Cayley fibrations in the Bryant–Salamon Spin(7) manifold

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Abstract

On each complete asymptotically conical Spin (7) manifold constructed by Bryant and Salamon, including the asymptotic cone, we consider a natural family of SU (2) actions preserving the Cayley form. For each element of this family, we study the (possibly singular) invariant Cayley fibration, which we describe explicitly, if possible. These can be reckoned as generalizations of the trivial flat fibration of $\mathbb{R}^8$ and the product of a line with the Harvey–Lawson coassociative fibration of $\mathbb{R}^7$. The fibres will provide new examples of asymptotically conical Cayley submanifolds in the Bryant–Salamon manifolds of topology $\mathbb{R}^4$, $\mathbb{R} \times S^3$ and $\mathcal{O}_{\mathbb{CP}^1}(-1)$.

1 Introduction

In 1926, Cartan showed how to associate a group to any Riemannian manifold through parallel transport [7]. He called such a group the holonomy group of the Riemannian manifold, and he used it to classify symmetric spaces. Almost 30 years later, Berger found all the groups that could appear as the holonomy of a simply-connected, non-symmetric, and irreducible Riemannian manifold [4]. The exceptional holonomy groups G_2 and Spin (7) belonged to this list. The existence of Riemannian manifolds with such holonomy was unknown until Bryant [5] provided incomplete examples and Bryant–Salamon [6] provided complete ones. In particular, Bryant and Salamon constructed a 1-parameter family of torsion-free G_2-structures on $\Lambda^2_-(T^*S^4)$, $\Lambda^2_-(T^*\mathbb{CP}^2)$, $\mathcal{S}(S^3)$, and a 1-parameter family of torsion-free Spin (7)-structures on $\mathcal{S}_-(S^4)$. The holonomy principle implies that the holonomy group of these manifolds is contained in G_2 and Spin (7), respectively. As Bryant and Salamon proved that their examples have full holonomy, the problem of the classification of Riemannian holonomy groups is settled.

Manifolds with exceptional holonomy are Ricci-flat and admit natural calibrated submanifolds. These are the associative threefold and the coassociative fourfold in the G_2 case,

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while they are the Cayley fourfold in the Spin (7) one. A crucial aspect of the study of manifolds with exceptional holonomy regards fibrations through these natural submanifolds. One of the main reasons for the interest in calibrated fibrations comes from mathematical physics. Indeed, analogously to the SYZ conjecture [25] that relates special Lagrangian fibrations in mirror Calabi–Yau manifolds, one would expect similar dualities for coassociative fibrations in the G_2 case and Cayley fibrations in the Spin (7) one. We refer the reader to [1, 9] for further details. Another reason lies in the attempt to understand and construct new compact manifolds with exceptional holonomy through these fibrations [8].

Some work has been carried out in the G_2 case (see f.i. [2, 3, 8, 15, 19]), while little is known in the Spin (7) setting. In particular, Karigiannis and Lotay [15] constructed an explicit coassociative fibration on each G_2 Bryant–Salamon manifold and the relative asymptotic cone. To do so, they chose a 3-dimensional Lie group acting through isometries preserving the G_2-structure, and they imposed the fibres to be invariant under this group action. In this way, the coassociative condition is reduced to a system of tractable ODEs defining the fibration. Previously, this idea was used to study cohomogeneity one calibrated submanifolds related to exceptional holonomy in the flat case by Lotay [21] and in $\Lambda_-^2(T^*S^4)$ by Kawai [16]. Analogously, we consider Cayley fibrations on each Spin (7) Bryant–Salamon manifold and the relative asymptotic cone, which are invariant under a natural family of structure-preserving SU (2) actions.

The first key observation, due to Bryant and Salamon [6], is that $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ is contained in the subgroup of the isometry group that preserves the Spin (7)-structure. Indeed, one can lift an action of SO (5) on S^4 to an action of $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ on the spinor bundle of S^4. The $\mathrm{Sp}(1)$ factor of $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ comes from a twisting of the fibre. Clearly, this group admits plenty of 3-dimensional subgroups. The family we consider consists of the subgroups that respect the direct product, i.e. that do not sit diagonally in $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$. Through Lie group theory, it is easy to find these subgroups. Indeed, they either are the whole $\mathrm{Sp}(1)$, appearing in the second factor or the lift of one of the following subgroups of SO (5), which are going to be contained in the first factor:

$$\mathrm{SO}(3) \times \mathrm{Id}_2, \quad \mathrm{Sp}(1) \times \mathrm{Id}_1,$$
$$\mathrm{SO}(3) \quad \text{acting irreducibly on } \mathbb{R}^5,$$

where $\mathrm{Sp}(1) \times \mathrm{Id}_1$ denotes both the subgroup acting on $\mathbb{H} \times \mathbb{R}$ by left multiplication and by right multiplication of the quaternionic conjugate. Observe that their lifts to $\mathrm{Sp}(2)$ are all diffeomorphic to $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$. Moreover, the $\mathrm{Sp}(1)$ contained in the second factor will only act on the fibres of $\mathscr{S}_-(S^4)$, leaving the base fixed.

1.1 Summary of results and organization of the paper

In Sect. 2, we briefly review some basic results on Spin (7) and Riemannian geometry. In particular, once fixed the convention for the Spin (7)-structure, we recall the definition of Cayley submanifolds, together with Karigiannis–Min-Oo's characterization [12, Proposition 2.5], and Cayley fibrations. Similarly to [15, Definition 1.2], our notion of Cayley fibrations allows the fibres to be singular and to self-intersect. Finally, we provide the definitions of asymptotically conical and conically singular manifolds.

Section 3 contains a detailed description of the 1-parameter family of Spin (7) manifolds constructed by Bryant–Salamon. Here, we also discuss the automorphism group. In particular, we briefly explain why the system of ODEs characterizing the fibration induced by the irreducible action of SO (3) on S^4 is going to be too complicated to be solved.

Starting from Sect. 4, we deal with Cayley fibrations. Here, we study the fibration invariant under the SU(2) acting only on the fibres of $\mathcal{J}_-(S^4)$. In this case, the fibration is trivial, i.e. coincide with the usual projection map from $\mathcal{J}_-(S^4)$ to S^4. We compute the multi-moment map in the sense of [22, 23], which is a polynomial depending on the square of the distance function. Blowing-up at any point of the zero section, the fibration becomes the trivial flat fibration of $\mathbb{R}^8$.

In Sect. 5, we consider the action on $\mathcal{J}_-(S^4)$ induced by $SO(3) \times \text{Id}_2 \subset SO(5)$ acting on S^4. Under a suitable choice of metric-diagonalizing coframe on an open, dense set $\mathcal{U}$, the system of ODEs characterizing the Cayley condition is completely integrable, and hence, we obtain a locally trivial fibration on $\mathcal{U}$ whose fibres are Cayley submanifolds. Extending by continuity the fibration to the whole $\mathcal{J}_-(S^4)$, we prove that the parameter is S^4 and the fibres are topological $\mathbb{R}^4$s, $\mathcal{O}_{\mathbb{CP}^1}(-1)$s or $\mathbb{R} \times S^3$s. Through a asymptotic analysis, it is easy to see that the $\mathbb{R}^4$s separating the Cayleys of different topology are the only singular ones. The singularity is asymptotic to the Lawson–Osserman cone [18]. Each Cayley intersects at least another one in the zero section of $\mathcal{J}_-(S^4)$, and, at infinity, they are asymptotic to a non-flat cone with link S^3 endowed with either the round metric or a squashed metric. While in the G_2 case [15, Sects. 5.7, 6.7], the multi-moment map they explicitly compute has a clear geometrical interpretation, it does not in our case. Finally, keeping track of the Cayley fibration, blowing-up at the north pole, we obtain the fibration on $\mathbb{R}^8$, which is given by the product of the SU(2)-invariant coassociative fibration constructed by Harvey and Lawson [10, Section IV.3] with a line.

We deal with the Cayley fibration invariant under the SU(2) action induced from $\text{Sp}(1) \times \text{Id}_1$ in Sect. 6. The left quaternionic multiplication gives the same fibration as the conjugate right quaternionic multiplication up to orientation. Contrary to the previous case, we cannot completely integrate the system of ODEs we obtain on an open, dense set $\mathcal{U}$. However, we deduce all the information we are interested in via a dynamical system argument. In particular, we show that the fibres are parametrized by a 4-dimensional sphere and that they are smooth submanifolds of topology $S^3 \times \mathbb{R}$, $\mathbb{R}^4$ or S^4. The unique point of intersection is the south (north) pole of the zero section, where all fibres of topology $\mathbb{R}^4$ and the sole Cayley of topology S^4 (i.e. the zero section) intersect. It is easy to show that all Cayleys are asymptotic to a non-flat cone with round link S^3. We also compute the multi-moment map and show that the fibration converges to the trivial flat fibration of $\mathbb{R}^8$ when we blow-up at the north pole.

The last group action that would be natural to study is the lift of $SO(3)$ acting irreducibly on $\mathbb{R}^5$. However, in this case the ODEs become extremely complicated and cannot be solved explicitly. Moreover, the analogous action on the flat Spin(7) space and on the Bryant–Salomon G_2 manifold $\Lambda_-^2(T^*S^4)$ was studied by Lotay [20, Sect. 5.3.3] and Kawai [16], respectively. In both cases, the defining ODEs for Cayley submanifolds and coassociative submanifolds were too complicated.

2 Preliminaries

In this section, we recall some basic results concerning Spin(7) manifolds, Cayley submanifolds and Riemannian conifolds.

2.1 Spin(7) manifolds

We use the same convention of [6] and [10] to define $\mathrm{Spin}(7)$-structures and $\mathrm{Spin}(7)$ manifolds.

The local model is $\mathbb{R}^8 \cong \mathbb{R}^4 \oplus \mathbb{R}^4$ with coordinates $(x_0, ..., x_3, a_0, ..., a_3)$, and Cayley form:

$$\Phi_{\mathbb{R}^8} = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 + da_0 \wedge da_1 \wedge da_2 \wedge da_3 + \sum_{i=1}^{3} \omega_i \wedge \eta_i,$$

where $\omega_i = dx_0 \wedge dx_i - dx_j \wedge dx_k$, $\eta_i = da_0 \wedge da_i - da_j \wedge da_k$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Note that $\{\omega_i\}_{i=1}^3$ and $\{\eta_i\}_{i=1}^3$ are the standard basis of the anti-self-dual 2-forms on the two copies of $\mathbb{R}^4$. It is well known that $\mathrm{Spin}(7)$ is isomorphic to the stabilizer of $\Phi_{\mathbb{R}^8}$ in $\mathrm{GL}(8, \mathbb{R})$.

Remark 2.1 This choice of convention for $\Phi_{\mathbb{R}^8}$ is compatible with the fact that we will be working on $\displaystyle{\not\!S_-(S^4)}$. Indeed, we can identify our local model with $\displaystyle{\not\!S_-(\mathbb{R}^4)}$. Further details regarding the sign conventions and orientations for $\mathrm{Spin}(7)$-structures can be found in [13].

Definition 2.2 Let M be a manifold, and let Φ be a 4-form on M. We say that Φ is admissible if, for every $x \in M$, there exists an oriented isomorphism $i_x : \mathbb{R}^8 \to T_x M$ such that $i_x^* \Phi = \Phi_{\mathbb{R}^8}$. We also refer to Φ as a $\mathrm{Spin}(7)$-structure on M.

The $\mathrm{Spin}(7)$-structure on M also induces a Riemannian metric, g_Φ, and an orientation, vol_Φ, on M. With respect to these structures Φ is self-dual. We refer the reader to [26] for further details.

Definition 2.3 Let M be a manifold, and let Φ be a $\mathrm{Spin}(7)$-structure on M. We say that (M, Φ) is a $\mathrm{Spin}(7)$ manifold if the $\mathrm{Spin}(7)$-structure is torsion-free, i.e. $d\Phi = 0$. In this case, $\mathrm{Hol}(g_\Phi) \subseteq \mathrm{Spin}(7)$.

2.2 Cayley submanifolds and Cayley fibrations

Given (M, Φ), $\mathrm{Spin}(7)$ manifold, it is clear that Φ has comass one, and hence, it is a calibration.

Definition 2.4 We say that a 4-dimensional oriented submanifold is Cayley if it is calibrated by Φ, i.e. if $\Phi|_{N^4} = \mathrm{vol}_{N^4}$. Fixed a point $p \in M$, a 4-dimensional oriented vector subspace H of $T_p M$ is said to be a Cayley 4-plane if $\Phi|_p$ calibrates H.

Remark 2.5 Observe that N is a Cayley submanifold if and only if $T_p N$ is a Cayley 4-plane for all $p \in N$.

We now give Karigiannis and Min-Oo characterization of the Cayley condition.

Proposition 2.6 (Karigiannis–Min-Oo [12, Proposition 2.5]) *The subspace spanned by tangent vectors* u, v, w, y *is a Cayley* 4-*plane, up to orientation, if and only if the following form vanishes*:

$$\eta = \pi_7\Big(u^\flat \wedge B(v, w, y) + v^\flat \wedge B(w, u, y) + w^\flat \wedge B(u, v, y) + y^\flat \wedge B(v, u, w)\Big),$$

where

$$B(u, v, w) := w \lrcorner\, v \lrcorner\, u \lrcorner\, \Phi$$

and

$$\pi_7(u^\flat \wedge v^\flat) := \frac{1}{4}\Big(u^\flat \wedge v^\flat + u \lrcorner\, v \lrcorner\, \Phi\Big).$$

Remark 2.7 The reduction of the structure group of M to $\mathrm{Spin}\,(7)$ induces an orthogonal decomposition of the space of differential k-forms for every k, which corresponds to an irreducible representation of $\mathrm{Spin}\,(7)$. In particular, if $k = 2$, the irreducible representations of $\mathrm{Spin}\,(7)$ are of dimension 7 and 21. At each point $x \in M$, these representations induce the decomposition of $\Lambda^2(T_x^* M)$ into two subspaces, which we denote by Λ_7^2 and Λ_{21}^2, respectively. The map π_7 defined in Proposition 2.6 is precisely the projection map from the space of two forms to Λ_7^2. Further details can be found in [26].

Following [15], we extend the definition of Cayley fibration so that it may admit intersecting fibres and singular fibres.

Definition 2.8 Let (M, Φ) be a $\mathrm{Spin}\,(7)$ manifold. M admits a Cayley fibration if there exists a family of Cayley submanifolds N_b (possibly singular) parametrized by a 4-dimensional space $\mathcal{B}$ satisfying the following properties:

- M is covered by the family $\{N_b\}_{b \in \mathcal{B}}$;
- there exists an open dense set $\mathcal{B}^\circ \subset \mathcal{B}$ such that N_b is smooth for all $b \in \mathcal{B}^\circ$;
- there exists an open dense set $M' \subset M$ and a smooth fibration $\pi : M' \to \mathcal{B}$ with fibre N_b for all $b \in \mathcal{B}$.

Remark 2.9 The last point allows the Cayley submanifolds in the family $\mathcal{B}$ to intersect. Indeed, this may happen in $M \setminus M'$. Moreover, we may lose information (e.g. completeness and topology) when we restrict the Cayley fibres to M'.

We conclude this subsection explaining how we determine the topology of $\mathbb{R}^2$ bundles over $S^2 \cong \mathbb{CP}^1$ arising as the smooth fibres of a Cayley fibration. This is the same discussion used in [15]. Let N be the total space of an $\mathbb{R}^2$-bundle over $\mathbb{CP}^1$ which is also a Cayley submanifold of a $\mathrm{Spin}\,(7)$ manifold (M, Φ). Since N is orientable and it is the total space of a bundle over an oriented base, it is an orientable bundle. We deduce that N is homeomorphic to a holomorphic line bundle over $\mathbb{CP}^1$. These objects are classified by an integer $k \in \mathbb{Z}$ and are denoted by $\mathcal{O}_{\mathbb{CP}^1}(k)$. Moreover, for $k > 0$ we have the following topological characterization of $\mathcal{O}_{\mathbb{CP}^1}(-k)$:

$$\mathcal{O}_{\mathbb{CP}^1}(-k) \setminus \mathbb{CP}^1 \cong \mathbb{C}^2/\mathbb{Z}_k \cong \mathbb{R}^+ \times (S^3/\mathbb{Z}_k).$$

In the situation we will consider, the submanifolds we construct have the form $N \setminus S^2 = \mathbb{R}^+ \times S^3$. Hence, the only possibility is to obtain topological $\mathcal{O}_{\mathbb{CP}^1}(-1)$s.

2.3 Riemannian conifolds

We now recall the definitions of asymptotically conical and conically singular Riemannian manifolds.

Definition 2.10 A Riemannian cone is a Riemannian manifold (M_0, g_0) with $M_0 = \mathbb{R}^+ \times \Sigma$ and $g_0 = dr^2 + r^2 g_\Sigma$, where r is the coordinate on $\mathbb{R}^+$ and g_Σ is a Riemannian metric on the link of the cone, Σ.

Definition 2.11 We say that a Riemannian manifold (M, g) is asymptotically conical (AC) with rate $\lambda < 0$ if there exists a Riemannian cone (M_0, g_0) and a diffeomorphism $\Psi : (R, \infty) \times \Sigma \to M \setminus K$ satisfying:

$$|\nabla^j(\Psi^* g - g_0)| = O(r^{\lambda - j}) \qquad r \to \infty \quad \forall j \in \mathbb{N},$$

where K is a compact set of M and $R > 0$. (M_0, g_0) is the asymptotic cone of (M, g) at infinity.

Definition 2.12 We say that a Riemannian manifold (M, g) is conically singular with rate $\mu > 0$ if there exists a Riemannian cone (M_0, g_0) and a diffeomorphism $\Psi : (0, \epsilon) \times \Sigma \to M \setminus K$ satisfying:

$$|\nabla^j(\Psi^* g - g_0)| = O(r^{\mu - j}) \qquad r \to 0 \quad \forall j \in \mathbb{N},$$

where K is a closed subset of M and $\epsilon > 0$. (M_0, g_0) is the asymptotic cone of (M, g) at the singularities.

Remark 2.13 As Σ does not need to be connected, AC manifolds may admit more than one end and asymptotically singular manifolds may admit more than one singular point.

3 Bryant–Salamon $\mathrm{Spin}(7)$ manifolds

In this section we will describe the central objects of this work, i.e. the $\mathrm{Spin}(7)$ manifolds constructed by Bryant and Salamon in [6]. There, they provided a 1-parameter family of torsion-free $\mathrm{Spin}(7)$-structures on $M := \mathcal{S}_-(S^4)$, the negative spinor bundle on S^4. The 4-dimensional sphere is endowed with the metric of constant sectional curvature k, which is the unique spin self-dual Einstein 4-manifolds with positive scalar curvature [11]. Without loss of generality, we rescale the sphere so that $k = 1$.

Remark 3.1 The Bryant–Salamon construction on S^4 also works on spin 4-manifolds with self-dual Einstein metric, but negative scalar curvature, and on spin orbifolds with self-dual Einstein metric. However, in these cases, the metric is not complete or smooth.

3.1 The negative spinor bundle of S^4

Let S^4 be the 4-sphere endowed with the Riemannian metric of constant sectional curvature 1. As S^4 is clearly spin, given $P_{SO(4)}$ frame bundle of S^4 we can find the spin structure $P_{Spin(4)}$ together with the spin representation:

$$\mu := (\mu_+, \mu_-) : Sp(1) \times Sp(1) \cong Spin(4) \to GL(\mathbb{H}) \times GL(\mathbb{H}),$$

where $\mu_\pm(p_\pm)(v) := v\bar{p}_\pm$. Let $\tilde{\pi} : P_{Spin(4)} \to P_{SO(4)}$ be the double cover in the definition of spin structure, and let $\tilde{\pi}_0^n : Spin(n) \to SO(n)$ be the double (universal) covering map for all $n \geq 3$. The negative spinor bundle over S^4 is defined as the associated bundle:

$$\not{S}_-(S^4) := P_{Spin(4)} \times_{\mu_-} \mathbb{H}.$$

The positive spinor bundle is defined analogously, taking μ_+ instead.

Given an oriented local orthonormal frame for S^4, $\{e_0, e_1, e_2, e_3\}$, the real volume element $e_0 \cdot e_1 \cdot e_2 \cdot e_3$ acts as the identity on the negative spinors and as minus the identity on the positive ones. Now, let $\{b_0, b_1, b_2, b_3\}$ be the dual coframe of $\{e_0, e_1, e_2, e_3\}$, let $\tilde{\omega}$ be the connection 1-form relative to the Levi-Civita connection of S^4 with respect to the frame $\{e_0, e_1, e_2, e_3\}$ and let $\{\sigma_1, \sigma_i, \sigma_j, \sigma_k\}$ be a local orthonormal frame for the negative spinor bundle corresponding to the standard basis of $\{1, i, j, k\}$ in this trivialization. Hence, we can define the linear coordinates (a_0, a_1, a_2, a_3) which parametrize a point in the fibre as $a_0\sigma_1 + a_1\sigma_i + a_2\sigma_j + a_3\sigma_k$.

By the properties of the spin connection and the fact we are working on the negative spinor bundle, we can write:

$$\nabla\sigma_\alpha = \left(\rho_1\mu_-(e_2 \cdot e_3) + \rho_2\mu_-(e_3 \cdot e_1) + \rho_3\mu_-(e_1 \cdot e_2)\right)\sigma_\alpha$$
$$= \left(\rho_1\mu_-(i) + \rho_2\mu_-(j) + \rho_3\mu_-(k)\right)\sigma_\alpha,$$

where $2\rho_1 = \tilde{\omega}_2^3 - \tilde{\omega}_0^1$, $2\rho_2 = -\tilde{\omega}_0^2 - \tilde{\omega}_1^3$ and $2\rho_3 = \tilde{\omega}_1^2 - \tilde{\omega}_0^3$. It is well known that these are the connection forms on the bundle of anti-self-dual 2-forms, with respect to the connection induced by the Levi-Civita connection on S^4 and the frame given by $\Omega_i := b_0 \wedge b_i - b_j \wedge b_k$. As usual, (i, j, k) is a cyclic permutation of $(1, 2, 3)$. The ρ_is are characterized by:

$$d\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = -\begin{pmatrix} 0 & -2\rho_3 & 2\rho_2 \\ 2\rho_3 & 0 & -2\rho_1 \\ -2\rho_2 & 2\rho_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}, \tag{3.1}$$

and the vertical one forms are:

$$\begin{aligned} \xi_0 &= da_0 + \rho_1 a_1 + \rho_2 a_2 + \rho_3 a_3, & \xi_1 &= da_1 - \rho_1 a_0 - \rho_3 a_2 + \rho_2 a_3, \\ \xi_2 &= da_2 - \rho_2 a_0 + \rho_3 a_1 - \rho_1 a_3, & \xi_3 &= da_3 - \rho_3 a_0 - \rho_2 a_1 + \rho_1 a_2. \end{aligned} \tag{3.2}$$

Remark 3.2 If we denote by π_{S^4} the vector bundle projection map from $\not{S}_-(S^4)$ to S^4, we can obtain horizontal forms on $\not{S}_-(S^4)$ via pullback. For example, $\{\pi_{S^4}^*(b_i)\}_{i=1}^4$ and the linear combinations of their wedge product are horizontal forms on $\not{S}_-(S^4)$. In order to keep our notation light, we will omit the pullback from now on.

As S^4 is self-dual and with scalar curvature equal to $12k$, we have:

$$d\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = -2\begin{pmatrix} \rho_2 \wedge \rho_3 \\ \rho_3 \wedge \rho_1 \\ \rho_1 \wedge \rho_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix},$$

which is equivalent to [6, p. 842] and [15, 3.24]. We can use it to compute:

$$\begin{aligned}
d\xi_0 &= \xi_1 \wedge \rho_1 + \xi_2 \wedge \rho_2 + \xi_3 \wedge \rho_3 + 1/2\big(a_1\Omega_1 + a_2\Omega_2 + a_3\Omega_3\big), \\
d\xi_1 &= -\xi_0 \wedge \rho_1 - \xi_2 \wedge \rho_3 + \xi_3 \wedge \rho_2 + 1/2\big(-a_0\Omega_1 - a_2\Omega_3 + a_3\Omega_2\big), \\
d\xi_2 &= -\xi_0 \wedge \rho_2 + \xi_1 \wedge \rho_3 - \xi_3 \wedge \rho_1 + 1/2\big(-a_0\Omega_2 + a_1\Omega_3 - a_3\Omega_1\big), \\
d\xi_3 &= -\xi_0 \wedge \rho_3 - \xi_1 \wedge \rho_2 + \xi_2 \wedge \rho_1 + 1/2\big(-a_0\Omega_3 - a_1\Omega_2 + a_2\Omega_1\big),
\end{aligned}$$

(3.3)

that is going to be useful below.

Remark 3.3 A detailed account of spin geometry can be found in [17]. Observe that, there, the definition of positive and negative spinors is interchanged. We opted to stay consistent with [6]. Indeed, the vertical 1-form we obtain coincide with the ones obtained by Bryant and Salamon, up to renaming the ρ_is. The same holds for the relative exterior derivatives.

3.2 The Spin (7)-structures

If $r^2 := a_0^2 + a_1^2 + a_2^2 + a_3^2$ is the square of the distance function from the zero section and c is a positive constant, then the Spin (7)-structures defined by Bryant and Salamon are:

$$\begin{aligned}
\Phi_c :=\ &16(c + r^2)^{-4/5}\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 + 25(c + r^2)^{6/5}b_0 \wedge b_1 \wedge b_2 \wedge b_3 \\
&+ 20(c + r^2)^{1/5}(A_1 \wedge \Omega_1 + A_2 \wedge \Omega_2 + A_3 \wedge \Omega_3),
\end{aligned}$$

(3.4)

where $A_i := \xi_0 \wedge \xi_i - \xi_j \wedge \xi_k$. As usual, (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

The metric induced by Φ_c is

$$g_c := 4(c + r^2)^{-2/5}(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2) + 5(c + r^2)^{3/5}(b_0^2 + b_1^2 + b_2^2 + b_3^2), \quad (3.5)$$

while the induced volume element is

$$\text{vol}_c := (20)^2(c + r^2)^{2/5}(\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \wedge b_0 \wedge b_1 \wedge b_2 \wedge b_3). \quad (3.6)$$

Setting $c = 0$ and $M_0 := \mathcal{S}_-(S^4) \setminus S^4 \cong \mathbb{R}^+ \times S^7$, we obtain a Spin (7) cone (M_0, Φ_0), i.e. M_0 with the metric induced by the Spin (7)-structure Φ_0 is a Riemannian cone.

Theorem 3.4 (Bryant–Salamon [6, p. 847]) *The* Spin (7)*-structure* Φ_c *is torsion-free for all* $c \geq 0$. *Moreover, these manifolds have full holonomy* Spin (7).

It is well known that the Bryant–Salamon Spin (7) manifolds we have just described are asymptotically conical (see, for instance, [24, p.184]); hence, we state here the main results concerning their asymptotic geometry.

Theorem 3.5 *For every* $c \geq 0$, (M, Φ_c) *is an asymptotically conical Riemannian manifold with rate* $\lambda = -10/3$ *and asymptotic cone* (M_0, Φ_0).

3.3 Automorphism Group

A natural subset of the diffeomorphism group of a $\mathrm{Spin}(7)$-manifold is the automorphism group, i.e. the subgroup that preserves the $\mathrm{Spin}(7)$-structure. Clearly, the automorphism group is contained in the isometry group with respect to the induced metric.

In the setting we are considering, Bryant and Salamon noticed that the diffeomorphisms given by the $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-action described as follows are actually in the automorphism group [6, Theorem 2]. Consider $\mathrm{SO}(5)$ acting on S^4 in the standard way. This induces an action on the frame bundle of S^4 via the differential, which easily lifts to a $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ action on $P_{\mathrm{Spin}(4)}$. If we combine it with the standard quaternionic left multiplication by unit vectors on $\mathbb{H}$, we have defined an $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ action on $P_{\mathrm{Spin}(4)} \times \mathbb{H}$. As it commutes with μ_-, it passes to the quotient $\mathcal{S}_-(S^4)$.

By Lie group theory [16, Appendix B], we know that the 3-dimensional connected closed subgroups of $\mathrm{Sp}(2)$ are the lift of one of the following subgroups of $\mathrm{SO}(5)$:

$$\mathrm{SO}(3) \times \mathrm{Id}_2, \qquad \mathrm{Sp}(1) \times \mathrm{Id}_1,$$

$$\mathrm{SO}(3) \quad \text{acting irreducibly on } \mathbb{R}^5,$$

where $\mathrm{Sp}(1) \times \mathrm{Id}_1$ denotes both the subgroup acting on $\mathbb{H} \times \mathbb{R}$ by left multiplication and by right multiplication of the quaternionic conjugate. Observe that they are all diffeomorphic to $\mathrm{SU}(2)$. In particular, the family of 3-dimensional subgroups that do not sit diagonally in $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ consists of

$$G \times 1_{\mathrm{Sp}(1)} \subset \mathrm{Sp}(2) \times \mathrm{Sp}(1)$$

and

$$1_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2) \times \mathrm{Sp}(1),$$

where G is one of the lifts above. These are going to be the subgroups of the automorphism group that we will take into consideration.

4 The Cayley fibration invariant under the $\mathrm{Sp}(1)$ action on the fibre

Let $M := \mathcal{S}_-(S^4)$ and $M_0 := \mathbb{R}^+ \times S^7$ be endowed with the torsion-free $\mathrm{Spin}(7)$-structures Φ_c constructed by Bryant and Salamon and described in Sect. 3.

Observe that (M, Φ_c) and (M_0, Φ_0) admit a trivial Cayley Fibration. Indeed, it is straightforward to see that the natural projection to S^4 realizes both spaces as honest Cayley fibrations with smooth fibres diffeomorphic to $\mathbb{R}^4$ and $\mathbb{R}^4 \setminus \{0\}$, respectively. In both cases, the parametrizing family is clearly S^4

The fibres are asymptotically conical to the cone of link S^3 and metric:

$$ds^2 + \frac{9}{25} g_{S^3},$$

where $s = r^{3/5} 10/3$ and g_{S^3} is the standard unit round metric.

Since $\mathrm{Id}_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1)$ acts trivially on the basis, and as $\mathrm{Sp}(1)$ on the fibres of $\mathcal{S}_-(S^4)$ identified with $\mathbb{H}$, it is clear that the trivial fibration is invariant under $\mathrm{Id}_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1)$.

Remark 4.1 We compute the associated multi-moment map, v_c, in the sense of Madsen and Swann [22, 23]. This is:

$$v_c := \frac{20}{3}(r^2 - 5c)(c + r^2)^{1/5} + \frac{100}{3}c^{6/5},$$

where we subtracted $c^{6/5}100/3$ so that the range of the multi-moment map is $[0, \infty)$. Observe that the level sets of v_c coincide with the level sets of the distance function from the zero section.

Remark 4.2 As in [15, Sect. 4.4], this fibration becomes the trivial Cayley fibration of $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$ when we blow-up at any point of the zero section.

5 The Cayley fibration invariant under the lift of the SO(3) × Id₂ action on S^4

Let $M := \not{\mathcal{S}}_-(S^4)$ and $M_0 := \mathbb{R}^+ \times S^7$ be endowed with the torsion-free Spin(7)-structures Φ_c constructed by Bryant and Salamon that we described in Sect. 3. On each Spin(7) manifold, we construct the Cayley Fibration which is invariant under the lift to M (or M_0) of the standard SO(3) × Id₂ action on $S^4 \subset \mathbb{R}^3 \oplus \mathbb{R}^2$.

5.1 The choice of coframe on S^4

As in [15], we choose an adapted orthonormal coframe on S^4 which is compatible with the symmetries we will impose. Since the action coincides, when restricted to S^4, with the one used by Karigiannis and Lotay on $\Lambda^2_-(T^*S^4)$ [15, Sect. 5], it is natural to employ the same coframe, which we now recall.

We split $\mathbb{R}^5$ into the direct sum of a 3-dimensional vector subspace $P \cong \mathbb{R}^3$ and its orthogonal complement $P^\perp \cong \mathbb{R}^2$. As S^4 is the unit sphere in $\mathbb{R}^5$, we can write, with respect to this splitting:

$$S^4 = \left\{ (\mathbf{x}, \mathbf{y}) \in P \oplus P^\perp : |\mathbf{x}|^2 + |\mathbf{y}|^2 = 1 \right\}.$$

Now, for all $(\mathbf{x}, \mathbf{y}) \in S^4$ there exists a unique $\alpha \in [0, \pi/2]$, some $\mathbf{u} \in S^2 \subset P$ and some $\mathbf{v} \in S^1 \subset P^\perp$ such that:

$$\mathbf{x} = \cos\alpha\,\mathbf{u}, \qquad \mathbf{y} = \sin\alpha\,\mathbf{v}.$$

Observe that $\mathbf{u}$ and $\mathbf{v}$ are uniquely determined when $\alpha \in (0, \pi/2)$, while when $\alpha = 0, \pi/2$, $\mathbf{v}$ can be any unit vector in $P^\perp$ ($\mathbf{y} = 0$) and $\mathbf{u}$ can be any unit vector in P ($\mathbf{x} = 0$), respectively. Hence, we are writing S^4 as the disjoint union of an S^2, corresponding to $\alpha = 0$, of an S^1, corresponding to $\alpha = \pi/2$, and of $S^2 \times S^1 \times (0, \pi/2)$.

If we put spherical coordinates on S^2 and polar coordinates on S^1, then we can write

$$\mathbf{u} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi),$$

and

$$\mathbf{v} = (\cos\beta, \sin\beta),$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$. As usual, ϕ is not unique when $\theta = 0, \pi$.

It follows that if we take out the points where $\theta = 0, \pi$ from $S^2 \times S^1 \times (0, \pi/2)$, we have constructed a coordinate patch U parametrized by $(\alpha, \beta, \theta, \phi)$ on S^4. Explicitly, U is S^4 minus two totally geodesic S^2:

$$S^2_{y_1, y_2 = 0} = \left\{ (\mathbf{x}, \mathbf{0}) \in P \oplus P^\perp : |x|^2 = 1 \right\},$$

corresponding to $\alpha = 0$, and

$$S^2_{x_2, x_3 = 0} = \left\{ (\cos \alpha, 0, 0, \sin \alpha \cos \beta, \sin \alpha \sin \beta) \in P \oplus P^\perp : \alpha \in (0, \pi) \right\},$$

corresponding to $\theta = 0$ and $\theta = \pi$. Observe, that the S^1 corresponding to $\alpha = \pi/2$ is a totally geodesic equator in $S^2_{x_2, x_3 = 0}$.

A straightforward computation shows that the coordinate frame $\{\partial_\alpha, \partial_\beta, \partial_\theta, \partial_\phi\}$ is orthogonal and can be easily normalized obtaining:

$$f_0 := \partial_\alpha, \qquad f_1 := \frac{\partial_\beta}{\sin \alpha}, \qquad f_2 := \frac{\partial_\theta}{\cos \alpha}, \qquad f_3 := \frac{\partial_\phi}{\cos \alpha \sin \theta}.$$

The dual orthonormal coframe is given by:

$$b_0 := d\alpha, \qquad b_1 := \sin \alpha \, d\beta, \qquad b_2 := \cos \alpha \, d\theta, \qquad b_3 := \cos \alpha \sin \theta \, d\phi. \quad (5.1)$$

Observe that $\{b_0, b_1, b_2, b_3\}$ is positively oriented with respect to the outward pointing normal of S^4, hence, the volume form is:

$$\mathrm{vol}\,_{S^4} = \sin \alpha \cos^2 \alpha \sin \theta \, d\alpha \wedge d\beta \wedge d\theta \wedge d\phi.$$

5.2 The horizontal and the vertical space

As in [15, Sect. 5.2], we use (3.1) to compute the ρ_i's in the coordinate frame we have just defined. Indeed, (5.1) implies that:

$$\begin{aligned}
\Omega_1 &= \sin \alpha \, d\alpha \wedge d\beta - \cos^2 \alpha \sin \theta \, d\theta \wedge d\phi, \\
\Omega_2 &= \cos \alpha \, d\alpha \wedge d\theta - \sin \alpha \cos \alpha \sin \theta \, d\phi \wedge d\beta, \\
\Omega_3 &= \cos \alpha \sin \theta \, d\alpha \wedge d\phi - \sin \alpha \cos \alpha \, d\beta \wedge d\theta;
\end{aligned} \quad (5.2)$$

hence, we deduce that:

$$\begin{aligned}
d\Omega_1 &= 2 \sin \alpha \cos \alpha \sin \theta \, d\alpha \wedge d\theta \wedge d\phi, \\
d\Omega_2 &= (\sin^2 \alpha - \cos^2 \alpha) \sin \theta \, d\alpha \wedge d\phi \wedge d\beta - \sin \alpha \cos \alpha \cos \theta \, d\theta \wedge d\phi \wedge d\beta, \\
d\Omega_3 &= \cos \alpha \cos \theta \, d\theta \wedge d\alpha \wedge d\phi + (\sin^2 \alpha - \cos^2 \alpha) \, d\alpha \wedge d\beta \wedge d\theta.
\end{aligned}$$

We conclude that in these coordinates we have:

$$2\rho_1 = -\cos \alpha \, d\beta + \cos \theta \, d\phi; \quad 2\rho_2 = \sin \alpha \, d\theta; \quad 2\rho_3 = \sin \alpha \sin \theta \, d\phi.$$

Now that we have computed the connection forms, we immediately see from (3.2) that the vertical one forms are:

$$\xi_0 = da_0 + a_1\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right) + a_2\frac{\sin\alpha}{2}d\theta + a_3\frac{\sin\alpha\sin\theta}{2}d\phi,$$
$$\xi_1 = da_1 - a_0\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right) - a_2\frac{\sin\alpha\sin\theta}{2}d\phi + a_3\frac{\sin\alpha}{2}d\theta,$$
$$\xi_2 = da_2 - a_0\frac{\sin\alpha}{2}d\theta + a_1\frac{\sin\alpha\sin\theta}{2}d\phi - a_3\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right),$$
$$\xi_3 = da_3 - a_0\frac{\sin\alpha\sin\theta}{2}d\phi - a_1\frac{\sin\alpha}{2}d\theta + a_2\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right).$$

(5.3)

5.3 The $SU(2)$ action

Given the splitting of Sect. 5.1, $\mathbb{R}^5 = P \oplus P^\perp$, since $P \cong \mathbb{R}^3$ and $P^\perp \cong \mathbb{R}^2$, we can consider $SO(3)$ acting in the usual way on P and trivially on $P^\perp$. In other words, we see $SO(3) \cong SO(P) \times \mathrm{Id}_{P^\perp} \subset SO(P \oplus P^\perp) \cong SO(5)$. Obviously, this is also an action on S^4.

By taking the differential, $SO(3)$ acts on the frame bundle $P_{SO(4)}$ of S^4. The theory of covering spaces implies that this action lifts to a $\mathrm{Spin}(3) \cong SU(2)$ action on the spin structure $P_{\mathrm{Spin}(4)}$ of S^4. In particular, the following diagram is commutative:

$$
\begin{array}{ccc}
 & & P_{\mathrm{Spin}(4)} \\
 & \nearrow & \downarrow{\tilde{\pi}} \\
\mathrm{Spin}(3) \times P_{\mathrm{Spin}(4)} \xrightarrow[\tilde{\pi}_0^3 \times \tilde{\pi}]{} SO(3) \times P_{SO(4)} \longrightarrow & & P_{SO(4)}
\end{array}
$$

(5.4)

Finally, if $\mathrm{Spin}(3)$ acts trivially on $\mathbb{H}$, we can combine the two $\mathrm{Spin}(3)$ actions to obtain one on $P_{\mathrm{Spin}(4)} \times \mathbb{H}$, which descends to the quotient $P_{\mathrm{Spin}(4)} \times_{\mu_-} \mathbb{H} = \mathcal{S}_-(S^4)$.

Remark 5.1 Recall that $TS^4 = P_{SO(4)} \times . \mathbb{R}^4$, where $\cdot$ is the standard representation of $SO(4)$ on $\mathbb{R}^4$. Let G be a subgroup of $SO(5)$ which acts on $P_{SO(4)} \times . \mathbb{R}^4$ via the differential on the first term and trivially on the second. It is straightforward to verify that this action passes to the quotient and that it coincides with the differential on TS^4.

Now, we describe the geometry of this $\mathrm{Spin}(3)$ action on $\mathcal{S}_-(S^4)$. Since $\tilde{\pi}$ is fibre-preserving and (5.4) represents a commutative diagram, we observe that, fixed a point $p = (\mathbf{x}, \mathbf{y}) \in S^4 \subset P \oplus P^\perp$, the subgroup of $\mathrm{Spin}(3)$ that preserves the fibre of $P_{\mathrm{Spin}(4)}$ over p is the lift of the subgroup of $SO(3)$ that fixes the fibre of $P_{SO(4)}$ over p.

We first assume $\alpha \neq \pi/2$. The subgroup of $SO(3)$ that preserves the fibres of $P_{SO(4)}$ rotates the tangent space of $S^2 \subset P$ and fixes the other vectors tangent to S^4. Explicitly, if $\{e_i\}_{i=0}^3$ is the oriented orthonormal frame of Sect. 5.1 (or an analogous frame when $\alpha = 0$, $\theta = 0, \pi$), the transformation matrix under the action is:

$$
h_\gamma := \left[\begin{array}{c|c} \mathrm{Id}_2 & \\ \hline & \begin{array}{cc} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{array} \end{array}\right] \in SO(4),
$$

(5.5)

for some $\gamma \in [0, 2\pi)$.

Claim 1 *For all* $\gamma \in [0, 4\pi)$, *under the isomorphism* $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, *we have*:

$$\tilde{\pi}_0^4(\tilde{h}_\gamma) = h_\gamma,$$

where $\tilde{h}_\gamma = (\cos(\gamma/2) + i\sin(\gamma/2), \cos(\gamma/2) + i\sin(\gamma/2))$.

Proof It is well known that, in this context, $\tilde{\pi}_0^4((l, r)) \cdot a = la\bar{r}$ for all $(l, r) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and all $a \in \mathbb{H} \cong \mathbb{R}^4$. The claim follows from a straightforward computation. □

Using once again the commutativity of (5.4) and Claim 1, we deduce that the action in the trivialization of $\mathcal{S}_-(S^4)$ induced by $\{e_i\}_{i=0}^3$ is as follows:

$$
\begin{array}{ccccccc}
U \times \mathbb{H} & \xrightarrow{\cong} & (U \times \mathrm{Spin}(4)) \times_{\mu_-} \mathbb{H} & \longrightarrow & (U \times \mathrm{Spin}(4)) \times_{\mu_-} \mathbb{H} & \xrightarrow{\cong} & U \times \mathbb{H} \\
(p, a) & \longmapsto & \left[(p, 1_{\mathrm{Spin}(4)}), a\right] & \longmapsto & \left[(p, \tilde{h}_\gamma), a\right] & \longmapsto & \left(p, a\hat{h}_\gamma\right)
\end{array}
$$

where $\hat{h}_\gamma := \cos(\gamma/2) - i\sin(\gamma/2)$ and where $a \in \mathbb{H}$. If we write both $\mathbb{R}^2$ factors of $\mathbb{H} \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ in polar coordinates, i.e.

$$a = s\cos(\gamma_-/2) + is\sin(\gamma_-/2) + jt\cos(\gamma_+/2) + kt\sin(\gamma_+/2),$$

for $s, t \in [0, \infty)$ and $\gamma_\pm \in [0, 4\pi)$, we observe that

$$a\hat{h}_\gamma = s\cos\left((\gamma_- - \gamma)/2\right) + is\sin\left((\gamma_- - \gamma)/2\right) + jt\cos\left((\gamma_+ + \gamma)/2\right) + kt\sin((\gamma_+ + \gamma)/2).$$

Geometrically, this is a rotation of angle $-\gamma/2$ on the first $\mathbb{R}^2$ and of angle $\gamma/2$ on the second.

Now, we assume $\alpha = \pi/2$. In this case, the whole $\mathrm{Spin}(3)$ fixes the fibre of $\mathcal{S}_-(S^4)$.

Claim 2 $\mathrm{Spin}(3)$ *acts on the fibre of* $\mathcal{S}_-(S^4)$ *as* $\mathrm{Sp}(1)$ *acts on* $\mathbb{H}$ *via right multiplication of the quaternionic conjugate*.

Proof Consider an orthonormal frame such that, at $p = (0, \cos\beta, \sin\beta)$, has the form:

$$e_0 = -\sin\beta\partial_3 + \cos\beta\partial_4; \qquad e_1 = \partial_0; \qquad e_2 = \partial_1; \qquad e_3 = \partial_2,$$

where ∂_i are the coordinate vectors of $\mathbb{R}^5 \cong P \oplus P^\perp$. Observe that the $\mathrm{SO}(3)$ action fixes e_0 and acts on e_1, e_2, e_3 via matrix multiplication. In particular, given $G \in \mathrm{SO}(3)$, the transformation matrix of the frame at p is:

$$\left[\begin{array}{c|c} 1 & \\ \hline & G \end{array}\right].$$

Moreover, for all $g \in \mathrm{Sp}(1) \cong \mathrm{Spin}(3)$ and $(g, g) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{Spin}(4)$, then

$$\tilde{\pi}_0^4((g, g)) = \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{\pi}_0^3(g) \end{array}\right],$$

where we recall that $\tilde{\pi}_0^3(l) \cdot x = lx\bar{l}$ for all $l \in \mathrm{Sp}(1)$ and $x \in \mathrm{Im}\mathbb{H} \cong \mathbb{R}^3$. Indeed, the left-hand side reads:

$$\tilde{\pi}_0^4((g, g)) \cdot a = ga\overline{g} = g\mathrm{Re}a\overline{g} + g\mathrm{Im}a\overline{g} = \mathrm{Re}a + g\mathrm{Im}a\overline{g},$$

while the right-hand side is:

$$\left[\begin{array}{c|c} 1 & \\ \hline & \tilde{\pi}_0^3(g) \end{array} \right] a = \left(\begin{array}{c} \mathrm{Re}a \\ g\mathrm{Im}a\overline{g} \end{array} \right).$$

We conclude the proof through the commutativity of (5.4). □

We put all these observations in a lemma.

Lemma 5.2 *The orbits of the* $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ *action on* $\mathcal{S}_-(S^4)$ *are given in Table* 1.

5.4 Spin(**3**) **adapted coordinates**

The description of the $\mathrm{SU}(2)$ action that we carried out in Sect. 5.3 suggests the following reparametrization of the linear coordinates (a_0, a_1, a_2, a_3) on the fibres of $\mathcal{S}_-(S^4)$:

$$a_0 = s\cos\left(\frac{\delta - \gamma}{2}\right); \quad a_1 = s\sin\left(\frac{\delta - \gamma}{2}\right); \quad a_2 = t\cos\left(\frac{\delta + \gamma}{2}\right); \quad a_3 = t\sin\left(\frac{\delta + \gamma}{2}\right),$$
(5.6)

where $s, t \in [0, \infty)$, $\gamma \in [0, 4\pi)$ and $\delta \in [0, 2\pi)$. This is a well-defined coordinate system when s and t are strictly positive; we will assume this from now on. Geometrically, γ represents the $\mathrm{SU}(2)$ action, while δ can be either seen as the phase in the orbit of the action when $(a_0, a_1) = (s, 0)$ or as twice the common angle in $[0, \pi)$ that the suitable point in the orbit makes with $(s, 0)$ and $(t, 0)$. These interpretations can be recovered by putting $\gamma = \delta$ and $\gamma = 0$, respectively.

Similarly to [15], we introduce the standard left-invariant coframe on $\mathrm{SU}(2)$ of coordinates γ, θ, ϕ defined on the same intervals as above:

$$\sigma_1 = d\gamma + \cos\theta d\phi; \quad \sigma_2 = \cos\gamma d\theta + \sin\gamma\sin\theta d\phi; \quad \sigma_3 = \sin\gamma d\theta - \cos\gamma\sin\theta d\phi.$$
(5.7)

Observe that:

$$\sigma_2 \wedge \sigma_3 = -\sin\theta d\theta \wedge d\phi.$$
(5.8)

Our choice of parametrization of $\mathcal{S}_-(S^4)$ implies that (5.7) is a coframe on the 3-dimensional orbits of the $\mathrm{SU}(2)$ action.

Table 1 Spin(3) Orbits

α	(s, t)	Orbit
$\neq \frac{\pi}{2}$	$= (0, 0)$	S^2
$\neq \frac{\pi}{2}$	$\neq (0, 0)$	S^3
$= \frac{\pi}{2}$	$= (0, 0)$	Point
$= \frac{\pi}{2}$	$\neq (0, 0)$	S^3

So far, we have constructed a coordinate system $\alpha, \beta, \theta, \phi, s, t, \delta, \gamma$ defining a chart $\mathcal{U}$ of $\mathcal{S}_-(S^4)$ and a coframe $\{\sigma_1, \sigma_2, \sigma_3, d\alpha, d\beta, ds, dt, d\delta\}$ on that chart. These coordinates and coframe are such that γ, θ, ϕ parametrize the orbits of the SU(2) action and $\{\sigma_1, \sigma_2, \sigma_3\}$ forms a coframe on these orbits. Let $\{\partial_1, \partial_2, \partial_3, \partial_\alpha, \partial_\beta, \partial_s, \partial_t, \partial_\delta\}$ be the relative dual frame.

5.5 The Spin(7) geometry in the adapted coordinates

In this subsection, we write the Cayley form Φ_c, as in (3.4), and the relative metric g_c, as in (3.5), with respect to the SU(2) adapted coordinates defined in Sect. 5.4.

Lemma 5.3 *The horizontal 2-forms* $\Omega_1, \Omega_2, \Omega_3$, *in the adapted frame defined in Sect*. 5.4, *satisfy*:

$$\Omega_1 = \sin \alpha \, d\alpha \wedge d\beta + \cos^2 \alpha \, \sigma_2 \wedge \sigma_3$$

and

$$\cos \gamma \Omega_2 + \sin \gamma \Omega_3 = \cos \alpha (d\alpha \wedge \sigma_2 - \sin \alpha d\beta \wedge \sigma_3),$$
$$- \sin \gamma \Omega_2 + \cos \gamma \Omega_3 = \cos \alpha (-d\alpha \wedge \sigma_3 - \sin \alpha d\beta \wedge \sigma_2).$$

Proof The equations follow from (5.2), (5.7) and (5.8). □

Lemma 5.4 *The vertical 2-forms* A_1, A_2, A_3, *in the adapted frame defined in Sect*. 5.4, *have the form*:

$$
\begin{aligned}
A_1 =& \frac{1}{2}(sds - tdt) \wedge d\delta + \frac{\cos \alpha}{2}(sds + tdt) \wedge d\beta - \frac{1}{2}(sds + tdt) \wedge \sigma_1 \\
& + \frac{\sin \alpha}{2}(tds - sdt) \wedge \sigma_3 + (s^2 + t^2)\frac{\sin^2 \alpha}{4}\sigma_2 \wedge \sigma_3 + \frac{st \sin \alpha}{2}\sigma_2 \wedge d\delta,
\end{aligned}
$$

$$
\begin{aligned}
A_2 =& \cos \gamma ds \wedge dt - \frac{t}{2}\sin \gamma ds \wedge (d\gamma + d\delta) - \frac{s}{2}\sin \gamma dt \wedge (d\delta - d\gamma) - \frac{st}{2}\cos \gamma d\gamma \wedge d\delta \\
& - (s^2 + t^2)\frac{\sin \alpha \cos \alpha}{4}\sin \theta d\beta \wedge d\phi + \sin \gamma (sdt - tds) \wedge \left(-\frac{\cos \alpha}{2}d\beta + \frac{\cos \theta}{2}d\phi\right) \\
& + st \cos \gamma d\delta \wedge \left(-\frac{\cos \alpha}{2}d\beta + \frac{\cos \theta}{2}d\phi\right) + \frac{\sin \alpha}{2}d\theta \wedge (tdt + sds) \\
& + \frac{t^2 \sin \alpha \sin \theta}{4}(d\gamma + d\delta) \wedge d\phi + \frac{s^2 \sin \alpha \sin \theta}{4}d\phi \wedge (d\delta - d\gamma); \\
A_3 =& \sin \gamma ds \wedge dt + \frac{t}{2}\cos \gamma ds \wedge (d\gamma + d\delta) + \frac{s}{2}\cos \gamma dt \wedge (d\delta - d\gamma) + \frac{st}{2}\sin \gamma d\delta \wedge d\gamma \\
& + (s^2 + t^2)\frac{\sin \alpha}{4}(\cos \alpha d\beta \wedge d\theta + \cos \theta d\theta \wedge d\phi) \\
& - \cos \gamma (sdt - tds) \wedge \left(-\frac{\cos \alpha}{2}d\beta + \frac{\cos \theta}{2}d\phi\right) \\
& + st \sin \gamma d\delta \wedge \left(-\frac{\cos \alpha}{2}d\beta + \frac{\cos \theta}{2}d\phi\right) + \frac{\sin \alpha \sin \theta}{2}d\phi \wedge (tdt + sds) \\
& + \frac{t^2 \sin \alpha}{4}d\theta \wedge (d\gamma + d\delta) + \frac{s^2 \sin \alpha}{4}(d\delta - d\gamma) \wedge d\theta.
\end{aligned}
$$

$$(5.9)$$

Proof Computing the exterior derivatives of the a_i's in the coordinates (5.6), we can reduce our statement to a long computation based on (5.3). ☐

Corollary 5.5 *The vertical 2-forms A_1, A_2, A_3, in the adapted frame defined in Sect.* 5.4, *satisfy*:

$$
\begin{aligned}
A_1 = {} & \left(ds + \frac{t \sin \alpha}{2} \sigma_2 \right) \wedge \left(\frac{s}{2} d\delta + \frac{s \cos \alpha}{2} d\beta - \frac{s}{2} \sigma_1 + \frac{t \sin \alpha}{2} \sigma_3 \right) \\
& - \left(dt - \frac{s \sin \alpha}{2} \sigma_2 \right) \wedge \left(\frac{t}{2} d\delta - \frac{t \cos \alpha}{2} d\beta + \frac{t}{2} \sigma_1 + \frac{s \sin \alpha}{2} \sigma_3 \right)
\end{aligned}
\tag{5.10}
$$

and

$$
\begin{aligned}
\cos \gamma A_2 + \sin \gamma A_3 = {} & \left(ds + \frac{t \sin \alpha}{2} \sigma_2 \right) \wedge \left(dt - \frac{s \sin \alpha}{2} \sigma_2 \right) \\
& + \left(\frac{s}{2} d\delta + \frac{s \cos \alpha}{2} d\beta - \frac{s}{2} \sigma_1 + \frac{t \sin \alpha}{2} \sigma_3 \right) \\
& \wedge \left(\frac{t}{2} d\delta - \frac{t \cos \alpha}{2} d\beta + \frac{t}{2} \sigma_1 + \frac{s \sin \alpha}{2} \sigma_3 \right)
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
\cos \gamma A_3 - \sin \gamma A_2 = {} & \left(ds + \frac{t \sin \alpha}{2} \sigma_2 \right) \wedge \left(\frac{t}{2} d\delta - \frac{t \cos \alpha}{2} d\beta + \frac{t}{2} \sigma_1 + \frac{s \sin \alpha}{2} \sigma_3 \right) \\
& + \left(dt - \frac{s \sin \alpha}{2} \sigma_2 \right) \wedge \left(\frac{s}{2} d\delta + \frac{s \cos \alpha}{2} d\beta - \frac{s}{2} \sigma_1 + \frac{t \sin \alpha}{2} \sigma_3 \right).
\end{aligned}
\tag{5.12}
$$

Proof The first equation in Lemma 5.4 is exactly the development of (5.10).

A straightforward computation involving (5.9) gives:

$$
\begin{aligned}
\cos \gamma A_2 + \sin \gamma A_3 = {} & ds \wedge dt + \frac{st}{2} d\delta \wedge \sigma_1 - \frac{st}{2} \cos \alpha d\delta \wedge d\beta + (s^2 + t^2) \frac{\sin \alpha \cos \alpha}{4} d\beta \wedge \sigma_3 \\
& + \frac{\sin \alpha}{2} \sigma_2 \wedge (t dt + s ds) + \frac{(t^2 - s^2) \sin \alpha}{4} \sigma_3 \wedge d\delta - \frac{(t^2 + s^2) \sin \alpha}{4} \sigma_1 \wedge \sigma_3;
\end{aligned}
$$

$$
\begin{aligned}
\cos \gamma A_3 - \sin \gamma A_2 = {} & \frac{1}{2} (t ds - s dt) \wedge \sigma_1 + \frac{1}{2} (t ds + s dt) \wedge d\delta + (s^2 + t^2) \frac{\sin \alpha \cos \alpha}{4} d\beta \wedge \sigma_2 \\
& + \frac{\cos \alpha}{2} (s dt - t ds) \wedge d\beta + \frac{\sin \alpha}{2} (t dt + s ds) \wedge \sigma_3 - \frac{(s^2 + t^2) \sin \alpha}{4} \sigma_1 \wedge \sigma_2 \\
& + \frac{(t^2 - s^2) \sin \alpha}{4} \sigma_2 \wedge d\delta;
\end{aligned}
$$

which coincide with the development of (5.11) and (5.12), respectively. ☐

Remark 5.6 Using the identities:

$$
\begin{aligned}
b_0 \wedge b_1 \wedge b_2 \wedge b_3 &= -\frac{1}{2} \Omega_1 \wedge \Omega_1, \\
\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 &= -\frac{1}{2} A_1 \wedge A_1
\end{aligned}
\tag{5.13}
$$

and

$$\sum_{i=1}^{3} A_i \wedge \Omega_i = A_1 \wedge \Omega_1 + (\cos \gamma \Omega_2 + \sin \gamma \Omega_3) \wedge (\cos \gamma A_2 + \sin \gamma A_3)$$
$$+ (-\sin \gamma \Omega_2 + \cos \gamma \Omega_3) \wedge (-\sin \gamma A_2 + \cos \gamma A_3), \tag{5.14}$$

one could easily find Φ_c in the adapted frame of Sect. 5.4. It is clear from Corollary 5.5 that it is not going to be in a nice form.

Lemma 5.7 *Given $c \geq 0$, the Riemannian metric g_c, in the adapted frame of Sect. 5.4, takes the form*:

$$g_c = 5(c + r^2)^{3/5} \left(d\alpha^2 + \sin^2 \alpha \, d\beta^2 + \cos^2 \alpha (\sigma_2^2 + \sigma_3^2) \right)$$
$$+ 4(c + r^2)^{-2/5} \left(ds^2 + dt^2 + \frac{r^2 \cos^2 \alpha}{4} d\beta^2 + \frac{r^2}{4} \sigma_1^2 - \frac{r^2 \cos \alpha}{2} d\beta\sigma_1 + \frac{r^2 \sin^2 \alpha}{4}(\sigma_2^2 + \sigma_3^2) \right.$$
$$+ \frac{(t^2 - s^2)}{2} d\delta\sigma_1 + (st \sin \alpha) d\delta\sigma_3 + \frac{r^2}{4} d\delta^2 + \sin \alpha (tds - sdt)\sigma_2 - \frac{(t^2 - s^2) \cos \alpha}{2} d\delta d\beta \right),$$

where $r^2 = s^2 + t^2$.

Proof Combining (3.5), (5.1), (5.3) and (5.7), it is easy to obtain the Riemannian metric in the claimed form. □

5.6 The diagonalizing coframe and frame

In this subsection we define the last coframe on $\cancel{g}_-(S^4)$ that we will use. The motivation comes from the form of A_1, $\cos \gamma A_2 + \sin \gamma A_3$ and $\cos \gamma A_3 - \sin \gamma A_2$ that we obtained in (5.10), (5.11) and (5.12), respectively. We let:

$$\tilde{ds} = ds + \frac{t \sin \alpha}{2} \sigma_2; \qquad\qquad \tilde{dt} = dt - \frac{s \sin \alpha}{2} \sigma_2;$$
$$\omega_1 = sd\delta + s \cos \alpha d\beta - s\sigma_1 + t \sin \alpha \sigma_3; \qquad \omega_2 = td\delta - t \cos \alpha d\beta + t\sigma_1 + s \sin \alpha \sigma_3. \tag{5.15}$$

Since $t\omega_1 + s\omega_2 = 2tsd\delta + (t^2 + s^2) \sin \alpha \sigma_3$ and $s\omega_2 - t\omega_1 = 2st\sigma_1 - 2st \cos \alpha d\beta + (s^2 - t^2) \sin \alpha \sigma_3$, it is clear that $\{\sigma_2, \sigma_3, d\alpha, d\beta, \omega_1, \omega_2, \tilde{ds}, \tilde{dt}\}$ is a coframe on $\mathcal{U}$. Let $\{e_2, e_3, e_\alpha, e_\beta, e_{\omega_1}, e_{\omega_2}, e_s, e_t\}$ denote the relative dual frame.

Corollary 5.8 *The vertical 2-forms A_1, A_2, A_3, in the coframe defined in this subsection, satisfy*:

$$A_1 = \frac{1}{2} \left(\tilde{ds} \wedge \omega_1 - \tilde{dt} \wedge \omega_2 \right) \tag{5.16}$$

and

$$\cos \gamma A_2 + \sin \gamma A_3 = \tilde{ds} \wedge \tilde{dt} + \frac{1}{4} \omega_1 \wedge \omega_2; \tag{5.17}$$

$$\cos \gamma A_3 - \sin \gamma A_2 = \frac{1}{2} \left(\tilde{d}s \wedge \omega_2 + \tilde{d}t \wedge \omega_1 \right).$$ (5.18)

Proof It follows immediately from Corollary 5.5 and (5.15). □

Proposition 5.9 *Given* $c \geq 0$, *the Cayley form* Φ_c, *in the coframe defined in this subsection, satisfies*:

$$\Phi_c = 4(c + r^2)^{-4/5} \tilde{d}s \wedge \tilde{d}t \wedge \omega_2 \wedge \omega_1 + 25(c + r^2)^{6/5} \sin \alpha \cos^2 \alpha d\alpha \wedge d\beta \wedge \sigma_3 \wedge \sigma_2$$

$$10(c + r^2)^{1/5} \bigg(\left(\tilde{d}s \wedge \omega_1 - \tilde{d}t \wedge \omega_2 \right) \wedge \left(\sin \alpha d\alpha \wedge d\beta + \cos^2 \alpha \sigma_2 \wedge \sigma_3 \right)$$

$$+ \frac{1}{2} \left(4 \tilde{d}s \wedge \tilde{d}t + \omega_1 \wedge \omega_2 \right) \wedge \left(\cos \alpha (d\alpha \wedge \sigma_2 - \sin \alpha d\beta \wedge \sigma_3) \right)$$

$$+ \left(\tilde{d}s \wedge \omega_2 + \tilde{d}t \wedge \omega_1 \right) \wedge \cos \alpha \left(-d\alpha \wedge \sigma_3 - \sin \alpha d\beta \wedge \sigma_2 \right) \bigg),$$

(5.19)

where $r^2 = s^2 + t^2$.

Proof This is a straightforward consequence of Lemma 5.3, (5.13), (5.14) and Corollary 5.8. □

Proposition 5.10 *Given* $c \geq 0$, *the Riemannian metric* g_c, *in the coframe defined in this subsection, satisfies*:

$$g_c = 5(c + r^2)^{3/5} \left(d\alpha^2 + \sin^2 \alpha d\beta^2 + \cos^2 \alpha \left(\sigma_2^2 + \sigma_3^2 \right) \right)$$

$$+ 4(c + r^2)^{-2/5} \left(\tilde{d}s^2 + \tilde{d}t^2 + \frac{(\omega_1^2 + \omega_2^2)}{4} \right),$$

(5.20)

where $r^2 = s^2 + t^2$.

Proof The first addendum remains invariant from Lemma 5.7, while (5.15) implies that the remaining part is equal to the second addendum in Lemma 5.7. □

In particular, using this coframe, we sacrifice compatibility with the group action to obtain a simpler form for Φ_c and a diagonal metric.

We conclude this subsection by computing the dual frame with respect to the SU(2) adapted frame $\{ \partial_1, \partial_2, \partial_3, \partial_\alpha, \partial_\beta, \partial_s, \partial_t, \partial_\delta \}$.

Lemma 5.11 *The dual frame* $\{ e_2, e_3, e_\alpha, e_\beta, e_{\omega_1}, e_{\omega_2}, e_s, e_t \}$ *satisfies*:

$$e_\alpha = \partial_\alpha; \qquad\qquad e_\beta = \partial_\beta + \cos \alpha \partial_1;$$

$$e_2 = \partial_2 - \frac{t \sin \alpha}{2} \partial_s + \frac{s \sin \alpha}{2} \partial_t; \qquad e_3 = \partial_3 - \frac{(s^2 + t^2) \sin \alpha}{2st} \partial_\delta + \frac{(t^2 - s^2) \sin \alpha}{2st} \partial_1;$$

$$e_s = \partial_s; \qquad\qquad e_t = \partial_t;$$

$$e_{\omega_1} = \frac{1}{2s} \partial_\delta - \frac{1}{2s} \partial_1; \qquad\qquad e_{\omega_2} = \frac{1}{2t} \partial_\delta + \frac{1}{2t} \partial_1;$$

(5.21)

where $\{\partial_1, \partial_2, \partial_3, \partial_\alpha, \partial_\beta, \partial_s, \partial_t, \partial_\delta\}$ *is the dual frame with respect to the* SU(2) *adapted coordinates of Sect.* 5.4.

Proof It is straightforward to verify these identities from (5.15) and the definition of dual frame. □

5.7 The Cayley condition

As the generic orbit of the SU(2) action we are considering is 3-dimensional (see Lemma 5.2), it is sensible to look for SU(2)-invariant Cayley submanifolds. Indeed, Harvey and Lawson theorem [10, Theorem IV .4.3] guarantees the local existence and uniqueness of a Cayley passing through any given generic orbit. To construct such a submanifold N, we consider a 1-parameter family of 3-dimensional SU(2)-orbits in M. Hence, the coordinates that do not describe the orbits, i.e. α, β, s, t and δ, need to be functions of a parameter τ. Explicitly, we have:

$$N = \left\{ \left((\cos\alpha(\tau)\mathbf{u}, \sin\alpha(\tau)\mathbf{v}), \left(s(\tau)\cos\left(\frac{\delta(\tau)-\gamma}{2}\right), s(\tau)\sin\left(\frac{\delta(\tau)-\gamma}{2}\right), \right. \right. \right.$$
$$\left. \left. \left. t(\tau)\cos\left(\frac{\delta(\tau)+\gamma}{2}\right), t(\tau)\sin\left(\frac{\delta(\tau)+\gamma}{2}\right) \right) \right) : |\mathbf{u}| = 1, |\mathbf{v}| = 1, \gamma \in [0, 4\pi), \tau \in (-\epsilon, \epsilon) \right\},$$
$$(5.22)$$

and its tangent space is spanned by: $\{\partial_1, \partial_2, \partial_3, \dot{s}\partial_s + \dot{t}\partial_t + \dot{\alpha}\partial_\alpha + \dot{\beta}\partial_\beta + \dot{\delta}\partial_\delta\}$, where the dots denotes the derivative with respect to τ. The Cayley condition imposed on this tangent space (see Proposition 2.6) generates a system of ODEs on $\alpha, \beta, s, t, \delta$.

Theorem 5.12 *Let N be an* SU(2)-*invariant submanifold as described at the beginning of this subsection. Then, N is Cayley in the chart $\mathcal{U}$ if and only if the following system of ODEs is satisfied*:

$$\begin{cases} (s^2 + t^2)\sin^2\alpha\cos\alpha\dot{\beta} = 0 \\ \cos^2\alpha(t\dot{s} - s\dot{t}) = 0 \\ \cos^2\alpha st\dot{\delta} = 0 \\ -5(c+r^2)\cos^2\alpha s\dot{\alpha} + r^2\sin^2\alpha\dot{\alpha}s - 2\sin\alpha\cos\alpha t^2\dot{s} - 4\cos\alpha\sin\alpha s^2\dot{s} - 2\sin\alpha\cos\alpha st\dot{t} = 0, \\ 5(c+r^2)\cos^2\alpha t\dot{\alpha} - r^2\sin^2\alpha\dot{\alpha}t + 2\sin\alpha\cos\alpha s^2\dot{t} + 4\cos\alpha\sin\alpha t^2\dot{t} + 2\sin\alpha\cos\alpha st\dot{s} = 0 \\ 5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}s - 2\sin\alpha\cos\alpha t^2 s\dot{\delta} - r^2\sin^3\alpha\dot{\beta}s = 0 \\ -5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}t - 2\sin\alpha\cos\alpha ts^2\dot{\delta} + r^2\sin^3\alpha\dot{\beta}t = 0 \end{cases}$$
$$(5.23)$$

where $r^2 = s^2 + t^2$ as usual.

As it mainly consists of computations, we leave the proof of Theorem 5.12 to Appendix A .

Corollary 5.13 *Let N be an* SU(2)-*invariant submanifold as described at the beginning of this subsection. Then, N is Cayley in the chart $\mathcal{U}$ if and only if the following system of ODEs is satisfied*:

$$\begin{cases} \dot{\beta} = 0 \\ (t\dot{s} - s\dot{t}) = 0 \\ \dot{\delta} = 0 \\ 5(c + r^2)\cos^2\alpha st\dot{\alpha} - (s^2 + t^2)st\sin^2\alpha\dot{\alpha} + 2\sin\alpha\cos\alpha(s^2 + t^2)(\dot{s}t + t\dot{s}) = 0 \end{cases},$$

where $r^2 = s^2 + t^2$ as usual.

Proof As $\alpha \in (0, \pi/2)$ and $s, t > 0$, we get immediately the first three equations from the first three equations of (5.23). The last two equations of (5.23) are superfluous as $\dot{\beta} = 0$ and $\dot{\delta} = 0$. The same holds for t times the fourth equation plus s times the fifth equation of (5.23), where we use $t\dot{s} - s\dot{t} = 0$ this time. We conclude by considering s times the fifth equation minus t times the fourth equation of (5.23). □

5.8 The Cayley fibration

In the previous section we found the condition that makes N, SU (2)-invariant submanifold, a Cayley submanifold. Explicitly, it consists of a system of ODEs that is completely integrable; these solutions will give us the desired fibration.

Theorem 5.14 *Let N be an* SU (2)-*invariant submanifold as described at the beginning of Sect.* 5.7. *Then, N is Cayley in $\mathcal{U}$ if and only if the following quantities are constant:*

$$\beta, \qquad \delta, \qquad \frac{s}{t}, \qquad F := 2\sin^{5/2}\alpha\cos^{1/2}\alpha st + 5c\frac{st}{(s^2 + t^2)}H(\alpha),$$

where $H(\alpha)$ is the primitive function of $h(\alpha) := (\cos\alpha\sin\alpha)^{3/2}$.

Proof The condition on β and δ follows immediately from Corollary 5.13. Taking the derivative in τ of s/t, we see that

$$0 = \frac{d}{d\tau}\left(\frac{s}{t}\right) = \frac{\dot{s}t - \dot{t}s}{t^2},$$

which is equivalent to the second equation in Corollary 5.13, as $t > 0$. Analogously, one can see that the derivative with respect to τ of F is equivalent to the last equation of Corollary 5.13 if we assume that s/t is constant. □

Setting

$$v := \frac{s}{t}, \qquad u := st,$$

the preserved quantities transform to:

$$\beta, \qquad \delta, \qquad v, \qquad F := 2\sin^{5/2}\alpha\cos^{1/2}\alpha(v^2 + 1)u + 5cvH(\alpha),$$

where we multiplied F by the constant $(v^2 + 1)$. Observe that this is an admissible transformation from $s, t \in (0, \infty)$ to $u, v \in (0, \infty)$. Moreover, fixed β, δ, v, we can represent the SU (2)-invariant Cayley submanifolds as the level sets of F reckoned as a $\mathbb{R}$-valued function

of α and u. An easy analysis of F shows that these level sets can be represented as in Fig. 1.

The dashed lines in the two graphs correspond to the curves formed by the u-minimums of each level set and to the two vertical lines: $\alpha = \arccos(1/\sqrt{6})$. For $c = 0$, these coincide, while in the generic case the locus of the u-minimum is:

$$\alpha = \arccos\left(\sqrt{\frac{u(v^2+1)}{6u(v^2+1)+5cv}}\right),$$

which is only asymptotic to $\alpha = \arccos(1/\sqrt{6})$ for $u \to \infty$.

5.9 The conical version

We first consider the easier case, i.e. when $c = 0$. It is clear from the graph that the SU(2)-invariant Cayleys passing through $\mathcal{U}$ are contained in $\mathcal{U}$, have topology $S^3 \times \mathbb{R}$, and are smooth. Moreover, we can construct a Cayley fibration on the chart $\mathcal{U}$ with base an open subset of $\mathbb{R}^4$. To do so, we associate with each point of $\mathcal{U}$ the value of β, δ, s/t and F of the Cayley passing through that point. This SU(2)-invariant fibration naturally extends to the whole M_0 via continuity. Using Table 1 and Harvey and Lawson uniqueness theorem [10, Theorem IV.4.3], we can describe the extension precisely. Indeed, when $\alpha = \pi/2$, the fibres of π_{S^4} are SU(2)-invariant Cayley submanifolds; when $\alpha \neq \pi/2$ and $s = 0$ or $t = 0$, the suitable Cayley submanifolds constructed by Karigiannis and Min-Oo [12] are SU(2)-invariant; finally, when $\alpha = 0$ and $(s, t) \neq 0$, the fibres are given by an extension of [14]. The topology of these Cayley submanifolds that are not contained in $\mathcal{U}$ is $\mathbb{R}^4 \setminus \{0\}$ in the first case and $\mathbb{R} \times S^3$ in the remaining ones. Observe that this fibration does not admit singular or intersecting fibres.
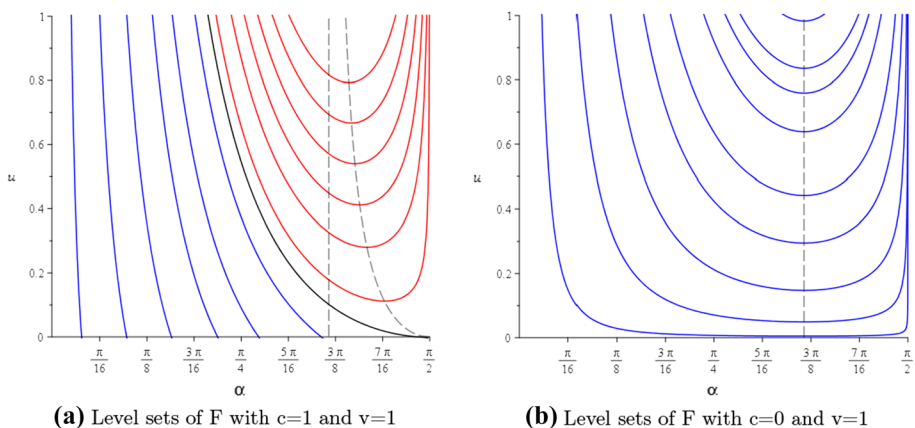

(a) Level sets of F with c=1 and v=1   (b) Level sets of F with c=0 and v=1

Fig. 1 Level sets of F in the generic and in the conical case

5.10 The smooth version

Now, we consider the generic case, i.e. when $c > 0$. Differently from the cone, the graph of the level sets of F shows that the $SU(2)$-invariant Cayley submanifolds passing through $\mathcal{U}$ do not remain contained in it, and they admit three different topologies in the extension. The red, black, and blue lines correspond to submanifolds with topology $\mathbb{R} \times S^3$, $\mathbb{R}^4$ and $\mathcal{O}_{\mathbb{CP}^1}(-1)$, respectively. Indeed, the first two cases are obvious, while if we assume smoothness, we can deduce the third one through the argument of Sect. 2.2. We define an $SU(2)$-invariant Cayley fibration on $\mathcal{U}$ that extends to the whole M exactly as above. If we fix a value of F corresponding to a Cayley of topology $\mathcal{O}_{\mathbb{CP}^1}(-1)$, then, for every δ, v, all the different Cayleys will intersect in a $\mathbb{CP}^1 \subset S^4$, where S^4 is the zero section of $\not{S}_-(S^4)$. In particular, the M' of Definition 2.8 is equal to M_0 in this context, i.e. we can assume $u > 0$.

5.11 The parametrizing space

Using Fig. 1, we can study the parametrizing space $\mathcal{B}$ of the Cayley fibrations we have just described. We will only deal with the smooth version, as the conical case is going to be completely analogous.

Ignoring β for a moment, it is immediate to see that if we restrict our attention to the fibres that are topologically $\mathcal{O}_{\mathbb{CP}^1}(-1)$ and the ones corresponding to the black line, the parametrizing space is homeomorphic to $S^2 \times [0, 1]$. The remaining fibres are parametrized by $B^3(1)$, open unit ball of $\mathbb{R}^3$. As we removed the zero section of $\not{S}_-(S^4)$, it is clear that we can glue these partial parametrizations together to obtain $\overline{B^3(2)}$. Now, β gives a circle action on $\overline{B^3(2)}$ that vanishes on its boundary. We conclude that the parametrizing space $\mathcal{B}$ of the smooth Cayley fibration is S^4. Indeed, this is essentially the same way to describe S^4 as we did in Sect. 5.1.

5.12 The smoothness of the fibres (the asymptotic analysis)

In this subsection, we study the smoothness of the fibres. Observe that this property is obviously satisfied as long as they are contained in the chart $\mathcal{U}$. Hence, the Cayleys of topology $S^3 \times \mathbb{R}$ are smooth, and we only need to check the remaining ones in the points where they meet the zero section, i.e. when the $SU(2)$ group action degenerates. To this purpose, we carry out an asymptotic analysis (Fig. 2).

Let β_0, v_0, δ_0 and F_0 be the constants determining a Cayley fibre N. By the explicit formula for F, we see that N is given by:

Fig. 2 Approximation of a Cayley at $u = 0$ when $\alpha_0 \in (0, \pi/2)$
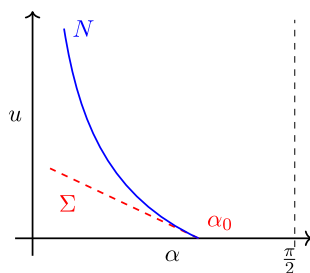
$$u = \frac{F_0 - 5cv_0 H(\alpha)}{2\sin^{5/2}\alpha\cos^{1/2}\alpha(v_0^2 + 1)}$$

We first check the smoothness of the fibres that meet the zero section ($u = 0$) at some $\alpha_0 \in (0, \pi/2)$, i.e. the ones of topology $\mathcal{O}_{\mathbb{CP}^1}(-1)$. For this purpose, if we expand near α_0^- and we obtain the linear approximation of N at that point. Explicitly, this is the SU(2)-invariant 4-dimensional submanifold Σ characterized by the equation

$$u = -\frac{5cv_0}{2\tan\alpha_0(v_0^2 + 1)}(\alpha - \alpha_0),$$

and where v, δ, β are constantly equal to v_0, δ_0, β_0.

Now, we want to study the asymptotic behaviour of the metric g_c when restricted to Σ, and then, we let α tends to α_0 from the left. To do so, it is convenient to compute the following identities using the definition of $u := st$ and $v := s/t$:

$$
\begin{aligned}
dt &= \frac{1}{2\sqrt{uv}}du - \frac{1}{2v}\sqrt{\frac{u}{v}}dv, \\
ds &= \frac{\sqrt{v}}{2\sqrt{u}}du + \frac{\sqrt{u}}{2\sqrt{v}}dv, \\
ds^2 &= \frac{v}{4u}du^2 + \frac{u}{4v}dv^2 + \frac{1}{2}dudv, \\
dt^2 &= \frac{1}{4uv}du^2 + \frac{u}{4v^3}dv^2 - \frac{1}{2v^2}dudv.
\end{aligned}
\tag{5.24}
$$

The metric g_c, in the coframe $\{\sigma_1, \sigma_2, \sigma_3, d\alpha, d\beta, du, dv, d\delta\}$, then can be rewritten as:

$$
\begin{aligned}
g_c = {}&5\left(c + \frac{u}{v}(1 + v^2)\right)^{3/5}\left(d\alpha^2 + \sin^2\alpha\, d\beta^2 + \cos^2\alpha(\sigma_2^2 + \sigma_3^2)\right) \\
&+ 4\left(c + \frac{u}{v}(1 + v^2)\right)^{-2/5}\left(\frac{1}{4uv}(1 + v^2)du^2 + \frac{u}{4v^3}(1 + v^2)dv^2 + \frac{1}{2v^2}(v^2 - 1)dudv\right. \\
&+ \frac{u}{v}(1 + v^2)\frac{\cos^2\alpha}{4}d\beta^2 + \frac{u}{4v}(1 + v^2)\sigma_1^2 - \frac{\cos\alpha}{2}\frac{u}{v}(1 + v^2)d\beta\sigma_1 \\
&+ \frac{u}{v}(1 + v^2)\frac{\sin^2\alpha}{4}(\sigma_2^2 + \sigma_3^2) \\
&+ \frac{u(1 - v^2)}{2v}d\delta\sigma_1 + u\sin\alpha\, d\delta\sigma_3 + \frac{u}{4v}(1 + v^2)d\delta^2 + \sin\alpha\frac{u}{v}dv\sigma_2 - \frac{u(1 - v^2)\cos\alpha}{2v}\left.d\delta d\beta\right),
\end{aligned}
\tag{5.25}
$$

where we used (5.24) and Lemma 5.7. Now, if we restrict (5.25) to Σ, and we let α tend to α_0 from the left, we get:

$$
\begin{aligned}
g_c|_N &\sim \frac{c^{-2/5}}{v_0}(1 + v_0^2)\left(\frac{du^2}{u} + u\sigma_1^2\right) + 5c^{3/5}\cos^2\alpha_0(\sigma_2^2 + \sigma_3^2) \\
&\sim dr^2 + r^2\frac{\sigma_1^2}{4} + 5c^{3/5}\cos^2\alpha_0(\sigma_2^2 + \sigma_3^2),
\end{aligned}
$$

where

$$r = \sqrt{\frac{1 + v_0^2}{v_0 c^{2/5}}} 2\sqrt{u}.$$

As the length of σ_1 is 4π, we deduce that the metric g_c extends smoothly to the $\mathbb{CP}^1 \cong S^2$ contained in the zero section (Fig. 3). This two-dimensional sphere corresponds to the base of the bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)$.

Finally, we check the smoothness of the fibres meeting the zero section at $\alpha_0 = \pi/2$, i.e. the ones with topology $\mathbb{R}^4$. Expanding for $\alpha \to \pi/2^-$, we immediately see that the first order is not enough and we need to pass to second order. Explicitly, this is the $SU(2)$-invariant 4-dimensional submanifold Σ of equation:

$$u = A(\alpha - \pi/2)^2,$$

where $A := cv(1 + v^2)^{-1}$ is the constant depending on c, v determined by the expansion. As above, the remaining parameters v, δ, β are constantly equal to v_0, δ_0, β_0.

If we restrict g_c as defined in (5.25) to Σ, and we let α tend to $\pi/2$, then we obtain:

$$\begin{aligned}
g_c|_N &\sim 5c^{3/5}(\alpha - \pi/2)^2(\sigma_2^2 + \sigma_3^2) + Ac^{-2/5}\left(\frac{1 + v^2}{v}\right)(\alpha - \pi/2)^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\
&\quad + \left(5c^{3/5} + 4Ac^{-2/5}\frac{1 + v^2}{v}\right)d\alpha^2 \\
&\sim c^{3/5}(\alpha - \alpha_0)^2\left(\sigma_1^2 + 6(\sigma_2^2 + \sigma_3^2)\right) + 9c^{3/5}d\alpha^2,
\end{aligned}$$

where we also used the expansion of $\cos\alpha$ around $\pi/2$ and the explicit value of A. We conclude that N is not smooth when it meets the zero section, and it develops an asymptotically conical singularity at that point.

Remark 5.15 The singularity is asymptotic to the Lawson–Osserman cone [18].

5.13 The main theorems

We collect all these results in the following theorems. Observe that we are using the notion of Cayley fibration given in Definition 2.8.

Theorem 5.16 (Generic case) *Let (M, Φ_c) be the Bryant–Salamon manifold constructed over the round sphere S^4 for some $c > 0$, and let $SU(2)$ act on M as in Sect. 5.3. Then, M admits an $SU(2)$-invariant Cayley fibration parametrized by $\mathcal{B} \cong S^4$. The fibres are topologically $\mathcal{O}_{\mathbb{CP}^1}(-1)$, $S^3 \times \mathbb{R}$ and $\mathbb{R}^4$. Apart from the non-vertical fibres of topology $\mathbb{R}^4$, all*

Fig. 3 Approximation of a Cayley at $u = 0$ when $\alpha_0 = \pi/2$
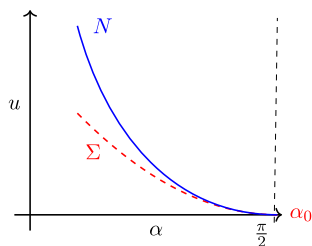
the others are smooth. The singular fibres of the Cayley fibration have a conically singular point and are parametrized by $(\mathcal{B}^\circ)^c \cong S^2 \times S^1$ (β, δ, v in our description). Moreover, at each point of the zero section $S^4 \subset \mathcal{J}_-(S^4)$, infinitely many Cayley fibres intersect.

Theorem 5.17 (Conical case) *Let (M_0, Φ_0) be the conical Bryant–Salamon manifold constructed over the round sphere S^4, and let $\mathrm{SU}(2)$ act on M_0 as in Sect. 5.3. Then, M_0 admits an $\mathrm{SU}(2)$-invariant Cayley fibration parametrized by $\mathcal{B} \cong S^4$. The fibres are topologically $S^3 \times \mathbb{R}$ and are all smooth. Moreover, as these do not intersect, the $\mathrm{SU}(2)$-invariant Cayley fibration is a fibration in the usual differential geometric sense with fibres Cayley submanifolds.*

Remark 5.18 It is interesting to observe that, in the generic case, the family of singular $\mathbb{R}^4$s separates the fibres of topology $S^3 \times \mathbb{R}$ from the ones of topology $\mathcal{O}_{\mathbb{CP}^1}(-1)$.

Remark 5.19 Similarly to [15, Sect. 5.11.1], one can blow-up at the north pole and argue that in the limit the Cayley fibration splits into the product of a line $\mathbb{R}$ and of an $\mathrm{SU}(2)$-invariant coassociative fibration on $\mathbb{R}^7$. By the uniqueness of the $\mathrm{SU}(2)$-invariant coassociative fibrations of $\mathbb{R}^7$, we deduce that the latter is the Harvey and Lawson coassociative fibration [10, Section IV.3] up to a reparametrization.

Remark 5.20 From the computations that we have carried out, it is easy to give an explicit formula for the multi-moment map v_c associated with this action. Indeed, this is:

$$v_c = 5(c + s^2 + t^2)^{1/5}\left((s^2 + t^2)\cos^2\alpha - \frac{1}{6}(s^2 + t^2 - 5c)\right) - \frac{25}{6}c^{6/5} \qquad c \geq 0.$$

Obviously, the range of v_c is the whole $\mathbb{R}$. Under the usual transformation $u = st$ and $v = s/t$, the multi-moment map becomes:

$$v_c = \frac{5}{6}\left(c + \frac{u(1+v^2)}{v}\right)^{1/5}\left(6\frac{u(1+v^2)}{v}\cos^2\alpha - \frac{u(1+v^2)}{v} + 5c\right) - \frac{25}{6}c^{6/5}.$$

We draw the level sets of v_c in Fig. 4.

The black lines correspond to the level set relative to zero, the red lines correspond to negative values, while the blue lines correspond to the positive ones.

Differently from the conical case, the 0-level set of v_c for $c > 0$ does not coincide with the locus of u-minimum of each level set of F. Moreover, for every $c \geq 0$, it does not even coincide with the set of $\mathrm{SU}(2)$-orbits of minimum volume in each fibre.

5.14 Asymptotic geometry

Inspecting the geometry of the Cayley fibration (see Fig. 1), we deduce that there are two asymptotic behaviours for the fibres: one for $\alpha \sim 0$ and one for $\alpha \sim \pi/2$. In both cases, as $u \to \infty$, the tangent space of the Cayley fibre N tends to be spanned by $\partial_u, \partial_1, \partial_2, \partial_3$. We can use the formula for the metric (5.25) to obtain, for $\alpha \sim 0$:
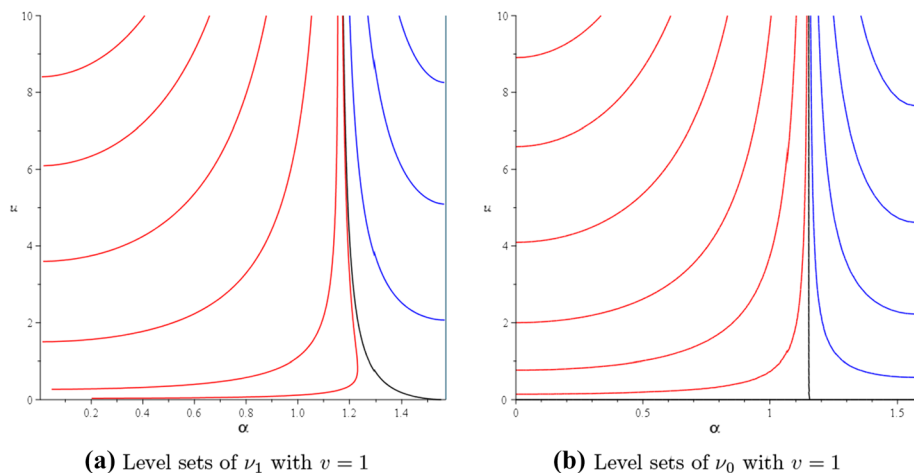
(a) Level sets of ν_1 with $v = 1$ **(b)** Level sets of ν_0 with $v = 1$

Fig. 4 Level sets of the multi-moment map in the generic and conical case

$$g_c|_N \sim 5\left(\frac{1+v^2}{v}\right)^{3/5} u^{3/5}(\sigma_2^2 + \sigma_3^2) + u^{-2/5}\left(\frac{1+v^2}{v}\right)^{-2/5}\left(\frac{1+v^2}{v}\right)\left(\frac{du^2}{u} + u\sigma_1^2\right)$$

$$= \left(\frac{1+v^2}{v}\right)^{3/5}\left(u^{3/5}(5(\sigma_2^2 + \sigma_3^2) + \sigma_1^2) + u^{-7/5}du^2\right)$$

$$= dr^2 + \frac{9}{25}r^2\frac{(\sigma_1^2 + 5(\sigma_2^2 + \sigma_3^2))}{4},$$

and, for $\alpha \sim \pi/2$:

$$g_c|_N \sim \left(\frac{1+v^2}{v}\right)^{3/5}\left(u^{3/5}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + u^{-7/5}du^2\right)$$

$$= dr^2 + \frac{9}{25}r^2\frac{(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)}{4},$$

where, in both cases,

$$r := \frac{10}{3}\left(\frac{1+v^2}{v}\right)^{3/10}u^{3/10}.$$

When $\alpha \sim \pi/2$, the link S^3 is endowed with the round metric, while when $\alpha \sim 0$, the round sphere is squashed by a factor $1/5$.

Remark 5.21 Observe that $1/5$ is also the squashing factor on the round metric of S^7 that makes the space homogeneous, non-round, and Einstein. It is well known that there are no other metrics satisfying these properties [27].

6 The Cayley fibration invariant under the lift of the $\mathrm{Sp}\,(1) \times \mathrm{Id}_1$ action on S^4

Let $M := \mathcal{S}_-(S^4)$ and $M_0 := \mathbb{R}^+ \times S^7$ be endowed with the torsion-free Spin (7)-structures Φ_c constructed by Bryant and Salamon that we described in Sect. 3. On each Spin (7) manifold, we construct the Cayley Fibration which is invariant under the lift to M (or M_0) of the standard (left multiplication) $\mathrm{Sp}\,(1) \times \mathrm{Id}_1$ action on $S^4 \subset \mathbb{H} \oplus \mathbb{R}$.

Remark 6.1 The exact same computations will work for the $\mathrm{Sp}\,(1) \times \mathrm{Id}_1$ action given by right multiplication of the quaternionic conjugate. In this case, the role of the north and of the south pole will be interchanged.

6.1 The choice of coframe on S^4

As in Sect. 5, we choose an adapted orthonormal coframe on S^4 which is compatible with the symmetries we will impose.

Consider $\mathbb{R}^5$ as the sum of a 4-dimensional space $P \cong \mathbb{H}$ and its orthogonal complement $P^\perp \cong \mathbb{R}$. With respect to this splitting, we can write the 4-dimensional unit sphere in the following fashion:

$$S^4 = \left\{ (\mathbf{x}, y) \in P \oplus P^\perp \ : \ |\mathbf{x}|^2 + |y|^2 = 1 \right\}.$$

Now, for all $(\mathbf{x}, y) \in S^4$ there exists a unique $\alpha \in [-\pi/2, \pi/2]$ such that

$$\mathbf{x} = \cos \alpha \mathbf{u}, \qquad y = \sin \alpha,$$

for some $\mathbf{u} \in S^3$. Note that $\mathbf{u}$ is uniquely determined when $\alpha \neq \pm \pi/2$. Essentially, we are writing S^4 as a 1-parameter family of S^3s that are collapsing to a point on each end of the parametrization.

Let $\{\partial_1, \partial_2, \partial_3\}$ be the standard left-invariant orthonormal frame on $S^3 \cong \mathrm{Sp}\,(1)$. Considering this frame in the description of S^4 above, we deduce that

$$f_0 := \partial_\alpha, \qquad f_1 := \frac{\partial_1}{\cos \alpha}, \qquad f_2 := \frac{\partial_2}{\cos \alpha}, \qquad f_3 := \frac{\partial_3}{\cos \alpha},$$

is an oriented orthonormal frame of $S^4 \setminus \{\alpha = \pm \pi/2\}$. The dual coframe is:

$$b_0 := d\alpha; \qquad b_1 := \cos \alpha \sigma_1; \qquad b_2 := \cos \alpha \sigma_2; \qquad b_3 := \cos \alpha \sigma_3, \qquad (6.1)$$

where $\{\sigma_i\}_{i=1}^3$ is the dual coframe of $\{\partial_i\}_{i=1}^3$ in S^3, which is well known to satisfy:

$$d \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = 2 \begin{pmatrix} \sigma_2 \wedge \sigma_3 \\ \sigma_3 \wedge \sigma_1 \\ \sigma_1 \wedge \sigma_2 \end{pmatrix}. \qquad (6.2)$$

We deduce that the round metric on the unit sphere S^4 can be written as:

$$g_{S^4} = d\alpha^2 + \cos^2 \alpha g_{S^3},$$

and the volume form is:

$$\text{vol}_{S^4} = \cos^3 \alpha d\alpha \wedge \text{vol}_{S^3},$$

where $g_{S^3} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$ and $\text{vol}_{S^3} = \sigma_1 \wedge \sigma_2 \wedge \sigma_3$.

6.2 The horizontal and the vertical space

Exactly as in Sect. 5.2, we can compute the connection 1-forms ρ_i for $i = 1, 2, 3$ with respect to the coframe we have constructed. Indeed, a straightforward computation involving (3.1), (6.1) and (6.2) implies that $\rho_i = l\sigma_i$ for all $i = 1, 2, 3$, where

$$l := \frac{\sin \alpha - 1}{2}.$$

Hence, we can deduce from (3.2) that the vertical 1-forms in these coordinates are:

$$
\begin{aligned}
\xi_0 &= da_0 + l(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3), & \xi_1 &= da_1 + l(-a_0\sigma_1 - a_2\sigma_3 + a_3\sigma_2), \\
\xi_2 &= da_2 + l(-a_0\sigma_2 + a_1\sigma_3 - a_3\sigma_1), & \xi_3 &= da_3 + l(-a_0\sigma_3 - a_1\sigma_2 + a_2\sigma_1).
\end{aligned}
$$

(6.3)

6.3 The SU (2) action

Given the splitting of $\mathbb{R}^5$ into $P \cong \mathbb{H}$ and its orthogonal complement $P^\perp$, we can consider $SU(2) \cong Sp(1)$ acting via left multiplication on P and trivially on $P^\perp$. Equivalently, we are considering $Sp(1) \cong Sp(P) \times \text{Id}_{P^\perp} \subset SO(5)$. Being a subgroup of $SO(5)$, the action descends to the unit sphere S^4.

We first consider $\alpha \neq -\pi/2$, where we trivialize $S^4 \setminus \{southpole\}$ using homogeneous quaternionic coordinates on $\mathbb{HP}^1 \cong S^4$. In this chart, diffeomorphic to $\mathbb{H}$, the action is given by standard left multiplication.

We extend the action on S^4 to the tangent bundle of S^4 via the differential. In this trivialization, $\mathbb{H} \times \mathbb{H}$, the action is given by left multiplication on both factors. Hence, if we pick the trivialization of $P_{SO(4)}$ induced by $\{1, i, j, k\}$, the action of $p \in Sp(1)$ maps the element $(x, \text{Id}_{SO(4)}) \in \mathbb{H} \times SO(4)$ to $(p \cdot x, \tilde{p})$, where

$$
\tilde{p} = \begin{bmatrix}
p_0 & -p_1 & -p_2 & -p_3 \\
p_1 & p_0 & -p_3 & p_2 \\
p_2 & p_3 & p_0 & -p_1 \\
p_3 & -p_2 & p_1 & p_0
\end{bmatrix}.
$$

By the simply connectedness of $Sp(1) \cong \text{Spin}(3)$, we can lift the action to the spin structure $P_{\text{Spin}(4)}$ of S^4. Using a similar diagram to (5.4) and the fact that the lift of $\tilde{p}$ is $(p, \text{Id}_{Sp(1)}) \in Sp(1) \times Sp(1)$, we can show that in the trivialization of $P_{\text{Spin}(4)}$, $\mathbb{H} \times Sp(1) \times Sp(1)$, the element $(x, (\text{Id}_{Sp(1)}, \text{Id}_{Sp(1)}))$ is mapped to $(p \cdot x, (p, \text{Id}_{Sp(1)}))$.

As in Sect. 5, this passes to the quotient space: $\mathcal{S}_-(S^4)$, and, in the induced trivialization, $\mathbb{H} \times \mathbb{H}$, the action of $Sp(1)$ is only given by left multiplication on the first factor by definition of μ_-.

A similar argument works for the other chart of $\mathbb{HP}^1$. However, the left multiplication becomes right multiplication of the conjugate, and the lift of the new $\tilde{p}$ is $(\text{Id}_{Sp(1)}, p)$. It follows that $Sp(1)$ acts on the fibre over the south pole as it acts on $\mathbb{H}$.

In particular, we proved the following lemma.

Lemma 6.2 *The orbits of the* SU (2) *action on* $\mathcal{\$}_-(S^4)$ *are given in Table* 2.

When $\alpha \neq \pm\pi/2$, we can use the orthonormal frame of Sect. 6.1. Obviously, it is invariant under the action. Hence, in the induced trivialization of $\mathcal{\$}_-(S^4)$, Sp (1) acts only on the component of the basis. In particular, it follows that $\{\sigma_1, \sigma_2, \sigma_3\}$ is a coframe on the orbits of the SU (2) action, and, $\{\partial_1, \partial_2, \partial_3\}$ is the relative frame. Observe that we are working on the coframe $\{d\alpha, \sigma_1, \sigma_2, \sigma_3, da_0, da_1, da_2, da_3\}$.

6.4 The choice of frame and the Spin (7) geometry in the adapted coordinates

Since the considered SU (2) action only moves the base of the vector bundle $\mathcal{\$}_-(S^4)$ in the trivialization of Sect. 6.1, it is natural to use: $\{d\alpha, \sigma_1, \sigma_2, \sigma_3, \xi_0, \xi_1, \xi_2, \xi_3\}$. The metrics g_c and the Cayley forms Φ_c admit a nice formula with respect to this coframe. Recall that we are working on the chart $\mathcal{U} := \mathcal{\$}_-(S^4) \setminus \{\alpha = \pm\pi/2\}$.

Proposition 6.3 *Given $c \geq 0$, the Riemannian metric g_c, in the coframe considered in this subsection, satisfies:*

$$g_c = 5(c + r^2)^{3/5}\left(d\alpha^2 + \cos^2 \alpha\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)\right) + 4(c + r^2)^{-2/5}\left(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2\right),$$
(6.4)

where $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Given $c \geq 0$, the Cayley form Φ_c, in the coframe considered in this subsection, satisfies:

$$\begin{aligned}\Phi_c =\ &16(c + r^2)^{-4/5}\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 + 25(c + r^2)^{6/5} \cos^3 \alpha\, d\alpha \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &+ 20(c + r^2)^{1/5} \cos \alpha\left(\sum_{i=1}^{3}(\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k) \wedge (d\alpha \wedge \sigma_i - \cos \alpha\sigma_j \wedge \sigma_k)\right),\end{aligned}$$
(6.5)

where $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Proof It follows immediately from (3.4), (3.5) and the choice of the coframe. ☐

If we denote by $\{e_\alpha, e_1, e_2, e_3, e_{\xi_0}, e_{\xi_1}, e_{\xi_2}, e_{\xi_3}\}$ the frame dual to $\{d\alpha, \sigma_1, \sigma_2, \sigma_3, \xi_0, \xi_1, \xi_2, \xi_3\}$, it is straightforward to relate these vectors to $\partial_\alpha, \partial_1, \partial_2, \partial_3, \partial_{a_0}, \partial_{a_1}, \partial_{a_2}, \partial_{a_3}$.

Table 2 Spin (3) Orbits

α	a	Orbit
$\neq \pm\frac{\pi}{2}$		S^3
$= -\frac{\pi}{2}$	$\neq 0$	S^3
$= -\frac{\pi}{2}$	$= 0$	Point
$= \frac{\pi}{2}$		Point

Lemma 6.4 *The dual frame* $\{e_\alpha, e_1, e_2, e_3, e_{\xi_0}, e_{\xi_1}, e_{\xi_2}, e_{\xi_3}\}$ *satisfies*:

$$e_\alpha = \partial_\alpha; \qquad\qquad\qquad e_1 = \partial_1 + l\left(-a_1\partial_{a_0} + a_0\partial_{a_1} + a_3\partial_{a_2} - a_2\partial_{a_3}\right);$$
$$e_2 = \partial_2 + l\left(-a_2\partial_{a_0} - a_3\partial_{a_1} + a_0\partial_{a_2} + a_1\partial_{a_3}\right); \qquad e_3 = \partial_3 + l\left(-a_3\partial_{a_0} + a_2\partial_{a_1} - a_1\partial_{a_2} + a_0\partial_{a_3}\right);$$
$$e_{\xi_i} = \partial_{a_i} \qquad \forall i = 0, 1, 2, 3,$$

where l is as defined in Sect. 6.2.

Proof It is straightforward from the definition of dual frame and (6.3). $\qquad\qquad\square$

6.5 The Cayley condition

Analogously to the case carried out in Sect. 5, the generic orbits of the considered $SU(2)$ action are 3-dimensional (see Lemma 6.2). Hence, it is sensible to look for invariant Cayley submanifolds. To this purpose, we assume that the submanifold N consists of a 1-param-eter family of 3-dimensional $SU(2)$-orbits in M. In particular, the coordinates that do not describe the orbits, i.e. a_0, a_1, a_2, a_3 and α, need to be functions of a parameter τ. This means that we can write:

$$N = \left\{ \left((\cos\alpha(\tau)\mathbf{u}, \sin\alpha(\tau)), (a_0(\tau), a_1(\tau), a_2(\tau), a_3(\tau)) \right) : |u| = 1, \tau \in (-\epsilon, \epsilon) \right\}. \quad (6.6)$$

The tangent space is spanned by $\{\partial_1, \partial_2, \partial_3, \dot{\alpha}\partial_\alpha + \sum_{i=0}^3 \dot{a}_i\partial_{a_i}\}$, where the dots denote the derivatives with respect to τ. The condition under which N is Cayley becomes a system of ODEs.

Theorem 6.5 *Let N be an* $SU(2)$-*invariant submanifold as described at the beginning of this subsection. Then, N is Cayley in the chart* $\mathcal{U}$ *if and only if the following system of ODEs is satisfied*:

$$\begin{cases} \dot{a}_0 a_1 - \dot{a}_1 a_0 - \dot{a}_2 a_3 + \dot{a}_3 a_2 = 0 \\ \dot{a}_0 a_2 + \dot{a}_1 a_3 - \dot{a}_2 a_0 - \dot{a}_3 a_1 = 0 \\ \dot{a}_0 a_3 - \dot{a}_1 a_2 + \dot{a}_2 a_1 - \dot{a}_3 a_0 = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 gr^2)\dot{a}_0 - l(l^2 gr^2 - 3f\cos^2\alpha)a_0\dot{\alpha} = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 gr^2)\dot{a}_1 - l(l^2 gr^2 - 3f\cos^2\alpha)a_1\dot{\alpha} = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 gr^2)\dot{a}_2 - l(l^2 gr^2 - 3f\cos^2\alpha)a_2\dot{\alpha} = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 gr^2)\dot{a}_3 - l(l^2 gr^2 - 3f\cos^2\alpha)a_3\dot{\alpha} = 0 \end{cases}$$

where $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$, $l = (\sin\alpha - 1)/2$, $f = 5(c + r^2)^{3/5}$ *and* $g = 4(c + r^2)^{-2/5}$.

Proof We first write the tangent space of N, which is spanned by $\{\partial_1, \partial_2, \partial_3, \dot{\alpha}\partial_\alpha + \sum_{i=0}^3 \dot{a}_i\partial_{a_i}\}$, in terms of the frame $\{e_\alpha, e_1, e_2, e_3, e_{\xi_0}, e_{\xi_1}, e_{\xi_2}, e_{\xi_3}\}$. This can be easily done using Lemma 6.4. Through a long computation analogous to the one carried out in Appendix A, we can apply Proposition 2.6 to this case, and we obtain the system of ODEs. $\qquad\square$

Remark 6.6 It is interesting to point out that, exactly as in the $SO(3) \times \mathrm{Id}_2$ case (see Lemma A.4), the projection π_7 of Proposition 2.6 will just be the identity in the proof of Theorem 6.5.

6.6 The Cayley fibration

In the previous section, we found the condition that makes N, SU (2)-invariant submanifold, Cayley. This consists of a system of ODEs, which will characterize the desired Cayley fibration.

Harvey and Lawson local existence and uniqueness theorem implies that any SU (2)-invariant Cayley can meet the zero section only when $\alpha = \pm\pi/2$, i.e. outside of $\mathcal{U}$. Otherwise, the zero section of $\mathcal{J}_-(S^4)$, which is Cayley, would intersect such an N in a 3-dimensional submanifold, contradicting Harvey and Lawson theorem. It follows that the initial value of one of the a_is is different from zero. We take $a_0(0) \neq 0$, as the other cases will follow similarly. Now, it is straightforward to notice that:

$$a_1 = \frac{a_1(0)}{a_0(0)}a_0; \qquad a_2 = \frac{a_2(0)}{a_0(0)}a_0; \qquad a_3 = \frac{a_3(0)}{a_0(0)}a_0; \tag{6.7}$$

solves the first 3 equations of the system given in Theorem 6.5. Moreover, it also reduces the remaining equations to the ODE:

$$\cos\alpha(-f\cos^2\alpha + 3l^2gr^2)\dot{a}_0 - l(l^2gr^2 - 3f\cos^2\alpha)a_0\dot{\alpha} = 0,$$

where, as usual, $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$, $l = (\sin\alpha - 1)/2$, $f = 5(c + r^2)^{3/5}$ and $g = 4(c + r^2)^{-2/5}$. As (6.7) implies that $a_0 = p^{-1}r$, where p is the positive real number satisfying $p^2 = 1 + \sum_{i=1}^{3}(a_i(0)/a_0(0))^2$, we can rewrite the previous ODE as:

$$\cos\alpha(-f\cos^2\alpha + 3l^2gr^2)\dot{r} - l(l^2gr^2 - 3f\cos^2\alpha)r\dot{\alpha} = 0. \tag{6.8}$$

Remark 6.7 It is easy to verify that (6.8) is not in exact form. Hence, it cannot be easily integrated. It is a non-trivial open task to verify whether, possibly up to change of coordinates, (6.8) can be integrated in closed form.

In order to understand the SU (2)-invariant Cayley fibrations, we analyse the ODE (6.8). First, we deduce the sign of $f_1 := \cos\alpha(-f\cos^2\alpha + 3l^2gr^2)$. If we let

$$\alpha_c(r) := \arcsin\left(-\frac{2r^2 + 5c}{8r^2 + 5c}\right),$$

it is easy to verify that f_1 is positive on the left of α_c for $(\alpha, r) \in (-\pi/2, \pi/2) \times \mathbb{R}^+$, and negative otherwise. Moreover, f_1 vanishes along the 3 curves $\alpha_c, \alpha = \pm\pi/2$; there, f_1 changes sign. Note that $\alpha_c \to \arcsin(-1/4)$ as $r \to \infty$.

Now, we consider $f_2 := l(l^2gr^2 - 3f\cos^2\alpha)r$. Letting

$$\beta_c(r) := \arcsin\left(-\frac{14r^2 + 15c}{16r^2 + 15c}\right),$$

then f_2 is positive on the right of β_c for $(\alpha, r) \in (-\pi/2, \pi/2) \times \mathbb{R}^+$, and it is negative otherwise. Obviously, f_2 vanishes along the curve β_c and the vertical line $\alpha = \pi/2$. Note that $\beta_c \to \arcsin(7/8)$ as $r \to \infty$. The last key observation is that f_2/f_1 tends to zero as α tends to $\pi/2$.

Putting what said so far together, and observing that $\beta_c(r) < \alpha_c(r)$ for all $r > 0$, we can draw the flow lines for (6.8) (see Fig. 5).

Fig. 5 Flow lines for (6.8)
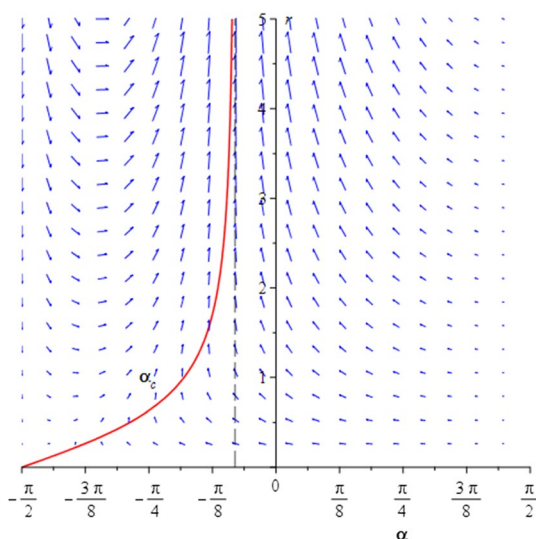
Finally, we can use these to deduce the form of the solutions from standard arguments (see Fig. 6). We give further details in Appendix B .

6.7 The conical version

We consider the easier conical case first. From a topological point of view, it is obvious that the red and green Cayleys of Fig. 6b are homeomorphic to $S^3 \times \mathbb{R}$. As the group action becomes trivial on $\alpha = \pi/2$, the topology of the fibres in blue cannot be recovered from the picture. However, it will be clear from the asymptotic analysis that these are smooth topological $\mathbb{R}^4$s. As a consequence, we have constructed a Cayley fibration on the chart $\mathcal{U} \cap M_0$, which extends to the whole M_0 by continuity (i.e. we complete the Cayleys in blue and we

(a) generic case

(b) conical case
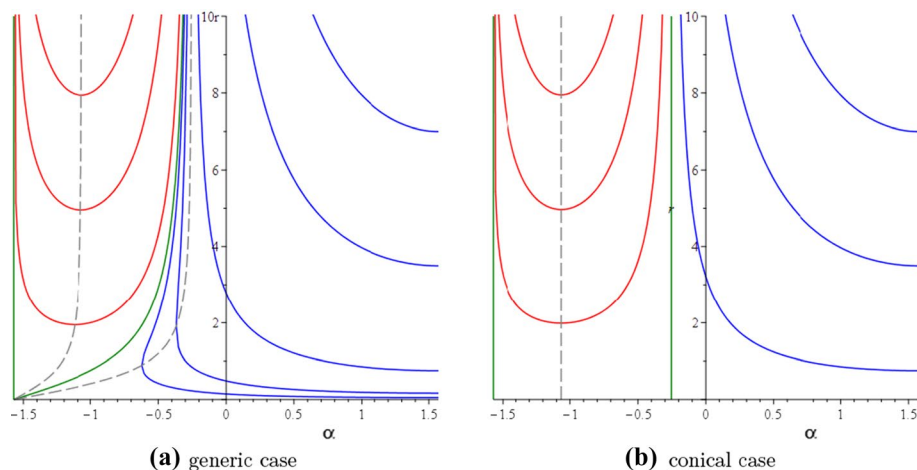
Fig. 6 Solutions of (6.8)

add the whole π_0-fibre at $\alpha = -\pi/2$). On M_0 the Cayley fibration remains a fibration in the classical sense. A reasoning similar to the one of Sect. 5 shows that the parametrizing space $\mathcal{B}$ of the Cayley fibration is $\mathbb{R}^4$.

6.8 The smooth version

Now, we deal with the generic case $c > 0$. As above, the topology of the red Cayleys of Fig. 6a is $S^3 \times \mathbb{R}$; the blue ones have topology $\mathbb{R}^4$. In the latter, we use the same asymptotic analysis argument of the conical case. Finally, the submanifolds in green are smooth topological $\mathbb{R}^4$s. As usual, we extend the Cayley fibration on $\mathcal{U}$ to the whole M by continuity (i.e. we add the whole π_c-fibre over $\alpha = -\pi/2$, we complete the Cayleys in blue and green, and we add the zero section S^4). Observe that the zero section, the π_c-fibre over $\alpha = -\pi/2$ and the green Cayleys all intersect in a point p. It follows that the M' given in Definition 2.8 is equal to $M \setminus \{p\}$. Once again, a reasoning similar to the one of Sect. 5 shows that the parametrizing space $\mathcal{B}$ of the Cayley fibration is S^4.

6.9 The smoothness of the fibres (the asymptotic analysis)

In this subsection, we study the smoothness of the fibres. This is trivial as long as the submanifolds are contained in $\mathcal{U}$; hence, the Cayleys of topology $S^3 \times \mathbb{R}$ are smooth, and we only need to check the others at the points where they meet $\partial \mathcal{U}$. To this purpose, we carry out a asymptotic analysis similar to the one of Sect. 5.

As a first step, we restrict the metric g_c to N. Combining (6.4) together with (6.7) and its consequence $a_0 = p^{-1}r$ for p positive real number satisfying $p^2 = 1 + \sum_{i=1}^{3}(a_i(0)/a_0(0))^2$, we can write the restriction as follows:

$$g_c|_N = \left(5(c + r^2)^{3/5}\cos^2\alpha + 4(c + r^2)^{-2/5}l^2r^2\right)\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)$$
$$+4(c + r^2)^{-2/5}dr^2 + 5(c + r^2)^{3/5}d\alpha^2, \tag{6.9}$$

where α and r are related by the differential equation (6.8) and, as usual, $l = (\sin\alpha - 1)/2$.

Recall that $f_2/f_1 \to 0$ as $\alpha \to \pi/2$. Therefore, the Cayleys around $\alpha = \pi/2$ are asymptotic to the horizontal line $\alpha = r_0$ for some constant $r_0 \geq 0$. By (6.9), the metric in this first order linear approximation becomes:

$$g_c|_N \sim 5(c + r_0^2)^{3/5}\left(d(\alpha - \pi/2)^2 + (\alpha - \pi/2)^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)\right).$$

In this way, we have proved that near $\alpha = \pi/2$ every Cayley we have constructed is smooth. Moreover, we can also deduce that the blue Cayleys of Fig. 6 are topologically $\mathbb{R}^4$s.

Finally, we need to check whether the remaining Cayleys of topology $\mathbb{R}^4$ are smooth or not. In this situation, we can approximate them near $\alpha = -\pi/2$ with the submanifold associated with the line:

$$\alpha = Ar - \frac{\pi}{2},$$

where A is some positive constant (as the lines corresponding to the Cayleys live between α_c and β_c). The metric in the linear approximation is asymptotic to:

$$g_c|_N \sim c^{-2/5}(5cA^2 + 4)\left(dr^2 + r^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)\right),$$

and hence, we conclude that these submanifolds are smooth as well.

6.10 The main theorems

Putting all these results together we obtain the following theorems.

Theorem 6.8 (Generic case) *Let* (M, Φ_c) *be the Bryant–Salamon manifold constructed over the round sphere* S^4 *for some* $c > 0$, *and let* SU (2) *act on M as in Sect.* 6.3. *Then, M admits an* SU (2)-*invariant Cayley fibration parametrized by* $\mathcal{B} \cong S^4$. *The fibres are topologically* $S^3 \times \mathbb{R}$, S^4 *and* $\mathbb{R}^4$. *All the Cayleys are smooth. There is only one point where multiple fibres intersect. This point lies in the zero section of* $\mathcal{S}_-(S^4)$, *and there are* $S^3 \sqcup \{two points\}$ *Cayleys passing through it.*

Theorem 6.9 (Conical case) *Let* (M_0, Φ_0) *be the conical Bryant–Salamon manifold constructed over the round sphere* S^4, *and let* SU (2) *act on* M_0 *as in Sect.* 6.3. *Then,* M_0 *admits an* SU (2)-*invariant Cayley fibration parametrized by* $\mathcal{B} \cong \mathbb{R}^4$. *The fibres are topologically* $S^3 \times \mathbb{R}$ *or* $\mathbb{R}^4$ *and are all smooth. Moreover, as these do not intersect, the* SU (2)-*invariant Cayley fibration is a fibration in the usual differential geometric sense with fibres Cayley submanifolds.*

Remark 6.10 Blowing-up at the north pole, it is easy to see that the Cayley fibration becomes trivial in the limit.

Remark 6.11 As in the previous section, we are able to compute the multi-moment maps relative to this action explicitly. Indeed, this is:

$$v_c := \frac{5}{6}(r^2 - 5c)(c + r^2)^{1/5}(\sin \alpha - 1)^3 - \frac{25}{2}(c + r^2)^{6/5} \cos^2 \alpha(\sin \alpha - 1).$$

In order to provide an idea on how the multi-moment maps behave, we draw the level sets of v_1 and v_0 (see Fig. 7).

6.11 Asymptotic geometry

The first observation we need to make is that there are only two asymptotic behaviours for the Cayleys constructed in Theorem 6.8 and in Theorem 6.9: one corresponding to $\alpha \sim -\pi/2$ and the other to $\alpha \sim \arcsin(-1/4)$. In both cases, we can use (6.9) to obtain the asymptotic cone, which is:

$$g_c|_N \sim ds^2 + \frac{9}{25}s^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2),$$

for $\alpha \sim \pi/2$, and it is

$$g_c|_N \sim ds^2 + \frac{9}{16}s^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2),$$

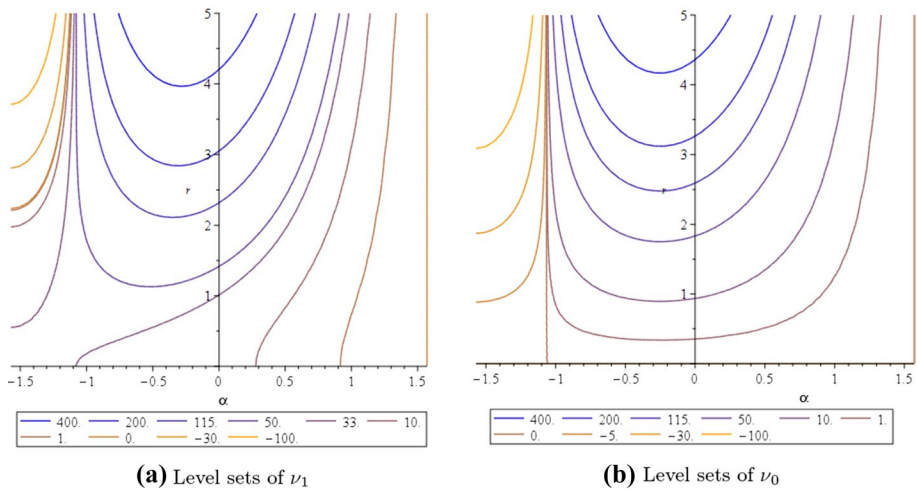for $\alpha \sim \arcsin(-1/4)$, where $s := (10/3)r^{3/5}$.

(a) Level sets of ν_1 **(b)** Level sets of ν_0

Fig. 7 Level sets of the multi-moment map in the generic and conical case

Appendix A

In this appendix, we prove Theorem 5.12. First, we need to rewrite the tangent space of N in the diagonalizing frame of Sect. 5.6.

Lemma A.1 *The tangent space of N is spanned by*:

$$u := te_{\omega_2} - se_{\omega_1}, \quad v := e_2 + \frac{\sin\alpha}{2}(te_s - se_t), \quad w := e_3 + \sin\alpha\left(te_{\omega_1} + se_{\omega_2}\right)$$

and

$$y := \dot{s}e_s + \dot{t}e_t + \dot{\alpha}e_\alpha + \dot{\beta}e_\beta + \dot{\delta}\left(se_{\omega_1} + te_{\omega_2}\right).$$

Moreover, through the musical isomorphism, we have:

$$u^\flat = (c+r^2)^{-2/5}(t\omega_2 - s\omega_1), \quad v^\flat = 5(c+r^2)^{3/5}\cos^2\alpha\sigma_2 + 2(c+r^2)^{-2/5}\sin\alpha(t\tilde{d}s - s\tilde{d}t),$$
$$w^\flat = 5(c+r^2)^{3/5}\cos^2\alpha\sigma_3 + (c+r^2)^{-2/5}\sin\alpha\left(t\omega_1 + s\omega_2\right)$$

and

$$y^\flat = 5(c+r^2)^{3/5}(\dot{\alpha}d\alpha + \sin^2\alpha\,\dot{\beta}d\beta) + 4(c+r^2)^{-2/5}(\dot{s}\tilde{d}s + \dot{t}\tilde{d}t) + (c+r^2)^{-2/5}\dot{\delta}(s\omega_1 + t\omega_2),$$

where $r^2 = s^2 + t^2$.

Proof One can immediately see from Lemma 5.11 that $\partial_1 = u$, $\partial_2 = v$ and $\partial_\delta = se_{\omega_1} + te_{\omega_2}$. We use these equalities to obtain:

$$(s^2+t^2)\partial_\delta - (t^2-s^2)\partial_1 = (s^2+t^2)(se_{\omega_1} + te_{\omega_2}) - (t^2-s^2)(te_{\omega_2} - se_{\omega_1})$$
$$= 2st(te_{\omega_1} + se_{\omega_2}),$$

which implies that $\partial_3 = w$. We conclude noticing that $\dot{s}\partial_s + \dot{t}\partial_t + \dot{\alpha}\partial_\alpha + \dot{\beta}\partial_\beta + \dot{\delta}\partial_\delta = y - \dot{\beta}\cos\alpha\partial_1$, where we used once again Lemma 5.11. Obviously, the space spanned by $\{u, v, w, y\}$ coincides with the one spanned by $\{u, v, w, y - \dot{\beta}\cos\alpha\partial_1\}$.

The second part of the Lemma follows immediately from Proposition 5.10, where we proved that the metric is diagonal in this frame. $\qquad\square$

Let B be as in Proposition 2.6. We compute the terms of B in the basis $\{u, v, w, y\}$.

Lemma A.2 *Let u, v, w, y as in Lemma A.1. Then, we have*:

$$
\begin{aligned}
B(v, w, y) = {}& 25(c + r^2)^{6/5}\sin\alpha\cos^2\alpha(\dot{\beta}d\alpha - \dot{\alpha}d\beta) \\
&+ 2\sin^2\alpha(c + r^2)^{-4/5}\big((t\dot{t} + s\dot{s})(t\omega_2 - s\omega_1) - (t^2 - s^2)\dot{\delta}(t\tilde{d}t + s\tilde{d}s)\big) \\
&+ 5(c + r^2)^{1/5}\Big(2\cos^2\alpha\big(\dot{s}\omega_1 - \dot{t}\omega_2 + \dot{\delta}(t\tilde{d}t - s\tilde{d}s) + \sin\alpha(ts\dot{\delta}\sigma_2 + (s\dot{t} - t\dot{s})\sigma_3)\big) \\
&+ 2\sin\alpha\cos\alpha\big((s^2 - t^2)\dot{\delta}d\alpha + \dot{\alpha}(t\omega_2 - s\omega_1)\big) + (s^2 + t^2)\sin^3\alpha(\dot{\alpha}d\beta - \dot{\beta}d\alpha) \\
&+ 4\cos\alpha\sin^2\alpha\big(\dot{\beta}(s\tilde{d}s + t\tilde{d}t) - (s\dot{s} + t\dot{t})d\beta\big)\Big), \\
B(w, u, y) = {}& 4(c + r^2)^{-4/5}(t^2 + s^2)\sin\alpha(\dot{t}\tilde{d}s - \dot{s}\tilde{d}t) \\
&+ 5(c + r^2)^{1/5}\big(-2\cos^2\alpha(s\dot{s} + t\dot{t})\sigma_2 - 2\cos\alpha\sin\alpha st\dot{\delta}d\beta + \cos\alpha\sin\alpha\dot{\beta}(t\omega_1 + s\omega_2) \\
&+ 2\cos\alpha(s\dot{t} - t\dot{s})d\alpha + 2\cos\alpha\dot{\alpha}(t\tilde{d}s - s\tilde{d}t) + \cos\alpha\sin\alpha(t^2 + s^2)\dot{\alpha}\sigma_2 \\
&- \cos\alpha\sin^2\alpha(t^2 + s^2)\dot{\beta}\sigma_3\big), \\
B(u, v, y) = {}& 2(c + r^2)^{-4/5}\sin\alpha(-2\dot{\delta}st(t\tilde{d}t + s\tilde{d}s) + (t\dot{t} + s\dot{s})(t\omega_1 + s\omega_2)) \\
&5(c + r^2)^{1/5}\big(-2\cos^2\alpha(s\dot{s} + t\dot{t})\sigma_3 - 2\cos\alpha st\dot{\delta}d\alpha + \cos\alpha\dot{\alpha}(s\omega_2 + t\omega_1) \\
&+ 2\cos\alpha\sin\alpha(t\dot{s} - s\dot{t})d\beta + 2\cos\alpha\sin\alpha\dot{\beta}(s\tilde{d}t - t\tilde{d}s) + (s^2 + t^2)\cos\alpha\sin\alpha\dot{\alpha}\sigma_3 \\
&+ (s^2 + t^2)\cos\alpha\sin^2\alpha\dot{\beta}\sigma_2\big), \\
B(v, u, w) = {}& 2(c + r^2)^{-4/5}\sin^2\alpha(t^2 + s^2)(t\tilde{d}t + s\tilde{d}s) + 10(c + r^2)^{1/5}\big(-\cos^2\alpha(s\tilde{d}s + t\tilde{d}t) \\
&+ \sin\alpha\cos\alpha(t^2 + s^2)d\alpha\big),
\end{aligned}
$$

where B is defined in Proposition 2.6 and $r^2 = s^2 + t^2$.

Proof The multi-linearity of the Cayley form Φ_c implies that the same property holds for B. Now, expanding the formula (5.19) for Φ_c, we obtain:

$$
\begin{aligned}
\Phi_c = {}& 4(c + r^2)^{-4/5}\tilde{d}s \wedge \tilde{d}t \wedge \omega_2 \wedge \omega_1 + 25(c + r^2)^{6/5}\sin\alpha\cos^2\alpha d\alpha \wedge d\beta \wedge \sigma_3 \wedge \sigma_2 \\
&10(c + r^2)^{1/5}\Big(\sin\alpha\tilde{d}s \wedge \omega_1 \wedge d\alpha \wedge d\beta + \cos^2\alpha\tilde{d}s \wedge \omega_1 \wedge \sigma_2 \wedge \sigma_3 - \sin\alpha\tilde{d}t \wedge \omega_2 \wedge d\alpha \wedge d\beta \\
&- \cos^2\alpha\tilde{d}t \wedge \omega_2 \wedge \sigma_2 \wedge \sigma_3 + 2\cos\alpha\tilde{d}s \wedge \tilde{d}t \wedge d\alpha \wedge \sigma_2 - 2\cos\alpha\sin\alpha\tilde{d}s \wedge \tilde{d}t \wedge d\beta \wedge \sigma_3 \\
&+ \frac{\cos\alpha}{2}\omega_1 \wedge \omega_2 \wedge d\alpha \wedge \sigma_2 - \frac{\cos\alpha\sin\alpha}{2}\omega_1 \wedge \omega_2 \wedge d\beta \wedge \sigma_3 - \cos\alpha\tilde{d}s \wedge \omega_2 \wedge d\alpha \wedge \sigma_3 \\
&- \cos\alpha\sin\alpha\tilde{d}s \wedge \omega_2 \wedge d\beta \wedge \sigma_2 - \cos\alpha\tilde{d}t \wedge \omega_1 \wedge d\alpha \wedge \sigma_3 - \cos\alpha\sin\alpha\tilde{d}t \wedge \omega_1 \wedge d\beta \wedge \sigma_2\Big).
\end{aligned}
$$

It is straightforward to conclude using the definition of B. $\qquad\square$

Consider the two-form given in Proposition 2.6 that projects to η through π_7. The summands of such two-forms can be computed through a direct computation involving the terms obtained in Lemma A.1 and Lemma A.2.

Corollary A.3 *Let* u, v, w, y *as in Lemma* A.1 *and let* $\Psi_1 := u^\flat \wedge B(v, w, y)$, $\Psi_2 = v^\flat \wedge B(w, u, y)$, $\Psi_3 = w^\flat \wedge B(u, v, y)$, $\Psi_4 = y^\flat \wedge B(v, u, w)$, *where* B *is as defined in Proposition* 2.6. *Then, we have*:

$$
\begin{aligned}
\Psi_1 = {} & 25(c + r^2)^{4/5} \sin\alpha \cos^2\alpha (t\omega_2 - s\omega_1) \wedge (\dot\beta d\alpha - \dot\alpha d\beta) \\
& - (c + r^2)^{-6/5} 2\sin^2\alpha (t^2 - s^2)\dot\delta(t\omega_2 - s\omega_1) \wedge (t\tilde{d}t + s\tilde{d}s) \\
& + 5(c + r^2)^{-1/5}\Big(2\cos^2\alpha\big((t\dot s - s\dot t)\omega_2 \wedge \omega_1 + \dot\delta(t\omega_2 - s\omega_1) \wedge (t\tilde{d}t - s\tilde{d}s)\big) \\
& + 2\sin\alpha\cos^2\alpha\big(ts\dot\delta(t\omega_2 - s\omega_1) \wedge \sigma_2 + (st - t\dot s)(t\omega_2 - s\omega_1) \wedge \sigma_3\big) \\
& + 2\sin\alpha\cos\alpha(s^2 - t^2)\dot\delta(t\omega_2 - s\omega_1) \wedge d\alpha + (t^2 + s^2)\sin^3\alpha(t\omega_2 - s\omega_1) \wedge (\dot\alpha d\beta - \dot\beta d\alpha) \\
& + 4\cos\alpha\sin^2\alpha\big(\dot\beta(t\omega_2 - s\omega_1) \wedge (s\tilde{d}s + t\tilde{d}t) - (s\dot s + t\dot t)(t\omega_2 - s\omega_1) \wedge d\beta\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_2 = {} & 25(c + r^2)^{4/5}\big(-2\cos^3\alpha\sin\alpha st\dot\delta\sigma_2 \wedge d\beta + \cos^3\alpha\sin\alpha\dot\beta\sigma_2 \wedge (t\omega_1 + s\omega_2) \\
& + 2\cos^3\alpha(s\dot t - t\dot s)\sigma_2 \wedge d\alpha + 2\cos^3\alpha\dot\alpha\sigma_2 \wedge (t\tilde{d}s - s\tilde{d}t) - \cos^3\alpha\sin^2\alpha(t^2 + s^2)\dot\beta\sigma_2 \wedge \sigma_3\big) \\
& + 10(c + r^2)^{-1/5}\Big(2\sin\alpha\cos^2\alpha(t^2 + s^2)\sigma_2 \wedge(t\tilde{d}s - \dot s\tilde{d}t) - 2\cos^2\alpha\sin\alpha(s\dot s + t\dot t)(t\tilde{d}s - s\tilde{d}t) \wedge \sigma_2 \\
& - 2\cos\alpha\sin^2\alpha st\dot\delta(t\tilde{d}s - s\tilde{d}t) \wedge d\beta + \cos\alpha\sin^2\alpha\dot\beta(t\tilde{d}s - s\tilde{d}t) \wedge (t\omega_1 + s\omega_2) \\
& + 2\cos\alpha\sin\alpha(s\dot t - t\dot s)(t\tilde{d}s - s\tilde{d}t) \wedge d\alpha - \cos\alpha\sin^2\alpha(t^s + s^2)\dot\alpha\sigma_2 \wedge (t\tilde{d}s - s\tilde{d}t) \\
& + \cos\alpha\sin^3\alpha(t^2 + s^2)\dot\beta\sigma_3 \wedge (t\tilde{d}s - s\tilde{d}t)\Big) + 8(c + r^2)^{-6/5}\sin^2\alpha(t^2 + s^2)(s\dot t - t\dot s)\tilde{d}s \wedge \tilde{d}t,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3 = {} & 25(c + r^2)^{4/5}\big(-2\cos^3\alpha st\dot\delta\sigma_3 \wedge d\alpha + \cos^3\alpha\dot\alpha\sigma_3 \wedge (s\omega_2 + t\omega_1) + 2\cos^3\alpha\sin\alpha(t\dot s - s\dot t)\sigma_3 \wedge d\beta \\
& + 2\cos^3\alpha\sin\alpha\dot\beta\sigma_3 \wedge (s\tilde{d}t - t\tilde{d}s) + (s^2 + t^2)\cos^3\alpha\sin^2\alpha\dot\beta\sigma_3 \wedge \sigma_2\big) \\
& - 4(c + r^2)^{-6/5}\sin^2\alpha\dot\delta st(t\omega_1 + s\omega_2) \wedge (t\tilde{d}t + s\tilde{d}s) \\
& + 5(c + r^2)^{-1/5}\Big(2\sin\alpha\cos^2\alpha\big((t\dot t + s\dot s)\sigma_3 \wedge (t\omega_1 + s\omega_2) - 2\dot\delta st\sigma_3 \wedge (t\tilde{d}t + s\tilde{d}s)\big) \\
& - 2\sin\alpha\cos^2\alpha(s\dot s + t\dot t)(t\omega_1 + s\omega_2) \wedge \sigma_3 - 2\cos\alpha\sin\alpha st\dot\delta(t\omega_1 + s\omega_2) \wedge d\alpha \\
& + 2\cos\alpha\sin^2\alpha(t\dot s - s\dot t)(t\omega_1 + s\omega_2) \wedge d\beta + 2\cos\alpha\sin^2\alpha\dot\beta(t\omega_1 + s\omega_2) \wedge (s\tilde{d}t - t\tilde{d}s) \\
& + (s^2 + t^2)\cos\alpha\sin^2\alpha\dot\alpha(t\omega_1 + s\omega_2) \wedge \sigma_3 + (s^2 + t^2)\cos\alpha\sin^3\alpha\dot\beta(t\omega_1 + s\omega_2) \wedge \sigma_2\Big),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_4 = {} & 2(c + r^2)^{-6/5}\sin^2\alpha(t^2 + s^2)\big(\dot\delta(s\omega_1 + t\omega_2) \wedge (t\tilde{d}t + s\tilde{d}s) + 4(\dot t s - s\dot t)\tilde{d}t \wedge \tilde{d}s\big) \\
& + 50(c + r^2)^{4/5}\big(-\cos^2\alpha(\dot\alpha d\alpha + \sin^2\alpha\dot\beta d\beta) \wedge (s\tilde{d}s + t\tilde{d}t) + \cos\alpha\sin^3\alpha(t^2 + s^2)\dot\beta d\beta \wedge d\alpha\big) \\
& + 10(c + r^2)^{-1/5}\Big(\sin^2\alpha(t^2 + s^2)(\dot\alpha d\alpha + \sin^2\alpha\dot\beta d\beta) \wedge (t\tilde{d}t + s\tilde{d}s) - 4\cos^2\alpha(\dot s t - t s)\tilde{d}s \wedge \tilde{d}t \\
& + 4\sin\alpha\cos\alpha(t^2 + s^2)(\dot s\tilde{d}s + \dot t\tilde{d}t) \wedge d\alpha - \cos^2\alpha\dot\delta(s\omega_1 + t\omega_2) \wedge (s\tilde{d}s + t\tilde{d}t) \\
& + \sin\alpha\cos\alpha(t^2 + s^2)\dot\delta(s\omega_1 + t\omega_2) \wedge d\alpha\Big),
\end{aligned}
$$

where $r^2 = s^2 + t^2$.

Moreover,

$$\eta = \pi_7(\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4),$$

where η and π_7 are defined in Proposition 2.6.

Finally, we turn our attention to the map π_7. As recalled in Remark 2.7, this map is the projection to the linear subspace Λ^2_7 of the space of 2-forms on M.

Lemma A.4 *In the coframe $\{\sigma_2, \sigma_3, d\alpha, d\beta, \omega_1, \omega_2, \tilde{d}s, \tilde{d}t\}$, a basis for Λ^2_7 is given by the following 2-forms:*

$$\lambda_1 := -\cos\alpha\sigma_2 \wedge \omega_1 + d\alpha \wedge \omega_2 + 2\sin\alpha d\beta \wedge \tilde{d}t + 2\cos\alpha\sigma_3 \wedge \tilde{d}s,$$

$$\lambda_2 := \cos\alpha\sigma_2 \wedge \omega_2 + d\alpha \wedge \omega_1 - 2\sin\alpha d\beta \wedge \tilde{d}s + 2\cos\alpha\sigma_3 \wedge \tilde{d}t,$$

$$\lambda_3 := \cos\alpha\sigma_3 \wedge \omega_1 + \sin\alpha d\beta \wedge \omega_2 + 2\cos\alpha\sigma_2 \wedge \tilde{d}s - 2d\alpha \wedge \tilde{d}t,$$

$$\lambda_4 := -\cos\alpha\sigma_3 \wedge \omega_2 + \sin\alpha d\beta \wedge \omega_1 + 2\cos\alpha\sigma_2 \wedge \tilde{d}t + 2d\alpha \wedge \tilde{d}s,$$

$$\lambda_5 := 5(c+r^2)\cos\alpha\sigma_3 \wedge d\alpha + 5(c+r^2)\sin\alpha\cos\alpha\sigma_2 \wedge d\beta + 2\omega_2 \wedge \tilde{d}s + 2\omega_1 \wedge \tilde{d}t,$$

$$\lambda_6 := 5(c+r^2)\sin\alpha\cos\alpha\sigma_3 \wedge d\beta - 5(c+r^2)\cos\alpha\sigma_2 \wedge d\alpha + \omega_2 \wedge \omega_1 + 4\tilde{d}t \wedge \tilde{d}s,$$

$$\lambda_7 := 5(c+r^2)\sin\alpha d\beta \wedge d\alpha + 5(c+r^2)\cos^2\alpha\sigma_3 \wedge \sigma_2 + 2\tilde{d}s \wedge \omega_1 - 2\tilde{d}t \wedge \omega_2.$$

Proof Using the explicit formula for π_7 given in Proposition 2.6, it is easy to verify that $\pi_7(\lambda_i) = \lambda_i$ for all $i = 1...7$. We deduce that the λ_is form a basis of Λ^7_2 as they are linearly independent and the dimension of Λ^7_2 is 7. $\square$

At this point, the proof of Theorem 5.12 follows easily. Indeed, we can rewrite the sum of the Ψ_i given in Corollary A.3 as follows:

$$\begin{aligned}
\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 =& 5(c+r^2)^{-1/5}\left(-5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}t + r^2\sin^3\alpha\dot{\beta}t - 2\sin\alpha\cos\alpha ts^2\dot{\delta}\right)\lambda_1 \\
& + 5(c+r^2)^{-1/5}\left(5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}s - r^2\sin^3\alpha\dot{\beta}s - 2\sin\alpha\cos\alpha t^2 s\dot{\delta}\right)\lambda_2 \\
& + 5(c+r^2)^{-1/5}\left(5(c+r^2)\cos^2\alpha t\dot{\alpha} + 4\cos\alpha\sin\alpha t^2\dot{t} + 2\sin\alpha\cos\alpha sts\dot{s}\right.\\
& \left. + 2\sin\alpha\cos\alpha s^2\dot{t} - r^2\sin^2\alpha\dot{\alpha}t\right)\lambda_3 + 5(c+r^2)^{-1/5}\left(-5(c+r^2)\cos^2\alpha s\dot{\alpha}\right.\\
& \left. - 4\cos\alpha\sin\alpha s^2\dot{s} - 2\sin\alpha\cos\alpha st\dot{t} - 2\sin\alpha\cos\alpha t^2\dot{s} + r^2\sin^2\alpha\dot{\alpha}s\right)\lambda_4 \\
& - 2\cos^2\alpha st\dot{\delta}\left(25(c+r^2)^{-1/5}\lambda_5\right) \\
& + 2\cos^2\alpha(t\dot{s} - s\dot{t})\left(25(c+r^2)^{-1/5}\lambda_6\right) \\
& + 2(s^2 + t^2)\sin^2\alpha\cos\alpha\dot{\beta}\left(25(c+r^2)^{-1/5}\lambda_7\right).
\end{aligned}$$

From Corollary A.3 and Lemma A.4, we deduce the ODEs of Theorem 5.12.

Appendix B

In this appendix, we study in detail the ODE (6.8). First, observe that in the chart we are considering the orbits are 3-dimensional; hence, the derivative $(\dot{\alpha}, \dot{r})$ cannot vanish. In particular, we can reparametrize the curve such that $\dot{\alpha} = f_1$ and deduce from (6.8) that $\dot{r} = f_2$. Indeed, we recall that (6.8) can be rewritten as:

$$f_1 \dot{r} - f_2 \dot{\alpha} = 0.$$

Since $(\dot{\alpha}, \dot{r}) \parallel (f_1, f_2)$, we recast the problem into finding the integral curves of the vector field $X = (f_1, f_2)$. Observe that X makes sense on the whole strip $\{r \geq 0, \alpha \in [-\pi/2, \pi/2]\}$ and vanishes at $(-\pi/2, 0)$ or along the curve $\alpha = \pi/2$. It follows that two solutions of the ODE can only intersect there. Moreover, $\{r = 0\}$ and $\{\alpha = -\pi/2\}$ are solutions.

We split our analysis in 3 parts, corresponding to the different coupled signs of f_1 and f_2:

(1) $\alpha \leq \beta_c(r)$;
(2) $\alpha_c(r) \leq \alpha$;
(3) $\beta_c(r) < \alpha < \alpha_c(r)$.

B.1. The set $\alpha \leq \beta_c(r)$

Since in this set $f_1 > 0$ and $f_2 < 0$, starting from an initial point and going forward in time the solution needs to decrease in r and increase in α in a monotonic way, until it hits β_c. There, $\dot{r} = 0$, so, the solution intersects the curve horizontally.

If we instead go backwards in time, α decreases, while r increases. Hence, the solution can either meet the vertical line $\alpha = -\pi/2$ at some $r_0 > 0$ or explode at infinity. However, the first instance cannot occur since the vertical line $\alpha = -\pi/2$ is a solution of the system of ODEs as well.

B.2. The set $\alpha_c(r) \leq \alpha$

In this case, we have $f_1 < 0$ and $f_2 > 0$, hence, if we take a point and study the solution going backwards in time the solution needs to decrease in r and increase in α. We deduce that it passes through the vertical line $\alpha = \pi/2$ horizontally at some $r_0 > 0$. Indeed, it does not meet $\{r = 0\}$, as the zero section is another solution of the system of ODEs. Moreover, if we reparametrize (6.8) such that $\dot{\alpha} = 1$, which we can do in the complement of α_c, we see that the solution is $r = r(\alpha)$ in this region and that $dr/d\alpha = f_2/f_1 < 0$. Since $f_2/f_1 \to 0$ as $\alpha \to \pi/2$, each solution tends to the vertical line horizontally, and hence, they cannot intersect there.

If we go forward in time, we either have $r \to \infty$ or we pass through α_c vertically. Under the same reparametrization as before, we deduce that the solutions $r(\alpha)$ with initial conditions along the line $\{\alpha = \arcsin(-1/4)\}$ cannot explode and they need to intersect α_c. Moreover, each point of α_c can be reached by such a solution.

B.3. The set $\beta_c(r) < \alpha < \alpha_c(r)$

As before, we pick a point and we see what happens to the solution going forwards and backwards in time. From the fact that $f_1, f_2 > 0$, there are only two possibilities forward in time: we either have $r \to +\infty$ as $\alpha \to \arcsin(-1/4)$ or we meet α_c vertically. The latter case will not happen, otherwise, we would have a solution with a cuspid singularity.

If we go backwards in time, we either intersect α_c, β_c or $(-\pi/2, 0)$. It is obvious that there are solutions intersecting α_c and β_c. In order to prove the existence of the last case, consider the segment given by an horizontal line restricted to this set. Let K_{α_c} be the subset

from which the solutions will meet α_c backwards in time, and let K_{β_c} the one relative to β_c. It is easy to show that these subsets are disjoint connected open subintervals arbitrarily close to each other, using continuity of the initial data. As this cannot cover the starting interval, we conclude.

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