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ABSTRACT

We prove phase-space mixing for solutions to Liouville's equation for integrable systems. Under a natural non-harmonicity condition, we obtain weak convergence of the distribution function with rate $\langle \text{time} \rangle^{-1}$. In one dimension, we also study the case where this condition fails at a certain energy, showing that mixing still holds but with a slower rate. When the condition holds and functions have higher regularity, the rate can be faster.

I. INTRODUCTION

We study Liouville's equation in action-angle coordinates,

$$\partial_t f(t,q,k) + \omega(k) \cdot \nabla_q f(t,q,k) = 0, \tag{1}$$

where $q \in \mathbb{T}^d$, the d-dimensional (flat) torus, are the *angles* and $(k_1, \dots, k_d) \in K$ are the conserved quantities in a suitable open set K. The function $\omega : K \mapsto \mathbb{R}^d$ gives the frequencies associated with each angle. We think of Liouville's equation as describing the evolution of a large number of gas molecules or collisionless kinetic gas without interactions. Liouville's theorem guarantees that, if the motion of a single particle in this system is integrable, there exist coordinates (q,k) that bring the equation in the form (1).

In classical mechanics, this occurs, for instance, in a potential well in one space-dimension or for spherically symmetric potentials in dimensions 2 and 3. In these cases, in *physical* coordinates, Liouville's equation reads

$$\partial_t F(t,x,p) = -p \cdot \nabla_x F + \nabla_x V \cdot \nabla_p F, \tag{2}$$

which can be transformed (for an open set $K \subset \mathbb{R}^d$ of values of the conserved energy and angular momentum) into the form (1).

The applications are not restricted to classical systems. In a relativistic context, geodesic motion in the Kerr family of space-times is integrable, and Liouville's equation (or the collisionless Boltzmann equation) can be written in the form (1).

If the system is anharmonic, in the sense that points with nearby energies move at different angular speeds $\omega(k)$, regular initial distributions will eventually stretch out to thin filaments that cover the region of phase space allowed by the conservation laws, as illustrated in Fig. 1.

This phenomenon is called phase-space mixing. It leads to weak convergence in the sense that, for any measurement of a macroscopic quantity encoded in a *test function* $\phi(q,k)$, its value satisfies

$$\lim_{t\to\infty} \int_K \int_{\mathbb{T}^d} f(t,q,k)\phi(q,k)dqdk = \int_K \bar{f}_0(k) \int_{\mathbb{T}^d} \phi(q,k)dqdk.$$

Here and throughout this paper, we use a bar to denote the average over the periodic variables, i.e.,

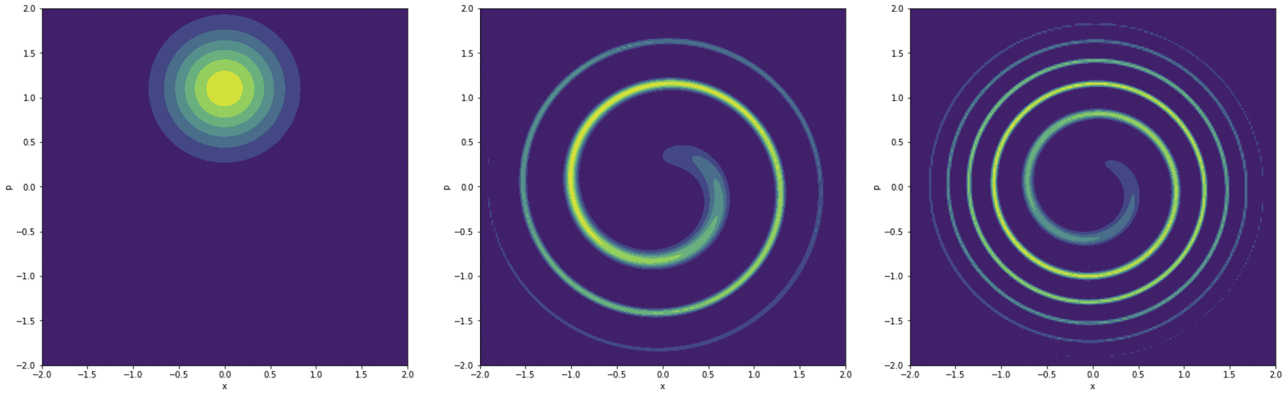

FIG. 1. Snapshots at times $t = 0$, 20π, and 40π of the evolution of a Gaussian initial condition in a perturbed harmonic oscillator with Hamiltonian $H = p^2/2 + x^2/2 + 0.3x^4$, approximate at first order in perturbation theory.

$$\bar{f}_0(k) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f_0(q,k)dq.$$

Also, we use this term throughout the this paper, we will use the term "test function" loosely to designate "a function against which the distribution function is tested," and these functions need not belong to C_c^∞. Mathematically, the roles of ϕ and f are symmetric by the time-reversal invariance of the evolution, while conceptually f is an empirical density associated with a large number of particles, and only its averages over regions of phase space (i.e., the support of ϕ) have a physical significance.

The relevance of phase-space mixing in clusters of stars had been pointed out historically by Lynden–Bell[1,2] and was highlighted more recently in Mouhot and Villani's proof of Landau damping in the Vlasov–Poisson system on the torus.[3] This sparked interest in proving mixing in linear models that describe astrophysical systems[4] or are reasonable *toy models* for these systems.[5] A recent paper[6] studied the one-dimensional Liouville equation with a slightly anharmonic potential $V(x) = \frac{x^2}{2} + \varepsilon\frac{x^4}{2}$ and proved the time convergence of the (one-dimensional) Coulomb potential generated by this distribution. The authors of Ref. 6 used the so-called vector-field method and obtained a rate of convergence. The vector-field method has been introduced by Klainerman[7] in the context of wave equations and has been applied to transport equations or the Vlasov–Poisson system, for instance, in Refs. 8–10.

In this paper, we apply the vector-field method to general integrable systems and any choice of test function ϕ. We obtain power-like convergence to the equilibrium value $\int \phi \bar{f}_0$. Since we study essentially a transport equation in $\mathbb{T}^d$ rather than $\mathbb{R}^d$, the rate of decay does not improve with the dimension. We will assume throughout that $\omega : K \mapsto \mathbb{R}^d$ is of class C^2 and use the notation $D\omega$ for its Jacobian matrix, i.e.,

$$(D\omega)_{j,l}(k) = \partial_j \omega_l(k).$$

With these preliminaries in place, we can state our main theorem.

Theorem I.1. *Let $f(t,q,k)$ be the solution to (1) with initial datum $f_0 \in C^1(\mathbb{T}^d \times K)$. Assume that $\phi \in C_c^1(\mathbb{T}^d \times K)$ is bounded and that*

$$\omega \in C^2(K) \quad \text{and} \quad \det D\omega(k) \neq 0 \quad \text{for all} \quad k \in K. \tag{3}$$

Then there exists C depending on ω, f_0, and ϕ such that

$$\left| \int_K \int_{\mathbb{T}^d} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk \right| \leq \frac{C}{1+|t|}.$$

Remark 1. The constant C depends on the initial data, on the test function ϕ, and on the inverse of $D\omega$. In Propositions II.1 and III.1, we give more precise statements that allow us to relax the hypotheses and estimate the constant for concrete cases.

Remark 2. For the case of a particle in a one-dimensional potential well, $2\pi/\omega(k)$ is the inverse of the period T of the trajectories, which, in turn, is the derivative of the area function Π [see, e.g., Ref. 11 (Sec. 50)]. In terms of the potential V, we have

$$T(h) = \Pi'(h), \qquad \Pi(h) := \oint_{\frac{p^2}{2}+V(x)=h} p dx.$$

Apart from regularity issues, condition (3) is simply $T'(h) = \Pi''(h) \neq 0$. For one-dimensional systems (2-dimensional phase space), there is an extensive literature on the monotonicity properties of the period function $T(h)$. For many potentials, $T'(h)$ has a definite sign [see, e.g., Refs. 12–15]. In Appendixes B and C in Ref. 5, these conditions are specified to several potentials relevant in astrophysics.

Even if the condition $\det D\omega \neq 0$ fails at some points, mixing may still hold. For simplicity, we state this result in the one-dimensional case and for a linearly vanishing ω'.

Theorem I.2. *Fix f_0 and ϕ of class C^1, with compact support, and let f denote the corresponding solution to Liouville's equation. Assume that $\omega \in C^2(K)$ and $\omega'(k) \neq 0$ except for k in the finite set $\{k_1, \dots, k_N\}$ and that $\omega''(k_i) \neq 0$. Then, there is $C > 0$ such that*

$$\left| \int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}(k))\phi(q,k) dq dk \right| \leq \frac{C}{1 + |t|^{1/3}}.$$

In a different direction, if condition (3) holds and the functions involved have a better regularity, the strategy in the proof of Theorem I.1 can be iterated to obtain a better rate of decay. Again, we state the one-dimensional result for simplicity.

Theorem I.3. *For $d = 1$ and under the hypotheses of Theorem* I.1*, assume that additionally, $\omega'(k)^{-1} \in C^l(K)$ and $f_0, \phi \in C^l(\mathbb{T} \times K)$ for some $l \geq 2$. Then, there exists $C > 0$ depending on ω, f_0, and ϕ such that*

$$\left| \int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)\, \mathrm{d}q\mathrm{d}k \right| \leq \frac{C}{1 + |t|^l}.$$

A striking consequence is that mixing is actually super-polynomial when ω, f_0, and ϕ are of class C^∞.

Finally, we study the Coulomb potential generated by a particle density F. We will use the notation F for the density in the physical phase space $\mathbb{R}^d \times \mathbb{R}^d$ and $f = F \circ N$ for the density in action-angle coordinates. The motivation to consider the Coulomb potential, in particular, is to take into account the gravitational self-interaction (the Vlasov–Poisson system). We generalize the main result of Ref. 6 to any dimension d and any integrable system. The $\langle t \rangle^{-1}$-decay that we prove remains insufficient to treat the nonlinear equation. This is natural since we do not expect, in general, that $\bar{f}_0$ is a stationary state for the Vlasov–Poisson system.

The Coulomb potential can be written as the integral of F against a *test function* with a singularity, which can be compensated by requiring some extra regularity of F. For a given F defined in Euclidean space, we define the Coulomb potential generated by its particle density as the unique solution to

$$-\Delta V_F(x) = \int_{\mathbb{R}^d} F(x,p)\, \mathrm{d}p \text{ and } \begin{cases} V_F(0) = 0 & \text{if } d = 1, \\ \lim_{|x| \to \infty} V_F(x) = 0 & \text{if } d \geq 2. \end{cases} \tag{4}$$

We will assume that the system with Hamiltonian $H(x,p) = |p|^2/2 + V(x)$ is integrable and denotes $N : \mathbb{T}^d \times K \mapsto G \subset \mathbb{R}^d \times \mathbb{R}^d$ for the transformation from action-angle variables to the position and momentum, where G is the open set of values of position and momenta for which this transformation is well defined and invertible. Then, we have the following corollary.

Corollary I.4. Assume that N is a C^1-diffeomorphism and that the frequencies $\omega(k)$ satisfy (3). Let $F_0 \in C^1_c(G) \cap L^1(G)$. Denote by F the solution to Liouville's equation (2), and then

$$\left\| V_F - V_{\widetilde{F}_0} \right\|_{L^\infty} \leq \frac{C}{1 + |t|},$$

where

$$\widetilde{F}_0 := \overline{(F_0 \circ N)} \circ N^{-1}.$$

The remainder of this paper is organized as follows. In Sec. II, we prove the one-dimensional case of Theorems I.1–I.3. In Sec. III, we prove the general case of Theorem I.1 and its Corollary I.4.

II. ONE-DIMENSIONAL CASE

The main tool in the proof of this theorem is the *vector field*,

$$W := \omega'(k) t \partial_q + \partial_k. \tag{5}$$

A straightforward calculation shows that W commutes with the Liouville operator,

$$L := \partial_t + \omega(k)\partial_q.$$

Therefore, if f solves the transport equation (1), the same goes for $W^n f$ (and $|W^n f|$) for any $n = 0, 1, 2, \ldots$, and thus,

$$\iint |W^n f|(t,q,k)g(k)dkdq = \iint |W^n f_0|(q,k)g(k)dkdq \tag{6}$$

for sufficiently regular functions f and g. As usual, we will use this property to obtain time-independent bounds.

Proposition II.1. Let f denote the solution to (1) with initial data $f_0 \in L^1$ and fix $\phi \in L^\infty$. Assume that either f or ϕ has compact support in $\mathbb{T}^d \times K$. Then, provided that all terms on the right-hand-side are finite,

$$\begin{aligned}&\left|\int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk\right| \\ &\quad \le \frac{2\pi}{t}\left(\left\|\frac{\bar{\phi}}{\omega'}\partial_k f_0\right\|_{L^1} + \left\|\bar{f}_0 \partial_k \frac{\phi}{\omega'}\right\|_{L^1}\right).\end{aligned} \tag{7}$$

Remark 3. The hypotheses on f, ϕ, and ω of Theorem I.1 imply directly that the terms in the upper bound are indeed finite. Since it is sufficient to prove the decay for large values of t, this proposition implies the one-dimensional case of Theorem I.1.

Remark 4. The hypothesis on compact support is only needed to ensure the absence of boundary terms when integrating by parts. It can be weakened by adding the value(s) of $\frac{\bar{f}_0\bar{\phi}}{\omega'}$ at ∂K to the right-hand-side, provided that these values are well defined.

Proof. First, note that as a limiting case of (6) or by using the exact time evolution and a change of variables,

$$\begin{aligned}\bar{f}_0(k) &:= \frac{1}{2\pi}\int_{\mathbb{T}} f_0(q,k)dq \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f_0(q - \omega(k)t, k)\,\mathrm{d}q =: \frac{1}{2\pi}\int_{\mathbb{T}} f(t,q,k)\,\mathrm{d}q.\end{aligned}$$

We insert this in the expression that we need to estimate and use the fundamental theorem of calculus to write

$$\begin{aligned}\left|\int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right| &= \left|\frac{1}{2\pi}\int_K \int_{\mathbb{T}}\int_{\mathbb{T}} (f(t,q,k) - f(t,q',k))\phi(q,k)\,\mathrm{d}q'\,\mathrm{d}q\mathrm{d}k\right| \\ &= \left|\frac{1}{2\pi}\int_K \int_{\mathbb{T}}\int_{\mathbb{T}}\int_{q'}^{q} \partial_q f(t,\tilde{q},k)\phi(q,k)\,\mathrm{d}\tilde{q}\,\mathrm{d}q'\,\mathrm{d}q\mathrm{d}k\right| \\ &\le \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\int_K \partial_q f(t,\tilde{q},k)\phi(q,k)\,\mathrm{d}k\right|\mathrm{d}\tilde{q}\,\mathrm{d}q'\,\mathrm{d}q. \\ &= \int_{\mathbb{T}}\int_{\mathbb{T}}\left|\int_K [\partial_q f(t,\tilde{q},k)]\phi(q,k)\,\mathrm{d}k\right|\mathrm{d}\tilde{q}\,\mathrm{d}q.\end{aligned}$$

To obtain this inequality, we first used Fubini's theorem to perform the k-integral before the others and then extended the range of $\tilde{q}$ (which requires inserting the absolute value). The last line is just the observation that the q'-dependence has disappeared from the integrand.

We now use W defined in (5) to write $\partial_q = (\omega'(k)t)^{-1}(W - \partial_k)$. The first term will have the required form to apply (6), and we can integrate by parts (the boundary terms disappear due to the assumptions on f and ϕ) to bring the second term in this form as well. This gives

$$\int_K [\partial_q f(t,\tilde{q},k)]\phi(q,k)\,k = t^{-1}\int_K [Wf](t,\tilde{q},k)\frac{\phi(q,k)}{\omega'(k)}\,\mathrm{d}k + t^{-1}\int_K f(t,\tilde{q},k)\partial_k\left[\frac{\phi(q,k)}{\omega'(k)}\right]\mathrm{d}k.$$

Inserting this in the bound, we found

$$\begin{aligned}\left|\int_K\int_{\mathbb{T}}(f(t,q,k)-\bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right| &\\ \leq t^{-1}\int_K\int_{\mathbb{T}}\frac{|Wf|(t,\tilde{q},k)}{|\omega'(k)|}\int_{\mathbb{T}}|\phi(q,k)|\,\mathrm{d}q\mathrm{d}\tilde{q}\mathrm{d}k &\\ + t^{-1}\int_K\int_{\mathbb{T}}f(t,\tilde{q},k)\int_{\mathbb{T}}\left|\partial_k\frac{\phi(q,k)}{\omega'(k)}\right|\mathrm{d}q\mathrm{d}\tilde{q}\mathrm{d}k.&\end{aligned}$$

In both terms, we can apply (6), and thus, using furthermore that $W|_{t=0}=\partial_k$, we have obtained

$$\begin{aligned}\left|\int_K\int_{\mathbb{T}}(f(t,q,k)-\bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right| &\\ \leq t^{-1}\int_K\int_{\mathbb{T}}\frac{|\partial_k f_0|(\tilde{q},k)}{|\omega'(k)|}\int_{\mathbb{T}}|\phi(q,k)|\,\mathrm{d}q\mathrm{d}\tilde{q}\mathrm{d}k &\\ + t^{-1}\int_K\int_{\mathbb{T}}f_0(\tilde{q},k)\int_{\mathbb{T}}\left|\partial_k\frac{\phi(q,k)}{\omega'(k)}\right|\mathrm{d}q\mathrm{d}\tilde{q}\mathrm{d}k.&\end{aligned}$$

This can be rewritten in terms of the averages over $\mathbb{T}$ to give (7). □

A. Localization argument

In this section, we prove Theorem I.2. We use the explicit rate of decay and the expression for the upper bound obtained in Proposition II.1 allows for extensions when $\omega'(k)$ vanishes at some *energies* in the support of ϕ. We use a simple localization argument to treat the case where $\omega'(k)$ vanishes linearly.

Proof of Theorem I.2. Let $0<\varepsilon<1$ to be fixed later. We fix a smooth cutoff function χ with support in $(-1,1)$ and values in $[0,1]$ such that $\chi\equiv 1$ in $[-1/2,1/2]$. We define $\chi_{i,\varepsilon}:=\chi(\frac{k-k_i}{\varepsilon})$ and $\eta_\varepsilon:=\prod_{i=1}^N(1-\chi_{i,\varepsilon})$. Then, we write $\phi(q,k)=\eta_\varepsilon(k)\phi(q,k)+(1-\eta_\varepsilon(k))\phi(q,k)$. Note that $\eta_\varepsilon(k)\phi(q,k)$ satisfies the hypotheses of Proposition II.1. Thus, (7) gives

$$\begin{aligned}\left|\int_K\int_{\mathbb{T}}(f(t,q,k)-\bar{f}_0(k))\eta_\varepsilon(k)\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right| &\\ \leq\frac{2\pi}{t}\left\|\frac{\eta_\varepsilon}{\omega'}\right\|_{L^\infty}\left(\|\bar{\phi}\partial_k f_0\|_{L^1}+\left\|\bar{f}_0\partial_k\phi\right\|_{L^1}\right) &\\ +\frac{2\pi}{t}\left\|\partial_k\left(\frac{\eta_\varepsilon}{\omega'}\right)\right\|_{L^\infty}\left\|\phi\bar{f}_0\right\|_{L^1}.&\end{aligned}$$

Now, we need to extract the ε-dependence from the L^∞-norms. Since $\omega''(k_i)\neq 0$, for some $C>0$ and all $\varepsilon\leq 1$, we have

$$\inf_{\mathrm{supp}(\eta_\varepsilon)\cap\mathrm{supp}(f)}|\omega'(k)|\geq\frac{\varepsilon}{C}.$$

This gives the bounds

$$\left\|\frac{\eta_\varepsilon}{\omega'}\right\|_{L^\infty}\leq\frac{C}{\varepsilon},\qquad\left\|\partial_k\left(\frac{\eta_\varepsilon}{\omega'}\right)\right\|_{L^\infty}\leq\frac{C}{\varepsilon^2}.$$

We have obtained

$$\left|\int_K\int_{\mathbb{T}}(f(t,q,k)-\bar{f}_0(k))\eta_\varepsilon(k)\phi(q,k)\right|\leq\frac{C}{t\varepsilon^2}. \tag{8}$$

On the other hand,

$$\begin{aligned}\left|\int_K\int_{\mathbb{T}}(f(t,q,k)-\bar{f}_0(k))(1-\eta_\varepsilon(k))\phi(q,k)\right| &\\ \leq 2\int_{\mathbb{R}^+}(1-\eta_\varepsilon(k))\int_{\mathbb{T}}f(t,q,k)\,\mathrm{d}q\mathrm{d}k &\\ \leq C\varepsilon\left\|\bar{f}_0\right\|_{L^\infty}.&\end{aligned}$$

We sum with (8), evaluate at some $T > 1$, and pick $\varepsilon = T^{-1/3}$ to obtain

$$\left| \int_K \int_{\mathbb{T}} (f(T,q,k) - \bar{f}_0(k))\phi(q,k) \right| \leq CT^{-1/3}.$$

for small T, both terms are bounded, this implies the result. □

B. Improved decay

If the initial condition is more regular, we can improve the estimate on the decay. To this end, we use the following L^1-version of Poincaré inequality.

Lemma II.2 (Poincaré inequality). Assume that $g : \mathbb{T} \mapsto \mathbb{R}$ is a periodic function of class C^l and $g(x) = 0$ for some $x \in [0, 2\pi)$. Then, for all $l \in \mathbb{N}$,

$$\int_{\mathbb{T}} |g(q)| \, \mathrm{d}q \leq (\pi)^l \int_{\mathbb{T}} |g^{(l)}(q)| \, \mathrm{d}q.$$

Proof. Without loss of generality, we may assume that $x = 0$. Then,

$$\begin{aligned} \int_0^\pi |g(q)| \, \mathrm{d}q &= \int_0^\pi \left| \int_0^s g'(r) \, \mathrm{d}r \right| \mathrm{d}s \\ &\leq \int_0^\pi |g'(r)| \int_r^\pi \mathrm{d}s \, \mathrm{d}r \leq \pi \int_0^\pi |g'(r)| \, \mathrm{d}r. \end{aligned}$$

Treating the contribution to the L^1-norm of the interval $[\pi, 2\pi]$ analogously, we find that

$$\int_{\mathbb{T}} |g(q)| \, \mathrm{d}q \leq \pi \int_{\mathbb{T}} |g'(q)| \, \mathrm{d}q.$$

For the case $l \geq 2$, we proceed by induction. By periodicity, $\int_{\mathbb{T}} g^{(l-1)} = 0$, so $g^{(l-1)}(x) = 0$ for some $x \in \mathbb{T}$, and we can iterate the argument. □

As a consequence, we can obtain a faster rate of decay for more regular initial data and observables. For the sake of readability, we assume that the support of ϕ is compact (bounded away from the boundary of K) although it is possible to relax this to suitable decay of the functions and their derivatives.

Proof of Theorem I.3. As in the proof of Proposition II.1, we bound

$$\left| \int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k) \, \mathrm{d}q\mathrm{d}k \right| \leq \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \partial_{\tilde{q}} \int_K f(t,\tilde{q},k)\phi(q,k) \, \mathrm{d}k \right| \mathrm{d}\tilde{q}\mathrm{d}q.$$

We then use Lemma II.2 to insert $l - 1$ additional derivatives as follows:

$$\begin{aligned} \left| \int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k) \, \mathrm{d}q\mathrm{d}k \right| &\leq \pi^{l-1} \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \partial_{\tilde{q}}^l \int_K f(t,\tilde{q},k)\phi(q,k) \, \mathrm{d}k \right| \mathrm{d}\tilde{q}\mathrm{d}q \\ &\leq t^{-l}\pi^{l-1} \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \int_K \left(\frac{W - \partial_k}{\omega'(k)} \right)^l f(t,\tilde{q},k)\phi(q,k) \, \mathrm{d}k \right| \mathrm{d}\tilde{q}\mathrm{d}q. \end{aligned}$$

In the previous expression, we keep in mind that the operator W only affects the variables denoted by k and $\tilde{q}$, not q. Expanding the product makes 2^l terms appear. In order to integrate by parts, we iterate the identities

$$[\partial_K, W] = \omega''(k)t\partial_q = \frac{\omega''(k)}{\omega'(k)}(W - \partial_k),$$

and for any sufficiently regular function $g(k)$,

$$[W, g(k)] = [\partial_K, g(k)] = g'(k).$$

This allows us to obtain an identity of the form

$$\left(\frac{W - \partial_k}{\omega'(k)} \right)^l = \sum_{j=0}^{l} \sum_{m=0}^{l} g_{j,m}(k)\partial_k^j W^m,$$

where each of the functions $g_{n,m}(k)$ is a complicated expression containing powers of $(\omega')^{-1}$ and its derivatives up to order $l-(m+j)$. In each term, we integrate by parts in K to obtain

$$\left|\int_K \left(\frac{W-\partial_k}{\omega'(k)}\right)^l f(t,\tilde{q},k)\phi(q,k)\,\mathrm{d}k\right| \le \sum_{j=0}^{l}\sum_{m=0}^{l}\left|\int_K W^m f(t,\tilde{q},k)\partial_k^j(g_{j,m}(k)\phi(k))\,\mathrm{d}k\right|.$$
$$\le \sum_{j=0}^{l}\sum_{m=0}^{l}\left\|\partial_k^j(g_{j,m}\phi)\right\|_{L^\infty}\int_K |W^m f(t,\tilde{q},k)|\,\mathrm{d}k.$$

Finally, we apply (6) to bound

$$\left|\int_K\int_{\mathbb{T}}(f(t,q,k)-\bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right| \le t^{-l}\pi^l\sum_{j=0}^{l}\sum_{m=0}^{l}\left\|\partial_k^j(g_{j,m}\phi)\right\|_{L^\infty}\int_{\mathbb{T}}\int_K|W^m f(t,\tilde{q},k)|\,\mathrm{d}\tilde{q}$$
$$= t^{-l}\pi^l\sum_{j=0}^{l}\sum_{m=0}^{l}\left\|\partial_k^j(g_{j,m}\phi)\right\|_{L^\infty}\left\|\partial_k^m f_0\right\|_{L^1}. \qquad \square$$

III. PROOFS FOR $d \ge 2$

In this section, we prove mixing for the d-dimensional case. In this case, we define the following d vector fields:

$$W = tD\omega(k)\nabla_q + \nabla_k \qquad \text{or} \qquad W_j = t\sum_{i=1}^{d}(\partial_{k_j}\omega_i(k))\partial_{q_i} + \partial_{k_j}. \tag{9}$$

As before, each component of W commutes with the Liouville operator. If $D\omega$ is an invertible matrix, most of the proof goes through as before. For the sake of completeness, we state Theorem I.1 with an explicit bound on the right-hand-side. To this end, we define the matrix norm

$$|M|_\infty = \max_{i,j}|M_{i,j}|.$$

Proposition III.1. Let $f(t,q,k)$ be the solution to (1) with initial datum $f_0 \in C^1(\mathbb{T}^d\times K)$. Assume that $\phi \in C_c^1(\mathbb{T}^d\times K)$ and that

$$\det D\omega(k) \neq 0 \qquad \textit{for all } k \in K.$$

Then, $M := (D\omega(k))^{-1}$ is well defined and

$$\left|\int_K\int_{\mathbb{T}}(f(t,q,k)-\bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right|$$
$$\le \frac{2\pi d}{t}\left(\||M|_\infty\phi\|_{L^\infty}\sum_{i=1}^{d}\left\|\partial_{k_i}\bar{f}_0\right\|_{L^1} + \|\nabla_k\cdot M\phi\hat{e}_j\|_{L^\infty}\left\|\bar{f}_0\right\|_{L^1}\right).$$

Proof. As before, we express the left-hand-side as

$$\left|\int_K\int_{\mathbb{T}^d}(f(t,q,k)-\bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right|$$
$$= \frac{1}{(2\pi)^d}\left|\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\int_K(f(t,q,k)-f(t,q',k))\phi(q,k)\,\mathrm{d}k\,\mathrm{d}q\mathrm{d}q'\right|.$$

We then write

$$f(t,q,k)-f(t,q',k) = \sum_{j=1}^{d}\int_{q_j'}^{q_j}\partial_{q_j}f(t,q_1,\dots q_{j-1},s,q_{j+1}',\dots,q_d',k)\,\mathrm{d}s$$

and bound

$$\left|\int_K\int_{\mathbb{T}^d}(f(t,q,k)-\bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right| \le \frac{1}{(2\pi)^d}\sum_{j=1}^{d}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\int_0^{2\pi}\left|\int_K \partial_{q_j} f(t,q_1,\dots q_{j-1},s,q'_{j+1},\dots,q'_d,k)\phi(q,k)\,\mathrm{d}k\right|\mathrm{d}s\mathrm{d}q\mathrm{d}q'.$$

We now write $\partial_{q_j} = t^{-1}[M(W-\nabla_k)]_j$. Thus, by using the divergence theorem for the second term (and using the compact support of ϕ to conclude the absence of boundary terms), we obtain

$$\begin{aligned}
&\left|\int_K \partial_{q_j} f(t,q_1,\dots q_{j-1},s,q'_{j+1},\dots,q'_d,k)\phi(q,k)\,\mathrm{d}k\right| \\
&\quad\le \frac{1}{t}\int_K\left|[M(k)W]_j f(t,q_1,\dots q_{j-1},s,q'_{j+1},\dots,q'_d,k)\phi(q,k)\right|\mathrm{d}k \\
&\qquad+\frac{1}{t}\int_K\left|f(t,q_1,\dots q_{j-1},s,q'_{j+1},\dots,q'_d,k)(\nabla_k\cdot M(k)^\top\phi(q,k)\hat{e}_j)\right|\mathrm{d}k \\
&\quad\le \frac{\||M|_\infty\phi\|_{L^\infty}}{t}\sum_{i=1}^{d}\int_{\mathbb{R}^d_+}\left|W_i f(t,q_1,\dots q_{j-1},s,q'_{j+1},\dots,q'_d,k)\right|\mathrm{d}k \\
&\qquad+\frac{\left\|\nabla_k\cdot M^\top\phi\hat{e}_j\right\|_{L^\infty}}{t}\int_K\left|f(t,q_1,\dots q_{j-1},s,q'_{j+1},\dots,q'_d,k)\right|\mathrm{d}k.
\end{aligned}$$

Inserting this in the previous bound and using (6), we obtain

$$\begin{aligned}
&\left|\int_K\int_{\mathbb{T}^d}(f(t,q,k)-\bar{f}_0(k))\phi(q,k)\,\mathrm{d}q\mathrm{d}k\right| \\
&\le \frac{2\pi d}{t}\left(\||M|_\infty\phi\|_{L^\infty}\sum_{i=1}^{d}\left\|\partial_{k_i}\bar{f}_0\right\|_{L^1}+\left\|\nabla_k\cdot M^\top\phi\hat{e}_j\right\|_{L^\infty}\left\|\bar{f}_0\right\|_{L^1}\right).
\end{aligned}$$

□

Finally, we prove Corollary I.4.

Proof of Corollary I.4. For fixed $x_0\in\mathbb{R}^d$, we write ϕ_{x_0} for the fundamental solution to Poisson's equation in dimension d. In particular,

$$\phi_{x_0}(x)=\begin{cases}(x-x_0)\mathbb{I}_{[-\infty,x_0]}+\max(0,x_0) & \text{if } d=1,\\ -(2\pi)^{-1}\ln(|x-x_0|) & \text{if } d=2,\\ \kappa_d|x-x_0|^{-d+2} & \text{if } d\ge 3\end{cases}$$

for a suitable constant κ_d. Hence, we can write

$$V_F(x_0)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi_{x_0}(x)F(t,x,v)\,\mathrm{d}x\,\mathrm{d}v=\int_{\mathbb{T}^d}\int_K\varphi_{x_0}(q,k)f(t,q,k)\,\mathrm{d}k\,\mathrm{d}q,$$

where we used the notation $f=F\circ N$ and $\varphi_{x_0}=\phi_{x_0}\circ N$. Now, the integral is in a suitable form to apply the arguments in the proofs of Theorems II.1 and III.1, provided that φ_{x_0} is sufficiently regular. The Coulomb kernel ϕ_{x_0} belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}^d)$ since the integral of its derivative in a ball is finite. Outside a sufficiently large ball, the function and its derivatives are bounded. Since we assume that N is of class C^1, φ_{x_0} inherits these properties. Thus, $\nabla_k\varphi_{x_0}\in L^1+L^\infty$. For the L^∞-part, we can apply Proposition III.1 directly, and for the L^1 part, we switch the roles of f and ϕ in the Proof of Proposition III.1. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

M. Moreno: Formal analysis (equal); Writing-original-draft (equal); Writing-review-editing (equal). **P. Rioseco:** Formal analysis (equal); Writing-original-draft (equal); Writing-review-editing (equal). **H. Van Den Bosch:** Formal analysis (equal); Writing-original-draft (equal); Writing-review-editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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