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Eisenstein cohomology classes for GL_N over imaginary quadratic fields

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Abstract. We study the arithmetic of degree $N-1$ Eisenstein cohomology classes for the locally symmetric spaces attached to GL_N over an imaginary quadratic field k. Under natural conditions we evaluate these classes on $(N-1)$-cycles associated to degree N extensions L/k as linear combinations of generalized Dedekind sums. As a consequence we prove a remarkable conjecture of Sczech and Colmez expressing critical values of L-functions attached to Hecke characters of L as polynomials in Kronecker–Eisenstein series evaluated at torsion points on elliptic curves with complex multiplication by k. We recover in particular the algebraicity of these critical values.

1. Introduction

The relationship between Eisenstein series, the cohomology of arithmetic groups and special values of L-functions has been studied extensively.

A classical example is that of weight 2 Eisenstein series attached to $(\alpha,\beta)\in(\mathbb{Q}/\mathbb{Z})^2$. Such a series can be defined as limits of finite sums:

$$E_{2,(\alpha,\beta)}(\tau)=\lim_{M\to+\infty}\sum_{m=-M}^{M}\left(\lim_{N\to+\infty}{\sum_{n=-N}^{N}}'\frac{e^{2i\pi(m\alpha+n\beta)}}{(m\tau+n)^2}\right)\quad(\tau\in\mathcal{H}).$$

The prime on the sum means that we exclude the term $(m,n)=(0,0)$. When $(\alpha,\beta)\neq(0,0)$, the holomorphic 1-form $E_{2,(\alpha,\beta)}(\tau)d\tau$ on Poincaré's upper half-plane $\mathcal{H}$ is invariant under any subgroup $\Gamma\subset\mathrm{SL}_2(\mathbb{Z})$ that fixes (α,β) modulo $\mathbb{Z}^2$. This holomorphic form then represents a cohomology class in $H^1(\Gamma,\mathbb{C})$. A remarkable feature of these classes is that they are rational and even almost integral.

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A convenient and compact way to state the precise integrality properties of these cohomology classes is to consider for each prime integer the "p-smoothed Eisenstein series"

$$\begin{aligned} E^{(p)}_{2,(\alpha,\beta)}(\tau) &= \sum_{j=1}^{p} E_{2,(\alpha+j/p,\beta)}(\tau) - pE_{2,(\alpha,\beta)}(\tau) \\ &= p(E_{2,(p\alpha,\beta)}(p\tau) - E_{2,(\alpha,\beta)}(\tau)), \end{aligned}$$

and suppose furthermore that $\Gamma \subset \Gamma_0(p)$. Then $E^{(p)}_{2,(\alpha,\beta)}(\tau)$ yields a homomorphism

$$\Phi^{(p)}_{(\alpha,\beta)} : \Gamma \to \mathbb{C}$$

by the rule

$$\Phi^{(p)}_{(\alpha,\beta)}(\gamma) = \int_{\tau_0}^{\gamma\tau_0} E^{(p)}_{2,(\alpha,\beta)}(\tau)\, d\tau,$$

for any base point $\tau_0 \in \mathcal{H}$, and it is classical (see e.g. [36, Theorem 13]) that we have

$$\Phi^{(p)}_{(\alpha,\beta)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} 0 & \text{if } c = 0, \\ (2\pi)^2 \cdot \mathrm{sign}(c) \cdot \big(D_{p\alpha,\beta}\big(\frac{pa}{|c|}\big) - pD_{\alpha,\beta}\big(\frac{a}{|c|}\big)\big) & \text{otherwise.} \end{cases} \tag{1.1}$$

Here $D_{\alpha,\beta}$ denotes the *generalized Dedekind sum*

$$D_{\alpha,\beta}\left(\frac{a}{c}\right) = \sum_{j=1}^{c} \left(\left(\frac{j-\beta}{c}\right)\right)\left(\left(\frac{a(j-\beta)}{c} - \alpha\right)\right) \quad \text{for } c > 0 \text{ and } (a,c) = 1,$$

where the symbol $((x))$ is defined by

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

These sums define rational numbers and enjoy many beautiful arithmetical properties, see e.g. [29]. On the other hand a formula of Siegel [36] expresses the values at non-positive integers of the ζ-functions attached to real quadratic fields as periods of Eisenstein series. The expression (1.1) can therefore be turned into a very explicit expression for these special values. This implies in particular that they are essentially integral which is the key input in the construction by Coates and Sinnott [10] of p-adic L-functions over real quadratic fields.

Using Selberg's and Langlands' theory of Eisenstein series Harder has vastly generalized the above mentioned "Eisenstein cohomology classes." In [18] he constructed a complement to the cuspidal cohomology for the group GL_2 over number fields and managed to construct rational representatives. In a more recent work Harder even managed to address some integrality properties of these classes, see [20]. However for the group GL_N it is hard to check that Eisenstein classes are rational and the automorphic form theory is not yet adapted to the study of integrality properties of these classes.

For GL_N over the field of rational numbers, Nori [27] and Sczech [32] have proposed constructions of Eisenstein cohomology classes that have turned out to be very efficient in practice to study the fine arithmetical properties of L-functions over totally real number fields, see e.g. [2, 7, 8, 17]. Sczech's approach more generally gives formulas analogous to (1.1). The goal of this paper is to prove similar formulas for the group GL_N over an imaginary quadratic field k. As a consequence we prove a remarkable conjecture of Sczech and Colmez [11, Conjecture, p. 205] expressing critical values of L-functions attached to Hecke characters of finite extension of k as polynomials in Kronecker–Eisenstein series evaluated at torsion points on elliptic curves with complex multiplication by k.

We now describe in more details our main results.

1.1. An Eisenstein cocycle for imaginary quadratic fields. Fix a positive integer $N \geq 2$. Let k be a quadratic imaginary field with ring of integers $\mathcal{O}$ and let $\mathfrak{p} \subset \mathcal{O}$ be an ideal of prime norm $\mathrm{N}\mathfrak{p}$. Our first main result is the construction of an $(N-1)$-cocycle for the level $\mathfrak{p}$ congruence subgroup

$$\Gamma_0(\mathfrak{p}) = \left\{ \begin{pmatrix} a & {}^t b \\ c & D \end{pmatrix} \in \mathrm{SL}_N(\mathcal{O}) : a \in \mathcal{O},\ b \in \mathcal{O}^{N-1},\ c \in \mathfrak{p}^{N-1},\ D \in M_{N-1}(\mathcal{O}) \right\}$$

of $\mathrm{SL}_N(\mathcal{O})$ taking values in the space of polynomials in certain classical series called Kronecker–Eisenstein series.

Let us recall the definition of Kronecker–Eisenstein series (see [37, VIII, Section 12]). We fix once and for all an embedding $\sigma : k \to \mathbb{C}$. For a fractional ideal $\mathfrak{I}$ of k and non-negative integers p and q, define

$$K^{p,q}(z, \mathfrak{I}, s) = p! \sum_{\lambda \in \sigma(\mathfrak{I})} \frac{\overline{z+\lambda}^{\,q}}{(z+\lambda)^{p+1}|z+\lambda|^s},$$

where we assume that $z \in \mathbb{C}$ satisfies $z \notin \sigma(\mathfrak{I})$. The series converges when $\mathrm{Re}(s) > 1 + q - p$ and has analytic continuation to $s \in \mathbb{C}$ that is regular at $s = 0$. It is a classical result due to Damerell [13] that the values at $s = 0$ have the following algebraicity property:

$$K^{p,q}(z_0, \mathfrak{I}, 0) \in \Omega_\infty^{1+p+q} \pi^{-q} \overline{\mathbb{Q}} \quad \text{for } z_0 \in k \backslash \sigma(\mathfrak{I}). \tag{1.2}$$

Here Ω_∞ denotes any period of a $\overline{\mathbb{Q}}$-rational holomorphic differential against a non-zero rational homology class on an elliptic curve with CM by k defined over $\overline{\mathbb{Q}}$. In fact, these series have almost integral values; we refer to [24] for precise results.

We introduce polynomials in the series $K^{p,q}(z, \mathfrak{I}, s)$. For fractional ideals $\mathfrak{I}_1, \ldots, \mathfrak{I}_N$ and multi-indices $I = (i_1, \ldots, i_N) \in \mathbb{Z}_{\geq 0}^N$ and $J = (j_1, \ldots, j_N) \in \mathbb{Z}_{\geq 0}^N$, we set

$$K^{I,J}(z, \mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N, s) = K^{i_1,j_1}(z_1, \mathfrak{I}_1, s) \cdots K^{i_N,j_N}(z_N, \mathfrak{I}_N, s).$$

More generally, for an $\mathcal{O}$-lattice $\Lambda \subset k^N$, we pick fractional ideals $\mathfrak{I}_1, \ldots, \mathfrak{I}_N$ such that $\mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N$ has finite index in Λ and set

$$\begin{aligned} K^{I,J}(z, \Lambda, s) &= \sum_{\lambda \in \Lambda / \mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N} K^{I,J}(z + \sigma(\lambda), \mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N, s) \\ &= \sum_{\lambda \in \Lambda} \prod_{1 \leq k \leq N} i_k! \frac{\overline{z_k + \sigma(\lambda_k)}^{\,j_k}}{(z_k + \sigma(\lambda_k))^{i_k+1}|z_k + \sigma(\lambda_k)|^s}. \end{aligned}$$

As the last expression shows, $K^{I,J}(z, \Lambda, s)$ does not depend on the choice of the fractional ideals $\mathfrak{I}_k$. We set

$$K^{I,J}(z, \Lambda) := K^{I,J}(z, \Lambda, 0).$$

Each $K^{I,J}(z,\Lambda)$ defines a smooth function on an open subset of $\mathbb{C}^N$ obtained by removing all Λ-translates of a finite number of hyperplanes. We write

$$\mathcal{F} = \langle K^{I,J}(\gamma z,\Lambda) : \gamma \in \mathrm{SL}_N(k),\ \Lambda \subset k^N \text{ an } \mathcal{O}\text{-lattice}\rangle$$

for the $\mathbb{C}$-span of $\mathrm{SL}_N(k)$-translates of all functions $K^{I,J}(z,\Lambda)$.

Next we introduce the $\mathfrak{p}$-smoothed series

$$K_{\mathfrak{p}}^{I,J}(z,\mathcal{O}^N) = K^{I,J}(z,\mathfrak{p}^{-1}\oplus\mathcal{O}^{N-1}) - \mathrm{N}\mathfrak{p}\cdot K^{I,J}(z,\mathcal{O}^N)$$

and, for $A \in M_N(\mathcal{O})$, we define the generalized Dedekind sum $D_{\mathfrak{p}}^{I,J}(z,A)$ by

$$D_{\mathfrak{p}}^{I,J}(z,A) = \det A^{-1} K_{\mathfrak{p}}^{I,J}(A^{-1}z, A^{-1}\mathcal{O}^N)$$

if A is invertible and set $D_{\mathfrak{p}}^{I,J}(z,A) = 0$ otherwise. These sums are natural generalizations of Dedekind sums for imaginary quadratic fields and N variables.

Our first theorem shows that the series $D_{\mathfrak{p}}^{I,J}(z,A)$ can be combined into a homogeneous $(N-1)$-cocycle for $\Gamma_0(\mathfrak{p})$. In the following statement, for a multi-index $I \in \mathbb{Z}_{\geq 0}^N$, we write $|I| = i_1 + \cdots + i_N$. When I (resp. J) runs over multi-indices with $|I| = p$ (resp. $|J| = q$), the vectors

$$e^I := e_1^{i_1}\cdots e_N^{i_N} \in \mathrm{Sym}^p\mathbb{C}^N \quad (\text{resp. } \bar{e}^J := \overline{e_1}^{j_1}\cdots\overline{e_N}^{j_N} \in \overline{\mathrm{Sym}^q\mathbb{C}^N})$$

form a basis of $\mathrm{Sym}^p\mathbb{C}^N$ (resp. of $\overline{\mathrm{Sym}^q\mathbb{C}^N}$).

Theorem 1.1. *Given $\gamma_1,\ldots,\gamma_N \in \Gamma_0(\mathfrak{p})$, define*

$$A(\underline{\gamma}) = (\gamma_1 e_1|\cdots|\gamma_N e_1) \in M_N(\mathcal{O}).$$

The map

$$\mathbf{\Phi}_{\mathfrak{p}}^{p,q} : \Gamma_0(\mathfrak{p})^N \to \mathcal{F}\otimes\mathrm{Sym}^p\mathbb{C}^N\otimes\overline{\mathrm{Sym}^q\mathbb{C}^N}$$

given by

$$\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(z,\underline{\gamma}) = \sum_{|I|=p,|J|=q} D_{\mathfrak{p}}^{I,J}(z,A(\underline{\gamma}))\otimes A(\underline{\gamma})(e^I\otimes\bar{e}^J)$$

is a homogeneous $(N-1)$-cocycle. Here the sum runs over all multi-indices $I, J \in \mathbb{Z}_{\geq 0}^N$ with $|I| = p$ and $|J| = q$.

More concretely, the cocycle property of $\mathbf{\Phi}_{\mathfrak{p}}^{p,q}$ means that

$$\begin{aligned}\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\gamma z,\gamma\gamma_1,\ldots,\gamma\gamma_N) &= \gamma\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\gamma z,\gamma_1,\ldots,\gamma_N)\\ &= \sum_{|I|=p,\ |J|=q} D_{\mathfrak{p}}^{I,J}(z,A(\underline{\gamma}))\otimes\gamma A(\underline{\gamma})(e^I\otimes\bar{e}^J)\end{aligned}$$

for any $\gamma,\gamma_1,\ldots,\gamma_N \in \Gamma_0(\mathfrak{p})$, and

$$\sum_{1\leq k\leq N+1}(-1)^{k-1}\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(z,\gamma_1,\ldots,\widehat{\gamma_k},\ldots,\gamma_{N+1}) = 0$$

for any $\gamma_1,\ldots,\gamma_{N+1} \in \Gamma_0(\mathfrak{p})$ (here as usual the notation $\widehat{\gamma_k}$ means that the term γ_k is to be omitted).

More generally, in the body of the paper we introduce a cocycle $\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(z,\underline{\gamma},\Lambda(\mathfrak{I}))$ for the $\mathcal{O}$-lattice $\Lambda(\mathfrak{I}) = \mathfrak{I}^{-1}\oplus\mathcal{O}^{N-1}$ of k^N.

1.2. Application to critical values of Hecke L-functions. We refer to [15, Section 1] or [9, 30, 34] for generalities on Hecke characters. Let L/k be a field extension of degree $N > 1$ and let $n : L \to k$ denote the norm map. We fix an algebraic Hecke character ψ_k of k of infinity type $(p, q) \in \mathbb{Z}^2$ and a Dirichlet character χ of L, and consider the algebraic Hecke character

$$\phi = \chi \cdot (\psi_k \circ n) \tag{1.3}$$

of L. We denote the conductor of ϕ by $\mathfrak{f}$, so that for $\alpha \equiv 1 \bmod \mathfrak{f}$ we have

$$\phi((\alpha)) = n(\alpha)^p \overline{n(\alpha)}^q.$$

Note that if k is a maximal CM field in L, then any algebraic Hecke character ϕ of L is of the above form.

The Hecke L-function of ϕ is

$$L(\phi, s) = \prod_{(\mathfrak{P},\mathfrak{f})=1} (1 - \phi(\mathfrak{P})\mathrm{N}\mathfrak{P}^{-s})^{-1} = \sum_{(\mathfrak{a},\mathfrak{f})=1} \phi(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-s},$$

where the sum, resp. the product, runs over integral ideals $\mathfrak{a}$, resp. prime ideals $\mathfrak{P}$, of $\mathcal{O}_L$ coprime to $\mathfrak{f}$. The global L-function of ϕ is

$$\Lambda(\phi, s) = L_\infty(\phi, s) L(\phi, s),$$

where

$$L_\infty(\phi, s) = \prod_{v|\infty} \Gamma(\phi_v, s).$$

Here each ϕ_v with $v|\infty$ is of the form

$$z^p \bar{z}^q = (z\bar{z})^{\frac{w}{2}} \left(\frac{z}{\bar{z}}\right)^{\frac{p-q}{2}},$$

with $w = p + q$ (the weight), and

$$\Gamma(\phi_v, s) = 2(2\pi)^{-(s - \frac{w}{2} + \frac{|p-q|}{2})} \Gamma\left(s - \frac{w}{2} + \frac{|p-q|}{2}\right).$$

The value $L(\phi, s_0)$ at an integer $s_0 \in \mathbb{Z}$ is said to be critical if and only if

$$\mathrm{ord}_{s=s_0} L_\infty(\phi, s) = \mathrm{ord}_{s=s_0} L_\infty(\phi^{-1}, 1 - s) = 0.$$

In our case this is equivalent to

$$\frac{w}{2} - \frac{|p-q|}{2} < s_0 < 1 + \frac{w}{2} + \frac{|p-q|}{2}.$$

Our second main result is that for critical s_0 the value $L(\phi, s_0)$ can be expressed as an explicit polynomial in Kronecker–Eisenstein series; answering positively a conjecture of Sczech and Colmez [11, Conjecture, p. 205]. Note that the complex conjugate $\overline{\psi_k}$ has weight (q, p) and that multiplying ϕ by an integral power of the norm character shifts s by an integer. Thus we may assume that $p < 0$ and $q \geq 0$ and consider only the critical value $L(\phi, 0)$.

Our result is more conveniently expressed in terms of partial zeta functions, as follows. For integers p, q and integral ideals $\mathfrak{a}, \mathfrak{f}$ of $\mathcal{O}_L$, define

$$\zeta_{\mathfrak{f}}^{p,q}(\mathfrak{a}, s) = \sum_{x \in U(\mathfrak{f}) \backslash 1 + \mathfrak{f}\mathfrak{a}^{-1}}{}' \frac{\overline{n(x)}^q}{n(x)^{p+1} |n(x)|^{2s}}, \quad \mathrm{Re}(s) \gg 0.$$

Here $U(\mathfrak{f})$ denotes the group of units of $\mathcal{O}_L^\times$ that are congruent to 1 modulo $\mathfrak{f}$. (Since $u\bar{u} = 1$ for every $u \in \mathcal{O}^\times$, this is well-defined provided that $p + q + 1$ is divisible by the order of the subgroup $n(U(\mathfrak{f}))$ of $\mathcal{O}^\times$, which we assume.) Choosing integral ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ giving a system of representatives for the ray class group $C_{\mathfrak{f}}$, we can write

$$L(\phi, s) = \sum_j \phi(\mathfrak{a}_j) \mathrm{N}\mathfrak{a}_j^{-s} \zeta_{\mathfrak{f}}^{-p-1,q}(\mathfrak{a}_j, s).$$

Given two distinct prime ideals $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$ of $\mathcal{O}_L$ coprime to $\mathfrak{f}$ and $\mathfrak{a}$, we define also the "smoothed" partial zeta functions

$$\zeta_{\mathfrak{f},\mathfrak{P}}^{p,q}(\mathfrak{a}, s) = \mathrm{N}\mathfrak{P}^{-s} \zeta_{\mathfrak{f}}^{p,q}(\mathfrak{a}\mathfrak{P}, s) - \mathrm{N}\mathfrak{P}^{1-s} \zeta_{\mathfrak{f}}^{p,q}(\mathfrak{a}, s),$$
$$\zeta_{\mathfrak{f},\mathfrak{P},\widetilde{\mathfrak{P}}}^{p,q}(\mathfrak{a}, s) = \mathrm{N}\widetilde{\mathfrak{P}}^{-s} \zeta_{\mathfrak{f},\mathfrak{P}}^{p,q}(\mathfrak{a}\widetilde{\mathfrak{P}}, s) - \mathrm{N}\widetilde{\mathfrak{P}}^{-s} \zeta_{\mathfrak{f},\mathfrak{P}}^{p,q}(\mathfrak{a}, s).$$

These modified zeta functions appear in an expression for $L(\phi, s)$ with modified Euler factors at $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$. Namely, setting

$$L_{\mathfrak{P}}(\phi \cdot \mathrm{N}, s) = (1 - \phi(\mathfrak{P})\mathrm{N}\mathfrak{P}^{1-s})^{-1}, \quad L_{\widetilde{\mathfrak{P}}}(\phi, s) = (1 - \phi(\widetilde{\mathfrak{P}})\mathrm{N}\widetilde{\mathfrak{P}}^{-s})^{-1},$$

and using the fact that $\mathfrak{a}_1\mathfrak{P}, \dots, \mathfrak{a}_r\mathfrak{P}$ is also a system of representatives of $C_{\mathfrak{f}}$, we have

$$L_{\mathfrak{P}}(\phi \cdot \mathrm{N}, s)^{-1} L(\phi, s) = \sum_j \phi(\mathfrak{a}_j\mathfrak{P}) \mathrm{N}\mathfrak{a}_j^{-s} \zeta_{\mathfrak{f},\mathfrak{P}}^{-p-1,q}(\mathfrak{a}_j, s),$$
$$L_{\widetilde{\mathfrak{P}}}(\phi, s)^{-1} L_{\mathfrak{P}}(\phi \cdot \mathrm{N}, s)^{-1} L(\phi, s) = \sum_j \phi(\mathfrak{a}_j\mathfrak{P}\widetilde{\mathfrak{P}}) \mathrm{N}\mathfrak{a}_j^{-s} \zeta_{\mathfrak{f},\mathfrak{P},\widetilde{\mathfrak{P}}}^{-p-1,q}(\mathfrak{a}_j, s),$$

Theorem 1.2 below shows that, for appropriate choices of $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$, the zeta function $\zeta_{\mathfrak{f},\mathfrak{P},\widetilde{\mathfrak{P}}}^{-p-1,q}(\mathfrak{a}_j, s)$ can be expressed using the Eisenstein cocycle of Theorem 1.1.

Let $U(\mathfrak{f})^1 \subseteq U(\mathfrak{f})$ be the subgroup of units of relative norm one, and let $U(\mathfrak{f})' \subseteq U(\mathfrak{f})^1$ be a torsion–free subgroup that maps bijectively to $U(\mathfrak{f})^1/U(\mathfrak{f})^1_{\mathrm{tors}}$. We also fix a isomorphism $\alpha : L \to k^N$ of k-vector spaces and denote the $\mathcal{O}$-lattice $\alpha(\mathfrak{f}\mathfrak{a}^{-1}) \subset k^N$ by $\Lambda(\mathfrak{f}\mathfrak{a}^{-1})$.[1)] Through the isomorphism α the automorphism of L defined by multiplication by u_i corresponds to a matrix U_i that belongs to the intersection $\Gamma(\Lambda(\mathfrak{f}\mathfrak{a}^{-1}))$ of $\mathrm{Aut}_{\mathcal{O}}(\Lambda(\mathfrak{f}\mathfrak{a}^{-1}))$ with $\mathrm{SL}_N(k)$. Moreover, given a prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$ coprime to $\mathfrak{f}$ and $\mathfrak{a}$ and of prime norm $\mathfrak{p} = n(\mathfrak{P})$, the matrices U_i belong to $\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{f}\mathfrak{a}^{-1}))$.

We denote by $\sigma_1, \dots, \sigma_N$ the embeddings of L into $\mathbb{C}$ that restrict to the fixed embedding $\sigma : k \to \mathbb{C}$.

Theorem 1.2. *Let p, q be non-negative integers, $\mathfrak{f}$ be an ideal of $\mathcal{O}_L$ and let $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ be integral ideals that form a system of representatives of the ray class group $C_{\mathfrak{f}}$. Then there*

1) In Lemma 4.1 we prove that $\Lambda(\mathfrak{f}\mathfrak{a}^{-1})$ is of type $\Lambda(\mathfrak{J})$.

exist $v_0 \in k^N$ and two prime ideals $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$ with $\mathfrak{p} = n(\mathfrak{P})$ prime, such that

$$[U(\mathfrak{f}) : U(\mathfrak{f})'] \det(\sigma_i(\alpha(e_j)))\zeta^{p,q}_{\mathfrak{f},\mathfrak{P},\widetilde{\mathfrak{P}}}(\mathfrak{a}, 0)$$
$$= \frac{1}{(p!)^N} \sum_{\sigma \in S_{N-1}} \mathrm{sgn}(\sigma) \cdot \sum_{\substack{x \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f} \\ x \neq 0}} \langle \mathbf{\Phi}^{pN,qN}_{\mathfrak{p}}(v_0 + \alpha(x), \underline{u}_\sigma, \Lambda(\mathfrak{f}\mathfrak{a}^{-1})), (n \circ \alpha^{-1})^p \otimes \overline{(n \circ \alpha^{-1})}^q \rangle$$

for $\mathfrak{a} \in \{\mathfrak{a}_1, \dots, \mathfrak{a}_r\}$. Here the first sum runs over the symmetric group S_{N-1} of permutations on $N-1$ letters and for $\sigma \in S_{N-1}$ we set $\underline{u}_\sigma = (1, U_{\sigma(1)}, U_{\sigma(1)}U_{\sigma(2)}, \dots, U_{\sigma(1)} \cdots U_{\sigma(N-1)})$.

Note that each term of the double sum on the right hand side is a generalized Dedekind sum and therefore a polynomial in Kronecker–Eisenstein series evaluated at torsion points on elliptic curves with complex multiplication by k. From (1.2) we deduce the following corollary. Note that $\phi(\mathfrak{a}_j) \in \overline{\mathbb{Q}}$, so that the algebraicity of $L(\phi, 0)$ follows from that of the $\zeta^{-p-1,q}_{\mathfrak{f}}(\mathfrak{a}_j, 0)$.

Corollary 1.3. *Let Ω_∞ be any non–zero period of a $\overline{\mathbb{Q}}$-rational global differential on an elliptic curve with complex multiplication by k, defined over $\overline{\mathbb{Q}}$. Let ϕ be an algebraic Hecke character of the form* (1.3)*. Assume that $p < 0$ and $q \geq 0$. Then*

$$L(\phi, 0) \in \Omega_\infty^{N(q-p)} \pi^{-Nq} \overline{\mathbb{Q}}.$$

Remark. As was pointed out to us by Don Blasius, one can take Ω_∞ to be a period of a holomorphic differential on an elliptic curve defined over k_{ab} – the maximal abelian extension of k. Then

$$L(\phi, 0) \in \Omega_\infty^{N(q-p)} \pi^{-Nq} k_{ab} E,$$

where E is the CM field generated by the values of ϕ. This follows from the fact that the ratio of two arithmetic automorphic functions with Fourier coefficients in $\mathbb{Q}_{ab}$, and having the same weight, takes value in k_{ab} when evaluated at a CM point.

In fact, one can be more precise: Blasius [5] proves a reciprocity law for values at CM points of modular forms which generalizes that of Shimura for functions. According to it, if a value transforms by a Hecke character, then it is the Deligne period of the motive attached to the Hecke character. Since Theorem 1.2 expresses the $L(\phi, 0)$ as a linear combination of products of values of L-functions of modular Eisenstein series, the general law of Blasius should apply to show the following: Let $M(\phi)$ be the motive – defined over L, with coefficients in E, and of rank one – attached to ϕ and let $c^+\mathrm{Res}_{L/\mathbb{Q}} M(\phi)$ be the period attached by Deligne [14, Section 8], we have

$$L(\mathrm{Res}_{L/\mathbb{Q}} M(\phi)) = c^+\mathrm{Res}_{L/\mathbb{Q}} M(\phi) \in (E \otimes \mathbb{C})^\times / E^\times$$

as conjectured by Deligne [14] as part of a much more general picture.

Relation to other works. In the case $N = 2$ Theorem 1.1 is proved by Sczech [31] and Ito [22], in case $(p, q) = (0, 0)$, and Obaisi [28] proved the corresponding Theorem 1.2. In general, partial results towards both Theorem 1.2 and Corollary 1.3 are obtained by Colmez in [11]; see also [16] for related works.

Corollary 1.3 is not new. In the case $L = k$ it is due to Damerell [13]. In the case $N = 2$ it is due to Ito [22]. In general, it is a particular case of a theorem announced by Harder in [19,21] that deals with Hecke L-functions associated to extensions L/k with k an arbitrary CM fields. When $L = k$ is CM, this was known before thanks to works of Shimura [35] and Blasius [5]. Harder provided a proof of his theorem for $N = 2$ in [18], but to the authors' knowledge the full details of Harder's proof for $N > 2$ have never appeared in print. However, the fact that the (regularized) L-value of the Hecke character divided by the "Katz period" is algebraic has recently been fully proved by Kings and Sprang [25] using completely different techniques that allow them to deduce good integrality results. This generalizes works of Shimura and Katz in the case of a CM field to arbitrary extensions of CM fields. In the case where k is a quadratic imaginary field, integrality results of the same quality could be deduced from Theorem 1.2 and works of Katz [24] showing that certain regularization ("smoothing") of the expression (1.2) are algebraic integers. We expect that, in combination with recent work of Andreatta and Iovita [1], the explicit formula of Theorem 1.2, conjectured by Sczech and Colmez, could be used to p-adically interpolate L-values of algebraic Hecke characters of F in the non-split case.

Note that, quite similarly as in the work of Kings and Sprang, the cohomology class studied in this paper takes its roots in a certain equivariant cohomology class; we discuss the latter in [4]. The topological origin of this class is enough to give an elementary direct proof of the integrality of critical values of Hecke L-functions associated to totally real fields, see [2,3,27].

To conclude let us mention that it is not clear to us if the formula of Theorem 1.2 can be generalized to the case where k is an arbitrary CM field.

1.3. Notation and conventions. We write $|S|$ for the cardinality of a set S. Throughout the paper we fix an integer $N \geq 2$ and let

$$V = \mathbb{C}^N \quad \text{(column vectors)}.$$

We write $\overline{V} = V \otimes_{\mathbb{C}} \overline{\mathbb{C}}$ for the complex conjugate of V and $V^\vee$ for the ($\mathbb{C}$-linear) dual of V; we identify $V^\vee$ with the space of length N row vectors using the standard dot product. We write $e_1, \dots, e_N$ for the standard basis of V and $z_1, \dots, z_N$ for the standard coordinates on V and set $\partial_{z_i} = \partial/\partial z_i$. For a multi-index $I = (i_1, \dots, i_N) \in \mathbb{Z}^N_{\geq 0}$, we write

$$\begin{aligned} e^I &= e_1^{i_1} \cdots e_N^{i_N} \in \mathrm{Sym}^N V, \\ z^I &= z_1^{i_1} \cdots z_N^{i_N} \in \mathrm{Sym}^N V^\vee, \\ \overline{z}^I &= \overline{z_1}^{i_1} \cdots \overline{z_N}^{i_N} \in \mathrm{Sym}^N \overline{V}^\vee. \end{aligned}$$

We denote the transpose of a matrix X by ${}^tX$ and set $X^* = {}^t\overline{X}$. We denote by 1_N the identity matrix of rank N and by $\mathrm{diag}(t_1, \dots, t_N)$ a diagonal matrix with diagonal entries $t_1, \dots, t_N$. Let

$$\begin{aligned} G &= \mathrm{SL}_N(\mathbb{C}), \\ K &= \mathrm{SU}(N), \\ X &= \mathrm{SL}_N(\mathbb{C})/\mathrm{SU}(N). \end{aligned}$$

The Lie algebras of G and K are denoted by $\mathfrak{g}$ and $\mathfrak{k}$ respectively.

We write $A^*(X)$ for the space of smooth differential forms on X.

Throughout the paper we fix an imaginary quadratic field k and an embedding $\sigma : k \to \mathbb{C}$. We write $\mathrm{N}\mathfrak{p}$ for the norm of a prime ideal $\mathfrak{p}$. We denote by $\mathcal{O}$ the ring of integers of k and define

$$\begin{aligned} V_k &= k^N \quad \text{(column vectors)},\\ G_k &= \mathrm{SL}_N(k),\\ \mathbf{G} &= \mathrm{Res}_{k/\mathbb{Q}}\mathrm{SL}_{N,k} \end{aligned}$$

The standard simplex (a simplicial set) is denoted by Δ_N, and its geometric realization by $|\Delta_N|$. We write $\Delta_k * \Delta_r$ for the join of two simplices.

2. Differential forms on the symmetric space of $\mathrm{SL}_N(\mathbb{C})$

Fix an integer $N \geq 2$ and let $V = \mathbb{C}^N$ (column vectors). We identify the points of the symmetric space

$$X := G/K$$

of $G = \mathrm{SL}_N(\mathbb{C})$ with positive definite hermitian N-by-N matrices h of unit determinant via the map

$$gK \mapsto h := {}^t\overline{g}^{-1} \cdot g^{-1};$$

under this identification the action of $g \in G$ on X by left multiplication corresponds to the action $g \cdot h := {}^t\overline{g}^{-1} h g^{-1}$. A matrix $h \in X$ defines a positive definite hermitian form on $\mathbb{C}^N$ given by $v \mapsto v^* h v$. The entries h_{ij} $(1 \leq i, j \leq N)$ of h define smooth functions $h_{ij} : X \to \mathbb{C}$.

We write $\mathcal{S}(V)$ for the space of Schwartz functions on V. For $p, q \geq 0$, let

$$V^{p,q} = \mathrm{Sym}^p V^\vee \otimes \mathrm{Sym}^q \overline{V}; \tag{2.1}$$

it is naturally a representation of G. We will identify elements of $V^{p,q}$ with linear functionals on the tensor product of the complex vector spaces $\mathrm{Sym}^p V$ (homogeneous holomorphic polynomials of degree p on $V^\vee$) and $\mathrm{Sym}^q \overline{V}^\vee$ (homogeneous anti-holomorphic polynomials of degree q on V).

The natural action of G on $\mathcal{S}(V)$ defined by $(g \cdot f)(v) = f(g^{-1}v)$ turns $\mathcal{S}(V) \otimes V^{p,q}$ into a smooth G-module. Let $A^*(X; \mathcal{S}(V) \otimes V^{p,q})$ be the space of differential forms on X valued in $\mathcal{S}(V) \otimes V^{p,q}$. This space carries an action of G given by

$$(g, \omega(x, Y)) \mapsto g \cdot \omega(g^{-1}x, g^{-1}Y), \quad x \in X,\ Y \in \wedge T_x X.$$

In this section we introduce G-invariant differential forms

$$\psi^{p,q} \in A^{N-1}(X; \mathcal{S}(V) \otimes V^{p,q})^G$$

valued in this G-module.

2.1. Polynomial forms. Fix a vector $v \in V$. We write $(hv)_1, \ldots, (hv)_N$ (respectively $(dhv)_1, \ldots, (dhv)_N$) for the components of the vector hv (respectively dhv):

$$\begin{aligned} (hv)_i &= \sum_{1 \leq j \leq N} h_{ij} v_j \in C^\infty(X),\\ (dhv)_i &= \sum_{1 \leq j \leq N} dh_{ij} v_j \in A^1(X). \end{aligned}$$

Define

$$p(v) = 2(-1)^{\frac{N(N-1)}{2}} \sum_{i \geq 1} (-1)^{i-1} (hv)_i (dhv)_N \wedge \cdots \wedge \widehat{(dhv)_i} \wedge \cdots \wedge (dhv)_1 \in A^{N-1}(X).$$

(Here, as usual, the term under the symbol $\widehat{\ }$ is to be omitted.) Note that, as a function of v, p is a (holomorphic) polynomial of degree N and so defines a form

$$p \in A^{N-1}(X; \mathbb{C}[V]).$$

The conjugate polynomial $\overline{p(v)}$ defines a form in $A^{N-1}(X; \mathbb{C}[\overline{V}])$. Since h is hermitian, we have

$$\overline{(hv)_i} = \sum_j \overline{h_{ij} v_j} = \sum_j \overline{v_j} h_{ji} = (v^* h)_i$$

and so we can write

$$\overline{p(v)} = 2(-1)^{\frac{N(N-1)}{2}} \sum_{i \geq 1} (-1)^{i-1} (v^*h)_i (v^* dh)_N \wedge \cdots \wedge \widehat{(v^* dh)_i} \wedge \cdots \wedge (v^* dh)_1. \tag{2.2}$$

Lemma 2.1. *The form $\overline{p}$ is G-invariant. That is, for $g \in G$ we have*

$$g^* \overline{p(gv)} = \overline{p(v)}, \quad v \in V.$$

Proof. Let us assume that $v \neq 0$ (the case $v = 0$ is obvious). The statement then follows from the fact that given a representation of a group G on an N-dimensional complex vector space W, and a basis $e_1, \ldots, e_N$ of W with dual basis $e_1^\vee, \ldots, e_N^\vee \in W^\vee$, the element $e_1 \otimes e_1^\vee + \cdots + e_N \otimes e_N^\vee$ of $W \otimes W^\vee$ is G-invariant.

Namely, consider the $\mathbb{C}$-vector space $W \subset \mathcal{C}^\infty(X)$ spanned by $(v^*h)_1, \ldots, (v^*h)_N$. For $g \in G$ we have

$$(gv)^*(g \cdot h) = v^{*t}\overline{g}({}^t\overline{g}^{-1} h g^{-1}) = v^* h g^{-1}.$$

This shows that W is naturally a representation of G that is isomorphic to the dual $V^\vee$ of V. The same statement (with same proof) holds for the $\mathbb{C}$-vector space $\widetilde{W} \subset A^1(X)$ spanned by $(v^*dh)_1, \ldots, (v^*dh)_N$.

Consider the map

$$W \otimes \wedge^{N-1} \widetilde{W} \to \wedge^N \widetilde{W}, \quad w \otimes \widetilde{w} = dw \wedge \widetilde{w}.$$

Here $\wedge^N \widetilde{W} \simeq \mathbb{C} \cdot (v^*dh)_N \wedge \cdots \wedge (v^*dh)_1$ is isomorphic to the trivial G-representation via the map $z \cdot (v^*dh)_N \wedge \cdots \wedge (v^*dh)_1 \mapsto z$. Thus we obtain a pairing

$$W \otimes \wedge^{N-1} \widetilde{W} \to \mathbb{C}.$$

A direct check shows that the basis

$$(-1)^{N-i} (v^*dh)_N \wedge \cdots \wedge \widehat{(v^*dh)_i} \wedge \cdots \wedge (v^*dh)_1 \quad (1 \leq i \leq N)$$

of $\wedge^{N-1} \widetilde{W}$ is dual to the basis $(v^*h)_i$ $(1 \leq i \leq N)$ of W, and the lemma follows. □

Lemma 2.2. *Let $v \neq 0$. Then the form*

$$(v^*hv)^{-N}\overline{p(v)} \in A^{N-1}(X)$$

is closed.

Proof. An equivalent statement is the equality

$$Nd(v^*hv) \wedge \overline{p(v)} = (v^*hv)d\,\overline{p(v)}. \tag{2.3}$$

Differentiating (2.2) we obtain

$$\begin{aligned} d\,\overline{p(v)} &= 2(-1)^{\frac{N(N-1)}{2}} d\Big(\sum_{i\geq 1}(-1)^{i-1}(v^*h)_i(v^*dh)_N \\ &\qquad\qquad \wedge\cdots\wedge\widehat{(v^*dh)_i}\wedge\cdots\wedge(v^*dh)_1\Big) \\ &= 2N(-1)^{\frac{(N+2)(N-1)}{2}}(v^*dh)_N\wedge\cdots\wedge(v^*dh)_1. \end{aligned} \tag{2.4}$$

On the other hand we have $d(v^*hv) = \sum_j (v^*dh)_j v_j$ and hence

$$\begin{aligned} Nd(v^*hv)\wedge\overline{p(v)} &= 2N(-1)^{\frac{N(N-1)}{2}}\Big(\sum_j (v^*dh)_j v_j\Big)\wedge\Big(\sum_{i\geq 1}(-1)^{i-1}(v^*h)_i(v^*dh)_N \\ &\qquad\qquad \wedge\cdots\wedge\widehat{(v^*dh)_i}\wedge\cdots\wedge(v^*dh)_1\Big) \\ &= 2N(-1)^{\frac{N(N-1)}{2}}\sum_j(-1)^{j-1}(v^*dh)_j v_j (v^*h)_j (v^*dh)_N \\ &\qquad\qquad \wedge\cdots\wedge\widehat{(v^*dh)_j}\wedge\cdots\wedge(v^*dh)_1 \\ &= 2N(-1)^{\frac{(N+2)(N-1)}{2}}\Big(\sum_j v_j(v^*h)_j\Big)(v^*dh)_N\wedge\cdots\wedge(v^*dh)_1 \\ &= (v^*hv)d\,\overline{p(v)} \end{aligned}$$

and the assertion follows. □

2.2. Schwartz forms. We can now define the forms $\psi^{p,q}$ mentioned in the introduction to this section. First we consider the case $p = q = 0$: for $v \in V$, we define

$$\psi^{0,0}(v) = e^{-v^*hv}\overline{p(v)}.$$

Remark. The form $\psi^{0,0}$ arises naturally as a component of a characteristic form defined by Mathai and Quillen. More precisely, the vector bundle $\mathcal{V} = X \times V$ over X with fiber V carries a tautological metric, and the main result of [26] is the construction of a canonical Thom form $U \in A^{2N}(X \times V)$ and an infinitesimal transgression $\widetilde{U}$ of U in $A^{2N-1}(X)$ (denoted $-i_X U_t$ in [26, Section 7]). A vector $v \in V$ defines a section of $\mathcal{V}$ over X, and the form $\psi^{0,0}(v)$ is essentially obtained from $\widetilde{U}$ by contracting with the vector fields $\partial_{z_1}, \ldots, \partial_{z_N}$ (this gives a form in $A^{N-1}(X \times V)$) and then pulling back by this section. We refer to [3] for more details on this perspective.

Note that the hermitian form $v \mapsto v^* h v$ on V is positive definite and so $\psi^{0,0}$, as a function of v, belongs to the Schwartz space $\mathcal{S}(V)$. Also note that, for any $g \in G$, the expression $v^* h v$ is invariant upon replacing h with $g^* h$ and v with gv, and so Lemma 2.1 implies that $\psi^{0,0}$ is G-invariant:

$$g^* \psi^{0,0}(gv) = \psi^{0,0}(v), \quad g \in G. \tag{2.5}$$

Thus $\psi^{0,0} \in A^{N-1}(X; \mathcal{S}(V))^G$.

For arbitrary $p, q \geq 0$ we define

$$\psi^{p,q} \in A^{N-1}(X; \mathcal{S}(V) \otimes V^{p,q})^G$$

so that its value on $P \otimes \overline{Q}$, where P (resp. Q) is a holomorphic polynomial of degree p on $V^\vee$ (resp. a holomorphic polynomial of degree q on V), is given by

$$\psi^{p,q}(v, P \otimes \overline{Q}) = \overline{Q}(\overline{v}) P(-\partial_{z_1}, \dots, -\partial_{z_N}) \psi^{0,0}(v).$$

From now on we often omit the indices p, q and simply write $\psi(v, P \otimes \overline{Q})$. One can give a more explicit expression for $\psi(v, P \otimes \overline{Q})$: the identity

$$-\partial_{z_i}\left(e^{-v^* h v}\right) = (v^* h)_i e^{-v^* h v}$$

gives

$$P(-\partial_{z_1}, \dots, -\partial_{z_N})\left(e^{-v^* h v}\right) = P((v^* h)_1, \dots, (v^* h)_N) e^{-v^* h v};$$

since if $\overline{p(v)}$ is an anti-holomorphic polynomial, we have $\partial_{z_i} \overline{p(v)} = 0$ for all i, and we conclude that

$$\psi(v, P \otimes \overline{Q}) = e^{-v^* h v} \overline{p(v, P, Q)},$$

with

$$\overline{p(v, P, Q)} := \overline{Q(v)} P(v^* h) \overline{p(v)}.$$

Note that $\overline{p(\cdot, P, Q)}$ is an anti-holomorphic polynomial in v of degree $N + p + q$. This expression shows that, generalizing the invariance property (2.5), we have

$$g^* \psi(gv, gP \otimes \overline{gQ}) = \psi(v, P \otimes \overline{Q}). \tag{2.6}$$

The following is a generalization of Lemma 2.2.

Lemma 2.3. *Let $v \neq 0$. For any $P \in \mathrm{Sym}^p V$ and $Q \in \mathrm{Sym}^q V^\vee$, the form*

$$(v^* h v)^{-N-p} \overline{p(v, P, Q)} \in A^{N-1}(X)$$

is closed.

Proof. Since $d\overline{Q(v)} = 0$, it suffices to assume that $Q = 1$ and that P is monomial, say $P = e^I$ for some multi-index I of degree p. Then $P(v^* h) = (v^* h)_1^{i_1} \cdots (v^* h)_N^{i_N}$ and

$$dP(v^* h) = \left(\sum_j i_j \frac{d(v^* h)_j}{(v^* h)_j} \right) P(v^* h),$$

and so

$$\begin{aligned}
dP(v^*h)\wedge\overline{p(v)} &= 2(-1)^{\frac{N(N-1)}{2}}P(v^*h)\Big(\sum_j i_j\frac{d(v^*h)_j}{(v^*h)_j}\Big)\\
&\qquad\wedge\Big(\sum_{i\ge 1}(-1)^{i-1}(v^*h)_i(v^*dh)_N\wedge\cdots\wedge\widehat{(v^*dh)_i}\wedge\cdots\wedge(v^*dh)_1\Big)\\
&= 2(-1)^{\frac{N(N-1)}{2}}P(v^*h)\sum_j i_j(-1)^{j-1}(v^*dh)_j\wedge(v^*dh)_N\\
&\qquad\qquad\wedge\cdots\wedge\widehat{(v^*dh)_j}\wedge\cdots\wedge(v^*dh)_1\\
&= 2(-1)^{\frac{(N+2)(N-1)}{2}}\Big(\sum_j i_j\Big)P(v^*h)(v^*dh)_N\wedge\cdots\wedge(v^*dh)_1\\
&= pN^{-1}P(v^*h)d\,\overline{p(v)},
\end{aligned}$$

where the last equality follows from (2.4). Using (2.3) we compute

$$\begin{aligned}
&d((v^*hv)^{-N-p}P(v^*h)\overline{p(v)})\\
&\quad= (v^*hv)^{-N-p-1}\big[-(N+p)d(v^*hv)\wedge P(v^*h)\overline{p(v)}\\
&\qquad\qquad + (v^*hv)dP(v^*h)\wedge\overline{p(v)} + (v^*hv)P(v^*h)d\,\overline{p(v)}\,\big]\\
&\quad= (v^*hv)^{-N-p}\big[-(N+p)N^{-1}+pN^{-1}+1\big]P(v^*h)d\,\overline{p(v)}\\
&\quad= 0.
\end{aligned}$$

□

2.3. Mellin transform. We define $\eta(v,s)$ to be the Mellin transform of $\psi(v)$; that is, for holomorphic polynomials P and Q define

$$\eta(v,P\otimes\overline{Q},s)=\int_0^\infty \psi(tv,P\otimes\overline{Q})t^{s+N+p-q}\frac{dt}{t}. \tag{2.7}$$

Then

$$g^*\eta(gv,gP\otimes\overline{gQ},s)=\eta(v,P\otimes\overline{Q},s),\quad g\in G, \tag{2.8}$$

because ψ is G-invariant. Since $\overline{p(tv,P,Q)}=t^{N+p+q}\overline{p(v,P,Q)}$, we have

$$\begin{aligned}
\eta(v,P\otimes\overline{Q},s) &= \int_0^\infty e^{-t^2v^*hv}t^{s+2N+2p}\frac{dt}{t}\overline{p(v,P,Q)}\\
&= 2^{-1}\Gamma(N+p+\tfrac{s}{2})(v^*hv)^{-\frac{s}{2}-N-p}\overline{p(v,P,Q)}.
\end{aligned}\tag{2.9}$$

Lemma 2.4. *We have*

$$d\eta(v,P\otimes\overline{Q},s)=c(s)(v^*hv)^{-\frac{s}{2}-N-p}\overline{Q(v)}P(v^*h)d\,\overline{p(v)},$$

where $c(s)=(-4N)^{-1}s\Gamma(N+p+\frac{s}{2})$.

Proof. By Lemma 2.3 we have

$$d((v^*hv)^{-N-p}\overline{p(v,P,Q)})=0.$$

Using (2.3), we compute

$$\begin{aligned}
&2\Gamma(N+p+\tfrac{s}{2})^{-1} d\eta(v, P\otimes\overline{Q}, s)\\
&\quad = d((v^*hv)^{-\frac{s}{2}}(v^*hv)^{-N-p}\overline{p(v,P,Q)})\\
&\quad = -2^{-1}s(v^*hv)^{-1-\frac{s}{2}} d(v^*hv)\wedge(v^*hv)^{-N-p}\overline{p(v,P,Q)}\\
&\quad = -(2N)^{-1}s(v^*hv)^{-\frac{s}{2}-N-p}\overline{Q(v)}P(v^*h)d\,\overline{p(v)}. \qquad \square
\end{aligned}$$

Using Lemma 2.4, we can represent the form $d\eta(v, P\otimes\overline{Q}, s)$ as a Mellin transform: we define

$$\phi \in A^N(X; \mathcal{S}(V)\otimes V^{p,q})^G$$

by

$$\phi(v, P\otimes\overline{Q}) = e^{-v^*hv}\overline{Q(v)}P(v^*h)d\,\overline{p(v)}; \tag{2.10}$$

then the above lemma implies that

$$d\eta(v, P\otimes\overline{Q}, s) = -\frac{s}{2N}\int_0^\infty \phi(tv, P\otimes\overline{Q})t^{s+N+p-q}\frac{dt}{t}. \tag{2.11}$$

For further reference we note the homogeneity property (which follows from (2.9)):

$$\eta(zv, P(z\,\cdot)\otimes\overline{Q(z^{-1}\,\cdot)}, s) = |z|^{-s}z^{-N}\eta(v, P\otimes\overline{Q}, s) \tag{2.12}$$

for $z\in\mathbb{C}^\times$.

2.4. Example: The case $N = 2$. We compute the form $\psi^{0,0}(v)$ when $N = 2$. We have

$$\begin{aligned}
\psi^{0,0}(v) &= -2e^{-v^*hv}(\overline{(hv)}_1(\overline{dhv})_2 - \overline{(hv)}_2(\overline{dhv})_1)\\
&= -2e^{-v^*hv}(\omega_{11}\overline{v_1}^2 + \omega_{12}\overline{v_1}\,\overline{v_2} + \omega_{22}\overline{v_2}^2),
\end{aligned}$$

with

$$\begin{aligned}
\omega_{11} &= \overline{h_{11}}d\overline{h_{21}} - \overline{h_{21}}d\overline{h_{11}} = h_{11}dh_{12} - h_{12}dh_{11},\\
\omega_{12} &= h_{11}dh_{22} - h_{12}dh_{21} + h_{21}dh_{12} - h_{22}dh_{11},\\
\omega_{22} &= h_{21}dh_{22} - h_{22}dh_{21}.
\end{aligned}$$

Let us rewrite the expression in classical coordinates. For $\tau = (z, y)\in\mathbb{H}^3 = \mathbb{C}\times\mathbb{R}_{>0}$, write

$$g_\tau = \begin{pmatrix} y^{\frac12} & zy^{-\frac12}\\ 0 & y^{-\frac12}\end{pmatrix}.$$

The map $\tau\mapsto g_\tau K$ identifies $\mathbb{H}^3$ with $X = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$. In these coordinates we have

$$\begin{aligned}
h_\tau = {}^t\overline{g_\tau}^{-1}g_\tau^{-1} &= \begin{pmatrix} y^{-\frac12} & 0\\ -\overline{z}y^{-\frac12} & y^{\frac12}\end{pmatrix}\begin{pmatrix} y^{-\frac12} & -zy^{-\frac12}\\ 0 & y^{\frac12}\end{pmatrix}\\
&= y^{-1}\begin{pmatrix} 1 & -z\\ -\overline{z} & y^2+|z|^2\end{pmatrix},
\end{aligned}$$

and

$$v^* h_\tau v = |g_\tau^{-1} v|^2 = y^{-1}(|v_1 - z v_2|^2 + |y v_2|^2).$$

Hence

$$\begin{aligned} dh_\tau &= -y^{-2} dy \begin{pmatrix} 1 & -z \\ -\overline{z} & y^2 + |z|^2 \end{pmatrix} + y^{-1} d \begin{pmatrix} 1 & -z \\ -\overline{z} & y^2 + |z|^2 \end{pmatrix} \\ &= -h_\tau y^{-1} dy + y^{-1} \begin{pmatrix} 0 & -dz \\ -d\overline{z} & 2y dy + z d\overline{z} + \overline{z} dz \end{pmatrix} \end{aligned}$$

and we compute

$$\begin{aligned} \omega_{11} &= y^{-2} dz, \\ \omega_{12} &= -2(y^{-1} dy + y^{-2} \overline{z} dz), \\ \omega_{22} &= 2\overline{z} y^{-1} dy - d\overline{z} + \overline{z}^2 y^{-2} dz. \end{aligned}$$

Writing $\psi^{0,0}(v) = \psi(v)_y dy + \psi(v)_z dz + \psi(v)_{\overline{z}} d\overline{z}$, we obtain

$$\begin{aligned} \psi(v)_{\overline{z}} &= 2e^{-v^* h_\tau v} \cdot \overline{v_2}^2, \\ \psi(v)_y &= -2e^{-v^* h_\tau v}(-2y^{-1} \overline{v_1 v_2} + 2\overline{z} y^{-1} \overline{v_2}^2) \\ &= 4y^{-1} e^{-v^* h_\tau v} \overline{(v_1 - z v_2) v_2}, \\ \psi(v)_z &= -2y^{-2} e^{-v^* h_\tau v}(\overline{v_1}^2 - 2\overline{z v_1 v_2} + \overline{z}^2 \overline{v_2}^2) \\ &= -2y^{-2} e^{-v^* h_\tau v} \overline{(v_1 - z v_2)}^2. \end{aligned}$$

The Mellin transform $\eta^{0,0}(v,s) = \eta(v,s)_y dy + \eta(v,s)_z dz + \eta(v,s)_{\overline{z}} d\overline{z}$ (defined in (2.7) below) is then given by

$$\begin{aligned} \eta(v,s)_{\overline{z}} &= \Gamma(\tfrac{s}{2} + 2) y^{\frac{s}{2}} \frac{(\overline{y v_2})^2}{(|v_1 - z v_2|^2 + |y v_2|^2)^{\frac{s}{2}+2}}, \\ \eta(v,s)_y &= 2\Gamma(\tfrac{s}{2} + 2) y^{\frac{s}{2}} \frac{\overline{(v_1 - z v_2) y v_2}}{(|v_1 - z v_2|^2 + |y v_2|^2)^{\frac{s}{2}+2}}, \\ \eta(v,s)_z &= -\Gamma(\tfrac{s}{2} + 2) y^{\frac{s}{2}} \frac{\overline{(v_1 - z v_2)}^2}{(|v_1 - z v_2|^2 + |y v_2|^2)^{\frac{s}{2}+2}}. \end{aligned}$$

Thus we recover the form introduced by Ito in [22].

2.5. Fourier transform. Recall that the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ identifies the tangent space $T_{eK} X$ at the point $eK \in X$ with $\mathfrak{p}$. Given $Y \in \wedge^{N-1} \mathfrak{p}$ and polynomials P and Q, evaluation at Y defines a Schwartz function

$$\psi(Y, P \otimes \overline{Q}) \in \mathcal{S}(V)$$

given explicitly by

$$\psi(v, Y; P \otimes \overline{Q}) = e^{-v^* v} \overline{p(v, Y; P, Q)},$$

with $\overline{p(v, Y; P, Q)} = \overline{Q(v)} P(v^*) \overline{p(v, Y)}$.

We write $\langle\cdot,\cdot\rangle$ for the scalar product on V defined by $\langle v,w\rangle = 2\,\mathrm{Re}(w^*v)$ and given a Schwartz form $f \in \mathcal{S}(V)$, we define its Fourier transform $\mathcal{F} f \in \mathcal{S}(V)$ by

$$\mathcal{F} f(v) = \int_V f(w) e^{2\pi i \langle v,w\rangle}\, dw,$$

where dw denotes the Lebesgue measure on $\mathbb{C}^N$. Since the polynomial $\overline{p(v,Y;P,Q)}$ is antiholomorphic, it is also harmonic and hence we have

$$\mathcal{F}\psi(Y, P\otimes\overline{Q}) = C\psi(Y, P\otimes\overline{Q}) \tag{2.13}$$

for some constant C satisfying $C^4 = 1$. In particular,

$$\mathcal{F}\psi(0,Y;P\otimes\overline{Q}) = \psi(0,Y;P\otimes\overline{Q}) = 0.$$

Similar statements hold for $\phi(Y, P\otimes\overline{Q})$ for any $Y \in \wedge^N\mathfrak{p}$.

2.6. Integral on a maximal torus. Let $T \subset G$ be the torus of diagonal matrices. The inclusion of T in G induces an embedding

$$T/T\cap K \to X$$

identifying $T/T\cap K$ with the submanifold of X consisting of diagonal hermitian matrices. This submanifold is diffeomorphic to $\mathbb{R}^{N-1}_{>0}$: writing

$$C = \{(t_1,\dots,t_N) \in \mathbb{R}^N_{>0} : t_1\cdots t_N = 1\}, \tag{2.14}$$

the map $(t_1,\dots,t_N) \mapsto \mathrm{diag}(t_1^{-1},\dots,t_N^{-1})T\cap K$ identifies $C \simeq T/T\cap K$. We use this identification to orient $T/T\cap K$ as follows: forgetting the coordinate t_N gives a diffeomorphism $C \simeq \mathbb{R}^{N-1}_{>0}$. We orient C, and hence $T/T\cap K$, by pulling back the standard orientation of $\mathbb{R}^{N-1}_{>0}$ (given by the volume form $\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_{N-1}}{t_{N-1}}$).

Lemma 2.5. *Let $v \neq 0$. If $\mathrm{Re}(s) + 2N + 2p > 0$, the form $\eta(v, P\otimes\overline{Q}, s)$ is integrable on $T/T\cap K$. For $P = e^I$ and $Q = z^J$ monomial with $I = (i_1,\dots,i_N)$ and $J = (j_1,\dots,j_N)$ multi-indices, we have*

$$\int_{T/T\cap K} \eta(v, P\otimes\overline{Q}, s) = \prod_{k=1}^N \Gamma\Big(\frac{s}{2N} + 1 + i_k\Big)\frac{\overline{v_k}^{j_k}}{|v_k|^{\frac{s}{N}} v_k^{i_k+1}}.$$

Proof. Since $\eta(v, P\otimes\overline{Q}, s) = \overline{Q(v)}\eta(v, P\otimes 1, s)$, we may assume that $Q = 1$. In the above coordinates for C we have $h = \mathrm{diag}(t_1^2,\dots,t_N^2)$ and $dh = 2\mathrm{diag}(t_1dt_1,\dots,t_Ndt_N)$, and so the restriction of $P(v^*h)\overline{p(v)}$ to $T/T\cap K$ is given by

$$\begin{aligned}
&\prod_{j=1}^N (t_j^2\overline{v_j})^{i_j}\cdot 2(-1)^{\frac{N(N-1)}{2}}\sum_j (-1)^{j-1}\overline{v_j}t_j^2(\overline{v_N}2t_N\,dt_N) \\
&\qquad\qquad \wedge\cdots\wedge\widehat{(\overline{v_j}2t_j\,dt_j)}\wedge\cdots\wedge(\overline{v_1}2t_1\,dt_1) \\
&= 2^N(-1)^{\frac{N(N-1)}{2}}\prod_{j=1}^N t_j^{2i_j}\overline{v_j}^{i_j+1}\sum_j(-1)^{j-1}\frac{dt_N}{t_N}\wedge\cdots\wedge\widehat{\frac{dt_j}{t_j}}\wedge\cdots\wedge\frac{dt_1}{t_1}
\end{aligned}$$

since $t_1 \cdots t_N = 1$. For $t \in C$ and $u > 0$, set $u_i = t_i u$. This gives

$$u_1 \cdots u_N = u^N \quad \text{and} \quad \frac{du_i}{u_i} = \frac{dt_i}{t_i} + \frac{du}{u},$$

and hence

$$\left(\sum_j (-1)^{j-1} \frac{dt_N}{t_N} \wedge \cdots \wedge \widehat{\frac{dt_j}{t_j}} \wedge \cdots \wedge \frac{dt_1}{t_1} \right) \wedge \frac{du}{u} = \frac{du_N}{u_N} \wedge \cdots \wedge \frac{du_1}{u_1}.$$

The map $((t_1, \ldots, t_N), u) \mapsto (u_1, \ldots, u_N)$ induces a diffeomorphism $C \times \mathbb{R}_{>0} \simeq \mathbb{R}^N_{>0}$. Using this as change of variables, we compute

$$\begin{aligned}
\int_{T/T\cap K} \eta(v, P \otimes 1, s) &= \int_C \int_0^\infty \psi(uv, P \otimes 1) u^{s+N+p} \frac{du}{u} \\
&= 2^N (-1)^{\frac{N(N-1)}{2}} \int_{\mathbb{R}^N_{>0}} e^{-\sum u_j^2 |v_j|^2} \left(\prod_{j=1}^N t_j^{2i_j} u^{i_j+1} \overline{v_j}^{i_j+1} \right) \\
&\qquad \cdot u^{s+N+p} \frac{du_N}{u_N} \wedge \cdots \wedge \frac{du_1}{u_1} \\
&= 2^N \prod_{j=1}^N \overline{v_j}^{i_j+1} \int_0^\infty e^{-u_j^2 |v_j|^2} u_j^{\frac{s+2N}{N}+2i_j} \frac{du_j}{u_j} \\
&= \prod_{j=1}^N \Gamma(\tfrac{s}{2N} + 1 + i_j) \frac{\overline{v_j}^{i_j+1}}{|v_j|^{\frac{s}{N}+2+2i_j}}. \qquad \square
\end{aligned}$$

The above lemma shows that the integral of $\eta(v, P \otimes \overline{Q}, s)$ on $T/T \cap K$ has meromorphic continuation to $s \in \mathbb{C}$ that is regular at $s = 0$. Its value at $s = 0$ for $P = e^I$ and $Q = z^J$ is

$$\int_{T/T\cap K} \eta(v, P \otimes \overline{Q}, s) \Big|_{s=0} = \prod_{k=1}^N i_k! \frac{\overline{v_k}^{j_k}}{v_k^{i_k+1}}.$$

It follows easily that for arbitrary P we can write

$$\int_{T/T\cap K} \eta(v, P \otimes \overline{Q}, s) = C(s) \overline{Q(v)} P(-\partial_{z_1}, \ldots, -\partial_{z_N}) \left(\prod_{j=1}^N \frac{\overline{v_j}}{|v_j|^{\frac{s}{N}+2}} \right),$$

for some meromorphic function $C(s)$ such that $C(0) = 1$.

3. Eisenstein cocycle

Let k be an imaginary quadratic field with ring of integers $\mathcal{O}$. We fix an integer $N \geq 2$ and let $V_k = k^N$ and $G_k = SL_N(k)$ (recall that $V = \mathbb{C}^N$ and $G = SL_N(\mathbb{C})$). We also fix an embedding $\sigma : k \to \mathbb{C}$, which makes V a k-module and induces inclusions $V_k \subset V$ and $G_k \subset G$.

Given a non-zero ideal $\mathfrak{I}$ of $\mathcal{O}$, define

$$\Lambda(\mathfrak{I}) = \mathfrak{I}^{-1} \oplus \mathcal{O}^{N-1}. \tag{3.1}$$

It is an $\mathcal{O}$-submodule of k^N that we regard as a lattice in $\mathbb{C}^N$ via the embedding $k^N \to \mathbb{C}^N$ induced by σ. We write $\Gamma(\Lambda(\mathfrak{I}))$ for the intersection of $\mathrm{Aut}_{\mathcal{O}}(\Lambda(\mathfrak{I}))$ with $\mathrm{SL}_N(k)$; more explicitly,

$$\Gamma(\Lambda(\mathfrak{I})) = \left\{ \begin{pmatrix} a & {}^t b \\ c & D \end{pmatrix} \in \mathrm{SL}_N(k) : a \in \mathcal{O},\ D \in M_{N-1}(\mathcal{O}), \right.$$
$$\left. b \in (\mathfrak{I}^{-1})^{N-1},\ c \in \mathfrak{I}^{N-1} \right\}.$$

Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal coprime to $\mathfrak{I}$. We define a congruence subgroup $\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$ of $\Gamma(\Lambda(\mathfrak{I}))$ by

$$\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I})) = \left\{ \begin{pmatrix} a & {}^t b \\ c & D \end{pmatrix} \in \Gamma(\Lambda(\mathfrak{I})) : c \in (\mathfrak{p}\mathfrak{I})^{N-1} \right\};$$

thus $\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I})) = \Gamma(\Lambda(\mathfrak{p}\mathfrak{I})) \cap \Gamma(\Lambda(\mathfrak{I}))$. When $\mathfrak{I} = \mathcal{O}$, we have $\Gamma(\mathcal{O}) = \mathrm{SL}_N(\mathcal{O})$ and $\Gamma_0(\mathfrak{p}, \Lambda(\mathcal{O})) = \Gamma_0(\mathfrak{p})$ is the standard level $\mathfrak{p}$ subgroup of $\mathrm{SL}_N(\mathcal{O})$.

In this section we prove Theorem 1.1. We first define a more general cocycle

$$\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\Lambda(\mathfrak{I})) : \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))^N \to \mathcal{F} \otimes V^{p,q},$$

where $V^{p,q}$ is given in (2.1) and $\mathcal{F}$ is a certain space of functions defined on complements of unions of affine hyperplanes in V, endowed with a natural action of $\mathrm{SL}_N(k)$. In the last section we will show that its cohomology class is non-trivial by computing its value explicitly on the units of degree N field extensions of k.

3.1. Definition of the cocycle. Let $\mathfrak{I} \subset k$ be a fractional ideal. Then $\sigma(\mathfrak{I}) \subset \mathbb{C}$ is a lattice. Given a pair of integers $p, q \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{C}$, define the Kronecker–Eisenstein series

$$K^{p,q}(z, \mathfrak{I}, s) = p! \sum_{a \in \sigma(\mathfrak{I})} \frac{\overline{z+a}^q}{(z+a)^{p+1}|z+a|^s}, \quad z \notin \sigma(\mathfrak{I}).$$

The sum converges absolutely for $\mathrm{Re}(s) > 1 + q - p$ and for z in a compact subset of $\mathbb{C}$. The series $K^{p,q}(z, \mathfrak{a}, s)$ has an analytic continuation to the whole s-plane that is regular at $s = 0$, see e.g. [11, 12, 37].

More generally, for an $\mathcal{O}$-lattice $\Lambda \subset k^N$, let $U(\Lambda)$ be the open subset of $\mathbb{C}^N$ obtained by removing all translates of coordinate hyperplanes by $\lambda \in \sigma(\Lambda)$. For $I = (i_1, \ldots, i_N)$ and $J = (j_1, \ldots, j_N)$ in $\mathbb{Z}_{\geq 0}^N$ and $z \in U(\Lambda)$, define

$$K^{I,J}(z, \Lambda, s) = \sum_{\lambda \in \sigma(\Lambda)} \prod_{1 \leq k \leq N} i_k! \frac{\overline{z_k + \lambda_k}^{j_k}}{(z_k + \lambda_k)^{i_k+1}|z_k + \lambda_k|^s}.$$

The function $K^{I,J}(z, \Lambda, s)$ can be expressed as a homogeneous degree N polynomial of Kronecker–Eisenstein series: pick non-zero fractional ideals $\mathfrak{I}_1, \ldots, \mathfrak{I}_N$ of k such that

$$\Lambda \supseteq \mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N.$$

Then

$$K^{I,J}(z, \Lambda, s) = \sum_{\lambda \in \Lambda / \mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N} K^{I,J}(z + \sigma(\lambda), \mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N, s)$$

and

$$K^{I,J}(z, \mathfrak{F}_1 \oplus \cdots \oplus \mathfrak{F}_N, s) = \prod_{1 \le k \le N} K^{i_k, j_k}(z_k, \mathfrak{F}_k, s).$$

Thus $K^{I,J}(z, \Lambda, s)$ converges absolutely for $\mathrm{Re}(s) > 1 + \max\{j_k - i_k\}$ and has analytic continuation to all $s \in \mathbb{C}$ that is regular at $s = 0$. We set

$$K^{I,J}(z, \Lambda) = K^{I,J}(z, \Lambda, 0)$$

and define

$$\mathscr{F} := \mathrm{span}\langle K^{I,J}(\gamma^{-1} z, \Lambda) : \gamma \in \mathrm{SL}_N(k), \Lambda \text{ an } \mathcal{O}\text{-lattice in } k^N \rangle,$$

which carries a natural action of $\mathrm{SL}_N(k)$.

Definition 3.1. Let A be a matrix in $\mathrm{End}(\Lambda(\mathfrak{F})) \cap \mathrm{GL}_N(k)$. Then

$$A^{-1}\Lambda(\mathfrak{F}) \supseteq \Lambda(\mathfrak{F})$$

and we define the generalized Dedekind sum

$$\begin{aligned} D^{I,J}(z, A, \Lambda(\mathfrak{F})) &= \det A^{-1} K^{I,J}(A^{-1} z, A^{-1}\Lambda(\mathfrak{F})) \\ &= \det A^{-1} \sum_{\lambda \in \Lambda(\mathfrak{F})/A\Lambda(\mathfrak{F})} K^{I,J}(A^{-1}(z + \sigma(\lambda)), \Lambda(\mathfrak{F})). \end{aligned}$$

Let $\mathfrak{p}$ be a proper ideal of $\mathcal{O}$ coprime to $\mathfrak{F}$ and let $\mathrm{N}\mathfrak{p}$ be its norm. Define

$$D_{\mathfrak{p}}^{I,J}(z, A, \Lambda(\mathfrak{F})) = D^{I,J}(z, A, \Lambda(\mathfrak{p}\mathfrak{F})) - \mathrm{N}\mathfrak{p} \cdot D^{I,J}(z, A, \Lambda(\mathfrak{F})).$$

If $A \in \mathrm{End}(\Lambda(\mathfrak{F}))$ but A is not invertible, set

$$D^{I,J}(z, A, \Lambda(\mathfrak{F})) = D_{\mathfrak{p}}^{I,J}(z, A, \Lambda(\mathfrak{F})) = 0.$$

For $p, q \in \mathbb{Z}_{\ge 0}$, recall the G-representation $V^{p,q}$ introduced in (2.1). A basis of $V^{p,q}$ is given by the vectors

$$e^{I,J} := (({}^t e_1)^{i_1} \cdots ({}^t e_N)^{i_N}) \otimes (\overline{e_1}^{j_1} \cdots \overline{e_N}^{j_N}),$$

where $I, J \in \mathbb{Z}_{\ge 0}^N$ satisfy $i_1 + \cdots + i_N = p$ and $j_1 + \cdots + j_N = q$.

Recall that given a group Γ and a $\mathbb{Z}[\Gamma]$-module M, a map $\alpha : \Gamma^N \to M$ is said to be a homogeneous $(N-1)$-cocycle if it is equivariant, that is,

$$\alpha(\gamma\gamma_1, \ldots, \gamma\gamma_N) = \gamma\alpha(\gamma_1, \ldots, \gamma_N), \quad \gamma, \gamma_1, \ldots, \gamma_N \in \Gamma, \tag{3.2}$$

and satisfies

$$\sum_{1 \le i \le N+1} (-1)^{i-1} \alpha(\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{N+1}) = 0, \quad \gamma_1, \ldots, \gamma_{N+1} \in \Gamma. \tag{3.3}$$

Theorem 3.2. *Let $\mathfrak{F}, \mathfrak{p} \subseteq \mathcal{O}$ be non-zero coprime ideals of $\mathcal{O}$ and assume that $\mathfrak{p} \neq \mathcal{O}$. Given $\underline{\gamma} = (\gamma_1, \ldots, \gamma_N) \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{F}))^N$, let*

$$A(\underline{\gamma}) = (\gamma_1 e_1 | \cdots | \gamma_N e_1) \in M_N(\mathcal{O})$$

be the matrix formed by the first columns of $\gamma_1, \dots, \gamma_N$. *Then* $A(\underline{\gamma}) \in \mathrm{End}(\Lambda(\mathfrak{I}))$. *For fixed* p, q *in* $\mathbb{Z}_{\geq 0}$, *define a map*

$$\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\Lambda(\mathfrak{I})) : \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))^N \to \mathcal{F} \otimes V^{p,q}$$

by

$$\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(z, \underline{\gamma}, \Lambda(\mathfrak{I})) = \sum_{\substack{|I|=p \\ |J|=q}} D_{\mathfrak{p}}^{I,J}(z, A(\underline{\gamma}), \Lambda(\mathfrak{I})) \otimes A(\underline{\gamma}) e^{I,J}.$$

Then $\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\Lambda(\mathfrak{I}))$ *is a homogeneous* $(N-1)$*-cocycle.*

Note that the first row of the matrix $A(\underline{\gamma})$ in the statement has entries in $\mathcal{O}$ whereas all its other rows have entries in $\mathfrak{I}$. The statement that $A(\underline{\gamma}) \in \mathrm{End}(\Lambda(\mathfrak{I}))$ follows. Note also that

$$A(\gamma\gamma_1, \dots, \gamma\gamma_N) = \gamma A(\gamma_1, \dots, \gamma_N), \quad \gamma, \gamma_1, \dots, \gamma_N \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I})).$$

The equivariance property (3.2) of $\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\Lambda(\mathfrak{I}))$ follows from this. Thus it remains to show the cocycle property (3.3). To prove it we will next define – as an Eisenstein series – a closed $\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$-invariant differential form

$$E_{\mathfrak{p}}(z, \psi^{p,q}, \Lambda(\mathfrak{I})) \in A^{N-1}(X) \otimes V^{p,q}$$

and $(N-1)$-dimensional submanifolds

$$\Delta(\underline{\gamma}) \subset X, \quad \underline{\gamma} \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))^N,$$

such that

$$\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(z, \underline{\gamma}, \Lambda(\mathfrak{I})) = \int_{\Delta(\underline{\gamma})} E_{\mathfrak{p}}(z, \psi^{p,q}, \Lambda(\mathfrak{I})).$$

The cocycle property will follow from the fact that for $\gamma_1, \dots, \gamma_{N+1} \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$ we can find a simplex

$$\Delta(\gamma_1, \dots, \gamma_{N+1}) \subset X$$

with boundary

$$\partial\Delta(\gamma_1, \dots, \gamma_{N+1}) = \sum_{1 \leq i \leq N+1} (-1)^{i-1} \Delta(\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_{N+1})$$

and such that $E_{\mathfrak{p}}(z, \psi^{p,q}, \Lambda(\mathfrak{I}))$ decreases rapidly on $\Delta(\gamma_1, \dots, \gamma_{N+1})$ for fixed z.

3.2. Eisenstein series. For $v \in V$, an $\mathcal{O}$-lattice $\Lambda \subset V_k$ and a holomorphic polynomial P (resp. Q) on $V^\vee$ (resp. on V), consider the theta series

$$\theta(v, P \otimes \overline{Q}; \psi, \Lambda) := \sum_{\lambda \in \Lambda} \psi(v + \lambda, P \otimes \overline{Q}).$$

The series converges rapidly as $\psi(v, P \otimes \overline{Q})$ is rapidly decreasing. By (2.6), we obtain a differential form $\theta(v, P \otimes \overline{Q}; \psi, \Lambda) \in A^{N-1}(X)$ satisfying

$$\gamma^*\theta(\gamma v, \gamma P \otimes \overline{\gamma Q}; \psi, \Lambda) = \theta(v, P \otimes \overline{Q}; \psi, \Lambda), \quad \gamma \in \Gamma(\Lambda),$$

where $\Gamma(\Lambda) := \mathrm{Aut}_{\mathcal{O}}(\Lambda) \cap \mathrm{SL}_N(k)$. The Mellin transform of $\theta(v, P \otimes \overline{Q}; \psi, \Lambda)$ is the Eisenstein series

$$\begin{aligned} E(v, P \otimes \overline{Q}; \psi, \Lambda, s) &:= \int_0^{+\infty} \theta(tv, P \otimes \overline{Q}; \psi, t\Lambda) t^{s+N+p-q} \frac{dt}{t} \\ &= \sum_{\lambda \in \Lambda} \eta(v + \lambda, P \otimes \overline{Q}, s), \end{aligned}$$

where $\eta(v, s)$ is given by (2.9). Here the sum converges when $\mathrm{Re}(s) \gg 0$ but can be analytically continued to the whole s-plane in a standard way using Poisson summation. To do this, consider the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^N$ given by

$$\langle v, w \rangle = 2\,\mathrm{Re}(v \cdot w^*)$$

and define

$$\Lambda^\vee = \{w \in \mathbb{C}^N : \langle v, w \rangle \in \mathbb{Z} \text{ for all } v \in \Lambda\}.$$

Given $g \in G$ and a tangent vector $Y \in \mathfrak{p} = T_{eK} X$ we can define a vector $g_* Y \in T_{gK} X$. The invariance property (2.6) can be rewritten as

$$\psi(gv, g_* Y; gP \otimes \overline{gQ}) = \psi(v, Y; P \otimes \overline{Q}).$$

By (2.13), $\psi(\cdot, Y; P \otimes \overline{Q}) \in \mathcal{S}(V)$ is an eigenvector for the Fourier transform and so Poisson summation gives

$$\begin{aligned} &\sum_{\lambda \in \Lambda} \psi(t(v + \lambda), g_* Y; P \otimes \overline{Q}) \\ &\quad = \sum_{\lambda \in \Lambda} \psi(tg^{-1}(v + \lambda), Y; g^{-1}(P \otimes \overline{Q})) \\ &\quad = C\,\mathrm{Vol}(\mathbb{C}^N/\Lambda)^{-1} t^{-2N} \sum_{\lambda \in \Lambda^\vee} e^{2\pi i \langle v, \lambda \rangle} \psi(t^{-1} g^* \lambda, Y; g^{-1}(P \otimes \overline{Q})). \end{aligned}$$

Using this, we can write

$$\begin{aligned} &E(v, g_* Y; P \otimes \overline{Q}; \psi, \Lambda, s) \\ &\quad = \int_0^{\infty} \theta(tv, g_* Y; P \otimes \overline{Q}; \psi, t\Lambda) t^{s+N+p-q} \frac{dt}{t} \\ &\quad = \int_1^{\infty} \theta(tv, g_* Y; P \otimes \overline{Q}; \psi, t\Lambda) t^{s+N+p-q} \frac{dt}{t} + C\ \mathrm{Vol}(\mathbb{C}^N/\Lambda)^{-1} \\ &\qquad\quad \cdot \sum_{\lambda \in \Lambda^\vee} e^{2\pi i \langle v, \lambda \rangle} \int_1^{\infty} \psi(tg^* \lambda, Y; g^{-1}(P \otimes \overline{Q})) t^{-s+N+q-p} \frac{dt}{t}. \end{aligned}$$

The last expression converges for all $s \in \mathbb{C}$ and gives the desired analytic continuation (with no poles since $\psi(0) = \mathcal{F}\psi(0) = 0$) of $E(v, P \otimes \overline{Q}; \psi, \Lambda, s)$. We set

$$E(v, P \otimes \overline{Q}; \psi, \Lambda) = E(v, P \otimes \overline{Q}; \psi, \Lambda, 0) \in A^{N-1}(X).$$

Proposition 3.3. *For a fixed column vector $v \in \mathbb{C}^N$ and polynomials P and Q, the form $E(v, P \otimes \overline{Q}; \psi, \Lambda)$ is closed.*

Proof. For $t > 0$ define the theta series

$$\theta(tv, P \otimes \overline{Q}; \phi, t\Lambda) = \sum_{\lambda \in \Lambda} \phi(t(v + \lambda), P \otimes \overline{Q}),$$

where ϕ is given in (2.10). The same argument used above shows that

$$E(v, P \otimes \overline{Q}; \phi, \Lambda, s) := \int_0^\infty \theta(tv, P \otimes \overline{Q}; \phi, t\Lambda) t^{s+N+p-q} \frac{dt}{t}, \quad \mathrm{Re}(s) \gg 0,$$

admits analytic continuation to $s \in \mathbb{C}$ (with no poles). The relation

$$dE(v, P \otimes \overline{Q}; \psi, \Lambda, s) = -\frac{s}{2N} E(v, P \otimes \overline{Q}; \phi, \Lambda, s),$$

which follows from (2.11), proves the claim. □

Thus we may regard $E(\cdot, P \otimes \overline{Q}; \psi, \Lambda)$ as a closed differential $(N-1)$-form on X valued on the space of smooth functions $\mathcal{C}^\infty(V)$, and by (2.8) we have the equivariance property

$$\gamma^* E(\gamma v, \gamma P \otimes \overline{\gamma Q}; \psi, \Lambda) = E(v, P \otimes \overline{Q}; \psi, \Lambda), \quad \gamma \in \Gamma(\Lambda).$$

For $\Lambda = \Lambda(\mathfrak{J})$ defined in (3.1), we set

$$\begin{aligned} E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{J}), s) = E(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{p}\mathfrak{J}), s) \\ - \mathrm{N}\mathfrak{p} \cdot E(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{J}), s) \end{aligned}$$

and

$$E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{J})) = E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{J}), 0).$$

Again we regard $E_{\mathfrak{p}}(\cdot, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{J}))$ as a closed differential $(N-1)$-form on X valued in $\mathcal{C}^\infty(V)$, equivariant under $\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{J}))(= \Gamma(\Lambda(\mathfrak{p}\mathfrak{J})) \cap \Gamma(\Lambda(\mathfrak{J})))$.

3.3. Behavior on Siegel sets. Fix two coprime ideals $\mathfrak{p}$ and $\mathfrak{J}$ of $\mathcal{O}$ with $\mathfrak{p}$ of prime norm. Recall that proper rational parabolics of $G_k = \mathrm{SL}_N(k)$ are in bijection with proper flags

$$W_\bullet : 0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_r \subsetneq k^N, \quad r \geq 0.$$

Before stating our next result we recall the definition of Siegel sets. For a strictly increasing sequence $\mathbf{J} = \{j_1 < \cdots < j_r\}$ of integers in $\{1, \ldots, N-1\}$, let $W_{j_k} = \langle e_1, \ldots, e_{j_k} \rangle$ and let $P_{\mathbf{J}}$ be the standard parabolic of $\mathrm{SL}_N(k)$ stabilizing the flag

$$W_{\mathbf{J}} : 0 \subsetneq W_{j_1} \subsetneq \cdots \subsetneq W_{j_r} \subsetneq V.$$

We can write $P_{\mathbf{J}} = NMA$, where (setting $j_0 = 0$ and $j_{r+1} = N$)

$$N = N_{\mathbf{J}} = \left\{ \begin{pmatrix} 1_{j_1} & * & \cdots & * \\ 0 & 1_{j_2 - j_1} & \cdots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 1_{j_{r+1} - j_r} \end{pmatrix} \right\}, \tag{3.4}$$

$$M = M_{\mathbf{J}} = \left\{ \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_{r+1} \end{pmatrix} : A_k \in \mathrm{GL}_{j_k - j_{k-1}}(\mathbb{C}),\ |\det(A_k)| = 1 \right\},$$

$$A = A_{\mathbf{J}} = \{a(t_1, \ldots, t_{r+1}) : t_k > 0,\ \det a(t_1, \ldots, t_{r+1}) = 1\}.$$

where

$$a(t_1, \ldots, t_{r+1}) := \begin{pmatrix} t_1 1_{j_1} & 0 & \cdots & 0 \\ 0 & t_2 1_{j_2 - j_1} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_{r+1} 1_{j_{r+1} - j_r} \end{pmatrix}.$$

An element $g \in G$ can be written as

$$g = nmak, \quad n \in N,\ m \in M,\ a \in A,\ k \in \mathrm{SU}(N).$$

In this decomposition n and a are uniquely determined by g and m and k are determined up to an element of $M \cap \mathrm{SU}(N)$.

For $t \in \mathbb{R}_{>0}$, let

$$A_t = \left\{ a(t_1, \ldots, t_{r+1}) \in A : \frac{t_k}{t_{k+1}} \geq t \text{ for all } k \right\}.$$

The Siegel set determined by $t > 0$ and a relatively compact set $\omega \subset NM$ is

$$S(t, \omega) := \omega A_t \cdot \mathrm{SU}(N) \subset \mathrm{SL}_N(\mathbb{C}); \tag{3.5}$$

we refer to its image in X also as a Siegel set.

More generally, suppose that $W_\bullet$ is a proper flag of k^N. A *Siegel set for the cusp defined by* $W_\bullet$ is a set of the form

$$S(g, t, \omega) := g^{-1} \omega A_t \cdot \mathrm{SU}(N),$$

where $g \in \mathrm{SL}_N(k)$ is such that $gW_\bullet$ is a standard flag (i.e. of the form $W_{\mathbf{J}}$ for some $\mathbf{J}$).

We say that $W_\bullet$ *defines a good cusp* if $\gamma e_1 \in W_0$ for some $\gamma \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$.

Proposition 3.4. *Suppose that $W_\bullet$ defines a good cusp. If $v \in k^N$ satisfies*

$$(v + \Lambda(\mathfrak{p}\mathfrak{I})) \cap W_r = \emptyset,$$

then $E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{I}))$ is rapidly decreasing on every Siegel set for the cusp defined by $W_\bullet$.

For the proof it will be convenient to work with adeles. Given a finite Schwartz function $\phi_f \in \mathcal{S}(V_k(\mathbb{A}_f))$ and $t > 0$, let

$$\begin{aligned} \theta(v, t, P \otimes \overline{Q}; \phi_f \otimes \psi) &= \sum_{\lambda \in k^N} \phi_f(\lambda) \psi(t(v + \lambda), P \otimes \overline{Q}), \\ E(v, P \otimes \overline{Q}; \phi_f \otimes \psi, s) &= \int_0^\infty \theta(v, t, P \otimes \overline{Q}; \phi_f \otimes \psi) t^{s+N+p-q} \frac{dt}{t} \\ &= \sum_{\lambda \in k^N} \phi_f(\lambda) \eta(v + \lambda, P \otimes \overline{Q}, s). \end{aligned}$$

Using Poisson summation as in Section 3.2, for $Y \in \wedge^{N-1}\mathfrak{p}$, we may write

$$\begin{aligned} &\theta(v, g_* Y, t, P \otimes \overline{Q}; \phi_f \otimes \psi) \\ &\quad = C \operatorname{Vol}(\mathbb{C}^N / \Lambda(\mathfrak{I}))^{-1} t^{-2N} \\ &\qquad \cdot \sum_{\lambda \in V_k} \widehat{\phi_f}(\lambda) e^{2\pi i \langle v, \lambda \rangle} \psi(t^{-1} g^* \lambda, Y; g^{-1}(P \otimes \overline{Q})) \end{aligned}$$

and

$$\begin{aligned}
&E(v, g_*Y, P \otimes \overline{Q}; \phi_f \otimes \psi, s) \\
&\quad = \sum_{\lambda \in V_k} \phi_f(\lambda) \int_1^\infty \psi(tg^{-1}(v+\lambda), Y; g^{-1}(P \otimes \overline{Q})) t^{s+N+p-q} \frac{dt}{t} \\
&\qquad + C \ \mathrm{Vol}(\mathbb{C}^N/\Lambda(\mathfrak{I}))^{-1} \sum_{\lambda \in V_k} \widehat{\phi_f}(\lambda) e^{2\pi i \langle v, \lambda \rangle} \\
&\qquad\quad \cdot \int_1^\infty \psi(tg^*\lambda, Y; g^{-1}(P \otimes \overline{Q})) t^{-s+N+q-p} \frac{dt}{t},
\end{aligned}$$

showing that $E(v, P \otimes \overline{Q}; \phi_f \otimes \psi, s)$ admits analytic continuation to $s \in \mathbb{C}$ that is regular at $s = 0$. Note that we can write

$$E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{I})) = E(v, P \otimes \overline{Q}; \phi_f(\mathfrak{p}, \mathfrak{I}) \otimes \psi, s)|_{s=0},$$

where $\phi_f(\mathfrak{p}, \mathfrak{I}) \in \mathcal{S}(V_k(\mathbb{A}_f))$ is given by

$$\phi_f(\lambda; \mathfrak{p}, \mathfrak{I}) = \begin{cases} 0 & \text{if } \lambda \notin \Lambda(\mathfrak{p}\mathfrak{I}) \otimes_{\mathcal{O}} \widehat{\mathcal{O}}, \\ 1 & \text{if } \lambda \in \Lambda(\mathfrak{p}\mathfrak{I}) \otimes_{\mathcal{O}} \widehat{\mathcal{O}} \text{ and } \lambda_1 \notin \mathfrak{I}^{-1} \otimes_{\mathcal{O}} \widehat{\mathcal{O}}, \\ 1 - \mathrm{N}(\mathfrak{p}) & \text{if } \lambda \in \Lambda(\mathfrak{p}\mathfrak{I}) \otimes_{\mathcal{O}} \widehat{\mathcal{O}} \text{ and } \lambda_1 \in \mathfrak{I}^{-1} \otimes_{\mathcal{O}} \widehat{\mathcal{O}}. \end{cases}$$

Proof of Proposition 3.4. Fix $Y \in \wedge^{N-1}\mathfrak{p}$ and a vector $v \in k^N$. Define $\widetilde{\phi_f} \in \mathcal{S}(V_k(\mathbb{A}_f))$ by $\widetilde{\phi_f}(\lambda) = \phi_f(\lambda - v; \mathfrak{p}, \mathfrak{I})$. For $g = (g_f, g_\infty) \in \mathrm{SL}_N(\mathbb{A}_k)$, polynomials P and $\overline{Q}$ and $t > 0$, define

$$\theta_{P\otimes\overline{Q}}(g, t) = \sum_{\lambda \in k^N} \widetilde{\phi_f}(g_f^{-1}\lambda) \psi(tg_\infty^{-1}\lambda, Y; P \otimes \overline{Q})$$

and

$$\begin{aligned}
E_{P\otimes\overline{Q}}(g, s) &= \int_0^\infty \theta_{P\otimes\overline{Q}}(g, t) t^{s+N+p-q} \frac{dt}{t} \\
&= \sum_{\lambda \in k^N} \widetilde{\phi_f}(g_f^{-1}\lambda) \eta(tg_\infty^{-1}\lambda, Y; P \otimes \overline{Q}, s).
\end{aligned}$$

Then our Eisenstein series $E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{I}))$ satisfies

$$E_{\mathfrak{p}}(v, (g_\infty)_*Y; P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{I}), s) = E_{g_\infty^{-1}(P\otimes\overline{Q})}((g_f = 1, g_\infty), s).$$

Since $E_{P\otimes\overline{Q}}$ is linear in P and Q and $V^{p,q}$ is a polynomial representation of G, it follows that $g_\infty^{-1}(P \otimes \overline{Q})$ grows at most polynomially on any Siegel set. It follows that it suffices to show that $E_{P\otimes\overline{Q}}((g_f = 1, g_\infty), s)$ is rapidly decreasing on every Siegel set for the cusp defined by $W_\bullet$ for all $P \otimes \overline{Q} \in V^{p,q}$; since $E_{P\otimes\overline{Q}}(g, s)$ is an automorphic form, we can check this by showing that the constant term

$$E_{P\otimes\overline{Q}}((g_f = 1, g_\infty), s)_{\mathbf{N}} = \int_{\mathbf{N}(\mathbb{Q})\backslash\mathbf{N}(\mathbb{A})} E_{P\otimes\overline{Q}}(n(g_f = 1, g_\infty), s)\, dn$$

vanishes, where $\mathbf{N}$ denotes the unipotent radical of the parabolic $\mathbf{P}$ corresponding to $W_\bullet$. Let us fix P and Q and drop $P \otimes \overline{Q}$ from the notation and write simply $E(g, s)$ and $E(g, s)_{\mathbf{N}}$. By transitivity of constant terms, we may assume that $\mathbf{P}$ is maximal, i.e. that the flag $W_\bullet$ consists of just one proper subspace W_0 of k^N. Note that under our assumptions on v we have $\widetilde{\phi_f}(\lambda) = 0$ for $\lambda \in W_0$. For $\lambda \in k^N - W_0$, the orbit of λ under $\mathbf{N}(\mathbb{Q})$ is $\lambda + W_0$. Writing $\mathbf{N}_\lambda(\mathbb{Q})$ for the stabilizer of λ in $\mathbf{N}(\mathbb{Q})$, we have

$$\begin{aligned}
E((g_f = 1, g_\infty), s)_{\mathbf{N}} &= \int_{\mathbf{N}(\mathbb{Q})\backslash\mathbf{N}(\mathbb{A})} \Bigg(\sum_{\lambda \in k^N - W_0} \widetilde{\phi_f}(n_f^{-1}\lambda) \\
&\qquad \cdot \int_0^\infty \psi(tg_\infty^{-1} n_\infty^{-1}\lambda, Y; P \otimes \overline{Q}) t^{s+N+p-q} \frac{dt}{t} \Bigg) dn \\
&= \sum_{\substack{\lambda \in k^N / W_0 \\ \lambda \neq 0}} \int_{\mathbf{N}(\mathbb{Q})\backslash\mathbf{N}(\mathbb{A})} \Bigg(\sum_{n' \in \mathbf{N}_\lambda(\mathbb{Q})\backslash\mathbf{N}(\mathbb{Q})} \widetilde{\phi_f}((n'n_f)^{-1}\lambda) \\
&\qquad \cdot \eta(tg_\infty^{-1}(n'n_\infty)^{-1}\lambda, Y; P \otimes \overline{Q}, s) \Bigg) dn \\
&= \mathrm{Vol}(\mathbf{N}_\lambda(\mathbb{Q})\backslash\mathbf{N}_\lambda(\mathbb{A}_f)) \sum_{\substack{\lambda \in k^N / W_0 \\ \lambda \neq 0}} \int_{\mathbf{N}_\lambda(\mathbb{A}_f)\backslash\mathbf{N}(\mathbb{A}_f)} \widetilde{\phi_f}(n_f^{-1}\lambda)\, dn_f \\
&\qquad \cdot \int_{\mathbf{N}_\lambda(\mathbb{R})\backslash\mathbf{N}(\mathbb{R})} \eta(tg_\infty^{-1} n_\infty^{-1}\lambda, Y; P \otimes \overline{Q}, s)\, dn_\infty.
\end{aligned}$$

Let $\gamma \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$ such that $l := \langle \gamma e_1 \rangle \subseteq W_0$. As the Schwartz function $\phi_{\mathfrak{p}}(\mathfrak{I})$, the $\mathfrak{p}$-component of ϕ_f, satisfies

$$\int_{k_{\mathfrak{p}}} \phi_{\mathfrak{p}}(w + xe_1; \mathfrak{I})\, dx = 0, \quad w \in V_k(k_{\mathfrak{p}}),$$

using that $\phi_{\mathfrak{p}}(\mathfrak{I})$ is invariant under $\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$, we compute

$$\begin{aligned}
\int_{\mathbf{N}_\lambda(\mathbb{A}_f)\backslash\mathbf{N}(\mathbb{A}_f)} \widetilde{\phi_f}(n_f^{-1}\lambda)\, dn_{\mathfrak{p}} &= \int_{W_0(\mathbb{A}_f)} \widetilde{\phi_f}(\lambda + w)\, dw \\
&= \int_{W_0(\mathbb{A}_f)} \phi_f(-v + \lambda + w)\, dw \\
&= \int_{W_0(\mathbb{A}_f)/l(\mathbb{A}_f)} \int_{\mathbb{A}_f} \phi_f(-v + \lambda + w' + x\gamma e_1)\, dx\, dw' \\
&= 0,
\end{aligned}$$

showing that indeed the constant term is zero. □

3.4. Tits compactification and modular symbols. First recall that the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ is a simplicial set whose non-degenerate simplices are in bijection with (proper) rational parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$, or equivalently with proper k-rational flags

$$W_\bullet : 0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_r \subsetneq k^N, \quad r \geq 0.$$

The stabilizer $\mathbf{P}(W_\bullet)$ of this flag is a rational parabolic of $\mathbf{G}$ that defines an r-simplex in $\Delta_{\mathbb{Q}}(\mathbf{G})$. Its i-th face is the simplex corresponding to the flag obtained from $W_\bullet$ by deleting W_i (degenerate simplices correspond to proper flags where we allow $W_i = W_{i+1}$ for any i).

For a parabolic subgroup $\mathbf{P}$, denote by $\mathbf{N_P}$ its unipotent radical and write $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$ for its Levi quotient, $\mathbf{S_P}$ for the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{L_P}$ and $A_\mathbf{P} = \mathbf{S_P}(\mathbb{R})^0$ for the identity component of the real points of $\mathbf{S_P}$. Writing $X(\mathbf{L_P})_\mathbb{Q}$ for the group of rational characters of $\mathbf{L_P}$, we define $\mathbf{M_P} = \bigcap_{\alpha \in X(\mathbf{L_P})_\mathbb{Q}} \ker \alpha^2$. Then we have the direct product decomposition

$$\mathbf{L_P}(\mathbb{R}) = \mathbf{M_P}(\mathbb{R}) A_\mathbf{P}.$$

The simplex in $\Delta_\mathbb{Q}(\mathbf{G})$ corresponding to $\mathbf{P}$ admits a natural geometric realization. To define it, let $\mathfrak{a}_\mathbf{P}$ and $\mathfrak{n}_\mathbf{P}$ be the Lie algebras of $A_\mathbf{P}$ and $N_\mathbf{P}$ respectively, and let $\Phi^+(P, A_\mathbf{P})$ be the set of roots for the adjoint action of $\mathfrak{a}_\mathbf{P}$ on $\mathfrak{n}_\mathbf{P}$. These roots define a positive chamber

$$\mathfrak{a}_\mathbf{P}^+ = \{H \in \mathfrak{a}_\mathbf{P} : \alpha(H) > 0,\ \alpha \in \Phi^+(P, A_\mathbf{P})\}.$$

Writing $\langle \cdot, \cdot \rangle$ for the Killing form on $\mathfrak{g}$, we define an open simplex

$$\mathfrak{a}_\mathbf{P}^+(\infty) = \{H \in \mathfrak{a}_\mathbf{P}^+ : \langle H, H \rangle = 1\} \subset \mathfrak{a}_\mathbf{P}^+$$

and a closed simplex

$$\overline{\mathfrak{a}_\mathbf{P}^+}(\infty) = \{H \in \mathfrak{a}_\mathbf{P} : \alpha(H) \geq 0,\ \langle H, H \rangle = 1,\ \alpha \in \Phi^+(P, A_\mathbf{P})\}$$

in $\mathfrak{a}_\mathbf{P}$. Note that for $\mathbf{P}$ maximal the Lie algebra $\mathfrak{a}_\mathbf{P}$ is one-dimensional and so $\overline{\mathfrak{a}_\mathbf{P}^+}(\infty)$ is just a point. Moreover, if $\mathbf{Q}$ is another rational parabolic, then $\mathfrak{a}_\mathbf{Q}^+(\infty)$ is a face of the closed simplex $\mathfrak{a}_\mathbf{P}^+(\infty)$ if and only if $\mathbf{P} \subseteq \mathbf{Q}$. It follows that $\mathfrak{a}_\mathbf{P}^+(\infty)$ gives a geometric realization of the simplex in $\Delta_\mathbb{Q}(\mathbf{G})$ corresponding to $\mathbf{P}$, and so the Tits building $\Delta_\mathbb{Q}(\mathbf{G})$ admits the geometric realization

$$|\Delta_\mathbb{Q}(\mathbf{G})| \sim \coprod_\mathbf{P} \overline{\mathfrak{a}_\mathbf{P}^+}(\infty)/\sim, \tag{3.6}$$

where the union runs over all proper rational parabolics $\mathbf{P}$ of $\mathbf{G}$ and $\sim$ is the equivalence relation induced by the identification of $\mathfrak{a}_\mathbf{Q}^+(\infty)$ with a face of $\mathfrak{a}_\mathbf{P}^+(\infty)$ whenever $\mathbf{P} \subseteq \mathbf{Q}$. As a set we may write

$$|\Delta_\mathbb{Q}(\mathbf{G})| = \coprod_\mathbf{P} \mathfrak{a}_\mathbf{P}^+(\infty)$$

as a disjoint union of open simplexes $\mathfrak{a}_\mathbf{P}^+(\infty)$.

3.4.1. Tits compactification. Here we follow [23] and [6, Section III.12]. The Tits compactification ${}_\mathbb{Q}\overline{X}^T$ has boundary $|\Delta_\mathbb{Q}(\mathbf{G})|$: as a set we have

$${}_\mathbb{Q}\overline{X}^T = X \cup \coprod_\mathbf{P} \mathfrak{a}_\mathbf{P}^+(\infty).$$

The topology on ${}_\mathbb{Q}\overline{X}^T$ can be described in terms of convergent sequences (for a full description see [6]). Note that we have fixed $x_0 \in X$ corresponding to the maximal compact subgroup $K = \mathrm{SU}(N) \subset G = \mathbf{G}(\mathbb{R})$ and hence a unique Cartan involution θ of G that fixes K and extends to $\mathbf{G}$ (namely, $\theta(g) = {}^t\overline{g}^{-1}$). There is a unique section $i_0 : \mathbf{L_P} \to \mathbf{P}$ of the quotient map $\mathbf{P} \to \mathbf{L_P}$ with image invariant under θ. We write

$$P = \mathbf{P}(\mathbb{R}), \quad N_\mathbf{P} = \mathbf{N_P}(\mathbb{R}), \quad A_\mathbf{P}(x_0) = i_0(A_\mathbf{P}), \quad M_\mathbf{P}(x_0) = i_0(\mathbf{M_P}(\mathbb{R}))$$

and obtain the Langlands decomposition (explicitly given by (3.4) for standard parabolics)

$$P = N_{\mathbf{P}} A_{\mathbf{P}}(x_0) \mathbf{M}_{\mathbf{P}}(x_0).$$

Writing $X_{\mathbf{P}} = \mathbf{M}_{\mathbf{P}}(x_0)/(K \cap \mathbf{M}_{\mathbf{P}}(x_0))$, this induces a diffeomorphism

$$N_{\mathbf{P}} \times A_{\mathbf{P}}(x_0) \times X_{\mathbf{P}} \to X, \quad (n, a, mK) \mapsto namK. \tag{3.7}$$

The topology on ${}_{\mathbb{Q}}\overline{X}^T$ is characterized by the following properties:

(1) The subspace topology on the boundary $|\Delta_{\mathbb{Q}}(\mathbf{G})|$ is the quotient topology given by (3.6).
(2) Let $x \in X$. A sequence $x_n \in {}_{\mathbb{Q}}\overline{X}^T$, $n \geq 1$, converges to x if and only if $x_n \in X$ for $n \gg 0$ and x_n converges to x in the usual topology of X.
(3) Let $H_\infty \in \mathfrak{a}_{\mathbf{P}}^+(\infty)$ and let $(x_j)_{j \geq 1}$ be a sequence in X. Write $x_j = n_j \exp(H_j) m_j$ for unique $n_j \in N_{\mathbf{P}}$, $H_j \in \mathfrak{a}_{\mathbf{P}}$ and $m_j \in X_{\mathbf{P}}$ according to the horospherical decomposition (3.7). Then $x_j \to H_\infty$ if and only if x_j is unbounded and
   (i) $H_j/\|H_j\| \to H_\infty$ in $\mathfrak{a}_{\mathbf{P}}$,
   (ii) $d(n_j m_j x_0, x_0)/\|H_j\| \to 0$,

   where d denotes the Riemannian distance on X.

With this topology, ${}_{\mathbb{Q}}\overline{X}^T$ is a Hausdorff space on which $\mathbf{G}(\mathbb{Q})$ acts continuously.

Given points $x \in X$ and $x' \in {}_{\mathbb{Q}}\overline{X}^T$, we denote by $[x, x']$ the unique oriented geodesic segment starting at x and ending at x'. More explicitly, if $x' \in X$, we define $[x, x']$ to be the image of

$$s(x, x') : [0,1] \to {}_{\mathbb{Q}}\overline{X}^T, \quad t \mapsto s(t; x, x'), \tag{3.8}$$

the constant speed parametrization by the unit interval of the unique oriented geodesic segment with $s(0; x, x') = x$ and $s(1; x, x') = x'$. If x' belongs to the boundary of ${}_{\mathbb{Q}}\overline{X}^T$, then there exists a unique parabolic subgroup $\mathbf{P}$ such that x' corresponds to $H_\infty \in \mathfrak{a}_{\mathbf{P}}^+(\infty)$. In the coordinates given by (3.7), we have $x = n \exp(H) m$, and we define $[x, x']$ to be the image of the map

$$s(x, x') : [0,1] \to {}_{\mathbb{Q}}\overline{X}^T, \quad t \mapsto s(t; x, x') = \begin{cases} n \exp(H + \frac{t}{1-t} H_\infty) m & \text{if } t < 1, \\ x' & \text{if } t = 1. \end{cases} \tag{3.9}$$

Given subsets $S \subset X$ and $S' \subset {}_{\mathbb{Q}}\overline{X}^T$, the cone $C(S, S')$ (also known as the join $S * S'$) is the subset of ${}_{\mathbb{Q}}\overline{X}^T$ defined as

$$C(S, S') = \bigcup_{\substack{x \in S \\ x' \in S'}} [x, x'].$$

If $S = \{x\}$, we say that $C(S, S')$ is the cone on S' with vertex x. When S' is given by a simplicial map $\Delta_r \to \Delta_{\mathbb{Q}}(\mathbf{G})$ into the Tits boundary, the cone on S' with vertex x is naturally the image of an $(r+1)$-simplex $|\Delta_{r+1}| \to {}_{\mathbb{Q}}\overline{X}^T$ (oriented so that the boundary orientation agrees with that of S'). More generally, if S is given by a simplicial map $|\Delta_k| \to X$ and S' is given by a simplicial map $\Delta_r \to \Delta_{\mathbb{Q}}(\mathbf{G})$, then the cone $C(S, S')$ is the image of a map

$$|\Delta_k| \times |\Delta_r| \times [0,1] \to_{\mathbb{Q}} \overline{X}^T$$

that factors through the join

$$|\Delta_{k+r+1}| \simeq |\Delta_k * \Delta_r| \to_{\mathbb{Q}} \overline{X}^T. \tag{3.10}$$

3.4.2. Modular symbols. For $k \geq 0$, let Δ'_k be the first barycentric subdivision of the standard k-simplex. Its vertices are in bijection with the non-empty subsets of $\{0, \dots, k\}$, and a set of vertices $\{v_0, \dots, v_r\}$ forms an r-simplex if and only if they are linearly ordered, i.e. $v_0 \subseteq \cdots \subseteq v_r$. Denote this simplex by $\Delta_{v_0,\dots,v_r}$.

For a collection $\underline{\gamma} = (\gamma_0, \dots, \gamma_{k-1})$ of $k \leq N$ elements of $\Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$, let us define a continuous map

$$\Delta(\underline{\gamma}) : |\Delta'_{k-1}| \to {}_{\mathbb{Q}}\overline{X}^T.$$

Assume first $\langle \gamma_0 e_1, \dots, \gamma_{k-1} e_1 \rangle \neq k^N$. For each chain $v_0 \subseteq \cdots \subseteq v_r$ defining an r-simplex in Δ'_{k-1}, the flag

$$0 \subsetneq \langle \gamma_i e_1 \mid i \in v_0 \rangle \subseteq \langle \gamma_i e_1 \mid i \in v_1 \rangle \subseteq \cdots \subseteq \langle \gamma_i e_1 \mid i \in v_r \rangle \subsetneq k^N \tag{3.11}$$

is a proper flag of length r. We define $\Delta(\underline{\gamma})(\Delta_{v_0,\dots,v_r})$ to be the corresponding (possibly degenerate) r-simplex in $\Delta_{\mathbb{Q}}(\mathbf{G})$; we give this simplex the orientation induced by $\Delta(\underline{\gamma})$ by the standard orientation on Δ'_{k-1}. This assignment preserves faces and degeneracies and so defines a simplicial map $\Delta(\underline{\gamma})$.

Next assume that $k = N$ and the vectors $\gamma_0 e_1, \dots, \gamma_{N-1} e_1$ are linearly independent. Define

$$A(\underline{\gamma}) = (\gamma_0 e_1 | \cdots | \gamma_{N-1} e_1) \in M_N(\mathcal{O}) \cap \mathrm{GL}_N(k)$$

to be the matrix formed by the first columns of $\gamma_0, \dots, \gamma_{N-1}$. Fix an N-th root $(\det A(\underline{\gamma}))^{-\frac{1}{N}}$ of $\det A(\underline{\gamma})^{-1}$ and let $a(\underline{\gamma}) = (\det A(\underline{\gamma}))^{-\frac{1}{N}} A(\underline{\gamma})$; the matrix $a(\underline{\gamma})$ has determinant one and defines a point

$$x_0(\underline{\gamma}) = a(\underline{\gamma}) K \in X \tag{3.12}$$

(independent of the choice of N-th root above). Suppose that $v_0 \subsetneq \cdots \subsetneq v_r$ is a chain defining a non-degenerate r-simplex in Δ'_{N-1}. If $v_r \neq \{0, \dots, N-1\}$, then we define $\Delta(\underline{\gamma})(\Delta_{v_0,\dots,v_r})$ to be the r-simplex of $\Delta_{\mathbb{Q}}(\mathbf{G})$ corresponding to the flag (3.11). If $v_r = \{0, \dots, N-1\}$, then we define

$$\Delta(\underline{\gamma})(\Delta_{v_0,\dots,v_r}) = \text{cone on } \Delta(\underline{\gamma})(\Delta_{v_0,\dots,v_{r-1}}) \text{ with vertex } x_0(\underline{\gamma}). \tag{3.13}$$

These assignments are compatible with face maps and therefore give rise to a well-defined continuous map $\Delta(\underline{\gamma}) : \Delta'_{N-1} \to {}_{\mathbb{Q}}\overline{X}^T$. By induction on k one shows that

$$\Delta(\gamma'\gamma_0, \dots, \gamma'\gamma_{k-1}) = \gamma' \Delta(\gamma_0, \dots, \gamma_{k-1}), \qquad \text{for } \gamma' \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I})).$$

Note that when the vectors $\gamma_i e_1$ are linearly dependent, the image of the map $\Delta(\underline{\gamma})$ is contained in the boundary of ${}_{\mathbb{Q}}\overline{X}^T$. When they are linearly independent, the intersection

$$\Delta^\circ(\underline{\gamma}) := X \cap \mathrm{Im}(\Delta(\underline{\gamma}))$$

of the image of $\Delta(\underline{\gamma})$ with the interior X of ${}_{\mathbb{Q}}\overline{X}^T$ is a submanifold of dimension $N-1$, namely

$$\Delta^\circ(\underline{\gamma}) = \left\{ a(\underline{\gamma}) \mathrm{diag}(t_1, \dots, t_N) K : t_i \in \mathbb{R}_{>0}, \prod_{i=1}^{N} t_i = 1 \right\} \subset G/K = X. \tag{3.14}$$

(To see this, we may assume that $\gamma_i e_1 = e_i$, so that $a(\underline{\gamma})$ is the identity matrix 1_N. Consider first a non-degenerate simplex $\Delta_{v_0,\dots,v_r}$ in Δ'_{N-1} with $v_k = \{1, \dots, |v_k|\}$. If $|v_r| < N$, then the r-simplex $\Delta(1_N)(\Delta_{v_0,\dots,v_r})$ in the Tits compactification of X corresponds to the standard proper flag

$$0 \subsetneq \langle e_i \mid i \le |v_0|\rangle \subsetneq \cdots \subsetneq \langle e_i \mid i \le |v_r|\rangle \neq V_k,$$

and the subgroup $A_{\mathbf{P}}$ of the corresponding parabolic $\mathbf{P}$ is

$$A_{\mathbf{P}} = \{a(t_0,\dots,t_{r+1}) : t_i > 0,\ \det a(t_0,\dots,t_{r+1}) = 1\},$$

where

$$a(t_0,\dots,t_{r+1}) := \begin{pmatrix} t_0\cdot 1_{|v_0|} & & & \\ & t_1\cdot 1_{|v_1|-|v_0|} & & \\ & & \ddots & \\ & & & t_{r+1}1_{N-|v_r|} \end{pmatrix}.$$

If $|v_r| = N$, then the cone of $\Delta(1_N)(\Delta_{v_0,\dots,v_{r-1}})$ with vertex x_0 is

$$\{a(t_0,\dots,t_{r+1})K \mid a(t_0,\dots,t_{r+1}) \in A_{\mathbf{P}}, t_0 \ge \cdots \ge t_{r+1}\}.$$

The statement follows since any r-simplex $\Delta(1_N)(\Delta_{v_0,\dots,v_r})$ can be obtained as a translate of a simplex corresponding to a standard flag as above by a Weyl group element.)

The coordinates

$$a(\underline{\gamma})\mathrm{diag}(t_1,\dots,t_N)K \mapsto t_i$$

identify $\Delta^\circ(\underline{\gamma})$ with the manifold C defined in (2.14). This isomorphism is orientation preserving.[2)] For convenience we define $\Delta^\circ(\underline{\gamma}) = \emptyset$ if $A(\underline{\gamma})$ is not invertible.

Note that (3.14) implies that $\Delta^\circ(\underline{\gamma})$ admits a finite cover by $\mathrm{SL}_N(k)$-translates of (images in X of) standard Siegel sets of the form (3.5). One may take this cover to consist of one Siegel set for every parabolic $\mathbf{P}$ stabilizing a flag consisting of subspaces of the form $\langle \gamma_i e_1 \mid i \in I\rangle$ for $I \subsetneq \{0,\dots,N-1\}$.

3.5. Evaluation on modular symbols and the cocycle property. We can now relate the Eisenstein series $E_{\mathfrak{p}}(v, P\otimes\overline{Q};\psi,\Lambda(\mathfrak{I}))$ and the Eisenstein cocycle.

Proposition 3.5. *Assume that v does not lie in any $\Lambda(\mathfrak{p}\mathfrak{I})$-translate of a proper subspace of V of the form $\langle\gamma_i e_1 \mid i \in I\rangle$ for $I \subseteq \{0,\dots,N-1\}$. Then $\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\,\cdot\,,\underline{\gamma},\Lambda(\mathfrak{I}))$ is defined at v and*

$$\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(v,\underline{\gamma},\Lambda(\mathfrak{I}))(P\otimes\overline{Q}) = \int_{\Delta^\circ(\underline{\gamma})} E_{\mathfrak{p}}(v, P\otimes\overline{Q};\psi^{p,q},\Lambda(\mathfrak{I})).$$

Proof. Consider the matrix $A(\underline{\gamma}) = (\gamma_0 e_1|\dots|\gamma_{N-1}e_1)$. If $A(\underline{\gamma})$ is not invertible, then both sides are zero by definition. Now assume that $A(\underline{\gamma})$ is invertible and take $A(\underline{\gamma})^{-1}P$ and $\overline{A(\underline{\gamma})^{-1}Q}$ to be monomial, say

$$A(\underline{\gamma})^{-1}P(z) = z^I = z_1^{i_1}\cdots z_N^{i_N} \quad\text{and}\quad \overline{A(\underline{\gamma})^{-1}Q}(\overline{z}) = \overline{z}^J = \overline{z_1}^{j_1}\cdots\overline{z_N}^{j_N};$$

it suffices to show that with this choice of P and $\overline{Q}$ we have

$$\int_{\Delta^\circ(\underline{\gamma})} E_{\mathfrak{p}}(v, P\otimes\overline{Q};\psi^{p,q},\Lambda(\mathfrak{I})) = D_{\mathfrak{p}}^{I,J}(v, A(\underline{\gamma}),\Lambda(\mathfrak{I})).$$

[2)] Recall that we have defined the orientation of $\Delta^\circ(\underline{\gamma})$ to be induced by the boundary and that the orientation on C is fixed in Section 2.6.

Note that the proof of Proposition 3.4 shows that, for any s, the Eisenstein series $E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{I}), s)$ is rapidly decreasing on any Siegel set corresponding to a good cusp; since $\Delta^\circ(\underline{\gamma})$ admits a finite cover by such Siegel sets, it follows that $E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi, \Lambda(\mathfrak{I}), s)$ is integrable over $\Delta^\circ(\underline{\gamma})$. For $\mathrm{Re}(s) \gg 0$, we compute

$$\begin{aligned}
&\int_{\Delta^\circ(\underline{\gamma})} E(v, P \otimes \overline{Q}; \psi^{p,q}, \Lambda(\mathfrak{I}), s) \\
&\quad = \sum_{\lambda \in \Lambda(\mathfrak{I})} \int_{\Delta^\circ(\underline{\gamma})} \eta^{p,q}(v + \lambda, P \otimes \overline{Q}, s) \\
&\quad = \sum_{\lambda \in \Lambda(\mathfrak{I})} \int_{T/T\cap K} a(\underline{\gamma})^* \eta^{p,q}(v + \lambda, P \otimes \overline{Q}, s) \\
&\quad = \sum_{\lambda \in \Lambda(\mathfrak{I})} \int_{T/T\cap K} \eta^{p,q}(a(\underline{\gamma})^{-1}(v + \lambda), a(\underline{\gamma})^{-1}(P \otimes \overline{Q}), s) \\
&\quad = |\det A(\underline{\gamma})|^{-\frac{s}{N}} \det A(\underline{\gamma})^{-1} \sum_{\lambda \in \Lambda(\mathfrak{I})} \int_{T/T\cap K} \eta^{p,q}(A(\underline{\gamma})^{-1}(v + \lambda), A(\underline{\gamma})^{-1}(P \otimes \overline{Q}), s),
\end{aligned}$$

where the last equality follows from the homogeneity property (2.12). The desired identity follows by analytic continuation from Lemma 2.5. □

We can now use Proposition 3.5 to prove that $\mathbf{\Phi}_{\mathfrak{p}}^{p,q}(\Lambda(\mathfrak{I}))$ is indeed an $(N-1)$-cocycle, i.e. that it satisfies property (3.3).

Given $N+1$ elements $\gamma_0, \dots, \gamma_N \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$, write S_j for $\Delta(\gamma_0, \dots, \widehat{\gamma_j}, \dots, \gamma_N)$ and fix $x \in X$ such that

$$x \notin \bigcup_{0 \le j \le N} S_j.$$

Then S_j is an (oriented) $(N-1)$-simplex in the boundary of ${}_{\mathbb{Q}}\overline{X}^T$, and we denote by $(-1)^j S_j$ the same simplex with opposite orientation if j is odd. Note that $\sum_j (-1)^j S_j$ is a cycle, i.e. $\sum (-1)^j \partial S_j = 0$. For each j with $0 \le j \le N$, we next define an N-simplex

$$C_j : |\Delta_N| \to {}_{\mathbb{Q}}\overline{X}^T.$$

Assume first that $\langle \gamma_0 e_1, \dots, \widehat{\gamma_j e_1}, \dots, \gamma_N e_1 \rangle \neq k^N$. We define C_j to be the cone on $(-1)^j S_j$ with vertex x (cf. Section 3.4.1); its boundary is $\partial C_j = (-1)^j S_j - (-1)^j C(x, \partial S_j)$.

Now assume that $\langle \gamma_0 e_1, \dots, \widehat{\gamma_j e_1}, \dots, \gamma_N e_1 \rangle = k^N$. Then the intersection S_j° of S_j with X is non-empty, and in (3.12) we have defined a barycenter $x_j := x_0(\gamma_0, \dots, \widehat{\gamma_j}, \dots, \gamma_N) \in S_j^\circ$ such that S_j is the cone on ∂S_j with vertex x_j. We define C_j to be the cone $C([x, x_j], \partial S_j)$, where $[x, x_j]$ denotes the oriented geodesic segment from x to x_j. More explicitly, let

$$s : [0,1] \to [x, x_j]$$

be the constant speed parametrization of the geodesic segment joining $s(0) = x$ to $s(1) = x_j$. For each simplex $\Delta(\gamma_0, \dots, \widehat{\gamma_j}, \dots, \gamma_N)(\Delta_{v_0,\dots,v_r})$ contained in ∂S_j, let $\mathbf{P}$ be the corresponding parabolic; writing $s(t) = n(t)\exp(H_t)m(t)$, we obtain a map

$$[0,1] \times \mathfrak{a}_{\mathbf{P}}^+ \to X, \quad (t, H') \mapsto n(t)\exp(H_t + H')m(t), \tag{3.15}$$

whose closure is $C([x, x_j], \Delta(\gamma_0, \dots, \widehat{\gamma_j}, \dots, \gamma_N)(\Delta_{v_0,\dots,v_r}))$. Since ∂S_j has empty boundary, the boundary of C_j is the union of S_j (= the cone on S_j° with vertex x_j) and the cone

$C(x, \partial S_j)$; we orient C_j so that the induced orientation on S_j is that given by (3.13), so that $\partial C_j = (-1)^j S_j - (-1)^j C(x, \partial S_j)$.

It follows that the sum $\Delta(\gamma_0, \ldots, \gamma_N) = \sum_j C_j$ has boundary $\sum_j (-1)^j S_j$. The cocycle property (3.3) follows immediately from Stokes' theorem and the following lemma.

Lemma 3.6. *Assume that v does not lie in any $\Lambda(\mathfrak{p}\mathfrak{I})$-translate of a proper subspace of V of the form $\langle \gamma_i e_1 \mid i \in I \rangle$ for $I \subseteq \{0, \ldots, N\}$. Then the Eisenstein series $E_{\mathfrak{p}}(v, P \otimes \overline{Q}; \psi^{p,q}, \Lambda(\mathfrak{I}))$ is rapidly decreasing on $\Delta(\gamma_0, \ldots, \gamma_N)$.*

Proof. By Theorem 3.4, it suffices to show that each C_j can be covered by finitely many Siegel sets of good cusps, which is obvious from the explicit description (3.15). □

4. Eisenstein cocycle and critical values of Hecke L-series

4.1. Units of extensions of k. Let L be a field extension of k of degree $N \geq 2$. We denote its ring of integers by $\mathcal{O}_L$ and write $\sigma_1, \ldots, \sigma_N$ for the complex embeddings of L in $\mathbb{C}$ extending σ. We obtain an embedding

$$\underline{\sigma} \in \mathrm{Hom}_{\mathcal{O}}(L, \mathbb{C}^N), \quad \underline{\sigma}(l) = (\sigma_1(l), \ldots, \sigma_N(l)).$$

Let $n : L^\times \to k^\times$ be the norm map and L^1 be the kernel of n. We fix ideals $\mathfrak{a}$, $\mathfrak{P}$ and $\mathfrak{f}$ of $\mathcal{O}_L$ that are pairwise coprime and such that $\mathfrak{p} := n(\mathfrak{P})$ is prime. We let $U(\mathfrak{f}) = \mathcal{O}_L^\times \cap (1 + \mathfrak{f})$ and $U(\mathfrak{f})^1 = U(\mathfrak{f}) \cap L^1$. We denote by $U(\mathfrak{f})^1_{\mathrm{tors}}$ the torsion subgroup of $U(\mathfrak{f})^1$ and fix units $u_1, \ldots, u_{N-1} \in U(\mathfrak{f})$ that generate a subgroup $U(\mathfrak{f})' := \langle u_1, \ldots, u_{N-1} \rangle$ of $U(\mathfrak{f})^1$ that is free abelian of rank $N - 1$ and maps bijectively to $U(\mathfrak{f})^1 / U(\mathfrak{f})^1_{\mathrm{tors}}$ via the quotient map.

Lemma 4.1. *Let $\mathfrak{I}$ be a fractional ideal of $\mathcal{O}$ coprime to $\mathfrak{p}$. Assume that $\mathfrak{I}$ is isomorphic to $(\det_{\mathcal{O}}(\mathfrak{f}\mathfrak{a}^{-1}))^{-1}$. There exists a k-isomorphism $\alpha : L \xrightarrow{\sim} k^N$ making the diagram*

$$\begin{array}{ccc} \mathfrak{f}\mathfrak{a}^{-1} & \xrightarrow[\alpha]{\sim} & \Lambda(\mathfrak{I}) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{f}(\mathfrak{a}\mathfrak{P})^{-1} & \xrightarrow[\alpha]{\sim} & \Lambda(\mathfrak{p}\mathfrak{I}) \end{array}$$

commute.

Proof. Fix an isomorphism $\tilde{\alpha} : \mathfrak{f}\mathfrak{a}^{-1} \xrightarrow{\sim} \Lambda(\mathfrak{I})$. Then the $\mathcal{O}$-lattices $\Lambda_1 = \Lambda(\mathfrak{p}\mathfrak{I})$ and $\Lambda_2 = \tilde{\alpha}(\mathfrak{f}(\mathfrak{a}\mathfrak{P})^{-1})$ contain $\Lambda(\mathfrak{I})$ and $\Lambda_i / \Lambda(\mathfrak{I}) \simeq \mathcal{O}/\mathfrak{p}$ for $i = 1, 2$. For each finite place v of k and $i = 1, 2$ we obtain an $\mathcal{O}_v$-lattice $\Lambda_{i,v} = \Lambda_i \otimes_{\mathcal{O}} \mathcal{O}_v$ in k_v^N, and we have $\Lambda_{1,v} = \Lambda_{2,v}$ for all $v \neq \mathfrak{p}$. Pick $g_{\mathfrak{p}} \in \mathrm{SL}_N(\Lambda(\mathfrak{I})_{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \Lambda_{2,\mathfrak{p}} = \Lambda_{1,\mathfrak{p}}$ and let

$$U_{\mathfrak{p}} = \mathrm{SL}_N(\Lambda(\mathfrak{I})_{\mathfrak{p}}) \cap \mathrm{SL}_N(\Lambda_{2,\mathfrak{p}}).$$

Then $U_{\mathfrak{p}}$ is an open compact subgroup of $\mathrm{SL}_N(k_{\mathfrak{p}})$ and $U = g_{\mathfrak{p}} U_{\mathfrak{p}} \times \prod_{v \neq \mathfrak{p}} \mathrm{SL}_N(\Lambda(\mathfrak{I})_v)$ is an open subset of $\mathrm{SL}_N(\mathbb{A}_{k,f})$. As $\mathrm{SL}_N(k)$ is dense in $\mathrm{SL}_N(\mathbb{A}_{k,f})$, we may find $g \in \mathrm{SL}_N(k) \cap U$. Then g stabilizes $\Lambda(\mathfrak{I})$ and $g\Lambda_2 = \Lambda_1$ (since $g\Lambda_{2,v} = \Lambda_{1,v}$ for every finite place v), and so $\alpha := g \circ \tilde{\alpha}$ makes the diagram in the statement commute. □

From now on we fix an isomorphism α as in the above lemma. These choices define:

- a vector
$$v_0 = \alpha(1) \in k^N,$$
- a k-basis $\alpha_j = \alpha^{-1}(e_j)$ $(j = 1, \dots, N)$ of L and a matrix
$$a_\alpha = (\sigma_i(\alpha_j))^{-1} \det(\sigma_i(\alpha_j))^{\frac{1}{N}} \in \mathrm{SL}_N(\mathbb{C})$$
(here $\det(\sigma_i(\alpha_j))^{\frac{1}{N}}$ denotes a fixed N-th root of $\det(\sigma_i(\alpha_j))$),
- an inclusion
$$\iota_\alpha : L^\times \to \mathrm{GL}_N(k)$$
sending $l \in L^\times$ to the map $x \mapsto \alpha(l\alpha^{-1}(x))$, that can be described using a_α:
$$\iota_\alpha(l) = a_\alpha \mathrm{diag}(\underline{\sigma}(l)) a_\alpha^{-1}.$$

Define
$$\Gamma_1(v_0, \Lambda(\mathfrak{I})) = \{\gamma \in \Gamma(\Lambda(\mathfrak{I})) : (\gamma - 1)v_0 \in \Lambda(\mathfrak{I})\}$$
and
$$\Gamma := \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I})) \cap \Gamma_1(v_0, \Lambda(\mathfrak{I})).$$
As multiplication by $u \in U(\mathfrak{f})^1$ induces an $\mathcal{O}$-linear automorphism of $\mathfrak{f}\mathfrak{a}^{-1}$ and $\mathfrak{f}(\mathfrak{a}\mathfrak{P})^{-1}$ of determinant 1 that preserves $1 + \mathfrak{f}\mathfrak{a}^{-1}$, the restriction of ι_α to $U(\mathfrak{f})^1$ defines an inclusion
$$\iota_\alpha : U(\mathfrak{f})^1 \to \Gamma.$$

- Write $(L \otimes_{k,\sigma} \mathbb{C})^1$ for the elements of $(L \otimes_{k,\sigma} \mathbb{C})^\times$ of norm 1 and $(L \otimes_{k,\sigma} \mathbb{C})^1_c$ for the maximal compact subgroup of $(L \otimes_{k,\sigma} \mathbb{C})^1$. The map $u \mapsto \iota_\alpha(u)a_\alpha (= a_\alpha \mathrm{diag}(\underline{\sigma}(u)))$ induces an embedding

$$\iota_\alpha : (L \otimes_{k,\sigma} \mathbb{C})^1/(L \otimes_{k,\sigma} \mathbb{C})^1_c \to X \tag{4.1}$$

and hence a basepoint $x_\alpha = a_\alpha \mathrm{SU}(N) \in X$ and a map
$$\iota_\alpha : X(\mathfrak{f}) := U(\mathfrak{f})^1 \backslash (L \otimes_{k,\sigma} \mathbb{C})^1/(L \otimes_{k,\sigma} \mathbb{C})^1_c \to \Gamma \backslash X.$$

By Kronecker's theorem ("algebraic integers all of whose conjugates are of norm one are roots of unity"), the kernel of the action of $U(\mathfrak{f})^1$ on $(L \otimes_{k,\sigma} \mathbb{C})^1/(L \otimes_{k,\sigma} \mathbb{C})^1_c$ equals the torsion subgroup $U(\mathfrak{f})^1_{\mathrm{tors}}$ of $U(\mathfrak{f})^1$, and the action of
$$U(\mathfrak{f})' = \langle u_1, \dots, u_{N-1} \rangle \simeq U(\mathfrak{f})^1/U(\mathfrak{f})^1_{\mathrm{tors}}$$
on $(L \otimes_{k,\sigma} \mathbb{C})^1/(L \otimes_{k,\sigma} \mathbb{C})^1_c$ is free. Fix the orientation on $(L \otimes_{k,\sigma} \mathbb{C})^1/(L \otimes_{k,\sigma} \mathbb{C})^1_c$ associated to the canonical orientation of $\mathbb{C}^N$ and write
$$[X(\mathfrak{f})] \in \mathrm{H}_{N-1}(X(\mathfrak{f}), \mathbb{Z}) \simeq \mathrm{H}_{N-1}(\langle u_1, \dots, u_{N-1} \rangle, \mathbb{Z})$$
for the fundamental class of the (compact, oriented) $(N-1)$-manifold $X(\mathfrak{f})$. We write
$$\mathrm{cor} : \mathrm{H}_*(\langle u_1, \dots, u_{N-1} \rangle, \mathbb{Q}) \to \mathrm{H}_*(U(\mathfrak{f})^1, \mathbb{Q})$$
and
$$\mathrm{res} : \mathrm{H}_*(U(\mathfrak{f})^1, \mathbb{Q}) \to \mathrm{H}_*(\langle u_1, \dots, u_{N-1} \rangle, \mathbb{Q})$$

for the corestriction and restriction maps respectively and set

$$Z_{\mathfrak{f}} = [U(\mathfrak{f}) : U(\mathfrak{f})']^{-1} \mathrm{cor}[X(\mathfrak{f})] \in \mathrm{H}_{N-1}(U(\mathfrak{f})^1, \mathbb{Q}).$$

- The embeddings $\sigma_1, \ldots, \sigma_N : L \to \mathbb{C}$ give a basis for $(L \otimes_{k,\sigma} \mathbb{C})^\vee$; let us denote by $\frac{\partial}{\partial \sigma_1}, \ldots, \frac{\partial}{\partial \sigma_N}$ the dual basis of $(L \otimes_{k,\sigma} \mathbb{C})^{\vee\vee} \simeq L \otimes_{k,\sigma} \mathbb{C}$. Writing $\alpha_{\mathbb{C}} = \alpha \otimes 1$ for the extension of $\alpha : L \to k^N$ to an isomorphism $L \otimes_{k,\sigma} \mathbb{C} \to V$, we define polynomials

$$\begin{aligned} P_\alpha &= \alpha_{\mathbb{C}}\left(\frac{\partial}{\partial \sigma_1}\right) \cdots \alpha_{\mathbb{C}}\left(\frac{\partial}{\partial \sigma_N}\right) \in \mathrm{Sym}^N V, \\ \overline{Q_\alpha} &= \overline{n} \circ \alpha_{\mathbb{C}}^{-1} \in \mathrm{Sym}^N \overline{V}^\vee. \end{aligned}$$

Note that these polynomials satisfy

$$a_\alpha^{-1} P_\alpha = \det(\sigma_i(\alpha_j))^{-1}(e_1 \cdots e_N), \quad \overline{a_\alpha^{-1} Q_\alpha} = \overline{\det(\sigma_i(\alpha_j))} \cdot \overline{z_1 \cdots z_N}. \tag{4.2}$$

For non-negative integers p, q, we define

$$P_\alpha^{p,q} = p!^{-N} \cdot P_\alpha^p \otimes \overline{Q_\alpha}^q \in (V^{pN,qN})^\vee.$$

Then $P_\alpha^{p,q}$ is invariant under $\iota_\alpha(U(\mathfrak{f})^1)$. We define

$$Z_{\mathfrak{f}}^{p,q} = Z_{\mathfrak{f}} \otimes P_\alpha^{p,q} \in \mathrm{H}_{N-1}(U(\mathfrak{f})^1, (V^{pN,qN})^\vee).$$

Let

$$\mathrm{res}(E_{\mathfrak{p}}(v_0; \psi^{pN,qN}, \Lambda(\mathfrak{I}))) \in \mathrm{H}^{N-1}(U(\mathfrak{f})^1, V^{pN,qN})$$

be the cohomology class defined by the restriction of the closed form $E_{\mathfrak{p}}(v_0; \psi^{pN,qN}, \Lambda(\mathfrak{I}))$ and define

$$\begin{aligned} &\langle E_{\mathfrak{p}}(v_0; \psi^{pN,qN}, \Lambda(\mathfrak{I})), Z_{\mathfrak{f}}^{p,q} \rangle \\ &\quad = \mathrm{res}(E_{\mathfrak{p}}(v_0; \psi^{pN,qN}, \Lambda(\mathfrak{I}))) \cap Z_{\mathfrak{f}}^{p,q} \\ &\quad = [U(\mathfrak{f}) : U(\mathfrak{f})']^{-1} \int_{X(\mathfrak{f})} \iota_\alpha^* E_{\mathfrak{p}}(v_0, P_\alpha^{p,q}; \psi^{pN,qN}, \Lambda(\mathfrak{I}), s)\Big|_{s=0}. \end{aligned}$$

4.2. Partial zeta functions. Given integers $p, q \geq 0$, define the partial zeta function

$$\zeta_{\mathfrak{f}}^{p,q}(\mathfrak{a}, s) = \sum_{x \in U(\mathfrak{f}) \backslash 1 + \mathfrak{f}\mathfrak{a}^{-1}}' \frac{\overline{n(x)}^q}{n(x)^{p+1} |n(x)|^{2s}}, \quad \mathrm{Re}(s) \gg 0.$$

(Since $u\overline{u} = 1$ for every $u \in \mathcal{O}^\times$, this is well-defined provided that $p + q + 1$ is divisible by the order of the subgroup $n(U(\mathfrak{f}))$ of $\mathcal{O}^\times$, which we assume.) Define also the "$\mathfrak{P}$-smoothed" partial zeta function

$$\zeta_{\mathfrak{f},\mathfrak{P}}^{p,q}(\mathfrak{a}, s) = \mathrm{N}\mathfrak{P}^{-s} \zeta_{\mathfrak{f}}^{p,q}(\mathfrak{a}\mathfrak{P}, s) - \mathrm{N}\mathfrak{P}^{1-s} \zeta_{\mathfrak{f}}^{p,q}(\mathfrak{a}, s).$$

These partial zeta functions admit meromorphic continuation to $s \in \mathbb{C}$ that is regular at $s = 0$.

Proposition 4.2. *We have*

$$\langle E_{\mathfrak{p}}(v_0; \psi^{pN,qN}, \Lambda(\mathfrak{I})), Z_{\mathfrak{f}}^{p,q} \rangle = \det(\sigma_i(\alpha_j)) \zeta_{\mathfrak{f},\mathfrak{P}}^{p,q}(\mathfrak{a}, 0).$$

Proof. For s in the range of convergence of the Eisenstein series, we compute

$$\begin{aligned}
&\int_{X(\mathfrak{f})} \iota_\alpha^* E(v_0, P_\alpha^{p,q}; \psi^{pN,qN}, \Lambda(\mathfrak{I}), s) \\
&= \int_{U(\mathfrak{f})'\backslash (L\otimes_{k,\sigma}\mathbb{C})^1/(L\otimes_{k,\sigma}\mathbb{C})^1_c} \iota_\alpha^* \Big(\sum_{v\in v_0+\Lambda(\mathfrak{I})} \eta^{pN,qN}(v, P_\alpha^{p,q}, s) \Big) \\
&= \int_{U(\mathfrak{f})'\backslash (L\otimes_{k,\sigma}\mathbb{C})^1/(L\otimes_{k,\sigma}\mathbb{C})^1_c} \iota_\alpha^* \Big(\sum_{x\in 1+\mathfrak{f}\mathfrak{a}^{-1}} \eta^{pN,qN}(\alpha(x), P_\alpha^{p,q}, s) \Big) \\
&= \int_{U(\mathfrak{f})'\backslash (L\otimes_{k,\sigma}\mathbb{C})^1/(L\otimes_{k,\sigma}\mathbb{C})^1_c} \iota_\alpha^* \Big(\sum_{x\in U(\mathfrak{f})'\backslash 1+\mathfrak{f}\mathfrak{a}^{-1}} \sum_{u\in U(\mathfrak{f})'} \eta^{pN,qN}(\alpha(ux), P_\alpha^{p,q}, s) \Big) \\
&= \sum_{x\in U(\mathfrak{f})'\backslash 1+\mathfrak{f}\mathfrak{a}^{-1}} \int_{(L\otimes_{k,\sigma}\mathbb{C})^1/(L\otimes_{k,\sigma}\mathbb{C})^1_c} \iota_\alpha^* \eta^{pN,qN}(\alpha(x), P_\alpha^{p,q}, s).
\end{aligned}$$

Writing T for the torus of diagonal matrices in G, note that the image of ι_α is identified with the translate $a_\alpha(T/T\cap K) \subset X$. Since

$$(a_\alpha^{-1} v)_i = \det(\sigma_i(\alpha_j))^{-\frac{1}{N}} \sigma_i(\alpha^{-1}(v)),$$

using (4.2) and Lemma 2.5 and writing $\Delta = \det(\sigma_i(\alpha_j))^{-\frac{1}{N}}$, we compute

$$\begin{aligned}
&\int_{(L\otimes_{k,\sigma}\mathbb{C})^1/(L\otimes_{k,\sigma}\mathbb{C})^1_c} \iota_\alpha^* \eta^{pN,qN}(\alpha(x), P_\alpha^{p,q}, s) \\
&= \int_{T/T\cap K} \eta^{pN,qN}(a_\alpha^{-1}\alpha(x), a_\alpha^{-1}(P_\alpha^{p,q}), s) \\
&= p!^{-N} \int_{T/T\cap K} \eta^{pN,qN}(\Delta\underline{\sigma}(x), \Delta^{pN}(e_1\cdots e_N)^p \otimes \overline{\Delta}^{-qN}\overline{z_1\cdots z_N}^q, s) \\
&= p!^{-N}\Delta^{pN}\overline{\Delta}^{-qN} \int_{T/T\cap K} \eta^{pN,qN}(\Delta\underline{\sigma}(x), (e_1\cdots e_N)^p \otimes \overline{z_1\cdots z_N}^q, s) \\
&= \Delta^{pN}\overline{\Delta}^{-qN} p!^{-N}\Gamma(\tfrac{s}{2N}+1+p)^N \prod_{k=1}^N \frac{(\overline{\Delta\sigma_k(x)})^q}{|\Delta\sigma_k(x)|^{\frac{s}{N}}(\Delta\sigma_k(x))^{p+1}} \\
&= |\Delta|^{-s}\Delta^{-N} p!^{-N}\Gamma(\tfrac{s}{2N}+1+p)^N \frac{\overline{n(x)}^q}{|n(x)|^{\frac{s}{N}} n(x)^{p+1}}
\end{aligned}$$

and the statement follows. □

4.3. Moving cycles to the Tits boundary. We now give a fundamental domain $\mathcal{D}$ for the action of $\langle u_1, \dots, u_{N-1}\rangle$ on the image of the map ι_α defined in (4.1), and a decomposition of $\mathcal{D}$ into $(N-1)$-simplices indexed by the symmetric group S_{N-1}.

4.3.1. Simplices. Let us first define the relevant simplices. For any integer $k \geq 0$, $x \in X$ and $\underline{\gamma} = (\gamma_0, \dots, \gamma_k) \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))^{k+1}$, we define a continuous map

$$\widetilde{\Delta}(\underline{\gamma}, x) : |\Delta_k| \to X$$

inductively on k as follows:

- For $k = 0$ we have $\Delta_k = \{*\}$ and we set $\widetilde{\Delta}(\gamma_0, x)(*) = \gamma_0 x$.
- Assume that $k \geq 1$ and that we have defined $\widetilde{\Delta}(\gamma_0, \dots, \gamma_{k-1}, x)$ for every collection of elements $\gamma_0, \dots, \gamma_{k-1} \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{F}))$. We define $\widetilde{\Delta}(\gamma_0, \dots, \gamma_k, x)$ to be the cone on $\widetilde{\Delta}(\gamma_0, \dots, \gamma_{k-1}, x)$ with vertex $\gamma_k x$ (we orient this cone by declaring that its vertices $\gamma_0 x, \dots, \gamma_k x$ are in increasing order).

By induction on k one shows that

$$\widetilde{\Delta}(\gamma'\gamma_0, \dots, \gamma'\gamma_{k-1}, x) = \gamma'\widetilde{\Delta}(\gamma_0, \dots, \gamma_{k-1}, x) \quad \text{for } \gamma' \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{F})).$$

4.3.2. Fundamental domain. Consider now the fundamental domain $\mathcal{D}$ for the action of $\langle u_1, \dots, u_{N-1}\rangle$ on the image of the map ι_α defined as follows: for $\underline{t} \in [0,1]^{N-1}$, let

$$\sigma(\underline{u})(\underline{t}) = (\sigma_1(u_1)^{t_1} \cdots \sigma_1(u_{N-1})^{t_{N-1}}, \dots, \sigma_N(u_1)^{t_1} \cdots \sigma_N(u_{N-1})^{t_{N-1}}) \in (\mathbb{C}^\times)^N$$

and let

$$\mathcal{D} = \{a_\alpha \sigma(\underline{u})(\underline{t})K : \underline{t} \in [0,1]^{N-1}\} \subset X.$$

There is a standard decomposition of $[0,1]^{N-1}$ into $(N-1)$-simplices:

$$[0,1]^{N-1} = \bigcup_{\sigma \in S_{N-1}} \{(t_1, \dots, t_{N-1}) \in [0,1]^{N-1} : t_{\sigma(1)} \leq \cdots \leq t_{\sigma(N-1)}\}.$$

This induces a corresponding simplicial decomposition of $\mathcal{D}$: writing

$$U_i = \iota_\alpha(u_i) \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{F}))$$

and

$$\underline{u}_\sigma = (1, U_{\sigma(1)}, U_{\sigma(1)}U_{\sigma(2)}, \dots, U_{\sigma(1)} \cdots U_{\sigma(N-1)}), \tag{4.3}$$

we have

$$\mathcal{D} = \sum_{\sigma \in S_{N-1}} \operatorname{sgn}(\sigma)\widetilde{\Delta}(\underline{u}_\sigma, x_\alpha). \tag{4.4}$$

4.3.3. Deforming $\widetilde{\Delta}(\underline{\gamma}, x)$. Let $k \geq 0$, $x \in X$ and $\underline{\gamma} \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{F}))^{k+1}$ with $k < N$. In this subsection we define a homotopy between the simplices $\widetilde{\Delta}(\underline{\gamma}, x)$ and $\Delta(\underline{\gamma})$; that is, a map

$$H(\underline{\gamma}, x) : |\Delta_k| \times [0,1] \to {}_{\mathbb{Q}}\overline{X}^T$$

such that

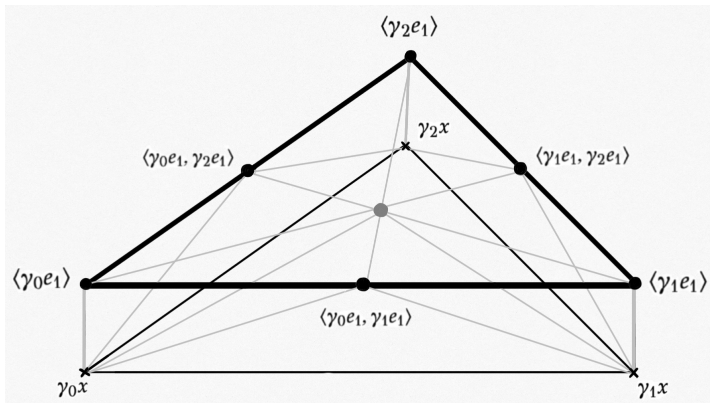
$$\begin{aligned} H(\underline{\gamma}, x)|_{|\Delta_k| \times \{0\}} &= \widetilde{\Delta}(\underline{\gamma}, x), \\ H(\underline{\gamma}, x)|_{|\Delta_k| \times \{1\}} &= \Delta(\underline{\gamma}). \end{aligned} \tag{4.5}$$

Since the cones in X depend on the order of the vertices we try to define this homotopy with a bit of care.

Moreover, we will show that H can be covered by a finite number of Siegel sets attached to good cusps (recall that we say that a cusp corresponding to a rational flag $W_\bullet$ is good if we can find $\gamma \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{F}))$ such that $\gamma e_1 \in W_0$) and has the equivariance property

$$H(\gamma'\gamma_0, \dots, \gamma'\gamma_k, x) = \gamma' H(\gamma_0, \dots, \gamma_k, x) \quad \text{for } \gamma' \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{F})). \tag{4.6}$$

To define H, we use a decomposition of $|\Delta_k| \times [0,1]$ defined inductively as follows. For $k = 0$ we take the decomposition of $\{*\} \times [0,1] \simeq [0,1]$ with 0-simplices $\{0\}$ and $\{1\}$ and the 1-simplex $(0,1)$. The decomposition of $|\Delta_k| \times [0,1]$ is defined inductively on k by joining every simplex of $(\Delta_{k-1} \times \{0\}) \cup (\partial\Delta_k \times [0,1])$ with the barycenter of $\Delta_k \times \{1\}$, as in the following figure.



More precisely, we define, inductively on k, a subset S_k of $\Delta_k \times \Delta'_k$ satisfying

$$|\Delta_k| \times |[0,1]| = \overset{\circ}{\bigsqcup_{(s,s')\in S_k}} C(|s| \times \{0\}, |s'| \times \{1\}) \tag{4.7}$$

(here the symbol $\bigsqcup^\circ$ denotes an almost disjoint union: the cones indexed by different pairs in S_k have disjoint interiors). We define S_k as follows:

- For $k = 0$ we have $\Delta_0 = \{*\}$ and we set $S_0 = \Delta_0 \times \Delta_0$.
- Let $k > 0$ and assume that S_{k-1} has been defined. Let x_0 be the barycenter of Δ_k. To describe which pairs (s, s') belong to S_k, recall that every simplex $s' \in \Delta'_k$ either (i) is the 0-simplex $\{x_0\}$, or (ii) lies on a face of $\partial\Delta_k$, or (iii) is the cone $C(\{x_0\}, s'')$ with vertex x_0 for a unique simplex s'' of Δ'_k contained in the boundary $\partial\Delta_k$. In case (i) we declare that for every $s \in \Delta_k$ we have $(s, \{x_0\}) \in S_k$. In case (ii), the vertex s' lies on a face $\Delta_{k-1} \subset \partial\Delta_k$. We declare that $(s, s') \in S_k$ if and only if s belongs to the same face of $\partial\Delta_k$ as s' and $(s, s') \in S_{k-1}$. In case (iii), we declare that $(s, s') \in S_k$ with $s' = C(\{x_0\}, s'')$ if and only if s and s'' belong to the same face of $\partial\Delta_k$ and $(s, s'') \in S_{k-1}$. Property (4.7) follows by induction on k.

With this decomposition of $|\Delta_k| \times |[0,1]|$ in hand, we can now define H by induction on k. For $k = 0$ we recall the definition of $s(x, x') : [0,1] \to {}_{\mathbb{Q}}\overline{X}^T$ (see (3.8) and (3.9)). Writing $\langle \gamma_0 e_1 \rangle$ for the point of the boundary of ${}_{\mathbb{Q}}\overline{X}^T$ corresponding to the flag given by the line $\langle \gamma_0 e_1 \rangle$, we set

$$H(\gamma_0, x) = s(\gamma_0 x, \langle \gamma_0 e_1 \rangle).$$

Note that the image of $H(\gamma_0, x)$ is the cone $C(\{\gamma_0 x\}, \{\langle \gamma_0 e_1 \rangle\})$.

Assume that $k \geq 1$ and we have defined $H(\gamma_0, \ldots, \gamma_{k-1}, x)$ for every collection of elements $\gamma_0, \ldots, \gamma_{k-1} \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))$. Let $\underline{\gamma} = (\gamma_0, \ldots, \gamma_k) \in \Gamma_0(\mathfrak{p}, \Lambda(\mathfrak{I}))^{k+1}$. Assume first that $\langle \gamma_0 e_1, \ldots, \gamma_k e_1 \rangle \neq k^N$. Then $\Delta(\underline{\gamma})$ corresponds to a simplex in $\Delta_{\mathbb{Q}}(\mathbf{G})$. We define $H(\underline{\gamma}, x)$ using the decomposition (4.7) by taking the restriction of $H(\underline{\gamma}, x)$ to $C(|s| \times \{0\}, |s'| \times \{1\})$ to be the simplicial map (3.10) whose image is the cone $C(\widetilde{\Delta}(\underline{\gamma}, x)(s), \Delta(\underline{\gamma})(s'))$.

Now assume that $\langle \gamma_0 e_1, \ldots, \gamma_k e_1 \rangle = k^N$ (and hence $k + 1 = N$). Given $(s, s') \in S_{N-1}$, we define the restriction of $H(\underline{\gamma}, x)$ to $C(|s| \times \{0\}, |s'| \times \{1\})$ to be the simplicial map (3.10) whose image is the cone defined as follows:

- If s' is the barycenter x_0 of Δ_{N-1}, take the cone to be $C(\widetilde{\Delta}(\underline{\gamma}, x)(s), \{x_0(\underline{\gamma})\})$ (recall that $x_0(\underline{\gamma}) \in X$ denotes the barycenter of the modular symbol $\Delta(\underline{\gamma})$).
- If s' belongs to the boundary $\partial\Delta_{N-1}$, then s and s' belong to the same face of $\partial\Delta_{N-1}$, and the restriction of $H(\underline{\gamma}, x)$ to $C(|s| \times \{0\}, |s'| \times \{1\})$ has already been defined to be the map whose image is the cone $C(\widetilde{\Delta}(\underline{\gamma}, x)(s), \Delta(\underline{\gamma})(s'))$.
- In the remaining case we have $s' = C(\{x_0\}, s'')$ for a unique simplex $s'' \in \partial\Delta_{N-1}$. In this case we form the cone $C' := C(\widetilde{\Delta}(\underline{\gamma}, x)(s), \{x_0(\underline{\gamma})\}) \subset X$ and take the cone to be $C(C', \Delta(\underline{\gamma})(s''))$.

By induction on k one shows that $H(\underline{\gamma}, x)$ is well-defined and continuous[3)] and satisfies (4.5) and (4.6). Note that the image of $H(\underline{\gamma}, x)$ is given by a finite union of cones of the form $C(S, S')$, where S is a compact subset of X and S' is a simplex in the boundary of ${}_{\mathbb{Q}}\overline{X}^T$ corresponding to a good cusp; it follows that the image of $H(\underline{\gamma}, x)$ can be covered by finitely many Siegel sets attached to these cusps.

4.4. Smoothing and evaluation. We can use the above results to express values of partial zeta functions as polynomials in Kronecker–Eisenstein series, by using the fact that the Eisenstein series $E_{\mathfrak{p}}(v_0; \psi, \Lambda(\mathfrak{I}))$ is closed and moving the simplices in (4.4) to the Tits boundary. In order to guarantee that the Eisenstein series is rapidly decreasing, we will use the following lemma due to Colmez and Schneps [12, Lemma 5]. In its statement we write $v_{\widetilde{\mathfrak{P}}}$ for the valuation defined by a prime ideal $\widetilde{\mathfrak{P}}$ of $\mathcal{O}_L$ and denote by $S_{L/k}$ the set of all non–zero prime ideals $\widetilde{\mathfrak{P}}$ of $\mathcal{O}_L$ such that the residue field $\mathcal{O}_L/\widetilde{\mathfrak{P}}$ has degree one over $\mathcal{O}/(\widetilde{\mathfrak{P}} \cap \mathcal{O})$.

Lemma 4.3. *Let $\{\phi_i\}_{i \in I}$ be a finite collection of non-zero k-linear forms on L. There exists a constant C such that if $\widetilde{\mathfrak{P}} \in S_{L/k}$ satisfies $\mathrm{N}\widetilde{\mathfrak{P}} > C$ and $l \in L$ satisfies $v_{\widetilde{\mathfrak{P}}}(l) < 0$ and $v_{\mathfrak{P}'}(l) \geq 0$ for every other prime divisor $\mathfrak{P}'$ of $(\widetilde{\mathfrak{P}} \cap \mathcal{O})\mathcal{O}_L$, then $\phi_i(l) \neq 0$ for every $i \in I$.*

In particular, if $\widetilde{\mathfrak{P}} \in S_{L/k}$ satisfies $\mathrm{N}\widetilde{\mathfrak{P}} > C$, $\mathfrak{a}$ is a fractional ideal of L coprime to $(\widetilde{\mathfrak{P}} \cap \mathcal{O})\mathcal{O}_L$ and $l \in \mathfrak{a}\widetilde{\mathfrak{P}}^{-1} - \mathfrak{a}$, then the forms ϕ_i are all non-vanishing on the coset $l + \mathfrak{a}$.

Proof. We can write $\phi_i(l) = \mathrm{tr}_{L/k}(l_i l)$ for unique $l_i \in L^\times$. Take C so that $\mathrm{N}\widetilde{\mathfrak{P}} > C$ implies that $\tilde{\mathfrak{p}} = \widetilde{\mathfrak{P}} \cap \mathcal{O}$ is unramified in L and for every prime divisor $\mathfrak{P}'$ of $\tilde{\mathfrak{p}}\mathcal{O}_L$ we have $v_{\mathfrak{P}'}(l_i) = 0$ for all i. For $i \in I$ and l as in the statement, we have $v_{\widetilde{\mathfrak{P}}}(l_i l) < 0$ and $v_{\mathfrak{P}'}(l_i l) \geq 0$ for every other prime divisor $\mathfrak{P}'$ of $\tilde{\mathfrak{p}}\mathcal{O}_L$. This implies ([33, II, Section 3, Corollaire 2]) that $\mathrm{tr}_{L/k}(l_i l)$ is not a $\tilde{\mathfrak{p}}$-integer, and hence is not zero. □

For a prime ideal $\widetilde{\mathfrak{P}}$ of $\mathcal{O}_L$ coprime to $\mathfrak{f}$, $\mathfrak{a}$ and $\mathfrak{P}$, define the "$(\mathfrak{P}, \widetilde{\mathfrak{P}})$-smoothed" zeta function

$$\zeta^{p,q}_{\mathfrak{f},\mathfrak{P},\widetilde{\mathfrak{P}}}(\mathfrak{a}, s) = \mathrm{N}\widetilde{\mathfrak{P}}^{-s} \zeta^{p,q}_{\mathfrak{f},\mathfrak{P}}(\mathfrak{a}\widetilde{\mathfrak{P}}, s) - \mathrm{N}\widetilde{\mathfrak{P}}^{-s} \zeta^{p,q}_{\mathfrak{f},\mathfrak{P}}(\mathfrak{a}, s).$$

The following theorem implies Theorem 1.2 of the introduction.

[3)] This latter statement can be deduced from the following general principle: let X and Y two topological spaces, $(F_i)_{i \in I}$ a finite cover of X by closed sets, and $f_i : F_i \to Y$ continuous maps. If f_i and f_j coincides on $F_i \cap F_j$ for all i, j, then there exists a (unique) continuous map $f : X \to Y$ that is equal to f_i on F_i for each i.

Theorem 4.4. *There exists a constant C such that if $\widetilde{\mathfrak{P}}$ is a prime ideal of $\mathcal{O}_L$ such that the residue field $\mathcal{O}_L/\widetilde{\mathfrak{P}}$ has degree one over $\mathcal{O}/\widetilde{\mathfrak{p}}$ and $\mathrm{N}\widetilde{\mathfrak{P}} > C$, then*

$$\det(\sigma_i(\alpha_j))\zeta^{p,q}_{\mathfrak{f},\mathfrak{P},\widetilde{\mathfrak{P}}}(\mathfrak{a},0) = [U(\mathfrak{f}) : U(\mathfrak{f})']^{-1} \sum_{\sigma \in S_{N-1}} \mathrm{sgn}(\sigma) \cdot \sum_{\substack{l \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f} \\ l \neq 0}} \mathbf{\Phi}^{pN,qN}_{\mathfrak{p}}(v_0 + \alpha(l), \underline{u}_\sigma, \Lambda(\mathfrak{I}))(P^{p,q}_\alpha).$$

Proof. Let us define a collection $\{\phi_i\}_{i\in I}$ as in Lemma 4.3. Writing $\underline{u}_{\sigma,j}$ $(0 \leq j < N)$ for the components of the N-tuple $\underline{u}_\sigma$ in (4.3), we consider the finite set $\{W_i\}_{i\in I}$ of all proper subspaces W_i of V_k of the form $\langle \underline{u}_{\sigma,j} e_1 \mid j \in J\rangle$, for all $\sigma \in S_{N-1}$ and all $J \subseteq \{0,\dots,N-1\}$. For each subspace W_i we choose a non-zero linear form ϕ_i on L such that $W_i \subseteq \ker(\phi_i \circ \alpha^{-1})$. Let $C(I)$ be the constant provided by Lemma 4.3.

Now take $C > C(I)$ such that any prime ideal $\widetilde{\mathfrak{P}}$ with $\mathrm{N}\widetilde{\mathfrak{P}} > C$ is coprime to $\mathfrak{a}$, $\mathfrak{f}$ and $\mathfrak{P}$; then Lemma 4.3 and Theorem 3.4 show that, for any $l \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f} - \mathfrak{f}$, the Eisenstein series $E_{\mathfrak{p}}(v_0 + \alpha(l), P^{pN,qN}_\alpha; \psi^{p,q}, \Lambda(\mathfrak{I}))$ is rapidly decreasing on every Siegel set of every cusp corresponding to a flag $W_\bullet$ given by a chain of subspaces W_i with $i \in I$; in particular, for such l we have

$$\int_{\partial H(\underline{u}_\sigma, x_\alpha)} E_{\mathfrak{p}}(v_0 + \alpha(l), P^{p,q}_\alpha; \psi^{pN,qN}, \Lambda(\mathfrak{I})) = 0.$$

Since

$$\sum_{\substack{l \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f} \\ l \neq 0}} E_{\mathfrak{p}}(v_0 + \alpha(l), P^{p,q}_\alpha; \psi^{pN,qN}, \Lambda(\mathfrak{I}))$$

is invariant under $U(\mathfrak{f})^1$, the equivariance property (4.6) shows that

$$\begin{aligned} (4.8)\quad 0 &= \sum_{\sigma \in S_{N-1}} \mathrm{sgn}(\sigma) \int_{\partial H(\underline{u}_\sigma, x_\alpha)} \sum_{\substack{l \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f} \\ l \neq 0}} E_{\mathfrak{p}}(v_0 + \alpha(l), P^{p,q}_\alpha; \psi^{pN,qN}, \Lambda(\mathfrak{I})) \\ &= \sum_{\sigma \in S_{N-1}} \mathrm{sgn}(\sigma) \int_{\Delta^\circ(\underline{u}_\sigma)} \sum_{\substack{l \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f} \\ l \neq 0}} E_{\mathfrak{p}}(v_0 + \alpha(l), P^{p,q}_\alpha; \psi^{pN,qN}, \Lambda(\mathfrak{I})) \\ &\quad - \sum_{\sigma \in S_{N-1}} \mathrm{sgn}(\sigma) \int_{\widetilde{\Delta}(\underline{u}_\sigma, x_\alpha)} \sum_{\substack{l \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f} \\ l \neq 0}} E_{\mathfrak{p}}(v_0 + \alpha(l), P^{p,q}_\alpha; \psi^{pN,qN}, \Lambda(\mathfrak{I})). \end{aligned}$$

The proof of Proposition 4.2 shows that

$$\begin{aligned} &[U(\mathfrak{f}) : U(\mathfrak{f})']^{-1} \int_{X(\mathfrak{f})} \sum_{\substack{l \in \widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f} \\ l \neq 0}} E_{\mathfrak{p}}(v_0 + \alpha(l), P^{p,q}_\alpha; \psi^{pN,qN}, \Lambda(\mathfrak{I})) \\ &= \det(\sigma_i(\alpha_j))\zeta^{p,q}_{\mathfrak{f},\mathfrak{P},\widetilde{\mathfrak{P}}}(\mathfrak{a},0). \end{aligned}$$

We compute

$$\begin{aligned}
&\int_{X(\mathfrak{f})} \sum_{\substack{l\in\widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f}\\ l\neq 0}} E_{\mathfrak{p}}(v_0+\alpha(l), P_{\alpha}^{p,q}; \psi^{pN,qN}, \Lambda(\mathfrak{I}))\\
&= \sum_{\sigma\in S_{N-1}} \mathrm{sgn}(\sigma) \int_{\widetilde{\Delta}(\underline{u}_\sigma, x_\alpha)} \sum_{\substack{l\in\widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f}\\ l\neq 0}} E_{\mathfrak{p}}(v_0+\alpha(l), P_{\alpha}^{p,q}; \psi^{pN,qN}, \Lambda(\mathfrak{I}))\\
&= \sum_{\sigma\in S_{N-1}} \mathrm{sgn}(\sigma) \int_{\Delta^\circ(\underline{u}_\sigma)} \sum_{\substack{l\in\widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f}\\ l\neq 0}} E_{\mathfrak{p}}(v_0+\alpha(l), P_{\alpha}^{p,q}; \psi^{pN,qN}, \Lambda(\mathfrak{I}))\\
&= \sum_{\sigma\in S_{N-1}} \mathrm{sgn}(\sigma) \sum_{\substack{l\in\widetilde{\mathfrak{P}}^{-1}\mathfrak{f}/\mathfrak{f}\\ l\neq 0}} \mathbf{\Phi}_{\mathfrak{p}}^{pN,qN}(v_0+\alpha(l), \underline{u}_\sigma, \Lambda(\mathfrak{I}))(P_{\alpha}^{p,q}).
\end{aligned}$$

using (4.4), (4.8) and Proposition 3.5, respectively. □

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