9

The Mathematicians' Use of Diagrams in Plato[1]

Tamsin de Waal

1. Introduction

I was lucky enough to have Keith as one of my supervisors, whilst a doctoral student at King's College London. My thesis was primarily focused on Plato and the problem of how we can come to know forms, but related to the access problem in contemporary philosophy of mathematics. Keith introduced me to a world that I was previously unfamiliar with, with infectious enthusiasm. Our discussions were always wide-ranging, but, somehow, we always seemed to come back to Euclid. So it seemed only appropriate that my chapter in this volume should be about diagrams, and the role they play in coming to know forms.

It is clear that mathematics, on Plato's view, has an important role to play in facilitating progress towards knowledge of forms. This is particularly clear in the *Republic*. Here, mathematics constitutes a significant part of the prospective philosopher-rulers' education programme – both in the sense that it takes up a lot of space in the description of the programme, and in the sense that it takes up so many (ten) years of training. And it is proposed on the grounds that such studies have the power to effect 'a conversion and turning about of the soul' (521c), and to 'draw the soul away from the world of becoming to the world of being' (521d).

This raises a number of broad questions. Why should Plato set such mathematical store on mathematics? How does he see it effecting cognitive development? How much of a cognitive shift is it capable of effecting? How do the different mathematical disciplines contribute to this shift? Is mathematics the only thing that can prompt this cognitive development? Whilst I engage with these broad questions here, the focus of the chapter is a relatively narrow one: whether, on Plato's account, the use of diagrams in mathematics plays a part in facilitating cognitive development.

Plato highlights the use of diagrams in mathematics in the context of the *Republic* divided line image (509dff), itself, of course, a diagram. Following immediately on from the sun simile (506eff), the divided line picks up on the distinction drawn there between a visible and an intelligible realm: 'Well, suppose you have a line divided into two unequal parts, and then divide the two parts again in the same ratio, to represent the visible and the intelligible realms' (509d).[2] The four sub-divisions correspond to four distinct cognitive states – intelligence (*noēsis*) and understanding (*dianoia*), which are correlated to the intelligible realm, and belief (*pistis*) and imagination (*eikasia*), which are correlated to the visible realm (511d–e).[3] These are to be arranged 'in a proportion, considering that they participate in clearness and precision in the same degree as their objects partake of truth and reality' (511e). It is generally assumed that we should picture a vertical line, here,[4] and that the larger part is the upper part, and represents the intelligible realm. But it is notable that none of this is explicitly specified.

Dianoia is distinguished from *noēsis* on two methodological grounds:

- Whilst those in a state of *noēsis* treat hypotheses as just that and seek 'that which requires no assumption and is the starting-point of all' (511b), those in a state of *dianoia* make use of hypotheses, using them as first principles and deeming them unnecessary to explain (510c).[5]
- Further, whilst those in a state of *noēsis* rely on forms only and progress systematically through these, those in a state of *dianoia* make use of sensible objects – they investigate by using these (the originals of the visible realm) as images (510b).

In order to clarify the distinction between *dianoia* and *noēsis*, Socrates identifies the latter with dialectic, and the former with mathematics, the sort of reasoning that mathematicians engage in.[6] Mathematicians, 'students of geometry and reckoning and such subjects' (510c), rely on hypotheses, and:

> They use the visible forms (*tois horōmenois eidesi*) besides, and make their accounts about them, not thinking about/intending them, but those things to which these are like (*eoike*), making their accounts for the sake of the square itself and the diagonal itself, not for the sake of this (diagonal) which they draw (*graphousin*), and likewise in regard to the other things: these very things which they mould and draw, of which there are shadows and images (*eikones*) in water, they are using in their turn as images (*hōs eikosin*), while they are seeking to get a view of the things which one can see in no other way than with thought.
>
> 510d–e

The references to drawing, and, more particularly, to the drawing of squares and diagonals, make it clear that 'the visible forms', the sensible objects, in question are, first and foremost, geometrical diagrams.[7] Mathematicians use diagrams and treat these sensible objects as images or likenesses. The reference to 'students of geometry *and reckoning and such subjects*' makes it clear that the use of geometrical diagrams is not confined to geometry.

The use of diagrams here has received very little attention. Commentators have tended to pass over this, focusing instead on the use of hypotheses.[8] In as much as the use of diagrams is noted, it is generally cast in a predominantly negative light – as either a defect of mathematical method, or a failure of certain mathematicians.[9] Plato is clear, it is true, that knowledge of forms can only be achieved where sensible objects and hypotheses are relinquished, so, in this sense, the use of diagrams in mathematics *is* cognitively limiting. However, as I aim to demonstrate in this chapter, the use of diagrams in mathematics also has a highly positive cognitive effect, on Plato's account, and plays a crucial role in facilitating progress towards knowledge of forms.

2. What mathematics is Plato recommending?

The first point to consider is whether the mathematical method described in the divided line, complete with the use of sensible objects, is the same mathematics recommended for the prospective philosopher-rulers, the guardians, in Bk VII of the *Republic*. To put this another way, should we identify the dianoetic method of the divided line image with the mathematical method, correctly applied?

(i) The dianoetic versus mathematical method

The mathematical curriculum recommended for the guardians begins with arithmetic and logistic.[10] This is to be followed by plane geometry (526c–527c), three-dimensional (solid) geometry (528a–d), astronomy (527d–528a) and harmonics (530–531c). Socrates pointedly insists that the five mathematical subjects be studied in this order.[11]

In setting out the mathematical curriculum, Socrates distinguishes the mathematics to be studied by the guardians both from applied mathematics (the sort of mathematics generals might use – i.e. what the *Philebus* classifies as popular, as opposed to philosophical, mathematics),[12] and from mathematics as practised by contemporary mathematicians. With respect to the latter, Socrates picks out astronomy (528eff) and harmonics (531aff) for particular criticism – these concern themselves with the visible and the audible, respectively, proceeding by means of the senses, rather than by reasoning. Solid geometry, he observes, is altogether neglected by contemporary mathematicians (528b–d). The mathematical method recommended (i.e. the philosophical mathematical method) is one that proceeds by means of problems (530b–c),[13] concerning itself with intelligible entities,[14] and one that is pursued for the sake of knowledge (525c–d).

As to whether we should identify this method with the dianoetic method, on one possible reading, the dianoetic method is a *mis*application of the mathematical method by certain mathematicians – for example, the mathematicians of the day.

On this reading, the correctly applied mathematical method does not take its hypotheses to be first principles and does not make use of sensible objects. But in this case, what *is* mathematics, as distinct from dialectic? On a more moderate version of this reading, the correctly applied mathematical method does make use of sensible objects, but does not, like the dianoetic method, make *inappropriate* use of these.[15] It is not obvious, though, what might constitute 'inappropriate' use of sensible objects. The most significant sense in which sensible objects might be deemed to be misused is where they are treated not as the likenesses they are, but as originals. This is what the astronomers of the day are accused of doing at 528eff. Plato, though, is clear that the dianoetic mathematicians treat sensible objects as likenesses.[16] And it is notable that Plato does not criticize the geometers (or arithmeticians) of the day in the way he criticizes the astronomers of the day, and those who practise harmonics.

Alternatively, then, we might take it that the dianoetic method *is* the mathematical method, correctly applied – after all, the philosophical mathematician of *Republic* Bk VII, is, as we shall see, characterized as being in a state of *dianoia*.

On this reading, the use of hypotheses and diagrams are simply necessary features of the mathematical method. And in this case, Plato cannot be criticizing (any) mathematicians, for relying on these. Might he nonetheless mean to be critical of these features? It is hard to see why he would so strongly advocate the study of mathematics if he meant to criticize features that are fundamental to the practice of mathematics. On this reading, then, whilst it doesn't preclude the possibility that Plato has reservations about the use of diagrams (of the sort suggested by Patterson, for instance),[17] the use of diagrams is neither a defect of mathematical method, nor a failure of certain mathematicians. And likewise with respect to the use of hypotheses. The fact that mathematics does not go back to first principles and explain hypotheses is not a deficiency of mathematics. It is simply not its job to do so. This is the job of dialectic. Thus the mathematician hands over the realities he discovers to the dialectician (*Euthydemus* 290b–d).

I take this second reading to be the most compelling. A key proponent of this view is Myles Burnyeat. With respect to the use of hypotheses, Burnyeat, pointing to Plato's use of the word 'compelled' (*anankazetai*/*anankazomenēn*) at 510b and 511a, argues that 'the soul is compelled to start from [hypotheses] because there is no other way of doing deductive mathematics than by deriving theorems and solutions from what is laid down at the beginning'. Whilst Burnyeat's focus is on the use of hypotheses in Greek mathematics, he does also note that the use of diagrams and constructions is plausibly essential.[18]

(ii) The necessity of diagrams

That diagrams were, at any rate, perceived to be essential to Greek mathematics is suggested by the use of the Greek word *diagramma*. As Reviel Netz notes in *The Shaping of Deduction in Greek Mathematics*, where *diagramma* literally means 'figure marked out by lines', it is generally used in mathematical contexts to mean proposition or proof.[19] This, he argues, is a reflection of the fact that 'diagrams are considered by the Greeks not as appendages to propositions, but as the core of a proposition'.[20] Thus Netz claims: 'Diagrams are the metonyms of propositions; in effect, the metonyms of mathematics'.[21]

Greek mathematics was geometry-heavy. On Plato's own account, plane and solid geometry form a substantial part of the *Republic* mathematical curriculum. And the use of geometry extends beyond these disciplines. Where, today, we turn geometry into arithmetic, the Greeks tackled arithmetic by means of geometry – as we see from the fact that Plato ascribes the operations of 'squaring, applying and adding and the like' to the geometers (*Rep.* 527a–b). Although Euclid is later than Plato (*c.* 300 BC), the general consensus is that already in Plato's time mathematics includes most of what would come to be Euclidean geometry.[22]

Given the predominance of geometry, it is no surprise that the diagram should be a predominant feature of Greek mathematics, as Netz argues it is. Greek mathematical exchanges would, as a rule, he notes, be accompanied by a diagram or diagrams:[23] 'When mathematical results were presented in anything other than the most informal, private contexts, lettered diagrams were used. These would typically have been prepared prior to the mathematical reasoning. Rulers and compasses may have been used.'[24] As to the role of diagrams in Greek mathematics, they very clearly do a lot of work as pedagogical or heuristic aids.[25] But diagrams also do real work in the reasoning – whether the mathematician is using them to present or to work out a proof, whether looking at or constructing them. As Patterson notes, natural language and diagrams are part of a hybrid notational system, where both make essential contributions to proofs.[26] Often, the diagram supplies information that is not supplied by the collection of statements about the diagram.[27] (And it is worth noting that the *Republic* divided line, as a diagram, is a case in point.[28]) In these cases, the diagram is very obviously indispensable. Arguably, too, diagrams play an indispensable role in the process of discovery. This is an argument that is increasingly made in the context of contemporary philosophy of mathematics.[29] In this context, of course, discussion is not confined to Euclidean geometry. But it is notable that the geometrical demonstration in Plato's *Meno* (82b–85d) is often used to illustrate the epistemologically significant role diagrams play in mathematical reasoning.[30] I will return to the *Meno* in the final section of this chapter.

Suffice it to say here that the use of diagrams in the *Meno*[31] and elsewhere in Plato,[32] taken together with the characterization of mathematical method in the *Republic*, suggests that Plato himself shares the perception that diagrams are a fundamental and necessary feature of mathematics – as fundamental and necessary as hypotheses. And in fact, these two features of mathematics are bound together. As Plato says at *Republic* 510b: 'In using those things which then were imitated as (themselves) images, the soul is compelled to seek from hypotheses'. The soul is compelled to seek from hypotheses in as much as it is viewing something as an image. An image is, by definition, an image of something else. So, in using something as an image, one necessarily introduces a supposition (the something else which it is *supposed* to be an image of). Using a given drawn line as an image of the diagonal is to operate with a hypothesis (that the diagonal itself exists and is like this).

3. What is the cognitive shift effected by mathematics in general?

Allowing, then, that we can identify the dianoetic method of the divided line with the mathematical method that Plato is recommending for the guardians, and that the use of diagrams is both a necessary and a predominant feature of this method, we should be predisposed to think that the use of diagrams in mathematics plays an important role in facilitating cognitive development, on Plato's account. Determining what this role might be requires us to first consider the nature of the cognitive shift effected by mathematics more generally.

(i) The intermediate cognitive state of the mathematician

Earlier in the *Republic*, at 473ff, Socrates distinguishes between sightlovers (lovers of sights and sounds) and philosophers, and their relative cognitive states. Philosophers recognize the one, a single form; they recognize *to kalon*. They therefore have knowledge. Sightlovers, on the other hand, only recognize the many – *ta kala* – 'beautiful sounds, colours, and shapes, and everything fashioned out of them' (476b), and thus only have belief (*doxa*). The philosopher, because he knows what beauty is, can distinguish between beautiful and ugly things, and can distinguish between the visible and the intelligible. The sightlover cannot. Furthermore, the sightlover misidentifies the many beautiful things as beauty itself. He confuses one and many, taking the many to be the one, the likeness to be the original: '[He] thinks that what is like something is not like, but is the very thing to which it is like' (476c). For this reason, the sightlover is described as being in a dream state (476c–d). The philosopher, by contrast, is 'capable of seeing both [beauty] itself and the things which have a share of it and thinks neither that the things which have a share of it are it, nor that it is the things which have a share of it' (476d). The philosopher does not confuse one and many, likeness and original, and is thus described as awake.

The (philosophical) mathematician is characterized as occupying the middle ground between the sightlover and philosopher. 'I think,' says Glaucon to Socrates, at 511c, 'you call the cognitive state of geometers and the like *dianoia* and not *noēsis*, meaning by *dianoia* something midway between *doxa* and *nous*.' Socrates reiterates the point at 533d–e, adding that *dianoia* has an intermediate degree of clarity.

We have already seen that Plato distinguishes *dianoia* and *noēsis* on methodological grounds. Whether or not *dianoia* and *noēsis* are also distinguishable on ontological grounds, such that the intermediate nature of *dianoia* is reflected in its objects, is a much-debated point – one fueled by Aristotle's assertion that Plato posits intermediate mathematical objects (*Metaphysics* I.5.987b14–18).[33] I proceed here on the basis that *dianoia* does not have a distinct object.[34] Allowing that the 'upper' half of the divided line is broadly analogous with the 'lower' half, the implication is that *dianoia* and *noēsis* have the same object, forms, but differ in terms of the degree to which they grasp reality.

With respect to the lower half of the divided line, the objects of *eikasia* and *pistis* are essentially the same. The objects of *pistis* are sensible objects, whilst the objects of *eikasia* are images, which is to say, shadows and reflections of sensible objects. What distinguishes the two cognitive states is the clarity of their view of reality. *Pistis* apprehends reality more clearly, more directly, than *eikasia*. The person in a state of *pistis* views images of forms (sensible objects), as opposed to images of images, and recognizes that shadows and reflections of sensible objects are just that. However, he fails to recognize that those sensible objects are themselves images.

With respect to the upper half of the line, Plato describes those in a state of *dianoia* as 'dreaming' (*oneirōttousi*) about reality/being (*to on*) (533c). This language attributes the same sense of indirect perception or apprehension of originals to *dianoia* as is attributed to *eikasia*. The implication is that the objects of *dianoia* are reflections or shadows of forms. Whilst *eikasia* does not have a clear view of the sensible objects that cast reflections, *dianoia* does not have a clear, a direct view, of forms. Only *noēsis* does. *Dianoia* does not have a clear view of forms, because it aims at forms on the basis of hypotheses that it treats as first principles, and does not attain a truly synoptic view of forms – it does not see each form in relation to all other things.[35] This means *dianoia* cannot have a true account of each form, any more than *eikasia* can have a true account of the physical object that casts the reflection.

On this reading of the divided line, what, above all, distinguishes the mathematician from the philosopher is his grasp of forms. The mathematician has some grasp of forms, such as the square itself, and the diagonal itself, but does not, as the philosopher does, fully grasp these. What, above all, distinguishes the mathematician from the sightlover is his cognitive attitude to sensible objects. Both mathematician and sightlover are described as being in a dream state, but whilst the sightlover's dream state is associated with his confusion of likeness and original, the mathematician's dream state is associated with his failure to go back to first principles (it doesn't necessarily follow from this that the mathematician *mistakes* his hypotheses for first principles, just that he treats them *as if* they were true). Notably, his dream state is not associated with his use of diagrams. The mathematician, in treating diagrams as images, treats the many as the likenesses they are.[36] He does not make the mistake the sightlover makes – he, like the philosopher, does not confuse the many with the one, he does not confuse likeness and original. So the mathematician has made a significant cognitive advance on the sightlover.

The intermediate cognitive state attributed to the mathematician in this way is reflected in what the mathematician does – in particular, the fact that he treats sensible objects as images – and what he *implicitly* thinks. When we are told that the mathematician is thinking about the square itself, for instance, this seems to be about his implicit, rather than explicit, view of such objects. It is not reflected in what the mathematician says:

> Their language is most ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas in fact the whole study is engaged in for the sake of knowledge (*gnōseōs*) . . . knowledge of something always existing, and not of something coming into being at a certain time and perishing.
>
> 527a–b[37]

(ii) Conversion and ascent

The divided line, as a static image, places the mathematician/mathematics. It indicates that the study of mathematics cultivates the intermediate cognitive state of *dianoia*. The language of conversion and ascent used of mathematics in the setting out of the mathematical curriculum, indicates that the study of mathematics is also responsible for bringing about the initial shift from a state of *pistis* to a state of *dianoia*, and for facilitating progress towards *noēsis*. It indicates, that is, that mathematics is responsible for bringing about the conversion from the sightlover's cognitive state to that of the mathematician, and for facilitating ascent towards the philosopher's cognitive state.

Conversion is a matter of being turned towards more real things (see e.g. *Rep.* 515d). It evidently requires cognitive access (up to a point) to the one, the original, that the sightlover has no access to. But, as Harte highlights, it cannot simply be a matter of gaining cognitive access to the original, since having cognitive access to the original doesn't guarantee not confusing likeness and original. One could know Simmias, and nonetheless take a picture of Simmias to be the real Simmias. Avoiding the sightlover's confusion apparently, therefore, involves simultaneously being put in mind of the one, the original, and recognizing that what we are perceiving is a likeness of that – something which falls short of it.[38] This involves grasping the distinction between the visible and intelligible, likeness and original, and so developing a different cognitive attitude to sensible objects.

If mathematics is to initiate such a conversion, then clearly it must do so at the level of *pistis*, within the cave.[39] We have already seen that Plato posits a popular mathematics that operates at this level. Whilst he is keen to distinguish this sort of mathematics from philosophical mathematics, Plato is clear that popular mathematics can itself be educational.[40] That arithmetic at this level, simple applied arithmetic involving sensible units, is capable of prompting thought, is made explicit at *Republic* 522cff. Here, we are told that the study of arithmetic (and logistic), 'this trivial matter of distinguishing one and two and three' (522e), naturally leads to thought.

Socrates explains that certain perceptions, ones that issue in a contradictory perception, are thought-provoking. 'Number' (*arithmos*) and 'the one' belong to that class of things that impinge upon the senses with their opposite (524dff). We perceive the same thing as one, in one context, but as a many, in another context.[41] To take an example from the *Parmenides*: Socrates is one, in the sense that he is one of seven men present, but also many, in the sense that he is constituted of many parts (129b–d). Since one cannot be grasped separately, or adequately, from its opposite through perception, but only confounded with many, we are puzzled as to what one is, and resort to thought to resolve the compresence of opposites that sight presents us with. 'And this (i.e. experiencing contradictory perceptions of this sort),' says Socrates, 'is how we came to use the terms "the intelligible" (*to noēton*) and "the visible" (*to horaton*)' (*Rep.* 524d).

'If this is true of the one, the same holds of all number' (525a), Socrates reasons.[42] And since arithmetic is wholly concerned with number (525a), the study of it prompts inquiry into the one, and prompts students to distinguish it from the many. This inquiry is initiated at the level of popular arithmetic, through operations with sensible units. Subsequently, at the level of philosophical arithmetic, it is numbers understood as collections of units, where these are intelligible units that are equal to each other and admit no division into parts, that continue the conversion work.[43] Ultimately, the study of arithmetic leads to contemplation of the nature of numbers (525c), directing the soul upwards to discourse of the numbers themselves (*autōn tōn arithmōn*) (525d–e).[44]

A striking feature of this passage is the emphasis on the physical manifestation of number/the one. Glaucon observes at 525a that it is especially the visual perception of one that provokes thought. Plato's point, I take it, is not simply that all sensible units, like army units, can be cut up such that they are at once both one and many, and so provoke thought of an intelligible one. But also, and, perhaps, especially, that the study of arithmetic can only play its part in conversion where it is, in the first instance, engaging with sensible units. Only where it engages with sensible objects can it facilitate recognition of the deficiency of sensible objects, hence recognition of the visible–intelligible distinction.

This is a point underlined in the *Phaedo*. Here, perceiving equal sticks and stones is a catalyst for the recollection process (74bff): 'It must be as a result of the senses that we obtained the notion that all sensible equals are striving to realize actual equality but falling short of it' (75a–b).

It is presumably above all for this reason that Socrates insists that the mathematical studies on the guardians' curriculum, and especially arithmetic,[45] have practical utility – that they be useful to soldiers (*Rep.* 521d). This ensures that the guardians, who will train as soldiers prior to embarking on their mathematical studies, engage in popular arithmetic, operating with sensible units (for the purposes of ordering troops, for instance), prior to engaging in philosophical arithmetic.[46]

The emphasis on the order in which the mathematical subjects should be studied by the guardians might give rise to the view that arithmetic – first, at the popular level, and then, at the philosophical level – is solely responsible for conversion from *pistis* to *dianoia*. On such a view, the other disciplines might then be seen to facilitate a gradual ascent from *dianoia* to *noēsis*. And indeed the different disciplines *can* be seen to build on one another in this way. As the student of mathematics progresses through the different mathematical disciplines, he comes to perceive mathematical phenomena in contexts that gradually become more complex and broader, and thus in relation to more and more things.[47] In this way, the mathematician gradually builds towards the synoptic view of the dialectician. But Socrates indicates, too, that arithmetic is not alone in effecting conversion.[48] The suggestion is that all the subjects together effect a gradual turning around, where this is a difficult and painful process (as underlined in the cave image at 515c–516a).[49]

In this case, arithmetic should rather be seen as initiating the process of conversion, instilling a low-level recognition of the visible–intelligible distinction. The other disciplines then reinforce awareness of this distinction. Geometry (plane and solid), I want to suggest, does so particularly effectively, on Plato's account, and achieves this through its use of diagrams.[50]

4. How does the use of diagrams facilitate conversion and ascent?

In the famous geometrical demonstration at 82b–85d in the *Meno*, Socrates offers to show that what we call learning is actually recollection of things we already know, and proceeds to question a household servant with no knowledge of geometry on a geometrical problem: how to double a square. When, eventually, the boy discovers the theorem (that the square on the diagonal of a given square is double the area of the original square), Socrates claims that the boy has discovered this for himself. He, Socrates, has merely drawn true beliefs out of the boy that were already in him. Here, as in the *Republic*, the practice of mathematics (and for all that the boy is new to geometry, the geometry he is engaged in is nonetheless philosophical geometry) is associated with a state of dreaming: 'At this moment, those beliefs have just been stirred up in him (the boy), *like a dream*'. And, here, Plato suggests that it will be continued dialectical questioning 'about the same matters on many occasions and in many ways' (85c–d) that will lead to the awakening of the mathematician.

The process of questioning in the demonstration involves the use of diagrams. Although diagrams are not explicitly referenced, it is clear from the use of demonstratives, and the references to drawing (e.g. at 83b), that they are being used. *Phaedo* 73a–b, which refers us back to the *Meno* passage, highlights both the use of diagrams in this context, and their capacity to prompt thought:

> When people are being asked questions, if someone asks the right questions, they of themselves tell everything as it is – and yet if knowledge and right account had not happened to be within them, they would not have been able to do this – for example, if someone brings them to diagrams (*diagrammata*) or something like that,[51] here it most clearly affirms that this is so.

Taken together, the *Meno* and *Phaedo* suggest that the geometrical diagram is the paradigm case of a sensible object that has the power to prompt thought and recollection.

Why should Plato grant diagrams this status? Consider, for example, Euclid 1.5. The proposition here is first stated as a general claim (the *protasis* (enunciation): 'In isosceles triangles the angles at the base are equal to one another . . .'). So it is universal. The argument, though, does not proceed by thinking about universals (isosceles triangularity). It works instead by thinking about a typical instance. The diagram is the figure you (are instructed to) draw or imagine[52] in the setting out of the particular instance (the *ekthesis* (exposition): 'Let ABC be an isosceles triangle having the side AB equal to the side AC . . .'). The proposition is then proved true of the particular instance. Finally, on the basis that the instance was typical, the general proposition is taken to have been proved.[53]

The diagram in a Euclidean proof is the particular instance the geometer is thinking about, whether looking at or constructing it, and the argument is about instances (of line, triangle, even number). But the argument proceeds by logical inferences, from hypotheses (Euclidean postulates and definitions, such as the definition of 'odd number'), and proves a universal truth (conditional on the hypotheses being true). The geometer thus uses the diagram to prove something that holds not only for the drawn/imagined triangle, the instance, but for any triangle as defined in his initial hypotheses. In so doing, he treats the diagram of an isosceles triangle, say, as representative of all isosceles triangles. As Proclus observes, geometers 'use the objects set out in the diagram not as these particular figures, but as figures resembling others of the same sort'.[54] In the language of the *Republic*, geometers treat diagrams as images (*eikones*). They use diagrams as patterns (*paradeigmata*) to aid the study of ideal mathematical entities and relations (529d–e). Thus in the *Euthydemus* (290b–c), Plato characterizes geometers (along with arithmeticians and astronomers – presumably philosophical astronomers) – as 'hunters' who make use of diagrams (*diagrammata*), but are hunting mathematical truths: 'they are not in each case diagram-makers, but discover realities (*ta onta*)'.

The geometer is clear that he is using the diagram in this way, as a representation – and Plato in the *Republic* indicates as much in not ascribing the mathematician's dream state to his use of diagrams. For one thing, a single instance is insufficient support for a universal claim. For another, the diagram is not like a picture of Simmias – it is schematic, illustrating only the topological features of the object represented. The geometer understands that the diagram is only an instance, and only approximate – that, as Plato says in the *Seventh Letter*, 'every one of the circles which are drawn in geometric exercises or are turned by the lathe is full what is opposite to the fifth [i.e. the form of circle], since it is in contact with the straight everywhere' (343a). He understands that solving a problem, proving a proposition, and arriving at a geometrical truth cannot be achieved through straightforward observation or measurement of the particular diagram (and just a few attempts at trying this would confirm as much). Thus Socrates in the *Republic* observes that any geometer with experience 'would think it absurd to examine [geometrical diagrams] seriously in the expectation of finding in them the absolute truth with regard to equals or doubles or any other ratio' (529d–530a).

So the geometer, in using the diagram as a representation, distinguishes between the diagram, the instance, and the mathematical object as defined in his initial hypotheses, and recognizes that the former falls short – that it is approximate and suffers compresence of opposites.

Of course, using the diagram as a representation also requires the geometer to observe the similarity between diagram and mathematical object. It requires him to see the inexact diagram as the (exact) mathematical object. To put it another way, it requires him to see the inexact diagram as a picture or reminder of the (exact) mathematical object.[55] In the *Meno* geometrical demonstration, Socrates begins by asking the boy: 'Do you know that a square figure is *like this*?' (*toiouton estin*) (82b).[56] As Sherry puts it, Socrates' initial line of questioning here is concerned with finding out whether the boy, as someone who is new to geometry, 'has mastered the skill of *treating* a diagram *as* a square, in other words, the skill of *letting* a diagram be a square'.[57] It is a 'skill' honed through the practice of philosophical mathematics, and one demonstrated by the *Republic* (philosophical) mathematicians: they make use of diagrams, making their accounts about them, but think about and make their accounts for the sake of those things of which their diagrams are likenesses (510d–e). Ultimately, it is a skill that allows the geometer to access the intelligible through perception and the visible.[58]

Engaging with a diagram, then, whether looking at it or constructing it, simultaneously (a) puts the geometer in mind of the ideal objects and properties instantiated there (the one, the original) (albeit that he does not fully grasp the nature of these entities); and (b) prompts the recognition that the instance(s) perceived, though similar, are distinct, and only likenesses of these, where a likeness is something that falls short of the original. In line with Harte's reading of how the sightlover's mistake is to be avoided, recognition of the original, and recognition of what one perceives as merely a likeness of this, occur in one and the same act.

With repeated practice, the use of diagrams will reinforce the recognition of a visible–intelligible distinction effected by arithmetic. And it will reinforce recognition of the likeness–original distinction this entails. Or, perhaps better, (rein)force. Plato repeatedly emphasizes the element of compulsion in mathematics. At *Republic* 510b and 511a (see also 511c), as we have seen, the soul is said to be compelled to start from hypotheses. At 523aff, contradictory perceptions of one and many, hence arithmetic, are said to compel thought (524c and 524e). At 526e, it is twice claimed that mathematics compels the soul to contemplate being. And the cave image emphasizes the role of compulsion more generally in conversion and ascent (515c–e, 519c). Plato does not explicitly state that the use of diagrams compels thought, but the *Meno, Phaedo*, and *Republic*, together suggest that it does so in two ways. First, simply in virtue of the fact that the approximate diagram will issue in a contradictory perception: the diagram is at once circular and straight, one and many. Second, and above all, in virtue of the fact that the role of diagrams in Euclidean geometry renders them obviously representational. Diagrams therefore compel thought of the intelligible entities they represent, and compel the mathematician to recognize their similarity to and difference from these entities.

Because Euclidean geometry has grown out of a subject that is concerned with land measurement, about the physical world and sensible objects, it models the properties of the three-dimensional space that the geometer perceives around him. That Plato is very conscious of this is demonstrated in the *Meno* at 74a–76d. As Wedberg observes: 'The concepts of Euclidean geometry are, for Plato, not variables, but certain properties and relations intimately related to our sense perception of space. In the *Meno* the general notion of geometrical figure (*schēma*) is explained in a manner which clearly shows its origin in visual perception.'[59] What is important about this (aside from the fact that it means physical diagrams can very easily be used to reason about Euclidean properties and relations) is that the geometer will associate the diagram with sensible objects – he will recognize the diagram as one among other sensible objects. Since the guardians' study of geometry proceeds (from popular to philosophical geometry) as geometry has historically evolved,[60] this association is easily made. And it is underscored by some of the practices he engages in as a geometer: in particular, congruence proofs, where one shows that things which coincide with one another are equal to one another, by taking one figure, moving it, and placing it on another (superposition).

Recognizing that the diagram is one among other sensible objects, in this way, will prompt the further recognition that sensible objects in general are likenesses, hence that sensible objects in general are deficient. The fact that the guardians are to study astronomy (and harmonics) after geometry, is presumably, in part, because geometry – above all through its use of diagrams – will prepare them to treat stars as representations, where these are not obviously so in the way that diagrams are. As Socrates observes, the study of geometry will make for the better reception of all studies (*Rep.* 527c).

5. Conclusion

My aim here has been to demonstrate that Plato views the use of diagrams in mathematics in a highly positive light – that he sees their use as crucial and central to its capacity to facilitate conversion and progress towards knowledge of forms.

One point I have so far made very little of is the fact that the divided line image, which highlights the mathematicians' use of diagrams, is itself a diagram – a diagram to be treated as an image (*eikon*). As such, it draws attention to key characteristics of diagrams – not least, their approximate and representational nature. Above all, though, the divided line image, as a diagram, highlights the role of the diagram in bringing out the visible–intelligible, likeness–original distinctions. Its explicit purpose is to bring both the visible and intelligible into view at the same time (offering a synoptic view), and to highlight the differences between them, bringing the likeness–original relation to the fore.[61]

Notes

1 I presented earlier versions of this chapter at the NYU Ancient Philosophy Work in Progress Seminar, the Institute of Classical Studies Ancient Philosophy Seminar and the Oxford Ancient Philosophy Workshop. I am very grateful to audiences at these seminars for helpful comments. I am especially grateful to Hugh Benson for comments on a draft of this chapter, and to Paul Pritchard.

2 The exact proportions, here, are not clear.

3 Socrates has previously distinguished knowledge (*epistēmē*) and opinion (*doxa*). Here, knowledge encompasses intelligence and understanding, while opinion encompasses belief and imagination.

4 And Socrates' references to the section 'below' (*katō*), at *Rep.* 511a, and the 'highest' (*anōtatō*) section, at 511d, suggest that this is the case.

5 A more literal translation of 510c reads: 'In regard to these things, as if knowing [them], having made the hypotheses for themselves, they deem necessary no further account about them either for themselves or others, as being apparent to everyone'.

6 Plato leaves open the possibility that *dianoia* encompasses more than just mathematical reasoning. See Fine ('Knowledge & Belief in *Republic* V–VII', 2003): 107: '[Mathematics] is just one example of L3-type reasoning – Plato's moral reasoning in the *Republic* is another example of it'. The suggestion, though, is that mathematics is the most systematic way to achieve/cultivate a cognitive state of *dianoia*. See *Rep.* 518d: 'Then this turning around of the mind itself might be made a subject of professional skill (*techne*), which would effect the conversion as easily and effectively as possible'.

7 With respect to other sensible objects that might be used by the (philosophical) mathematician, the reference to moulding at *Rep.* 510d–e points to the use of models of solids. And astronomy uses the stars as patterns to aid them in the study of realities (529e). It also makes use of planetaria or armillary spheres (*Timaeus* 40c–d).

8 A notable exception is Patterson (2007). His approach, however, is importantly different from my own approach here. He considers the positive role of diagrams in the context of the *Meno* geometrical demonstration, by way of establishing what it is in the use of diagrams that Socrates/Plato is concerned about in the divided line.

9 See, for instance, Cross and Woozley (1964): 'First, the mathematician uses sensible images, and second, he is compelled to employ assumptions which remain unproved assumptions. These Plato regards as defects in mathematical method which the philosopher, pursuing his method of dialectic, is able to avoid' (232); and Annas (1981) asserts that mathematicians, on Plato's view, are 'complacent' and that 'mathematics has two defects compared with dialectic'. (277ff)

10 Klein (1968): 19–20: suggests that arithmetic is 'first and foremost the art of correct counting', while logistic is calculating, where this entails 'knowledge of the relations which connect the single numbers'. But he notes that the two terms frequently occur together and are hard to distinguish on this primary level.

11 See *Rep.* 528a–b and 528d.

12 See *Philebus* 56d: 'The mathematical arts can be divided into two kinds: that of the many (*tōn pollōn*), and that of the philosophers'. The distinction is essentially one between applied and theoretical mathematics, where the latter is more accurate.

13 As to how to understand this, Burnyeat (2000): 15, n.18, suggests that 'problem' here should not be translated 'in such a way as to *confine* Platonic astronomy and harmonics to problems in the technical sense' – i.e. constructions, as distinct from theorems. See also Benson (2012): 188.

14 For instance, units *as such*, as opposed to units of oxen or army.

15 See esp. Benson (2012); also Benson (2010).

16 Benson (2012) allows that 'it is difficult to see why the use of sensible diagrams and constructions should be thought to be problematic' (194), but suggests that inappropriate use of ordinary sensible objects might roughly consist in 'mistaking contingent or artificial consequences of one's hypothesis for genuine or natural consequences . . .. For example, one might take the hypothesis that the circumference of a circle is equivalent to twice the product of π and the radius . . . to be refuted by measuring the circumference of a given circle' (194–195). He concedes that no accomplished mathematician would do this, but argues that unaccomplished ones, beginners of geometry, might be inclined to make this mistake – these geometers must learn not to allow accidental features of the diagram too much weight, and to recognize the essential features of the diagram. Benson's view is compatible with the view that diagrams are a fundamental feature of Greek mathematics, and that they play a key role in conversion, as I argue here, but he takes the line that it is because they are fundamental that Plato emphasizes their misuse in *dianoia*.

17 Patterson (2007) suggests that Plato is worried 'the use of diagrams reflects, promotes, and to a large extent constitutes a thinking of the abstract in the sensible and particular . . .. The use of diagrams, if not understood from the larger perspective of Platonic metaphysics and theory of knowledge, condemns the practitioner to ignorance of the true foundations both of mathematical truth and mathematical cognition' (2). But he is also clear that it does not follow from this that Plato would want to ban the use of diagrams or reform the mathematical method (30, 33).

18 Burnyeat (1987): 218–219; and (2000): 37–38.

19 Netz (1999): esp. 35–40.

20 Netz (1999): 35.

21. Netz (1999): 40.

22. See Heath (1921): esp. 216–217. Heath concludes that 'there is probably little in the whole compass of the *Elements* of Euclid, except the new theory of proportion due to Eudoxus and its consequences, which was not in substance included in the recognized content of geometry and arithmetic by Plato's time' (217). See also Burnyeat (2000): 24: 'Euclid's *Elements* incorporates much previous work, from two main sources: first, earlier *Elements* by Leon and Theudius, both fourth-century mathematicians who spent time in the Academy, and, second, the works of Theaetetus and Eudoxus, two outstanding mathematicians with whom Plato had significant contact. If we could read the mathematics available at the time Plato wrote the Republic, a good deal of it would look like an early draft of Euclid's *Elements*'.

23 Netz (1999): 14.

24 Netz (1999): 19. Netz suggests that diagrams would have been prepared on a range of media, including dusted surfaces, wax tablets, papyri and whiteboards (15–16). He notes that his own efforts to draw diagrams in the sand were an unmitigated disaster – which raises questions regarding the standard interpretation of the geometrical

demonstration in the *Meno*, on which Socrates draws diagrams in the sand as he speaks.

25 See Patterson (2007) for examples.

26 Patterson (2007): 14.

27 See Patterson (2007): 14–17. And Netz (1998) argues that: 'Greek mathematics relies upon diagrams in an essential, logical way. Without diagrams, objects lose their truth-value. *Ergo*, part of the content is supplied by the diagram, and not solely by the text. The diagram is not just a pedagogic aid, it is a necessary, logical component' (34).

28 It emerges through construction of the diagram that the two middle sections of the line are equal in size. This is not something that is specified in the text.

29 See, for instance, Brown (2008): esp. chs. 3 and 12; and Giaquinto (1993) and (2008).

30 See esp. Giaquinto (1993).

31 Both in the geometrical demonstration, and also at 86dff.

32 For instance, at *Theaetetus* 147d–148b, and esp. at *Timaeus* 40c–d: 'To describe all this (the movements of the heavens) without an inspection of models (*mimēmatōn*) of these movements would be labour in vain' (40d).

33 For the view that *dianoia* and *noēsis* are *not* distinguishable on ontological grounds, see, for instance, Pritchard (1995): 108–111. See also Fine ('Knowledge & Belief in *Republic* V–VII', 2003): 99–104 who argues against an objects analysis and for a contents analysis of the *Republic* divided line. For the view that *dianoia* and *noēsis* are distinguishable on ontological grounds, and that the objects of mathematics are intermediates, see, for instance Annas (1988): esp. 251, Denyer (2006): 303, and Burnyeat (2000): 33ff.

34 It is clear that Plato *should* posit intermediate mathematical entities – it is not possible to do mathematics with forms – and there are indications outside of the *Republic* divided line passage that he does (see esp. *Philebus* 55c–59c). But I see no compelling evidence for this view within the divided line passage. *Rep.* 534a suggests that Plato is deliberately side-stepping discussion of mathematical objects, in order to bring out the cognitive differences between mathematician and philosopher, by means of the image–original relation.

35 Cross and Woozley (1964): 238 argue that the objects of *dianoia* and *noēsis* differ in as much as the forms that *noēsis* grasps, unlike the forms that *dianoia* grasps, are grasped in all their interconnectedness and in the light of the Good.

36 Some mathematicians do make the sightlover's mistake – for instance, the astronomers criticized for focusing their attention on the stars, and not what is beyond. But these are the mathematicians of the day, not the philosophical mathematician.

37 Or: '. . . knowledge of something which is always the case, and not of something which is sometimes the case and sometimes not'.

38 See Harte (2006). See esp. 37: 'On my reading, if one recognizes the equality of perceptible equals in the manner that my interpretation of the crucial experience requires, recognition of the form, and recognition of what one perceives as merely a likeness of the form must occur in one and the same act'.

39 See Pritchard (1995): 106; also Gregory (1996): 453.

40 Plato distinguishes the use of arithmetic and logistic in war from its use for the purposes of buying and selling (*Rep.* 525b–c), thus distinguishing between educational practical pursuits and purely practical ones. Rosen (2005): 290–291 suggests that this distinction is made on the basis that moneymaking is base. At 536d–537a, the practice of mathematics is advocated for young children. See also *Laws* 819.

41 There are many different interpretations of this famous passage, but on one possible reading Plato is identifying relative properties, such as large and small, and one, also, as things that impinge upon the senses together with their opposite because they are 'incomplete' predicates, context-dependent.

42 The point seems to be that all numbers attached to visible and tangible bodies suffer the compresence of one and many, hence provoke thought. Presumably because every number is made up of ones. So three, say, is one number ('one' in a series), but also three units, and one and many in this sense (see Craig 1994: 282). Thus where we perceive three oxen, we see *a* three, but also three units, hence one compresent with many. And so it is with all numbers, that they appear no more one than its opposite, many.

43 See e.g. Pritchard (1995): 65: 'a Greek number must be some definite plurality (collection) of units (monads) where the unit is either some physical object or an abstract unit'.

44 Some commentators (see, for instance, Cross and Woozley 1964: 237) have taken this to mean that the study of arithmetic prompts contemplation of *forms of numbers – all* numbers, not just One – so the form of Two and the form of Three etc. If, however, all numbers suffer the same compresence of one and many, then even if there *are* forms of all numbers (*Phaedo* 101b–c, for instance, gives the impression that there are), observation of this compresence of one and many will always provoke contemplation and discourse of the form of One, at least in the first instance. In this case, Socrates must mean that we are prompted to contemplation and discourse of 'the nature of numbers', where it is in the nature of every number to be one.

45 See esp. *Rep.* 522c–e; also 525b and 526c.

46 The requirement for practical utility is often dismissed as insignificant, but is surely crucial for the purposes of conversion. Shorey (1935) 148, n.(c), for instance, notes: 'This further prerequisite of the higher education follows naturally from the plan of the *Republic*; but it does not interest Plato much and is, after one or two repetitions, dropped'. See also Burnyeat (2000): 10–11: 'Plato is not serious about justifying the study [of arithmetic] on grounds of its practical utility'.

47 Plane geometry is concerned with two dimensions, solid geometry, three dimensions, and astronomy, solids in revolution (*Rep.* 528a–b, 528d–e). Harmonics, like astronomy, is a study of motion – 'as the eyes fasten on astronomical motions, so the ears fasten on harmonic ones' (530d). Understanding something in three dimensions requires recognizing how it stands in relation to more other things, for instance depth, than if it were two-dimensional. And understanding a three-dimensional object in motion will require recognizing how it stands in relation to even more things, such as time.

48 See e.g. *Rep.* 526e, 527b, 529a, 533d. Socrates' insistence on the practical utility of mathematical subjects other than arithmetic, might also be taken to demonstrate this – plane geometry applies to the conduct of war (526c); and astronomy is 'serviceable not only to agriculture and navigation, but still more to the military art' (527d).

49 See Craig (1994): 285: 'The sequence of five studies . . . is not simply and exclusively an anabasis ("ascent"), although rhetorically that is the dominant impression created. Partly this is a carry-over from the Cave allegory . . .. This impression is reinforced by the natural pedagogic progression of the studies themselves . . .. But the progression of study from arithmetic to harmony doesn't bring one ever closer to Being, nor is it said to'.

50 One might also make the case that popular geometry – geometry applied to the sensible world, e.g. for building houses or organizing troops – already plays a

conversion role, just as popular arithmetic does. But I want to argue here that it is especially the use of diagrams that effects conversion.

51 For instance, models of solids, where these have the same function as diagrams.

52 We don't have to cash out the use of visible objects in terms of physical diagrams: we could just as well be *imagining* a particular triangle – this will involve imagining a triangle to be a particular size, say.

53 See Mueller (1981): 13: 'The *protasis* is formulated without letters to make the generality of what is being proved apparent. The *ekthesis* starts the proof, but, before the proof is continued, the *diorismos* insists that it is only necessary to establish something particular to establish the *protasis*. When the particular thing has been established, the *sumperasma* repeats what was insisted upon in the *diorismos*'.

54 *A Commentary on the First Book of Euclid's Elements*, trans. G. R. Morrow (1970): 207.

55 Giaquinto (1993) 90: 'The diagram must look similar to something which appears exactly square. Seeing the diagram as a square involves both seeing it and observing this similarity of appearance'.

56 Although the *Meno* demonstration is not set out like a Euclidean proposition, as Patterson notes (2007) 5ff, it roughly speaking constitutes a Euclidean production proof.

57 Sherry (2008): 64.

58 See Giaquinto (1993) on the *Meno* demonstration: 'Although the process includes bits of deductive sentential reasoning, the use of diagrams in this process is clearly not a superfluous adjunct to a proof (a valid sequence of sentences), since no proof of the theorem was followed or constructed. On the other hand, the use of diagrams was not empirical: the visual experience that resulted from the use of diagrams was not used as a source of observational evidence for this or that proposition. In this case vision was a means of getting information about things that were not before one's eyes. Seeing the diagram as a geometrical figure of a certain sort, seeing parts of it as related in certain geometrical ways and visualizing motions of the parts, enabled us to tap our geometrical concepts in a way which feels clear and immediate'. See also Brown (2008), who argues that diagrams (in the context of contemporary mathematics) can be 'windows to Plato's heaven' (40).

59 Wedberg (1955): 47.

60 I owe this point to Andy Gregory.

61 Brumbaugh (1968) observes that Plato in general makes prominent use of diagrams to represent the intelligible–visible or knowledge–opinion relationship, suggesting that he sees diagrams as naturally fitted to bring out this relationship.

References

Annas, J. (1981). *An Introduction to Plato's* Republic. Oxford: Oxford University Press.

Benson, H. H. (2010). 'Plato's philosophical method in the *Republic*: The divided line (510b-511d)'. In M. McPherran (ed.), *Cambridge Critical Guide to the* Republic. Cambridge: Cambridge University Press, pp. 188–208.

Benson, H. H. (2012). 'The problem is not mathematics, but mathematicians: Plato and the mathematicians again'. *Philosophia Mathematica* (3)20: 170–199.

Brown, J. R. (2008). *Philosophy of Mathematics*. London and New York: Routledge.

Brumbaugh, R. S. (1968). *Plato's Mathematical Imagination: The Mathematical Passages in the Dialogues and Their Interpretation*. Bloomington, IN: Indiana University Press.

Burnyeat, M. F. (2000). 'Plato on why mathematics is good for the soul'. *Proceedings of the British Academy* 103: 1–81.

Burnyeat, M. F. (1987). 'Platonism and mathematics: A prelude to discussion'. In A. Graeser (ed.), *Mathematics and Metaphysics in Aristotle*. Bern: Haupt, pp. 213–240.

Craig, L. H. (1994). *The War Lover: A Study of Plato's* Republic. Toronto: University of Toronto Press.

Cross, R. C. and Woozley, A. D. (1964). *Plato's Republic: A Philosophical Commentary*. New York: Macmillan.

Denyer, N. (2007). 'Sun and line: The role of the good'. In G. R. F Ferrari (ed.), *The Cambridge Companion to Plato's* Republic. Cambridge: Cambridge University Press, pp. 284–309.

Fine, G. (2003). *Plato on Knowledge and Forms*. Oxford: Oxford University Press.

Harte, V. (2006). 'Beware of imitations: Image recognition in Plato'. In F. G. Hermann (ed.), *New Essays on Plato*. Swansea: Classical Press of Wales, pp. 21–42.

Heath, T. (1921). *A History of Greek Mathematics*, vol. 2. Oxford: Clarendon Press.

Giaquinto, M. (1993). 'Diagrams: Socrates and Meno's slave'. *International Journal of Philosophical Studies* 1: 81–97.

Giaquinto, M. (2008). 'Visualizing in mathematics'. In P. Mancosu (ed.), *The Philosophy of Mathematical Practice*. Oxford: Oxford University Press, pp. 22–64.

Gregory, A. (1996). 'Astronomy and observation in Plato's *Republic*'. *Studies in History and Philosophy of Science* 27(4): 451–471.

Klein, J. (1968). *Greek Mathematical Thought and the Origin of Algebra*. Cambridge, MA: Massachusetts Institute of Technology Press.

Mueller, I. (1981). *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. New York: Dover Publications.

Netz, R. (1998). 'Greek mathematical diagrams: Their use and their meaning'. *For the Learning of Mathematics* 18(3): 33–39.

Netz, R. (1999). *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*. Cambridge: Cambridge University Press.

Patterson, R. (2007). 'Diagrams, dialectic, and mathematical foundations in Plato'. *Apeiron* 40: 1–33.

Pritchard, P. (1995). *Plato's Philosophy of Mathematics*. Berlin: Academia-Verlag.

Proclus (1970). *A Commentary on the First Book of Euclid's Elements*, trans. G. R. Morrow. Princeton, NJ: Princeton University Press.

Rosen, S. (2005). *Plato's* Republic. New Haven, CT: Yale University Press.

Sherry, D. (2008). 'The role of diagrams in mathematical arguments'. *Foundations of Science* 14(1–2): 59–74.

Shorey, P. (1935). *Plato: The Republic, Books VI-X*. Cambridge, MA: Harvard University Press.

Wedberg, A. (1955). *Plato's Philosophy of Mathematics*. Stockholm: Almqvist & Wiksell.