Stability properties of Lane-Emden and Goldreich-Weber solutions to the Euler-Poisson system

King Ming Lam

UCL

A thesis presented for the degree of
Doctor of Philosophy

I, King Ming Lam, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

The Euler-Poisson equation (or system) is the compressible Euler equation (which describes a compressible fluid) coupled with the Poisson equation (which describes Newtonian gravity). In a free boundary context where the domain evolved according to the fluid motion, this system describes a classical model of a star – a lump of compressible gas or liquid surrounded by a vacuum subject to forces created by its own pressure and gravity – and has been long studied by astrophysicists.

Two classes of important special solutions to this system are the Lane-Emden stars and the Goldreich-Weber stars. The former is a class of spherically symmetric static solutions describing a static star, while the latter is a class of expanding/collapsing solution which could describe for instance a supernova expansion or gravitational collapse. An important question regarding these solutions are their stability properties under perturbations.

The expanding Goldreich-Weber stars consist of two types – one that expands at a linear rate and one that expands at a self-similar rate. In this thesis we prove that the former is non-linearly stable under perturbations (allowed to be non-radial), and the latter class is codimension-1 non-linearly stable under irrotational perturbations (also allowed to be non-radial).

In the next part of the thesis, we will establish the linear stability properties of the liquid Lane-Emden stars, in particular we found that it differs from that of gaseous Lane-Emden stars. We establish various qualitative properties of the liquid Lane-Emden stars and using them we show that their linear stability properties depend not only on the adiabatic index but also the central density of the star. Such dependence on central density is not seen in the gaseous Lane-Emden stars.

Impact Statement

Partial differential equations (PDEs) are ubiquitous in almost every area of science, whether natural science like physics, chemistry and biology, or social or applied science like engineering, epidemiology and economics. It is tremendously important and useful to science as it gives a description of how quantities change or evolve according to rules.

The physics of how our world behaves are described by various partial differential equations – Maxwell's equations in electromagnetism, Navier–Stokes equations in fluid dynamics, Schrödinger equation in quantum mechanics or Einstein's equations in general relativity which describes gravity and spacetime. Hence for the advancement and understanding of science, it is important to have a good mathematical understanding of PDEs.

My study of the stability properties to solutions of the Euler-Poisson equation not only further the understanding of this particular PDE, but has the potential to shed light on other important PDEs that are similar or related. The Euler equation with the Euler-Poisson system is in particular a special case of the Navier–Stokes equations, whose problem of existence and smoothness is one of the biggest and most important open problems in mathematics, one of the Millennium Prize Problems.

Euler-Poisson equations also share similarities to many other equations for being a wave-like equation, as is for example Einstein's equations. And it was through the understanding of PDEs, by discovering a certain solution to Einstein's equations and proving its stability using mathematics, that physicists were able to predict the existence of black holes and gravitational waves before they were even directly observed in our universe.

For many discoveries in pure mathematics, its direct impact to the wider world might not be immediately apparent, but like number theory in computer cryptography, many were to find unexpected use many years later in previously unimagined places and go on to hugely impact and benefit the wider world. Therefore, I believe my study of the Euler-Poisson equation here not only contributes to the field of PDE analysis, but has the potential to have a wider impact in our world too, even if not as much as the distinguished examples that I mentioned.

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Chapter 1

Introduction

1.1 The Euler-Poisson system

A classical model for stars is given by the *Euler-Poisson system*, as a lump of compressible gas or liquid surrounded by vacuum, subjected to forces created by its own pressure and gravity. This is a system widely studied by (astro)physicists, for example by Chandrasekhar [4]. The fluid is modelled by the *compressible Euler equations*, which model perfect fluids with no heat conduction and no viscosity. It is given by

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} \tag{1.1}$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.2}$$

where ρ is the fluid density, $\mathbf{u}$ the fluid velocity, $\mathbf{g}$ the body accelerations acting on the fluid (e.g. gravity, inertial accelerations, electric field acceleration etc.), p the fluid pressure. All these quantities are functions of space $\mathbf{x}$ (in $\mathbb{R}^d$) and time t. Here $\frac{D}{Dt} := \partial_t + \mathbf{u} \cdot \nabla$ denotes the material derivative. Equation (1.1) is the so-called *momentum equation* which describes the forces on the fluid particles and conservation of momentum. Equation (1.2) is the so-called *continuity equation* which describes the conservation of mass.

We take $\mathbf{g}$ to be the Newtonian gravity

$$\mathbf{g} = \nabla \phi$$

where ϕ is the gravitational potential is defined by the Poisson equation

$$\Delta\phi = 4\pi\rho \qquad \text{and} \qquad \lim_{\|\mathbf{x}\|\to\infty} \phi(t,\mathbf{x}) = 0. \tag{1.3}$$

Here the boundary condition at infinity $\lim_{\|\mathbf{x}\|\to\infty} \phi(t,\mathbf{x}) = 0$ ensures that we have a unique solution for ϕ, and furthermore we require this boundary condition to be zero since we want to model an isolated star where no energy is coming from infinity. This case describes a *self-gravitating star*, subject to gravity induced by its own mass as well as force coming from its own pressure.

The lump of liquid does not have a fixed shape, its shape and volume can change - the pressure of the liquid will try to force the liquid to occupy more space/volume and expend the domain, while any gravitational force will try to do the opposite. As a result we have a *free surface problem* where the domain $\Omega(t) = \operatorname{supp}\rho$ changes with time, being deformed by the fluid motion itself. We have the following boundary conditions:

a) the pressure on the boundary matches that of the vacuum outside, i.e. $p = 0$ on $\partial\Omega$;

b) the normal velocity at which the boundary changes is equal to $\mathbf{u}\cdot\mathbf{n}$ at any point on the boundary, where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$.

We now have $d+1$ equations for our fluid (the scalar continuity equation (1.2) plus the vector momentum equation (1.1)), but $d+2$ unknowns (ρ, p and $\mathbf{u}$). To close the system, we need to specify an *equation of state*. The classical model which we will consider is that of a *barotropic fluid* where pressure is a function of density only, wherein we have a equation of state $p = P(\rho)$ that relates the pressure to the density. Under this assumption, we shall consider the classical case of *polytropic fluid* where p is related to ρ via

$$p = \begin{cases} K\rho^\gamma & \text{for gasous stars} \\ K\rho^\gamma - C & \text{for liquid stars.} \end{cases} \tag{1.4}$$

for some constant $K, C > 0$ and $2 \geq \gamma \geq 1$. Note that under (1.4), since pressure is 0 on the boundary $\partial\Omega$, the density of the fluid will be 0 on the boundary for the gas case, and $(C/K)^{1/\gamma}$ on the boundary for the liquid case. Without loss of generality, we can take $K = C = 1$. The constant for the power γ, called the *adiabatic index*, makes a difference however since it defines the fundamental relationship between p and ρ. With respect to the natural scaling of the system, the parameter γ plays a crucial role as natural criticality parameter, with different behaviour to be expected from different values of γ. We will refer to the Euler-Poisson system with adiabatic index γ as the (EP)$_\gamma$-system.

1.1.1 Physical vacuum condition

Generically, the body of fluid will have non-zero pressure away from the vacuum boundary. Since the vacuum is the area of zero pressure, we should expect the forces from the pressure to work to expand the domain $\Omega(t)$ (while gravity work to shrink it). In other words, heuristically, the acceleration due to pressure of the vacuum boundary should be non-zero. And in order for the movement of the boundary to be well-defined it should also be finite. The acceleration of fluid particles due to pressure is $-\rho^{-1}\nabla p$ (see momentum equation (1.1)). Note that

$$-\rho^{-1}\nabla p \sim -\nabla w \qquad \text{where} \qquad w = \begin{cases} \rho^{\gamma-1} & \text{when } \gamma > 1 \\ \ln \rho & \text{when } \gamma = 1. \end{cases}$$

Hence a "fluid particle" at the vacuum boundary $\mathbf{x}_0 \in \partial\Omega(t)$ is accelerated by the pressure by an amount proportional to $-\mathbf{n} \cdot \nabla w(\mathbf{x}_0)$. Hence in order for the free boundary problem to be well defined, heuristically we expect the condition

$$0 < -\mathbf{n} \cdot \nabla w|_{\partial\Omega} < \infty$$

to be generic and natural to the free boundary problem. This specifies the rate the density decay towards the vacuum boundary. This condition is known as the *physical vacuum condition*. Indeed, it has been shown that outside of this condition, the problem is in general ill-posed; and when solution exist, it should come to satisfy the physical vacuum condition in finite time. Whereas within the physical vacuum condition framework, well-posedness of the problem has been established via, for example, the methods of Jang and Masmoudi [33] and Coutand and Shkoller [7].

Note that gas with $\gamma = 1$ can never satisfy the physical vacuum condition, so it does not admit well-defined solutions with a vacuum region within the physical vacuum condition framework. For more discussions on the physical vacuum condition and well-posedness of solutions with a vacuum region, see [32] and [33].

1.1.2 Conserved quantities and scaling

The Euler-Poisson system (1.1)-(1.3) possesses the following important conserved quantities – mass, momentum and energy, given respectively by:

$$M[\rho] := \int_{\Omega(t)} \rho \, \mathbf{dx}, \tag{1.5}$$

$$\mathbf{W}[\rho, \mathbf{u}] := \int_{\Omega(t)} \rho \mathbf{u} \, \mathbf{dx}, \tag{1.6}$$

$$E[\rho, \mathbf{u}] := \begin{cases} \int_{\Omega(t)} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}\rho\phi + \frac{1}{\gamma - 1}\rho^\gamma \right) \mathbf{dx} & \text{for gas, } \gamma > 1 \\ \int_{\Omega(t)} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}\rho\phi + \rho \ln \rho \right) \mathbf{dx} & \text{for gas, } \gamma = 1 \\ \int_{\Omega(t)} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}\rho\phi + \frac{1}{\gamma - 1}\rho^\gamma + 1 \right) \mathbf{dx} & \text{for liquid, } \gamma > 1 \\ \int_{\Omega(t)} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}\rho\phi + \rho \ln \rho + 1 \right) \mathbf{dx} & \text{for liquid, } \gamma = 1 \end{cases} \tag{1.7}$$

The Euler-Poisson system possess scaling structures so that if $(\rho, \mathbf{u})$ is a classical solution of the $(\mathrm{EP})_\gamma$-system, then $(\tilde{\rho}, \tilde{\mathbf{u}})$ defined by

$$\tilde{\rho}(t, \mathbf{x}) = \lambda^{-\frac{2}{2-\gamma}} \rho \left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{\mathbf{x}}{\lambda} \right) \tag{1.8}$$

$$\tilde{\mathbf{u}}(t, \mathbf{x}) = \lambda^{-\frac{\gamma-1}{2-\gamma}} \mathbf{u} \left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{\mathbf{x}}{\lambda} \right) \tag{1.9}$$

is also a solution to the $(\mathrm{EP})_\gamma$-system for any fixed $\lambda > 0$. And we have

$$M[\tilde{\rho}] = \lambda^{d - \frac{2}{2-\gamma}} M[\rho]$$
$$E[\tilde{\rho}, \tilde{\mathbf{u}}] = \lambda^{d - \frac{2\gamma}{2-\gamma}} E[\rho, \mathbf{u}].$$

The mass critical and energy critical indices are the indices γ for which the mass and energy respectively remains unchanged under this scaling. For $d = 3$, the mass critical index is $\gamma = 4/3$ and the energy critical index is $\gamma = 6/5$.

A γ larger than the mass critical index is said to be mass *sub-critical* where good behaviour is expected; while a γ smaller than the mass critical index is said to be mass *super-critical* where bad behaviour is expected. A heuristic way to see this is as follows. Imagine we have a solution in equilibrium where pressure and gravity are in perfect balance. When $\lambda > 1$, the density/mass of $(\rho, \mathbf{u})$ is more concentrated in space (has larger maximum value and smaller support) than that of

$(\tilde{\rho}, \tilde{\mathbf{u}})$. Heuristically this should be due to a stronger gravity effect due to a larger overall mass. In the mass sub-critical range, the mass of $(\rho, \mathbf{u})$ is larger than $(\tilde{\rho}, \tilde{\mathbf{u}})$ as expected, hence good behaviour is expected. Whereas in the mass super-critical range, the mass of $(\rho, \mathbf{u})$ is smaller than $(\tilde{\rho}, \tilde{\mathbf{u}})$, this strangeness suggest behaviours might be not so good, at least when it comes to the stability of such a equilibrium solution.

Similarly, a γ larger than the energy critical index is said to be energy *sub-critical*; while a γ smaller than the energy critical index is said to be energy *super-critical*.

1.1.3 Lagrangian formulation

A changing domain is difficult to do analysis on, but since we are dealing with a fluid made up of "fluid particles", using this structure we can convert to Lagrangian coordinates by tracing "fluid particles" and obtain an equivalent formulation where the domain is fixed. This formulation is particularly well suited to the analysis of fluids featuring a vacuum boundary.

The Lagrangian formulation is as follows. Let $\boldsymbol{\eta}(t, \mathbf{x})$ be the the fluid flow map, defined through

$$\partial_t \boldsymbol{\eta} = \mathbf{u} \circ \boldsymbol{\eta} \qquad \text{with} \qquad \boldsymbol{\eta}(0, \mathbf{x}) = \boldsymbol{\eta}_0(\mathbf{x}),$$

where $\mathbf{u} \circ \boldsymbol{\eta}(t, \mathbf{x}) = \mathbf{u}(t, \boldsymbol{\eta}(t, \mathbf{x}))$. The spatial domain is then fixed for all time as $\Omega_0 := \boldsymbol{\eta}_0^{-1}(\Omega(0))$. To reformulate the (EP)$_\gamma$-system in the new variables, we introduce

$$\begin{aligned}
&\mathbf{v} = \mathbf{u} \circ \boldsymbol{\eta} && \text{(Lagrangian velocity)}\\
&f = \rho \circ \boldsymbol{\eta} && \text{(Lagrangian density)}\\
&\psi = \phi \circ \boldsymbol{\eta} && \text{(Lagrangian potential)}\\
&A = (\nabla \boldsymbol{\eta})^{-1} && \text{(inverse of the deformation tensor)}\\
&J = \det(\nabla \boldsymbol{\eta}) && \text{(Jacobian determinant)}\\
&a = JA && \text{(cofactor matrix of the deformation tensor)}
\end{aligned}$$

Under this change of coordinates, the continuity equation becomes $fJ = f_0 J_0$ and the momentum equation (1.1) in the domain Ω_0 reads

$$\partial_t \mathbf{v} + \frac{1}{f_0 J_0} \partial_k (A^k (f_0 J_0)^\gamma J^{1-\gamma}) + A \nabla \psi = \mathbf{0}, \tag{1.10}$$

where Einstein summation convention is used (see Definition 1.4.1). Moreover, ψ solves the Poisson equation

$$(A\nabla) \cdot (A\nabla)\psi = 4\pi f_0 J_0 J^{-1}. \tag{1.11}$$

For details of the Lagrangian description of the Euler-Poisson system, we refer to [22].

1.2 Lane–Emden and Goldreich-Weber stars

There are two classes of special solutions to the Euler-Poisson system of much physical importance.

1) The first is the *Lane–Emden stars* (LE stars) [4], which are time independent spherically symmetric solutions modelling a static star where pressure and gravity are in perfect balance. Lane–Emden stars exist for all $2 \geq \gamma \geq 1$ and for both gas and liquid (see section 4.1).

2) The second is the *Goldreich-Weber stars* (GW stars) [17], which are spherically symmetric expanding or collapsing solutions modelling a star that is expanding or collapsing in time. Goldreich-Weber stars exist at the mass critical index $\gamma = 4/3$ (when $d = 3$) for gas.

To obtain these solutions, we look for spherically symmetric solutions to the Euler-Poisson system of the form $\boldsymbol{\eta}(t, \mathbf{x}) = \lambda(t)\mathbf{x}$, and assume without loss of generality that $\lambda(0) = 1$. Under this affine ansatz, the momentum equation (1.10) in $\mathbb{R}^3$ reduces to

$$f_0 \ddot{\lambda}\mathbf{x} + \lambda^{-1+3(1-\gamma)} \nabla f_0^{\gamma} + \lambda^{-2} f_0 \nabla \mathcal{K} f_0 = \mathbf{0} \tag{1.12}$$

where

$$\mathcal{K} := 4\pi\Delta^{-1} \tag{1.13}$$

i.e. $\mathcal{K} f(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\mathbf{y}$ for any $f \in L^2(\mathbb{R}^3)$; and we observe that

$$\begin{aligned}\psi(\mathbf{x}) = (\mathcal{K}\rho)(\boldsymbol{\eta}(\mathbf{x})) &= -\int \frac{\rho(\mathbf{y})}{|\boldsymbol{\eta}(\mathbf{x}) - \mathbf{y}|} \mathrm{d}\mathbf{y} = -\int \frac{f(\mathbf{z})J(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x}) - \boldsymbol{\eta}(\mathbf{z})|} \mathrm{d}\mathbf{z} \\ &= -\int \frac{f_0(\mathbf{z})J_0(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x}) - \boldsymbol{\eta}(\mathbf{z})|} \mathrm{d}\mathbf{z} = -\frac{1}{\lambda}\int \frac{f_0(\mathbf{z})}{|\mathbf{x} - \mathbf{z}|} \mathrm{d}\mathbf{z} = \frac{1}{\lambda}\mathcal{K}(f_0)(\mathbf{x}).\end{aligned}$$

1.2.1 Lane–Emden stars

To look for time independent solutions like the Lane–Emden stars, we further take $\lambda(t) = 1$ in our ansatz. This reduce (1.12) to

$$\frac{1}{f_0}\nabla f_0^\gamma + \nabla \mathscr{K} f_0 = \mathbf{0}$$

or equivalently

$$0 = \begin{cases} \frac{\gamma}{\gamma-1}\Delta f_0^{\gamma-1} + 4\pi f_0 & \text{when} \quad \gamma > 1 \\ \Delta \ln f_0 + 4\pi f_0 & \text{when} \quad \gamma = 1 \end{cases}$$

The assumes $d = 3$, but a slight modification to the derivation shows the same formula works for general $d \geq 3$. Assuming spherically symmetry, we have $\Delta = r^{-(d-1)}\partial_r(r^{d-1}\partial_r)$, this gives the ODE which defines the Lane–Emden stars.

Definition 1.2.1 (Lane–Emden solutions)**.** *Lane–Emden (LE) stars are time-independent static solutions of the free-boundary Euler-Poisson system* (1.1)–(1.4)*, its density ρ is given by the ODE:*

i. when $\gamma > 1$,

$$\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = -4\pi\frac{\gamma-1}{\gamma}w^\alpha. \tag{1.14}$$

or equivalently

$$\frac{\mathrm{d}w}{\mathrm{d}r} = -4\pi\frac{\gamma-1}{\gamma}\frac{1}{r^{d-1}}\int_0^r y^{d-1}w(y)^\alpha \mathrm{d}y \tag{1.15}$$

where $w = \rho^{\gamma-1}$ and $\alpha = (\gamma-1)^{-1}$.

ii. when $\gamma = 1$,

$$\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{\mathrm{d}h}{\mathrm{d}r}\right) = -4\pi e^h. \tag{1.16}$$

or equivalently

$$\frac{\mathrm{d}h}{\mathrm{d}r} = -4\pi\frac{1}{r^{d-1}}\int_0^r y^{d-1}e^{h(y)}\mathrm{d}y \tag{1.17}$$

where $h = \ln\rho$.

We will also refer to these ODEs as the *steady state equation* since they define steady state solutions to the Euler-Poisson system. We also call w the *enthalpy* and the ODE for it the *enthalpy equation*. Solution to these ODEs exist (see Theorem B.1.3). Moreover, the gaseous solutions has compact support when $\gamma > 2d/(d+2)$ (energy sub-critical indices), and infinite support otherwise (see Theorem 4.1.4). Liquid solutions always have compact support - they are the gaseous solutions truncated at $\rho = 1$.

1.2.2 Goldreich-Weber stars

The mass-criticality of the problem allows for the existence of a special class of expanding solutions, known as the Goldreich-Weber stars [17]. The reason such solutions exist is, roughly speaking, because the scaling properties of the Euler-Poisson system in the mass critical case allows us to scale solutions while maintain the overall mass. This suggests that natural solutions that evolve in time under this scaling exist (note that solutions must conserve overall mass in time). For reader's convenience we provide a brief summary of this special class of solutions of (1.1)–(1.4) which has been analysed in [17, 46, 15, 11]. A comprehensive overview can be found in [20].

For notational simplicity, here we will restrict to the physically relevant case of $d = 3$. In the mass-criticality index of $\gamma = 4/3$, the equation (1.12) becomes separable. Indeed, we get

$$\ddot{\lambda}\lambda^2\mathbf{x} + \frac{1}{f_0}\nabla(f_0^{\frac{4}{3}}) + \nabla\mathcal{K}f_0 = \mathbf{0}$$

Assuming spherically symmetry we get

$$\ddot{\lambda}\lambda^2 + \frac{1}{rf_0}\partial_r(f_0^{\frac{4}{3}}) + \frac{1}{r}\partial_r\mathcal{K}f_0 = 0.$$

Since we can separate variables above, we look for a $\delta \in \mathbb{R}$ and a δ-dependent solution $(\lambda, f_0) = (\lambda_\delta, f_0^\delta)$ so that

$$\ddot{\lambda}(t)\lambda(t)^2 = \delta, \tag{1.18}$$

$$\frac{4}{r}\partial_r\bar{w}_\delta + \frac{1}{r}\partial_r\mathcal{K}(\bar{w}_\delta^3) = -\delta, \tag{1.19}$$

where $\bar{w}_\delta$ is the enthalpy associated with f_0^δ satisfying

$$(\bar{w}_\delta)^3 := f_0^\delta. \tag{1.20}$$

We also equip (1.18) with initial data

$$\lambda(0) = 1, \qquad \dot{\lambda}(0) = \lambda_1 \in \mathbb{R}. \tag{1.21}$$

It can be shown that there exists a negative constant $\tilde{\delta} < 0$ such that the solution $(\lambda_\delta(t), \bar{w}_\delta)$ to (1.18)–(1.21) exists for all $\delta \geq \tilde{\delta}$, see [15, 20], whereby $\lambda_\delta(\cdot)$ either blows up in finite positive time, or exists globally for all $t \geq 0$. Moreover, for any such $\delta \geq \tilde{\delta}$, the enthalpy profile $\bar{w}_\delta$ is compactly supported, has finite total mass, and by adapting the value $\bar{w}_\delta(0)$ it can be normalised to be supported on the interval $r \in [0, R]$ for a fixed $R > 0$. At the vacuum boundary, by analogy to the classical Lane-Emden stars [20], the Goldreich-Weber star satisfies the so-called physical vacuum condition, which in this context reads

$$\bar{w}_\delta'(r)\Big|_{r=R} < 0. \tag{1.22}$$

Self-similarly expanding Goldreich-Weber stars

The self-similarly expanding Goldreich-Weber stars are the subclass of solutions to (1.18)–(1.21) of total energy 0, for which $\lambda_\delta(\cdot)$ exists for all $t \geq 0$. Since the total conserved energy of the above affine motion is easily seen to be

$$E_\delta(t) = \left(\lambda_1^2 + 2\delta\right) \int 2\pi f_0^\delta z^4 \, \mathrm{d}z, \tag{1.23}$$

solutions with vanishing energy necessitate $\delta < 0$. For any such $\tilde{\delta} \leq \delta < 0$, equation (1.18) with (1.21) is explicitly solvable with

$$\lambda_\delta(t) = \left(1 + \frac{3}{2}\lambda_1 t\right)^{2/3}, \qquad \lambda_1^2 = -2\delta. \tag{1.24}$$

In particular, for any $\lambda_1 > 0$ we obtain an expanding solution with the explicit rate of expansion $\lambda_\delta(t) \sim_{t\to\infty} t^{\frac{2}{3}}$. This is the self-similarly expanding Goldreich-Weber solution.

Definition 1.2.2 (Self-similarly expanding Goldreich-Weber solutions)**.** *To any $\delta \in [\tilde{\delta}, 0)$ we associate the Goldreich-Weber (GW) star which constitutes a solution of the mass-critical free-boundary Euler-Poisson system* (1.1)–(1.4)*:*

$$\bar{\rho}(t, \mathbf{x}) = \lambda_\delta(t)^{-3} \bar{w}_\delta^3 \left(\frac{|\mathbf{x}|}{\lambda_\delta(t)}\right), \qquad \bar{\mathbf{u}}(t, \mathbf{x}) = \frac{\dot{\lambda}_\delta(t)}{\lambda_\delta(t)} \mathbf{x}, \qquad \bar{\Omega}(t) = B_{\lambda_\delta(t)}(\mathbf{0}), \tag{1.25}$$

with $\lambda_\delta(t)$ given by (1.24) *with $\lambda_1 > 0$ and $\bar{w}_\delta$ the normalised solution to* (1.19) *as above.*

These solutions are spherical symmetric about the origin, have zero momentum $\mathbf{W}[\bar{\rho}, \bar{\mathbf{u}}] = 0$ and zero energy $E[\bar{\rho}, \bar{\mathbf{u}}] = 0$. Without loss of generality, this can be assumed by setting our frame of reference.

Remark 1.2.3. *The Galilean invariance of the Euler-Poisson system* (1.1)–(1.3) *implies the conservation of momentum. If we change our frame of reference, we can obtain an enlarged family of the GW-solutions with arbitrary momentum $\bar{\mathbf{W}} \in \mathbb{R}^3$. More precisely, for any motion $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{p}_1$ we can obtain a new solution via*

$$\begin{aligned} \bar{\rho}_{\mathbf{p}}(t, \mathbf{x}) &= \bar{\rho}(t, \mathbf{x} - \mathbf{p}(t)), \\ \bar{\mathbf{u}}_{\mathbf{p}}(t, \mathbf{x}) &= \bar{\mathbf{u}}(t, \mathbf{x} - \mathbf{p}(t)) + \mathbf{p}_1, \end{aligned}$$

or equivalently $\boldsymbol{\eta}_{\mathbf{p}}(t, \mathbf{x}) = \boldsymbol{\eta}(t, \mathbf{x}) + \mathbf{p}(t)$ in Lagrangian coordinates. It is easy to verify that $(\bar{\rho}_{\mathbf{p}}, \bar{\mathbf{u}}_{\mathbf{p}})$ so obtained solves the Euler-Poisson system with the total momentum $\mathbf{W}[\bar{\rho}_{\mathbf{p}}, \bar{\mathbf{u}}_{\mathbf{p}}] = M[\bar{\rho}, \bar{\mathbf{u}}]\mathbf{p}_1$ and energy $E[\bar{\rho}_{\mathbf{p}}, \bar{\mathbf{u}}_{\mathbf{p}}] = \frac{1}{2}M[\bar{\rho}, \bar{\mathbf{u}}]|\mathbf{p}_1|^2$. The freedom to choose $\mathbf{p}_1 \in \mathbb{R}^3$ thus parametrises the three degrees of freedom associated with the total linear momentum, and this will play a role in our analysis.

Linearly expanding Goldreich-Weber stars

In the case

$$\delta > 0 \qquad \text{or} \qquad \delta = 0 \text{ with } \lambda_1 > 0 \qquad \text{or} \qquad \delta \in (\tilde{\delta}, 0) \text{ with } \lambda_1 > \sqrt{2|\delta|}, \tag{1.26}$$

the solution $\lambda_\delta(\cdot)$ exists for all $t \geq 0$ and expands indefinitely at a linear rate, i.e. there exists a constant $c > 0$ such that

$$\lim_{t \to \infty} \dot{\lambda}(t) = c.$$

These solutions have strictly positive energy

$$E_{\delta, \lambda_1}(t) = \left(\lambda_1^2 + 2\delta\right) \int 2\pi f_0^\delta z^4 \, \mathrm{d}z > 0. \tag{1.27}$$

This subclass of solutions to (1.18)–(1.21) is the linearly expanding Goldreich-Weber solution.

Definition 1.2.4 (Linearly expanding Goldreich-Weber solutions)**.** *To any δ, λ_1 sat-*

isfying (1.26) *we associate the Goldreich-Weber (GW) star which constitutes a solution of the mass-critical free-boundary Euler-Poisson system* (1.1)–(1.4)*:*

$$\bar{\rho}(t,\mathbf{x}) = \lambda_{\delta,\lambda_1}(t)^{-3}\bar{w}_\delta^3\left(\frac{|\mathbf{x}|}{\lambda_{\delta,\lambda_1}(t)}\right), \quad \bar{\mathbf{u}}(t,\mathbf{x}) = \frac{\dot{\lambda}_{\delta,\lambda_1}(t)}{\lambda_{\delta,\lambda_1}(t)}\mathbf{x}, \quad \bar{\Omega}(t) = B_{\lambda_{\delta,\lambda_1}(t)}(\mathbf{0}), \tag{1.28}$$

with $\lambda_\delta(t)$ *the solution to* (1.18)(1.21) *and* $\bar{w}_\delta$ *the normalised solution to* (1.19) *as above.*

Unless stated otherwise, we shall drop the subscript δ in the definition of the GW-solution, as this will create no confusion in the analysis.

1.3 History and results

1.3.1 History and background

The Euler equations were first described by Euler in 1757, but even in the absence of vacuum it is a highly non-trivial problem. In particular, the Euler equations are prone to singularity/shock formation even from quite regular data, see for example [6] by Christodoulou and [61] by Sideris. This made the study of existence and behaviour and global solutions non-trivial. In the presence of vacuum the problem is even trickier due to the degeneracy of the boundary and its movements. When the density is bounded away from zero, the system is strictly hyperbolic and the classical theory of hyperbolic systems for local existence applies: [45]. In the presence of the vacuum, we have singularity or discontinuity across the boundary and this no longer applies, and so even local existence is non-trivial, with various local existence theory only establish recently.

The Euler-Poisson equations with the polytropic equation of state has long been used to model stellar structure and evolution by astrophysics such as Chandrasekhar [4], Shapiro and Teukolsky [59]. Despite and because of its simplicity, it is a good approximation for certain regions of various types of stars, and so it serves as the classical model of stars. In particular, many equations of state that are of astrophysical interest behave like polytropes for low and for high pressures. One of the most important class of solutions to the Euler-Poisson equation with the polytropic equation of state are the time independent steady state solutions known as the *Lane–Emden stars* (for any $\gamma \in [1,2]$). It is the classical model of a spherically symmetric stationary steady star where pressure and gravity are in perfect balance. In 1980 astrophysicists Goldreich and Weber [17] found a special class of expanding and collapsing solutions for $\gamma = 4/3$, the mass critical index where the natural scaling preserves the mass. This is another important class of solutions as it models stellar collapse and expansion such as supernova expansion. Importantly, both the Lane–Emden stars and the Goldreich-Weber stars in the gaseous case has $w \sim \mathrm{dist}(\,\cdot\,, \Omega^c)$ near the vacuum boundary.

Therefore, given these special solutions of physical interest, it is important to have an existence theory for the Euler and Euler-Poisson equation that is compatible and includes this particular boundary behaviour – the so-called physical boundary condition. However, this is not easy due to the degeneracy near the vacuum boundary and the fact that w is not smooth across it. There are works, for example [47] by Makino, Ukai and Kawashima in the 1980s, which established an existence theory for compact solutions to the compressible Euler equations. However, their solution can only exist for a finite time, and moreover do not apply to data satisfying the physical boundary conditions since their method requires a degree of smoothness across the boundary not satisfied by the physical boundary conditions.

It was only recently that progress on local existence theory in this area has been made. In this formulation, one has a moving domain $\Omega(t)$, treating $\partial\Omega(t)$ as an unknown that evolves with the flow, prescribing the so-called *physical boundary condition* $0 < -\mathbf{n} \cdot \nabla w < \infty$ on the boundary where $\mathbf{n}$ is the outward normal of the boundary, i.e. $w \sim \mathrm{dist}(\,\cdot\,, \Omega^c)$ near the boundary. For the Euler equations in the gaseous case in this setting, Coutand and Shkoller [7], and Jang and Masmoudi [33] have recently independently proved local existence. A different proof based on Eulerian approach (rather than using Lagrangian coordinates as previous work) was later given by Ifrim and Tataru [28]. Extensions of local existence result to the Euler-Poisson equations has been done by using the fact that the gravity term is of lower order - see [19] by Gu and Lei, and [22] by Hadžić and Jang. For the liquid case, local-existence for the Euler system is established by Lindblad [37] using Nash–Moser construction; Trakhinin [63] using the theory of symmetric hyperbolic systems; and Coutand, Hole and Shkoller [8] using the vanishing viscosity method with a parabolic regularization together with some time-differentiated a priori estimates. Local existence for the liquid Euler-Poisson equation has been proven by Ginsberg, Lindblad and Luo [16]. In the relativistic setting, this has been done by Oliynyk [49] and Miao, Shahshahani and Wu [48].

Since already the Euler equation alone is prone to singularity/shocks which made global-in-time behaviour non-trivial, when coupled with an additional attractive gravitational force, we expect the Euler-Poisson to be even more prone to singularity or blow ups. Indeed, it has been shown that solutions could blow up in a finite time - [11, 46]. As a result there are not many global-in-time existence and uniqueness results for the Euler-Poisson system outside of the aforementioned special solutions. One new result in this direction is [5] a global existence result with radial symmetry in the class of weak solutions. As an aside, we should mention that in the presence of viscosity (which we do not have here), the parabolic effect takes over and various asymptotic stability results are available [42, 43].

Due to their physical significance, it is important to understand what happens in the vicinity of these special Lane–Emden and Goldreich-Weber solutions, whether they are linearly and non-linearly stable. On the problem of stability of gaseous Lane–Emden stars it is known [38, 34, 39] that the Lane–Emden stars in $\mathbb{R}^3$ are linearly unstable in the mass super-critical range when $1 < \gamma < 4/3$ and linearly stable when $4/3 \leq \gamma < 2$ (The nonradial linear stability question is treated by [34, 39] from the Lagrangian and the Eulerian perspective respectively). Recently, nonlinear instability in the range $6/5 \leq \gamma < 4/3$ has been proven in [30, 31] by Jang. Nonlinear stability in the range $4/3 < \gamma < 2$ has not been fully proven yet and remains open, but under the assumption that a global-in-time solution exists, nonlinear stability in the range $4/3 < \gamma < 2$ has been shown by variation arguments, see [53] by Rein, and [40] by Luo and Smoller. In the critical case $\gamma = 4/3$ the Lane-Emden star is nonlinearly unstable despite the conditional linear stability - in fact the family of expanding/collapsing Goldreich-Weber stars that exist for this γ can get arbitrarily close to the Lane-Emden star. Nonlinear stability of the expanding gaseous Goldreich-Weber stars against radially symmetric perturbations was proven by Hadžić and Jang [20].

1.3.2 Statement of main results and plan for the thesis

The first two results to be presented in this thesis are the generalisation of Goldreich-Weber stars' nonlinear stability under radial perturbations proven in [20] by Hadžić and Jang to nonlinear stability under non-radial perturbations. In Chapter 2 and 3 respectively, we will prove our results, the following two theorems, which are joint work with Hadžić and Jang.

Theorem 1.3.1. *The self-similarly expanding Goldreich-Weber stars are co-dimension 4 non-linearly stable under irrotational perturbations. The class of self-similarly expanding Goldreich-Weber stars is co-dimension 1 non-linearly stable under irrotational perturbations.*

Theorem 1.3.2. *The linearly expanding Goldreich-Weber stars are non-linearly stable (under general perturbations).*

The precise statement of these theorems will be stated (formulated in detailed notations) in Chapter 2 and 3 respectively. More precisely, Theorem 1.3.1 corresponds to Theorem 2.1.7 and Corollary 2.1.8, while Theorem 1.3.2 corresponds to Theorem 3.1.3. The content of Chapter 2 proving Theorem 1.3.1 is the content of our paper [23].

Since in the mass critical case $\gamma = \frac{4}{3}$, the Lane–Emden solutions are embedded in a larger family of collapsing/expanding Goldreich-Weber solutions, our results can be viewed as a definitive nonradial instability statement about the mass-critical LE-solutions.

The driving stabilisation mechanism that allows for the global existence in Theorem 2.1.7 is the expansion of the support of the background GW-star. Intuitively, this is because density must goes to zero as the support of the star expanding since overall mass of the star is conserved. When there is no vacuum boundary present, the dispersion induced by the expansion was used by Grassin [18], Serre [58], and Rozanova [57] to give examples of global-in-time solutions to the compressible Euler flows.

The GW-stars belong to a class of so-called affine motions. In the context of compressible flows the notion of an affine motion goes back to the works of Ovsiannikov [50] and Dyson [13]. In the presence of vacuum, Sideris [62] showed the existence of a finite-parameter family of compactly supported expanding affine flows, whose nonlinear stability was shown by Hadžić and Jang [21] and Shkoller and Sideris [60] for the pure Euler flows. For expanding profiles with small initial densities, but not necessarily close to the Sideris solutions, see [52]. Further results in this direction, in the nonisentropic setting and in the presence of heat convection can be found in [54, 55, 56]. A similar method works for the Euler-Poisson system and global-in-time flows were shown to exist in both the gravitational and electrostatic case [22], where the Euler part of the flow entirely dominates the gravitational/electrostatic response of the model. Another application of an expansion-induced stabilisation is the work of Parmeshwar [51] where an N-body configuration of expanding stars is shown to exist globally in-time. If damping is present in Euler flows it can drive sublinear expansion of Barenblatt-like solutions, see [44, 64, 65].

Our result concerning the self-similarly expanding GW stars in Theorem 1.3.1 has one notable difference to the above results. Our result for the linearly expanding

GW stars in Theorem 1.3.2 is in fact easier than that for self-similarly expanding GW stars in Theorem 1.3.1. The reason for that is that the linearly expanding GW stars expands at a faster rate than the self-similarly expanding GW stars, as a result the stability resulting from the dispersion due to expansion is stronger. As a result in particular, the gravitational forces in the linearly expanding case are of secondary order. The various results we mentioned above, interestingly, like this linearly expanding GW case in Theorem 1.3.2 do not depend on the attractive/repulsive nature of the force field. Our result on the self-similarly expanding GW stars in Theorem 1.3.1 are for this reason in a different dynamic regimes.

My final result is on the *liquid* Lane–Emden stars. Despite the linear stability results on the gaseous Lane–Emden stars, the question of linear stability of *liquid* Lane–Emden stars has not been established. My third result presented in the final Chapter 4 is thus on exactly this, and it is the content of my paper [36].

The linear stability of the gaseous Lane–Emden stars does not depend on its central density. However, notably, we found that in the liquid case, the stability of the Lane–Emden stars does depend on its central density.

Definition 1.3.3. *For a liquid Lane–Emden stars with density profile ρ, we say that it has* small relative central density *if $\rho(0) - 1$ is close to 0.*

The main theorem we will prove in the final Chapter 4 is the following.

Theorem 1.3.4. *Assume $d < 10$. The liquid Lane–Emden stars are, against radial perturbations,*

i. *linearly stable when $\gamma \geq 2(d-1)/d$;*
ii. *linearly stable when $\gamma < 2(d-1)/d$ for stars with small relative central density (see Definition 1.3.3);*
iii. *linearly unstable when $\gamma < 2(d-1)/d$ for stars with large central density.*

This kind of dependence on central density is seen in relativistic stars described by the Einstein–Euler equations, where it was found in [24] by Hadžić, Lin and Rein that strongly relativistic steady stars (those with very large central density) are unstable. It turns out the imposition of a liquid boundary creates extrema points on the mass-radius plot when $\gamma < 2(d-1)/d$ so that they end up similar to those in the relativistic case, where a turning point principle proven by Hadžić, Lin and Rein [24, 25] dictates that such extrema points introduces unstable modes (see diagram). The two problems share similarity in such a way that the methods of dynamical system utilised in [24] can be adapted for use in our study of liquid Lane–Emden stars - we use that to prove part iii of our theorem.

Making use of my linear (in)stability analysis concerning liquid Lane–Emden stars, very recently Hao and Miao have proven in [26] non-linear instability of liquid Lane–Emden stars in $\mathbb{R}^3$ with large central density in the regime $1 \leq \gamma < 4/3$.

1.3.3 Overview of methods and results in this thesis

In this subsection we will mainly only outline the connections, relationships, similarities and differences between the three main results of this thesis. More detail outline of methods specific to each results will be described in their own respective chapters.

Chapter 2 and 3 are on the non-linear and non-radial stability of expanding GW stars, whereas Chapter 4 is on the linear radial (in)stability of liquid LE stars.

The mechanism that allows for proving non-linear stability for GW stars is the dispersion effect coming from the expansion of the GW star. But this relies on having linear stability firstly, hence in both Chapter 2 and 4, we have to prove positivity of the linear operator $\mathbf{L}$ and L associated with the Euler-Poisson system linearised around the self-similarly expanding GW and liquid LE stars respectively.

Since for GW stars we seek to establish *non-radial* stability, this requires the analysis of a PDE operator $\mathbf{L}$ in Chapter 2, rather than just an ODE operator L for the radial (in)stability results for the liquid LE stars in Chapter 4. For this we adapt and extend the methods by Jang and Makino in [34], where they proved non-negativity for the linear PDE operator associated with the gaseous LE stars in the mass-subcritical range, to the GW star. In this thesis we have not extended this non-radial analysis to liquid LE stars — in that case there will be extra challenges arising from boundary terms coming from the discontinuity of density near the vacuum boundary — this deserved to be studied in future works.

Thus our results on liquid LE stars in Chapter 4 were just on radial (in)stability. However, the work to establish this radial (in)stability is not trivial, since we have to obtain precise estimates regarding the profile of the liquid stars in order to prove properties of the linear operator L for the liquid LE stars.

These kind of precise linear analysis is not needed for the linearly expanding GW star in Chapter 3 because there the rate of expansion is fast enough that gravitational effects essentially becomes lower order in the dynamics and which means the pressure effect automatically gave the desired positivity.

The radius of the expanding GW stars is proportional to $\lambda(t)$. For the self-similarly expanding GW star $\lambda(t) \sim t^{2/3}$, while for the linearly expanding GW star $\lambda(t) \sim t$. The faster rate of expansion for the latter case means that the problem is easier since there is a stronger dispersive effect that we can harness. Using the *high order energy method*, this dispersive effect allow us to upgrade linear stability into non-linear stability for the expanding GW stars.

For both cases we convert to a rescaled time coordinate s such that $\ln \lambda(s) \sim s$, and rescaled all variables so that they centred around the expanding GW profile. In the new coordinate, the deviation $\boldsymbol{\theta}$ away from the expanding GW star satisfies ((2.7) and (3.7))

$$\partial_s^2 \boldsymbol{\theta} \underbrace{- \frac{1}{2} \mathscr{b} \partial_s \boldsymbol{\theta}}_{V} + \delta \boldsymbol{\theta} + \mathbf{P} + \mathbf{G} = \mathbf{0} \qquad \text{(self-similarly expanding GW star)}$$

$$\underbrace{\lambda \partial_s^2 \boldsymbol{\theta} + \lambda' \partial_s \boldsymbol{\theta}}_{W} + \delta \boldsymbol{\theta} + \mathbf{P} + \mathbf{G} = \mathbf{0} \qquad \text{(linearly expanding GW star)}$$

where $\mathscr{b}$ is some negative constant, $\mathbf{P}$ is the pressure term and $\mathbf{G}$ is the gravity term.

In the first equation for the self-similarly expanding GW star, the term V comes from and represents the dispersive effect of the expansion. It gives an effect analogous to the "damping term" $\dot{x}$ in a damped harmonic oscillator $\ddot{x} + \dot{x} + x = 0$. Its stabilising behaviour allows us to close the estimate to prove non-linear stability for the self-similarly expanding GW star.

In the second equation for the linearly expanding GW star, the even stronger dispersive effect is expressed in W. Since $\lambda \sim e^{Cs}$ for some $C > 0$, for large s, the equation is approximately just $\partial_s^2 \boldsymbol{\theta} + C \partial_s \boldsymbol{\theta} = \mathbf{0}$. So heuristically we can expect $\|\partial_s \boldsymbol{\theta}\|_* \sim e^{-Cs}$ in some appropriate norm $\| \cdot \|_*$, i.e. we have great decay on the "velocity level". And again this stabilising behaviour allows us to close the estimate to prove non-linear stability for the linearly expanding GW star. However, the term $\mathbf{P}$ in this case is not negligible and is in fact very important, the reason being that it contains the highest number of space derivatives with $\mathbf{P}$ containing something like $\nabla \nabla \cdot \boldsymbol{\theta}$. To obtain control for space derivatives of $\boldsymbol{\theta}$, it is thus necessary to use the structure of $\mathbf{P}$.

Due to the good decaying effect of the "velocity level" $\|\partial_s \boldsymbol{\theta}\|_*$, control for $\|\boldsymbol{\theta}\|_*$ comes basically for free in a process analogous to the fundamental theorem of calculus. This is also the reason why the gravity term $\mathbf{G}$, being not top order in space derivatives unlike $\mathbf{P}$, is basically negligible in the dynamics in this case. We do not have this very strong decaying effect of the "velocity level" for the self-similarly expanding GW star however, so obtaining control for $\|\boldsymbol{\theta}\|_*$ is more difficult for the self-similarly expanding case. In this case we need to assume the fluid to be irrotational and use this extra structure to help control $\|\boldsymbol{\theta}\|_*$. It is not very clear whether this restriction is just technical or more fundamental, future work could probably investigate if the irrotational condition can be dropped in this case.

1.4 Notation

As Chapter 2, 3 and 4 are devoted to the self-similarly expanding GW stars, linearly expanding GW stars and liquid LE stars respectively, we will write $\bar{w}$ to denote the the self-similarly expanding GW stars, linearly expanding GW stars and liquid LE stars enthalpy profile respectively (see Definition 1.2.2, 1.2.4 and 1.2.1) in these chapters.

Since the gaseous Euler-Poisson system is degenerate near the vacuum boundary, we will need to make use of weighted Sobolev spaces. Let $L^2(B_R, w)$ denote the L^2 space on B_R weighted by a non-negative weight w. Of crucial importance in this thesis are the weighted inner products

$$\langle g, h \rangle_k := \int_{B_R} g h \bar{w}^k \mathbf{dx}, \tag{1.29}$$

$$\langle \mathbf{g}, \mathbf{h} \rangle_k := \int_{B_R} \mathbf{g} \cdot \mathbf{h} \bar{w}^k \mathbf{dx}, \tag{1.30}$$

defined for any scalar fields $g, h \in L^2(B_R, \bar{w}^k)$ and vector fields $\mathbf{g}, \mathbf{h} \in L^2(B_R, \bar{w}^k)^3$. The weighted inner product for tensor fields are defined in the same way. The associated norm is then given by

$$\|f\|_k^2 = \int_{B_R} |f(\mathbf{x})|^2 \bar{w}(\mathbf{x})^k \mathbf{dx}. \tag{1.31}$$

To capture the structure of the roughly spherical stars, we will need to use the following specially defined radial and tangential derivatives in our analysis. We define

$$X_r := x^i \partial_i = r\partial_r$$
$$\not{\partial}_i := \epsilon_{ijk} x^j \partial_k$$
$$\not{\partial}_{ij} := x^i \partial_j - x^j \partial_i$$

where ϵ_{ijk} is the alternating symbol (see Definition 1.4.1). Note that $\not{\partial}_{ij} = \epsilon_{ijk}\not{\partial}_k$. We denote

$$\begin{aligned}\operatorname{div}\boldsymbol{\theta} &:= \nabla\cdot\boldsymbol{\theta}\\ [\operatorname{curl}\boldsymbol{\theta}]^k_l &:= \partial_l\theta^k - \partial_k\theta^l\\ \operatorname{div}_{\mathscr{A}}\boldsymbol{\theta} &:= (\mathscr{A}\nabla)\cdot\boldsymbol{\theta}\\ [\operatorname{curl}_{\mathscr{A}}\boldsymbol{\theta}]^k_l &:= \mathscr{A}\partial_l\theta^k - \mathscr{A}\partial_k\theta^l\end{aligned}$$

where $\mathscr{A}\nabla := \mathscr{A}^k\partial_k$ and $\mathscr{A}\partial_i := \mathscr{A}^k_i\partial_k$.

In this thesis, especially Chapter 2 and 3, we will be using some fairly standard notations in the following that the reader will have seen elsewhere.

Definition 1.4.1 (Standard notations)**.**

i. *Greek letter superscript on derivatives are multi-index notation for derivatives. For example,* $\not{\partial}^\alpha = \not{\partial}_1^{\alpha_1}\not{\partial}_2^{\alpha_2}\not{\partial}_3^{\alpha_3}$ *where* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$*. And* $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

ii. *Roman letter indices such as* i, j, k, l, m *on derivatives and vector or tensor fields are assumed to range over* $\{1, 2, 3\}$*. However, this does not apply to* s *which we reserved to denote the rescaled time variable. Also, it does not apply when they are indices of non-vector or non-tensor objects, for example* Ψ_{lm} *and* Λ_{lm} *in Chapter 2.*

iii. *The Einstein summation convention will be used, i.e. repeated indices on derivatives and vector or tensor fields are summed over. For example,* $\partial_i\theta^i = \sum_{i=1}^3 \partial_i\theta^i$*. However, this does not apply to non-vector or non-tensor objects, for example* Ψ_{lm} *and* Λ_{lm} *in Chapter 2.*

iv. I *denotes the identity matrix,* δ_{ij} *or* δ^i_j *the Kronecker delta,* ϵ_{ijk} *the alternating symbol (Levi-Civita symbol).*

v. C *will denote generic "analyst's constant", whose exact value can change from line to line and term to term. When appearing in equalities, it can potentially denote any real constant, but when appearing in inequalities, it is generally assumed to be non-negative. We will use subscript to emphasise its dependence on certain variables, for example* C_δ *is a constant that depends on* δ.

vi. $\mathbf{e}_i$ *(*$i = 1, 2, 3$*) denotes the standard basis of* $\mathbb{R}^3$*, while* $\mathbf{e}_r$ *denotes the radial unit vector* $\mathbf{x}/|\mathbf{x}|$.

Now we will define some important special new notations that the reader probably will not have seen before.

Definition 1.4.2 (Special notations)**.**

i. *We will denote $\partial_\bullet$ as a generic derivative, so it can be any of ∂_s, ∂_i, $\not\partial$ or X_r.*

ii. *We will use $\bullet$ to denote an unspecified index, or to emphasise the vectorial/tensorial nature of non-scalar quantities. For example if A is a matrix, we can write $A^k_\bullet$.*

iii. *When the exact value/ordering of the indices is not important, we shall often write $\langle\star\rangle$ for a generic term that looks like $\star$ to avoid invoking indices. For example, $\langle C\mathscr{A}\nabla\boldsymbol{\theta}\rangle$ could represent a term like $C\mathscr{A}^i_j\partial_k\theta^l$ for some i, j, k, l and constant $C \in \mathbb{R}$.*

iv. *We will write $\mathscr{R}[\star]$ to denote terms that can be bounded by $\star$, e.g. $|\mathscr{R}[S_nE_n]| \lesssim S_nE_n$.*

v. *We will write $\mathbf{1}_\star$ to denote the usual indicator function, and write $\mathbf{1}[\star]$ to denote the Iverson bracket. For example, $\mathbf{1}_A(\mathbf{x}) = \mathbf{1}[\mathbf{x} \in A]$.*

Chapter 2

Nonradial stability of self-similarly expanding Goldreich-Weber stars

2.1 Introduction

2.1.1 Equation in self-similar coordinates

Let $(\bar{\rho}, \bar{\mathbf{u}})$ be a given self-similarly expanding GW-flow from Definition 1.2.2 with the corresponding radius $R\lambda(t)$ and the associated enthalpy $\bar{w} : [0, R] \to \mathbb{R}_+$. In order to study the stability of the flow, we follow the strategy introduced in [20, 21] and renormalise the equation by introducing a new unknown

$$\boldsymbol{\xi}(t, \mathbf{x}) = \frac{\boldsymbol{\eta}(t, \mathbf{x})}{\lambda(t)}. \tag{2.1}$$

We suitably renormalise the inverse of the Jacobian gradient and the Jacobian determinant, so that

$$\begin{aligned}
\mathscr{A} &:= (\nabla \boldsymbol{\xi})^{-1} && = \lambda A \\
\mathscr{J} &:= \det(\nabla \boldsymbol{\xi}) && = \lambda^{-3} J \\
\mathscr{a} &:= \mathscr{J} \mathscr{A} && = \lambda^{-2} a \\
\Phi &:= -\int \frac{f_0(\mathbf{z}) \mathscr{J}_0(\mathbf{z})}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} \mathrm{d}\mathbf{z} && = \lambda \psi
\end{aligned}$$

We next formulate the problem in self-similar variables. To this end we introduce the self-similar time coordinate s adapted to the expanding profile via

$$\frac{ds}{dt} = \lambda(t)^{-\frac{3}{2}}.$$

We then have the following change of coordinate formula $\partial_t = \lambda^{-3/2}\partial_s$. The condition $\ddot{\lambda}\lambda^2 = \delta$ (1.18) becomes

$$\delta = \lambda^{1/2}\partial_s(\lambda^{-3/2}\partial_s\lambda) = \frac{\partial_s^2\lambda}{\lambda} - \frac{3}{2}\frac{(\partial_s\lambda)^2}{\lambda^2} = \partial_s\left(\frac{\partial_s\lambda}{\lambda}\right) - \frac{1}{2}\frac{(\partial_s\lambda)^2}{\lambda^2} = -\frac{1}{2}\mathscr{b}^2 \tag{2.2}$$

where

$$\mathscr{b} := -\frac{\partial_s\lambda}{\lambda} = -\sqrt{2|\delta|} < 0. \tag{2.3}$$

Then the Euler-Poisson equations (1.10) becomes

$$\begin{aligned}\mathbf{0} &= \partial_t\mathbf{v} + (f_0J_0)^{-1}\partial_k(A^k(f_0J_0)^{4/3}J^{-1/3}) + A\nabla\psi \\ &= \lambda^{-3/2}\partial_s(\lambda^{-3/2}\partial_s(\lambda\boldsymbol{\xi})) + \lambda^{-2}(f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \lambda^{-2}\mathscr{A}\nabla\Phi\end{aligned}$$

Times the equation by λ^2 we get

$$\begin{aligned}\mathbf{0} &= \lambda^{1/2}\partial_s(\lambda^{-3/2}\partial_s(\lambda\boldsymbol{\xi})) + (f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \mathscr{A}\nabla\Phi \\ &= \left(\partial_s^2\boldsymbol{\xi} + \frac{1}{2}\frac{\partial_s\lambda}{\lambda}\partial_s\boldsymbol{\xi} + \left(\frac{\partial_s^2\lambda}{\lambda} - \frac{3}{2}\frac{(\partial_s\lambda)^2}{\lambda^2}\right)\boldsymbol{\xi}\right) + (f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) \\ &\quad + \mathscr{A}\nabla\Phi \\ &= \left(\partial_s^2\boldsymbol{\xi} - \frac{1}{2}\mathscr{b}\partial_s\boldsymbol{\xi} + \delta\boldsymbol{\xi}\right) + (f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \mathscr{A}\nabla\Phi\end{aligned}$$

So the Euler-Poisson equations in terms of $\boldsymbol{\xi}$ (2.1) is:

$$\partial_s^2\boldsymbol{\xi} - \frac{1}{2}\mathscr{b}\partial_s\boldsymbol{\xi} + \delta\boldsymbol{\xi} + \frac{1}{f_0J_0}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \mathscr{A}\nabla\Phi = \mathbf{0}. \tag{2.4}$$

The self-similarly expanding GW-star is a particular s-independent solution of (2.4) of the form $\boldsymbol{\xi}(\mathbf{x}) \equiv \mathbf{x}$ and $f_0 = \bar{w}^3$. Before formulating the stability problem, we must first make the use of the labelling gauge freedom and fix the choice of the initial enthalpy $(f_0J_0)^{1/3}$ for the general perturbation to be exactly identical to the background enthalpy $\bar{w}$, i.e. we set

$$(f_0J_0)^{1/3} = \bar{w} \qquad \text{on } B_R(\mathbf{0}). \tag{2.5}$$

Equation (2.5) can be re-written in the form $\rho_0 \circ \boldsymbol{\eta}_0 \det[\nabla\boldsymbol{\eta}_0] = \bar{w}^3$ on the initial domain $B_R(\mathbf{0})$. By a result of Dacorogna-Moser [10] and similarly to [20, 21] there exists a choice of an initial bijective map $\boldsymbol{\eta}_0 : B_R(\mathbf{0}) \to \Omega(\mathbf{0})$ so that (2.5) holds true. The gauge fixing condition (2.5) is necessary as it constrains the freedom to arbitrary relabel the particles at the initial time.

We now introduce the perturbation

$$\boldsymbol{\theta}(\mathbf{x}) := \boldsymbol{\xi}(\mathbf{x}) - \mathbf{x}, \tag{2.6}$$

which measures the deviation of the nonlinear flow to the background Goldreich-Weber profile.

Lemma 2.1.1 (Euler-Poisson in self-similar coordinate)**.** *The perturbation $\boldsymbol{\theta}$ defined in* (2.6) *formally solves*

$$\partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G} = \mathbf{0}, \tag{2.7}$$

where the nonlinear pressure operator $\mathbf{P}$ and the nonlinear gravity operator $\mathbf{G}$ read

$$\mathbf{P} := \bar{w}^{-3}\partial_k(\bar{w}^4(\mathscr{A}^k_\bullet \mathscr{J}^{-1/3} - I^k_\bullet)), \tag{2.8}$$

$$\mathbf{G} := \mathscr{A}\nabla\Phi - \mathscr{K}\nabla\bar{w}^3 = \mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}_\bullet\bar{w}^3) - \mathscr{K}\nabla\bar{w}^3 \tag{2.9}$$

$$= \mathscr{K}_{\boldsymbol{\xi}}((\mathscr{A} - I)\nabla\bar{w}^3 - \bar{w}^3\mathscr{A}^i_m\mathscr{A}^l_\bullet\partial_i\partial_l\theta^m) + (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K})\nabla\bar{w}^3, \tag{2.10}$$

and

$$(\mathscr{K}_{\boldsymbol{\xi}} g)(\mathbf{x}) := -\int \frac{g(\mathbf{z})}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\mathrm{d}\mathbf{z} \tag{2.11}$$

Proof. Recall that the GW-enthalpy satisfies

$$\mathbf{0} = \delta\mathbf{x} + 4\nabla\bar{w} + \nabla\mathscr{K}\bar{w}^3 \tag{2.12}$$

Using the gauge condition (2.5), the momentum equation (2.4) becomes

$$\bar{w}^3\left(\partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta}\right) + \partial_k(\bar{w}^4(\mathscr{A}^k\mathscr{J}^{-1/3} - I^k)) + \bar{w}^3(\mathscr{A}\nabla\Phi - \nabla\mathscr{K}\bar{w}^3) = \mathbf{0}.$$

Note that formally

$$(\nabla\mathscr{K}\rho)(\mathbf{x}) = -\int\nabla_{\mathbf{x}}\frac{\rho(\mathbf{z})}{|\mathbf{x} - \mathbf{z}|}\mathrm{d}\mathbf{z} = \int\nabla_{\mathbf{z}}\frac{\rho(\mathbf{z})}{|\mathbf{x} - \mathbf{z}|}\mathrm{d}\mathbf{z} = -\int\frac{\nabla\rho(\mathbf{z})}{|\mathbf{x} - \mathbf{z}|}\mathrm{d}\mathbf{z}$$
$$= (\mathscr{K}\nabla\rho)(\mathbf{x})$$

and so

$$\begin{aligned}
A\nabla\psi(\mathbf{x}) &= (\nabla\phi)(\boldsymbol{\eta}(\mathbf{x})) = (\nabla\mathcal{K}\rho)(\boldsymbol{\eta}(\mathbf{x})) = (\mathcal{K}\nabla\rho)(\boldsymbol{\eta}(\mathbf{x})) \\
&= -\int \frac{\nabla\rho(\mathbf{y})}{|\boldsymbol{\eta}(\mathbf{x})-\mathbf{y}|}\mathrm{d}\mathbf{x} = -\int \frac{A\nabla f(\mathbf{z})J(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} \\
&= -\int \frac{a\nabla(fJJ^{-1})(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} = -\int \frac{a\nabla(\bar{w}^3J^{-1})(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} \\
&= -\int \frac{\nabla\cdot(a\bar{w}^3J^{-1})(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} = -\int \frac{\nabla\cdot(A\bar{w}^3)(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} \\
&= \frac{1}{\lambda^2}(\mathcal{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathcal{A}\bar{w}^3))(\mathbf{x}),
\end{aligned}$$

where we denote $\nabla\cdot M = \partial_i M^i$ for a matrix M and recall (2.11). We then have

$$\mathcal{A}\nabla\Phi = \lambda^2 A\nabla\psi(\mathbf{x}) = \mathcal{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathcal{A}\bar{w}^3).$$

Hence, we can write the momentum equation as

$$\begin{aligned}
\mathbf{0} &= \partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathcal{b}\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta} + \bar{w}^{-3}\partial_k(\bar{w}^4(\mathcal{A}^k\mathcal{J}^{-1/3} - I^k)) + \mathcal{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathcal{A}\bar{w}^3) - \mathcal{K}\nabla\bar{w}^3 \\
&= \partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathcal{b}\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G},
\end{aligned}$$

where we have also made use of (2.12). Note that we can write

$$\begin{aligned}
\mathbf{G} &= \mathcal{K}_{\boldsymbol{\xi}}(\nabla\cdot(\mathcal{A}\bar{w}^3) - \nabla\bar{w}^3) + (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K})\nabla\bar{w}^3 \\
&= \mathcal{K}_{\boldsymbol{\xi}}((\mathcal{A}-I)\nabla\bar{w}^3 - \bar{w}^3\mathcal{A}^i_m\mathcal{A}^l_{\bullet}\partial_i\partial_l\theta^m) + (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K})\nabla\bar{w}^3.
\end{aligned}$$

□

2.1.2 Total energy and momentum

Next we will give expressions for the total momentum and energy in terms of $\boldsymbol{\theta}$. We will write the expressions in a way that separates the linear and non-linear terms of $\boldsymbol{\theta}$ clearly. To that end, we first derive the following identity.

Lemma 2.1.2. *For any $\boldsymbol{\theta}$ sufficiently smooth we have the identity*

$$\int_{B_R}\left(\bar{w}^4\nabla\cdot\boldsymbol{\theta} + \frac{1}{2}\mathcal{b}^2\bar{w}^3\boldsymbol{\theta}\cdot\mathbf{x} - \frac{1}{2}\bar{w}^3(\mathcal{K}^{(1)}_{\boldsymbol{\xi}}\bar{w}^3)\right)\mathrm{d}\mathbf{x} = 0,$$

where

$$(\mathcal{K}^{(1)}_{\boldsymbol{\xi}} g)(\mathbf{x}) := \int \frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^3} g(\mathbf{z}) \mathrm{d}\mathbf{z}, \tag{2.13}$$

and we recall (2.3).

Proof. We have

$$\begin{aligned}
\int \bar{w}(\mathbf{x})^3 (\mathcal{K}^{(1)}_{\boldsymbol{\xi}} \bar{w}^3)(\mathbf{x}) \mathrm{d}\mathbf{x} &= \int\int \bar{w}(\mathbf{x})^3 \bar{w}(\mathbf{z})^3 \frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^3} \mathrm{d}\mathbf{z}\mathrm{d}\mathbf{x} \\
&= 2\int\int \bar{w}(\mathbf{x})^3 \bar{w}(\mathbf{z})^3 \frac{(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\theta}(\mathbf{x})}{|\mathbf{x}-\mathbf{z}|^3} \mathrm{d}\mathbf{z}\mathrm{d}\mathbf{x} \\
&= 2\int \bar{w}(\mathbf{x})^3 \boldsymbol{\theta}(\mathbf{x}) \cdot \int \bar{w}(\mathbf{z})^3 \frac{\mathbf{x}-\mathbf{z}}{|\mathbf{x}-\mathbf{z}|^3} \mathrm{d}\mathbf{z}\mathrm{d}\mathbf{x} \\
&= 2\int \bar{w}^3 \boldsymbol{\theta} \cdot \nabla \mathcal{K} \bar{w}^3 \mathrm{d}\mathbf{x}
\end{aligned}$$

Also, using (2.12), we have

$$\begin{aligned}
\int \left(\bar{w}^4 \nabla\cdot\boldsymbol{\theta} - w^3\boldsymbol{\theta}\cdot\nabla\mathcal{K}\bar{w}^3\right) \mathrm{d}\mathbf{x} &= \int \left(-\boldsymbol{\theta}\cdot\nabla\bar{w}^4 - w^3\boldsymbol{\theta}\cdot\nabla\mathcal{K}\bar{w}^3\right) \mathrm{d}\mathbf{x} \\
&= \int \delta\bar{w}^3\boldsymbol{\theta}\cdot\mathbf{x}\mathrm{d}\mathbf{x} = -\frac{1}{2}\mathscr{b}^2 \int \bar{w}^3\boldsymbol{\theta}\cdot\mathbf{x}\mathrm{d}\mathbf{x}.
\end{aligned}$$

□

With this identity we can now derive the expression for the momentum and the energy in terms of $\boldsymbol{\theta}$.

Lemma 2.1.3 (Momentum and energy in self-similar coordinate)**.** *Fix a* $\delta \in [\tilde{\delta}, 0)$*. In self-similar Lagrangian coordinates introduced above, the total momentum* (1.6) *and energy* (1.7) *are respectively denoted by*

$$\begin{aligned}
\mathbf{W}_{\delta}[\boldsymbol{\theta}](s) &:= \mathbf{W}_{\delta}(s, \boldsymbol{\theta}(s), \partial_s\boldsymbol{\theta}(s)), \\
E_{\delta}[\boldsymbol{\theta}](s) &:= E_{\delta}(s, \boldsymbol{\theta}(s), \partial_s\boldsymbol{\theta}(s)),
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{W}_{\delta}(s, \boldsymbol{\theta}(s), \partial_s\boldsymbol{\theta}(s)) \\
&= \bar{\mathbf{W}} + \frac{1}{\lambda(s)^{1/2}} \int (\partial_s\boldsymbol{\theta}(s) - \mathscr{b}\boldsymbol{\theta}(s))\bar{w}^3 \mathrm{d}\mathbf{x}, \\
&E_{\delta}(s, \boldsymbol{\theta}(s), \partial_s\boldsymbol{\theta}(s))
\end{aligned}$$

$$= \bar{E} + \frac{1}{\lambda(s)} \int \left(\frac{1}{2} \bar{w}^3 \left(|\partial_s \boldsymbol{\theta}(s) - \mathscr{b}\boldsymbol{\theta}(s)|^2 - \mathscr{b}\mathbf{x} \cdot \left(2\partial_s \boldsymbol{\theta}(s) - \frac{5}{2}\mathscr{b}\boldsymbol{\theta}(s) \right) \right) \right) \mathrm{d}\mathbf{x}$$
$$+ \frac{1}{\lambda(s)} \int \left(3\bar{w}^4 \left(\mathscr{J}(s)^{-\frac{1}{3}} - 1 + \frac{1}{3} \nabla \cdot \boldsymbol{\theta}(s) \right) + \frac{1}{2} \bar{w}^3 (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K} - \mathscr{K}_{\boldsymbol{\xi}}^{(1)}) \bar{w}^3(s) \right) \mathrm{d}\mathbf{x},$$

and $\bar{\mathbf{W}} := \mathbf{W}_\delta[\mathbf{0}] = \mathbf{0}$ *and* $\bar{E} := E_\delta[\mathbf{0}] = 0$ *are respectively the momentum and energy of the GW star given by Definition 1.2.2.*

Proof. We clearly have

$$\begin{aligned}
\mathbf{W}[\rho, \mathbf{u}] &= \int f J \partial_t \boldsymbol{\eta} \, \mathrm{d}\mathbf{x} = \int f_0 J_0 \partial_t \boldsymbol{\eta} \, \mathrm{d}\mathbf{x} = \int \bar{w}^3 \lambda^{-3/2} \partial_s (\lambda(\mathbf{x} + \boldsymbol{\theta})) \mathrm{d}\mathbf{x} \\
&= \bar{\mathbf{W}} + \frac{1}{\lambda^{1/2}} \int (\partial_s \boldsymbol{\theta} - \mathscr{b}\boldsymbol{\theta}) \bar{w}^3 \mathrm{d}\mathbf{x}, \\
E[\rho, \mathbf{u}] &= \int \left(\frac{1}{2} f |\partial_t \boldsymbol{\eta}|^2 + 3 f^{\frac{4}{3}} + \frac{1}{2} f \psi \right) J \mathrm{d}\mathbf{x} \\
&= \int \left(\frac{1}{2} f_0 J_0 |\partial_t \boldsymbol{\eta}|^2 + 3 J^{-\frac{1}{3}} (f_0 J_0)^{\frac{4}{3}} + \frac{1}{2} f_0 J_0 \psi \right) \mathrm{d}\mathbf{x} \\
&= \int \left(\frac{1}{2} \bar{w}^3 |\lambda^{-3/2} \partial_s (\lambda(\mathbf{x} + \boldsymbol{\theta}))|^2 + \frac{3}{\lambda} \mathscr{J}^{-\frac{1}{3}} \bar{w}^4 + \frac{1}{2\lambda} \bar{w}^3 \Phi \right) \mathrm{d}\mathbf{x} \\
&= \bar{E} + \frac{1}{\lambda} \int \Bigg(\frac{1}{2} \bar{w}^3 (|\partial_s \boldsymbol{\theta} - \mathscr{b}\boldsymbol{\theta}|^2 - 2\mathscr{b}\mathbf{x} \cdot (\partial_s \boldsymbol{\theta} - \mathscr{b}\boldsymbol{\theta})) \\
&\qquad\qquad + 3(\mathscr{J}^{-\frac{1}{3}} - 1) \bar{w}^4 + \frac{1}{2} \bar{w}^3 (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K}) \bar{w}^3 \Bigg) \mathrm{d}\mathbf{x} \\
&= \bar{E} + \frac{1}{\lambda} \int \left(\frac{1}{2} \bar{w}^3 \left(|\partial_s \boldsymbol{\theta} - \mathscr{b}\boldsymbol{\theta}|^2 - \mathscr{b}\mathbf{x} \cdot \left(2\partial_s \boldsymbol{\theta} - \frac{5}{2} \mathscr{b}\boldsymbol{\theta} \right) \right) \right) \mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{\lambda} \int \left(3\bar{w}^4 \left(\mathscr{J}^{-\frac{1}{3}} - 1 + \frac{1}{3} \nabla \cdot \boldsymbol{\theta} \right) + \frac{1}{2} \bar{w}^3 (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K} - \mathscr{K}_{\boldsymbol{\xi}}^{(1)}) \bar{w}^3 \right) \mathrm{d}\mathbf{x}.
\end{aligned}$$

When re-writing $E[\rho, \mathbf{u}]$ above, we have used Lemma 2.1.2 and (2.12). □

Remark 2.1.4. *If instead we consider the GW solutions* $\boldsymbol{\eta}_{\mathbf{p}}$ *translated at constant velocity* $\mathbf{p}_1$ *as in Remark 1.2.3, we will get* $\bar{\mathbf{W}} = M\mathbf{p}_1$ *and* $\bar{E} = \frac{1}{2} M |\mathbf{p}_1|^2$ *instead of* $\bar{\mathbf{W}} = \mathbf{0}$ *and* $\bar{E} = 0$.

2.1.3 High-order energies and the main theorem

We now introduce high-order weighted Sobolev norm that measures the size of the deviation $\boldsymbol{\theta}$ *without* time derivatives. Recall the notation in section 1.4 in Chapter 1. Assuming that $(s, \mathbf{y}) \mapsto \boldsymbol{\theta}(s, \mathbf{y})$ is a sufficiently smooth field, for any $n \in \mathbb{N}_0$

and $s \geq 0$ we let

$$Z_n(s) := \sum_{|\beta|+b\leq n} \|X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b} + \sum_{c\leq n} \|\nabla^c \boldsymbol{\theta}\|_{3+2c} \tag{2.14}$$

$$\mathfrak{X}_n(s) := \sup_{\tau\in[0,s]} Z_n(\tau). \tag{2.15}$$

Next we define energy norms *with* time-derivatives - they will be a basis of our high-order energy method explained in Section 2.5.

$$S_n(s) := \sum_{\substack{a+|\beta|+b\leq n \\ a>0}} \left(\|\partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a \nabla X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{4+b}^2\right)$$

$$S_{n,c}(s) := \sum_{\substack{a+|\beta|+b\leq n \\ a>0 \\ |\beta|+b\leq c}} \left(\|\partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a \nabla X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{4+b}^2\right)$$

$$S_{n,c,d}(s) := \sum_{\substack{a+|\beta|+b\leq n \\ a>0 \\ |\beta|+b\leq c \\ b\leq d}} \left(\|\partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a \nabla X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{4+b}^2\right)$$

$$Q_n(s) := \sum_{\substack{a+c\leq n+1 \\ a>0}} \|\partial_s^a \nabla^c \boldsymbol{\theta}\|_{3+2c}^2$$

$$Q_{n,d}(s) := \sum_{\substack{a+c\leq n+1 \\ a>0 \\ c\leq d+1}} \|\partial_s^a \nabla^c \boldsymbol{\theta}\|_{3+2c}^2$$

Note that $S_{n,n,n} = S_n$. We will also use the convention that $S_{n,-1} = 0$ etc.

Remark 2.1.5. *The indexing above is needed to describe qualitative differences between taking the time, the angular, and the radial derivatives. We shall need this distinction to later close our estimates via a delicate induction argument in high-order spaces, which not only depends on the number of derivatives, but also on the order in which the derivatives are taken.*

We define the total instant energy via

$$E_n := S_n + Q_n. \tag{2.16}$$

We shall run the energy identity using E_n; energies S_n and Q_n will be used for high-order estimates near the vacuum boundary and near the origin respectively. In particular, the control afforded by Q_n is stronger near the origin, while S_n is stronger near the boundary. Finally we define

$$\mathcal{S}_\bullet(s) := \sup_{\tau\in[0,s]} S_\bullet(\tau) + \int_0^s S_\bullet(\tau)\mathrm{d}\tau, \tag{2.17}$$

$$\mathbb{Q}_\bullet(s) := \sup_{\tau\in[0,s]} Q_\bullet(\tau) + \int_0^s Q_\bullet(\tau)\mathrm{d}\tau, \tag{2.18}$$

$$\mathcal{E}_n(s) := \sup_{\tau\in[0,s]} E_n(\tau) + \int_0^s E_n(\tau)\mathrm{d}\tau, \tag{2.19}$$

where $\bullet$ stands for indices of the form n; n, c; n, c, d in (2.17), and of the form n; n, c in (2.18). The norms (2.17)–(2.19) will play the role of the "left hand side" in the high-order energy identities.

Remark 2.1.6. *We emphasise that the higher order energies E_n we defined (always with a subscript $n \in \mathbb{N}_0$) are different from the total conserved energy E (and E_δ) defined in* (1.7)*. Where no confusion arises, we will refer to both as "energy".*

In this chapter, we make the following a priori assumption:

A priori assumption: $E_\bullet, Z_\bullet \leq \epsilon$ where $\epsilon > 0$ is some small constant. (2.20)

We now state our main theorem.

Theorem 2.1.7 (Nonlinear stability of GW stars)**.** *Let $n \geq 21$. There exists $\tilde{\delta} \leq \delta^* < 0$ such that for any $\delta \in (\delta^*, 0)$ the associated GW expanding star from Definition 1.2.2 is codimension-4 nonlinearly stable in the class of irrotational perturbations. More precisely, there exists an $\epsilon_0 > 0$ such that for any initial data $(\boldsymbol{\theta}(0), \partial_s\boldsymbol{\theta}(0))$ satisfying*

$$E_n(0) + Z_n(0)^2 \leq \epsilon_0 \tag{2.21}$$

$$\mathbf{W}_\delta(0, \boldsymbol{\theta}(0), \partial_s\boldsymbol{\theta}(0)) = \mathbf{W}_\delta[\mathbf{0}] =: \bar{\mathbf{W}} = \mathbf{0} \tag{2.22}$$

$$E_\delta(0, \boldsymbol{\theta}(0), \partial_s\boldsymbol{\theta}(0)) = E_\delta[\mathbf{0}] =: \bar{E} = 0 \tag{2.23}$$

$$\mathrm{curl}_A\, \partial_t\boldsymbol{\eta}(0) = \mathbf{0}, \tag{2.24}$$

the associated solution $s \mapsto (\boldsymbol{\theta}(s, \cdot), \partial_s\boldsymbol{\theta}(s, \cdot))$ to (2.7) *exists for all $s \geq 0$ and is unique in the class of all data with finite norm $E_n + Z_n^2$. Moreover, there exists a constant $C > 0$ such that*

$$E_n(s) + Z_n(s)^2 \leq C\epsilon_0 \qquad \textit{for all} \qquad s \geq 0,$$

and $E_n(s)$ decays exponentially fast in s.

Note that condition (2.24) is the Lagrangian statement that the fluid velocity is irrotational. The momentum and energy constraints (2.22)–(2.23) define the codimension-4 "manifold" of initial data.

Heuristically speaking, since momentum and energy are conserved quantities for the Euler-Poisson system, it is necessary that our perturbation does not alter the momentum and energy if our background solution were to be the right asymptotic-in-time "limit". Indeed, the momentum and energy constraints (2.22)–(2.23) are necessary in our stability analysis due to the presence of growing modes of the linearised operator in self-similar coordinates induced by the conservation of the energy and momentum. However if our perturbation does alter the momentum $\mathbf{W}$ and energy E away from $\mathbf{0}$ and 0 such that $E = \frac{1}{2}|\mathbf{W}|^2/M$, our proof can be easily adapted to show that it still leads to global existence with the solution staying close to a GW star for all time, but one translated at constant velocity $\mathbf{p}_1$ with $\mathbf{W} = M\mathbf{p}_1$ and $E = \frac{1}{2}M|\mathbf{p}_1|^2$ as described in Remarks 1.2.3 and 2.1.4. In this sense the "manifold" of GW-solutions is codimension-1 nonlinearly stable in the class of irrotational perturbations, even though each individual GW-star is only codimension-4 stable. In particular, given any initial data $(\rho_0, \mathbf{u}_0)$ such that $E[\rho_0, \mathbf{u}_0] = \frac{1}{2}|\mathbf{W}[\rho_0, \mathbf{u}_0]|^2/M[\rho_0, \mathbf{u}_0]$, we can change our frame of reference and subtract a constant velocity of $\mathbf{p}_1 = \mathbf{W}[\rho_0, \mathbf{u}_0]|/M[\rho_0, \mathbf{u}_0]$ from $\mathbf{u}_0$ to obtain

$$
\begin{aligned}
\mathbf{W}[\rho_0, \mathbf{u}_0 - \mathbf{p}_1] &= \mathbf{W}[\rho_0, \mathbf{u}_0] - M[\rho_0, \mathbf{u}_0]\mathbf{p}_1 = \mathbf{0} \\
E[\rho_0, \mathbf{u}_0 - \mathbf{p}_1] &= E[\rho_0, \mathbf{u}_0] - \int_{\mathbb{R}^3} \rho_0 \mathbf{u}_0 \cdot \mathbf{p}_1 d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^3} \rho_0 |\mathbf{p}_1|^2 d\mathbf{x} \\
&= E[\rho_0, \mathbf{u}_0] - \mathbf{p}_1 \cdot \mathbf{W}[\rho_0, \mathbf{u}_0] + \frac{1}{2}|\mathbf{p}_1|^2 M[\rho_0, \mathbf{u}_0] = 0
\end{aligned}
$$

So in this new frame of reference, the constraints (2.22) and (2.23) are satisfied.

To formalise this, note that for any $\mathbf{p}_1 \in \mathbb{R}^3$, $\boldsymbol{\theta} = \mathbf{p}_1 t \lambda(t)^{-1}$ is a global-in-time solution to (2.7) which corresponds to a Lagrangian description of a GW-star translated by a constant velocity. Then, as a corollary of Theorem 2.1.7 we have the following result.

Corollary 2.1.8. *Let $n \geq 21$. There exists $\tilde{\delta} \leq \delta^* < 0$ such that for any $\delta \in (\delta^*, 0)$, the "manifold" of GW-stars $(\bar{\rho}_{\mathbf{p}}, \bar{\mathbf{u}}_{\mathbf{p}})$, $\mathbf{p} = \mathbf{p}_1 t$, $\mathbf{p}_1 \in \mathbb{R}^3$ (from Remark 1.2.3) is codimension-1 nonlinearly stable in the class of irrotational perturbations. More precisely, for given any initial data $(\tilde{\boldsymbol{\theta}}(0), \partial_s \tilde{\boldsymbol{\theta}}(0))$ define*

$$\boldsymbol{\theta}_0 = \tilde{\boldsymbol{\theta}}(0)$$

$$(\partial_s\boldsymbol{\theta})_0 = \partial_s\tilde{\boldsymbol{\theta}}(0) - \mathbf{p}_1,$$

where

$$\mathbf{p}_1 = \frac{W_\delta(0,\tilde{\boldsymbol{\theta}}(0),\partial_s\tilde{\boldsymbol{\theta}}(0))}{M[\bar{w}^3]}.$$

Then, there exists an $\epsilon_0 > 0$ *such that for any initial data* $(\tilde{\boldsymbol{\theta}}(0),\partial_s\tilde{\boldsymbol{\theta}}(0))$ *such that*

$$\begin{aligned}(E_n + Z_n^2)[\boldsymbol{\theta}_0,(\partial_s\boldsymbol{\theta})_0] &\le \epsilon_0\\ E_\delta(0,\tilde{\boldsymbol{\theta}}(0),\partial_s\tilde{\boldsymbol{\theta}}(0)) &= \frac{1}{2}|\mathbf{W}_\delta(0,\tilde{\boldsymbol{\theta}}(0),\partial_s\tilde{\boldsymbol{\theta}}(0))|^2/M[\bar{w}^3]\\ \operatorname{curl}_A \partial_t\boldsymbol{\eta}(0) &= \mathbf{0}\end{aligned}$$

the associated solution $s \mapsto (\tilde{\boldsymbol{\theta}}(s,\,\cdot\,),\partial_s\tilde{\boldsymbol{\theta}}(s,\,\cdot\,))$ *to* (2.7) *exists for all* $s \ge 0$ *and is unique in the class of all data with finite norm* $(E_n + Z_n^2)[\tilde{\boldsymbol{\theta}}]$. *Moreover, there exists a constant* $C > 0$ *such that*

$$(E_n + Z_n^2)[\boldsymbol{\theta}](s) \le C\epsilon_0 \qquad \textit{for all} \qquad s \ge 0,$$

where $\boldsymbol{\theta} = \tilde{\boldsymbol{\theta}} - \mathbf{p}_1 t\lambda(t)^{-1}$, *and* $E_n[\boldsymbol{\theta}](s)$ *decays exponentially fast in* s.

Remark 2.1.9. *Our goal is not to optimise the number* n *of derivatives in our spaces. As usual, the size of* n *is conditioned by the Hardy-Sobolev type embeddings, which allow us to bound the* L^∞*-norms of contributions with less than* $\lfloor\frac{n}{2}\rfloor$ *derivatives by* $\bar{w}^k$*-weighted Sobolev norms.*

Remark 2.1.10. *The subclass of expanding GW-stars with non-zero total energy (in the frame of reference of* $\mathbf{0}$ *momentum) consists of stars that expand at a linear rate in time, i.e.* not *at the self-similar rate considered above. This problem leads to a stronger damping effect which allows the "Euler part" of the flow to dominate the dynamics. The stability of such GW-stars is the content of Chapter 3.*

Local-in-time well-posedness. The presence of vacuum is known to pose challenges in the well-posedness theory for compressible fluid flows. To develop a satisfactory local existence and uniqueness theory, one needs to impose an additional assumption on the initial data - the so-called physical vacuum condition (1.1.1). In the works of Jang and Masmoudi [33] and independently Coutand and Shkoller [7] the local well-posedness for the compressible Euler equations was shown in the Lagrangian coordinates (for a more recent treatment in Eulerian coordinates see [28]). From the point of view of regularity theory, gravity represents a lower order term, so the techniques from [33, 7] can be adapted to obtain a local-in-time well-posedness result for the free boundary EP-system [31, 19, 41, 22]. In particular, a simple adaptation of the methods in [33, 22] yields the following local well-posedness result in the weighted high-order energy space $E_n + Z_n^2$.

Theorem 2.1.11 (Local well-posedness)**.** *Let* $n \geq 21$. *Then for any given initial data* $(\boldsymbol{\theta}(0), \partial_s \boldsymbol{\theta}(0))$ *such that* $E_n(0) + Z_n(0)^2 < \infty$, *there exist some* $T > 0$ *and a unique solution* $(\boldsymbol{\theta}, \partial_s \boldsymbol{\theta}) : [0, T] \times B_R \to \mathbb{R}^3 \times \mathbb{R}^3$ *to* (2.7) *such that* $E_n(s) + Z_n(s)^2 \leq 2(E_n(0) + Z_n(0)^2)$ *for all* $s \in [0, T]$.

Theorem 2.1.11 is a starting point for the continuity argument that will culminate in the proof of Theorem 2.1.7.

2.1.4 Proof strategy

The basic idea behind the global existence in Theorem 2.1.7 is the presence of the damping term $-\frac{1}{2}\mathscr{b}\partial_s \boldsymbol{\theta}$ in (2.7), which clearly suggests a stabilising mechanism in the problem. Such a term appears as a direct consequence of the expanding character of the underlying GW-motion (and it would be of the opposite sign if we were linearising about a collapsing GW-star). This stabilisation effect was first exhibited in [20] where the purely radial version of Theorem 2.1.7 was established.

Since the problem features the vacuum free boundary satisfying the physical vacuum condition (1.1.1), we use weighted high-order energy spaces introduced by Jang and Masmoudi [33]. The key idea to overcome a possible loss of derivatives is to introduce increasing powers of $\bar{w}$ into the function spaces, as we increase the number of radial derivatives, but not the tangential ones. In particular, the proof of the main result is based on a high-order energy method which necessitates commuting the equation (2.7) with operators of the form $\partial_s^a X_r^n \not\partial^\beta$. To understand the energy contribution from the combined pressure and gravity term $\mathbf{P} + \mathbf{G}$ (see (2.8)–(2.9)), we must linearise (2.7). As shown in Lemma 2.2.1, this linearisation reads

$$\partial_s^2 \boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s \boldsymbol{\theta} + \mathbf{L}\boldsymbol{\theta} = \mathbf{0}, \tag{2.25}$$

where the linearised operator $\mathbf{L}$ takes the form

$$\mathbf{L}\boldsymbol{\theta} := -\frac{4}{3}\nabla(\bar{w}^{-2}\nabla \cdot (\bar{w}^3 \boldsymbol{\theta})) - \nabla \mathscr{K} \nabla \cdot (\bar{w}^3 \boldsymbol{\theta}), \tag{2.26}$$

where we recall (2.3) and (1.13). The fundamental challenge with respect to the radial result [20] is to show the coercivity of the operator $\mathbf{L}$ in suitably weighted spaces, dictated by our local-in-time well-posedness theory.

The difficulty in proving a useful coercivity bound for the operator $\mathbf{L}$ lies in the antagonism between the nonlocal nature of the gravitational interaction described by $\mathbf{G}$ in (2.9), and the Lagrangian perspective, which is naturally imposed on us by the problem. The operator $\mathbf{L}$ has a nontrivial unstable space, spanned by the eigenvectors $\mathbf{x}$ and the standard basis $\mathbf{e}_i$, $i = 1, 2, 3$. The 4-dimensional nature of the unstable space is a reflection of the energy and momentum conservation laws, which in self-similar variables induce formally unstable modes.

Nonradial linearised analysis around the Lane-Emden stars ($\delta = \lambda_1 = 0$) is given in [34] where the non-negativity of the associated quadratic form is shown using the expansion in spherical harmonics. In this work we work in a similar spirit, but our linear analysis around the GW-stars improves upon [34] considerably, as we show strict quantitative coercivity bound

$$\langle \mathbf{L}\boldsymbol{\theta}, \boldsymbol{\theta}\rangle_3 \gtrsim \int_{B_R} \bar{w}^{-2}|\Delta\Psi|^2 \mathbf{dx} + \int_{\mathbb{R}^3} |\nabla\Psi|^2 \mathbf{dx}, \tag{2.27}$$

under the crucial orthogonality conditions

$$\langle \boldsymbol{\theta}, \mathbf{x}\rangle_3 = 0 = \langle \boldsymbol{\theta}, \mathbf{e}_i\rangle_3, \qquad i = 1, 2, 3, \tag{2.28}$$

where $\Delta\Psi = \operatorname{div}(\bar{w}^3\boldsymbol{\theta})$. This is the central estimate of Section 2.2 (see Theorem 2.2.5) and it relies on a careful decomposition in spherical harmonics. It is non-trivial as it requires a careful use of the above orthogonality conditions to obtain quantitative lower bounds for the 0-th and the 1-st order spherical harmonics. In the former case, the problem essentially reduces to the radial coercivity bound from [20], while the analysis of the projection of $\mathbf{L}$ onto 1-st order spherical harmonics requires a careful use of Sturm-Liouville theory, see Lemma 2.2.10, a related argument was used in [34].

One of the main challenges is that the quantity $\int_{B_R} \bar{w}^{-2}|\Delta\Psi|^2 \mathbf{dx} + \int_{\mathbb{R}^3} |\nabla\Psi|^2 \mathbf{dx}$ on the right-hand side of (2.27) a priori does not appear useful for the energy estimates as we need to control the norms $\|\boldsymbol{\theta}\|_3^2 + \|\nabla\boldsymbol{\theta}\|_4^2$, which are localised to the set B_R by definition, see (1.31). An intermediate step towards a resolution of this issue is to relate the general estimate (2.27) (which holds for any sufficiently smooth map $\boldsymbol{\theta}$), to the nonlinear dynamics. In Section 2.3, by linearising the nonlinear energy-momentum constraints

$$E[\rho, \mathbf{u}] = \bar{E}, \qquad \mathbf{W}[\rho, \mathbf{u}] = \bar{\mathbf{W}}, \tag{2.29}$$

we obtain effective ODEs (modulo lower order nonlinear terms) that allow to dynamically control the inner products $\langle \boldsymbol{\theta}, \mathrm{x}\rangle_3$ and $\langle \boldsymbol{\theta}, \mathrm{e}_i\rangle_3$. With this in hand we prove in Proposition 2.3.6 a high-order differentiated version of the bound (2.27) for the solutions of (2.25) satisfying the constraints (2.29):

$$\int_{B_R} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \partial_s^a \not{\partial}^\beta \boldsymbol{\theta})|^2 \mathrm{d}\mathbf{x} + \|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3^2 \lesssim \langle \mathbf{L} \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 + \|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3^2 + \text{l.o.t.} \tag{2.30}$$

The final and crucial step toward useful lower bounds is to exploit the irrotationality assumption $\nabla \times \mathbf{u} = 0$ to obtain a dynamic control over $\|\boldsymbol{\theta}\|_3^2 + \|\nabla \boldsymbol{\theta}\|_4^2$. Looking at (2.30), this necessitates a careful examination of the $\bar{w}$-weighted divergence appearing on the left-hand side. It is clear that any vectorfield such that $\nabla \cdot (\bar{w}^3 \boldsymbol{\theta}) = 0$ formally belongs to the kernel of $\mathbf{L}$ and therefore, to obtain strict coercivity, we must mod out this infinite-dimensional kernel. The orthogonal complement with respect to the $\langle \cdot, \cdot\rangle_3$-inner product consists precisely of the gradients, so the first key observation is the content of Lemma 2.4.1, which roughly states that $\partial_s^a \boldsymbol{\theta}$ is a gradient modulo "good" terms for $a \geq 1$, assuming $\nabla \times \mathbf{u} = 0$. Here, in simplest possible terms, the issue is that the irrotationality in Lagrangian variables creates error terms that a priori seem problematic, but luckily all such terms can be absorbed into a pure gradient. The second key ingredient is Lemma 2.4.6, which is an exact identity relating the norm of the weighted divergence of $\boldsymbol{\theta}$ to the weighted norms of the derivative of $\boldsymbol{\theta}$. This can be viewed as a form of "elliptic regularity". Finally, we use these ingredients in the central statement of Section 2.4 - Proposition 2.4.7 - to show that natural energy norms obtained via integration-by-parts from (2.25) control the weighted norms of the pure time derivatives of $\boldsymbol{\theta}$. In Proposition 2.4.9 we treat also the angular derivatives in our operators, and the same statement as in the previous proposition holds, modulo the presence of a linear (and therefore not small) contribution, which fortunately involves one angular derivative less. This decoupling structure enables us to use a careful inductive procedure to eventually close the nonlinear estimates.

The nonlinear arguments are presented in Sections 2.5 and 2.6. The global nonlinear stability will follow from the bound

$$\mathscr{E}_n \lesssim \mathscr{E}_n(0) + (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2} \mathscr{E}_n \tag{2.31}$$

in the regime where $E_n + Z_n^2$ is sufficiently small. To prove such a bound, we commute (2.7) with high-order derivatives and while the discussion above refers to the extraction of coercive bounds for the linear part of the operator, we are still left with the nonlinear estimates. Propositions 2.5.5, 2.5.6, and 2.5.12 show that the deviation of the pressure term $\mathbf{P}$ and the gravity term $\mathbf{G}$ from its linearisation, can be controlled by the good trilinear error $(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n$ modulo some terms that scale like the linear norms, but always decouple at the top order of differentiation, so that they involve, for example, "one spatial derivative less and one time derivative more". This decoupling is crucial for the closure of the estimates, and the key effective reduction to the linear problem is formulated in Theorem 2.5.14. This feature of the problem suggests that we can show (2.31) inductively by taking derivatives in the right order. Key energy bounds for the nonlinear contributions from the pressure and the gravity are presented in Sections 2.6.1 and 2.6.2 respectively. The final continuity argument and the exponential decay based on (2.31) is presented in Section 2.6.3.

2.2 Linearisation and coercivity

2.2.1 The linear and non-linear part of Euler-Poisson system

The proof of Theorem 2.1.7 crucially relies on good coercive properties of the linearisation around the background GW-star. In the next lemma we formally derive the linearised Euler-Poisson system.

Lemma 2.2.1 (Linearised Euler-Poisson)**.** *The formal linearisation of (2.7) reads*

$$\partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s\boldsymbol{\theta} + \mathbf{L}\boldsymbol{\theta} = \mathbf{0} \tag{2.32}$$

where

$$\mathbf{L}\boldsymbol{\theta} := -\frac{4}{3}\nabla(\bar{w}^{-2}\nabla\cdot(\bar{w}^3\boldsymbol{\theta})) - \nabla\mathscr{K}\nabla\cdot(\bar{w}^3\boldsymbol{\theta}) \tag{2.33}$$

and we recall (2.3)*. Moreover, the formal linearisation of the gravitational contribution* $\mathbf{G}$ *(2.9) is given by the operator*

$$\mathbf{G}_L\boldsymbol{\theta} := \boldsymbol{\theta}\cdot\nabla\nabla\mathscr{K}\bar{w}^3 - \nabla\mathscr{K}\nabla\cdot(\bar{w}^3\boldsymbol{\theta}) = \mathscr{K}^{(1)}_{\boldsymbol{\xi}}\nabla\bar{w}^3 - \mathscr{K}\partial_i(\bar{w}^3\nabla\theta^i), \tag{2.34}$$

where we recall (2.11) *and* (2.13)*.*

Proof. Since $\nabla\boldsymbol{\xi} = I + \nabla\boldsymbol{\theta}$, to first order (in $\boldsymbol{\theta}$) we have $\mathscr{A} = I - \nabla\boldsymbol{\theta}$ and

$\mathscr{J} = 1 + \nabla \cdot \boldsymbol{\theta}$. So to first order we have

$$\begin{aligned}\mathscr{A}\mathscr{J}^{-1/3} &= (I - \nabla\boldsymbol{\theta})(1 + \nabla \cdot \boldsymbol{\theta})^{-1/3} = (I - \nabla\boldsymbol{\theta}) \left(1 - \frac{1}{3}\nabla \cdot \boldsymbol{\theta}\right) \\ &= \left(1 - \frac{1}{3}\nabla \cdot \boldsymbol{\theta}\right) I - \nabla\boldsymbol{\theta}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\bar{w}^3}\partial_k(\bar{w}^4(\mathscr{A}^k\mathscr{J}^{-1/3} - I^k)) &= -\frac{1}{3\bar{w}^3}\nabla(\bar{w}^4\nabla \cdot \boldsymbol{\theta}) - \frac{1}{\bar{w}^3}\partial_k(\bar{w}^4\nabla\theta^k) \\ &= -\frac{4}{3}\nabla(\bar{w}\nabla \cdot \boldsymbol{\theta}) - 4\nabla(\boldsymbol{\theta} \cdot \nabla\bar{w}) + 4\boldsymbol{\theta} \cdot \nabla\nabla\bar{w} \\ &= -\frac{4}{3}\nabla(\bar{w}^{-2}\nabla \cdot (\bar{w}^3\boldsymbol{\theta})) + 4\boldsymbol{\theta} \cdot \nabla\nabla\bar{w} \\ &= -\frac{4}{3}\nabla(\bar{w}^{-2}\nabla \cdot (\bar{w}^3\boldsymbol{\theta})) - \boldsymbol{\theta} \cdot \nabla(\delta\mathbf{x} + \nabla\mathscr{K}\bar{w}^3) \\ &= -\frac{4}{3}\nabla(\bar{w}^{-2}\nabla \cdot (\bar{w}^3\boldsymbol{\theta})) - \delta\boldsymbol{\theta} - \boldsymbol{\theta} \cdot \nabla\nabla\mathscr{K}\bar{w}^3\end{aligned}$$

Since

$$\begin{aligned}|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|^2 &= |\mathbf{x} - \mathbf{z} + \boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2 \\ &= |\mathbf{x} - \mathbf{z}|^2 + 2(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})) + |\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2,\end{aligned}$$

to first order we have

$$\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} = \frac{1}{|\mathbf{x} - \mathbf{z}|}\left(1 - \frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^2}\right).$$

So to first order we have

$$\begin{aligned}\Phi(\mathbf{x}) &= -\int \frac{\bar{w}(\mathbf{z})^3}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\mathrm{d}\mathbf{z} \\ &= -\int \frac{\bar{w}(\mathbf{z})^3}{|\mathbf{x} - \mathbf{z}|}\mathrm{d}\mathbf{z} + \int \frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^3}\bar{w}(\mathbf{z})^3\mathrm{d}\mathbf{z} \\ &= (\mathscr{K}\bar{w}^3)(\mathbf{x}) + \int \left(-\boldsymbol{\theta}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\frac{1}{|\mathbf{x} - \mathbf{z}|} - \boldsymbol{\theta}(\mathbf{z}) \cdot \nabla_{\mathbf{z}}\frac{1}{|\mathbf{x} - \mathbf{z}|}\right)\bar{w}(\mathbf{z})^3\mathrm{d}\mathbf{z} \\ &= (\mathscr{K}\bar{w}^3)(\mathbf{x}) + \boldsymbol{\theta} \cdot \nabla(\mathscr{K}\bar{w}^3)(\mathbf{x}) - (\mathscr{K}\nabla \cdot (\bar{w}^3\boldsymbol{\theta}))(\mathbf{x})\end{aligned}$$

and

$$\begin{aligned}\mathscr{A}\nabla\Phi &= (I^i - \nabla\theta^i)\partial_i(\mathscr{K}\bar{w}^3 + \boldsymbol{\theta} \cdot \nabla\mathscr{K}\bar{w}^3 - \mathscr{K}\nabla \cdot (\bar{w}^3\boldsymbol{\theta})) \\ &= \nabla\mathscr{K}\bar{w}^3 - (\nabla\theta^i)\partial_i\mathscr{K}\bar{w}^3 + \nabla(\boldsymbol{\theta} \cdot \nabla\mathscr{K}\bar{w}^3) - \nabla\mathscr{K}\nabla \cdot (\bar{w}^3\boldsymbol{\theta})\end{aligned}$$

$$
\begin{aligned}
&= \nabla \mathcal{K} \bar{w}^3 + \boldsymbol{\theta} \cdot \nabla \nabla \mathcal{K} \bar{w}^3 - \nabla \mathcal{K} \nabla \cdot (\bar{w}^3 \boldsymbol{\theta}) \\
&= \nabla \mathcal{K} \bar{w}^3 + \mathbf{G}_L \boldsymbol{\theta},
\end{aligned}
$$

where we have used (2.34) in the last line. Therefore the linearisation of the momentum equation (2.7) takes the form (2.32). Note that

$$
\begin{aligned}
\mathbf{G}_L \boldsymbol{\theta} &= \boldsymbol{\theta} \cdot \nabla \mathcal{K} \nabla \bar{w}^3 - \mathcal{K} \partial_i (\nabla \bar{w}^3 \theta^i) - \mathcal{K} \partial_i (\bar{w}^3 \nabla \theta^i) \\
&= \int \frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^3} \nabla \bar{w}(\mathbf{z})^3 \mathrm{d}\mathbf{z} - \mathcal{K} \partial_i (\bar{w}^3 \nabla \theta^i) \\
&= \mathcal{K}_{\boldsymbol{\xi}}^{(1)} \nabla \bar{w}^3 - \mathcal{K} \partial_i (\bar{w}^3 \nabla \theta^i),
\end{aligned}
$$

which completes the proof of the lemma. □

Finally, it will be important to keep track of the precise structure of the nonlinear correction $\mathbf{G} - \mathbf{G}_L \boldsymbol{\theta}$, which is given in the next lemma.

Lemma 2.2.2 (Non-linear part of gravity term)**.** *We have*

$$
\begin{aligned}
\mathbf{G} - \mathbf{G}_L \boldsymbol{\theta} = \mathcal{K}_{\boldsymbol{\xi}} (\mathcal{A}^i_l (\partial_k \theta^l)(\nabla \theta^k) \partial_i \bar{w}^3 - \bar{w}^3 (\mathcal{A}^i_m \mathcal{A}^l_\bullet - I^i_m I^l_\bullet) \partial_i \partial_l \theta^m) \\
- (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K}) \partial_i (\bar{w}^3 \nabla \theta^i) + (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K} - \mathcal{K}_{\boldsymbol{\xi}}^{(1)}) \nabla \bar{w}^3 \qquad (2.35)
\end{aligned}
$$

Proof. Since $\mathcal{A} = (\nabla \boldsymbol{\xi})^{-1}$, we have

$$
I^i_j = \mathcal{A}^i_k \partial_j \xi^k = \mathcal{A}^i_k (I^k_j + \partial_j \theta^k).
$$

Therefore $\mathcal{A}^i_j - I^i_j = -\mathcal{A}^i_k \partial_j \theta^k$. We have

$$
\begin{aligned}
\mathbf{G} - \mathbf{G}_L \boldsymbol{\theta} &= \mathcal{K}_{\boldsymbol{\xi}} ((\mathcal{A} - I) \nabla \bar{w}^3 - \bar{w}^3 \mathcal{A}^i_m \mathcal{A}^l_\bullet \partial_i \partial_l \theta^m) + (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K} - \mathcal{K}_{\boldsymbol{\xi}}^{(1)}) \nabla \bar{w}^3 \\
&\quad + \mathcal{K} \partial_i (\bar{w}^3 \nabla \theta^i) \\
&= \mathcal{K}_{\boldsymbol{\xi}} ((\nabla \theta^i - \mathcal{A}^i_k \nabla \theta^k) \partial_i \bar{w}^3 - \bar{w}^3 (\mathcal{A}^i_m \mathcal{A}^l_\bullet - I^i_m I^l_\bullet) \partial_i \partial_l \theta^m) \\
&\quad + (\mathcal{K} - \mathcal{K}_{\boldsymbol{\xi}}) \partial_i (\bar{w}^3 \nabla \theta^i) + (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K} - \mathcal{K}_{\boldsymbol{\xi}}^{(1)}) \nabla \bar{w}^3 \\
&= \mathcal{K}_{\boldsymbol{\xi}} (\mathcal{A}^i_l (\partial_k \theta^l)(\nabla \theta^k) \partial_i \bar{w}^3 - \bar{w}^3 (\mathcal{A}^i_m \mathcal{A}^l_\bullet - I^i_m I^l_\bullet) \partial_i \partial_l \theta^m) \\
&\quad - (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K}) \partial_i (\bar{w}^3 \nabla \theta^i) + (\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K} - \mathcal{K}_{\boldsymbol{\xi}}^{(1)}) \nabla \bar{w}^3.
\end{aligned}
$$

□

We next derive helpful identities for the operators $\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K}$ and $\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K} - \mathcal{K}_{\boldsymbol{\xi}}^{(1)}$

appearing on the right-hand side of (2.35). We first note that

$$(\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K})g(\mathbf{x}) = -\int_{\mathbb{R}^3} K_1(\mathbf{x},\mathbf{z})g(\mathbf{z})\mathrm{d}\mathbf{z}, \tag{2.36}$$

$$(\mathcal{K}_{\boldsymbol{\xi}} - \mathcal{K} - \mathcal{K}_{\boldsymbol{\xi}}^{(1)})g(\mathbf{x}) = -\int_{\mathbb{R}^3} K_2(\mathbf{x},\mathbf{z})g(\mathbf{z})\mathrm{d}\mathbf{z}, \tag{2.37}$$

where

$$K_1(\mathbf{x},\mathbf{z}) := \frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} - \frac{1}{|\mathbf{x}-\mathbf{z}|}, \tag{2.38}$$

$$K_2(\mathbf{x},\mathbf{z}) := \frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} - \frac{1}{|\mathbf{x}-\mathbf{z}|} + \frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^3}. \tag{2.39}$$

In the following lemma, we write K_1 and K_2 explicitly in terms of $\boldsymbol{\theta}$, which will play a role in our energy estimates. In particular, we see that $\boldsymbol{\theta}$ appears at least linearly in K_1, and at least quadratically in K_2.

Lemma 2.2.3. *We have*

$$\begin{aligned} K_2(\mathbf{x},\mathbf{z}) &= K_1(\mathbf{x},\mathbf{z}) + \frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^3} \\ &= -\frac{1}{2}\frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^3} && (2.40) \\ &\quad + \frac{3}{4|\mathbf{x}-\mathbf{z}|}\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right)^2 \\ &\qquad \varpi_{\frac{1}{2}}\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right), && (2.41) \end{aligned}$$

where

$$\varpi_q(y) := \int_0^1 \frac{1-z}{(1+yz)^{q+2}}\mathrm{d}z, \qquad y > -1, \quad q \in \mathbb{R}. \tag{2.42}$$

Proof. Let $q \in \mathbb{R} \setminus \{-1, 0\}$. Then for $y > -1$ and $y \neq 0$

$$\begin{aligned} \int_0^1 \frac{1-z}{(1+yz)^{q+2}}\mathrm{d}z &= -\frac{1}{(q+1)y}\left[\frac{1-z}{(1+yz)^{q+1}}\right]_0^1 - \frac{1}{(q+1)y}\int_0^1 \frac{1}{(1+yz)^{q+1}}\mathrm{d}z \\ &= \frac{1}{(q+1)y}\left(1 + \frac{1}{qy}\left[\frac{1}{(1+yz)^q}\right]_0^1\right) \\ &= \frac{1}{q(q+1)y^2}\left(-1 + qy + \frac{1}{(1+y)^q}\right) \end{aligned}$$

and thus

$$\frac{1}{(1+y)^q} = 1 - qy + q(q+1)y^2\varpi_q(y), \qquad y > -1 \tag{2.43}$$

where we note that (2.43) trivially holds for $y = 0$ and $q = -1, 0$. Since

$$\begin{aligned}|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|^2 &= |\mathbf{x} - \mathbf{z} + \boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2 \\ &= |\mathbf{x} - \mathbf{z}|^2 + 2(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})) + |\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2,\end{aligned}$$

we have

$$\frac{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^2} = 1 + 2\frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^2}.$$

Hence by applying (2.43) with $y = 2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}$ and $q = \frac{1}{2}$, we see that

$$\begin{aligned}\frac{|\mathbf{x} - \mathbf{z}|}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} = 1 &- \left(\frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^2} + \frac{1}{2}\frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^2}\right) \\ &+ \frac{3}{4}\left(2\frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^2}\right)^2 \\ &\quad \varpi_{\frac{1}{2}}\left(2\frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^2}\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}K_2(\mathbf{x}, \mathbf{z}) &= \underbrace{\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} - \frac{1}{|\mathbf{x} - \mathbf{z}|}}_{=K_1(\mathbf{x},\mathbf{z})} + \frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^3} \\ &= -\frac{1}{2}\frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^3} \\ &\quad + \frac{3}{4|\mathbf{x} - \mathbf{z}|}\left(2\frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^2}\right)^2 \\ &\quad\quad \varpi_{\frac{1}{2}}\left(2\frac{(\mathbf{x} - \mathbf{z}) \cdot (\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x} - \mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x} - \mathbf{z}|^2}\right).\end{aligned}$$

□

2.2.2 Coercivity of L

A fundamental prerequisite for the understanding of the nonlinear stability is a good linear stability theory. This entails a precise understanding of the coercivity properties of the operator $\mathbf{L}$ and this is the subject of this section.

For sufficiently smooth $\boldsymbol{\theta}$, we have

$$\langle \mathbf{L}\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\rangle_3 = \int_{B_R} \left(\frac{4}{3}\bar{w}^{-2}\nabla\cdot(\bar{w}^3\boldsymbol{\theta}_2)\nabla\cdot(\bar{w}^3\boldsymbol{\theta}_1) + \nabla\cdot(\bar{w}^3\boldsymbol{\theta}_2)\mathscr{K}\nabla\cdot(\bar{w}^3\boldsymbol{\theta}_1)\right)\mathrm{d}\mathbf{x}$$

Note this is defined in a weak sense for $\boldsymbol{\theta}_i$ $(i = 1,2)$ such that $\nabla\cdot(\bar{w}^3\boldsymbol{\theta}_i) \in L^2(B_R, \bar{w}^{-2})$. We see that $\mathbf{L}$ is symmetric under $\langle\cdot,\cdot\rangle_3$ since

$$\int \nabla\cdot(\bar{w}^3\boldsymbol{\theta}_2)\mathscr{K}\nabla\cdot(\bar{w}^3\boldsymbol{\theta}_1)\mathrm{d}\mathbf{x} = -\int\int \frac{\nabla\cdot(\bar{w}^3\boldsymbol{\theta}_2)(\mathbf{x})\nabla\cdot(\bar{w}^3\boldsymbol{\theta}_1)(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.$$

Before stating the main theorem, we first characterise the growing modes for the linearised dynamics.

Proposition 2.2.4 (Growing modes). *Let* e_i *(*$i = 1,2,3$*) be the standard basis of* $\mathbb{R}^3$. *Then* e_i *and* $\mathbf{x}$ *are eigenfunctions for* $\mathbf{L}$ *with eigenvalue* δ *and* 3δ *respectively.*

Proof. Let $\mathbf{f} \in \mathbb{R}^3$ be a constant vector. Since $\mathbf{0} = \delta\mathbf{x} + 4\nabla\bar{w} + \nabla\mathscr{K}\bar{w}^3$, we have

$$\begin{aligned}
\mathbf{L}\mathbf{f} &= -\nabla\left(\frac{4}{3}\bar{w}^{-2}\nabla\cdot(\bar{w}^3\mathbf{f}) + \mathscr{K}\nabla\cdot(\bar{w}^3\mathbf{f})\right) = -\nabla\left(\frac{4}{3}\bar{w}^{-2}\mathbf{f}\cdot\nabla\bar{w}^3 + \mathscr{K}\mathbf{f}\cdot\nabla\bar{w}^3\right)\\
&= -\nabla\left(4\mathbf{f}\cdot\nabla\bar{w} + \mathbf{f}\cdot\nabla\mathscr{K}\bar{w}^3\right) = \nabla\left(\delta\mathbf{f}\cdot\mathbf{x}\right)\\
&= \delta\mathbf{f}
\end{aligned}$$

And

$$\begin{aligned}
\mathbf{L}\mathbf{x} &= -\nabla\left(\frac{4}{3}\bar{w}^{-2}\nabla\cdot(\bar{w}^3\mathbf{x}) + \mathscr{K}\nabla\cdot(\bar{w}^3\mathbf{x})\right)\\
&= -\nabla\left(4\mathbf{x}\cdot\nabla\bar{w} + \mathscr{K}\mathbf{x}\cdot\nabla\bar{w}^3 + 4\bar{w} + 3\mathscr{K}\bar{w}^3\right)\\
&= -\nabla\left(4\mathbf{x}\cdot\nabla\bar{w} + \mathbf{x}\cdot\nabla\mathscr{K}\bar{w}^3 + 4\bar{w} + \mathscr{K}\bar{w}^3\right)\\
&= \nabla\left(\delta\mathbf{x}\cdot\mathbf{x} - 4\bar{w} - \mathscr{K}\bar{w}^3\right) = 2\delta\mathbf{x} + \delta\mathbf{x} = 3\delta\mathbf{x}
\end{aligned}$$

where we have used

$$\begin{aligned}
\mathscr{K}(\mathbf{x}\cdot\nabla\bar{w}^3)(\mathbf{y}) &= -\int \frac{\mathbf{x}\cdot\nabla\bar{w}(\mathbf{x})^3}{|\mathbf{y}-\mathbf{x}|}\mathrm{d}\mathbf{x}\\
&= \int\left(\frac{\bar{w}(\mathbf{x})^3\nabla\cdot\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} + \bar{w}(\mathbf{x})^3\mathbf{x}\cdot\nabla_{\mathbf{x}}\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)\mathrm{d}\mathbf{x}\\
&= \int\left(\frac{3\bar{w}(\mathbf{x})^3}{|\mathbf{y}-\mathbf{x}|} + \bar{w}(\mathbf{x})^3\mathbf{x}\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}\right)\mathrm{d}\mathbf{x}
\end{aligned}$$

$$= \int \left(\frac{2\bar{w}(\mathbf{x})^3}{|\mathbf{y}-\mathbf{x}|} + \bar{w}(\mathbf{x})^3 \mathbf{y} \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \right) \mathrm{d}\mathbf{x}$$
$$= -2\mathscr{K}\bar{w}^3 + \mathbf{y} \cdot \nabla \mathscr{K}\bar{w}^3.$$

□

The main result of this section states that if the perturbation $\boldsymbol{\theta}$ is orthogonal to the four eigenvectors from Proposition 2.2.4, then the operator $\mathbf{L}$ is non-negative and we provide a quantitative lower bound.

Theorem 2.2.5 (Non-negativity of $\mathbf{L}$)**.** *Recall that $\bar{w} = \bar{w}_\delta$ and $\mathbf{L}$ (2.33) depends on δ. There exists $\epsilon > 0$ such that for any $\delta \in (-\epsilon, 0)$ the following holds. If $\boldsymbol{\theta}$ is such that $\|\boldsymbol{\theta}\|_3 + \|\nabla\boldsymbol{\theta}\|_4 < \infty$ and*

$$\langle \boldsymbol{\theta}, \mathbf{x} \rangle_3 = 0 = \langle \boldsymbol{\theta}, \mathbf{e}_i \rangle_3, \qquad i = 1, 2, 3 \tag{2.44}$$

then we have

$$\langle \mathbf{L}\boldsymbol{\theta}, \boldsymbol{\theta} \rangle_3 \gtrsim \int_{B_R} \bar{w}^{-2} |\Delta\Psi|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3} |\nabla\Psi|^2 \mathrm{d}\mathbf{x} \tag{2.45}$$

where the constants do not depend on δ, and Ψ is the gravitational potential induced by the flow disturbance $\bar{w}^3\boldsymbol{\theta}$:

$$\Psi := \frac{1}{4\pi} \mathscr{K} \nabla \cdot (\bar{w}^3 \boldsymbol{\theta}) \in H^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$$
$$\Delta\Psi = \nabla \cdot (\bar{w}^3 \boldsymbol{\theta}) \in L^2(B_R, \bar{w}^{-2})$$

The proof of Theorem 2.2.5 is a simple consequence of Lemmas 2.2.8–2.2.11. Our strategy is to use spherical harmonics to break down the problem into a sequence of scalar problems for each individual mode, by analogy to [34]. The modes $l = 0, 1$ correspond to radial and translational motion, and therefore, although formally unstable, can be factored out from the dynamics through suitable orthogonality conditions.

Lemma 2.2.6 (Spherical harmonics decomposition)**.** *Suppose $\boldsymbol{\theta}$ is such that $\|\boldsymbol{\theta}\|_3 + \|\nabla\boldsymbol{\theta}\|_4 < \infty$. Then*

$$g := \nabla \cdot (\bar{w}^3 \boldsymbol{\theta}) \in L^2(B_R, \bar{w}^{-2}) \tag{2.46}$$

$$\Psi(\mathbf{x}) := \frac{1}{4\pi} \mathscr{K} g(\mathbf{x}) \in H^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3), \tag{2.47}$$

and they can be expanded in spherical harmonics

$$g(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm}(r) Y_{lm}(\mathbf{x}) \qquad \text{on } B_R, \tag{2.48}$$

$$\Psi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Psi_{lm}(r) Y_{lm}(\mathbf{x}) \qquad \text{on } \mathbb{R}^3, \tag{2.49}$$

that converge in $L^2(B_R, \bar{w}^{-2})$ and $L^2(\mathbb{R}^3)$ respectively, where the spherical harmonics Y_{lm} are introduced in Appendix A.2. Moreover, Ψ_{lm} are related to g_{lm} by

$$\Psi_{lm}(r) = \frac{-1}{2l+1} \left(\int_0^r \frac{y^{l+2}}{r^{l+1}} g_{lm}(y) \mathrm{d}y + \int_r^R \frac{r^l}{y^{l-1}} g_{lm}(y) \mathrm{d}y \right) \tag{2.50}$$

$$g_{lm} = \Delta^{\langle l \rangle} \Psi_{lm} := \left(\frac{1}{r^2} \left(r^2 \Psi_{lm}' \right)' - \frac{l(l+1)}{r^2} \Psi_{lm} \right). \tag{2.51}$$

With this, the following identity holds:

$$\langle \mathbf{L}\boldsymbol{\theta}, \boldsymbol{\theta} \rangle_3 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Lambda_{lm}, \tag{2.52}$$

where

$$\Lambda_{lm} := \int_0^R \left(\frac{4}{3} \bar{w}^{-2} g_{lm}^2 + 4\pi g_{lm} \Psi_{lm} \right) r^2 \mathrm{d}r, \qquad l \geq 0,\ m \in \{-l, \dots, l\}. \tag{2.53}$$

Proof. From $\|\boldsymbol{\theta}\|_3 + \|\nabla\boldsymbol{\theta}\|_4 < \infty$, Corollary A.3.2 of Hardy-Poincaré inequality means that we have $\|\boldsymbol{\theta}\|_2 + \|\nabla\boldsymbol{\theta}\|_4 < \infty$. This immediately gives that $g \in L^2(B_R, \bar{w}^{-2})$. Since Ψ is a convolution of g with the kernel $\| \cdot \|^{-1}$, where g is trivially extended by 0 on $\mathbb{R}^3 \backslash B_R$, standard computation shows $\Psi \in C^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. Since spherical harmonics form an L^2 basis (see [1, 29, 9] and Appendix A.2), we have the spherical harmonics expansion (2.48)-(2.49) for g and Ψ in L^2.

By Lemma A.2.1 we have

$$\frac{1}{|\mathbf{x} - \mathbf{y}|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|, |\mathbf{y}|\}^l}{\max\{|\mathbf{x}|, |\mathbf{y}|\}^{l+1}} Y_{lm}(\mathbf{y}) Y_{lm}(\mathbf{x})$$

which converge uniformly on all compact set in $\{(\mathbf{x}, \mathbf{y}) : |\mathbf{x}| \neq |\mathbf{y}|\}$. So we have

$\mathscr{K} g(\mathbf{x})$

$$= -4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_{lm}(\mathbf{x}) \left(\int_{B_{|\mathbf{x}|}(\mathbf{0})} \frac{|\mathbf{y}|^l}{|\mathbf{x}|^{l+1}} g Y_{lm} \mathrm{d}\mathbf{y} + \int_{B_{|\mathbf{x}|}(\mathbf{0})^c} \frac{|\mathbf{x}|^l}{|\mathbf{y}|^{l+1}} g Y_{lm} \mathrm{d}\mathbf{y} \right)$$

$$= -4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_{lm}(\mathbf{x}) \left(\int_0^{|\mathbf{x}|} \frac{y^{l+2}}{|\mathbf{x}|^{l+1}} g_{lm} \mathrm{d}y + \int_{|\mathbf{x}|}^{R} \frac{|\mathbf{x}|^l}{y^{l-1}} g_{lm} \mathrm{d}y \right).$$

We therefore conclude that

$$\Psi_{lm}(r) = \frac{-1}{2l+1} \left(\int_0^r \frac{y^{l+2}}{r^{l+1}} g_{lm}(y) \mathrm{d}y + \int_r^R \frac{r^l}{y^{l-1}} g_{lm}(y) \mathrm{d}y \right)$$

since spherical harmonics expansion is unique (using standard Hilbert space theory and the fact that spherical harmonics forms a L^2 basis for L^2 functions on the sphere). Inverting this expression, we get (2.51).

Now using the spherical harmonics expansion for g and Ψ, we get

$$\begin{aligned} \langle \mathbf{L}\boldsymbol{\theta}, \boldsymbol{\theta} \rangle_3 &= \int \left(\frac{4}{3} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \boldsymbol{\theta})|^2 + \nabla \cdot (\bar{w}^3 \boldsymbol{\theta}) \mathscr{K} \nabla \cdot (\bar{w}^3 \boldsymbol{\theta}) \right) \mathrm{d}\mathbf{x} \\ &= \int \left(\frac{4}{3} \bar{w}^{-2} |g|^2 + 4\pi g \Psi \right) \mathrm{d}\mathbf{x} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Lambda_{lm}, \end{aligned}$$

with Λ_{lm} as in (2.53). □

From [20] we have the following lemma.

Lemma 2.2.7. *There exists* $\epsilon > 0$ *such that for any* $\delta \in (-\epsilon, 0)$ *and the associated* $\bar{w} = \bar{w}_\delta$, *we have*

$$\langle \mathscr{L}\varphi, \varphi \rangle_{\bar{w}^3 r^4} \gtrsim \|\varphi'\|^2_{\bar{w}^4 r^4} + \|\varphi\|^2_{\bar{w}^3 r^4} \qquad \textit{whenever} \qquad \langle \varphi, 1 \rangle_{\bar{w}^3 r^4} = 0$$

where the constants do not depend on δ, *and*

$$\begin{aligned} \mathscr{L}\varphi &:= -\frac{4}{3\bar{w}^3 r^4} \partial_r \left(\bar{w}^4 r^4 \partial_r \varphi \right) + 3\delta\varphi \\ \langle f, g \rangle_{\bar{w}^k r^4} &:= \int_0^R f(r) g(r) \bar{w}(r)^k r^4 \mathrm{d}r. \end{aligned}$$

We shall use Lemma 2.2.7 to obtain coercivity for the quadratic form Λ_{00} under the orthogonality assumption $\langle \boldsymbol{\theta}, \mathbf{x} \rangle_3 = 0$.

Lemma 2.2.8 ($l = 0$ mode bound)**.** *Suppose* $\boldsymbol{\theta}$ *is as in Lemma 2.2.6 and* $\langle \boldsymbol{\theta}, \mathbf{x} \rangle_3 =$

0. *Then we have* $\Psi_{00}'(r) = 0$ *for* $r \geq R$ *and*

$$\Lambda_{00} \gtrsim \int_0^R \left(\bar{w}^{-2} g_{00}^2 + (\Psi_{00}')^2\right) r^2 \mathrm{d}r, \tag{2.54}$$

where we recall (2.53).

Proof. From $\|\boldsymbol{\theta}\|_3 + \|\nabla\boldsymbol{\theta}\|_4 < \infty$, Corollary A.3.2 of Hardy-Poincaré inequality means that we have $\|\boldsymbol{\theta}\|_2 + \|\nabla\boldsymbol{\theta}\|_4 < \infty$. It follows that $\bar{w}^3\boldsymbol{\theta}$ is well defined on ∂B_R (trace theorem) and must vanish there. Since $\bar{w}^3\boldsymbol{\theta} = \nabla\Psi + \mathbf{C}$ where $\mathbf{C}$ is divergence-free, we have

$$\int_{\partial B_R} \partial_r \Psi \, \mathrm{d}S = \int_{\partial B_R} \nabla\Psi \cdot \mathrm{d}\mathbf{S} = \int_{\partial B_R} \bar{w}^3 \boldsymbol{\theta} \cdot \mathrm{d}\mathbf{S} = 0.$$

It follows that $\Psi_{00}'(R) = 0$. Now taking the derivative of (2.50) and using $g_{00}(r) = 0$ for $r > R$, we see that in fact we must have

$$\Psi_{00}'(r) = 0 \qquad \text{for} \qquad r \geq R. \tag{2.55}$$

From the orthogonality condition $\langle \boldsymbol{\theta}, \mathbf{x} \rangle_3 = 0$ we infer that

$$0 = \langle \boldsymbol{\theta}, \mathbf{x} \rangle_3 = \frac{1}{2} \int \bar{w}^3 \boldsymbol{\theta} \cdot \nabla |\mathbf{x}|^2 \, \mathrm{d}\mathbf{x} = \frac{1}{2} \int g |\mathbf{x}|^2 \, \mathrm{d}\mathbf{x}.$$

This means

$$\int_0^R g_{00}(r) r^4 \mathrm{d}r = 0. \tag{2.56}$$

and therefore by (2.51) and (2.55) in terms of Ψ_{00},

$$\int_0^R \Psi_{00}'(r) r^3 \mathrm{d}r = 0. \tag{2.57}$$

Since $\Psi \in H^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$, we have that $\partial_r \Psi \in L^2(\mathbb{R}^3) \cap C(\mathbb{R}^3 \setminus \{\mathbf{0}\})$. So $\partial_r \Psi$ has spherical harmonics expansion $\partial_r \Psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Psi_{r,lm} Y_{lm}$ in $L^2(\mathbb{R}^3)$ with

$$\Psi_{r,lm}(r) = \frac{1}{4\pi r^2} \int_{\partial B_r} (\partial_r \Psi) Y_{lm} \mathrm{d}S = \frac{1}{4\pi r^2} \partial_r \int_{B_r} \Psi Y_{lm} \mathrm{d}S = \partial_r \Psi_{lm}(r) = \Psi_{lm}'(r). \tag{2.58}$$

If we denote

$$\varphi := \Psi'_{00}/(r\bar{w}^3), \tag{2.59}$$

then by (2.57) we have

$$0 = \int_0^R \Psi'_{00}(r) r^3 \mathrm{d}r = \int_0^R \varphi(r)\bar{w}^3 r^4 \mathrm{d}r$$

and thus $\langle \varphi, 1\rangle_{\bar{w}^3 r^4} = 0$. Using (2.51) and (2.55), we get

$$\begin{aligned}
\Lambda_{00} &= \int_0^R \left(\frac{4}{3}\bar{w}^{-2} g_{00}^2 + 4\pi g_{00}\Psi_{00}\right) r^2 \mathrm{d}r \\
&= \int_0^R \left(\frac{4}{3r^2}\bar{w}^{-2}\left(\left(r^2\Psi'_{00}\right)'\right)^2 + 4\pi\left(r^2\Psi'_{00}\right)'\Psi_{00}\right)\mathrm{d}r \\
&= \int_0^R \left(\frac{4}{3r^2}\bar{w}^{-2}\left(\left(r^2\Psi'_{00}\right)'\right)^2 - 4\pi r^2(\Psi'_{00})^2\right)\mathrm{d}r + 4\pi R^2 \Psi'_{00}(R)\Psi_{00}(R) \\
&= \int_0^R \left(\frac{4}{3r^2}\bar{w}^{-2}\left(\left(r^3\bar{w}^3\varphi\right)'\right)^2 - 4\pi\varphi^2\bar{w}^6 r^4\right)\mathrm{d}r
\end{aligned}$$

Now since $0 = 3\delta + 4\Delta\bar{w} + 4\pi\bar{w}^3$ as in (2.12), we see that

$$\begin{aligned}
\Lambda_{00} &= \int_0^R \left(\frac{4}{3r^2}\bar{w}^{-2}\left(\left(r^3\bar{w}^3\varphi\right)'\right)^2 + (3\delta + 4\Delta\bar{w})\varphi^2\bar{w}^3 r^4\right)\mathrm{d}r \\
&= \int_0^R \Bigg(\frac{4}{3r^2}\bar{w}^{-2}\left(3r^2\bar{w}^3\varphi + 3r^3\bar{w}^2\bar{w}'\varphi + r^3\bar{w}^3\varphi'\right)^2 \\
&\qquad + 4(r^2\bar{w}')'\varphi^2\bar{w}^3 r^2 + 3\delta\varphi^2\bar{w}^3 r^4\Bigg)\mathrm{d}r \\
&= \int_0^R \Bigg(\frac{4}{3}\left(3r\bar{w}^2\varphi + 3r^2\bar{w}\bar{w}'\varphi + r^2\bar{w}^2\varphi'\right)^2 \\
&\qquad - 4r^2\bar{w}'(\varphi^2\bar{w}^3 r^2)' + 3\delta\varphi^2\bar{w}^3 r^4\Bigg)\mathrm{d}r \\
&= \int_0^R \Bigg(\frac{4}{3}\big(9r^2\bar{w}^4\varphi^2 + 9r^4\bar{w}^2(\bar{w}')^2\varphi^2 + r^4\bar{w}^4(\varphi')^2 \\
&\qquad\quad + 18r^3\bar{w}^3\bar{w}'\varphi^2 + 6r^4\bar{w}^3\bar{w}'\varphi\varphi' + 6r^3\bar{w}^4\varphi\varphi'\big) \\
&\qquad - 4(2\varphi\varphi'\bar{w}^3\bar{w}' r^4 + 3\varphi^2\bar{w}^2(\bar{w}')^2 r^4 + 2\varphi^2\bar{w}^3\bar{w}' r^3) + 3\delta\varphi^2\bar{w}^3 r^4\Bigg)\mathrm{d}r \\
&= \int_0^R \Bigg(\frac{4}{3}\left(9r^2\bar{w}^4\varphi^2 + r^4\bar{w}^4(\varphi')^2 + 12r^3\bar{w}^3\bar{w}'\varphi^2 + 6r^3\bar{w}^4\varphi\varphi'\right) \\
&\qquad + 3\delta\varphi^2\bar{w}^3 r^4\Bigg)\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^R \left(\frac{4}{3} r^4 \bar{w}^4 (\varphi')^2 + 3\delta \varphi^2 \bar{w}^3 r^4\right) \mathrm{d}r = \langle \mathscr{L}\varphi, \varphi \rangle_{\bar{w}^3 r^4} \\
&\gtrsim \|\varphi'\|^2_{\bar{w}^4 r^4} + \|\varphi\|^2_{\bar{w}^3 r^4} = \int_0^R ((\varphi')^2 \bar{w}^4 + \varphi^2 \bar{w}^3) r^4 \mathrm{d}r
\end{aligned}
$$

Then for ϵ small enough, we get

$$
\begin{aligned}
\Lambda_{00} &\gtrsim \int_0^R \left(\epsilon \left(\frac{4}{3} r^4 \bar{w}^4 (\varphi')^2 + 3\delta \varphi^2 \bar{w}^3 r^4 \right) + \varphi^2 \bar{w}^3 r^4 \right) \mathrm{d}r \\
&= \int_0^R \left(\epsilon \left(\frac{4}{3} \bar{w}^{-2} g_{00}^2 - 4\pi (\Psi'_{00})^2 \right) + \bar{w}^{-3} (\Psi'_{00})^2 \right) r^2 \mathrm{d}r.
\end{aligned}
$$

Finally by choosing ϵ small enough we get (2.54). □

In order to prove positivity of the higher modes, we will need the following lemma which provides an estimate from below for Λ_{lm} by an elliptic operator; a related bound was also used in [34].

Lemma 2.2.9. *Suppose $\boldsymbol{\theta}$ is as in Lemma 2.2.6. Then for any $l \geq 0, m \in \{-l, \dots, l\}$, we have*

$$
\Lambda_{lm} \geq 4\pi \int_0^R \left(-\Delta^{\langle l \rangle} - 3\pi \bar{w}^2\right) (\Psi_{lm}) \Psi_{lm} r^2 \mathrm{d}r.
$$

Proof. We have

$$
\begin{aligned}
\Lambda_{lm} &= \int_0^R \left(\frac{4}{3} \bar{w}^{-2} g_{lm}^2 + 4\pi g_{lm} \Psi_{lm} \right) r^2 \mathrm{d}r \\
&= \int_0^R \left| \frac{2}{\sqrt{3}\bar{w}} g_{lm} + 2\pi\sqrt{3}\bar{w}\Psi_{lm} \right|^2 r^2 \mathrm{d}r - 4\pi \int_0^R \left(g_{lm}\Psi_{lm} + 3\pi \bar{w}^2 \Psi_{lm}^2 \right) r^2 \mathrm{d}r \\
&\geq 4\pi \int_0^R \left(-\Delta^{\langle l \rangle} - 3\pi \bar{w}^2\right) (\Psi_{lm}) \Psi_{lm} r^2 \mathrm{d}r.
\end{aligned}
$$

□

With this bound from below by an elliptic operator, we can prove the positivity of Λ_{lm} using elliptic ODE theory.

Lemma 2.2.10 ($l = 1$ modes bound)**.** *Suppose $\boldsymbol{\theta}$ is as in Lemma 2.2.6 and $\langle \boldsymbol{\theta}, \mathrm{e}_i \rangle_3 = 0$ for $i = 1, 2, 3$. Then we have $\Psi_{1m}(r) = 0$ for $r \geq R$ and*

$$
\Lambda_{1m} \gtrsim \int_0^R \left(\bar{w}^{-2} g_{1m}^2 r^2 + \Psi'^2_{1m} r^2 + \Psi^2_{1m} \right) \mathrm{d}r, \qquad m = -1, 0, 1, \tag{2.60}
$$

where we recall (2.53).

Proof. For these modes, we adapt the method of proof as found in [34] that makes use of the Sturm-Liouville theory. We have by Lemma 2.2.9

$$\Lambda_{1m} \geq 4\pi \langle A_1 \Psi_{1m}, \Psi_{1m} \rangle_{r^2}$$

where $\langle y_1, y_2 \rangle_{r^2} := \int_0^R y_1 y_2 r^2 \mathrm{d}r$ and

$$A_1 := -\Delta^{\langle 1 \rangle} - 3\pi \bar{w}^2. \tag{2.61}$$

As this operator A_1 resembles the operator A analyzed in [34] (cf. (7.15) of [34]), by arguing analogously, we deduce that the operator A_1 has the Friedrichs extension in the Hilbert space induced by the inner product $\langle y_1, y_2 \rangle_{r^2}$, denoted by the same A_1. Moreover it is of Sturm-Liouville type and the eigenvalues are simple under the Dirichlet boundary condition on $r = R$, i.e. $y(R) = 0$ (cf. Section VII of [34]).

We next claim the least eigenvalue μ_1 of A_1 is strictly positive. Let ϕ_1 be an associated eigenfunction such that $A_1\phi_1 = \mu_1\phi_1$. Since ϕ_1 must have no zeros on $(0, R)$ by Sturm-Liouville theory, we may assume that $\phi_1(r) > 0$ for $r \in (0, R)$ so that $\phi_1'(R) \leq 0$ and $\phi_1(R) = 0$. In fact we must have $\phi_1'(R) < 0$, for if $\phi_1'(R) = 0$, then ϕ_1 must be the zero function, which is a contradiction. To see the latter assertion, note that A_1 is a second order ODE operator with C^1 coefficients away from the origin. Picard-Lindelöf existence theorem implies that for any $\epsilon > 0$ the solution u on $(\epsilon, R]$ satisfying $u'(R) = u(R) = 0$ must be unique. Since $u = 0$ is such a solution, we must have $\phi_1' = u = 0$. On the other hand, recalling $\Delta(4\bar{w}) = -3\delta - 4\pi\bar{w}^3$, we see that $A_1\bar{w}' = 0$. Note that $\bar{w}'(R) \neq 0$, so $\bar{w}' \notin \operatorname{Dom} A_1$ where $\operatorname{Dom} A_1$ denotes the domain of A_1 under the Sturm-Liouville theory framework. By using $A_1\bar{w}' = 0$, the properties of ϕ_1 and integration by parts, we have

$$\begin{aligned} 0 = \langle A_1 \bar{w}', \phi_1 \rangle_{r^2} &= \langle \bar{w}', A_1\phi_1 \rangle_{r^2} + R^2 \bar{w}'(R)\phi_1'(R) \\ &= \mu_1 \langle \bar{w}', \phi_1 \rangle_{r^2} + R^2 \bar{w}'(R)\phi_1'(R). \end{aligned}$$

Since $\bar{w}'(r) < 0$ for $r \in (0, R]$, we see that $\langle \bar{w}', \phi_1 \rangle_{r^2} < 0$. Also $R^2\bar{w}'(R)\phi_1'(R) > 0$. Therefore we must have $\mu_1 > 0$.

By the orthogonality condition

$$0 = \langle \boldsymbol{\theta}, \mathrm{e}_i \rangle_3 = \int_{B_R} \bar{w}^3 \boldsymbol{\theta} \cdot \nabla x^i \, \mathrm{d}\mathbf{x} = \int_{B_R} g x^i \, \mathrm{d}\mathbf{x},$$

we conclude that $0 = \int_0^R g_{1m} r^3 \mathrm{d}r$ and therefore

$$\Psi_{1m}(r) = 0 \qquad \text{for} \qquad r \geq R. \tag{2.62}$$

This (2.62) means that $\Psi_{1m} \in \operatorname{Dom} A_1$, it follows that

$$\Lambda_{1m} \geq 4\pi \langle A_1 \Psi_{1m}, \Psi_{1m} \rangle_{r^2} \geq 4\pi \mu_1 \langle \Psi_{1m}, \Psi_{1m} \rangle_{r^2} \geq 0. \tag{2.63}$$

The second inequality of (2.63) implies

$$\int_0^R \left(\Psi_{1m}'^2 + \frac{2}{r^2} \Psi_{1m}^2 - 3\pi \bar{w}^2 \Psi_{1m}^2 \right) r^2 \mathrm{d}r \geq \mu_1 \int_0^R \Psi_{1m}^2 r^2 \mathrm{d}r$$

which we can rewrite as

$$\begin{aligned}
&(1+\epsilon) \int_0^R \left(\Psi_{1m}'^2 + \frac{2}{r^2} \Psi_{1m}^2 - 3\pi \bar{w}^2 \Psi_{1m}^2 \right) r^2 \mathrm{d}r \\
&\geq \epsilon \int_0^R \left(\Psi_{1m}'^2 + \frac{2}{r^2} \Psi_{1m}^2 - 3\pi \bar{w}^2 \Psi_{1m}^2 \right) r^2 \mathrm{d}r + \mu_1 \int_0^R \Psi_{1m}^2 r^2 \mathrm{d}r \\
&\geq \epsilon \int_0^R \left(\Psi_{1m}'^2 r^2 + 2\Psi_{1m}^2 \right) \mathrm{d}r + (\mu_1 - 3\epsilon\pi \bar{w}(0)^2) \int_0^R \Psi_{1m}^2 r^2 \mathrm{d}r
\end{aligned}$$

Chose ϵ small enough so that the last term is non-negative. Hence we see that

$$\int_0^R \left(\Psi_{1m}'^2 + \frac{2}{r^2} \Psi_{1m}^2 - 3\pi \bar{w}^2 \Psi_{1m}^2 \right) r^2 \mathrm{d}r \gtrsim \int_0^R \left(\Psi_{1m}'^2 r^2 + \Psi_{1m}^2 \right) \mathrm{d}r$$

Together with (2.63) we deduce that

$$\Lambda_{1m} = \int_0^R \left(\frac{4}{3} \bar{w}^{-2} g_{1m}^2 + 4\pi g_{1m} \Psi_{1m} \right) r^2 \mathrm{d}r \gtrsim \int_0^R \left(\Psi_{1m}'^2 r^2 + \Psi_{1m}^2 \right) \mathrm{d}r$$

We can rewrite this as, for some $C > 0$,

$$\begin{aligned}
(1+\epsilon)\Lambda_{1m} &\geq \epsilon \int_0^R \left(\frac{4}{3} \bar{w}^{-2} g_{1m}^2 + 4\pi g_{1m} \Psi_{1m} \right) r^2 \mathrm{d}r + C \int_0^R \left(\Psi_{1m}'^2 r^2 + \Psi_{1m}^2 \right) \mathrm{d}r \\
&= \epsilon \int_0^R \left(\frac{4}{3} \bar{w}^{-2} g_{1m}^2 r^2 + 4\pi \left((r^2 \Psi_{1m}')' - 2\Psi_{1m} \right) \Psi_{1m} \right) \mathrm{d}r \\
&\quad + C \int_0^R \left(\Psi_{1m}'^2 r^2 + \Psi_{1m}^2 \right) \mathrm{d}r \\
&= \epsilon \int_0^R \left(\frac{4}{3} \bar{w}^{-2} g_{1m}^2 r^2 - 4\pi \left(\Psi_{1m}'^2 r^2 + 2\Psi_{1m}^2 \right) \right) \mathrm{d}r
\end{aligned}$$

$$+ C \int_0^R \left(\Psi_{1m}'^2 r^2 + \Psi_{1m}^2\right) \mathrm{d}r$$

Choosing ϵ small enough we obtain (2.60). □

Lemma 2.2.11 ($l \geq 2$ modes bound)**.** *Suppose $\boldsymbol{\theta}$ is as in Lemma 2.2.6. Then for $l \geq 2$,*

$$\Lambda_{lm} \gtrsim \int_0^R \bar{w}^{-2} g_{lm}^2 r^2 \mathrm{d}r + \int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2\right) \mathrm{d}r, \qquad m \in \{-l, \dots, l\}. \tag{2.64}$$

Proof. For these higher modes, we use a continuity argument. We have by Lemma 2.2.9

$$\begin{aligned}
\Lambda_{lm} &\geq 4\pi \int_0^R \left(-\Delta^{\langle l\rangle} - 3\pi\bar{w}^2\right)(\Psi_{lm})\Psi_{lm} r^2 \mathrm{d}r \\
&= 4\pi \int_0^R \left(\Psi_{lm}'^2 + \frac{l(l+1)}{r^2}\Psi_{lm}^2 - 3\pi\bar{w}^2\Psi_{lm}^2\right) r^2 \mathrm{d}r - 4\pi R^2 \Psi_{lm}(R)\Psi_{lm}'(R) \\
&= 4\pi \int_0^R \left(\Psi_{lm}'^2 + \frac{l(l+1)}{r^2}\Psi_{lm}^2 - 3\pi\bar{w}^2\Psi_{lm}^2\right) r^2 \mathrm{d}r \qquad (2.65) \\
&\quad + 4\pi \left(\int_R^\infty \left(\Psi_{lm}'^2 r^2 + (r^2\Psi_{lm}')'\Psi_{lm}\right) \mathrm{d}r\right) \\
&= 4\pi \int_0^\infty \left(\Psi_{lm}'^2 + \frac{l(l+1)}{r^2}\Psi_{lm}^2 - 3\pi\bar{w}^2\Psi_{lm}^2\right) r^2 \mathrm{d}r \qquad (2.66) \\
&= 4\pi \int_0^\infty \left(-\Delta^{\langle 1\rangle} - 3\pi\bar{w}^2\right)(\Psi_{lm})\Psi_{lm} r^2 \mathrm{d}r + 4\pi \int_0^\infty (l(l+1) - 2)\Psi_{lm}^2 \mathrm{d}r \qquad (2.67)
\end{aligned}$$

where we used

$$g_{lm}(r) = \Delta^{\langle l\rangle}\Psi_{lm}(r) = \frac{1}{r^2}(r^2\Psi_{lm}')' - \frac{l(l+1)}{r^2}\Psi_{lm} = 0 \qquad \text{for} \qquad r > R.$$

Recall that $\bar{w} = \bar{w}_\delta$ depends on δ. In the proof of Lemma 2.2.10 we have shown that

$$\int_0^R \left(-\Delta^{\langle 1\rangle} - 3\pi\bar{w}_\delta^2\right)(y) y r^2 \mathrm{d}r \geq 0$$

for all $y \in H^2([0, R], r^2)$ such that $y(R) = 0$. In fact when $\bar{w}_\delta = \bar{w}_0$ (the Lane-

Emden star), the same analysis can be extended to any $R' \geq R$ to give rise to

$$\int_0^{R'} \left(-\Delta^{\langle 1\rangle} - 3\pi\bar{w}_0^2\right)(y) y r^2 \mathrm{d}r \geq 0 \tag{2.68}$$

for all $y \in H^2([0,R'], r^2)$ such that $y(R') = 0$. To do so, we replace $\bar{w}_0'$ (used to argue the non-negativity of the least eigenvalue) with $\tilde{w}'$, where $\tilde{w} := -\frac{1}{4}\mathcal{K}\bar{w}_0^3$ (recall $\bar{w}_0 = 0$ for $r > R$). Note that $\tilde{w}$ is $C^3(\mathbb{R}^3)$, or $C^3([0,\infty))$ as a function of the radial variable. By (1.19), we see that $\tilde{w}' = \bar{w}_0'$ on $[0,R]$. Moreover, $\tilde{w}' < 0$ on $(0,\infty)$. Since $\Delta\tilde{w} = -\pi\bar{w}_0^3$, taking ∂_r we get $\Delta^{\langle 1\rangle}\tilde{w} = -3\pi\bar{w}_0^2\bar{w}_0' = -3\pi\bar{w}_0^2\tilde{w}'$. So we have $(-\Delta^{\langle 1\rangle} - 3\pi\bar{w}_0^2)\tilde{w}' = 0$ on $[0,\infty)$ which allows us to apply the same proof in Lemma 2.2.10.

Let

$$y_{R'}(r) = \Psi_{lm}(r) - \Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r}{R'}$$

From (2.50) we see that $y_{R'}(R') = 0$. By using $\Delta^{\langle 1\rangle} r = 0$ and applying (2.68) with $y = y_{R'}$, we obtain

$$\begin{aligned}
&\int_0^{R'} \left(-\Delta^{\langle 1\rangle} - 3\pi\bar{w}_0^2\right)(\Psi_{lm})\Psi_{lm} r^2 \mathrm{d}r \\
&= \int_0^{R'} \left(-\Delta^{\langle 1\rangle} - 3\pi\bar{w}_0^2\right)(y_{R'}(r))\left(y_{R'}(r) + \Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r}{R'}\right) r^2 \mathrm{d}r \\
&\quad - \int_0^{R} 3\pi\bar{w}_0^2 \Psi_{lm}(r)\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r}{R'} r^2 \mathrm{d}r \\
&\geq \int_0^{R'} \left(-\Delta^{\langle 1\rangle} - 3\pi\bar{w}_0^2\right)(y_{R'}(r))\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r^3}{R'} \mathrm{d}r \\
&\quad - \int_0^{R} 3\pi\bar{w}_0^2 \Psi_{lm}(r)\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r^3}{R'} \mathrm{d}r
\end{aligned}$$

Denote the last two integral terms by K. By integrating by parts and using the boundary condition $y_{R'}(R') = 0$,

$$\begin{aligned}
K = &-R'^2 y_{R'}'(R')\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1} \\
&- \int_0^{R} 3\pi\bar{w}_0^2 y_{R'}(r)\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r^3}{R'} \mathrm{d}r \\
&- \int_0^{R} 3\pi\bar{w}_0^2 \Psi_{lm}(r)\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r^3}{R'} \mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= -R'^2\Psi'_{lm}(R')\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1} + R'(\Psi_{lm}(R))^2\left(\frac{R}{R'}\right)^{2l+2} \\
&\quad + 3\pi\int_0^R \bar{w}_0^2(\Psi_{lm}(R))^2\left(\frac{R}{R'}\right)^{2l+2}\frac{r^4}{R'^2}\mathrm{d}r \\
&\quad - 6\pi\int_0^R \bar{w}_0^2\Psi_{lm}(r)\Psi_{lm}(R)\left(\frac{R}{R'}\right)^{l+1}\frac{r^3}{R'}\mathrm{d}r \\
&\to 0 \qquad \text{as} \qquad R'\to\infty
\end{aligned}
$$

when $l \geq 1$, where we used (2.50) to see for example that $\Psi'_{lm}(R') \to 0$ as $R' \to \infty$.

Therefore we have proven[1] that for any $l \geq 1$,

$$
\begin{aligned}
&\int_0^\infty \left(\Psi'^2_{lm} + \frac{l(l+1)}{r^2}\Psi^2_{lm} - 3\pi\bar{w}_0^2\Psi^2_{lm}\right)r^2\mathrm{d}r \\
&\qquad\qquad \geq \int_0^\infty (l(l+1)-2)\Psi^2_{lm}\mathrm{d}r \quad \text{for all} \quad \Psi_{lm}. \quad (2.69)
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\int_0^\infty \left(\Psi'^2_{lm} + \frac{l(l+1)}{r^2}\Psi^2_{lm} - 3\pi\bar{w}_\delta^2\Psi^2_{lm}\right)r^2\mathrm{d}r \\
&\qquad \geq \underbrace{\int_0^\infty (l(l+1)-2)\Psi^2_{lm}\mathrm{d}r - 3\pi\left\|(\bar{w}_\delta^2-\bar{w}_0^2)r^2\right\|_{L^\infty}\int_0^\infty \Psi^2_{lm}\mathrm{d}r}_{=M}
\end{aligned}
$$

For sufficiently small δ we have

$$
M \geq (l(l+1)-3)\int_0^\infty \Psi^2_{lm}\mathrm{d}r
$$

which leads to

$$
\Lambda_{lm} \geq 4\pi(l(l+1)-3)\int_0^\infty \Psi^2_{lm}\mathrm{d}r \geq 0.
$$

Observe that

$$
\begin{aligned}
&(1+\epsilon)\int_0^\infty \left(\Psi'^2_{lm} + \frac{l(l+1)}{r^2}\Psi^2_{lm} - 3\pi\bar{w}_\delta^2\Psi^2_{lm}\right)r^2\mathrm{d}r \\
&\geq \epsilon\int_0^\infty \left(\Psi'^2_{lm} + \frac{l(l+1)}{r^2}\Psi^2_{lm} - 3\pi\bar{w}_\delta^2\Psi^2_{lm}\right)r^2\mathrm{d}r + (l(l+1)-3)\int_0^\infty \Psi^2_{lm}\mathrm{d}r.
\end{aligned}
$$

[1] The proof of (2.69) can be easily adapted to correct an inconsistency appearing in [34] and establish the non-negativity of the quadratic form $\langle \mathbf{L}\boldsymbol{\theta}, \boldsymbol{\theta}\rangle_3$ around the Lane-Emden stars.

Choosing $\epsilon > 0$ small enough we see that

$$\begin{aligned}
&\int_0^\infty \left(\Psi_{lm}'^2 + \frac{l(l+1)}{r^2}\Psi_{lm}^2 - 3\pi\bar{w}_\delta^2\Psi_{lm}^2 \right) r^2 \mathrm{d}r \\
&\gtrsim \int_0^\infty \left(\Psi_{lm}'^2 r^2 + (l(l+1) - 4)\Psi_{lm}^2 \right) \mathrm{d}r \\
&\gtrsim \int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2 \right) \mathrm{d}r.
\end{aligned}$$

We have

$$\Lambda_{lm} = \int_0^R \left(\frac{4}{3}\bar{w}^{-2} g_{lm}^2 + 4\pi g_{lm}\Psi_{lm} \right) r^2 \mathrm{d}r \gtrsim \int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2 \right) \mathrm{d}r.$$

We can rewrite this as, for some $C > 0$,

$$\begin{aligned}
(1+\epsilon)\Lambda_{lm} &\geq \epsilon \int_0^R \left(\frac{4}{3}\bar{w}^{-2} g_{lm}^2 + 4\pi g_{lm}\Psi_{lm} \right) r^2 \mathrm{d}r \\
&\quad + C \int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2 \right) \mathrm{d}r \\
&= \epsilon \int_0^R \left(\frac{4}{3}\bar{w}^{-2} g_{lm}^2 r^2 + 4\pi \left((r^2\Psi_{lm}')' - l(l+1)\Psi_{lm} \right) \Psi_{lm} \right) \mathrm{d}r \\
&\quad + C \int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2 \right) \mathrm{d}r \\
&= \frac{4}{3}\epsilon \int_0^R \bar{w}^{-2} g_{lm}^2 r^2 \mathrm{d}r + 4\pi\epsilon \int_0^\infty \left(\Psi_{lm}'^2 r^2 - l(l+1)\Psi_{lm}^2 \right) \mathrm{d}r \\
&\quad + C \int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2 \right) \mathrm{d}r
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
4\pi \int_0^R \left(\Psi_{lm}'^2 + \frac{l(l+1)}{r^2}\Psi_{lm}^2 \right) r^2 \mathrm{d}r - 4\pi R^2 \Psi_{lm}(R)\Psi_{lm}'(R) \\
= 4\pi \int_0^\infty \left(\Psi_{lm}'^2 + \frac{l(l+1)}{r^2}\Psi_{lm}^2 \right) r^2 \mathrm{d}r
\end{aligned}$$

proved in (2.66). Choosing $\epsilon > 0$ small enough we obtain the desired (2.64). □

Proof of Theorem 2.2.5. Combining all the bounds we have for each l, m from Lemmas 2.2.8–2.2.11, we have

$$\langle \mathbf{L}\boldsymbol{\theta}, \boldsymbol{\theta} \rangle_3 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Lambda_{lm}$$

$$\gtrsim \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^R \bar{w}^{-2}g_{lm}^2 r^2 \mathrm{d}r + \int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2\right)\mathrm{d}r.$$

We know

$$\int_{B_R} \bar{w}^{-2}|\Delta\Psi|^2 \mathrm{d}\mathbf{x} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^R \bar{w}^{-2}g_{lm}^2 r^2\mathrm{d}r.$$

It remains to show that

$$\int_{\mathbb{R}^3} |\nabla\Psi|^2 \mathrm{d}\mathbf{x} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^\infty \left(\Psi_{lm}'^2 r^2 + l(l+1)\Psi_{lm}^2\right)\mathrm{d}r \tag{2.70}$$

Since $\nabla\Psi \in L^2(\mathbb{R}^3)^3$, it has a vector spherical harmonics expansion in $L^2(\mathbb{R}^3)^3$ [2, 14],

$$\nabla\Psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\Psi_{lm}^{[0]}\mathbf{Y}_{lm}^{[0]} + \Psi_{lm}^{[1]}\mathbf{Y}_{lm}^{[1]} + \Psi_{lm}^{[2]}\mathbf{Y}_{lm}^{[2]}\right). \tag{2.71}$$

where

$$\mathbf{Y}_{lm}^{[0]} = Y_{lm}\hat{\mathbf{r}}, \qquad \mathbf{Y}_{lm}^{[1]} = r\nabla Y_{lm}, \qquad \mathbf{Y}_{lm}^{[2]} = \mathbf{r}\times\nabla Y_{lm}$$

are the vector spherical harmonics [2, 14]. We have

$$\Psi_{lm}^{[0]}(r) = \frac{1}{r^2}\int_{\partial B_r}\nabla\Psi\cdot\mathbf{Y}_{lm}^{[0]}\mathrm{d}S = \frac{1}{r^2}\int_{\partial B_r}(\partial_r\Psi)Y_{lm}\mathrm{d}S = \Psi_{lm}'(r)$$

using (2.58). And

$$\begin{aligned}\Psi_{lm}^{[1]}(r) &= \frac{1}{l(l+1)r^2}\int_{\partial B_r}\nabla\Psi\cdot\mathbf{Y}_{lm}^{[1]}\mathrm{d}S\\ &= -\frac{1}{l(l+1)r^2}\int_{\partial B_r}(\Psi r\Delta Y_{lm} + \Psi\hat{\mathbf{r}}\cdot\nabla Y_{lm})\mathrm{d}S\\ &= \frac{1}{r^3}\int_{\partial B_r}\Psi Y_{lm}\mathrm{d}S = \frac{1}{r}\Psi_{lm}(r)\end{aligned}$$

where we used the fact that $\Delta Y_{lm} = -l(l+1)r^{-2}Y_{lm}$. Also,

$$\begin{aligned}\Psi_{lm}^{[2]}(r) &= \frac{1}{l(l+1)r^2}\int_{\partial B_r}\nabla\Psi\cdot\mathbf{Y}_{lm}^{[2]}\mathrm{d}S\\ &= -\frac{1}{l(l+1)r^2}\int_{\partial B_r}\Psi\nabla\cdot(\mathbf{r}\times\nabla Y_{lm})\mathrm{d}S = 0.\end{aligned}$$

Evaluating $\int_{\mathbb{R}^3} |\nabla\Psi|^2 \mathrm{dx}$ using (2.71) we get (2.70). This completes the proof of (2.70). □

2.3 Momentum and energy

The energy and momentum conservation account for a four-dimensional freedom in the parameter space of the self-similar Goldreich-Weber solutions, see Definition 1.2.2. We shall require that the initial perturbation belongs to a codimension 4 "manifold" of initial data so that they have the same total momentum and total energy as the background GW star, i.e. (2.22) and (2.23). We will show that the linearisation of this requirement allows us to dynamically control the inner products

$$\langle \boldsymbol{\theta}, \mathbf{x}\rangle_3, \;\; \langle \boldsymbol{\theta}, \mathrm{e}_i\rangle_3, \qquad i = 1, 2, 3,$$

modulo nonlinear terms, which is necessary for the proof of linear coercivity in Theorem 2.2.5. Hence, by fixing the total momentum and energy, we will be able to apply the non-negativity results we have for the linear operator $\mathbf{L}$ to control $\int_{B_R} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \partial_s^a \not{\partial}^\beta \boldsymbol{\theta})|^2 \mathrm{dx}$ with $\langle \mathbf{L}\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 + \|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3^2$ modulo a correction involving non-linear terms. This is the main result of this section, stated and proved in Proposition 2.3.6.

Firstly, the momentum condition (2.22) gives us the following.

Lemma 2.3.1. *Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, and such that* $\mathbf{W} = \bar{\mathbf{W}}$ (2.22)*. Then*

$$-\frac{1}{2}\langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}, \mathrm{e}_i\rangle_3^2 = \delta \langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathrm{e}_i\rangle_3^2, \qquad a \geq 0,\ |\beta| \geq 0.$$

Proof. From Lemma 2.1.3, we see that when $\mathbf{W}_\delta[\boldsymbol{\theta}] = \bar{\mathbf{W}}$ we have

$$\langle \partial_s \boldsymbol{\theta}, \mathrm{e}_i\rangle_3 = \not{b} \langle \boldsymbol{\theta}, \mathrm{e}_i\rangle_3 \qquad \text{for} \qquad i = 1, 2, 3.$$

and hence for any a with $\boldsymbol{\theta}$ sufficiently smooth,

$$\langle \partial_s^{a+1} \boldsymbol{\theta}, \mathrm{e}_i\rangle_3 = \not{b} \langle \partial_s^a \boldsymbol{\theta}, \mathrm{e}_i\rangle_3 \qquad \text{for} \qquad i = 1, 2, 3. \tag{2.72}$$

Now note that, using integration by parts,

$$\begin{aligned} \langle \partial_s^{a+1} \not{\partial}_j \not{\partial}^{\beta'} \boldsymbol{\theta}, \mathrm{e}_i\rangle_3 &= -\langle \partial_s^{a+1} \not{\partial}^{\beta'} \boldsymbol{\theta}, \not{\partial}_j \mathrm{e}_i\rangle_3 &(2.73)\\ &= 0 = \not{b} \langle \partial_s^a \not{\partial}_j \not{\partial}^{\beta'} \boldsymbol{\theta}, \mathrm{e}_i\rangle_3 \qquad \text{for} \qquad i = 1, 2, 3. &(2.74) \end{aligned}$$

We are done noting $\delta = -\frac{1}{2}\mathscr{b}^2$ (2.2). $\square$

We now turn our attention to the energy condition (2.23).

Lemma 2.3.2. *Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, and such that $E = \bar{E}$* (2.23)*. Then*

$$\begin{aligned}\frac{5}{2}\mathscr{b}^2\langle\partial_s^a\boldsymbol{\theta},\mathbf{x}\rangle_3 &= 2\mathscr{b}\langle\partial_s^{a+1}\boldsymbol{\theta},\mathbf{x}\rangle_3 \\ &\quad - \int\left(\bar{w}^3\partial_s^a|\partial_s\boldsymbol{\theta}-\mathscr{b}\boldsymbol{\theta}|^2 + 6\bar{w}^4\partial_s^a\left(\mathscr{J}^{-\frac{1}{3}} - 1 + \frac{1}{3}\nabla\cdot\boldsymbol{\theta}\right)\right)\mathrm{d}\mathbf{x} \\ &\quad - \int\bar{w}^3\partial_s^a(\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K} - \mathscr{K}_{\boldsymbol{\xi}}^{(1)})\bar{w}^3\mathrm{d}\mathbf{x}. \qquad (2.75)\end{aligned}$$

Proof. From Lemma 2.1.3, we see that when $E_\delta[\boldsymbol{\theta}] = \bar{E}$ we have

$$\begin{aligned}\frac{5}{2}\mathscr{b}^2\langle\boldsymbol{\theta},\mathbf{x}\rangle_3 &= 2\mathscr{b}\langle\partial_s\boldsymbol{\theta},\mathbf{x}\rangle_3 \\ &\quad - \int\left(\bar{w}^3|\partial_s\boldsymbol{\theta}-\mathscr{b}\boldsymbol{\theta}|^2 + 6\bar{w}^4\left(\mathscr{J}^{-\frac{1}{3}} - 1 + \frac{1}{3}\nabla\cdot\boldsymbol{\theta}\right)\right. \\ &\qquad\left. + \bar{w}^3(\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K} - \mathscr{K}_{\boldsymbol{\xi}}^{(1)})\bar{w}^3\right)\mathrm{d}\mathbf{x}.\end{aligned}$$

And hence for any $a \geq 0$ the identity (2.75) easily follows. $\square$

We collect a few more easy statements in the next lemma.

Lemma 2.3.3. (i) *For any $K : B_R \times B_R \to \mathbb{R}$ sufficiently nice and $g \in H_0^1(B_R)$ we have*

$$\not{\partial}_{i,\mathbf{x}}\int_{B_R}K(\mathbf{x},\mathbf{z})g(\mathbf{z})\mathrm{d}\mathbf{z} = \int_{B_R}(g(\mathbf{z})(\not{\partial}_{i,\mathbf{x}} + \not{\partial}_{i,\mathbf{z}})K(\mathbf{x},\mathbf{z}) + K(\mathbf{x},\mathbf{z})\not{\partial}_{i,\mathbf{z}}g(\mathbf{z}))\,\mathrm{d}\mathbf{z}$$

(ii) *For any $\boldsymbol{\theta} : B_R \to \mathbb{R}^3$ sufficiently smooth and $\mathbf{x},\mathbf{y} \in B_R$ we have*

$$|\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}(\mathbf{x}) - \partial_s^a\not{\partial}^\beta\boldsymbol{\theta}(\mathbf{z})| \leq \|\nabla\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\|_{L^\infty(B_R)}|\mathbf{x}-\mathbf{z}| \qquad (2.76)$$

$$|\not{\partial}^\beta\mathbf{x} - \not{\partial}^\beta\mathbf{z}| \leq |\mathbf{x}-\mathbf{z}| \qquad (2.77)$$

Proof. For part (i), integrate by parts to get

$$\int_{B_R}K(\mathbf{x},\mathbf{z})\not{\partial}_{i,\mathbf{z}}g(\mathbf{z})\mathrm{d}\mathbf{z} = -\int_{B_R}g(\mathbf{z})\not{\partial}_{i,\mathbf{z}}K(\mathbf{x},\mathbf{z})\mathrm{d}\mathbf{z}.$$

For part (ii), use the mean value inequality to get (2.76). Bound (2.77) follows from $\not{\partial}_{ij}x^k = x^i\delta_j^k - x^j\delta_i^k$. $\square$

Lemma 2.3.4. *Let $n \geq 20$ and $a + |\beta| \leq n$ with $a > 0$. We have*

$$|\partial_s^a(\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta K_2(\mathbf{x},\mathbf{z})| \lesssim \frac{(E_n + Z_n^2)^{1/2}}{|\mathbf{x}-\mathbf{z}|^2} \sum_{\substack{0<a'\leq a \\ \beta' \leq \beta}} |\partial_s^{a'} \not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{x}) - \partial_s^{a'} \not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{z})| + \frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2} \sum_{\beta' \leq \beta} |\not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{x}) - \not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{z})|,$$

where we recall K_2 (2.39).

Proof. From Lemma 2.2.3,

$$K_2(\mathbf{x},\mathbf{z}) = -\frac{1}{2}\frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^3} + \frac{3}{4|\mathbf{x}-\mathbf{z}|}\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right)^2 \varpi_{\frac{1}{2}} \underbrace{\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x}) - \boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right)}_{:=y(\mathbf{x},\mathbf{z})}$$

Note that $|y(\mathbf{x},\mathbf{z})| \lesssim \|\nabla\boldsymbol{\theta}\|_{L^\infty}$. Our a priori assumption (2.20) together with the embedding theorems A.3.5 and A.3.6 mean that $\|\nabla\boldsymbol{\theta}\|_{L^\infty}$ is bounded by a small constant. So we can assume $|y(\mathbf{x},\mathbf{z})| \leq 1/2$. Then from the definition of ϖ_q (2.42) we can see that

$$\varpi_{\frac{1}{2}}^{(k)}(y(\mathbf{x},\mathbf{z})) \lesssim 1 \qquad \text{for any} \qquad k \geq 0.$$

Now using part (ii) of Lemma 2.3.3, chain and product rule for derivatives and the embedding theorems A.3.5 and A.3.6, we can see that $\partial_s^a(\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta K_2(\mathbf{x},\mathbf{z})$ satisfies the stated bounds. □

Lemma 2.3.5. *Let $n \geq 20$ and $a+|\beta| \leq n$ with $a > 0, |\beta| \geq 0$. Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, and such that $E = \bar{E}$* (2.23)*. Then*

$$\left|3\delta\langle\partial_s^a\not{\partial}^\beta\boldsymbol{\theta},\mathbf{x}\rangle_3^2 + \frac{24}{25}\langle\partial_s^{a+1}\not{\partial}^\beta\boldsymbol{\theta},\mathbf{x}\rangle_3^2\right| \lesssim S_{n,|\beta|-1,0} + C_\delta(E_n + Z_n^2)^{1/2}E_n.$$

Proof. First we deal with the case $|\beta| = 0$. From (2.75) we get

$$2\langle\partial_s^{a+1}\boldsymbol{\theta},\mathbf{x}\rangle_3 = \frac{5}{2}\not{\partial}\langle\partial_s^a\boldsymbol{\theta},\mathbf{x}\rangle_3$$

$$+ \mathscr{b}^{-1} \int \left(\bar{w}^3 \partial_s^a |\partial_s \boldsymbol{\theta} - \mathscr{b}\boldsymbol{\theta}|^2 + 6\bar{w}^4 \partial_s^a \left(\mathscr{J}^{-\frac{1}{3}} - 1 + \frac{1}{3}\nabla \cdot \boldsymbol{\theta} \right) \right) \mathrm{d}\mathbf{x}$$
$$+ \mathscr{b}^{-1} \int \bar{w}^3 \partial_s^a (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K} - \mathscr{K}_{\boldsymbol{\xi}}^{(1)}) \bar{w}^3 \mathrm{d}\mathbf{x}. \tag{2.78}$$

With the embedding theorems A.3.5 and A.3.6, it is easy to see that

$$\left| \int \left(\bar{w}^3 \partial_s^a |\partial_s \boldsymbol{\theta} - \mathscr{b}\boldsymbol{\theta}|^2 + 6\bar{w}^4 \partial_s^a \left(\mathscr{J}^{-\frac{1}{3}} - 1 + \frac{1}{3}\nabla \cdot \boldsymbol{\theta} \right) \right) \mathrm{d}\mathbf{x} \right| \lesssim_\delta (E_n + Z_n^2)^{1/2} E_n^{1/2}.$$

Now using Lemmas 2.3.3–2.3.4 and Young's convolution inequality we have

$$\left| \int \bar{w}^3 \partial_s^a (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K} - \mathscr{K}_{\boldsymbol{\xi}}^{(1)}) \bar{w}^3 \mathrm{d}\mathbf{x} \right| = \left| \int \int \bar{w}^3(\mathbf{x}) \bar{w}^3(\mathbf{z}) \partial_s^a K_2(\mathbf{x}, \mathbf{z}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{z} \right| \lesssim (E_n + Z_n^2)^{1/2} E_n^{1/2}. \tag{2.79}$$

Therefore, upon taking the square of (2.78) and using the simple bound $|\langle \partial_s^a \boldsymbol{\theta}, \mathbf{x} \rangle_3| \lesssim E_n^{\frac{1}{2}}$, we obtain

$$\left| 4\langle \partial_s^{a+1} \boldsymbol{\theta}, \mathbf{x} \rangle_3^2 - \frac{25}{4} \mathscr{b}^2 \langle \partial_s^a \boldsymbol{\theta}, \mathbf{x} \rangle_3^2 \right| \lesssim_\delta (E_n + Z_n^2)^{1/2} E_n,$$

which concludes the proof for when $|\beta| = 0$ since $\delta = -\frac{1}{2}\mathscr{b}^2$ (recall (2.2)).

Now note that, using integration by parts,

$$|\langle \partial_s^{a+1} \not{\partial}_j \not{\partial}^{\beta'} \boldsymbol{\theta}, \mathbf{x} \rangle_3| = |\langle \partial_s^{a+1} \not{\partial}^{\beta'} \boldsymbol{\theta}, \not{\partial}_j \mathbf{x} \rangle_3| \lesssim S_{n,|\beta'|,0}^{1/2}.$$

Similarly we have $|\langle \partial_s^a \not{\partial}_j \not{\partial}^{\beta'} \boldsymbol{\theta}, \mathbf{x} \rangle_3| \lesssim S_{n,|\beta'|,0}^{1/2}$. Noting that $|\delta| \lesssim 1$ and we are done. □

Proposition 2.3.6. *Let $n \geq 20$ and $a + |\beta| \leq n$ with $a > 0$. Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11 such that* $\mathbf{W} = \bar{\mathbf{W}}$ (2.22) *and* $E = \bar{E}$ (2.23). *Then*

$$|\mathscr{b}|^2 \int_{B_R} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \partial_s^a \not{\partial}^\beta \boldsymbol{\theta})|^2 \mathrm{d}\mathbf{x} \lesssim \langle \mathbf{L} \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta} \rangle_3 + \frac{49}{50} \|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3^2 + S_{n,|\beta|-1,0} + C_\delta (E_n + Z_n^2)^{1/2} E_n. \tag{2.80}$$

Proof. Let

$$\tilde{\boldsymbol{\theta}} = \partial_s^a \not{\partial}^\beta \boldsymbol{\theta} - \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3}{\|\mathbf{x}\|_3^2}\mathbf{x} - \sum_{i=1}^3 \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3}{\|\mathbf{e}_i\|_3^2}\mathbf{e}_i \tag{2.81}$$

$$\boldsymbol{\theta}' = \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta} - \frac{\langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3}{\|\mathbf{x}\|_3^2}\mathbf{x} - \sum_{i=1}^3 \frac{\langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3}{\|\mathbf{e}_i\|_3^2}\mathbf{e}_i \tag{2.82}$$

Then $\langle \tilde{\boldsymbol{\theta}}, \mathbf{x}\rangle_3 = 0 = \langle \tilde{\boldsymbol{\theta}}, \mathbf{e}_i\rangle_3$ for $i = 1, 2, 3$, and

$$\|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3^2 = \|\boldsymbol{\theta}'\|_3^2 + \frac{\langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3^2}{\|\mathbf{x}\|_3^2} + \sum_{i=1}^3 \frac{\langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3^2}{\|\mathbf{e}_i\|_3^2}. \tag{2.83}$$

Since $\mathbf{x}$ and $\mathbf{e}_i$ are eigenfunctions of $\mathbf{L}$ with eigenvalues 3δ and δ respectively, we have

$$\begin{aligned}
\langle \mathbf{L}\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 &= \Bigg\langle \mathbf{L}\tilde{\boldsymbol{\theta}} + 3\delta \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3}{\|\mathbf{x}\|_3^2}\mathbf{x} + \sum_{i=1}^3 \delta \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3}{\|\mathbf{e}_i\|_3^2}\mathbf{e}_i, \\
&\qquad \tilde{\boldsymbol{\theta}} + \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3}{\|\mathbf{x}\|_3^2}\mathbf{x} + \sum_{i=1}^3 \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3}{\|\mathbf{e}_i\|_3^2}\mathbf{e}_i \Bigg\rangle_3 \\
&= \langle \mathbf{L}\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}\rangle_3 + 3\delta \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3^2}{\|\mathbf{x}\|_3^2} + \sum_{i=1}^3 \delta \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3^2}{\|\mathbf{e}_i\|_3^2}. \qquad (2.84)
\end{aligned}$$

We use Lemmas 2.3.1 and 2.3.5 to control the last two terms on the right-most side of (2.84) to get

$$\begin{aligned}
\langle \mathbf{L}\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}\rangle_3 &\leq \langle \mathbf{L}\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 + \frac{24}{25}\frac{\langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3^2}{\|\mathbf{x}\|_3^2} + \frac{1}{2}\sum_{i=1}^3 \frac{\langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3^2}{\|\mathbf{e}_i\|_3^2} \\
&\quad + CS_{n,|\beta|-1,0} + C_\delta (E_n + Z_n^2)^{1/2} E_n \qquad (2.85) \\
&\leq \langle \mathbf{L}\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 + \frac{24}{25}\|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3^2 \\
&\quad + CS_{n,|\beta|-1,0} + C_\delta (E_n + Z_n^2)^{1/2} E_n, \qquad (2.86)
\end{aligned}$$

where we have used (2.83) in the last line. We now use the decomposition (2.81) and then apply Theorem 2.2.5 (with $\boldsymbol{\theta} = \tilde{\boldsymbol{\theta}}$) to obtain

$$\begin{aligned}
&\epsilon \int_{B_R} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \partial_s^a \not{\partial}^\beta \boldsymbol{\theta})|^2 \mathrm{d}\mathbf{x} \\
&\leq C\epsilon \int_{B_R} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \tilde{\boldsymbol{\theta}})|^2 \mathrm{d}\mathbf{x} + C\epsilon \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{x}\rangle_3^2}{\|\mathbf{x}\|_3^2} + C\epsilon \sum_{i=1}^3 \frac{\langle \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \mathbf{e}_i\rangle_3^2}{\|\mathbf{e}_i\|_3^2}
\end{aligned}$$

$$
\begin{aligned}
&\le \langle \mathbf{L}\tilde{\boldsymbol\theta}, \tilde{\boldsymbol\theta}\rangle_3 + \frac{C\epsilon}{|\delta|}\|\partial_s^{a+1}\not\partial^\beta\boldsymbol\theta\|_3^2 + CS_{n,|\beta|-1,0} + C_\delta(E_n+Z_n^2)^{1/2}E_n \\
&\le \langle \mathbf{L}\partial_s^a\not\partial^\beta\boldsymbol\theta, \partial_s^a\not\partial^\beta\boldsymbol\theta\rangle_3 + \frac{24}{25}\|\partial_s^{a+1}\not\partial^\beta\boldsymbol\theta\|_3^2 + \frac{C\epsilon}{|\delta|}\|\partial_s^{a+1}\not\partial^\beta\boldsymbol\theta\|_3^2 \\
&\quad + CS_{n,|\beta|-1,0} + C_\delta(E_n+Z_n^2)^{1/2}E_n \\
&\le \langle \mathbf{L}\partial_s^a\not\partial^\beta\boldsymbol\theta, \partial_s^a\not\partial^\beta\boldsymbol\theta\rangle_3 + \frac{49}{50}\|\partial_s^{a+1}\not\partial^\beta\boldsymbol\theta\|_3^2 + CS_{n,|\beta|-1,0} + C_\delta(E_n+Z_n^2)^{1/2}E_n,
\end{aligned}
\tag{2.87}
$$

where we have chosen ϵ small enough so that $C\epsilon \lesssim 1$ in the second line and then further shrink ϵ so that $\frac{C\epsilon}{|\delta|} < \frac{1}{50}$ in the fourth line. Note that since $\delta = -\frac{1}{2}\not b^2$ (recall (2.2)), the dependence of ϵ on $\not b$ is $\epsilon \sim |\not b|^2$. We have used Lemmas 2.3.1 and 2.3.5 in the second bound, and (2.86) in the third bound. □

2.4 Coercivity via irrotationality

Note that Proposition 2.3.6 only controls the weighted divergence $g = \nabla\cdot(\bar w^3\boldsymbol\theta)$ and not the norms of $\boldsymbol\theta$ in our energy spaces. It is therefore still not strong enough for our energy estimates in Sections 2.5–2.6. To derive the coercivity we seek, we must mod out the kernel of $\mathbf{L}$, i.e. the subspace of $\boldsymbol\theta$ with weighted divergence $g = 0$. This is naturally linked to the assumption of irrotationality (2.24) which guarantees, we show this in the key result of this section – Proposition 2.4.9, that we can in fact dynamically control $\|\partial_s^a\not\partial^\beta\boldsymbol\theta\|_3^2 + \|\partial_s^a\nabla\not\partial^\beta\boldsymbol\theta\|_4^2$ modulo lower order nonlinear terms.

2.4.1 Lagrangian description of irrotationality

From (2.33) it is clear that any H^2 vectorfield $\boldsymbol\theta$ such that $g = \nabla\cdot(\bar w^3\boldsymbol\theta) = 0$ is in the kernel of the operator $\mathbf{L}$. In particular, to obtain strict coercivity of $\mathbf{L}$ we restrict ourselves to $\langle\cdot,\cdot\rangle_3$-orthogonal complement of $K = \{\boldsymbol\theta : \nabla\cdot(\bar w^3\boldsymbol\theta) = 0\}$. Note that $\{\boldsymbol\theta = \nabla\vartheta\} \subseteq K^\perp$ since for any $\boldsymbol\theta_0 \in K$ we have

$$
\langle\nabla\vartheta, \boldsymbol\theta_0\rangle_3 = \int \nabla\vartheta\cdot\boldsymbol\theta_0\bar w^3 \mathrm{dx} = \int \vartheta\nabla\cdot(\boldsymbol\theta_0\bar w^3)\mathrm{dx} = 0.
$$

Therefore, the natural assumption to hope for the strict coercivity of the term on the left-hand side of (2.80) is that $\boldsymbol\theta$ is in fact a gradient. In this section we show that this is true to the top order if we assume that the fluid is irrotational. The challenge is that the irrotationality condition in the Lagrangian variables (2.24) is expressed at the level of the s-derivative of the flow map, and a careful analysis is necessary to obtain satisfactory lower bounds.

Lemma 2.4.1. *Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the fluid is irrotational, i.e. initially* (2.24) *holds. Then for $a > 0$ we have*

$$\partial_s \boldsymbol{\theta} = \nabla \left(\tilde{H} + \frac{1}{2}\mathscr{b}|\boldsymbol{\theta} + \mathbf{x}|^2 \right) - (\partial_s \theta^k)\nabla\theta^k \tag{2.88}$$

$$\partial_s^a \boldsymbol{\theta} = \nabla H_a - \sum_{j=0}^{\lfloor \frac{a-1}{2} \rfloor} C_{a,j}(\partial_s^{a-j}\theta^k)\nabla\partial_s^j\theta^k \tag{2.89}$$

for some real constants $C_{a,j}$, $j \in \{1, \dots, \lfloor \frac{a-1}{2} \rfloor\}$ and H^1-functions H_a and $\tilde{H}$.

Proof. Since the Euler-Poisson equation preserves the fluid irrotational condition, (2.24) implies that $\operatorname{curl}_A \partial_t \boldsymbol{\eta} = \mathbf{0}$ for t, or equivalently in Eulerian coordinates $\nabla \times \mathbf{u} = 0$. Since any curl-free vector field can be written as a gradient, we have $\mathbf{u} = \nabla \hat{H}$ for some $\hat{H}$, or equivalently $\partial_t \boldsymbol{\eta} = A\nabla H$ for some H in Lagrangian coordinates. Since

$$\partial_t \boldsymbol{\eta} = \lambda^{-3/2}\partial_s(\lambda(\boldsymbol{\theta} + \mathbf{x})) = \lambda^{-3/2}((\boldsymbol{\theta} + \mathbf{x})\partial_s\lambda + \lambda\partial_s\boldsymbol{\theta}) = \lambda^{-1/2}(\partial_s\boldsymbol{\theta} - \mathscr{b}(\boldsymbol{\theta} + \mathbf{x})),$$

this means on the level of $\boldsymbol{\theta}$ we have

$$\partial_s\boldsymbol{\theta} - \mathscr{b}(\boldsymbol{\theta} + \mathbf{x}) = \mathscr{A}\nabla\tilde{H}.$$

Hence we have

$$\begin{aligned}
\partial_s\boldsymbol{\theta} &= \nabla\tilde{H} + (\mathscr{A} - I)\nabla\tilde{H} + \mathscr{b}(\boldsymbol{\theta} + \mathbf{x}) = \nabla\tilde{H} + (I - \mathscr{A}^{-1})\mathscr{A}\nabla\tilde{H} + \mathscr{b}(\boldsymbol{\theta} + \mathbf{x}) \\
&= \nabla\tilde{H} - (\partial_s\theta^k - \mathscr{b}(\theta^k + x^k))\nabla\theta^k + \mathscr{b}(\boldsymbol{\theta} + \mathbf{x}) \\
&= \nabla \left(\tilde{H} + \frac{1}{2}\mathscr{b}|\boldsymbol{\theta} + \mathbf{x}|^2 \right) - (\partial_s\theta^k)\nabla\theta^k.
\end{aligned}$$

This proves (2.88). To prove (2.89), we will use induction. We have shown that it is true for $a = 1$. Suppose it is true for some $a \geq 1$. Then

$$\partial_s^{a+1}\boldsymbol{\theta} = \nabla\partial_s H_a - \partial_s \sum_{j=0}^{\lfloor \frac{a-1}{2} \rfloor} C_{a,j}(\partial_s^{a-j}\theta^k)\nabla\partial_s^j\theta^k$$

$$= \nabla\partial_s H_a - \sum_{j=0}^{\lfloor\frac{a-1}{2}\rfloor} C_{a,j}(\partial_s^{a+1-j}\theta^k)\nabla\partial_s^j\theta^k - \sum_{j=0}^{\lfloor\frac{a-1}{2}\rfloor} C_{a,j}(\partial_s^{a-j}\theta^k)\nabla\partial_s^{j+1}\theta^k$$
$$= \nabla\partial_s H_a - \sum_{j=0}^{\lfloor\frac{a-1}{2}\rfloor} C_{a,j}(\partial_s^{a+1-j}\theta^k)\nabla\partial_s^j\theta^k - \sum_{j=1}^{\lfloor\frac{a-1}{2}\rfloor+1} C_{a,j-1}(\partial_s^{a+1-j}\theta^k)\nabla\partial_s^j\theta^k.$$

Note that $\lfloor\frac{a-1}{2}\rfloor+1 > \lfloor\frac{a}{2}\rfloor$ if and only if a is odd. Assume therefore that $a = 2a'+1$ for some $a' \in \mathbb{N}\cup\{0\}$. Then $\lfloor\frac{a-1}{2}\rfloor + 1 = a'+1$ and $\lfloor\frac{a}{2}\rfloor = a'$. When $j = a'+1$ in the last sum, we have

$$\begin{aligned} C_{2a'+1,a'}(\partial_s^{2a'+2-(a'+1)}\theta^k)\nabla\partial_s^{a'+1}\theta^k &= C_{2a'+1,a'}(\partial_s^{a'+1}\theta^k)\nabla\partial_s^{a'+1}\theta^k \\ &= \frac{1}{2}C_{2a'+1,a'}\nabla(\partial_s^{a'+1}\theta^k)^2 \end{aligned}$$

which can be absorbed into H_{a+1}. Therefore

$$\begin{aligned} \partial_s^{a+1}\boldsymbol{\theta} = \nabla &\left(\partial_s H_a + \frac{1}{2}\mathbf{1}[a \text{ odd}]C_{a,a'}\nabla(\partial_s^{a'+1}\theta^k)^2\right) \\ &- \sum_{j=0}^{\lfloor\frac{a-1}{2}\rfloor} C_{a,j}(\partial_s^{a+1-j}\theta^k)\nabla\partial_s^j\theta^k - \sum_{j=1}^{\lfloor\frac{a}{2}\rfloor} C_{a,j-1}(\partial_s^{a+1-j}\theta^k)\nabla\partial_s^j\theta^k \\ = \nabla &H_{a+1} - \sum_{j=0}^{\lfloor\frac{a}{2}\rfloor} C_{a+1,j}(\partial_s^{a-j}\theta^k)\nabla\partial_s^j\theta^k, \end{aligned}$$

where $\mathbf{1}[\star]$ denotes the Iverson bracket (see Definition 1.4.2). This completes the induction argument. □

Remark 2.4.2. *The above lemma is a purely structural statement about (suitably smooth) irrotational fields. Strictly speaking we do not need $\boldsymbol{\theta}$ to be a solution of the Euler-Poisson system* (2.7).

With this we can now show that the curl of $\partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}$ equals lower order terms and non-linear terms.

Lemma 2.4.3. *Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the fluid is irrotational, i.e. initially* (2.24) *holds. Then for $a > 0$ we have*

$$\nabla \times \partial_s^a\boldsymbol{\theta} = -\partial_s^{a-1}((\nabla\partial_s\theta^k)\times\nabla\theta^k). \tag{2.90}$$

Moreover, for some constants $C_{\gamma,\beta} > 0$ we have

$$\nabla \times \partial_s^a \not{\partial}^\beta \boldsymbol{\theta} = -\partial_s^{a-1} \not{\partial}^\beta ((\nabla \partial_s \theta^k) \times \nabla \theta^k) + \sum_{|\gamma|<|\beta|} C_{\gamma,\beta} \langle \nabla \partial_s^a \not{\partial}^\gamma \boldsymbol{\theta} \rangle, \tag{2.91}$$

$$\nabla \times \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta} = -\partial_s^{a-1} X_r^b \not{\partial}^\beta ((\nabla \partial_s \theta^k) \times \nabla \theta^k) + \sum_{|\gamma|+d<|\beta|+b} C_{\gamma,\beta} \langle \nabla \partial_s^a X_r^d \not{\partial}^\gamma \boldsymbol{\theta} \rangle, \tag{2.92}$$

where we recall notations defined in Definition 1.4.2.

Proof. Apply $\nabla \times \partial_s^{a-1}$ to (2.88) to get (2.90). Formulas (2.91)–(2.92) follow trivially when $|\beta| = 0 = b$. Now assume formula (2.91) is true for a multi-index β, $|\beta| \geq 0$. Then

$$\begin{aligned}
\nabla \times \partial_s^a \not{\partial}_j \not{\partial}^\beta \boldsymbol{\theta} &= \langle \nabla \partial_s^a \not{\partial}^\beta \boldsymbol{\theta} \rangle + \not{\partial}_j \nabla \times \partial_s^a \not{\partial}^\beta \boldsymbol{\theta} \\
&= \langle \nabla \partial_s^a \not{\partial}^\beta \boldsymbol{\theta} \rangle - \partial_s^{a-1} \not{\partial}_j \not{\partial}^\beta ((\nabla \partial_s \theta^k) \times \nabla \theta^k) + \sum_{|\gamma|<|\beta|} C_{\gamma,\beta} \langle \not{\partial}_j \nabla \partial_s^a \not{\partial}^\gamma \boldsymbol{\theta} \rangle \\
&= -\partial_s^{a-1} \not{\partial}_j \not{\partial}^\beta ((\nabla \partial_s \theta^k) \times \nabla \theta^k) + \sum_{|\gamma|<|\beta|+1} C'_{\gamma,\beta} \langle \nabla \partial_s^a \not{\partial}^\gamma \boldsymbol{\theta} \rangle,
\end{aligned}$$

where we recall the notation from Definition 1.4.2 and the commutation relation $[\not{\partial}_j, \nabla] = \langle \nabla \rangle$ from Lemma A.1.2. The proof then follows by induction. The proof of (2.92) is similar, using the commutation relation $[X_r, \nabla] = \langle \nabla \rangle$ from Lemma A.1.2. □

Corollary 2.4.4. *Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the fluid is irrotational, i.e. initially* (2.24) *holds. Let $n \geq 20$.*

i. For $a + |\beta| \leq n$ with $a > 0$ we have

$$\|\nabla \times \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\|_4^2 \lesssim S_{n,|\beta|-1,0} + (E_n + Z_n^2) E_n.$$

ii. For $a + |\beta| + b \leq n$ with $a > 0$ we have

$$\|\nabla \times \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\|_{4+b}^2 \lesssim S_{n,|\beta|+b-1} + (E_n + Z_n^2) E_n.$$

Proof. Use Lemma 2.4.3 and note that

$$\left\| \partial_s^{a-1} \not{\partial}^\beta ((\nabla \partial_s \theta^k) \times \nabla \theta^k) \right\|_4^2 \lesssim (E_n + Z_n^2) E_n$$

$$\left\| \sum_{|\gamma|<|\beta|} C_{\gamma,\beta} \langle \nabla \partial_s^a \not\partial^\gamma \boldsymbol{\theta} \rangle \right\|_4^2 \lesssim S_{n,|\beta|-1,0},$$

which yields the first claim. The second claim follows similarly. □

2.4.2 Coercivity of L

The lemmas in the last subsection showed that $\partial_s^a \boldsymbol{\theta}$ is a gradient on the linear level, which will ultimately help us show that $\|\partial_s^a \not\partial^\beta \boldsymbol{\theta}\|_3^2 + \|\partial_s^a \nabla \not\partial^\beta \boldsymbol{\theta}\|_4^2$ can be "controlled" by the linearised dynamics. We start by showing we can control $\|\partial_s^a \boldsymbol{\theta}\|_3^2$ in the following lemma.

Lemma 2.4.5. *Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the fluid is irrotational, i.e. initially* (2.24) *holds. Let $n \geq 20$. Then we have the bound*

$$\|\partial_s^a \boldsymbol{\theta}\|_3^2 \lesssim \int_{B_R} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \partial_s^a \boldsymbol{\theta})|^2 \mathrm{dx} + (E_n + Z_n^2)^{1/2} E_n \qquad \textit{for} \qquad 0 < a \leq n. \tag{2.93}$$

Proof. Let $g = \nabla \cdot (\bar{w}^3 \partial_s^a \boldsymbol{\theta})$. Multiply both sides of this equation by H_a and integrate over B_R to get

$$\begin{aligned} \int_{B_R} \bar{w}^3 (\nabla H_a) \cdot \partial_s^a \boldsymbol{\theta} \, \mathrm{dx} &= -\int_{B_R} g H_a \mathrm{dx} = -\int_{B_R} g(H_a - (H_a)_{B_{2R/3}}) \mathrm{dx} \\ &\leq \epsilon^{-1} \int_{B_R} \bar{w}^{-2} g^2 \mathrm{dx} + \epsilon \int_{B_R} (H_a - (H_a)_{B_{2R/3}})^2 \bar{w}^2 \mathrm{dx} \\ &\leq \epsilon^{-1} \int_{B_R} \bar{w}^{-2} g^2 \mathrm{dx} + \epsilon C' \int_{B_R} |\nabla H_a|^2 \bar{w}^4 \mathrm{dx} \end{aligned}$$

where we have used the Hardy-Poincaré inequality in the last line, see Theorem A.3.1. From this and Lemma 2.4.1 we get

$$\begin{aligned} \int_{B_R} \bar{w}^3 |\partial_s^a \boldsymbol{\theta}|^2 \mathrm{dx} \leq \epsilon^{-1} \int_{B_R} \bar{w}^{-2} g^2 \mathrm{dx} + \epsilon C' \int_{B_R} |\partial_s^a \boldsymbol{\theta}|^2 \bar{w}^4 \mathrm{dx} \\ + (1+\epsilon) C'' (E_n + Z_n^2)^{1/2} E_n \end{aligned}$$

where we bound for example

$$\left| \int_{B_R} \bar{w}^3 \left(\sum_{j=0}^{\lfloor \frac{a-1}{2} \rfloor} C_{a,j} (\partial_s^{a-j} \theta^k) \nabla \partial_s^j \theta^k \right) \cdot \partial_s^a \boldsymbol{\theta} \, \mathrm{dx} \right|$$

$$\lesssim (E_n + Z_n^2)^{1/2} \left| \int_{B_R} \bar{w}^3 \left(\sum_{j=0}^{\lfloor \frac{a-1}{2} \rfloor} C_{a,j} (\partial_s^{a-j} \theta^k) \right) \cdot \partial_s^a \boldsymbol{\theta} \, \mathrm{d}\mathbf{x} \right|$$
$$\lesssim (E_n + Z_n^2)^{1/2} S_n$$

Choosing ϵ small enough, we get (2.93). □

Before proving the key result of this section, we have the following structural decomposition, which holds for any sufficiently smooth vectorfield $\boldsymbol{\theta}$.

Lemma 2.4.6. *For any $\boldsymbol{\theta}$ such that $\|\boldsymbol{\theta}\|_3 + \|\nabla\boldsymbol{\theta}\|_4 < \infty$ we have*

$$\begin{aligned}
&\int_{B_R} \frac{4}{3} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \boldsymbol{\theta})|^2 \mathrm{d}\mathbf{x} \\
&= \int_{B_R} \left(\bar{w}^4 \left(\frac{1}{3} |\nabla \cdot \boldsymbol{\theta}|^2 + |\nabla \boldsymbol{\theta}|^2 + [\operatorname{curl} \boldsymbol{\theta}]_l^k \partial_k \theta^l \right) - 4 \bar{w}^3 \theta^k \theta^l \partial_k \partial_l \bar{w} \right) \mathrm{d}\mathbf{x} \\
&= \int_{B_R} \Bigg(\bar{w}^4 \left(\frac{1}{3} |\nabla \cdot \boldsymbol{\theta}|^2 + |\nabla \boldsymbol{\theta}|^2 - \frac{1}{2} |\operatorname{curl} \boldsymbol{\theta}|^2 \right) \\
&\qquad - \bar{w}^3 \left(\bar{w}'' |\boldsymbol{\theta} \cdot \mathbf{e}_r|^2 + \frac{\bar{w}'}{r} (|\boldsymbol{\theta}|^2 - |\boldsymbol{\theta} \cdot \mathbf{e}_r|^2) \right) \Bigg) \mathrm{d}\mathbf{x}.
\end{aligned}$$

where $\mathbf{e}_r$ denotes the radial unit vector $\mathbf{x}/|\mathbf{x}|$.

Proof. The first line follows from the following identity:

$$\begin{aligned}
-\frac{4}{3} \nabla (\bar{w}^{-2} \nabla \cdot (\bar{w}^3 \boldsymbol{\theta})) &= -\frac{1}{3\bar{w}^3} \nabla (\bar{w}^4 \nabla \cdot \boldsymbol{\theta}) - \frac{1}{\bar{w}^3} \partial_k (\bar{w}^4 \nabla \theta^k) - 4 \boldsymbol{\theta} \cdot \nabla \nabla \bar{w} \\
&= -\frac{1}{3\bar{w}^3} \nabla (\bar{w}^4 \nabla \cdot \boldsymbol{\theta}) - \frac{1}{\bar{w}^3} \partial_k (\bar{w}^4 (\partial_k \boldsymbol{\theta} + [\operatorname{curl} \boldsymbol{\theta}]_\bullet^k) \\
&\quad - 4 \boldsymbol{\theta} \cdot \nabla \nabla \bar{w}.
\end{aligned}$$

And then the second line follows from

$$\begin{aligned}
[\operatorname{curl} \boldsymbol{\theta}]_l^k [\operatorname{curl} \boldsymbol{\theta}]_l^k &= (\partial_l \theta^k - \partial_k \theta^l)(\partial_l \theta^k - \partial_k \theta^l) \\
&= (\partial_k \theta^l - \partial_l \theta^k) \partial_k \theta^l - (\partial_l \theta^k - \partial_k \theta^l) \partial_k \theta^l \\
&= -2 (\partial_l \theta^k - \partial_k \theta^l) \partial_k \theta^l = -2 [\operatorname{curl} \boldsymbol{\theta}]_l^k \partial_k \theta^l
\end{aligned}$$

and

$$\begin{aligned}
\theta^k \theta^l \partial_k \partial_l \bar{w} &= \theta^k \theta^l \partial_k \left(\bar{w}' \frac{x^l}{r} \right) = \theta^k \theta^l \left(\bar{w}'' \frac{x^l x^k}{r^2} + \bar{w}' \frac{\delta_k^l}{r} - \bar{w}' \frac{x^l x^k}{r^3} \right) \\
&= \bar{w}'' |\boldsymbol{\theta} \cdot \mathbf{e}_r|^2 + \frac{\bar{w}'}{r} (|\boldsymbol{\theta}|^2 - |\boldsymbol{\theta} \cdot \mathbf{e}_r|^2).
\end{aligned}$$

□

Using this, we can now prove that we can control $\|\partial_s^a \boldsymbol{\theta}\|_3^2 + \|\partial_s^a \nabla \boldsymbol{\theta}\|_4^2$.

Proposition 2.4.7. *Let* $n \geq 20$. *Let* $\boldsymbol{\theta}$ *be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the energy, momentum, and irrotationality constraints* (2.22), (2.23), *and* (2.24) *hold respectively. Then for any* $0 < a \leq n$ *we have*

$$\|\partial_s^a \boldsymbol{\theta}\|_3^2 + \|\partial_s^a \nabla \boldsymbol{\theta}\|_4^2 \lesssim |\mathscr{b}|^{-2} \left(\frac{49}{50} \left\|\partial_s^{a+1} \boldsymbol{\theta}\right\|_3^2 + \langle \mathbf{L} \partial_s^a \boldsymbol{\theta}, \partial_s^a \boldsymbol{\theta} \rangle \right) + C_\delta (E_n + Z_n^2)^{1/2} E_n \tag{2.94}$$

Proof. Combining Proposition 2.3.6 and Lemma 2.4.5 we have, for small ϵ,

$$\underbrace{\epsilon \int_{B_R} \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \partial_s^a \boldsymbol{\theta})|^2 \mathrm{d}\mathbf{x} + \|\partial_s^a \boldsymbol{\theta}\|_3^2}_{:=M}$$
$$\lesssim |\mathscr{b}|^{-2} \left(\langle \mathbf{L} \partial_s^a \boldsymbol{\theta}, \partial_s^a \boldsymbol{\theta} \rangle_3 + \frac{49}{50} \|\partial_s^{a+1} \boldsymbol{\theta}\|_3^2 \right) + C_\delta (E_n + Z_n^2)^{1/2} E_n.$$

Note that by Corollary 2.4.4 $\|\operatorname{curl} \partial_s^a \boldsymbol{\theta}\|_4^2 \lesssim (E_n + Z_n^2) E_n$. Using Lemma 2.4.6 we have

$$\begin{aligned}
M &= \int_{\mathbb{R}^3} \left(\epsilon \bar{w}^{-2} |\nabla \cdot (\bar{w}^3 \partial_s^a \boldsymbol{\theta})|^2 + \bar{w}^3 |\partial_s^a \boldsymbol{\theta}|^2 \right) \mathrm{d}\mathbf{x} \\
&\geq \int_{\mathbb{R}^3} \left(\epsilon \frac{3}{4} \bar{w}^4 \left(|\nabla \partial_s^a \boldsymbol{\theta}|^2 - \frac{1}{2} |\operatorname{curl} \partial_s^a \boldsymbol{\theta}|^2 \right) - \epsilon \frac{3}{4} \bar{w}^3 \bar{w}'' |\partial_s^a \boldsymbol{\theta} \cdot \mathbf{e}_r|^2 + \bar{w}^3 |\partial_s^a \boldsymbol{\theta}|^2 \right) \mathrm{d}\mathbf{x} \\
&\geq \int_{\mathbb{R}^3} \left(\epsilon \frac{3}{4} \bar{w}^4 |\nabla \partial_s^a \boldsymbol{\theta}|^2 - \epsilon \frac{3}{4} \bar{w}^3 \bar{w}'' |\partial_s^a \boldsymbol{\theta} \cdot \mathbf{e}_r|^2 + \bar{w}^3 |\partial_s^a \boldsymbol{\theta}|^2 \right) \mathrm{d}\mathbf{x} - \epsilon C (E_n + Z_n^2) E_n
\end{aligned}$$

Choosing ϵ small enough, we then have

$$M + (E_n + Z_n^2) S_n \gtrsim \int_{\mathbb{R}^3} \left(\bar{w}^4 |\nabla \partial_s^a \boldsymbol{\theta}|^2 + \bar{w}^3 |\partial_s^a \boldsymbol{\theta}|^2 \right) \mathrm{d}\mathbf{x}.$$

□

Next we will upgrade our estimate to control $\|\partial_s^a \slashed{\partial}^\beta \boldsymbol{\theta}\|_3^2 + \|\partial_s^a \nabla \slashed{\partial}^\beta \boldsymbol{\theta}\|_4^2$ for $|\beta| > 0$. First we will need the following lemma.

Lemma 2.4.8. *For any vector field* $\boldsymbol{\theta}$ *and* $\epsilon > 0$,

$$\|\boldsymbol{\theta}\|_3^2 \lesssim \epsilon \|X_r \boldsymbol{\theta}\|_4^2 + (1 + \epsilon^{-1}) \|\boldsymbol{\theta}\|_4^2.$$

Proof. We have

$$
\begin{aligned}
-\int_{B_R} |\boldsymbol{\theta}|^2 r\bar{w}'\bar{w}^3 \mathrm{d}\mathbf{x} &= -\frac{1}{4}\int_{B_R} |\boldsymbol{\theta}|^2 X_r \bar{w}^4 \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_{B_R} \bar{w}^4 \boldsymbol{\theta}\cdot X_r\boldsymbol{\theta}\ \mathrm{d}\mathbf{x} + \frac{3}{4}\int_{B_R} |\boldsymbol{\theta}|^2 \bar{w}^4 \mathrm{d}\mathbf{x} \\
&\le \epsilon\|X_r\boldsymbol{\theta}\|_4^2 + \frac{1}{4}(3+\epsilon^{-1})\|\boldsymbol{\theta}\|_4^2.
\end{aligned}
$$

Now

$$
\|\boldsymbol{\theta}\|_3^2 \lesssim \|\boldsymbol{\theta}\|_4^2 - \int_{B_R} |\boldsymbol{\theta}|^2 r\bar{w}'\bar{w}^3 \mathrm{d}\mathbf{x} \le \epsilon\|X_r\boldsymbol{\theta}\|_4^2 + \frac{1}{4}(7+\epsilon^{-1})\|\boldsymbol{\theta}\|_4^2.
$$

□

Proposition 2.4.9. *Let $n \ge 20$. Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the energy, momentum, and irrotationality constraints* (2.22), (2.23), *and* (2.24) *hold respectively. Then for any $a+|\beta| \le n$ with $a, |\beta| > 0$ we have*

$$
\begin{aligned}
&\left\|\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a \nabla \not{\partial}^\beta \boldsymbol{\theta}\right\|_4^2 \\
&\lesssim |\not{b}|^{-2}\left(\frac{49}{50}\left\|\partial_s^{a+1}\not{\partial}^\beta\boldsymbol{\theta}\right\|_3^2 + \langle \mathbf{L}\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}, \partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\rangle\right) + CS_{n,|\beta|-1,0} + C_\delta(E_n+Z_n^2)^{1/2}E_n
\end{aligned}
\tag{2.95}
$$

Proof. By Proposition 2.3.6 and Lemma 2.4.6 we have

$$
\begin{aligned}
&\|\nabla\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\|_4^2 - \frac{1}{2}\|\operatorname{curl}\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\|_4^2 - \int_{B_R}\bar{w}''\bar{w}^3|\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\cdot\mathbf{e}_r|^2\mathrm{d}\mathbf{x} \\
&\le \int_{B_R}\frac{4}{3}\bar{w}^{-2}|\nabla\cdot(\bar{w}^3\partial_s^a\not{\partial}^\beta\boldsymbol{\theta})|^2\mathrm{d}\mathbf{x} \\
&\lesssim |\not{b}|^{-2}\left(\langle\mathbf{L}\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}, \partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\rangle_3 + \frac{49}{50}\|\partial_s^{a+1}\not{\partial}^\beta\boldsymbol{\theta}\|_3^2\right) + C_\delta(E_n+Z_n^2)E_n.
\end{aligned}
$$

By Corollary 2.4.4 we have

$$
\begin{aligned}
&\|\nabla\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\|_4^2 - \int_{B_R}\bar{w}''\bar{w}^3|\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\cdot\mathbf{e}_r|^2\mathrm{d}\mathbf{x} \\
&\lesssim |\not{b}|^{-2}\left(\langle\mathbf{L}\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}, \partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\rangle_3 + \frac{49}{50}\|\partial_s^{a+1}\not{\partial}^\beta\boldsymbol{\theta}\|_3^2\right) + CS_{n,|\beta|-1,0} + C_\delta(E_n+Z_n^2)E_n.
\end{aligned}
$$

Now by Lemma 2.4.8, we have $\frac{1}{2\epsilon}\|\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\|_3^2 - \frac{1}{2}\|\nabla\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\|_4^2 \lesssim_\epsilon \|\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\|_4^2 \le$

$S_{n,|\beta|-1,0}$. Adding this and the above equation, and chosing ϵ small enough we get

$$\|\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\|_3^2 + \|\nabla \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\|_4^2 \\ \lesssim |\delta|^{-2} \left(\langle \mathbf{L} \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta} \rangle_3 + \frac{49}{50} \|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3^2 \right) + C S_{n,|\beta|-1,0} + C_\delta (E_n + Z_n^2) E_n.$$

□

Remark 2.4.10. *Estimate* (2.95) *features an order 1 term $CS_{n,|\beta|-1,0}$ on the right-hand side. This could be problematic for the closure of the estimates, but the key point is that this term is effectively decoupled, as it features one tangential derivative less. This will allows us later to close the estimates via induction on the order of derivatives in the problem.*

2.5 Reduction to linear problem

In order to prove the bound (2.31), we will need to apply the coercivity estimates from Section 2.4. In particular, we must control the non-linear terms in order to effectively reduce the problem to a linear one. In Sections 2.5.1 and 2.5.2 we will prove high-order energy bounds for the nonlinear contributions from the pressure and the gravity term respectively. We will also prove high-order energy bounds for the full gravity term (including the linear part) in Section 2.5.2 that we will need for induction on radial derivatives. Then using these, we will reduce the full non-linear problem to the linear one in Section 2.5.3. This will then allow us to prove energy estimates and our main theorem in Section 2.6.

2.5.1 Estimating the non-linear part of the pressure term

In this subsection we will estimate the non-linear part of the pressure term $\partial_s^a X_r^b \not{\partial}^\beta \mathbf{P}$ (2.7), and show that it can be bounded by $(\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2} \mathscr{E}_n$.

Recall from (2.8) that $\mathbf{P} := \bar{w}^{-3} \partial_k (\bar{w}^4 (\mathscr{A}^k \mathscr{J}^{-1/3} - I^k))$. In the next two lemmas, we will compute the commutators between the operator $\partial_s^a X_r^b \not{\partial}^\beta$ and the weighted derivative $\bar{w}^{-3} \partial_k (\bar{w}^4 \cdot)$. Lemma 2.5.1 deals with the case $b = 0$ (no radial derivatives) while Lemma 2.5.2 includes the radial derivatives. Lemmas 2.5.1 and 2.5.2 are necessary to control all the non-“top-order” contributions coming from $\partial_s^a X_r^b \not{\partial}^\beta \mathbf{P}$ by our energy norms.

Lemma 2.5.1. *For any tensor field T_i^k sufficiently smooth, we have*

$$\not{\partial}^\beta \left(\bar{w}^{-3}\partial_k(\bar{w}^4 T_i^k)\right) = \bar{w}^{-3}\partial_k(\bar{w}^4 \not{\partial}^\beta T_i^k) + \sum_{|\beta'|\le|\beta|-1} \langle C\bar{w}^{-3}\nabla(\bar{w}^4\not{\partial}^{\beta'}T)\rangle \tag{2.96}$$

for $i = 1, 2, 3$, where we recall notations defined in Definition 1.4.2.

Proof. We will prove this by induction. Assume this is true for β, then we have

$$\begin{aligned}
\not{\partial}_j\not{\partial}^\beta \left(\bar{w}^{-3}\partial_k(\bar{w}^4 T_i^k)\right) &= \bar{w}^{-3}\partial_k(\bar{w}^4\not{\partial}_j\not{\partial}^\beta T_i^k) - \bar{w}^{-3}\epsilon_{jkl}\partial_l(\bar{w}^4\not{\partial}^\beta T_i^k) \\
&\quad + \not{\partial}_j \sum_{|\beta'|\le|\beta|-1} \langle C\bar{w}^{-3}\nabla(\bar{w}^4\not{\partial}^{\beta'}T)\rangle \\
&= \bar{w}^{-3}\partial_k(\bar{w}^4\not{\partial}_j\not{\partial}^\beta T_i^k) + \sum_{|\beta'|\le|\beta|} \langle C\bar{w}^{-3}\nabla(\bar{w}^4\not{\partial}^{\beta'}T)\rangle.
\end{aligned}$$

where we used the commutation relation for $[\not{\partial}_j, \partial_k]$ from Lemma A.1.2. □

The use of radial derivatives naturally changes the weighting structure, which is one of the key observations that makes the high-order energy argument possible and goes back to [33].

Lemma 2.5.2. *For any tensor field T_i^k sufficiently smooth, we have*

$$\begin{aligned}
X_r \left(\bar{w}^{-c}\partial_k(\bar{w}^{1+c}T_i^k)\right) &= \bar{w}^{-(1+c)}\partial_k(\bar{w}^{2+c}X_r T_i^k) \\
&\quad + (1+c)(T_i^k X_r\partial_k\bar{w}) + (\partial_k\bar{w})\not{\partial}_{kj}T_i^j - \bar{w}\partial_k T_i^k \\
\not{\partial}_j \left(\bar{w}^{-c}\partial_k(\bar{w}^{1+c}T_i^k)\right) &= \bar{w}^{-c}\partial_k(\bar{w}^{1+c}\not{\partial}_j T_i^k) \\
&\quad - \epsilon_{jkl}((1+c)(\partial_l\bar{w})T_i^k + \bar{w}\partial_l T_i^k)) \\
X_r^d\not{\partial}^\beta \left(\bar{w}^{-3}\partial_k(\bar{w}^4 T_i^k)\right) &= \bar{w}^{-(3+d)}\partial_k(\bar{w}^{4+d}X_r^d\not{\partial}^\beta T_i^k) \\
&\quad + \left(\sum_{\substack{d'\le d\\ |\beta'|\le|\beta|-1}} + \sum_{\substack{d'\le d-1\\ |\beta'|\le|\beta|+1}}\right)\langle C\omega X_r^{d'}\not{\partial}^{\beta'}T\rangle \\
&\quad + \left(\sum_{\substack{d'\le d-1\\ |\beta'|\le|\beta|-1}} + \sum_{\substack{d'\le d-2\\ |\beta'|\le|\beta|}}\right)\langle C X_r^{d'}\not{\partial}^{\beta'}\nabla T\rangle \\
&\quad + \left(\sum_{\substack{d'\le d\\ |\beta'|\le|\beta|-1}} + \sum_{\substack{d'\le d-1\\ |\beta'|\le|\beta|}}\right)\langle C\bar{w} X_r^{d'}\not{\partial}^{\beta'}\nabla T\rangle
\end{aligned} \tag{2.97}$$

for any $c \ge 0$ and $i = 1, 2, 3$, where ω denotes some derivatives of $\bar{w}$. Here we *used notations defined in Definition 1.4.2.*

Proof. First note that

$$x^i X_r T^i = x^i x^j \partial_j T^i = r^2 \partial_i T^i + x^j (x^i \partial_j - x^j \partial_i) T^i = r^2 \partial_i T^i - x^i \not{\partial}_{ij} T^j.$$

Using this we have

$$\begin{aligned}
X_r \left(\bar{w}^{-c} \partial_k (\bar{w}^{1+c} T_i^k) \right) &= X_r \left((1+c) T_i^k \partial_k \bar{w} + \bar{w} \partial_k T_i^k \right) \\
&= (1+c) \left((X_r T_i^k) \partial_k \bar{w} + T_i^k X_r \partial_k \bar{w} \right) + (\mathbf{x} \cdot \nabla \bar{w}) \partial_k T_i^k \\
&\quad + \bar{w} X_r \partial_k T_i^k \\
&= (1+c) \left((X_r T_i^k) \partial_k \bar{w} + T_i^k X_r \partial_k \bar{w} \right) \\
&\quad + (\mathbf{x} \cdot \nabla \bar{w}) r^{-2} (x^k X_r T_i^k + x^k \not{\partial}_{kj} T_i^j) \\
&\quad + \bar{w} \partial_k X_r T_i^k - \bar{w} \partial_k T_i^k \\
&= (1+c) \left((X_r T_i^k) \partial_k \bar{w} + T_i^k X_r \partial_k \bar{w} \right) \\
&\quad + (\partial_k \bar{w})(X_r T_i^k + \not{\partial}_{kj} T_i^j) + \bar{w} \partial_k X_r T_i^k - \bar{w} \partial_k T_i^k \\
&= \bar{w}^{-(1+c)} \partial_k (\bar{w}^{2+c} X_r T_i^k) + (1+c)(T_i^k X_r \partial_k \bar{w}) \\
&\quad + (\partial_k \bar{w}) \not{\partial}_{kj} T_i^j - \bar{w} \partial_k T_i^k \\
\not{\partial}_j \left(\bar{w}^{-c} \partial_k (\bar{w}^{1+c} T_i^k) \right) &= \bar{w}^{-c} \partial_k (\bar{w}^{1+c} \not{\partial}_j T_i^k) - \epsilon_{jkl} \bar{w}^{-c} \partial_l (\bar{w}^{1+c} T_i^k).
\end{aligned}$$

where we used commutation relations from Lemma A.1.2. The final formula can be proven by induction. □

The next lemma deals with the terms we get when we apply $\partial_s^a X_r^b \not{\partial}^\beta$ to $\mathscr{A} \mathscr{J}^{-1/3} - I$.

Lemma 2.5.3. *Let*

$$T := \mathscr{A} \mathscr{J}^{-1/3} - I. \tag{2.98}$$

Recall notations defined in Definition 1.4.2. For $a > 0$ and $|\gamma| > 0$, we have

$$\partial_\bullet T = T_T[\partial_\bullet \nabla \theta], \tag{2.99}$$

$$\partial_s^a X_r^d \not{\partial}^\beta T = T_T[\partial_s^a X_r^b \not{\partial}^\beta \nabla \theta] + T_{R:a,\beta,d} \tag{2.100}$$

$$\partial^\gamma T = T_T[\partial^\gamma \nabla \theta] + T_{R:\gamma}. \tag{2.101}$$

where

$$T_T[M]^k := -\mathscr{J}^{-1/3}\left(\mathscr{A}^k_m\mathscr{A}^l + \frac{1}{3}\mathscr{A}^k\mathscr{A}^l_m\right)M^m_l, \qquad k = 1,2,3 \tag{2.102}$$

$$T_{R:a,\beta,d} := \mathscr{J}^{-1/3}\sum_{c=2}^{a+d+|\beta|}\sum_{\substack{\sum_{i=1}^c(a_i,d_i,\beta_i)=(a,d,\beta)\\|a_i|+|d_i|+|\beta_i|>0}}\langle C\rangle\langle\mathscr{A}\rangle^{1+c}\prod_{i=1}^c\langle\partial_s^{a_i}X_r^{d_i}\not\partial^{\beta_i}\nabla\boldsymbol{\theta}\rangle \tag{2.103}$$

$$T_{R:\gamma} := \mathscr{J}^{-1/3}\sum_{c=2}^{|\gamma|}\sum_{\substack{\sum_{i=1}^c\gamma_i=\gamma\\|\gamma_i|>0}}\langle C\rangle\langle\mathscr{A}\rangle^{1+c}\prod_{i=1}^c\langle\partial^{\gamma_i}\nabla\boldsymbol{\theta}\rangle. \tag{2.104}$$

Proof. Applying Lemma A.1.1 we get that

$$\begin{aligned}\partial_\bullet(\mathscr{A}^k\mathscr{J}^{-1/3} - I^k) &= -\mathscr{J}^{-1/3}\mathscr{A}^k_m\mathscr{A}^l\partial\partial_l\theta^m - \frac{1}{3}\mathscr{J}^{-1/3}\mathscr{A}^k\mathscr{A}^l_m\partial_\bullet\partial_l\theta^m\\&= -\mathscr{J}^{-1/3}\left(\mathscr{A}^k_m\mathscr{A}^l + \frac{1}{3}\mathscr{A}^k\mathscr{A}^l_m\right)\partial_\bullet\partial_l\theta^m.\end{aligned}$$

Hence $\partial_\bullet T^k = T_T[\partial_\bullet\nabla\theta]^k$. By repeated application of this we get the next two formulas. ☐

We have from (2.7)

$$\partial_s^aX_r^b\not\partial^\beta\mathbf{P} = \partial_s^aX_r^b\not\partial^\beta\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right)$$

Let

$$\begin{aligned}\mathbf{P}_d\boldsymbol{\theta} &:= \bar{w}^{-3-d}\partial_k(\bar{w}^{4+d}T_T[\nabla\boldsymbol{\theta}]^k) &(2.105)\\&= -\bar{w}^{-(3+d)}\partial_k\left(\bar{w}^{4+d}\left(\mathscr{A}^k_m\mathscr{A}^l + \frac{1}{3}\mathscr{A}^k\mathscr{A}^l_m\right)\partial_l\theta^m\right) &(2.106)\end{aligned}$$

Let $\mathbf{P}_{d,L}$ be the linear part of $\mathbf{P}_d$, i.e.

$$\begin{aligned}\mathbf{P}_{d,L}\boldsymbol{\theta} &:= -\bar{w}^{-(3+d)}\partial_k\left(\bar{w}^{4+d}\left(I^k_mI^l + \frac{1}{3}I^kI^l_m\right)\partial_l\theta^m\right)\\&= -\frac{1}{3\bar{w}^{3+d}}\nabla(\bar{w}^{4+d}\nabla\cdot\boldsymbol{\theta}) - \frac{1}{\bar{w}^{3+d}}\partial_k(\bar{w}^{4+d}\nabla\theta^k) &(2.107)\end{aligned}$$

In doing energy estimates, the term $\langle\partial_s^aX_r^b\not\partial^\beta\mathbf{P}, \partial_s^aX_r^b\not\partial^\beta\boldsymbol{\theta}\rangle$ and $\langle\partial_s^aX_r^b\not\partial^\beta\mathbf{P}, \partial_s^{a+1}X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle$ will arise. Using the lemmas in this subsection, we will next show that $\mathbf{P}$ here can be reduced to $\mathbf{P}_{d,L}$ modulo remainder terms that can be estimated.

To that end we will first derive the following identity.

Lemma 2.5.4. *For any vector field $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ sufficiently smooth we have*

$$\begin{aligned}\langle \mathbf{P}_d\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\rangle_{3+d} &= \int \Bigg((\mathscr{A}\partial_m \boldsymbol{\theta}_1)\cdot(\mathscr{A}\partial_m \boldsymbol{\theta}_2) + \frac{1}{3}(\operatorname{div}_{\mathscr{A}} \boldsymbol{\theta}_1)(\operatorname{div}_{\mathscr{A}} \boldsymbol{\theta}_2) \\ &\qquad - \frac{1}{2}[\operatorname{curl}_{\mathscr{A}} \boldsymbol{\theta}_1]^m_j [\operatorname{curl}_{\mathscr{A}} \boldsymbol{\theta}_2]^m_j \Bigg) \mathscr{J}^{-1/3}\bar{w}^{4+d}\mathbf{dx} \end{aligned} \tag{2.108}$$

$$\begin{aligned}\langle \mathbf{P}_{d,L}\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\rangle_{3+d} &= \int \left((\partial_m \boldsymbol{\theta}_1)\cdot(\partial_m \boldsymbol{\theta}_2) + \frac{1}{3}(\operatorname{div} \boldsymbol{\theta}_1)(\operatorname{div} \boldsymbol{\theta}_2) - \frac{1}{2}[\operatorname{curl} \boldsymbol{\theta}_1]^m_j [\operatorname{curl} \boldsymbol{\theta}_2]^m_j \right) \bar{w}^{4+d}\mathbf{dx}\end{aligned} \tag{2.109}$$

Proof. We have

$$\langle \mathbf{P}_d\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\rangle_{3+d} = \int \left((\mathscr{A}\partial_l \theta_1^m)\cdot(\mathscr{A}\partial_m \theta_2^l) + \frac{1}{3}(\operatorname{div}_{\mathscr{A}} \boldsymbol{\theta}_1)(\operatorname{div}_{\mathscr{A}} \boldsymbol{\theta}_2)\right) \mathscr{J}^{-1/3}\bar{w}^{4+d}\mathbf{dx}.$$

We are done for $\mathbf{P}_d$ noting that

$$\begin{aligned}[\operatorname{curl}_{\mathscr{A}} \boldsymbol{\theta}_1]^m_j [\operatorname{curl}_{\mathscr{A}} \boldsymbol{\theta}_2]^m_j &= (\mathscr{A}\partial_j \theta_1^m - \mathscr{A}\partial_m \theta_1^j)(\mathscr{A}\partial_j \theta_2^m - \mathscr{A}\partial_m \theta_2^j) \\ &= 2(\mathscr{A}\partial_j \theta_1^m)(\mathscr{A}\partial_j \theta_2^m) - 2(\mathscr{A}\partial_j \theta_1^m)(\mathscr{A}\partial_m \theta_2^j).\end{aligned}$$

Similarly for $\mathbf{P}_{d,L}$. □

Using this lemma, we will estimate the difference between "$\mathbf{P}_b$" and "$\mathbf{P}_{b,L}$".

Proposition 2.5.5. *Let $n \geq 20$ and $a + |\beta| + b \leq n$ with $a > 0$. For any $\boldsymbol{\theta}$ that satisfies our a priori assumption* (2.20) *we have*

$$\left| \int_0^s \langle \mathbf{P}_b \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a+1} X_r^b \not{\partial}^\beta \boldsymbol{\theta}\rangle_{3+b} \mathrm{d}\tau - \frac{1}{2}\left\langle \mathbf{P}_{b,L} \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\rangle_{3+b}\right|_0^s \right| \lesssim (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n$$

$$\left| \int_0^s \langle \mathbf{P}_b \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\rangle_{3+b} \mathrm{d}\tau - \int_0^s \langle \mathbf{P}_{b,L} \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\rangle_{3+b} \mathrm{d}\tau \right| \lesssim (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n$$

Proof. By Lemma 2.5.4

$$
\begin{aligned}
&\langle \mathbf{P}_b \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}, \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}\rangle_{3+b} \\
&= \int \Big((\mathscr{A}\partial_m \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}) \cdot (\mathscr{A}\partial_m \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}) \\
&\qquad + \frac{1}{3}(\operatorname{div}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta})(\operatorname{div}_{\mathscr{A}} \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}) \\
&\qquad - \frac{1}{2}[\operatorname{curl}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}]_j^m [\operatorname{curl}_{\mathscr{A}} \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}]_j^m \Big) \mathscr{J}^{-1/3} \bar{w}^{4+b} \mathbf{dx} \\
&= \int \Big((\mathscr{A}\partial_m \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}) \cdot \partial_s (\mathscr{A}\partial_m \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}) \\
&\qquad + \frac{1}{3}(\operatorname{div}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta})\partial_s(\operatorname{div}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}) \\
&\qquad - \frac{1}{2}[\operatorname{curl}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}]_j^m \partial_s[\operatorname{curl}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}]_j^m \Big) \mathscr{J}^{-1/3} \bar{w}^{4+b} \mathbf{dx} \\
&\quad + \mathscr{R}[(E_n + Z_n^2)^{1/2} E_n] \\
&= \frac{1}{2}\partial_s \int \Big(|\mathscr{A}\partial_m \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}|^2 + \frac{1}{3}|\operatorname{div}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}|^2 \\
&\qquad - \frac{1}{2}|[\operatorname{curl}_{\mathscr{A}} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}]|^2 \Big) \mathscr{J}^{-1/3} \bar{w}^{4+b} \mathbf{dx} \\
&\quad + \mathscr{R}[(E_n + Z_n^2)^{1/2} E_n] \\
&= \frac{1}{2}\partial_s \int \Big(|\partial_m \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}|^2 + \frac{1}{3}|\operatorname{div} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}|^2 - \frac{1}{2}|[\operatorname{curl} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}]|^2 \Big) \bar{w}^{4+b} \mathbf{dx} \\
&\quad + \partial_s \mathscr{R}[(E_n + Z_n^2)^{1/2} E_n] + \mathscr{R}[(E_n + Z_n^2)^{1/2} E_n] \\
&= \frac{1}{2}\partial_s \langle \mathbf{P}_{b,L} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\rangle_{3+b} + \partial_s \mathscr{R}[(E_n + Z_n^2)^{1/2} E_n] \\
&\quad + \mathscr{R}[(E_n + Z_n^2)^{1/2} E_n]
\end{aligned}
$$

where we recall notation $\mathscr{R}[\star]$ introduced in Definition 1.4.2. Integrating in time we get the first equation. For the second equation, note that

$$
\langle \mathbf{P}_b \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\rangle_{3+b} = \langle \mathbf{P}_{b,L} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\rangle_{3} + \mathscr{R}[(E_n + Z_n^2)^{1/2} E_n].
$$

Integrating in time we get the second equation. □

And finally we will estimate the difference between "$\mathbf{P}$" and "$\mathbf{P}_b$".

Proposition 2.5.6. *Let $n \geq 20$. For any $\boldsymbol{\theta}$ that satisfies our a priori assumption* (2.20) *we have* *i. For $a+|\beta|\le n$ with $a>0$ we have*

$$\left|\int_0^s \langle \partial_s^a \not{\partial}^\beta \mathbf{P} - \mathbf{P}_0 \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 \mathrm{d}\tau\right| \lesssim \mathcal{S}_{n,|\beta|-1,0}^{1/2} \mathcal{S}_{n,|\beta|,0}^{1/2} + (\mathcal{E}_n + \mathcal{Z}_n^2)^{1/2} \mathcal{E}_n$$

$$\left|\int_0^s \langle \partial_s^a \not{\partial}^\beta \mathbf{P} - \mathbf{P}_0 \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a} \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 \mathrm{d}\tau\right| \lesssim \mathcal{S}_{n,|\beta|-1,0}^{1/2} \mathcal{S}_{n,|\beta|,0}^{1/2} + (\mathcal{E}_n + \mathcal{Z}_n^2)^{1/2} \mathcal{E}_n$$

ii. For $a+|\beta|+b\le n$ with $a>0$ we have

$$\left|\int_0^s \langle \partial_s^a X_r^b \not{\partial}^\beta \mathbf{P} - \mathbf{P}_b \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a+1} X_r^b \not{\partial}^\beta \boldsymbol{\theta}\rangle_{3+b} \mathrm{d}\tau\right| \\ \lesssim (\mathcal{S}_{n,|\beta|+b-1}^{1/2} + \mathcal{S}_{n,|\beta|+b,b-1}^{1/2}) \mathcal{S}_{n,|\beta|+b}^{1/2} + (\mathcal{E}_n + \mathcal{Z}_n^2)^{1/2} \mathcal{E}_n$$

$$\left|\int_0^s \langle \partial_s^a X_r^b \not{\partial}^\beta \mathbf{P} - \mathbf{P}_b \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a} X_r^b \not{\partial}^\beta \boldsymbol{\theta}\rangle_{3+b} \mathrm{d}\tau\right| \\ \lesssim (\mathcal{S}_{n,|\beta|+b-1}^{1/2} + \mathcal{S}_{n,|\beta|+b,b-1}^{1/2}) \mathcal{S}_{n,|\beta|+b}^{1/2} + (\mathcal{E}_n + \mathcal{Z}_n^2)^{1/2} \mathcal{E}_n$$

Proof. i. Using Lemma 2.5.1 we have

$$\begin{aligned}
&\left|\int_0^s \langle \partial_s^a \not{\partial}^\beta \mathbf{P} - \mathbf{P}_L \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 \mathrm{d}\tau\right| \\
&\le \left|\int_0^s \int_{B_R} \partial_k (\bar{w}^4 (T_{R:a,\beta})_i^k) \partial_s^{a+1} \not{\partial}^\beta \theta^i \ \mathbf{dx}\mathrm{d}\tau\right| \\
&+ \left|\int_0^s \int_{B_R} \sum_{|\beta'|\le|\beta|-1} \langle C\nabla(\bar{w}^4 \partial_s^a \not{\partial}^{\beta'} T)\rangle \langle \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\rangle \ \mathbf{dx}\mathrm{d}\tau\right| \\
&\le \left|\int_0^s \int_{B_R} \bar{w}^4 \partial_s (T_{R:a,\beta})_i^k \partial_s^a \partial_k \not{\partial}^\beta \theta^i \ \mathbf{dx}\mathrm{d}\tau\right| \\
&+ \left|\int_0^s \int_{B_R} \sum_{|\beta'|\le|\beta|-1} \bar{w}^4 \langle C \partial_s^{a+1} \not{\partial}^{\beta'} T\rangle \langle \partial_s^a \nabla \not{\partial}^\beta \boldsymbol{\theta}\rangle \ \mathbf{dx}\mathrm{d}\tau\right| \\
&\quad + (E_n + Z_n^2)(0)^{1/2} E_n(0) + (E_n + Z_n^2)(s)^{1/2} E_n(s) \\
&\quad + S_{n,|\beta|-1,0}(0)^{1/2} S_{n,|\beta|,0}(0)^{1/2} + S_{n,|\beta|-1,0}(s)^{1/2} S_{n,|\beta|,0}(s)^{1/2} \\
&\lesssim \mathcal{S}_{n,|\beta|-1,0}^{1/2} \mathcal{S}_{n,|\beta|,0}^{1/2} + (\mathcal{E}_n + \mathcal{Z}_n^2)^{1/2} \mathcal{E}_n.
\end{aligned}$$

Proof of the second formula is similar and easier.

ii. By Lemma 2.5.2 we need to estimate the following.

$$\begin{aligned}
&\left|\int_0^s\int_{B_R}\left(\sum_{\substack{b'\le b\\|\beta'|\le|\beta|-1}}+\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|+1}}\right)\langle C\omega\partial_s^a X_r^{b'}\not\partial^{\beta'}T\rangle\langle\partial_s^{a+1}X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{3+b}\mathrm{dxd}\tau\right|\\
&\lesssim\left|\int_0^s\int_{B_R}\sum_{\substack{b'\le b\\|\beta'|\le|\beta|-1}}\langle C\omega T_T[\partial_s^{a+1}X_r^{b'}\not\partial^{\beta'}\nabla\boldsymbol{\theta}]\rangle\langle\partial_s^a X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{3+b}\mathrm{dxd}\tau\right|\\
&\quad+\left|\int_0^s\int_{B_R}\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|+1}}\langle C\omega T_T[\partial_s^a X_r^{b'}\not\partial^{\beta'}\nabla\boldsymbol{\theta}]\rangle\langle\partial_s^{a+1}X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{3+b}\mathrm{dxd}\tau\right|\\
&\quad+\mathcal{S}_{n,|\beta|+b-1}^{1/2}\mathcal{S}_{n,|\beta|+b}^{1/2}+(\mathscr{E}_n+\mathscr{Z}_n^2)^{1/2}\mathscr{E}_n\\
&\lesssim(\mathcal{S}_{n,|\beta|+b-1}^{1/2}+\mathcal{S}_{n,|\beta|+b,b-1}^{1/2})\mathcal{S}_{n,|\beta|+b}^{1/2}+(\mathscr{E}_n+\mathscr{Z}_n^2)^{1/2}\mathscr{E}_n
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^s\int_{B_R}\left(\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|-1}}+\sum_{\substack{b'\le b-2\\|\beta'|\le|\beta|}}\right)\langle C\partial_s^a X_r^{b'}\not\partial^{\beta'}\nabla T\rangle\langle\partial_s^{a+1}X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{3+b}\mathrm{dxd}\tau\right|\\
&\lesssim\left|\int_0^s\int_{B_R}\left(\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|-1}}+\sum_{\substack{b'\le b-2\\|\beta'|\le|\beta|}}\right)\langle C\partial_s^{a+1}X_r^{b'}\not\partial^{\beta'}T\rangle\langle\partial_s^a\nabla X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{3+b}\mathrm{dxd}\tau\right|\\
&\quad+\left|\int_0^s\int_{B_R}\left(\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|-1}}+\sum_{\substack{b'\le b-2\\|\beta'|\le|\beta|}}\right)\langle C\partial_s^{a+1}X_r^{b'}\not\partial^{\beta'}T\rangle\langle\omega\partial_s^a X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{2+b}\mathrm{dxd}\tau\right|\\
&\quad+\mathcal{S}_{n,|\beta|+b-1}^{1/2}\mathcal{S}_{n,|\beta|+n}^{1/2}+(\mathscr{E}_n+\mathscr{Z}_n^2)^{1/2}\mathscr{E}_n\\
&\lesssim\mathcal{S}_{n,|\beta|+b-1}^{1/2}\mathcal{S}_{n,|\beta|+n}^{1/2}+(\mathscr{E}_n+\mathscr{Z}_n^2)^{1/2}\mathscr{E}_n
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^s\int_{B_R}\left(\sum_{\substack{b'\le b\\|\beta'|\le|\beta|-1}}+\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|}}\right)\langle C\bar{w}X_r^{b'}\not\partial^{\beta'}\nabla T\rangle\langle\partial_s^{a+1}X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{3+b}\mathrm{dxd}\tau\right|\\
&\lesssim\left|\int_0^s\int_{B_R}\left(\sum_{\substack{b'\le b\\|\beta'|\le|\beta|-1}}+\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|}}\right)\langle C\partial_s^{a+1}X_r^{b'}\not\partial^{\beta'}T\rangle\langle\partial_s^a\nabla X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle\bar{w}^{4+b}\mathrm{dxd}\tau\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^s\int_{B_R}\left(\sum_{\substack{b'\le b\\|\beta'|\le|\beta|-1}}+\sum_{\substack{b'\le b-1\\|\beta'|\le|\beta|}}\right)\langle C\partial_s^{a+1}X_r^{b'}\not{\partial}^{\beta'}T\rangle\langle\omega\partial_s^aX_r^b\not{\partial}^\beta\boldsymbol{\theta}\rangle\bar{w}^{3+b}\mathrm{d}\mathrm{x}\mathrm{d}\tau\right|\\
&+\mathcal{S}_{n,|\beta|+b-1}^{1/2}\mathcal{S}_{n,|\beta|+n}^{1/2}+(\mathcal{E}_n+\mathcal{Z}_n^2)^{1/2}\mathcal{E}_n\\
\lesssim\ &\mathcal{S}_{n,|\beta|+b-1}^{1/2}\mathcal{S}_{n,|\beta|+n}^{1/2}+(\mathcal{E}_n+\mathcal{Z}_n^2)^{1/2}\mathcal{E}_n
\end{aligned}
$$

This proves the first formula. Proof of the second formula is similar and easier.

□

2.5.2 Estimating the linear and non-linear part of the gravity term

In this subsection we will estimate the gravity term $\partial_s^aX_r^b\not{\partial}^\beta\mathbf{G}$ (2.7) and show that it can be bounded by E_n. We will also estimate the non-linear part of $\partial_s^a\not{\partial}^\beta\mathbf{G}$, and show that it can be bounded by $(\mathcal{E}_n+\mathcal{Z}_n^2)^{1/2}\mathcal{E}_n$.

Since the gravity term is a non-local term, we need to estimate convolution-like operator. However, rather than the convolution kernel $|\mathbf{x}-\mathbf{z}|^{-1}$ we actually need to estimate $|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|^{-1}$. The next two lemmas tell us how to reduce the latter to the former, which will allows us to estimate using the Young's convolution inequality.

Lemma 2.5.7. *Let $\boldsymbol{\xi}$ be as in* (2.6)*. For any $\mathbf{x},\mathbf{y}\in B_R$ we have*

$$
\begin{aligned}
|\mathbf{x}-\mathbf{z}|&\le\|\mathcal{A}\|_{L^\infty(B_R)}|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|\\
|\partial_s^a\not{\partial}_{\mathbf{x}}^\beta\boldsymbol{\xi}(\mathbf{x})-\partial_s^a\not{\partial}_{\mathbf{z}}^\beta\boldsymbol{\xi}(\mathbf{z})|&\le\|\nabla\partial_s^a\not{\partial}^\beta\boldsymbol{\xi}\|_{L^\infty(B_R)}|\mathbf{x}-\mathbf{z}|
\end{aligned}
$$

Proof. Using the mean value inequality we have

$$
\begin{aligned}
|\mathbf{x}-\mathbf{z}|&=|\boldsymbol{\xi}^{-1}\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}^{-1}\boldsymbol{\xi}(\mathbf{z})|\\
&\le\|\nabla\boldsymbol{\xi}^{-1}\|_{L^\infty(B_R)}|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|=\|\mathcal{A}\|_{L^\infty(B_R)}|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|
\end{aligned}
$$

and

$$
|\partial_s^{a_i}\not{\partial}_{\mathbf{x}}^{\beta_i}\boldsymbol{\xi}(\mathbf{x})-\partial_s^{a_i}\not{\partial}_{\mathbf{z}}^{\beta_i}\boldsymbol{\xi}(\mathbf{z})|\le\|\nabla\partial_s^{a_i}\not{\partial}^{\beta_i}\boldsymbol{\xi}\|_{L^\infty(B_R)}|\mathbf{x}-\mathbf{z}|.
$$

□

Lemma 2.5.8. *Let $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$ be as in* (2.6)*, and $\boldsymbol{\theta}$ satisfies our a priori assumption*

(2.20). *Let* $n \geq 21$ *and* $a + |\beta| \leq n$ *with* $a > 0$.

i. When $a + |\beta| > n/2$ *we have*

$$\begin{aligned}
&\left|\partial_s^a (\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\right)\right| \\
&\lesssim \frac{1}{|\mathbf{x}-\mathbf{z}|^2} \sum_{\substack{n/2 < a' + |\gamma| \leq n \\ a' > 0}} |\partial_s^{a'} \not{\partial}_{\mathbf{x}}^\gamma \boldsymbol{\theta}(\mathbf{x}) - \partial_s^{a'} \not{\partial}_{\mathbf{z}}^\gamma \boldsymbol{\theta}(\mathbf{z})| \\
&\quad + \frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2} \sum_{n/2 < |\gamma| \leq n} |\not{\partial}_{\mathbf{x}}^\gamma \boldsymbol{\xi}(\mathbf{x}) - \not{\partial}_{\mathbf{z}}^\gamma \boldsymbol{\xi}(\mathbf{z})|
\end{aligned}$$

ii. When $|\beta| > n/2$ *we have*

$$\left|(\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\right)\right| \lesssim \frac{1}{|\mathbf{x}-\mathbf{z}|^2} \sum_{n/2 < |\gamma| \leq n} |\not{\partial}_{\mathbf{x}}^\gamma \boldsymbol{\xi}(\mathbf{x}) - \not{\partial}_{\mathbf{z}}^\gamma \boldsymbol{\xi}(\mathbf{z})|$$

iii. When $a + |\beta| \leq n/2$ *we have*

$$\left|\partial_{i,\mathbf{z}} \partial_s^a (\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\right)\right| \lesssim \frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2}$$

iv. When $|\beta| \leq n/2$ *we have*

$$\left|\partial_{i,\mathbf{z}} (\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\right)\right| \lesssim \frac{1}{|\mathbf{x}-\mathbf{z}|^2}$$

Proof. These follows from Lemma 2.5.7, the embedding theorems A.3.5 and A.3.6, the a priori bounds $E_n, Z_n \lesssim 1$ (2.20), and the following.

$$\begin{aligned}
&\partial_s^a (\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\right) \\
&= \sum_{m=1}^{a+|\beta|} \sum_{\substack{\sum_{i=1}^m (a_i + a_i') = a \\ \sum_{i=1}^m (\beta_i + \beta_i') = \beta \\ |a_i| + |\beta_i| > 0}} \frac{(-1)^m (2m)!}{m! 2^m} \frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|^{1+2m}} \\
&\qquad \prod_{i=1}^m (\partial_s^{a_i} \not{\partial}_{\mathbf{x}}^{\beta_i} \boldsymbol{\xi}(\mathbf{x}) - \partial_s^{a_i} \not{\partial}_{\mathbf{z}}^{\beta_i} \boldsymbol{\xi}(\mathbf{z})) \cdot (\partial_s^{a_i'} \not{\partial}_{\mathbf{x}}^{\beta_i'} \boldsymbol{\xi}(\mathbf{x}) - \partial_s^{a_i'} \not{\partial}_{\mathbf{z}}^{\beta_i'} \boldsymbol{\xi}(\mathbf{z})).
\end{aligned}$$

□

Since we cannot commute extra weights into the non-local gravity term, the radial derivatives which eat up weight need to be estimated differently in a way that would negate the non-local integral and allow extra weights to be used. Using methods from [22], the following two lemmas provide the way to do this. More precisely, the radial derivative can be estimated with curl, divergence and tangential derivatives. And this is useful because the curl and divergence of the gravity term consist only of local or non-linear terms, which we can estimate.

Lemma 2.5.9. *For any vector field* $\tilde{\mathbf{G}} \in H^1_{\text{loc}}$

$$|X_r\tilde{\mathbf{G}}|^2 \lesssim |r\nabla\cdot\tilde{\mathbf{G}}|^2 + |r\nabla\times\tilde{\mathbf{G}}|^2 + \sum_{k=1}^{3}|\not{\partial}_k\tilde{\mathbf{G}}|^2.$$

Proof. Note that

$$\begin{aligned}|\mathbf{x}\cdot\tilde{\mathbf{G}}|^2 &= (x^i\tilde{G}^i)(x^j\tilde{G}^j) = (x^j\tilde{G}^i)(x^i\tilde{G}^j) = (x^j\tilde{G}^i)(x^j\tilde{G}^i) + (x^j\tilde{G}^i)(x^i\tilde{G}^j - x^j\tilde{G}^i)\\ &= |\mathbf{x}|^2|\tilde{\mathbf{G}}|^2 - \frac{1}{2}(x^i\tilde{G}^j - x^j\tilde{G}^i)(x^i\tilde{G}^j - x^j\tilde{G}^i) = r^2|\tilde{\mathbf{G}}|^2 - |\mathbf{x}\times\tilde{\mathbf{G}}|^2\end{aligned}$$

We have by definition

$$\partial_i = \frac{x^j}{r^2}\not{\partial}_{ji} + \frac{x^i}{r^2}X_r, \ \ i = 1, 2, 3.$$

The divergence and the curl of $\tilde{\mathbf{G}}$ can be written as

$$r^2\nabla\cdot\tilde{\mathbf{G}} = x^j\not{\partial}_{ji}\tilde{\mathbf{G}}^i + \mathbf{x}\cdot X_r\tilde{\mathbf{G}} \qquad \text{and} \qquad r^2\nabla\times\tilde{\mathbf{G}} = x^j\not{\partial}_{j\bullet}\times\tilde{\mathbf{G}} + \mathbf{x}\times X_r\tilde{\mathbf{G}}$$

We then obtain

$$r^2|X_r\tilde{\mathbf{G}}|^2 = |r^2\nabla\cdot\tilde{\mathbf{G}} - x^j\not{\partial}_{ji}G^i|^2 + |r^2\nabla\times\tilde{\mathbf{G}} - x^j\not{\partial}_{j\bullet}\times\tilde{\mathbf{G}}|^2$$

from which we deduce the result. □

Lemma 2.5.10. *Let* $\mathbf{G}$ *be as in* (2.9). *We have*

$$\begin{aligned}\nabla\cdot\mathbf{G} &= (I-\mathscr{A})\nabla\cdot\mathbf{G} + (I-\mathscr{A})\nabla\cdot\nabla\mathscr{K}\bar{w}^3 + 4\pi\bar{w}^3(\mathscr{J}^{-1}-1)\\ \nabla\times\mathbf{G} &= (I-\mathscr{A})\nabla\times\mathbf{G} + (I-\mathscr{A})\nabla\times\nabla\mathscr{K}\bar{w}^3.\end{aligned}$$

Proof. By definition $\mathbf{G} = \mathscr{A}\nabla\Phi - \nabla\mathscr{K}\bar{w}^3$, so

$$\begin{aligned}\mathscr{A}\nabla\cdot\mathbf{G} &= (\mathscr{A}\nabla)\cdot(\mathscr{A}\nabla)\Phi - \mathscr{A}\nabla\cdot\nabla\mathscr{K}\bar{w}^3\\ &= (\mathscr{A}\nabla)\cdot(\mathscr{A}\nabla)\Phi + (I-\mathscr{A})\nabla\cdot\nabla\mathscr{K}\bar{w}^3 - 4\pi\bar{w}^3\end{aligned}$$

And we have

$$(\mathscr{A}\nabla)\cdot(\mathscr{A}\nabla)\Phi(\mathbf{x}) = \lambda^3(A\nabla)\cdot(A\nabla)\psi(\mathbf{x}) = \lambda^3(\nabla\cdot\nabla\mathscr{K}\rho)(\boldsymbol{\eta}(\mathbf{x}))$$
$$= \lambda^3 4\pi\rho(\boldsymbol{\eta}(\mathbf{x})) = \lambda^3 4\pi f(\mathbf{x}) = \lambda^3 4\pi\bar{w}^3 J^{-1} = 4\pi\bar{w}^3\mathscr{J}^{-1}. \tag{2.110}$$

So we get the first formula. Proof for the second formula is similar but we use $(\mathscr{A}\nabla)\times(\mathscr{A}\nabla) = 0$. □

Finally we can prove the main results of this subsection.

Proposition 2.5.11. *Let $n \geq 21$ and suppose $\boldsymbol{\theta}$ satisfies our a priori assumption* (2.20). *For $a + |\beta| + b \leq n$ with $a > 0$ we have*

$$\|\partial_s^a X_r^b \not\partial^\beta \mathbf{G}\|_{3+b}^2 \lesssim E_n. \tag{2.111}$$

Proof. By definition

$$\mathbf{G} = \mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}\bar{w}^3) - \mathscr{K}\nabla\bar{w}^3 = -\int_{\mathbb{R}^3}\frac{\partial_k(\mathscr{A}^k\bar{w}^3)}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}d\mathbf{z} + \int_{\mathbb{R}^3}\frac{\nabla\bar{w}^3}{|\mathbf{x}-\mathbf{z}|}d\mathbf{z}$$

Consider first when $b = 0$. Since $a > 0$, by Lemma 2.3.3 we have

$$\begin{aligned}
&\partial_s^a \not\partial^\beta \mathbf{G}(\mathbf{x}) \\
&= \partial_s^a \not\partial^\beta \mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}\bar{w}^3)(\mathbf{x}) = -\partial_s^a\not\partial^\beta\int_{\mathbb{R}^3}\frac{\partial_k(\mathscr{A}^k\bar{w}^3)}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}d\mathbf{z} \\
&= -\int_{\mathbb{R}^3}\sum_{\substack{a_1+a_2=a\\ \beta_1+\beta_2=\beta}}\partial_s^{a_1}(\not\partial_{\mathbf{x}}+\not\partial_{\mathbf{z}})^{\beta_1}\left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\right)\partial_s^{a_2}\not\partial_{\mathbf{z}}^{\beta_2}\partial_k(\mathscr{A}^k\bar{w}^3)(\mathbf{z})d\mathbf{z} \\
&= -\int_{\mathbb{R}^3}\sum_{\substack{a_1+a_2=a\\ \beta_1+\beta_2=\beta\\ a_1+|\beta_1|>n/2}}\partial_s^{a_1}(\not\partial_{\mathbf{x}}+\not\partial_{\mathbf{z}})^{\beta_1}\left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\right)\partial_s^{a_2}\not\partial_{\mathbf{z}}^{\beta_2}\partial_k(\mathscr{A}^k\bar{w}^3)(\mathbf{z})d\mathbf{z} \\
&\quad -\int_{\mathbb{R}^3}\sum_{\substack{a_1+a_2=a\\ \beta_1+\beta_2=\beta\\ a_1+|\beta_1|\leq n/2}}\partial_s^{a_1}(\not\partial_{\mathbf{x}}+\not\partial_{\mathbf{z}})^{\beta_1}\left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\right)\partial_s^{a_2}\not\partial_{\mathbf{z}}^{\beta_2}\partial_k(\mathscr{A}^k\bar{w}^3)(\mathbf{z})d\mathbf{z} \\
&= -\int_{\mathbb{R}^3}\sum_{\substack{a_1+a_2=a\\ \beta_1+\beta_2=\beta\\ a_1+|\beta_1|>n/2}}\partial_s^{a_1}(\not\partial_{\mathbf{x}}+\not\partial_{\mathbf{z}})^{\beta_1}\left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\right)\partial_s^{a_2}\not\partial_{\mathbf{z}}^{\beta_2}\partial_k(\mathscr{A}^k\bar{w}^3)(\mathbf{z})d\mathbf{z} \\
&\quad +\int_{\mathbb{R}^3}\sum_{\substack{a_1+a_2=a\\ \beta_1+\beta_2\leq\beta\\ a_1+|\beta_1|\leq n/2}}\langle\nabla_{\mathbf{z}}\rangle\partial_s^{a_1}(\not\partial_{\mathbf{x}}+\not\partial_{\mathbf{z}})^{\beta_1}\left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\right)(\langle\partial_s^{a_2}\not\partial_{\mathbf{z}}^{\beta_2}\mathscr{A}\rangle\bar{w}^3)(\mathbf{z})d\mathbf{z}
\end{aligned}$$

Now using Lemma 2.5.8, we get

$$
\begin{aligned}
|\partial_s^a \not{\partial}^\beta \mathbf{G}(\mathbf{x})| \lesssim & \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{z}|^2} \sum_{\substack{n/2<a'+|\gamma|\leq n \\ a'>0}} |\partial_s^{a'} \not{\partial}_{\mathbf{x}}^\gamma \boldsymbol{\theta}(\mathbf{x}) - \partial_s^{a'} \not{\partial}_{\mathbf{z}}^\gamma \boldsymbol{\theta}(\mathbf{z})| \mathrm{d}\mathbf{z} \\
& + \int_{\mathbb{R}^3} \frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2} \sum_{n/2<|\gamma|\leq n} |\not{\partial}_{\mathbf{x}}^\gamma \boldsymbol{\xi}(\mathbf{x}) - \not{\partial}_{\mathbf{z}}^\gamma \boldsymbol{\xi}(\mathbf{z})| \mathrm{d}\mathbf{z} \\
& + \int_{\mathbb{R}^3} \sum_{\substack{0<a_2\leq a \\ \beta_2\leq\beta}} \frac{1}{|\mathbf{x}-\mathbf{z}|^2} (\langle \partial_s^{a_2} \not{\partial}_{\mathbf{z}}^{\beta_2} \mathscr{A} \rangle \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
& + \int_{\mathbb{R}^3} \sum_{\beta_2\leq\beta} \frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2} (\langle \not{\partial}_{\mathbf{z}}^{\beta_2} \mathscr{A} \rangle \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z}
\end{aligned}
$$

Now using Young's convolution inequality we get

$$
\|\partial_s^a \not{\partial}^\beta \mathbf{G}(\mathbf{x})\|_{L^2(\mathbb{R}^3)} \lesssim E_n^{1/2}.
$$

Hence $\|\partial_s^a \not{\partial}^\beta \mathbf{G}\|_3^2 \lesssim E_n$. From the above proof, with small modification, we can further see that

$$
\begin{aligned}
\|\partial_s^a \not{\partial}^\beta \mathbf{G}(\mathbf{x})\|_{L^\infty(\mathbb{R}^3)} &\lesssim E_n^{1/2} \qquad \text{when} \qquad a+|\beta| \leq n/2 \\
\|\not{\partial}^\beta \mathbf{G}(\mathbf{x})\|_{L^\infty(\mathbb{R}^3)} &\lesssim 1 \qquad\quad\;\; \text{when} \qquad |\beta| \leq n/2.
\end{aligned}
$$

Now we deal with the case $b>0$. Let

$$
\begin{aligned}
W_{n,c} &= \sum_{\substack{a+|\beta|+b\leq n \\ a>0 \\ |\beta|+b\leq c}} \|\partial_s^a X_r^b \not{\partial}^\beta \mathbf{G}\|_{3+b}^2 \\
W_{n,c,d} &= \sum_{\substack{a+|\beta|+b\leq n \\ a>0 \\ |\beta|+b\leq c \\ b\leq d}} \|\partial_s^a X_r^b \not{\partial}^\beta \mathbf{G}\|_{3+b}^2 \\
V_{n,c} &= \sum_{\substack{a+|\beta|+b\leq n \\ a>0 \\ |\beta|+b\leq c}} \sup_{\mathbb{R}^3} \left(\bar{w}^b |\partial_s^a X_r^b \not{\partial}^\beta \mathbf{G}|^2 \right) \\
V_{n,c,d} &= \sum_{\substack{a+|\beta|+b\leq n \\ a>0 \\ |\beta|+b\leq c \\ b\leq d}} \sup_{\mathbb{R}^3} \left(\bar{w}^b |\partial_s^a X_r^b \not{\partial}^\beta \mathbf{G}|^2 \right).
\end{aligned}
$$

For $a + |\beta| + b \leq n/2$, using the above lemmas 2.5.9 and 2.5.10 we have

$$
\begin{aligned}
&\bar{w}^b |\partial_s^a X_r^b \not\partial^\beta \mathbf{G}|^2 \\
&\lesssim \bar{w}^b |r \partial_s^a \nabla \cdot X_r^{b-1} \not\partial^\beta \mathbf{G}|^2 + \bar{w}^b |r \partial_s^a \nabla \times X_r^{b-1} \not\partial^\beta \mathbf{G}|^2 + \sum_{k=1}^{3} \bar{w}^b |\partial_s^a X_r^{b-1} \not\partial_k \not\partial^\beta \mathbf{G}|^2 \\
&\lesssim \bar{w}^b |r \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \cdot \mathbf{G}|^2 + \bar{w}^b |r \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \times \mathbf{G}|^2 + V_{n,b+|\beta|-1} + V_{n,b+|\beta|,b-1} \\
&\lesssim \bar{w}^b |r \partial_s^a X_r^{b-1} \not\partial^\beta (I - \mathscr{A}) \nabla \cdot \mathbf{G}|^2 + \bar{w}^b |r \partial_s^a X_r^{b-1} \not\partial^\beta (I - \mathscr{A}) \nabla \times \mathbf{G}|^2 \\
&\quad + E_n + V_{n,b+|\beta|-1} + V_{n,b+|\beta|,b-1} \\
&\lesssim \bar{w}^b |r (I - \mathscr{A}) \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \cdot \mathbf{G}|^2 + \bar{w}^b |r (I - \mathscr{A}) \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \times \mathbf{G}|^2 \\
&\quad + E_n + V_{n,b+|\beta|-1} + V_{n,b+|\beta|,b-1} \\
&\lesssim \bar{w}^b (E_n + Z_n^2) |r \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \mathbf{G}|^2 + E_n + V_{n,b+|\beta|-1} + V_{n,b+|\beta|,b-1}
\end{aligned}
$$

So

$$
V_{n,b+|\beta|,b} \lesssim (E_n + Z_n^2) V_{n,b+|\beta|,b} + E_n + V_{n,b+|\beta|-1} + V_{n,b+|\beta|,b-1}
$$

By a priori assumption (2.20), we have $E_n + Z_n^2 \ll 1$, so

$$
V_{n,b+|\beta|,b} \lesssim E_n + V_{n,b+|\beta|-1} + V_{n,b+|\beta|,b-1}.
$$

We know $V_{n',c,0} \lesssim E_n$ for all $c \leq n' \leq n/2$, so by induction we get $V_{n',c,d} \lesssim E_n$ for all $d \leq c \leq n' \leq n/2$.

Now for $a + |\beta| + b \leq n$, using the above lemmas 2.5.9 and 2.5.10 and results for V we have

$$
\begin{aligned}
&\|\partial_s^a X_r^b \not\partial^\beta \mathbf{G}\|_{3+b}^2 \\
&\lesssim \|r \partial_s^a \nabla \cdot X_r^{b-1} \not\partial^\beta \mathbf{G}\|_{3+b}^2 + \|r \partial_s^a \nabla \times X_r^{b-1} \not\partial^\beta \mathbf{G}\|_{3+b}^2 + \sum_{k=1}^{3} \|\partial_s^a X_r^{b-1} \not\partial_k \not\partial^\beta \mathbf{G}\|_{3+b}^2 \\
&\lesssim \|r \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \cdot \mathbf{G}\|_{3+b}^2 + \|r \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \times \mathbf{G}\|_{3+b}^2 + W_{n,b+|\beta|-1} + W_{n,b+|\beta|,b-1} \\
&\lesssim \|r \partial_s^a X_r^{b-1} \not\partial^\beta (1 - \mathscr{A}) \nabla \cdot \mathbf{G}\|_{3+b}^2 + \|r \partial_s^a X_r^{b-1} \not\partial^\beta (1 - \mathscr{A}) \nabla \times \mathbf{G}\|_{3+b}^2 \\
&\quad + E_n + W_{n,b+|\beta|-1} + W_{n,b+|\beta|,b-1} \\
&\lesssim \|r (1 - \mathscr{A}) \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \cdot \mathbf{G}\|_{3+b}^2 + \|r (1 - \mathscr{A}) \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \times \mathbf{G}\|_{3+b}^2 \\
&\quad + E_n + W_{n,b+|\beta|-1} + W_{n,b+|\beta|,b-1} \\
&\lesssim (E_n + Z_n^2) \|r \partial_s^a X_r^{b-1} \not\partial^\beta \nabla \mathbf{G}\|_{3+b}^2 + E_n + W_{n,b+|\beta|-1} + W_{n,b+|\beta|,b-1}
\end{aligned}
$$

So

$$W_{n,b+|\beta|,b} \lesssim (E_n + Z_n^2) W_{n,b+|\beta|,b} + E_n + W_{n,b+|\beta|-1} + W_{n,b+|\beta|,b-1}$$

By a priori assumption (2.20), we have $E_n + Z_n^2 \ll 1$, so

$$W_{n,b+|\beta|,b} \lesssim E_n + W_{n,b+|\beta|-1} + W_{n,b+|\beta|,b-1}.$$

We know $W_{n,c,0} \lesssim E_n$ for all c, so by induction we get $W_{n,c,d} \lesssim E_n$ for all $d \leq c \leq n$. □

We are now in the position to estimate the difference between the high order derivatives of nonlinear gravity term $\mathbf{G}$ (2.9) and its linearised part $\mathbf{G}_L$ (2.34).

Proposition 2.5.12. *Let* $n \geq 21$ *and suppose* $\boldsymbol{\theta}$ *satisfies our a priori assumption* (2.20). *For* $a + |\beta| \leq n$ *with* $a > 0$ *we have*

$$\left| \int_0^s \langle \partial_s^a \not{\partial}^\beta \mathbf{G} - \mathbf{G}_L \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta} \rangle_3 \mathrm{d}\tau \right| \lesssim (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2} \mathscr{E}_n$$
$$\left| \int_0^s \langle \partial_s^a \not{\partial}^\beta \mathbf{G} - \mathbf{G}_L \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^{a} \not{\partial}^\beta \boldsymbol{\theta} \rangle_3 \mathrm{d}\tau \right| \lesssim (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2} \mathscr{E}_n$$

Proof. Since $\|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_3 + \|\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\|_3 \lesssim E_n^{1/2}$, it suffice to prove that

$$\|\partial_s^a \not{\partial}^\beta \mathbf{G} - \mathbf{G}_L \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\|_3 \lesssim (E_n + Z_n^2)^{1/2} E_n^{1/2}.$$

Recall from Lemma 2.2.2 that

$$\mathbf{G} - \mathbf{G}_L \boldsymbol{\theta} = \underbrace{\mathscr{K}_{\boldsymbol{\xi}}(\mathscr{A}_l^i (\partial_k \theta^l)(\nabla \theta^k) \partial_i \bar{w}^3 - \bar{w}^3 (\mathscr{A}_m^i \mathscr{A}_\bullet^l - I_m^i I_\bullet^l) \partial_i \partial_l \theta^m)}_{:=M_1}$$
$$\underbrace{-(\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K}) \partial_i (\bar{w}^3 \nabla \theta^i)}_{:=M_2} + \underbrace{(\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K} - \mathscr{K}_{\boldsymbol{\xi}}^{(1)}) \nabla \bar{w}^3}_{:=M_3}.$$

Now $\|\partial_s^a \not{\partial}^\beta M_1\|_3$ can be estimated in a similar way as the previous Proposition 2.5.11, and $\|\partial_s^a \not{\partial}^\beta M_3\|_3$ can be estimated in the same way as in Lemma 2.3.5 in equation (2.79). Now in the same way as in Lemma 2.3.4 and recalling K_1 (2.38) we can show that

$$|\partial_s^a (\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta K_1(\mathbf{x}, \mathbf{z})| \lesssim \frac{1}{|\mathbf{x} - \mathbf{z}|^2} \sum_{\substack{0 < a' \leq a \\ \beta' \leq \beta}} |\partial_s^{a'} \not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{x}) - \partial_s^{a'} \not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{z})|$$

$$+\frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2}\sum_{\beta'\leq\beta}|\partial\!\!\!/^{\beta'}\boldsymbol{\theta}(\mathbf{x})-\partial\!\!\!/^{\beta'}\boldsymbol{\theta}(\mathbf{z})|$$

$$|(\partial\!\!\!/_{\mathbf{x}}+\partial\!\!\!/_{\mathbf{z}})^\beta K_1(\mathbf{x},\mathbf{z})|\lesssim\frac{1}{|\mathbf{x}-\mathbf{z}|^2}\sum_{\beta'\leq\beta}|\partial\!\!\!/^{\beta'}\boldsymbol{\theta}(\mathbf{x})-\partial\!\!\!/^{\beta'}\boldsymbol{\theta}(\mathbf{z})|$$

And when $a+|\beta|\leq n/2$,

$$|\partial_{i,\mathbf{z}}\partial_s^a(\partial\!\!\!/_{\mathbf{x}}+\partial\!\!\!/_{\mathbf{z}})^\beta K_1(\mathbf{x},\mathbf{z})|\lesssim\frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2}$$

and when $|\beta|\leq n/2$,

$$|\partial_{i,\mathbf{z}}(\partial\!\!\!/_{\mathbf{x}}+\partial\!\!\!/_{\mathbf{z}})^\beta K_1(\mathbf{x},\mathbf{z})|\lesssim\frac{(E_n+Z_n^2)^{1/2}}{|\mathbf{x}-\mathbf{z}|^2}.$$

Using these bounds (in the same way we use Lemma 2.5.8 in the proof of the previous Proposition 2.5.11), we can estimate $\|\partial_s^a\partial\!\!\!/^\beta M_2\|_3$. □

2.5.3 Reduction to linear problem

Having estimated the non-linear parts of the equation in the last two subsections, in this section we will use them to reduce our problem to the linear problem for which we have the coercivity result that we can apply. We only need to do this for the case with no radial derivatives, the case with radial derivatives can be obtained by induction.

Lemma 2.5.13. *For any $\boldsymbol{\theta}$ that satisfies our a priori assumption* (2.20) *we have*

$$\int_0^s\langle\mathbf{G}_L\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta},\partial_s^{a+1}\partial\!\!\!/^\beta\boldsymbol{\theta}\rangle_3\mathrm{d}\tau=\frac{1}{2}\left.\langle\mathbf{G}_L\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta},\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta}\rangle_3\right|_0^s$$

$$\langle\mathbf{L}\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta},\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta}\rangle_3=\delta\|\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta}\|_3^2+\langle\mathbf{P}_{0,L}\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta},\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta}\rangle_3+\langle\mathbf{G}_L\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta},\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta}\rangle_3,$$

where we recall (2.33)*,* (2.34) *and* (2.107)*.*

Proof. We have from (2.34)

$$\begin{aligned}&\langle\mathbf{G}_L\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta},\partial_s^{a+1}\partial\!\!\!/^\beta\boldsymbol{\theta}\rangle_3\\&=\int\Big((\partial_s^a\partial\!\!\!/^\beta\theta^i)(\partial_s^{a+1}\partial\!\!\!/^\beta\theta^j)\bar{w}^3\partial_i\partial_j\mathcal{K}\bar{w}^3\\&\qquad-(4\pi)^{-1}(\nabla\mathcal{K}\nabla\cdot(\bar{w}^3\partial_s^a\partial\!\!\!/^\beta\boldsymbol{\theta}))\cdot(\nabla\mathcal{K}\nabla\cdot(\bar{w}^3\partial_s^{a+1}\partial\!\!\!/^\beta\boldsymbol{\theta}))\Big)\mathrm{d}\mathbf{x}\end{aligned}$$

$$= \frac{1}{2}\partial_s \int \left((\partial_s^a \not{\partial}^\beta \theta^i)(\partial_s^a \theta^j)\bar{w}^3 \partial_i \partial_j \mathscr{K} \bar{w}^3 - (4\pi)^{-1}|\nabla \mathscr{K} \nabla \cdot (\bar{w}^3 \partial_s^a \not{\partial}^\beta \boldsymbol{\theta})|^2\right) \mathrm{dx}$$
$$= \frac{1}{2}\partial_s \langle \mathbf{G}_L \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3.$$

The second formula follows from the definition of $\mathbf{L}$, $\mathbf{P}_{0,L}$ and $\mathbf{G}_L$. □

The following theorem reduces the full non-linear problem to the linear one.

Theorem 2.5.14. *Let $n \geq 20$ and suppose $\boldsymbol{\theta}$ satisfies our a priori assumption* (2.20)*. For $a + |\beta| \leq n$ with $a > 0$ we have*

$$\left| \int_0^s \langle \partial_s^a \not{\partial}^\beta (\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 \mathrm{d}\tau - \frac{1}{2} \left. \langle \mathbf{L} \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 \right|_0^s \right|$$
$$\lesssim \mathcal{S}_{n,|\beta|-1,0}^{1/2} \mathcal{S}_{n,|\beta|,0}^{1/2} + (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2} \mathscr{E}_n \quad (2.112)$$

$$\left| \int_0^s \langle \partial_s^a \not{\partial}^\beta (\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s^a \boldsymbol{\theta}\rangle_3 \mathrm{d}\tau - \int_0^s \langle \mathbf{L} \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}, \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 \mathrm{d}\tau \right|$$
$$\lesssim \mathcal{S}_{n,|\beta|-1,0}^{1/2} \mathcal{S}_{n,|\beta|,0}^{1/2} + (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2} \mathscr{E}_n \quad (2.113)$$

Proof. Using Lemma 2.5.13 and Propositions 2.5.5, 2.5.6, and 2.5.12, we conclude the proof. □

This theorem above reduces the non-linear problem for time and tangential derivatives to the linear problem. Now applying our linear coercivity results from before, we get the following coercivity result for our non-linear problem, allowing us to control $\|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a \nabla \not{\partial}^\beta \boldsymbol{\theta}\|_{4+b}^2$.

Corollary 2.5.15. *Let $n \geq 20$. Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the energy, momentum, and irrotationality constraints* (2.22)*,* (2.23)*, and* (2.24) *hold respectively. Then for $a + |\beta| \leq n$ with $a > 0$ we have*

$$\left\|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a \nabla \not{\partial}^\beta \boldsymbol{\theta}\right\|_4^2$$
$$\lesssim |\mathscr{b}|^{-2} \left(C S_{n,|\beta|,0}(0) + \frac{1}{2} \left. \left\|\partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\right\|_3^2 \right|_0^s + \int_0^s \langle \partial_s^a \not{\partial}^\beta (\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s^{a+1} \not{\partial}^\beta \boldsymbol{\theta}\rangle_3 \mathrm{d}\tau \right)$$
$$+ C\left(\mathcal{S}_{n,|\beta|-1,0} + |\mathscr{b}|^{-2} \mathcal{S}_{n,|\beta|-1,0}^{1/2} \mathcal{S}_{n,|\beta|,0}^{1/2}\right) + C_\delta (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2} \mathscr{E}_n \quad (2.114)$$

$$\int_0^s \left(\left\|\partial_s^{a+1}\not{\partial}^\beta\boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a\nabla\not{\partial}^\beta\boldsymbol{\theta}\right\|_4^2\right)\mathrm{d}\tau$$
$$\lesssim |\mathscr{b}|^{-2}\int_0^s \left(\left\|\partial_s^{a+1}\not{\partial}^\beta\boldsymbol{\theta}\right\|_3^2 + \langle\partial_s^a\not{\partial}^\beta(\delta\boldsymbol{\theta}+\mathbf{P}+\mathbf{G}), \partial_s^a\not{\partial}^\beta\boldsymbol{\theta}\rangle_3\right)\mathrm{d}\tau$$
$$+ C\left(\mathcal{S}_{n,|\beta|-1,0} + |\mathscr{b}|^{-2}\mathcal{S}_{n,|\beta|-1,0}^{1/2}\mathcal{S}_{n,|\beta|,0}^{1/2}\right) + C_\delta(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n \qquad (2.115)$$

Proof. Combining Theorem 2.5.14 and Propositions 2.4.7 and 2.4.9 we conclude the proof. □

To control the version with radial derivative $\|\partial_s^{a+1}X_r^b\not{\partial}^\beta\boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a X_r^b\not{\partial}^\beta\boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a\nabla X_r^b\not{\partial}^\beta\boldsymbol{\theta}\|_{4+b}^2$, we do not need to apply the linear coercivity result like Theorem 2.5.14 above. This is because we get control of $\|\partial_s^a\nabla X_r^b\not{\partial}^\beta\boldsymbol{\theta}\|_{4+b}^2$ directly from the pressure term, while the control of $\|\partial_s^{a+1}X_r^b\not{\partial}^\beta\boldsymbol{\theta}\|_{3+b}^2 + \|\partial_s^a X_r^b\not{\partial}^\beta\boldsymbol{\theta}\|_{3+b}^2$ and the gravity term we get automatically from induction from the step with one less space derivative, as follows.

Corollary 2.5.16. *Let $n \geq 21$ and suppose $\boldsymbol{\theta}$ satisfies our a priori assumption* (2.20). *For $a + |\beta| + b \leq n$ with $a, b > 0$ we have*

$$\left\|\partial_s^{a+1}X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{3+b}^2 + \left\|\partial_s^a X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{3+b}^2 + \left\|\partial_s^a\nabla X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{4+b}^2$$
$$\lesssim CS_{n,|\beta|+b,b}(0) + \frac{1}{2}\left\|\partial_s^{a+1}X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{3+b}^2 \qquad (2.116)$$
$$+ \int_0^s \langle\partial_s^a X_r^b\not{\partial}^\beta(\delta\boldsymbol{\theta}+\mathbf{P}+\mathbf{G}), \partial_s^{a+1}X_r^b\not{\partial}^\beta\boldsymbol{\theta}\rangle_{3+b}\mathrm{d}\tau$$
$$+ C\left(\mathcal{S}_{n,|\beta|+b-1} + (\mathcal{S}_{n,|\beta|+b-1}^{1/2} + \mathcal{S}_{n,|\beta|+b,b-1}^{1/2})\mathscr{E}_n^{1/2} + (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n\right) \qquad (2.117)$$

$$\int_0^s \left(\left\|\partial_s^{a+1}X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{3+b}^2 + \left\|\partial_s^a X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{3+b}^2 + \left\|\partial_s^a\nabla X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{4+b}^2\right)\mathrm{d}\tau$$
$$\lesssim \int_0^s \left(\left\|\partial_s^{a+1}X_r^b\not{\partial}^\beta\boldsymbol{\theta}\right\|_{3+b}^2 + \langle\partial_s^a X_r^b\not{\partial}^\beta(\delta\boldsymbol{\theta}+\mathbf{P}+\mathbf{G}), \partial_s^a X_r^b\not{\partial}^\beta\boldsymbol{\theta}\rangle_{3+b}\right)\mathrm{d}\tau$$
$$+ C\left(\mathcal{S}_{n,|\beta|+b-1} + (\mathcal{S}_{n,|\beta|+b-1}^{1/2} + \mathcal{S}_{n,|\beta|+b,b-1}^{1/2})\mathscr{E}_n^{1/2} + (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n\right) \qquad (2.118)$$

Proof. By Propositions 2.5.6 and 2.5.5, we can replace $\partial_s^a X_r^b\not{\partial}^\beta\mathbf{P}$ by $\mathbf{P}_{b,L}\partial_s^a X_r^b\not{\partial}^\beta\boldsymbol{\theta}$. Now by Lemma 2.5.4 we have

$$\frac{1}{2}\left[\|\partial_s^a\nabla X_r^b\not{\partial}^\beta\boldsymbol{\theta}\|_{4+b}^2 + \frac{1}{3}\|\partial_s^a\nabla\cdot(X_r^b\not{\partial}^\beta\boldsymbol{\theta})\|_{4+b}^2 - \frac{1}{2}\|[\partial_s^a\operatorname{curl}X_r^b\not{\partial}^\beta\boldsymbol{\theta}]\|_{4+b}^2\right]_0^s$$

$$= \int_0^s \langle \mathbf{P}_{b,L} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}, \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta} \rangle_{3+b} \mathrm{d}\tau$$

Now using Corollary 2.4.4 we get

$$\|\partial_s^a \nabla X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{4+b}^2 \lesssim C S_{n,|\beta|+b,b}(0) + \int_0^s \langle \mathbf{P}_{b,L} \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}, \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta} \rangle_{3+b} \mathrm{d}\tau \\ + C\left(S_{n,|\beta|+b-1} + (E_n + Z_n^2) E_n\right).$$

Furthermore, note that

$$\left\|\partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta}\right\|_{3+b}^2 + \left\|\partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\right\|_{3+b}^2 \\ \lesssim \left\|\partial_s^{a+1} \nabla X_r^{b-1} \not\partial^\beta \boldsymbol{\theta}\right\|_{4+(b-1)}^2 + \left\|\partial_s^a \nabla X_r^{b-1} \not\partial^\beta \boldsymbol{\theta}\right\|_{4+(b-1)}^2 \lesssim S_{n,|\beta|+b-1}.$$

Now note that, using Proposition 2.5.11,

$$\left|\int_0^s \langle \delta \partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}, \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta} \rangle_{3+b} \mathrm{d}\tau\right| \lesssim \mathcal{S}_{n,|\beta|+b-1}$$
$$\left|\int_0^s \langle \partial_s^a X_r^b \not\partial^\beta \mathbf{G}, \partial_s^{a+1} X_r^b \not\partial^\beta \boldsymbol{\theta} \rangle_{3+b} \mathrm{d}\tau\right| \lesssim \mathcal{S}_{n,|\beta|+b-1}^{1/2} \mathcal{E}_n^{1/2}$$

then we are done for the first formula. Proof for the second formula is similar. □

2.6 Energy estimates and proof of the main theorem

In this section we finally commute the momentum equation (2.7) and then derive the high-order energy estimates. Since the bounds near the vacuum boundary are more delicate as they are sensitive to the weights, we present them in Section 2.6.1 and the estimates away from the vacuum boundary in Section 2.6.2. Then finally we will prove our main theorem in section 2.6.3 using the energy estimates.

2.6.1 Near boundary energy estimate

In this subsection we will prove the energy estimate for $\mathcal{S}_n$ (recall (2.17)).

Theorem 2.6.1 (Near boundary energy estimate)**.** *Let* $n \geq 21$*, and assume that* $\epsilon > 0$ *and* $|\delta|$ *are sufficiently small. Let* $\boldsymbol{\theta}$ *be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the energy, momentum, and irrotationality constraints* (2.22)*,* (2.23)*, and* (2.24) *hold* *respectively. Then there exist* $m > 0$ *such that*

$$\mathcal{S}_n - C\epsilon\mathcal{E}_n \lesssim_\epsilon |\mathscr{b}|^{-m}\mathcal{S}_n(0) + C_\delta(\mathcal{E}_n + \mathcal{I}_n^2)^{1/2}\mathcal{E}_n \tag{2.119}$$

whenever our a priori assumption (2.20) *is satisfied. Here we recall Definition* (2.19) *of the total norm* $\mathcal{E}_n$.

Proof. Let $a + |\beta| + b \le n$. Apply $\partial_s^a X_r^b \partial\!\!\!/^\beta$ to the momentum equation (2.7) to get

$$\partial_s^{a+2} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta} - \frac{\mathscr{b}}{2}\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta} + \partial_s^a X_r^b \partial\!\!\!/^\beta(\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}) = 0$$

Taking the $\langle\cdot,\cdot\rangle_{3+b}$-inner with $\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}$ we get

$$\begin{aligned} 0 = &\frac{1}{2}\partial_s\|\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2 + \langle \partial_s^a X_r^b \partial\!\!\!/^\beta(\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_{3+b} \\ &- \frac{\mathscr{b}}{2}\|\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2. \end{aligned}$$

On the other hand, taking inner product of the equation with $\partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}$ we get

$$\begin{aligned} 0 = &\partial_s\langle \partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_{3+b} - \|\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2 - \frac{\mathscr{b}}{4}\partial_{3+b}\|\partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2 \\ &+ \langle \partial_s^a X_r^b \partial\!\!\!/^\beta(\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_{3+b} \end{aligned}$$

where we used the identiity $\langle \partial_s^{a+2} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle = \partial_s\langle \partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle - \|\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|^2$. Multiply the latter equation by c, add to it two times the equation before, and then integrate w.r.t. s to obtain

$$\begin{aligned} 0 = &\left(\frac{1}{2}\|\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2 + c\langle \partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}, \partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_{3+b} - \frac{c\mathscr{b}}{4}\|\partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2\right)\Bigg|_0^s \\ &+\int_0^s \Bigg(\langle \partial_s^a X_r^b \partial\!\!\!/^\beta(\delta\boldsymbol{\theta}+\mathbf{P}+\mathbf{G}), \partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_{3+b} + c\langle \partial_s^a X_r^b \partial\!\!\!/^\beta(\delta\boldsymbol{\theta}+\mathbf{P}+\mathbf{G}), \partial_s^a X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_{3+b} \\ &\qquad - \left(c + \frac{\mathscr{b}}{2}\right)\|\partial_s^{a+1} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2\Bigg)\mathrm{d}\tau. \end{aligned}$$

i. When $b = 0$, using Corollary 2.5.15 we get

$$\begin{aligned} &\left\|\partial_s^{a+1} \partial\!\!\!/^\beta \boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a \partial\!\!\!/^\beta \boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a \nabla \partial\!\!\!/^\beta \boldsymbol{\theta}\right\|_4^2 \\ &\quad + c\int_0^s \left(\left\|\partial_s^{a+1} \partial\!\!\!/^\beta \boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a \partial\!\!\!/^\beta \boldsymbol{\theta}\right\|_3^2 + \left\|\partial_s^a \nabla \partial\!\!\!/^\beta \boldsymbol{\theta}\right\|_4^2\right)\mathrm{d}\tau \\ &\quad + |\mathscr{b}|^{-2}\left(c\langle \partial_s^{a+1} \partial\!\!\!/^\beta \boldsymbol{\theta}, \partial_s^a \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_3 - \frac{c\mathscr{b}}{4}\|\partial_s^a \partial\!\!\!/^\beta \boldsymbol{\theta}\|_3^2\right)\Bigg|_0^s \end{aligned}$$

$$
\begin{aligned}
&- |\mathcal{b}|^{-2}\int_0^s \left(2c+\frac{\mathcal{b}}{2}\right)\|\partial_s^{a+1}\partial\!\!\!/^{\beta}\boldsymbol{\theta}\|_3^2 \mathrm{d}\tau \\
&\lesssim |\mathcal{b}|^{-2}S_{n,|\beta|,0}(0)+\mathcal{S}_{n,|\beta|-1,0}+|\mathcal{b}|^{-2}\mathcal{S}_{n,|\beta|-1,0}^{1/2}\mathcal{S}_{n,|\beta|,0}^{1/2}+C_\delta(\mathcal{E}_n+\mathcal{X}_n^2)^{1/2}\mathcal{E}_n.
\end{aligned}
$$

Choosing c small enough (e.g. $c=|\mathcal{b}|^2/100$ when $\mathcal{b}\ll 1$), we get

$$
\begin{aligned}
&\left\|\partial_s^{a+1}\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_3^2+\left\|\partial_s^{a}\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_3^2+\left\|\partial_s^{a}\nabla\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_4^2 \\
&\quad+\int_0^s\left(\left\|\partial_s^{a+1}\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_3^2+\left\|\partial_s^{a}\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_3^2+\left\|\partial_s^{a}\nabla\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_4^2\right)\mathrm{d}\tau \\
&\lesssim |\mathcal{b}|^{-2}(|\mathcal{b}|^{-2}S_{n,|\beta|,0}(0)+\mathcal{S}_{n,|\beta|-1,0}+|\mathcal{b}|^{-2}\mathcal{S}_{n,|\beta|-1,0}^{1/2}\mathcal{S}_{n,|\beta|,0}^{1/2} \\
&\quad+C_\delta(\mathcal{E}_n+\mathcal{X}_n^2)^{1/2}\mathcal{E}_n),
\end{aligned}
$$

and so (noting that the constant implicit in the notation $\lesssim$ do not depend on s)

$$
\begin{aligned}
\mathcal{S}_{n,|\beta|,0}\lesssim{}& |\mathcal{b}|^{-4}\mathcal{S}_{n,|\beta|,0}(0)+|\mathcal{b}|^{-2}\mathcal{S}_{n,|\beta|-1,0}+|\mathcal{b}|^{-4}\mathcal{S}_{n,|\beta|-1,0}^{1/2}\mathcal{S}_{n,|\beta|,0}^{1/2} \\
&+C_\delta(\mathcal{E}_n+\mathcal{X}_n^2)^{1/2}\mathcal{E}_n.
\end{aligned}
$$

In particular when $|\beta|=0$ we have $\mathcal{S}_{n,0,0}\lesssim|\mathcal{b}|^{-4}\mathcal{S}_{n,0,0}(0)+C_\delta(\mathcal{E}_n+\mathcal{X}_n^2)^{1/2}\mathcal{E}_n$. And so using Young's inequality and by induction on $|\beta|$ we have

$$
\mathcal{S}_{n,|\beta|,0}\lesssim|\mathcal{b}|^{-4-8|\beta|}\mathcal{S}_{n,|\beta|,0}(0)+C_\delta(\mathcal{E}_n+\mathcal{X}_n^2)^{1/2}\mathcal{E}_n \tag{2.120}
$$

for all $|\beta|\le n$.

ii. When $b>0$, using Corollary 2.5.16 we get

$$
\begin{aligned}
&\left\|\partial_s^{a+1}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{3+b}^2+\left\|\partial_s^{a}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{3+b}^2+\left\|\partial_s^{a}\nabla X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{4+b}^2 \\
&\quad+c\int_0^s\left(\left\|\partial_s^{a+1}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{3+b}^2+\left\|\partial_s^{a}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{3+b}^2+\left\|\partial_s^{a}\nabla X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{4+b}^2\right)\mathrm{d}\tau \\
&\quad+\left(c\langle\partial_s^{a+1}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta},\partial_s^{a}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\rangle_3-\frac{c\mathcal{b}}{4}\|\partial_s^{a}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\|_3^2\right)\Bigg|_0^s \\
&\quad-\int_0^s\left(2c+\frac{\mathcal{b}}{2}\right)\|\partial_s^{a+1}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\|_3^2\mathrm{d}\tau \\
&\lesssim S_{n,|\beta|+b,b}(0)+\mathcal{S}_{n,|\beta|+b-1}+(\mathcal{S}_{n,|\beta|+b-1}^{1/2}+\mathcal{S}_{n,|\beta|+b,b-1}^{1/2})\mathcal{E}_n^{1/2} \\
&\quad+C_\delta(\mathcal{E}_n+\mathcal{X}_n^2)^{1/2}\mathcal{E}_n
\end{aligned}
$$

Choosing c small enough we get

$$
\left\|\partial_s^{a+1}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{3+b}^2+\left\|\partial_s^{a}X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{3+b}^2+\left\|\partial_s^{a}\nabla X_r^b\partial\!\!\!/^{\beta}\boldsymbol{\theta}\right\|_{4+b}^2
$$

$$
\begin{aligned}
&+ \int_0^s \left(\left\|\partial_s^{a+1} X_r^b \not{\partial}^\beta \boldsymbol{\theta}\right\|_{3+b}^2 + \left\|\partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\right\|_{3+b}^2 + \left\|\partial_s^a \nabla X_r^b \not{\partial}^\beta \boldsymbol{\theta}\right\|_{4+b}^2 \right) \mathrm{d}\tau \\
&\lesssim |\mathscr{b}|^{-1} (S_{n,|\beta|+b,b}(0) + \mathcal{S}_{n,|\beta|+b-1} + (\mathcal{S}_{n,|\beta|+b-1}^{1/2} + \mathcal{S}_{n,|\beta|+b,b-1}^{1/2}) \mathscr{E}_n^{1/2} \\
&\quad + C_\delta (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2} \mathscr{E}_n)
\end{aligned}
$$

and so

$$
\begin{aligned}
\mathcal{S}_{n,|\beta|+b,b} \lesssim |\mathscr{b}|^{-1} (&\mathcal{S}_{n,|\beta|+b,b}(0) + \mathcal{S}_{n,|\beta|+b-1} + \mathcal{S}_{n,|\beta|+b,b-1} \\
&+ (\mathcal{S}_{n,|\beta|+b-1}^{1/2} + \mathcal{S}_{n,|\beta|+b,b-1}^{1/2}) \mathscr{E}_n^{1/2} + C_\delta (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2} \mathscr{E}_n).
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\mathcal{S}_{n,c,d} \lesssim |\mathscr{b}|^{-1} (&\mathcal{S}_{n,c,d}(0) + \mathcal{S}_{n,c-1} + \mathcal{S}_{n,c,d-1} \\
&+ (\mathcal{S}_{n,c-1}^{1/2} + \mathcal{S}_{n,c,d-1}^{1/2}) \mathscr{E}_n^{1/2} + C_\delta (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2} \mathscr{E}_n).
\end{aligned}
$$

We already know $\mathcal{S}_{n,c,0} \lesssim |\mathscr{b}|^{-4-8c} \mathcal{S}_{n,c,0}(0) + C_\delta (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2} \mathscr{E}_n$. And so using Young's inequality and by induction on c and d we have

$$
\mathcal{S}_{n,c,d} - C\epsilon \mathscr{E}_n \lesssim_\epsilon |\mathscr{b}|^{-m} \mathcal{S}_{n,c,d}(0) + C_\delta (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2} \mathscr{E}_n.
$$

for all $d \le c \le n$. This means we have $\mathcal{S}_n - C\epsilon \mathscr{E}_n \lesssim_\epsilon |\mathscr{b}|^{-m} \mathcal{S}_n(0) + C_\delta (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2} \mathscr{E}_n$.

□

2.6.2 Near origin energy estimate

In this subsection we will prove the energy estimate for $\mathbb{Q}_n$, see (2.18).

Lemma 2.6.2. *Let $k \ge 0$. For any $\boldsymbol{\theta}$ we have*

$$
\|\nabla \boldsymbol{\theta}\|_{k+2}^2 \lesssim \|\nabla \cdot \boldsymbol{\theta}\|_{k+2}^2 + \|\nabla \times \boldsymbol{\theta}\|_{k+2}^2 + \|\boldsymbol{\theta}\|_k^2
$$

Proof. We have

$$
\begin{aligned}
&\int |\nabla \boldsymbol{\theta}|^2 \bar{w}^{k+2} \mathrm{d}\mathbf{x} \\
&= \int (\partial_j \theta^i)(\partial_j \theta^i) \bar{w}^{k+2} \mathrm{d}\mathbf{x} = \int (\partial_j \theta^i)(\partial_i \theta^j + [\operatorname{curl} \boldsymbol{\theta}]_j^i) \bar{w}^{k+2} \mathrm{d}\mathbf{x} \\
&= \int (\partial_j \theta^i)(\partial_i \theta^j) \bar{w}^{k+2} \mathrm{d}\mathbf{x} + \int (\partial_j \theta^i) [\operatorname{curl} \boldsymbol{\theta}]_j^i \bar{w}^{k+2} \mathrm{d}\mathbf{x}
\end{aligned}
$$

$$
\begin{aligned}
&= -\int (\partial_i \partial_j \theta^i)(\theta^j)\bar{w}^{k+2}\mathbf{dx} + \int (\partial_j \theta^i)[\operatorname{curl}\boldsymbol{\theta}]^i_j \bar{w}^{k+2}\mathbf{dx} \\
&\quad - (k+2)\int (\partial_j \theta^i)(\theta^j)\bar{w}^{k+1}\partial_i \bar{w}\mathbf{dx} \\
&= \int (\partial_i \theta^i)(\partial_j \theta^j)\bar{w}^{k+2}\mathbf{dx} + \int (\partial_j \theta^i)[\operatorname{curl}\boldsymbol{\theta}]^i_j \bar{w}^{k+2}\mathbf{dx} \\
&\quad + (k+2)\left(\int (\partial_i \theta^i)(\theta^j)\bar{w}^{k+1}\partial_j \bar{w}\mathbf{dx} - \int (\partial_j \theta^i)(\theta^j)\bar{w}^{k+1}\partial_i \bar{w}\mathbf{dx}\right) \\
&\lesssim \delta' \int |\nabla \boldsymbol{\theta}|^2 \bar{w}^{k+2}\mathbf{dx} + \frac{1}{\delta'}\left(\int (|\nabla\cdot\boldsymbol{\theta}|^2 + |\nabla\times\boldsymbol{\theta}|^2)\bar{w}^{k+2}\mathbf{dx} + \int |\boldsymbol{\theta}|^2 \bar{w}^k \mathbf{dx}\right).
\end{aligned}
$$

Picking δ' small enough, we are done. □

Using this lemma, the following lemma shows that in fact we only need to control the divergence $\partial_s^a \nabla^c \nabla \cdot \boldsymbol{\theta}$ in order to control the near origin energy Q_n.

Lemma 2.6.3. *Let $n \geq 20$ and $c \leq n$. Let $\boldsymbol{\theta}$ be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the irrotationality constraint* (2.24)*. Then*

$$
Q_{n,c} \lesssim \|\partial_s^a \nabla^c \nabla \cdot \boldsymbol{\theta}\|^2_{3+2(c+1)} + Q_{n,c-1} + (E_n + Z_n^2)E_n.
$$

Proof. Let $a + c \leq n$ with $a > 0$. Using the previous Lemma 2.6.2, we have

$$
\begin{aligned}
\|\partial_s^a \nabla^{c+1}\boldsymbol{\theta}\|^2_{3+2(c+1)} &\lesssim \|\partial_s^a \nabla^c \nabla\cdot\boldsymbol{\theta}\|^2_{3+2(c+1)} + \|\partial_s^a \nabla^c \nabla\times\boldsymbol{\theta}\|^2_{3+2(c+1)} \\
&\quad + \|\partial_s^a \nabla^c \boldsymbol{\theta}\|^2_{3+2c} \\
&\leq \|\partial_s^a \nabla^c \nabla\cdot\boldsymbol{\theta}\|^2_{3+2(c+1)} + \|\partial_s^a \nabla^c \nabla\times\boldsymbol{\theta}\|^2_{3+2(c+1)} + Q_{n,c-1}.
\end{aligned}
$$

Recalling (2.88) we have $\partial_s^a \nabla^c \nabla \times \boldsymbol{\theta} = -\partial_s^{a-1}\nabla^c((\partial_s \nabla\theta^k) \times \nabla\theta^k)$ and therefore

$$
\|\partial_s^a \nabla^c \nabla\times\boldsymbol{\theta}\|^2_{3+2(c+1)} \lesssim (E_n + Z_n^2)E_n.
$$

□

Lemma 2.6.4. *For any tensor field T smooth enough we have*

$$
\partial^\gamma\left(\bar{w}^{-3}\partial_k(\bar{w}^4 T^k)\right) = \bar{w}\partial^\gamma \partial_k T^k + \sum_{|\gamma'|\leq|\gamma|} \bar{w}^{|\gamma|-|\gamma'|}\langle C\partial^{\gamma'}T^k\rangle.
$$

where we recall notations introduced in Definition 1.4.2.

Proof. The statement follows easily by induction. □

Theorem 2.6.5 (Near origin energy estimate)**.** *Let* $n \geq 21$ *and* δ *small. Let* $\boldsymbol{\theta}$ *be a solution of* (2.7) *in the sense of Theorem 2.1.11, given on its maximal interval of existence. Assume further that the energy, momentum, and irrotationality constraints* (2.22), (2.23), *and* (2.24) *hold respectively. Then we have*

$$\mathcal{Q}_n \lesssim |\mathscr{b}|^{-4}\mathscr{E}_n(0) + C_\delta(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n \tag{2.121}$$

whenever our a priori assumption (2.20) *is satisfied.*

Proof. Recall the momentum equation (2.7) is

$$\mathbf{0} = \partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta} + \bar{w}^{-3}\partial_k(\bar{w}^4(\underbrace{\mathscr{A}^k\mathscr{J}^{-1/3} - I^k}_{=T})) + \mathscr{A}\nabla\Phi - \nabla\mathscr{K}\bar{w}^3.$$

where we recall T in (2.98). Also recall from (2.110) $(\mathscr{A}\nabla)\cdot(\mathscr{A}\nabla)\Phi(\mathbf{x}) = 4\pi\bar{w}^3\mathscr{J}^{-1}$. So taking the divergence of the gravity term makes it easy to estimate. From Lemma 2.6.3 we also know that to control Q_n it suffices to estimate the divergence. Let $a + |\gamma| + 1 \leq n$. Evaluating the dot product of (2.7) with $\partial_s^a\partial^\gamma\mathscr{A}\nabla$ we get

$$\begin{aligned}\mathbf{0} &= \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s\boldsymbol{\theta} + \delta\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\\ &\quad + \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right) + 4\pi\partial_s^a\partial^\gamma(\bar{w}^3\mathscr{J}^{-1}) - (\partial_s^a\partial^\gamma\mathscr{A}\nabla)\cdot\nabla\mathscr{K}\bar{w}^3\\ &= \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s\boldsymbol{\theta} + \delta\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\\ &\quad + \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right) + 4\pi\partial_s^a\partial^\gamma(\bar{w}^3(\mathscr{J}^{-1}-1))\\ &\quad - (\partial_s^a\partial^\gamma(\mathscr{A}-I)\nabla)\cdot\nabla\mathscr{K}\bar{w}^3\end{aligned}$$

From here we will the do two things (i) and (ii) as follows

(i) Times the equation with $\bar{w}^{6+2|\gamma|}\partial_s^{a+1}\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}$ and integrate in time and space we get

$$\begin{aligned}0 = \int_0^s \Bigg(&\Big\langle \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s\boldsymbol{\theta} + \delta\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta},\\ &\qquad\qquad \partial_s^{a+1}\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\Big\rangle_{6+2|\gamma|}\\ &+ \left\langle\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right), \partial_s^{a+1}\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\right\rangle_{6+2|\gamma|}\\ &+ \big\langle 4\pi\partial_s^a\partial^\gamma(\bar{w}^3(\mathscr{J}^{-1}-1)) - (\partial_s^a\partial^\gamma(\mathscr{A}-I)\nabla)\cdot\nabla\mathscr{K}\bar{w}^3,\\ &\qquad\qquad \partial_s^{a+1}\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\big\rangle_{6+2|\gamma|}\Bigg)\mathrm{d}\tau\end{aligned}$$

Now commuting $\mathscr{A}$ with space and time derivatives, we get a non-linear remainder $\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]$ (recall notation $\mathscr{R}[\star]$ defined in Definition 1.4.2),

$$\begin{aligned}0 = \int_0^s \Bigg(&\Big\langle \partial_s \mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta}, \mathscr{A}\nabla\partial_s^{a+1}\partial^\gamma\cdot\boldsymbol{\theta}\Big\rangle_{6+2|\gamma|} \\ &+ \langle \delta\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}, \partial_s\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\rangle_{6+2|\gamma|} \\ &+ \left\langle \mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right), \partial_s\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\rangle_{6+2|\gamma|} \\ &+ \big\langle 4\pi(\mathscr{J}^{-1}-1)\partial_s^a\partial^\gamma(\bar{w}^3) - (\mathscr{A}-I)\nabla\cdot\nabla\mathscr{K}\partial_s^a\partial^\gamma\bar{w}^3, \\ &\qquad \partial_s^{a+1}\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\big\rangle_{6+2|\gamma|}\Bigg)\mathrm{d}\tau \\ &+ \mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]\end{aligned}$$

Now terms in the first two line we factorised, and terms in the last line in the integral we can estimate by $\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}$ and $\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}$,

$$\begin{aligned}0 = \int_0^s \Bigg(&\frac{1}{2}\partial_s\left\|\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta}\right\|_{6+2|\gamma|}^2 - \frac{1}{2}\mathscr{b}\left\|\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta}\right\|_{6+2|\gamma|}^2 \\ &+ \frac{1}{2}\delta\partial_s\left\|\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\|_{6+2|\gamma|}^2 \\ &+ \left\langle \mathscr{A}\nabla\cdot\left(\bar{w}\partial_s^a\partial^\gamma\partial_kT^k\right), \partial_s\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\rangle_{6+2|\gamma|}\Bigg)\mathrm{d}\tau \\ &+ \mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]\end{aligned}$$

Now terms that are full time derivatives can be evaluated, and $\partial_s^a\partial^\gamma\partial_kT^k$ can be converted to $T_T[\partial_s^a\partial^\gamma\partial_k\nabla\boldsymbol{\theta}]^k$ (recall Lemma 2.5.3) leaving a reminder that we can estimate with $(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n$.

$$\begin{aligned}0 = &\frac{1}{2}\left(\left\|\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta}\right\|_{6+2|\gamma|}^2 + \delta\left\|\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\|_{6+2|\gamma|}^2\right)\Big|_0^s \\ &+ \int_0^s \Bigg(-\frac{1}{2}\mathscr{b}\left\|\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta}\right\|_{6+2|\gamma|}^2 \\ &\qquad + \left\langle \mathscr{A}\nabla\cdot\left(\bar{w}T_T[\partial_s^a\partial^\gamma\partial_k\nabla\boldsymbol{\theta}]^k\right), \partial_s\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\rangle_{6+2|\gamma|}\Bigg)\mathrm{d}\tau \\ &+ \mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]\end{aligned}$$

Now all the term before the term with T_T can be bounded by $\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}$ and $\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}$, and we integrate by parts on the term with T_T,

$$0 = -\int_0^s \left\langle \bar{w}T_T[\partial_s^a\partial^\gamma\partial_k\nabla\boldsymbol{\theta}]^k, \partial_s\mathscr{A}\nabla(\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta})\right\rangle_{6+2|\gamma|}\mathrm{d}\tau$$

$$+\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]+\mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]$$

Now we expend the terms by definition and simplify,

$$\begin{aligned}
&0\\
&=\int_0^s\left\langle \mathscr{J}^{-1/3}\left(\mathscr{A}_m^k\mathscr{A}^l+\frac{1}{3}\mathscr{A}^k\mathscr{A}_m^l\right)\partial_s^a\partial^\gamma\partial_k\partial_l\theta^m,\partial_s(\mathscr{A}^j\mathscr{A}_i^\ell\partial_j\partial_\ell\partial_s^a\partial^\gamma\theta^i)\right\rangle_{7+2|\gamma|}\mathrm{d}\tau\\
&\quad+\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]+\mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]\\
&=\int_0^s\left\langle \mathscr{J}^{-1/3}\left(\mathscr{A}_m^k\mathscr{A}_o^l+\frac{1}{3}\mathscr{A}_o^k\mathscr{A}_m^l\right)\partial_k\partial_l\partial_s^a\partial^\gamma\theta^m,\partial_s(\mathscr{A}_o^j\mathscr{A}_i^\ell\partial_j\partial_\ell\partial_s^a\partial^\gamma\theta^i)\right\rangle_{7+2|\gamma|}\mathrm{d}\tau\\
&\quad+\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]+\mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]\\
&=\int_0^s\frac{4}{3}\left\langle \mathscr{J}^{-1/3}\mathscr{A}_o^k\mathscr{A}_m^l\partial_k\partial_l\partial_s^a\partial^\gamma\theta^m,\partial_s(\mathscr{A}_o^j\mathscr{A}_i^\ell\partial_j\partial_\ell\partial_s^a\partial^\gamma\theta^i)\right\rangle_{7+2|\gamma|}\mathrm{d}\tau\\
&\quad+\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]+\mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]
\end{aligned}$$

Now the term in the integral can be factorised into a time derivative,

$$\begin{aligned}
0&=\frac{2}{3}\int_0^s\int_{B_R}\mathscr{J}^{-1/3}\partial_s\left\|\mathscr{A}^k\mathscr{A}_m^l\partial_k\partial_l\partial_s^a\partial^\gamma\theta^m\right\|^2\bar{w}^{7+2|\gamma|}\mathrm{d}\mathbf{x}\mathrm{d}\tau\\
&\quad+\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]+\mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]
\end{aligned}$$

Now we can evaluate the time integral using integration by parts, leaving a remainder term that can be estimated with $(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n$ when the time derivative falls on $\mathscr{J}^{-1/3}$,

$$\begin{aligned}
0&=\frac{2}{3}\int_{B_R}\mathscr{J}^{-1/3}\left\|\mathscr{A}^k\mathscr{A}_m^l\partial_k\partial_l\partial_s^a\partial^\gamma\theta^m\right\|^2\bar{w}^{7+2|\gamma|}\mathrm{d}\mathbf{x}\Big|_0^s\\
&\quad+\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]+\mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]\\
&=\frac{2}{3}\int_{B_R}\left\|\nabla\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\|^2\bar{w}^{7+2|\gamma|}\mathrm{d}\mathbf{x}\Big|_0^s\\
&\quad+\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]+\mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left\|\nabla\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\|_{3+2(2+|\gamma|)}^2\\
&\lesssim\mathbb{Q}_{n,|\gamma|+1}(0)+\mathbb{Q}_{n,|\gamma|}+\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}+\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}+(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n.
\end{aligned}$$

Using Lemma 2.6.3 we get

$$Q_{n,|\gamma|+1} \lesssim \mathbb{Q}_{n,|\gamma|+1}(0)+\mathbb{Q}_{n,|\gamma|}+\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}+\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}+(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n$$

(ii) Times the equation with $\bar{w}^{6+2|\gamma|}\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}$ and integrate in time and space we get

$$\begin{aligned}0 = \int_0^s \Bigg(\Bigg\langle &\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s^2\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\partial_s\boldsymbol{\theta} + \delta\partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}, \\ &\qquad\qquad \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\Bigg\rangle_{6+2|\gamma|} \\ &+ \left\langle \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right), \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\right\rangle_{6+2|\gamma|} \\ &+ \Big\langle 4\pi\partial_s^a\partial^\gamma(\bar{w}^3(\mathscr{J}^{-1}-1)) - (\partial_s^a\partial^\gamma(\mathscr{A}-I)\nabla)\cdot\nabla\mathscr{K}\bar{w}^3, \\ &\qquad\qquad \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\Big\rangle_{6+2|\gamma|}\Bigg)\mathrm{d}\tau\end{aligned}$$

Now commuting $\mathscr{A}$ with space and time derivatives, we get a non-linear remainder $\mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]$,

$$\begin{aligned}0 = \int_0^s \Bigg(\Bigg\langle &\partial_s\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta} - \frac{1}{2}\mathscr{b}\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta}, \mathscr{A}\nabla\partial_s^a\partial^\gamma\cdot\boldsymbol{\theta}\Bigg\rangle_{6+2|\gamma|} \\ &+ \langle\delta\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}, \mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\rangle_{6+2|\gamma|} \\ &+ \left\langle\mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right), \mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\rangle_{6+2|\gamma|} \\ &+ \Big\langle 4\pi(\mathscr{J}^{-1}-1)\partial_s^a\partial^\gamma(\bar{w}^3) - (\mathscr{A}-I)\nabla\cdot\nabla\mathscr{K}\partial_s^a\partial^\gamma\bar{w}^3, \\ &\qquad\qquad \partial_s^a\partial^\gamma\mathscr{A}\nabla\cdot\boldsymbol{\theta}\Big\rangle_{6+2|\gamma|}\Bigg)\mathrm{d}\tau \\ &+ \mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]\end{aligned}$$

Now all the terms, apart from the top order term involving T_T from the pressure, can be bounded by $(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n]+\mathscr{R}[\mathbb{Q}_{n,|\gamma|}+\mathbb{Q}_{n,|\gamma|}+\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}+\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}$,

$$\begin{aligned}0 = -\int_0^s \Big(&\langle\mathscr{A}\nabla\cdot\partial_s^{a+1}\partial^\gamma\boldsymbol{\theta}, \partial_s\mathscr{A}\nabla\partial_s^a\partial^\gamma\cdot\boldsymbol{\theta}\rangle_{6+2|\gamma|} \\ &+ \left\langle\mathscr{A}\nabla\cdot\left(\bar{w}\partial_s^a\partial^\gamma\partial_kT^k\right), \mathscr{A}\nabla\cdot\partial_s^a\partial^\gamma\boldsymbol{\theta}\right\rangle_{6+2|\gamma|}\Big)\mathrm{d}\tau \\ &+ \mathscr{R}[(\mathscr{E}_n+\mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]\end{aligned}$$

$$
\begin{aligned}
&= \int_0^s \left\langle \mathscr{A}\nabla \cdot \left(\bar{w}T_T[\partial_s^a \partial^\gamma \partial_k \nabla \boldsymbol{\theta}]^k\right), \mathscr{A}\nabla \cdot \partial_s^a \partial^\gamma \boldsymbol{\theta} \right\rangle_{6+2|\gamma|} \mathrm{d}\tau \\
&\quad + \mathscr{R}[(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]
\end{aligned}
$$

Now we integrate by parts,

$$
\begin{aligned}
0 &= -\int_0^s \left\langle \bar{w}T_T[\partial_s^a \partial^\gamma \partial_k \nabla \boldsymbol{\theta}]^k, \mathscr{A}\nabla(\mathscr{A}\nabla \cdot \partial_s^a \partial^\gamma \boldsymbol{\theta}) \right\rangle_{6+2|\gamma|} \mathrm{d}\tau \\
&\quad + \mathscr{R}[(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]
\end{aligned}
$$

Now we expend the terms by defintion and simplify,

$$
\begin{aligned}
0 &= \int_0^s \left\langle \mathscr{J}^{-1/3}\left(\mathscr{A}_m^k \mathscr{A}^l + \frac{1}{3}\mathscr{A}^k \mathscr{A}_m^l\right) \partial_s^a \partial^\gamma \partial_k \partial_l \theta^m, \mathscr{A}^j \mathscr{A}_i^\ell \partial_j \partial_\ell \partial_s^a \partial^\gamma \theta^i \right\rangle_{7+2|\gamma|} \mathrm{d}\tau \\
&\quad + \mathscr{R}[(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] \\
&= \int_0^s \left\langle \mathscr{J}^{-1/3}\left(\mathscr{A}_m^k \mathscr{A}_o^l + \frac{1}{3}\mathscr{A}_o^k \mathscr{A}_m^l\right) \partial_k \partial_l \partial_s^a \partial^\gamma \theta^m, \mathscr{A}_o^j \mathscr{A}_i^\ell \partial_j \partial_\ell \partial_s^a \partial^\gamma \theta^i \right\rangle_{7+2|\gamma|} \mathrm{d}\tau \\
&\quad + \mathscr{R}[(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] \\
&= \int_0^s \frac{4}{3} \left\langle \mathscr{J}^{-1/3} \mathscr{A}_o^k \mathscr{A}_m^l \partial_k \partial_l \partial_s^a \partial^\gamma \theta^m, \mathscr{A}_o^j \mathscr{A}_i^\ell \partial_j \partial_\ell \partial_s^a \partial^\gamma \theta^i \right\rangle_{7+2|\gamma|} \mathrm{d}\tau \\
&\quad + \mathscr{R}[(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] \\
&= \frac{4}{3}\int_0^s \int_{B_R} \mathscr{J}^{-1/3} \left\| \mathscr{A}^k \mathscr{A}_m^l \partial_k \partial_l \partial_s^a \partial^\gamma \theta^m \right\|^2 \bar{w}^{7+2|\gamma|} \mathrm{dx}\mathrm{d}\tau \\
&\quad + \mathscr{R}[(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] \\
&= \frac{4}{3}\int_0^s \int_{B_R} \left\| \nabla\nabla \cdot \partial_s^a \partial^\gamma \boldsymbol{\theta} \right\|^2 \bar{w}^{7+2|\gamma|} \mathrm{dx}\mathrm{d}\tau \\
&\quad + \mathscr{R}[(\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}] + \mathscr{R}[\mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}] + \mathscr{R}[\mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2}]
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^s \left\| \nabla\nabla \cdot \partial_s^a \partial^\gamma \boldsymbol{\theta} \right\|_{3+2(2+|\gamma|)}^2 \mathrm{d}\tau \\
&\lesssim \mathbb{Q}_{n,|\gamma|+1}(0) + \mathbb{Q}_{n,|\gamma|} + \mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + \mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n.
\end{aligned}
$$

Using Lemma 2.6.3 we get

$$
\begin{aligned}
&\int_0^s Q_{n,|\gamma|+1} \mathrm{d}\tau \\
&\lesssim \mathbb{Q}_{n,|\gamma|+1}(0) + \mathbb{Q}_{n,|\gamma|} + \mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + \mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + (\mathscr{E}_n + \mathscr{X}_n^2)^{1/2}\mathscr{E}_n
\end{aligned}
$$

Combining the results of (i) and (ii) and noting that $\lesssim$ does not depend on s, we get that

$$\begin{aligned}\mathbb{Q}_{n,|\gamma|+1} &\lesssim \mathbb{Q}_{n,|\gamma|+1}(0) + \mathbb{Q}_{n,|\gamma|} + \mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + \mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2}\mathscr{E}_n \\ &\lesssim \mathscr{E}_n(0) + \mathbb{Q}_{n,|\gamma|} + \mathbb{Q}_{n,|\gamma|}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + \mathbb{Q}_{n-1}^{1/2}\mathbb{Q}_{n,|\gamma|+1}^{1/2} + (\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2}\mathscr{E}_n\end{aligned}$$

We have by definition and equation (2.120) in the previous theorem

$$\begin{aligned}\mathbb{Q}_{n,0} \lesssim \mathcal{S}_{n,0} &\lesssim |\mathscr{b}|^{-4}\mathcal{S}_n(0) + C_\delta(\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2}\mathscr{E}_n \\ &\le |\mathscr{b}|^{-4}\mathscr{E}_n(0) + C_\delta(\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2}\mathscr{E}_n\end{aligned}$$

And so using Young's inequality and by induction we have

$$\mathbb{Q}_{n,d} \lesssim |\mathscr{b}|^{-4}\mathscr{E}_n(0) + C_\delta(\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2}\mathscr{E}_n$$

for all $d \le n$. Therefore we have $\mathbb{Q}_n \lesssim |\mathscr{b}|^{-4}\mathscr{E}_n(0) + C_\delta(\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2}\mathscr{E}_n$. □

2.6.3 Bootstrapping scheme and final theorem

In this subsection we will prove our main theorem that the energy E_n decays exponentially while Z_n remains bounded. To do so we will use the bootstrapping scheme in the following lemma and proposition.

Lemma 2.6.6. *Suppose $E : [0,T] \to [0,\infty]$ is continuous and*

$$E(t) \le C_1 E(0) + C_2 E(t)^{3/2} \qquad \textit{whenever} \qquad \sup_{\tau\in[0,t]} E(\tau) \le C_3.$$

where $C_1 \ge 1$. Then $E \le 2C_1E(0)$ whenever $E(0) \le \min\{(2^5 C_1 C_2^2)^{-1}, C_3/2C_1\}$.

Proof. We will prove this by a standard bootstrap argument. Let

$$I = \left\{ t \in [0,T] : \sup_{\tau\in[0,t]} E(\tau) \le \min\{2C_1E(0), C_3\} \right\}.$$

Then I is non-empty (since $0 \in I$) and closed (since E is continuous). If $I = [0,T]$, then we are done. Otherwise, let $t_0 = \inf\{t \in [0,T] : t \notin I\}$. We must have $t_0 \in I$ since $0 \in I$ and I is closed. Then we have

$$E(t_0) \le C_1E(0) + C_2(2C_1E(0))^{3/2} \le \frac{3}{2}C_1E(0) \le \frac{3}{4}C_3.$$

So by continuity of E, a neighbourhood of t_0 must lie in I. But this contradicts the definition of t_0. So we must have $I = [0, T]$. □

Proposition 2.6.7. *Suppose $E, Z : [0, \infty) \to [0, \infty]$ are continuous and for all $t \geq t_0 \geq 0$ we have*

$$\begin{aligned} \mathscr{E}_{t_0}(t) &\leq C_0 \mathscr{E}_{t_0}(t_0) + C_1 Z(t_0) \mathscr{E}_{t_0}(t) + C_2 (1 + (t - t_0)^k) \mathscr{E}_{t_0}(t)^{3/2} \\ Z(t) &\leq Z(t_0) + C_3 (t - t_0)^l \mathscr{E}_{t_0}^{1/2}(t) \end{aligned}$$

whenever $\sup_{\tau \in [t_0, t]} (E(\tau) + Z(\tau)) \leq C_4$*, where* $k, l \geq 0$ *and*

$$\mathscr{E}_{t_0}(t) = \sup_{\tau \in [t_0, t]} E(\tau) + \int_{t_0}^{t} E(\tau) \mathrm{d}\tau.$$

Then there exist $\epsilon > 0$ such that $\mathscr{E}_0 \leq 6 C_0 \mathscr{E}_0(0)$ whenever $\mathscr{E}_0(0), Z(0) \leq \epsilon$. Moreover, $E(t) \leq 16(4^{-t/32C_0}) C_0 E(0)$.

Proof. Let $T = 32 C_0$ and $C_1 Z(0) < \min\{1/4, C_1 C_4/4\} \leq 1/2$. Then by the above Lemma 2.6.6, for small enough ϵ, we have $\mathscr{E}_0 \leq 4 C_0 \mathscr{E}_0(0)$ on $[0, T]$ whenever $\mathscr{E}_0(0) \leq \epsilon$. On $[T/2, T]$, there must exist a point T_1 such that $E(T_1) \leq \frac{1}{4} \mathscr{E}_0(0)$, otherwise $\mathscr{E}_0(T) > 4 C_0 \mathscr{E}_0(0)$.

By having a small enough ϵ, we can assume $2 C_0^{1/2} C_1 C_3 T^l \mathscr{E}_0(0)^{1/2} < \min\{1/8, C_1 C_4/8\}$. Now

$$\begin{aligned} C_1 Z(T_1) &\leq C_1 Z(0) + 2 C_0^{1/2} C_1 C_3 T^l \mathscr{E}_0(0)^{1/2} \\ &\leq \min\left\{\frac{1}{4}, \frac{C_1 C_4}{4}\right\} + \min\left\{\frac{1}{8}, \frac{C_1 C_4}{8}\right\} \leq \min\left\{\frac{1}{2}, \frac{C_1 C_4}{2}\right\}. \end{aligned}$$

Then by the above Lemma 2.6.6, we get that $\mathscr{E}_{T_1} \leq 4 C_0 \mathscr{E}_{T_1}(T_1) = 4 C_0 E(T_1) \leq C_0 \mathscr{E}_0(0)$ on $[T_1, T_1 + T]$. On $[T_1 + T/2, T_1 + T]$, there must exist a point T_2 such that $E(T_2) \leq \frac{1}{4} \mathscr{E}_{T_1}(T_1) = \frac{1}{4} E(T_1)$, otherwise $\mathscr{E}_{T_1}(T) > 4 C_0 \mathscr{E}_{T_1}(T_1)$.

Repeating inductively, we can get $T_n \in [T_{n-1} + T/2, T_{n-1} + T]$ such that

$$\begin{aligned} C_1 Z(T_n) &\leq C_1 Z(T_{n-1}) + C_1 C_3 T^l \mathscr{E}_{T_{n-1}}(T_n)^{1/2} \\ &\leq \left(\frac{1}{4} \sum_{m=0}^{n-1} \frac{1}{2^m} + \frac{1}{4} \frac{1}{2^n} \right) \min\{1, C_1 C_4\} \leq \frac{1}{2} \min\{1, C_1 C_4\} \\ \mathscr{E}_{T_n} &\leq 4^{1-n} C_0 \mathscr{E}_0(0) \qquad \text{on} \qquad [T_n, T_n + T]. \end{aligned}$$

Now

$$\mathscr{E}_0 \le \mathscr{E}_0(\infty) \le \mathscr{E}_0(T_1) + \mathscr{E}_{T_1}(T_2) + \mathscr{E}_{T_2}(T_3) + \cdots \le 6C_0\mathscr{E}_0(0).$$

□

Finally, before proving the main theorem, we provide a simple lemma based on the fundamental theorem of calculus, which relates the $\mathscr{Z}_n$-norm (2.15) to the total energy norm $\mathscr{E}_n$ (2.19).

Lemma 2.6.8. *We have*

$$\mathscr{Z}_1(s) \le \mathscr{Z}_1(0) + Cs^{1/2}\mathscr{S}_1^{1/2}(s)$$
$$\mathscr{Z}_n(s) \le \mathscr{Z}_n(0) + Cs^{1/2}\mathscr{E}_n^{1/2}(s)$$

Proof. We have $h(s) - h(0) = \int_0^s \partial_s h(\tau)\mathrm{d}\tau$ and therefore

$$(h(s) - h(0))^2 = \left(\int_0^s \partial_s h(\tau)\mathrm{d}\tau\right)^2 \le s\int_0^s (\partial_s h(\tau))^2\mathrm{d}\tau.$$

This easily gives $\|h(s) - h(0)\|_k^2 \le s\int_0^s \|\partial_s h(\tau)\|_k^2\mathrm{d}\tau$ and thus $\|h(s)\|_k \le \|h(0)\|_k + s^{1/2}\left(\int_0^s \|\partial_s h(\tau)\|_k^2\mathrm{d}\tau\right)^{1/2}$, which concludes the proof. □

Theorem 2.6.9. *Let $n \ge 21$ and δ small. Let $(\boldsymbol{\theta}, \partial_s\boldsymbol{\theta})$ be a solution of* (2.7) *in the sense of Theorem 2.1.11 and that satisfies* (2.22)*,* (2.23) *and* (2.24) *(i.e. the perturbation does not change the momentum or energy of the star, and correspond to an irrotational flow). Then there is some $m > 0$ such that we have*

$$\mathscr{E}_n(s) \lesssim |\mathscr{b}|^{-m}\mathscr{E}_n(0) + C_\delta\left(\mathscr{Z}_n(0)\mathscr{E}_n(s) + (1 + s^{1/2})\mathscr{E}_n(s)^{3/2}\right) \tag{2.122}$$

whenever our a priori assumption (2.20) *is satisfied. Moreover, there exists $\epsilon_0 > 0$ such that if $E_n(0) + Z_n(0)^2 \le \epsilon_0$, then we have $\mathscr{E}_n \lesssim \mathscr{E}_n(0)$ with $E_n(s) \lesssim e^{-C|\mathscr{b}|^m s}$ (decaying exponentially on $[0, \infty)$) and Z_n bounded on $[0, \infty)$.*

Proof. By the energy estimates in Theorem 2.6.1 and 2.6.5 (for $\mathscr{S}_n$ and $\mathscr{Q}_n$) in the last two subsections and Lemma 2.6.8 we have (choosing ϵ small enough)

$$\mathscr{E}_n \lesssim \mathscr{S}_n + \mathscr{Q}_n - C\epsilon\mathscr{E}_n \lesssim_\epsilon |\mathscr{b}|^{-m}\mathscr{E}_n(0) + C_\delta(\mathscr{E}_n + \mathscr{Z}_n^2)^{1/2}\mathscr{E}_n$$
$$\lesssim |\mathscr{b}|^{-m}\mathscr{E}_n(0) + C_\delta\left(\mathscr{Z}_n(0)\mathscr{E}_n(s) + (1 + s^{1/2})\mathscr{E}_n(s)^{3/2}\right)$$

Using Proposition 2.6.7 above, we get $\mathscr{E}_n \lesssim \mathscr{E}_n(0)$ with $E_n(s) \lesssim e^{-C|\mathscr{b}|^m s}$ and Z_n

bounded on $[0, \infty)$. $\square$

Chapter 3

Nonradial stability of linearly expanding Goldreich-Weber stars

3.1 Introduction

3.1.1 Equation in linearly-expanding coordinates

Let $(\bar{\rho}, \bar{\mathbf{u}})$ be a given linearly expanding GW-flow from Definition 1.2.4 with the corresponding radius $R\lambda(t)$ and the associated enthalpy $\bar{w} : [0, R] \to \mathbb{R}_+$. In order to study the stability of the flow, like in the last chapter renormalise the equation by introducing a new unknown

$$\boldsymbol{\xi}(t, \mathbf{x}) = \frac{\boldsymbol{\eta}(t, \mathbf{x})}{\lambda(t)}. \tag{3.1}$$

We suitably renormalise the inverse of the Jacobian gradient and the Jacobian determinant, so that

$$\begin{aligned}
\mathscr{A} &:= (\nabla \boldsymbol{\xi})^{-1} &&= \lambda A \\
\mathscr{J} &:= \det(\nabla \boldsymbol{\xi}) &&= \lambda^{-3} J \\
\mathscr{a} &:= \mathscr{J} \mathscr{A} &&= \lambda^{-2} a \\
\Phi &:= -\int \frac{f_0(\mathbf{z}) \mathscr{J}_0(\mathbf{z})}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} \mathrm{d}\mathbf{z} &&= \lambda \psi
\end{aligned}$$

We next formulate the problem in "linear" variables. To this end we introduce the linear time coordinate s adapted to the expanding profile via

$$\frac{ds}{dt} = \lambda(t)^{-1}.$$

In this new coordinate, $\lambda(s)$ is an increasing function such that

$$\lambda(s) \sim e^{s\sqrt{\lambda_1^2+2\delta}} \qquad \text{as} \qquad s \to \infty. \tag{3.2}$$

We have the following change of coordinate formula $\partial_t = \lambda^{-1}\partial_s$. The condition $\ddot{\lambda}\lambda^2 = \delta$ (1.18) becomes

$$\delta = \lambda\partial_s(\lambda^{-1}\partial_s\lambda) = \partial_s^2\lambda - \frac{(\partial_s\lambda)^2}{\lambda} \tag{3.3}$$

Then the Euler-Poisson equations (1.10) becomes

$$\begin{aligned}\mathbf{0} &= \partial_t\mathbf{v} + (f_0J_0)^{-1}\partial_k(A^k(f_0J_0)^{4/3}J^{-1/3}) + A\nabla\psi \\ &= \lambda^{-1}\partial_s(\lambda^{-1}\partial_s(\lambda\boldsymbol{\xi})) + \lambda^{-2}(f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \lambda^{-2}\mathscr{A}\nabla\Phi\end{aligned}$$

Times the equation by λ^2 we get

$$\begin{aligned}\mathbf{0} &= \lambda\partial_s(\lambda^{-1}\partial_s(\lambda\boldsymbol{\xi})) + (f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \mathscr{A}\nabla\Phi \\ &= \lambda\left(\partial_s^2\boldsymbol{\xi} + \frac{\partial_s\lambda}{\lambda}\partial_s\boldsymbol{\xi} + \left(\frac{\partial_s^2\lambda}{\lambda} - \frac{(\partial_s\lambda)^2}{\lambda^2}\right)\boldsymbol{\xi}\right) + (f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) \\ &\quad + \mathscr{A}\nabla\Phi \\ &= \left(\lambda\partial_s^2\boldsymbol{\xi} + (\partial_s\lambda)\partial_s\boldsymbol{\xi} + \delta\boldsymbol{\xi}\right) + (f_0J_0)^{-1}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \mathscr{A}\nabla\Phi\end{aligned}$$

So the Euler-Poisson equations in terms of $\boldsymbol{\xi}$ (3.1) is:

$$\lambda\partial_s^2\boldsymbol{\xi} + \lambda'\partial_s\boldsymbol{\xi} + \delta\boldsymbol{\xi} + \frac{1}{f_0J_0}\partial_k(\mathscr{A}^k(f_0J_0)^{4/3}\mathscr{J}^{-1/3}) + \mathscr{A}\nabla\Phi = \mathbf{0}, \tag{3.4}$$

where $\lambda' := \partial_s\lambda$.

The GW-star is a particular s-independent solution of (3.4) of the form $\boldsymbol{\xi}(\mathbf{x}) \equiv \mathbf{x}$ and $f_0 = \bar{w}^3$. Before formulating the stability problem, we must first make the use of the labelling gauge freedom and fix the choice of the initial enthalpy $(f_0J_0)^{1/3}$ for the general perturbation to be exactly identical to the background enthalpy $\bar{w}$, i.e. we set

$$(f_0J_0)^{1/3} = \bar{w} \qquad \text{on } B_R(\mathbf{0}). \tag{3.5}$$

Equation (3.5) can be re-written in the form $\rho_0 \circ \boldsymbol{\eta}_0 \det[\nabla\boldsymbol{\eta}_0] = \bar{w}^3$ on the initial domain $B_R(\mathbf{0})$. By a result of Dacorogna-Moser [10] and similarly to [20, 21] there exists a choice of an initial bijective map $\boldsymbol{\eta}_0 : B_R(\mathbf{0}) \to \Omega(\mathbf{0})$ so that (3.5) holds true. The gauge fixing condition (3.5) is necessary as it constraints the freedom to arbitrary relabel the particles at the initial time.

We now introduce the perturbation

$$\boldsymbol{\theta}(\mathbf{x}) := \boldsymbol{\xi}(\mathbf{x}) - \mathbf{x}, \tag{3.6}$$

which measures the deviation of the nonlinear flow to the background Goldreich-Weber profile.

Lemma 3.1.1 (Euler-Poisson in linearly-expanding coordinate)**.** *The perturbation $\boldsymbol{\theta}$ defined in* (3.6) *formally solves*

$$\lambda\partial_s^2\boldsymbol{\theta} + \lambda'\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G} = \mathbf{0}, \tag{3.7}$$

where the nonlinear pressure operator $\mathbf{P}$ *and the nonlinear gravity operator* $\mathbf{G}$ *read*

$$\mathbf{P} := \bar{w}^{-3}\partial_k(\bar{w}^4(\mathscr{A}^k_\bullet\mathscr{J}^{-1/3} - I^k_\bullet)), \tag{3.8}$$

$$\mathbf{G} := \mathscr{A}\nabla\Phi - \mathscr{K}\nabla\bar{w}^3 = \mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}_\bullet\bar{w}^3) - \mathscr{K}\nabla\bar{w}^3 \tag{3.9}$$

$$= \mathscr{K}_{\boldsymbol{\xi}}((\mathscr{A} - I)\nabla\bar{w}^3 - \bar{w}^3\mathscr{A}^i_m\mathscr{A}^l_\bullet\partial_i\partial_l\theta^m) + (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K})\nabla\bar{w}^3, \tag{3.10}$$

and

$$(\mathscr{K}_{\boldsymbol{\xi}}g)(\mathbf{x}) := -\int\frac{g(\mathbf{z})}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|}\mathrm{d}\mathbf{z} \tag{3.11}$$

Proof. Recall that the GW-enthalpy satisfies

$$\mathbf{0} = \delta\mathbf{x} + 4\nabla\bar{w} + \nabla\mathscr{K}\bar{w}^3 \tag{3.12}$$

Using the gauge condition (3.5), the momentum equation (3.4) becomes

$$\bar{w}^3\left(\lambda\partial_s^2\boldsymbol{\theta} + \lambda'\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta}\right) + \partial_k(\bar{w}^4(\mathscr{A}^k\mathscr{J}^{-1/3} - I^k)) + \bar{w}^3(\mathscr{A}\nabla\Phi - \nabla\mathscr{K}\bar{w}^3) = \mathbf{0}.$$

Note that formally

$$(\nabla\mathscr{K}\rho)(\mathbf{x}) = -\int\nabla_{\mathbf{x}}\frac{\rho(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|}\mathrm{d}\mathbf{z} = \int\nabla_{\mathbf{z}}\frac{\rho(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|}\mathrm{d}\mathbf{z} = -\int\frac{\nabla\rho(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|}\mathrm{d}\mathbf{z}$$
$$= (\mathscr{K}\nabla\rho)(\mathbf{x})$$

and so

$$\begin{aligned}
A\nabla\psi(\mathbf{x}) &= (\nabla\phi)(\boldsymbol{\eta}(\mathbf{x})) = (\nabla\mathscr{K}\rho)(\boldsymbol{\eta}(\mathbf{x})) = (\mathscr{K}\nabla\rho)(\boldsymbol{\eta}(\mathbf{x})) \\
&= -\int \frac{\nabla\rho(\mathbf{y})}{|\boldsymbol{\eta}(\mathbf{x})-\mathbf{y}|}\mathrm{d}\mathbf{x} = -\int \frac{A\nabla f(\mathbf{z})J(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} \\
&= -\int \frac{a\nabla(fJJ^{-1})(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} = -\int \frac{a\nabla(\bar{w}^3J^{-1})(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} \\
&= -\int \frac{\nabla\cdot(a\bar{w}^3J^{-1})(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} = -\int \frac{\nabla\cdot(A\bar{w}^3)(\mathbf{z})}{|\boldsymbol{\eta}(\mathbf{x})-\boldsymbol{\eta}(\mathbf{z})|}\mathrm{d}\mathbf{z} \\
&= \frac{1}{\lambda^2}(\mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}\bar{w}^3))(\mathbf{x}),
\end{aligned}$$

where we denote $\nabla\cdot M = \partial_i M^i$ for a matrix M and recall (3.11). We then have

$$\mathscr{A}\nabla\Phi = \lambda^2 A\nabla\psi(\mathbf{x}) = \mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}\bar{w}^3).$$

Hence, we can write the momentum equation as

$$\begin{aligned}
\mathbf{0} &= \lambda\partial_s^2\boldsymbol{\theta} + \lambda'\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta} + \bar{w}^{-3}\partial_k(\bar{w}^4(\mathscr{A}^k\mathscr{J}^{-1/3} - I^k)) + \mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}\bar{w}^3) - \mathscr{K}\nabla\bar{w}^3 \\
&= \lambda\partial_s^2\boldsymbol{\theta} + \lambda'\partial_s\boldsymbol{\theta} + \delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G},
\end{aligned}$$

where we have also made use of (3.12). Note that we can write

$$\begin{aligned}
\mathbf{G} &= \mathscr{K}_{\boldsymbol{\xi}}(\nabla\cdot(\mathscr{A}\bar{w}^3) - \nabla\bar{w}^3) + (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K})\nabla\bar{w}^3 \\
&= \mathscr{K}_{\boldsymbol{\xi}}((\mathscr{A} - I)\nabla\bar{w}^3 - \bar{w}^3\mathscr{A}^i_m\mathscr{A}^l_{\bullet}\partial_i\partial_l\theta^m) + (\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K})\nabla\bar{w}^3.
\end{aligned}$$

□

3.1.2 High-order energies and the main theorem

We now introduce high-order weighted Sobolev norm that we will use for our high-order energy method explained in Section 3.5. Recall the notation in section 1.4 in Chapter 1. Assuming that $(s,\mathbf{y}) \mapsto \boldsymbol{\theta}(s,\mathbf{y})$ is a sufficiently smooth field, for any $n \in \mathbb{N}_0$ we let

$$\begin{aligned}
S_n(s) &:= \sum_{|\beta|+b\le n} \left(\lambda\|X_r^b\not\partial^\beta\partial_s\boldsymbol{\theta}\|^2_{3+b} + \|X_r^b\not\partial^\beta\boldsymbol{\theta}\|^2_{3+b} + \|\nabla X_r^b\not\partial^\beta\boldsymbol{\theta}\|^2_{4+b}\right) \\
Q_n(s) &:= \sum_{c\le n} \left(\lambda\|\nabla^c\partial_s\boldsymbol{\theta}\|^2_{3+2c} + \|\nabla^c\boldsymbol{\theta}\|^2_{3+2c} + \|\nabla^{c+1}\boldsymbol{\theta}\|^2_{4+2c}\right) \\
Z_n(s) &:= \sum_{|\beta|+b=n} \lambda\|X_r^b\not\partial^\beta(\mathscr{A}\nabla\times\partial_s\boldsymbol{\theta})\|^2_{4+b} + \lambda\|\nabla^n(\mathscr{A}\nabla\times\partial_s\boldsymbol{\theta})\|^2_{4+2n}
\end{aligned}$$

We define the total instant energy via

$$E_n := S_n + Q_n + Z_n. \tag{3.13}$$

We shall run the energy identity using E_n; Z_n controls the curl of the velocity, while the energies S_n and Q_n will be used for high-order estimates near the vacuum boundary and near the origin respectively. In particular, the control afforded by Q_n is stronger near the origin, while S_n is stronger near the boundary. Finally we define

$$\mathcal{S}_n(s) := \sup_{\tau\in[0,s]} S_n(\tau), \tag{3.14}$$

$$\mathbb{Q}_n(s) := \sup_{\tau\in[0,s]} Q_n(\tau), \tag{3.15}$$

$$\mathcal{E}_n(s) := \sup_{\tau\in[0,s]} E_n(\tau), \tag{3.16}$$

The norms (3.14)–(3.16) will play the role of the "left hand side" in the high-order energy identities.

Remark 3.1.2. *We emphasise that the higher order energies E_n we defined (always with a subscript $n \in \mathbb{N}_0$) are different from the total conserved energy E (and E_δ) defined in* (1.7). *Where no confusion arises, we will refer to both as "energy".*

In this chapter, we make the following a priori assumption:

A priori assumption: $E_n \leq \epsilon$ where $\epsilon > 0$ is some small constant. (3.17)

We now state our main theorem.

Theorem 3.1.3 (Nonlinear stability of GW stars)**.** *Let $n \geq 21$. The linearly expanding GW star from Definition 1.2.4 nonlinearly stable. More precisely, there exists an $\epsilon^* > 0$ such that for any initial data $(\boldsymbol{\theta}(0), \partial_s\boldsymbol{\theta}(0))$ satisfying*

$$E_n(0) \leq \epsilon^*, \tag{3.18}$$

the associated solution $s \mapsto (\boldsymbol{\theta}(s,\,\cdot\,), \partial_s\boldsymbol{\theta}(s,\,\cdot\,))$ to (3.7) *exists for all $s \geq 0$ and is unique in the class of all data with finite norm E_n. Moreover, there exists a constant $C > 0$ such that*

$$E_n(s) \leq C\epsilon^* \qquad \textit{for all} \qquad s \geq 0.$$

Remark 3.1.4. *Like in the last chapter (cf. Remark 2.1.9), it is not our goal to optimise the number n of derivatives in our spaces.*

Local-in-time well-posedness. The same process as described in section 2.1.3 for the self-similarly expanding GW star can be use to obtain the equivalent well-posedness result in the weighted high-order energy space E_n defined in the current section for the linearly expanding GW star.

Theorem 3.1.5 (Local well-posedness)**.** *Let $n \geq 21$. Then for any given initial data $(\boldsymbol{\theta}(0), \partial_s\boldsymbol{\theta}(0))$ such that $E_n(0) < \infty$, there exist some $T > 0$ and a unique solution $(\boldsymbol{\theta}, \partial_s\boldsymbol{\theta}) : [0,T] \times B_R \to \mathbb{R}^3 \times \mathbb{R}^3$ to* (3.7) *such that $E_n(s) \leq 2E_n(0)$ for all $s \in [0,T]$.*

Theorem 3.1.5 is a starting point for the continuity argument that will culminate in the proof of Theorem 3.1.3.

3.1.3 Proof strategy

The basic idea behind the global existence in Theorem 3.1.3 is similar to that of the self-similarly expanding GW case in the last chapter. In fact, it is easier here than in the last chapter owing to the fact the linearly expanding GW expands at a faster rate that the self-similarly expanding GW star, and hence there is a stronger dispersion effect. Therefore, the basic outline and method of the proof is similar to the last chapter, but it is simpler here.

In particular, we have the exponentially increasing $\lambda(t)$ factor in the first term in (3.7), this means we get a λ factor in front of the "velocity" terms in the higher order energy in (3.13). This gives terms on the velocity level (terms with at least one time derivatives) an extra decay that in effect made it of secondary importance on par with the non-linear term and hence negligible in the dynamics.

This in particular renders the effect of gravity in the dynamics to be secondary:

$$\int_{s_0}^{s} \langle X_r^b \not\partial^\beta \mathbf{G}, \partial_s X_r^b \not\partial^\beta \boldsymbol{\theta} \rangle_{3+b} \mathrm{d}\tau \lesssim \mathscr{E}_n(s) \int_{s_0}^{s} \lambda^{-1/2} \mathrm{d}\tau.$$

Here $\mathscr{E}_n(s) \int_{s_0}^{s} \lambda^{-1/2}\mathrm{d}\tau$ have effect on par with non-linear term $\mathscr{E}^{3/2}$, see Proposition 3.5.3 and Theorem 3.5.4.

These also means we do not need a precise coercivity result like in the self-similarly expanding GW case for $\mathbf{L}$ in the last chapter. Hence in particular we do not need the fluid to be irrotational in this case. Instead it is sufficient to just do a vorticity estimate (section 3.4) to estimate the curl, similar to what is done in [21].

Note also that linear motion is secondary (bounded) in a linearly expanding coordinate. So a non-zero momentum in the initial data, which in theory should make the overall GW star to travel at constant speed in the direction of the momentum, is automatically encapsulated by the linear expanding coordinate about a linearly expanding GW star centred at the origin.

Many terms that appeared on the primary "linear level" in self-similarly expanding case of the last chapter now thus appear in the linearly expanding case in the secondary "non-linear" level. This also means that we do not need higher time derivatives in our higher order energy in (3.13), and also do not need the sophisticated triple induction scheme on the higher order energies that we have to do in the last chapter.

3.2 Pressure estimates

In this section we will estimate the non-linear part of the pressure term $X_r^b \not\partial^\beta \mathbf{P}$ and $\partial^\gamma \mathbf{P}$ (3.7).

Recall from (3.8) that $\mathbf{P} := \bar{w}^{-3}\partial_k(\bar{w}^4(\mathscr{A}^k \mathscr{J}^{-1/3} - I^k))$. In the next two lemmas, we will compute the commutators between the operator $X_r^b \not\partial^\beta$ (and ∂^γ) and the weighted derivative $\bar{w}^{-3}\partial_k(\bar{w}^4\cdot)$. Lemmas 3.2.1 are necessary to control all the non-"top-order" contributions coming from $X_r^b \not\partial^\beta \mathbf{P}$ and $\partial^\gamma \mathbf{P}$ by our energy norms.

The use of radial derivatives naturally changes the weighting structure, which is one of the key observations that makes the high-order energy argument possible and goes back to [33]. And for Cartesian derivatives, we need two powers of weight for every derivatives, which gives a weaker control than those given by the radial and tangential derivatives.

Lemma 3.2.1. *For any tensor field T_i^k sufficiently smooth, we have*

$$
\begin{aligned}
X_r\left(\bar{w}^{-c}\partial_k(\bar{w}^{1+c}T_i^k)\right) &= \bar{w}^{-(1+c)}\partial_k(\bar{w}^{2+c}X_r T_i^k) \\
&\quad + (1+c)(T_i^k X_r \partial_k \bar{w}) + (\partial_k \bar{w})\not\partial_{kj} T_i^j - \bar{w}\partial_k T_i^k \\
\not\partial_j\left(\bar{w}^{-c}\partial_k(\bar{w}^{1+c}T_i^k)\right) &= \bar{w}^{-c}\partial_k(\bar{w}^{1+c}\not\partial_j T_i^k) \\
&\quad - \epsilon_{jkl}((1+c)(\partial_l \bar{w})T_i^k + \bar{w}\partial_l T_i^k)) \\
\partial_j\left(\bar{w}^{-c}\partial_k(\bar{w}^{1+c}T_i^k)\right) &= \bar{w}^{-(2+c)}\partial_k(\bar{w}^{3+c}\partial_j T_i^k) \\
&\quad + (1+c)(T_i^k \partial_j \partial_k \bar{w}) + (\partial_j \bar{w})\partial_k T_i^k - 2(\partial_j T_i^k)\partial_k \bar{w} \\
X_r^d \not\partial^\beta\left(\bar{w}^{-3}\partial_k(\bar{w}^4 T_i^k)\right) &= \bar{w}^{-(3+d)}\partial_k(\bar{w}^{4+d} X_r^d \not\partial^\beta T_i^k)
\end{aligned}
$$

$$+\left(\sum_{\substack{d'\le d\\|\beta'|\le|\beta|-1}}+\sum_{\substack{d'\le d-1\\|\beta'|\le|\beta|+1}}\right)\langle C\omega X_r^{d'}\not\partial^{\beta'}T\rangle$$
$$+\left(\sum_{\substack{d'\le d-1\\|\beta'|\le|\beta|-1}}+\sum_{\substack{d'\le d-2\\|\beta'|\le|\beta|}}\right)\langle CX_r^{d'}\not\partial^{\beta'}\nabla T\rangle$$
$$+\left(\sum_{\substack{d'\le d\\|\beta'|\le|\beta|-1}}+\sum_{\substack{d'\le d-1\\|\beta'|\le|\beta|}}\right)\langle C\bar{w}X_r^{d'}\not\partial^{\beta'}\nabla T\rangle \tag{3.19}$$
$$\partial^\gamma\left(\bar{w}^{-3}\partial_k(\bar{w}^4T_i^k)\right)=\bar{w}^{-(3+2|\gamma|)}\partial_k(\bar{w}^{4+2|\gamma|}\partial^\gamma T_i^k)$$
$$+\sum_{|\gamma'|\le|\gamma|-1}\left(\langle C\omega\partial^{\gamma'}T\rangle+\langle C\omega\partial^{\gamma'}\nabla T\rangle\right)$$

for any $c\ge 0$ and $i=1,2,3$, where ω denotes some derivatives of $\bar{w}$. Here we used notations defined in Definition 1.4.2.

Proof. Proofs for all but the last equation are the same as Lemma 2.5.2. For the last equation we have

$$\begin{aligned}
&\partial_j\left(\bar{w}^{-c}\partial_k(\bar{w}^{1+c}T_i^k)\right)\\
&=\partial_j\left((1+c)T_i^k\partial_k\bar{w}+\bar{w}\partial_kT_i^k\right)\\
&=(1+c)\left((\partial_jT_i^k)\partial_k\bar{w}+T_i^k\partial_j\partial_k\bar{w}\right)+(\partial_j\bar{w})\partial_kT_i^k+\bar{w}\partial_j\partial_kT_i^k\\
&=\bar{w}^{-(2+c)}\partial_k(\bar{w}^{3+c}\partial_jT_i^k)+(1+c)(T_i^k\partial_j\partial_k\bar{w})+(\partial_j\bar{w})\partial_kT_i^k-2(\partial_jT_i^k)\partial_k\bar{w}
\end{aligned}$$

where we used commutation relations from Lemma A.1.2. The final two formulas can be proven by induction. □

The next lemma deals with the terms we get when we apply $X_r^b\not\partial^\beta$ or ∂^γ to $\mathscr{A}\mathscr{J}^{-1/3}-I$.

Lemma 3.2.2. *Let*

$$T:=\mathscr{A}\mathscr{J}^{-1/3}-I. \tag{3.20}$$

Recall notations defined in Definition 1.4.2. For $|\gamma|>0$, we have

$$\partial_\bullet T=T_T[\partial_\bullet\nabla\theta], \tag{3.21}$$
$$X_r^d\not\partial^\beta T=T_T[X_r^b\not\partial^\beta\nabla\theta]+T_{R:\beta,d} \tag{3.22}$$

$$\partial^\gamma T = T_T[\partial^\gamma \nabla\theta] + T_{R:\gamma}. \tag{3.23}$$

where

$$T_T[M]^k := -\mathcal{J}^{-1/3}\left(\mathcal{A}^k_m\mathcal{A}^l + \frac{1}{3}\mathcal{A}^k\mathcal{A}^l_m\right)M^m_l, \qquad k=1,2,3 \tag{3.24}$$

$$T_{R:\beta,d} := \mathcal{J}^{-1/3}\sum_{c=2}^{d+|\beta|}\sum_{\substack{\sum_{i=1}^c (d_i,\beta_i)=(d,\beta)\\ |d_i|+|\beta_i|>0}}\langle C\rangle\langle\mathcal{A}\rangle^{1+c}\prod_{i=1}^c\langle X_r^{d_i}\partial\!\!\!/^{\beta_i}\nabla\boldsymbol{\theta}\rangle \tag{3.25}$$

$$T_{R:\gamma} := \mathcal{J}^{-1/3}\sum_{c=2}^{|\gamma|}\sum_{\substack{\sum_{i=1}^c \gamma_i=\gamma\\ |\gamma_i|>0}}\langle C\rangle\langle\mathcal{A}\rangle^{1+c}\prod_{i=1}^c\langle\partial^{\gamma_i}\nabla\boldsymbol{\theta}\rangle. \tag{3.26}$$

Proof. Applying Lemma A.1.1 we get that

$$\begin{aligned}\partial_\bullet(\mathcal{A}^k\mathcal{J}^{-1/3} - I^k) &= -\mathcal{J}^{-1/3}\mathcal{A}^k_m\mathcal{A}^l\partial\partial_l\theta^m - \frac{1}{3}\mathcal{J}^{-1/3}\mathcal{A}^k\mathcal{A}^l_m\partial_\bullet\partial_l\theta^m\\ &= -\mathcal{J}^{-1/3}\left(\mathcal{A}^k_m\mathcal{A}^l + \frac{1}{3}\mathcal{A}^k\mathcal{A}^l_m\right)\partial_\bullet\partial_l\theta^m.\end{aligned}$$

Hence $\partial_\bullet T^k = T_T[\partial_\bullet\nabla\theta]^k$. By repeated application of this we get the next two formulas. □

We have from (3.7)

$$X_r^b\partial\!\!\!/^\beta\mathbf{P} = X_r^b\partial\!\!\!/^\beta\left(\bar{w}^{-3}\partial_k(\bar{w}^4T^k)\right)$$

Let

$$\begin{aligned}\mathbf{P}_d\boldsymbol{\theta} &:= \bar{w}^{-3-d}\partial_k(\bar{w}^{4+d}T_T[\nabla\boldsymbol{\theta}]^k)\\ &= -\bar{w}^{-(3+d)}\partial_k\left(\bar{w}^{4+d}\left(\mathcal{A}^k_m\mathcal{A}^l + \frac{1}{3}\mathcal{A}^k\mathcal{A}^l_m\right)\partial_l\theta^m\right)\end{aligned} \tag{3.27}$$

Let $\mathbf{P}_{d,L}$ be the linear part of $\mathbf{P}_d$, i.e.

$$\begin{aligned}\mathbf{P}_{d,L}\boldsymbol{\theta} &:= -\bar{w}^{-(3+d)}\partial_k\left(\bar{w}^{4+d}\left(I^k_mI^l + \frac{1}{3}I^kI^l_m\right)\partial_l\theta^m\right)\\ &= -\frac{1}{3\bar{w}^{3+d}}\nabla(\bar{w}^{4+d}\nabla\cdot\boldsymbol{\theta}) - \frac{1}{\bar{w}^{3+d}}\partial_k(\bar{w}^{4+d}\nabla\theta^k)\end{aligned} \tag{3.28}$$

In doing energy estimates, terms like $\langle X_r^b\partial\!\!\!/^\beta\mathbf{P}, \partial_s X_r^b\partial\!\!\!/^\beta\boldsymbol{\theta}\rangle$ will arise. Using the lemmas in this subsection, we will next show that $\mathbf{P}$ here can be reduced to $\mathbf{P}_{d,L}$ modulo remainder terms that can be estimated.

To that end we will first derive the following identity.

Lemma 3.2.3. *For any vector field $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ sufficiently smooth we have*

$$\langle \mathbf{P}_d \boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \rangle_{3+d}$$
$$= \int \Big((\mathscr{A}\partial_m \boldsymbol{\theta}_1) \cdot (\mathscr{A}\partial_m \boldsymbol{\theta}_2) + \frac{1}{3}(\operatorname{div}_{\mathscr{A}} \boldsymbol{\theta}_1)(\operatorname{div}_{\mathscr{A}} \boldsymbol{\theta}_2) \tag{3.29}$$
$$- \frac{1}{2}[\operatorname{curl}_{\mathscr{A}} \boldsymbol{\theta}_1]^m_j [\operatorname{curl}_{\mathscr{A}} \boldsymbol{\theta}_2]^m_j \Big) \mathscr{J}^{-1/3} \bar{w}^{4+d} \mathrm{d}\mathbf{x} \tag{3.30}$$

$$\langle \mathbf{P}_{d,L} \boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \rangle_{3+d}$$
$$= \int \Big((\partial_m \boldsymbol{\theta}_1) \cdot (\partial_m \boldsymbol{\theta}_2) + \frac{1}{3}(\operatorname{div} \boldsymbol{\theta}_1)(\operatorname{div} \boldsymbol{\theta}_2) - \frac{1}{2}[\operatorname{curl} \boldsymbol{\theta}_1]^m_j [\operatorname{curl} \boldsymbol{\theta}_2]^m_j \Big) \bar{w}^{4+d} \mathrm{d}\mathbf{x} \tag{3.31}$$

Proof. Same as Lemma 2.5.4 (except here $\bar{w}$ is the linearly expanding GW star profile rather than the self-similarly expanding one). □

Using this lemma, we will estimate the difference between "$\mathbf{P}_b$" and "$\mathbf{P}_{b,L}$".

Proposition 3.2.4. *Let $n \geq 20$. Let $|\beta| + b \leq n$ and $|\gamma| \leq n$. For any $\boldsymbol{\theta}$ that satisfies our a priori assumption* (3.17) *we have*

$$\left| \int_{s_0}^{s} \langle \mathbf{P}_b X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s X_r^b \not{\partial}^\beta \boldsymbol{\theta} \rangle_{3+b} \mathrm{d}\tau - \frac{1}{2} \left\langle \mathbf{P}_{b,L} X_r^b \not{\partial}^\beta \boldsymbol{\theta}, X_r^b \not{\partial}^\beta \boldsymbol{\theta} \right\rangle_{3+b} \Big|_{s_0}^{s} \right| \lesssim \mathscr{E}_n(s)^{3/2}$$
$$\left| \int_{s_0}^{s} \langle \mathbf{P}_{2|\gamma|} \partial^\gamma \boldsymbol{\theta}, \partial_s \partial^\gamma \boldsymbol{\theta} \rangle_{3+2|\gamma|} \mathrm{d}\tau - \frac{1}{2} \left\langle \mathbf{P}_{2|\gamma|,L} \partial^\gamma \boldsymbol{\theta}, \partial^\gamma \boldsymbol{\theta} \right\rangle_{3+2|\gamma|} \Big|_{s_0}^{s} \right| \lesssim \mathscr{E}_n(s)^{3/2}$$

Proof. By Lemma 3.2.3

$$\langle \mathbf{P}_b X_r^b \not{\partial}^\beta \boldsymbol{\theta}, \partial_s X_r^b \not{\partial}^\beta \boldsymbol{\theta} \rangle_{3+b}$$
$$= \int \Big((\mathscr{A}\partial_m X_r^b \not{\partial}^\beta \boldsymbol{\theta}) \cdot (\mathscr{A}\partial_m \partial_s X_r^b \not{\partial}^\beta \boldsymbol{\theta}) + \frac{1}{3}(\operatorname{div}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta})(\operatorname{div}_{\mathscr{A}} \partial_s X_r^b \not{\partial}^\beta \boldsymbol{\theta})$$
$$- \frac{1}{2}[\operatorname{curl}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta}]^m_j [\operatorname{curl}_{\mathscr{A}} \partial_s X_r^b \not{\partial}^\beta \boldsymbol{\theta}]^m_j \Big) \mathscr{J}^{-1/3} \bar{w}^{4+b} \mathrm{d}\mathbf{x}$$
$$= \int \Big((\mathscr{A}\partial_m X_r^b \not{\partial}^\beta \boldsymbol{\theta}) \cdot \partial_s (\mathscr{A}\partial_m X_r^b \not{\partial}^\beta \boldsymbol{\theta}) + \frac{1}{3}(\operatorname{div}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta}) \partial_s (\operatorname{div}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta})$$
$$- \frac{1}{2}[\operatorname{curl}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta}]^m_j \partial_s [\operatorname{curl}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta}]^m_j \Big) \mathscr{J}^{-1/3} \bar{w}^{4+b} \mathrm{d}\mathbf{x} + \mathscr{R}[\lambda^{-1/2} E_n^{3/2}]$$
$$= \frac{1}{2}\partial_s \int \Big(|\mathscr{A}\partial_m X_r^b \not{\partial}^\beta \boldsymbol{\theta}|^2 + \frac{1}{3}|\operatorname{div}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta}|^2 - \frac{1}{2}|[\operatorname{curl}_{\mathscr{A}} X_r^b \not{\partial}^\beta \boldsymbol{\theta}]|^2 \Big) \mathscr{J}^{-1/3} \bar{w}^{4+b} \mathrm{d}\mathbf{x}$$
$$+ \mathscr{R}[\lambda^{-1/2} E_n^{3/2}]$$

$$= \frac{1}{2}\partial_s \int \left(|\partial_m X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}|^2 + \frac{1}{3}|\operatorname{div} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}|^2 - \frac{1}{2}|[\operatorname{curl} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}]|^2 \right) \bar{w}^{4+b} \mathrm{d}\mathbf{x}$$
$$+ \partial_s \mathscr{R}[E_n^{3/2}] + \mathscr{R}[\lambda^{-1/2} E_n^{3/2}]$$
$$= \frac{1}{2}\partial_s \langle \mathbf{P}_{b,L} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}, X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta} \rangle_{3+b} + \partial_s \mathscr{R}[E_n^{3/2}] + \mathscr{R}[\lambda^{-1/2} E_n^{3/2}]$$

where we recall notation $\mathscr{R}[\star]$ introduced in Definition 1.4.2. Since $\int_0^\infty \lambda^{1/2} \mathrm{d}s < \infty$, integrating in time we get the first equation. Proof for the second formula is similar. □

And finally we will estimate the difference between "$\mathbf{P}$" and "$\mathbf{P}_b$".

Proposition 3.2.5. *Let $n \geq 20$ and $|\beta| + b \leq n$. Suppose $\boldsymbol{\theta}$ satisfies our a priori assumption* (3.17)*. Then we have*

$$\left| \int_{s_0}^{s} \langle X_r^b \partial\!\!\!/^\beta \mathbf{P} - \mathbf{P}_b X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}, \partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta} \rangle_{3+b} \mathrm{d}\tau \right| \lesssim \int_{s_0}^{s} \lambda^{-\frac{1}{2}} E_n \mathrm{d}\tau$$
$$\left| \int_{s_0}^{s} \langle \partial^\gamma \mathbf{P} - \mathbf{P}_{2|\gamma|} \partial^\gamma \boldsymbol{\theta}, \partial_s \partial^\gamma \boldsymbol{\theta} \rangle_{3+2|\gamma|} \mathrm{d}\tau \right| \lesssim \int_{s_0}^{s} \lambda^{-\frac{1}{2}} E_n \mathrm{d}\tau.$$

Proof. By Lemma 3.2.1 we need to estimate the following.

$$\left| \int_{s_0}^{s} \int_{B_R} \left(\sum_{\substack{b' \leq b \\ |\beta'| \leq |\beta| - 1}} + \sum_{\substack{b' \leq b-1 \\ |\beta'| \leq |\beta| + 1}} \right) \langle C \omega X_r^{b'} \partial\!\!\!/^{\beta'} T \rangle \langle \partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta} \rangle \bar{w}^{3+b} \mathrm{d}\mathbf{x} \mathrm{d}\tau \right|$$
$$+ \left| \int_{s_0}^{s} \int_{B_R} \left(\sum_{\substack{b' \leq b-1 \\ |\beta'| \leq |\beta| - 1}} + \sum_{\substack{b' \leq b-2 \\ |\beta'| \leq |\beta|}} \right) \langle C X_r^{b'} \partial\!\!\!/^{\beta'} \nabla T \rangle \langle \partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta} \rangle \bar{w}^{3+b} \mathrm{d}\mathbf{x} \mathrm{d}\tau \right|$$
$$+ \left| \int_{s_0}^{s} \int_{B_R} \left(\sum_{\substack{b' \leq b \\ |\beta'| \leq |\beta| - 1}} + \sum_{\substack{b' \leq b-1 \\ |\beta'| \leq |\beta|}} \right) \langle C \bar{w} X_r^{b'} \partial\!\!\!/^{\beta'} \nabla T \rangle \langle \partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta} \rangle \bar{w}^{3+b} \mathrm{d}\mathbf{x} \mathrm{d}\tau \right|$$
$$\lesssim \int_{s_0}^{s} \lambda^{-\frac{1}{2}} E_n \mathrm{d}\tau$$

where the $\lambda^{-1/2}$ factor comes from estimating $\|\partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b} \lesssim \lambda^{-1/2} E_n^{1/2}$. The terms with T can be estimated noting the structure given in Lemma 3.2.2. This proves the first formula. The proof for the second formula is similar. □

3.3 Gravity estimates

In this subsection we will estimate the gravity term $X_r^b \partial\!\!\!/^\beta \mathbf{G}$ and $\partial^\gamma \mathbf{G}$ (3.7) and show that it can be bounded by E_n.

Since the gravity term is a non-local term, we need to estimate convolution-like operator. However, rather than the convolution kernel $|\mathbf{x}-\mathbf{z}|^{-1}$ we actually need to estimate $|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|^{-1}$. The next two lemmas tell us how to reduce the latter to the former, which will allows us to estimate using the Young's convolution inequality.

Lemma 3.3.1. *Let $\boldsymbol{\xi}$ be as in* (3.6)*. For any $\mathbf{x}, \mathbf{y} \in B_R$ we have*

$$|\mathbf{x}-\mathbf{z}| \le \|\mathscr{A}\|_{L^\infty(B_R)} |\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|$$
$$|\partial\!\!\!/_{\mathbf{x}}^\beta \boldsymbol{\xi}(\mathbf{x}) - \partial\!\!\!/_{\mathbf{z}}^\beta \boldsymbol{\xi}(\mathbf{z})| \le \|\nabla \partial\!\!\!/^\beta \boldsymbol{\xi}\|_{L^\infty(B_R)} |\mathbf{x}-\mathbf{z}|$$

Proof. Same as Lemma 2.5.7. □

Lemma 3.3.2. *Let $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$ be as in* (3.6)*, and $\boldsymbol{\theta}$ satisfies our a priori assumption* (3.17)*. Let $n \ge 21$ and $|\beta| \le n$.*

i. When $|\beta| > n/2$ we have

$$\left| (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|} \right) \right| \lesssim \frac{1}{|\mathbf{x}-\mathbf{z}|^2} \sum_{n/2<|\gamma|\le n} |\partial\!\!\!/_{\mathbf{x}}^\gamma \boldsymbol{\xi}(\mathbf{x}) - \partial\!\!\!/_{\mathbf{z}}^\gamma \boldsymbol{\xi}(\mathbf{z})|$$

ii. When $|\beta| \le n/2$ we have

$$\left| \partial_{i,\mathbf{z}} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|} \right) \right| \lesssim \frac{1}{|\mathbf{x}-\mathbf{z}|^2}$$

Proof. These follows from Lemma 3.3.1, the embedding theorems A.3.7 and A.3.8, the a priori bounds $E_n \lesssim 1$ (3.17), and the following.

$$\begin{aligned} &(\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^\beta \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|} \right) \\ &= \sum_{m=1}^{|\beta|} \sum_{\substack{\sum_{i=1}^m (\beta_i+\beta_i')=\beta \\ |\beta_i|>0}} \frac{(-1)^m (2m)!}{m! 2^m} \frac{1}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|^{1+2m}} \\ &\qquad \prod_{i=1}^m (\partial\!\!\!/_{\mathbf{x}}^{\beta_i} \boldsymbol{\xi}(\mathbf{x}) - \partial\!\!\!/_{\mathbf{z}}^{\beta_i} \boldsymbol{\xi}(\mathbf{z})) \cdot (\partial\!\!\!/_{\mathbf{x}}^{\beta_i'} \boldsymbol{\xi}(\mathbf{x}) - \partial\!\!\!/_{\mathbf{z}}^{\beta_i'} \boldsymbol{\xi}(\mathbf{z})). \end{aligned}$$

□

Since we cannot commute extra weights into the non-local gravity term, the radial derivatives which eat up weight need to be estimated differently in a way that would negate the non-local integral and allow extra weights to be used. Using methods from [22], the following two lemmas provide the way to do this. More precisely, the radial derivative can be estimated with curl, divergence and tangential derivatives. And this is useful because the curl and divergence of the gravity term consist only of local or non-linear terms, which we can estimate.

Lemma 3.3.3. *For any vector field* $\tilde{\mathbf{G}} \in H^1_{\text{loc}}$

$$|X_r\tilde{\mathbf{G}}|^2 \lesssim |r\nabla \cdot \tilde{\mathbf{G}}|^2 + |r\nabla \times \tilde{\mathbf{G}}|^2 + \sum_{k=1}^{3} |\not{\partial}_k\tilde{\mathbf{G}}|^2.$$

Proof. Same as Lemma 2.5.9. □

The corresponding version for Cartesian derivatives is as follows.

Lemma 3.3.4. *Let* $k \geq 0$. *For any vector field* $\tilde{\mathbf{G}} \in H^1_{\text{loc}}$,

$$\|\nabla\tilde{\mathbf{G}}\|^2_{k+2} \lesssim \|\nabla \cdot \tilde{\mathbf{G}}\|^2_{k+2} + \|\nabla \times \tilde{\mathbf{G}}\|^2_{k+2} + \|\tilde{\mathbf{G}}\|^2_k$$

Proof. Same as Lemma 2.6.2. □

The div and curl of the gravity term can be written as the following form that will allow us to estimate them later.

Lemma 3.3.5. *Let* $\mathbf{G}$ *be as in* (3.9). *We have*

$$\nabla \cdot \mathbf{G} = (I - \mathscr{A})\nabla \cdot \mathbf{G} + (I - \mathscr{A})\nabla \cdot \nabla\mathscr{K}\bar{w}^3 + 4\pi\bar{w}^3(\mathscr{J}^{-1} - 1)$$
$$\nabla \times \mathbf{G} = (I - \mathscr{A})\nabla \times \mathbf{G} + (I - \mathscr{A})\nabla \times \nabla\mathscr{K}\bar{w}^3.$$

Proof. Same as Lemma 2.5.10 (except here $\bar{w}$ is the linearly expanding GW star profile rather than the self-similarly expanding one). □

We next derive helpful identities for the operators $\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K}$. We first note that

$$(\mathscr{K}_{\boldsymbol{\xi}} - \mathscr{K})g(\mathbf{x}) = -\int_{\mathbb{R}^3} K_1(\mathbf{x}, \mathbf{z})g(\mathbf{z})\text{d}\mathbf{z}, \tag{3.32}$$

where

$$K_1(\mathbf{x}, \mathbf{z}) := \frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} - \frac{1}{|\mathbf{x} - \mathbf{z}|}, \tag{3.33}$$

In the following lemma, we write K_1 explicitly in terms of $\boldsymbol{\theta}$, which will play a role in our energy estimates. In particular, we see that $\boldsymbol{\theta}$ appears at least linearly in K_1.

Lemma 3.3.6. *We have*

$$\begin{aligned} K_1(\mathbf{x},\mathbf{z}) = &-\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^3} - \frac{1}{2}\frac{|\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^3} \\ &+ \frac{3}{4|\mathbf{x}-\mathbf{z}|}\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right)^2 \\ &\varpi_{\frac{1}{2}}\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right), \end{aligned} \tag{3.34}$$

where

$$\varpi_q(y) := \int_0^1 \frac{1-z}{(1+yz)^{q+2}}\mathrm{d}z, \qquad y > -1, \quad q \in \mathbb{R}. \tag{3.35}$$

Proof. Same as Lemma 2.2.3. □

We collect a few more easy statements in the next lemma.

Lemma 3.3.7. (i) *For any $K : B_R \times B_R \to \mathbb{R}$ sufficiently nice and $g \in H_0^1(B_R)$ we have*

$$\not{\partial}_{i,\mathbf{x}} \int_{B_R} K(\mathbf{x},\mathbf{z})g(\mathbf{z})\mathrm{d}\mathbf{z} = \int_{B_R} \left(g(\mathbf{z})(\not{\partial}_{i,\mathbf{x}} + \not{\partial}_{i,\mathbf{z}})K(\mathbf{x},\mathbf{z}) + K(\mathbf{x},\mathbf{z})\not{\partial}_{i,\mathbf{z}}g(\mathbf{z})\right)\mathrm{d}\mathbf{z}$$

(ii) *For any $\boldsymbol{\theta} : B_R \to \mathbb{R}^3$ sufficiently smooth and $\mathbf{x}, \mathbf{y} \in B_R$ we have*

$$|\partial_s^a \not{\partial}^\beta \boldsymbol{\theta}(\mathbf{x}) - \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}(\mathbf{z})| \leq \|\nabla \partial_s^a \not{\partial}^\beta \boldsymbol{\theta}\|_{L^\infty(B_R)} |\mathbf{x}-\mathbf{z}| \tag{3.36}$$

$$|\not{\partial}^\beta \mathbf{x} - \not{\partial}^\beta \mathbf{z}| \leq |\mathbf{x}-\mathbf{z}| \tag{3.37}$$

Proof. Same as Lemma 2.3.3. □

Lemma 3.3.8. *Let $n \geq 20$ and $|\beta| \leq n$. We have*

$$|(\not{\partial}_{\mathbf{x}} + \not{\partial}_{\mathbf{z}})^\beta K_1(\mathbf{x},\mathbf{z})| \lesssim \frac{1}{|\mathbf{x}-\mathbf{z}|^2} \sum_{\beta' \leq \beta} |\not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{x}) - \not{\partial}^{\beta'} \boldsymbol{\theta}(\mathbf{z})|$$

where we recall K_1 (3.33).

Proof. From Lemma 3.3.6,

$$\begin{aligned}K_1(\mathbf{x},\mathbf{z}) = &-\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^3} - \frac{1}{2}\frac{|\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^3}\\ &+ \frac{3}{4|\mathbf{x}-\mathbf{z}|}\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right)^2\\ &\varpi_{\frac{1}{2}}\underbrace{\left(2\frac{(\mathbf{x}-\mathbf{z})\cdot(\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} + \frac{|\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\theta}(\mathbf{z})|^2}{|\mathbf{x}-\mathbf{z}|^2}\right)}_{:=y(\mathbf{x},\mathbf{z})}\end{aligned}$$

Note that $|y(\mathbf{x},\mathbf{z})| \lesssim \|\nabla\boldsymbol{\theta}\|_{L^\infty}$. Our a priori assumption (3.17) together with the embedding theorems A.3.7 and A.3.8 mean that $\|\nabla\boldsymbol{\theta}\|_{L^\infty}$ is bounded by a small constant. So we can assume $|y(\mathbf{x},\mathbf{z})| \le 1/2$. Then from the definition of ϖ_q (3.35) we can see that

$$\varpi^{(k)}_{\frac{1}{2}}(y(\mathbf{x},\mathbf{z})) \lesssim 1 \qquad \text{for any} \qquad k \ge 0.$$

Now using part (ii) of Lemma 3.3.7, chain and product rule for derivatives and the embedding theorems A.3.7 and A.3.8, we can see that $\partial_s^a(\not\partial_{\mathbf{x}} + \not\partial_{\mathbf{z}})^\beta K_1(\mathbf{x},\mathbf{z})$ satisfies the stated bounds. □

Finally we can prove the main results of this subsection.

Proposition 3.3.9 (Gravity estimates)**.** *Let $n \ge 21$ and suppose $\boldsymbol{\theta}$ satisfies our a priori assumption* (3.17)*. Then we have*

$$\begin{aligned}\|X_r^b\not\partial^\beta\mathbf{G}\|^2_{3+b} &\lesssim E_n &&\textit{when} && |\beta|+b\le n,\\ \|\bar{w}^{b/2}X_r^b\not\partial^\beta\mathbf{G}\|^2_{L^\infty(\mathbb{R}^3)} &\lesssim E_n &&\textit{when} && |\beta|+b\le n/2,\\ \|\nabla^c\mathbf{G}\|^2_{3+2c} &\lesssim E_n &&\textit{when} && c\le n,\\ \|\bar{w}^{c/2}\nabla^c\mathbf{G}\|^2_{L^\infty(\mathbb{R}^3)} &\lesssim E_n &&\textit{when} && c\le n/2.\end{aligned}$$

Proof. By definition

$$\begin{aligned}\mathbf{G} = \mathscr{K}_{\boldsymbol{\xi}}\nabla\cdot(\mathscr{A}\bar{w}^3) - \mathscr{K}\nabla\bar{w}^3 &= -\int_{\mathbb{R}^3}\frac{\partial_k(\mathscr{A}^k\bar{w}^3)}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\mathbf{dz} + \int_{\mathbb{R}^3}\frac{\nabla\bar{w}^3}{|\mathbf{x}-\mathbf{z}|}\mathbf{dz}\\ &= -\int_{\mathbb{R}^3}\frac{\partial_k((\mathscr{A}^k - I^k)\bar{w}^3)}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\mathbf{dz} - \int_{\mathbb{R}^3}K_1(\mathbf{x},\mathbf{z})\nabla\bar{w}^3\mathbf{dz}\end{aligned}$$

By Lemma 3.3.7 we have

$$\not\partial^\beta\mathbf{G}(\mathbf{x}) = -\not\partial^\beta\left(\int_{\mathbb{R}^3}\frac{\partial_k((\mathscr{A}^k - I^k)\bar{w}^3)}{|\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{z})|}\mathbf{dz} + \int_{\mathbb{R}^3}K_1(\mathbf{x},\mathbf{z})\nabla\bar{w}^3\mathbf{dz}\right)$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} \sum_{\beta_1+\beta_2=\beta} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} \right) \partial\!\!\!/_{\mathbf{z}}^{\beta_2} \partial_k ((\mathscr{A}^k - I^k)\bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&\quad - \int_{\mathbb{R}^3} \sum_{\beta_1+\beta_2=\beta} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} K_1(\mathbf{x},\mathbf{z}) (\partial\!\!\!/_{\mathbf{z}}^{\beta_2} \nabla \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&= -\int_{\mathbb{R}^3} \sum_{\substack{\beta_1+\beta_2=\beta \\ |\beta_1|>n/2}} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} \right) \partial\!\!\!/_{\mathbf{z}}^{\beta_2} \partial_k ((\mathscr{A}^k - I^k)\bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&\quad - \int_{\mathbb{R}^3} \sum_{\substack{\beta_1+\beta_2=\beta \\ |\beta_1|\le n/2}} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} \right) \partial\!\!\!/_{\mathbf{z}}^{\beta_2} \partial_k ((\mathscr{A}^k - I^k)\bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&\quad - \int_{\mathbb{R}^3} \sum_{\beta_1+\beta_2=\beta} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} K_1(\mathbf{x},\mathbf{z}) (\partial\!\!\!/_{\mathbf{z}}^{\beta_2} \nabla \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&= -\int_{\mathbb{R}^3} \sum_{\substack{\beta_1+\beta_2=\beta \\ |\beta_1|>n/2}} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} \right) \partial\!\!\!/_{\mathbf{z}}^{\beta_2} \partial_k ((\mathscr{A}^k - I^k)\bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&\quad + \int_{\mathbb{R}^3} \sum_{\substack{\beta_1+\beta_2\le\beta \\ |\beta_1|\le n/2}} \langle \nabla_{\mathbf{z}} \rangle (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} \left(\frac{1}{|\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\xi}(\mathbf{z})|} \right) (\langle \partial\!\!\!/_{\mathbf{z}}^{\beta_2} (\mathscr{A} - I) \rangle \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&\quad - \int_{\mathbb{R}^3} \sum_{\beta_1+\beta_2=\beta} (\partial\!\!\!/_{\mathbf{x}} + \partial\!\!\!/_{\mathbf{z}})^{\beta_1} K_1(\mathbf{x},\mathbf{z}) (\partial\!\!\!/_{\mathbf{z}}^{\beta_2} \nabla \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z}
\end{aligned}
$$

Now using Lemma 3.3.2 and 3.3.8, we get

$$
\begin{aligned}
|\partial_s^a \partial\!\!\!/^\beta \mathbf{G}(\mathbf{x})| &\lesssim \int_{\mathbb{R}^3} \frac{E_n^{1/2}}{|\mathbf{x}-\mathbf{z}|^2} \sum_{n/2<|\gamma|\le n} |\partial\!\!\!/_{\mathbf{x}}^\gamma \boldsymbol{\xi}(\mathbf{x}) - \partial\!\!\!/_{\mathbf{z}}^\gamma \boldsymbol{\xi}(\mathbf{z})| \bar{w}^2 \mathrm{d}\mathbf{z} \\
&\quad + \int_{\mathbb{R}^3} \sum_{\beta_2\le\beta} \frac{1}{|\mathbf{x}-\mathbf{z}|^2} (\langle \partial\!\!\!/_{\mathbf{z}}^{\beta_2} (\mathscr{A} - I) \rangle \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&\quad + \int_{\mathbb{R}^3} \sum_{\beta_1+\beta_2=\beta} \sum_{\beta'\le\beta_1} \frac{|\partial\!\!\!/^{\beta'} \boldsymbol{\theta}(\mathbf{x}) - \partial\!\!\!/^{\beta'} \boldsymbol{\theta}(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^2} (\partial\!\!\!/_{\mathbf{z}}^{\beta_2} \nabla \bar{w}^3)(\mathbf{z}) \mathrm{d}\mathbf{z}
\end{aligned}
$$

Now using Young's convolution inequality we get

$$
\|\partial\!\!\!/^\beta \mathbf{G}(\mathbf{x})\|_{L^2(\mathbb{R}^3)} \lesssim E_n^{1/2}.
$$

Hence $\|\partial\!\!\!/^\beta \mathbf{G}\|_3^2 \lesssim E_n$. From the above proof, with small modification, we can further see that

$$
\|\partial\!\!\!/^\beta \mathbf{G}(\mathbf{x})\|_{L^\infty(\mathbb{R}^3)} \lesssim E_n^{1/2} \qquad \text{when} \qquad |\beta| \le n/2
$$

Now we deal with the case $b > 0$. Let

$$\begin{aligned} W_n &= \sum_{|\beta|+b\le n} \|X_r^b \partial\!\!\!/^\beta \mathbf{G}\|_{3+b}^2 \\ W_{n,d} &= \sum_{\substack{|\beta|+b\le n \\ b\le d}} \|X_r^b \partial\!\!\!/^\beta \mathbf{G}\|_{3+b}^2 \\ V_n &= \sum_{|\beta|+b\le n} \sup_{\mathbb{R}^3} \left(\bar{w}^b |X_r^b \partial\!\!\!/^\beta \mathbf{G}|^2\right) \\ V_{n,d} &= \sum_{\substack{|\beta|+b\le n \\ b\le d}} \sup_{\mathbb{R}^3} \left(\bar{w}^b |X_r^b \partial\!\!\!/^\beta \mathbf{G}|^2\right). \end{aligned}$$

For $|\beta| + b \le n/2$, using the above lemmas 3.3.3 and 3.3.5 we have

$$\begin{aligned} &\bar{w}^b |X_r^b \partial\!\!\!/^\beta \mathbf{G}|^2 \\ &\lesssim \bar{w}^b |r\nabla \cdot X_r^{b-1} \partial\!\!\!/^\beta \mathbf{G}|^2 + \bar{w}^b |r\nabla \times X_r^{b-1} \partial\!\!\!/^\beta \mathbf{G}|^2 + \sum_{k=1}^{3} \bar{w}^b |X_r^{b-1} \partial\!\!\!/_k \partial\!\!\!/^\beta \mathbf{G}|^2 \\ &\lesssim \bar{w}^b |r X_r^{b-1} \partial\!\!\!/^\beta \nabla \cdot \mathbf{G}|^2 + \bar{w}^b |r X_r^{b-1} \partial\!\!\!/^\beta \nabla \times \mathbf{G}|^2 + V_{b+|\beta|-1} + V_{b+|\beta|,b-1} \\ &\lesssim \bar{w}^b |r X_r^{b-1} \partial\!\!\!/^\beta ((I - \mathscr{A})\nabla \cdot \mathbf{G})|^2 + \bar{w}^b |r X_r^{b-1} \partial\!\!\!/^\beta ((I - \mathscr{A})\nabla \times \mathbf{G})|^2 \\ &\quad + E_n + V_{b+|\beta|-1} + V_{b+|\beta|,b-1} \\ &\lesssim \bar{w}^b |r(I - \mathscr{A}) X_r^{b-1} \partial\!\!\!/^\beta \nabla \cdot \mathbf{G}|^2 + \bar{w}^b |r(I - \mathscr{A}) X_r^{b-1} \partial\!\!\!/^\beta \nabla \times \mathbf{G}|^2 \\ &\quad + E_n + V_{b+|\beta|-1} + V_{b+|\beta|,b-1} \\ &\lesssim \bar{w}^b E_n |r X_r^{b-1} \partial\!\!\!/^\beta \nabla \mathbf{G}|^2 + E_n + V_{b+|\beta|-1} + V_{b+|\beta|,b-1} \end{aligned}$$

So

$$V_{b+|\beta|,b} \lesssim E_n V_{b+|\beta|,b} + E_n + V_{b+|\beta|-1} + V_{b+|\beta|,b-1}$$

By a priori assumption (3.17), we have $E_n \ll 1$, so

$$V_{b+|\beta|,b} \lesssim E_n + V_{b+|\beta|-1} + V_{b+|\beta|,b-1}.$$

We know $V_{n',0} \lesssim E_n$ for all $n' \le n/2$, so by induction we get $V_{n',d} \lesssim E_n$ for all $d \le n' \le n/2$.

Now for $|\beta| + b \le n$, using the above lemmas 3.3.3 and 3.3.5 and results for V we have

$$\|X_r^b \partial\!\!\!/^\beta \mathbf{G}\|_{3+b}^2$$

$$\begin{aligned}
&\lesssim \|r\nabla\cdot X_r^{b-1}\not{\partial}^\beta \mathbf{G}\|_{3+b}^2 + \|r\nabla\times X_r^{b-1}\not{\partial}^\beta \mathbf{G}\|_{3+b}^2 + \sum_{k=1}^{3}\|X_r^{b-1}\not{\partial}_k\not{\partial}^\beta \mathbf{G}\|_{3+b}^2 \\
&\lesssim \|rX_r^{b-1}\not{\partial}^\beta\nabla\cdot \mathbf{G}\|_{3+b}^2 + \|rX_r^{b-1}\not{\partial}^\beta\nabla\times \mathbf{G}\|_{3+b}^2 + W_{b+|\beta|-1} + W_{b+|\beta|,b-1} \\
&\lesssim \|rX_r^{b-1}\not{\partial}^\beta((1-\mathscr{A})\nabla\cdot \mathbf{G})\|_{3+b}^2 + \|rX_r^{b-1}\not{\partial}^\beta((1-\mathscr{A})\nabla\times \mathbf{G})\|_{3+b}^2 \\
&\quad + E_n + W_{b+|\beta|-1} + W_{b+|\beta|,b-1} \\
&\lesssim \|r(1-\mathscr{A})X_r^{b-1}\not{\partial}^\beta\nabla\cdot \mathbf{G}\|_{3+b}^2 + \|r(1-\mathscr{A})X_r^{b-1}\not{\partial}^\beta\nabla\times \mathbf{G}\|_{3+b}^2 \\
&\quad + E_n + W_{b+|\beta|-1} + W_{b+|\beta|,b-1} \\
&\lesssim E_n\|rX_r^{b-1}\not{\partial}^\beta\nabla\mathbf{G}\|_{3+b}^2 + E_n + W_{b+|\beta|-1} + W_{b+|\beta|,b-1}
\end{aligned}$$

So

$$W_{b+|\beta|,b} \lesssim E_n W_{b+|\beta|,b} + E_n + W_{b+|\beta|-1} + W_{b+|\beta|,b-1}$$

By a priori assumption (3.17), we have $E_n \ll 1$, so

$$W_{b+|\beta|,b} \lesssim E_n + W_{b+|\beta|-1} + W_{b+|\beta|,b-1}.$$

We know $W_{n,0} \lesssim E_n$, so by induction we get $W_{n,d} \lesssim E_n$ for all $d \le n$.

Let

$$Y_n = \sum_{c\le n}\|\nabla^c \mathbf{G}\|_{3+2c}^2$$

By Sobolev embeddings like those used to prove the embedding theorems A.3.7 and A.3.8, we have that

$$\|\bar{w}^{c/2}\nabla^c\mathbf{G}\|_{L^\infty(\mathbb{R}^3)}^2 \lesssim E_n + Y_n \qquad \text{when} \qquad c \le n/2.$$

Now for $c \le n$, using the above lemmas 3.3.4 and 3.3.5 we have

$$\begin{aligned}
&\|\nabla^c\mathbf{G}\|_{3+2c}^2 \\
&\lesssim \|\nabla^{c-1}\nabla\cdot\mathbf{G}\|_{3+2c}^2 + \|\nabla^{c-1}\nabla\times\mathbf{G}\|_{3+2c}^2 + \|\nabla^{c-1}\mathbf{G}\|_{3+2(c-1)}^2 \\
&\lesssim \|\nabla^{c-1}((1-\mathscr{A})\nabla\cdot\mathbf{G})\|_{3+2c}^2 + \|\nabla^{c-1}((1-\mathscr{A})\nabla\times\mathbf{G})\|_{3+2c}^2 + E_n + Y_{c-1} \\
&\lesssim \|(1-\mathscr{A})\nabla^{c-1}\nabla\cdot\mathbf{G}\|_{3+2c}^2 + \|(1-\mathscr{A})\nabla^{c-1}\nabla\times\mathbf{G}\|_{3+2c}^2 + E_n(1+Y_n) + Y_{c-1} \\
&\lesssim E_n\|\nabla^{c-1}\nabla\mathbf{G}\|_{3+2c}^2 + E_n(1+Y_n) + Y_{c-1}
\end{aligned}$$

So

$$Y_c \lesssim E_n Y_n + E_n + Y_{c-1}$$

By induction on $c \leq n$ we have

$$Y_n \lesssim E_n Y_n + E_n$$

By a priori assumption (3.17), we have $E_n \ll 1$, so we get $Y_n \lesssim E_n$. $\square$

3.4 Vorticity estimates

In this section we will estimate the vorticity Z_n, $\|\nabla \times X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{4+b}$ and $\|\nabla \times \partial^\gamma \boldsymbol{\theta}\|_{4+2|\gamma|}$ which will be needed to control the "curl" part of the pressure term as seem in Lemma 3.2.3.

By taking curl to the Euler-Poisson equation (3.7) we can essentially get rid of the pressure and gravity terms, which allows us to estimate the curl separately.

Lemma 3.4.1. *Let $\boldsymbol{\theta}$ be a solution of* (3.7) *in the sense of Theorem 3.1.5. Then for any $s \geq s_0 \geq 0$ we have*

$$\lambda(s)^{\frac{1}{2}}(\mathscr{A}\nabla \times \partial_s \boldsymbol{\theta})(s) = \frac{\lambda(s_0)}{\lambda(s)^{1/2}}(\mathscr{A}\nabla \times \partial_s \boldsymbol{\theta})(s_0) + \lambda(s)^{-\frac{1}{2}} \int_{s_0}^{s} (\partial_s \mathscr{A})\nabla \times \partial_s \boldsymbol{\theta} \mathrm{d}s'$$

$$\begin{aligned}(\mathscr{A}\nabla \times \boldsymbol{\theta})(s) = (\mathscr{A}\nabla \times \boldsymbol{\theta})(s_0) &+ \int_{s_0}^{s} (\partial_s \mathscr{A})\nabla \times \boldsymbol{\theta} \, \mathrm{d}s' \\ &+ \int_{s_0}^{s} \frac{\lambda(s_0)}{\lambda(s)} \mathrm{d}s' (\mathscr{A}\nabla \times \partial_s \boldsymbol{\theta})(s_0) \\ &+ \int_{s_0}^{s} \lambda(s')^{-1} \int_{s_0}^{s'} (\partial_s \mathscr{A})\nabla \times \partial_s \boldsymbol{\theta} \mathrm{d}s'' \mathrm{d}s'.\end{aligned}$$

Proof. Recall (3.4) is

$$\mathbf{0} = \lambda \partial_s^2 \boldsymbol{\xi} + \lambda' \partial_s \boldsymbol{\xi} + \delta \boldsymbol{\xi} + \frac{1}{w^3}\partial_k(\mathscr{A}^k w^4 \mathscr{J}^{-1/3}) + \mathscr{A}\nabla\Phi \tag{3.38}$$

Now note that

$$\begin{aligned}\frac{1}{w^3}\partial_k(\mathscr{A}^k w^4 \mathscr{J}^{-1/3}) &= \bar{w}^{-3}\partial_k(\bar{w}^4 a^k \mathscr{J}^{-4/3}) = \bar{w}^{-3} a^k \partial_k(\bar{w}^4 \mathscr{J}^{-4/3}) \\ &= (\bar{w}\mathscr{J}^{-1/3})^{-1}\mathscr{A}^k \partial_k(\bar{w}^4 \mathscr{J}^{-4/3}) = \frac{4}{3}\mathscr{A}\nabla(\bar{w}^3 \mathscr{J}^{-1})\end{aligned}$$

and $\mathscr{A}\nabla \times \boldsymbol{\xi} = \epsilon_{\bullet jk}\mathscr{A}^l_j\partial_l\xi_k = \epsilon_{\bullet jk}\delta_{jk} = \mathbf{0}$. So taking $\mathscr{A}\nabla\times$ to (3.4) we get

$$\begin{aligned}\mathbf{0} &= \lambda\mathscr{A}\nabla \times \partial_s^2\boldsymbol{\theta} + \lambda'\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta} \\ &= \lambda\left(\partial_s(\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta}) - (\partial_s\mathscr{A})\nabla \times \partial_s\boldsymbol{\theta}\right) + \lambda'\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta} \\ &= \partial_s(\lambda\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta}) - (\partial_s\mathscr{A})\nabla \times \partial_s\boldsymbol{\theta}\end{aligned}$$

So

$$\lambda(s)(\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta})(s) = \lambda(s_0)(\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta})(s_0) + \int_{s_0}^{s}(\partial_s\mathscr{A})\nabla \times \partial_s\boldsymbol{\theta}\mathrm{d}s'$$

So

$$\begin{aligned}\lambda\partial_s(\mathscr{A}\nabla \times \boldsymbol{\theta})(s) = \lambda(\partial_s\mathscr{A})\nabla \times \boldsymbol{\theta}(s) + \lambda(s_0)(\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta})(s_0) \\ + \int_{s_0}^{s}(\partial_s\mathscr{A})\nabla \times \partial_s\boldsymbol{\theta}\mathrm{d}s'\end{aligned}$$

So

$$\begin{aligned}(\mathscr{A}\nabla \times \boldsymbol{\theta})(s) = (\mathscr{A}\nabla \times \boldsymbol{\theta})(s_0) &+ \int_{s_0}^{s}(\partial_s\mathscr{A})\nabla \times \boldsymbol{\theta}\ \mathrm{d}s' \\ &+ \int_{s_0}^{s}\frac{\lambda(s_0)}{\lambda(s)}\mathrm{d}s'(\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta})(s_0) \\ &+ \int_{s_0}^{s}\lambda(s')^{-1}\int_{s_0}^{s'}(\partial_s\mathscr{A})\nabla \times \partial_s\boldsymbol{\theta}\mathrm{d}s''\mathrm{d}s'.\end{aligned}$$

□

Proposition 3.4.2. *Let $\boldsymbol{\theta}$ be a solution of* (3.7) *in the sense of Theorem 3.1.5. Let* $n \geq 21$. *Then for any* $s \geq s_0 \geq 0$, *we have*

$$Z_n(s) \lesssim Z_n(s_0) + \frac{(s-s_0)^2}{\lambda(s)}\mathscr{E}_n(s) + \lambda(s)^{-1}\mathscr{E}_n(s)^2.$$

Proof. Take $X_r^b\slashed{\partial}^\beta$ $(b+\beta = n)$ to the first equation in Lemma 3.4.1 we get

$$\begin{aligned}\lambda(s)^{\frac{1}{2}}X_r^b\slashed{\partial}^\beta(\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta})(s) = \frac{\lambda(s_0)}{\lambda(s)^{1/2}}X_r^b\slashed{\partial}^\beta(\mathscr{A}\nabla \times \partial_s\boldsymbol{\theta})(s_0) \\ + \lambda(s)^{-\frac{1}{2}}X_r^b\slashed{\partial}^\beta\int_{s_0}^{s}(\partial_s\mathscr{A})\nabla \times \partial_s\boldsymbol{\theta}\mathrm{d}s'\end{aligned}$$

Since λ is an increasing function (3.2), we have $\lambda(s_0)^{1/2}/\lambda(s)^{1/2} \leq q$ which we can used to bound the first term on the RHS. And for the other terms we can estimate for example

$$\begin{aligned}
&\int_{B_R}\left|\lambda(s)^{-\frac{1}{2}}\int_{s_0}^{s}(X_r^b\not{\partial}^\beta\partial_s\mathscr{A})\nabla\times\partial_s\boldsymbol{\theta}\,\mathrm{d}s'\right|^2\bar{w}^{4+b}\mathbf{dx}\\
&\leq\lambda(s)^{-1}\int_{B_R}\left|\int_{s_0}^{s}(X_r^b\not{\partial}^\beta\partial_s\mathscr{A})\nabla\times\partial_s\boldsymbol{\theta}\,\mathrm{d}s'\right|^2\bar{w}^{4+b}\mathbf{dx}\\
&\lesssim\lambda(s)^{-1}\int_{B_R}\left|\int_{s_0}^{s}(X_r^b\not{\partial}^\beta\mathscr{A})\nabla\times\partial_s^2\boldsymbol{\theta}\,\mathrm{d}s'\right|^2\bar{w}^{4+b}\mathbf{dx}+\lambda(s)^{-1}\mathscr{E}_n(s)^2\\
&\lesssim\frac{(s-s_0)^2}{\lambda(s)}\mathscr{E}_n(s)+\lambda(s)^{-1}\mathscr{E}_n(s)^2
\end{aligned}$$

where we used $\lambda\partial_s^2\boldsymbol{\theta}=-\lambda'\partial_s\boldsymbol{\theta}-\delta\boldsymbol{\theta}-\mathbf{P}-\mathbf{G}$ (3.7).

We can then repeat this process for ∂^γ ($|\gamma|=n$) in place of $X_r^b\not{\partial}^\beta$ to complete the proof. □

Proposition 3.4.3. *Let $\boldsymbol{\theta}$ be a solution of* (3.7) *in the sense of Theorem 3.1.5. Let $n\geq 21$. Then for any $s\geq s_0\geq 0$ and $n'\leq n$, we have*

$$\begin{aligned}
\sum_{b+\beta\leq n'}\|\nabla\times X_r^b\not{\partial}^\beta\boldsymbol{\theta}(s)\|_{4+b}^2&\lesssim E_n(s_0)+S_{n'-1}(s)+\mathscr{E}_n(s)^2\\
&\quad+\int_{s_0}^{s}\frac{1+(s'-s_0)^2}{\lambda(s')}\mathrm{d}s'\,\mathscr{E}_n(s)\\
\sum_{|\gamma|\leq n}\|\nabla\times\partial^\gamma\boldsymbol{\theta}(s)\|_{4+2|\gamma|}^2&\lesssim E_n(s_0)+\mathscr{E}_n(s)^2+\int_{s_0}^{s}\frac{1+(s'-s_0)^2}{\lambda(s')}\mathrm{d}s'\,\mathscr{E}_n(s).
\end{aligned}$$

Proof. Take $X_r^b\not{\partial}^\beta$ ($b+\beta=n$) to the second equation in Lemma 3.4.1 and then estimate in a similar way to Lemma 3.4.2. We estimate

$$\begin{aligned}
&\left\|\int_{s_0}^{s}\lambda(s')^{-1}\int_{s_0}^{s'}\star\,\mathrm{d}s''\mathrm{d}s'\right\|_{4+b}^2\\
&\leq\int_{B_R}\left(\int_{s_0}^{s}\lambda(s')^{-1}\mathrm{d}s'\right)\left(\int_{s_0}^{s}\lambda(s')^{-1}\left|\int_{s_0}^{s'}\star\,\mathrm{d}s''\right|^2\mathrm{d}s'\right)\bar{w}^{4+b}\mathbf{dx}\\
&\lesssim\int_{s_0}^{s}\lambda(s')^{-1}\int_{B_R}\left|\int_{s_0}^{s'}\star\,\mathrm{d}s''\right|^2\bar{w}^{4+b}\mathbf{dx}\mathrm{d}s'\\
&\lesssim\int_{s_0}^{s}\frac{1+(s'-s_0)^2}{\lambda(s')}\mathrm{d}s'\,\mathscr{E}_n(s)
\end{aligned}$$

where we used the Cauchy–Schwarz inequality to get the first inequality, and the fact that $\int_0^\infty \lambda^{-1}\mathrm{d}s' < \infty$ for the second inequality (see (3.2)). And

$$\sum_{b+\beta\le n} \|[\nabla\times, X_r^b\not\partial^\beta]\boldsymbol{\theta}(s)\|_{4+b}^2 \lesssim S_{n'-1}(s)$$

$$\sum_{b+\beta\le n} \|X_r^b\not\partial^\beta((\mathscr{A}-I)\nabla\times\boldsymbol{\theta})(s)\|_{4+b}^2 \lesssim \mathscr{E}_n(s)^2.$$

Then we get the first formula. Proof for the second formula is similar. □

3.5 Energy estimates and proof of the main theorem

In this section we finally commute the momentum equation (3.7) and then derive the high-order energy estimates. Then finally we will prove our main theorem using the energy estimates.

Theorem 3.5.1 (Energy estimates)**.** *Let* $n \ge 21$. *Let* $\boldsymbol{\theta}$ *be a solution of* (3.7) *in the sense of Theorem 3.1.5, given on its maximal interval of existence. Then*

$$\mathscr{E}_n(s) \lesssim \mathscr{E}_n(s_0) + \mathscr{E}_n(s)^{3/2} + \mathscr{E}_n(s)\int_{s_0}^{s} \lambda^{-1/2}\mathrm{d}\tau. \tag{3.39}$$

for any $s \ge s' \ge 0$ *whenever our a priori assumption* (3.17) *is satisfied. Here we recall Definition* (3.16) *of the total norm* $\mathscr{E}_n$.

Proof. Since $\mathscr{E}_n = \mathscr{S}_n + \mathscr{Q}_n + \mathscr{X}_n$, we need to prove the formula with LHS each of these three component terms.

We first deal with the $\mathscr{S}_n$ part. Let $|\beta| + b \le n$. Apply $X_r^b\not\partial^\beta$ to the momentum equation (3.7) to get

$$\lambda\partial_s^2 X_r^b\not\partial^\beta\boldsymbol{\theta} + \lambda'\partial_s X_r^b\not\partial^\beta\boldsymbol{\theta} + X_r^b\not\partial^\beta(\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}) = 0$$

Taking the $\langle\cdot,\cdot\rangle_{3+b}$-inner with $\partial_s X_r^b\not\partial^\beta\boldsymbol{\theta}$ we get

$$\begin{aligned} 0 &= \frac{1}{2}\lambda\partial_s\|\partial_s X_r^b\not\partial^\beta\boldsymbol{\theta}\|_{3+b}^2 + \lambda'\|\partial_s X_r^b\not\partial^\beta\boldsymbol{\theta}\|_{3+b}^2 \\ &\quad + \langle X_r^b\not\partial^\beta(\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle_{3+b} \\ &= \frac{1}{2}\partial_s\left(\lambda\|\partial_s X_r^b\not\partial^\beta\boldsymbol{\theta}\|_{3+b}^2\right) + \frac{1}{2}\lambda'\|\partial_s X_r^b\not\partial^\beta\boldsymbol{\theta}\|_{3+b}^2 \\ &\quad + \langle X_r^b\not\partial^\beta(\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s X_r^b\not\partial^\beta\boldsymbol{\theta}\rangle_{3+b} \end{aligned}$$

Integrate in time we get

$$0 = \frac{1}{2}\left(\lambda\|\partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2\right)\Big|_{s_0}^{s} + \frac{1}{2}\int_{s_0}^{s} \lambda' \|\partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2 \mathrm{d}\tau$$
$$+ \int_{s_0}^{s} \langle X_r^b \partial\!\!\!/^\beta(\delta\boldsymbol{\theta} + \mathbf{P} + \mathbf{G}), \partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\rangle_{3+b} \mathrm{d}\tau$$

By Proposition 3.2.5, 3.2.4, 3.3.9 and Lemma 3.2.3 we get

$$\frac{1}{2}\Bigg(\lambda\|\partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{3+b}^2 + \|\nabla X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{4+b}^2 + \frac{1}{3}\|\operatorname{div} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{4+b}^2$$
$$- \frac{1}{2}\|\operatorname{curl} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_{4+b}^2\Bigg)\Bigg|_{s_0}^{s} + \frac{1}{2}\int_{s_0}^{s} \lambda' \|\partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|_3^2 \mathrm{d}\tau$$
$$\lesssim \mathscr{E}_n(s)^{3/2} + \int_{s_0}^{s} \lambda^{-1/2} E_n \mathrm{d}\tau$$

Using Proposition 3.4.3 we get

$$\lambda(s)\|\partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s)\|_{3+b}^2 + \|\nabla X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s)\|_{4+b}^2$$
$$\lesssim E_n(s_0) + E_{n-1}(s) + \mathscr{E}_n(s)^2 + \int_{s_0}^{s} \frac{1 + (s' - s_0)^2}{\lambda(s')} \mathrm{d}\tau\, \mathscr{E}_n(s) + \mathscr{E}_n(s)^{3/2}$$
$$+ \int_{s_0}^{s} \lambda^{-1/2} E_n \mathrm{d}\tau$$
$$\lesssim E_n(s_0) + E_{n-1}(s) + \mathscr{E}_n(s)^{3/2} + \mathscr{E}_n(s) \int_{s_0}^{s} \lambda^{-1/2} \mathrm{d}\tau$$

where we used (3.2) and (3.17). Add to it

$$\|X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s)\|_{3+b}^2 = \|X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s_0)\|_{3+b}^2 + 2\int_{s_0}^{s} \langle X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s), \partial_s X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s)\rangle_{3+b} \mathrm{d}\tau$$
$$\lesssim E_n(s_0) + \int_{s_0}^{s} \lambda^{-1/2} E_n \mathrm{d}\tau,$$

sum over $|\beta| + b \leq n' \leq n$ and we get

$$S_{n'}(s) \lesssim E_n(s_0) + S_{n'-1}(s) + \mathscr{E}_n(s)^{3/2} + \mathscr{E}_n(s) \int_{s_0}^{s} \lambda^{-1/2} \mathrm{d}\tau.$$

Induct on n' we get

$$S_n(s) \lesssim E_n(s_0) + \mathscr{E}_n(s)^{3/2} + \mathscr{E}_n(s) \int_{s_0}^{s} \lambda^{-1/2} \mathrm{d}\tau.$$

To prove the $\mathbb{Q}_n$ part, we repeat the above with ∂^γ in place of $X_r^b \partial\!\!\!/^\beta$, and weight

$3 + 2|\gamma|$ instead of weight $3 + b$.

Finally the $\mathfrak{X}_n$ part is given by Proposition 3.4.2 noting that

$$\lambda(s)^{-1}\mathscr{E}_n(s)^2 + \frac{(s-s_0)^2}{\lambda(s)}\mathscr{E}_n(s) \lesssim \mathscr{E}_n(s)^{3/2} + \mathscr{E}_n(s)\int_{s_0}^{s} \lambda^{-1/2}\mathrm{d}\tau.$$

where we used (3.2) and (3.17). □

To proof our main theorem that the energy E_n remains bounded, we will use the bootstrapping scheme in the following lemma and proposition.

Lemma 3.5.2. *Suppose* $E : [0,T] \to [0,\infty]$ *is continuous and*

$$E(t) \leq C_1 E(0) + C_2 E(t)^{3/2} \qquad \textit{whenever} \qquad \sup_{\tau\in[0,t]} E(\tau) \leq C_3.$$

where $C_1 \geq 1$. *Then* $E \leq 2C_1 E(0)$ *whenever* $E(0) \leq \min\{(2^5 C_1 C_2^2)^{-1}, C_3/2C_1\}$.

Proof. Same as Lemma 2.6.6. □

Proposition 3.5.3. *Suppose* $E : [0,\infty) \to [0,\infty]$ *are continuous and for all* $s \geq s_0 \geq 0$ *we have*

$$E(s) \leq C_1 E(s_0) + C_2 E(s)^{3/2} + C_3 F(s_0,s)E(s) \quad \textit{whenever} \quad \sup_{\tau\in[s_0,s]} E(\tau) \leq C_4$$

where $F : \{(s_0,s) \in [0,\infty)\times[0,\infty) : s \geq s_0\} \to [0,\infty)$ *is a function such that*

i. $\lim_{s_0\to\infty} \sup_{s\geq s_0} F(s_0,s) = 0$*;*

ii. $\lim_{\delta'\to 0} \sup_{|s-s_0|\leq\delta'} F(s_0,s) = 0$.

Then there exist $\epsilon^* > 0$ *such that* $E \lesssim_{C_1} E(0)$ *whenever* $E(0) \leq \epsilon^*$.

Proof. Pick s_∞ large enough so that

$$C_3 \sup_{s\geq s_\infty} F(s_\infty,s) < \frac{1}{2}.$$

Then by Lemma 3.5.2 there exist $\epsilon_\infty > 0$ such that $\sup_{s\in[s_\infty,\infty)} \leq 4C_1 E(s_\infty)$ whenever $E(s_\infty) \leq \epsilon_\infty$.

Now pick δ' small enough so that

$$C_3 \sup_{|s-s_0|\leq\delta'} F(s_0,s) < \frac{1}{2}.$$

Then by Lemma 3.5.2 there exist $\epsilon_0 > 0$ such that $\sup_{s\in[m\delta',(m+1)\delta']} \leq 4C_1 E(m\delta')$ whenever $E(m\delta') \leq \epsilon_0$.

Let $\epsilon^* \leq \min\{\epsilon_0, \epsilon_\infty\}/(4C_1)^{\lceil s_\infty/\delta' \rceil}$. Then $E(s) \leq (4C_1)^{\lceil s_\infty/\delta' \rceil + 1} E(0)$ for all $s \geq 0$ whenever $E(0) \leq \epsilon^*$. □

Theorem 3.5.4. *Let $n \geq 21$. Let $(\boldsymbol{\theta}, \partial_s \boldsymbol{\theta})$ be a solution of* (3.7) *in the sense of Theorem 3.1.5. Then there exists $\epsilon^* > 0$ such that if $E_n(0) \leq \epsilon^*$, then we have $\mathscr{E}_n \lesssim \mathscr{E}_n(0)$.*

Proof. By the energy estimates in Theorem 3.5.1 we have

$$\mathscr{E}_n(s) \lesssim \mathscr{E}_n(s_0) + \mathscr{E}_n(s)^{3/2} + \mathscr{E}_n(s) \int_{s_0}^{s} \lambda^{-1/2} \mathrm{d}\tau. \tag{3.40}$$

Applying Proposition 3.5.3 above with $E = \mathscr{E}_n$ and $F(s_0, s) = \int_{s_0}^{s} \lambda^{-1/2} \mathrm{d}\tau$ (which satisfies the properties required for the proposition because of (3.2)) we get the desired result. □

Chapter 4

Linear stability of liquid Lane-Emden stars

In order to be able to prove linear (in)stability of liquid Lane-Emden stars, we will need detail information about the Lane–Emden stars. So in the first section of this chapter, we will establish various qualitative properties of the liquid Lane–Emden stars, including bounds for its density profile and radius. And using these results, in second section we will prove Theorem 1.3.4 concerning the linear stability of Lane-Emden stars, where we will prove the existence and non-existence of unstable modes using the variational formulation.

4.1 Lane–Emden stars and its properties

4.1.1 Basic properties of Lane–Emden stars

Theorem 4.1.1. *Let $\gamma \geq 1$. For every $\rho_0 > 0$, the ODE defining the LE stars (Definition 1.2.1) admits a unique solution $\rho \geq 0$ such that $\rho(0) = \rho_0$. The interval of existence $[0, R)$ is such that either $R = \infty$ or $\lim_{r\to R} \rho(r) = 0$.*

Proof. This is a know standard result, we included the proof in the appendix for completeness. □

In fact, due to the following decay estimate, we must have $\lim_{r\to R} \rho(r) = 0$ even if $R = \infty$. This is a elementary decay estimate that we will be using in many places later on.

Lemma 4.1.2 (Decay estimates)**.** *Suppose ρ is a solution to the Lane–Emden ODE*

(Definition 1.2.1) on $[0, R)$ *with* $\rho(0) = \rho_0$*. Then for* $r \in [0, R)$ *we have*

$$\rho(r) \leq \begin{cases} \left(\rho_0^{-(2-\gamma)} + \frac{2\pi}{d}\frac{2-\gamma}{\gamma}r^2\right)^{-\frac{1}{2-\gamma}} & \textit{when} \quad \gamma \neq 2 \\ \exp\left(\ln(\rho_0) - \frac{2\pi}{d}\frac{1}{\gamma}r^2\right) & \textit{when} \quad \gamma = 2 \end{cases}.$$

Equivalently, we have when $\gamma > 1$

$$w(r) \leq \begin{cases} \left(w_0^{-(\alpha-1)} + \frac{2\pi}{d}\frac{2-\gamma}{\gamma}r^2\right)^{-\frac{1}{\alpha-1}} & \textit{when} \quad \alpha \neq 1 \\ \exp\left(\ln(w_0) - \frac{2\pi}{d}\frac{\gamma-1}{\gamma}r^2\right) & \textit{when} \quad \alpha = 1 \end{cases}$$

and when $\gamma = 1$*,*

$$h(r) \leq -\ln\left(e^{-h_0} + \frac{2\pi}{d}r^2\right).$$

Proof. **Case 1,** $\gamma > 1$ From (1.15) we have

$$\begin{aligned} w'(r) &= -4\pi\frac{\gamma-1}{\gamma}\frac{1}{r^{d-1}}\int_0^r y^{d-1}w(y)^\alpha \mathrm{d}y \\ &\leq -4\pi\frac{\gamma-1}{\gamma}\frac{1}{r^{d-1}}w(r)^\alpha\int_0^r y^{d-1}\mathrm{d}y = -\frac{4\pi}{d}\frac{\gamma-1}{\gamma}w(r)^\alpha r \end{aligned}$$

where we have the first inequality because w is decreasing. Rearranging and integrating we get

$$\begin{aligned} \frac{2\pi}{d}\frac{\gamma-1}{\gamma}r^2 &\leq -\int_0^r \frac{w'(y)}{w(y)^\alpha}\mathrm{d}y = \int_{w(r)}^{w_0}\frac{1}{z^\alpha}\mathrm{d}z \\ &= \begin{cases} \frac{1}{\alpha-1}\left(\frac{1}{w(r)^{\alpha-1}} - \frac{1}{w_0^{\alpha-1}}\right) & \text{when} \quad \alpha \neq 1 \\ \ln w_0 - \ln w(r) & \text{when} \quad \alpha = 1 \end{cases}. \end{aligned}$$

Rearranging gives the desired result.

Case 2, $\gamma = 1$ From (1.17) we have

$$h'(r) = -4\pi\frac{1}{r^{d-1}}\int_0^r y^{d-1}e^{h(y)}\mathrm{d}y \leq -4\pi\frac{1}{r^{d-1}}e^{h(r)}\int_0^r y^{d-1}\mathrm{d}y = -\frac{4\pi}{d}e^{h(r)}r$$

where we have the first inequality because w is decreasing. Rearranging and integrating we get

$$\frac{2\pi}{d}r^2 \leq -\int_0^r h'(y)e^{-h(y)}\mathrm{d}y = e^{-h(r)} - e^{-h_0}.$$

So

$$-h(r) \geq \ln\left(e^{-h_0} + \frac{2\pi}{d} r^2\right).$$

□

In particular if $\rho_0 > 1$, then ρ would reach 1 at some $r = r^* < \infty$ with $w'(r^*) < 0$ (by (1.15)) or $h'(r^*) < 0$ (by (1.17)). We can cut off the solution at this point to obtain a state state for fluid stars.

Note that if we consider gas stars, i.e. $p = \rho^\gamma$, then whether $R = \infty$ or not correspond to whether the star is compactly supported or not.

Following is the so-called Pohozaev integral for the Lane–Emden stars, more generally considered in [27].

Proposition 4.1.3 (Pohozaev integral)**.** *Suppose ρ is a solution to the Lane–Emden ODE (Definition 1.2.1) on $[0, R)$. Then for $r \in [0, R)$ we have*

$$\begin{aligned}
&2\pi\left(2 - \frac{2-\gamma}{\gamma} d\right) \int_0^r \rho(y)^\gamma y^{d-1} \mathrm{d}y \\
&\qquad = \frac{1}{2}\gamma(\gamma-1)\left(\rho(r)^{-(2-\gamma)}\rho'(r)\right)^2 r^d + 4\pi\frac{\gamma-1}{\gamma}\rho(r)^\gamma r^d \\
&\qquad\quad + \frac{1}{2}(d-2)\gamma\rho(r)^{2\gamma-3}\rho'(r) r^{d-1}.
\end{aligned}$$

Equivalently, we have for $\gamma > 1$,

$$\begin{aligned}
&2\pi\frac{\gamma-1}{\gamma}\left(\frac{2d}{1+\alpha} - (d-2)\right)\int_0^r w(y)^{\alpha+1} y^{d-1}\mathrm{d}y \\
&= \frac{1}{2}w'(r)^2 r^d + 4\pi\left(\frac{\gamma-1}{\gamma}\right)^2 w(r)^{\alpha+1} r^d + \frac{1}{2}(d-2)w'(r)w(r)r^{d-1}.
\end{aligned}$$

and for $\gamma = 1$,

$$-4\pi\int_0^r e^{h(y)} y^{d-1}\mathrm{d}y = h'(r) r^{d-1}.$$

Proof. **Case 1,** $\gamma > 1$ Recall w satisfies the Lane–Emden ODE (Definition 1.2.1)

$$-4\pi\frac{\gamma-1}{\gamma} w(r)^\alpha = \frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = w''(r) + \frac{d-1}{r} w'(r).$$

for $r \in [0, R)$. Times this by w and integrating w.r.t. $r^{d-1}\mathrm{d}r$ we have

$$-4\pi\frac{\gamma-1}{\gamma}\int_0^r w^{\alpha+1} y^{d-1}\mathrm{d}y = \int_0^r w'' w y^{d-1}\mathrm{d}y + (d-1)\int_0^r w' w y^{d-2}\mathrm{d}y$$

$$= w'(r)w(r)r^{d-1} - \int_0^r (w')^2 y^{d-1} \mathrm{d}y \tag{4.1}$$

On the other hand times by rw' and integrating w.r.t. $r^{d-1}\mathrm{d}r$ we have

$$-\frac{4\pi}{1+\alpha}\frac{\gamma-1}{\gamma}\left(w(r)^{\alpha+1}r^d - d\int_0^r w^{\alpha+1}y^{d-1}\mathrm{d}y\right) \tag{4.2}$$

$$= -4\pi\frac{\gamma-1}{\gamma}\int_0^r w^\alpha w' y^d \mathrm{d}y$$

$$= \int_0^r w''w'y^d\mathrm{d}y + (d-1)\int_0^r (w')^2 y^{d-1}\mathrm{d}y$$

$$= \frac{1}{2}w'(r)^2 r^d + (d/2-1)\int_0^r (w')^2 y^{d-1}\mathrm{d}y \tag{4.3}$$

where we used

$$\int_0^r w^\alpha w' y^d \mathrm{d}y = w(r)^{\alpha+1}r^d - \int_0^r (\alpha w^\alpha w' y^d + d w^{\alpha+1} y^{d-1})\mathrm{d}y.$$

Using (4.1) to eliminate the $\int (w')^2 y^{d-1}\mathrm{d}y$ term in (4.3) we get the Pohozaev integral.

Case 2, $\gamma = 1$ Integrating the h-equation $-4\pi e^h = h'' + (d-1)r^{-1}h'$ w.r.t. $r^{d-1}\mathrm{d}r$ we have

$$-4\pi\int_0^r e^{h(y)}y^{d-1}\mathrm{d}y = \int_0^r h''(y)y^{d-1}\mathrm{d}y + (d-1)\int_0^r h'(y)y^{d-2}\mathrm{d}y = h'(r)r^{d-1}.$$

□

Theorem 4.1.4 (Support of Lane–Emden stars)**.**

i. Suppose w is a gas star. Then w has compact support if

$$\gamma > \frac{2d}{d+2} \qquad \textit{or equivalently} \qquad \alpha < \frac{d+2}{d-2}$$

and infinitely support otherwise.

ii. (Explicit solution when $\gamma = \frac{2d}{d+2}$) When $\gamma = 2d/(d+2)$ we have explicit steady state solution

$$w(r) = A\left(1 + \frac{2\pi}{d^2}A^{\frac{4}{d-2}}r^2\right)^{1-d/2}$$

or equivalently

$$\rho(r) = C\left(1 + \frac{2\pi}{d^2}C^{\frac{4}{d+2}}r^2\right)^{-1-d/2}.$$

And the support of the liquid star is

$$R = \left(\frac{d^2}{2\pi} C^{-\frac{4}{d+2}} (C^{\frac{2}{d+2}} - 1) \right)^{\frac{1}{2}}.$$

Proof. Both of these are known standard results (see for example [27] for i.). i. can be proven from the Pohozaev integral. We included the proof in the appendix for completeness. □

Proposition 4.1.5 (Self-similarity of solutions). *Let ρ be a gaseous steady state. Then $\rho_\kappa(r) = \kappa\rho(\kappa^{1-\gamma/2} r)$ is a gaseous steady state for any $\kappa > 0$, and the corresponding liquid star has support $R = \kappa^{-(1-\gamma/2)} \rho^{-1}(1/\kappa)$.*

Proof. This is standard result based on scaling argument, we included the proof in the appendix for completeness. □

4.1.2 Singular solutions to the steady state equation

Proposition 4.1.6 (Singular star). *When $2(d-1) - d\gamma \geq 0$, the following solve the Lane–Emden ODE (Definition 1.2.1) on $(0, \infty)$*

$$\rho(r) = \left(\frac{1}{2\pi} \frac{-d\gamma^2 + 2(d-1)\gamma}{(2-\gamma)^2} \right)^{\frac{1}{2-\gamma}} r^{-\frac{2}{2-\gamma}}.$$

And this is the only solution of the form $\rho(r) = Ar^a$.

Proof. First we consider the $\gamma > 1$ case. Consider $w(r) = Ar^a$. Substitute into the Lane–Emden ODE (Definition 1.2.1)

$$w'' + (d-1)r^{-1}w' = -4\pi \frac{\gamma - 1}{\gamma} w^\alpha$$

we get

$$Aa(a-1)r^{a-2} + A(d-1)ar^{a-2} = -4\pi \frac{\gamma-1}{\gamma} A^\alpha r^{a\alpha}$$

For this to have the possibility of holding, we need

$$a\alpha = a - 2 \qquad \Longleftrightarrow \qquad a = -\frac{2}{\alpha - 1} = -\frac{1}{\beta} = -2\frac{\gamma - 1}{2 - \gamma}.$$

Then the equation becomes

$$\begin{aligned} -2\frac{\gamma-1}{2-\gamma}\frac{-\gamma}{2-\gamma} + -2(d-1)\frac{\gamma-1}{2-\gamma} &= -4\pi\frac{\gamma-1}{\gamma}A^{\alpha-1} \\ \iff \qquad -\gamma^2 + (d-1)(2-\gamma)\gamma &= 2\pi(2-\gamma)^2 A^{2\beta} \end{aligned}$$

So we need

$$A = \left(\frac{1}{2\pi}\frac{-d\gamma^2 + 2(d-1)\gamma}{(2-\gamma)^2}\right)^{\frac{1}{2\beta}}$$

We also need

$$-d\gamma^2 + 2(d-1)\gamma \geq 0 \qquad \iff \qquad 2(d-1) - d\gamma \geq 0.$$

For the $\gamma = 1$ case, substituting $\rho(r) = Ar^a$ in the Lane–Emden ODE (Definition 1.2.1)

$$0 = \Delta(\ln\rho) + 4\pi\rho = \frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho^2} + (d-1)\frac{1}{r}\frac{\rho'}{\rho} + 4\pi\rho$$

we get

$$0 = (a(a-1) - a^2 + (d-1)a)r^{-2} + 4\pi Ar^a.$$

So we need $a = -2$ and $6 - 4 - 2(d-1) + 4\pi A = 0$. □

4.1.3 Dynamical system formulation of the steady state equation

In order to prove our main (in)stability theorem for liquid stars, we will need a precise estimate for the radius of the liquid star, and due to the self-similar scaling of steady state solutions, this means we need to understand the precise tail behaviour of gaseous stars. And to do that we reformulate the Lane–Emden stars as a solutions to a dynamical system, and utilise methods of dynamical system analogous to that in [24].

Let

$$m(r) = 4\pi\int_0^r y^{d-1}\rho(y)\mathrm{d}y.$$

When $\gamma > 1$, the steady state equation (1.15) is

$$\frac{\mathrm{d}w}{\mathrm{d}r} = -\frac{\gamma - 1}{\gamma}\frac{m(r)}{r^{d-1}} \qquad \text{where} \qquad w = \rho^{\gamma-1}$$

When $\gamma = 1$, the steady state equation (1.17) is

$$\frac{\mathrm{d}h}{\mathrm{d}r} = -\frac{m(r)}{r^{d-1}} \qquad \text{where} \qquad h = \ln\rho.$$

Let

$$\begin{aligned} u_1(r) &= r^{\frac{2}{2-\gamma}}\rho(r) \\ u_2(r) &= r^{\frac{2}{2-\gamma}-d}m(r) \end{aligned}$$

The steady state equation for $\gamma > 1$ is then

$$\begin{aligned} \frac{\mathrm{d}u_1}{\mathrm{d}r} &= r^{\frac{2}{2-\gamma}}\alpha w^{\alpha-1}\frac{\mathrm{d}w}{\mathrm{d}r} + \frac{2}{2-\gamma}r^{\frac{2}{2-\gamma}-1}\rho &&= -\frac{1}{\gamma}r^{-1}u_1^{2-\gamma}u_2 + \frac{2}{2-\gamma}r^{-1}u_1 \\ \frac{\mathrm{d}u_2}{\mathrm{d}r} &= 4\pi r^{\frac{2}{2-\gamma}-1}\rho - \left(d - \frac{2}{2-\gamma}\right)r^{\frac{2}{2-\gamma}-d-1}m &&= 4\pi r^{-1}u_1 - \left(d - \frac{2}{2-\gamma}\right)r^{-1}u_2 \end{aligned}$$

And the steady state equation for $\gamma = 1$ is

$$\begin{aligned} \frac{\mathrm{d}u_1}{\mathrm{d}r} &= -r^{3-d}\rho m + 2r\rho &&= -r^{-1}u_1u_2 + 2r^{-1}u_1 \\ \frac{\mathrm{d}u_2}{\mathrm{d}r} &= 4\pi r\rho - (d-2)r^{1-d}m &&= 4\pi r^{-1}u_1 - (d-2)r^{-1}u_2 \end{aligned}$$

Now let $v_j(\tau) = u_j(e^\tau)$, i.e. the change of variable $\tau = \ln r$, then we obtain the planar autonomous dynamical system

$$\begin{aligned} \frac{\mathrm{d}v_1}{\mathrm{d}\tau} &= -\frac{1}{\gamma}v_1^{2-\gamma}v_2 + \frac{2}{2-\gamma}v_1 \\ \frac{\mathrm{d}v_2}{\mathrm{d}\tau} &= 4\pi v_1 - \left(d - \frac{2}{2-\gamma}\right)v_2 \end{aligned}$$

or equivalently

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} = \mathbf{F}(\mathbf{v}) \qquad \text{where} \qquad \mathbf{F}(\mathbf{v}) = \begin{pmatrix} -\frac{1}{\gamma}v_1^{2-\gamma}v_2 + \frac{2}{2-\gamma}v_1 \\ 4\pi v_1 - \left(d - \frac{2}{2-\gamma}\right)v_2 \end{pmatrix}.$$

Note that $\mathbf{F} \in C^\infty((0,\infty)\times\mathbb{R}) \cap C([0,\infty)\times\mathbb{R})$.

The following two propositions established bounds for ρ, u_1, u_2, which will be needed later on when we apply results from dynamical systems to prove the tail behaviour of ρ.

Proposition 4.1.7. *Suppose $\gamma < 2$. Then*

$$\rho(r) \leq \left(\frac{2\pi}{d}\frac{2-\gamma}{\gamma}\right)^{-\frac{1}{2-\gamma}} r^{-\frac{2}{2-\gamma}}$$
$$m(r) \leq 4\pi\left(d - \frac{2}{2-\gamma}\right)^{-1}\left(\frac{2\pi}{d}\frac{2-\gamma}{\gamma}\right)^{-\frac{1}{2-\gamma}} r^{d-\frac{2}{2-\gamma}}$$

Proof. The first inequality follows from the decay estimates. Then we have

$$\begin{aligned} m(r) &= 4\pi\int_0^r y^{d-1}\rho(y)\mathrm{d}y \\ &\leq 4\pi\left(\frac{2\pi}{d}\frac{2-\gamma}{\gamma}\right)^{-\frac{1}{2-\gamma}}\int_0^r y^{d-1-\frac{2}{2-\gamma}}\mathrm{d}y \\ &= 4\pi\left(d-\frac{2}{2-\gamma}\right)^{-1}\left(\frac{2\pi}{d}\frac{2-\gamma}{\gamma}\right)^{-\frac{1}{2-\gamma}} r^{d-\frac{2}{2-\gamma}}. \end{aligned}$$

□

Proposition 4.1.8. *We have*

$$u_1(r) \sim r^{\frac{2}{2-\gamma}}\rho(0) \qquad \text{as} \qquad r \to 0$$
$$u_2(r) \sim r^{\frac{2}{2-\gamma}}\rho(0)\frac{4\pi}{d} \qquad \text{as} \qquad r \to 0$$

Proof. We have

$$\frac{u_1(r)}{r^{\frac{2}{2-\gamma}}\rho(0)} = \frac{\rho(r)}{\rho(0)} \to 1 \qquad \text{as} \qquad r \to 0$$

$$\begin{aligned} \frac{u_2(r)}{r^{\frac{2}{2-\gamma}}\rho(0)\frac{4\pi}{d}} &= \frac{d}{\rho(0)}r^{-d}\int_0^r y^{d-1}\rho(y)\mathrm{d}y = \frac{d}{\rho(0)}r^{-d}\int_0^r y^{d-1}(\rho(0)+o(1))\mathrm{d}y \\ &= \frac{d}{\rho(0)}\left(\frac{\rho(0)}{d} + r^{-d}o(r^d)\right) \to 1 \qquad \text{as} \qquad r \to 0. \end{aligned}$$

□

Steady states of the dynamical system

In order to apply results from dynamical systems to our case, we need to know the steady states of the dynamical system. The following lemma detailed the steady states of the dynamical system and their property.

Lemma 4.1.9. *Let* $2(d-1)-d\gamma>0$*. The dynamical system*

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau}=\mathbf{F}(\mathbf{v}) \qquad \textit{where} \qquad \mathbf{F}(\mathbf{v})=\begin{pmatrix} -\dfrac{1}{\gamma}v_1^{2-\gamma}v_2+\dfrac{2}{2-\gamma}v_1 \\ 4\pi v_1-\left(d-\dfrac{2}{2-\gamma}\right)v_2 \end{pmatrix}$$

has two steady states

$$\begin{aligned} \mathbf{v}=\mathbf{0} \qquad & \textit{and} \\ \mathbf{v}=\mathbf{v}^*:=&\begin{pmatrix} \left(\dfrac{1}{2\pi}\dfrac{\gamma}{2-\gamma}\left(d-\dfrac{2}{2-\gamma}\right)\right)^{\frac{1}{2-\gamma}} \\ \dfrac{2\gamma}{2-\gamma}\left(\dfrac{1}{2\pi}\dfrac{\gamma}{2-\gamma}\left(d-\dfrac{2}{2-\gamma}\right)\right)^{\frac{\gamma-1}{2-\gamma}} \end{pmatrix} \\ =&\begin{pmatrix} \left(\dfrac{1}{2\pi}\dfrac{-d\gamma^2+2(d-1)\gamma}{(2-\gamma)^2}\right)^{\frac{1}{2-\gamma}} \\ \dfrac{2\gamma}{2-\gamma}\left(\dfrac{1}{2\pi}\dfrac{-d\gamma^2+2(d-1)\gamma}{(2-\gamma)^2}\right)^{\frac{\gamma-1}{2-\gamma}} \end{pmatrix}. \end{aligned}$$

Moreover, if $\gamma<\frac{2d}{d+2}$*, then* $\mathbf{v}^*$ *is (exponentially) stable.*

Proof. It can be checked directly that $\mathbf{F}(\mathbf{v})=\mathbf{0}$ iff $\mathbf{v}=\mathbf{0}$ or $\mathbf{v}=\mathbf{v}^*$. Note that $\mathbf{F}$ is differentiable at $\mathbf{v}^*$, but not at $\mathbf{0}$ (unless $\gamma=1$). On $(0,\infty)\times\mathbb{R}$ we have

$$\nabla\mathbf{F}(\mathbf{v})=\begin{pmatrix} -\dfrac{2-\gamma}{\gamma}v_1^{1-\gamma}v_2+\dfrac{2}{2-\gamma} & -\dfrac{1}{\gamma}v_1^{2-\gamma} \\ 4\pi & -\left(d-\dfrac{2}{2-\gamma}\right) \end{pmatrix}$$

So

$$\nabla\mathbf{F}(\mathbf{v}^*)=\begin{pmatrix} \dfrac{2}{2-\gamma}-2 & -\dfrac{1}{2\pi}\dfrac{-d\gamma+2(d-1)}{(2-\gamma)^2} \\ 4\pi & -\left(d-\dfrac{2}{2-\gamma}\right) \end{pmatrix}$$

The eigenvalues of $\nabla \mathbf{F}(\mathbf{v}^*)$ are

$$\lambda = \frac{2}{2-\gamma} - 1 - \frac{d}{2} \pm \frac{1}{2}\sqrt{(d-2)^2 - 8\frac{-d\gamma + 2(d-1)}{(2-\gamma)^2}}.$$

So assuming

$$d + 2 > \frac{4}{2-\gamma} \qquad \iff \qquad \gamma < \frac{2d}{d+2}$$

the system at $\mathbf{v}^*$ is (exponentially) stable provided

$$0 > \text{Re}\left(\frac{2}{2-\gamma} - 1 - \frac{d}{2} + \frac{1}{2}\sqrt{(d-2)^2 - 8\frac{-d\gamma + 2(d-1)}{(2-\gamma)^2}}\right)$$

$$\begin{aligned}
\iff \qquad & \left(d + 2 - \frac{4}{2-\gamma}\right)^2 > (d-2)^2 - 8\frac{-d\gamma + 2(d-1)}{(2-\gamma)^2} \\
\iff \qquad & 8d > \frac{8}{(2-\gamma)^2}\left(d\gamma - 2(d-1) - 2\right) \\
\iff \qquad & d(2-\gamma)^2 > d\gamma - 2(d-1) - 2 \\
\iff \qquad & d(2-\gamma)((2-\gamma)+1) > 0 \\
\iff \qquad & d(2-\gamma)(3-\gamma) > 0
\end{aligned}$$

which is true. □

4.1.4 Tail behaviour for gaseous star

The following result gives detailed estimate for the tail behaviour for gaseous stars, and hence the boundary behaviour of liquid stars, that is crucial to prove our main (in)stability result in the next section.

Theorem 4.1.10. *Suppose $\gamma < 2d/(d+2)$. Then the gas star tends asymptotically to the singular star as $r \to \infty$. More precisely, there exist $c > 0$ such that*

$$\left| r^{\frac{2}{2-\gamma}}\rho(r) - \left(\frac{1}{2\pi}\frac{-d\gamma^2 + 2(d-1)\gamma}{(2-\gamma)^2}\right)^{\frac{1}{2-\gamma}} \right| = |u_1(r) - v_1^*| \lesssim r^{-c}$$

$$\left| r^{\frac{2}{2-\gamma}-d}m(r) - \frac{2\gamma}{2-\gamma}\left(\frac{1}{2\pi}\frac{-d\gamma^2 + 2(d-1)\gamma}{(2-\gamma)^2}\right)^{\frac{\gamma-1}{2-\gamma}} \right| = |u_2(r) - v_2^*| \lesssim r^{-c}$$

Proof. We will work with the dynamical system formulation formulated in the previous section. From Proposition 4.1.7 and the non-negativity of ρ and m we have

$$
\begin{aligned}
0 < v_1 &\leq \left(\frac{2\pi}{d}\frac{2-\gamma}{\gamma}\right)^{-\frac{1}{2-\gamma}} \\
0 < v_2 &\leq 4\pi\left(d - \frac{2}{2-\gamma}\right)^{-1}\left(\frac{2\pi}{d}\frac{2-\gamma}{\gamma}\right)^{-\frac{1}{2-\gamma}}
\end{aligned}
$$

We will show that there exist $\epsilon' > 0$ and $T \in \mathbb{R}$ such that $v_1(\tau) \geq \epsilon'$ for all $\tau \geq T$. By Proposition 4.1.8 we have $v_1(\tau) \sim e^{2\tau/(2-\gamma)}\rho(0)$ as $\tau \to -\infty$. So for $\epsilon > 0$ small enough, we can find $\tau_1 \in \mathbb{R}$ such that $v_1(\tau_1) = \epsilon$ and $v_1(\tau_1 + \delta) > \epsilon$ for small $\delta > 0$. If $v_1(\tau) > \epsilon$ for all $\tau > \tau_1$ then we are done. Otherwise there exist a least $\tau_2 > \tau_1$ such that $v_1(\tau_2) = \epsilon$. Since $F_1(\epsilon, v_2(\tau_1)) > 0$, from the expression of F_1 we see that $F_1(\epsilon, y) > 0$ for all $y \in [0, v_2(\tau_1)]$. So we must have $v_2(\tau_2) > v_2(\tau_1)$. We must have $F_1(\epsilon, v_2(\tau_2)) \leq 0$. We claim that there exist $\tau_3 \geq \tau_2$ such that $F_1(\mathbf{v}(\tau_3)) = 0$. Suppose no such point exist, then $F_1(\mathbf{v}(\tau)) < 0$ for all $\tau \geq \tau_2$, in other words

$$
\frac{1}{\gamma}v_1(\tau)^{2-\gamma}v_2(\tau) > \frac{2}{2-\gamma}v_1(\tau) \qquad \text{for all} \qquad \tau \geq \tau_2. \tag{4.4}
$$

We will show that this is impossible. Using (4.4) we have for all $\tau \geq \tau_2$

$$
F_2(\mathbf{v}(\tau)) = 4\pi v_1(\tau) - \left(d - \frac{2}{2-\gamma}\right)v_2(\tau) < 4\pi\epsilon - \left(d - \frac{2}{2-\gamma}\right)\epsilon^{\gamma-1}\frac{2\gamma}{2-\gamma}.
$$

By choosing ϵ small enough, we can make $F_2(\mathbf{v}(\tau))$ less than a fix strictly negative number for all $\tau \geq \tau_2$. This means $v_2(\tau) \to 0$ as $\tau \to \infty$.

i. When $\gamma = 1$, (4.4) gives $v_2(\tau) > 2$ for all $\tau \geq \tau_2$. This is a contradiction.

ii. When $\gamma > 1$, (4.4) gives

$$
\left(\frac{2-\gamma}{2\gamma}v_2(\tau)\right)^{\frac{1}{\gamma-1}} > v_1(\tau) \qquad \text{for all} \qquad \tau \geq \tau_2.
$$

So for all $\tau \geq \tau_2$ we have

$$
\begin{aligned}
\frac{\mathrm{d}v_2}{\mathrm{d}\tau} &= 4\pi v_1 - \left(d - \frac{2}{2-\gamma}\right)v_2 \\
&< 4\pi\left(\frac{2-\gamma}{2\gamma}v_2\right)^{\frac{1}{\gamma-1}} - \left(d - \frac{2}{2-\gamma}\right)v_2 \\
&\sim -\left(d - \frac{2}{2-\gamma}\right)v_2 \qquad \text{as} \qquad v_2 \to 0.
\end{aligned}
$$

So we must have

$$\begin{aligned} |v_1(\tau)| &\to 0 && \text{as} \quad \tau \to \infty \\ |v_2(\tau)| &= O(e^{-(d-\frac{2}{2-\gamma})\tau}) && \text{as} \quad \tau \to \infty. \end{aligned}$$

In other words we have

$$\begin{aligned} |u_1(r)| &\to 0 && \text{as} \quad r \to \infty \\ |u_2(r)| &= O(r^{-(d-\frac{2}{2-\gamma})}) && \text{as} \quad r \to \infty. \end{aligned}$$

The Pohozaev integral gives

$$\begin{aligned} 2\pi\frac{\gamma-1}{\gamma}\left(2-\frac{2-\gamma}{\gamma}d\right)&\int_0^r \rho(y)^\gamma y^{d-1}\mathrm{d}y \\ &\geq \frac{1}{2}(d-2)w'(r)w(r)r^{d-1} \\ &= -\frac{1}{2}\frac{\gamma-1}{\gamma}(d-2)u_1(r)^{\gamma-1}u_2(r)r^{d-\frac{2}{2-\gamma}-2\frac{\gamma-1}{2-\gamma}} \\ &\to 0 \qquad \text{as} \qquad r \to \infty \end{aligned}$$

but the LHS of the equation becomes more and more negative as $r \to \infty$. This is a contradiction

Therefore we have $\tau_3 \geq \tau_2$ such that $F_1(\mathbf{v}(\tau_3)) = 0$ and $F_2(\mathbf{v}(\tau_3)) < 0$. From the expression for $\mathbf{F}$ we see that $F_1(v_1(\tau_3), x) \geq 0$ for all $x \in [0, v_2(\tau_3)]$. Let $\tau_0 \leq \tau_1$ be the point such that $v_1(\tau_0) = v_1(\tau_3)$. Since $F_1(\mathbf{v}(\tau_0)) > 0$ and $F_1(\mathbf{v}(\tau_3)) = 0$, we see from the expression of $\mathbf{F}$ that we must have $v_2(\tau_3) > v_2(\tau_0)$. Hence for $\tau > \tau_3$, $v(\tau)$ stays in the region bounded by the arc $v([v_0, v_3])$ and the line $\{(v_1(\tau_3), x) : x \in [v_2(\tau_0), v_2(\tau_3)]\}$. Hence $T = \tau_3$ works.

So $\{\mathbf{v}(\tau) : \tau \geq T\}$ lies in a compact set within the region $v_1 > \epsilon/2$ where $\mathbf{F}$ is C^1. This means its ω-limit set is non-empty, compact and connected by a standard result in dynamical systems. By the Poincaré-Bendixson theorem, the ω-limit set must be either

(a) $\{\mathbf{v}^*\}$;

(b) a periodic orbit;

(c) homoclinic obits connecting $\mathbf{v}^*$.

But by Bendixson-Dulac theorem, there is no periodic orbits in $(0,\infty)\times(0,\infty)$

since on this region

$$\nabla \cdot \left(\frac{1}{v_1^{2-\gamma}} \mathbf{F}(\mathbf{v}) \right) = 2\frac{\gamma-1}{2-\gamma} v_1^{-(2-\gamma)} - \left(d - \frac{2}{2-\gamma} \right) v_1^{-(2-\gamma)}$$
$$= -\left(d - \frac{2\gamma}{2-\gamma} \right) v_1^{-(2-\gamma)} < 0$$

where we note that

$$d > \frac{2\gamma}{2-\gamma} \qquad \Longleftrightarrow \qquad \gamma < \frac{2d}{d+2}.$$

Since $\mathbf{v}^*$ is (exponentially) stable, the ω-limit set cannot have homoclinic orbits connecting $\mathbf{v}^*$ either. So the ω-limit set of $\mathbf{v}$ must be $\{\mathbf{v}^*\}$. So $\mathbf{v}(\tau) \to \mathbf{v}^*$ as $\tau \to \infty$. Since the fixed point $\mathbf{v}^*$ is exponentially stable, we have $\|\mathbf{v}(\tau) - \mathbf{v}^*\| \lesssim e^{-c\tau}$ for some $c > 0$. Converting to the variable r gives us the desired result. □

Corollary 4.1.11. *Suppose $\gamma < 2d/(d+2)$. Let ρ be a gaseous steady state and $\rho_\kappa(r) = \kappa\rho(\kappa^{1-\gamma/2}r)$. Then there exist $c > 0$ such that*

$$\left| r^{\frac{2}{2-\gamma}} \rho_\kappa(r) - \left(\frac{1}{2\pi} \frac{-d\gamma^2 + 2(d-1)\gamma}{(2-\gamma)^2} \right)^{\frac{1}{2-\gamma}} \right| = \left| r^{\frac{2}{2-\gamma}} \rho_\kappa(r) - v_1^* \right|$$
$$\lesssim (\kappa^{1-\gamma/2} r)^{-c}$$
$$\left| r^{\frac{2}{2-\gamma}-d} m_\kappa(r) - \frac{2\gamma}{2-\gamma} \left(\frac{1}{2\pi} \frac{-d\gamma^2 + 2(d-1)\gamma}{(2-\gamma)^2} \right)^{\frac{\gamma-1}{2-\gamma}} \right| = \left| r^{\frac{2}{2-\gamma}-d} m_\kappa(r) - v_2^* \right|$$
$$\lesssim (\kappa^{1-\gamma/2} r)^{-c}$$

Proof. Using the last theorem we have

$$\left| r^{\frac{2}{2-\gamma}} \rho_\kappa(r) - v_1^* \right| = \left| (\kappa^{1-\gamma/2} r)^{\frac{2}{2-\gamma}} \rho(\kappa^{1-\gamma/2} r) - v_1^* \right| \lesssim (\kappa^{1-\gamma/2} r)^{-c}.$$

We have

$$m_\kappa(r) = 4\pi \int_0^r y^{d-1} \rho_\kappa(y) \mathrm{d}y = 4\pi\kappa^{1-(d-1)(1-\gamma/2)} \int_0^r (\kappa^{1-\gamma/2} y)^{d-1} \rho(\kappa^{1-\gamma/2} y) \mathrm{d}y$$
$$= 4\pi\kappa^{1-d(1-\gamma/2)} \int_0^{\kappa^{1-\gamma/2} r} z^{d-1} \rho(z) \mathrm{d}z = \kappa^{1-d(1-\gamma/2)} m(\kappa^{1-\gamma/2} r)$$

So using the last theorem we have

$$\left| r^{\frac{2}{2-\gamma}-d} m_\kappa(r) - v_2^* \right| = \left| (\kappa^{1-\gamma/2} r)^{\frac{2}{2-\gamma}-d} m(\kappa^{1-\gamma/2} r) - v_2^* \right| \lesssim (\kappa^{1-\gamma/2} r)^{-c}.$$

□

Corollary 4.1.12. *Suppose $\gamma < 2d/(d+2)$. Let ρ be a gaseous steady state and $\rho_\kappa(r) = \kappa\rho(\kappa^{1-\gamma/2}r)$. Then the liquid star ρ_κ has radius*

$$R_\kappa \to R_\infty := (v_1^*)^{1-\gamma/2} = \left(\frac{1}{2\pi}\frac{-d\gamma^2 + 2(d-1)\gamma}{(2-\gamma)^2}\right)^{\frac{1}{2}} \qquad \textit{as} \qquad \kappa \to \infty.$$

Proof. From the last theorem, we have

$$\left|\rho^{-1}\left(\frac{1}{\kappa}\right)^{\frac{2}{2-\gamma}}\frac{1}{\kappa} - v_1^*\right| \lesssim \rho^{-1}\left(\frac{1}{\kappa}\right)^{-c} \to 0 \qquad \text{as} \qquad \kappa \to \infty.$$

So we have

$$R_\kappa = \kappa^{-(1-\gamma/2)}\rho^{-1}(1/\kappa) = \left(\rho^{-1}\left(\frac{1}{\kappa}\right)^{\frac{2}{2-\gamma}}\frac{1}{\kappa}\right)^{1-\gamma/2} \to (v_1^*)^{1-\gamma/2} \quad \text{as} \quad \kappa \to \infty.$$

□

4.2 Linear stability for radially symmetric perturbations

4.2.1 Equations with spherical symmetry

Assuming spherical symmetry, so that $\mathbf{u}(r,t) = u(r,t)\hat{\mathbf{r}}$, the continuity equation $\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0$ becomes

$$\begin{aligned} 0 &= \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = \partial_t\rho + \rho u\nabla\cdot\hat{\mathbf{r}} + \hat{\mathbf{r}}\cdot\nabla(\rho u) \\ &= \partial_t\rho + (d-1)\rho u\frac{1}{r} + \hat{\mathbf{r}}\cdot\partial_r(\rho u)\nabla r \\ &= \partial_t\rho + (d-1)\rho u\frac{1}{r} + \partial_r(\rho u) = D_t\rho + \rho\left((d-1)u\frac{1}{r} + \partial_r u\right) \\ &= D_t\rho + \rho\frac{\partial_r(r^{d-1}u)}{r^{d-1}}. \end{aligned} \tag{4.5}$$

The momentum equation (1.1) reads

$$\begin{aligned} \mathbf{0} &= \rho\frac{D\mathbf{u}}{Dt} + \nabla p + \rho\nabla\phi = \rho(\partial_t u\hat{\mathbf{r}} + (u\hat{\mathbf{r}}\cdot\nabla)(u\hat{\mathbf{r}})) + (\partial_r p)\hat{\mathbf{r}} + \rho(\partial_r\phi)\hat{\mathbf{r}} \\ &= \rho(\partial_t u\hat{\mathbf{r}} + u\partial_r(u\hat{\mathbf{r}})) + (\partial_r p)\hat{\mathbf{r}} + \rho(\partial_r\phi)\hat{\mathbf{r}} \end{aligned}$$

$$= (\rho(\partial_t u + u\partial_r u + \partial_r \phi) + \partial_r p)\, \hat{\mathbf{r}}$$

So the momentum equation is

$$D_t u + \partial_r \phi + \frac{1}{\rho}\partial_r p = \partial_t u + u\partial_r u + \partial_r \phi + \frac{1}{\rho}\partial_r p = 0.$$

The Poisson equation reads

$$4\pi\rho = \Delta\phi = \nabla \cdot ((\partial_r \phi)\hat{\mathbf{r}}) = (\partial_r \phi)\frac{d-1}{r} + \partial_r^2 \phi = \frac{1}{r^{d-1}}\partial_r(r^{d-1}\partial_r \phi)).$$

We can put this into the momentum equation to get

$$\partial_t u + u\partial_r u + \frac{4\pi}{r^{d-1}}\int_0^r s^{d-1}\rho(s)\mathrm{d}s + \frac{1}{\rho}\partial_r p = 0. \tag{4.6}$$

4.2.2 Equations in Lagrangian coordinates

Let $\eta(y,t)$ be the (radial) location of the fluid particle that was at $\eta_0(y)$ at time 0. η is given by

$$\partial_t \eta = u \circ \eta \qquad \text{with} \qquad \eta(y,0) = \eta_0(y)$$

where $u \circ \eta(y,t) = u(\eta(y,t),t)$. The spacial domain is then fixed for all time as $[0,R] := \eta_0^{-1}(\{r : r\hat{\mathbf{r}} \in \Omega_0\})$. We then have the Lagrangian variables

$$v = u \circ \eta \qquad \text{(Lagrangian velocity)}$$
$$f = \rho \circ \eta \qquad \text{(Lagrangian density)}$$
$$\psi = \phi \circ \eta \qquad \text{(Lagrangian potential)}$$

We have for any h,

$$\partial_y(h \circ \eta) = ((\partial_r h) \circ \eta)\partial_y \eta \qquad \text{and so} \qquad (\partial_r h) \circ \eta = (\partial_y \eta)^{-1}\partial_y(h \circ \eta).$$

$$\partial_t(h \circ \eta) = (\partial_t h) \circ \eta + ((\partial_r h) \circ \eta)\partial_t \eta = (D_t h) \circ \eta.$$

Let

$$J = \frac{\eta^{d-1}}{y^{d-1}}\partial_y \eta = \frac{1}{dy^{d-1}}\partial_y(\eta^d).$$

Then

$$\partial_t J = \frac{1}{dy^{d-1}}\partial_y(d\eta^{d-1}\partial_t \eta) = \frac{1}{y^{d-1}}\left((d-1)\eta^{d-2}v\partial_y \eta + \eta^{d-1}\partial_y v\right)$$

$$= \frac{1}{y^{d-1}}\left((d-1)\eta^{d-2}v\partial_y\eta + \eta^{d-1}((\partial_r u)\circ\eta)\partial_y\eta\right)$$
$$= (d-1)J\frac{v}{\eta} + J((\partial_r u)\circ\eta) = J\left(\frac{\partial_r(r^{d-1}u)}{r^{d-1}}\right)\circ\eta$$

So the continuity equation (4.5) in Lagrangian is

$$0 = \partial_t f + f\frac{\partial_t J}{J} \qquad \text{and so} \qquad \partial_t \ln(fJ) = 0 \qquad \text{and so} \qquad fJ = f_0 J_0.$$

The Poisson equation is

$$4\pi f = \frac{1}{\eta^{d-1}\partial_y\eta}\partial_y\left(\frac{\eta^{d-1}}{\partial_y\eta}\partial_y\psi\right).$$

And so

$$\frac{1}{\partial_y\eta}\partial_y\psi = \frac{4\pi}{d\eta^{d-1}}\int_0^y f(s)\partial_y\eta^d(s)\mathrm{d}s = \frac{4\pi}{\eta^{d-1}}\int_0^y s^{d-1}(f_0J_0)(s)\mathrm{d}s.$$

And the momentum equation (4.6) is

$$0 = \partial_t v + \frac{1}{\partial_y\eta}\partial_y\psi + \frac{1}{f\partial_y\eta}\partial_y f^\gamma$$
$$= \partial_t^2\eta + \frac{4\pi}{\eta^{d-1}}\int_0^y s^{d-1}(f_0J_0)(s)\mathrm{d}s + \frac{1}{f_0J_0}\frac{\eta^{d-1}}{y^{d-1}}\partial_y\left(f_0J_0\frac{dy^{d-1}}{\partial_y\eta^d}\right)^\gamma$$

4.2.3 The eigenvalue problem for the linear stability of steady states

Let $\bar\rho$ be a Lane–Emden steady state solution. Before formulating the stability problem, we must first make the use of the labelling gauge freedom and fix the choice of f_0J_0 for the general perturbation to be exactly identical to the background enthalpy $\bar\rho$, or equivalently $(\rho_0\circ\eta_0)(\eta_0/y)^{d-1}\eta_0' = \bar\rho$ on the initial domain $[0,R]$. By a result of Dacorogna-Moser [10] and similarly to [20, 21] there exists a choice of an initial bijective map $\eta_0 : [0,R] \to \overline{\operatorname{supp}\rho_0}$ so that this holds true. Then our equation becomes

$$\partial_t^2\eta + \frac{4\pi}{\eta^{d-1}}\int_0^y s^{d-1}\bar\rho(s)\mathrm{d}s + \frac{1}{\bar\rho}\frac{\eta^{d-1}}{y^{d-1}}\partial_y\left(\bar\rho\frac{dy^{d-1}}{\partial_y\eta^d}\right)^\gamma = 0. \tag{4.7}$$

Since $\bar\rho$ is a steady state solution, we have

$$\frac{4\pi}{y^{d-1}}\int_0^y s^{d-1}\bar\rho(s)\mathrm{d}s + \frac{1}{\bar\rho}\partial_y\bar\rho^\gamma = 0.$$

To study linear stability, we linearised the equation about the steady state as follows.

Proposition 4.2.1 (Linearised momentum equation for perturbation)**.** *Let $\eta(y,t) = y$ correspond to the steady state solution $\rho = \bar{\rho}$. Consider a perturbation of this given by $\eta(y,t) = y(1+\zeta(y,t))$. Then the perturbation variable $\zeta(y,t)$ satisfies, to first order,*

$$y\partial_t^2\zeta + \frac{2(d-1)}{\bar{\rho}}\zeta\partial_y\bar{\rho}^\gamma - \gamma\frac{1}{\bar{\rho}}\partial_y\left(\bar{\rho}^\gamma(d\zeta + y\partial_y\zeta)\right) = 0$$
$$d\zeta(R,t) + R\partial_y\zeta(R,t) = 0.$$

Furthermore, if $\zeta(y,t) = e^{\lambda t}\chi(y)$, then χ satisfies

$$\underbrace{-\gamma\partial_y(\bar{\rho}^\gamma y^{d+1}\partial_y\chi) + (2(d-1) - d\gamma)y^d\chi\partial_y\bar{\rho}^\gamma}_{:=L\chi} = -\lambda^2 y^{d+1}\bar{\rho}\chi \tag{4.8}$$
$$d\chi(R) + R\partial_y\chi(R) = 0. \tag{4.9}$$

Proof. We have $\eta(y,t) = y(1+\zeta(y,t))$, so

$$\partial_y\eta = 1 + \zeta + y\partial_y\zeta.$$

Assuming ζ and $\partial_y\zeta$ is small, we have

$$\begin{aligned}
\eta^{d-1} &= y^{d-1}(1 + (d-1)\zeta + o(\zeta)) \\
\eta^{-(d-1)} &= y^{-(d-1)}(1 - (d-1)\zeta + o(\zeta)) \\
\partial_y\eta^d &= d\eta^{d-1}\partial_y\eta = dy^{d-1}(1 + (d-1)\zeta + o(\zeta))(1 + \zeta + y\partial_y\zeta) \\
&= dy^{d-1}(1 + d\zeta + y\partial_y\zeta + o(|\zeta| + |\partial_y\zeta|)) \\
(\partial_y\eta^d)^{-\gamma} &= (dy^{d-1})^{-\gamma}\left(1 - \gamma(d\zeta + y\partial_y\zeta) + o(|\zeta| + |\partial_y\zeta|)\right)
\end{aligned}$$

So the momentum equation (4.7) is

$$\begin{aligned}
0 = {} & y\partial_t^2\zeta + \frac{4\pi}{y^{d-1}}(1 - (d-1)\zeta)\int_0^y s^{d-1}\bar{\rho}(s)\mathrm{d}s \\
& + \frac{1}{\bar{\rho}}(1 + (d-1)\zeta)\partial_y\left(\bar{\rho}^\gamma(1 - \gamma(d\zeta + y\partial_y\zeta))\right) + o(|\zeta| + |\partial_y\zeta|).
\end{aligned}$$

Discarding non-linear terms and simplify we get the linearised momentum equation

$$0 = y\partial_t^2\zeta + \frac{2(d-1)}{\bar{\rho}}\zeta\partial_y\bar{\rho}^\gamma - \gamma\frac{1}{\bar{\rho}}\partial_y\left(\bar{\rho}^\gamma(d\zeta + y\partial_y\zeta)\right).$$

For solutions of the form $\zeta(y,t) = e^{\lambda t}\chi(y)$, we have that χ satisfies

$$
\begin{aligned}
0 &= \lambda^2 y\bar{\rho}\chi + 2(d-1)\chi\partial_y\bar{\rho}^\gamma - \gamma\partial_y\left(\rho^\gamma(d\chi + y\partial_y\chi)\right) \\
&= \lambda^2 y\bar{\rho}\chi + (2(d-1) - d\gamma)\chi\partial_y\bar{\rho}^\gamma - \gamma y(\partial_y\chi)\partial_y\rho^\gamma - \gamma\rho^\gamma\partial_y(d\chi + y\partial_y\chi) \\
&= \lambda^2 y\bar{\rho}\chi + (2(d-1) - d\gamma)\chi\partial_y\bar{\rho}^\gamma - \frac{\gamma}{y^d}\partial_y(\bar{\rho}^\gamma y^{d+1}\partial_y\chi)
\end{aligned}
$$

So we want to solve a Sturm-Liouville type equation

$$
\lambda^2 y^{d+1}\bar{\rho}\chi = \gamma\partial_y(\bar{\rho}^\gamma y^{d+1}\partial_y\chi) - (2(d-1) - d\gamma)y^d\chi\partial_y\bar{\rho}^\gamma =: -L\chi
$$

Since $fJ = f_0J_0 = \bar{\rho}$ and $f(R) = \bar{\rho}(R) = 1$, we have $J(R) = 1$. Now

$$
\begin{aligned}
J &= \frac{\eta^{d-1}}{y^{d-1}}\partial_y\eta = (1+\zeta)^{d-1}\partial_y(y(1+\zeta)) = (1+\zeta)^{d-1}(1+\zeta+y\partial_y\zeta) \\
&= 1 + d\zeta + y\partial_y\zeta + o(|\zeta| + |\partial_y\zeta|).
\end{aligned}
$$

Discarding the non-linear terms and evaluating at R we get a Robin type boundary condition $d\zeta(R) + R\partial_y\zeta(R) = 0$. In terms of χ this condition reads

$$
d\chi(R) + R\partial_y\chi(R) = 0.
$$

□

In this chapter, we say the system is linearly unstable to mean that the linearised equation admits an growing mode solution of the form $\zeta(y,t) = e^{\lambda t}\chi(y)$ with $\lambda > 0$. Otherwise we call the system linearly stable.

Given $\chi_1, \chi_2 \in C^2([0,R])$ satisfying the boundary condition (4.9), under the usual L^2 inner product, we have using integration by parts

$$
\begin{aligned}
\langle L\chi_1, \chi_2\rangle &= \int_0^R \gamma\bar{\rho}^\gamma y^{d+1}(\partial_y\chi_1)(\partial_y\chi_2) + (2(d-1) - d\gamma)y^d\chi_1\chi_2\partial_y\bar{\rho}^\gamma \mathrm{d}y \\
&\quad - \gamma\bar{\rho}(R)^\gamma R^{d+1}\chi_2(R)\partial_y\chi_1(R) \\
&= -\int_0^R \chi_1\gamma\partial_y(\bar{\rho}^\gamma y^{d+1}\partial_y\chi_2) - (2(d-1) - d\gamma)y^d\chi_1\chi_2\partial_y\bar{\rho}^\gamma \mathrm{d}y \\
&\quad - \gamma\bar{\rho}(R)^\gamma R^{d+1}(\chi_2(R)\partial_y\chi_1(R) - \chi_1(R)\partial_y\chi_2(R)) \\
&= \langle\chi_1, L\chi_2\rangle
\end{aligned}
$$

So L is symmetric under $C^2([0,R])$ functions satisfying the (4.9). Note that in particular

$$\langle L\chi, \chi\rangle = \int_0^R \gamma\bar{\rho}^\gamma y^{d+1}(\partial_y\chi)^2 + (2(d-1) - d\gamma)y^d\chi^2\partial_y\bar{\rho}^\gamma \mathrm{d}y + d\gamma R^d\chi(R)^2. \tag{4.10}$$

Define the bilinear form

$$Q[\chi_1, \chi_2] = \int_0^R \gamma\bar{\rho}^\gamma y^{d+1}(\partial_y\chi_1)(\partial_y\chi_2) + (2(d-1) - d\gamma)y^d\chi_1\chi_2\partial_y\bar{\rho}^\gamma \mathrm{d}y \\ + d\gamma R^d\chi_1(R)\chi_2(R).$$

Note that this equation is well defined for spherically symmetric functions in $H^1(B_R(\mathbb{R}^{d+2}))$, where we consider y the radial variable, because 1) the trace theorem for Sobolev space which meant that $\chi(R)$ is well defined; and 2) the fact that $\partial_y\bar{\rho}^\gamma \sim y$ which means the the integral was weighted by $\sim y^{d+1}$.

We want to solve $L\chi = -\lambda^2 y^{d+1}\bar{\rho}\chi$ on $[0, R]$ with the boundary condition $d\chi(R) + R\partial_y\chi(R) = 0$. If there exist a negative eigenvalue $\mu = -\lambda^2 < 0$, the we found a growing mode of the original linearised problem, so the system is unstable. So we want to find the smallest eigenvalue μ. The following lemma help to make a criterion for finding μ in terms of the quadratic form Q.

Lemma 4.2.2. *Let $H_r^1(B_R(\mathbb{R}^{d+2}))$ denote the subspace of spherically symmetric functions in $H^1(B_R(\mathbb{R}^{d+2}))$. We consider functions in $H_r^1(B_R(\mathbb{R}^{d+2}))$ to be functions of one variable defined by radial distance $y \in [0, R]$. In this space we have*

$$\inf_{\|\chi\|_{y^{d+1}\bar{\rho}}=1} \langle L\chi, \chi\rangle_{L^2([0,R])} =: \mu_* = \inf\{\mu : \exists\chi \neq 0 \text{ s.t. } L\chi = \mu y^{d+1}\bar{\rho}\chi\}$$

where $\|\chi\|_w^2 = \langle\chi, w\chi\rangle_{L^2([0,R])}$. Moreover, the infimum is attained by some $\chi_* \in H_r^1(\mathbb{R}^{d+2})$ which is an eigenfunction of L with eigenvalue μ_*.

Proof. It is clear that

$$\mu_* = \inf_{\|\chi\|_{y^{d+1}\bar{\rho}}=1} \langle L\chi, \chi\rangle \leq \inf\{\mu : \exists\chi \neq 0 \text{ s.t. } L\chi = \mu y^{d+1}\bar{\rho}\chi\}.$$

To prove equality, it suffice then to prove that μ_* is an eigenvalue of L.

Pick χ_n with $\|\chi_n\|_{y^{d+1}\bar{\rho}} = 1$ such that $\langle L\chi_n, \chi_n\rangle \to \inf_{\|\chi\|_{y^{d+1}\bar{\rho}}=1}\langle L\chi, \chi\rangle$. Since

$$\int_0^R y^d\chi_n^2\partial_y\bar{\rho}^\gamma \mathrm{d}y \sim \|\chi_n\|_{y^{d+1}\bar{\rho}} = 1,$$

and the first and last term in (4.10) are positive, $\inf_{\|\chi\|_{y^{d+1}\bar{\rho}}=1}\langle L\chi, \chi\rangle$ is finite. Now

$$\begin{aligned}\|\partial_y \chi_n\|^2_{L^2(B_R(\mathbb{R}^{d+2}))} &\lesssim \int_0^R \gamma\bar{\rho}^\gamma y^{d+1}(\partial_y\chi_n)^2 \mathrm{d}y + d\gamma R^d \chi_n(R)^2 \\ &= \langle L\chi_n, \chi_n\rangle - (2(d-1) - d\gamma)\int_0^R y^d \chi^2 \partial_y \bar{\rho}^\gamma \mathrm{d}y \\ &\lesssim |\langle L\chi_n, \chi_n\rangle| + \|\chi_n\|_{y^{d+1}\bar{\rho}}.\end{aligned}$$

And obviously

$$\|\chi_n\|_{L^2(B_R(\mathbb{R}^{d+2}))} \lesssim \|\chi_n\|_{y^{d+1}\bar{\rho}}.$$

Hence χ_n is bounded in $H^1(B_R(\mathbb{R}^{d+2}))$. Wlog, picking an appropriate subsequence, we can assume χ_n converge weakly to some χ_*. By the Rellich-Kondrachov theorem, $\chi_n \to \chi$ in $L^2(B_R(\mathbb{R}^{d+2}))$. It follows that $\|\chi_*\|_{y^{d+1}\bar{\rho}} = 1$ By the lower semi-continuity of weak convergence, we have $\liminf \|\chi_n\|_{H^1(B_R(\mathbb{R}^{d+2}))} \geq \|\chi_*\|$. Since $\|\chi_n\|_{L^2(B_R(\mathbb{R}^{d+2}))} \to \|\chi_*\|_{L^2(B_R(\mathbb{R}^{d+2}))}$, we must have

$$\liminf \|\partial_y \chi_n\|^2_{L^2(B_R(\mathbb{R}^{d+2}))} \geq \|\partial_y \chi_*\|^2_{L^2(B_R(\mathbb{R}^{d+2}))}.$$

Since $\|\cdot\|_{y^{d+1}\bar{\rho}}$ is an (equivalent) norm for $L^2(B_R(\mathbb{R}^{d+2}))$, we have

$$\liminf \int_0^R \gamma\bar{\rho}^\gamma y^{d+1}(\partial_y\chi_n)^2 \mathrm{d}y \geq \int_0^R \gamma\bar{\rho}^\gamma y^{d+1}(\partial_y\chi_*)^2 \mathrm{d}y.$$

Since the trace operator T is continuous and linear, we also have $T\chi_n \rightharpoonup T\chi_*$ and so by the lower semi-continuity of weak convergence, $\liminf \chi_n(R)^2 \geq \chi_*(R)^2$. It follows that $\langle L\chi_*, \chi_*\rangle \leq \inf_{\|\chi\|_{y^{d+1}\bar{\rho}}=1}\langle L\chi, \chi\rangle$, and that means we must have equality and the infimum is attained.

For any f we must have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\frac{Q[\chi_* + \epsilon f, \chi_* + \epsilon f]}{\langle \chi_* + \epsilon f, \chi_* + \epsilon f\rangle_{y^{d+1}\bar{\rho}}}\right)\bigg|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\frac{Q[\chi_*, \chi_*] + 2\epsilon Q[\chi_*, f] + \epsilon^2 Q[f, f]}{\langle \chi_*, \chi_*\rangle_{y^{d+1}\bar{\rho}} + 2\epsilon\langle \chi_*, f\rangle_{y^{d+1}\bar{\rho}} + \epsilon^2\langle f, f\rangle_{y^{d+1}\bar{\rho}}}\right)\bigg|_{\epsilon=0} \\ &= \frac{2Q[\chi_*, f]}{\langle \chi_*, \chi_*\rangle_{y^{d+1}\bar{\rho}}} - \frac{2Q[\chi_*, \chi_*]\langle \chi_*, f\rangle_{y^{d+1}\bar{\rho}}}{\langle \chi_*, \chi_*\rangle^2_{y^{d+1}\bar{\rho}}}\end{aligned}$$

and so $Q[\chi_*, f] = \mu_*\langle \chi_*, f\rangle_{y^{d+1}\bar{\rho}}$. Hence χ_* is a weak solution to $L\chi = \mu_*\chi$. By elliptic regularity, χ_* is smooth on $(0, R]$, and so the weak solution is in fact a classical solution. Therefore χ_* is in fact an eigenfunction of L with eigenvalue μ_*. □

From this we get the linear stability criterion in terms of whether the quadratic form Q is non-negative definite.

Proposition 4.2.3. *If $\langle L\chi, \chi\rangle \geq 0$ for all χ, then the corresponding liquid Lane-Emden star is linearly stable under radial perturbations. And if there exist χ such that $\langle L\chi, \chi\rangle < 0$, then it must be linearly unstable.*

Proof. If there exist χ such that $\langle L\chi, \chi\rangle < 0$, then by the previous lemma there exist $-\mu < 0$ and χ_* such that $L\chi_* = -\mu y^{d+1}\bar{\rho}\chi_*$. This, by the last proposition, means the linearised momentum equation admits a solution of the form $\zeta(y,t) = e^{\sqrt{\mu}t}\chi_*(y)$. This grows exponentially in time, and hence the corresponding liquid Lane-Emden star is linearly unstable. Conversely, if $\langle L\chi, \chi\rangle \geq 0$ for all χ, then no such growing solutions exist and hence the corresponding liquid Lane-Emden star is linearly stable under radial perturbations. □

4.2.4 (In)stability results

We will now prove our main Theorem 1.3.4 on the (in)stability results for liquid Lane–Emden stars. This will be split into three theorems below, that together will established Theorem 1.3.4.

Theorem 4.2.4. *The liquid Lane–Emden stars is, against radial perturbations, linearly stable when $\gamma \geq 2(d-1)/d$.*

Proof. Since $\partial_y\bar{\rho}^\gamma < 0$, if $2(d-1) - d\gamma \leq 0$, then it is clear from equation (4.10) that $\langle L\chi, \chi\rangle > 0$ for all $\chi \neq 0$, hence the system must be stable. □

When $\gamma < 2(d-1)/d$, the proof of linear stability for stars of small relative central density require Poincaré-Hardy-type inequalities in the following two lemmas and proposition.

Lemma 4.2.5. *Let $v \in C^1([a,b])$, then*

$$\int_a^b |v(z)|^2 \mathrm{d}z \lesssim_{a,b} |v(a)|^2 + \int_a^b |v'(z)|^2 \mathrm{d}z$$
$$\int_a^b |v(z)|^2 \mathrm{d}z \lesssim_{a,b} |v(b)|^2 + \int_a^b |v'(z)|^2 \mathrm{d}z.$$

Proof. We will prove the second statement, the first is proven in the same way. By the fundamental theorem of calculus

$$v(z) = v(b) - \int_z^b v'(y)\mathrm{d}y.$$

Using the fact that $|x+y|^2 \leq |x|^2 + 2|x||y| + |y|^2 \leq 2|x|^2 + 2|y|^2$ and Hölder's inequality we have

$$|v(z)|^2 \leq 2|v(b)|^2 + 2\left|\int_z^b v'(y)\mathrm{d}y\right|^2 \leq 2|v(b)|^2 + 2(b-z)^2 \int_z^b |v'(y)|^2\mathrm{d}y.$$

So

$$\begin{aligned}\int_a^b |v(z)|^2\mathrm{d}z &\leq \int_a^b \left(2|v(b)|^2 + 2(b-a)^2 \int_z^b |v'(y)|^2\mathrm{d}y\right)\mathrm{d}z \\ &\leq 2(b-a)|v(b)|^2 + 2(b-a)^3 \int_a^b |v'(y)|^2\mathrm{d}y.\end{aligned}$$

□

Lemma 4.2.6. *Let $a \geq 2$ and $0 < b < c < \infty$. Then for any $v \in C^1([0,c])$ we have*

$$\int_0^b z^a|v(z)|^2\mathrm{d}z \lesssim_{a,b,c} \int_0^c z^a|v'(z)|^2\mathrm{d}z + \int_b^c z^a|v(z)|^2\mathrm{d}z.$$

Proof. Let $\phi \in C^\infty([0,\infty))$ be a decreasing function such that $\phi(z) = 1$ for $z \leq b$ and $\phi(z) = 0$ for $z \geq c$. First note that integration by parts tells us that

$$\int_0^c z^{a-1}\phi v(\phi v)'\mathrm{d}z = -\int_0^c (a-1)z^{a-2}(\phi v)^2 + z^{a-1}(\phi v)'\phi v\,\mathrm{d}z.$$

Using the lemma above, we have

$$\begin{aligned}\int_0^b z^a|\phi v|^2\mathrm{d}z &= \int_0^b \left|z^{a/2}\phi v\right|^2\mathrm{d}z \lesssim_c \int_0^c \left|(z^{a/2}\phi v)'\right|^2\mathrm{d}z \\ &\leq \frac{a^2}{4}\int_0^c z^{a-2}|\phi v|^2\mathrm{d}z + a\int_0^c z^{a-1}\phi v(\phi v)'\mathrm{d}z + \int_0^c z^a|(\phi v)'|^2\mathrm{d}z \\ &\leq \frac{1}{4}(a^2 - 2a(a-1))\int_0^c z^{a-2}|\phi v|^2\mathrm{d}z + \int_0^c z^a|(\phi v)'|^2\mathrm{d}z \\ &= -\frac{1}{4}a(a-2)\int_0^c z^{a-2}|\phi v|^2\mathrm{d}z + \int_0^c z^a|(\phi v)'|^2\mathrm{d}z \\ &\leq \int_0^c z^a|(\phi v)'|^2\mathrm{d}z\end{aligned}$$

Using the fact that $|x+y|^2 \le |x|^2 + 2|x||y| + |y|^2 \le 2|x|^2 + 2|y|^2$ we have

$$\begin{aligned}\int_0^b z^a |v|^2 \mathrm{d}z \le \int_0^b z^a |\phi v|^2 \mathrm{d}z \lesssim_c \int_0^c z^a |(\phi v)'|^2 \mathrm{d}z = \int_0^c z^a |\phi' v + \phi v'|^2 \mathrm{d}z \\ \le 2\int_0^c z^a (|\phi' v|^2 + |\phi v'|^2) \mathrm{d}z \\ \le 2\|\phi'\|_\infty^2 \int_b^c z^a |v|^2 \mathrm{d}z + 2\int_0^c z^a |v'|^2 \mathrm{d}z.\end{aligned}$$

□

Proposition 4.2.7. *Let $a \ge 2$. We have*

$$\int_0^1 z^a |v(z)|^2 \mathrm{d}z \lesssim_a \int_0^1 z^a |v'(z)|^2 \mathrm{d}z + |v(1)|^2 \qquad \textit{for all} \qquad v \in C^1([0,1]).$$

Proof. Using the first of the above two lemmas we have

$$\begin{aligned}\int_{\frac{1}{2}}^1 z^a |v(z)|^2 \mathrm{d}z \le \int_{\frac{1}{2}}^1 |v(z)|^2 \mathrm{d}z \lesssim |v(1)|^2 + \int_{\frac{1}{2}}^1 |v'(y)|^2 \mathrm{d}y \\ \le |v(1)|^2 + 2^a \int_0^1 z^a |v'(z)|^2 \mathrm{d}z\end{aligned}$$

And using the second of the above two lemmas we have

$$\int_0^{\frac{1}{2}} z^a |v(z)|^2 \mathrm{d}z \lesssim_a \int_0^1 z^a |v'(z)|^2 \mathrm{d}z + \int_{\frac{1}{2}}^1 z^a |v(z)|^2 \mathrm{d}z$$

but the rightmost term we have already estimated in the right form. Hence we are done. □

With this Poincaré-Hardy-type inequality, we can prove linear stability for stars of small relative central density (see Definition 1.3.3) when $\gamma < 2(d-1)/d$.

Theorem 4.2.8. *Suppose $\gamma < 2(d-1)/d$. There exist $\epsilon > 0$ such that the liquid Lane–Emden stars are linearly stable against radial perturbations whenever $\bar\rho(0) - 1 < \epsilon$ (i.e. small relative central density, Definition 1.3.3).*

Proof. Let $z = y/R$. Let $\tilde\chi(z) = \chi(Rz)$ and $\tilde\rho(z) = \bar\rho(Rz)$. Then (4.10) becomes

$$\begin{aligned}\langle L\chi, \chi\rangle = \int_0^1 \Bigg(\gamma\tilde\rho(z)^\gamma (Rz)^{d+1} \frac{1}{R^2} \partial_z \tilde\chi(z)^2 \\ + (2(d-1) - d\gamma)(Rz)^d \tilde\chi(z)^2 \frac{1}{R} \partial_z \tilde\rho^\gamma(z) R \Bigg) \mathrm{d}z + d\gamma R^d \tilde\chi(1)^2\end{aligned}$$

$$= R^d \int_0^1 \left(\gamma \tilde{\rho}^\gamma z^{d+1} (\partial_z \tilde{\chi})^2 + (2(d-1) - d\gamma) z^d \tilde{\chi}^2 \partial_z \tilde{\rho}^\gamma \right) \mathrm{d}z + R^d d\gamma \tilde{\chi}(1)^2.$$

We know from our derivation of the existence of steady states that

$$\begin{aligned} \frac{\mathrm{d}\bar{\rho}^{\gamma-1}}{\mathrm{d}y} &= -4\pi \frac{\gamma-1}{\gamma} \frac{1}{y^{d-1}} \int_0^y r^{d-1} \bar{\rho}(r) \mathrm{d}r \\ &\geq -4\pi\bar{\rho}(0) \frac{\gamma-1}{\gamma} \frac{1}{y^{d-1}} \int_0^y r^{d-1} \mathrm{d}r = -4\pi\bar{\rho}(0) \frac{\gamma-1}{\gamma} \frac{1}{d} y \end{aligned}$$

So

$$\begin{aligned} \partial_y \bar{\rho}^\gamma &= \gamma \bar{\rho}^{\gamma-1} \partial_y \bar{\gamma} = \frac{\gamma}{\gamma-1} \bar{\rho} \partial_y \bar{\rho}^{\gamma-1} \geq -\frac{4\pi}{d} \bar{\rho}(0)^2 y \\ \partial_z \tilde{\rho}^\gamma &\geq -R \frac{4\pi}{d} \bar{\rho}(0)^2 (Rz) = -\frac{4\pi}{d} \bar{\rho}(0)^2 R^2 z. \end{aligned}$$

So when $2(d-1) - d\gamma \geq 0$ we have

$$\begin{aligned} \langle L\chi, \chi \rangle \geq R^d \int_0^1 &\left(\gamma z^{d+1} (\partial_z \tilde{\chi})^2 - \frac{4\pi}{d} \bar{\rho}(0)^2 R^2 (2(d-1) - d\gamma) z^{d+1} \tilde{\chi}^2 \right) \mathrm{d}z \\ &+ R^d d\gamma \tilde{\chi}(1)^2. \end{aligned}$$

From the decay estimates we have for the steady states we see that $R \to 0$ as $\bar{\rho}(0) \searrow 1$. So the above proposition tells us that for small enough relative central density $\bar{\rho}(0) - 1$, we have

$$\frac{4\pi}{d} \bar{\rho}(0)^2 R^2 (2(d-1) - d\gamma) \int_0^1 z^{d+1} \tilde{\chi}^2 \mathrm{d}z < \gamma \int_0^1 z^{d+1} (\partial_z \tilde{\chi})^2 \mathrm{d}z + d\gamma \tilde{\chi}(1)^2$$

for any $\tilde{\chi}$. It follows that we have stability. □

Finally, it remains to prove linear instability for stars of large central density when $\gamma < 2(d-1)/d$.

Theorem 4.2.9. *Suppose $\gamma < 2(d-1)/d$ and $d < 10$. There exist $C > 0$ such that the liquid Lane–Emden stars are linearly unstable whenever $\bar{\rho}(0) > C$ (i.e. large central density).*

Proof. We deal with three sub-cases individually.

Case 1: $\gamma > 2d/(d+2)$ We saw that the family of gaseous steady states are self-similar, so that the family is given by $\bar{\rho}_\kappa(y) = \kappa \bar{\rho}_*(\kappa^{1-\gamma/2} y)$ where $\bar{\rho}_*$ is a steady state. The corresponding liquid star has $R = \kappa^{-(1-\gamma/2)} \bar{\rho}_*^{-1}(1/\kappa)$. So

$\tilde{\rho}_\kappa(z) = \kappa\bar{\rho}_*(\bar{\rho}_*^{-1}(1/\kappa)z)$. With this, (4.10) reads

$$\begin{aligned}\langle L_\kappa \chi, \chi\rangle = R^d\kappa^\gamma \int_0^1 &\gamma\bar{\rho}_*(\bar{\rho}_*^{-1}(1/\kappa)z)^\gamma z^{d+1}(\partial_z\tilde{\chi})^2 \\ &+ (2(d-1)-d\gamma)\bar{\rho}_*^{-1}(1/\kappa)z^d\tilde{\chi}^2(\bar{\rho}_*^\gamma)'(\bar{\rho}_*^{-1}(1/\kappa)z)\mathrm{d}z \\ &+ R^d d\gamma\tilde{\chi}(1)^2\end{aligned}$$

When $\gamma > 2d/(d+2)$, the gaseous steady state $\bar{\rho}_*$ has compact support. Then $\bar{\rho}_*^{-1}(1/\kappa) \to \bar{\rho}_*^{-1}(0) =: R_*$ as $\kappa \to \infty$. So

$$\int_0^1 z^d\partial_z\tilde{\rho}_\kappa^\gamma \mathrm{d}z = \kappa^\gamma \underbrace{\bar{\rho}_*^{-1}(1/\kappa)\int_0^1 z^d(\bar{\rho}_*^\gamma)'(\bar{\rho}_*^{-1}(1/\kappa)z)\mathrm{d}z}_{\to R_*\int_0^1 z^d(\bar{\rho}_*^\gamma)'(R_* z)\mathrm{d}z \quad \text{as} \quad \kappa\to\infty}$$

by dominated convergence where the integrand is dominated by $z^d\|(\bar{\rho}_*^\gamma)'\|_\infty$. So

$$\begin{aligned}&\langle L_\kappa 1, 1\rangle \\ &= R^d(2(d-1)-d\gamma)\kappa^\gamma\bar{\rho}_*^{-1}(1/\kappa)\int_0^1 z^d\tilde{\chi}^2(\bar{\rho}_*^\gamma)'(\bar{\rho}_*^{-1}(1/\kappa)z)\mathrm{d}z + R^d d\gamma \\ &\to -\infty \qquad \text{as} \qquad \kappa\to\infty.\end{aligned}$$

Hence we have instability for large central density.

Case 2: $\gamma = 2d/(d+2)$ From the explicit formula, we have

$$\begin{aligned}\rho_\kappa^\gamma(y) &= \kappa^{\frac{2d}{d+2}}\left(1+\frac{2\pi}{d^2}\kappa^{\frac{4}{d+2}}y^2\right)^{-d} = \left(\kappa^{-\frac{2}{d+2}}+\frac{2\pi}{d^2}\kappa^{\frac{2}{d+2}}y^2\right)^{-d} \\ \partial_y\rho_\kappa^\gamma(y) &= -\frac{4\pi}{d}\kappa^{\frac{2}{d+2}}y\left(\kappa^{-\frac{2}{d+2}}+\frac{2\pi}{d^2}\kappa^{\frac{2}{d+2}}y^2\right)^{-d-1}\end{aligned}$$

and

$$R_\kappa = \frac{d}{\sqrt{2\pi}}\kappa^{-\frac{2}{d+2}}\left(\kappa^{\frac{2}{d+2}}-1\right)^{\frac{1}{2}}.$$

Let $\chi_\kappa(y) = \kappa^{\frac{d}{d+2}}\chi(\kappa^{\frac{1}{d+2}}y)$. From (4.10) we have

$$\begin{aligned}&\langle L_\kappa\chi_\kappa, \chi_\kappa\rangle \\ &= \int_0^{R_\kappa}\gamma\bar{\rho}_\kappa^\gamma y^{d+1}(\partial_y\chi_\kappa)^2 + (2(d-1)-d\gamma)y^d\chi_\kappa^2\partial_y\bar{\rho}_\kappa^\gamma\mathrm{d}y + d\gamma R_\kappa^d\chi_\kappa(R_\kappa)^2 \\ &= \kappa\int_0^{R_\kappa}\frac{2d}{d+2}\left(\kappa^{-\frac{2}{d+2}}+\frac{2\pi}{d^2}\kappa^{\frac{2}{d+2}}y^2\right)^{-d} y^{d+1}\chi'(\kappa^{\frac{1}{d+2}}y)^2\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&- \kappa\frac{4\pi}{d}\left(2(d-1)-\frac{2d^2}{d+2}\right) \\
&\qquad\qquad \int_0^{R_\kappa} y^{d+1}\left(\kappa^{-\frac{2}{d+2}}+\frac{2\pi}{d^2}\kappa^{\frac{2}{d+2}}y^2\right)^{-d-1}\chi(\kappa^{\frac{1}{d+2}}y)^2\mathrm{d}y \\
&+ d\gamma\kappa^{\frac{d}{d+2}}R_\kappa^d\chi(\kappa^{\frac{1}{d+2}}R_\kappa)^2 \\
= {} & \frac{2d}{d+2}\int_0^{\kappa^{\frac{1}{d+2}}R_\kappa}\left(\kappa^{-\frac{2}{d+2}}+\frac{2\pi}{d^2}z^2\right)^{-d} z^{d+1}\chi'(z)^2\mathrm{d}z \\
&- \frac{8\pi}{d}\frac{d-2}{d+2}\int_0^{\kappa^{\frac{1}{d+2}}R_\kappa}\left(\kappa^{-\frac{2}{d+2}}+\frac{2\pi}{d^2}z^2\right)^{-d-1} z^{d+1}\chi(z)^2\mathrm{d}z \\
&+ \frac{2d^2}{d+2}\kappa^{\frac{d}{d+2}}R_\kappa^d\chi(\kappa^{\frac{1}{d+2}}R_\kappa)^2 \\
\to {} & \frac{2d}{d+2}\left(\frac{d^2}{2\pi}\right)^d\int_0^{\frac{d}{\sqrt{2\pi}}} z^{-d+1}\chi'(z)^2\mathrm{d}z \\
&- \frac{8\pi}{d}\frac{d-2}{d+2}\left(\frac{d^2}{2\pi}\right)^{d+1}\int_0^{\frac{d}{\sqrt{2\pi}}} z^{-d-1}\chi(z)^2\mathrm{d}z \\
&+ \frac{2d^2}{d+2}\left(\frac{d^2}{2\pi}\right)^{\frac{d}{2}}\chi\left(\frac{d}{\sqrt{2\pi}}\right)^2 \qquad \text{as} \qquad \kappa\to\infty
\end{aligned}
$$

Let $\chi(z)=1$, then we see that $\langle L_\kappa\chi_\kappa,\chi_\kappa\rangle\to-\infty$ as $\kappa\to\infty$. Hence we have instability.

Case 3: $\gamma<2d/(d+2)$ From (4.10) we have

$$
\begin{aligned}
&\langle L_\kappa\chi,\chi\rangle \\
&= \int_0^{R_\kappa}\gamma\bar\rho_\kappa^\gamma y^{d+1}(\partial_y\chi)^2+(2(d-1)-d\gamma)y^d\chi^2\partial_y\bar\rho_\kappa^\gamma\mathrm{d}y+d\gamma R_\kappa^d\chi(R_\kappa)^2 \\
&= \int_0^{R_\kappa}\gamma\bar\rho_\kappa^\gamma y^{d+1}(\partial_y\chi)^2-(2(d-1)-d\gamma)y\chi^2\bar m_\kappa\bar\rho_\kappa\mathrm{d}y+d\gamma R_\kappa^d\chi(R_\kappa)^2.
\end{aligned}
$$

Fix $\delta>0$, and suppose χ is constant on $[0,\epsilon]$. Then we have

$$
\begin{aligned}
&\langle L_\kappa\chi,\chi\rangle \\
&\le \int_\epsilon^{R_\kappa}\gamma\bar\rho_\kappa^\gamma y^{d+1}(\partial_y\chi)^2-(2(d-1)-d\gamma)y\chi^2\bar m_\kappa\bar\rho_\kappa\mathrm{d}y+d\gamma R_\kappa^d\chi(R_\kappa)^2 \\
&\to \int_\epsilon^{R_\infty}\gamma(v_1^*)^\gamma y^{d+1-\frac{2\gamma}{2-\gamma}}(\partial_y\chi)^2-(2(d-1)-d\gamma)y^{d+1-\frac{4}{2-\gamma}}\chi^2v_2^*v_1^*\mathrm{d}y \\
&\quad + d\gamma R_\infty^d\chi(R_\infty)^2 \\
&= \gamma(v_1^*)^\gamma\int_\epsilon^{R_\infty} y^{d+1-\frac{2\gamma}{2-\gamma}}(\partial_y\chi)^2-2\left(d-\frac{2}{2-\gamma}\right)y^{d-\frac{2+\gamma}{2-\gamma}}\chi^2\mathrm{d}y
\end{aligned}
$$

$$+ d\gamma R_\infty^d \chi(R_\infty)^2$$

as $\kappa \to \infty$. Let $\chi(y) = \epsilon^{-a} \wedge y^{-a}$. Then

$$\gamma(v_1^*)^\gamma \int_\epsilon^{R_\infty} y^{d+1-\frac{2\gamma}{2-\gamma}} (\partial_y \chi)^2 - 2\left(d - \frac{2}{2-\gamma}\right) y^{d-\frac{2+\gamma}{2-\gamma}} \chi^2 \mathrm{d}y + d\gamma R_\infty^d \chi(R_\infty)^2$$
$$= \gamma(v_1^*)^\gamma \left(a^2 - 2\left(d - \frac{2}{2-\gamma}\right)\right) \int_\epsilon^{R_\infty} y^{d-\frac{2+\gamma}{2-\gamma}-2a} \mathrm{d}y + d\gamma R_\infty^{d-2a}$$

Let $a = \frac{1}{2}(d - \frac{2\gamma}{2-\gamma})$ so that $d - \frac{2+\gamma}{2-\gamma} - 2a = -1$. By choosing ϵ small enough we can make the integral large enough in magnitude. So if we have instability if

$$\left(a^2 - 2\left(d - \frac{2}{2-\gamma}\right)\right) < 0.$$

We have

$$(d/2 - a)(2-\gamma) = \gamma \qquad \Longleftrightarrow \qquad \gamma = \frac{d-2a}{1 + d/2 - a}$$

So the above condition is

$$0 > a^2 - 2\left(d - (1 + d/2 - a)\right) = a^2 - 2a - (d-2)$$

By definition $a \le \frac{1}{2}(d-2)$, so it suffice to have

$$a^2 - 2a - (d-2) \le a^2 - 4a < 0$$

$a^2 - 4a$ is negative when $a \in (0,4)$. Hence we are done if $\frac{1}{2}(d-2) < 4$, or equivalently $d < 10$.

□

This completes the proof of Theorem 1.3.4.

Appendix A

Goldreich-Weber stars appendix

A.1 Differentiation and commutation properties

Here we first collect some standard results on how derivatives interact with $\mathscr{J}$ and $\mathscr{A}$, which can be found for example in [33]. After that we state various derivative commutators frequently used in the article.

Lemma A.1.1. *Recall notations defined in Definition 1.4.2. We have*

$$\begin{aligned}\mathscr{A}^i_j - I^i_j &= -\mathscr{A}^i_k \partial_j \theta^k \\ \partial_\bullet \mathscr{J} &= \mathscr{J} \mathscr{A}^l_m \partial_\bullet \partial_l \theta^m \\ \partial_\bullet \mathscr{A}^i_j &= -\mathscr{A}^i_m \mathscr{A}^l_j \partial_\bullet \partial_l \theta^m\end{aligned}$$

Proof. Since $\mathscr{A} = (\nabla \boldsymbol{\xi})^{-1}$, we have

$$I^i_j = \mathscr{A}^i_k \partial_j \xi^k = \mathscr{A}^i_k (I^k_j + \partial_j \theta^k).$$

It can be proven that if $U : t \mapsto U(t)$ is a differentiable map of invertible square matrices, then

i. $\dfrac{\mathrm{d} \det U}{\mathrm{d}t} = \det(U) \tilde{\rho} \left(U^{-1} \dfrac{\mathrm{d}U}{\mathrm{d}t} \right);$

ii. $\dfrac{\mathrm{d}U^{-1}}{\mathrm{d}t} = -U^{-1} \dfrac{\mathrm{d}A}{\mathrm{d}t} U^{-1}.$

Using i. we get $\partial_\bullet J = J A^k_i \partial_\bullet \partial_k \eta^i$, and using ii. we get $\partial_\bullet A^i_j = -A^i_k (\partial_\bullet \partial_l \eta^k) A^l_j$. Converting to $\mathscr{A}$ and $\mathscr{J}$ by tracing the definition and keeping track of the factors of λ, we get the stated formulas. □

We commonly use various commutation properties between the Cartesian, radial, angular derivatives, and their Lagrangian counterparts.

Lemma A.1.2 (Commutation relations)**.** *We have the following commutation relations*

$$\begin{aligned}
[X_r, \nabla] &= -\nabla \\
[X_r, \mathbf{x}] &= \mathbf{x} \\
[\not\partial_i, \partial_j] &= -\epsilon_{ijk}\partial_k \\
[\not\partial_i, x^j] &= -\epsilon_{ijk}x^k \\
[X_r, \mathscr{K}] &= 2\mathscr{K} \\
[\not\partial_i, \mathscr{K}] &= 0 \\
[\partial_s, \mathscr{A}\partial_j] &= -(\mathscr{A}\partial_j\partial_s\theta^m)\mathscr{A}\partial_m \\
[\nabla, \mathscr{A}\partial_j] &= -(\mathscr{A}\partial_j\nabla\theta^m)\mathscr{A}\partial_m \\
[X_r, \mathscr{A}\partial_j] &= -(\mathscr{A}^l_j X_r\partial_l\theta^m)\mathscr{A}\partial_m - \mathscr{A}\partial_j \\
[\not\partial_i, \mathscr{A}\partial_j] &= -(\mathscr{A}^l_j\not\partial_i\partial_l\theta^m)\mathscr{A}\partial_m - \epsilon_{ikl}\mathscr{A}^k_j\partial_l \\
[X_r, \not\partial_i] &= 0 \\
[\not\partial_i, \not\partial_{i'}] &= -\not\partial_{ii'}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
[X_r, \partial_j] &= x^i\partial_i\partial_j - \partial_j x^i\partial_i = -\partial_j \\
[X_r, x^j] &= x^i\partial_i x^j - x^j x^i\partial_i = x^j \\
[\not\partial_i, \partial_j] &= \epsilon_{ilk}x^l\partial_k\partial_j - \epsilon_{ilk}\partial_j x^l\partial_k = -\epsilon_{ijk}\partial_k \\
[\not\partial_i, x^j] &= \epsilon_{ilk}x^l\partial_k x^j - \epsilon_{ilk}x^j x^l\partial_k = \epsilon_{ilj}x^l \\
[\partial_s, \mathscr{A}\partial_j] &= \partial_s\mathscr{A}^i_j\partial_i - \mathscr{A}^i_j\partial_i\partial_s = -\mathscr{A}^i_m\mathscr{A}^l_j(\partial_l\partial_s\theta^m)\partial_i \\
[\partial_i, \mathscr{A}\partial_j] &= \partial_i\mathscr{A}^k_j\partial_k - \mathscr{A}^k_j\partial_k\partial_i = -\mathscr{A}^k_m\mathscr{A}^l_j(\partial_l\partial_i\theta^m)\partial_k \\
[X_r, \mathscr{A}\partial_j] &= X_r(\mathscr{A}^k_j\partial_k) - \mathscr{A}^k_j\partial_k X_r = -(\mathscr{A}^k_m\mathscr{A}^l_j X_r\partial_l\theta^m)\partial_k + \mathscr{A}^k_j X_r\partial_k - \mathscr{A}^k_j\partial_k X_r \\
&= -(\mathscr{A}^k_m\mathscr{A}^l_j X_r\partial_l\theta^m)\partial_k - \mathscr{A}^k_j\partial_k \\
[\not\partial_i, \mathscr{A}\partial_j] &= \not\partial_i(\mathscr{A}^k_j\partial_k) - \mathscr{A}^k_j\partial_k\not\partial_i = -(\mathscr{A}^k_m\mathscr{A}^l_j\not\partial_i\partial_l\theta^m)\partial_k + \mathscr{A}^k_j\not\partial_i\partial_k - \mathscr{A}^k_j\partial_k\not\partial_i \\
&= -(\mathscr{A}^l_j\not\partial_i\partial_l\theta^m)\mathscr{A}\partial_m - \epsilon_{ikl}\mathscr{A}^k_j\partial_l \\
[X_r, \not\partial_i] &= \epsilon_{ijk}x^l\partial_l(x^j\partial_k) - \epsilon_{ijk}x^j\partial_k(x^l\partial_l) \\
&= \epsilon_{ijk}\left(\delta^j_l x^l\partial_k + x^l x^j\partial_l\partial_k - \delta^l_k x^j\partial_l - x^j x^l\partial_k\partial_l\right) = \epsilon_{ijk}\left(x^j\partial_k - x^j\partial_k\right) \\
&= 0 \\
[\not\partial_i, \not\partial_{i'}] &= \epsilon_{ijk}\epsilon_{i'j'k'}\Big(x^j\partial_k(x^{j'}\partial_{k'}) - x^{j'}\partial_{k'}(x^j\partial_k)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \epsilon_{ijk}\epsilon_{i'j'k'}\left(\delta_k^{j'}x^j\partial_{k'} - \delta_{k'}^{j}x^{j'}\partial_k\right) \\
&= \epsilon_{ijk}\epsilon_{i'kk'}x^j\partial_{k'} - \epsilon_{ik'k}\epsilon_{i'j'k'}x^{j'}\partial_k = \epsilon_{kij}\epsilon_{kk'i'}x^j\partial_{k'} - \epsilon_{k'ki}\epsilon_{k'i'j'}x^{j'}\partial_k \\
&= (\delta_{ik'}\delta_{ji'} - \delta_{ii'}\delta_{jk'})x^j\partial_{k'} - (\delta_{ki'}\delta_{ij'} - \delta_{kj'}\delta_{ii'})x^{j'}\partial_k \\
&= x^{i'}\partial_i - x^i\partial_{i'} = -\not{\partial}_{ii'} \\
\mathscr{K}(\mathbf{x}\cdot\nabla g)(\mathbf{y}) &= -\int \frac{\mathbf{x}\cdot\nabla g(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\mathrm{d}\mathbf{x} = \int\left(\frac{g(\mathbf{x})\nabla\cdot\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} + g(\mathbf{x})\mathbf{x}\cdot\nabla_{\mathbf{x}}\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)\mathrm{d}\mathbf{x} \\
&= \int\left(\frac{3g(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} + g(\mathbf{x})\mathbf{x}\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}\right)\mathrm{d}\mathbf{x} \\
&= \int\left(\frac{2g(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} + g(\mathbf{x})\mathbf{y}\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}\right)\mathrm{d}\mathbf{x} \\
&= -2\mathscr{K}g + \mathbf{y}\cdot\nabla\mathscr{K}g \\
\mathscr{K}(\not{\partial}_i g)(\mathbf{y}) &= -\int \frac{\epsilon_{ijk}x^j\partial_k g(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\mathrm{d}\mathbf{x} = \int g(\mathbf{x})\epsilon_{ijk}x^j\frac{y^k-x^k}{|\mathbf{y}-\mathbf{x}|^3}\mathrm{d}\mathbf{x} \\
&= \int g(\mathbf{x})\epsilon_{ijk}x^j\frac{y^k}{|\mathbf{y}-\mathbf{x}|^3}\mathrm{d}\mathbf{x} = \int g(\mathbf{x})\epsilon_{ijk}y^j\frac{-x^k}{|\mathbf{y}-\mathbf{x}|^3}\mathrm{d}\mathbf{x} \\
&= \int g(\mathbf{x})\epsilon_{ijk}y^j\frac{y^k-x^k}{|\mathbf{y}-\mathbf{x}|^3}\mathrm{d}\mathbf{x} = (\not{\partial}_i\mathscr{K}g)(\mathbf{y}).
\end{aligned}
$$

□

A.2 Spherical harmonics

Spherical harmonics has a real as well as complex version. For the definition and basic properties of the complex version, see [29]. The relation between complex spherical harmonics $Y_l^m : S^2 \to \mathbb{C}$ and real spherical harmonics $Y_{lm} : S^2 \to \mathbb{R}$ are

$$
Y_l^m = \begin{cases} \frac{1}{\sqrt{2}}(Y_{l,-m} - iY_{lm}) & m < 0 \\ Y_{l0} & m = 0 \\ \frac{(-1)^m}{\sqrt{2}}(Y_{lm} + iY_{l,-m}) & m > 0 \end{cases}
$$

We also have the relation $(Y_l^m)^* = (-1)^m Y_l^{-m}$. The zeroth and first order real spherical harmonics are given by

$$
\begin{aligned}
Y_{0,0}(\mathbf{x}) &= \frac{1}{\sqrt{4\pi}}, & Y_{1,-1}(\mathbf{x}) &= \sqrt{\frac{3}{4\pi}}\frac{x^2}{|\mathbf{x}|}, \\
Y_{1,0}(\mathbf{x}) &= \sqrt{\frac{3}{4\pi}}\frac{x^3}{|\mathbf{x}|}, & Y_{1,1}(\mathbf{x}) &= \sqrt{\frac{3}{4\pi}}\frac{x^1}{|\mathbf{x}|}.
\end{aligned}
$$

The spherical harmonics satisfy the following orthonormal conditions

$$\int_{S^2} Y_{lm} Y_{l'm'} \mathrm{d}S = \delta_{ll'}\delta_{mm'} = \int_{S^2} Y_m^m (Y_{l'}^{m'})^* \mathrm{d}S$$

and they form a basis for $L^2(S^2)$ [1] so that, in particular, any function $g \in L^2(S^2)$ has a spherical harmonics expansion

$$g = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm} Y_{lm}, \qquad\qquad g_{lm} \in \mathbb{R}$$

that converge in $L^2(S^2)$. More generally, a function $g \in L^2(B_R)$ has a spherical harmonics expansion in $L^2(B_R)$,

$$g = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm}(r) Y_{lm}, \qquad\qquad g_{lm} : [0, R] \to \mathbb{R}. \tag{A.1}$$

Indeed, since $L^2(B_R) = L^2([0,R]; L^2(S^2), r^2) = L^2([0,R]; L^2(\partial B_r))$, or in other words

$$\int_{B_R} |\cdot| \,\mathrm{d}\mathbf{x} = \int_0^R \int_{\partial B_r} |\cdot| \,\mathrm{d}S\mathrm{d}r,$$

$g|_{\partial B_r}$ must be in $L^2(\partial B_r)$ for almost every $r \in [0, R]$. So a spherical harmonics expansion exist for almost every r. Now

$$\left\| g - \sum_{l=0}^{N} \sum_{m=-l}^{l} g_{lm} Y_{lm} \right\|_{L^2(B_R)}^2 = \int_0^R \left\| g - \sum_{l=0}^{N} \sum_{m=-l}^{l} g_{lm} Y_{lm} \right\|_{L^2(\partial B_r)}^2 \mathrm{d}r$$
$$\to 0 \qquad \text{as} \qquad N \to \infty$$

by dominated convergence theorem (where the dominating function is $4\|g\|^2_{L^2(\partial B_r)}$). Hence (A.1) converge in $L^2(B_R)$. Similarly, functions in $L^2(B_R, \bar{w}^{-2})$ and $L^2(\mathbb{R}^3)$ have a spherical harmonics expansion.

The following lemma allows us to expand gravitational potentials in spherical harmonics.

Lemma A.2.1. *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ *we have*

$$\frac{1}{|\mathbf{x} - \mathbf{y}|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|, |\mathbf{y}|\}^l}{\max\{|\mathbf{x}|, |\mathbf{y}|\}^{l+1}} Y_{lm}(\mathbf{y}) Y_{lm}(\mathbf{x})$$

and this expression converge uniformly for $(\mathbf{x}, \mathbf{y})$ *in any compact set in* $\{(\mathbf{r}, \mathbf{r}') \in$

$\mathbb{R}^6 : |\mathbf{r}| \neq |\mathbf{r}'|\}$.

Proof. From [29] we have

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|,|\mathbf{y}|\}^l}{\max\{|\mathbf{x}|,|\mathbf{y}|\}^{l+1}} Y_l^m(\mathbf{y})^* Y_l^m(\mathbf{x})$$

One derivation of this formula is as follows. Assume $r' = |\mathbf{r}'| < |\mathbf{r}| = r$, otherwise swap $\mathbf{r}'$ and $\mathbf{r}$. By the law of cosines,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\gamma}} = \frac{1}{r\sqrt{1+h^2-2h\cos\gamma}} \quad \text{with} \quad h := \frac{r'}{r}.$$

We find here the generating function of the Legendre polynomials $P_\ell(\cos\gamma)$:

$$\frac{1}{\sqrt{1+h^2-2h\cos\gamma}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(\cos\gamma). \tag{A.2}$$

Use of the spherical harmonic addition theorem

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} (-1)^m Y_\ell^{-m}(\theta,\varphi) Y_\ell^m(\theta',\varphi')$$

gives our first formula. Since $|P_\ell(\cos\gamma)| \leq 1$ for all ℓ, the power series in (A.2) has radius of convergence 1, and uniform convergence for any compact set in B_1. By Identity theorem for analytic functions, the equality of (A.2) holds for $h < 1$. We thus conclude that the expansion for $|\mathbf{r}-\mathbf{r}'|^{-1}$ converge uniformly on any compact set in $\{(\mathbf{r},\mathbf{r}') \in \mathbb{R}^6 : |\mathbf{r}| \neq |\mathbf{r}'|\}$. Moreover, in real spherical harmonics,

$$\begin{aligned}
&\frac{1}{|\mathbf{x}-\mathbf{y}|} \\
&= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|,|\mathbf{y}|\}^l}{\max\{|\mathbf{x}|,|\mathbf{y}|\}^{l+1}} Y_l^m(\mathbf{y})^* Y_l^m(\mathbf{x}) \\
&= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(-1)^m}{2l+1} \frac{\min\{|\mathbf{x}|,|\mathbf{y}|\}^l}{\max\{|\mathbf{x}|,|\mathbf{y}|\}^{l+1}} Y_l^{-m}(\mathbf{y}) Y_l^m(\mathbf{x}) \\
&= 4\pi \sum_{l=0}^{\infty} \sum_{m=1}^{l} \frac{1}{2} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|,|\mathbf{y}|\}^l}{\max\{|\mathbf{x}|,|\mathbf{y}|\}^{l+1}} (Y_{l,m}(\mathbf{y}) - iY_{l,-m}(\mathbf{y}))(Y_{lm}(\mathbf{x}) + iY_{l,-m}(\mathbf{x})) \\
&\quad + 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|,|\mathbf{y}|\}^l}{\max\{|\mathbf{x}|,|\mathbf{y}|\}^{l+1}} Y_{l0}(\mathbf{y}) Y_{l0}(\mathbf{x})
\end{aligned}$$

$$+ 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{-1} \frac{1}{2} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|, |\mathbf{y}|\}^l}{\max\{|\mathbf{x}|, |\mathbf{y}|\}^{l+1}} (Y_{l,-m}(\mathbf{y}) + iY_{lm}(\mathbf{y}))(Y_{l,-m}(\mathbf{x}) - iY_{lm}(\mathbf{x}))$$

$$= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{\min\{|\mathbf{x}|, |\mathbf{y}|\}^l}{\max\{|\mathbf{x}|, |\mathbf{y}|\}^{l+1}} Y_{lm}(\mathbf{y}) Y_{lm}(\mathbf{x})$$

□

There also exist a vectorial version of spherical harmonics which allow the expansion of L^2 vector fields, details can be found in [2, 14].

A.3 Hardy-Poincaré inequality and embeddings

Here we will state and prove a version of the Hardy-Poincaré inequality and the embedding theorems, which are important for our analysis.

Denote $B_R = B_R(\mathbb{R}^m)$ the ball of radius R in $\mathbb{R}^m$, and $L^2(B_R, w)$ the L^2 space on B_R weighted by w. Denote $d_{\partial B_R}$ the distance function to ∂B_R.

Theorem A.3.1 (Hardy-Poincaré inequality)**.** *Let $R > 0$ and $k \geq 0$. For any $\theta \in H^1_{\text{loc}}(B_R(\mathbb{R}^m))$ we have*

$$\|\theta - \theta_{B_{(m-1)R/m}}\|_{L^2(B_R, d^k_{\partial B_R})} \lesssim \|\nabla\theta\|_{L^2(B_R, d^{k+2}_{\partial B_R})}, \tag{A.3}$$

where θ_{B_r} denotes the average of θ on B_r.

Proof. Using only the standard Poincaré inequality and elementary methods, we will provide here a proof for the case $k > 0$. However, in this thesis we also used the case $k = 0$. A slightly different version of the case $k = 0$ was first proven in [3] which makes use of the Hardy inequality. A proof of the $k = 0$ case for the version here (and a more general form) can be found in [12] by Drelichman and Durán, the proof of which (with very slight modification) will work for the $k = 0$ as well as the $k > 0$ case.

It suffice to show this for $\theta \in C^1(\bar{B}_R)$ since $C^1(\bar{B}_R)$ is dense in the type of Sobolev spaces we are considering [35]. In particular we can assume $\lim_{|\mathbf{x}|\to R} \theta(\mathbf{x}) d_{\partial B_R}(\mathbf{x})^{(k+1)/2} = 0$ for integration by parts later.

Let w be a smooth function on B_R such that $d_{\partial B_R} \lesssim w \lesssim d_{\partial B_R}$ and $w = d_{\partial B_R}$ on $B_R \setminus B_{(m-1)R/m}$. Let $G = \theta w^{k/2}$. Then

$$\int_{B_R} |\nabla\theta|^2 w^{k+2} \mathrm{d}\mathbf{x} = \int_{B_R} |\nabla(w^{-k/2}G)|^2 w^{k+2} \mathrm{d}\mathbf{x}$$

$$
\begin{aligned}
&= \int_{B_R} |w^{-k/2}\nabla G - \frac{k}{2} G w^{-k/2-1}\nabla w|^2 w^{k+2}\mathbf{dx} \\
&= \int_{B_R} |w\nabla G - \frac{k}{2} G\nabla w|^2\mathbf{dx} \\
&= \int_{B_R} \left(w^2|\nabla G|^2 - k(G\nabla G)(w\nabla w) + \frac{k^2}{4}|\nabla w|^2 G^2 \right)\mathbf{dx} \\
&= \int_{B_R} \left(w^2|\nabla G|^2 - \frac{k}{4}(\nabla w^2)(\nabla G^2) + \frac{k^2}{4}|\nabla w|^2 G^2 \right)\mathbf{dx} \\
&= \int_{B_R} \left(w^2|\nabla G|^2 + \frac{k}{4}\left(\Delta w^2 + k|\nabla w|^2\right) G^2 \right)\mathbf{dx}
\end{aligned}
$$

On $B_R \setminus B_{(m-1)R/m}$ we have

$$
\begin{aligned}
k(\Delta w^2 + k|\nabla w|^2) &= \frac{k}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}\frac{\mathrm{d}(R-r)^2}{\mathrm{d}r} \right) + k^2\left(\frac{\mathrm{d}(R-r)}{\mathrm{d}r} \right)^2 \\
&= \frac{k}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}(-2R+2r) \right) + k^2 \\
&= k\left(-2(n-1)\frac{R}{r} + 2n \right) + k^2 \\
&\geq \begin{cases} k^2 & k \geq 0 \\ k^2 + 2k & k \leq 0 \end{cases}
\end{aligned}
$$

which is strictly positive when $k \in \mathbb{R} \setminus [-2, 0]$. So we have

$$
\begin{aligned}
&\frac{k^2 + \min\{0, 2k\}}{4}\int_{B_R} \theta^2 w^k \mathbf{dx} \\
&= \frac{k^2 + \min\{0, 2k\}}{4}\int_{B_R} G^2 \mathbf{dx} \\
&\leq \int_{B_R} |\nabla\theta|^2 w^{k+2}\mathbf{dx} + \frac{k}{4}(k + \|\Delta w^2\|_\infty + k\|\nabla w\|_\infty)\int_{B_{(m-1)R/m}} G^2\mathbf{dx} \\
&\lesssim \int_{B_R} |\nabla\theta|^2 w^{k+2}\mathbf{dx} + \int_{B_{(m-1)R/m}} \theta^2\mathbf{dx} \\
&\lesssim \int_{B_R} |\nabla\theta|^2 w^{k+2}\mathbf{dx} + \left(\int_{B_{(m-1)R/m}} \theta\mathbf{dx} \right)^2 + \int_{B_{(m-1)R/m}} |\nabla\theta|^2\mathbf{dx} \\
&\lesssim \int_{B_R} |\nabla\theta|^2 w^{k+2}\mathbf{dx} + \left(\int_{B_{(m-1)R/m}} \theta\mathbf{dx} \right)^2
\end{aligned}
$$

where we used the Poincaré inequality on $B_{(m-1)R/m}$. Replacing θ with $\theta -$

$\theta_{B_{(m-1)R/m}}$ (valid when $k > -1$) we see that

$$\int_{B_R} (\theta - \theta_{B_{(m-1)R/m}})^2 w^k \mathbf{dx} \lesssim \int_{B_R} |\nabla\theta|^2 w^{k+2} \mathbf{dx}.$$

□

An immediate corollary of the Hardy-Poincaré inequality is the following.

Corollary A.3.2. *Let* $k, l \geq 0$. *We have*

$$\|\theta\|_k \lesssim \|\theta\|_l + \|\nabla\theta\|_{k+2} \tag{A.4}$$

Proof. We have

$$\begin{aligned}\|\theta\|_k &\leq \|\theta - \theta_{B_{(m-1)R/m}}\|_k + \|\theta_{B_{(m-1)R/m}}\|_k \lesssim \theta_{B_{(m-1)R/m}}\|1\|_k + \|\nabla\theta\|_{k+2} \\ &\leq \|\theta\|_{L^2(B_{(m-1)R/m})}\|1\|_{L^2(B_{(m-1)R/m})}\|1\|_k + \|\nabla\theta\|_{k+2} \\ &\lesssim \|\theta\|_l + \|\nabla\theta\|_{k+2}.\end{aligned}$$

□

Using this corollary we will next derive the embedding theorems. We will show that terms with less than $n/2$ the derivatives can be estimated in the L^∞ norm by E_n or $E_n + Z_n^2$, which is relevant for the energy estimates. For this we will need the following lemmas.

Lemma A.3.3. *We have*

$$\sum_{c=0}^{n} \|\nabla^{n-c} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|^2_{\max\{3+b+n-2c,0\}} \lesssim \sum_{c=0}^{n} \|\nabla^c X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|^2_{3+b+c}$$

$$\|\bar{w}^{\lfloor b/2 \rfloor} X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s)\|^2_{H^n} \lesssim \sum_{c=0}^{4+2n} \|\nabla^c X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|^2_{3+b+c}$$

$$\|\bar{w}^{\lfloor b/2 \rfloor} \nabla X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}(s)\|^2_{H^n} \lesssim \sum_{c=1}^{6+2n} \|\nabla^c X_r^b \partial\!\!\!/^\beta \boldsymbol{\theta}\|^2_{3+b+c}$$

Proof. The first formula follows from repeated application of the above Corollary A.3.2. The formulas after follows from the first with n replaced by $4 + 2n$ and $6 + 2n$ respectively. □

Lemma A.3.4. *We have*

$$
\begin{aligned}
\|\bar{w}^{\lfloor b/2 \rfloor} X_r^b \not{\partial}^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 &\lesssim \sum_{c=0}^{8} \|\nabla^c X_r^b \not{\partial}^\beta \boldsymbol{\theta}\|_{3+b+c}^2 \\
&\lesssim \sum_{b'+|\beta'| \le 8+b+|\beta|} \|X_r^{b'} \not{\partial}^{\beta'} \boldsymbol{\theta}\|_{3+b'}^2 + \sum_{c \le 8+b+|\beta|} \|\nabla^c \boldsymbol{\theta}\|_{3+2c}^2 \\
\|\bar{w}^{\lfloor b/2 \rfloor} \nabla X_r^b \not{\partial}^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 &\lesssim \sum_{c=1}^{10} \|\nabla^c X_r^b \not{\partial}^\beta \boldsymbol{\theta}\|_{3+b+c}^2 \\
&\lesssim \sum_{b'+|\beta'| \le 10+b+|\beta|} \|X_r^{b'} \not{\partial}^{\beta'} \boldsymbol{\theta}\|_{3+b'}^2 + \sum_{c \le 10+b+|\beta|} \|\nabla^c \boldsymbol{\theta}\|_{3+2c}^2
\end{aligned}
$$

Proof. This follows from Lemma A.3.3 and the embedding $H^2 \hookrightarrow L^\infty$. □

A.3.1 Embedding theorems for self-similarly expanding GW stars

Using Lemmas A.3.3 and A.3.4 we can derive the Embedding theorems for self-similarly expanding GW stars, relevant for Chapter 2.

Theorem A.3.5 (Near boundary embedding theorem)**.** *We have*

$$
\begin{aligned}
&\sum_{a+|\beta|+b \le n} \|\bar{w}^{\lfloor b/2 \rfloor} \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 \\
&\qquad \lesssim \sum_{a+|\beta|+b \le 8+n} \|\partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\|_{3+b}^2 + \sum_{a+c \le 8+n} \|\partial_s^a \nabla^c \boldsymbol{\theta}\|_{3+2c}^2 \\
&\qquad \lesssim E_{n+8} + Z_{n+8}^2 \\
&\sum_{\substack{a+|\beta|+b \le n \\ a>0}} \|\bar{w}^{\lfloor b/2 \rfloor} \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 \\
&\qquad \lesssim \sum_{\substack{a+|\beta|+b \le 8+n \\ a>0}} \|\partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\|_{3+b}^2 + \sum_{\substack{a+c \le 8+n \\ a>0}} \|\partial_s^a \nabla^c \boldsymbol{\theta}\|_{3+2c}^2 \\
&\qquad \lesssim E_{n+8} \\
&\sum_{a+|\beta|+b \le n} \|\bar{w}^{\lfloor b/2 \rfloor} \nabla \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 \\
&\qquad \lesssim \sum_{a+|\beta|+b \le 10+n} \|\partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}\|_{3+b}^2 + \sum_{a+c \le 10+n} \|\partial_s^a \nabla^c \boldsymbol{\theta}\|_{3+2c}^2 \\
&\qquad \lesssim E_{n+10} + Z_{n+10}^2 \\
&\sum_{\substack{a+|\beta|+b \le n \\ a>0}} \|\bar{w}^{\lfloor b/2 \rfloor} \nabla \partial_s^a X_r^b \not{\partial}^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2
\end{aligned}
$$

$$\lesssim \sum_{\substack{a+|\beta|+b\leq 10+n \\ a>0}} \|\partial_s^a X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \sum_{\substack{a+c\leq 10+n \\ a>0}} \|\partial_s^a \nabla^c \boldsymbol{\theta}\|_{3+2c}^2$$
$$\lesssim E_{n+10}$$

Proof. The proof is a direct consequence of Lemma A.3.4. □

Theorem A.3.6 (Near origin embedding theorem)**.** *We have*

$$\sum_{a+c\leq n+1} \|\bar{w}^c \partial_s^a \nabla^c \boldsymbol{\theta}(s)\|_{L^\infty}^2 \lesssim E_{n+10} + Z_{n+10}^2$$

Proof. Similar to above A.3.5. □

A.3.2 Embedding theorems for linearly expanding GW stars

Using Lemmas A.3.3 and A.3.4 we can derive the Embedding theorems for linearly expanding GW stars, relevant for Chapter 3.

Theorem A.3.7 (Near boundary embedding theorem)**.** *We have*

$$\sum_{|\beta|+b\leq n} \|\bar{w}^{\lfloor b/2\rfloor} X_r^b \not\partial^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 \lesssim \sum_{|\beta|+b\leq 8+n} \|X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \sum_{c\leq 8+n} \|\nabla^c \boldsymbol{\theta}\|_{3+2c}^2$$
$$\lesssim E_{n+8}$$
$$\sum_{|\beta|+b\leq n} \|\bar{w}^{\lfloor b/2\rfloor} \partial_s X_r^b \not\partial^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 \lesssim \sum_{|\beta|+b\leq 8+n} \|\partial_s X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \sum_{c\leq 8+n} \|\partial_s \nabla^c \boldsymbol{\theta}\|_{3+2c}^2$$
$$\lesssim \lambda^{-1} E_{n+8}$$
$$\sum_{|\beta|+b\leq n} \|\bar{w}^{\lfloor b/2\rfloor} \nabla X_r^b \not\partial^\beta \boldsymbol{\theta}(s)\|_{L^\infty}^2 \lesssim \sum_{|\beta|+b\leq 10+n} \|X_r^b \not\partial^\beta \boldsymbol{\theta}\|_{3+b}^2 + \sum_{c\leq 10+n} \|\nabla^c \boldsymbol{\theta}\|_{3+2c}^2$$
$$\lesssim E_{n+10}$$

Proof. The proof is a direct consequence of Lemma A.3.4. □

Theorem A.3.8 (Near origin embedding theorem)**.** *We have*

$$\sum_{c\leq n+1} \|\bar{w}^c \nabla^c \boldsymbol{\theta}(s)\|_{L^\infty}^2 \lesssim E_{n+10}$$
$$\sum_{c\leq n} \|\bar{w}^c \partial_s \nabla^c \boldsymbol{\theta}(s)\|_{L^\infty}^2 \lesssim \lambda^{-1} E_{n+10}$$

Proof. Similar to above A.3.7. □

Appendix B

Lane-Emden stars appendix

B.1 Standard results for gaseous stars

Lemma B.1.1. *Let* $x, y \in [a, b]$ *where* $0 < a < b$.

i. If $\alpha \in [0, 1]$*, then* $|x^\alpha - y^\alpha| \leq \alpha a^{\alpha-1}|x - y|$.

ii. If $\alpha \in [1, \infty)$*, then* $|x^\alpha - y^\alpha| \leq \alpha b^{\alpha-1}|x - y|$.

Proof. This follows from the mean value inequality apply to the function $f(x) = x^\alpha$. In case i, $|f'(x)|$ is bounded by $\alpha a^{\alpha-1}$ on $[a, b]$. In case ii, $|f'(x)|$ is bounded by $\alpha b^{\alpha-1}$ on $[a, b]$. □

Lemma B.1.2. *Let* $x, y \in \mathbb{R}$*, then*

$$|e^x - e^y| \leq \max\{e^x, y^y\}|x - y|.$$

Proof. This follows from the mean value inequality apply to the function $f(x) = e^x$, noting that $|f'(x)|$ is bounded by $\max\{e^x, y^y\}$ on $[x, y]$. □

Theorem B.1.3. *For every* $\rho_0 > 0$*, the Lane–Emden ODE (Definition 1.2.1) admits a unique solution* $\rho \geq 0$ *such that* $\rho(0) = \rho_0$*. The interval of existence* $[0, R)$ *is such that either* $R = \infty$ *or* $\lim_{r \to R} \rho(r) = 0$.

Proof. **When** $\gamma > 1$**:** (1.15) is equivalent to

$$w(r) = w_0 - 4\pi \frac{\gamma - 1}{\gamma} \int_0^r \frac{1}{z^{d-1}} \int_0^z y^{d-1} w(y)^\alpha \mathrm{d}y \mathrm{d}z := T(w)(r).$$

Assume $w_0 > \epsilon > 0$, we claim that for small enough $\delta > 0$, T maps $C([0, \delta], [\epsilon, w_0])$ to itself. Note that for $r \in [0, \delta]$ and $w \in C([0, \delta], [\epsilon, w_0])$

we have

$$\begin{aligned}
0 &\le 4\pi\frac{\gamma-1}{\gamma}\int_0^r \frac{1}{z^{d-1}}\int_0^z y^{d-1}w(y)^\alpha \mathrm{d}y\mathrm{d}z \\
&\le 4\pi\frac{\gamma-1}{\gamma}w_0\int_0^r \frac{1}{z^{d-1}}\int_0^z y^{d-1}\mathrm{d}y\mathrm{d}z = 4\pi\frac{\gamma-1}{\gamma}w_0\int_0^r \frac{1}{z^{d-1}}\frac{1}{d}z^d\mathrm{d}z \\
&= \frac{2\pi}{d}\frac{\gamma-1}{\gamma}w_0\delta^2
\end{aligned}$$

Choosing δ small enough we can make this smaller than $w_0-\epsilon$. It follows that $T(w)\in C([0,\delta],[\epsilon,w_0])$. Now for $u,v\in C([0,\delta],[\epsilon,w_0])$ we have using the first lemma

$$\begin{aligned}
\|T(u)-&T(v)\|_{C([0,\delta])} \\
&\le 4\pi\frac{\gamma-1}{\gamma}\sup_{r\in[0,\delta]}\left|\int_0^r \frac{1}{z^{d-1}}\int_0^z y^{d-1}(u(y)^\alpha - v(y)^\alpha)\mathrm{d}y\mathrm{d}z\right| \\
&\le \frac{2\pi}{d}\frac{\gamma-1}{\gamma}\delta^2\max\{\alpha\epsilon^{\alpha-1},\alpha w_0^{\alpha-1}\}\|u-v\|_{C([0,\delta])}
\end{aligned}$$

By shrinking $\delta>0$ further, we can make T a contraction map. Hence a unique fixed point for T exist. It follows that our equation has a unique solution in a small interval $[0,\delta]$. For $r_0>0$, we claim that as long as $w(r_0)>0$, we can extend the solution beyond r_0. Indeed, our equation

$$\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = -4\pi\frac{\gamma-1}{\gamma}w^\alpha$$

is equivalent to

$$\begin{aligned}
&w(r) \\
&= w_{r_0} + \int_{r_0}^r \frac{1}{z^{d-1}}\left(r_0^{d-1}w'_{r_0} - 4\pi\frac{\gamma-1}{\gamma}\int_{r_0}^z y^{d-1}w(y)^\alpha\mathrm{d}y\right)\mathrm{d}z \\
&= w_{r_0} + \frac{r_0^{d-1}w'_{r_0}}{d-2}\left(\frac{1}{r_0^{d-2}} - \frac{1}{r^{d-2}}\right) - 4\pi\frac{\gamma-1}{\gamma}\int_{r_0}^r \frac{1}{z^{d-1}}\int_{r_0}^z y^{d-1}w(y)^\alpha\mathrm{d}y\mathrm{d}z \\
&:= H(w)(r)
\end{aligned}$$

Fix $\epsilon>0$ such that $\epsilon<w_{r_0}$. A similar computation as above show that for small enough $\delta>0$, T maps $C(\bar{B}_\delta(r_0),[\epsilon,w_0])$ to itself. Furthermore,

$$\|H(u)-H(v)\|_{C(\bar{B}_\delta(r_0))}$$

$$\leq 4\pi\frac{\gamma-1}{\gamma}\sup_{r\in[0,\delta]}\left|\int_{r_0}^{r}\frac{1}{z^{d-1}}\int_{r_0}^{z}y^{d-1}(u(y)^\alpha-v(y)^\alpha)\mathrm{d}y\mathrm{d}z\right|$$
$$\leq 4\pi\frac{\gamma-1}{\gamma}\delta^2\left(\frac{r_0+\delta}{r_0-\delta}\right)^{d-1}\max\{\alpha\epsilon^{\alpha-1},\alpha w_0^{\alpha-1}\}\|u-v\|_{C(\bar{B}_\delta(r_0))}$$

By shrinking $\delta > 0$ further, we can make H a contraction map. Hence a unique fixed point for H exist. It follows that our equation has a unique solution on small interval $\bar{B}_\delta(r_0)$. The overlapping bit with the previous solution agrees, so we have extended our solution.

It follows that a solution our equation exist on a maximal interval $[0, R)$ such that either $R = \infty$ or $\lim_{r\to R} w(r) = 0$, noting that w is a deceasing function because w' is always non-positive as can be seen from (1.15).

When $\gamma = 1$**:** This time we consider the (1.16) and (1.17). Using the second lemma instead of the first, the same proof of existence carries over, without having to cap h at 0. It follows that our equation exist on a maximal interval $[0, R)$ such that either $R = \infty$ or $\lim_{r\to R} h(r) = -\infty$.

□

Theorem B.1.4. *Suppose w is a gas star. Then w has compact support if*

$$\gamma > \frac{2d}{d+2} \qquad \text{or equivalently} \qquad \alpha < \frac{d+2}{d-2}$$

and infinitely support otherwise.

Proof. When $\gamma = 1$, (1.17) gives

$$h'(r) \geq -4\pi\frac{1}{r^{d-1}}e^{h_0}\int_0^r y^{d-1}\mathrm{d}y = -\frac{4\pi}{d}e^{h_0}r.$$

So

$$h(r) \geq h_0 - \frac{4\pi}{d}e^{h_0}\int_0^r y\mathrm{d}y = h_0 - \frac{2\pi}{d}e^{h_0}r^2.$$

So the gaseous star cannot have compact support.

We will prove the $\gamma > 1$ case in three steps:

Step 1 If the gaseous solution w has compact support, then the Pohozaev integral gives

$$2\pi\frac{\gamma-1}{\gamma}\left(\frac{2d}{1+\alpha}-(d-2)\right)\int_0^R w^{\alpha+1}y^{d-1}\mathrm{d}y = \frac{1}{2}w'(R)^2R^d > 0$$

This means w cannot have compact support if

$$\frac{2d}{1+\alpha} \leq d-2 \quad \Longleftrightarrow \quad \alpha \geq \frac{d+2}{d-2} \quad \Longleftrightarrow \quad \gamma \leq \frac{d-2}{d+2}+1 = \frac{2d}{d+2}.$$

Step 2 Now suppose the condition in the proposition hold but w has infinite support. Fix some $\epsilon > 0$. From (1.15) we have

$$w'(r) \leq -\frac{1}{r^{d-1}} \underbrace{4\pi\frac{\gamma-1}{\gamma}\int_0^\epsilon y^{d-1}w(y)^\alpha \mathrm{d}y}_{:=m_\epsilon} \qquad \text{for} \qquad r \geq \epsilon.$$

So

$$w(r) = -\int_r^\infty w'(y)\mathrm{d}y \geq m_\epsilon \int_r^\infty \frac{1}{y^{d-1}}\mathrm{d}y = \frac{m_\epsilon}{d-2}\frac{1}{r^{d-2}}.$$

Combining this and the decay estimate for w we get

$$\frac{2\pi}{d}\frac{\gamma-1}{\gamma} \leq \begin{cases} \frac{1}{\alpha-1}\left(\frac{1}{w(r)^{\alpha-1}} - \frac{1}{w_0^{\alpha-1}}\right)\left(\frac{d-2}{m_\epsilon}\right)^{\frac{2}{d-2}} w(r)^{\frac{2}{d-2}} & \text{when} \quad \alpha \neq 1 \\ (\ln w_0 - \ln w(r))\left(\frac{d-2}{m_\epsilon}\right)^{\frac{2}{d-2}} w(r)^{\frac{2}{d-2}} & \text{when} \quad \alpha = 1 \end{cases}.$$

If

$$\frac{2}{d-2} - (\alpha-1) > 0 \qquad \text{or equivently} \qquad (\alpha-1)(d-2) < 2,$$

then the RHS tends to 0 as $r \to \infty$ but not the LHS, this is a contradiction. So $R < \infty$. So we know there exist compactly supported gas solutions at least when $(\alpha-1)(d-2) < 2$.

Step 3 Last step, we prove compact support for when $(\alpha-1)(d-2) < 2$. So we can assume now

$$(\alpha-1)(d-2) \geq 2 \qquad \Longleftrightarrow \qquad d \geq \frac{2\alpha}{\alpha-1}.$$

In particular $\alpha > 1$. Note that by the decay estimate we must have

$$w(r)^{\alpha-1}r^\sigma \to 0 \qquad \text{as} \qquad r \to \infty \qquad \text{for any} \qquad \sigma \in [0,2).$$

The decay estimate also gives us

$$\begin{aligned} w(r) &\le \left(w_0^{-(\alpha-1)} + (\alpha-1)\frac{2\pi}{d}\frac{\gamma-1}{\gamma}r^2 \right)^{-\frac{1}{\alpha-1}} \\ &\le \left((\alpha-1)\frac{2\pi}{d}\frac{\gamma-1}{\gamma} \right)^{-\frac{1}{\alpha-1}} r^{-\frac{2}{\alpha-1}}. \end{aligned}$$

Now using $(*)$, we can bound w' as follows

$$\begin{aligned} 0 \ge w'(r) &= -4\pi\frac{\gamma-1}{\gamma}\frac{1}{r^{d-1}}\int_0^r y^{d-1}w(y)^\alpha \mathrm{d}y \\ &\gtrsim -\frac{1}{r^{d-1}}\left(w_0^\alpha + \int_1^r y^{d-1-\frac{2\alpha}{\alpha-1}}\mathrm{d}y \right) \\ &\gtrsim \begin{cases} -\dfrac{C + r^{d-\frac{2\alpha}{\alpha-1}}}{r^{d-1}} \\ -\dfrac{C+\ln r}{r^{d-1}} \end{cases} \\ &\gtrsim \begin{cases} -r^{-(d-1)} - r^{1-\frac{2\alpha}{\alpha-1}} & \text{when} \quad d \ne \dfrac{2\alpha}{\alpha-1} \\ -r^{-(d-1)} - r^{-(d-1)}\ln r & \text{when} \quad d = \dfrac{2\alpha}{\alpha-1} \end{cases} \end{aligned}$$

where C is some constant. Recall we assume in this step

$$d \ge \frac{2\alpha}{\alpha-1} \qquad \iff \qquad d-1 \ge \frac{2\alpha}{\alpha-1} - 1.$$

So the estimate means that

$$w'(r)r^\sigma \to 0 \qquad \text{as} \qquad r \to \infty \qquad \text{for any} \qquad \sigma \in \left[0, \frac{2\alpha}{\alpha-1} - 1\right).$$

Using these estimates, we see that $w'(r)^2 r^d \to 0$ as $r \to \infty$ if

$$\frac{4\alpha}{\alpha-1} - 2 > d \quad \iff \quad 4\alpha - 2\alpha + 2 > d\alpha - d \quad \iff \quad \alpha < \frac{d+2}{d-2}.$$

Also $w(r)^{\alpha+1}r^d \to 0$ as $r \to \infty$ if

$$2\frac{\alpha+1}{\alpha-1} > d \quad \iff \quad 2\alpha + 2 > d\alpha - d \quad \iff \quad \alpha < \frac{d+2}{d-2}.$$

And finally, $w'(r)w(r)r^{d-1} \to 0$ as $r \to \infty$ if

$$\frac{2\alpha}{\alpha-1} - 1 + \frac{2}{\alpha-1} > d-1 \quad \iff \quad 2\frac{\alpha+1}{\alpha-1} > d \quad \iff \quad \alpha < \frac{d+2}{d-2}.$$

Since we assumed w has infinite support, the Pohozaev integral holds for $r \in [0, \infty)$. Taking limit $r \to \infty$ in the Pohozaev integral then gives

$$2\pi \frac{\gamma - 1}{\gamma} \left(\frac{2d}{1+\alpha} - (d-2) \right) \int_0^\infty w^{\alpha+1} y^{d-1} \mathrm{d}y = 0.$$

But the LHS must be strictly positive, this is a contraction. Hence w must be compactly supported.

□

Proposition B.1.5. *When $\gamma = 2d/(d+2)$ we have explicit Lane–Emden steady state solution*

$$w(r) = A \left(1 + \frac{2\pi}{d^2} A^{\frac{4}{d-2}} r^2 \right)^{1-d/2}$$

or equivalently

$$\rho(r) = C \left(1 + \frac{2\pi}{d^2} C^{\frac{4}{d+2}} r^2 \right)^{-1-d/2}.$$

And the support of the liquid star is

$$R = \left(\frac{d^2}{2\pi} C^{-\frac{4}{d+2}} (C^{\frac{2}{d+2}} - 1) \right)^{\frac{1}{2}}.$$

Proof. Consider

$$\begin{aligned} w(r) &= A(1+Br^b)^a \\ w'(r) &= ABabr^{b-1}(1+Br^b)^{a-1} \\ w''(r) &= ABab(b-1)r^{b-2}(1+Br^b)^{a-1} + AB^2ab^2(a-1)r^{2(b-1)}(1+Br^b)^{a-2} \end{aligned}$$

Substitute into the Lane–Emden ODE (Definition 1.2.1)

$$w'' + (d-1)r^{-1}w' = -4\pi \frac{\gamma-1}{\gamma} w^\alpha$$

we get

$$\begin{aligned} &-4\pi \frac{\gamma-1}{\gamma} A^\alpha (1+Br^b)^{\alpha a} \\ &= ABab(b-1+d-1)r^{b-2}(1+Br^b)^{a-1} + AB^2ab^2(a-1)r^{2(b-1)}(1+Br^b)^{a-2} \\ &= ABab(1+Br^b)^{a-2} \\ &\qquad \big((b-1+d-1)r^{b-2} + B(b(a-1) + (b-1+d-1))r^{2(b-1)} \big) \end{aligned}$$

For this to have the possibility to hold, we need $b = 2$ and $a = 1 - d/2$. Then the equation becomes

$$-4\pi \frac{\gamma - 1}{\gamma} A^{\alpha}(1 + Br^2)^{\alpha(1-d/2)} = ABd(2-d)(1 + Br^2)^{-(1+d/2)}$$

For this to have the possibility to hold, we need

$$\frac{1}{\gamma - 1} = \alpha = -\frac{2+d}{2-d} = \frac{d+2}{d-2} \qquad \iff \qquad \gamma = \frac{d-2}{d+2} + 1 = \frac{2d}{d+2}.$$

And then we also need

$$\begin{aligned} -4\pi \frac{d-2}{2d} A^{\frac{d+2}{d-2}} = ABd(2-d) \qquad &\iff \qquad 2\pi A^{\frac{4}{d-2}} = Bd^2 \\ &\iff \qquad B = \frac{2\pi}{d^2} A^{\frac{4}{d-2}}. \end{aligned}$$

So when $\gamma = 2d/(d+2)$ we have explicit steady state solution

$$w(r) = A\left(1 + \frac{2\pi}{d^2} A^{\frac{4}{d-2}} r^2\right)^{1-d/2}$$

or equivalently

$$\rho(r) = C\left(1 + \frac{2\pi}{d^2} C^{\frac{4}{d+2}} r^2\right)^{-1-d/2}.$$

The support R of the liquid star is then given by

$$\begin{aligned} 1 = C\left(1 + \frac{2\pi}{d^2} C^{\frac{4}{d+2}} R^2\right)^{-1-d/2} \qquad &\iff \qquad 1 + \frac{2\pi}{d^2} C^{\frac{4}{d+2}} R^2 = C^{\frac{2}{d+2}} \\ &\iff \qquad R = \left(\frac{d^2}{2\pi} C^{-\frac{4}{d+2}} (C^{\frac{2}{d+2}} - 1)\right)^{\frac{1}{2}}. \end{aligned}$$

□

Proposition B.1.6 (Self-similarity of solutions)**.** *Let ρ be a gaseous steady state. Then $\rho_\kappa(r) = \kappa\rho(\kappa^{1-\gamma/2} r)$ is a gaseous steady state for any $\kappa > 0$, and the corresponding liquid star has support $R = \kappa^{-(1-\gamma/2)}\rho^{-1}(1/\kappa)$.*

Proof. First we deal with the $\gamma > 1$ case. Recall the Lane–Emden ODE (Definition 1.2.1)

$$w''(r) + (d-1)r^{-1}w'(r) = -4\pi \frac{\gamma - 1}{\gamma} w(r)^{\alpha}.$$

Let $v(r) = \kappa w(\kappa^\beta r)$. Then $v'(r) = \kappa^{\beta+1} w'(\kappa^\beta r)$ and $v''(r) = \kappa^{2\beta+1} w''(\kappa^\beta r)$. So

$$\begin{aligned} v''(r) + (d-1)r^{-1}v'(r) &= \kappa^{2\beta+1} w''(\kappa^\beta r) + (d-1)(\kappa^\beta r)^{-1}\kappa^{2\beta+1} w'(\kappa^\beta r) \\ &= -\kappa^{2\beta+1} 4\pi \frac{\gamma-1}{\gamma} w(\kappa^\beta r)^\alpha = -\kappa^{2\beta+1-\alpha} 4\pi \frac{\gamma-1}{\gamma} v(r)^\alpha \end{aligned}$$

So if we choose

$$\beta = \frac{1}{2}(\alpha - 1) = \frac{1}{2}\frac{2-\gamma}{\gamma-1},$$

then v is again a solution to the Lane–Emden ODE (Definition 1.2.1). Converting back to ρ gives the desired result.

For $\gamma = 1$, it is clear by substitution that $\rho_\kappa(r) = \kappa\rho(\sqrt{\kappa} r)$ is a solution to the Lane–Emden ODE (Definition 1.2.1)

$$0 = \Delta(\ln \rho) + 4\pi\rho = \frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho^2} + (d-1)\frac{1}{r}\frac{\rho'}{\rho} + 4\pi\rho.$$

□

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