

$$q_{\mu-(N_0+1)} - a_\mu s_{-(N_0+1)} + (e_0^* a_{\mu-1} e_0)\tau + \tau(e_0^* a_{\mu-1} e_0) + (e_0^* a_{\mu-1} e_0)\alpha_{-N_0} + \alpha^*_{-N_0}(e_0^* a_{\mu-1} e_0).$$

As before, we next split (5.13) into two (hermitian) parts, where the first part is given by

$$q_{\mu-(N_0+1)} - a_\mu s_{-(N_0+1)} + (e_0^* a_{\mu-1} e_0)\tau + \tau(e_0^* a_{\mu-1} e_0) = \left[\begin{array}{c|c} \tilde{u}_1 & \tilde{\gamma} \\ \hline \tilde{\gamma}^* & \tilde{u}_2 \end{array}\right],$$

where $u_j = u_j^*$ are blocks of sizes $N_j \times N_j$, and the second one by

$$(e_0^* a_{\mu-1} e_0)\alpha_{-N_0} + \alpha^*_{-N_0}(e_0^* a_{\mu-1} e_0) = \left[e_0^* a_{\mu-1} e_0, \alpha_{-N_0}\right]$$

(recall that α_{-N_0} is skew-hermitian by (5.18)). Hence, we look for α_{-N_0} in the form

$$\alpha_{-N_0} = \left[\begin{array}{c|c} 0 & \tilde{\delta} \\ \hline -\tilde{\delta}^* & 0 \end{array}\right], \tag{5.24}$$

and, using as before the fact that $e_0^* a_{\mu-1} e_0$ is blockwise diagonal with blocks $\lambda_{\mu-1,1}$ and $\lambda_{\mu-1,2}$, we are therefore led to the equation

$$\lambda_{\mu-1,1}\tilde{\delta} - \tilde{\delta}\lambda_{\mu-1,2} = -\tilde{\gamma}. \tag{5.25}$$

As before, the equation as a solution and we complete the proof by induction. Once more, it is important to note that the only condition that $\alpha_{-(N_0+1)}$ must satisfy in the $N_0 + 1$-st step of the induction is $\alpha_{-(N_0+1)} + \alpha^*_{-(N_0+1)} = 0$. □

We state (without proof, since it follows the lines of the foregoing proof) the blockwise diagonalization theorem in the case of semiregular symbols.

Theorem 5.3. *Let $\mu > 0$, and let $A = A^* \sim \sum_{j\geq 0} a_{\mu-j} \in S_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$. Suppose $a_\mu = p_\mu I_N$ with $p_\mu \in C^\infty(\mathbb{R}^{2n} \setminus \{0\})$ positively homogeneous of degree μ, and that $a_{\mu-1}$, for some $e_0 \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathsf{M}_N)$ positively homogenous of degree 0 and such that $e_0 e_0^* = e_0^* e_0 = I_N$, $X \neq 0$, can be written as*

$$a_{\mu-1} = e_0 b_{\mu-1} e_0^*, \quad \text{where} \;\; b_{\mu-1} = b_{\mu-1}^* = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right], \quad X \neq 0,$$

where $\lambda_{\mu-1,j} \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathsf{M}_{N_j})$, $j = 1, 2$ and $N = N_1 + N_2$, are positively homogeneous of degree $\mu - 1$, and are such that

$$\operatorname{Spec}(\lambda_{\mu-1,1}(X)) \cap \operatorname{Spec}(\lambda_{\mu-1,2}(X)) = \emptyset, \quad \forall X \in \mathbb{R}^{2n}, \;\; |X| = 1.$$

Then there exists $E \in S_{\rm sreg}(1, g; \mathsf{M}_N)$ with $E \sim \sum_{j\geq 0} e_{-j}$ and principal symbol e_0 (hence $e_{-k} \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathsf{M}_N)$ is positively homogeneous of degree $-k$) such that

$$E^w(x, D)^* E^w(x, D) - I, \quad E^w(x, D) E^w(x, D)^* - I \in S(m^{-\infty}, g; \mathsf{M}_N), \tag{5.26}$$

and

$$E^w(x, D)^* A^w(x, D) E^w(x, D) - B^w(x, D) \in S(m^{-\infty}, g; \mathsf{M}_N), \tag{5.27}$$

where the symbol $B \sim \sum_{j\geq 0} b_{2-j} \in S_{\rm sreg}(m^\mu, g; \mathsf{M}_N)$ is blockwise diagonal, with

$$b_{2-j}(X) = \left[\begin{array}{c|c} b_{2-j,1}(X) & 0 \\ \hline 0 & b_{2-j,2}(X) \end{array}\right], \forall X \neq 0, \forall j \geq 0,$$

the blocks $b_{k,2-j}$ being of sizes $N_k \times N_k$, $k = 1, 2$, with

$$b_\mu = a_\mu = p_\mu I_N, \quad b_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right], \;\; X \neq 0.$$

6. The Subprincipal Symbol

In spectral asymptotics, the subprincipal symbol plays an important role. We shall in this section study its structure and the transformation laws under different diagonalizers of the principal part. From the decoupling theorems (semiclassical case as well as semiregular one) we obtain the following general form for the subprincipal term of the diagonalized symbol.

Proposition 6.1. *For the subprincipal part $b_{\mu-2}$ of the h^∞-diagonalization given in Theorem 5.1, or in the semiregular case by Theorem 5.3, one has, by (5.21), the formula (recall that $a_\mu = p_\mu I_N$):*

$$\begin{aligned} b_{\mu-2} =& e_{-2}^* e_0 b_\mu + b_\mu e_0^* e_{-2} + e_0^* a_{\mu-2} e_0 - \frac{i}{2}\left(e_0^*\left\{a_\mu, e_0\right\} + \left\{e_0^*, a_\mu e_0\right\}\right) \\ &+ e_{-1}^* a_\mu e_{-1} + b_{\mu-1} e_0^* e_{-1} + e_{-1}^* e_0 b_{\mu-1}, \end{aligned}$$

where

$$e_{-2} = \frac{i}{4} e_0 \left\{e_0^*, e_0\right\} - \frac{1}{2} e_0 \alpha_{-1}^* \alpha_{-1} + e_0 \alpha_{-2},$$

with $\alpha_{-2}^* = -\alpha_{-2}$ and $\alpha_{-1}^* = -\alpha_{-1}$ determined by (5.24) and (5.25).

Remark 6.2. It is important to note in e_{-2} the presence of the term $-e_0\alpha^*_{-1}\alpha_{-1}/2$ which depends on the symbol of order $\mu - 1$ (i.e. the semiprincipal part) of our system.

To study the structure of the subprincipal term, for the sake of clarity we will be considering in the first place the case $N = 2$ and afterwards the case of a general N.

All of the results below hold also true in the *semiregular* case of Theorem 5.3 above, the only change being that where in the case of the h^∞-diagonalization we have $X \in \mathbb{R}^{2n}$, in the semiregular case we have $X \neq 0$.

6.1. The case $N = 2$. Suppose hence that $N = 2$, that $a_\mu = a^*_\mu = p_\mu I_2$, $p_\mu \in S(m^\mu, g)$ scalar, and that $a_{\mu-1} = a^*_{\mu-1}$. Let (by slightly changing notation) $\lambda^+_{\mu-1}, \lambda^-_{\mu-1} \in S(m^{\mu-1}, g)$ be the eigenvalues of $a_{\mu-1}$, and suppose that

$$|\lambda^+_{\mu-1}(X) - \lambda^-_{\mu-1}(X)| \gtrsim m(X)^{\mu-1}, \forall X \in \mathbb{R}^{2n}, \tag{6.1}$$

whence the existence of a smooth unitary matrix e_0 such that

$$e_0(X)^* a_{\mu-1}(X) e_0(X) = \begin{bmatrix} \lambda^+_{\mu-1}(X) & 0 \\ 0 & \lambda^-_{\mu-1}(X) \end{bmatrix}, \quad \forall X \in \mathbb{R}^{2n}.$$

We have the following corollary.

Corollary 6.3. *Suppose that $a_\mu = a^*_\mu = p_\mu I_2$ is a scalar matrix and that $a_{\mu-1} = a^*_{\mu-1}$ possesses smooth eigenvalues $\lambda^\pm_{\mu-1}$ satisfying (6.1). Let $\{w_+, w_-\}$ be the* ***canonical*** *basis of $\mathbb{C}^2$, namely $w_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so that for the semiprincipal symbol $b_{\mu-1}$ of the (h^∞-)diagonalization we have $b_{\mu-1}(X) w_\pm = \lambda^\pm_{\mu-1}(X) w_\pm$, $\pm$-respectively, for all $X \in \mathbb{R}^{2n}$. Then for the subprincipal symbol $b_{\mu-2} = \begin{bmatrix} b^+_{\mu-2} & 0 \\ 0 & b^-_{\mu-2} \end{bmatrix}$ we have with $j = \pm$*

$$\begin{aligned} b^{(j)}_{\mu-2} &= \langle b_{\mu-2} w_j, w_j \rangle \\ &= \langle e^*_0 a_{\mu-2} e_0 w_j, w_j \rangle + \frac{1}{2} \mathrm{Im} \left(\langle \{e^*_0, p_\mu\} e_0 w_j, w_j \rangle \right) + \frac{1}{2} \mathrm{Im} \left(\langle e^*_0 \{p_\mu, e_0\} w_j, w_j \rangle \right). \end{aligned}$$

In addition, as for the term δ determined by equation (5.16) one has

$$\delta = -\frac{1}{\lambda^+_{\mu-1} - \lambda^-_{\mu-1}} \langle (e^*_0 a_{\mu-2} e_0 - i e^*_0 \{p_\mu, e_0\}) w_-, w_+ \rangle \tag{6.2}$$

Proof. Recall that $b_\mu = a_\mu = p_\mu I_2$. We write the subprincipal term $b_{\mu-2}$ as

$$b_{\mu-2} = b'_{\mu-2} + b''_{\mu-2},$$

where

$$b'_{\mu-2} := e^*_{-2} e_0 b_\mu + b_\mu e^*_0 e_{-2} + e^*_0 a_{\mu-2} e_0 - \frac{i}{2} \left(e^*_0 \left\{ p_\mu, e_0 \right\} + \left\{ e^*_0, p_\mu e_0 \right\} \right),$$

and

$$b''_{\mu-2} := e^*_{-1} a_\mu e_{-1} + b_{\mu-1} e^*_0 e_{-1} + e^*_{-1} e_0 b_{\mu-1}.$$

As for $b'_{\mu-2}$ we have, by Corollary 9.2.6 in [14] (used to deal with the terms coming from the "regular" step in the order of the symbols), that for $j = \pm$, respectively,

$$\langle b'_{\mu-2} w_j, w_j \rangle = \left\langle e_0^* a_{\mu-2} e_0 w_j, w_j \right\rangle + \frac{1}{2} \mathrm{Im} \left(\left\langle \{e_0^*, p_\mu\} e_0 w_j, w_j \right\rangle \right) \\ + \frac{1}{2} \mathrm{Im} \left(\left\langle e_0^* \{p_\mu, e_0\} w_j, w_j \right\rangle \right) - p_\mu \langle \alpha_{-1}^* \alpha_{-1} w_j, w_j \rangle$$

(the last term in the above expression is due to the form of e_{-2}, see Remark 6.2).

As for $b''_{\mu-2}$, on the other hand, we have (since $e_{-1} = e_0 \alpha_{-1}$)

$$\left\langle e_{-1}^* a_\mu e_{-1} w_j, w_j \right\rangle = \left\langle \alpha_{-1}^* b_\mu \alpha_{-1} w_j, w_j \right\rangle = p_\mu \left\langle \alpha_{-1} w_j, \alpha_{-1} w_j \right\rangle = \left\langle p_\mu \alpha_{-1}^* \alpha_{-1} w_j, w_j \right\rangle,$$

and

$$\left\langle \left(b_{\mu-1} e_0^* e_{-1} + e_{-1}^* e_0 b_{\mu-1} \right) w_j, w_j \right\rangle = \left\langle \left(b_{\mu-1} \alpha_{-1} + \alpha_{-1}^* b_{\mu-1} \right) w_j, w_j \right\rangle = 0$$

because $b_{\mu-1} \alpha_{-1} w_\pm = r w_\mp$ with $r \in C^\infty(\mathbb{R}^{2n}; \mathbb{C})$ (the same happens for $\alpha_{-1}^* b_{\mu-1}$ for a different r) and $\langle w_\mp, w_\pm \rangle = 0$ ($\pm$-respectively). It follows that

$$\langle b''_{\mu-2} w_j, w_j \rangle = p_\mu \langle \alpha_{-1}^* \alpha_{-1} w_j, w_j \rangle, \quad j = \pm.$$

Therefore, adding the expressions for $\langle b'_{\mu-2} w_j, w_j \rangle$ and for $\langle b''_{\mu-2} w_j, w_j \rangle$, we obtain the formula for the subprincipal part $b_{\mu-2}$.

We finally prove (6.2). By (5.16) we have that $\delta = -\gamma/(\lambda^+_{\mu-1} - \lambda^-_{\mu-1})$. Therefore we have to compute γ by means of equation (5.14). Hence, recalling that $\alpha_{-1} = \begin{bmatrix} 0 & \delta \\ -\delta^* & 0 \end{bmatrix}$ and that $e_{-1} = e_0 \alpha_{-1}$, we have

$$\gamma = \left\langle \left(e_0^* a_{\mu-2} e_0 - \frac{i}{2} \left(e_0^* \left\{ a_\mu, e_0 \right\} + \left\{ e_0^*, a_\mu e_0 \right\} \right) - p_\mu \left(-\frac{i}{2} \{e_0^*, e_0\} \right) w_-, w_+ \right\rangle \\ = \langle (e_0^* a_{\mu-2} e_0 - i e_0^* \{p_\mu, e_0\}) w_-, w_+ \rangle,$$

which gives (6.2) and concludes the proof. □

We must now study (still remaining in the 2×2 case) the transformation properties of the subprincipal term depending on the choice of e_0. We have the following proposition.

Proposition 6.4. *Suppose that $a_\mu = a_\mu^* = p_\mu I_2$ is a scalar matrix and that $a_{\mu-1} = a_{\mu-1}^*$ possesses smooth eigenvalues $\lambda^+_{\mu-1}$ and $\lambda^-_{\mu-1}$ satisfying (6.1). Let e_0 and $\tilde{e}_0$ be smooth, unitary 2×2 matrices in $S(1, g; \mathsf{M}_2)$ such that*

$$e_0^* a_{\mu-1} e_0 = \tilde{e}_0^* a_{\mu-1} \tilde{e}_0 = b_{\mu-1} = \begin{bmatrix} \lambda^+_{\mu-1} & 0 \\ 0 & \lambda^-_{\mu-1} \end{bmatrix}.$$

Denote by $b_{\mu-2}$ and $\tilde{b}_{\mu-2}$, respectively, the subprincipal terms given in Corollary 6.3, associated respectively with e_0 and $\tilde{e}_0$. Let hence $f \in S(1, g; \mathsf{M}_2)$ be the unitary matrix such that $e_0 = \tilde{e}_0 f$, so that (since $\lambda^+_{\mu-1} \neq \lambda^-_{\mu-1}$)

$$f = \begin{bmatrix} f_+ & 0 \\ 0 & f_- \end{bmatrix}$$

with the $f_j \in S(1, g)$ *and* $|f_j(X)| = 1$*, for all* $X \in \mathbb{R}^{2n}$*,* $j = \pm$*. Then, with* $\{w_+, w_-\}$ *the canonical basis of* $\mathbb{C}^2$ *as before,*

$$b_{\mu-2}^{(j)} = \langle b_{\mu-2} w_j, w_j \rangle = \left\langle \tilde{b}_{\mu-2} w_j, w_j \right\rangle + \operatorname{Im}\left(f_j \left\{ \bar{f}_j, p_\mu \right\} \right), \; j = \pm. \tag{6.3}$$

Moreover, one has

$$\delta = \bar{f}_+ f_- \tilde{\delta} \tag{6.4}$$

where $\tilde{\delta}$ *is determined by equation* (5.16) *with* $\tilde{e}_0$ *in place of* e_0*.*

Proof. By Corollary 6.3 and by the proof of Proposition 9.2.7 in [14] we have (6.3). Hence, we only need to show that $\delta = \bar{f}_+ f_- \tilde{\delta}$.
On the one hand,

$$\begin{aligned} \left\langle e_0^* \{a_\mu, e_0\} w_-, w_+ \right\rangle &= \bar{f}_+ \left\langle \{a_\mu, \tilde{e}_0 f\} w_-, \tilde{e}_0 w_+ \right\rangle \\ &= \bar{f}_+ f_- \left\langle \left\{ p_\mu, \tilde{e}_0 \right\} w_-, \tilde{e}_0 w_+ \right\rangle + \bar{f}_+ \left\langle \left\{ p_\mu, f_- \right\} \tilde{e}_0 w_-, \tilde{e}_0 w_+ \right\rangle \\ &= \bar{f}_+ f_- \left\langle \tilde{e}_0^* \left\{ p_\mu, \tilde{e}_0 \right\} w_-, w_+ \right\rangle + \bar{f}_+ \left\langle \left\{ p_\mu, f_- \right\} \tilde{e}_0 w_-, \tilde{e}_0 w_+ \right\rangle . \end{aligned} \tag{6.5}$$

On the other hand,

$$\bar{f}_+ \left\langle \left\{ p_\mu, f_- \right\} \tilde{e}_0 w_-, \tilde{e}_0 w_+ \right\rangle = \bar{f}_+ \{p_\mu, f_-\} \langle \tilde{e}_0 w_-, \tilde{e}_0 w_+ \rangle = 0. \tag{6.6}$$

Now, (6.5) and (6.6) give

$$\begin{aligned} \delta &= -\frac{1}{\lambda_{\mu-1}^+ - \lambda_{\mu-1}^-} \langle (e_0^* a_{\mu-2} e_0 - i e_0^* \{p_\mu, e_0\}) w_-, w_+ \rangle \\ &= -\frac{\bar{f}_+ f_-}{\lambda_{\mu-1}^+ - \lambda_{\mu-1}^-} \langle (\tilde{e}_0^* a_{\mu-2} \tilde{e}_0 - i \tilde{e}_0^* \{p_\mu, \tilde{e}_0\}) w_-, w_+ \rangle \\ &= \bar{f}_+ f_- \tilde{\delta}, \end{aligned}$$

which concludes the proof. □

6.2. The case of blockwise matrices. We pass in this subsection to the study of the subprincipal symbol in the more general case of a diagonalization into 2 blocks, with $N > 2$.

Suppose now that:

(i) $a_\mu = a_\mu^* = p_\mu I_N$ is a scalar matrix with $\mu > 0$, $p_\mu \in S(m^\mu, g)$;
(ii) $a_{\mu-1} = a_{\mu-1}^*$ is such that (as in Theorem 5.1) there exists a smooth unitary matrix $e_0 \in S(1, g; \mathsf{M}_N)$ such that

$$e_0(X)^* a_{\mu-1}(X) e_0(X) = \left[\begin{array}{c|c} \lambda_{\mu-1}^+(X) & 0 \\ \hline 0 & \lambda_{\mu-1}^-(X) \end{array} \right], \quad \forall X \in \mathbb{R}^{2n}, \tag{6.7}$$

where, writing $N = N_+ + N_-$, we have that $\pm$-respectively $\lambda_{\mu-1}^\pm \in S(m^{\mu-1}, g, \mathsf{M}_{N_\pm})$, with

$$\inf \left\{ |\zeta_1 - \zeta_2|; \zeta_1 \in \operatorname{Spec}(\lambda_{\mu-1}^+), \zeta_2 \in \operatorname{Spec}(\lambda_{\mu-1}^-) \right\} \gtrsim m(X)^{\mu-1}, \forall X \in \mathbb{R}^{2n}. \tag{6.8}$$

We have the following corollary.

Corollary 6.5. *Suppose that a_μ and $a_{\mu-1}$ satisfy the conditions (i) and (ii) above. Consider, $\pm$-respectively, the orthogonal projectors $\pi_\pm\colon \mathbb{C}^N \longrightarrow \mathbb{C}^N = \mathbb{C}^{N_+} \oplus \mathbb{C}^{N_-}$ onto $\mathbb{C}^{N_+} \oplus \{0\}$ and $\{0\} \oplus \mathbb{C}^{N_-}$ respectively (that is, $\pi_+ = [I_{N_+}|0_{N_-}]$ and $\pi_- = [0_{N_-}|I_{N_+}]$), so that for the semiprincipal symbol $b_{\mu-1}$ of the h^∞-diagonalization we have $\pi_\pm b_{\mu-1}(X)\pi_\pm^* = \lambda^\pm_{\mu-1}(X)$, $\pm$-respectively, for all $X \in \mathbb{R}^{2n}$. Then for the subprincipal symbol $b_{\mu-2} = \left[\begin{array}{c|c} b^+_{\mu-2} & 0 \\ \hline 0 & b^-_{\mu-2}\end{array}\right]$ we have, with $j = \pm$,*

$$\begin{aligned}
b^{(j)}_{\mu-2} &= \pi_j b_{\mu-2}\pi_j^* \\
&= \pi_j e_0^* a_{\mu-2} e_0 \pi_j^* - \frac{i}{2}\pi_j\{e_0^*, a_\mu\}e_0\pi_j^* - \frac{i}{2}\pi_j e_0^*\{a_\mu, e_0\}\pi_j^* \\
&= \pi_j e_0^* a_{\mu-2} e_0 \pi_j^* - i\pi_j e_0^*\{p_\mu, e_0\}\pi_j^*.
\end{aligned}$$

In addition, for δ (see (5.16)) one has $\lambda^+_{\mu-1}\delta - \delta\lambda^-_{\mu-1} = -\gamma$ where

$$\gamma = \pi_+\left(e_0^* a_{\mu-2} e_0 - i e_0^*\{p_\mu, e_0\}\right)\pi_-^*. \tag{6.9}$$

Proof. By Proposition 6.1, the terms in $\pi_j b_{\mu-2}\pi_j^*$ for $j = \pm$ are given by (recall that $b_\mu = a_\mu = p_\mu I_N$)

$$\begin{aligned}
&\pi_j\left(e_{-2}^* e_0 b_\mu + b_\mu e_0^* e_{-2}\right)\pi_j^* \\
&\quad= \frac{i}{4}\pi_j\left(b_\mu\{e_0^*, e_0\} - \{e_0^*, e_0\}^* b_\mu\right)\pi_j^* \qquad \text{(1st)} \\
&\qquad + \pi_j\left(-\frac{1}{2}\alpha_{-1}^*\alpha_{-1} + \alpha_{-2}^*\right)b_\mu\pi_j^* + \pi_j b_\mu\left(-\frac{1}{2}\alpha_{-1}^*\alpha_{-1} + \alpha_{-2}\right)\pi_j^*
\end{aligned}$$

(since $\{e_0^*, e_0\}^* = -\{e_0^*, e_0\}$ and $\alpha_{-2}^* = -\alpha_{-2}$)

$$\begin{aligned}
&= \frac{i}{2}\pi_j\left(b_\mu\{e_0^*, e_0\}\right)\pi_j^* - b_\mu\pi_j\alpha_{-1}^*\alpha_{-1}\pi_j^*, \\
\pi_j\{e_0^*, a_\mu e_0\}\pi_j^* &= \pi_j\{e_0^*, e_0\}p_\mu\pi_j^* + \pi_j\{e_0^*, p_\mu\}e_0\pi_j^*, \qquad \text{(2nd)} \\
\pi_j e_{-1}^* a_\mu e_{-1}\pi_j^* &= p_\mu\pi_j\alpha_{-1}^*\alpha_{-1}\pi_j^*, \qquad \text{(3rd)}
\end{aligned}$$

and finally

$$\pi_j\left(b_{\mu-1}e_0^* e_{-1} + e_{-1}e_0 b_{\mu-1}\right)\pi_j^* = \pi_j b_{\mu-1}\alpha_{-1}\pi_j^* + \pi_j\alpha_{-1}^* b_{\mu-1}\pi_j^* = 0, \qquad \text{(4th)}$$

since $b_{\mu-1}\alpha_{-1}$ is blockwise anti-diagonal. Summing the above terms gives the expression of $b^{(j)}_{\mu-2}$.

We next show (6.9). By (5.16) we have that $\lambda^+_{\mu-1}\delta - \delta\lambda^-_{\mu-1} = -\gamma$. Therefore, we just need to compute γ by means of equation (5.14). Hence, recalling that $\alpha_{-1} = \left[\begin{array}{c|c} 0 & \delta \\ \hline -\delta^* & 0\end{array}\right]$ and that $e_{-1} = e_0\alpha_{-1}$, we have

$$\begin{aligned}
\gamma &= \pi_+\left(e_0^* a_{\mu-2} e_0 - \frac{i}{2}\left(e_0^*\left\{a_\mu, e_0\right\} + \left\{e_0^*, a_\mu e_0\right\}\right) - a_\mu\left(-\frac{i}{2}\{e_0^*, e_0\}\right)\right)\pi_-^* \\
&= \pi_+\left(e_0^* a_{\mu-2} e_0 - i e_0^*\{p_\mu, e_0\}\right)\pi_-^*.
\end{aligned}$$

which gives (6.9) and concludes the proof. ◻

As before, we must now study (still remaining in the $N \times N$ case, $N > 2$, with 2 blocks) the transformation properties of the subprincipal terms depending on the choice of e_0. We have the following proposition.

Proposition 6.6. *Suppose that a_μ and $a_{\mu-1}$ satisfy the above conditions (i) and (ii). Let e_0 and $\tilde{e}_0$ be smooth, unitary $N \times N$ matrices in $S(1, g; \mathsf{M}_N)$ such that*

$$e_0^* a_{\mu-1} e_0 = \tilde{e}_0^* a_{\mu-1} \tilde{e}_0 = b_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1}^+ & 0 \\ \hline 0 & \lambda_{\mu-1}^- \end{array}\right],$$

with the blocks $\lambda_{\mu-1}^\pm$ satisfying (6.7). Denote by $b_{\mu-2}$ and $\tilde{b}_{\mu-2}$, respectively, the subprincipal terms given in Corollary 6.3, associated respectively with e_0 and $\tilde{e}_0$. Let hence $f \in S(1, g; \mathsf{M}_N)$ be the unitary matrix such that $e_0 = \tilde{e}_0 f$, so that (by the spacing property of the spectra of $\lambda_{\mu-1}^\pm$)

$$f = \left[\begin{array}{c|c} f_+ & 0 \\ \hline 0 & f_- \end{array}\right]$$

with the $f_\pm \in S(1, g; \mathsf{M}_{N_\pm})$ being themselves unitary matrices. As before, consider $\pi_\pm$ the projectors of $\mathbb{C}^N = \mathbb{C}^{N_+} \oplus \mathbb{C}^{N_-}$ respectively onto $\mathbb{C}^{N_+} \oplus \{0\}$ and $\{0\} \oplus \mathbb{C}^{N_-}$. Then, for $j = \pm$,

$$b_{\mu-2}^{(j)} = \pi_j b_{\mu-2} \pi_j^* = f_j^* \pi_j \tilde{b}_{\mu-2} \pi_j^* f_j - \frac{i}{2}\left(f_j^* \{p_\mu, f_j\} - \{p_\mu, f_j\}^* f_j\right) \quad (6.10)$$

$$= f_j^* \pi_j \tilde{b}_{\mu-2} \pi_j^* f_j + \mathsf{Im}(f_j^* \{p_\mu, f_j\})$$

(where, for a matrix A, we put $2i\,\mathsf{Im}(A) = A - A^*$ for its skew-Hermitian part). Moreover,

$$\lambda_{\mu-1}^+ \tilde{\delta} - \tilde{\delta} \lambda_{\mu-1}^- = f_+ \left(\lambda_{\mu-1}^+ \delta - \delta \lambda_{\mu-1}^-\right) f_-^* \quad (6.11)$$

where $\tilde{\delta}$ is determined by equation (5.16) with $\tilde{e}_0$ in place of e_0.

Proof. We prove (6.10) by following the scheme of proof of Corollary 9.2.6 in [14]. One has

$$\pi_j e_0^* a_{\mu-2} e_0 \pi_j^* = \underbrace{\pi_j f^*}_{= f_j^* \pi_j} \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 \underbrace{f \pi_j^*}_{= \pi_j^* f_j} = f_j^* \pi_j \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 \pi_j^* f_j,$$

$$\begin{aligned} \pi_j \{e_0^*, a_\mu\} e_0 \pi_j^* &= \pi_j \{f^* \tilde{e}_0^*, p_\mu\} \tilde{e}_0 f \pi_j^* \\ &= f_j^* \pi_j \{\tilde{e}_0^*, p_\mu\} \tilde{e}_0 \pi_j^* f_j + \underbrace{\{f_j^*, p_\mu\}}_{= -\{p_\mu, f_j\}^*} \underbrace{\pi_j \tilde{e}_0^* \tilde{e}_0 \pi_j^*}_{I_{N_j}} f_j \\ &= f_j^* \pi_j \{\tilde{e}_0^*, p_\mu\} \tilde{e}_0 \pi_j^* f_j - \{p_\mu, f_j\}^* f_j, \end{aligned}$$

$$\pi_j e_0^* \{a_\mu, e_0\} \pi_j^* = \pi_j f^* \tilde{e}_0^* \{p_\mu, \tilde{e}_0 f\} \pi_j^*$$

$$= f_j^* \pi_j \tilde{e}_0^* \{p_\mu, \tilde{e}_0\} \pi_j^* f_j + f_j^* \underbrace{\pi_j \tilde{e}_0^* \tilde{e}_0 \pi_j^*}_{I_{N_j}} \{p_\mu, f_j\}$$
$$= f_j^* \pi_j \tilde{e}_0^* \{p_\mu, \tilde{e}_0\} \pi_j^* f_j + f_j^* \{p_\mu, f_j\}.$$

Corollary 6.5 gives the formula.
We next prove (6.11). By (5.16) we have that $\lambda_{\mu-1}^+ \delta - \delta \lambda_{\mu-1}^- = -\gamma$ and by (6.9) that

$$\gamma = \pi_+ \left(e_0^* a_{\mu-2} e_0 - i e_0^* \{p_\mu, e_0\} \right) \pi_-^*.$$

We have therefore to study the transformation properties of γ. One has

$$\begin{aligned} \gamma &= \pi_+ \big(f^* \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 f - i f^* \tilde{e}_0^* \{p_\mu, \tilde{e}_0 f\} \big) \pi_-^* \\ &= f_+^* \pi_+ \big(\tilde{e}_0^* a_{\mu-2} \tilde{e}_0 - i \tilde{e}_0^* \{p_\mu, \tilde{e}_0\} \big) \pi_-^* f_- - i f_+^* \underbrace{\pi_+ \tilde{e}_0^* \tilde{e}_0 \pi_-^*}_{=0} \{p_\mu, f_-\} = f_+^* \tilde{\gamma} f_-, \end{aligned}$$

whence (6.11). This concludes the proof. □

7. The Weyl Law

In this section, we prove for a system $A \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, a semiregular metric globally elliptic system of the kind introduced in Definition 2.4 (i.e., an SMGES), a "classical Weyl-Law" and a "refined Weyl-Law" result of the kind proved for *scalar* semiregular operators, respectively, by Helffer and Robert [6] and by Doll, Gannot and Wunsch [4]. We follow the approach in [4] for both the results. As is classical, the approach is based on the construction of an FIO (Fourier integral operator) parametrix of the Schrödinger unitary group generated by A^{w}. We will hence have to exploit our diagonalization result (in the semiregular setting) developed earlier in the paper. In fact, we construct a parametrix for the diagonalized system and thus obtain a parametrix by conjugating with the operator $E^{\mathrm{w}}(x, D)$ constructed in Sect. 5. However, because of that conjugation we need to have a better control on the compositions occurring in conjugations. Hence, it will be convenient to construct a parametrix following the idea of Doll and Zelditch in [5], that is, by exploiting the fact that the parametrix FIO can in fact be written as a Weyl-quantization. Having the parametrix for $e^{-itA^{\mathrm{w}}}$, we then follow the classical approach, in that we will be able to consider the trace of its Schwartz distribution kernel and obtain our results through the asymptotics of the convolution of the counting function with a suitable scalar function (with compactly supported Fourier transform) and classical Tauberian arguments.

Throughout the section ds denotes the Riemannian metric induced on $\{p_2 = 1\}$ or on $\{p_2 = \lambda\}$ (it will be clear from the context) by the Euclidean one with $\lambda > 0$, and $ds/|\nabla p_2|$ denotes the associated Leray-Liouville measure.

In the proofs of Proposition 7.2 and of Theorem 7.8 it will be fundamental that the angular gradients of the X-ray transform of the eigenvalues of the semiprincipal symbol vanish to infinite order exactly on a subset of measure zero of $\mathbb{S}^{2n-1}$. Namely, if we denote by ∂_ω^α, $\alpha \in \mathbb{N}^{2n-1}$, the tangential derivatives to $\mathbb{S}^{2n-1}$ and by $\lambda_{1,j}$ $(1 \le j \le r)$ the eigenvalues of the semiprincipal symbol a_1 of the SMGES under study, we require that for all $1 \le j \le r$ (recall that r is the number of *distinct* eigenvalues of a_1)

- **Condition DGW:**

$$\Pi_{2\pi,j} := \left\{\omega \in \mathbb{S}^{2n-1};\ \partial_\omega^\alpha \int_0^{2\pi} (\lambda_{1,j} \circ \exp tH_{p_2})(\omega)dt = 0,\ \forall\alpha \in \mathbb{N}^{2n-1} \setminus \{0\}\right\}$$
has measure zero. (7.1)

Remark 7.1. Note that to impose **Condition DGW** (7.1) we need a certain kind of knowledge of the eigenvalues of the semiprincipal symbol a_1 of the SMGES we are considering. By definition of that class, we already know that the eigenvalues are smooth functions, whose graphs never cross. However, we may think of **Condition DGW** (7.1) as a condition on the logarithmic derivative of the characteristic polynomial P of the semiprincipal symbol, and, for that, Rouché's Theorem is very useful. In fact, by the Definition 2.4 of SMGES, for all $\omega_0 \in \mathbb{S}^{2n-1}$ and all $1 \le j \le r$, there is a closed disk in the complex plane $B_{\omega_0,j}$ centered at $\lambda_j(\omega_0)$ and containing no other $\lambda_{j'}(\omega_0)$ with $j' \neq j$. In particular, $B_{\omega_0,j} \cap B_{\omega_0,j'} = \emptyset$ for $j \neq j'$. Hence, by Rouché's Theorem there is an open neighborhood U_{ω_0} of ω_0 on the sphere $\mathbb{S}^{2n-1}$ such that

$$\lambda_j(\omega) = \frac{1}{2\pi i N_j} \int_{\partial B_{\omega_0,j}} \lambda \frac{\partial_\lambda P(\omega;\lambda)}{P(\omega;\lambda)} d\lambda,\ \ \forall\omega \in U_{\omega_0},\ \ j = 1, \dots, r, \tag{7.2}$$

where N_j is the multiplicity of λ_j and $P(\omega;\lambda) := \det(a_1(\omega) - \lambda I_N)$. Thus, we can give a local representation of λ_j around every $\omega \in \mathbb{S}^{2n-1}$ and, therefore, by the compactness of $\mathbb{S}^{2n-1}$ there is a finite open covering $\{U_{\omega_k}\}_{k=1,\dots,\bar{k}}$ of $\mathbb{S}^{2n-1}$ such that on each open set of the cover the identity (7.2) holds. Finally, a partition of unity argument subordinated to the covering $\{U_{\omega_k}\}_{k=1,\dots,\bar{k}}$ gives the function $\mathbb{S}^{2n-1} \ni \omega \mapsto \lambda_j(\omega)$ for all j. Therefore **Condition DGW** (7.1) may be described in terms of derivatives of the logarithmic derivative of the characteristic polynomial P of the semiprincipal symbol.

For clarity of exposition, we first prove a result in the fully *diagonal* case which serves as a guide to guess what the result should look like in the more general, *nondiagonal* case.

Proposition 7.2. *Let $B = B^* \sim \sum_{j\ge 0} b_{2-j} \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$ be a **diagonal** SMGES symbol. Hence, in particular, $b_2 = p_2 I_N$ with $p_2 \in S(m^2, g)$ the scalar harmonic oscillator. Let $\mathbb{R} \ni \lambda \longmapsto \mathsf{N}(\lambda)$ denote the spectral counting function associated with B^{w}. We have the following asymptotics*

$$\mathsf{N}(\lambda) = \left(\frac{N}{(2\pi)^n}\int_{p_2\le 1} dX\right)\lambda^n - \left((2\pi)^{-n}\int_{p_2=1} \mathsf{Tr}\,(b_1)\frac{ds}{|\nabla p_2|}\right)\lambda^{n-1/2} + O(\lambda^{n-1}),\ \lambda \to +\infty. \tag{7.3}$$

*Furthermore, if **Condition DGW** (7.1) is satisfied, then (7.3) can be refined to*

$$\mathsf{N}(\lambda) = (2\pi)^{-n}\left(\sum_{j=1}^{N}\left(\int_{p_2+b_{1,j}\le\lambda} dX\right) - \int_{p_2=\lambda} \mathsf{Tr}\,(b_0)\frac{ds}{|\nabla p_2|}\right) + o(\lambda^{n-1}),\ \lambda \to +\infty, \tag{7.4}$$

where $b_{1,j}$ is the j-th term of the diagonal of b_1 with $j = 1, \dots, N$.

Proof. Of course, we may write the counting function as

$$\mathbb{R} \ni \lambda \longmapsto \mathsf{N}(\lambda) = \sum_{j=1}^{N} \mathsf{N}_j(\lambda),$$

where N_j is the counting function given by the jth diagonal term of B^{w}. Applying then the scalar results by Doll, Gannot and Wunsch [4] to get the asymptotics of each of the contributions in the two cases of the statement, we sum up the asymptotics of all contributions to get the asymptotics of $\mathsf{N}(\lambda)$.

To obtain (7.3) for each $1 \le j \le N$, let $\rho \in \mathscr{S}(\mathbb{R})$ such that $\hat{\rho}$ has compact support in $(-\varepsilon, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ and $\rho = 1$ on a neighborhood of 0. We have

$$(\mathsf{N}_j * \rho)(\lambda) = (2\pi)^{-n} \left(\int_{p_2 + b_{1,j} \le \lambda} dX - \int_{p_2 = \lambda} \mathrm{Tr}\,(b_0) \frac{ds}{|\nabla p_2|} \right) + O(\lambda^{n-3/2}),\ \lambda \to +\infty, \tag{7.5}$$

by [4], Proposition 6.1. Since

$$\mathrm{Vol}(\{p_2 + b_{1,j} \le \lambda\}) = \lambda^n \mathrm{Vol}(\{p_2 + \lambda^{-1/2} b_{1,j} \le 1\}),$$

a Taylor-expansion in powers of $\lambda^{-1/2}$ and Lemma IV.7 of [6] give the asymptotics

$$\mathsf{N}_j(\lambda) = (2\pi)^{-n} \left(\Big(\int_{p_2 \le 1} dX \Big) \lambda^n - \Big(\int_{p_2 = 1} b_{1,j} \frac{ds}{|\nabla p_2|} \Big) \lambda^{n-1/2} \right) + O(\lambda^{n-1}),\ \lambda \to +\infty.$$

Therefore, as $\lambda \to +\infty$,

$$\begin{aligned}
\mathsf{N}(\lambda) &= \sum_{j=1}^{N} \mathsf{N}_j(\lambda) \\
&= \sum_{j=1}^{N} \left(\left((2\pi)^{-n} \int_{p_2 \le 1} dX \right) \lambda^n - \left((2\pi)^{-n} \int_{p_2 = 1} b_{1,j} \frac{ds}{|\nabla p_2|} \right) \lambda^{n-1/2} \right) + O(\lambda^{n-1}) \\
&= \left(\frac{N}{(2\pi)^n} \int_{p_2 \le 1} dX \right) \lambda^n - \left((2\pi)^{-n} \int_{p_2 = 1} \mathsf{Tr}(b_1) \frac{ds}{|\nabla p_2|} \right) \lambda^{n-1/2} + O(\lambda^{n-1}),
\end{aligned}$$

which gives (7.3).

We next prove (7.4). In fact, by virtue of **Condition DGW** (7.1) we are in a position to apply Theorem 1.2 of [4] to each diagonal term of B and obtain that

$$\mathsf{N}_j(\lambda) = (2\pi)^{-n} \left(\left(\int_{p_2 + b_{1,j} \le \lambda} dX \right) - \int_{p_2 = \lambda} b_{0,j} \frac{ds}{|\nabla p_2|} \right) + o(\lambda^{n-1}),\ \lambda \to +\infty,$$

for all $1 \le j \le N$, whence

$$\begin{aligned}\mathsf{N}(\lambda) &= \sum_{j=1}^{N} \mathsf{N}_j(\lambda)\\ &=(2\pi)^{-n}\left(\sum_{j=1}^{N}\int_{p_2+b_{1,j}\leq\lambda} dX - \int_{p_2=\lambda} \mathsf{Tr}\,(b_0)\,\frac{ds}{|\nabla p_2|}\right) + o(\lambda^{n-1}),\ \lambda\to+\infty,\end{aligned}$$

which concludes the proof. □

As already mentioned, the fundamental tool to obtain the Weyl law for the class of semiregular ψdo *systems* we are interested in, is a parametrix of the unitary group $t \mapsto e^{-itA^{\mathrm{w}}}$. In our vector-valued situation, by the diagonalization result Theorem 5.3 this goes through the construction of the parametrix in the case of a semiregular system with scalar principal part, *blockwise scalar semiprincipal part*, and a full blockwise subprincipal part.

For the parametrix construction in the diagonal case, we will first construct a parametrix of the *reduced propagator* (see Lemma 7.3 below) and will then compose the latter with the unitary group of the harmonic oscillator (which is the Weyl-quantization of an exponential, see Hörmander in [11]). The main advantage of such a construction is that, following the approach of Doll and Zelditch [5], the parametrix is a Weyl-quantitation. This is crucial, for we have to compose the FIOs by the diagonalizers to obtain a parametrix for $t \mapsto e^{-itA^{\mathrm{w}}}$, and this is a *delicate* point.
We next follow the approach as in the scalar case by Doll, Gannot and Wunsch [4] (which is in turn inspired by Hörmander [10]), which gives a result that generalizes their Proposition 6.1, hence yielding an asymptotics for $\mathsf{N} * \rho$ for a suitable localizing function ρ (belonging to $\mathscr{S}(\mathbb{R})$ such that $\hat{\rho}$ has compact support in $(-\varepsilon, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ and $\hat{\rho} = 1$ in a neighborhood of 0). The refined Weyl-Law estimate will then follow from that by a Tauberian argument.

We consider at first the construction of the reduced propagator in the case of a system B with *scalar principal and semiprincipal symbols* (note that we allow a matrix-valued subprincipal symbol and lower order terms).

Note that we will have to consider Weyl-quantizations of the kind $(e^{i\phi_1}\alpha)^{\mathrm{w}}$, where $\alpha \in S_{\mathrm{sreg}}(1, g; \mathsf{M}_N)$ and ϕ_1 is an isotropic symbol of order 1. This is done according to the Weyl-Hörmander calculus with metric $|dX|^2$ whose Planck constant is 1.

Lemma 7.3. *Let* $B = B^* \sim \sum_{j\geq 0} b_{2-j} \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, *where the* $b_j = b_j^*$ *are positively homogeneous of degree* j *and* b_2 *and* b_1 *are scalar:* $b_2 = p_2 I_N$ *and* $b_1 = p_1 I_N$, *where* p_2 *is the harmonic oscillator and* p_1 *is homogeneous of degree* 1. *For* $t \in \mathbb{R}$ *consider*

$$P(t) := e^{itp_2^{\mathrm{w}}}(B^{\mathrm{w}} - p_2^{\mathrm{w}})e^{-itp_2^{\mathrm{w}}}.$$

Let H_{p_2} *be the Hamilton field of* p_2 *and* $t \mapsto \exp\left(tH_{p_2}\right)(X)$ *be its bicharacteristic flow. Consider the phase-function*

$$\mathbb{R}_t \times \mathbb{R}^{2n}_X \ni (t, X) \mapsto \tilde{\phi}_1(t, X) := -\int_0^t p_1 \circ \exp\left(sH_{p_2}\right)(X)\, ds. \tag{7.6}$$

Then there is $\tilde{\alpha} \in C^\infty(\mathbb{R}_t; S_{\mathrm{sreg}}(1, g; \mathsf{M}_N))$ *such that* $\mathbb{R} \ni t \mapsto \tilde{F}(t) := (e^{i\tilde{\phi}_1(t)}\tilde{\alpha}(t))^{\mathrm{w}}$ *solves*

$$(i\partial_t - P)\tilde{F} \in C^\infty(\mathbb{R}_t; \mathscr{L}(\mathscr{S}', \mathscr{S}) \otimes \mathsf{M}_N),\ \ \tilde{F}|_{t=0} = I_N + R,$$

where R is smoothing.

Proof. As usual, we make a WKB construction, the main point being that the eikonal equation and the transport equations are globally solvable in time. Note that in the transport equations we have a matrix term of order zero (generated by the in general non-scalar subprincipal part b_0), but this is harmless in solving them.

Observe that since H_{p_2} is linear, $X \mapsto \exp(tH_{p_2})(X)$ is a global linear diffeomorphism for all t, so that by Egorov's Theorem (or Hörmander's theorem on the invariance of the Weyl calculus through linear symplectomorphisms) we have that the Weyl symbol of $P(t)$ is given by $(B - p_2) \circ \exp\big(tH_{p_2}\big)$. Therefore the principal term of $P(t)$ is

$$\tilde{p}_1(t) := p_1 \circ \exp(tH_{p_2}), \quad t \in \mathbb{R},$$

and the semiprincipal one is

$$\tilde{b}_0(t) := b_0 \circ \exp(tH_{p_2}), \quad t \in \mathbb{R}.$$

The eikonal equation is

$$\begin{cases} \partial_t \tilde{\phi}_1 + \tilde{p}_1 = 0, \\ \tilde{\phi}_1|_{t=0} = 0, \end{cases}$$

and it is solved for all t and X by $\tilde{\phi}_1$ given in (7.6).

As for the terms of the WKB expansion of $\tilde{\alpha} \sim \sum_{j\geq 0} \tilde{\alpha}_{-j}$ we have a sequence of transport equations, the first of which has the form

$$\begin{cases} \partial_t \tilde{\alpha}_0 = (\tilde{b}_0 - \frac{1}{2}\{\tilde{p}_1, \tilde{\phi}_1\} I_N)\tilde{\alpha}_0, \\ \tilde{\alpha}_0|_{t=0} = I_N. \end{cases}$$

Since the characteristics are straight lines, the solution exists for all times, the matrix-valued term $\tilde{b}_0$ being, as already mentioned, harmless. One proceeds similarly for the other transport equations (which have the same structure, with initial condition the zero-matrix and source terms depending on the $\tilde{\alpha}_{-j}$s already constructed, as usual). Observe that $\tilde{b}_0 - \frac{1}{2}\{\tilde{p}_1, \tilde{\phi}_1\} I_N$ is homogeneous of degree 0 and that the higher transport equations for $\tilde{\alpha}_{-j}$ preserve homogeneity ($\tilde{\alpha}_{-j}$ is homogeneous of degree $-j$). The characteristics being straight lines, the $\tilde{\alpha}_{-j}(t)$ exist for all times. Taking $\tilde{\alpha} \sim \sum_{j\geq 0} \tilde{\alpha}_{-j}$ concludes the proof. □

Next, we need a composition result for quadratic phase functions (analogous to Proposition 4.2 in [5]).

Proposition 7.4. *Let $A \in \mathsf{M}_{2n}$ be a real symmetric matrix and $a, b \in \mathscr{S}(\mathbb{R}^{2n})$. We have*

$$(e^{i\langle A\cdot,\cdot\rangle} a \# b)(X) = \pi^{-4n} e^{i\langle AX,X\rangle} \int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle} a(X+Y_1) b(X + JAX + Y_2)\, dY_1\, dY_2,$$

where $Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{R}^{4n}$, $X \in \mathbb{R}^{2n}$, the $4n \times 4n$ matrix Q given by

$$Q := \left[\begin{array}{c|c} -A & -J \\ \hline J & 0 \end{array}\right], \tag{7.7}$$

and where $J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$ *is the standard* $2n \times 2n$ *symplectic matrix, # being the composition operator in the Weyl calculus.*

Proof. The proof follows by the integral representation for the composition of Schwartz symbols (see Zworski [23]) and a change of coordinates in the integral. In fact, by [23] Theorem 4.11,

$$
\begin{aligned}
&(e^{i\langle A\cdot,\cdot\rangle}a\#b)(X) \\
&\quad=\pi^{-4n}e^{i\langle AX,X\rangle}\int_{\mathbb{R}^{4n}} e^{-2i\sigma(Y_1,Y_2)+i\langle A(X+Y_1),X+Y_1\rangle}a(X+Y_1)b(X+Y_2)\,dY_1\,dY_2,
\end{aligned} \tag{7.8}
$$

where

$$\sigma(Y_1, Y_2) = \frac{1}{2}\left\langle \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\right\rangle.$$

Now, the change of coordinates in (7.8)

$$Y_1 = \tilde{Y}_1, \quad Y_2 = \tilde{Y}_2 + JAX,$$

leads to (using Y_1, Y_2 again)

$$
\begin{aligned}
&(e^{i\langle A\cdot,\cdot\rangle}a\#b)(X) \\
&\quad=\pi^{-4n}e^{i\langle AX,X\rangle}\int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle}a(X+Y_1)b(X+JAX+Y_2)\,dY_1\,dY_2.
\end{aligned}
$$

In fact,

$$2\sigma(Y_1, Y_2) = \left\langle \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix}\tilde{Y}, \tilde{Y}\right\rangle + \left\langle AX, \tilde{Y}_1\right\rangle + \left\langle A\tilde{Y}_1, X\right\rangle,$$

where $\tilde{Y} := \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix} \in \mathbb{R}^{4n}$. Hence, $-2\sigma(Y_1, Y_2) + \langle A(X+Y_1), X+Y_1\rangle = -\langle QY, Y\rangle$ and the proof is complete. □

By Proposition 7.4 we may compute how quadratic exponentials act on oscillating functions.

Proposition 7.5. *Let* ϕ_1 *be real, homogeneous of degree* 1 *and smooth on* $\mathbb{R}^{2n} \setminus \{0\}$. *Let* $a \in S(m^{\mu_1}, g; \mathsf{M}_N)$*, and* $b \in S(m^{\mu_2}, g; \mathsf{M}_N)$*. For any given real symmetric and positive-definite (resp. negative-definite) matrix* $A \in \mathsf{M}_{2n}$ *we have*

$$(e^{i\langle A\cdot,\cdot\rangle}a\#e^{i\phi_1}b)(X) = e^{i\langle AX,X\rangle+i\phi_1(X+JAX)}c,$$

where $X \in \mathbb{R}^{2n}$ *and* $c \in S(m^{\mu_1+\mu_2}, g; \mathsf{M}_N)$

Proof. Since the linear map defined by Q (see (7.7)) is injective we may use the usual approximation argument and a non-stationary phase argument to extend the previous approach to semiregular symbols. We may hence consider

$$(e^{i\langle A\cdot,\cdot\rangle}a\sharp e^{i\phi_1}b)(X) = \pi^{-4n}e^{i\langle AX,X\rangle+i\phi_1(X+JAX)} \times \\ \times \int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle}a(X+Y_1)e^{i(\phi_1(X+JAX+Y_2)-\phi_1(X+JAX))}b(X+JAX+Y_2)dY_1dY_2. \tag{7.9}$$

We show that the integral in the right-hand side of (7.9) is a symbol $c \in S(m^{\mu_1+\mu_2}, g; \mathsf{M}_N)$. Note that when $A > 0$ (resp. $A < 0$), then $I_{2n} + JA$ is invertible. We next define, for $\lambda > 0$,

$$c_\lambda(X) := \pi^{-4n}\int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle}a(\sqrt{\lambda}X+Y_1)e^{i(\phi_1(\sqrt{\lambda}(I_{2n}+JA)X+Y_2)-\phi_1(\sqrt{\lambda}(I_{2n}+JA)X)} \\ \times b(\sqrt{\lambda}(I_{2n}+JA)X+Y_2)dY_1dY_2,$$

$X \in \mathbb{R}^{2n}$. In order to prove that c is a symbol it suffices to show that there is $\lambda_0 \geq 1$ such that for all $\lambda \geq \lambda_0$ and all $1 \leq |X| \leq 2$ we have

$$|\partial_X^\alpha c_\lambda(X)| \leq C_\alpha \lambda^{(\mu_1+\mu_2)/2}, \tag{7.10}$$

where C_α is independent of λ (and X). In fact, if (7.10) holds, then for any given $X \in \mathbb{R}^{2n}$ with $|X| \geq \lambda_0$ we may find $\lambda \geq \lambda_0$ for which $\sqrt{\lambda} \leq |X| \leq 2\sqrt{\lambda}$ and hence may find a unique $\tilde{X}$ with $1 \leq |\tilde{X}| \leq 2$ such that $X = \sqrt{\lambda}\,\tilde{X}$. Therefore

$$|\partial_X^\alpha c(X)| = \lambda^{-|\alpha|/2}|\partial_{\tilde{X}} c_\lambda(\tilde{X})| \leq C'_\alpha \lambda^{(\mu_1+\mu_2-|\alpha|)/2},$$

for $\lambda \geq \lambda_0$ and with $C'_\alpha \geq C_\alpha$. Since $\frac{1}{2}|X| \leq \sqrt{\lambda} \leq |X|$, we hence have that

$$|\partial_X^\alpha c(X)| \leq C''_\alpha \langle X\rangle^{\mu_1+\mu_2-|\alpha|},$$

for $|X| \geq \lambda_0$ and with $C''_\alpha \geq C'_\alpha$, that is, $c \in S(m^{\mu_1+\mu_2}, g; \mathsf{M}_N)$.

Now, for $\lambda \geq \lambda_0$, let

$$f_\lambda\colon \mathbb{R}^{2n}_X \times \mathbb{R}^{4n}_Y \ni (X,Y) \longmapsto a(\sqrt{\lambda}(X+Y_1))b(\sqrt{\lambda}((I_{2n}+JA)X+Y_2)).$$

Note that, for any fixed constant C with $0 < C < C_{\min} := \min_{1\leq|X|\leq 2}|(I_{2n}+JA)X|$, one has

$$|\partial_X^\alpha f_\lambda(X,Y)| \leq C_\alpha \lambda^{(\mu_1+\mu_2)/2}, \tag{7.11}$$

uniformly in $1 \leq |X| \leq 2$ and $|Y| \leq C$, because a and b are symbols.

For $\mu \in \mathbb{R}$ define

$$\Phi_\mu(X,Y) := -\langle QY,Y\rangle + \mu\phi_1((I_{2n}+JA)X+Y_2) - \mu\phi_1((I_{2n}+JA)X),$$

and

$$c_{\lambda,\mu}(X) = \pi^{-4n}\lambda^{2n}\int e^{i\lambda\Phi_\mu(X,Y)} f_\lambda(X,Y)dY. \tag{7.12}$$

By using the homogeneity of the phase and the dilation $Y \mapsto Y/\sqrt{\lambda}$, we have that

$$c_\lambda = c_{\lambda,\lambda^{-1/2}}.$$

We next study $c_{\lambda,\mu}$ as $\lambda \to +\infty$ and μ in a neighborhood of zero ($1 \le |X| \le 2$ and $|Y| \le C$). Let $C_\mu = \{Y \neq 0;\ d_Y \Phi_\mu = 0\}$ denote the set of stationary points. Thus $Y \in C_\mu$ iff

$$\begin{cases} -AY_1 - JY_2 = 0, \\ JY_1 + \mu \nabla \phi_1((I_{2n} + JA)X + Y_2) = 0. \end{cases}$$

By the Implicit Function Theorem, we may parametrize Y by (μ, X) near any fixed X_0 with $1 \le |X_0| \le 2$ for $|\mu|$ sufficiently small. In fact, the Jacobian of $d_Y \Phi_\mu$with respect to Y is

$$\begin{bmatrix} -A & -J \\ J & \mu(\partial_{Y_2} \nabla \phi_1((I_{2n} + JA)X + Y_2)) \end{bmatrix},$$

which is invertible when $\mu = 0$. Hence, we obtain that $|Y(\mu, X)| \le C'|\mu|$. In particular, $|Y(\mu, X)| \le C < C_{\min}$ for $|\mu|$ sufficiently small, and $1 \le |X| \le 2$, whence the bounds (7.11) for f_λ. Moreover, note that $(I_{2n} + JA)X + Y_2 \neq 0$ by taking C small enough, since $|Y(\mu, X)| \le C$ if $1 \le |X| \le 2$ when μ is sufficiently small.

Next, without loss of generality we may assume that f_λ vanishes on the complement of $\{(X, Y);\ \ 1 \le |X| \le 2,\ |Y| \le C/2\}$. In fact, we wish to prove (7.10) for $1 \le |X| \le 2$ and Φ_μ is stationary only if $|Y| \le C/2$ (by taking $|\mu|$ even smaller), so that the contribution to the integral $c_{\lambda,\mu}(X)$ when $|Y| > C/2$ (and $1 \le |X| \le 2$) is $O(\lambda^{-\infty})$ (uniformly in μ in a neighborhood of zero) by a non-stationary phase argument.

We may now estimate integral (7.12) and its derivatives. Consider $\partial_X^\gamma c_{\lambda,\mu}$. It is a sum of terms, where those with $\ell \le |\gamma|$ derivatives landing on the exponential factor can be written as

$$\pi^{-4n} \lambda^{2n} (\lambda\mu)^\ell \int e^{i\lambda \Phi_\mu(X,Y)} \left(\partial_z^{\gamma'} f_\lambda(X, Y)\right) \sum_{|\beta|=\ell} Y_2^\beta h_\beta(X, Y, \mu) dY, \tag{7.13}$$

for some smooth functions h_β and $|\gamma'| = |\gamma| - \ell$. In fact, expanding ϕ_1 at $(I_{2n} + JA)X$, with $|Y| \le C/2$ and $1 \le |X| \le 2$, we have

$$\phi_1((I_{2n} + JA)X + Y_2) = \phi_1((I_{2n} + JA)X) + \langle Y_2, \nabla \phi_1((I_{2n} + JA)X)\rangle + \sum_{|\alpha|=2} Y_2^\alpha \psi_\alpha(X, Y_2),$$

for some smooth functions ψ_α. Hence, for any given $\mu \in \mathbb{R}$,

$$\begin{aligned} \Phi_\mu(X, Y) :=& - \langle QY, Y\rangle + \mu\phi_1((I_{2n} + JA)X + Y_2) - \mu\phi_1((I_{2n} + JA)X) \\ &= - \langle QY, Y\rangle + \mu(\langle Y_2, \nabla \phi_1((I_{2n} + JA)X)\rangle + \mu \sum_{|\alpha|=2} Y_2^\alpha \psi_\alpha(X, Y_2)). \end{aligned}$$

Now, by the stationary-phase method, recalling the bounds (7.11), we have that at the critical set C_μ each term $Y_2^\beta h_\beta(X, Y, \mu)$ in (7.13) gives an additional factor of order

$O(|\mu|^{\ell})$, since $Y(\mu, X) = O(|\mu|)$. The stationary-phase formula eliminates the prefactor λ^{2n}, and setting $\mu = \lambda^{-1/2}$ gives

$$|\partial_X^{\alpha} c_{\lambda}(X)| \leq C_{\alpha} \lambda^{(\mu_1+\mu_2)/2},$$

in a neighborhood of X_0 (for λ large). Since $\{X \in \mathbb{R}^{2n};\ 1 \leq |X| \leq 2\}$ is compact, this implies the symbol estimates (7.10), and the proof is complete. □

We are now ready to use the preceding results to obtain a parametrix of the unitary group of B^{w} (still in the case where the principal and semiprincipal parts are scalar) by composing the parametrix of the unitary group of the harmonic oscillator obtained by Hörmander with that of the reduced propagator.

Lemma 7.6. *Let $B = B^*$ be as in Lemma 7.3. Then, for all $k \in \mathbb{Z}$ and for $\varepsilon > 0$ sufficiently small, putting $I_{\varepsilon}(k) := (2k\pi - \varepsilon, 2k\pi + \varepsilon) \subset \mathbb{R}_t$, there are functions*

$$\phi_j \in C^{\infty}(I_{\varepsilon}(k) \times \dot{\mathbb{R}}^{2n}; \mathbb{R}),\ \ j = 1, 2,$$

homogeneous of degree j in $X \neq 0$ and

$$\alpha \in C^{\infty}(\mathbb{R}_t; S(1, g; \mathsf{M}_N)),$$

such that

$$U - \tilde{U} \in C^{\infty}(I_{\varepsilon}(k); \mathscr{L}(\mathscr{S}', \mathscr{S}) \otimes \mathsf{M}_N),$$

where U is the unitary group of B^{w} and (recalling Proposition 7.5) $\tilde{U} := (e^{i(\phi_2+\phi_1)}\alpha)^{\mathrm{w}} = U_0\tilde{F}$, with

$$U_0(t) := \cos(t/2)^{-n}(e^{i\phi_2(t)})^{\mathrm{w}}$$

the unitary group of the harmonic oscillator with $t \notin \pi + 2\pi\mathbb{Z}$ and $\phi_2(t) := -2\tan(t/2)p_2$, and $\tilde{F} = (e^{i\tilde{\phi}_1}\tilde{\alpha})^{\mathrm{w}}$ the reduced parametrix obtained in Lemma 7.3. In addition,

$$\phi_2\colon\ I_{\varepsilon}(k) \times \mathbb{R}^{2n}_X \ni (t, X) \longmapsto -2\tan(t/2)p_2(X),$$
$$\phi_1\colon\ I_{\varepsilon}(k) \times \mathbb{R}^{2n}_X \ni (t, X) \longmapsto \tilde{\phi}_1(t, (I_{2n} - 2\tan(t/2)J)(X)).$$

Proof. One has that $\tilde{U} := U_0\tilde{F}$ is a parametrix on $I_{\varepsilon}(k)$ because

$$(i\partial_t - B^{\mathrm{w}})U_0\tilde{F} = \underbrace{((i\partial_t - p_2^{\mathrm{w}})U_0)}_{\in C^{\infty}(I_{\varepsilon}(k);\mathscr{L}(\mathscr{S}',\mathscr{S}))}\tilde{F} + U_0(i\partial_t\tilde{F}) - (B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F},$$

and

$$U_0(i\partial_t\tilde{F}) - (B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F} = U_0(i\partial_t\tilde{F} - U_0^{-1}(B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F}).$$

With P the reduced propagator, as

$$P - U_0^{-1}(B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0 \in C^{\infty}(I_{\varepsilon}(k); \mathscr{L}(\mathscr{S}', \mathscr{S}) \otimes \mathsf{M}_N),$$

we have

$$i\partial_t\tilde{F} - U_0^{-1}(B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F} \in C^{\infty}(I_{\varepsilon}(k); \mathscr{L}(\mathscr{S}', \mathscr{S}) \otimes \mathsf{M}_N),$$

and $(U_0\tilde{F})|_{t=0} = I_N + R$, which shows that $\tilde{U}$ is a parametrix of $e^{-itB^{\mathrm{w}}}$ on $I_{\varepsilon}(k)$. By Proposition 7.5 we finally have $\phi_1(t) = \tilde{\phi}_1 \circ (I_{2n} - 2\tan(t/2)J)$. □

We next consider a general ψdo system A^{w} whose symbol belongs to the class SMGES (see Definition 2.4). As already anticipated, we determine an asymptotic expansion of $\mathsf{N} * \rho$ with a suitable $\rho \in \mathscr{S}(\mathbb{R})$, which leads immediately to the Weyl law (see (7.15) below). We exploit the construction of the parametrix in the blockwise diagonal case to obtain a parametrix of the Schrödinger group $e^{-itA^{\mathrm{w}}}$.

In the sequel it will be useful to have the following elementary lemma.

Lemma 7.7. *Let $p_1 \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathbb{R})$ be positively homogeneous of degree 1. Recall that p_2 is the standard harmonic oscillator. For $\lambda \geq 0$ consider the volume of the set $\{X \in \mathbb{R}^{2n};\ \ p_2(X) + p_1(X) \leq \lambda\}$. We have, as $\lambda \to +\infty$,*

$$\int_{p_2+p_1\leq\lambda} dX = \lambda^n\Big(\int_{p_2\leq 1} dX - \int_{p_2=1} p_1 \frac{ds}{|\nabla p_2|}\,\lambda^{-1/2} + \frac{n}{2}\int_{p_2=1} p_1^2 \frac{ds}{|\nabla p_2|}\,\lambda^{-1} + o(\lambda^{-1})\Big).$$

Proof. In the first place

$$\int_{p_2+p_1\leq\lambda} dX = \lambda^n \int_{p_2+\lambda^{-1/2}p_1\leq 1} dX.$$

Using polar coordinates $X = \rho\omega$, $\rho \geq 0$ and $\omega \in \mathbb{S}^{2n-1}$ write

$$\int_{p_2+\lambda^{-1/2}p_1\leq 1} dX = \int_{\mathbb{S}^{2n-1}}\Big(\int_0^{q_+(\omega,\lambda)} \rho^{2n-1} d\rho\Big) d\sigma(\omega),$$

where $d\sigma(\omega)$ is the Riemannian measure induced by $\mathbb{R}^{2n}$ on the unit sphere and

$$q_+(\omega,\lambda) = \frac{1}{2p_2(\omega)}\Big(\sqrt{4p_2(\omega) + \lambda^{-1}p_1(\omega)^2} - \lambda^{-1/2}p_1(\omega)\Big).$$

A first-order Taylor expansion of q_+^{2n} as $\lambda \to +\infty$ gives the result. □

We may proceed to the Weyl law.

Theorem 7.8. *[Weyl law]. Let $A = A^*$, with $A \sim \sum_{j\geq 0} a_{2-j} \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, be a second-order SMGES, with principal symbol $p_2 I_N$, p_2 being the harmonic oscillator. Adopting the notation used in Definition 2.4, we hence denote by $\lambda_{1,j}$ (with multiplicity N_j), $1 \leq j \leq r$, the eigenvalues of the semiprincipal part. Then, if $\rho \in \mathscr{S}(\mathbb{R})$ is chosen such that $\hat{\rho}$ has compact support in $(-\varepsilon, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ and $\hat{\rho} = 1$ on a neighborhood of 0*

$$(\mathsf{N} * \rho)(\lambda) = \left(\sum_{j=1}^{r}\left(\frac{N_j}{(2\pi)^n}\int_{p_2+\lambda_{1,j}\leq\lambda} dX\right) - (2\pi)^{-n}\int_{p_2=\lambda} Tr(a_0)\,\frac{ds}{|\nabla p_2|}\right) + O(\lambda^{n-3/2}), \tag{7.14}$$

as $\lambda \to +\infty$ (recall that Tr is the matrix trace).
Therefore

$$\mathsf{N}(\lambda) = \left(\frac{N}{(2\pi)^n}\int_{p_2\leq 1} dX\right)\lambda^n - \left((2\pi)^{-n}\int_{p_2=1} \mathit{Tr}(a_1)\,\frac{ds}{|\nabla p_2|}\right)\lambda^{n-1/2} + O(\lambda^{n-1}),\ \ \lambda \to +\infty \tag{7.15}$$

Proof. In the first place we obtain a parametrix $U_A(t)$ of the unitary group $t \mapsto e^{-itA^{\mathrm{w}}}$ of A^{w} by a parametrix of the unitary group of its diagonalization B^{w}. Then we study the distribution $\hat{\rho}\mathrm{Tr}(U_A)$ where $\mathrm{Tr}(U_A) = \mathrm{Tr}_\Delta \mathsf{Tr}(U_A)$ denotes the trace of the Schwartz kernel of U_A (where Tr_Δ denotes the restriction to the diagonal). Since $N' * \rho = \mathcal{F}^{-1}\{\hat{\rho}\mathrm{Tr}(U_A)\}$, modulo a rapidly decreasing term, we finally get the result.

- **The parametrix** U_A. Recall that the decoupling Theorem 5.3 of Sect. 5 diagonalizes A^{w} (modulo smoothing operators), so that the principal symbol b_2 of the blockwise diagonal operator B^{w} is p_2 while the semiprincipal symbol $b_1 = \mathrm{diag}(\lambda_{1,j} I_{N_j};\ 1 \le j \le r)$ is blockwise scalar. Hence, there is an operator S with Schwartz kernel $\mathsf{K}_S \in C^\infty(\mathbb{R}_t; \mathscr{S}(\mathbb{R}^{2n}_{x,y}))$ such that

$$e^{-itA^{\mathrm{w}}} = E^{\mathrm{w}} e^{-itB^{\mathrm{w}}} (E^{\mathrm{w}})^* + S(t), \quad \forall t \in \mathbb{R}$$

(see, for instance, Lemma 5.2 of [12]).
For notational simplicity, we suppose that the number r of blocks is 2 (the proof extends to the case $r \ge 3$ with no difficulties). Hence, consider the symbols in blockwise form

$$B =: \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right],$$

where B_j is an $N_j \times N_j$ block ($j = 1, 2$), and

$$E =: \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array}\right],$$

where E_{kj} is an $N_k \times N_j$ block ($j, k = 1, 2$).
Since for $j = 1, 2$ the semiprincipal term $\lambda_{1,j}$ of B_j is scalar we obtain a parametrix U_{B_j} of the unitary group of B_j^{w} by Lemma 7.6. Thus

$$U_A(t) := E^{\mathrm{w}} \left[\begin{array}{c|c} U_{B_1}(t) & 0 \\ \hline 0 & U_{B_2}(t) \end{array}\right] (E^{\mathrm{w}})^*$$

is a parametrix of the unitary group U_A, and its entries on the principal diagonal are given by

$$E_{11}^{\mathrm{w}} U_{B_1}(t)(E_{11}^{\mathrm{w}})^* + E_{12}^{\mathrm{w}} U_{B_2}(t)(E_{12}^{\mathrm{w}})^* \quad \text{and} \quad E_{21}^{\mathrm{w}} U_{B_1}(t)(E_{21}^{\mathrm{w}})^* + E_{22}^{\mathrm{w}} U_{B_2}(t)(E_{22}^{\mathrm{w}})^*.$$

- **Use of the parametrix.** Recall that

$$(\mathcal{F}_{\lambda\to t}\mathsf{N}')(t) = \mathrm{Tr}(e^{-itA^{\mathrm{w}}}),$$

where $\mathrm{Tr}(e^{-itA^{\mathrm{w}}})$ is well defined as a tempered distribution. Hence,

$$\mathcal{F}_{\lambda\to t}\{\mathsf{N}' * \rho\}(t) = \hat{\rho}(t)\mathrm{Tr}(e^{-itA^{\mathrm{w}}}),$$

and we may consider the distribution

$$K(t) = \hat{\rho}(t)\mathrm{Tr}(U_A)(t) = \hat{\rho}(t)\mathrm{Tr}\Big(E_{11}^{\mathrm{w}} U_{B_1}(t)(E_{11}^{\mathrm{w}})^* + E_{12}^{\mathrm{w}} U_{B_2}(t)(E_{12}^{\mathrm{w}})^* + E_{21}^{\mathrm{w}} U_{B_1}(t)(E_{21}^{\mathrm{w}})^* + E_{22}^{\mathrm{w}} U_{B_2}(t)(E_{22}^{\mathrm{w}})^*\Big),$$

for $t \in (-\varepsilon, \varepsilon)$. Next, for $j, k = 1, 2$ let

$$K_{kj}(t) := \hat{\rho}(t)\mathrm{Tr}\left(E_{kj}^{\mathrm{w}} U_{B_j}(t)(E_{kj}^{\mathrm{w}})^*\right) = \hat{\rho}(t)\mathrm{Tr}_{\Delta}\mathsf{Tr}\left(E_{kj}^{\mathrm{w}} U_{B_j}(t)(E_{kj}^{\mathrm{w}})^*\right).$$

Denote by $\tilde{\phi}_{1,j}, \tilde{\alpha}_j, \alpha_j$ and $\phi_{1,j}$, $j = 1, 2$, respectively, the $\tilde{\phi}_1, \tilde{\alpha}, \alpha$ and ϕ_1 constructed in Lemmas 7.3 and 7.6 when $B = B_j$. Now,

$$E_{kj}^{\mathrm{w}} U_{B_j}(t)(E_{kj}^{\mathrm{w}})^* := E_{kj}^{\mathrm{w}} U_0(t) F_j(t)(E_{kj}^{\mathrm{w}})^*,$$

where $F_j(t)$ is the parametrix of the reduced propagator $e^{itp_2^{\mathrm{w}}}(B_j^{\mathrm{w}} - p_2^{\mathrm{w}})e^{-itp_2^{\mathrm{w}}}$. Hence,

$$E_{kj}^{\mathrm{w}} U_{B_j}(t)(E_{kj}^{\mathrm{w}})^* = U_0(t)(E_{kj} \circ \exp(tH_{p_2}))^{\mathrm{w}}\big(E_{kj}^{\mathrm{w}} F_j(t)^*\big)^*,$$

where $F_j(t)^* = (e^{-i\tilde{\phi}_{1,j}(t)}\tilde{\alpha}_j(t)^*)^{\mathrm{w}}$. By Proposition 4.1 of [5] and Lemma 7.6, we have

$$K_{kj}(t) := (2\pi)^{-n}\hat{\rho}(t)\int e^{i(\phi_2(t,X)+\phi_{1,j}(t,X))} c_{kj}(t, X)\, dX$$

(which makes sense since $\hat{\rho}$ has support on the interval where U_{B_j} is well defined). Now, by construction of ϕ_2 and $\phi_{1,j}$, we have $\phi_2(0, X) + \phi_{1,j}(0, X) = 0$, which yields by a Taylor's expansion

$$\phi_2(t, \cdot) + \phi_{1,j}(t, \cdot) = t\psi_j(t, \cdot),$$

where ψ_j is given to leading order in t by

$$\psi_j(t, \cdot) = -(p_2 + \lambda_{1,j}) + \frac{t}{2}\left(-H_{p_2}\lambda_{1,j}\right) + t^2 r_j(t, \cdot).$$

Following Hörmander [10], Lemma 29.1.3, we define

$$Q_{kj}(t, \lambda) := (2\pi)^{-n}\int_{\{-\psi_j(t,\cdot)\le\lambda\}} c_{kj}(t, X)\hat{\rho}(t)\; dX.$$

For sufficiently small $|t|$, the function $\psi_j(t, \cdot)$ is elliptic in $S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, and by the above lemma by Hörmander, Q_{kj} is a Kohn-Nirenberg symbol in $S^n(\mathbb{R}_\lambda)$ for $|t|$ sufficiently small, and, furthermore,

$$K_{kj}(t) = \int_{\mathbb{R}} e^{-it\lambda}\partial_\lambda Q_{kj}(t, \lambda)\; d\lambda.$$

Thus $K_{kj}(t)$ is a conormal distribution, which can be written as the Fourier transform of a symbol independent of t (see [9], Lemma 18.2.1). Defining

$$\tilde{Q}_{kj}(\lambda) := e^{iD_t D_\lambda} Q_{kj}(0, \lambda) \tag{7.16}$$

and recalling the definition of $K_{kj}(t)$,

$$\mathcal{F}^{-1}_{t\to\lambda}\Big(\hat{\rho}(t)\mathrm{Tr}(E_{kj}^{\mathrm{w}} U_{B_j}(t)(E_{kj}^{\mathrm{w}})^*)\Big)(\lambda) = \partial_\lambda \tilde{Q}_{kj}(\lambda).$$

From (7.16) we have

$$\tilde{Q}_{kj}(\lambda) = Q_{kj}(0, \lambda) - i\partial_t\partial_\lambda Q_{kj}(0, \lambda) + R_{kj}(\lambda), \quad R_{kj} \in S^{n-2}(\mathbb{R}_\lambda). \tag{7.17}$$

- **The expansion (7.17).** For the first term in the above expansion we have

$$Q_{kj}(0,\lambda) = (2\pi)^{-n} \int_{\{p_2+\lambda_{1,j}\leq\lambda\}} c_{kj}(0,X)\, dX.$$

Now,

$$c_{kj}(t,X) = \mathrm{Tr}\left(e^{-i(\phi_2+\phi_{1,j})}(E_{kj}\#(e^{i(\phi_2+\phi_{1,j})}\alpha_j)\#E^*_{kj})\right)(t,X)$$

where $t \in (-\varepsilon,\varepsilon)$, whence

$$c_{kj}(0,X) = \mathrm{Tr}\left(E_{kj}\#E^*_{kj}\right)(X), \tag{7.18}$$

since $U_0(0) = I$ and $F_j(0) = I_{N_j}$ by construction.
As for the next term in the expansion (7.17), with $\langle\cdot|\cdot\rangle$ denoting the distributional duality in the X variables and recalling that

$$Q_{kj}(t,\lambda) = (2\pi)^{-n}\langle H(\psi_j(t,\cdot)+\lambda)|c_{jk}(t,\cdot)\hat{\rho}(t)\rangle,$$

we have

$$\begin{aligned}
-i(\partial_t Q_{kj})(0,\lambda) &= (2\pi)^{-n}\left\langle H(\psi_j+\lambda)|-i\partial_t c_{kj}\right\rangle\Big|_{t=0} - i(2\pi)^{-n}\left\langle \delta(\psi_j+\lambda)|c_{kj}\partial_t\psi_j\right\rangle\Big|_{t=0}\\
&= -(2\pi)^{-n}\left\langle H(\lambda-p_2)|i\partial_t c_{kj}\right\rangle\Big|_{t=0} + \tilde{r}_{kj}(\lambda)
\end{aligned} \tag{7.19}$$

where $\tilde{r}_{kj} \in S^{n-1/2}(\mathbb{R}_\lambda)$ (and H and δ are the Heaviside and Delta distributions). Therefore, we need to compute $\partial_t c_{kj}(0,\cdot)$. Put $h_0(t,\cdot)$ for the (Weyl) symbol of $U_0(t)$, and for $j = 1,2$ denote by $h_j(t,\cdot)$ and by $f_j(t,\cdot)$ those of $U_{B_j}(t)$ and of $F_j(t)$, respectively. Recall that $h_0(0,\cdot) = 1$ and that $h_j(0,\cdot) = I_{N_j}$, $j = 1,2$. We then have

$$\partial_t c_{kj}(0,\cdot) = \partial_t \mathrm{Tr}\left(e^{-i(\phi_2(t)+\phi_{1,j}(t))}E_{kj}\#h_j\#E^*_{kj}\right)\Big|_{t=0}$$

$$\begin{aligned}
=&\mathrm{Tr}\left((\partial_t e^{-i(\phi_2(t)+\phi_{1,j}(t))})E_{kj}\#h_j\#E^*_{kj}\right)\Big|_{t=0} + \mathrm{Tr}\left(e^{-i(\phi_2(t)+\phi_{1,j}(t))}E_{kj}\#\partial_t h_0\#f_j\#E^*_{kj}\right)\Big|_{t=0}\\
&+\mathrm{Tr}\left(e^{-i(\phi_2(t)+\phi_{1,j}(t))}E_{kj}\#h_0\#\partial_t f_j\#E^*_{kj}\right)\Big|_{t=0}\\
=&\mathrm{Tr}\left(ip_2E_{kj}\#E^*_{kj}\right) + \mathrm{Tr}\left(i\lambda_{1,j}E_{kj}\#E^*_{kj}\right) + \mathrm{Tr}\left(-iE_{kj}\#p_2\#E^*_{kj}\right)\\
&+\mathrm{Tr}\left(-iE_{kj}\#\lambda_{1,j}\#E^*_{kj}\right) + \mathrm{Tr}\left(-iE_{kj}\#b_{0,j}\#E^*_{kj}\right).
\end{aligned}$$

Recalling that $b_{0,j}$ is the subprincipal term of B^{w}_j and denoting by $e_{0,kj}$ the principal symbol of E^{w}_{kj}, we therefore have (note that at $t = 0$ all the symbols involved are nice isotropic symbols)

$$\partial_t c_{kj}(0,\cdot) = -\frac{1}{2}\mathrm{Tr}\Big(e_{0,kj}\{p_2,e^*_{0,kj}\} + \{e_{0,kj},p_2\}e^*_{0,kj}\Big) - i\,\mathrm{Tr}(e_{0,kj}b_{0,j}e^*_{0,kj}) + s_{kj}, \tag{7.20}$$

where $s_{kj} \in S(m^{-1},g)$. By taking ∂_λ of $(\partial_t Q_{kj})(0,\cdot)$ in (7.19) we hence have

$$-i(\partial_\lambda\partial_t Q_{kj})(0,\lambda) = -i(2\pi)^{-n}\left\langle\delta(\lambda-p_2)|\partial_t c_{kj}\right\rangle\Big|_{t=0} + O(\lambda^{n-3/2}),\ \ \lambda\to+\infty. \tag{7.21}$$

- **The asymptotics of $\mathsf{N}' * \rho$.** To obtain the result we have to integrate the following equation, which holds for any given real exponent $\gamma > 0$ (see [6], Lemma IV.1)

$$
\begin{aligned}
(\mathsf{N}' * \rho)(\lambda) &= \mathcal{F}^{-1}_{t\to\lambda}\Big(\hat{\rho}\mathrm{Tr}(e^{-itA^{\mathrm{w}}})\Big)(\lambda)\\
&= \mathcal{F}^{-1}_{t\to\lambda}\Big(\hat{\rho}\mathrm{Tr}\left(E^{\mathrm{w}}_{11}U_{B_1}(E^{\mathrm{w}}_{11})^* + E^{\mathrm{w}}_{12}U_{B_2}(E^{\mathrm{w}}_{12})^*\right.\\
&\qquad \left. + E^{\mathrm{w}}_{21}U_{B_1}(E^{\mathrm{w}}_{21})^* + E^{\mathrm{w}}_{22}U_{B_2}(E^{\mathrm{w}}_{22})^*\right)\Big)(\lambda)\\
&\qquad + O(\lambda^{-\gamma}) = \sum_{k,j=1}^{2} \partial_\lambda \tilde{Q}_{kj}(\lambda) + O(\lambda^{-\gamma}).
\end{aligned}
$$

Hence, to obtain (7.14) we need to compute $\sum_{k,j=1}^{2}\partial_\lambda \tilde{Q}_{kj}(\lambda)$. In the first place we note that by (7.18) one has

$$
c_{1j}(0,\cdot) + c_{2j}(0,\cdot) = \mathsf{Tr}\left(E_{1j}\# E^*_{1j} + E_{2j}\# E^*_{2j}\right), \quad j = 1, 2.
$$

Hence, with $e_{-1,kj}$ denoting the semiprincipal symbol of E^{w}_{kj}, for $\tilde{r}$ a suitable symbol in $S_{\mathrm{sreg}}(m^{-2}, g)$, we have

$$
\begin{aligned}
(c_{1j} + c_{2j})(0,\cdot) &= \sum_{k=1}^{2} \mathrm{Tr}\left(e_{0,kj}e^*_{0,kj} + e_{-1,kj}e^*_{0,kj} + e_{0,kj}e^*_{-1,kj}\right) + \tilde{r}\\
&= \sum_{k=1}^{2} \mathrm{Tr}\left(e^*_{0,kj}e_{0,kj} + e^*_{0,kj}e_{-1,kj} + e^*_{-1,kj}e_{kj,0}\right) + \tilde{r}\\
&= \mathrm{Tr}\left(I_{N_j}\right) + \tilde{r} = N_j + \tilde{r},
\end{aligned}
\tag{7.22}
$$

where the third equality follows from the symbolic identity $E^*\# E = I_N$. Hence, by (7.21) we get

$$
\sum_{k=1}^{2} -i(\partial_\lambda\partial_t Q_{kj})(0,\lambda) = -i(2\pi)^{-n}\Big\langle\delta(\lambda - p_2)\Big|\sum_{k=1}^{2}\partial_t c_{kj}\Big\rangle\Big|_{t=0} + O(\lambda^{n-3/2}),
\tag{7.23}
$$

as $\lambda \to +\infty$. By (7.22) and (7.23), we have

$$
\begin{aligned}
\sum_{j=1}^{2}\sum_{k=1}^{2}\tilde{Q}_{kj}(\lambda) &= \sum_{j=1}^{2}\sum_{k=1}^{2}\partial_\lambda Q_{kj}(0,\lambda) - i\partial_t\partial_\lambda Q_{kj}(0,\lambda) + R_{kj}(\lambda)\\
&= (2\pi)^{-n}\sum_{j=1}^{2} N_j\int_{\{p_2+\lambda_{1,j}\le\lambda\}} dX - i(2\pi)^{-n}\Big\langle\delta(\lambda - p_2)\Big|\sum_{k,j=1}^{2}\partial_t c_{kj}\Big\rangle\Big|_{t=0}\\
&\quad + O(\lambda^{n-3/2}),
\end{aligned}
\tag{7.24}
$$

as $\lambda \to +\infty$, and by (7.20)

$$-i\sum_{k,j=1}^{2}\partial_t c_{kj}(0) = -\,\mathsf{Tr}\Big(e_0 b_0 e_0^* - \frac{i}{2}(e_0\{p_2, e_0^*\} + \{e_0, p_2\}e_0^*)\Big) + \sum_{k,j=1}^{2} s_{kj} \quad (7.25)$$

$$= -\,\mathsf{Tr}\Big(b_0 + \frac{i}{2}(e_0^*\{p_2, e_0\} + \{e_0^*, p_2\}e_0)\Big) + \sum_{k,j=1}^{2} s_{kj}$$

$$= -\,\mathsf{Tr}(a_0) + \sum_{k,j=1}^{2} s_{kj},$$

where the third equality follows from Corollary 6.5.

Hence (7.14) is obtained by substituting (7.25) into (7.24) and by recalling that $\delta(\lambda - p_2) = ds/|\nabla p_2|\big|_{p_2=\lambda}$.

From (7.14) one immediately gets (using the polynomial growth of N, as is well-known) the asymptotics (7.15) by using Lemma 7.7 and grouping the $O(\lambda^{n-1})$ terms.

□

We finally prove the refined asymptotics of $\mathsf{N}(\lambda)$ for a positive ψdo system A^{w} satisfying the hypotheses of Theorem 7.8 and the **Condition DGW** (7.1).

Theorem 7.9 *[Refined Weyl law]. Let $A = A^* \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$ be a second-order SMGES satisfying the hypotheses of Theorem 7.8. If* ***Condition DGW*** *(7.1) is satisfied, then*

$$\mathsf{N}(\lambda) = (2\pi)^{-n}\left(\sum_{j=1}^{r}\left(N_j\int_{p_2+\lambda_{1,j}\le\lambda} dX\right.\right.$$
$$\left.\left. - \int_{p_2=\lambda} Tr\,(a_0)\,\frac{ds}{|\nabla p_2|}\right)\right) + o(\lambda^{n-1}),\ \ \lambda \to +\infty. \quad (7.26)$$

In particular, as $\lambda \to +\infty$

$$\mathsf{N}(\lambda) = (2\pi)^{-n}\Big(N\lambda^n\int_{p_2\le 1} dX - \lambda^{n-1/2}\int_{p_2=1}\mathsf{Tr}(a_1)\frac{ds}{|\nabla p_2|}$$
$$+\lambda^{n-1}\int_{p_2=1}\Big(\frac{n}{2}\mathsf{Tr}(a_1^2) - \mathsf{Tr}(a_0)\Big)\frac{ds}{|\nabla p_2|}\Big) + o(\lambda^{n-1}). \quad (7.27)$$

Proof. Fix an even and positive cutoff function $\rho \in \mathscr{S}(\mathbb{R}^{2n})$ in the time variable such that $\hat\rho = 1$ on $(-\varepsilon, \varepsilon)$ for some $\varepsilon \in (0, \pi/2)$ and $\operatorname{supp}\hat\rho \subset (-\pi/2, \pi/2)$.

We have to show that, under our assumptions, $(\mathsf{N} * \rho)(\lambda) = \mathsf{N}(\lambda)$ modulo an error which is $o(\lambda^{n-1})$, so that the result follows from the asymptotics (7.14) by using the following Tauberian theorem (see [20], Theorem B.5.1) which allows the required comparison between N and $\mathsf{N} * \rho$.

Lemma 7.10. *Let ρ be fixed as above. If there is a real number γ such that $(\mathsf{N}' * \rho)(\lambda) = O(\lambda^\gamma)$ and $(\mathsf{N}' * \chi)(\lambda) = o(\lambda^\gamma)$ for all χ satisfying $\hat\chi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\hat\chi \subset (0, +\infty)$, then $\mathsf{N}(\lambda) = (\mathsf{N} * \rho)(\lambda) + o(\lambda^\gamma)$ as $\lambda \to +\infty$.*

We have therefore to prove that

$$\mathcal{F}^{-1}_{t\to\lambda}\big(\chi(t)\mathrm{Tr}\, e^{-itA^{\mathrm{w}}}\big)(\lambda) = o(\lambda^{n-1}), \tag{7.28}$$

for any given $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (0,\infty)$ (here χ is playing the role of $\hat{\chi}$ in Lemma 7.10). Since Theorem 7.8 in particular shows that

$$(\mathsf{N}' * \rho)(\lambda) = O(\lambda^{n-1}),$$

it follows that if we have (7.28) then the hypotheses of Lemma 7.10 are fulfilled. Now, by Proposition 1.1 and Sect. 3 in [4] we have that sing supp Tr $U(t) \subset 2\pi\mathbb{Z}$, whence we need to check (7.28) only for $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (2\pi k - \varepsilon, 2\pi k + \varepsilon)$ where $k \in \mathbb{Z}\setminus\{0\}$ and $\varepsilon \in (0, \pi/2)$. Now, for all real γ (again, we suppose without loss of generality $r = 2$) we have

$$\begin{aligned}\mathcal{F}^{-1}_{t\to\lambda}\big(\chi(t)\mathrm{Tr}\, e^{-itA^{\mathrm{w}}}\big)(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\Big(\chi \mathrm{Tr}\,\big(&E^{\mathrm{w}}_{11}U_{B_1}(E^{\mathrm{w}}_{11})^* + E^{\mathrm{w}}_{12}U_{B_2}(E^{\mathrm{w}}_{12})^* \\ &+E^{\mathrm{w}}_{21}U_{B_1}(E^{\mathrm{w}}_{21})^* + E^{\mathrm{w}}_{22}B_2(E^{\mathrm{w}}_{22})^*\big)\Big)(\lambda) + O(\lambda^{-\gamma})\end{aligned}$$

(see Lemma 4.7 in [4]) and for all j, k

$$\mathcal{F}^{-1}_{t\to\lambda}\left(\chi \mathrm{Tr}\left(E^{\mathrm{w}}_{kj}U_{B_j}(E^{\mathrm{w}}_{kj})^*\right)\right)(\lambda) = \int e^{it\lambda}e^{i(\phi_2(t,X)+\phi_{1,j}(t,X))}\chi(t)c(t,X)\,dt\,dX,$$

where c is a suitable amplitude and ϕ_2, $\phi_{1,j}$ are given as in the proof of Theorem 7.8. Hence, we are in a position to use Proposition 5.1 of [4], with $\psi_2 := \phi_2$ and $\psi_1 := \phi_{1,j}$. Since $\phi_2 := -2\tan(t/2)p_2$ and χ is supported close to $2\pi k$, the hypotheses of that proposition for the phases ψ_2, ψ_1 and amplitude c are satisfied (in the notation of that proposition, we take $t_0 = 2\pi k$ and $r_0 = \sqrt{2}$).

Now, since $\phi_{1,j}(2k\pi, X) = -k\int_0^{2\pi}(\lambda_{1,j}\circ \exp tH_{p_2})(X)dt$, **Condition DGW** (7.1) yields that the set of the $\omega \in \mathbb{S}^{2n-1}$ at which $\partial^\alpha_\omega \phi_{1,j}(2k\pi,\omega)$, $|\alpha| = 1$, vanish to infinite order ($j = 1,\dots,r$) has measure zero for all $k \in \mathbb{Z}\setminus\{0\}$. Thus, Proposition 5.1 in [4] shows that

$$\mathcal{F}^{-1}_{t\to\lambda}\left(\chi \mathrm{Tr}\left(E^{\mathrm{w}}_{kj}U_{B_j}(E^{\mathrm{w}}_{kj})^*\right)\right) = o(\lambda^{n-1}),$$

for all $j, k = 1,\dots, r$.

The final formula (7.27) is obtained by Taylor-expanding the volume term in (7.26) using once more Lemma 7.7.

The proof is complete. □

8. Some Examples

8.1. Weyl-asymptotics for the JC-model 3.1 ($n = 1$, $N = 2$). In this case we have that $\lambda_\pm(X) = \pm|\alpha||\psi(X)|$, and $a_0(X) = \gamma\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, where $\alpha \neq 0$ and γ are real numbers. Since the eigenvalues of a_1 are constant on the level sets of $p_2(X) = |X|^2/2$, **Condition DGW** does not hold and we may only have the classical Weyl-law:

$$\mathsf{N}(\lambda) = (2\pi)^{-1}2\lambda\int_{p_2\le 1} dX + O(1) = 2\lambda + O(1),$$

as $\lambda \to +\infty$.

In fact, that is the same for all the JC models in the various configurations along with the geometric realization $\Box_1^{(3)}$ (see Sects. 3 and 4), because of the presence of the 0 eigenvalue.

Because of that limitation in case of the JC models, it is interesting to give an example of a deformation of a JC-model which indeed satisfies the hypotheses for a refined Weyl law. This is done in the next subsection.

8.2. Refined Weyl-asymptotics for bigger size systems. Recall that in the JC-models with $n = N - 1$ atom levels (and their geometric generalizations) we have that an eigenvalue of the semiprincipal term is 0 with a fixed multiplicity. In this case $\Pi_{2\pi} = \mathbb{S}^{2n-1}$ for the 0-eigenvalue and we cannot conclude a refined Weyl-law. However, let us consider the following deformation of the JC-model in the Ξ-configuration 3.2. Let $n = 2$ and $N = 3$. Recall that $\psi_j(X) = (x_j + i\xi_j)/\sqrt{2}$ is the symbol of the annihilation operator in the x_j variable, $j = 1, 2$. For $\alpha_1, \alpha_2 \neq 0$ real, we put $\alpha\psi := (\alpha_1\psi_1, \alpha_2\psi_2)$ and consider the functions $f_j(X) = \alpha_j\psi_j(X)/|\alpha\psi(X)|$, $X \neq 0$, which are homogeneous of degree 0, $j = 1, 2$. Let

$$\lambda_+,\ \lambda_-,\ \mu \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$$

be homogeneous of degree 1, such that

$$\lambda_-(X) < \lambda_+(X),\ \lambda_+(X) - \lambda_-(X) \approx |X|,\ |\lambda_\pm(X) - \mu(X)| \approx |X|,\ X \neq 0,$$

with

$$\textit{either } \mu(X) \in (\lambda_-(X), \lambda_+(X)),\ \textit{or } \ \mu(X) \notin (\lambda_-(X), \lambda_+(X)),\ \forall X \neq 0.$$

We consider then

$$A_{1,\mu} = \begin{bmatrix} \mu|f_2|^2 + \frac{\lambda_+ + \lambda_-}{2}|f_1|^2 & \frac{\lambda_+ - \lambda_-}{2}\bar{f}_1 & (-\mu + \frac{\lambda_+ + \lambda_-}{2})\bar{f}_1\bar{f}_2 \\ \frac{\lambda_+ - \lambda_-}{2} f_1 & \frac{\lambda_+ + \lambda_-}{2} & \frac{\lambda_+ - \lambda_-}{2}\bar{f}_2 \\ (-\mu + \frac{\lambda_+ + \lambda_-}{2}) f_1 f_2 & \frac{\lambda_+ - \lambda_-}{2} f_2 & \mu|f_1|^2 + \frac{\lambda_+ + \lambda_-}{2}|f_2|^2 \end{bmatrix} = e_0 \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda_+ & 0 \\ 0 & 0 & \lambda_- \end{bmatrix} e_0^*,$$

where

$$e_0(X) = \begin{bmatrix} -\overline{f_2(X)} & \overline{f_1(X)}/\sqrt{2} & \overline{f_1(X)}/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ f_1(X) & f_2(X)/\sqrt{2} & f_2(X)/\sqrt{2} \end{bmatrix},$$

is smooth and unitary for $X \in \mathbb{R}^4$, $X \neq 0$, and homogeneous of degree 0. Then, if we require that the sets

$$\{\omega \in \mathbb{S}^3;\ \partial_\omega^\alpha \mu(\omega) = 0,\ \forall \alpha \in \mathbb{N}^{2n-1} \setminus \{0\}\},\ \{\omega \in \mathbb{S}^3;\ \partial_\omega^\alpha \lambda_\pm(\omega) = 0,\ \forall \alpha \in \mathbb{N}^{2n-1} \setminus \{0\}\}$$

have measure zero, we have a refined Weyl-law

$$\mathsf{N}(\lambda) = (2\pi)^{-2}\Big(3|\mathbb{S}^3|\lambda^2 - \lambda^{3/2}\int_{p_2=1}(\lambda_+ + \lambda_- + \mu)\frac{ds}{|\nabla p_2|}$$

$$+\lambda \int_{p_2=1} \Big(\lambda_+^2 + \lambda_-^2 + \mu^2 - (\gamma_1 + \gamma_2)\Big) \frac{ds}{|\nabla p_2|} + o(1)\Big), \ \lambda \to +\infty.$$

In particular, in the case of Sect. 3.2 we have $\lambda_\pm = \pm|\alpha\psi|$, and a computation shows that in coordinates $\omega = (\sin\theta_3 \sin\theta_2 \cos\theta_1, \sin\theta_3 \cos\theta_2, \sin\theta_3 \sin\theta_2 \sin\theta_1, \cos\theta_3)$, with $\theta_1 \in [0, 2\pi]$ and $\theta_2, \theta_3 \in [0, \pi]$,

$$\lambda_+(\omega)^2 = \frac{1}{2}\Big(\alpha_2^2 + \sin^2\theta_3 \sin^2\theta_2 (\alpha_1^2 - \alpha_2^2)\Big).$$

Therefore when $\alpha_1^2 \neq \alpha_2^2$, the sets $\Pi_{2\pi,\pm}$ have measure zero. Hence, considering

$$\mu(X) = \kappa\lambda_+(X) + (1-\kappa)\lambda_-(X), \text{ for some } \kappa \in (0, 1),$$

yields that for $\alpha_1^2 \neq \alpha_2^2$ and $\kappa \neq 1/2$, the system with the semiprincipal part

$$A_{1,\mu} = \begin{bmatrix} \mu|f_2|^2 & \frac{\lambda_+-\lambda_-}{2}\bar{f}_1 & -\mu\bar{f}_1\bar{f}_2 \\ \frac{\lambda_+-\lambda_-}{2} f_1 & 0 & \frac{\lambda_+-\lambda_-}{2}\bar{f}_2 \\ -\mu f_1 f_2 & \frac{\lambda_+-\lambda_-}{2} f_2 & \mu|f_1|^2 \end{bmatrix}$$

satisfies the hypotheses of the refined Weyl-law and we therefore have

$$\mathsf{N}(\lambda) = (2\pi)^{-2}\Big(3|\mathbb{S}^3|\lambda^2 - \lambda^{3/2}\int_{p_2=1} \mu \frac{ds}{|\nabla p_2|} + \\ +\lambda \int_{p_2=1} \Big(\lambda_+^2 + \lambda_-^2 + \mu^2 - (\gamma_1 + \gamma_2)\Big) \frac{ds}{|\nabla p_2|} + o(1)\Big), \ \lambda \to +\infty.$$

By tensorizing the symbols with I_2, one readily obtains in the same hypotheses on μ the refined Weyl-law for the 6×6 Laplacian $\square_1^{(3)}$ (see Sect. 4.1) with semiprincipal term $A_{1,\mu} \otimes I_2$.

Acknowledgement We wish to thank the referees for the crucial observations and the useful suggestions.

Funding Open access funding provided by Alma Mater Studiorum - Università di Bologna within the CRUI-CARE Agreement.

Data Availability All data is provided in full in this paper.

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Communicated by E. Smit