Enumerative combinatorics, continued fractions and total positivity

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A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

Supervised by
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To the memory of my grandfather
Sudhir Chandra Deb (1928–2017)

Declaration

I, Bishal Deb, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

Determining whether a given number is positive is a fundamental question in mathematics. This can sometimes be answered by showing that the number counts some collection of objects, and hence, must be positive. The work done in this dissertation is in the field of enumerative combinatorics, the branch of mathematics that deals with exact counting. We will consider several problems at the interface between enumerative combinatorics, continued fractions and total positivity.

In our first contribution, we exhibit a lower-triangular matrix of polynomials in six indeterminates that appears empirically to be coefficientwise totally positive, and which includes as a special case the Eulerian triangle. This generalises Brenti's conjecture from 1996. We prove the coefficientwise total positivity of a three-variable case which includes the reversed Stirling subset triangle.

Our next contribution is the study of two sequences whose Stieltjes-type continued fraction coefficients grow quadratically; we study the Genocchi and median Genocchi numbers. We find Stieltjes-type and Thron-type continued fractions for some multivariate polynomials that enumerate D-permutations, a class of permutations of $2n$, with respect to a very large (sometimes infinite) number of simultaneous statistics that measure cycle status, record status, crossings and nestings.

After this, we interpret the Foata–Zeilberger bijection in terms of Laguerre digraphs, which enables us to count cycles in permutations. Using this interpretation, we obtain Jacobi-type continued fractions for multivariate polynomials enumerating permutations, and also Thron-type and Stieltjes-type continued fractions for multivariate polynomials enumerating D-permutations, in both cases including the counting of cycles. This enables us to prove some conjectured continued fractions due to Sokal–Zeng from 2022, and Randrianarivony–Zeng from 1996.

Finally, we introduce the higher-order Stirling cycle and subset numbers; these generalise the Stirling cycle and subset numbers, respectively. We introduce some conjectures which involve different total-positivity questions for these triangular arrays and then answer some of them.

Impact Statement

The primary impact of this dissertation is to academia, and in particular, to study of enumerative combinatorics in mathematics (and also possibly to theoretical computer science).

Positivity problems form a central theme in enumerative combinatorics, see for example [Sta99]. One subclass of these problems is on total positivity, see for example [Bre89, Bre95, Bre96]. Another line of investigation is the study of continued fractions in combinatorics which was initiated by Flajolet [Fla80]. The two main threads in this thesis concern the interaction between enumerative combinatorics with total positivity and with continued fractions. The new contributions are presented in Chapters 3-6.

In Chapter 3 we present some conjectures generalising conjectures due to Brenti from 1996 [Bre96]. Our generalisation may suggest new ways of attacking this stubborn conjecture that has been open for over a quarter century. The total positivity of triangles generated by n-dependent or k-dependent recurrences were handled by Brenti in 1995 [Bre95]. However, our primary result in this chapter concerns the reversed Stirling subset triangle that has mixed n and k-dependence.

Our work in Chapters 4 and 5 is a sequel to the recent work of Sokal and Zeng [SZ22]. In Chapter 4, we take one step up in complexity and study continued fractions in which the continued fraction coefficients grow quadratically. We intend to categorise sequences according to the growth rate of their Stieltjes-type continued fraction coefficients and study each family separately. This line of investigation was initiated by Pollaczek [Pol56].

Then in Chapter 5 we show how one can count cycles in permutations using the

Foata–Zeilberger bijection [FZ90], thereby obtaining new continued fractions. Our novel method allows us to resolve conjectures due to Sokal and Zeng [SZ22] from 2022, and also a conjecture due to Randrianarivony and Zeng from [RZ96a] from 1996.

Finally, in Chapter 6 we present several conjectures on total positivity of various kinds concerning the higher-order Stirling cycle and subset numbers. We advertise four questions that one has whenever they have a lower-triangular array of real numbers. Also, see Section 2.1.6 where we provide several examples of important triangles in combinatorics in the context of these questions. All of these four questions are important positivity questions in combinatorics and we present them in a systematised way.

Acknowledgements

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Finally, I would conclude by thanking my elder brother, mentor and friend Manjil P. Saikia. You have mentored me since I was a high school student. Also, thank you to Khimpi bou for listening to my ramblings and advising me on big and small things. Thank you for providing me with a home in this country outside of London. I am immensely grateful for your friendship.

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   Content: The computer experiments for Conjectures 3.0.4 and 3.0.3 were performed by Alan Sokal. The combinatorial interpretations in Section 3.1 were constructed by Bishal Deb. The planar network in Figure 3.1(b) was discovered by Tomack Gilmore. The first proof for this planar network in Section 3.2.1 was jointly formulated by all the coauthors (20% contribution each). The second bijective proof in Section 3.2.3 is due to Bishal Deb.

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   The overall contribution of Bishal Deb is 75%.

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Chapter 1

Introduction

A *combinatorial family of objects* is a class of sets $\mathcal{S} = (S_i)_{i\in I}$ where I is an indexing set, and each set S_i is finite. (Most often $I = \mathbb{N}$.) *Enumerative combinatorics* is the branch of mathematics that deals with exact enumeration of combinatorial families, i.e., given a combinatorial class $\mathcal{S} = \left(S_i\right)_{i\in I}$, we are interested in computing $|S_i|$. When $I = \mathbb{N}$, we have several ways to enumerate $\mathcal{S}$; we are often looking for a nice closed-form formula for $|S_i|$ in terms of i. However, depending on our situation, we may prefer to work with the ordinary generating function (this is the formal power series $\sum_{i=0}^{\infty} |S_i| t^i$), or the exponential generating function (this is the formal power series $\sum_{i=0}^{\infty} |S_i| t^i / i!$). Again, the generating functions may have nice closed-form formulae. However, there are other ways of representing the generating functions such as a differential or a functional equation, a series expansion, or — most importantly for the present thesis — a continued fraction. Our prototypical example of a combinatorial family is the class of all permutations $\left(\mathfrak{S}_n\right)_{n\geq 0}$. Here, we have the nice closed-form formula $|\mathfrak{S}_n| = n!$. The ordinary generating function $\sum_{i=0}^{\infty} n! t^n$ does not have a nice closed-form expression (indeed, it has zero radius of convergence), but it does have the following continued fraction due to Euler [Eul60]:

$$\sum_{n=0}^{\infty} n!t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cfrac{3t}{1-\cfrac{3t}{1-\ddots}}}}}}}. \tag{1.1}$$

The exponential generating function is simply $1/(1-t)$.

It is common practice in enumerative combinatorics to refine the counting by keeping track of one or several *statistics* along with computing the cardinality of the sets of each size. Given a combinatorial family $\mathcal{S} = (S_n)_{n\in\mathbb{N}}$, a statistic is a function $f : \cup S_n \to \mathbb{N}$. Given a statistic f, we are interested in computing the generating polynomials $p_n(x) = \sum_{\tau\in S_n} x^{f(\tau)} = \sum_{i\in\mathbb{N}} \left|f^{-1}(i)\cap S_n\right| x^i$. Setting $x = 1$, we clearly obtain $p_n(1) = |S_n|$. Thus, we upgrade from counting using natural numbers to counting using polynomials with nonnegative integer coefficients to obtain a more refined enumeration. For example, when our class is $S_n = \mathfrak{S}_n$ and the statistic is the number of cycles in a permutation, the polynomials $p_n(x)$ are given by $p_n(x) = x(x+1)(x+2)\cdots(x+n-1)$. In fact, we need not restrict ourselves to only one statistic and univariate polynomials; we can count multiple statistics simultaneously and work with multivariate polynomials.

The two main threads in this thesis concern the interaction between enumerative combinatorics and the following themes: total positivity and continued fractions. We will introduce these themes in Sections 1.1 and 1.2, respectively; we will also mention how the investigations of these themes meet at the end of Section 1.1 (and in greater detail later in Sections 2.1.4 and 2.3). We will then end this introductory chapter by stating the outline of this thesis in Section 1.3.

1.1 Total positivity and combinatorics

Positivity problems form a central theme in enumerative combinatorics (see for example [Sta99]). The primary idea, which will be a recurring theme in this thesis, is that an integer can be proven to be nonnegative if it is the cardinality of a set. One such class of problems are *total positivity problems in combinatorics*.

Definition 1.1.1. A matrix of real numbers is said to be *totally positive* (TP in short) if all its minors are nonnegative.[1] We say that a matrix is *strictly totally positive* (STP in short) if all its minors are strictly positive.

Notice that the matrix in Definition 1.1.1 need not be a square matrix, or even finite. We can have an infinite number of rows and columns; all we need is the determinant of all finite square submatrices to be nonnegative. Nonnegative *bidiagonal matrices* are an example of a class of totally positive matrices; these are matrices with non-negative entries on two consecutive diagonals and zeroes everywhere else.

The study of total positivity was started independently in the 1930s by two different schools: Schoenberg and his school while studying the distribution of roots of polynomials in the complex plane, and Krein and his school while studying problems in mechanics. See the Foreword and the Remarks at the end of each chapter of [Pin09] for more detailed historical notes on the theory of total positivity.

The study of total positivity in combinatorics was initiated by Brenti [Bre89, Bre95, Bre96]. Several matrices whose entries are given by important combinatorial sequences or arrays are totally positive. Some examples of totally positive matrices are

- the binomial triangle $\left(\binom{n}{k}\right)_{n,k\geq 0}$,
- the binomial square matrix $\left(\binom{n+k}{k}\right)_{n,k\geq 0}$,
- the lower-triangular matrix of Stirling cycle numbers $\left(\left[{n \atop k}\right]\right)_{n,k\geq 0}$ (here the (n,k)-th entry counts the number of permutations of $[n]$ with k cycles, they are also called the unsigned Stirling numbers of the first kind),
- the lower-triangular matrix of Stirling subset numbers $\left(\left\{{n \atop k}\right\}\right)_{n,k\geq 0}$ (here the

[1] ***Warning***: We would like to warn the reader that there could be some confusion in terminology while looking at the relevant literature from different sources. In the sources such as [GK02, FJ11] the terms "total nonnegativity" and "total positivity" are used to refer to what we call "total positivity" and "strict total positivity", respectively. Here we follow the terminology in sources such as [Kar68, Pin09].

(n,k)-th entry counts the number of set partitions of $[n]$ into k nonempty blocks, they are also called the Stirling numbers of the second kind),

- and the Hankel matrix of factorials $((n+k)!)_{n,k\geq 0}$.

We will explain these examples with relevant citations in Section 2.1. An example of a matrix that has been conjectured to be totally positive is the triangle of Eulerian numbers $\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0}$ [Bre96]; here $\left\langle {n+1 \atop k} \right\rangle$ is the number of permutations on $[n+1]$ with k descents. We discuss this conjecture in more detail in Chapter 3.

In the spirit of introducing statistics, one might be interested in studying matrices whose entries are polynomials in one or several variables with nonnegative integer coefficients where the variables keep track of the distribution of one or several statistics. Thus, it makes sense to extend the notion of total positivity to matrices with polynomial entries.

Definition 1.1.2. We say that a polynomial with real coefficients (in one or several variables) is *coefficientwise-positive*, if all its coefficients are nonnegative. A matrix whose entries are polynomials with real coefficients is said to be *coefficientwise-totally positive* (*coefficientwise-TP* in short) if all its minors, which are polynomials themselves, are coefficientwise-positive.

A *bidiagonal matrix* is one where every entry is zero except for the entries in two consecutive diagonals. A bidiagonal matrix whose entries are polynomials with non-negative coefficients is an example of matrix that is coefficientwise-totally positive; this is not difficult to prove. Another example is the lower-triangular matrix $\left(\binom{n}{k}x^k y^{n-k}\right)_{n,k\geq 0}$. We will see more on these examples in Section 2.1.3.

Now let A be a lower-triangular matrix

$$A = \begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ \vdots & & & \ddots \end{bmatrix} \tag{1.2}$$

and let $A_n(x)$ be the n-th row-generating polynomial of A, i.e.,

$$A_n(x) \;=\; \sum_{k=0}^{n} a_{nk}x^k. \tag{1.3}$$

One can then ask the following different questions:

1a. Is the matrix A totally positive?

1b. Is the reversal of A, $A^{\text{rev}} := (a_{n,n-k})_{n,k\geq 0}$ totally positive? (Here $a_{n,k} = 0$ when $k < 0$.)

When the rows of A are palindromic, questions 1a. and 1b. are the same.

2. Is the lower-triangular Toeplitz matrix of the n-th row sequence, $(a_{n,i-j})_{i,j\geq 0}$, totally positive?

3. Is the Hankel matrix of the sequence $\big(A_n(x)\big)_{n\geq 0}$, which is the matrix $\big(A_{n+k}(x)\big)_{n,k\geq 0}$, coefficientwise-totally positive with respect to the variable x?

For several important combinatorial matrices, the answer to all four of these questions have either been proven to be true or are conjectured to be true. Some examples of matrices where all four questions are provably true are the binomial triangle, the Stirling cycle triangle and the Stirling subset triangle. Additionally, for the Eulerian triangle the questions 1a. and 1b. are conjectured to be true, while the answer to questions 2. and 3. is provably true. We will see more on these along with the relevant citations in Section 2.1.6.

The total positivity of a Toeplitz matrix depends on the distribution of the zeroes and poles of the associated ordinary generating function. In the particular case of question 2., the ordinary generating function $A_n(x)$ is a polynomial and the total positivity of the matrix $(a_{n,i-j})_{i,j\geq 0}$ is equivalent to the polynomial $A_n(x)$ being negative-real-rooted, i.e. all its (complex) zeroes lie in $(-\infty, 0]$. The study of real-rooted polynomials has been important in combinatorics, see for example [Brä15]. We will see more on this in Section 2.1.5.

On the other hand, the total positivity of a Hankel matrix is related to completely different behaviour of the associated ordinary generating function. This is where the study of continued fractions come in. More precisely, given a sequence of real numbers $(b_n)_{n\geq 0}$ its Hankel matrix $(b_{n+k})_{n,k\geq 0}$ is totally positive if and only if there exist real numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that the following identity holds (at the level of formal power series):

$$\sum_{n=0}^{\infty} b_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{\ddots}}} \,. \tag{1.4}$$

This result is combination of results due to Stieltjes [Sti94] and Gantmakher–Krein [GK37]. The continued fraction of the form as the one on the right-hand side of equation (1.4) is called a *Stieltjes-type continued fraction* (S-fraction in short).

If $(b_n)_{n\geq 0}$ is instead a sequence of polynomials in one or several variables, we no longer have any equivalent condition available to us. However, if an S-fraction such as the one in equation (1.4) exists where the $\alpha_0, \alpha_1, \ldots$ are all polynomials with non-negative coefficients, then that is a *sufficient* condition for the Hankel matrix $(b_{n+k})_{n,k\geq 0}$ to be coefficientwise-totally positive (see Theorem 2.1.3 below). However, there are other methods to prove coefficientwise total positivity for Hankel matrices such as existence of Thron-type continued fractions, branched S-fractions and the production-matrix method. We will employ some of these methods in this thesis. We will see more on total positivity and coefficientwise-total positivity of Hankel matrices in Sections 2.1.4 and 2.3.

Our contributions are presented in Chapters 3–6, where we try to attempt to answer some of the above four questions for some interesting combinatorial triangles. In some cases, we are successful in proving these, thereby obtaining new results. In others, we generalise existing conjectures and thus provide new conjectures. In Chapter 3, we will generalise questions 1a. and 1b. for the triangle of Eulerian numbers $\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0}$ thereby generalising a conjecture of [Bre96]. However, we can prove question 1b. for the Stirling subset triangle $\left(\left\{ {n \atop k} \right\}\right)_{n,k\geq 0}$. In Chapters 4 and 5 our problems are generalisations of question 3. Finally, in Chapter 6, we will ask all of these questions for some generalisations of the Stirling cycle and subset numbers. We will present some conjectures and prove some special cases of our conjectures.

1.2 Continued fractions and combinatorics

If $(a_n)_{n\geq 0}$ is a sequence of combinatorial numbers or polynomials with $a_0 = 1$, it is often fruitful to seek to express its ordinary generating function as a continued fraction. The most commonly studied types of continued fractions are Stieltjes-type (S-fractions),

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}}, \tag{1.5}$$

and Jacobi-type (J-fractions),

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cfrac{\beta_3 t^2}{1 - \cdots}}}}. \tag{1.6}$$

(In Section 1.1, we have already mentioned that the existence of S-fractions is important in the study of totally positive Hankel matrices and we will see this in even more detail in Section 2.1.4.) This line of investigation goes back at least to Euler [Eul60, Eul55a], but it gained impetus following Flajolet's influential paper [Fla80]. We recall that even though the ordinary generating function $\sum_{i=0}^{\infty} n! t^n$ does not have a nice closed-form expression, it does have the nice S-fraction stated in equation (1.1). Using some contraction formulae one can transform S-fractions into J-fractions, we will see this in Section 2.2.2.

The work that we present in Chapters 4 and 5 can be viewed as a sequel to the work of Sokal and Zeng [SZ22] which presented Stieltjes-type and Jacobi-type continued fractions for some "master polynomials" that enumerate permutations, set partitions or perfect matchings with respect to a large (sometimes infinite) number of independent statistics. In this paper, the authors systematized the study of the "linear family": namely, sequences in which the S-fraction coefficients $(\alpha_n)_{n\geq 1}$ grow linearly in n. More precisely, in the simplest case for permutations [SZ22, Theorem 2.1] the even and odd coefficients grow affinely in n:

$$\alpha_{2k-1} \;=\; x+(k-1)u, \qquad \alpha_{2k} \;=\; y+(k-1)v \tag{1.7}$$

When $x=y=u=v=1$, these coefficients $\alpha_{2k-1}=\alpha_{2k}=k$ correspond to Euler's continued fraction (1.1) for $n!$; so it is natural to expect that the resulting polynomials $P_n(x,y,u,v)$ can be interpreted as enumerating permutations of $[n]$ with respect to some suitable statistics.

In Chapter 4 we take one step up in complexity, to consider the "quadratic family", in which the $(\alpha_n)_{n\geq 1}$ grow quadratically in n. For instance, we could consider

$$\alpha_{2k-1} \;=\; [x_1+(k-1)u_1]\,[x_2+(k-1)u_2], \tag{1.8a}$$

$$\alpha_{2k} \;=\; [y_1+(k-1)v_1]\,[y_2+(k-1)v_2] \tag{1.8b}$$

With all parameters set to 1, these coefficients $\alpha_{2k-1}=\alpha_{2k}=k^2$ correspond to the continued fraction [Vie, eq. (9.7)] [Vie83, p. V-15] for the median Genocchi numbers [OEI19, A005439]; so it is natural to seek a combinatorial model that is enumerated by the median Genocchi numbers. We shall focus on a class of permutations of $[2n]$ called *D-permutations*. These were introduced by Lazar and Wachs [LW22, Laz20] and are defined by imposing some constraints concerning the parity (even/odd) of excedances and anti-excedances. Our most general results in Chapter 4 will involve a less commonly studied type of continued fraction called Thron-type continued fraction (T-fraction). We will enumerate D-permutations with respect to a large (sometimes infinite) number of independent statistics.

The continued fractions in [SZ22] and in Chapter 4 have been classified as "first" or "second" depending on whether they do not or do involve the count of cycles. In both of these works, the "second" continued fractions were proven using two specialisations, but conjectured using one specialisation. In Chapter 5, we prove the one-specialisation conjecture of Sokal–Zeng [SZ22, Conjecture 2.3] and then we use similar proof techniques to prove our corresponding one-specialisation conjecture for D-permutations (see Theorem 4.3.1 below). We will also prove a continued fraction that was conjectured by Randrianarivony and Zeng in 1996 [RZ96a, Conjecture 12] for D-o-semiderangements[2] (a subclass of D-permutations).

The continued fractions in Chapters 4 and 5 are obtained bijectively. We employ different bijections from permutations and D-permutations to labelled Motzkin and Schröder paths, respectively. Our bijections are variants of or are motivated by a bijection of Foata and Zeilberger [FZ90], and another bijection of Biane [Bia93]. The continued fractions are then immediately obtained by using Flajolet's general theory [Fla80] of interpreting S or J-fractions as weighted Dyck or Motzkin paths and its generalisation [FG17, OdJ15, JV17, Sok, PS20] to interpreting Thron-type continued fractions as weighted Schröder paths. We will review this in Sections 2.2.4 and 2.2.5. Finally, we carefully read off the various statistics in our bijections.

1.3 Outline of thesis

In Chapter 2, we will provide fundamental definitions and facts for total positivity and continued fractions in the context of enumerative combinatorics, and we set up notation for the rest of this thesis. The remaining four chapters discuss four different projects undertaken by the author, some of which are based on joint work with other researchers.

Chapter 3 is based on joint work with Xi Chen, Alexander Dyachenko, Tomack Gilmore and Alan D. Sokal [CDD$^+$21, CDD$^+$]. We exhibit a lower-triangular matrix of polynomials in six indeterminates that appears empirically to be coefficientwise totally positive, and which includes as a special case the Eulerian triangle. This generalises Brenti's conjecture from 1996 [Bre96]. We prove the

[2] In their paper [RZ96a], Randrianarivony and Zeng call these Genocchi permutations. We will explain our nomenclature in Section 2.4.3.

coefficientwise total positivity of a three-variable case which includes the reversed Stirling subset triangle. We do this by constructing a planar digraph and then use the Lindström–Gessel–Viennot lemma. We provide two proofs, one using a recurrence and the second is a direct bijection between paths in the digraph and set partitions.

Chapter 4 is on the study of continued fractions for Genocchi and median Genocchi numbers and is based on joint work with Alan D. Sokal [DS22]. The Genocchi and median Genocchi numbers have S-fractions whose coefficients grow quadratically, see (2.45) and (2.51), respectively. We find Stieltjes-type and Thron-type continued fractions for some multivariate polynomials that enumerate D-permutations, a class of permutations of $2n$, with respect to a very large (sometimes infinite) number of simultaneous statistics that measure cycle status, record status, crossings and nestings.

Our next chapter (Chapter 5) is based on [Deb23] which is the author's solo work. In this chapter, we interpret a bijection due to Foata and Zeilberger [FZ90] in terms of Laguerre digraphs [FS84, Sok22], which enables us to count cycles in permutations. Using this interpretation, we obtain Jacobi-type continued fractions for multivariate polynomials enumerating permutations, and also Thron-type and Stieltjes-type continued fractions for multivariate polynomials enumerating D-permutations, in both cases including the counting of cycles. This enables us to prove some conjectured continued fractions due to Sokal–Zeng from 2022 [SZ22], and Randrianarivony–Zeng from 1996 [RZ96a].

Our final chapter (Chapter 6) is based on joint work with Alan D. Sokal [DS]. Here we introduce the higher-order Stirling cycle and subset numbers; these generalise the Stirling cycle and subset numbers, respectively. We then ask the four questions for lower-triangular matrices introduced in Section 1.1 and thus, introduce some new conjectures; we then answer some of these.

Chapter 2

Preliminaries

This chapter provides the fundamental definitions and facts, and sets up notation for the rest of this thesis. This chapter is organised as follows: we begin by revisiting the notion of total positivity in Section 2.1 in which we mention basic definitions, provide examples and basic facts, and mention some proof techniques. Then in Section 2.2 we introduce continued fractions and then state associated formulae, and also the link to combinatorics. In Section 2.3 we state some results on coefficientwise Hankel-total positivity. We then define Genocchi and median Genocchi numbers, and D-permutations in Section 2.4 which will play a central role in Chapters 4 and 5. Finally, we define various permutation statistics that will play an important role in this thesis (Section 2.5).

2.1 Total positivity

We will recall the basic definitions of total positivity, provide examples and basic facts. We begin in Section 2.1.1 with the basic definitions, some examples and then mention operations that do and do not preserve total positivity. We then state a theorem due to Loewner and Whitney in Section 2.1.2. In Section 2.1.3, we introduce the upgraded notion of coefficientwise total positivity. We then focus our attention to the total positivity of some specific matrices: in Section 2.1.4 we look at Hankel matrices and this is where continued fractions come into the picture, and in Section 2.1.5 we look at Toeplitz matrices and we show the connection to real-rooted polynomials. We then ask some general questions on the total positivity of various kinds for any lower-triangular matrix in Section 2.1.6, we provide some examples where the answer to all four of these questions is either known or is conjectured to be true. Finally, in Section 2.1.7 we provide an overview of the celebrated Lindström–Gessel–Viennot lemma and mention how it is a very powerful tool to prove total positivity.

2.1.1 Definition and examples

We defined the notion of total positivity in definition 1.1.1 which we recall here: a matrix of real numbers is said to be *totally positive* (TP in short) if all its minors are nonnegative. We say that a matrix is *strictly totally positive* (STP in short) if all its minors are strictly positive.

In this thesis, rows and columns of infinite size will be indexed from 0, and finite rows and columns will be indexed from 1. The following are some standard examples of totally positive matrices:

1. *Diagonal matrices* with nonnegative diagonal entries constitute a family of TP matrices. For real numbers $\boldsymbol{d} = (d_i)_{i\geq 0}$, let $D(\boldsymbol{d})$ denote the diagonal matrix with entries $d_0, d_1, d_2, \ldots$.

2. *Bidiagonal matrices* are our next example. These are matrices with all entries 0, other than the entries in two consecutive diagonals. Bidiagonal matrices with nonnegative entries are totally positive as any submatrix is upper or lower triangular, and hence the corresponding minor is the product of its diagonal entries all of which are nonnegative.

   By a *lower-bidiagonal matrix* (*upper-bidiagonal matrix*), we shall refer to a matrix whose non-zero entries are only allowed to be on the diagonal or the first subdiagonal (first superdiagonal). For $\boldsymbol{d} = (d_i)_{i\geq 0}$ and $\boldsymbol{x} = (x_i)_{i\geq 1}$, let $L(\boldsymbol{d},\boldsymbol{x})$ be the lower-bidiagonal matrix with entries $d_0, d_1, d_2, \ldots$ on the diagonal and entries $x_1, x_2, \ldots$ on the subdiagonal and zero elsewhere, i.e.,

$$L(\boldsymbol{d},\boldsymbol{x}) = \begin{bmatrix} d_0 & 0 & 0 & 0 & \dots \\ x_1 & d_1 & 0 & 0 & \dots \\ 0 & x_2 & d_2 & 0 & \dots \\ 0 & 0 & x_3 & d_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Similarly, let $U(\boldsymbol{d},\boldsymbol{x})$ be the upper bidiagonal matrix with entries $d_0, d_1, d_2, \ldots$ on the diagonal and entries $x_1, x_2, \ldots$ on the superdiagonal and zero elsewhere. Thus, $U(\boldsymbol{d},\boldsymbol{x}) = L(\boldsymbol{d},\boldsymbol{x})^\intercal$.

We abbreviate $L(\mathbf{1},\boldsymbol{x})$ as $L(\boldsymbol{x})$ and $U(\mathbf{1},\boldsymbol{x})$ as $U(\boldsymbol{x})$ where $\mathbf{1}$ is the all-ones vector.

Given two totally positivity matrices A and B, we can show using the Cauchy–Binet formula that their product AB (if it exists) is also totally positive (see for e.g. [FJ11, Section 1.1]). This fact is so fundamental to the theory of total positivity that we shall henceforth use it without comment. However, the following matrix operations do not preserve total positivity:

1. The sum of two TP matrices need not be TP. For example, consider the two matrices,
$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$
Clearly A and B are both TP, but $\det(A+B) = -1 < 0$ and hence $A+B$ is not.

2. The Hadamard product (also called the entry-wise product) of two TP matrices need not be TP. For example, consider the two matrices,
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$
Their Hadamard product has determinant -1. However, the Hadamard product

preserves the non-negativity of the minors of size two.

Some other examples of totally positive matrices are the binomial triangle $B \stackrel{\text{def}}{=} \left(\binom{n}{k}\right)_{n,k\geq 0}$, the binomial square $\left(\binom{n+k}{k}\right)_{n,k\geq 0} = B \cdot B^{\mathsf{T}}$, the lower-triangular matrices of Stirling cycle numbers and the lower-triangular matrices of Stirling subset numbers, and the Hankel matrix of factorials

$$((n+k)!)_{n,k\geq 0} = D\big((n!)_{n\geq 0}\big) \cdot B \cdot B^{\mathsf{T}} \cdot D\big((n!)_{n\geq 0}\big) \,. \tag{2.1}$$

See for e.g. [Bre95, Bre96] for proofs of total positivity of the binomial, Stirling cycle and subset triangles.

2.1.2 Loewner–Whitney theorem and Neville elimination

For a fixed positive integer n, let $\mathcal{G} = \mathrm{GL}_n(\mathbb{R})$ be the set of all $n \times n$ invertible matrices with real entries. Let $\mathcal{G}_{\geq 0}$ be the subset consisting of all invertible TP matrices. Clearly, $\mathcal{G}_{\geq 0}$ is a submonoid of $\mathcal{G}$ under matrix multiplication. We know that invertible diagonal and bidiagonal matrices with nonnegative entries are elements of $\mathcal{G}_{\geq 0}$. Consider the following subset of diagonal and bidiagonal matrices:

$$\begin{aligned}
D_i(x) &= D(\underbrace{1,\ldots,1}_{i-1},x,\underbrace{1,\ldots,1}_{n-i}), && (2.2a)\\
L_i(x) &= L(\underbrace{0,\ldots,0}_{i-1},x,\underbrace{0,\ldots,0}_{n-i-1}), && (2.2b)\\
U_i(x) &= (L_i(x))^{\mathsf{T}}\,. && (2.2c)
\end{aligned}$$

For $x > 0$, [Whi52] first noticed and [Loe55] first stated that these matrices generate the entire monoid $\mathcal{G}_{\geq 0}$.

Theorem 2.1.1 ([Whi52, Loe55] Loewner–Whitney Theorem)**.** *For $x > 0$, the matrices $D_i(x)$, $L_i(x)$ and $U_i(x)$ generate $\mathcal{G}_{\geq 0}$.*

In fact, their proof is constructive: given $A \in \mathcal{G}_{\geq 0}$, they provide a factorisation of A into $D_i(x)$, $L_i(x)$ and $U_i(x)$. [GP96] studied the factorisations of TP matrices in depth and gave an efficient polynomial-time algorithm. In fact, their algorithm, which is often referred to as Neville elimination, is more general and works for singular matrices as well. See Section 6.4 in [Pin09] for historical remarks on this entire story. See [Gil21, Section 2.1] for more on Neville elimination and the relation to planar digraphs. This algorithm has been very useful for our experiments and we employ it to efficiently test the total positivity of matrices of real numbers.

2.1.3 Coefficientwise total positivity

We first recall the notion of coefficientwise-total positivity from Definition 1.1.2: we say that a polynomial with real coefficients (in one or several variables) is *coefficientwise-positive*, if all its coefficients are nonnegative. A matrix whose entries are polynomials with real coefficients is said to be *coefficientwise-totally positive* (*coefficientwise-TP* in short) if all its minors, which are polynomials themselves, are coefficientwise-positive.

Now we define the related notion of pointwise-total positivity:

Definition 2.1.1. We say that a polynomial (in one or several variables) is *pointwise-positive*, if it is nonnegative when substituted with non-negative real numbers. A matrix whose entries are polynomials with real coefficients is said to be *pointwise-totally positive* (*pointwise-TP* in short) if all its minors are pointwise-positive.

Note that coefficientwise-positivity implies pointwise-positivity. However, the converse is not true. For example, $x^2 - 2x + 1$ is not coefficientwise-positive even though it is pointwise-positive. Similarly, coefficientwise-total positivity implies pointwise-total positivity but the converse is not true.

For indeterminates $\boldsymbol{d}, \boldsymbol{x}$ the diagonal matrix $D(\boldsymbol{d})$, and the bidiagonal matrices $L(\boldsymbol{d}, \boldsymbol{x})$ and $U(\boldsymbol{d}, \boldsymbol{x})$ are clearly coefficientwise-TP with respect to the variables $\boldsymbol{d}, \boldsymbol{x}$. Also, the product of two coefficientwise-TP matrices is also coefficientwise-TP.

Another example of a matrix which is coefficientwise-TP is the weighted binomial matrix $B(x,y) \stackrel{\text{def}}{=} \left(\binom{n}{k} x^k y^{n-k}\right)_{n \geq 0, k \geq 0}$. In fact, entries satisfy the following identity

$$\binom{n}{k} x^k y^{n-k} = x\left(\binom{n-1}{k-1} x^{k-1} y^{n-k}\right) + y\left(\binom{n-1}{k} x^k y^{n-k-1}\right). \tag{2.3}$$

This gives us the matrix identity

$$B(x,y) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B(x,y) \end{array}\right] \cdot L\big((x,x,x,\ldots),(y,y,y,\ldots)\big) \tag{2.4}$$

which by induction shows that $B(x,y)$ is coefficientwise-TP in the variables x and y.

2.1.4 Total positivity of Hankel matrices, Stieltjes-moment problem, and continued fractions

We will now focus on Hankel matrices and state some equivalent conditions for these matrices to be totally positive. This will show the connection to continued fractions and motivate us to study continued fractions.

Given a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$, the matrix

$$H_\infty(\boldsymbol{a}) \stackrel{\text{def}}{=} (a_{n+k})_{n,k\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \\ a_2 & a_3 & a_4 & \\ \vdots & & & \ddots \end{bmatrix} \tag{2.5}$$

is called its *Hankel matrix*. If $\boldsymbol{a}$ is a sequence of real numbers, we say that $\boldsymbol{a}$ is *Hankel totally positive* (*Hankel-TP* in short) if $H_\infty(\boldsymbol{a})$ is TP. Similarly, if $\boldsymbol{a}$ is a sequence of polynomials with real coefficients, we say that $\boldsymbol{a}$ is *coefficientwise-Hankel-totally positive* (*coefficientwise-Hankel-TP* in short) if the matrix $H_\infty(\boldsymbol{a})$ is coefficientwise-TP.

The following theorem provides equivalent conditions for $\boldsymbol{a}$ to be Hankel-TP when $\boldsymbol{a}$ is a sequence of real numbers.

Theorem 2.1.2 ([Sti94], [GK37]). *For a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ of real numbers, the following are equivalent:*

1. *The Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.*

2. *There exists a positive measure* μ *supported on* $[0,\infty)$ *such that*

$$a_n = \int_0^\infty x^n d\mu(x) \tag{2.6}$$

for all $n \geq 0$*, i.e.* $\boldsymbol{a}$ *is a Stieltjes-moment sequence.*

3. *There exists real numbers* $\alpha_0, \alpha_1, \ldots \geq 0$ *such that the following formal power series identity holds:*

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{\ddots}}}\,. \tag{2.7}$$

The equivalence of *2.* and *3.* is due to Stieltjes [Sti94]; he also proved that the positive-semidefiniteness of the Hankel matrix $H_\infty(\boldsymbol{a})$ and the corresponding once-shifted Hankel matrix (this is *weaker* than the total positivity of $H_\infty(\boldsymbol{a})$) implies *2.* Gantmakher and Krein [GK37] proved that *2.* implies *1.*

Thus, Theorem 2.1.2 relates total positivity to the moment problem and to the study of continued fractions. Continued fractions having the form shown on the right-hand side of equation (2.7) are known as Stieltjes-type continued fractions, or S-fractions in short. For example let us look at the situation when $a_n = n!$:

(a) We showed in equation (2.1) that the Hankel matrix $H_\infty((n!)_{n\geq 0})$ is totally positive.

(b) The sequence $(n!)_{n\geq 0}$ can be obtained as the sequence of moments of the measure $e^{-x}dx$, i.e., we have

$$n! \;=\; \int_0^\infty x^n e^{-x} dx\,. \tag{2.8}$$

(c) We recall the continued fraction of Euler [Eul60] which we stated in (1.1):

$$\sum_{n=0}^{\infty} n!t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cfrac{3t}{1-\cfrac{3t}{1-\ddots}}}}}}} . \tag{2.9}$$

Let us now consider the situation when $\boldsymbol{a}$ is a sequence of polynomials. In this situation, *2.* in Theorem 2.1.2 no longer makes sense. However, we can still talk about *1.* and *3.* by replacing Hankel-TP with coefficientwise-Hankel-TP in *1.*, and if $\alpha_0, \alpha_1, \ldots$ are a sequence of coefficientwise-positive polynomials. In this situation, the following theorem holds:

Theorem 2.1.3 ([Fla80], [Vie83], [Sok14]). *Let $\boldsymbol{a} = (a_n)_{n\geq 0}$ and $\alpha_0, \alpha_1, \ldots$ be two sequences of polynomials. If the polynomials $\alpha_0, \alpha_1, \ldots$ are coefficientwise-positive and they satisfy the following power series identity:*

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{\ddots}}} \tag{2.10}$$

then $\boldsymbol{a}$ is coefficientwise-Hankel totally positive.

The above theorem was first stated in this form by Sokal in [Sok14]. However, it is an easy corollary of [Fla80] and of [Vie83, Section 3, Chapter 4].

The converse of Theorem 2.1.3 is not true for two reasons. First of all, if $\boldsymbol{a}$ is a sequence of polynomials that is coefficientwise Hankel-totally positive, there may not exist *any* continued fraction (2.10) with coefficients $\alpha_0, \alpha_1, \ldots$ *in the ring of polynomials*; in general the coefficients $\alpha_0, \alpha_1, \ldots$ are rational functions. And secondly, even if a continued fraction (2.10) does exist in the ring of polynomials, the coefficientwise-positivity of $\alpha_0, \alpha_1, \ldots$ is merely a *sufficient* condition for the coefficientwise Hankel-total positivity of $\boldsymbol{a}$, not a necessary one. This has been discussed in [PSZ18, Section 6.1]. We will see more on continued fractions in Section 2.2. The above theorem has also been generalised to T-fractions and branched continued fractions. We will state these results in Section 2.3.

2.1.5 Total positivity of Toeplitz matrices

We will now focus on Toeplitz matrices and state an equivalent condition for these matrices to be totally positive. This will motivate us to also consider the distribution of zeroes of polynomials.

Let $\boldsymbol{a} = (a_n)_{n\geq 0}$ be a sequence of real numbers. The *Toeplitz matrix* of $\boldsymbol{a}$ is defined to be the lower-triangular matrix

$$T_\infty(\boldsymbol{a}) \stackrel{\text{def}}{=} (a_{n-k})_{n,k\geq 0} = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ a_2 & a_1 & a_0 & \\ \vdots & & & \ddots \end{bmatrix} \tag{2.11}$$

where $a_{n-k} = 0$ when $n < k$. We say that the sequence $\boldsymbol{a}$ is *Toeplitz-totally positive* (*Toeplitz-TP* in short) if $T_\infty(\boldsymbol{a})$ is totally positive[1].

The following theorem provides an equivalent condition for $\boldsymbol{a}$ to be Toeplitz-TP.

Theorem 2.1.4 (Aissen–Schoenberg–Whitney ([ASW52]), Edrei ([Edr52]))**.** *For a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ of real numbers with $a_0 = 1$, the following are equivalent:*

1. *The sequence $\boldsymbol{a}$ is Toeplitz-totally positive.*

2. *There exists sequences $\boldsymbol{\alpha} = (\alpha)_{n\geq 1}, \boldsymbol{\beta} = (\beta_n)_{n\geq 1}$ with $\alpha_n \geq 0$, $\beta \geq 0$, and a number $\gamma \geq 0$ such that*

$$f_{\boldsymbol{a}}(t) = e^{\gamma t} \cdot \frac{\prod_{n=1}^{\infty}(1+\alpha_n t)}{\prod_{n=1}^{\infty}(1-\beta_n t)} \,. \tag{2.12}$$

[1] These sequences have also been called Pólya-frequency sequences in the literature.

Note that if all but finitely many terms of the sequence $\boldsymbol{a}$ is 0, then $f_{\boldsymbol{a}}(t)$ is a polynomial. In this case, Theorem 2.1.4 says that $\boldsymbol{a}$ is Toeplitz-TP if and only if all the zeroes of $f_{\boldsymbol{a}}(t)$ are real and non-positive.

Remark. If $\boldsymbol{a}$ is instead a sequence of polynomials, then the ordinary generating function $f_{\boldsymbol{a}}(t)$ need not (and usually does not) have a representation (2.12) with $\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma$ being coefficientwise-positive polynomials. However, it can be shown that if they exist and are coefficientwise-positive, then the sequence $\boldsymbol{a}$ is coefficientwise Toeplitz-TP [Sok]. ■

2.1.6 Three total positivity questions for lower-triangular matrices

We now revisit the three total-positivity questions for lower-triangular matrices which were asked in Section 1.1 and we provide citations containing proofs or conjectures for our examples.

Recall that A is the lower-triangular matrix

$$A = \begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ \vdots & & & \ddots \end{bmatrix}$$

and $A_n(x)$ is the n-th row-generating polynomial of A, i.e.,

$$A_n(x) = \sum_{k=0}^{n} a_{nk} x^k. \tag{2.13}$$

We shall refer to a lower-triangular matrix as a *triangle*.

We then asked the following questions:

1a. Is the matrix A totally positive?

1b. Is the reversal of A, $A^{\text{rev}} := (a_{n,n-k})_{n,k \geq 0}$ totally positive? (Here $a_{n,k} = 0$ when $k < 0$.)

When the rows of A are palindromic, questions 1a. and 1b. are the same.

2. Are $A_n(x)$ negative-real-rooted, i.e., is the n-th row of A Toeplitz-totally positive?

3. Is the sequence $\big(A_n(x)\big)_{n\geq 0}$ coefficientwise Hankel-totally positive in the variable x?

For several important combinatorial triangles, the answer to all four of these questions have either been proven to be true or are conjectured to be true. Some examples are:

1. When $A = \big(\binom{n}{k}\big)_{n,k\geq 0}$ is the matrix of binomial coefficients, the matrix A is known to be totally positive (this follows from equation (2.4) specialised to $x = y = 1$). In this situtation, $A^{\text{rev}} = A$ as the rows are palindromic. The row-generating polynomials $A_n(x) = (x+1)^n$ are clearly negative-real-rooted. And finally, the sequence $\big((x+1)^n\big)_{n\geq 0}$ is coefficientwise Hankel-totally positive in the variable x; this is because they have the continued fraction $\sum_{n=0}^{\infty}(x+1)^n t^n = 1/(1-(1+x)t)$.

2. When $A = \big(\genfrac{[}{]}{0pt}{}{n}{k}\big)_{n,k\geq 0}$ is the matrix of Stirling cycle numbers, which count the number of permutations on $[n]$ with k cycles. The total positivity of the matrix A and its reversal A^{rev} were shown by Brenti in 1995 [Bre95]; they hold because the entries satisfy a binomial-like recurrence with nonnegative n-dependent coefficients. The row generating polynomials $A_n(x) = x(x+1)\cdots(x+n-1)$ are clearly real-rooted with non-positive roots. The sequence $(A_n(x))_{n\geq 0}$ has a Stieltjes-type continued fraction discovered by Euler in 1760 [Eul60], also see [SZ22, eq. (2.2), (2.6)].

3. When $A = \big(\genfrac{\{}{\}}{0pt}{}{n}{k}\big)_{n,k\geq 0}$ is the matrix of Stirling subset numbers, which count the number of set partitions on $[n]$ with k blocks. The total positivity of the matrix A is due to Brenti from 1995 [Bre95]; it holds because the entries $\genfrac{\{}{\}}{0pt}{}{n}{k}$ satisfy a binomial-like recurrence with nonnegative k-dependent coefficients. However, the total positivity of the reversal A^{rev} was only recently shown by us in [CDD$^+$21], and we will see this in greater detail in Chapter 3. The real-rootedness of the row generating polynomials $A_n(x)$ is due to Harper from

1967 [Har67]. The sequence $(A_n(x))_{n\geq 0}$ has a Stieltjes-type continued fraction, discovered by various people in the second half of the twentieth century, see [SZ22, eq. (3.2), (3.5) and footnote 19].

4. When $A = \left(\left\langle {n \atop k} \right\rangle\right)_{n,k\geq 0}$, is the matrix of Eulerian numbers, which count the number of permutations on $[n]$ containing k descents. The row-generating polynomials $A_n(x)$ were shown to be real-rooted by Frobenius in 1910 [Fro10]. The sequence $(A_n(x))_{n\geq 0}$ has a Stieltjes-type continued fraction, going back to Stieltjes [Sti94], also see [SZ22, eq. (2.2), (2.8) and footnote 4]. In 1996, Brenti [Bre96] conjectured that the matrix A is totally positive[2]. In this case, A and A^{rev} are not palindromic but the total positivity of one would imply the total positivity of the other. We will generalise Brenti's conjecture in Chapter 3.

We will study these questions for some new triangles generalising the Stirling cycle and subset triangles in Chapter 6.

2.1.7 Lindström–Gessel–Viennot Lemma

An often-used tool to prove total positivity is the well-known Lindström–Gessel–Viennot lemma (LGV lemma in short), which is a lemma that has been discovered and rediscovered several times most notably in [GV89]. We shall now provide a brief description of the LGV lemma in a setting that is useful for proving total positivity. See for example [AZ18, Chapter 32] for a proof of the LGV lemma.

Let $\mathcal{D}$ be a digraph and let R be a commutative ring with identity. Let each edge e of $\mathcal{D}$ be assigned a weight $w_e \in R$. Let $\mathcal{U} = \{u_0, u_1, \ldots\}$ and $\mathcal{V} = \{v_0, v_1, \ldots\}$ be two distinguished sets of vertices which we call the sources and sinks, respectively. For a path $\mathcal{P}$ in $\mathcal{D}$, we write $\mathcal{P} : u_n \to v_k$ to mean that $\mathcal{P}$ starts at the source vertex u_n and ends at the sink vertex v_k. The weight of a path $\mathcal{P}$ is the product of the weights

[2] Dyachenko [Dya] has verified this conjecture for the first 512×512 submatrix of the matrix $\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0}$.

of its edges and we denote it as $\mathrm{wt}(\mathcal{P})$. Now let

$$P(u_n \to v_k) \stackrel{\text{def}}{=} \sum_{\mathcal{P}: u_n \to v_k} \mathrm{wt}(\mathcal{P})\,. \tag{2.14}$$

We define the path matrix of $(\mathcal{D},\mathcal{U},\mathcal{V})$ to be the matrix

$$\boldsymbol{P}(\mathcal{D},\mathcal{U},\mathcal{V}) \stackrel{\text{def}}{=} \left(P(u_n \to v_k)\right)_{n\geq 0, k\geq 0}. \tag{2.15}$$

We call this set-up an *LGV network*.

Let $I = \{n_1, \dots, n_r\}$ and $J = \{k_1, \dots, k_r\}$ be two sets of indices. From the usual definition of determinant of a matrix we have that

$$\begin{aligned} \det \boldsymbol{P}(\mathcal{D},\mathcal{U},\mathcal{V})_{I,J} &= \det\left(P(u_{n_i} \to v_{k_i})\right)_{1\leq i\leq r} \\ &= \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sign}(\sigma) \prod_{i=1}^{r} P(u_{n_i} \to v_{k_{\sigma(i)}}) \\ &= \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sign}(\sigma) \sum_{(\mathcal{P}_1,\dots,\mathcal{P}_r)} \prod_{i=1}^{r} \mathrm{wt}(\mathcal{P}_i) \end{aligned} \tag{2.16}$$

where $\mathcal{P}_1 : u_{n_1} \to v_{n_{\sigma(1)}}, \dots, \mathcal{P}_r : u_{n_r} \to v_{n_{\sigma(r)}}$.

A sequence of paths $(\mathcal{P}_1, \dots, \mathcal{P}_r)$ is called *non-intersecting* if they have no vertices in common. The LGV lemma states that if the digraph $\mathcal{D}$ is acyclic, then

$$\det \boldsymbol{P}(\mathcal{D},\mathcal{U},\mathcal{V})_{I,J} = \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sign}(\sigma) \sum_{\substack{(\mathcal{P}_1,\dots,\mathcal{P}_r) \\ \text{non-intersecting}}} \prod_{i=1}^{r} \mathrm{wt}(\mathcal{P}_i) \tag{2.17}$$

where $\mathcal{P}_1 : u_{n_1} \to v_{n_{\sigma(1)}}, \dots, \mathcal{P}_r : u_{n_r} \to v_{n_{\sigma(r)}}$. Note that the only change in the expressions for the determinant in equations (2.16) and (2.17) is the phrase "non-intersecting" which may remove a lot of extra terms. We shall look at some examples soon.

We say that the triplet $(\mathcal{D},\mathcal{U},\mathcal{V})$ is *fully compatible* if for any subset of sources $\{u_{n_1}, \dots, u_{n_r}\}$, and any subset of sinks $\{v_{n_1}, \dots, v_{n_r}\}$, the only permutation $\sigma \in \mathfrak{S}_r$ which gives rise to a family of non-intersecting paths is the identity permutation. If

$(\mathcal{D},\mathcal{U},\mathcal{V})$ is fully compatible, equation (2.17) reduces to

$$\det \boldsymbol{P}(\mathcal{D},\mathcal{U},\mathcal{V})_{I,J} = \sum_{\substack{(\mathcal{P}_1,\ldots,\mathcal{P}_r) \\ \text{non-intersecting}}} \prod_{i=1}^{r} \mathrm{wt}(\mathcal{P}_i) \tag{2.18}$$

where $\mathcal{P}_1 : u_{n_1} \to v_{n_1}, \ldots, \mathcal{P}_r : u_{n_r} \to v_{n_r}$.

Let $\mathcal{D}$ be planar digraph with a planar embedding in which $\mathcal{U}$ and $\mathcal{V}$ lie on the boundary of a circle and all the other vertices lie within the circle, such that the sources $\mathcal{U}$ occur in clockwise order and the sinks $\mathcal{V}$ occur in anti-clockwise order such that the sources and sinks are not interspersed. We call $(\mathcal{D},\mathcal{U},\mathcal{V})$ with such an embedding a *planar network*.

From the descriptions of planar network and full compatibility, we can see that a planar network is fully compatible. This was first described by Brenti in [Bre95].

If the ring R is a ring of polynomials in one or several variables over $\mathbb{R}$ and if the edge weights w_e are coefficientwise-positive, then from the LGV lemma, we get that $\boldsymbol{P}(\mathcal{D},\mathcal{U},\mathcal{V})$ is coefficientwise-TP. Thus, to show that a matrix M is TP (or coefficientwise-TP), it suffices to construct a planar network $(\mathcal{D},\mathcal{U},\mathcal{V})$ whose path matrix is M. This is the set-up that one often uses to prove total positivity.

If a planar network $(\mathcal{D},\mathcal{U},\mathcal{V})$ is provided, it is not difficult to describe $\boldsymbol{P}(\mathcal{D},\mathcal{U},\mathcal{V})$. However, the other direction is difficult, i.e., if a matrix M is provided, it is highly nontrivial to construct a planar network $\mathcal{D}$ with $\boldsymbol{P}(\mathcal{D},\mathcal{U},\mathcal{V}) = M$. However, providing such a network for M, ensures total positivity of M.

A few examples of planar digraphs are given:

1. Planar network for the lower-bidiagonal matrix $L(\boldsymbol{d},\boldsymbol{x})$ is given in Figure 2.1.

2. Planar network for the weighted binomial triangle $\left(\binom{n}{k} x^k y^{n-k}\right)_{n,k\geq 0}$ is given in Figure 2.2.

2.2 Continued fractions

In this section, we will introduce some preliminaries and basic definitions on continued fractions. We begin in Section 2.2.1 with the basic definitions of classical continued fractions, which are the continued fractions of Stieltjes, Jacobi and Thron

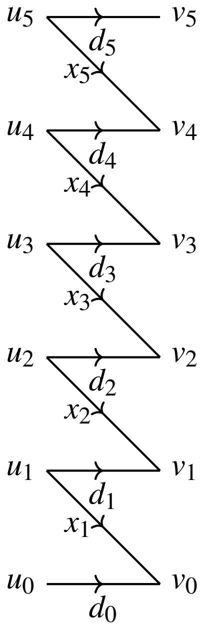

Figure 2.1: Planar network for the lower-bidiagonal matrix $L(\boldsymbol{d}, \boldsymbol{x})$.

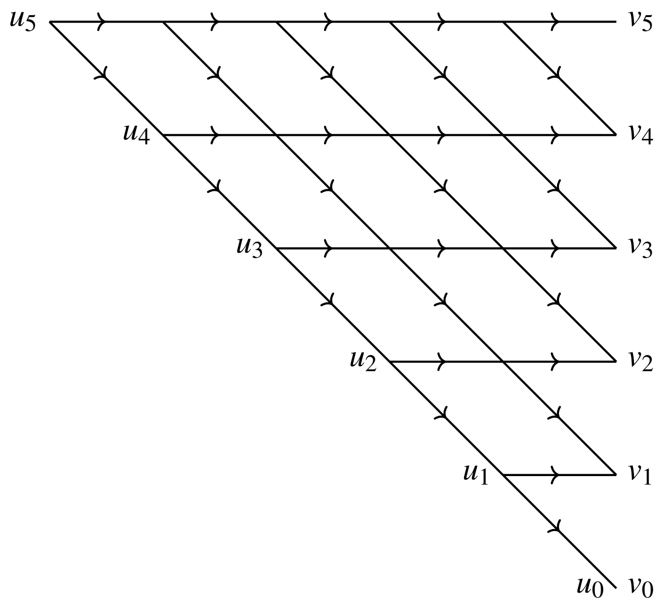

Figure 2.2: Planar network for the weighted binomial triangle $\left(\binom{n}{k} x^k y^{n-k}\right)_{n,k\geq 0}$. All the horizontal edges get weight x and all the diagonal edges get weight y.

types. We then state some contraction formulae to transform Stieltjes- and Thron-type continued fractions into Jacobi-type fractions in Section 2.2.2. In Section 2.2.3 we state some transformation formulae to transform one T-fraction to another.

Our proofs in Chapters 4 and 5 are based on Flajolet's [Fla80] combinatorial interpretation of continued fractions in terms of Dyck and Motzkin paths and its generalization [FG17, OdJ15, JV17, Sok, PS20] to Schröder paths, together with some bijections mapping combinatorial objects (in particular, D-permutations) to labelled Dyck, Motzkin or Schröder paths. We begin by reviewing briefly these two ingredients in Sections 2.2.4 and 2.2.5.

2.2.1 Classical continued fractions: S-fractions, J-fractions and T-fractions

Let $(a_n)_{n\geq 0}$ is a sequence of combinatorial numbers or polynomials with $a_0 = 1$. We recall that a continued fraction of Stieltjes-type (S-fraction) is

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}}\,, \tag{2.19}$$

and Jacobi-type (J-fraction) is

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cfrac{\beta_3 t^2}{1 - \cdots}}}}\,. \tag{2.20}$$

A less commonly studied type of continued fraction is the Thron-type (T-fraction):

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \delta_1 t - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \delta_3 t - \cfrac{\alpha_3 t}{1 - \cdots}}}}\,. \tag{2.21}$$

(Both sides of all these expressions are to be interpreted as formal power series in the indeterminate t.) This line of investigation goes back at least to Euler [Eul60, Eul55a], but it gained impetus following Flajolet's [Fla80] seminal discovery that any S-fraction (resp. J-fraction) can be interpreted combinatorially as a generating function for Dyck (resp. Motzkin) paths with suitable weights for each rise and fall (resp. each rise, fall and level step). More recently, several authors [FG17, OdJ15, JV17, Sok, PS20] have found a similar combinatorial interpretation of the general T-fraction: namely, as a generating function for Schröder paths with suitable weights for each rise, fall and long level step. These interpretations will be reviewed in Section 2.2.4 below.

2.2.2 Contraction formulae

The formulae for even and odd contraction of an S-fraction to an equivalent J-fraction are well known: see e.g. [DZ94, Lemmas 1 and 2] [Dum95, Lemma 1] for very simple algebraic proofs, and see [Vie83, pp. V-31–V-32] for enlightening combinatorial proofs based on grouping pairs of steps in a Dyck path. Here we will provide an extension of these formulae to suitable subclasses of T-fractions [Sok]:

Proposition 2.2.1 (Even contraction for T-fractions with $\delta_2 = \delta_4 = \delta_6 = \ldots = 0$)**.** We have

$$\cfrac{1}{1-\delta_1 t-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\delta_3 t-\cfrac{\alpha_3 t}{1-\cdots}}}} =$$

$$\cfrac{1}{1-(\alpha_1+\delta_1)t-\cfrac{\alpha_1\alpha_2 t^2}{1-(\alpha_2+\alpha_3+\delta_3)t-\cfrac{\alpha_3\alpha_4 t^2}{1-(\alpha_4+\alpha_5+\delta_5)t-\cfrac{\alpha_5\alpha_6 t^2}{1-\cdots}}}} \,. \qquad (2.22)$$

That is, the T-fraction on the left-hand side of (2.22) equals the J-fraction with coefficients

$$\begin{aligned} \gamma_0 &= \alpha_1 + \delta_1 && (2.23a) \\ \gamma_n &= \alpha_{2n} + \alpha_{2n+1} + \delta_{2n+1} \quad \text{for } n \geq 1 && (2.23b) \\ \beta_n &= \alpha_{2n-1}\alpha_{2n} && (2.23c) \end{aligned}$$

Here (2.22)/(2.23) holds as an identity in $\mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{\delta}_{\text{odd}}][[t]]$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$ and $\boldsymbol{\delta}_{\text{odd}} = (\delta_1, 0, \delta_3, 0, \ldots)$.

Proposition 2.2.2 (Odd contraction for T-fractions with $\delta_1 = \delta_3 = \delta_5 = \ldots = 0$)**.** We have

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \delta_4 t - \cfrac{\alpha_4 t}{1 - \cdots}}}}} =$$

$$1 + \cfrac{\alpha_1 t}{1 - (\alpha_1 + \alpha_2 + \delta_2)t - \cfrac{\alpha_2 \alpha_3 t^2}{1 - (\alpha_3 + \alpha_4 + \delta_4)t - \cfrac{\alpha_4 \alpha_5 t^2}{1 - \cdots}}} \,. \qquad (2.24)$$

That is, the T-fraction on the left-hand side of (2.24) equals 1 plus $\alpha_1 t$ times the J-fraction with coefficients

$$\begin{aligned} \gamma_n &= \alpha_{2n+1} + \alpha_{2n+2} + \delta_{2n+2} && (2.25a) \\ \beta_n &= \alpha_{2n}\alpha_{2n+1} && (2.25b) \end{aligned}$$

Here (2.24)/(2.25) holds as an identity in $\mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{\delta}_{\text{even}}][[t]]$, where $\boldsymbol{\delta}_{\text{even}} =$

$(0, \delta_2, 0, \delta_4, \ldots)$.

Both the algebraic and the combinatorial proofs of the contraction formulae for S-fractions can be easily generalized [Sok] to prove Propositions 2.2.1 and 2.2.2.

One consequence of Proposition 2.2.1 is that if two T-fractions with only odd deltas — say, one with coefficients $(\boldsymbol{\alpha}, \boldsymbol{\delta}_{\rm odd})$ and the other with coefficients $(\boldsymbol{\alpha}', \boldsymbol{\delta}'_{\rm odd})$ — give rise by contraction to the same J-fraction $(\boldsymbol{\beta}, \boldsymbol{\gamma})$, then they must be equal. In some cases this principle can be used to transform a T-fraction into an S-fraction (that is, $\boldsymbol{\delta}'_{\rm odd} = \mathbf{0}$): we will see an instance in Corollary 4.3.3 below.

By combining Propositions 2.2.1 and 2.2.2, we obtain:

Corollary 2.2.3 (Combining odd and even contraction)**.** If

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \delta_4 t - \cfrac{\alpha_4 t}{1 - \cdots}}}}}\,, \qquad (2.26)$$

then

$$\sum_{n=0}^{\infty} a_{n+1} t^n \;=\; \cfrac{a_1}{1 - \delta'_1 t - \cfrac{\alpha'_1 t}{1 - \cfrac{\alpha'_2 t}{1 - \delta'_3 t - \cfrac{\alpha'_3 t}{1 - \cdots}}}} \qquad (2.27)$$

whenever

$$\alpha_1 + \alpha_2 + \delta_2 \;=\; \alpha'_1 + \delta'_1 \qquad (2.28a)$$

$$\alpha_{2n+1} + \alpha_{2n+2} + \delta_{2n+2} \;=\; \alpha'_{2n} + \alpha'_{2n+1} + \delta'_{2n+1} \quad \text{for } n \geq 1 \qquad (2.28b)$$

$$\alpha_{2n}\alpha_{2n+1} \;=\; \alpha'_{2n-1}\alpha'_{2n} \qquad (2.28c)$$

2.2.3 Transformation formula

We now prove a useful transformation formula for T-fractions. First, a lemma:

Lemma 2.2.4. Let R be a commutative ring, and let $f(t), g(t) \in R[[t]]$. Then

$$1 + \frac{tf(t)}{1 - tf(t) - tg(t)} \;=\; \cfrac{1}{1 - \cfrac{tf(t)}{1 - tg(t)}} \tag{2.29}$$

as an identity in $R[[t]]$.

PROOF. Trivial: both sides equal $\dfrac{1 - tg(t)}{1 - tf(t) - tg(t)}$. □

Proposition 2.2.5 (Augmentation/restriction of T-fraction)**.**

$$1 + \cfrac{\delta_1 t}{1 - \delta_1 t - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \ldots}}} \;=\; \cfrac{1}{1 - \cfrac{\delta_1 t}{1 - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \ldots}}}}\,. \tag{2.30}$$

PROOF. Use the lemma with $f(t) = \delta_1$ and $g(t) = \cfrac{\alpha_1}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \ldots}}$. □

Reading the identity (2.30) from left to right, it says that if the ogf of a sequence $\boldsymbol{a} = (a_0, a_1, a_2, \ldots)$ with $a_0 = 1$ is given by a T-fraction with coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, then the ogf of the "augmented" sequence $\boldsymbol{a}' = (1, \delta_1 a_0, \delta_1 a_1, \delta_1 a_2, \ldots)$ is given by a T-fraction with coefficients $\boldsymbol{\alpha}'$ and $\boldsymbol{\delta}'$, where

$$\alpha'_1 \;=\; \delta_1 \tag{2.31a}$$

$$\alpha'_n \;=\; \alpha_{n-1} \quad \text{for } n \ge 2 \tag{2.31b}$$

$$\delta'_1 = \delta'_2 \;=\; 0 \tag{2.31c}$$

$$\delta'_n \;=\; \delta_{n-1} \quad \text{for } n \ge 3 \tag{2.31d}$$

In particular, if $\delta_2 = \delta_3 = \ldots = 0$, then the T-fraction on the right-hand side is an S-fraction. Of course, this transformation gives something interesting only when $\delta_1 \neq 0$.

Alternatively, reading the identity (2.30) from right to left, it says that if the ogf of a sequence $\boldsymbol{a}' = (a'_0, a'_1, a'_2, \ldots)$ with $a'_0 = 1$ is given by a T-fraction with coefficients $\boldsymbol{\alpha}'$ and $\boldsymbol{\delta}'$, where $\delta'_1 = \delta'_2 = 0$ (and of course $\alpha'_1 \neq 0$), then the ogf of the "restricted" sequence $\boldsymbol{a} = (a_1/\alpha'_1, a_2/\alpha'_1, \ldots)$ is given by a T-fraction with coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, where

$$\alpha_n = \alpha'_{n+1} \tag{2.32a}$$

$$\delta_1 = \alpha'_1 \tag{2.32b}$$

$$\delta_n = \delta'_{n+1} \quad \text{for } n \geq 2 \tag{2.32c}$$

2.2.4 Combinatorial interpretation of continued fractions

We will now mention Flajolet's seminal result [Fla80] on the combinatorial interpretation of continued fractions of Stieltjes and Jacobi type in terms of Dyck and Motzkin paths, respectively, and an analogous interpretation for Thron-type continued fractions in terms of Schröder paths. This will show that bijective combinatorics can be used to prove continued fractions. This is one of the key ingredients for our proofs in Chapters 4 and 5.

A *Motzkin path* of length $n \geq 0$ is a path $\omega = (\omega_0, \ldots, \omega_n)$ in the right quadrant $\mathbb{N} \times \mathbb{N}$, starting at $\omega_0 = (0,0)$ and ending at $\omega_n = (n,0)$, whose steps $s_j = \omega_j - \omega_{j-1}$ are $(1,1)$ ["rise" or "up step"], $(1,-1)$ ["fall" or "down step"] or $(1,0)$ ["level step"]. We write h_j for the *height* of the Motzkin path at abscissa j, i.e. $\omega_j = (j, h_j)$; note in particular that $h_0 = h_n = 0$. We write $\mathcal{M}_n$ for the set of Motzkin paths of length n, and $\mathcal{M} = \bigcup_{n=0}^{\infty} \mathcal{M}_n$. A Motzkin path is called a *Dyck path* if it has no level steps. A Dyck path always has even length; we write $\mathcal{D}_{2n}$ for the set of Dyck paths of length $2n$, and $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_{2n}$.

Let $\mathbf{a} = (a_i)_{i \geq 0}$, $\mathbf{b} = (b_i)_{i \geq 1}$ and $\mathbf{c} = (c_i)_{i \geq 0}$ be indeterminates; we will work in the ring $\mathbb{Z}[[\mathbf{a}, \mathbf{b}, \mathbf{c}]]$ of formal power series in these indeterminates. To each Motzkin path ω we assign a weight $W(\omega) \in \mathbb{Z}[\mathbf{a},\mathbf{b},\mathbf{c}]$ that is the product of the weights for the individual steps, where a rise starting at height i gets weight a_i, a fall starting at height i gets weight b_i, and a level step at height i gets weight c_i. Flajolet [Fla80] showed that the generating function of Motzkin paths can be expressed as a continued fraction:

Theorem 2.2.6 (Flajolet's master theorem)**.** We have

$$\sum_{\omega \in \mathcal{M}} W(\omega) \;=\; \cfrac{1}{1 - c_0 - \cfrac{a_0 b_1}{1 - c_1 - \cfrac{a_1 b_2}{1 - c_2 - \cfrac{a_2 b_3}{1 - \cdots}}}} \tag{2.33}$$

as an identity in $\mathbb{Z}[[\mathbf{a},\mathbf{b},\mathbf{c}]]$.

In particular, if $a_{i-1}b_i = \beta_i t^2$ and $c_i = \gamma_i t$ (note that the parameter t is conjugate to the length of the Motzkin path), we have

$$\sum_{n=0}^{\infty} t^n \sum_{\omega \in \mathcal{M}_n} W(\omega) \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}}\;, \tag{2.34}$$

so that the generating function of Motzkin paths with height-dependent weights is given by the J-type continued fraction (2.20). Similarly, if $a_{i-1}b_i = \alpha_i t$ and $c_i = 0$ (note that t is now conjugate to the semi-length of the Dyck path), we have

$$\sum_{n=0}^{\infty} t^n \sum_{\omega \in \mathcal{D}_{2n}} W(\omega) \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}\;, \tag{2.35}$$

so that the generating function of Dyck paths with height-dependent weights is given by the S-type continued fraction (2.19).

Let us now show how to handle Schröder paths within this framework. A *Schröder path* of length $2n$ ($n \geq 0$) is a path $\omega = (\omega_0, \ldots, \omega_{2n})$ in the right quadrant $\mathbb{N} \times \mathbb{N}$, starting at $\omega_0 = (0,0)$ and ending at $\omega_{2n} = (2n,0)$, whose steps are $(1,1)$ ["rise" or "up step"], $(1,-1)$ ["fall" or "down step"] or $(2,0)$ ["long level step"]. We write s_j for the step starting at abscissa $j-1$. If the step s_j is a rise or a fall, we set $s_j = \omega_j - \omega_{j-1}$ as before. If the step s_j is a long level step, we set $s_j = \omega_{j+1} - \omega_{j-1}$ and leave ω_j undefined; furthermore, in this case there is no step s_{j+1}. We write h_j for the height of the Schröder path at abscissa j whenever this is defined, i.e. $\omega_j = (j, h_j)$. Please note that $\omega_{2n} = (2n,0)$ and $h_{2n} = 0$ are always well-defined, because there cannot be a long level step starting at abscissa $2n-1$. Note also that a long level step at even (resp. odd) height can occur only at an odd-numbered (resp. even-numbered) step. We write $\mathcal{S}_{2n}$ for the set of Schröder paths of length $2n$, and $\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{S}_{2n}$.

There is an obvious bijection between Schröder paths and Motzkin paths: namely, every long level step is mapped onto a level step. If we apply Flajolet's master theorem with $a_{i-1}b_i = \alpha_i t$ and $c_i = \delta_{i+1} t$ to the resulting Motzkin path (note that t is now conjugate to the semi-length of the underlying Schröder path), we obtain

$$\sum_{n=0}^{\infty} t^n \sum_{\omega \in \mathcal{S}_{2n}} W(\omega) \;=\; \cfrac{1}{1 - \delta_1 t - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{1 - \cdots}}}\,, \tag{2.36}$$

so that the generating function of Schröder paths with height-dependent weights is given by the T-type continued fraction (2.21). More precisely, every rise gets a weight 1, every fall starting at height i gets a weight α_i, and every long level step at height i gets a weight δ_{i+1}. This combinatorial interpretation of T-fractions in terms of Schröder paths was found recently by several authors [FG17, OdJ15, JV17, Sok].

2.2.5 Labelled Dyck, Motzkin and Schröder paths

We have looked at the combinatorial interpretations of weighted Dyck, Motzkin and Schröder paths in terms of continued fractions in Section 2.2.4. As our next step in this approach, we will now look at Dyck, Motzkin and Schröder paths with height-dependent labels on each step which will lead us to some continued fractions.

Let $\mathcal{A} = (\mathcal{A}_h)_{h\geq 0}$, $\mathcal{B} = (\mathcal{B}_h)_{h\geq 1}$ and $\mathcal{C} = (\mathcal{C}_h)_{h\geq 0}$ be sequences of finite sets. An $(\mathcal{A},\mathcal{B},\mathcal{C})$*-labelled Motzkin path of length* $\boldsymbol{n}$ is a pair (ω,ξ) where $\omega = (\omega_0,\ldots,\omega_n)$ is a Motzkin path of length n, and $\xi = (\xi_1,\ldots,\xi_n)$ is a sequence satisfying

$$\xi_i \in \begin{cases} \mathcal{A}(h_{i-1}) & \text{if step } i \text{ is a rise (i.e. } h_i = h_{i-1}+1) \\ \mathcal{B}(h_{i-1}) & \text{if step } i \text{ is a fall (i.e. } h_i = h_{i-1}-1) \\ \mathcal{C}(h_{i-1}) & \text{if step } i \text{ is a level step (i.e. } h_i = h_{i-1}) \end{cases} \tag{2.37}$$

where h_{i-1} (resp. h_i) is the height of the Motzkin path before (resp. after) step i. [For typographical clarity we have here written $\mathcal{A}(h)$ as a synonym for $\mathcal{A}_h$, etc.] We call ξ_i the ***label*** associated to step i. We call the pair (ω,ξ) an $(\mathcal{A},\mathcal{B})$*-labelled Dyck path* if ω is a Dyck path (in this case $\mathcal{C}$ plays no role). We denote by $\mathcal{M}_n(\mathcal{A},\mathcal{B},\mathcal{C})$ the set of $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled Motzkin paths of length n, and by $\mathcal{D}_{2n}(\mathcal{A},\mathcal{B})$ the set of $(\mathcal{A},\mathcal{B})$-labelled Dyck paths of length $2n$.

We define a $(\mathcal{A},\mathcal{B},\mathcal{C})$*-labelled Schröder path* in an analogous way; now the sets $\mathcal{C}_h$ refer to long level steps. We denote by $\mathcal{S}_{2n}(\mathcal{A},\mathcal{B},\mathcal{C})$ the set of $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled Schröder paths of length $2n$.

Let us stress that the sets $\mathcal{A}_h$, $\mathcal{B}_h$ and $\mathcal{C}_h$ are allowed to be empty. Whenever this happens, the path ω is forbidden to take a step of the specified kind starting at the specified height.

Remark. What we have called an $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled Motzkin path is (up to small changes in notation) called a *path diagramme* by Flajolet [Fla80, p. 136] and a *history* by Viennot [Vie83, p. II-9]. Often the label sets $\mathcal{A}_h,\mathcal{B}_h,\mathcal{C}_h$ are intervals of integers, e.g. $\mathcal{A}_h = \{1,\ldots,A_h\}$ or $\{0,\ldots,A_h\}$; in this case the triplet $(\mathbf{A},\mathbf{B},\mathbf{C})$ of sequences of maximum values is called a *possibility function*. On the other hand, it is sometimes useful to employ labels that are *pairs* of integers (e.g. [SZ22, Section 6.2] and Section 4.6 below). It therefore seems preferable to state the general theory without any specific assumption about the nature of the label sets. ■

Following Flajolet [Fla80, Proposition 7A], we can state a "master J-fraction" for $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled Motzkin paths. Let $\mathbf{a} = (a_{h,\xi})_{h\ge 0,\, \xi\in\mathcal{A}(h)}$, $\mathbf{b} = (b_{h,\xi})_{h\ge 1,\, \xi\in\mathcal{B}(h)}$ and $\mathbf{c} = (c_{h,\xi})_{h\ge 0,\, \xi\in\mathcal{C}(h)}$ be indeterminates; we give an $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled Motzkin path (ω,ξ) a weight $W(\omega,\xi)$ that is the product of the weights for the individual steps, where a rise starting at height h with label ξ gets weight $a_{h,\xi}$, a fall starting at height h with label ξ gets weight $b_{h,\xi}$, and a level step at height h with label ξ gets weight $c_{h,\xi}$. Then:

Theorem 2.2.7 (Flajolet's master theorem for labelled Motzkin paths)**.** We have

$$\sum_{n=0}^{\infty} t^n \sum_{(\omega,\xi)\in\mathcal{M}_n(\mathcal{A},\mathcal{B},\mathcal{C})} W(\omega,\xi) \;=\; \cfrac{1}{1 - c_0 t - \cfrac{a_0 b_1 t^2}{1 - c_1 t - \cfrac{a_1 b_2 t^2}{1 - c_2 t - \cfrac{a_2 b_3 t^2}{1 - \cdots}}}} \tag{2.38}$$

as an identity in $\mathbb{Z}[\mathbf{a},\mathbf{b},\mathbf{c}][[t]]$, where

$$a_h \;=\; \sum_{\xi\in\mathcal{A}(h)} a_{h,\xi}\;, \qquad b_h \;=\; \sum_{\xi\in\mathcal{B}(h)} b_{h,\xi}\;, \qquad c_h \;=\; \sum_{\xi\in\mathcal{C}(h)} c_{h,\xi}\;. \tag{2.39}$$

This is an immediate consequence of Theorem 2.2.6 together with the definitions.

By specializing to $\mathbf{c} = \mathbf{0}$ and replacing t^2 by t, we obtain the corresponding theorem for $(\mathcal{A},\mathcal{B})$-labelled Dyck paths:

Corollary 2.2.8 (Flajolet's master theorem for labelled Dyck paths)**.** We have

$$\sum_{n=0}^{\infty} t^n \sum_{(\omega,\xi)\in\mathcal{D}_{2n}(\mathcal{A},\mathcal{B})} W(\omega,\xi) \;=\; \cfrac{1}{1-\cfrac{a_0b_1t}{1-\cfrac{a_1b_2t}{1-\cfrac{a_2b_3t}{1-\cdots}}}} \tag{2.40}$$

as an identity in $\mathbb{Z}[\mathbf{a},\mathbf{b}][[t]]$, where a_h and b_h are defined by (2.39).

Similarly, for labelled Schröder paths we have:

Theorem 2.2.9 (Flajolet's master theorem for labelled Schröder paths)**.** We have

$$\sum_{n=0}^{\infty} t^n \sum_{(\omega,\xi)\in\mathcal{S}_{2n}(\mathcal{A},\mathcal{B},\mathcal{C})} W(\omega,\xi) \;=\; \cfrac{1}{1-c_0t-\cfrac{a_0b_1t}{1-c_1t-\cfrac{a_1b_2t}{1-c_2t-\cfrac{a_2b_3t}{1-\cdots}}}} \tag{2.41}$$

as an identity in $\mathbb{Z}[\mathbf{a},\mathbf{b},\mathbf{c}][[t]]$, where a_h, b_h, c_h are defined by (2.39), with $c_{h,\xi}$ now referring to long level steps.

2.3 Some sufficient conditions for coefficientwise Hankel-total positivity

In Section 2.1.4 we discussed the total positivity of Hankel matrices. In Theorem 2.1.2 we saw that the existence of a Stieltjes-type continued fraction is an equivalent condition for a Hankel matrix of *real numbers* to be totally positive. This equivalence does not hold in the coefficientwise setting; however, we saw in Theorem 2.1.3 that the existence of an S-fraction with $\alpha_0, \alpha_1, \ldots$ that are coefficientwise-positive polynomials is a *sufficient* condition for coefficientwise Hankel-total positivity, although it is far from necessary. We will now see some more general sufficient conditions for the coefficientwise total positivity of Hankel matrices.

We begin with $\boldsymbol{a}$ which is a sequence of coefficientwise-positive polynomials. Also, let $\alpha_1, \alpha_2, \ldots$ and $\delta_1, \delta_2, \ldots$ be two more sequences of coefficientwise-positive polynomials. We will now state a result which will be very useful in this thesis: it generalises Theorem 2.1.3 from S-fractions to T-fractions. It was first found by Sokal (unpublished) in 2015; it is also a special case obtained by setting $m = 1$ in a far more general result by Pétréolle, Sokal and Zhu [PSZ18, Theorem 9.9] which we will also state in a while (see Theorem 2.3.3 below):

Theorem 2.3.1 (Coefficientwise Hankel-total postivity of T-fractions)**.** *Let* $\boldsymbol{a} = (a_n)_{n\geq 0}$, $\alpha_1, \ldots$ *and* $\delta_1, \delta_2, \ldots$ *be three sequences of polynomials with* $a_0 = 1$. *If the polynomials* $\alpha_1, \alpha_2, \ldots$ *and* $\delta_1, \delta_2, \ldots$ *are coefficientwise-positive and they satisfy the following power series identity:*

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \delta_1 t - \cfrac{\alpha_1 t}{1 - \delta_2 t - \cfrac{\alpha_2 t}{\ddots}}} \tag{2.42}$$

then $\boldsymbol{a}$ *is coefficientwise-Hankel totally positive.*

It is clear that if we set $\delta_n = 0$ for all $n \geq 1$, we obtain Theorem 2.1.3 as a special case. We have stated Theorem 2.3.1 separately as we will use it several times in this thesis.

We will now state another generalisation of Theorem 2.1.3 for which we need to recall a few definitions from [PSZ18]. Let $m \geq 1$ be a fixed integer. An ***m*-Dyck path** is a path in the upper half-plane $\mathbb{Z} \times \mathbb{N}$, starting and ending on the horizontal axis, using steps $(1,1)$ ["rise" or "up step"] and $(1,-m)$ ["m-fall" or "down step"]. Since the number of up steps must equal m times the number of down steps, the length of an m-Dyck path must be a multiple of $m+1$. Thus, a 1-Dyck path is simply a Dyck path.

Now let $\boldsymbol{\alpha} = (\alpha_i)_{i\geq m}$ be an infinite set of indeterminates. Then [PSZ18] the ***m*-Stieltjes–Rogers polynomial** of order n, denoted $S_n^{(m)}(\boldsymbol{\alpha})$, is the generating polynomial for m-Dyck paths from $(0,0)$ to $((m+1)n, 0)$ in which each rise gets weight 1 and each m-fall from height i gets weight α_i. Clearly $S_n^{(m)}(\boldsymbol{\alpha})$ is a homogeneous polynomial of degree n with nonnegative integer coefficients. Thus, the 1-Stieltjes–Rogers polynomial of order n is the polynomial $\sum_{\omega \in \mathcal{D}_{2n}} W(\omega)$ in the left-hand side of equation (2.35). We are now ready to state [PSZ18, Theorem 9.8]:

Theorem 2.3.2 ([PSZ18, Theorem 9.8])**.** *For each integer $m \geq 1$, the sequence $\mathbf{S}^{(m)} = \left(S_n^{(m)}(\boldsymbol{\alpha})\right)_{n\geq 0}$ of m-Stieltjes–Rogers polynomials is coefficientwise Hankel-totally positive sequence with respect to the indeterminates $\boldsymbol{\alpha}$.*

Finally, for the sake of completeness let us also introduce the ***m-Thron–Rogers polynomials***, even though we will not use them in this thesis. An ***m-Schröder path*** is a path in the upper half-plane $\mathbb{Z} \times \mathbb{N}$, starting and ending on the horizontal axis, using steps $(1,1)$ ["rise" or "up step"], $(1,-m)$ ["m-fall" or "down step"] and $(2,-(m-1))$ ["m-long step"]. We define the length of an m-Schröder path to be the number of rises plus the number of m-falls plus twice the number of m-long steps. It is not difficult to see that the length of an m-Schröder path must be a multiple of $m+1$. Also, it is clear that a 1-Schröder path is simply a Schröder path.

Now let $\boldsymbol{\alpha} = (\alpha_i)_{i\geq m}$ and $\boldsymbol{\delta} = (\delta_i)_{i\geq m}$ be infinite sets of indeterminates. Then [PSZ18] the ***m-Thron–Rogers polynomial*** of order n, denoted $T_n^{(m)}(\boldsymbol{\alpha}, \boldsymbol{\delta})$, is the generating polynomial for m-Schröder paths from $(0,0)$ to $((m+1)n, 0)$ in which each rise gets weight 1, each m-fall from height i gets weight α_i, and each m-long step from height i gets weight δ_{i+1}. Thus, the 1-Thron–Rogers polynomial of order n is the polynomial $\sum_{\omega \in \mathcal{S}_{2n}} W(\omega)$ in the left-hand side of equation (2.36). We are now ready to state [PSZ18, Theorem 9.9] which is a common generalisation of Theorems 2.1.3, 2.3.1 and 2.3.2:

Theorem 2.3.3 ([PSZ18, Theorem 9.9])**.** *For each integer $m \geq 1$, the sequence $\mathbf{T}^{(m)} = \left(T_n^{(m)}(\boldsymbol{\alpha}, \boldsymbol{\delta})\right)_{n\geq 0}$ of m-Thron–Rogers polynomials is coefficientwise Hankel-totally positive sequence with respect to the indeterminates $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$.*

2.4 Genocchi, median Genocchi numbers, and D-permutations

The Genocchi numbers appear already in Euler's book *Foundations of Differential Calculus, with Applications to Finite Analysis and Series*, first published in 1755 [Eul55b, paragraphs 181 and 182]; this book is E212 in Eneström's [Ene13] catalogue. These numbers were revisited by Genocchi [Gen52] in 1852. The beautiful survey article of Viennot [Vie] contains a wealth of useful information.

Our notation for the Genocchi and median Genocchi numbers is nonstandard but, we believe, sensible and logical. Later we will present a translation dictionary with respect to (the plethora of) previous notations: see footnotes 3 and 5.

We will work with objects enumerated by the Genocchi and median Genocchi numbers in Chapters 4 and 5. In this section, we first introduce the Genocchi numbers in Subsection 2.4.1. We then introduce the median Genocchi numbers in Section 2.4.2. Finally, we will define D-permutations and its various subclasses in Section 2.4.3 which will be one of our primary objects of study.

2.4.1 Genocchi numbers

The Genocchi numbers [OEI19, A110501][3]

$$(g_n)_{n\geq 0} \;=\; 1,1,3,17,155,2073,38227,929569,28820619,1109652905,\ldots \tag{2.43}$$

[3]Our g_n is usually written by combinatorialists as G_{2n+2}. However, many texts — particularly older ones, or those in the analysis and special-functions literature — define the (*signed*) Genocchi numbers $(G_n)_{n\geq 0}$ by [OLBC10, 24.15.1]

$$\frac{2t}{e^t+1} \;=\; \sum_{n=1}^{\infty} G_n \, \frac{t^n}{n!} \,,$$

which leads to $G_0 = 0$, $G_1 = 1$, $G_n = 0$ for odd $n \geq 3$, and $G_{2n+2} = (-1)^{n+1} g_n$.

Warning: Our g_n is denoted g_{n+1} by Lazar and Wachs [LW22, Laz20] and Eu, Fu, Lai and Lo [EFLL22].

are odd positive integers [Luc77, Bar81, HL18] [FH08, pp. 217–218] defined by the exponential generating function

$$t\,\tan(t/2) \;=\; \sum_{n=0}^{\infty} g_n \,\frac{t^{2n+2}}{(2n+2)!}\;. \tag{2.44}$$

The ordinary generating function of the Genocchi numbers has a classical S-fraction expansion [Vie, eq. (7.5)] [Vie83, p. V-9] [DZ94, eqns. (1.4) and (3.9)]

$$\sum_{n=0}^{\infty} g_n t^n \;=\; \cfrac{1}{1 - \cfrac{1\cdot 1t}{1 - \cfrac{1\cdot 2t}{1 - \cfrac{2\cdot 2t}{1 - \cfrac{2\cdot 3t}{1-\cdots}}}}} \tag{2.45}$$

with coefficients

$$\alpha_{2k-1} \;=\; k^2\,, \quad \alpha_{2k} \;=\; k(k+1)\;. \tag{2.46}$$

It then follows from Proposition 2.2.5 that the once-shifted Genocchi numbers $(g_{n+1})_{n\ge 0}$ have a T-fraction

$$\sum_{n=0}^{\infty} g_{n+1} t^n \;=\; \cfrac{1}{1 - t - \cfrac{1\cdot 2t}{1 - \cfrac{2\cdot 2t}{1 - \cfrac{2\cdot 3t}{1 - \cfrac{3\cdot 3t}{1-\cdots}}}}} \tag{2.47}$$

with coefficients

$$\alpha_{2k-1} \;=\; k(k+1)\,, \quad \alpha_{2k} \;=\; (k+1)^2\,, \quad \delta_1 \;=\; 1\,, \quad \delta_n \;=\; 0 \text{ for } n \ge 2\;. \tag{2.48}$$

Remark. Some generalizations of (2.45)/(2.46), incorporating additional parameters, are known in a variety of algebraic or combinatorial models: see [Dum86, Section 6] [DR95] [Zen96, eq. (3.3) and Corollaire 8] [RZ96b, Théorème 3 and Proposition 13] [RZ96a, Proposition 10] [Ran97, Théorème 1.2] [HZ99b, eq. (5) and Théorème 2]. See also [HZ99b, Corollaire 3] for a generalization of the T-fraction (2.47)/(2.48).[4] ■

2.4.2 Median Genocchi numbers

The median Genocchi numbers (or Genocchi medians for short) [OEI19, A005439][5] are defined by [HZ99a, p. 63]

$$h_n \;=\; \sum_{i=0}^{n-1} (-1)^i \binom{n}{2i+1} g_{n-1-i} \,. \tag{2.49}$$

The Genocchi medians

$$(h_n)_{n\ge 0} \;=\; 1, 1, 2, 8, 56, 608, 9440, 198272, 5410688, 186043904, 7867739648, \ldots \tag{2.50}$$

do not have any known exponential generating function. However, their ordinary generating function has a nice classical S-fraction expansion [Vie, eq. (9.7)] [Vie83, p. V-15] [DZ94, eqns. (1.5) and (3.8)]:

$$\sum_{n=0}^{\infty} h_n t^n \;=\; \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{4t}{1 - \cfrac{4t}{1 - \cdots}}}}} \tag{2.51}$$

---

[4] There is a typographical error in [HZ99b, eq. (12)]: on the left-hand side, t^n should be t^{n-1}.

[5] Our h_n is usually written by combinatorialists as H_{2n+1}.

Warning: Lazar and Wachs' [LW22, Laz20] h_n equals our h_{n+1}. Pan and Zeng's [PZ23] h_n is our h_{n+1} divided by 2^n.

with coefficients

$$\alpha_{2k-1} \;=\; \alpha_{2k} \;=\; k^2 \;. \tag{2.52}$$

It then follows from Corollary 2.2.3 that the once-shifted median Genocchi numbers $(h_{n+1})_{n\geq 0}$ have an S-fraction

$$\sum_{n=0}^{\infty} h_{n+1}\, t^n \;=\; \cfrac{1}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{6t}{1 - \cfrac{6t}{1 - \cdots}}}}} \tag{2.53}$$

with coefficients

$$\alpha_{2k-1} \;=\; \alpha_{2k} \;=\; k(k+1) \;. \tag{2.54}$$

Moreover, from Proposition 2.2.5 they also a T-fraction

$$\sum_{n=0}^{\infty} h_{n+1}\, t^n \;=\; \cfrac{1}{1 - t - \cfrac{1t}{1 - \cfrac{4t}{1 - \cfrac{4t}{1 - \cfrac{9t}{1 - \cdots}}}}} \tag{2.55}$$

with coefficients

$$\alpha_{2k-1} \;=\; k^2 \,, \quad \alpha_{2k} \;=\; (k+1)^2 \,, \quad \delta_1 \;=\; 1 \,, \quad \delta_n \;=\; 0 \text{ for } n \geq 2 \;. \tag{2.56}$$

Finally, let us define, for future reference, the sequence $(h^{\flat}_{n+1})_{n\geq 0}$ corresponding to the S-fraction underlying (2.55):

$$\sum_{n=0}^{\infty} h_{n+1}^{\flat} t^n \;=\; \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{4t}{1 - \cfrac{4t}{1 - \cfrac{9t}{1 - \cdots}}}}} \tag{2.57}$$

with coefficients

$$\alpha_{2k-1} \;=\; k^2 \,, \quad \alpha_{2k} \;=\; (k+1)^2 \,. \tag{2.58}$$

This sequence begins

$$(h_{n+1}^{\flat})_{n\ge 0} \;=\; 1,1,5,41,493,8161,178469,4998905,174914077,7487810257,\ldots \tag{2.59}$$

and cannot be found, at present, in [OEI19]. In Section 4.2.2 we will give its combinatorial interpretation. Using Proposition 2.2.5, we also get a T-fraction for the once-shifted sequence $(h_{n+2}^{\flat})_{n\ge 0}$:

$$\sum_{n=0}^{\infty} h_{n+2}^{\flat} t^n \;=\; \cfrac{1}{1 - t - \cfrac{4t}{1 - \cfrac{4t}{1 - \cfrac{9t}{1 - \cdots}}}} \tag{2.60}$$

with coefficients

$$\alpha_{2k-1} \;=\; (k+1)^2 \,, \quad \alpha_{2k} \;=\; (k+1)^2 \,, \quad \delta_1 \;=\; 1 \,, \quad \delta_n \;=\; 0 \text{ for } n \ge 2 \,. \tag{2.61}$$

Remark. Some generalizations of (2.51)/(2.52) or (2.53)/(2.54), incorporating additional parameters, are known in a variety of algebraic or combinatorial models: see [Dum86, Section 6] [DR95] [Zen96, eq. (3.3) and Corollaire 8] [RZ96b, Théorème 3 and Proposition 13] [RZ96a, Proposition 10 and Corollary 13] [Fei12, Theorem 0.1] [PZ23, Corollary 6]. ■

2.4.3 D-permutations

The median Genocchi numbers enumerate a class of permutations called D-permutations (short for Dumont-like permutations), they were introduced by Lazar and Wachs in [LW22, Laz20]. A permutation of $[2n]$ is called a D-permutation in case $2k-1 \le \sigma(2k-1)$ and $2k \ge \sigma(2k)$ for all k, i.e., it contains no even excedances and no odd anti-excedances. Let us say also that a permutation is an *e-semiderangement* (resp. *o-semiderangement*) in case in contains no even (resp. odd) fixed points; it is a *derangement* in case it contains no fixed points at all. A D-permutation that is also an e-semiderangement (resp. o-semiderangement, derangement) will be called a ***D-e-semiderangement*** (resp. ***D-o-semiderangement***, ***D-derangement***).[6] A D-permutation that contains exactly one cycle is called a ***D-cycle***. Notice that a D-cycle is also a D-derangement. Let $\mathfrak{D}_{2n}$ (resp. $\mathfrak{D}^{\mathrm{e}}_{2n}, \mathfrak{D}^{\mathrm{o}}_{2n}, \mathfrak{D}^{\mathrm{eo}}_{2n}, \mathfrak{DC}_{2n}$) denote the set of all D-permutations (resp. D-e-semiderangements, D-o-semiderangements, D-derangements, D-cycles) of $[2n]$. For instance,

$$\mathfrak{D}_2 = \{12, 21^{\mathrm{eo}}\} \tag{2.62a}$$

$$\mathfrak{D}_4 = \{1234, 1243, 2134, 2143^{\mathrm{eo}}, 3142^{\mathrm{eo}}, 3241^{\mathrm{o}}, 4132^{\mathrm{e}}, 4231\} \tag{2.62b}$$

$$\mathfrak{DC}_2 = \{21\} \tag{2.62c}$$

$$\mathfrak{DC}_4 = \{3142\} \tag{2.62d}$$

[6] In the past, D-o-semiderangements have been called *Genocchi permutations* [RZ96a, Ran97, HZ99b] or *excedance-alternating permutations* [ES00]; D-e-semiderangements [Dum74, p. 316, Corollaire 1] have been called *Dumont permutations* [LW22, Laz20, PZ23] or *Dumont permutations of the second kind* [BEM06, BJ21]; D-derangements have been called *Dumont derangements* [LW22, Laz20, PZ23]. D-permutations that are not semiderangements were apparently first considered in the recent work of Lazar and Wachs [LW22, Laz20].

Note also that the involution of $\mathfrak{S}_{2n}$ defined by $\sigma \mapsto R \circ \sigma \circ R$, where $R(i) = 2n+1-i$ is the reversal map, takes D-permutations into D-permutations and interchanges e-semiderangements with o-semiderangements; it therefore yields a bijection between D-e-semiderangements and D-o-semiderangements. Therefore, any result about one of the two types of D-semiderangements can be expressed equivalently in terms of the other.

where $^{\mathrm{e}}$ denotes e-semiderangements that are not derangements, $^{\mathrm{o}}$ denotes o-semiderangements that are not derangements, and $^{\mathrm{eo}}$ denotes derangements.

Remark. Three natural variants of this setup lead to nothing new:

1) In a D-permutation of $[2n+1]$, $2n+1$ must be a fixed point, and the rest is a D-permutation of $[2n]$.

2) Suppose we define an *anti-D-permutation* to be one in which $2k-1 \geq \sigma(2k-1)$ and $2k \leq \sigma(2k)$ for all k. Then, in an anti-D-permutation of $[2n]$, 1 and $2n$ must be fixed points, and the rest is, after renumbering, a D-permutation of $[2n-2]$.

3) In an anti-D-permutation of $[2n+1]$, 1 must be a fixed point, and the rest is, after renumbering, a D-permutation of $[2n]$.

So there is no loss of generality in studying only D-permutations of $[2n]$. ■

It is known [Dum74, DR94, LW22, Laz20] — and we will recover as part of our work — that

$$|\mathfrak{D}_{2n}| = h_{n+1} \tag{2.63a}$$

$$|\mathfrak{D}^{\mathrm{e}}_{2n}| = |\mathfrak{D}^{\mathrm{o}}_{2n}| = g_n \tag{2.63b}$$

$$|\mathfrak{D}^{\mathrm{eo}}_{2n}| = h_n \tag{2.63c}$$

$$|\mathfrak{DC}_{2n}| = g_{n-1} \tag{2.63d}$$

2.5 Permutation statistics

We will define various permutation statistics which will play an important role in this thesis, especially in Chapters 4 and 5. We intend this section to be useful as a glossary of statistics should the reader need to check the description of the large number of statistics which we will later simultaneously study. We will introduce our record and cycle classification in Section 2.5.1. Then we will define nesting and crossing statistics in Section 2.5.2.

2.5.1 Record and cycle classification

Given a permutation $\sigma \in \mathfrak{S}_N$, we will introduce various ways of classifying the indices and values $i \in [N]$.

Cycle classification:

Given a permutation $\sigma \in \mathfrak{S}_N$, an index $i \in [N]$ is called

- *cycle peak* (cpeak) if $\sigma^{-1}(i) < i > \sigma(i)$;
- *cycle valley* (cval) if $\sigma^{-1}(i) > i < \sigma(i)$;
- *cycle double rise* (cdrise) if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle double fall* (cdfall) if $\sigma^{-1}(i) > i > \sigma(i)$;
- *fixed point* (fix) if $\sigma^{-1}(i) = i = \sigma(i)$.

Clearly every index i belongs to exactly one of these five types; we refer to this classification as the ***cycle classification***.

Now suppose that σ is a D-permutation. Then the cycle classification of a non-fixed-point index i is equivalent to recording the parities of $\sigma^{-1}(i)$ and i:

- *cycle peak*: $\sigma^{-1}(i) < i > \sigma(i) \implies \sigma^{-1}(i)$ odd, i even
- *cycle valley*: $\sigma^{-1}(i) > i < \sigma(i) \implies \sigma^{-1}(i)$ even, i odd
- *cycle double rise*: $\sigma^{-1}(i) < i < \sigma(i) \implies \sigma^{-1}(i)$ odd, i odd
- *cycle double fall*: $\sigma^{-1}(i) > i > \sigma(i) \implies \sigma^{-1}(i)$ even, i even

(2.64)

For a fixed point i, we will later in (2.65) explicitly record the parity of i by distinguishing even and odd fixed points.

Record classification:

An index $i \in [N]$ is called a

- *record* (rec) (or *left-to-right maximum*) if $\sigma(j) < \sigma(i)$ for all $j < i$

  [note in particular that the indices 1 and $\sigma^{-1}(N)$ are always records];
- *antirecord* (arec) (or *right-to-left minimum*) if $\sigma(j) > \sigma(i)$ for all $j > i$

[note in particular that the indices N and $\sigma^{-1}(1)$ are always antirecords];

- *exclusive record* (erec) if it is a record and not also an antirecord;
- *exclusive antirecord* (earec) if it is an antirecord and not also a record;
- *record-antirecord* (rar) if it is both a record and an antirecord;
- *neither-record-antirecord* (nrar) if it is neither a record nor an antirecord.

Every index i thus belongs to exactly one of the latter four types; we refer to this classification as the ***record classification***.

Record-and-cycle classification:

One can apply the record and cycle classifications simultaneously, to obtain 10, and not 20, disjoint categories which we name in Table 2.1.

	cpeak	cval	cdrise	cdfall	fix
erec		ereccval	ereccdrise		
earec	eareccpeak			eareccdfall	
rar					rar
nrar	nrcpeak	nrcval	nrcdrise	nrcdfall	nrfix

Table 2.1: The 10 types in record-and-cycle classification

Clearly every index i belongs to exactly one of these 10 types; we call this the ***record-and-cycle classification***.

Variant record-and-cycle classification:

We will also use a variant of this classification involving record and antirecord values rather than indices. A value $i \in [N]$ is called a

- *record value* (rec′) (or *left-to-right maximum value*) if $\sigma(j) < i$ for all $j < \sigma^{-1}(i)$ [note in particular that the values $\sigma(1)$ and N are always record values];
- *antirecord value* (arec′) (or *right-to-left minimum value*) if $\sigma(j) > i$ for all $j > \sigma^{-1}(i)$ [note in particular that the values $\sigma^{-1}(N)$ and 1 are always antirecord values];

We also analogously define *exclusive record value* (erec′), *exclusive antirecord value* (earec′), *record-antirecord value* (rar′), *neither-record-antirecord value* (nrar′). Every index i thus belongs to exactly one of these four types; we refer to this classification as the ***variant record classification***.

We can similarly introduce the ***variant record-and-cycle classification*** consisting of the 10 disjoint categories which we mention in Table 2.2.

	cpeak	cval	cdrise	cdfall	fix
erec′	ereccpeak′		ereccdrise′		
earec′		eareccval′		eareccdfall′	
rar′					rar′
nrar′	nrcpeak′	nrcval′	nrcdrise′	nrcdfall′	nrfix′

Table 2.2: The 10 types in variant record-and-cycle classification

Notice that in record-and-cycle classification, cycle valleys (cycle peaks) can be exclusive record (exclusive anti-record) indices, whereas in the variant record-and-cycle classification, cycle valleys (cycle peaks) can now be exclusive anti-record (exclusive record) values.

Parity-refined record-and-cycle classification (for D-permutations):
Now let $\sigma \in \mathfrak{D}_{2N}$ be a D-permutation. We further refine the fixed points according to their parity.

- *even fixed point* (evenfix): $\sigma^{-1}(i) = i = \sigma(i)$ is even
- *odd fixed point* (oddfix): $\sigma^{-1}(i) = i = \sigma(i)$ is odd

(2.65)

We therefore refine the record-and-cycle classification by distinguishing even and odd fixed points which we mention in table 2.3.

	even	odd
rar	evenrar	oddrar
nrfix	evennrfix	oddnrfix

Table 2.3: Distinguishing fixed points by parity and record status

This leads to the ***parity-refined record-and-cycle classification***, in which each index i belongs to exactly one of 12 types. More precisely, each even i belongs to exactly one of the 6 types eareccpeak, nrcpeak, eareccdfall, nrcdfall, evenrar, evennrfix,

while each odd i belongs to exactly one of the 6 types

ereccval, nrcval, ereccdrise, nrcdrise, oddrar, oddnrfix.

We can also similarly introduce ***variant parity-refined record-and-cycle classification*** where each even i belongs to exactly one of the 6 types

ereccpeak′, nrcpeak′, eareccdfall′, nrcdfall′, evenrar′, evennrfix′,

while each odd i belongs to exactly one of the 6 types

eareccval′, nrcval′, ereccdrise′, nrcdrise′, oddrar′, oddnrfix′.

Minimum and maximum elements in a cycle:
Notice that each non-singleton cycle in a permutation $\sigma \in \mathfrak{S}_N$ consists of exactly one minimum element, which must be a cycle valley, and one maximum element, which must be a cycle peak. With this observation, we introduce the following four statistics:

- cycle valley minimum (minval): cycle valley that is the minimum in its cycle;
- cycle peak maximum (maxpeak): cycle peak that is the maximum in its cycle;
- cycle valley non-mimimum (nminval): cycle valley that is not the minimum in its cycle;
- cycle peak non-maximum (nmaxpeak): cycle peak that is not the maximum in its cycle.

Convention for set of elements:
Whenever we use the name of a statistic but with its first letter in capital, we will refer to the set of elements that belong to that statistic (in case that makes sense). For example, we use Cval to denote the set of all cycle valleys, and Evenfix to denote the set of even fixed points.

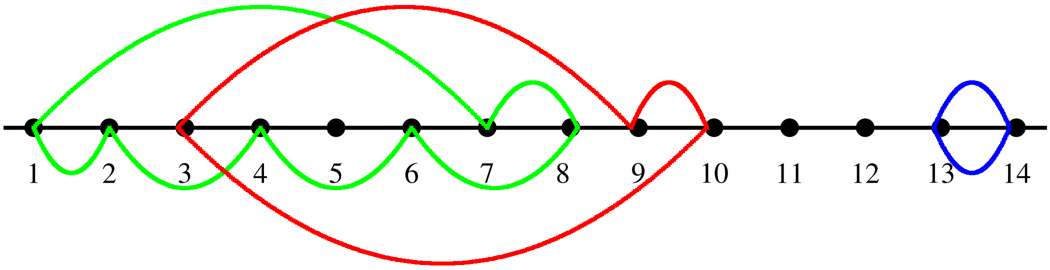

Figure 2.3: An example of a permutation $\sigma = 7\,1\,9\,2\,5\,4\,8\,6\,10\,3\,11\,12\,14\,13 = (1,7,8,6,4,2)\,(3,9,10)\,(5)\,(11)\,(12)\,(13,14) \in \mathfrak{S}_{14}$. Notice that σ is also a D-permutation.

2.5.2 Crossings and nestings

We now define (following [SZ22]) some permutation statistics that count ***crossings*** and ***nestings***.

Pictorial representation for permutations:

First we associate to each permutation $\sigma \in \mathfrak{S}_N$ a pictorial representation by placing vertices $1,2,\ldots,N$ along a horizontal axis and then drawing an arc from i to $\sigma(i)$ above (resp. below) the horizontal axis in case $\sigma(i) > i$ [resp. $\sigma(i) < i$]; if $\sigma(i) = i$ we do not draw any arc. This idea was first introduced by Corteel in [Cor07]. See Figure 2.3 for an example. Each vertex thus has either out-degree = in-degree = 1 (if it is not a fixed point) or out-degree = in-degree = 0 (if it is a fixed point). Of course, the arrows on the arcs are redundant, because the arrow on an arc above (resp. below) the axis always points to the right (resp. left); we therefore omit the arrows for simplicity.

Crossings and nestings:

We say that a quadruplet $i < j < k < l$ forms an

- *upper crossing* (ucross) if $k = \sigma(i)$ and $l = \sigma(j)$;
- *lower crossing* (lcross) if $i = \sigma(k)$ and $j = \sigma(l)$;
- *upper nesting* (unest) if $l = \sigma(i)$ and $k = \sigma(j)$;
- *lower nesting* (lnest) if $i = \sigma(l)$ and $j = \sigma(k)$.

Notice that for any such quadruplet, none of the four indices i, j, k, l is a fixed point. We also additionally consider the following degenerate cases where $j = k$ is a fixed point, we say that a triple $i < j < l$ forms an:

- *upper pseudo-nesting* (upsnest) if $l = \sigma(i)$ and $j = \sigma(j)$;

- *lower pseudo-nesting* (lpsnest) if $i = \sigma(l)$ and $j = \sigma(j)$.

Note that $\mathrm{upsnest}(\sigma) = \mathrm{lpsnest}(\sigma)$ for all σ, since for each fixed point j, the number of pairs (i,l) with $i < j < l$ such that $l = \sigma(i)$ has to equal the number of such pairs with $i = \sigma(l)$; we therefore write these two statistics simply as

$$\mathrm{psnest}(\sigma) \stackrel{\mathrm{def}}{=} \mathrm{upsnest}(\sigma) \;=\; \mathrm{lpsnest}(\sigma)\,. \tag{2.66}$$

If σ is a D-permutation, then its diagram has a special property: all arrows emanating from odd (resp. even) vertices are upper (resp. lower) arrows. Otherwise put, the leftmost (resp. rightmost) vertex of an upper (resp. lower) arc is always odd (resp. even). It follows that in an upper crossing or nesting $i < j < k < l$, the indices i and j must be odd; and in a lower crossing or nesting $i < j < k < l$, the indices k and l must be even. Similar comments apply to upper and lower joinings and pseudo-nestings.

We can further refine the four crossing/nesting categories by examining more closely the status of the inner index (j or k) whose *outgoing* arc belonged to the crossing or nesting: that is, j for an upper crossing or nesting, and k for a lower crossing or nesting. We state this in table 2.4.

	ucross	unest	lcross	lnest
$j \in$ Cval	ucrosscval	unestcval		
$j \in$ Cdrise	ucrosscdrise	unestcdrise		
$k \in$ Cpeak			lcrosscpeak	lnestcpeak
$k \in$ Cdfall			lcrosscdfall	lnestcdfall

Table 2.4: We consider a quadruplet $i < j < k < l$ and refine the four crossing/nesting categories by considering the status of j for upper crossings/nestings and k for lower crossings/nestings.

Index-refined crossing and nesting statistics:

A central role in our work will be played (just as in [SZ22]) by a refinement of these statistics: rather than counting the *total* numbers of quadruplets $i < j < k < l$ that form upper (resp. lower) crossings or nestings, we will count the number of upper (resp. lower) crossings or nestings that use a particular vertex j (resp. k) in second (resp. third) position. More precisely, we define the ***index-refined crossing and nesting statistics***

$$\begin{aligned}
\mathrm{ucross}(j,\sigma) &= \#\{i<j<k<l\colon\; k=\sigma(i) \text{ and } l=\sigma(j)\} && \text{(2.67a)}\\
\mathrm{unest}(j,\sigma) &= \#\{i<j<k<l\colon\; k=\sigma(j) \text{ and } l=\sigma(i)\} && \text{(2.67b)}\\
\mathrm{lcross}(k,\sigma) &= \#\{i<j<k<l\colon\; i=\sigma(k) \text{ and } j=\sigma(l)\} && \text{(2.67c)}\\
\mathrm{lnest}(k,\sigma) &= \#\{i<j<k<l\colon\; i=\sigma(l) \text{ and } j=\sigma(k)\} && \text{(2.67d)}
\end{aligned}$$

Note that $\mathrm{ucross}(j,\sigma)$ and $\mathrm{unest}(j,\sigma)$ can be nonzero only when j is an excedance (that is, a cycle valley or a cycle double rise), while $\mathrm{lcross}(k,\sigma)$ and $\mathrm{lnest}(k,\sigma)$ can be nonzero only when k is an anti-excedance (that is, a cycle peak or a cycle double fall). In a D-permutation, this means that $\mathrm{ucross}(j,\sigma)$ and $\mathrm{unest}(j,\sigma)$ can be nonzero only when j is odd and not a fixed point, while $\mathrm{lcross}(k,\sigma)$ and $\mathrm{lnest}(k,\sigma)$ can be nonzero only when k is even and not a fixed point.

When j is a fixed point, we also define the analogous quantity for pseudo-nestings:

$$\mathrm{psnest}(j,\sigma) \stackrel{\text{def}}{=} \#\{i<j\colon\; \sigma(i)>j\} \;=\; \#\{i>j\colon\; \sigma(i)<j\}\;. \qquad \text{(2.68)}$$

(Here the two expressions are equal because σ is a bijection from $[1,j)\cup(j,n]$ to itself.) In [SZ22, eq. (2.20)] this quantity was called the *level* of the fixed point j and was denoted $\mathrm{lev}(j,\sigma)$. Here we prefer psnest.

Variant index-refined crossing and nesting statistics

We also use a variant of (2.67) in which the roles of second and third position are interchanged:

$$\text{ucross}'(k,\sigma) = \#\{i<j<k<l\colon\ k=\sigma(i) \text{ and } l=\sigma(j)\} \qquad (2.69a)$$

$$\text{unest}'(k,\sigma) = \#\{i<j<k<l\colon\ k=\sigma(j) \text{ and } l=\sigma(i)\} \qquad (2.69b)$$

$$\text{lcross}'(j,\sigma) = \#\{i<j<k<l\colon\ i=\sigma(k) \text{ and } j=\sigma(l)\} \qquad (2.69c)$$

$$\text{lnest}'(j,\sigma) = \#\{i<j<k<l\colon\ i=\sigma(l) \text{ and } j=\sigma(k)\} \qquad (2.69d)$$

We remark that since nestings join the vertices in second and third positions, we have

$$\text{unest}'(k,\sigma) = \text{unest}(\sigma^{-1}(k),\sigma) \qquad (2.70a)$$

$$\text{lnest}'(j,\sigma) = \text{lnest}(\sigma^{-1}(j),\sigma) \qquad (2.70b)$$

Note that $\text{ucross}'(k,\sigma)$ and $\text{unest}'(k,\sigma)$ can be nonzero only when $\sigma^{-1}(k)$ is an excedance (that is, when k is a cycle peak or a cycle double rise), while $\text{lcross}'(j,\sigma)$ and $\text{lnest}'(j,\sigma)$ can be nonzero only when $\sigma^{-1}(j)$ is an anti-excedance (that is, j is a cycle valley or a cycle double fall). In a D-permutation, this means that $\text{ucross}'(k,\sigma)$ and $\text{unest}'(k,\sigma)$ can be nonzero only when $\sigma^{-1}(k)$ is odd and not a fixed point, while $\text{lcross}'(j,\sigma)$ and $\text{lnest}'(j,\sigma)$ can be nonzero only when $\sigma^{-1}(j)$ is even and not a fixed point. We call (2.69) the ***variant index-refined crossing and nesting statistics***.

We can also analogously define the statistics ucrosscpeak$'$, unestcpeak$'$, lcrosscval$'$, lnestcval$'$, lcrosscdfall$'$, lnestcdfall$'$, ucrosscdrise$'$, ucrosscdrise$'$. We omit the details.

Chapter 3

Coefficientwise total positivity of some matrices defined by linear recurrences [1]

In Chapter 1 we have seen that many interesting lower-triangular matrices (hereafter simply referred to as *triangles*) that arise in combinatorics have been shown to be totally positive: well-known examples include the binomial coefficients $\binom{n}{k}$, the Stirling cycle numbers $\left[{n \atop k}\right]$, and the Stirling subset numbers $\left\{{n \atop k}\right\}$.

But there are also many other combinatorially interesting triangles that appear to be totally positive but for which we have no proof. Foremost among these is what we call the "clean Eulerian triangle"

$$\boldsymbol{A} = \left(\left\langle {n \atop k} \right\rangle^{\text{clean}} \right)_{n,k \ge 0} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ 1 & 57 & 302 & 302 & 57 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.1}$$

which was conjectured by Brenti [Bre96] to be totally positive, already a quarter

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[1] The work in this chapter was done largely in collaboration with Xi Chen, Alexander Dyachenko, Tomack Gilmore and Alan D. Sokal. See pp. 11-13 for details.

of a century ago.[2][3] Here $\left\langle{n \atop k}\right\rangle^{\text{clean}}$ is the number of permutations of $[n+1]$ with k excedances (or k descents), or the number of increasing binary trees on the vertex set $[n+1]$ with k left children. These numbers satisfy the recurrence

$$\left\langle{n \atop k}\right\rangle^{\text{clean}} = (n-k+1)\left\langle{n-1 \atop k-1}\right\rangle^{\text{clean}} + (k+1)\left\langle{n-1 \atop k}\right\rangle^{\text{clean}} \tag{3.2}$$

for $n \geq 1$, with initial condition $\left\langle{0 \atop k}\right\rangle^{\text{clean}} = \delta_{k0}$.

Conjecture 3.0.1 ([Bre96, Conjecture 6.10])**.** The clean Eulerian triangle $\boldsymbol{A}$ is totally positive.

A similar problem concerns the reversed Stirling subset triangle. Recall that the Stirling subset number $\left\{{n \atop k}\right\}$ is the number of partitions of an n-element set into k non-empty blocks [OEI19, A048993/A008277]. We then write $\left\{{n \atop k}\right\}^{\text{rev}} = \left\{{n \atop n-k}\right\}$. The reversed Stirling subset triangle is [OEI19, A008278]

$$\boldsymbol{S}^{\text{rev}} = \left(\left\{{n \atop k}\right\}^{\text{rev}}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & \\ 1 & 0 & & & & & \\ 1 & 1 & 0 & & & & \\ 1 & 3 & 1 & 0 & & & \\ 1 & 6 & 7 & 1 & 0 & & \\ 1 & 10 & 25 & 15 & 1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.3}$$

These numbers satisfy the recurrence

$$\left\{{n \atop k}\right\}^{\text{rev}} = (n-k)\left\{{n-1 \atop k-1}\right\}^{\text{rev}} + \left\{{n-1 \atop k}\right\}^{\text{rev}} \tag{3.4}$$

for $n \geq 1$, with initial condition $\left\{{0 \atop k}\right\}^{\text{rev}} = \delta_{k0}$. Please note that the total positivity of a lower-triangular matrix does *not* in general imply the total positivity of its reversal.

[2] Note that there exist several different conventions for the Eulerian triangle. For our purposes, the “clean” version defined here is the most convenient, as it has 1’s both on the diagonal and in the zeroth column and is reversal-symmetric (i.e. $\left\langle{n \atop k}\right\rangle^{\text{clean}} = \left\langle{n \atop n-k}\right\rangle^{\text{clean}}$). It is easy to see that the other versions are totally positive if and only if the “clean” one is.

[3] Dyachenko [Dya] has verified this conjecture for the first 512×512 submatrix of the matrix $\left(\left\langle{n+1 \atop k}\right\rangle\right)_{n,k\geq 0}$.

Nevertheless we conjecture:

Conjecture 3.0.2. The reversed Stirling subset triangle $\boldsymbol{S}^{\mathrm{rev}}$ is totally positive.

In this chapter we present a more general triangle comprised of *polynomial* entries in *six* indeterminates that appears empirically to be *coefficientwise* totally positive and that yields, under suitable specialisations, both $\boldsymbol{A}$ and $\boldsymbol{S}^{\mathrm{rev}}$. We do not yet have any proof that this more general triangle is totally positive; indeed, we do not yet have any proof of Conjecture 3.0.1. But we are able to prove a special case that includes a generalisation of Conjecture 3.0.2.

Comparing recurrences (3.2) and (3.4) invites us to consider the more general linear recurrence

$$T(n,k) \;=\; [a(n-k)+c]\,T(n-1,k-1) \,+\, (dk+e)\,T(n-1,k) \tag{3.5}$$

for $n \geq 1$, with initial condition $T(0,k) = \delta_{k0}$. Here a,c,d,e could be integers or real numbers, but we prefer to treat them as algebraic indeterminates. Thus, the elements of the matrix $\boldsymbol{T} = (T(n,k))_{n,k\geq 0}$ belong to the polynomial ring $\mathbb{Z}[a,c,d,e]$, and we conjecture:

Conjecture 3.0.3. The lower-triangular matrix $\boldsymbol{T} = \big(T(n,k)\big)_{n,k\geq 0}$ defined by (3.5) is coefficientwise totally positive in the indeterminates a,c,d,e.

In particular, Conjecture 3.0.1 would follow by specialising $(a,c,d,e) = (1,1,1,1)$, while Conjecture 3.0.2 would follow by specialising $(a,c,d,e) = (1,0,0,1)$.

This, however, is not the end of the story. Inspired partly by the work of Brenti [Bre95] and partly by our own experiments, we were led to consider the *more* general recurrence

$$\begin{aligned} T(n,k) \;&=\; [a(n-k)+c]\,T(n-1,k-1) \,+\, (dk+e)\,T(n-1,k) \\ &\quad + [f(n-2)+g]\,T(n-2,k-1) \end{aligned} \tag{3.6}$$

for $n \geq 1$, with initial conditions $T(0,k) = \delta_{k0}$ and $T(-1,k) = 0$.

Again, we treat a,c,d,e,f,g as algebraic indeterminates, so that the matrix elements $T(n,k)$ belong to the polynomial ring $\mathbb{Z}[a,c,d,e,f,g]$. Note that this family is invariant under the reversal $k \to n-k$ by interchanging $(a,c) \leftrightarrow (d,e)$ and leaving f and g unchanged:

$$T(n,k;a,c,d,e,f,g) \;=\; T(n,n-k;d,e,a,c,f,g)\,. \tag{3.7}$$

Our main conjecture is the following:

Conjecture 3.0.4. The lower-triangular matrix $\boldsymbol{T} = \big(T(n,k)\big)_{n,k\geq 0}$ defined by (3.6) is coefficientwise totally positive in the indeterminates a,c,d,e,f,g.

Unfortunately, for the time being, Conjectures 3.0.1, 3.0.3 and 3.0.4 remain unproven. (We have verified Conjecture 3.0.4 up to 13×13; this computation took 109 days CPU time.) The rest of this extended abstract is devoted to proving the following special case of Conjecture 3.0.3, which is of some interest in its own right:

Theorem 3.0.5. The matrix $\boldsymbol{T} = (T(n,k))_{n,k\geq 0}$ specialised to $d = f = g = 0$ is coefficientwise totally positive.

The triangle that appears in Theorem 3.0.5 is a generalisation of the reversed Stirling subset triangle, and reduces to it when $(a,c,e) = (1,0,1)$; this proves Conjecture 3.0.2. In what follows we write $\boldsymbol{T}(a,c,d,e,f,g)$ for the matrix defined by (3.6), and $\boldsymbol{T}(a,c,d,e) = \boldsymbol{T}(a,c,d,e,0,0)$ for the matrix defined by (3.5).

It is possible to prove Theorem 3.0.5 in at least two different ways: one algebraic (unpublished [CDD$^+$]), the other combinatorial.

Section 3.1 establishes combinatorial interpretations of the entries of $\boldsymbol{T}(a,c,0,e)$ and $\boldsymbol{T}(0,c,d,e)$ as generating polynomials for set partitions with suitable weights. In Section 3.2 we present a planar network D' and show — by two different arguments — that the corresponding path matrix is equal to $\boldsymbol{T}(a,c,0,e)$; Theorem 3.0.5 then follows by the Lindström–Gessel–Viennot lemma.

3.1 Set partitions and the matrices $T(a,c,0,e)$ and $T(0,c,d,e)$

From the fundamental recurrence $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$ for the Stirling subset numbers and its consequence (3.4) for the reversed Stirling subset numbers, we see that the Stirling and reversed Stirling numbers correspond to the matrix $\boldsymbol{T}(a,c,d,e)$ with $(a,c,d,e) = (0,1,1,0)$ and $(1,0,0,1)$, respectively. Moreover, if one considers instead $\left\{ {n+1 \atop k+1} \right\}$ and $\left\{ {n+1 \atop k} \right\}^{\text{rev}}$, then these matrices correspond to $\boldsymbol{T}(a,c,d,e)$ with $(a,c,d,e) = (0,1,1,1)$ and $(1,1,0,1)$, respectively. We will now show how to generalise the combinatorial interpretations of $\left\{ {n+1 \atop k+1} \right\}$ and $\left\{ {n+1 \atop k} \right\}^{\text{rev}}$ in terms of set partitions to $\boldsymbol{T}(0,c,d,e)$ and $\boldsymbol{T}(a,c,0,e)$.

We write Π_n (resp. $\Pi_{n,k}$) for the set of all partitions of the set $[n]$ into nonempty blocks (resp. into exactly k nonempty blocks). For $i \in [n]$ and $\pi \in \Pi_n$, we write $\text{smallest}(\pi, i)$ for the smallest element of the block of π that contains i. We then have:

Proposition 3.1.1 (Interpretation of $\boldsymbol{T}(0,c,d,e)$ and $\boldsymbol{T}(a,c,0,e)$ in terms of set partitions)**.**

(i) The matrix $\boldsymbol{T} = \boldsymbol{T}(0,c,d,e)$ has the combinatorial interpretation

$$T(n,k) \;=\; \sum_{\pi \in \Pi_{n+1,k+1}} \prod_{i=2}^{n+1} w_\pi(i) \tag{3.8}$$

where

$$w_\pi(i) \;=\; \begin{cases} e & \text{if smallest}(\pi,i) = 1 \\ c & \text{if smallest}(\pi,i) = i \\ d & \text{if smallest}(\pi,i) \neq 1, i \end{cases} \tag{3.9}$$

(ii) The matrix $\boldsymbol{T} = \boldsymbol{T}(a,c,0,e)$ has the combinatorial interpretation

$$T(n,k) \;=\; \sum_{\pi \in \Pi_{n+1,n+1-k}} \prod_{i=2}^{n+1} w_\pi(i) \tag{3.10}$$

where

$$w_\pi(i) \;=\; \begin{cases} c & \text{if smallest}(\pi,i) = 1 \\ e & \text{if smallest}(\pi,i) = i \\ a & \text{if smallest}(\pi,i) \neq 1, i \end{cases} \tag{3.11}$$

Please note that if one restricts a partition $\pi \in \Pi_{n+1}$ to $[m]$ for some $m < n+1$ — let us call the result $\pi_m \in \Pi_m$ — then $w_\pi(i) = w_{\pi_m}(i)$ for $2 \le i \le m$, because smallest(π,i) = smallest(π_m,i). This fact will play a key role in justifying the recurrences.

PROOF OF PROOF OF PROPOSITION 3.1.1. To prove (i) we will show that the quantities $T(n,k)$ defined by (3.8)/(3.9) satisfy the desired recurrence. Part (ii) follows immediately from (i) by way of the reversal identity (3.7) with $f = g = 0$.

In a partition $\pi \in \Pi_{n+1,k+1}$, consider the status of the element $n+1$ and what remains when it is deleted. If $n+1$ is a singleton, then it gets a weight c, and what remains is a partition of $[n]$ with k blocks, in which each element gets the same weight as it did in π. This gives a term $c\,T(n-1,k-1)$. If instead $n+1$ belongs to the block containing 1, then it gets a weight e, and what remains is a partition of $[n]$ with $k+1$ blocks, in which each element gets the same weight as it did in π. This gives a term $e\,T(n-1,k)$. Finally, if $n+1$ belongs to a block whose smallest element lies in $\{2,3,\ldots,n\}$, then it gets a weight d, and what remains is a partition of $[n]$ with $k+1$ blocks, in which each element gets the same weight as it did in π. There are k blocks not containing 1 to which the element $n+1$ could have been attached. This gives a term $dk\,T(n-1,k)$. Summing these terms gives the desired recurrence. □

Here is another recurrence satisfied by these matrices, which will be useful later:

Lemma 3.1.2 (Alternate recurrences for $\boldsymbol{T}(0,c,d,e)$ and $\boldsymbol{T}(a,c,0,e)$)**.**

(i) The matrix $\boldsymbol{T} = \boldsymbol{T}(0,c,d,e)$ satisfies the recurrence

$$T(n,k) \;=\; eT(n-1,k) + \sum_{m=0}^{n-1} \binom{n-1}{m} d^m c\, T(n-1-m,k-1) \qquad (3.12)$$

for $n \geq 1$, where $T(n,k) \stackrel{\text{def}}{=} 0$ if $n < 0$ or $k < 0$.

(ii) The matrix $\boldsymbol{T} = \boldsymbol{T}(a,c,0,e)$ satisfies the recurrence

$$T(n,k) \;=\; cT(n-1,k-1) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e\, T(n-1-m,k-m) \qquad (3.13)$$

for $n \geq 1$, where $T(n,k) \stackrel{\text{def}}{=} 0$ if $n < 0$ or $k < 0$.

PROOF. (i) Use the interpretation of Proposition 3.1.1(i), and consider the status of element $n+1$. If it belongs to the block containing 1, then it gets a weight e, and what remains is a partition of $[n]$ with $k+1$ blocks; this gives a term $eT(n-1,k)$. Otherwise, it belongs to a block of size $m+1$ where $0 \leq m \leq n-1$. We choose the other m elements of this block in $\binom{n-1}{m}$ ways; then the smallest element of this block gets weight c, and the other m elements get weight d. What remains is a partition of an $(n-m)$-element set with k blocks, corresponding to $T(n-1-m,k-1)$.

(ii) follows immediately from (i) by the reversal identity. □

We remark that these recurrences, supplemented by the initial condition $T(0,k) = \delta_{k0}$, completely determine the matrices.

3.2 Planar networks and total positivity

Figure 3.1(a) shows what we call the *standard binomial-like planar network*, which we denote D. We label the vertices of D by pairs (i,j) with $0 \leq i \leq j$, where i increases from right to left and j increases from bottom to top. The horizontal directed edge from (i,j) to $(i-1,j)$ [where $1 \leq i \leq j$] is given a weight $\alpha_{i,j-i+1}$, while the diagonal directed edge from (i,j) to $(i-1,j-1)$ [where $1 \leq i \leq j$] is given a weight $\beta_{i,j-i}$. The source vertices are $u_n = (n,n)$ and the sink vertices are $v_k = (0,k)$.

It is easy to see that if the weights are purely i-dependent, then

$$P(u_n \to v_k) \;=\; \alpha_{n,\bullet} P(u_{n-1} \to v_{k-1}) + \beta_{n,\bullet} P(u_{n-1} \to v_k)\;, \qquad (3.14)$$

so that the entries of the corresponding path matrix satisfy a purely n-dependent linear recurrence. Similarly, if the weights are purely j-dependent, then

$$P(u_n \to v_k) \;=\; \alpha_{\bullet,k} P(u_{n-1} \to v_{k-1}) + \beta_{\bullet,k} P(u_{n-1} \to v_k)\;, \qquad (3.15)$$

so that the entries of the corresponding path matrix satisfy a purely k-dependent recurrence. In particular, by setting $\alpha_{i,j} = 1$ and $\beta_{i,j} = j$, we recover a digraph yielding the Stirling subset triangle $P(u_n \to v_k) = \left\{ {n \atop k} \right\}$; and more generally, by setting $\alpha_{i,j} = c$ and $\beta_{i,j} = jd + e$, we recover $\boldsymbol{T}(0,c,d,e)$ and prove its coefficientwise total positivity. This too goes back to Brenti [Bre95].

3.2.1 The planar network D'

We will now describe a digraph D' that is obtained from D by deleting certain edges (or equivalently, setting their weights to 0), setting some of the other weights to 1, and relabelling the remaining weights. A special role will be played by the *triangular numbers* $\triangle(n) := \binom{n+1}{2}$. We also define the "triangular ceiling" $\lceil k \rceil^{\text{tri}}$ to be the smallest triangular number that is $\geq k$, and the "triangular defect" $\{k\}^{\text{tri}} := \lceil k \rceil^{\text{tri}} - k$.

For the diagonal edges, we set

$$\beta_{i,l} \;=\; \begin{cases} e_{\triangle^{-1}(i+l-1)-l,\,l} & \text{if } i+l-1 \text{ is triangular and } i+l-1 \geq \triangle(l) \\ 1 & \text{if } i+l-1 \text{ is not triangular and } i+l-1 \geq \triangle(l) \\ 0 & \text{in all other cases} \end{cases} \qquad (3.16)$$

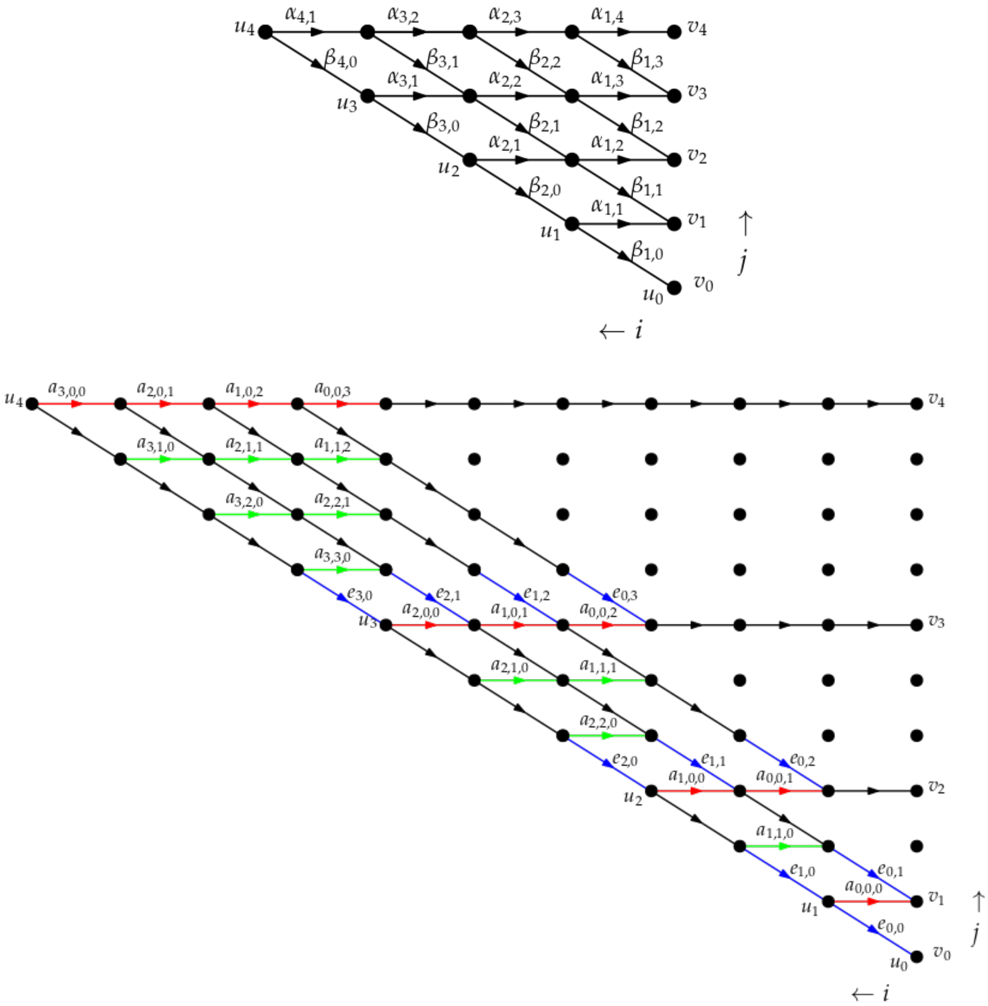


Figure 3.1: (a) The standard binomial-like planar network D (above), and (b) the planar network D' (below), each shown up to source u_4 and sink v_4.

for $i \geq 1$ and $l \geq 0$. For the horizontal edges, we set

$$\alpha_{i,l} = \begin{cases} a_{\triangle^{-1}(\lceil i+l-1 \rceil^{\text{tri}})-l,\,\{i+l-1\}^{\text{tri}},\,l-1} & \text{if } \triangle^{-1}(\lceil i+l-1 \rceil^{\text{tri}}) - l \geq \{i+l-1\}^{\text{tri}} \\ 1 & \text{if } i+l-1 \text{ is triangular and } i+l-1 < \triangle(l) \\ 0 & \text{in all other cases} \end{cases} \tag{3.17}$$

for $i, l \geq 1$. We then delete the edges with zero weight. Finally, we take the source vertices to be $u_n := (\triangle(n), \triangle(n))$ and the sink vertices to be $v_k := (0, \triangle(k))$. The resulting planar network D' is shown in Figure 3.1(b).

It is clear that every edge of D' either has weight 1 (we call these *black edges*) or else has a unique weight in the set $\mathcal{A} \cup \mathcal{E}$, where $\mathcal{A} := \{a_{i,j,l} \colon (i,j,l) \in \mathbb{N}^3 \text{ and } j \le i\}$ and $\mathcal{E} := \{e_{i,l} \colon (i,l) \in \mathbb{N}^2\}$ (we call these *coloured edges*). Each path $\mathcal{P}$ has a weight $w(\mathcal{P})$ that is a monomial in $\mathbb{Z}[\mathcal{A},\mathcal{E}]$.

Let $\boldsymbol{P}_{n,k}$ be the set of all paths in D' from u_n to v_k. It is easy to see that $\boldsymbol{P}_{n,k}$ is nonempty if and only if $n \ge k$. Furthermore, for any two distinct paths $\mathcal{P},\mathcal{P}'$ from U to V in D', we have $w(\mathcal{P}) \ne w(\mathcal{P}')$. Lastly, note that each path $\mathcal{P} \in \boldsymbol{P}_{n,k}$ traverses precisely n coloured edges, so $w(\mathcal{P})$ is a monomial of total degree n.

Applying the Lindström–Gessel–Viennot lemma to the digraph D', we can immediately conclude:

Proposition 3.2.1. The matrix $\boldsymbol{T} = (T(n,k))_{n,k\ge 0}$ defined by $T(n,k) = \sum_{\mathcal{P}\in \boldsymbol{P}_{n,k}} w(\mathcal{P})$, with entries in $\mathbb{Z}[\mathcal{A},\mathcal{E}]$, is coefficientwise totally positive.

The trouble with Proposition 3.2.1 — as with many applications of Lindström–Gessel–Viennot — is that the set of paths in a digraph can be a rather complicated object; our goal is to find a simpler combinatorial interpretation. This can be done either by obtaining a recurrence that can be compared with Lemma 3.1.2, or by constructing an explicit bijection between paths and set partitions. We shall now describe these approaches.

3.2.2 Proof by recurrence

For $0 \le m \le n$, let $u_{n,m} \stackrel{\text{def}}{=} (\triangle(n) - m, \triangle(n))$ be the vertex that lies m steps to the right of u_n. We observe that the subnetwork of D' reachable from $u_{n,m}$ is isomorphic — after contraction of some black edges, relabelling $u_n \to u_{n-m}$ and $v_k \to v_{k-m}$ of source and sink vertices, and relabelling of edge weights — to the subnetwork reachable from u_{n-m}. It follows that

$$P(u_{n,m} \to v_k) \;=\; P(u_{n-m} \to v_{k-m})\big|_{a_{i,j,l} \to a_{i,j,l+m},\, e_{i,l} \to e_{i,l+m}}\,. \tag{3.18}$$

Now consider a path $\mathcal{P}$ from u_n to v_k. If the first step is to the right, we obtain $a_{n-1,0,0}$ times $P(u_{n,1} \to v_k)$. If the first step is diagonally downwards, we enter a binomial-like network of size $n-1$, from which we can emerge on the right wall at some point $\widehat{u}_{n-1,m} \stackrel{\text{def}}{=} (\triangle(n-1), \triangle(n-1)+m)$ for $0 \le m \le n-1$; from there we follow edges diagonally downwards, arriving at the point $u_{n-1,m}$ and picking up an extra factor $e_{n-1-m,m}$. The contribution of the binomial-like network is a bit complicated, but if we make the specialisation $a_{i,j,l} \to a$ whenever $j > 0$, then its weight is just $\binom{n-1}{m} a^m$. We also specialise $a_{i,0,l} \to c_i$ and $e_{i,l} \to e_i$ in order to trivialise the relabellings in (3.18). It follows that with these specialisations the matrix $\boldsymbol{T}$ satisfies the recurrence

$$T(n,k) \;=\; c_{n-1}\, T(n-1,k-1) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m\, e_{n-1-m}\, T(n-1-m,k-m) \tag{3.19}$$

for $n \geq 1$. In particular, if $c_i = c$ and $e_i = e$ for all i, then we recover the recurrence (3.13). Applying Lemma 3.1.2(ii), we conclude:

Theorem 3.2.2. The path matrix $\boldsymbol{T} = (T(n,k))_{n,k\geq 0}$ defined by $T(n,k) = \sum_{\mathcal{P}\in \boldsymbol{P}_{n,k}} w(\mathcal{P})$, with the specialisations $e_{i,l} \to e$, $a_{i,0,l} \to c$, and $a_{i,j,l} \to a$ for $j > 0$, coincides with the matrix $\boldsymbol{T}(a,c,0,e)$.

Combining Proposition 3.2.1 with Theorem 3.2.2 proves Theorem 3.0.5. More generally, Proposition 3.2.1 shows that the matrix $\boldsymbol{T}$ defined by the recurrence (3.19) is coefficientwise totally positive in the indeterminates a, $(c_i)_{i\geq 0}$ and $(e_i)_{i\geq 0}$.

3.2.3 Proof by bijection between paths and set partitions

We now provide a bijective proof of Theorem 3.2.2. Our proof will consist of the following steps:

1. The first step is provided by Lemma 3.2.3 in which we characterise $\mathrm{wt}(\mathcal{P})$, the non-commutative product of the weights of a path $\mathcal{P}$, taken in the order of traversal (thus $\mathrm{wt}(\mathcal{P})$ is a word in the alphabet $\mathcal{A} \cup \mathcal{E}$). This will enable us to represent paths in D' as a class of words and we will then construct a bijection between these words and set partitions. We will use $\boldsymbol{W}_{n,k}$ to denote the set of words in bijective correspondence with the paths in $\boldsymbol{P}_{n,k}$.

2. In our second step, we will associate to every set partition $\pi \in \Pi_{n+1,n-k+1}$ a word $W(\pi)$. We will then show in Lemma 3.2.4 that $W(\pi) \in \boldsymbol{W}_{n,k}$. By doing

this, we will construct a map (via Lemma 3.2.3) $\Phi_{n,k}\colon \Pi_{n+1,n-k+1} \to \boldsymbol{P}_{n,k}$. We will also show in Lemma 3.2.5 that the map $\pi \mapsto W(\pi)$ is injective.

3. Then in our third step, for every word $\boldsymbol{w} \in \boldsymbol{W}_{n,k}$ we will construct a set partition $S(\boldsymbol{w})$.

4. Finally in our fourth step, we will show that the map $\pi \mapsto W(\pi)$ and the map $\boldsymbol{w} \mapsto S(\boldsymbol{w})$ are inverses of each other. As a consequence, this will also prove that the map $\Phi_{n,k}$ that we constructed in Step 2 is a bijection. We will state this in Theorem 3.2.7. We will then obtain our second proof of Theorem 3.2.2.

The reader might find it convenient to refer to Figure 3.1(b) while reading the rest of this section.

Step 1: Characterisation of words $\mathrm{wt}(\mathcal{P})$

We recall that $\mathrm{wt}(\mathcal{P})$ denotes the *non-commutative* product of the weights of $\mathcal{P}$, taken in the order of traversal. We will now characterise these words which is based on a detailed analysis of the paths in the set $\boldsymbol{P}_{n,k}$.

Lemma 3.2.3. Fix integers $n \geq k \geq 0$, and let $\boldsymbol{w}$ be a word in the alphabet $\mathcal{A} \cup \mathcal{E}$. Then $\boldsymbol{w} = \mathrm{wt}(\mathcal{P})$ for some path $\mathcal{P} \in \boldsymbol{P}_{n,k}$ if and only if and all of the following conditions hold:

(i) The first letter of $\boldsymbol{w}$ is either $e_{n-1,0}$ or $a_{n-1,j,0}$ where $0 \leq j \leq n-1$.

(ii) The last letter of $\boldsymbol{w}$ is either $e_{0,k}$ or $a_{0,0,k-1}$.

(iii) The letter following $a_{i,j,l}$ is either $e_{i-1,l+1}$ or $a_{i-1,j',l+1}$ where $j \leq j' \leq i-1$.

(iv) The letter following $e_{i,l}$ is either $e_{i-1,l}$ or $a_{i-1,j,l}$ where $0 \leq j \leq i-1$.

Furthermore, in this case the word $\boldsymbol{w}$ has length n and the path $\mathcal{P}$ is unique.

PROOF. Parts (i) and (ii) follow from examining the indices of the first (resp. last) coloured edge in $\mathcal{P}$. Parts (iii) and (iv) follow from the observation that whenever a horizontal coloured edge is traversed, l increases by 1 and i decreases by 1; and similarly, when a diagonal coloured edge is traversed, l remains unchanged and i decreases by 1. Furthermore, whenever a horizontal coloured edge is followed immediately by another horizontal coloured edge, the index j is weakly increasing. This shows the necessity for conditions (i)-(iv).

On the other hand, if we have a word $\boldsymbol{w}$ satisfying conditions (i)-(iv) then we can construct a path $\mathcal{P}$ that starts at u_n and ends at v_k. The letters in the word $\boldsymbol{w}$ will correspond to the coloured edges traversed in order and it is not difficult to see from the description (i)-(iv) that the black edges between two consecutive coloured edges given by letters in $\boldsymbol{w}$ are uniquely determined. This shows the sufficiency and also that $\mathcal{P}$ is unique. Finally, notice that the first indices in $\boldsymbol{w}$ start at $n-1$, end at 0 and decrease by 1 as we go from one letter to the next in the word and thus, $\boldsymbol{w}$ has length n. $\square$

Henceforth, we let $\boldsymbol{W}_{n,k} \stackrel{\text{def}}{=} \{\text{wt}(\mathcal{P}) \colon \mathcal{P} \in \boldsymbol{P}_{n,k}\}$.

Step 2: Construction of the map $\boldsymbol{\pi} \mapsto \boldsymbol{W(\pi)}$

We will now describe a map $W : \Pi_{n+1,n-k+1} \to \boldsymbol{W}_{n,k}$. For a set partition $\pi \in \Pi_{n+1,n-k+1}$, we will first define a total order $<_\pi$ on $[n+1]$. We will then use this total order to describe a word $W(\pi)$. Then in Lemma 3.2.4 we will show that $W(\pi) \in \boldsymbol{W}_{n,k}$. Finally, we will finish this Step by showing that the map $\pi \mapsto W(\pi)$ is injective (Lemma 3.2.5).

We first introduce some terms for set partitions following [SZ22, Section 3.1]. Given a set partition $\pi \in \Pi_n$, we say that an element $i \in [n]$ is

- an *opener* if it is the smallest element of a block of size ≥ 2;
- a *closer* if it is the largest element of a block of size ≥ 2;
- an *insider* if it is a non-opener non-closer element of a block of size ≥ 3;
- a *singleton* if it is the sole element of a block of size 1.

Also, for $i \in [n]$ and $\pi \in \Pi_n$, we write $\text{smallest}(\pi, i)$ for the smallest element of the block of π that contains i.

Given a set partition $\pi \in \Pi_{n+1,k}$ consisting of blocks $B_1, \ldots, B_k$, we define a total order $<_\pi$ on $[n+1]$ by the following procedure: start by taking the block containing 1 (we call it B_1) together with the largest elements of all the other blocks, and put them in increasing order; then insert all the remaining elements of each block (other than B_1) in increasing order immediately preceding its largest element. For example, for $\pi = \{\{1,5,8\},\{2,3,9\},\{4,7\},\{6\}\} \in \Pi_{9,4}$, the order is $15\underline{6}\,\underline{47}8\underline{239}$, where we have underlined each of the blocks other than B_1.

Under this total order, 1 is the smallest element and $n+1$ is the largest; it can therefore be written as $1p_1p_2\cdots p_n$ where $p_n = n+1$. We then define the word associated to a set partition $\pi \in \Pi_{n+1,n-k+1}$ to be $W(\pi) \stackrel{\text{def}}{=} w_n \cdots w_1$ where

$$w_i \stackrel{\text{def}}{=} \begin{cases} e_{i-1,l_i} & \text{if smallest}(\pi, p_i) = p_i \\ a_{i-1,0,l_i} & \text{if } p_i \in B_1 \ \ \text{[i.e. smallest}(\pi, p_i) = 1] \\ a_{i-1,j,l_i} & \text{if smallest}(\pi, p_i) \neq 1, p_i \text{ and largest}(\boldsymbol{p}, i)_j = p_{i-1} \end{cases} \tag{3.20}$$

Here $\text{largest}(\boldsymbol{p},i)_j$ denotes the jth largest element of the set $[2,n+1] \setminus \{p_i,\dots,p_n\}$. The index l_i is defined recursively: we set $l_n = 0$, and for $i < n$ we define $l_{i-1} = l_i$ if $\text{smallest}(\pi, p_i) = p_i$, and $l_{i-1} = l_i + 1$ otherwise. We say that the letter p_i ($2 \leq p_i \leq n+1$) has *type* $e, a_0, a_{\neq 0}$ depending on whether w_i is $e_{i-1,l_i}, a_{i-1,0,l_i}$ or a_{i-1,j,l_i} for $j \neq 0$, respectively. Thus, p_i has type e if it is the smallest in its block (note that $p_i \neq 1$); p_i has type a_0 if it is in block B_1 (note again that $p_i \neq 1$); and p_i has type $a_{\neq 0}$ otherwise.

For example, consider again $\pi = \{\{1,5,8\},\{2,3,9\},\{4,7\},\{6\}\} \in \Pi_{9,4}$ with total order 156478239. Table 3.1 illustrates how we obtain the word $W(\pi)$.
Thus, we get that

$$W(\pi) \ = \ a_{7,6,0}\, a_{6,6,1}\, e_{5,2}\, a_{4,0,2}\, a_{3,3,3}\, e_{2,4}\, e_{1,4}\, a_{0,0,4}\, . \tag{3.21}$$

We can check that here $W(\pi) \in \boldsymbol{W}_{8,5}$. In fact, this is true in general due to the following lemma:

Lemma 3.2.4. Given $\pi \in \Pi_{n+1,n-k+1}$, the word $W(\pi)$ consists of letters from $\mathcal{A} \cup \mathcal{E}$ and satisfies the conditions in Lemma 3.2.3, and thus we have $W(\pi) \in \boldsymbol{W}_{n,k}$.

i	p_i	$i-1$ (First index)	Type	l_i (Last index)	$[2,n+1]\setminus\{p_i,\dots,p_n\}$	j (Middle index)	w_i
8	9	7	$a_{\neq 0}$	0	$\{8,7,6,5,4,\underline{3},2\}$	6	$a_{7,6,0}$
7	3	6	$a_{\neq 0}$	1	$\{8,7,6,5,4,\underline{2}\}$	6	$a_{6,6,1}$
6	2	5	e	2	$\{8,7,6,5,4\}$		$e_{5,2}$
5	8	4	a_0	2	$\{7,6,5,4\}$	0	$a_{4,0,2}$
4	7	3	$a_{\neq 0}$	3	$\{6,5,\underline{4}\}$	3	$a_{3,3,3}$
3	4	2	e	4	$\{6,5\}$		$e_{2,4}$
2	6	1	e	4	$\{5\}$		$e_{1,4}$
1	5	0	a_0	4	$\varnothing$	0	$a_{0,0,4}$

Table 3.1: Obtaining the word $W(\pi)$ when $\pi = \{\{1,5,8\},\{2,3,9\},\{4,7\},\{6\}\} \in \Pi_{9,4}$. Here p_{i-1} has been underlined whenever p_i is of type $a_{\neq 0}$.

PROOF. To check that $w_i \in \mathcal{A}\cup\mathcal{E}$, we need only verify that if $w_i = a_{i-1,j,l}$ then $j \leq i-1$. This is clearly true if $p_i \in B_1$, in which case $j=0$. Otherwise, since $[2,n+1]\setminus\{p_i,\dots,p_n\}$ has cardinality $i-1$, and $\text{largest}(\boldsymbol{p},i)_j = p_{i-1}$ is the j-th largest element in this set, j can be at most $i-1$. Now we check the four conditions of Lemma 3.2.3.

(i) The element $p_n = n+1$ is the largest element of some block B. If it is a singleton we have $w_n = e_{n-1,0}$, and if $B = B_1$ we have $w_n = a_{n-1,0,0}$. Otherwise, the next-largest element in B is $p_{n-1} = n+1-j$ for some $1 \leq j \leq n-1$, so p_{n-1} is the j-th largest element in $[2,n+1]\setminus\{n+1\} = [2,n]$, and $w_n = a_{n-1,j,0}$.

(ii) Since π comprises $n-k+1$ blocks (one of which is B_1), there are exactly $n-k$ indices i for which $\text{smallest}(\pi,p_i) = p_i$. These are exactly the indices for which $l_{i-1} = l_i$; and for them we have $w_i = e_{i-1,l_i}$. Note that p_1 is either the smallest element of its block or the second-smallest element of B_1. In the first case $l_1 = k$ and $w_1 = e_{0,k}$, while in the second case $l_1 = k-1$ and $w_1 = a_{0,0,k-1}$.

(iii) If $w_{i+1} = a_{i,j,l}$ (where $l = l_{i+1}$), we have $\text{smallest}(\pi,p_{i+1}) \neq p_{i+1}$ and hence $l_i = l+1$; we also have $\text{largest}(\boldsymbol{p},i+1)_j = p_i$. We treat the cases $j=0$ and $j \neq 0$ separately. If $j=0$, then $p_{i+1} \in B_1$. If p_i is a singleton, then $w_i = e_{i-1,l+1}$. Otherwise, $w_i = a_{i-1,j',l+1}$ for some $0 \leq j' \leq i-1$, where the inequality follows from the fact that $w_i \in \mathcal{A}$. If instead $j \neq 0$, then p_{i+1} is contained in a block $B \neq B_1$ and $\min(B) \neq p_{i+1}$. The element p_i is thus the next-largest element of B (just below

$p_{i+1})$. If $p_i = \min(B)$, then $w_i = e_{i-1,l+1}$. Otherwise, we have $p_{i-1} < p_i$ and we have $w_i = a_{i-1,j',l+1}$. Now we only need to check that $j \leq j' \leq i-1$. The upper bound is clearly true since $w_i \in \mathcal{A}$. For the lower bound, if we instead have $j' < j$, then $\text{largest}(\boldsymbol{p}, i+1)_{j'} > p_i$, which implies $\text{largest}(\boldsymbol{p}, i)_{j'} = \text{largest}(\boldsymbol{p}, i+1)_{j'}$, that is, $p_{i-1} > p_i$, a contradiction.

(iv) If $w_{i+1} = e_{i,l}$ (where $l = l_{i+1}$), we have $\text{smallest}(\pi, p_{i+1}) = p_{i+1}$ and hence $l_i = l$. If p_i is a singleton, then $w_i = e_{i-1,l}$; otherwise, $w_i = a_{i-1,j,l}$, where $0 \leq j \leq i-1$ because $w_i \in \mathcal{A}$. This completes the proof. □

Thus, we have that Lemmas 3.2.3 and 3.2.4 together define a map $\Phi_{n,k}\colon \Pi_{n+1,n-k+1} \to \boldsymbol{P}_{n,k}$. We will now also show that this map is injective.

Lemma 3.2.5. The map $\pi \mapsto W(\pi)$ is injective.

PROOF. Let π and π' be set partitions with $W(\pi) = W(\pi') = \boldsymbol{w}$, and let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be the sequences associated to the total orders $<_\pi$ and $<_{\pi'}$, respectively. We will show inductively that $p_i = p'_i$ and $\pi|\{1, p_i, \ldots, p_n\} = \pi'|\{1, p'_i, \ldots, p'_n\}$.

For the base case $i = n$, $p_n = p'_n = n+1$ is obvious. Moreover, from Lemmas 3.2.4 and 3.2.3, w_n is either $e_{n-1,0}$ or $a_{n-1,j,0}$ for $0 \leq j \leq n-1$. From (3.20), if w_n is of type e or $a_{\neq 0}$, then $\pi|\{1, p_n\} = \{\{1\}, \{n+1\}\} = \pi'|\{1, p'_n\}$; and if w_n is of type a_0, then $\pi|\{1, p_n\} = \{\{1, n+1\}\} = \pi'|\{1, p'_n\}$. This settles the base case.

Now consider $i < n$: we have two cases, depending on whether w_{i+1} is of type $a_{\neq 0}$ or types a_0, e.

• If $w_{i+1} = a_{i,j,l}$ for some $1 \leq j \leq i$, then $p_i = p'_i$ is clearly determined by the induction hypothesis and (3.20); moreover, $p_i = p'_i$ and $p_{i+1} = p'_{i+1}$ are in the same block $B \neq B_1$, which establishes $\pi|\{1, p_i, \ldots, p_n\} = \pi'|\{1, p'_i, \ldots, p'_n\}$.

• If w_{i+1} is of type e or a_0, then by definition of the total order, p_i and p'_i are both equal to the largest element of $[2, n+1] \setminus \{p_i, \ldots, p_n\}$; furthermore, this element $p_i = p'_i$ can either be in B_1 or else be the largest element of some block $B \neq B_1$. From (3.20), we see that the former (resp. latter) will be true if w_i is of type a_0 (resp. e or $a_{\neq 0}$). In both cases we have $\pi|\{1, p_i, \ldots, p_n\} = \pi'|\{1, p'_i, \ldots, p'_n\}$. This concludes the proof. □

Step 3: Construction of the map $w \mapsto S(w)$

Given a word $\boldsymbol{w} \in \boldsymbol{W}_{n,k}$ with $\boldsymbol{w} = w_n \cdots w_1$ (i.e., each letter $w_i \in \mathcal{A} \cup \mathcal{E}$ and the word $\boldsymbol{w}$ satisfies the conditions in Lemma 3.2.3), we will construct a set partition $S(\boldsymbol{w})$. Our construction here will comprise of two steps:

(a) We will first construct a sequence $\boldsymbol{q} = q_n \cdots q_1$ where $q_i \in [2, n+1]$ and $q_n = n+1$. Every time we read a letter w_i, we will choose $q_i \in [2, n+1]$.

(b) We will build up $S(\boldsymbol{w})$ by inserting elements into its blocks, one at a time. We begin from $\mu_0 = \{\{1\}\}$ and end with $\mu_n = S(\boldsymbol{w})$. We will obtain μ_{n-i+1} by inserting q_i into μ_{n-i}.

Construction of sequence $\boldsymbol{q} = q_n \cdots q_1$:
We start from $q_n = n+1$. Then, for $i < n$, we proceed inductively: if $w_{i+1} = a_{i,j,l}$ with $j > 0$, we set q_i to be the jth largest element of the set $[2, n+1] \setminus \{q_{i+1}, \ldots, q_n\}$; otherwise we set q_i to be the largest element of $[2, n+1] \setminus \{q_{i+1}, \ldots, q_n\}$.

Construction of $S(\boldsymbol{w})$:
We will build up $S(\boldsymbol{w})$ by inserting elements into its blocks, one at a time, as we read the word $\boldsymbol{w}$ from left to right, beginning from $\mu_0 = \{\{1\}\}$ and ending with $\mu_n = S(\boldsymbol{w})$. Each block will be built up in decreasing order, starting with its largest element; indeed, each block other than B_1 will be built from start to finish in successive stages of the algorithm. Whenever we insert an element $q_i \in [2, n+1]$ into a block $B \neq B_1$, we will also declare whether that block is *finished* (i.e. q_i is an opener or a singleton in $S(\boldsymbol{w})$) or *unfinished* (i.e. q_i is a closer or an insider in $S(\boldsymbol{w})$).

Every time we read a letter w_i we insert q_i into μ_{n-i} in one of five ways:

(i) insert q_i into the block B_1;

(ii) insert q_i as an opener into an unfinished block $B \neq B_1$ (thereby declaring the block finished);

(iii) insert q_i as an insider into an unfinished block $B \neq B_1$ (thereby declaring the block still unfinished);

(iv) create a new block containing q_i as a singleton (thereby declaring the block finished);

(v) or create a new block containing q_i as a closer (thereby declaring the block unfinished).

The result is called μ_{n-i+1}.

We now explain how the elements $q_n, \ldots, q_1$ will be inserted into the set partition. At stage i (starting from $i = n$ and proceeding downwards), by Lemma 3.2.3 there are three possibilities for the letter w_i: $e_{i-1,l}$, $a_{i-1,0,l}$, or $a_{i-1,j,l}$ for some $1 \leq j \leq i-1$.

Case 1: $w_i = e_{i-1,l}$. If there is an unfinished block, we insert q_i into that block as an opener; otherwise, we create a new block with q_i as a singleton. In both cases the block is declared finished.

Case 2: $w_i = a_{i-1,0,l}$. We insert q_i into block B_1. (We remark that whenever this happens, π_{n-i} has no unfinished blocks.)

Case 3: $w_i = a_{i-1,j,l}$ for some $1 \leq j \leq i-1$. If there is an unfinished block, we insert q_i into that block as an insider; otherwise, we create a new block with q_i as a closer. In both cases the block is declared unfinished.

It is clear that each stage there is at most one unfinished block, so the algorithm is well-defined. Moreover, Lemma 3.2.3(iii) guarantees that last $a_{i,j,l}$ with $j > 0$ must be immediately followed by an $e_{i-1,l+1}$, so the algorithm will terminate with all blocks finished; and an element q_i inserted as an insider, closer, etc. will indeed have that status in the final set partition π_n.

Thus, for a word $\boldsymbol{w} \in \boldsymbol{W}_{n,k}$, we have constructed a map $\boldsymbol{w} \mapsto S(\boldsymbol{w})$ where $\boldsymbol{w} \in \Pi_{n+1,n-k+1}$.

Step 4: Proof of bijection

We will now prove that the map $\pi \mapsto W(\pi)$, constructed in Step 2, and the map $\boldsymbol{w} \mapsto S(\pi)$, constructed in Step 3, are bijective inverses of each other. However, we first require the following lemma:

Lemma 3.2.6. Given a word $\boldsymbol{w} \in \boldsymbol{W}_{n,k}$, the sequence $\boldsymbol{q}$ (in the construction of $S(\boldsymbol{w})$) coincides with the sequence $\boldsymbol{p}$ associated to the total order $<_{S(\boldsymbol{w})}$.

PROOF. For blocks $B \neq B_1$, we observe from Lemma 3.2.3(iii) that each maximal subword of $\boldsymbol{w}$ consisting of letters $a_{i,j,l}$ with $j > 0$ must be immediately followed by an $e_{i',l'}$, and in this subword the indices j must be weakly increasing. These facts ensure that the elements of B are inserted in decreasing order, and in successive stages. (The first element of B was the largest available element at the time of its insertion, because the letter w_{i+1} immediately preceding this subword was *not* of the form $a_{i,j,l}$ with $j > 0$.) For the block B_1, elements (other than 1) were also inserted in decreasing order: by Lemma 3.2.3(iii), $a_{i-1,0,l}$ cannot be immediately preceded by $a_{i,j,l}$ with $j > 0$, so again each element inserted into B_1 was the largest available element. Finally, the union of B_1 with the largest elements of the remaining blocks was also inserted in decreasing order, because each such element was the largest available element at the time of its insertion. All this taken together shows that the sequence $\boldsymbol{q}$ coincides with the sequence $\boldsymbol{p}$ associated to the total order $<_{S(\boldsymbol{w})}$. □

We are now ready to establish our bijection.

Theorem 3.2.7. The maps W and S are bijective inverses of each other. Thus, the map $\Phi_{n,k}$ is a bijection of $\Pi_{n+1,n-k+1}$ onto $\boldsymbol{P}_{n,k}$.

PROOF. From Lemma 3.2.5, we know that the map W is injective. Thus, it suffices to show that $W(S(\boldsymbol{w})) = \boldsymbol{w}$ which will show that W is surjective and that S is the bijective inverse of W.

From Lemma 3.2.6 we know that the sequence $\boldsymbol{q}$ for the word $\boldsymbol{w}$ coincides with the sequence $\boldsymbol{p}$ associated to the total order $<_{S(\boldsymbol{w})}$. Thus, we have $q_i = p_i$. Also, Cases 1–3 of construction of $S(\boldsymbol{w})$ correspond to inserting q_i with a status that corresponds to the three cases of (3.20). Thus, the type of w_i — that is, e, a_0 or $a_{\neq 0}$ — coincides with that of $W(S(\boldsymbol{w}))_i$.

We will now show that the indices i, j, l match. Lemmas 3.2.3 and 3.2.4 guarantee that the indices i and l in both $\boldsymbol{w}$ and $W(S(\boldsymbol{w}))$ are determined by the types, and therefore agree. The indices j matter only in the type $a_{\neq 0}$, and the third case of (3.20) coincides with how q_{i-1} is chosen. This proves that $W(S(\boldsymbol{w})) = \boldsymbol{w}$. □

Everything is now in place for us to prove Theorem 3.2.2 bijectively.

PROOF. [Second proof of Theorem 3.2.2] The definition (3.20) tells us that, within each word $\boldsymbol{w} \in \boldsymbol{W}_{n,k}$ (we know from Theorem 3.2.7 and Lemma 3.2.3 that these are in bijection with paths $\boldsymbol{P}_{n,k}$ and that their weights match), the letters $a_{i,0,l}$ correspond to elements in B_1, and the letters $e_{i,l}$ (resp. $a_{i,j,l}$ for $j > 0$) correspond to minimal (resp. non-minimal) elements of blocks $B \neq B_1$. After the specialisations $e_{i,l} \to e$, $a_{i,0,l} \to c$, and $a_{i,j,l} \to a$ for $j > 0$, by Proposition 3.1.1(ii) this is precisely the matrix $\boldsymbol{T}(a,c,0,e)$. □

Chapter 4

Classical continued fractions for some multivariate polynomials generalising the Genocchi and median Genocchi numbers[1]

4.1 Introduction

Our purpose here is to present continued fractions for some multivariate polynomials that generalize either the Genocchi numbers [OEI19, A110501] introduced in Section 2.4.1

$$(g_n)_{n\geq 0} \;=\; 1,1,3,17,155,2073,38227,929569,28820619,1109652905,\ldots \quad (4.1)$$

or the median Genocchi numbers [OEI19, A005439] introduced in Section 2.4.2

$$(h_n)_{n\geq 0} \;=\; 1,1,2,8,56,608,9440,198272,5410688,186043904,\ldots\,. \quad (4.2)$$

The present chapter can be viewed as a sequel to a recent paper by Sokal and Zeng [SZ22] in which they presented Stieltjes-type and Jacobi-type continued fractions

[1] The work in this chapter was largely in collaboration with Alan D. Sokal. See pp. 14-15 for details.

for some "master polynomials" that enumerate permutations, set partitions or perfect matchings with respect to a large (sometimes infinite) number of independent statistics. These polynomials systematize the "linear family": namely, sequences in which the Stieltjes continued-fraction coefficients $(\alpha_n)_{n\geq 1}$ grow linearly in n. More precisely, in the simplest case [SZ22, Theorem 2.1] the even and odd coefficients grow affinely in n:

$$\alpha_{2k-1} \;=\; x+(k-1)u, \qquad \alpha_{2k} \;=\; y+(k-1)v \tag{4.3}$$

When $x = y = u = v = 1$, these coefficients $\alpha_{2k-1} = \alpha_{2k} = k$ correspond to Euler's [Eul60, section 21] continued fraction (1.1) for the sequence $a_n = n!$; so it is natural to expect that the resulting polynomials $P_n(x,y,u,v)$ can be interpreted as enumerating permutations of $[n]$ with respect to some suitable statistics. The purpose of [SZ22] was to exhibit explicitly those statistics, and then to present some generalizations involving more refined statistics.

In the present chapter we take one step up in complexity, to consider the "quadratic family", in which the $(\alpha_n)_{n\geq 1}$ grow quadratically in n. For instance, we could consider

$$\alpha_{2k-1} \;=\; [x_1+(k-1)u_1]\,[x_2+(k-1)u_2] \tag{4.4a}$$

$$\alpha_{2k} \;=\; [y_1+(k-1)v_1]\,[y_2+(k-1)v_2] \tag{4.4b}$$

With all parameters set to 1, these coefficients $\alpha_{2k-1} = \alpha_{2k} = k^2$ correspond to the continued fraction (2.51) [Vie, eq. (9.7)] [Vie83, p. V-15] for the median Genocchi numbers; so it is natural to seek a combinatorial model that is enumerated by the median Genocchi numbers. Many such models are known. In this chapter we shall focus on a class of permutations of $[2n]$ called *D-permutations* [LW22, Laz20], which were defined in Section 2.4.3. We recall that a permutation is a D-permutation in case $2k-1 \leq \sigma(2k-1)$ and $2k \geq \sigma(2k)$ for all k. Let us also recall that a permutation is an *e-semiderangement* (resp. *o-semiderangement*) in case in contains no even (resp. odd) fixed points; it is a *derangement* in case it contains no fixed points at all. A D-permutation that is also an e-semiderangement (resp. o-semiderangement, derangement) will be called a ***D-e-semiderangement*** (resp. ***D-o-semiderangement***, ***D-derangement***).A D-permutation that contains exactly one cycle is called a ***D-cycle***. We also recall that $\mathfrak{D}_{2n}$ (resp. $\mathfrak{D}^{\mathrm{e}}_{2n}, \mathfrak{D}^{\mathrm{o}}_{2n}, \mathfrak{D}^{\mathrm{eo}}_{2n}, \mathfrak{DC}_{2n}$) denotes the set of all D-permutations (resp. D-e-semiderangements, D-o-semiderangements, D-derangements, D-cycles) of $[2n]$.

It is known [Dum74, DR94, LW22, Laz20] — and we will recover as part of our work — that

$$|\mathfrak{D}_{2n}| = h_{n+1} \tag{4.5a}$$

$$|\mathfrak{D}^{\mathrm{e}}_{2n}| = |\mathfrak{D}^{\mathrm{o}}_{2n}| = g_n \tag{4.5b}$$

$$|\mathfrak{D}^{\mathrm{eo}}_{2n}| = h_n \tag{4.5c}$$

$$|\mathfrak{DC}_{2n}| = g_{n-1} \tag{4.5d}$$

This suggests that we can obtain continued fractions for multivariate polynomials enumerating D-permutations, D-semiderangements, D-derangements or D-cycles by generalizing the known continued fractions for the Genocchi and median Genocchi numbers, analogously to what was done in [SZ22] by generalizing Euler's continued fraction for the factorials. That is, indeed, what we shall do in this chapter. These continued fractions will be of Stieltjes and Thron types; they can also be transformed by contraction into Jacobi type.[2] Our principal results will be:

1) A Thron-type continued fraction in 12 variables (Theorem 4.2.3) that enumerates D-permutations with respect to the parity-refined record-and-cycle classification (defined in Section 2.5.1). Specializations of this continued fraction give Stieltjes-type continued fractions that enumerate D-semiderangements and D-derangements.

2) A Thron-type continued fraction in 22 variables (Theorem 4.2.7) that gener-

[2] We call these *classical* continued fractions, in order to contrast them with the recently-introduced *branched* continued fractions [PSZ18], which are believed to apply to certain higher-order generalizations of the Genocchi numbers [PSZ18, Conjecture 16.1].

alizes the previous T-fraction by including four pairs of (p,q)-variables that count crossings and nestings and a pair of variables that count pseudo-nestings of fixed points (defined in Section 2.5.2).

3) A Thron-type continued fraction in six infinite families of indeterminates (Theorem 4.2.9) that generalizes the preceding two by further refining the counting of crossings, nestings and pseudo-nestings.

We call these the three versions of our "first T-fraction". Already the first version (Theorem 4.2.3) contains several known continued fractions for the Genocchi and median Genocchi numbers as special cases. We also have three variant versions of the first T-fraction (Theorems 4.2.10 and 4.2.11) using slightly different statistics.

We will also have three versions of a "second T-fraction" (Theorems 4.3.2, 4.3.6 and 4.3.9) that includes the counting of cycles, at the expense of partially renouncing the counting of records; as a corollary we obtain a continued fraction for D-cycles (Corollary 4.3.5). The second T-fractions look less symmetrical than the first ones; this defect seems to be inherent in including the counting of cycles (just as in [SZ22]). The proofs of both the first and second T-fractions will be based on bijections from D-permutations to labelled Schröder paths.

Our continued fractions will imply that our multivariate polynomial sequences generalising the Genocchi and median Genocchi numbers are coefficientwise-Hankel totally positive with respect to a large number of variables.

Before proceeding with the rest of this chapter, the reader may refer back to Section 2.4 to recall the necessary definitions and facts concerning the Genocchi and median Genocchi numbers; Sections 2.2.1, 2.2.2 and 2.2.3 for continued fractions, and transformation and contraction formulae, respectively; and Section 2.5 for various permutation statistics. The plan of this chapter is as follows: In Section 4.2 we state the three versions of the first T-fraction and note some of their corollaries, and in Section 4.3 we do the same for the second T-fraction. Then in Section 4.4 we write the consequence of our continued fractions to coefficientwise-Hankel total positivity of our polynomial sequences. In Section 4.5 we prove the first T-fraction by a bijection that combines ideas of Randrianarivony [Ran97] and Foata–Zeilberger [FZ90] together with some new ingredients. In Section 4.6 we prove the second T-fraction by a variant bijection that analogously combines ideas of Randrianarivony [Ran97] and Biane [Bia93].[3] Our proofs are based on the combinatorial theory of continued fractions introduced in Sections 2.2.4, 2.2.5. We conclude (Section 4.7) with some brief remarks on the relation of our work to [SZ22].

4.2 First T-fraction and its generalizations

In this section we state our first T-fraction for D-permutations, in three increasingly more general versions. The first and most basic version (Theorem 4.2.3) is a T-fraction in 12 variables that enumerates D-permutations with respect to the parity-refined record-and-cycle classification; it includes many previously known results as special cases. The second version (Theorem 4.2.7) is a (p,q)-generalization of the first one: it is a T-fraction in 22 variables that enumerates D-permutations with respect to the parity-refined record-and-cycle classification together with four pairs of (p,q)-variables counting the refined categories of crossing and nesting as well as two variables corresponding to pseudo-nestings of fixed points. Finally, our third version — what we call the "first master T-fraction" (Theorem 4.2.9) — is a T-fraction in six infinite families of indeterminates that generalizes the preceding two by employing the index-refined crossing and nesting statistics defined in (2.67).

The plan of this section is as follows: We begin (Section 4.2.1) by establishing some easy but important preliminary results concerning record-antirecord fixed points. Then, in Sections 4.2.2–4.2.4 we state the three versions of our first T-fraction and note some of their corollaries. Finally, in Section 4.2.5 we state variant forms of the three T-fractions. All these results will be proven in Section 4.5.

[3] Randrianarivony's proof [Ran97, Section 6] is already motivated by ideas of Foata and Zeilberger [FZ90], as Randrianarivony himself points out [Ran97, pp. 78, 88]. By contrast, our work here seems to be the first to apply a Biane-like [Bia93] bijection to D-permutations, D-semiderangements or D-derangements.

4.2.1 Preliminaries on record-antirecords

An important role in our study of D-permutations will be played by *record-antirecords*, i.e. indices i that are both a record and an antirecord. It is easy to see that, in any permutation, a record-antirecord must be a fixed point. More precisely:

Lemma 4.2.1. Consider a permutation $\sigma \in \mathfrak{S}_N$. An index $i \in [N]$ is a record-antirecord if and only if σ maps each of the sets $\{1,\ldots,i-1\}$, $\{i\}$ and $\{i+1,\ldots,N\}$ onto itself.

PROOF. If i is a record, then σ maps $\{1,\ldots,i-1\}$ injectively into $\{1,\ldots,\sigma(i)-1\}$, so that $\sigma(i) \geq i$. If i is an antirecord, then σ maps $\{i+1,\ldots,N\}$ injectively into $\{\sigma(i)+1,\ldots,N\}$, so that $\sigma(i) \leq i$. So if i is a record-antirecord, then i must be a fixed point and σ must map both $\{1,\ldots,i-1\}$ and $\{i+1,\ldots,N\}$ bijectively onto themselves.

The converse is obvious. □

For D-permutations, record-antirecords can occur only in pairs:

Lemma 4.2.2. Let $\sigma \in \mathfrak{D}_{2n}$ and let $i \in [n]$. Then the following are equivalent:

(a) $2i-1$ is a record-antirecord.

(b) $2i$ is a record-antirecord.

(c) σ maps each of the sets $\{1,\ldots,2i-1\}$ and $\{2i,\ldots,2n\}$ onto itself.

(d) σ maps each of the sets $\{1,\ldots,2i-2\}$, $\{2i-1\}$, $\{2i\}$ and $\{2i+1,\ldots,2n\}$ onto itself.

PROOF. By Lemma 4.2.1, (d) $\Longleftrightarrow$ "(a) and (b)", and "(a) or (b)" $\Longrightarrow$ (c). On the other hand, if σ is a D-permutation, then $\sigma(2i-1) \geq 2i-1$ and $\sigma(2i) \leq 2i$, so (c) $\Longrightarrow$ (d). □

Let us say that a permutation is ***pure*** if it has no record-antirecords. We write $\mathfrak{D}^{\text{pure}}_{2n}$ for the set of pure D-permutations of $[2n]$. From Lemma 4.2.1 we have

$$
\begin{array}{ccccccc}
 & & & \mathfrak{D}^{\text{e}}_{2n} & & & \\
 & & \subseteq & & \subseteq & & \\
\mathfrak{DC}_{2n} \subseteq \mathfrak{D}^{\text{eo}}_{2n} = \mathfrak{D}^{\text{e}}_{2n} \cap \mathfrak{D}^{\text{o}}_{2n} & & & & & & \mathfrak{D}^{\text{e}}_{2n} \cup \mathfrak{D}^{\text{o}}_{2n} \subseteq \mathfrak{D}^{\text{pure}}_{2n} \subseteq \mathfrak{D}_{2n}\,. \\
 & & \subseteq & & \subseteq & & \\
 & & & \mathfrak{D}^{\text{o}}_{2n} & & &
\end{array}
\tag{4.6}
$$

These inclusions are strict for $n \geq 2$:

	$\mathfrak{DC}_{2n}$	$\mathfrak{D}^{\text{eo}}_{2n}$	$\mathfrak{D}^{\text{e/o}}_{2n}$	$\mathfrak{D}^{\text{e}}_{2n} \cup \mathfrak{D}^{\text{o}}_{2n}$	$\mathfrak{D}^{\text{pure}}_{2n}$	$\mathfrak{D}_{2n}$
n	g_{n-1}	h_n	g_n	$2g_n - h_n$	$h^\flat_{n+1}$	h_{n+1}
0	0	1	1	1	1	1
1	1	1	1	1	1	2
2	1	2	3	4	5	8
3	3	8	17	26	41	56
4	17	56	155	254	493	608
5	155	608	2073	3538	8161	9440
6	2073	9440	38227	67014	178469	198272

For instance, the permutation 4231 belongs to $\mathfrak{D}^{\text{pure}}_4 \setminus (\mathfrak{D}^{\text{e}}_4 \cup \mathfrak{D}^{\text{o}}_4)$: it has both even and odd fixed points, but they are not record-antirecords. Likewise, the permutation 513462 belongs to $\mathfrak{D}^{\text{pure}}_6 \setminus (\mathfrak{D}^{\text{e}}_6 \cup \mathfrak{D}^{\text{o}}_6)$: it has an odd-even pair of fixed points 34, but they are not record-antirecords. We will show later that $|\mathfrak{D}^{\text{pure}}_{2n}| = h^\flat_{n+1}$; we recall that these numbers were defined by the S-fraction (2.57)/(2.58).

4.2.2 First T-fraction

We now introduce a polynomial in 12 variables that enumerates D-permutations according to the parity-refined record-and-cycle classification:

$$\begin{aligned} P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}) \;=& \\ \sum_{\sigma\in\mathfrak{D}_{2n}} & x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} y_1^{\rm ereccval(\sigma)} y_2^{\rm ereccdrise(\sigma)} \;\times \\ & u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} v_1^{\rm nrcval(\sigma)} v_2^{\rm nrcdrise(\sigma)} \;\times \\ & w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \;. \end{aligned} \tag{4.7}$$

Thus, the variables x_1 and u_1 are associated to cycle peaks, y_1 and v_1 to cycle valleys, x_2 and u_2 to cycle double falls, y_2 and v_2 to cycle double rises, $w_{\rm e}$ and $w_{\rm o}$ to neither-record-antirecord fixed points, and $z_{\rm e}$ and $z_{\rm o}$ to record-antirecord fixed points. We remark that (4.7) is the same as the polynomial introduced in [SZ22, eq. (2.19)], but restricted to D-permutations and refined to record the parity of fixed points. Since in a D-permutation each even (resp. odd) index $i \in [2n]$ belongs to exactly one of the 6 types mentioned at the end of Section 2.5.1, it follows that the polynomial P_n is homogeneous of degree n in $x_1,x_2,u_1,u_2,w_{\rm e},z_{\rm e}$ and also homogeneous of degree n in $y_1,y_2,v_1,v_2,w_{\rm o},z_{\rm o}$.

The polynomials (4.7) have a beautiful T-fraction:

Theorem 4.2.3 (First T-fraction for D-permutations)**.** The ordinary generating function of the polynomials (4.7) has the T-type continued fraction

$$\sum_{n=0}^{\infty} P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o})\, t^n \;=\;$$

$$\cfrac{1}{1 - z_{\rm e} z_{\rm o}\, t - \cfrac{x_1 y_1\, t}{1 - \cfrac{(x_2{+}w_{\rm e})(y_2{+}w_{\rm o})\, t}{1 - \cfrac{(x_1{+}u_1)(y_1{+}v_1)\, t}{1 - \cfrac{(x_2{+}u_2{+}w_{\rm e})(y_2{+}v_2{+}w_{\rm o})\, t}{1 - \cfrac{(x_1{+}2u_1)(y_1{+}2v_1)\, t}{1 - \cfrac{(x_2{+}2u_2{+}w_{\rm e})(y_2{+}2v_2{+}w_{\rm o})\, t}{1 - \cdots}}}}}}} \tag{4.8}$$

with coefficients

$$\alpha_{2k-1} \;=\; [x_1 + (k-1)u_1]\,[y_1 + (k-1)v_1] \tag{4.9a}$$

$$\alpha_{2k} \;=\; [x_2 + (k-1)u_2 + w_{\rm e}]\,[y_2 + (k-1)v_2 + w_{\rm o}] \tag{4.9b}$$

$$\delta_1 \;=\; z_{\rm e} z_{\rm o} \tag{4.9c}$$

$$\delta_n \;=\; 0 \qquad \text{for } n \ge 2 \tag{4.9d}$$

We will prove Theorem 4.2.3 in Section 4.5.

Note that each of the coefficients α_i and δ_i is homogeneous of degree 1 in $x_1,x_2,u_1,u_2,w_{\rm e},z_{\rm e}$ and also homogeneous of degree 1 in $y_1,y_2,v_1,v_2,w_{\rm o},z_{\rm o}$. This reflects the homogeneities of the P_n.

Note also that the involution of $\mathfrak{D}_{2n}$ defined by $\sigma \mapsto R\circ\sigma\circ R$ where $R(i) = 2n+1-i$ is the reversal map, interchanges $x_1 \leftrightarrow y_1$, $x_2 \leftrightarrow y_2$, $u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_2$, $z_{\rm e} \leftrightarrow z_{\rm o}$, $w_{\rm e} \leftrightarrow w_{\rm o}$. The T-fraction (4.8)/(4.9) is invariant under these simultaneous interchanges.

The T-fraction (4.8)/(4.9) has numerous interesting specialisations:

1) With all variables equal to 1, it gives the T-fraction (2.55)/(2.56) for the once-shifted median Genocchi numbers, and confirms that $|\mathfrak{D}_{2n}| = h_{n+1}$.

2) With $z_{\mathrm{e}} = 0$ and/or $z_{\mathrm{o}} = 0$ and all other variables equal to 1, it gives the S-fraction (2.57)/(2.58) and shows that $|\mathfrak{D}_{2n}^{\mathrm{pure}}| = h_{n+1}^{\flat}$. More generally, with $z_{\mathrm{e}} = 0$ and/or $z_{\mathrm{o}} = 0$ and the other variables retained, it gives an S-fraction for pure D-permutations according to the parity-refined record-and-cycle classification. Note that since record-antirecords occur in pairs by Lemma 4.2.2, setting either $z_{\mathrm{e}} = 0$ or $z_{\mathrm{o}} = 0$ suffices to suppress them; this explains why these variables occur in the T-fraction only as a product $z_{\mathrm{e}} z_{\mathrm{o}}$.

3) With $z_{\mathrm{e}} = w_{\mathrm{e}} = 0$ (resp. $z_{\mathrm{o}} = w_{\mathrm{o}} = 0$) and all other variables equal to 1, it gives the S-fraction (2.45)/(2.46) for the Genocchi numbers, and confirms that $|\mathfrak{D}_{2n}^{\mathrm{e}}| = |\mathfrak{D}_{2n}^{\mathrm{o}}| = g_n$. More generally, with $z_{\mathrm{e}} = w_{\mathrm{e}} = 0$ (resp. $z_{\mathrm{o}} = w_{\mathrm{o}} = 0$) and the other variables retained, it gives an S-fraction for D-e-semiderangements (resp. D-o-semiderangements) according to the parity-refined record-and-cycle classification.

4) With $z_{\mathrm{e}} = z_{\mathrm{o}} = w_{\mathrm{e}} = w_{\mathrm{o}} = 0$ and all other variables equal to 1, it gives the S-fraction (2.51)/(2.52) for the median Genocchi numbers, and confirms that $|\mathfrak{D}_{2n}^{\mathrm{eo}}| = h_n$. More generally, with $z_{\mathrm{e}} = z_{\mathrm{o}} = w_{\mathrm{e}} = w_{\mathrm{o}} = 0$ and the other variables retained, it gives an S-fraction for D-derangements according to the parity-refined record-and-cycle classification; this S-fraction is of precisely the form (4.4) that was proposed in the Introduction of this chapter.

5) If we specialise to $x_1 = u_1$, $y_1 = v_1$, $x_2 = u_2$, $y_2 = v_2$ — that is, renounce the counting of records — then the coefficients (4.9a,b) simplify to

$$\alpha_{2k-1} \;=\; k^2 x_1 y_1 \tag{4.10a}$$

$$\alpha_{2k} \;=\; (kx_2 + w_{\mathrm{e}})\,(ky_2 + w_{\mathrm{o}}) \tag{4.10b}$$

Using Proposition 2.2.5, we can alternatively rewrite the T-fraction (4.8) as an S-fraction for the ogf of an “augmented” sequence:

Corollary 4.2.4 (First S-fraction for augmented D-permutations)**.** The ordinary generating function of the "augmented" sequence of polynomials (4.7) has the S-type continued fraction

$$1 + z_{\mathrm{e}}z_{\mathrm{o}}t \sum_{n=0}^{\infty} P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\mathrm{e}},w_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}})\, t^n \;=\; \cfrac{1}{1 - \cfrac{z_{\mathrm{e}}z_{\mathrm{o}}t}{1 - \cfrac{x_1y_1t}{1 - \cfrac{(x_2+w_{\mathrm{e}})(y_2+w_{\mathrm{o}})t}{1 - \cfrac{(x_1+u_1)(y_1+v_1)t}{1 - \cfrac{(x_2+u_2+w_{\mathrm{e}})(y_2+v_2+w_{\mathrm{o}})t}{1 - \cfrac{(x_1+2u_1)(y_1+2v_1)t}{1 - \cfrac{(x_2+2u_2+w_{\mathrm{e}})(y_2+2v_2+w_{\mathrm{o}})t}{1 - \cdots}}}}}}}} \tag{4.11}$$

with coefficients

$$\alpha_1 \;=\; z_{\mathrm{e}}z_{\mathrm{o}} \tag{4.12a}$$

$$\alpha_{2k-1} \;=\; [x_2+(k-2)u_2+w_{\mathrm{e}}]\,[y_2+(k-2)v_2+w_{\mathrm{o}}] \quad \text{for } k \ge 2 \tag{4.12b}$$

$$\alpha_{2k} \;=\; [x_1+(k-1)u_1]\,[y_1+(k-1)v_1] \tag{4.12c}$$

With all variables equal to 1, this gives (since $|\mathfrak{D}_{2n}| = h_{n+1}$) the S-fraction (2.51)/(2.52) for the sequence $(h_n)_{n\ge 0}$.

Also, using Proposition 2.2.5 in the other direction, we can rewrite the T-fraction (4.8) specialised to $z_{\mathrm{e}} = 0$ and/or $z_{\mathrm{o}} = 0$ as a T-fraction for the ogf of the "restricted" sequence:

Corollary 4.2.5 (First T-fraction for restricted pure D-permutations)**.** The ordinary generating function of the "restricted" sequence of polynomials (4.7) specialised to

$z_e = 0$ and/or $z_o = 0$ has the T-type continued fraction

$$\sum_{n=0}^{\infty} P_{n+1}(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_e,w_o,0,z_o)\, t^n \;=$$
$$\sum_{n=0}^{\infty} P_{n+1}(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_e,w_o,z_e,0)\, t^n \;=$$
$$\cfrac{x_1y_1}{1 - x_1y_1\,t - \cfrac{(x_2+w_e)(y_2+w_o)\,t}{1 - \cfrac{(x_1+u_1)(y_1+v_1)\,t}{1 - \cfrac{(x_2+u_2+w_e)(y_2+v_2+w_o)\,t}{1 - \cfrac{(x_1+2u_1)(y_1+2v_1)\,t}{1 - \cfrac{(x_2+2u_2+w_e)(y_2+2v_2+w_o)\,t}{1-\cdots}}}}}} \tag{4.13}$$

with coefficients

$$\alpha_{2k-1} \;=\; [x_2+(k-1)u_2+w_e]\,[y_2+(k-1)v_2+w_o] \tag{4.14a}$$
$$\alpha_{2k} \;=\; [x_1+k\,u_1]\,[y_1+k\,v_1] \tag{4.14b}$$
$$\delta_1 \;=\; x_1y_1 \tag{4.14c}$$
$$\delta_n \;=\; 0 \qquad \text{for } n \ge 2 \tag{4.14d}$$

With all variables equal to 1, this gives (since $|\mathfrak{D}_{2n}^{\rm pure}| = h_{n+1}^{\flat}$) the T-fraction (2.60)/(2.61) for the sequence $(h_{n+2}^{\flat})_{n\ge 0}$. With $w_e = 0$ (or $w_o = 0$) and all other variables equal to 1, this gives the T-fraction (2.47)/(2.48) for the once-shifted Genocchi numbers. And finally, with $w_e = w_o = 0$ and all other variables equal to 1, this gives the T-fraction (2.55)/(2.56) for the once-shifted median Genocchi numbers.

If we specialise the polynomials (4.7) to $x_1 = x_2$, $u_1 = u_2$, $z_e = w_e$ and all other variables to 1, we obtain the polynomial $P^\star_n(x,u,w_\mathrm{e})$

$$P^\star_n(x,u,w_\mathrm{e}) \;=\; \sum_{\sigma\in\mathfrak{D}_{2n}} x^{\mathrm{earec}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)+\mathrm{nrcdfall}(\sigma)} w_\mathrm{e}^{\mathrm{evenfix}(\sigma)} \,. \qquad (4.15)$$

We have the following S-fraction:

Corollary 4.2.6 (S-fraction for specialised D-permutations)**.** The ordinary generating function of the polynomials $P^\star_n(x,u,w_\mathrm{e})$ defined in (4.15) has the S-type continued fraction

$$\sum_{n=0}^{\infty} P^\star_n(x,u,w_\mathrm{e})\, t^n \;=\; \cfrac{1}{1-\cfrac{(x+w_\mathrm{e})t}{1-\cfrac{2xt}{1-\cfrac{2(x+u+w_\mathrm{e})t}{1-\cfrac{3(x+u)t}{1-\cdots}}}}} \qquad (4.16)$$

with coefficients

$$\alpha_{2k-1} \;=\; k\,[x+(k-1)u+w_\mathrm{e}] \qquad (4.17\mathrm{a})$$

$$\alpha_{2k} \;=\; (k+1)\,[x+(k-1)u] \qquad (4.17\mathrm{b})$$

PROOF. It suffices to verify that the two T-fractions, the first obtained by specialising $x_1 = x_2$, $u_1 = u_2$, $z_\mathrm{e} = w_\mathrm{e}$ and all other variables to 1, and the second given by (4.16)/(4.17) contract by Proposition 2.2.1 to the same J-fraction. □

Remarks. 1. Specializing to $x = u = w_\mathrm{e} = 1$ gives the S-fraction (2.53)/(2.54) for the once-shifted median Genocchi numbers.

2. Specializing to $x = u = 1$, $w_\mathrm{e} = 0$ gives the S-fraction (2.45)/(2.46) for the Genocchi numbers.

3. See also the generalization in Corollary 4.3.3, as well as Open Problem 4.3.4.

■

4.2.3 p,q-generalizations of the first T-fraction

We can extend Theorem 4.2.3 by introducing a p,q-generalization. Recall that for integer $n \geq 0$ we define

$$[n]_{p,q} \;=\; \frac{p^n - q^n}{p-q} \;=\; \sum_{j=0}^{n-1} p^j q^{n-1-j} \tag{4.18}$$

where p and q are indeterminates; it is a homogeneous polynomial of degree $n-1$ in p and q, which is symmetric in p and q. In particular, $[0]_{p,q} = 0$ and $[1]_{p,q} = 1$; and for $n \geq 1$ we have the recurrence

$$[n]_{p,q} \;=\; p\,[n-1]_{p,q} + q^{n-1} \;=\; q\,[n-1]_{p,q} + p^{n-1}\,. \tag{4.19}$$

If $p = 1$, then $[n]_{1,q}$ is the well-known q-integer

$$[n]_q \;=\; [n]_{1,q} \;=\; \frac{1-q^n}{1-q} \;=\; \begin{cases} 0 & \text{if } n = 0 \\ 1+q+q^2+\ldots+q^{n-1} & \text{if } n \geq 1 \end{cases} \tag{4.20}$$

If $p = 0$, then

$$[n]_{0,q} \;=\; \begin{cases} 0 & \text{if } n = 0 \\ q^{n-1} & \text{if } n \geq 1 \end{cases} \tag{4.21}$$

The statistics on permutations corresponding to the variables p and q will be crossings and nestings, as defined in Section 2.5.2. More precisely, we define the following polynomial in 22 variables that generalizes (4.7) by including four pairs of (p,q)-variables corresponding to the four refined types of crossings and nestings, as well as two variables corresponding to pseudo-nestings of fixed points:

$$
\begin{aligned}
& P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},p_{-1},p_{-2},p_{+1},p_{+2},q_{-1},q_{-2},q_{+1},q_{+2},s_{\rm e},s_{\rm o}) \\
&\qquad = \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} y_1^{\rm ereccval(\sigma)} y_2^{\rm ereccdrise(\sigma)} \times \\
&\qquad\qquad u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} v_1^{\rm nrcval(\sigma)} v_2^{\rm nrcdrise(\sigma)} \times \\
&\qquad\qquad w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \times \\
&\qquad\qquad p_{-1}^{\rm lcrosscpeak(\sigma)} p_{-2}^{\rm lcrosscdfall(\sigma)} p_{+1}^{\rm ucrosscval(\sigma)} p_{+2}^{\rm ucrosscdrise(\sigma)} \times \\
&\qquad\qquad q_{-1}^{\rm lnestcpeak(\sigma)} q_{-2}^{\rm lnestcdfall(\sigma)} q_{+1}^{\rm unestcval(\sigma)} q_{+2}^{\rm unestcdrise(\sigma)} \times \\
&\qquad\qquad s_{\rm e}^{\rm epsnest(\sigma)} s_{\rm o}^{\rm opsnest(\sigma)}
\end{aligned}
\tag{4.22}
$$

where

$$
{\rm epsnest}(\sigma) \;=\; \sum_{i\in{\rm Evenfix}} {\rm psnest}(i,\sigma)\,, \qquad {\rm opsnest}(\sigma) \;=\; \sum_{i\in{\rm Oddfix}} {\rm psnest}(i,\sigma)\,. \tag{4.23}
$$

We remark that (4.22) is essentially the same as the polynomial introduced in [SZ22, eq. (2.51)], but restricted to D-permutations and refined to record the parity of fixed points.[4]

We then have the following p,q-generalization of Theorem 4.2.3:

Theorem 4.2.7 (First T-fraction for D-permutations, p,q-generalization)**.** The ordi-

[4] The polynomial in [SZ22, eq. (2.51)] also included a more refined stratification of fixed points by level (= psnest) as defined in (2.68) above. That refined stratification is omitted here for simplicity — instead we include only the simple factors $s^{\rm psnest}$ — but it is included in (4.31) below.

nary generating function of the polynomials (4.22) has the T-type continued fraction

$$\sum_{n=0}^{\infty} P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_\mathrm{e},w_\mathrm{o},z_\mathrm{e},z_\mathrm{o},p_{-1},p_{-2},p_{+1},p_{+2},q_{-1},q_{-2},q_{+1},q_{+2},s_\mathrm{e},s_\mathrm{o})\, t^n \;=$$

$$\cfrac{1}{1-z_\mathrm{e}z_\mathrm{o}t-\cfrac{x_1y_1t}{1-\cfrac{(x_2+s_\mathrm{e}w_\mathrm{e})(y_2+s_\mathrm{o}w_\mathrm{o})t}{1-\cfrac{(p_{-1}x_1+q_{-1}u_1)(p_{+1}y_1+q_{+1}v_1)t}{1-\cfrac{(p_{-2}x_2+q_{-2}u_2+s_\mathrm{e}^2w_\mathrm{e})(p_{+2}y_2+q_{+2}v_2+s_\mathrm{o}^2w_\mathrm{o})t}{1-\cfrac{(p_{-1}^2x_1+q_{-1}[2]_{p_{-1},q_{-1}}u_1)(p_{+1}^2y_1+q_{+1}[2]_{p_{+1},q_{+1}}v_1)t}{1-\cfrac{(p_{-2}^2x_2+q_{-2}[2]_{p_{-2},q_{-2}}u_2+s_\mathrm{e}^3w_\mathrm{e})(p_{+2}^2y_2+q_{+2}[2]_{p_{+2},q_{+2}}v_2+s_\mathrm{o}^3w_\mathrm{o})t}{1-\cdots}}}}}} \tag{4.24}$$

with coefficients

$$\alpha_{2k-1} = \big(p_{-1}^{k-1}x_1+q_{-1}[k-1]_{p_{-1},q_{-1}}u_1\big)\big(p_{+1}^{k-1}y_1+q_{+1}[k-1]_{p_{+1},q_{+1}}v_1\big) \tag{4.25a}$$

$$\alpha_{2k} = \big(p_{-2}^{k-1}x_2+q_{-2}[k-1]_{p_{-2},q_{-2}}u_2+s_\mathrm{e}^kw_\mathrm{e}\big)\times \big(p_{+2}^{k-1}y_2+q_{+2}[k-1]_{p_{+2},q_{+2}}v_2+s_\mathrm{o}^kw_\mathrm{o}\big) \tag{4.25b}$$

$$\delta_1 = z_\mathrm{e}z_\mathrm{o} \tag{4.25c}$$

$$\delta_n = 0 \qquad \text{for } n\ge 2 \tag{4.25d}$$

We will prove Theorem 4.2.7 in Section 4.5. Of course we reobtain Theorem 4.2.3 by making the specialisation $p_{-1}=p_{-2}=p_{+1}=p_{+2}=q_{-1}=q_{-2}=q_{+1}=q_{+2}=s_\mathrm{e}=s_\mathrm{o}=1$.

Remarks. 1. If we specialise to $x_1=u_1$, $y_1=v_1$, $x_2=u_2$, $y_2=v_2$ — that is, renounce the counting of records — then the coefficients (4.25) simplify to

$$\alpha_{2k-1} = [k]_{p_{-1},q_{-1}}\,[k]_{p_{+1},q_{+1}}\,x_1y_1 \tag{4.26a}$$

$$\alpha_{2k} = \big([k]_{p_{-2},q_{-2}}x_2+s_\mathrm{e}^kw_\mathrm{e}\big)\big([k]_{p_{+2},q_{+2}}y_2+s_\mathrm{o}^kw_\mathrm{o}\big) \tag{4.26b}$$

$$\delta_1 = z_\mathrm{e}z_\mathrm{o} \tag{4.26c}$$

$$\delta_n = 0 \qquad \text{for } n\ge 2 \tag{4.26d}$$

2. If we further specialise to $x_1 = 1$, $x_2 = y_1 = y_2 = q$, $p_{-1} = p_{-2} = p_{+1} = p_{+2} = q$ and $q_{-1} = q_{-2} = q_{+1} = q_{+2} = s_{\mathrm{e}} = s_{\mathrm{o}} = q^2$ and recall [SZ22, Proposition 2.24] that the number of inversions (inv) of a permutation satisfies

$$\mathrm{inv} \;=\; \mathrm{cval} + \mathrm{cdrise} + \mathrm{cdfall} + \mathrm{ucross} + \mathrm{lcross} + 2(\mathrm{unest} + \mathrm{lnest} + \mathrm{psnest}) \;, \tag{4.27}$$

we obtain a T-fraction for D-permutations according to the number of even and odd fixed points and number of inversions, with coefficients $\alpha_{2k-1} = q^{2k-1}[k]_q^2$, $\alpha_{2k} = q^{2k}([k]_q + q^k w_{\mathrm{e}})([k]_q + q^k w_{\mathrm{o}})$, $\delta_1 = z_{\mathrm{e}} z_{\mathrm{o}}$.

3. And if we further specialise to $w_{\mathrm{o}} = z_{\mathrm{o}} = 0$, we reobtain the S-fraction for D-o-semiderangements according to the number of fixed points and number of inversions [Ran97, Théorème 1.2], with coefficients $\alpha_{2k-1} = q^{2k-1}[k]_q^2$, $\alpha_{2k} = q^{2k}[k]_q([k]_q + q^k w_{\mathrm{e}})$. ■

We now give the p,q-generalization of Corollary 4.2.6. We need to make the specialisations $y_1 = y_2 = v_1 = v_2 = w_{\mathrm{o}} = z_{\mathrm{o}} = p_{+1} = p_{+2} = q_{+1} = q_{+2} = s_{\mathrm{o}} = 1$ — that is, set all the weights associated to odd indices to 1 — and also specialise $x_1 = x_2$, $u_1 = u_2$, $z_{\mathrm{e}} = w_{\mathrm{e}}$, $p_{-1} = p_{-2}$, $q_{-1} = q_{-2}$ and $s_{\mathrm{e}} = 1$:

Corollary 4.2.8 (S-fraction for specialised D-permutations, p,q-generalization)**.** The ordinary generating function of the polynomials $P_n^\star(x,u,w_{\mathrm{e}},p_-,q_-)$ defined by

$$P_n^\star(x,u,w_{\mathrm{e}},p_-,q_-) \;=\; \sum_{\sigma \in \mathfrak{D}_{2n}} x^{\mathrm{earec}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)+\mathrm{nrcdfall}(\sigma)} w_{\mathrm{e}}^{\mathrm{evenfix}(\sigma)} p_-^{\mathrm{lcross}(\sigma)} q_-^{\mathrm{lnest}(\sigma)} \tag{4.28}$$

has the S-type continued fraction

$$\sum_{n=0}^{\infty} P_n^\star(x,u,w_{\mathrm{e}})\, t^n \;=\; \cfrac{1}{1 - \cfrac{(x+w_{\mathrm{e}})t}{1 - \cfrac{2xt}{1 - \cfrac{2(p_-x + q_-u + w_{\mathrm{e}})t}{1 - \cfrac{3(p_-x + q_-u)t}{1 - \cdots}}}}} \tag{4.29}$$

with coefficients

$$\alpha_{2k-1} = k\left(p_-^{k-1}x + q_-[k-1]_{p_-,q_-}u + w_{\mathrm{e}}\right) \tag{4.30a}$$

$$\alpha_{2k} = (k+1)\left(p_-^{k-1}x + q_-[k-1]_{p_-,q_-}u\right) \tag{4.30b}$$

PROOF. It suffices to verify that the T-fraction (4.24)/(4.25) with the given specialisations and the S-fraction (4.29)/(4.30) contract by Proposition 2.2.1 to the same J-fraction. $\square$

4.2.4 First master T-fraction

In fact, we can go farther, and introduce a polynomial in six infinite families of indeterminates $\mathbf{a} = (\mathsf{a}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{b} = (\mathsf{b}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{c} = (\mathsf{c}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{d} = (\mathsf{d}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{e} = (\mathsf{e}_{\ell})_{\ell\geq 0}$, $\mathbf{f} = (\mathsf{f}_{\ell})_{\ell\geq 0}$ that will have a nice T-fraction and that will include the polynomials (4.7) and (4.22) as specialisations.

Using the index-refined crossing and nesting statistics defined in (2.67), we define the polynomial $Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f})$ by

$$\begin{aligned} Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}) \; = \\ & \sum_{\sigma\in\mathfrak{D}_{2n}} \prod_{i\in\mathrm{Cval}(\sigma)} \mathsf{a}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Cpeak}(\sigma)} \mathsf{b}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \;\times \\ & \qquad \prod_{i\in\mathrm{Cdfall}(\sigma)} \mathsf{c}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \prod_{i\in\mathrm{Cdrise}(\sigma)} \mathsf{d}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \;\times \\ & \qquad \prod_{i\in\mathrm{Evenfix}(\sigma)} \mathsf{e}_{\mathrm{psnest}(i,\sigma)} \prod_{i\in\mathrm{Oddfix}(\sigma)} \mathsf{f}_{\mathrm{psnest}(i,\sigma)} \;. \end{aligned} \tag{4.31}$$

where $\mathrm{Cval}(\sigma) = \{i\colon\, \sigma^{-1}(i) > i < \sigma(i)\}$ and likewise for the others. We remark that (4.31) is the same as the polynomial introduced in [SZ22, eq. (2.77)], but restricted to D-permutations and refined to record the parity of fixed points.

The polynomials (4.31) then have a beautiful T-fraction:

Theorem 4.2.9 (First master T-fraction for D-permutations)**.** The ordinary generating function of the polynomials $Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f})$ has the T-type continued fraction

$$\sum_{n=0}^{\infty} Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f})\, t^n \;=\; \cfrac{1}{1 - \mathsf{e}_0\mathsf{f}_0 t - \cfrac{\mathsf{a}_{00}\mathsf{b}_{00} t}{1 - \cfrac{(\mathsf{c}_{00}+\mathsf{e}_1)(\mathsf{d}_{00}+\mathsf{f}_1) t}{1 - \cfrac{(\mathsf{a}_{01}+\mathsf{a}_{10})(\mathsf{b}_{01}+\mathsf{b}_{10}) t}{1 - \cfrac{(\mathsf{c}_{01}+\mathsf{c}_{10}+\mathsf{e}_2)(\mathsf{d}_{01}+\mathsf{d}_{10}+\mathsf{f}_2) t}{1-\cdots}}}}} \tag{4.32}$$

with coefficients

$$\alpha_{2k-1} \;=\; \left(\sum_{\xi=0}^{k-1} \mathsf{a}_{k-1-\xi,\xi}\right)\left(\sum_{\xi=0}^{k-1} \mathsf{b}_{k-1-\xi,\xi}\right) \tag{4.33a}$$

$$\alpha_{2k} \;=\; \left(\mathsf{e}_k + \sum_{\xi=0}^{k-1} \mathsf{c}_{k-1-\xi,\xi}\right)\left(\mathsf{f}_k + \sum_{\xi=0}^{k-1} \mathsf{d}_{k-1-\xi,\xi}\right) \tag{4.33b}$$

$$\delta_1 \;=\; \mathsf{e}_0\mathsf{f}_0 \tag{4.33c}$$

$$\delta_n \;=\; 0 \qquad \text{for } n \ge 2 \tag{4.33d}$$

We will prove this theorem in Section 4.5; it is our “master” version of the first T-fraction. It implies Theorems 4.2.3 and 4.2.7 by straightforward specialisations.

4.2.5 Variant forms of the first T-fractions

Our first T-fractions (Theorems 4.2.3, 4.2.7 and 4.2.9) have variant forms in which we use the variant index-refined crossing and nesting statistics (2.69) in place of the original index-refined crossing and nesting statistics (2.67).

It is convenient to start with the master T-fraction. Introducing indeterminates

$\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}$ as before, we define the variant polynomial $Q'_n(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f})$ by

$$\begin{aligned} Q'_n(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}) \;=\; & \\ & \sum_{\sigma\in\mathfrak{D}_{2n}} \prod_{i\in\mathrm{Cval}(\sigma)} \mathsf{a}_{\mathrm{lcross}'(i,\sigma),\,\mathrm{lnest}'(i,\sigma)} \prod_{i\in\mathrm{Cpeak}(\sigma)} \mathsf{b}_{\mathrm{ucross}'(i,\sigma),\,\mathrm{unest}'(i,\sigma)} \;\times \\ & \qquad \prod_{i\in\mathrm{Cdfall}(\sigma)} \mathsf{c}_{\mathrm{lcross}'(i,\sigma),\,\mathrm{lnest}'(i,\sigma)} \prod_{i\in\mathrm{Cdrise}(\sigma)} \mathsf{d}_{\mathrm{ucross}'(i,\sigma),\,\mathrm{unest}'(i,\sigma)} \;\times \\ & \qquad \prod_{i\in\mathrm{Evenfix}(\sigma)} \mathsf{e}_{\mathrm{psnest}(i,\sigma)} \prod_{i\in\mathrm{Oddfix}(\sigma)} \mathsf{f}_{\mathrm{psnest}(i,\sigma)} \;. \end{aligned} \tag{4.34}$$

Note that for cycle valleys and cycle peaks, the "u" and "l" have here been interchanged relative to (4.31); this is in accordance with the explanation after (2.69) about which types of indices can have nonzero values for the primed crossing and nesting statistics.

We then have the following variant of Theorem 4.2.9:

Theorem 4.2.10 (Variant first master T-fraction for D-permutations)**.** The ordinary generating function of the polynomials $Q'_n(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f})$ has the same T-type continued fraction (4.32)/(4.33) as the polynomials $Q_n(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f})$. Therefore

$$Q_n(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}) \;=\; Q'_n(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f})\,. \tag{4.35}$$

We will prove this theorem in Section 4.5.5.

Next we define the variant p,q-generalized polynomials P'_n:

$$\begin{aligned} & P'_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\mathrm{e}},w_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}},p_{-1},p_{-2},p_{+1},p_{+2},q_{-1},q_{-2},q_{+1},q_{+2},s_{\mathrm{e}},s_{\mathrm{o}}) \\ & \quad = \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\mathrm{ereccpeak}'(\sigma)} x_2^{\mathrm{eareccdfall}'(\sigma)} y_1^{\mathrm{eareccval}'(\sigma)} y_2^{\mathrm{ereccdrise}'(\sigma)} \;\times \\ & \qquad\quad u_1^{\mathrm{nrcpeak}'(\sigma)} u_2^{\mathrm{nrcdfall}'(\sigma)} v_1^{\mathrm{nrcval}'(\sigma)} v_2^{\mathrm{nrcdrise}'(\sigma)} \;\times \\ & \qquad\quad w_{\mathrm{e}}^{\mathrm{evennrfix}(\sigma)} w_{\mathrm{o}}^{\mathrm{oddnrfix}(\sigma)} z_{\mathrm{e}}^{\mathrm{evenrar}(\sigma)} z_{\mathrm{o}}^{\mathrm{oddrar}(\sigma)} \;\times \\ & \qquad\quad p_{-1}^{\mathrm{ucrosscpeak}'(\sigma)} p_{-2}^{\mathrm{lcrosscdfall}'(\sigma)} p_{+1}^{\mathrm{lcrosscval}'(\sigma)} p_{+2}^{\mathrm{ucrosscdrise}'(\sigma)} \;\times \\ & \qquad\quad q_{-1}^{\mathrm{unestcpeak}'(\sigma)} q_{-2}^{\mathrm{lnestcdfall}'(\sigma)} q_{+1}^{\mathrm{lnestcval}'(\sigma)} q_{+2}^{\mathrm{unestcdrise}'(\sigma)} \;\times \\ & \qquad\quad s_{\mathrm{e}}^{\mathrm{epsnest}(\sigma)} s_{\mathrm{o}}^{\mathrm{opsnest}(\sigma)} \end{aligned} \tag{4.36}$$

where the statistics have been defined in Sections 2.5.1/2.5.2, in particular see Table 2.2. We define

$$\begin{aligned}\text{ucrosscpeak}'(\sigma) &= \sum_{k\in\text{Cpeak}(\sigma)} \text{ucross}'(k,\sigma) &\quad (4.37a)\\ \text{unestcpeak}'(\sigma) &= \sum_{k\in\text{Cpeak}(\sigma)} \text{unest}'(k,\sigma) &\quad (4.37b)\end{aligned}$$

where ucross′ and unest′ have been defined in (2.69a,b), and likewise for the others.

We then have the following variant of Theorem 4.2.7:

Theorem 4.2.11 (Variant first T-fraction for D-permutations, p,q-generalization)**.** The ordinary generating function of the polynomials P'_n has the same T-type continued fraction (4.24)/(4.25) as the polynomials P_n. Therefore

$$\begin{aligned}&P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\mathrm{e}},w_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}},p_{-1},p_{-2},p_{+1},p_{+2},q_{-1},q_{-2},q_{+1},q_{+2},s_{\mathrm{e}},s_{\mathrm{o}})\\ &= P'_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\mathrm{e}},w_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}},p_{-1},p_{-2},p_{+1},p_{+2},q_{-1},q_{-2},q_{+1},q_{+2},s_{\mathrm{e}},s_{\mathrm{o}})\;.\end{aligned} \quad (4.38)$$

And finally, we can define variant polynomials $P'_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,$ $w_{\mathrm{e}},w_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}})$ by specialising $p_{-1}=p_{-2}=p_{+1}=p_{+2}=q_{-1}=q_{-2}=q_{+1}=q_{+2}=$ $s_{\mathrm{e}}=s_{\mathrm{o}}=1$, and thereby obtain a variant version of Theorem 4.2.3. We leave the details to the reader.

Remark. A four-variable special case of Theorem 4.2.11 was found by Randrianarivony and Zeng [RZ96a, Proposition 10] for D-o-semiderangements. Note first that for a D-o-semiderangement, each fixed point (which is necessarily even) must be a neither-record-antirecord: for if a fixed point i were a record or antirecord, then it would be both; but by Lemma 4.2.2, $i-1$ would then also be a record-antirecord fixed point, which is impossible since σ is a D-o-semiderangement.

Let us now look at the statistics defined by Randrianarivony and Zeng [RZ96a, p. 2]. Their statistic "lema" is a left-to-right maximum (i.e. a record) whose *value* is even: thus we can say that $\sigma^{-1}(i)$ is a record *position* and that i is even. But a record is always a weak excedance, hence $\sigma^{-1}(i) \le i$; and i is even, hence $i \ge \sigma(i)$. Since i is a record, it cannot be a fixed point, so it must be a cycle peak, and we have $\mathrm{lema} = \mathrm{ereccpeak}'$. Similarly, their statistic "remi" (resp. "romi") is a right-to-left minimum (i.e. an antirecord) whose *value* is even (resp. odd); by similar reasoning we obtain $\mathrm{remi} = \mathrm{eareccdfall}'$ and $\mathrm{romi} = \mathrm{eareccval}'$. Their polynomial [RZ96a, eq. (3.3)]

$$R_n(x,y,\bar{x},\bar{y}) \;=\; \sum_{\sigma\in\mathfrak{D}^{\mathrm{o}}_{2n}} x^{\mathrm{lema}(\sigma)} y^{\mathrm{romi}(\sigma)} \bar{x}^{\mathrm{fix}(\sigma)} \bar{y}^{\mathrm{remi}(\sigma)} \tag{4.39}$$

thus corresponds to our P'_n specialised to $x_1 = x$, $y_1 = y$, $w_{\mathrm{e}} = \bar{x}$, $x_2 = \bar{y}$, $w_{\mathrm{o}} = z_{\mathrm{o}} = 0$, and all other variables equal to 1. Their S-fraction [RZ96a, Proposition 10] is then a special case of Theorem 4.2.11. We will use this in Chapter 5. ■

4.3 Second T-fraction and its generalizations

It is natural to want to refine the foregoing polynomials by keeping track also of the number of cycles (cyc). Unfortunately, cyc does not seem to mesh well with the record classification: even the three-variable polynomials

$$P_n(x,y,\lambda) \;=\; \sum_{\sigma\in\mathfrak{D}_{2n}} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)} \tag{4.40}$$

do not have a J-fraction with polynomial coefficients (see Appendix of this chapter). Using the contraction formula (Proposition 2.2.1), it follows from this that $P_n(x,y,\lambda)$ also cannot have a T-fraction with polynomial coefficients and $\delta_2 = \delta_4 = \ldots = 0$.[5]

Nevertheless, it turns out that cyc *almost* meshes with the complete parity-refined record-and-cycle classification: it suffices to make one (stated here but proved later in Chapter 5) or two specialisations in which we partially renounce

[5]This does not exclude that $P_n(x,y,\lambda)$ might have a general T-fraction with polynomial coefficients. Indeed, it is easy to see [Sok] that, for *any* sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$ with $a_0 = 1$ in a commutative ring R, and *any* sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\ge 1}$ of *invertible* elements of R, there exists a unique sequence $\boldsymbol{\delta} = (\delta_n)_{n\ge 1}$ in R such that the ordinary generating function $\sum_{n=0}^{\infty} a_n t^n$ is represented by the T-fraction with coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$. So the polynomials $P_n(x,y,\lambda)$ are in fact represented by *uncountably many* different T-fractions with coefficients $\alpha_n \in \mathbb{Z}\setminus\{0\}$ (or even $\alpha_n \in \{1,2\}$) and $\delta_n \in \mathbb{Q}[x,y,\lambda]$. Furthermore, such T-fractions might exist also for certain noninvertible $\boldsymbol{\alpha}$, subject to suitable divisibility conditions. But it is far from clear whether any of these T-fractions are simple enough to be found explicitly.

recording the record status. We explain this in the next subsection.

4.3.1 Second T-fraction

4.3.1.1 Statement of a T-fraction and main theorem

We begin by introducing a polynomial in 13 variables that generalizes (4.7) by keeping track also of the number of cycles (cyc):

$$\begin{aligned}\widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda) \;=\; \\ \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} y_1^{\rm ereccval(\sigma)} y_2^{\rm ereccdrise(\sigma)} \;\times \\ u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} v_1^{\rm nrcval(\sigma)} v_2^{\rm nrcdrise(\sigma)} \;\times \\ w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \lambda^{\rm cyc(\sigma)}\;. \end{aligned} \tag{4.41}$$

Of course there is no hope that $\widehat{P}_n$ has a J-fraction (or a T-fraction with $\delta_2 = \delta_4 = \ldots = 0$) with polynomial coefficients, because even the specialisation (4.40) does not have one. Nevertheless, we find *empirically* that we need to make only one specialisation — either $u_1 = x_1$ or $v_1 = y_1$ — to obtain a good T-fraction. In other words, it suffices to renounce distinguishing either the antirecord status of cycle peaks or the record status of cycle valleys. For concreteness we show the second of these:

Theorem 4.3.1. The ordinary generating function of the polynomials (4.41) specialised to $v_1 = y_1$ has the T-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda)\, t^n \;=\;
\cfrac{1}{1-\lambda^2 z_{\rm e}z_{\rm o}\,t-\cfrac{\lambda x_1 y_1\, t}{1-\cfrac{(x_2+\lambda w_{\rm e})(y_2+\lambda w_{\rm o})\,t}{1-\cfrac{(\lambda+1)(x_1+u_1)y_1\,t}{1-\cfrac{(x_2+u_2+\lambda w_{\rm e})(y_2+v_2+\lambda w_{\rm o})\,t}{1-\cfrac{(\lambda+2)(x_1+2u_1)y_1\,t}{1-\cfrac{(x_2+2u_2+\lambda w_{\rm e})(y_2+2v_2+\lambda w_{\rm o})\,t}{1-\cdots}}}}}}} \tag{4.42}$$

with coefficients

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\,[x_1+(k-1)u_1]\,y_1 \tag{4.43a}$$

$$\alpha_{2k} \;=\; [x_2+(k-1)u_2+\lambda w_{\rm e}]\,[y_2+(k-1)v_2+\lambda w_{\rm o}] \tag{4.43b}$$

$$\delta_1 \;=\; \lambda^2 z_{\rm e}z_{\rm o} \tag{4.43c}$$

$$\delta_n \;=\; 0 \qquad \text{for } n \ge 2 \tag{4.43d}$$

The full theorem 4.3.1 will only be proved in Chapter 5. Our approach in this chapter will only be able to prove a weaker version in which we make the *two* specialisations $v_1 = y_1$ *and* $v_2 = y_2$, i.e. we renounce distinguishing the record status of cycle valleys *and* cycle double rises. However, we can do better than this: namely, we replace the pair y_2, v_2 by a pair of new variables $\widehat{y}_2, \widehat{v}_2$ that measure, not whether a cycle double rise i is a record *position*, but rather whether it is a record *value*, i.e. whether $\sigma^{-1}(i)$ is a record. We use the statistics ereccdrise′ and nrcdrise′, defined in Section 2.5.1 and define the modified polynomials

$$\widehat{\widehat{P}}_n(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,v_1,\widehat{v}_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda) \;=\; \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} y_1^{\rm ereccval(\sigma)} \widehat{y}_2^{\rm ereccdrise'(\sigma)} \times u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} v_1^{\rm nrcval(\sigma)} \widehat{v}_2^{\rm nrcdrise'(\sigma)} \times w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \lambda^{\rm cyc(\sigma)} . \qquad (4.44)$$

We are then able to prove:

Theorem 4.3.2 (Second T-fraction for D-permutations)**.** The ordinary generating function of the polynomials (4.44) specialised to $v_1 = y_1$ has the T-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{\widehat{P}}_n(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,y_1,\widehat{v}_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda)\, t^n \;=\; \cfrac{1}{1-\lambda^2 z_{\rm e}z_{\rm o}t-\cfrac{\lambda x_1y_1t}{1-\cfrac{(x_2+\lambda w_{\rm e})(\widehat{y}_2+\lambda w_{\rm o})t}{1-\cfrac{(\lambda+1)(x_1+u_1)y_1t}{1-\cfrac{(x_2+u_2+\lambda w_{\rm e})(\widehat{y}_2+\widehat{v}_2+\lambda w_{\rm o})t}{1-\cfrac{(\lambda+2)(x_1+2u_1)y_1t}{1-\cfrac{(x_2+2u_2+\lambda w_{\rm e})(\widehat{y}_2+2\widehat{v}_2+\lambda w_{\rm o})t}{1-\cdots}}}}}}} \qquad (4.45)$$

with coefficients

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\,[x_1+(k-1)u_1]\,y_1 \qquad (4.46a)$$

$$\alpha_{2k} \;=\; [x_2+(k-1)u_2+\lambda w_{\rm e}]\,[\widehat{y}_2+(k-1)\widehat{v}_2+\lambda w_{\rm o}] \qquad (4.46b)$$

$$\delta_1 \;=\; \lambda^2 z_{\rm e}z_{\rm o} \qquad (4.46c)$$

$$\delta_n \;=\; 0 \qquad \text{for } n\ge 2 \qquad (4.46d)$$

This continued fraction has exactly the same form as Theorem 4.3.1, except that y_2, v_2 are replaced by $\widehat{y}_2, \widehat{v}_2$. We will prove Theorem 4.3.2 in Section 4.6.

Note that each of the coefficients α_i and δ_i in (4.46) is homogeneous of degree 1 in $x_1, x_2,\ u_1, u_2, w_{\rm e}, z_{\rm e}$ and also homogeneous of degree 1 in $y_1, \widehat{y}_2, \widehat{v}_2, w_{\rm o}, z_{\rm o}$. As before, this reflects the homogeneities of the $\widehat{\widehat{P}}_n$.

Remark. Specializing Theorem 4.3.2 to $w_{\rm e} = w_{\rm o} = z_{\rm e} = z_{\rm o} = 0$ and all other variables except λ to 1, we obtain an S-fraction with $\alpha_{2k-1} = k(\lambda + k - 1)$, $\alpha_{2k} = k^2$ for counting D-derangements by number of cycles [PZ23, eq. (4.4)]. ■

It is perhaps worth observing that, in view of Theorem 4.3.2, Theorem 4.3.1 is equivalent to the following assertion about the equidistribution of statistics:

Theorem 4.3.1′. There exists a bijection $\psi_n \colon \mathfrak{D}_{2n} \to \mathfrak{D}_{2n}$ that maps the 12-tuple

$$(\mathrm{eareccpeak}, \mathrm{eareccdfall}, \mathrm{cval}, \mathrm{ereccdrise}, \mathrm{nrcpeak}, \mathrm{nrcdfall}, \mathrm{nrcdrise}, \\ \mathrm{evennrfix}, \mathrm{oddnrfix}, \mathrm{evenrar}, \mathrm{oddrar}, \mathrm{cyc})$$

onto

$$(\mathrm{eareccpeak}, \mathrm{eareccdfall}, \mathrm{cval}, \mathrm{ereccdrise}', \mathrm{nrcpeak}, \mathrm{nrcdfall}, \mathrm{nrcdrise}', \\ \mathrm{evennrfix}, \mathrm{oddnrfix}, \mathrm{evenrar}, \mathrm{oddrar}, \mathrm{cyc})\ .$$

4.3.1.2 Specialisation leading to S-fraction

We now give the cycle-counting generalization of Corollary 4.2.6. If in the T-fraction (4.45)/(4.46) we specialise $y_1 = \widehat{y}_2 = \widehat{v}_2 = 1 = z_{\rm o} = w_{\rm o} = 1$ — that is, all the weights associated to odd indices are set to 1 — and also specialise $z_{\rm e} = w_{\rm e}$, $x_1 = x_2$ and $u_1 = u_2$, the resulting T-fraction can be rewritten as an S-fraction by noticing that both of them contract (Proposition 2.2.1) to the same J-fraction, and we obtain a generalization of Corollary 4.2.6 that includes counting the number of cycles:

Corollary 4.3.3 (S-fraction for specialised D-permutations, counting number of cycles)**.** The ordinary generating function of the polynomials $P_n^{\star}(x, u, w_{\rm e}, \lambda)$ defined by

$$P_n^\star(x,u,w_{\rm e},\lambda) \;=\; \sum_{\sigma\in\mathfrak{D}_{2n}} x^{\mathrm{earec}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)+\mathrm{nrcdfall}(\sigma)} w_{\rm e}^{\mathrm{evenfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)} \tag{4.47}$$

has the S-type continued fraction

$$\sum_{n=0}^\infty P_n^\star(x,u,w_{\rm e},\lambda)\, t^n \;=\; \cfrac{1}{1 - \cfrac{\lambda(x+\lambda w_{\rm e})t}{1 - \cfrac{(\lambda+1)xt}{1 - \cfrac{(\lambda+1)(x+u+\lambda w_{\rm e})t}{1 - \cfrac{(\lambda+2)(x+u)t}{1-\cdots}}}}} \tag{4.48}$$

with coefficients

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\,[x+(k-1)u+\lambda w_{\rm e}] \tag{4.49a}$$

$$\alpha_{2k} \;=\; (\lambda+k)\,[x+(k-1)u] \tag{4.49b}$$

Of course there is also an analogous S-fraction in which the roles of odd and even indices are reversed.

Remarks. 1. Setting $x=u=1$ and $w_{\rm e}=0$ in (4.48)/(4.49), we obtain the S-fraction for D-semiderangements enumerated by number of cycles, with coefficients $\alpha_{2k-1}=k(\lambda+k-1)$ and $\alpha_{2k}=k(\lambda+k)$, which was found a quarter-century ago by Randrianarivony and Zeng [RZ96a, Corollary 13].

2. Setting $x=u=w_{\rm e}=1$ in (4.48)/(4.49), we obtain an S-fraction for D-permutations enumerated by number of cycles, with coefficients $\alpha_{2k-1}=\alpha_{2k}=k(\lambda+k)$. This S-fraction appeared in [HZ99b, eq. (14)] (in fact in a q-generalization), but with a different combinatorial interpretation. Further specialising to $\lambda=1$, we obtain the S-fraction (2.53)/(2.54) for the once-shifted Genocchi medians h_{n+1}. ■

Since the once-shifted Genocchi medians h_{n+1} count D-permutations of $[2n]$, the preceding remark suggests that one should try to find an S-fraction for D-permutations that generalizes (2.53)/(2.54):

Open Problem 4.3.4. Find the statistics on D-permutations that would lead to a more general S-fraction of the form

$$\alpha_{2k-1} \;=\; [x_1+(k-1)u_1]\,[x_2+ku_2] \tag{4.50a}$$

$$\alpha_{2k} \;=\; [y_1+(k-1)v_1]\,[y_2+kv_2] \tag{4.50b}$$

or

$$\alpha_{2k-1} \;=\; [x_1+(k-1)u_1]\,[x_2+(k-1)u_2+w] \tag{4.51a}$$

$$\alpha_{2k} \;=\; [y_1+(k-1)v_1]\,[y_2+(k-1)v_2+w'] \tag{4.51b}$$

■

4.3.1.3 Reformulation using cycle valley minima

We can rephrase Theorems 4.3.1 and 4.3.2 in a more suggestive form by observing that each non-singleton cycle contains precisely one maximum element, which is necessarily a cycle peak, and precisely one minimum element, which is necessarily a cycle valley. We use the statistics minval, maxpeak, nminval, nmaxpeak introduced in Section 2.5.1. We now introduce a polynomial that is similar to (4.7) except that the classification of cycle valleys as records or non-records — which we have already renounced in Theorem 4.3.1 — is replaced by the classification of cycle valleys as minima or non-minima:

$$\begin{aligned}
&\widetilde{P}_n(x_1,x_2,\widetilde{y}_1,y_2,u_1,u_2,\widetilde{v}_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}) \;=\\
&\qquad\sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} \widetilde{y}_1^{\rm minval(\sigma)} y_2^{\rm ereccdrise(\sigma)} \times\\
&\qquad\qquad u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} \widetilde{v}_1^{\rm nminval(\sigma)} v_2^{\rm nrcdrise(\sigma)} \times\\
&\qquad\qquad w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)}\;.
\end{aligned} \tag{4.52}$$

Then the factor $(\lambda+k-1)y_1$ in Theorem 4.3.1 and Theorem 4.3.2 is replaced by $\widetilde{y}_1+(k-1)\widetilde{v}_1$, and Theorem 4.3.1 can be rewritten as:

Theorem 4.3.1″. The ordinary generating function of the polynomials (4.52) has the T-type continued fraction

$$\sum_{n=0}^{\infty} \widetilde{P}_n(x_1,x_2,\widetilde{y}_1,y_2,u_1,u_2,\widetilde{v}_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o})\,t^n \;=\; \cfrac{1}{1-z_{\rm e}z_{\rm o}t-\cfrac{x_1\widetilde{y}_1t}{1-\cfrac{(x_2+w_{\rm e})(y_2+w_{\rm o})t}{1-\cfrac{(x_1+u_1)(\widetilde{y}_1+\widetilde{v}_1)t}{1-\cfrac{(x_2+u_2+w_{\rm e})(y_2+v_2+w_{\rm o})t}{1-\cfrac{(x_1+2u_1)(\widetilde{y}_1+2\widetilde{v}_1)t}{1-\cfrac{(x_2+2u_2+w_{\rm e})(y_2+2v_2+w_{\rm o})t}{1-\cdots}}}}}}} \tag{4.53}$$

with coefficients

$$\alpha_{2k-1} = [x_1+(k-1)u_1]\,[\widetilde{y}_1+(k-1)\widetilde{v}_1] \tag{4.54a}$$

$$\alpha_{2k} = [x_2+(k-1)u_2+w_{\rm e}]\,[y_2+(k-1)v_2+w_{\rm o}] \tag{4.54b}$$

$$\delta_1 = z_{\rm e}z_{\rm o} \tag{4.54c}$$

$$\delta_n = 0 \qquad \text{for } n\geq 2 \tag{4.54d}$$

Note that the factors of λ multiplying $w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}$ in Theorem 4.3.1 have now disappeared because we are no longer giving such factors to singleton cycles. Even more strikingly, Theorem 4.3.1″ now looks identical to Theorem 4.2.3 except that y_1,v_1 have been replaced by $\widetilde{y}_1,\widetilde{v}_1$. Moreover, Theorem 4.3.2 can be rewritten in a similar way, replacing y_1,v_1 by $\widetilde{y}_1,\widetilde{v}_1$ as in (4.52) *and* replacing y_2,v_2 by $\widehat{y}_2,\widehat{v}_2$ as in (4.44); for brevity we leave this reformulation to the reader.

4.3.1.4 S-fraction for D-cycles

We can also enumerate D-cycles by extracting the coefficient of λ^1 in Theorem 4.3.2. We begin by defining the polynomials analogous to (4.44) but restricted to D-cycles (which we recall have no fixed points):

$$\widehat{P}_n^{\mathfrak{DC}}(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,v_1,\widehat{v}_2) \;=\; \sum_{\sigma\in\mathfrak{DC}_{2n}} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} \widehat{y}_2^{\mathrm{ereccdrise}'(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} \widehat{v}_2^{\mathrm{nrcdrise}'(\sigma)} . \tag{4.55}$$

Theorem 4.3.2 implies the following:

Corollary 4.3.5 (S-fraction for D-cycles)**.** The ordinary generating function of the polynomials (4.55) specialised to $v_1 = y_1$ has the S-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{P}_{n+1}^{\mathfrak{DC}}(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,y_1,\widehat{v}_2)\, t^n \;=\; \cfrac{x_1 y_1}{1-\cfrac{x_2\widehat{y}_2\, t}{1-\cfrac{(x_1+u_1)y_1\, t}{1-\cfrac{(x_2+u_2)(\widehat{y}_2+\widehat{v}_2)\, t}{1-\cfrac{(x_1+2u_1)2y_1\, t}{1-\cfrac{(x_2+2u_2)(\widehat{y}_2+2\widehat{v}_2)\, t}{1-\cdots}}}}}} \tag{4.56}$$

with coefficients

$$\alpha_{2k-1} \;=\; [x_2+(k-1)u_2]\,[\widehat{y}_2+(k-1)\widehat{v}_2] \tag{4.57a}$$

$$\alpha_{2k} \;=\; [x_1+ku_1]\,ky_1 \tag{4.57b}$$

Specializing Corollary 4.3.5 by setting all variables equal to 1, we obtain an S-fraction with $\alpha_{2k-1} = k^2$ and $\alpha_{2k} = k(k+1)$, which we recognize as the S-fraction (2.45)/(2.46) for the Genocchi numbers g_n. We have therefore recovered the known fact [Laz20, LW22] that $|\mathfrak{DC}_{2n+2}| = g_n$, or equivalently that $|\mathfrak{DC}_{2n}| = g_{n-1}$.

4.3.2 p,q-generalizations of the second T-fraction

We can also make a p,q-generalization of the second T-fraction in Theorem 4.3.2. Let us define the polynomial in 23 variables that generalizes (4.44) by including four pairs of (p,q)-variables corresponding to the four refined types of crossings and nestings as well as two variables corresponding to pseudo-nestings of fixed points:

$$\begin{aligned}\widehat{\widehat{P}}_n(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,v_1,\widehat{v}_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},p_{-1},p_{-2},p_{+1},\widehat{p}_{+2},q_{-1},q_{-2},q_{+1},\widehat{q}_{+2},s_{\rm e},s_{\rm o},\lambda) \;=\; \\ \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} y_1^{\rm ereccval(\sigma)} \widehat{y}_2^{\,\rm ereccdrise'(\sigma)} \times \\ u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} v_1^{\rm nrcval(\sigma)} \widehat{v}_2^{\,\rm nrcdrise'(\sigma)} \times \\ w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \times \\ p_{-1}^{\rm lcrosscpeak(\sigma)} p_{-2}^{\rm lcrosscdfall(\sigma)} p_{+1}^{\rm ucrosscval(\sigma)} \widehat{p}_{+2}^{\,\rm ucrosscdrise'(\sigma)} \times \\ q_{-1}^{\rm lnestcpeak(\sigma)} q_{-2}^{\rm lnestcdfall(\sigma)} q_{+1}^{\rm unestcval(\sigma)} \widehat{q}_{+2}^{\,\rm unestcdrise'(\sigma)} \times \\ s_{\rm e}^{\rm epsnest(\sigma)} s_{\rm o}^{\rm opsnest(\sigma)} \lambda^{\rm cyc(\sigma)} \,. \end{aligned} \qquad (4.58)$$

This is the same as (4.22), except for the inclusion of the factor $\lambda^{\rm cyc(\sigma)}$ and the replacement of $y_2^{\rm ereccdrise(\sigma)}, v_2^{\rm nrcdrise(\sigma)}, p_{+2}^{\rm ucrosscdrise(\sigma)}, q_{+2}^{\rm unestcdrise(\sigma)}$ by $\widehat{y}_2^{\,\rm ereccdrise'(\sigma)}, \widehat{v}_2^{\,\rm nrcdrise'(\sigma)}, \widehat{p}_{+2}^{\,\rm ucrosscdrise'(\sigma)}, \widehat{q}_{+2}^{\,\rm unestcdrise'(\sigma)}$, respectively, where the statistics ucrosscdrise′ and unestcdrise′ are defined as

$$\mathrm{ucrosscdrise}'(\sigma) \;=\; \sum_{k\in{\rm Cdrise}(\sigma)} \mathrm{ucross}'(k,\sigma) \qquad (4.59a)$$

$$\mathrm{unestcdrise}'(\sigma) \;=\; \sum_{k\in{\rm Cdrise}(\sigma)} \mathrm{unest}'(k,\sigma) \qquad (4.59b)$$

[These statistics have already been used in (4.36).]

We now state the p,q-generalization of Theorem 4.3.2 that we are able to prove. It turns out that we need to make the specialisations $v_1 = y_1$ and $q_{+1} = p_{+1}$. The result is then the following:

Theorem 4.3.6 (Second T-fraction for D-permutations, p,q-generalization)**.** The ordinary generating function of the polynomials (4.58) specialised to $v_1 = y_1$ and

$q_{+1} = p_{+1}$ has the T-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{\widehat{P}}_n(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,y_1,\widehat{v}_2,w_\mathrm{e},w_\mathrm{o},z_\mathrm{e},z_\mathrm{o},p_{-1},p_{-2},p_{+1},\widehat{p}_{+2},q_{-1},q_{-2},p_{+1},\widehat{q}_{+2},s_\mathrm{e},s_\mathrm{o},\lambda)\, t^n \;=$$

$$\cfrac{1}{1-\lambda^2 z_\mathrm{e}z_\mathrm{o}t-\cfrac{\lambda x_1 y_1 t}{1-\cfrac{(x_2+\lambda s_\mathrm{e}w_\mathrm{e})(\widehat{y}_2+\lambda s_\mathrm{o}w_\mathrm{o})t}{1-\cfrac{(\lambda+1)p_{+1}y_1(p_{-1}x_1+q_{-1}u_1)t}{1-\cfrac{(p_{-2}x_2+q_{-2}u_2+\lambda s_\mathrm{e}^2 w_\mathrm{e})(\widehat{p}_{+2}\widehat{y}_2+\widehat{q}_{+2}\widehat{v}_2+\lambda s_\mathrm{o}^2 w_\mathrm{o})t}{1-\cfrac{(\lambda+2)p_{+1}^2 y_1(p_{-1}^2 x_1+q_{-1}[2]_{p_{-1},q_{-1}}u_1)t}{1-\cfrac{(p_{-2}^2 x_2+q_{-2}[2]_{p_{-2},q_{-2}}u_2+\lambda s_\mathrm{e}^3 w_\mathrm{e})(\widehat{p}_{+2}^2\widehat{y}_2+\widehat{q}_{+2}[2]_{\widehat{p}_{+2},\widehat{q}_{+2}}\widehat{v}_2+\lambda s_\mathrm{o}^3 w_\mathrm{o})t}{1-\cdots}}}}}}} \qquad (4.60)$$

with coefficients

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\, p_{+1}^{k-1} y_1 \left(p_{-1}^{k-1}x_1 + q_{-1}[k-1]_{p_{-1},q_{-1}}u_1\right) \qquad (4.61a)$$

$$\alpha_{2k} \;=\; \left(p_{-2}^{k-1}x_2 + q_{-2}[k-1]_{p_{-2},q_{-2}}u_2 + \lambda s_\mathrm{e}^k w_\mathrm{e}\right) \times \left(\widehat{p}_{+2}^{k-1}\widehat{y}_2 + \widehat{q}_{+2}[k-1]_{\widehat{p}_{+2},\widehat{q}_{+2}}\widehat{v}_2 + \lambda s_\mathrm{o}^k w_\mathrm{o}\right) \qquad (4.61b)$$

$$\delta_1 \;=\; \lambda^2 z_\mathrm{e}z_\mathrm{o} \qquad (4.61c)$$

$$\delta_n \;=\; 0 \qquad \text{for } n \ge 2 \qquad (4.61d)$$

We will prove Theorem 4.3.6 in Section 4.6. Of course we reobtain Theorem 4.3.2 by making the specialisation $p_{-1} = p_{-2} = p_{+1} = \widehat{p}_{+2} = q_{-1} = q_{-2} = \widehat{q}_{+2} = s_\mathrm{e} = s_\mathrm{o} = 1$.

Remark. If we specialise to $x_1 = u_1$, $x_2 = u_2$, $\widehat{y}_2 = \widehat{v}_2$ — that is, renounce the counting of records — then the coefficients (4.61) simplify to

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\, p_{+1}^{k-1} y_1\, [k]_{p_{-1},q_{-1}} x_1 \qquad (4.62a)$$

$$\alpha_{2k} \;=\; \left([k]_{p_{-2},q_{-2}}x_2 + \lambda s_\mathrm{e}^k w_\mathrm{e}\right)\left([k]_{\widehat{p}_{+2},\widehat{q}_{+2}}\widehat{y}_2 + \lambda s_\mathrm{o}^k w_\mathrm{o}\right) \qquad (4.62b)$$

$$\delta_1 \;=\; z_\mathrm{e}z_\mathrm{o} \qquad (4.62c)$$

$$\delta_n \;=\; 0 \qquad \text{for } n \ge 2 \qquad (4.62d)$$

■

We can also obtain a p,q-generalization of Corollary 4.3.3, or equivalently a cycle-counting generalization of Corollary 4.2.8. If in the T-fraction (4.60)/(4.61) we specialise $z_{\mathrm{o}} = w_{\mathrm{o}} = y_1 = \widehat{y}_2 = \widehat{v}_2 = p_{+1} = \widehat{p}_{+2} = \widehat{q}_{+2} = s_{\mathrm{o}} = 1$ — that is, all the weights associated to odd indices are set to 1 — and also specialise $z_{\mathrm{e}} = w_{\mathrm{e}}$, $x_1 = x_2$, $u_1 = u_2$, $p_{-1} = p_{-2}$, $q_{-1} = q_{-2}$ and $s_{\mathrm{e}} = 1$, the resulting T-fraction can be rewritten as an S-fraction by noticing that they contract (Proposition 2.2.1) to the same J-fraction:

Corollary 4.3.7 (S-fraction for D-permutations counting number of cycles, p,q-generalization)**.** The ordinary generating function of the polynomials $P^{\star}_n(x,u,w_{\mathrm{e}},p_-,q_-,\lambda)$ defined by

$$P^{\star}_n(x,u,w_{\mathrm{e}},p_-,q_-,\lambda) \;=\; \sum_{\sigma\in\mathfrak{D}_{2n}} x^{\mathrm{earec}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)+\mathrm{nrcdfall}(\sigma)} w_{\mathrm{e}}^{\mathrm{evenfix}(\sigma)} p_-^{\mathrm{lcross}(\sigma)} q_-^{\mathrm{lnest}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)} \tag{4.63}$$

has the S-type continued fraction

$$\sum_{n=0}^{\infty} P^{\star}_n(x,u,w_{\mathrm{e}},p_-,q_-,\lambda)\, t^n \;=\; \cfrac{1}{1-\cfrac{\lambda(x+\lambda w_{\mathrm{e}})t}{1-\cfrac{(\lambda+1)xt}{1-\cfrac{(\lambda+1)(p_-x+q_-u+\lambda w_{\mathrm{e}})t}{1-\cfrac{(\lambda+2)(p_-x+q_-u)t}{1-\cdots}}}}} \tag{4.64}$$

with coefficients

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\left(p_-^{k-1}x + q_-[k-1]_{p_-,q_-}u + \lambda w_{\mathrm{e}}\right) \tag{4.65a}$$

$$\alpha_{2k} \;=\; (\lambda+k)\left(p_-^{k-1}x + q_-[k-1]_{p_-,q_-}u\right) \tag{4.65b}$$

Finally, we can enumerate D-cycles by extracting the coefficient of λ^1 in Theorem 4.3.6. We begin by defining the polynomials analogous to (4.58) but restricted to D-cycles (which we recall have no fixed points):

$$\widehat{\widehat{P}}_n^{\mathfrak{DC}}(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,v_1,\widehat{v}_2,p_{-1},p_{-2},p_{+1},\widehat{p}_{+2},q_{-1},q_{-2},q_{+1},\widehat{q}_{+2},\lambda) =$$

$$\sum_{\sigma\in\mathfrak{DC}_{2n}} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} \widehat{y}_2^{\mathrm{ereccdrise}'(\sigma)} \times$$
$$u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} \widehat{v}_2^{\mathrm{nrcdrise}'(\sigma)} \times$$
$$p_{-1}^{\mathrm{lcrosscpeak}(\sigma)} p_{-2}^{\mathrm{lcrosscdfall}(\sigma)} p_{+1}^{\mathrm{ucrosscval}(\sigma)} \widehat{p}_{+2}^{\mathrm{ucrosscdrise}'(\sigma)} \times$$
$$q_{-1}^{\mathrm{lnestcpeak}(\sigma)} q_{-2}^{\mathrm{lnestcdfall}(\sigma)} q_{+1}^{\mathrm{unestcval}(\sigma)} \widehat{q}_{+2}^{\mathrm{unestcdrise}'(\sigma)} \lambda^{\mathrm{cyc}(\sigma)} . \qquad (4.66)$$

Theorem 4.3.6 then implies the following generalization of Corollary 4.3.5:

Corollary 4.3.8 (S-fraction for D-cycles, p,q-generalization)**.** The ordinary generating function of the polynomials (4.66) specialised to $v_1 = y_1$ and $q_{+1} = p_{+1}$ has the S-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{\widehat{P}}_{n+1}^{\mathfrak{DC}}(x_1,x_2,y_1,\widehat{y}_2,u_1,u_2,y_1,\widehat{v}_2,p_{-1},p_{-2},p_{+1},\widehat{p}_{+2},q_{-1},q_{-2},p_{+1},\widehat{q}_{+2})\, t^n =$$

$$\cfrac{x_1 y_1}{1 - \cfrac{x_2\widehat{y}_2\, t}{1 - \cfrac{(p_{-1}x_1 + q_{-1}u_1)p_{+1}y_1\, t}{1 - \cfrac{(p_{-2}x_2 + q_{-2}u_2)(\widehat{p}_{+2}\widehat{y}_2 + \widehat{q}_{+2}\widehat{v}_2)\, t}{1 - \cfrac{2(p_{-1}^2 x_1 + q_{-1}[2]_{p_{-1},q_{-1}}u_1)p_{+1}^2 y_1\, t}{1 - \cfrac{(p_{-2}^2 x_2 + q_{-2}[2]_{p_{-2},q_{-2}}u_2)(p_{+2}^2\widehat{y}_2 + q_{+2}[2]_{p_{+2},q_{+2}}\widehat{v}_2)\, t}{1 - \cdots}}}}}} \qquad (4.67)$$

with coefficients

$$\alpha_{2k-1} = \big(p_{-2}^{k-1}x_2 + q_{-2}[k-1]_{p_{-2},q_{-2}}u_2\big) \times \big(\widehat{p}_{+2}^{k-1}\widehat{y}_2 + \widehat{q}_{+2}[k-1]_{\widehat{p}_{+2},\widehat{q}_{+2}}\widehat{v}_2\big) \qquad (4.68a)$$

$$\alpha_{2k} = k\big(p_{-1}^k x_1 + q_{-1}[k]_{p_{-1},q_{-1}}u_1\big)\, p_{+1}^k y_1 \qquad (4.68b)$$

4.3.3 Second master T-fraction

As with the first T-fraction, we can go much farther, and obtain a T-fraction in six infinite families of indeterminates: $\mathbf{a} = (\mathsf{a}_\ell)_{\ell\geq 0}$, $\mathbf{b} = (\mathsf{b}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{c} = (\mathsf{c}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{d} = (\mathsf{d}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{e} = (\mathsf{e}_\ell)_{\ell\geq 0}$, $\mathbf{f} = (\mathsf{f}_\ell)_{\ell\geq 0}$; please note that $\mathbf{a}$ now has one index rather than two. We now define a sequence of polynomials that will include the polynomials (4.44) and (4.58) as specialisations:

$$\widehat{\widehat{Q}}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\lambda) \;=\;$$

$$\sum_{\sigma\in\mathfrak{D}_{2n}} \lambda^{\mathrm{cyc}(\sigma)} \prod_{i\in\mathrm{Cval}(\sigma)} \mathsf{a}_{\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Cpeak}(\sigma)} \mathsf{b}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \;\times$$
$$\prod_{i\in\mathrm{Cdfall}(\sigma)} \mathsf{c}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \prod_{i\in\mathrm{Cdrise}(\sigma)} \mathsf{d}_{\mathrm{ucross}'(i,\sigma),\,\mathrm{unest}'(i,\sigma)} \;\times$$
$$\prod_{i\in\mathrm{Evenfix}(\sigma)} \mathsf{e}_{\mathrm{psnest}(i,\sigma)} \prod_{i\in\mathrm{Oddfix}(\sigma)} \mathsf{f}_{\mathrm{psnest}(i,\sigma)} \,. \tag{4.69}$$

We remark that (4.69) is *almost* the same as the polynomial introduced in [SZ22, eq. (2.100)], but restricted to D-permutations and refined to record the parity of fixed points; the main difference is that the treatment of d is a bit nicer here, using the statistics ucross' and unest'.

Note that here, in contrast to the first master T-fraction, $\widehat{\widehat{Q}}_n$ depends on $\mathrm{ucross}(i,\sigma)$ and $\mathrm{unest}(i,\sigma)$ only via their sum: that is the price we have to pay in order to include the statistic cyc. Furthermore, the indices of d involve ucross' and unest' instead of ucross and unest.

The polynomials (4.69) then have a nice T-fraction:

Theorem 4.3.9 (Second master T-fraction for D-permutations)**.** The ordinary generating function of the polynomials $\widehat{\widehat{Q}}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\lambda)$ has the T-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{\widehat{Q}}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f})\, t^n \;=\; \cfrac{1}{1-\lambda^2 \mathsf{e}_0\mathsf{f}_0 t - \cfrac{\lambda \mathsf{a}_0\mathsf{b}_{00} t}{1-\cfrac{(\mathsf{c}_{00}+\lambda\mathsf{e}_1)(\mathsf{d}_{00}+\lambda\mathsf{f}_1)t}{1-\cfrac{(\lambda+1)\mathsf{a}_1(\mathsf{b}_{01}+\mathsf{b}_{10})t}{1-\cfrac{(\mathsf{c}_{01}+\mathsf{c}_{10}+\lambda\mathsf{e}_2)(\mathsf{d}_{01}+\mathsf{d}_{10}+\lambda\mathsf{f}_2)t}{1-\cdots}}}}} \tag{4.70}$$

with coefficients

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\mathsf{a}_{k-1}\left(\sum_{\xi=0}^{k-1} \mathsf{b}_{k-1-\xi,\xi}\right) \tag{4.71a}$$

$$\alpha_{2k} \;=\; \left(\lambda\mathsf{e}_k + \sum_{\xi=0}^{k-1} \mathsf{c}_{k-1-\xi,\xi}\right)\left(\lambda\mathsf{f}_k + \sum_{\xi=0}^{k-1} \mathsf{d}_{k-1-\xi,\xi}\right) \tag{4.71b}$$

$$\delta_1 \;=\; \lambda^2 \mathsf{e}_0\mathsf{f}_0 \tag{4.71c}$$

$$\delta_n \;=\; 0 \qquad \text{for } n \geq 2 \tag{4.71d}$$

We will prove this theorem in Section 4.6. It implies Theorems 4.3.2 and 4.3.6 by straightforward specialisations.

4.4 Coefficientwise Hankel-total positivity of the multivariate polynomial sequences

Given indeterminates $\mathbf{x} = (x_1, x_2, \ldots)$ and a sequence of polynomials $(q_n(\mathbf{x}))_{n\geq 0}$, we know from Theorem 2.3.1 that if the ordinary generating function $\sum_{n=0}^{\infty} q_n(\mathbf{x})$ has a Thron-type continued fraction with coefficientwise-positive coefficients then the Hankel matrix $(q_{n+k}(\mathbf{x}))_{n,k\geq 0}$ is coefficientwise-totally positive with respect to the variables $\mathbf{x}$. Thus, the following theorem is an easy corollary of our first Theorems 4.2.3, 4.2.7, 4.2.9, 4.2.10 and 4.2.11, and of our second Theorems 4.3.2, 4.3.6 and 4.3.9:

Theorem 4.4.1. Let $(R_n)_{n\geq 0}$ be the polynomial sequence defined by one of the following equations in Section 4.2: (4.7), (4.22),(4.31), (4.34), and (4.36); or by one of the following equations in Section 4.3: (4.44) with the specialisation $v_1 = y_1$, (4.58) specialised to $v_1 = y_1$ and $q_{+1} = p_{+1}$, and (4.69). Then the sequence $(R_n)_{n\geq 0}$ is coefficientwise-totally positive with respect to all of their corresponding variables.

4.5 First T-fraction: Proof of Theorems 4.2.3, 4.2.7, 4.2.9, 4.2.10 and 4.2.11

In this section we prove the first master T-fraction (Theorem 4.2.9) by a bijection from D-permutations to labelled Schröder paths. Our construction combines ideas of Randrianarivony [Ran97] with a variant [SZ22, Section 6.1] of Foata–Zeilberger [FZ90], together with some new ingredients. After proving Theorem 4.2.9, we deduce Theorems 4.2.3 and 4.2.7 by specialisation. Then, in Section 4.5.5, we prove the variant T-fractions of Theorems 4.2.10 and 4.2.11.

Let us define an ***almost-Dyck path*** of length $2n$ to be a path $\omega = (\omega_0, \ldots, \omega_{2n})$ in the right half-plane $\mathbb{N} \times \mathbb{Z}$, starting at $\omega_0 = (0,0)$ and ending at $\omega_{2n} = (2n,0)$, using the steps $(1,1)$ and $(1,-1)$, that stays always at height ≥ -1. Thus, an almost-Dyck path is like a Dyck path *except that* a down step from height 0 to height -1 is allowed; note, however, that it must be immediately followed by an up step back to height 0. Each non-Dyck part of the path is therefore of the form $(h_{2i-2}, h_{2i-1}, h_{2i}) = (0,-1,0)$. We write $\mathcal{D}^{\sharp}_{2n}$ for the set of almost-Dyck paths of length $2n$.

Next let us define a ***0-Schröder path*** to be a Schröder path in which long level steps, if any, occur only at height 0. We write $\mathcal{S}^{0}_{2n}$ for the set of 0-Schröder paths of length $2n$. There is an obvious bijection $\psi \colon \mathcal{D}^{\sharp}_{2n} \to \mathcal{S}^{0}_{2n}$ from almost-Dyck paths to 0-Schröder paths: namely, we replace each down-up pair starting and ending at height 0 with a long level step at height 0.

In this section we will construct a bijection from D-permutations of $[2n]$ onto labelled 0-Schröder paths of length $2n$, as follows: We first define the path by constructing an almost-Dyck path ω and then transforming it into a 0-Schröder path

$\widehat{\omega} = \psi(\omega)$. Then we define the labels ξ_i, which will lie in the sets

$$\mathcal{A}_h = \{0,\ldots,\lceil h/2 \rceil\} \quad \text{for } h \geq 0 \tag{4.72a}$$

$$\mathcal{B}_h = \{0,\ldots,\lceil (h-1)/2 \rceil\} \quad \text{for } h \geq 1 \tag{4.72b}$$

$$\mathcal{C}_0 = \{0\} \tag{4.72c}$$

$$\mathcal{C}_h = \varnothing \quad \text{for } h \geq 1 \tag{4.72d}$$

We also interpret our crossing, nesting and record statistics in terms of the heights and labels. Next we prove that the map $\sigma \mapsto (\widehat{\omega}, \xi)$ really is a bijection from the set $\mathfrak{D}_{2n}$ of D-permutations of $[2n]$ onto the set $\mathcal{S}_{2n}(\mathcal{A},\mathcal{B},\mathcal{C})$ of $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled Schröder paths of length $2n$. Finally, we sum over the labels ξ to obtain the weight $W(\widehat{\omega})$ associated to a Schröder path $\widehat{\omega}$, which upon applying (2.36) will yield Theorem 4.2.9.

4.5.1 Step 1: Definition of the almost-Dyck path

Given a D-permutation $\sigma \in \mathfrak{D}_{2n}$, we define a path $\omega = (\omega_0,\ldots,\omega_{2n})$ starting at $\omega_0 = (0,0)$, with steps $s_1,\ldots,s_{2n}$ as follows:

- If $\sigma^{-1}(i)$ is even, then s_i is a rise. (Note that in this case we must have $\sigma^{-1}(i) \geq i$, by definition of D-permutation.)
- If $\sigma^{-1}(i)$ is odd, then s_i is a fall. (Note that in this case we must have $\sigma^{-1}(i) \leq i$, by definition of D-permutation.)

(See Figure 4.1 for an example.) An alternative way of saying this is:

- If $\sigma^{-1}(i) > i$, then s_i is a rise. (In this case $\sigma^{-1}(i)$ must be even.)
- If $\sigma^{-1}(i) < i$, then s_i is a fall. (In this case $\sigma^{-1}(i)$ must be odd.)
- If i is a fixed point, then s_i is a rise if i is even, and a fall if i is odd.

Yet another alternative way of saying this is:

- If i is a cycle valley, cycle double fall or even fixed point, then s_i is a rise.

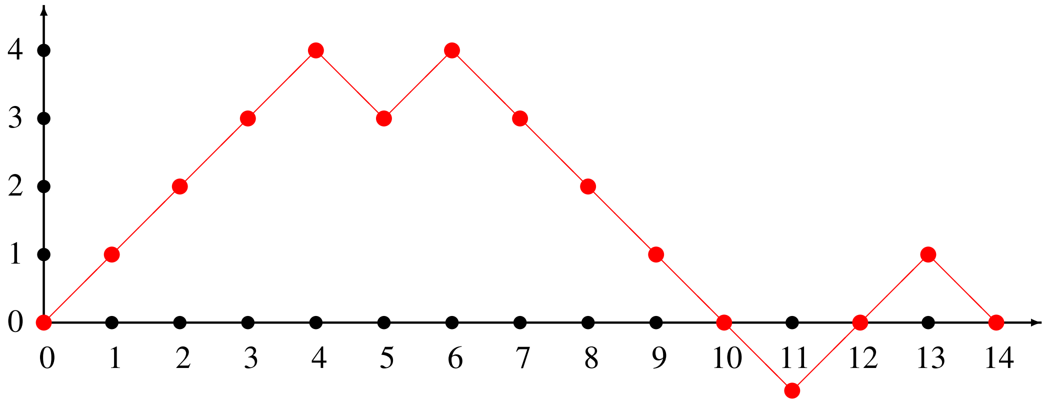


Figure 4.1: Almost-Dyck path ω corresponding to the D-permutation $\sigma = 7\,1\,9\,2\,5\,4\,8\,6\,10\,3\,11\,12\,14\,13 = (1,7,8,6,4,2)\,(3,9,10)\,(5)\,(11)\,(12)\,(13,14)$. This is the same D-permutation that was illustrated in Figure 2.3.

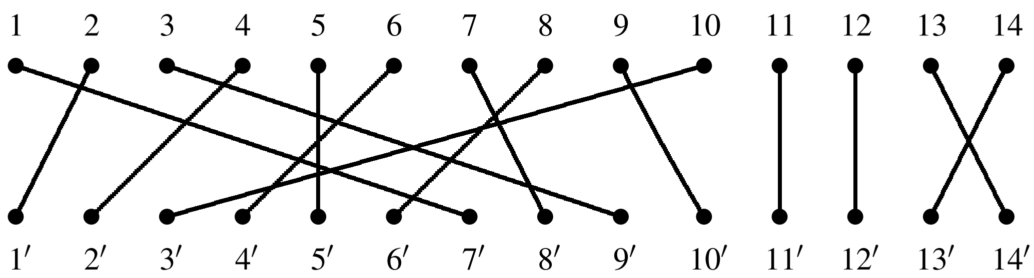


Figure 4.2: Bipartite digraph representing the permutation $\sigma = 7\,1\,9\,2\,5\,4\,8\,6\,10\,3\,11\,12\,14\,13 = (1,7,8,6,4,2)\,(3,9,10)\,(5)\,(11)\,(12)\,(13,14)$. Arrows run from the top row to the bottom row and are suppressed for clarity.

- If i is a cycle peak, cycle double rise or odd fixed point, then s_i is a fall.

Of course we need to prove that this ω is indeed an almost-Dyck path, i.e. that all the heights h_i are ≥ -1 and that $h_{2n} = 0$. We will do this by obtaining a precise interpretation of the heights h_i.

In what follows, it will be convenient to represent a permutation $\sigma \in \mathfrak{S}_N$ by a bipartite digraph $\Gamma = \Gamma(\sigma)$ in which the top row of vertices is labelled $1, \ldots, N$ and the bottom row $1', \ldots, N'$, and we draw an arrow from i to j' in case $\sigma(i) = j$ (see Figure 4.2). For $k \in [N]$, we denote by Γ_k the induced subgraph of Γ on the vertex set $\{1, \ldots, k\} \cup \{1', \ldots, k'\}$. Thus, the edges of Γ_k are arrows $i \to j'$ drawn whenever $\sigma(i) = j$ and $i \leq k$ and $j \leq k$. We say that a vertex of Γ_k is ***free*** if no arrow is incident on it. We write

$$f_k \stackrel{\text{def}}{=} \# \text{ of free vertices in the top row of } \Gamma_k \tag{4.73a}$$

$$= \#\{i \le k\colon\ \sigma(i) > k\}\,. \tag{4.73b}$$

Of course, we also have

$$f_k = \# \text{ of free vertices in the bottom row of } \Gamma_k \tag{4.74a}$$

$$= \#\{j \le k\colon\ \sigma^{-1}(j) > k\}\,. \tag{4.74b}$$

The total number of free vertices in Γ_k is therefore $2f_k$. Note that $f_k = 0$ if and only if σ maps $\{1,\ldots,k\}$ onto itself. Note also that

$$f_k - f_{k-1} = \begin{cases} +1 & \text{if } k \text{ is a cycle valley} \\ -1 & \text{if } k \text{ is a cycle peak} \\ 0 & \text{if } k \text{ is a cycle double rise, cycle double fall, or fixed point} \end{cases} \tag{4.75}$$

Remark. The sequence $(f_0,\ldots,f_{2n})$ is a Motzkin path; in fact, it is precisely the Motzkin path associated to the permutation σ by the Foata–Zeilberger (or Biane) bijection. To see this, compare (4.73)/(4.74) with [SZ22, eq. (6.4)]; or equivalently, compare (4.75) with [SZ22, definition of steps s_i preceding (6.2)]. ■

We can now give the promised interpretation of the heights:

Lemma 4.5.1 (Interpretation of the heights)**.** For $k \in [2n]$ we have

$$h_k = \begin{cases} 2f_k - 1 & \text{if } k \text{ is odd} \\ 2f_k & \text{if } k \text{ is even} \end{cases} \tag{4.76}$$

In particular, $h_k \ge -1$ and $h_{2n} = 0$, so that ω is an almost-Dyck path.

Furthermore, we have $(h_{2i-2}, h_{2i-1}, h_{2i}) = (0,-1,0)$ if and only if $2i-1$ and $2i$ are record-antirecords.

Please note that, by (4.76), the parity of h_k equals the parity of k; this reflects the fact that ω is an almost-Dyck path. Note also that (4.76) can be rewritten as

$$f_k \;=\; \left\lceil \frac{h_k}{2} \right\rceil \;=\; \begin{cases} (h_k+1)/2 & \text{if } k \text{ is odd} \\ h_k/2 & \text{if } k \text{ is even} \end{cases} \qquad (4.77)$$

And recall, finally, from Lemma 4.2.2 that record-antirecords come in pairs: $2i-1$ is a record-antirecord if and only if $2i$ is a record-antirecord.

PROOF OF LEMMA 4.5.1. We have

$$\begin{aligned} h_k &= \#\{j \le k\colon\; \sigma^{-1}(j) \text{ is even}\} \,-\, \#\{j \le k\colon\; \sigma^{-1}(j) \text{ is odd}\} && (4.78a) \\ &= 2\#\{j \le k\colon\; \sigma^{-1}(j) \text{ is even}\} \,-\, k && (4.78b) \\ &= 2\big[\#\{j \le k\colon\; \sigma^{-1}(j) > k\} \,+\, \#\{j \le k\colon\; k \ge \sigma^{-1}(j) > j\} \\ &\qquad\quad +\#\{j \le k\colon\; \sigma^{-1}(j) = j \text{ is even}\}\big] \,-\, k && (4.78c) \\ &= 2\big[\#\{j \le k : \sigma^{-1}(j) > k\} \,+\, \#\{i \le k\colon\; i > \sigma(i)\} \\ &\qquad\quad +\#\{i \le k\colon\; \sigma(i) = i \text{ is even}\}\big] \,-\, k && (4.78d) \\ &= 2\big[\#\{j \le k\colon\; \sigma^{-1}(j) > k\} \,+\, \#\{i \le k\colon\; i \text{ is even}\}\big] \,-\, k && (4.78e) \\ &= 2f_k + 2\left\lceil \frac{k-1}{2} \right\rceil \,-\, k\,, && (4.78f) \end{aligned}$$

which is (4.76).

Furthermore, $h_k = -1$ occurs when and only when k is odd (say, $k = 2i-1$) and $f_k = 0$. The latter statement means that σ maps $\{1,\ldots,2i-1\}$ onto itself. By Lemma 4.2.2(c) $\Longrightarrow$ (a,b) this means that $2i-1$ and $2i$ are record-antirecords. Conversely, if $2i-1$ is a record-antirecord, then by Lemma 4.2.2(a) $\Longrightarrow$ (d) we have $f_{2i-2} = f_{2i-1} = f_{2i} = 0$, so that $(h_{2i-2}, h_{2i-1}, h_{2i}) = (0,-1,0)$. $\square$

Remarks. 1. Randrianarivony [Ran97, Section 6] considered the special case of this construction in which σ is a D-o-semiderangement (in his terminology, a

"Genocchi permutation"); in this case ω is a Dyck path. Our proof of Lemma 4.5.1 is a very slight modification of his, designed to allow for fixed points of both parities. See also Han and Zeng [HZ99b, pp. 126–127] for the interpretation in terms of bipartite graphs and free vertices.

2. The number of almost-Dyck paths of length $2n$ is C_{n+1}, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number: it suffices to observe that an almost-Dyck path of semilength n can be converted to a Dyck path of semilength $n+1$ by adding a rise at the beginning and a fall at the end, and conversely. Equivalently, the number of 0-Schröder paths of length $2n$ is C_{n+1}: this follows from (2.36) with $\delta_1 = 1$, $\delta_n = 0$ for $n \geq 2$, and $\alpha_n = 1$ for $n \geq 1$ together with the identity

$$\frac{1}{1-t-tC(t)} \;=\; \frac{C(t)-1}{t} \tag{4.79}$$

for the Catalan generating function $C(t) = \sum\limits_{n=0}^{\infty} C_n t^n = (1-\sqrt{1-4t})/(2t)$. ■

4.5.2 Step 2: Definition of the labels ξ_i

We now define

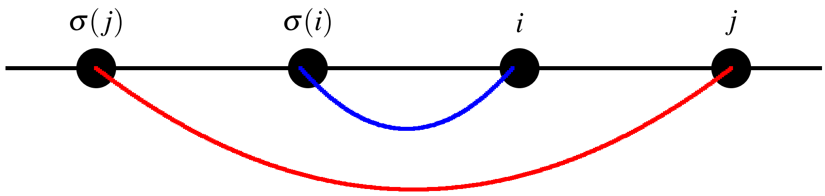
$$\xi_i \;=\; \begin{cases} \#\{j\colon\; \sigma(j) < \sigma(i) \leq i < j\} & \text{if } i \text{ is even} \\ \#\{j\colon\; j < i \leq \sigma(i) < \sigma(j)\} & \text{if } i \text{ is odd} \end{cases} \tag{4.80}$$

Note that the middle inequalities in this definition hold automatically: a D-permutation *always* has $\sigma(i) \leq i$ if i is even, and $i \leq \sigma(i)$ if i is odd. The definition (4.80) can be written equivalently as

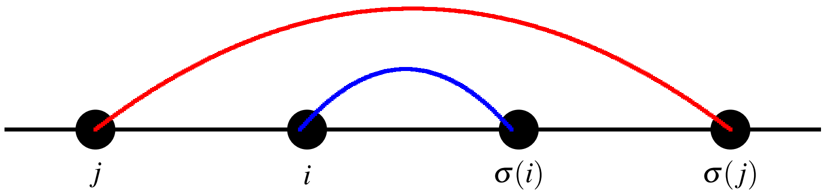
$$\xi_i \;=\; \begin{cases} \#\{2l > 2k\colon\; \sigma(2l) < \sigma(2k)\} & \text{if } i = 2k \\ \#\{2l-1 < 2k-1\colon\; \sigma(2l-1) > \sigma(2k-1)\} & \text{if } i = 2k-1 \end{cases} \tag{4.81}$$

since $\sigma(j) < j$ implies that j is even, and $j < \sigma(j)$ implies that j is odd.

It is worth remarking that the labels ξ_i defined in (4.80) are the same as those in the variant Foata–Zeilberger bijection [SZ22, eq. (6.5)] whenever i is not a fixed



When i is even and $\neq \sigma(i)$



When i is odd and $\neq \sigma(i)$

Figure 4.3: Nestings involved in the definition of the label ξ_i.

point.[6]

The label ξ_i has a simple interpretation in terms of the index-refined nesting statistics defined in (2.67)/(2.68):

Lemma 4.5.2 (Nesting statistics)**.** We have

$$\xi_i \;=\; \begin{cases} \mathrm{lnest}(i,\sigma) & \text{if } i \text{ is even and } \neq \sigma(i) \text{ [equivalently, } i > \sigma(i)] \\ \mathrm{unest}(i,\sigma) & \text{if } i \text{ is odd and } \neq \sigma(i) \text{ [equivalently, } i < \sigma(i)] \\ \mathrm{psnest}(i,\sigma) & \text{if } i = \sigma(i) \text{ (that is, } i \text{ is a fixed point)} \end{cases} \tag{4.82}$$

[6]With the only difference that in [SZ22] the labels were defined to start at 1, whereas here they start at 0.

See Figure 4.3. Note also that if i is a fixed point, we have

$$\xi_i \;=\; f_i \;=\; \left\lceil \frac{h_i}{2} \right\rceil \;=\; \begin{cases} h_i/2 & \text{if } i \text{ is even} \\ (h_i+1)/2 & \text{if } i \text{ is odd} \end{cases} \tag{4.83}$$

Now we shall show, as required by (4.72), that the following inequalities hold true:

Lemma 4.5.3 (Inequalities satisfied by the labels)**.** We have

$$0 \;\le\; \xi_i \;\le\; \left\lceil \frac{h_i - 1}{2} \right\rceil \;=\; \left\lceil \frac{h_{i-1}}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is even (i.e., } s_i \text{ is a rise)} \tag{4.84a}$$

$$0 \;\le\; \xi_i \;\le\; \left\lceil \frac{h_i}{2} \right\rceil \;=\; \left\lceil \frac{h_{i-1} - 1}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is odd (i.e., } s_i \text{ is a fall)} \tag{4.84b}$$

We remark that a fall starting at height $h_{i-1} = 0$, or a rise starting at height $h_{i-1} = -1$, always gets the label $\xi_i = 0$. When we pass from the almost-Dyck path ω to the 0-Schröder path $\widehat{\omega} = \psi(\omega)$, these labels become the label $\xi = 0$ for the long level step at height 0.

To prove the inequalities (4.84), we will interpret $\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i$ when s_i is a rise, and $\left\lceil \frac{h_i}{2} \right\rceil - \xi_i$ when s_i is a fall, in terms of crossing statistics, as follows:

Lemma 4.5.4 (Crossing statistics)**.**

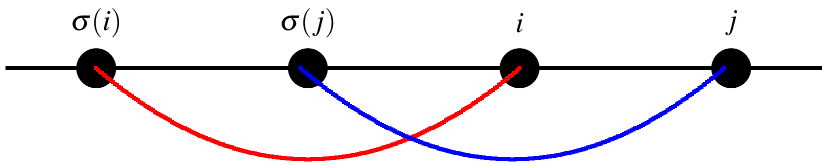
(a) If s_i a rise and i (hence also h_i) is odd, then

$$\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i \;=\; \mathrm{ucross}(i, \sigma)\,. \tag{4.85}$$

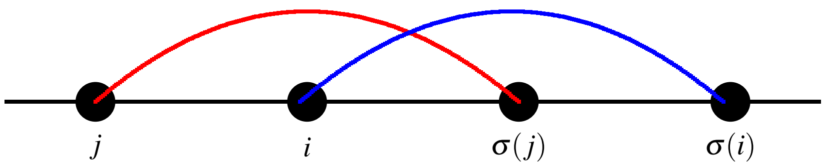
(b) If s_i a rise and i (hence also h_i) is even, then

$$\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i \;=\; \mathrm{lcross}(i, \sigma) + \mathrm{I}[\sigma(i) \neq i] \tag{4.86a}$$

$$=\; \mathrm{lcross}(i, \sigma) + \mathrm{I}[i \text{ is a cycle double fall}]\,. \tag{4.86b}$$



(b,d) When i is even and $\neq \sigma(i)$



(a,c) When i is odd and $\neq \sigma(i)$

Figure 4.4: Crossings involved in the identities for the label ξ_i.

(c) If s_i a fall and i (hence also h_i) is odd, then

$$\left\lceil \frac{h_i}{2} \right\rceil - \xi_i \;=\; \mathrm{ucross}(i,\sigma) + \mathrm{I}[\sigma(i) \neq i] \tag{4.87a}$$

$$\;=\; \mathrm{ucross}(i,\sigma) + \mathrm{I}[i \text{ is a cycle double rise}]\,. \tag{4.87b}$$

(d) If s_i a fall and i (hence also h_i) is even, then

$$\left\lceil \frac{h_i}{2} \right\rceil - \xi_i \;=\; \mathrm{lcross}(i,\sigma)\,. \tag{4.88}$$

(Here $\mathrm{I}[\mathit{proposition}] = 1$ if *proposition* is true, and 0 if it is false.)

Since the right-hand sides of (4.85)–(4.88) are manifestly nonnegative, Lemma 4.5.3 is an immediate consequence of Lemma 4.5.4.

PROOF OF LEMMA 4.5.4. (a,b) If s_i is a rise, then $\sigma^{-1}(i)$ is even and $i \le \sigma^{-1}(i)$. We now consider separately the cases of i odd and i even.

(a) If i is odd, then h_i is odd; moreover, since $\sigma^{-1}(i)$ is even, i cannot be a fixed point, and we have the strict inequalities $i < \sigma^{-1}(i)$ and $i < \sigma(i)$. Then

$$\begin{aligned}
\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i &= \frac{h_i - 1}{2} - \xi_i && \text{(4.89a)} \\
&= f_i - 1 - \xi_i && \text{(4.89b)} \\
&= \#\{j \le i\colon\; \sigma(j) > i\} - 1 - \#\{j\colon\; j < i \le \sigma(i) < \sigma(j)\} && \text{(4.89c)} \\
&= \#\{j < i\colon\; \sigma(j) > i\} - \#\{j\colon\; j < i < \sigma(i) < \sigma(j)\} && \text{(4.89d)} \\
&= \#\{j\colon\; j < i < \sigma(j) < \sigma(i)\} && \text{(4.89e)} \\
&= \mathrm{ucross}(i,\sigma)\;. && \text{(4.89f)}
\end{aligned}$$

See Figure 4.4(a,c).

(b) If i is even, then h_i is even, and $\sigma(i) \le i$. So either i is a fixed point (hence $\sigma(i) = i = \sigma^{-1}(i)$) or else $\sigma(i) < i < \sigma^{-1}(i)$. Then

$$\begin{aligned}
\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i &= \frac{h_i}{2} - \xi_i && \text{(4.90a)} \\
&= f_i - \xi_i && \text{(4.90b)} \\
&= \#\{j \le i\colon\; \sigma^{-1}(j) > i\} - \#\{j\colon\; \sigma(j) < \sigma(i) \le i < j\} && \text{(4.90c)} \\
&= \#\{j < i\colon\; \sigma^{-1}(j) > i\} + \mathrm{I}[\sigma^{-1}(i) > i] \\
&\qquad - \#\{j\colon\; \sigma(j) < \sigma(i) \le i < j\} && \text{(4.90d)} \\
&= \#\{j\colon\; \sigma(j) < i < j\} + \mathrm{I}[\sigma(i) \ne i] \\
&\qquad - \#\{j\colon\; \sigma(j) < \sigma(i) \le i < j\} && \text{(4.90e)} \\
&= \#\{j\colon\; \sigma(i) < \sigma(j) < i < j\} + \mathrm{I}[\sigma(i) \ne i] && \text{(4.90f)} \\
&= \mathrm{lcross}(i,\sigma) + \mathrm{I}[\sigma(i) \ne i]\;. && \text{(4.90g)}
\end{aligned}$$

See Figure 4.4(b,d). Note that the identity (4.90) holds also when $h_i = 0$, i.e. when the step s_i is a rise from height $h_{i-1} = -1$: in this case i is a record-antirecord fixed point and we have $f_i = \xi_i = 0$, so that both sides of (4.90) are zero.

(c,d) If s_i is a fall, then $\sigma^{-1}(i)$ is odd and $\sigma^{-1}(i) \le i$. We again consider separately the cases of i odd and i even.

(c) If i is odd, then h_i is odd, and $i \le \sigma(i)$. So either i is a fixed point (hence $\sigma^{-1}(i) = i = \sigma(i)$) or else $\sigma^{-1}(i) < i < \sigma(i)$. Then

$$\begin{aligned}
\left\lceil \frac{h_i}{2} \right\rceil - \xi_i &= \frac{h_i+1}{2} - \xi_i && (4.91a)\\
&= f_i - \xi_i && (4.91b)\\
&= \#\{j \le i\colon\; \sigma(j) > i\} - \#\{j\colon\; j < i \le \sigma(i) < \sigma(j)\} && (4.91c)\\
&= \#\{j < i\colon\; \sigma(j) > i\} + \mathrm{I}[\sigma(i) \ne i] \\
&\qquad\qquad - \#\{j\colon\; j < i < \sigma(i) < \sigma(j)\} && (4.91d)\\
&= \#\{j\colon\; j < i < \sigma(j) < \sigma(i)\} + \mathrm{I}[\sigma(i) \ne i] && (4.91e)\\
&= \mathrm{ucross}(i,\sigma) + \mathrm{I}[\sigma(i) \ne i]\,. && (4.91f)
\end{aligned}$$

See again Figure 4.4(a,c). Note that the identity (4.91) holds also when $h_i = -1$, i.e. when the step s_i is a fall from height $h_{i-1} = 0$; in this case i is a record-antirecord fixed point and we have $f_i = \xi_i = 0$, so that both sides of (4.91) are zero.

(d) If i is even, then h_i is even; moreover, since $\sigma^{-1}(i)$ is odd, i cannot be a fixed point, and we have the strict inequalities $\sigma^{-1}(i) < i$ and $\sigma(i) < i$. Then

$$\begin{aligned}
\left\lceil \frac{h_i}{2} \right\rceil - \xi_i &= \frac{h_i}{2} - \xi_i && (4.92a)\\
&= f_i - \xi_i && (4.92b)\\
&= \#\{j \le i\colon\; \sigma^{-1}(j) > i\} - \#\{j\colon\; \sigma(j) < \sigma(i) \le i < j\} && (4.92c)\\
&= \#\{j < i\colon\; \sigma^{-1}(j) > i\} - \#\{j\colon\; \sigma(j) < \sigma(i) < i < j\} && (4.92d)\\
&= \#\{j\colon\; \sigma(j) < i < j\} - \#\{j\colon\; \sigma(j) < \sigma(i) < i < j\} && (4.92e)\\
&= \#\{j\colon\; \sigma(i) < \sigma(j) < i < j\} && (4.92f)\\
&= \mathrm{lcross}(i,\sigma)\,. && (4.92g)
\end{aligned}$$

See again Figure 4.4(b,d). □

We now consider the four possible combinations of s_i (rise or fall) and parity of h_i (odd or even), and determine in each case the cycle classification of the index i. By definition s_i tells us the parity of $\sigma^{-1}(i)$, while the parity of h_i equals the parity of i. So these two pieces of information tell us what was recorded in (2.64)/(2.65):

- $\sigma^{-1}(i)$ even and i odd $\iff$ i is a cycle valley
- $\sigma^{-1}(i)$ even and i even $\iff$ i is either a cycle double fall or an even fixed point
- $\sigma^{-1}(i)$ odd and i odd $\iff$ i is either a cycle double rise or an odd fixed point
- $\sigma^{-1}(i)$ odd and i even $\iff$ i is a cycle peak

So we need only disambiguate the fixed points from the cycle double falls/rises in the middle two cases; we will see that in these cases i is a fixed point if and only if ξ_i takes its maximum allowed value. More precisely:

Lemma 4.5.5 (Cycle classification)**.**

(a) If s_i a rise and h_i is odd (hence h_{i-1} is even), then i is a cycle valley.

(b) If s_i a rise and h_i is even (hence h_{i-1} is odd), then i is an even fixed point in case $\xi_i = \left\lceil \frac{h_i - 1}{2} \right\rceil$ $(= h_i/2 = f_i)$; otherwise it is a cycle double fall.

(c) If s_i is a fall and h_i is odd (hence h_{i-1} is even), then i is an odd fixed point in case $\xi_i = \left\lceil \frac{h_i}{2} \right\rceil$ $(= (h_i+1)/2 = f_i)$; otherwise it is a cycle double rise.

(d) If s_i is a fall and h_i is even (hence h_{i-1} is odd), then i is a cycle peak.

PROOF. (a,d) follow immediately from (2.64)/(2.65).

(b) If s_i a rise and h_i is even (hence i is even), then by (4.86) we have

$$\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i \;=\; \mathrm{lcross}(i,\sigma) + \mathrm{I}[\sigma(i) \neq i]\,, \tag{4.93}$$

so i is a fixed point if and only if $\xi_i = \left\lceil \frac{h_i - 1}{2} \right\rceil$. Otherwise i is a cycle double fall.

(c) If s_i is a fall and h_i is odd (hence i is odd), by (4.87) we have

$$\left\lceil \frac{h_i}{2} \right\rceil - \xi_i \;=\; \mathrm{ucross}(i, \sigma) + \mathrm{I}[\sigma(i) \neq i] \,, \tag{4.94}$$

so i is a fixed point if and only if $\xi_i = \left\lceil \frac{h_i}{2} \right\rceil$. Otherwise i is a cycle double rise. □

Remark. We already saw in (4.83) that if i is a fixed point, then ξ_i takes the value specified in (b) or (c), which is also the maximum allowed value according to (4.84). Now we see the converse. ■

At the other extreme, it is easy to see that the index i is a record or antirecord if and only if ξ_i takes its minimum allowed value (namely, zero):

Lemma 4.5.6 (Record statistics)**.**

(a) If i is odd, then the index i is a record if and only if $\xi_i = 0$.

(b) If i is odd, then the index i is an antirecord if and only if $h_i = -1$ and $\xi_i = 0$, in which case i is a record-antirecord fixed point.

(c) If i is even, then the index i is an antirecord if and only if $\xi_i = 0$.

(d) If i is even, then the index i is a record if and only if $h_i = 0$ and $\xi_i = 0$, in which case i is a record-antirecord fixed point.

PROOF. (a,c) This is an immediate consequence of the definition (4.80).

(b) Every antirecord is a weak anti-excedance, so an odd index in a D-permutation can be an antirecord only if it is a fixed point, in which case it is a record-antirecord fixed point. This happens if and only if $h_i = -1$ and $\xi_i = 0$.

(d) Similar to (b), using the fact that every record is a weak excedance. □

4.5.3 Step 3: Proof of bijection

We prove that the map $\sigma \mapsto (\omega, \xi)$ is a bijection by explicitly describing the inverse map. That is, we let ω be any almost-Dyck path of length $2n$ and let ξ be any set of labels satisfying the inequalities (4.84), and we show how to reconstruct the unique D-permutation σ that gives rise to (ω, ξ) by the foregoing construction.

First, some preliminaries: Given a D-permutation $\sigma \in \mathfrak{D}_{2n}$ we can define four subsets of $[2n]$:

$$F = \{2,4,\ldots,2n\} = \text{even positions} \tag{4.95a}$$

$$F' = \{i\colon \sigma^{-1}(i) \text{ is even}\} = \{\sigma(2),\sigma(4),\ldots,\sigma(2n)\} \tag{4.95b}$$

$$G = \{1,3,\ldots,2n-1\} = \text{odd positions} \tag{4.95c}$$

$$G' = \{i\colon \sigma^{-1}(i) \text{ is odd}\} = \{\sigma(1),\sigma(3),\ldots,\sigma(2n-1)\} \tag{4.95d}$$

Note that F' (resp. G') are the positions of the rises (resp. falls) in the almost-Dyck path ω.

Let us observe that

$$F \cap F' = \text{cycle double falls and even fixed points} \tag{4.96a}$$

$$G \cap G' = \text{cycle double rises and odd fixed points} \tag{4.96b}$$

$$F \cap G' = \text{cycle peaks} \tag{4.96c}$$

$$F' \cap G = \text{cycle valleys} \tag{4.96d}$$

$$F \cap G = \varnothing \tag{4.96e}$$

$$F' \cap G' = \varnothing \tag{4.96f}$$

Let us also recall the notion of an *inversion table*: Let S be a totally ordered set of cardinality k, and let $\boldsymbol{x} = (x_1,\ldots,x_k)$ be a permutation of S (i.e., a word in which each element of S occurs exactly once); then the (left-to-right) inversion table corresponding to $\boldsymbol{x}$ is the sequence $\boldsymbol{p} = (p_1,\ldots,p_k)$ of nonnegative integers defined by $p_\alpha = \#\{\beta < \alpha\colon\ x_\beta > x_\alpha\}$. Note that $0 \le p_\alpha \le \alpha - 1$ for all $\alpha \in [k]$, so there are exactly $k!$ possible inversion tables. Given the inversion table $\boldsymbol{p}$, we can reconstruct the sequence $\boldsymbol{x}$ by working from right to left, as follows: There are p_k elements of S larger than x_k, so x_k must be the (p_k+1)th largest element of S. Then there are p_{k-1} elements of $S\setminus\{x_k\}$ larger than x_{k-1}, so x_{k-1} must be the $(p_{k-1}+1)$th largest element of $S\setminus\{x_k\}$. And so forth. [Analogously, the right-to-left inversion table corresponding to $\boldsymbol{x}$ is the sequence $\boldsymbol{q}=(q_1,\ldots,q_k)$ of nonnegative integers defined by $q_\alpha = \#\{\beta>\alpha\colon\ x_\beta<x_\alpha\}$.]

With these preliminaries out of the way, we can now describe the map $(\omega,\xi)\mapsto\sigma$. Given the almost-Dyck path ω, we can immediately reconstruct the sets F,F',G,G'. We now use the labels ξ to reconstruct the maps $\sigma\restriction F\colon\ F\to F'$ and $\sigma\restriction G\colon\ G\to G'$ as follows: The even subword $\sigma(2)\sigma(4)\cdots\sigma(2n)$ is a listing of F' whose right-to-left inversion table is given by $q_\alpha=\xi_{2\alpha}$; this is the content of (4.81a). Similarly, the odd subword $\sigma(1)\sigma(3)\cdots\sigma(2n-1)$ is a listing of G' whose left-to-right inversion table is given by $p_\alpha=\xi_{2\alpha-1}$; this is the content of (4.81b). See Figure 4.5 for an example.

The only thing that remains to be shown is that the σ thus constructed is indeed a D-permutation. For this, we need to show that the following inequalities hold:

$$\sigma(2k) \;\leq\; 2k \tag{4.97a}$$

$$\sigma(2k-1) \;\geq\; 2k-1 \tag{4.97b}$$

Let us do a double counting of the number of rises occuring after the step s_{2k}. As $2n-2k$ is the total number of steps after $2k$, the number of rises after s_{2k} is $(n-k)-h_{2k}/2$. Thus, we have that

$$\begin{aligned}
(n-k)-\frac{h_{2k}}{2} \;&=\; \#\{i>2k\colon\ \sigma^{-1}(i)\text{ is even}\} && (4.98a)\\
&\geq\; \#\{i>2k\colon\ \sigma^{-1}(i)>2k\text{ and }\sigma^{-1}(i)\text{ is even}\} && (4.98b)\\
&=\; \#\{2l>2k\colon\ \sigma(2l)>2k\} && (4.98c)\\
&=\; (n-k)-\#\{2l>2k\colon\ \sigma(2l)\leq 2k\} && (4.98d)
\end{aligned}$$

$$\begin{array}{r} F = \{2i \mid 1 \le i \le 7\} \\ F' = \{\sigma(2i) \mid 1 \le i \le 7\} \\ \text{Right-to-left inversion table: } \xi_{2i} \end{array} \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 1 & 2 & 4 & 6 & 3 & 12 & 13 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{r} G = \{2i-1 \mid 1 \le i \le 7\} \\ G' = \{\sigma(2i-1) \mid 1 \le i \le 7\} \\ \text{Left-to-right inversion table: } \xi_{2i-1} \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 7 & 9 & 5 & 8 & 10 & 11 & 14 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$2i$	ξ_{2i}	$F' \backslash \{\sigma(2),\ldots,\sigma(2i-2)\}$	$\sigma(2i)$
2	0	$\{\underline{1},2,3,4,6,12,13\}$	1
4	0	$\{\underline{2},3,4,6,12,13\}$	2
6	1	$\{3,\underline{4},6,12,13\}$	4
8	1	$\{3,\underline{6},12,13\}$	6
10	0	$\{\underline{3},12,13\}$	3
12	0	$\{\underline{12},13\}$	12
14	0	$\{\underline{13}\}$	13

$2i-1$	ξ_{2i-1}	$G' \backslash \{\sigma(13),\ldots,\sigma(2i+1)\}$	$\sigma(2i-1)$
13	0	$\{5,7,8,9,10,11,\underline{14}\}$	14
11	0	$\{5,7,8,9,10,\underline{11}\}$	11
9	0	$\{5,7,8,9,\underline{10}\}$	10
7	1	$\{5,7,\underline{8},9\}$	8
5	2	$\{\underline{5},7,9\}$	5
3	0	$\{7,\underline{9}\}$	9
1	0	$\{\underline{7}\}$	7

Figure 4.5: Reconstruction of the permutation $\sigma = 7\,1\,9\,2\,5\,4\,8\,6\,10\,3\,11\,12\,14\,13 = (1,7,8,6,4,2)\,(3,9,10)\,(5)\,(11)\,(12)\,(13,14)$ from its almost-Dyck path ω and labels ξ. The value $\sigma(2i)$ is chosen so that it has ξ_{2i} entries to its left in the remaining subset of F' in increasing order; the value $\sigma(2i-1)$ is chosen so that it has ξ_{2i-1} entries to its right in the remaining subset of G' in increasing order.

and hence,

$$\frac{h_{2k}}{2} \;\le\; \#\{2l > 2k\colon\; \sigma(2l) \le 2k\}\;. \tag{4.99}$$

If k is such that $\sigma(2k) > 2k$, then (4.99) becomes strict [because $i = \sigma(2k)$ contributes to (4.98a) but not to (4.98b)]. Furthermore, if $\sigma(2k) > 2k$, we also have

$$\#\{2l > 2k\colon\; \sigma(2l) \le 2k\} \;\le\; \#\{2l > 2k\colon\; \sigma(2l) < \sigma(2k)\} \;=\; \xi_{2k} \tag{4.100}$$

by (4.81). In this situation, equations (4.99) and (4.100) together give

$$\xi_{2k} \;>\; \frac{h_{2k}}{2} \;=\; \left\lceil \frac{h_{2k}}{2} \right\rceil \;=\; \left\lceil \frac{h_{2k}-1}{2} \right\rceil\,, \tag{4.101}$$

where the equalities occur because h_{2k} is even. This contradicts (4.84a,b), and proves that $\sigma(2k) \le 2k$.

The proof that $\sigma(2k-1) \ge 2k-1$ uses a similar double-counting argument for the number of falls before s_{2k-1}. $\square$

4.5.4 Step 4: Computation of the weights

We can now compute the weights associated to the 0-Schröder path $\widehat{\omega}$ in Theorem 2.2.9, which we recall are $a_{h,\xi}$ for a rise starting at height h with label ξ, $b_{h,\xi}$ for a fall starting at height h with label ξ, and $c_{h,\xi}$ for a long level step at height h with label ξ. (Of course, in the present case we have long level steps only at height 0.) We do this by putting together the information collected in Lemmas 4.5.2–4.5.6:

(a) Rise from height $h_{i-1} = 2k$ to height $h_i = 2k+1$ (hence i odd):

- By Lemma 4.5.3, the label satisfies $0 \le \xi_i \le k$.
- By Lemma 4.5.5(a), this is a cycle valley.
- By Lemma 4.5.2, $\mathrm{unest}(i,\sigma) = \xi_i$.
- By Lemma 4.5.4(a), $\mathrm{ucross}(i,\sigma) = k - \xi_i$.

Therefore, from (4.31), the weight for this step is

$$a_{2k,\xi} \;=\; \mathsf{a}_{k-\xi,\xi}\;. \tag{4.102}$$

(b) Rise from height $h_{i-1} = 2k-1$ to height $h_i = 2k$ (hence i even):

- By Lemma 4.5.3, the label satisfies $0 \le \xi_i \le k$.
- By Lemma 4.5.5(b), this is a cycle double fall if $0 \le \xi_i < k$, and a fixed point if $\xi_i = k$.
- By Lemma 4.5.2, $\xi_i = \begin{cases} \text{lnest}(i,\sigma) & \text{if } i \text{ is not a fixed point} \\ \text{psnest}(i,\sigma) & \text{if } i \text{ is a fixed point} \end{cases}$
- By Lemma 4.5.4(b), $\text{lcross}(i,\sigma) = k-1-\xi_i$ when i is not a fixed point.

Therefore, from (4.31), the weight for this step is

$$a_{2k-1,\xi} \;=\; \begin{cases} \mathsf{c}_{k-1-\xi,\xi} & \text{if } 0 \le \xi < k \\ \mathsf{e}_k & \text{if } \xi = k \end{cases} \tag{4.103}$$

(c) Fall from height $h_{i-1} = 2k$ to height $h_i = 2k-1$ (hence i odd):

- By Lemma 4.5.3, the label satisfies $0 \le \xi_i \le k$.
- By Lemma 4.5.5(c), this is a cycle double rise if $0 \le \xi_i < k$, and a fixed point if $\xi_i = k$.
- By Lemma 4.5.2, $\xi_i = \begin{cases} \text{unest}(i,\sigma) & \text{if } i \text{ is not a fixed point} \\ \text{psnest}(i,\sigma) & \text{if } i \text{ is a fixed point} \end{cases}$
- By Lemma 4.5.4(c), $\text{ucross}(i,\sigma) = k-1-\xi_i$ when i is not a fixed point.

Therefore, from (4.31), the weight for this step is

$$b_{2k,\xi} \;=\; \begin{cases} \mathsf{d}_{k-1-\xi,\xi} & \text{if } 0 \le \xi < k \\ \mathsf{f}_k & \text{if } \xi = k \end{cases} \tag{4.104}$$

(d) Fall from height $h_{i-1} = 2k+1$ to height $h_i = 2k$ (hence i even):

- By Lemma 4.5.3, the label satisfies $0 \le \xi_i \le k$.
- By Lemma 4.5.5(d), this is a cycle peak.

- By Lemma 4.5.2, $\mathrm{lnest}(i,\sigma) = \xi_i$.
- By Lemma 4.5.4(d), $\mathrm{lcross}(i,\sigma) = k - \xi_i$.

Therefore, from (4.31), the weight for this step is

$$b_{2k+1,\xi} \;=\; \mathsf{b}_{k-\xi,\xi} \,. \tag{4.105}$$

(e) Long level step at height 0:

This corresponds in the almost-Dyck path ω to a fall from height 0 to height -1, followed by a rise from height -1 to height 0. Applying case (d) with $k = 0$ and $\xi = 0$, followed by case (b) with $k = 0$ and $\xi = 0$, we obtain a weight

$$c_{0,0} \;=\; \mathsf{e}_0 \mathsf{f}_0 \,. \tag{4.106}$$

Putting this all together in Theorem 2.2.9, we obtain a T-fraction with

$$\begin{aligned}
\alpha_{2k-1} &= (\text{rise from } 2k-2 \text{ to } 2k-1) \times (\text{fall from } 2k-1 \text{ to } 2k-2) \\
&= \left(\sum_{\xi=0}^{k-1} \mathsf{a}_{k-1-\xi,\xi} \right) \left(\sum_{\xi=0}^{k-1} \mathsf{b}_{k-1-\xi,\xi} \right) && (4.107) \\
\alpha_{2k} &= (\text{rise from } 2k-1 \text{ to } 2k) \times (\text{fall from } 2k \text{ to } 2k-1) \\
&= \left(\mathsf{e}_k + \sum_{\xi=0}^{k-1} \mathsf{c}_{k-1-\xi,\xi} \right) \left(\mathsf{f}_k + \sum_{\xi=0}^{k-1} \mathsf{d}_{k-1-\xi,\xi} \right) && (4.108) \\
\delta_1 &= \mathsf{e}_0 \mathsf{f}_0 && (4.109) \\
\delta_n &= 0 \qquad \text{for } n \ge 2 && (4.110)
\end{aligned}$$

This completes the proof of Theorem 4.2.9. □

We can now deduce Theorem 4.2.7 as a corollary:

PROOF OF THEOREM 4.2.7. Comparing (4.22) with (4.31) and using Lemma 4.5.6, we see that the needed weights in (4.31) are

$$\mathsf{a}_{k-1-\xi,\xi} = p_{+1}^{k-1-\xi} q_{+1}^{\xi} \times \begin{cases} y_1 & \text{if } \xi = 0 \\ v_1 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{4.111}$$

$$\mathsf{b}_{k-1-\xi,\xi} = p_{-1}^{k-1-\xi} q_{-1}^{\xi} \times \begin{cases} x_1 & \text{if } \xi = 0 \\ u_1 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{4.112}$$

$$\mathsf{c}_{k-1-\xi,\xi} = p_{-2}^{k-1-\xi} q_{-2}^{\xi} \times \begin{cases} x_2 & \text{if } \xi = 0 \\ u_2 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{4.113}$$

$$\mathsf{d}_{k-1-\xi,\xi} = p_{+2}^{k-1-\xi} q_{+2}^{\xi} \times \begin{cases} y_2 & \text{if } \xi = 0 \\ v_2 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{4.114}$$

$$\mathsf{e}_k = \begin{cases} z_{\rm e} & \text{if } k = 0 \\ s_{\rm e}^k w_{\rm e} & \text{if } k \ge 1 \end{cases} \tag{4.115}$$

$$\mathsf{f}_k = \begin{cases} z_{\rm o} & \text{if } k = 0 \\ s_{\rm o}^k w_{\rm o} & \text{if } k \ge 1 \end{cases} \tag{4.116}$$

Inserting these into (4.107)–(4.110) yields the continued-fraction coefficients (4.25). □

PROOF OF THEOREM 4.2.3. Specialize Theorem 4.2.7 to $p_{-1} = p_{-2} = p_{+1} = p_{+2} = q_{-1} = q_{-2} = q_{+1} = q_{+2} = s_{\rm e} = s_{\rm o} = 1$. □

4.5.5 An alternative label $\widehat{\xi}_i$: Proof of Theorems 4.2.10 and 4.2.11

As mentioned earlier, Randrianarivony [Ran97, Section 6] employed a very similar construction in the special case where σ is a D-o-semiderangement. Our definition of the almost-Dyck path ω is essentially the same as his Dyck path, modified slightly to allow for fixed points of both parities. However, he used a very different definition of the labels, namely [Ran97, eq. (6.2)]

$$\widehat{\xi}_i \;=\; \begin{cases} \#\{2l > 2k\colon\; \sigma(2l) < \sigma(2k)\} & \text{if } i = \sigma(2k) \\ \#\{2l-1 < 2k-1\colon\; \sigma(2l-1) > \sigma(2k-1)\} & \text{if } i = \sigma(2k-1) \end{cases} \tag{4.117a}$$

$$= \begin{cases} \#\{j\colon\; j < i \le \sigma^{-1}(i) < \sigma^{-1}(j)\} & \text{if } \sigma^{-1}(i) \text{ is even} \\ \#\{j\colon\; \sigma^{-1}(j) < \sigma^{-1}(i) \le i < j\} & \text{if } \sigma^{-1}(i) \text{ is odd} \end{cases} \tag{4.117b}$$

We would now like to show how the alternative label $\widehat{\xi}_i$ can be use to prove the variant forms of our T-fractions (Theorems 4.2.10 and 4.2.11).

Just as the labels ξ_i are related to the index-refined crossing and nesting statistics (2.67), so the alternative labels $\widehat{\xi}_i$ are related to the variant index-refined crossing and nesting statistics (2.69). For the nesting statistics this is immediate from the definition (4.117b):

Lemma 4.5.7 (Nesting statistics for the alternative labels)**.** We have

$$\widehat{\xi}_i \;=\; \begin{cases} \mathrm{lnest}'(i,\sigma) & \text{if } \sigma^{-1}(i) \text{ is even (i.e., } s_i \text{ is a rise) and } \neq i \\ \mathrm{unest}'(i,\sigma) & \text{if } \sigma^{-1}(i) \text{ is odd (i.e., } s_i \text{ is a fall) and } \neq i \\ \mathrm{psnest}(i,\sigma) & \text{if } \sigma^{-1}(i) = i \text{ (i.e., } i \text{ is a fixed point)} \end{cases} \tag{4.118}$$

Remarks. 1. By comparing (4.117b) with (4.80), we see that

$$\widehat{\xi}_i \;=\; \xi_{\sigma^{-1}(i)}\,. \tag{4.119}$$

This explains (4.118): combine (4.82) with (2.70).

2. The distinction between ξ_i and $\widehat{\xi}_i$ is also related to the distinction between two different notions of "inversion table". In the approach used here and in [SZ22, Section 6.1, Step 3], the label ξ_i is the number of inversions associated to the *position* i. By contrast, in [Ran97, p. 88] and [FZ90, p. 51], the label $\widehat{\xi}_i$ is the number of

inversions associated to the *value* i, i.e. to the position $\sigma^{-1}(i)$. Thus, whenever i is not a fixed point, $\widehat{\xi}_i$ corresponds to the original Foata–Zeilberger [FZ90] labels, while ξ_i corresponds to the modified labels used in [SZ22, Section 6.1]. ■

The alternative labels $\widehat{\xi}_i$ satisfy the same inequalities as the original labels ξ_i:

Lemma 4.5.8 (Inequalities satisfied by the alternative labels)**.** We have

$$0 \;\le\; \widehat{\xi}_i \;\le\; \left\lceil \frac{h_i - 1}{2} \right\rceil \;=\; \left\lceil \frac{h_{i-1}}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is even (i.e., } s_i \text{ is a rise)} \tag{4.120a}$$

$$0 \;\le\; \widehat{\xi}_i \;\le\; \left\lceil \frac{h_i}{2} \right\rceil \;=\; \left\lceil \frac{h_{i-1} - 1}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is odd (i.e., } s_i \text{ is a fall)} \tag{4.120b}$$

Lemma 4.5.8 will be an immediate consequence of the following identities:

Lemma 4.5.9 (Crossing statistics for the alternative labels)**.**

(a) If s_i a rise (i.e. $\sigma^{-1}(i)$ is even), then

$$\left\lceil \frac{h_i - 1}{2} \right\rceil - \widehat{\xi}_i \;=\; \text{lcross}'(i,\sigma) + \mathrm{I}[i \text{ is even and } \sigma(i) \neq i] \tag{4.121a}$$

$$\;=\; \text{lcross}'(i,\sigma) + \mathrm{I}[i \text{ is a cycle double fall}]\,. \tag{4.121b}$$

(b) If s_i a fall (i.e. $\sigma^{-1}(i)$ is odd), then

$$\left\lceil \frac{h_i}{2} \right\rceil - \widehat{\xi}_i \;=\; \text{ucross}'(i,\sigma) + \mathrm{I}[i \text{ is odd and } \sigma(i) \neq i] \tag{4.122a}$$

$$\;=\; \text{ucross}'(i,\sigma) + \mathrm{I}[i \text{ is a cycle double rise}]\,. \tag{4.122b}$$

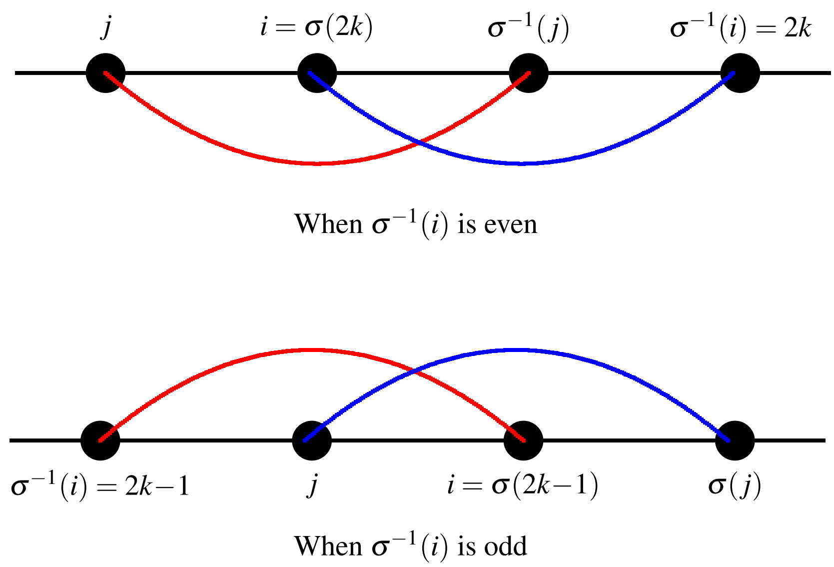


Figure 4.6: Crossings involved in the inequalities for the label $\widehat{\xi}_i$.

PROOF. (a) If s_i is a rise, then $\sigma^{-1}(i)$ is even, and $i \le \sigma^{-1}(i)$. We now consider separately the cases of i odd and i even.

(i) If i is odd, then h_i is odd; moreover, we have the strict inequality $i < \sigma^{-1}(i)$ (hence i is not a fixed point). Then

$$\begin{aligned}
\left\lceil \frac{h_i - 1}{2} \right\rceil - \widehat{\xi}_i \;&=\; \frac{h_i - 1}{2} - \widehat{\xi}_i && \text{(4.123a)} \\
&=\; f_i - 1 - \widehat{\xi}_i && \text{(4.123b)} \\
&=\; \#\{j \le i\colon\; \sigma^{-1}(j) > i\} - 1 \\
&\qquad\qquad - \#\{j < i\colon\; \sigma^{-1}(j) > \sigma^{-1}(i)\} && \text{(4.123c)} \\
&=\; \#\{j < i\colon\; \sigma^{-1}(j) > i\} - \#\{j < i\colon\; \sigma^{-1}(j) > \sigma^{-1}(i)\} && \text{(4.123d)} \\
&=\; \#\{j\colon\; j < i < \sigma^{-1}(j) \le \sigma^{-1}(i)\} && \text{(4.123e)} \\
&=\; \#\{j\colon\; j < i < \sigma^{-1}(j) < \sigma^{-1}(i)\} \\
&\qquad\qquad \text{since } j \ne i \text{ implies } \sigma^{-1}(j) \ne \sigma^{-1}(i) && \text{(4.123f)} \\
&=\; \mathrm{lcross}'(i, \sigma)\,. && \text{(4.123g)}
\end{aligned}$$

See Figure 4.6(a).

(ii) If i is even, then h_i is even, and

$$\begin{aligned}
\left\lceil \frac{h_i - 1}{2} \right\rceil - \widehat{\xi}_i &= \frac{h_i}{2} - \widehat{\xi}_i & (4.124a)\\
&= f_i - \widehat{\xi}_i & (4.124b)\\
&= \#\{j \le i\colon\ \sigma^{-1}(j) > i\} - \#\{j < i\colon\ \sigma^{-1}(j) > \sigma^{-1}(i)\} & (4.124c)\\
&= \#\{j < i\colon\ \sigma^{-1}(j) > i\} + \mathrm{I}[\sigma(i) \ne i] \\
&\qquad - \#\{j < i\colon\ \sigma^{-1}(j) > \sigma^{-1}(i)\} & (4.124d)\\
&= \#\{j\colon\ j < i < \sigma^{-1}(j) \le \sigma^{-1}(i)\} + \mathrm{I}[\sigma(i) \ne i] & (4.124e)\\
&= \#\{j\colon\ j < i < \sigma^{-1}(j) < \sigma^{-1}(i)\} + \mathrm{I}[\sigma(i) \ne i] \\
&\qquad \text{since } j \ne i \text{ implies } \sigma^{-1}(j) \ne \sigma^{-1}(i) & (4.124f)\\
&= \mathrm{lcross}'(i,\sigma) + \mathrm{I}[\sigma(i) \ne i]\,. & (4.124g)
\end{aligned}$$

See again Figure 4.6(a). Note that the identity (4.124) holds also when $h_i = 0$, i.e. when the step s_i is a rise from height $h_{i-1} = -1$; in this case i is a record-antirecord fixed point and we have $\widehat{\xi}_i = 0$, so that both sides of (4.124) are zero.

Combining (4.123) and (4.124) yields (4.121).

(b) If s_i is a fall, then $\sigma^{-1}(i)$ is odd, and $\sigma^{-1}(i) \le i$. We again consider separately the cases of i odd and i even.

(i) If i is odd, then h_i is odd, and

$$\begin{aligned}
\left\lceil \frac{h_i}{2} \right\rceil - \widehat{\xi}_i &= \frac{h_i + 1}{2} - \widehat{\xi}_i & (4.125a)\\
&= f_i - \widehat{\xi}_i & (4.125b)\\
&= \#\{j \le i\colon\ \sigma(j) > i\} - \#\{j\colon\ \sigma^{-1}(j) < \sigma^{-1}(i) \le i < j\} & (4.125c)\\
&= \#\{j \le i\colon\ \sigma(j) > i\} - \#\{j < \sigma^{-1}(i)\colon\ \sigma(j) > i\} & (4.125d)\\
&= \#\{j\colon\ \sigma^{-1}(i) \le j \le i < \sigma(j)\} & (4.125e)\\
&= \#\{j\colon\ \sigma^{-1}(i) < j \le i < \sigma(j)\} \\
&\qquad \text{since } i \ne \sigma(j) \text{ implies } j \ne \sigma^{-1}(i) & (4.125f)\\
&= \#\{j\colon\ \sigma^{-1}(i) < j < i < \sigma(j)\} + \mathrm{I}[\sigma^{-1}(i) < i < \sigma(i)] & (4.125g)\\
&= \mathrm{ucross}'(i,\sigma) + \mathrm{I}[\sigma(i) \ne i]\,. & (4.125h)
\end{aligned}$$

See Figure 4.6(b). Note that the identity (4.125) holds also when $h_i = -1$, i.e. when the step s_i is a fall from height $h_{i-1} = 0$; in this case i is a record-antirecord fixed point and we have $\widehat{\xi}_i = 0$, so that both sides of (4.125) are zero.

(ii) If i is even, then h_i is even; moreover, we have the strict inequality $\sigma^{-1}(i) < i$ (hence i is not a fixed point). Then

$$
\begin{aligned}
\left\lceil \frac{h_i}{2} \right\rceil - \widehat{\xi}_i &= \frac{h_i}{2} - \widehat{\xi}_i && \text{(4.126a)} \\
&= f_i - \widehat{\xi}_i && \text{(4.126b)} \\
&= \#\{j \le i \colon\; \sigma(j) > i\} - \#\{j < \sigma^{-1}(i) \colon\; \sigma(j) > i\} && \text{(4.126c)} \\
&= \#\{j \colon\; \sigma^{-1}(i) \le j \le i < \sigma(j)\} && \text{(4.126d)} \\
&= \#\{j \colon\; \sigma^{-1}(i) < j \le i < \sigma(j)\} \\
&\qquad\qquad \text{since } i \neq \sigma(j) \text{ implies } j \neq \sigma^{-1}(i) = \sigma^{-1}(i) && \text{(4.126e)} \\
&= \#\{j \colon\; \sigma^{-1}(i) < j < i < \sigma(j)\} \quad \text{since } i \ge \sigma(i) && \text{(4.126f)} \\
&= \mathrm{ucross}'(i, \sigma)\,. && \text{(4.126g)}
\end{aligned}
$$

See again Figure 4.6(b).

Combining (4.125) and (4.126) yields (4.122). □

Lemma 4.5.10 (Cycle classification for the alternative labels)**.**

(a) If s_i a rise and h_i is odd (hence h_{i-1} is even), then i is a cycle valley.

(b) If s_i a rise and h_i is even (hence h_{i-1} is odd), then i is an even fixed point in case $\widehat{\xi}_i = \left\lceil \frac{h_i - 1}{2} \right\rceil$ $(= h_i/2 = f_i)$; otherwise it is a cycle double fall.

(c) If s_i is a fall and h_i is odd (hence h_{i-1} is even), then i is an odd fixed point in case $\widehat{\xi}_i = \left\lceil \frac{h_i}{2} \right\rceil$ $(= (h_i + 1)/2 = f_i)$; otherwise it is a cycle double rise.

(d) If s_i is a fall and h_i is even (hence h_{i-1} is odd), then i is a cycle peak.

PROOF. Completely analogous to the proof of Lemma 4.5.5: just use Lemma 4.5.9 in place of Lemma 4.5.4. □

Lemma 4.5.11 (Record statistics for the alternative labels)**.**

(a) If $\sigma^{-1}(i)$ is odd, then the index $\sigma^{-1}(i)$ is a record if and only if $\widehat{\xi}_i = 0$.

(b) If $\sigma^{-1}(i)$ is even, then the index $\sigma^{-1}(i)$ is an antirecord if and only if $\widehat{\xi}_i = 0$.

PROOF. This is an immediate consequence of the definition (4.117b). □

Proof of bijection. The proof is similar to that presented in Section 4.5.3, but using a value-based rather than position-based notion of inversion table. Recall that if $S = \{s_1 < s_2 < \ldots < s_k\}$ is a totally ordered set of cardinality k, and $\boldsymbol{x} = (x_1, \ldots, x_k)$ is a permutation of S, then the (left-to-right) (position-based) inversion table corresponding to $\boldsymbol{x}$ is the sequence $\boldsymbol{p} = (p_1, \ldots, p_k)$ of nonnegative integers defined by $p_\alpha = \#\{\beta < \alpha \colon\ x_\beta > x_\alpha\}$. We now define the (left-to-right) *value-based* inversion table $\boldsymbol{p}'$ by $p'_{x_i} = p_i$; note that $\boldsymbol{p}'$ is a map from S to $\{0, \ldots, k-1\}$, such that p'_{x_i} is the number of entries to the left of x_i (in the word $\boldsymbol{x}$) that are larger than x_i. In particular, $0 \le p'_{s_i} \le k - i$. Given the value-based inversion table $\boldsymbol{p}'$, we can reconstruct the sequence $\boldsymbol{x}$ by working from largest to smallest value, as follows [Knu98, section 5.1.1]: We start from an empty sequence, and insert s_k. Then we insert s_{k-1} so that the resulting word has $p'_{s_{k-1}}$ entries to its left. Next we insert s_{k-2} so that the resulting word has $p'_{s_{k-2}}$ entries to its left, and so on. [The right-to-left value-based inversion table $\boldsymbol{q}'$ is defined analogously, and the reconstruction proceeds from smallest to largest.]

We now recall the definitions

$$F \;=\; \{2, 4, \ldots, 2n\} \;=\; \text{even positions} \tag{4.127a}$$

$$F' \;=\; \{i \colon\, \sigma^{-1}(i) \text{ is even}\} \;=\; \{\sigma(2), \sigma(4), \ldots, \sigma(2n)\} \tag{4.127b}$$

$$G \;=\; \{1, 3, \ldots, 2n-1\} \;=\; \text{odd positions} \tag{4.127c}$$

$$G' \;=\; \{i \colon\, \sigma^{-1}(i) \text{ is odd}\} \;=\; \{\sigma(1), \sigma(3), \ldots, \sigma(2n-1)\} \tag{4.127d}$$

Note that F' (resp. G') are the positions of the rises (resp. falls) in the almost-Dyck path ω.

We can now describe the map $(\omega, \widehat{\xi}) \mapsto \sigma$. Given the almost-Dyck path ω, we can immediately reconstruct the sets F, F', G, G'. We now use the labels $\widehat{\xi}$ to reconstruct the maps $\sigma \upharpoonright F\colon\ F \to F'$ and $\sigma \upharpoonright G\colon\ G \to G'$ as follows: The even subword $\sigma(2)\sigma(4)\cdots\sigma(2n)$ is a listing of F' whose right-to-left value-based inversion table is given by $q'_i = \widehat{\xi}_i$ for all $i \in F'$; this is the content of (4.117a). Similarly, the odd subword $\sigma(1)\sigma(3)\cdots\sigma(2n-1)$ is a listing of G' whose left-to-right value-based inversion table is given by $p'_i = \widehat{\xi}_i$ for all $i \in G'$; this again is the content of (4.117a). See Figure 4.7 for an example.

The only thing that remains to be shown is that the σ thus constructed is indeed a D-permutation. For this, we need to show that the following inequalities hold:

$$\sigma(2k-1) \;\geq\; 2k-1 \qquad (4.128a)$$

$$\sigma(2k) \;\leq\; 2k \qquad (4.128b)$$

We begin with a lemma:

Lemma 4.5.12.

(a) Let $f_1 < f_2 < \ldots < f_n$ be the elements of F' in increasing order. Then $f_j \leq 2j$ for every $j \in [n]$.

(b) Let $g_1 < g_2 < \ldots < g_n$ be the elements of G' in increasing order. Then $g_j \geq 2j-1$ for every $j \in [n]$.

PROOF. (a) Notice that the number of rises among the steps $s_1, s_2, \ldots, s_i$ is $(i+h_i)/2$ (and the number of falls is $(i-h_i)/2$). As there are exactly j rises among the steps $s_1, s_2, \ldots, s_{f_j}$, we have

$$\frac{f_j + h_{f_j}}{2} \;=\; j \qquad (4.129)$$

and hence

$$f_j \;=\; 2j - h_{f_j} \;\leq\; 2j \qquad (4.130)$$

since $h_{f_j} \geq 0$.

The proof of (b) is similar, using the fact that $h_{g_j} \geq -1$. $\square$

$$\begin{array}{r} F = \{2i \mid 1 \le i \le 7\} \\ F' = \{\sigma(2i) \mid 1 \le i \le 7\} \\ \text{Right-to-left inversion table: } \widehat{\xi}_{\sigma(2i)} \end{array} \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 1 & 2 & 4 & 6 & 3 & 12 & 13 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{r} G = \{2i-1 \mid 1 \le i \le 7\} \\ G' = \{\sigma(2i-1) \mid 1 \le i \le 7\} \\ \text{Left-to-right inversion table: } \widehat{\xi}_{\sigma(2i-1)} \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 7 & 9 & 5 & 8 & 10 & 11 & 14 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

f_i	$\widehat{\xi}_{f_i}$	Partial subword of F'
1	0	$\underline{1}$
2	0	$1\ \underline{2}$
3	0	$1\ 2\ \underline{3}$
4	1	$1\ 2\ \underline{4}\ 3$
6	1	$1\ 2\ 4\ \underline{6}\ 3$
12	0	$1\ 2\ 4\ 6\ 3\ \underline{12}$
13	0	$1\ 2\ 4\ 6\ 3\ 12\ \underline{13}$

g_i	$\widehat{\xi}_{g_i}$	Partial subword of G'
14	0	$\underline{14}$
11	0	$\underline{11}\ 14$
10	0	$\underline{10}\ 11\ 14$
9	0	$\underline{9}\ 10\ 11\ 14$
8	1	$9\ \underline{8}\ 10\ 11\ 14$
7	0	$\underline{7}\ 9\ 8\ 10\ 11\ 14$
5	2	$7\ 9\ \underline{5}\ 8\ 10\ 11\ 14$

Figure 4.7: Reconstruction of the permutation $\sigma = 7\,1\,9\,2\,5\,4\,8\,6\,10\,3\,11\,12\,14\,13 = (1,7,8,6,4,2)\,(3,9,10)\,(5)\,(11)\,(12)\,(13,14)$ from its almost-Dyck path ω and labels $\widehat{\xi}$. The value f_i is inserted so that it has $\widehat{\xi}_{f_i}$ entries to its right in the partial subword; the value g_i is inserted so that it has $\widehat{\xi}_{g_i}$ entries to its left in the partial subword.

Now consider any index $k \in [n]$. Let j' be the index for which $f_{j'} = \sigma(2k)$. From the definition of right-to-left inversion table, we know that there are $\widehat{\xi}_{\sigma(2k)}$ elements to the right of $\sigma(2k)$ in the word $\sigma(2)\sigma(4)\cdots\sigma(2n)$ which are smaller than $\sigma(2k)$. On the other hand, there are $n-k$ elements to the right of $\sigma(2k)$ in the word $\sigma(2)\sigma(4)\cdots\sigma(2n)$; and there are $n-j'$ elements in F' that are larger than $f_{j'}$. Therefore, there are at least $(n-k)-(n-j') = j'-k$ elements to the right of $\sigma(2k)$ in the word $\sigma(2)\sigma(4)\cdots\sigma(2n)$ that are smaller than $f_{j'} = \sigma(2k)$. Therefore,

$$\begin{aligned} \widehat{\xi}_{\sigma(2k)} &\geq j'-k && (4.131a)\\ &= \frac{f_{j'}+h_{f_{j'}}}{2} - k \qquad \text{[by (4.129)]} && (4.131b)\\ &= \frac{h_{\sigma(2k)}}{2} + \frac{\sigma(2k)-2k}{2}\,. && (4.131c) \end{aligned}$$

On the other hand, from (4.120a) we know that

$$\widehat{\xi}_{\sigma(2k)} \leq \left\lceil \frac{h_{\sigma(2k)}-1}{2} \right\rceil \leq \frac{h_{\sigma(2k)}}{2}\,. \tag{4.132}$$

Combining these two inequalities, we conclude that $\sigma(2k) \leq 2k$.

The proof that $\sigma(2k-1) \geq 2k-1$ is similar, using (4.120b). □

Remark. This proof of bijection is very close in spirit to that of Randrianarivony [Ran97, pp. 89–90]. ■

PROOF OF THEOREM 4.2.10. The computation of the weights is completely analogous to what was done in Section 4.5.4, but using Lemmas 4.5.7–4.5.11 in place of Lemmas 4.5.2–4.5.6. We leave the details to the reader: the upshot is that for cycle valleys and cycle peaks, “u” and “l” are interchanged compared to Section 4.5.4, and all statistics are primed. This is exactly what we have in (4.34). It therefore completes the proof of Theorem 4.2.10. □

PROOF OF THEOREM 4.2.11. Comparing (4.36) with (4.34) and using

Lemma 4.5.11, we see that the needed weights in (4.34) are the same as given in (4.111)–(4.116). Inserting these into Theorem 4.2.10 gives Theorem 4.2.11. □

4.6 Second T-fraction: Proof of Theorems 4.3.2, 4.3.6 and 4.3.9

In this section we prove the second master T-fraction (Theorem 4.3.9) by a bijection from D-permutations to labelled Schröder paths. Our construction combines ideas of Randrianarivony [Ran97] and Biane [Bia93] together with some new ingredients. After proving Theorem 4.3.9, we deduce Theorems 4.3.2 and 4.3.6 by specialisation.

Here we need to construct a bijection that will allow us to count the number of cycles (cyc), which is a global variable. To do this, we employ a modification of the Biane [Bia93] bijection, just as in Section 4.5 we employed a modification of the Foata–Zeilberger [FZ90] bijection. Our bijection maps $\mathfrak{D}_{2n}$ to the set of ***$(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled 0-Schröder paths of length $2n$***, where each label ξ_i is a *pair* of nonnegative integers $\xi_i = (\xi_i', \xi_i'')$ as follows:

$$
\begin{aligned}
\mathcal{A}_h &= \{0\}\times\{0\} && \text{for } h\geq 0 \text{ and } h \text{ even} && (4.133a)\\
\mathcal{A}_h &= \{0\}\times\{0,\ldots,\lceil h/2\rceil\} && \text{for } h\geq 0 \text{ and } h \text{ odd} && (4.133b)\\
\mathcal{B}_h &= \{0,\ldots,\lceil (h-1)/2\rceil\}\times\{0\} && \text{for } h\geq 1 \text{ and } h \text{ even} && (4.133c)\\
\mathcal{B}_h &= \{0,\ldots,\lceil (h-1)/2\rceil\}^2 && \text{for } h\geq 1 \text{ and } h \text{ odd} && (4.133d)\\
\mathcal{C}_0 &= \{0\}\times\{0\} && && (4.133e)\\
\mathcal{C}_h &= \varnothing && \text{for } h\geq 1 && (4.133f)
\end{aligned}
$$

or equivalently

$$\begin{aligned}
\mathcal{A}_h &= \{0\} \times \{0\} && \text{for } h \geq 0 \text{ and } h = 2k && \text{(4.134a)}\\
\mathcal{A}_h &= \{0\} \times \{0, \ldots, k\} && \text{for } h \geq 0 \text{ and } h = 2k-1 && \text{(4.134b)}\\
\mathcal{B}_h &= \{0, \ldots, k\} \times \{0\} && \text{for } h \geq 1 \text{ and } h = 2k && \text{(4.134c)}\\
\mathcal{B}_h &= \{0, \ldots, k\}^2 && \text{for } h \geq 1 \text{ and } h = 2k+1 && \text{(4.134d)}\\
\mathcal{C}_0 &= \{0\} \times \{0\} && && \text{(4.134e)}\\
\mathcal{C}_h &= \varnothing && \text{for } h \geq 1 && \text{(4.134f)}
\end{aligned}$$

Our presentation of this bijection will follow the same steps as in Section 4.5.

4.6.1 Step 1: Definition of the almost-Dyck path

The almost-Dyck path ω associated to a D-permutation $\sigma \in \mathfrak{D}_{2n}$ is identical to the one employed in Section 4.5. That is:

- If $\sigma^{-1}(i)$ is even, then s_i is a rise.
- If $\sigma^{-1}(i)$ is odd, then s_i is a fall.

The interpretation of the heights h_i is thus exactly as in Lemma 4.5.1. We then define the 0-Schröder path $\widehat{\omega} = \psi(\omega)$ as before.

4.6.2 Step 2: Definition of the labels $\xi_i = (\xi'_i, \xi''_i)$

We define the labels $\xi_i = (\xi'_i, \xi''_i)$ as follows:

$$\xi'_i = \begin{cases} 0 & \text{if } \sigma^{-1}(i) \text{ is even} \\ \#\{j\colon\ \sigma^{-1}(j) < \sigma^{-1}(i) \leq i < j\} & \text{if } \sigma^{-1}(i) \text{ is odd} \end{cases} \tag{4.135}$$

$$\xi''_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \#\{j\colon\ \sigma(j) < \sigma(i) \leq i < j\} & \text{if } i \text{ is even} \end{cases} \tag{4.136}$$

These labels ξ'_i, ξ''_i needed for the proof of the second T-fraction are related to the labels $\xi_i, \widehat{\xi}_i$ defined in (4.80) and (4.117) and employed in Section 4.5 for the proof of the first T-fraction, as follows:

$$\sigma^{-1}(i) \text{ is odd} \iff i \in \text{Cpeak} \cup \text{Cdrise} \cup \text{Oddfix}:$$
$$\xi_i'(\text{Second}) \;=\; \widehat{\xi}_i(\text{First}) \;=\; \xi_{\sigma^{-1}(i)}(\text{First}) \qquad (4.137a)$$

$$i \text{ is even} \iff i \in \text{Cpeak} \cup \text{Cdfall} \cup \text{Evenfix}:$$
$$\xi_i''(\text{Second}) = \xi_i(\text{First}) \qquad (4.137b)$$

Note that (4.137a) refers to the cases (4.133c,d) where ξ_i' is nontrivial [$\sigma^{-1}(i)$ odd means that s_i is a fall], while (4.137b) refers to the cases (4.133b,d) where ξ_i'' is nontrivial [i even $\iff$ h_i even $\iff$ $h = h_{i-1}$ odd]. Note in particular that cycle peaks belong to both cases [cf. (4.133d)], while cycle valleys belong to neither [cf. (4.133a)].

It is worth remarking that these labels $\xi_i = (\xi_i', \xi_i'')$ are the same as those in the Biane bijection [SZ22, Section 6.2, Step 2] whenever i is not a fixed point. Compare also (4.137) to [SZ22, eq. (6.24)].

We can give these labels a nice interpretation by using (as in Section 4.5) the representation of a permutation $\sigma \in \mathfrak{S}_N$ by a bipartite digraph $\Gamma = \Gamma(\sigma)$ in which the top row of vertices is labelled $1,\ldots,N$ and the bottom row $1',\ldots,N'$, and we draw an arrow from i to j' in case $\sigma(i) = j$. Recall that, for $k \in [N]$, we denote by Γ_k the induced subgraph of Γ on the vertex set $\{1,\ldots,k\} \cup \{1',\ldots,k'\}$. We can consider the "history" $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_N = \Gamma$ as a process of building up the permutation σ by successively considering the status of indices $1,2,\ldots,N$.

Since we have here a D-permutation $\sigma \in \mathfrak{D}_{2n}$, we will have vertices $1,\ldots,2n$ and $1',\ldots,2n'$. First recall from (4.73)/(4.74) that f_{i-1} is the number of free vertices in the top row of Γ_{i-1}, and also the number of free vertices in the bottom row of Γ_{i-1}; and recall from (4.77) that $f_{i-1} = \lceil h_{i-1}/2 \rceil$. We index the free vertices on each row of Γ_{i-1} starting from 0: the indices are thus $0,\ldots,f_{i-1}-1$. We then start from the digraph Γ_{i-1} and look at what happens at stage i (see Figure 4.8):

- If i is a cycle valley, then at stage i we add no arrows. Since no choices are being made at this stage, we set $\xi_i' = \xi_i'' = 0$.

- If i is a cycle double fall or an even fixed point, then at stage i we add an arrow from i on the top row to an unconnected dot j' on the bottom row, where $j = \sigma(i) \leq i$; then ξ_i'' is the index (left-to-right, counting from 0) of the unconnected dot j' among all the unconnected dots on the bottom row (together with the new dot i') — that is the content of (4.136). Note that $j = i$ (an even fixed point) corresponds to $\xi_i'' = f_{i-1}$.

  Since no unconnected dot on the top row was touched, we set $\xi_i' = 0$.

- Similarly, if i is a cycle double rise or an odd fixed point, we add an arrow from an unconnected dot j on the top row to i' on the bottom row, where $j = \sigma^{-1}(i) \leq i$; then ξ_i' is the index (left-to-right, counting from 0) of the unconnected dot j among all the unconnected dots on the top row (together with the new dot i) — that is the content of (4.135). Note that $j = i$ (an odd fixed point) corresponds to $\xi_i' = f_{i-1}$.

  Since no unconnected dot on the bottom row was touched, we set $\xi_i'' = 0$.

- If i is a cycle peak, then we add two arrows: from i on the top row to the unconnected dot j' on the bottom row, where $j = \sigma(i) < i$; and also from the unconnected dot k on the top row to i' on the bottom row, where $k = \sigma^{-1}(i) < i$. Then ξ_i' (resp. ξ_i'') is the index of k (resp. j') among the unconnected dots on the top (resp. bottom) row — that is the content of (4.135)/(4.136).

All this is closely analogous to what was done in [SZ22, Section 6.2, Step 2], but with fixed points treated differently.

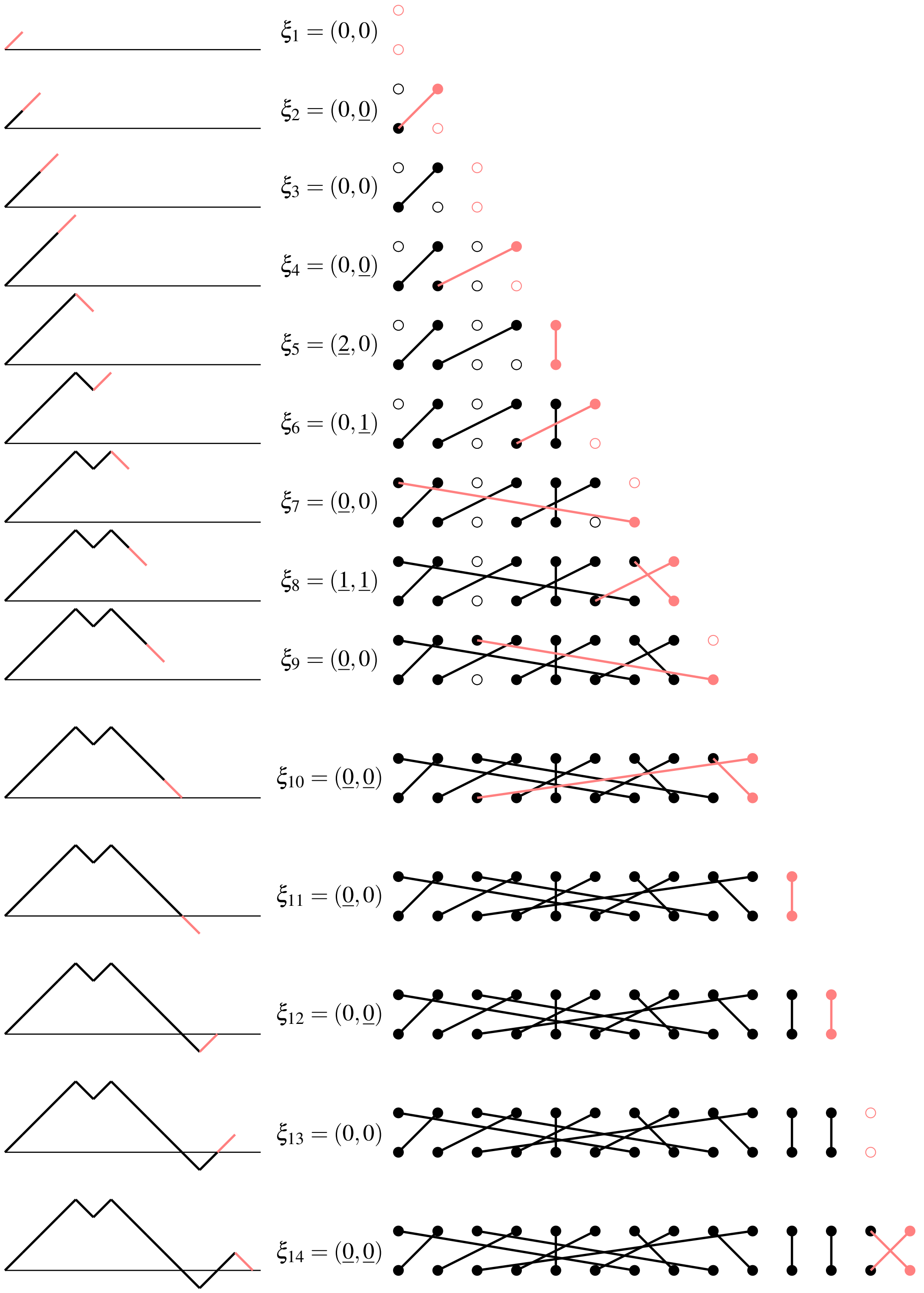


Figure 4.8: History of the permutation $\sigma = 7\,1\,9\,2\,5\,4\,8\,6\,10\,3\,11\,12\,14\,13 = (1,7,8,6,4,2)\,(3,9,10)\,(5)\,(11)\,(12)\,(13,14) \in \mathfrak{D}_{14}$ for the second bijection. Each stage of the construction shows: the partial almost-Dyck path $(\omega_0,\ldots,\omega_i)$, with the most recent step in pink; the label $\xi_i = (\xi_i', \xi_i'')$, with entries ξ_i' or ξ_i'' corresponding to new edges in the bipartite digraph being underlined; and the bipartite digraph Γ_i, with the new vertices and edges shown in pink. Note that the final almost-Dyck path is that of Figure 4.1, and the final bipartite digraph is that of Figure 4.2.

Lemma 4.6.1 (Nesting statistics for the second labels)**.**

(a) If $\sigma^{-1}(i)$ is odd (i.e., s_i is a fall), then

$$\xi'_i \;=\; \begin{cases} \mathrm{unest}'(i,\sigma) & \text{if } i \text{ is not a fixed point} \\[2mm] \mathrm{psnest}(i,\sigma) & \text{if } i \text{ is a fixed point} \end{cases} \tag{4.138}$$

(b) If i is even, then

$$\xi''_i \;=\; \begin{cases} \mathrm{lnest}(i,\sigma) & \text{if } i \text{ is not a fixed point} \\[2mm] \mathrm{psnest}(i,\sigma) & \text{if } i \text{ is a fixed point} \end{cases} \tag{4.139}$$

PROOF. (a) follows from (4.137a) and (4.118).

(b) follows from (4.137b) and (4.82). □

Lemma 4.6.2 (Inequalities satisfied by the second labels)**.**

(a) If $\sigma^{-1}(i)$ is odd (i.e., s_i is a fall), then

$$0 \;\le\; \xi'_i \;\le\; \left\lceil \frac{h_i}{2} \right\rceil \;=\; \left\lceil \frac{h_{i-1}-1}{2} \right\rceil \tag{4.140}$$

(b) If i is even, then

$$0 \;\le\; \xi''_i \;\le\; \frac{h_i}{2} \;=\; \begin{cases} \left\lceil \dfrac{h_{i-1}}{2} \right\rceil & \text{if } \sigma^{-1}(i) \text{ is even (i.e., } s_i \text{ is a rise)} \\[3mm] \dfrac{h_{i-1}-1}{2} & \text{if } \sigma^{-1}(i) \text{ is odd (i.e., } s_i \text{ is a fall)} \end{cases} \tag{4.141}$$

Lemma 4.6.2 will be an immediate consequence of the following identities:

Lemma 4.6.3 (Crossing statistics for the second labels)**.**

(a) If $\sigma^{-1}(i)$ is odd (i.e., s_i is a fall), then

$$\left\lceil \frac{h_i}{2} \right\rceil - \xi'_i \;=\; \mathrm{ucross}'(i,\sigma) + \mathrm{I}[i \text{ is odd and } \sigma(i) \neq i] \tag{4.142a}$$

$$\;=\; \mathrm{ucross}'(i,\sigma) + \mathrm{I}[i \text{ is a cycle double rise}]\;. \tag{4.142b}$$

(b) If i is even, then

$$\frac{h_i}{2} - \xi_i'' = \text{lcross}(i,\sigma) + \text{I}[\sigma^{-1}(i) \text{is even and } \sigma(i) \neq i] \quad (4.143a)$$

$$= \text{lcross}(i,\sigma) + \text{I}[i \text{ is a cycle double fall}] \,. \quad (4.143b)$$

PROOF. (a) Combining (4.137a) with (4.122) yields (4.142).

(b) If i is even, then h_i is also even. Combining (4.137b) with (4.86)/(4.88) yields (4.143). □

We now consider the four possible combinations of s_i (rise or fall) and parity of h_i (odd or even), and determine in each case the cycle classification of the index i. Exactly as with the first bijection, s_i tells us the parity of $\sigma^{-1}(i)$, while the parity of h_i equals the parity of i. So these two pieces of information again tell us what was recorded in (2.64)/(2.65):

- $\sigma^{-1}(i)$ even and i odd $\iff$ i is a cycle valley
- $\sigma^{-1}(i)$ even and i even $\iff$ i is either a cycle double fall or an even fixed point
- $\sigma^{-1}(i)$ odd and i odd $\iff$ i is either a cycle double rise or an odd fixed point
- $\sigma^{-1}(i)$ odd and i even $\iff$ i is a cycle peak

So, once again, we need only disambiguate the fixed points from the cycle double falls/rises in the middle two cases. We have:

Lemma 4.6.4 (Cycle classification for second bijection)**.**

(a) If s_i a rise and h_i is odd (hence h_{i-1} is even), then i is a cycle valley.

(b) If s_i a rise and h_i is even (hence h_{i-1} is odd), then i is an even fixed point in case $\xi_i'' = \left\lceil \frac{h_i - 1}{2} \right\rceil$ $(= h_i/2 = f_i)$; otherwise it is a cycle double fall.

(c) If s_i is a fall and h_i is odd (hence h_{i-1} is even), then i is an odd fixed point in case $\xi_i' = \left\lceil \frac{h_i}{2} \right\rceil$ $(= (h_i+1)/2 = f_i)$; otherwise it is a cycle double rise.

(d) If s_i is a fall and h_i is even (hence h_{i-1} is odd), then i is a cycle peak.

We remark that (b) also includes the case in which $(h_{i-1}, h_i) = (-1, 0)$, and (c) also includes the case in which $(h_{i-1}, h_i) = (0, -1)$.

PROOF. (a,d) follow immediately from (2.64)/(2.65).

(b) From Lemma 4.6.3(b), we have

$$\frac{h_i}{2} - \xi_i'' \begin{cases} \geq 1 & \text{if } i \text{ is a cycle double fall} \\ = 0 & \text{if } i \text{ is an even fixed point} \end{cases} \tag{4.144}$$

(c) Lemma 4.6.3(a), we have

$$\left\lceil \frac{h_i}{2} \right\rceil - \xi_i' \begin{cases} \geq 1 & \text{if } i \text{ is a cycle double rise} \\ = 0 & \text{if } i \text{ is an odd fixed point} \end{cases} \tag{4.145}$$

□

Lemma 4.6.5 (Record statistics for the second bijection)**.**

(a) If $\sigma^{-1}(i)$ is odd, then the index $\sigma^{-1}(i)$ is a record if and only if $\xi_i' = 0$.

(b) If i is even, then the index i is an antirecord if and only if $\xi_i'' = 0$.

PROOF. This is an immediate consequence of the definitions (4.135) and (4.136). □

4.6.3 Step 3: Proof of bijection

We prove that the map $\sigma \mapsto (\omega, \xi)$ is a bijection by explicitly describing the inverse map. That is, we let ω be any almost-Dyck path of length $2n$ and let ξ be any set of labels satisfying the inequalities (4.133)/(4.134), and we show how to reconstruct the unique D-permutation σ that gives rise to (ω, ξ) by the foregoing construction.

In fact, the interpretation as a bipartite digraph shows how to build the digraph, and hence reconstruct the D-permutation σ, by successively reading the steps s_i and labels ξ_i. Specifically, at stage i one starts from the digraph Γ_{i-1} and proceeds as follows (see again Figure 4.8):

(i) If s_i is a rise from height $h_{i-1} = 2k$ to height $h_i = 2k+1$ (by Lemma 4.6.4(a) this corresponds to i being a cycle valley), then we add no arrows.

(ii) If s_i is a rise from height $h_{i-1} = 2k-1$ to height $h_i = 2k$ and $\xi_i = (0,k)$ [by Lemma 4.6.4(b) this corresponds to i being an even fixed point], we add an arrow from i to i'.

(iii) If s_i is a rise from height $h_{i-1} = 2k-1$ to height $h_i = 2k$ with $\xi_i = (0,m)$ with $0 \le m < k$ (by Lemma 4.6.4(b) this corresponds to i being a cycle double fall), we add an arrow from i on the top row to the m-th free vertex (counting from 0) on the bottom row, call it j'. Of course $j < i$, because these are the only vertices visible in the digraph Γ_{i-1}.

(iv) If s_i is a fall from height $h_{i-1} = 2k$ to height $h_i = 2k-1$ and $\xi_i = (k,0)$ [by Lemma 4.6.4(c) this corresponds to i being an odd fixed point], we add an arrow from i to i'.

(v) If s_i is a fall from height $h_{i-1} = 2k$ to height $h_i = 2k-1$ and $\xi_i = (l,0)$ with $0 \le l < k$ (by Lemma 4.6.4(c) this corresponds to i being a cycle double rise), we add an arrow from the l-th free vertex (counting from 0) on the top row (which is of course $< i$) to i' on the bottom row.

(vi) If s_i is a fall from height $h_{i-1} = 2k+1$ to height $h_i = 2k$ and $\xi = (l,m)$ with $0 \le l,m \le k$ (by Lemma 4.6.4(d) this corresponds to i being a cycle peak), we add two arrows: one going from i on the top row to the m-th free vertex (counting from 0) on the bottom row (which is of course $< i$); and the other going from the l-th free vertex (counting from 0) on the top row to i' on the bottom row.

Clearly, once a vertex has become the source or sink of an arrow, it plays no further role in the construction and in particular receives no further arrows. Moreover, since $h_{2n} = 0$, at the end of the construction there are no unconnected vertices. The final result of the construction thus corresponds to a bijection between $\{1,\dots,2n\}$ and $\{1',\dots,2n'\}$, or in other words to permutation $\sigma \in \mathfrak{S}_{2n}$.

Note now that a free vertex i in the top row can be created only in situations (i) and (v), and in this case i is always odd. This means that a free vertex on the top row is always odd. Therefore, when in situations (v) and (vi) we connect a vertex $r < i$ on the top row to i' on the bottom row, r is always odd. On the other hand, we have already seen that in situations (iii) and (vi) when we connect the vertex i on the top row to a vertex j' with $j < i$ on the bottom row, i is always even. These two facts together show that σ is a D-permutation.

4.6.4 Step 4: Translation of the statistics

We begin by compiling the interpretations of heights and labels in terms of crossing and nesting statistics:

Lemma 4.6.6 (Crossing and nesting statistics)**.** We have

(a) When i is even and $h_i = 2k$,

$$\begin{aligned}
\xi''_i &= \text{lnest}(i,\sigma) && \text{if } i \in \text{Cpeak} \cup \text{Cdfall} && (4.146\text{a})\\
k - \xi''_i &= \text{lcross}(i,\sigma) && \text{if } i \in \text{Cpeak} && (4.146\text{b})\\
k - 1 - \xi''_i &= \text{lcross}(i,\sigma) && \text{if } i \in \text{Cdfall} && (4.146\text{c})\\
\xi''_i = k &= \text{psnest}(i,\sigma) && \text{if } i \in \text{Evenfix} && (4.146\text{d})\\
\xi'_i &= \text{unest}'(i,\sigma) && \text{if } i \in \text{Cpeak} && (4.146\text{e})\\
k - \xi'_i &= \text{ucross}'(i,\sigma) && \text{if } i \in \text{Cpeak} && (4.146\text{f})
\end{aligned}$$

(b) When i is odd and $h_{i-1} = 2k$,

$$\begin{aligned}
\xi'_i &= \text{unest}'(i,\sigma) && \text{if } i \in \text{Cdrise} && (4.147\text{a})\\
k - 1 - \xi'_i &= \text{ucross}'(i,\sigma) && \text{if } i \in \text{Cdrise} && (4.147\text{b})\\
k &= \text{ucross}(i,\sigma) + \text{unest}(i,\sigma) && \text{if } i \in \text{Cval} && (4.147\text{c})\\
\xi'_i = k &= \text{psnest}(i,\sigma) && \text{if } i \in \text{Oddfix} && (4.147\text{d})
\end{aligned}$$

PROOF. (a) When i is even, i can either be a cycle peak, a cycle double fall or an even fixed point.

- When i is a cycle double fall, $\sigma^{-1}(i)$ is even, so s_i is a rise from height $h_{i-1} = 2k-1$ to height $h_i = 2k$. Then (4.146a) follows from Lemma 4.6.1(b), and (4.146c) follows from Lemma 4.6.3(b).

- When i is a cycle peak, $\sigma^{-1}(i)$ is odd, so s_i is a fall from height $h_{i-1} = 2k+1$ to height $h_i = 2k$. Then (4.146a) follows from Lemma 4.6.1(b), (4.146b) follows from Lemma 4.6.3(b), (4.146e) follows from Lemma 4.6.1(a), and (4.146f) follows from Lemma 4.6.3(a).

- When i is an even fixed point, $\sigma^{-1}(i)$ is even, so s_i is a rise from height $h_{i-1} = 2k-1$ to height $h_i = 2k$. Then (4.146d) follows from Lemmas 4.6.1(b) and 4.6.4(b).

(b) When i is odd, i can either be a cycle valley, a cycle double rise or an odd fixed point.

- When i is a cycle double rise, $\sigma^{-1}(i)$ is odd, so s_i is a fall from height $h_{i-1} = 2k$ to height $h_i = 2k-1$. Then (4.147a) follows from Lemma 4.6.1(a), and (4.147b) follows from Lemma 4.6.3(a).

- When i is a cycle valley, $\sigma^{-1}(i)$ is even, so s_i is a rise from height $h_{i-1} = 2k$ to height $h_i = 2k+1$. As the almost-Dyck path is the same as in the first bijection, we can use Lemmas 4.5.2 and 4.5.4(a) to obtain $k = \mathrm{ucross}(i,\sigma) + \mathrm{unest}(i,\sigma)$, which is 4.147c).

- When i is an odd fixed point, $\sigma^{-1}(i)$ is odd, so s_i is a fall from height $h_{i-1} = 2k$ to height $h_i = 2k-1$. Then (4.147d) follows from Lemmas 4.6.1(a) and 4.6.4(c).

□

Remarks. 1. When i is a cycle valley, $\sigma^{-1}(i)$ is even and i is odd, so Lemmas 4.6.1 and 4.6.3 tell us nothing about ucross and unest; and the labels $\xi'_i = \xi''_i = 0$ carry no information. That is why in this case we learn only about the sum ucross + unest, which can be deduced from the heights alone.

2. Our treatment of cycle double rises is slightly nicer than that achieved in [SZ22, Lemma 6.4], thanks to the introduction of ucross′ and unest′.

3. In (4.146c) or (4.147b), $k = 0$ is impossible. These would correspond, respectively, to a rise from height -1 or a fall from height 0, which by Lemma 4.5.1 occur only when i is a record-antirecord fixed point, not when it is a cycle double fall or cycle double rise. ■

Finally, and most crucially, we come to the counting of cycles (cyc). We use the term *cycle closer* to denote the largest element in a non-singleton cycle. (This is the same as the "cycle peak maximum" defined in Section 2.5.1.) Obviously every non-singleton cycle has precisely one cycle closer. A cycle closer is always a cycle peak, but not conversely. So we need to know how many of the cycle peaks are the cycle closers. The answer is as follows:

Lemma 4.6.7 (Counting of cycles)**.** Fix $i = 2k$ such that $k \in [n]$, and fix $(s_1, \ldots, s_{i-1})$ and $(\xi_1, \ldots, \xi_{i-1})$. Consider all permutations $\sigma \in \mathfrak{D}_{2n}$ that have those given values for the first $i-1$ steps and labels and for which i is a cycle peak. Then:

(a) The value of $\xi_i = (\xi_i', \xi_i'')$ completely determines whether i is a cycle closer or not.

(b) For each value $\xi_i' \in [0, (h_{i-1} - 1)/2]$ there is precisely one value $\xi_i'' \in [0, (h_{i-1} - 1)/2]$ that makes i a cycle closer, and conversely.

The proof is similar to the proof of [SZ22, Lemma 6.5].

PROOF. We use once again the bipartite digraph of Figure 4.2, and let us also draw a vertical dotted line (with an upwards arrow) to connect each pair $j' \to j$. Now consider the restriction of this digraph to the vertex set $\{1, \ldots, i-1, 1', \ldots, (i-1)'\}$: as discussed in Step 3, this restriction can be reconstructed from the steps $(s_1, \ldots, s_{i-1})$ and the labels $(\xi_1, \ldots, \xi_{i-1})$. The connected components of this restriction are of two types: complete directed cycles and directed open chains; they correspond to cycles of σ whose cycle closers are, respectively, $\le i-1$ and $> i-1$. Each directed open chain runs from an unconnected dot on the bottom row to an unconnected dot on the top row.

Now suppose that i is a cycle peak. Then at stage i we add two arrows: from i on the top row to an unconnected dot j' on the bottom row; and also from an unconnected dot k on the top row to i' on the bottom row. Here ξ'_i (resp. ξ''_i) is the index of k (resp. j') among the unconnected dots on the top (resp. bottom) row.

Now the point is simply this: i is a cycle closer if and only if j' and k belong to the *same* directed open chain (with j' being its starting point and k being its ending point). So for each value $\xi'_i \in [0,(h_{i-1}-1)/2]$ there is precisely one value $\xi''_i \in [0,(h_{i-1}-1)/2]$ that makes i a cycle closer, and conversely. □

4.6.5 Step 5: Computation of the weights

We can now compute the weights associated to the 0-Schröder path $\widehat{\omega}$ in Theorem 2.2.9, which we recall are $a_{h,\xi}$ for a rise starting at height h with label ξ, $b_{h,\xi}$ for a fall starting at height h with label ξ, and $c_{h,\xi}$ for a long level step at height h with label ξ. (Of course, in the present case we have long level steps only at height 0.) We do this by putting together the information collected in Lemmas 4.6.2, 4.6.4, 4.6.6 and 4.6.7:

(a) Rise from height $h_{i-1} = 2k$ to height $h_i = 2k+1$ (hence i odd):

- By (4.134a), the label is $\xi_i = (0,0)$.
- By Lemma 4.6.4(a), this is a cycle valley.
- By (4.147c), $k = \mathrm{ucross}(i,\sigma) + \mathrm{unest}(i,\sigma)$.

Therefore, from (4.69), the weight for this step is

$$a_{2k,(0,0)} \;=\; \mathsf{a}_k \,. \tag{4.148}$$

(b) Rise from height $h_{i-1} = 2k-1$ to height $h_i = 2k$ (hence i even):

- By (4.134b) and Lemma 4.6.2(b), the label is $\xi_i = (0,\xi''_i)$ with $0 \le \xi''_i \le k$.
- By Lemma 4.6.4(b), this is a cycle double fall if $0 \le \xi''_i < k$, and an even fixed point if $\xi''_i = k$.

– By (4.146a,d),

$$\xi_i'' = \begin{cases} \mathrm{lnest}(i,\sigma) & \text{if } i \text{ is a cycle double fall} \\ \mathrm{psnest}(i,\sigma) & \text{if } i \text{ is an even fixed point} \end{cases} \tag{4.149}$$

– By (4.146c), $\mathrm{lcross}(i,\sigma) = k-1-\xi_i''$ when i is a cycle double fall.

Therefore, from (4.69), the weight for this step is

$$a_{2k-1,(0,\xi'')} = \begin{cases} \mathsf{c}_{k-1-\xi'',\xi''} & \text{if } 0 \le \xi'' < k \\ \lambda \mathsf{e}_k & \text{if } \xi'' = k \end{cases} \tag{4.150}$$

(c) Fall from height $h_{i-1} = 2k$ to height $h_i = 2k-1$ (hence i odd):

– By (4.134c) and Lemma 4.6.2(a), the label is $\xi_i = (\xi_i', 0)$ with $0 \le \xi_i' \le k$.

– By Lemma 4.6.4(c), this is a cycle double rise if $0 \le \xi_i' < k$, and an odd fixed point if $\xi_i' = k$.

– By (4.147a,d),

$$\xi_i' = \begin{cases} \mathrm{unest}'(i,\sigma) & \text{if } i \text{ is a cycle double rise} \\ \mathrm{psnest}(i,\sigma) & \text{if } i \text{ is an odd fixed point} \end{cases} \tag{4.151}$$

– By (4.147b), $\mathrm{ucross}'(i,\sigma) = k-1-\xi_i'$ when i is a cycle double rise.

Therefore, from (4.69), the weight for this step is

$$b_{2k,(\xi',0)} = \begin{cases} \mathsf{d}_{k-1-\xi',\xi'} & \text{if } 0 \le \xi' < k \\ \lambda \mathsf{f}_k & \text{if } \xi' = k \end{cases} \tag{4.152}$$

(d) Fall from height $h_{i-1} = 2k+1$ to height $h_i = 2k$ (hence i even):

– By (4.134d) and Lemma 4.6.2(a,b), the label is $\xi_i = (\xi_i', \xi_i'')$ with $0 \le \xi_i' \le k$ and $0 \le \xi_i'' \le k$.

- By Lemma 4.6.4(d), this is a cycle peak.
- By (4.146a), $\xi''_i = \text{lnest}(i, \sigma)$.
- By (4.146b), $k - \xi''_i = \text{lcross}(i, \sigma)$.

For each choice of $\xi''_i \in [0,k]$ there are $k+1$ possible choices of ξ'_i, of which one closes a cycle and the rest don't; this is the content of Lemma 4.6.7. Therefore, from (4.69), the total weight of all such steps is

$$b_{2k+1} \stackrel{\text{def}}{=} \sum_{\xi',\xi''} b_{2k+1,(\xi',\xi'')} \;=\; (\lambda+k) \sum_{\xi''=0}^{k} \mathsf{b}_{k-\xi'',\xi''} \,. \tag{4.153}$$

(e) Long level step at height 0:

This corresponds in the almost-Dyck path ω to a fall from height 0 to height -1, followed by a rise from height -1 to height 0. Applying case (c) with $k = 0$ and $\xi = 0$, followed by case (b) with $k = 0$ and $\xi = 0$, we obtain a weight

$$c_{0,0} \;=\; \lambda^2 \mathsf{e}_0 \mathsf{f}_0 \,. \tag{4.154}$$

Putting this all together in Theorem 2.2.9, we obtain a T-fraction with

$$\begin{aligned} \alpha_{2k-1} &= (\text{rise from } 2k-2 \text{ to } 2k-1) \times (\text{fall from } 2k-1 \text{ to } 2k-2) \\ &= \mathsf{a}_{k-1}\,(\lambda+k-1) \left(\sum_{\xi=0}^{k-1} \mathsf{b}_{k-1-\xi,\xi} \right) \end{aligned} \tag{4.155}$$

$$\begin{aligned} \alpha_{2k} &= (\text{rise from } 2k-1 \text{ to } 2k) \times (\text{fall from } 2k \text{ to } 2k-1) \\ &= \left(\lambda \mathsf{e}_k + \sum_{\xi=0}^{k-1} \mathsf{c}_{k-1-\xi,\xi} \right) \left(\lambda \mathsf{f}_k + \sum_{\xi=0}^{k-1} \mathsf{d}_{k-1-\xi,\xi} \right) \end{aligned} \tag{4.156}$$

$$\delta_1 = \lambda^2 \mathsf{e}_0 \mathsf{f}_0 \tag{4.157}$$

$$\delta_n = 0 \qquad \text{for } n \geq 2 \tag{4.158}$$

This completes the proof of Theorem 4.3.9. □

PROOF OF THEOREM 4.3.6. Comparing (4.58) with (4.69) and using Lemma 4.6.5, and recalling that we are making the specialisations $v_1 = y_1$ and $q_{+1} = p_{+1}$, we see that the needed weights in (4.69) are

$$\mathsf{a}_k = p_{+1}^k y_1 \tag{4.159}$$

$$\mathsf{b}_{k-1-\xi,\xi} = p_{-1}^{k-1-\xi} q_{-1}^{\xi} \times \begin{cases} x_1 & \text{if } \xi = 0 \\ u_1 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{4.160}$$

$$\mathsf{c}_{k-1-\xi,\xi} = p_{-2}^{k-1-\xi} q_{-2}^{\xi} \times \begin{cases} x_2 & \text{if } \xi = 0 \\ u_2 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{4.161}$$

$$\mathsf{d}_{k-1-\xi,\xi} = \widehat{p}_{+2}^{k-1-\xi} \widehat{q}_{+2}^{\xi} \times \begin{cases} \widehat{y}_2 & \text{if } \xi = 0 \\ \widehat{v}_2 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{4.162}$$

$$\mathsf{e}_k = \begin{cases} z_{\rm e} & \text{if } k = 0 \\ s_{\rm e}^k w_{\rm e} & \text{if } k \ge 1 \end{cases} \tag{4.163}$$

$$\mathsf{f}_k = \begin{cases} z_{\rm o} & \text{if } k = 0 \\ s_{\rm o}^k w_{\rm o} & \text{if } k \ge 1 \end{cases} \tag{4.164}$$

Inserting these into (4.155)–(4.158) yields the continued-fraction coefficients (4.61).
□

Remarks. 1. We needed to make the specialisation $v_1 = y_1$ because Lemma 4.6.5 tells us nothing about the record status of cycle valleys, for which $\sigma^{-1}(i)$ is even and i is odd. Similarly, we needed to make the specialisation $q_{+1} = p_{+1}$ because, for cycle valleys, Lemma 4.6.6(b) does not tell us about ucross and unest individually, but only their sum.

2. There is a variant master polynomial that we could have treated: instead of weighting cycle peaks using $\mathsf{b}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)}$ as in (4.69), we could instead weight them as $\mathsf{b}_{\mathrm{ucross}'(i,\sigma),\,\mathrm{unest}'(i,\sigma)}$. Then we would use (4.146e,f) instead of (4.146a,b);

the resulting T-fraction would be the same. ■

PROOF OF THEOREM 4.3.2. Specialize Theorem 4.3.6 to $p_{-1} = p_{-2} = p_{+1} = \widehat{p}_{+2} = q_{-1} = q_{-2} = \widehat{q}_{+2} = s_{\rm e} = s_{\rm o} = 1$. □

4.7 Some final remarks

When we began this work, we envisioned it as analogous to, though probably more complicated than, the study of the "linear family" [cf. (4.3)] that was undertaken in [SZ22]: our goal was to make a corresponding study of the "quadratic family" [cf. (4.4)]. What has surprised us is that the final results, as well as the associated methods of proof, turned out to be, not merely *analogous* to those of [SZ22], but in fact very closely *parallel* — much more closely parallel than we expected.

The key objects of [SZ22] were permutations of $[n]$; the key results were J-fractions; and the proofs involved bijections from permutations of $[n]$ to labelled Motzkin paths of length n, using the Foata–Zeilberger bijection for the first J-fraction (which did not involve the counting of cycles) and the Biane bijection for the second J-fraction (which involved the counting of cycles). The two bijections have the same paths but different labels.

By contrast, the key objects of the present chapter are D-permutations of $[2n]$; the key results are 0-T-fractions (that is, T-fractions that have $\delta_i = 0$ for all $i \geq 2$); and the proofs involve bijections from D-permutations of $[2n]$ to labelled 0-Schröder paths (or equivalently, labelled almost-Dyck paths) of length $2n$. Once again, the bijections for the first and second continued fractions have the same paths but different labels. Since these 0-Schröder paths are very different from the Motzkin paths of [SZ22], the definitions of the paths must also be very different, and indeed they are. The surprise was that the definitions of the *labels* in our constructions turned out to be almost identical to those employed in [SZ22] (where "almost" means that fixed points are treated differently): once again, we use the Foata–Zeilberger labels for the first T-fraction (which does not involve the counting of cycles) and the Biane labels for the second T-fraction (which involves the counting of cycles).

The final results also turned out to be amazingly similar. Compare, for instance, Theorem 4.2.7 with [SZ22, Theorem 2.7]: our coefficient α_{2k-1} is identical to the coefficient β_k in [SZ22, eq. (2.53c)]; our coefficient α_{2k} includes the same ingredients as the coefficient γ_k in [SZ22, eq. (2.53b)], but combined as a product rather than a sum. An analogous comparison holds between our first master T-fraction (Theorem 4.2.9) and the first master J-fraction in [SZ22, Theorem 2.9]. Furthermore, an analogous comparison holds between our T-fraction in Theorem 4.3.1 and the conjectured second J-fraction found [SZ22, Conjecture 2.3] (we will prove both of these in Chapter 5), and between our proved second T-fraction (Theorem 4.3.2) when specialised to $\widehat{v}_2 = \widehat{y}_2$ and the proved second J-fraction found in [SZ22, Theorem 2.4]. Finally, an almost analogous comparison holds between our second master T-fraction (Theorem 4.3.9) and the second master J-fraction [SZ22, Theorem 2.14], the only difference being that the treatment of d looks a bit more natural in the present work.

All this suggests to us that D-permutations are, among all the combinatorial models of the Genocchi and median Genocchi numbers, a particularly well-behaved one, which is closely analogous to ordinary permutations. It would be interesting to know whether similar continued fractions, in a large (or infinite) number of independent indeterminates, can be found for some of the other models of the Genocchi and median Genocchi numbers. Our approach in the present chapter has been to introduce a natural classification of indices into mutually exclusive categories — here the parity-refined record-and-cycle classification (Section 2.5.1) — and to build a homogeneous multivariate generating polynomial implementing this classification. It would be interesting to find analogous natural classifications for the other combinatorial models.

Appendix: J-fraction for the polynomials (4.40)

The first few polynomials

$$P_n(x,y,\lambda) \;=\; \sum_{\sigma \in \mathfrak{D}_{2n}} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)} \tag{4.165}$$

are too complicated to print even at $n = 3$. But we can give the first few J-fraction coefficients: they are

$$\begin{aligned}
\gamma_0 &= \lambda x(\lambda x + y) && \text{(4.166a)}\\
\beta_1 &= \lambda xy(\lambda + x)(\lambda + y) && \text{(4.166b)}\\
\gamma_1 &= (1+\lambda)(\lambda + x + y + xy) && \text{(4.166c)}\\
\beta_2 &= \lambda^3(1+xy) + \lambda^2(2+x+x^2+y+4xy+y^2) + \\
&\quad \lambda(1+3x+x^2+3y+4xy+x^2y+y^2+xy^2+x^2y^2) + \\
&\quad 2(x+y+x^2y+xy^2) && \text{(4.166d)}
\end{aligned}$$

followed by

$$\gamma_2 = \frac{N}{D} \tag{4.167}$$

where

$$\begin{aligned}
N \;=\; & (1+\lambda)\,\big[\lambda^4(1+xy) \;+\; \lambda^3(5+x+2x^2+y+10xy+2y^2+x^2y^2) \\
& + \lambda^2(7+8x+8x^2+8y+26xy+4x^2y+2x^3y+8y^2+4xy^2+7x^2y^2+2xy^3) \\
& + \lambda\,(3+15x+8x^2+2x^3+15y+22xy+18x^2y+5x^3y+8y^2+18xy^2 \\
& \qquad +13x^2y^2+x^3y^2+2y^3+5xy^3+x^2y^3) \\
& + (8x+2x^2+2x^3+8y+5xy+18x^2y+x^3y+2y^2+18xy^2+4x^2y^2+4x^3y^2 \\
& \qquad +2y^3+xy^3+4x^2y^3+x^3y^3)\big]
\end{aligned} \tag{4.168}$$

and

$$\begin{aligned}
D \;=\; & \lambda^3(1+xy) \;+\; \lambda^2(2+x+x^2+y+4xy+y^2) \\
& + \lambda\,(1+3x+x^2+3y+4xy+x^2y+y^2+xy^2+x^2y^2) \;+\; 2(x+y)(1+xy)
\end{aligned} \tag{4.169}$$

It can then be shown that

(a) γ_2 is not a polynomial in x (when y and λ are given fixed real values) unless $\lambda \in \{-2,-1,+1\}$ or $y \in \{-1,+1\}$.

(b) γ_2 is not a polynomial in y unless $\lambda \in \{-2,-1,+1\}$ or $x \in \{-1,+1\}$.

(c) γ_2 is not a polynomial in λ unless $x \in \{-1,+1\}$ or $y \in \{-1,+1\}$ or $x = y = 0$.

We state the polynomials obtained from the foregoing specialisations of γ_2:

Specialization	γ_2
$\lambda = +1$	$2(2+x)(2+y)$
$\lambda = -1$	0
$\lambda = -2$	$-1+x+y+xy/2$
$x = +1$	$(2+\lambda)(3+\lambda+2y)$
$x = -1$	$(1+\lambda)(4+\lambda-2y)$
$y = +1$	$(2+\lambda)(3+\lambda+2x)$
$y = -1$	$(1+\lambda)(4+\lambda-2x)$
$x = y = 0$	$(1+\lambda)(3+\lambda)$

These give rise to continued fractions as follows:

$\boldsymbol{\lambda = +1}$. By Theorem 4.2.3 specialised to $x_1 = x_2 = z_{\mathrm{e}} = z_{\mathrm{o}} = x$, $y_1 = y_2 = y$, $u_1 = u_2 = v_1 = v_2 = w_{\mathrm{e}} = w_{\mathrm{o}} = 1$, we obtain a T-fraction with $\alpha_{2k-1} = (x+k-1)(y+k-1)$, $\alpha_{2k} = (x+k)(y+k)$, $\delta_1 = x^2$ and hence by contraction (Proposition 2.2.1) a J-fraction with $\gamma_0 = x(x+y)$, $\gamma_n = 2(x+n)(y+n)$ for $n \geq 1$, $\beta_n = (x+n-1)(x+n)(y+n-1)(y+n)$.

$\boldsymbol{x = +1}$. By the reversal map $i \mapsto 2n+1-i$, the weight $x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)}$ is equivalent to $x^{\mathrm{rec}(\sigma)} y^{\mathrm{earec}(\sigma)}$. Now apply Theorem 4.3.2 specialised to $x_1 = x_2 = y$, $y_1 = \widehat{y}_2 = u_1 = u_2 = v_1 = \widehat{v}_2 = w_{\mathrm{e}} = w_{\mathrm{o}} = z_{\mathrm{e}} = z_{\mathrm{o}} = 1$: we obtain a T-fraction with $\alpha_{2k-1} = (\lambda+k-1)(y+k-1)$, $\alpha_{2k} = (\lambda+k)(y+k-1+\lambda)$, $\delta_1 = \lambda^2$. By contraction this yields a J-fraction with $\gamma_0 = \lambda(\lambda+y)$, $\gamma_n = (\lambda+n)(\lambda+2n-1+2y)$ for $n \geq 1$, $\beta_n = (\lambda+n-1)(\lambda+n)(y+n-1)(y+n-1+\lambda)$.

$\boldsymbol{y = +1}$. By Theorem 4.3.2 specialised to $x_1 = x_2 = z_{\mathrm{e}} = z_{\mathrm{o}} = x$, $y_1 = \widehat{y}_2 = u_1 = u_2 = v_1 = \widehat{v}_2 = w_{\mathrm{e}} = w_{\mathrm{o}} = 1$, we obtain a T-fraction with $\alpha_{2k-1} = (\lambda+k-1)(x+k-1)$, $\alpha_{2k} = (\lambda+k)(x+k-1+\lambda)$, $\delta_1 = \lambda^2 x^2$. By contraction this yields a J-fraction with $\gamma_0 = \lambda x(1+\lambda x)$, $\gamma_n = (\lambda+n)(\lambda+2n-1+2x)$ for $n \geq 1$, $\beta_n = (\lambda+n-1)(\lambda+n)(x+n-1)(x+n-1+\lambda)$.

$\boldsymbol{\lambda = -1}$. We *conjecture* a J-fraction with $\gamma_0 = x(x-y)$, $\gamma_n = 0$ for $n \geq 1$, $\beta_n = -xy(1-x)(1-y)$. We have proved this conjecture along with other continued fractions for permutations and D-permutations with weight $\lambda = -1$ in [DS23].

$\boldsymbol{\lambda = -2, x = -1, y = -1}$. These are polynomial through γ_2 and β_3, but we are unable to guess the general formula, and indeed we do not know whether they will continue to be polynomial at higher orders.

$\boldsymbol{x = 0}$. For all $n \geq 1$ and all $\sigma \in \mathfrak{S}_n$, the index n is an antirecord. Therefore, setting $x = 0$ suppresses all permutations for $n \geq 1$, and we have $P_n(0, y, \lambda) = \delta_{n0}$ (Kronecker delta). The value of γ_2 stated in the table above is completely irrelevant, because $\beta_1 = 0$.

$\boldsymbol{y = 0}$. It is not difficult to show that the only permutation $\sigma \in \mathfrak{S}_n$ with no exclusive records is the identity permutation. (The index 1 is always a record, so if there are no exclusive records it must be a fixed point; then the index 2 will be a record, and so forth.) Therefore $P_n(x, 0, \lambda) = \lambda^{2n} x^{2n}$. This gives an S-fraction with $\alpha_1 = \lambda^2 x^2$ and $\alpha_n = 0$ for $n \geq 2$. Equivalently, it gives a J-fraction with $\gamma_0 = \lambda^2 x^2$ and all other coefficients zero. The value of γ_2 stated in the table above is again completely irrelevant, because $\beta_1 = 0$.

Chapter 5

Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants

5.1 Introduction

5.1.1 Foreword

This chapter will introduce new results to the combinatorial theory of continued fractions for multivariate polynomials generalising the following three sequences of integers: factorials $(n!)_{n\geq 0}$, the Genocchi numbers [OEI19, A110501] introduced in Section 2.4.1

$$(g_n)_{n\geq 0} \;=\; 1,1,3,17,155,2073,38227,929569,28820619,1109652905,\ldots \tag{5.1}$$

and the median Genocchi numbers [OEI19, A005439] introduced in Section 2.4.2

$$(h_n)_{n\geq 0} \;=\; 1,1,2,8,56,608,9440,198272,5410688,186043904,\ldots\,. \tag{5.2}$$

We shall use permutations for studying factorials, and D-permutations [LW22, Laz20] and its subclasses (they were introduced in Section 2.4.3) for studying the Genocchi and median Genocchi numbers.

We shall consider continued fractions of Stieltjes-type (S-fraction),

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}, \tag{5.3}$$

as well as Jacobi-type and Thron-type (defined in (2.20),(2.21)). The ordinary generating functions of our integer sequences have S-fractions with coefficients $\alpha_{2k-1} = \alpha_{2k} = k$ for factorials [Eul60, section 21] (we saw this in equation (1.1)), $\alpha_{2k-1} = k^2$ and $\alpha_{2k} = k(k+1)$ for the Genocchi numbers [Vie, eq. (7.5)] [Vie83, p. V-9] [DZ94, eqns. (1.4) and (3.9)] (we saw this in equation (2.45)), and $\alpha_{2k-1} = \alpha_{2k} = k^2$ for the Genocchi medians [Vie, eq. (9.7)] [Vie83, p. V-15] [DZ94, eqns. (1.5) and (3.8)] (we saw this in equation (2.51)).

A systematic study of some combinatorial families whose associated S-fraction coefficients $(\alpha_n)_{n\geq 0}$ grow linearly in n was carried out by Sokal and Zeng in [SZ22]. They introduced various "master polynomials" enumerating permutations, set partitions and perfect matchings with respect to a large (sometimes infinite) number of simultaneous statistics. We carried out a similar study for D-permutations and its subclasses in Chapter 4: the associated T-fraction coefficients $(\alpha_n)_{n\geq 0}$ for these families grow quadratically in n. The continued fractions in [SZ22] and in our Chapter 4 were classified as "first" or "second" depending on whether they did not or did involve the count of cycles. Both in [SZ22] and in Chapter 4, the "second" continued fractions were proven using two specialisations. They were conjectured with only one specialisation ([SZ22, Conjecture 2.3] and Theorem 4.3.1), but a proof was lacking. Here we prove these conjectures. We will also prove a conjectured continued fraction of Randrianarivony and Zeng from 1996 [RZ96a, Conjecture 12] for D-o-semiderangements[1] (a subclass of D-permutations also introduced in Section 2.4.3).

[1]In their paper [RZ96a], Randrianarivony and Zeng call these Genocchi permutations. See Section 2.4.3 for our nomenclature.

Our proofs bring a surprising twist to this story. A common feature in the work of Sokal–Zeng [SZ22] and our work in Chapter 4 is that the proofs of the first and second continued fractions involve two different bijections: the first continued fractions were proved using bijections motivated from the Foata–Zeilberger bijection [FZ90], whereas the second continued fractions used the Biane bijection [Bia93] or a Biane-like bijection. However, in this chapter we will prove these conjectured second continued fractions by precisely the same bijections that were used to prove the first bijections in these papers. We will show, perhaps surprisingly, that these variants of the Foata–Zeilberger bijection can be used to obtain the counting of cycles.

Let us mention the historical context for our bijections. The Foata–Zeilberger bijection [FZ90] is a bijection between permutations and labelled Motzkin paths that has been very successfully employed to obtain continued fractions involving polynomial coefficients counting various permutation statistics (see for example [Ran98, Cor07, BS21, SZ22]). In a similar essence to the Foata–Zeilberger bijection, Randrianarivony [Ran97] introduced a bijection between D-o-semiderangements and labelled Dyck paths to obtain continued fractions counting various statistics on D-o-semiderangements. Motivated by Randrianarivony's bijection, in Section 4.5 we introduced two new bijections involving all D-permutations, one of which extends Randrianarivony's bijection.

The fundamental idea in this chapter is that we interpret the intermediate steps in these existing bijections in a new light in terms of *Laguerre digraphs*. A Laguerre digraph of size n is a directed graph where each vertex has a distinct label from the label set $[n]$ and has indegree 0 or 1 and outdegree 0 or 1.[2] Thus, the connected components in a Laguerre digraph are either directed paths or directed cycles. A path with one vertex and no edges will be called an isolated vertex, and a cycle with one vertex and one edge will be called a loop.

The Sokal–Zeng conjecture [SZ22, Conjecture 2.3] is a multivariate continued fraction containing 8 variables along with a one-parameter family of infinitely many

[2] Foata and Strehl [FS84] introduced an equivalent class of combinatorial objects called Laguerre configurations as a combinatorial interpretation of the Laguerre polynomials. Laguerre digraphs in the form that we use in this chapter were first introduced in [Sok22]. Also see [DDPS].

variables (the latter associated to fixed points) counting various simultaneous statistics for permutations. Our continued fraction in Theorem 4.3.1 is a multivariate continued fraction with 12 variables counting similar simultaneous statistics for D-permutations. The Randrianarivony–Zeng conjecture from 1996 [RZ96a, Conjecture 12] is a 4-variable continued fraction for D-o-semiderangements. In the same spirit as [PS20, SZ22] and Chapter 4, we will generalise these conjectured continued fractions and use our proofs to churn out continued fractions containing an infinite number of variables.

Before proceeding with the rest of this chapter, the reader may refer back to Section 2.2.1 to recall the necessary definitions and facts about continued fractions, Section 2.4 for the Genocchi numbers, median Genocchi numbers and D-permutations; and Section 2.5 for various permutation statistics. The rest of the introduction is organised as follows: We state the conjecture for permutations [SZ22, Conjecture 2.3] in Section 5.1.2. We then state the associated conjecture for Genocchi and median Genocchi numbers ([RZ96a, Conjecture 12] and Theorem 4.3.1) in Section 5.1.3. Then, in Section 5.1.4, we summarise our main ideas by providing an overview of the Foata–Zeilberger bijection and our interpretation of this bijection using Laguerre digraphs. The outline of the rest of this chapter is mentioned in Section 5.1.5.

5.1.2 Permutations: Statement of conjecture

The polynomial $\widehat{Q}_n$ was defined in [SZ22, Equation (2.29)]

$$\begin{aligned}\widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,\mathbf{w},\lambda) \;=\;\\ \sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \;\times\\ u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathrm{fix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)} \qquad (5.4)\end{aligned}$$

where $\mathbf{w}^{\mathrm{fix}(\sigma)}$ as defined in [SZ22, Equation (2.22)] is

$$\mathbf{w}^{\mathrm{fix}(\sigma)} \;=\; \prod_{i\in\mathrm{Fix}} w_{\mathrm{psnest}(i,\sigma)}. \qquad (5.5)$$

Sokal and Zeng stated the following conjecture in their paper:

Conjecture 5.1.1 ([SZ22, Conjecture 2.3]). *The ordinary generating function of the polynomials $\widehat{Q}_n$ specialised to $v_1 = y_1$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,\mathbf{w},\lambda)\, t^n \;=\; \cfrac{1}{1-\lambda w_0 t - \cfrac{\lambda x_1 y_1 t^2}{1-(x_2+y_2+\lambda w_1)t - \cfrac{(\lambda+1)(x_1+u_1)y_1 t^2}{1-(x_2+y_2+u_2+v_2+\lambda w_2)t - \cfrac{(\lambda+2)(x_1+2u_1)y_1 t^2}{1-\cdots}}}} \tag{5.6}$$

with coefficients

$$\gamma_0 \;=\; \lambda w_0 \tag{5.7a}$$

$$\gamma_n \;=\; [x_2+(n-1)u_2]+[y_2+(n-1)v_2]+\lambda w_n \qquad \textit{for } n \ge 1 \tag{5.7b}$$

$$\beta_n \;=\; (\lambda+n-1)[x_1+(n-1)u_1]y_1 \tag{5.7c}$$

Sokal and Zeng [SZ22, Theorem 2.4] proved this continued fraction subject to the further specialisation $v_2 = y_2$ using the Biane bijection. Here we will prove the full conjecture by using the Foata–Zeilberger bijection, suitably reinterpreted.

In Section 5.2, we will see that this conjecture is a special case of a more general J-fraction involving five families of infinitely many indeterminates and one additional variable. We will prove these results in Section 5.4.

5.1.3 D-Permutations: Statements of conjectures

In [RZ96a], Randrianarivony and Zeng introduced two sequences of polynomials for D-o-semiderangements [RZ96a, eq. (3.3)]

$$R_n(x,y,\bar{x},\bar{y}) \;=\; \sum_{\sigma\in\mathfrak{D}^{\mathrm{o}}_{2n}} x^{\mathrm{lema}(\sigma)} y^{\mathrm{romi}(\sigma)} \bar{x}^{\mathrm{fix}(\sigma)} \bar{y}^{\mathrm{remi}(\sigma)} \tag{5.8}$$

(we saw this polynomial in equation (4.39)) and [RZ96a, p. 9]

$$G_n(x,y,\bar{x},\bar{y}) \;=\; \sum_{\sigma\in\mathfrak{D}^{\mathrm{o}}_{2n}} x^{\mathrm{comi}(\sigma)} y^{\mathrm{lema}(\sigma)} \bar{x}^{\mathrm{cemi}(\sigma)} \bar{y}^{\mathrm{remi}(\sigma)} \tag{5.9}$$

where the statistics $\mathrm{lema}, \mathrm{romi}, \mathrm{remi}, \mathrm{comi}, \mathrm{cemi}$ are defined as follows:

- lema – left-to-right maxima whose value is even,
- romi – right-to-left minima whose value is odd,
- remi – right-to-left minima whose value is even,
- comi – odd cycle minima,
- cemi – even cycle minima;

for a permutation σ, $\mathrm{lema}(\sigma)$ denotes the number of left-to-right maxima (i.e. record) whose value $\sigma(i)$ is even, etc. See [RZ96a, p. 2] for a full description of these statistics.

In their paper, Randrianarivony and Zeng stated the following conjecture which we shall prove:

Conjecture 5.1.2 ([RZ96a, Conjecture 12])**.** *For $n \geq 1$ we have $R_n(x,y,\bar{x},\bar{y}) = G_n(x,y,\bar{x},\bar{y})$.*

Using [RZ96a, Proposition 10], Conjecture 5.1.2 can be equivalently stated as

Conjecture 5.1.2′. *The ordinary generating function of the polynomials $G_n(x,y,\bar{x},\bar{y})$*

defined in (5.8) *has the S-type continued fraction*

$$1+\sum_{n=1}^{\infty} G_n(x,y,\bar{x},\bar{y})t^n = \cfrac{1}{1-\cfrac{xyt}{1-\cfrac{1(\bar{x}+\bar{y})t}{1-\cfrac{(x+1)(y+1)t}{1-\cfrac{2(\bar{x}+\bar{y}+1)}{1-\cfrac{(x+2)(y+2)t}{1-\cfrac{3(\bar{x}+\bar{y}+2)t}{\cdots}}}}}}} \tag{5.10}$$

It is worthwhile to translate the statistics of Randrianarivony and Zeng to the statistics we introduced in Section 2.5.1. This was already done for the statistics involved in the polynomials $R_n(x,y,\bar{x},\bar{y})$ in Remark of Section 4.2.5; the following statistics are identical for D-o-semiderangements:

- lema = ereccpeak$'$
- remi = eareccdfall$'$
- romi = eareccval$'$
- fix = evennrfix.

It remains to translate the statistics cemi and comi.

Notice that the smallest element i of a cycle with at least two elements must be a cycle valley, and hence must be odd. As D-o-semiderangements do not have any odd fixed points, the cycle minima for fixed points are necessarily even. On the other hand, the smallest element i of a cycle with at least two elements must be a cycle valley, and hence must be odd. Thus, i is an even cycle minima if and only if it is an even fixed point, and i is an odd cycle minima if and only if it is the minimum valley of cycle with at least two elements. Thus, we have shown that

- cemi = evennrfix
- comi = minval.

This shows that

$$G_n(x,y,\bar{x},\bar{y}) \;=\; \sum_{\sigma\in\mathfrak{D}^{\mathrm{o}}_{2n}} x^{\mathrm{minval}(\sigma)} y^{\mathrm{ereccpeak}'(\sigma)} \bar{x}^{\mathrm{evennrfix}(\sigma)} \bar{y}^{\mathrm{eareccdfall}'(\sigma)}. \tag{5.11}$$

We will also look at the polynomials (4.41) introduced in Chapter 4 which we copy here for the convenience of the reader:

$$\begin{aligned}
&\widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\mathrm{e}},w_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}},\lambda) \;=\\
&\qquad \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times\\
&\qquad\qquad u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \times\\
&\qquad\qquad w_{\mathrm{e}}^{\mathrm{evennrfix}(\sigma)} w_{\mathrm{o}}^{\mathrm{oddnrfix}(\sigma)} z_{\mathrm{e}}^{\mathrm{evenrar}(\sigma)} z_{\mathrm{o}}^{\mathrm{oddrar}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}\,.
\end{aligned} \tag{5.12}$$

We will prove the Thron-type continued fraction in Theorem 4.3.1 involving the polynomials (5.12) which we also copy here:

Theorem 5.1.1. The ordinary generating function of the polynomials (5.12) specialised to $v_1 = y_1$ has the T-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,w_{\mathrm{e}},w_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}},\lambda)\, t^n \;=$$

$$\cfrac{1}{1-\lambda^2 z_{\mathrm{e}} z_{\mathrm{o}}\, t - \cfrac{\lambda x_1 y_1\, t}{1-\cfrac{(x_2+\lambda w_{\mathrm{e}})(y_2+\lambda w_{\mathrm{o}})\,t}{1-\cfrac{(\lambda+1)(x_1+u_1)y_1\,t}{1-\cfrac{(x_2+u_2+\lambda w_{\mathrm{e}})(y_2+v_2+\lambda w_{\mathrm{o}})\,t}{1-\cfrac{(\lambda+2)(x_1+2u_1)y_1\,t}{1-\cfrac{(x_2+2u_2+\lambda w_{\mathrm{e}})(y_2+2v_2+\lambda w_{\mathrm{o}})\,t}{1-\cdots}}}}}}} \tag{5.13}$$

with coefficients

$$\alpha_{2k-1} = (\lambda+k-1)\,[x_1+(k-1)u_1]\,y_1 \tag{5.14a}$$

$$\alpha_{2k} = [x_2+(k-1)u_2+\lambda w_{\mathrm{e}}]\,[y_2+(k-1)v_2+\lambda w_{\mathrm{o}}] \tag{5.14b}$$

$$\delta_1 = \lambda^2 z_{\mathrm{e}} z_{\mathrm{o}} \tag{5.14c}$$

$$\delta_n = 0 \qquad \text{for } n \geq 2 \tag{5.14d}$$

If this continued fraction is further specialised to $v_2 = y_2$, the resulting continued fraction is the same as our second T-fraction for D-permutations (Theorem 4.3.2) in Chapter 4, under the specialisation $\widehat{v}_2 = \widehat{y}_2$ and then identifying $\widehat{y}_2$ with y_2. We recall that Theorem 4.3.2 was proved using a Biane-like bijection. Here we will prove Theorem 5.1.1 using a Foata–Zeilberger-like bijection, suitably interpreted.

In Section 5.3, we will see that both Conjecture 5.1.2′ and Theorem 5.1.1 are special cases of general T-fractions involving six families of infinitely many indeterminates and one additional variable. We will prove these results in Section 5.5.

5.1.4 Overview of proof for results on permutations

We now summarise our proof for permutations (described in Section 5.4). Our results for D-permutations will also be obtained by using very similar ideas (described in Section 5.5). We first provide an overview of the Foata–Zeilberger bijection, and then briefly mention how we reinterpret it to obtain the count of cycles in a permutation.

Let $\sigma \in \mathfrak{S}_n$ be a permutation on n letters. This permutation σ partitions the set $[n]$ into excedance indices ($F = \{i \in [n] : \sigma(i) > i\}$), anti-excedance indices ($G = \{i \in [n] : \sigma(i) < i\}$), and fixed points ($H$). Similarly, σ also partitions $[n]$ into excedance values ($F' = \{i \in [n] : i > \sigma^{-1}(i)\}$), anti-excedance values ($G' = \{i \in [n] : i < \sigma^{-1}(i)\}$), and fixed points. Clearly, $\sigma \restriction F\colon\ F \to F'$, $\sigma \restriction G\colon\ G \to G'$, and $\sigma \restriction H\colon\ H \to H$ are bijections, and the permutation σ can be obtained from the following data:

- Two partitions of the set $[n] \;=\; F \cup G \cup H \;=\; F' \cup G' \cup H$.

- The two subwords of σ: $\sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)$ and $\sigma(y_1)\sigma(y_2)\ldots\sigma(y_l)$, where $G = \{x_1 < x_2 < \ldots < x_m\}$ and $F = \{y_1 < y_2 < \ldots < y_l\}$.

In their construction, Foata and Zeilberger [FZ90] use these data to describe a bijection from $\mathfrak{S}_n$ to a set of labelled Motzkin paths of length n (these have been defined in Sections 2.2.4, 2.2.5). One then uses Flajolet's theorem [Fla80] to obtain continued fractions from this bijection while keeping track of various simultaneous permutation statistics.

The Foata–Zeilberger bijection consists of the following steps (following [SZ22]):

- Step 1: A Motzkin path ω is constructed from σ. The path ω is fully determined by the sets F, F', G, G', H.

- Step 2: The labels ξ associated to ω are constructed from σ. It turns out that the labels depend on the maps $\sigma \restriction F\colon\ F \to F'$ and $\sigma \restriction G\colon\ G \to G'$ and the set H, separately.

- Step 3: This step describes the construction of the inverse map $(\omega, \xi) \mapsto \sigma$. This step is broken down as follows:

  - Step 3(a): The sets F, F', G, G', H are read off from the path ω.
  - Step 3(b): This description is the crucial part of the construction (at least for our purposes). We use *inversion tables* to construct the words $\sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)$ and $\sigma(y_1)\sigma(y_2)\ldots\sigma(y_l)$; the former is constructed using a "right-to-left" inversion table and the latter is constructed using a "left-to-right" inversion table.

It is, a priori, unclear how one might be able to track the number of cycles of σ in this construction. We resolve this issue by reinterpreting Step 3(b). We describe a "history" of this construction using Laguerre digraphs.

Recall that a Laguerre digraph of size n is a directed graph where each vertex has a distinct label from the label set $[n]$ and has indegree 0 or 1 and outdegree 0 or 1. It follows that the connected components in a Laguerre digraph are either directed paths or directed cycles. Clearly, any subgraph of a Laguerre digraph is also a Laguerre digraph. A permutation σ in cycle notation is equivalent to a Laguerre digraph L with no paths ([Sta09, pp. 22-23]). The directed edges of L are precisely $u \to \sigma(u)$. We will interpret Step 3(b) of the Foata–Zeilberger construction as building up a permutation as a sequence of Laguerre digraphs, starting from the empty digraph in which all vertices are isolated (i.e., have no adjacent edges), and ending with the digraph of the permutation σ in which there are no paths.

For a subset $S \subseteq [n]$, we let $L|_S$ denote the subgraph of L, containing the same set of vertices $[n]$, but only the edges $u \to \sigma(u)$, with $u \in S$ (we are allowed to have $\sigma(u) \notin S$). Let $u_1, \ldots, u_n$ be a rewriting of $[n]$. We consider the "history" $L|_\varnothing \subset L|_{\{u_1\}} \subset L|_{\{u_1,u_2\}} \subset \ldots \subset L|_{\{u_1,\ldots,u_n\}} = L$ as a process of building up the permutation σ by successively considering the status of vertices $u_1, u_2, \ldots, u_n$. At step u, we construct the edge $u \to \sigma(u)$. Thus, at each step we insert a new edge into the digraph, and at the end of this process, the resulting digraph obtained is the digraph of σ.

The crucial part of our construction is that we use a very special order $u_1, \ldots, u_n$: we first go through H in increasing order (we call this stage (a)), we then go through G in increasing order (stage (b)), finally we go through F but in decreasing order (stage (c)). This total order is suggested by the inversion tables. On building up the permutation σ using this history, we will see that the cycles can only be obtained during stage (c) and we can now count the number of cycles.

Our total order on $[n]$ only depends on the sets F, G, H, and hence, only on the path ω and not on the labels ξ. This is crucial for our proof to work.

5.1.5 Outline of chapter

The plan of this chapter is as follows: In Section 5.2 we state our results for permutations; this will include the continued fraction [SZ22, Conjecture 2.3] along with its generalisations. Next, we state our results for D-permutations in Section 5.3; this will include the continued fractions [RZ96a, Conjecture 12] and Theorem 5.1.1. In Section 5.4 we prove our continued fractions for permutations by reinterpreting Sokal and Zeng's variant of the Foata–Zeilberger bijection [SZ22, Section 6.1] using Laguerre digraphs. In Section 5.5 we prove our continued fractions for D-permutations by reinterpreting the two bijections stated in Sections 4.5.1-4.5.3 and Section 4.5.5 of Chapter 4, again by using Laguerre digraphs. We conclude (Section 5.6) with some brief remarks on our work.

5.2 Permutations: Statements of results

In this section, we state our continued fractions for permutations, in three increasingly more general versions. The first and most basic version (Theorem 5.2.1 is a J-fraction in 8 variables and another family of infinitely many variables that enumerates permutations with respect to the record-and-cycle classification except for the segregation of cycle valleys; it resolves [SZ22, Conjecture 2.3]. The second version (Theorem 5.2.2) is a (p,q)-generalisation of the first one: it is a J-fraction with 16 variables along with one family of infinitely many variables that enumerates permutations with respect to the record-and-cycle classification (introduced in Section 2.5.1) together with three pairs of (p,q)-variables counting the refined categories of crossing and nesting except for cycle valleys, and one variable corresponding to pseudo-nestings of fixed points. Finally, our third version (Theorem 5.2.3) — is a J-fraction in five infinite families of indeterminates along with one additional variable that keeps track of the number of cycles; this generalises the previous two by employing the index-refined crossing and nesting statistics (2.67). All these results will be proved in Section 5.4.

5.2.1 J-fraction (Sokal–Zeng conjecture)

Recall the polynomial $\widehat{Q}_n$ defined in equation (5.4)/[SZ22, equation (2.29)]

$$\widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,\mathbf{w},\lambda) \;=\; \sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathrm{fix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

where $\mathbf{w}^{\mathrm{fix}(\sigma)}$ is

$$\mathbf{w}^{\mathrm{fix}(\sigma)} \;=\; \prod_{i \in \mathrm{Fix}} w_{\mathrm{psnest}(i,\sigma)} \,.$$

Our first main result for permutations is [SZ22, Conjecture 2.3].

Theorem 5.2.1 ([SZ22, Conjecture 2.3], J-fraction for permutations)**.** *The ordinary generating function of the polynomials $\widehat{Q}_n$ specialised to $v_1 = y_1$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,\mathbf{w},\lambda)\, t^n \;=\;$$

$$\cfrac{1}{1-\lambda w_0 t-\cfrac{\lambda x_1 y_1 t^2}{1-(x_2+y_2+\lambda w_1)t-\cfrac{(\lambda+1)(x_1+u_1)y_1 t^2}{1-(x_2+y_2+u_2+v_2+\lambda w_2)t-\cfrac{(\lambda+2)(x_1+2u_1)y_1 t^2}{1-\cdots}}}} \tag{5.15}$$

with coefficients

$$\gamma_0 \;=\; \lambda w_0 \tag{5.16a}$$

$$\gamma_n \;=\; [x_2+(n-1)u_2]+[y_2+(n-1)v_2]+\lambda w_n \qquad \textit{for } n \ge 1 \tag{5.16b}$$

$$\beta_n \;=\; (\lambda+n-1)[x_1+(n-1)u_1]y_1 \tag{5.16c}$$

The continued fraction (5.15)/(5.16) requires only one specialisation, namely $v_1 = y_1$. This clearly generalises the second J-fraction for permutations of Sokal and Zeng [SZ22, Theorem 2.4] which also requires the specialisation $v_2 = y_2$.

We will prove Theorem 5.2.1 in Section 5.4.

5.2.2 p,q-generalisation

We now state a p,q-generalisation for Theorem 5.2.1 which also generalises [SZ22, Theorem 2.12]. Let us first recall the polynomial $\widehat{Q}_n$ defined in [SZ22, Equation (2.92)]

$$
\begin{aligned}
&\widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,\mathbf{w},p_{+1},p_{+2},p_{-1},p_{-2},q_{+1},q_{+2},q_{-1},q_{-2},s,\lambda) \;=\\
&\sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times\\
&\qquad u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathrm{fix}(\sigma)} \times\\
&\qquad p_{+1}^{\mathrm{ucrosscval}(\sigma)} p_{+2}^{\mathrm{ucrosscdrise}(\sigma)} p_{-1}^{\mathrm{lcrosscpeak}(\sigma)} p_{-2}^{\mathrm{lcrosscdfall}(\sigma)} \times\\
&\qquad q_{+1}^{\mathrm{unestcval}(\sigma)} q_{+2}^{\mathrm{unestcdrise}(\sigma)} q_{-1}^{\mathrm{lnestcpeak}(\sigma)} q_{-2}^{\mathrm{lnestcdfall}(\sigma)} s^{\mathrm{psnest}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}\;.
\end{aligned}
\tag{5.17}
$$

For the p,q-generalisation of their second J-fraction, involving the polynomials $\widehat{Q}_n$, Sokal and Zeng needed the specialisations $v_1 = y_1$, $v_2 = y_2$, $q_{+1} = p_{+1}$, and $q_{+2} = p_{+2}$. However, we now state a J-fraction that only requires the specialisations $v_1 = y_1$ and $q_{+1} = p_{+1}$.

Theorem 5.2.2 (J-fraction with p,q-generalisation for permutations)**.** *The ordinary generating function of the polynomials $\widehat{Q}_n$ specialised to $v_1 = y_1$ and $q_{+1} = p_{+1}$ has the J-type continued fraction*

$$
\sum_{n=0}^{\infty} \widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,\mathbf{w},p_{+1},p_{+2},p_{-1},p_{-2},p_{+1},q_{+2},q_{-1},q_{-2},s,\lambda)\, t^n \;=
$$

$$
\cfrac{1}{1-\lambda w_0 t-\cfrac{\lambda x_1 y_1 t^2}{1-(x_2+y_2+\lambda s w_1)t-\cfrac{(\lambda+1)(p_{-1}x_1+q_{-1}u_1)p_{+1}y_1t^2}{1-(p_{-2}x_2+q_{-2}u_2+p_{+2}y_2+q_{+2}v_2+\lambda s^2 w_2)t-\cfrac{(\lambda+2)(p_{-1}^2x_1+[q_{-1}p_{-1}+q_{-1}^2]u_1)p_{+1}^2y_1t^2}{1-\cdots}}}}
\tag{5.18}
$$

with coefficients

$$\gamma_0 \;=\; \lambda w_0 \tag{5.19a}$$

$$\gamma_n \;=\; (p_{-2}^{n-1}x_2+q_{-2}[n-1]_{p_{-2},q_{-2}}u_2)+(p_{+2}^{n-1}y_2+q_{+2}[n-1]_{p_{+2},q_{+2}}v_2)+\lambda s^n w_n \qquad \textit{for } n\geq 1 \tag{5.19b}$$

$$\beta_n \;=\; (\lambda+n-1)(p_{-1}^{n-1}x_1+q_{-1}[n-1]_{p_{-1},q_{-1}}u_1)p_{+1}^{n-1}y_1 \tag{5.19c}$$

We will prove this theorem in Section 5.4, as a special case of a more general result.

5.2.3 Master J-fraction

We can go much farther and obtain a more general J-fraction generalising Theorems 5.2.1 and 5.2.2. We obtain a J-fraction in the following five families of infinitely many indeterminates: $\mathbf{a} = (\mathsf{a}_\ell)_{\ell \ge 0}$, $\mathbf{b} = (\mathsf{b}_{\ell,\ell'})_{\ell,\ell' \ge 0}$, $\mathbf{c} = (\mathsf{c}_{\ell,\ell'})_{\ell,\ell' \ge 0}$, $\mathbf{d} = (\mathsf{d}_{\ell,\ell'})_{\ell,\ell' \ge 0}$, $\mathbf{e} = (\mathsf{e}_\ell)_{\ell \ge 0}$; please note that $\mathbf{a}$ and $\mathbf{e}$ have one index while $\mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ have two indices. Using the index-refined crossing and nesting statistics defined in (2.67), we define the polynomial $\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\lambda)$ by

$$\begin{aligned}
\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\lambda) \;=\; & \\
\sum_{\sigma \in \mathfrak{S}_n} \lambda^{\mathrm{cyc}(\sigma)} & \prod_{i \in \mathrm{Cval}(\sigma)} \mathsf{a}_{\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma)} \prod_{i \in \mathrm{Cpeak}(\sigma)} \mathsf{b}_{\mathrm{lcross}(i,\sigma),\, \mathrm{lnest}(i,\sigma)} \;\times \\
& \prod_{i \in \mathrm{Cdfall}(\sigma)} \mathsf{c}_{\mathrm{lcross}(i,\sigma),\, \mathrm{lnest}(i,\sigma)} \prod_{i \in \mathrm{Cdrise}(\sigma)} \mathsf{d}_{\mathrm{ucross}(i,\sigma),\, \mathrm{unest}(i,\sigma)} \prod_{i \in \mathrm{Fix}(\sigma)} \mathsf{e}_{\mathrm{psnest}(i,\sigma)}
\end{aligned} \tag{5.20}$$

where recall that $\mathrm{Cval}(\sigma) = \{i \colon \sigma^{-1}(i) > i < \sigma(i)\}$ and likewise for the others.

The polynomials (5.20) have a beautiful J-fraction:

Theorem 5.2.3 (Master J-fraction for permutations)**.** *The ordinary generating function of the polynomials $\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\lambda)$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\lambda)\, t^n \;=\; \cfrac{1}{1 - \lambda \mathsf{e}_0 t - \cfrac{\lambda \mathsf{a}_0 \mathsf{b}_{00} t^2}{1 - (\mathsf{c}_{00} + \mathsf{d}_{00} + \lambda \mathsf{e}_1)t - \cfrac{(\lambda+1)\mathsf{a}_1(\mathsf{b}_{01}+\mathsf{b}_{10})t^2}{1 - (\mathsf{c}_{01}+\mathsf{c}_{10}+\mathsf{d}_{01}+\mathsf{d}_{10}+\lambda \mathsf{e}_2)t - \cfrac{(\lambda+2)\mathsf{a}_2(\mathsf{b}_{02}+\mathsf{b}_{11}+\mathsf{b}_{20})t^2}{1 - \cdots}}}} \tag{5.21}$$

with coefficients

$$\gamma_0 = \lambda \mathsf{e}_0 \tag{5.22a}$$

$$\gamma_n = \left(\sum_{\xi=0}^{n-1} \mathsf{c}_{n-1-\xi,\xi}\right) + \left(\sum_{\xi=0}^{n-1} \mathsf{d}_{n-1-\xi,\xi}\right) + \lambda \mathsf{e}_n \qquad \textit{for } n \geq 1 \tag{5.22b}$$

$$\beta_n = (\lambda + n - 1)\, \mathsf{a}_{n-1} \left(\sum_{\xi=0}^{n-1} \mathsf{b}_{n-1-\xi,\xi}\right) \tag{5.22c}$$

$$\tag{5.22d}$$

We will prove this theorem in Section 5.4. It implies Theorems 5.2.1 and 5.2.2 by straightforward specialisations.

Remarks. 1. We remark that (5.20) is *almost* the same as the polynomial introduced in [SZ22, eq. (2.77)], except for the extra factor $\lambda^{\mathrm{cyc}(\sigma)}$ and the index of a depends on the sum $\mathrm{ucross}(i,\sigma) + \mathrm{unest}(i,\sigma)$. This is the price we have to pay in order to include the statistic cyc. See [SZ22, p. 13].

2. We also note that (5.20) is *almost* the same as the polynomial [SZ22, eq. (2.100)] as well, except our treatment of d is nicer as we are able to recover both $\mathrm{ucross}(i,\sigma)$, $\mathrm{unest}(i,\sigma)$, and not just their sum. In fact, this separation is what allows us to prove [SZ22, Conjecture 2.3] by using [SZ22, Lemma 2.10].

3. The continued fraction (5.21)/(5.22) is the same as [SZ22, eqs. (2.101),(2.102) and (2.103)] except for the indexing of d. ■

5.3 D-permutations: Statements of results

In this section, we state our continued fractions for D-permutations. Analogous to our T-fractions in Theorems 4.2.3, 4.2.7, 4.2.9 and the variant forms in Theorems 4.2.10, 4.2.11, our continued fractions for D-permutations here also have two variants: the first involve the record classification and the second involve the variant record classification, both have been introduced in Sections 2.5.1/2.5.2. The most basic versions in each variant are a T-fraction (Theorems 5.3.1 and 5.3.5) in 12 variables that enumerates D-permutations with respect to the parity-refined record-and-cycle classification and the variant parity-refined record-and-cycle classification respectively; except for the segregation of cycle valleys. Theorem 5.3.1 is the same as Theorem 4.3.1 and was conjectured in Chapter 4. The second versions (Theorems 5.3.3 and 5.3.6) are respective (p,q)-generalisations of the first versions: they are a T-fraction with 21 variables that enumerates D-permutations with respect to the parity-refined record-and-cycle classification and variant parity-refined record-and-cycle classification respectively, together with three pairs of (p,q)-variables counting the refined categories of crossings and nestings except for cycle valleys. Finally, our third versions (Theorems 5.3.4 and 5.3.7) — is a T-fraction in six infinite families of indeterminates and one additional variable; this generalises the previous versions by employing the index-refined crossing and nesting statistics (2.67) and the variant index-refined crossing and nesting statistics (2.69). One of the variables (λ) in each version counts the number of cycles.

The first variants will be stated in Section 5.3.1 and will be proved in Section 5.5.1. The second variants will be stated in Section 5.3.2 and will be proved in Section 5.5.2.

We then rephrase Theorems 5.3.1 and 5.3.5 using cycle valley minima in Section 5.3.3; our approach will be similar to what was done in Section 4.3.1.3 and will allow us to resolve [RZ96a, Conjecture 12].

5.3.1 Continued fractions using record classification

5.3.1.1 T-fraction

Recall the polynomial $\widehat{P}_n$ defined in (5.12)/(4.41)

$$\begin{aligned}\widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda) \;=\; \\ \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} y_1^{\rm ereccval(\sigma)} y_2^{\rm ereccdrise(\sigma)} \times \\ u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} v_1^{\rm nrcval(\sigma)} v_2^{\rm nrcdrise(\sigma)} \times \\ w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \lambda^{\rm cyc(\sigma)}.\end{aligned}$$

We will prove Theorem 4.3.1 as our first main result for D-permutations.

Theorem 5.3.1. *The ordinary generating function of the polynomials* (5.12) *specialised to $v_1 = y_1$ has the T-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,w_\mathrm{e},w_\mathrm{o},z_\mathrm{e},z_\mathrm{o},\lambda)\, t^n \;=\; \cfrac{1}{1-\lambda^2 z_\mathrm{e}z_\mathrm{o}t - \cfrac{\lambda x_1 y_1 t}{1-\cfrac{(x_2+\lambda w_\mathrm{e})(y_2+\lambda w_\mathrm{o})t}{1-\cfrac{(\lambda+1)(x_1+u_1)y_1 t}{1-\cfrac{(x_2+u_2+\lambda w_\mathrm{e})(y_2+v_2+\lambda w_\mathrm{o})t}{1-\cfrac{(\lambda+2)(x_1+2u_1)y_1 t}{1-\cfrac{(x_2+2u_2+\lambda w_\mathrm{e})(y_2+2v_2+\lambda w_\mathrm{o})t}{1-\cdots}}}}}}} \tag{5.23}$$

with coefficients

$$\alpha_{2k-1} = (\lambda+k-1)\,[x_1+(k-1)u_1]\,y_1 \tag{5.24a}$$
$$\alpha_{2k} = [x_2+(k-1)u_2+\lambda w_\mathrm{e}]\,[y_2+(k-1)v_2+\lambda w_\mathrm{o}] \tag{5.24b}$$
$$\delta_1 = \lambda^2 z_\mathrm{e}z_\mathrm{o} \tag{5.24c}$$
$$\delta_n = 0 \qquad \textit{for } n \ge 2 \tag{5.24d}$$

We will prove Theorem 5.3.1 in Section 5.5.1.

The continued fraction (5.23)/(5.24) is almost similar to our first T-fraction for D-permutations in equation (4.8)/(4.9) except for the extra factor λ and the specialisation $v_1 = y_1$. This continued fraction is also the same as our second T-fraction for D-permutations in equation (4.45)/(4.46) which proves the equidistribution of statistics for D-permutations in Theorem 4.3.1′.

We can also enumerate D-cycles by extracting the coefficient of λ^1 in Theorem 5.3.1. The analogous polynomials $P_n^{\mathfrak{DC}}$ are

$$P_n^{\mathfrak{DC}}(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2) \;=\; \sum_{\sigma\in\mathfrak{DC}_{2n}} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \,. \tag{5.25}$$

Corollary 5.3.2 (S-fraction for D-cycles)**.** *The ordinary generating function of the polynomials* (5.25) *specialised to $v_1 = y_1$ has the S-type continued fraction*

$$\sum_{n=0}^{\infty} P_{n+1}^{\mathfrak{DC}}(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2)\, t^n \;=\; \cfrac{x_1 y_1}{1-\cfrac{x_2y_2\,t}{1-\cfrac{(x_1+u_1)y_1\,t}{1-\cfrac{(x_2+u_2)(y_2+v_2)\,t}{1-\cfrac{(x_1+2u_1)2y_1\,t}{1-\cfrac{(x_2+2u_2)(y_2+2v_2)\,t}{1-\cdots}}}}}} \tag{5.26}$$

with coefficients

$$\alpha_{2k-1} \;=\; [x_2+(k-1)u_2]\,[y_2+(k-1)v_2] \tag{5.27a}$$

$$\alpha_{2k} \;=\; [x_1+ku_1]\,ky_1 \tag{5.27b}$$

5.3.1.2 p,q-generalisation

In this subsection, we shall provide a p,q-generalisation for Theorem 5.3.1 by including four pairs of (p,q)–variables corresponding to the four refined types of crossings and nestings, as well as two variables corresponding to pseudo-nestings for fixed points:

$$\widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},p_{-1},p_{-2},p_{+1},p_{+2},q_{-1},q_{-2},q_{+1},q_{+2},s_{\rm e},s_{\rm o},\lambda) =$$

$$\sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} y_1^{\rm ereccval(\sigma)} y_2^{\rm ereccdrise(\sigma)} \times$$
$$u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} v_1^{\rm nrcval(\sigma)} v_2^{\rm nrcdrise(\sigma)} \times$$
$$w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \times$$
$$p_{-1}^{\rm lcrosscpeak(\sigma)} p_{-2}^{\rm lcrosscdfall(\sigma)} p_{+1}^{\rm ucrosscval(\sigma)} p_{+2}^{\rm ucrosscdrise(\sigma)} \times$$
$$q_{-1}^{\rm lnestcpeak(\sigma)} q_{-2}^{\rm lnestcdfall(\sigma)} q_{+1}^{\rm unestcval(\sigma)} q_{+2}^{\rm unestcdrise(\sigma)} \times$$
$$s_{\rm e}^{\rm epsnest(\sigma)} s_{\rm o}^{\rm opsnest(\sigma)} \lambda^{\rm cyc(\sigma)} . \tag{5.28}$$

This is the same as (4.22) except for the extra factor of $\lambda^{\rm cyc(\sigma)}$. We now state a J-fraction under the specialisations $v_1 = y_1$ and $q_{+1} = p_{+1}$:

Theorem 5.3.3 (T-fraction for D-permutations, p,q-generalisation)**.** *The ordinary generating function of the polynomials* (5.28) *specialised to $v_1 = y_1$ and $q_{+1} = p_{+1}$ has the T-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},p_{-1},p_{-2},p_{+1},p_{+2},q_{-1},q_{-2},p_{+1},q_{+2},s_{\rm e},s_{\rm o},\lambda)\, t^n =$$

$$\cfrac{1}{1-\lambda^2 z_{\rm e}z_{\rm o}t - \cfrac{\lambda x_1 y_1 t}{1-\cfrac{(x_2+\lambda s_{\rm e}w_{\rm e})(y_2+\lambda s_{\rm o}w_{\rm o})t}{1-\cfrac{(\lambda+1)p_{+1}y_1(p_{-1}x_1+q_{-1}u_1)t}{1-\cfrac{(p_{-2}x_2+q_{-2}u_2+\lambda s_{\rm e}^2 w_{\rm e})(p_{+2}y_2+q_{+2}v_2+\lambda s_{\rm o}^2 w_{\rm o})t}{1-\cfrac{(\lambda+2)p_{+1}^2 y_1(p_{-1}^2 x_1+q_{-1}[2]_{p_{-1},q_{-1}}u_1)t}{1-\cfrac{(p_{-2}^2x_2+q_{-2}[2]_{p_{-2},q_{-2}}u_2+\lambda s_{\rm e}^3 w_{\rm e})(p_{+2}^2 y_2+q_{+2}[2]_{p_{+2},q_{+2}}v_2+\lambda s_{\rm o}^3 w_{\rm o})t}{1-\cdots}}}}}} \tag{5.29}$$

with coefficients

$$\alpha_{2k-1} = (\lambda+k-1)\, p_{+1}^{k-1} y_1 \left(p_{-1}^{k-1} x_1 + q_{-1} [k-1]_{p_{-1},q_{-1}} u_1 \right) \quad (5.30a)$$

$$\alpha_{2k} = \left(p_{-2}^{k-1} x_2 + q_{-2} [k-1]_{p_{-2},q_{-2}} u_2 + \lambda s_{\mathrm{e}}^{k} w_{\mathrm{e}} \right) \times \left(p_{+2}^{k-1} y_2 + q_{+2} [k-1]_{p_{+2},q_{+2}} v_2 + \lambda s_{\mathrm{o}}^{k} w_{\mathrm{o}} \right) \quad (5.30b)$$

$$\delta_1 = \lambda^2 z_{\mathrm{e}} z_{\mathrm{o}} \quad (5.30c)$$

$$\delta_n = 0 \qquad \textit{for } n \ge 2 \quad (5.30d)$$

We will prove this theorem in Section 5.5.1, as a special case of a more general result.

5.3.1.3 Master T-fraction

In this subsection, we shall provide a master T-fraction generalising Theorems 5.3.1 and 5.3.3. Let us introduce a polynomial in six infinite families of indeterminates $\mathbf{a} = (\mathsf{a}_\ell)_{\ell\ge 0}$, $\mathbf{b} = (\mathsf{b}_{\ell,\ell'})_{\ell,\ell'\ge 0}$, $\mathbf{c} = (\mathsf{c}_{\ell,\ell'})_{\ell,\ell'\ge 0}$, $\mathbf{d} = (\mathsf{d}_{\ell,\ell'})_{\ell,\ell'\ge 0}$, $\mathbf{e} = (\mathsf{e}_\ell)_{\ell\ge 0}$, $\mathbf{f} = (\mathsf{f}_\ell)_{\ell\ge 0}$ that will have a nice T-fraction and that will include the polynomials (5.12) and (5.28) as specialisations. Please note that $\mathbf{a}, \mathbf{e}$ and $\mathbf{f}$ have one index while $\mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ have two indices. Using the index-refined crossing and nesting statistics defined in (2.67), we define the polynomial $\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\lambda)$ by

$$\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\lambda) = \sum_{\sigma\in\mathfrak{D}_{2n}} \lambda^{\mathrm{cyc}(\sigma)} \prod_{i\in\mathrm{Cval}(\sigma)} \mathsf{a}_{\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Cpeak}(\sigma)} \mathsf{b}_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \times \prod_{i\in\mathrm{Cdfall}(\sigma)} \mathsf{c}_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \prod_{i\in\mathrm{Cdrise}(\sigma)} \mathsf{d}_{\mathrm{ucross}(i,\sigma),\mathrm{unest}(i,\sigma)} \times \prod_{i\in\mathrm{Evenfix}(\sigma)} \mathsf{e}_{\mathrm{psnest}(i,\sigma)} \prod_{i\in\mathrm{Oddfix}(\sigma)} \mathsf{f}_{\mathrm{psnest}(i,\sigma)} \,. \quad (5.31)$$

where recall that $\mathrm{Cval}(\sigma) = \{i\colon \sigma^{-1}(i) > i < \sigma(i)\}$ and likewise for the others.

We remark that (5.31) is *almost* the same as the polynomial (4.31) except for the extra factor $\lambda^{\mathrm{cyc}(\sigma)}$ and the index of a depends on the sum $\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma)$. That is the price we have to pay in order to include the statistic cyc. See Appendix in

Chapter 4. We also note that (5.31) is *almost* the same as the polynomial (5.20) as well, but restricted to D-permutations and refined to record the parity of fixed points.

The polynomials (5.31) have a beautiful T-fraction:

Theorem 5.3.4 (Master T-fraction for D-permutations)**.** *The ordinary generating function of the polynomials $\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\lambda)$ has the T-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\lambda)\, t^n \;=\; \cfrac{1}{1-\lambda^2 \mathsf{e}_0\mathsf{f}_0 t - \cfrac{\lambda \mathsf{a}_0\mathsf{b}_{00} t}{1-\cfrac{(\mathsf{c}_{00}+\lambda\mathsf{e}_1)(\mathsf{d}_{00}+\lambda\mathsf{f}_1)t}{1-\cfrac{(\lambda+1)\mathsf{a}_1(\mathsf{b}_{01}+\mathsf{b}_{10})t}{1-\cfrac{(\mathsf{c}_{01}+\mathsf{c}_{10}+\lambda\mathsf{e}_2)(\mathsf{d}_{01}+\mathsf{d}_{10}+\lambda\mathsf{f}_2)t}{1-\cdots}}}}} \tag{5.32}$$

with coefficients

$$\alpha_{2k-1} \;=\; (\lambda+k-1)\mathsf{a}_{k-1}\left(\sum_{\xi=0}^{k-1}\mathsf{b}_{k-1-\xi,\xi}\right) \tag{5.33a}$$

$$\alpha_{2k} \;=\; \left(\lambda\mathsf{e}_k+\sum_{\xi=0}^{k-1}\mathsf{c}_{k-1-\xi,\xi}\right)\left(\lambda\mathsf{f}_k+\sum_{\xi=0}^{k-1}\mathsf{d}_{k-1-\xi,\xi}\right) \tag{5.33b}$$

$$\delta_1 \;=\; \lambda^2\mathsf{e}_0\mathsf{f}_0 \tag{5.33c}$$

$$\delta_n \;=\; 0 \qquad \textit{for } n\ge 2 \tag{5.33d}$$

We will prove this theorem in Section 5.5.1. It implies Theorems 5.3.1 and 5.3.3 by straightforward specialisations.

Remark. We remark that (5.31) is *almost* the same as the polynomial (4.69) as well, except our treatment of d is different: we are able to recover the statistics $\mathrm{ucross}(i,\sigma)$, $\mathrm{unest}(i,\sigma)$, and not the statistics $\mathrm{ucross}'(i,\sigma)$, $\mathrm{unest}'(i,\sigma)$. In fact, this separation is crucial for our work and it allows us to prove this result. ■

5.3.2 Continued fractions using variant record classification

Similar to Section 4.2.5 in Chapter 4, our T-fractions for D-permutations (Theorems 5.3.1, 5.3.3, 5.3.4) in this chapter also have variant forms in which we use the variant index-refined crossing and nesting statistics (2.69). We shall state these variants in this subsection.

5.3.2.1 T-fraction

Let $\widehat{P}'_n$ be the polynomials defined as follows:

$$\begin{aligned}\widehat{P}'_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda) \;=\; \\ \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm ereccpeak'(\sigma)} x_2^{\rm eareccdfall'(\sigma)} y_1^{\rm eareccval'(\sigma)} y_2^{\rm ereccdrise'(\sigma)} \times \\ u_1^{\rm nrcpeak'(\sigma)} u_2^{\rm nrcdfall'(\sigma)} v_1^{\rm nrcval'(\sigma)} v_2^{\rm nrcdrise'(\sigma)} \times \\ w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \lambda^{\rm cyc(\sigma)}.\end{aligned} \tag{5.34}$$

We have the following variant of Theorem 5.3.1.

Theorem 5.3.5. *The ordinary generating function of the polynomials $\widehat{P}'_n$ defined in* (5.34) *specialised to $v_1 = y_1$ has the same T-type continued fraction* (5.23)/(5.24) *as the polynomials $\widehat{P}_n$ defined in* (5.12). *Therefore,*

$$\widehat{P}'_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda) \;=\; \widehat{P}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o},\lambda)\,. \tag{5.35}$$

We will prove Theorem 5.3.5 in Section 5.5.2.

5.3.2.2 p,q-generalisation

We shall now provide a p,q-generalisation for Theorem 5.3.5 by including four pairs of (p,q)–variables corresponding to the four variants of the refined types of crossings and nestings, as well as two variables corresponding to pseudo-nestings for fixed points:

$$\widehat{P}'_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w_{\rm e}, w_{\rm o}, z_{\rm e}, z_{\rm o}, p_{-1}, p_{-2}, p_{+1}, p_{+2}, q_{-1}, q_{-2}, q_{+1}, q_{+2}, s_{\rm e}, s_{\rm o}, \lambda) =$$

$$\sum_{\sigma \in \mathfrak{D}_{2n}} x_1^{\rm ereccpeak'(\sigma)} x_2^{\rm eareccdfall'(\sigma)} y_1^{\rm eareccval'(\sigma)} y_2^{\rm ereccdrise'(\sigma)} \times$$
$$u_1^{\rm nrcpeak'(\sigma)} u_2^{\rm nrcdfall'(\sigma)} v_1^{\rm nrcval'(\sigma)} v_2^{\rm nrcdrise'(\sigma)} \times$$
$$w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \times$$
$$p_{-1}^{\rm ucrosscpeak'(\sigma)} p_{-2}^{\rm lcrosscdfall'(\sigma)} p_{+1}^{\rm lcrosscval'(\sigma)} p_{+2}^{\rm ucrosscdrise'(\sigma)} \times$$
$$q_{-1}^{\rm unestcpeak'(\sigma)} q_{-2}^{\rm lnestcdfall'(\sigma)} q_{+1}^{\rm lnestcval'(\sigma)} q_{+2}^{\rm unestcdrise'(\sigma)} \times$$
$$s_{\rm e}^{\rm epsnest(\sigma)} s_{\rm o}^{\rm opsnest(\sigma)} \lambda^{\rm cyc(\sigma)} . \tag{5.36}$$

This is the same as the polynomial (4.36) except for the extra factor of $\lambda^{\rm cyc(\sigma)}$. We now state a J-fraction under the specialisations $v_1 = y_1$ and $q_{+1} = p_{+1}$:

Theorem 5.3.6. *The ordinary generating function of the polynomials $\widehat{P}'_n$ defined in* (5.36) *specialised to $v_1 = y_1$ and $q_{+1} = p_{+1}$ has the same T-type continued fraction* (5.29)/(5.30) *as the polynomials $\widehat{P}_n$ defined in* (5.28)*. Therefore*

$$\widehat{P}'_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w_{\rm e}, w_{\rm o}, z_{\rm e}, z_{\rm o}, p_{-1}, p_{-2}, p_{+1}, p_{+2}, q_{-1}, q_{-2}, p_{+1}, q_{+2}, s_{\rm e}, s_{\rm o}, \lambda)$$
$$= \widehat{P}_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w_{\rm e}, w_{\rm o}, z_{\rm e}, z_{\rm o}, p_{-1}, p_{-2}, p_{+1}, p_{+2}, q_{-1}, q_{-2}, p_{+1}, q_{+2}, s_{\rm e}, s_{\rm o}, \lambda) . \tag{5.37}$$

We will prove this theorem in Section 5.5.2, as a special case of a more general result.

5.3.2.3 Master T-fraction

We introduce a polynomial in six infinite families of indeterminates $\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e},\mathsf{f}$ as before but by using the variant index-refined crossing and nesting statistics (2.69):

$$\widehat{Q}'_n(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e},\mathsf{f},\lambda) \;=\;$$

$$\sum_{\sigma\in\mathfrak{D}_{2n}} \lambda^{\mathrm{cyc}(\sigma)} \prod_{i\in\mathrm{Cval}(\sigma)} \mathsf{a}_{\mathrm{lcross}'(i,\sigma)+\mathrm{lnest}'(i,\sigma)} \prod_{i\in\mathrm{Cpeak}(\sigma)} \mathsf{b}_{\mathrm{ucross}'(i,\sigma),\,\mathrm{unest}'(i,\sigma)} \;\times$$
$$\prod_{i\in\mathrm{Cdfall}(\sigma)} \mathsf{c}_{\mathrm{lcross}'(i,\sigma),\,\mathrm{lnest}'(i,\sigma)} \prod_{i\in\mathrm{Cdrise}(\sigma)} \mathsf{d}_{\mathrm{ucross}'(i,\sigma),\,\mathrm{unest}'(i,\sigma)} \;\times$$
$$\prod_{i\in\mathrm{Evenfix}(\sigma)} \mathsf{e}_{\mathrm{psnest}(i,\sigma)} \prod_{i\in\mathrm{Oddfix}(\sigma)} \mathsf{f}_{\mathrm{psnest}(i,\sigma)} \;. \tag{5.38}$$

We remark that (5.38) is *almost* the same as the polynomial introduced in (4.34), except for the extra factor $\lambda^{\mathrm{cyc}(\sigma)}$ and the index of a depends on the sum $\mathrm{lcross}'(i,\sigma)+\mathrm{lnest}'(i,\sigma)$.

We have the following variant of Theorem 5.3.4

Theorem 5.3.7. *The ordinary generating function of the polynomials $\widehat{Q}'_n$ defined in* (5.38) *has the same T-type continued fraction* (5.32)/(5.33) *as the polynomials $\widehat{Q}_n$ defined in* (5.31). *Therefore*

$$\widehat{Q}'_n(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e},\mathsf{f},\lambda) \;=\; \widehat{Q}_n(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e},\mathsf{f},\lambda)\,. \tag{5.39}$$

Theorem 5.3.7 will be proved in Section 5.5.2. It implies Theorems 5.3.5 and 5.3.6 by straightforward specialisations.

5.3.3 Reformulation of results using cycle valley minima and the Randrianarivony–Zeng conjecture

We will now rephrase our results in this section by identifying minimum elements in a cycle; this was introduced in Section 2.5.1. This will help us to prove Conjecture 5.1.2′ as a corollary. Our approach here will be the same as that in Section 4.3.1.3.

We notice that the number of cycles in a permutation can be recovered if we know the number of (even and odd) fixed points and the number of cycle valley minima (or the number of cycle peak maxima). We will rephrase our results by distributing the weight λ, that we had been using for the number of cycles, among fixed points and cycle valley minima. In Chapter 4 we did this by introducing the polynomial $\widetilde{P}_n$ in (4.52) which we recall here:

$$\widetilde{P}_n(x_1,x_2,\widetilde{y}_1,y_2,u_1,u_2,\widetilde{v}_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}) \;=\; \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm eareccpeak(\sigma)} x_2^{\rm eareccdfall(\sigma)} \widetilde{y}_1^{\rm minval(\sigma)} y_2^{\rm ereccdrise(\sigma)} \times u_1^{\rm nrcpeak(\sigma)} u_2^{\rm nrcdfall(\sigma)} \widetilde{v}_1^{\rm nminval(\sigma)} v_2^{\rm nrcdrise(\sigma)} \times w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \,. \qquad (5.40)$$

We can rephrase Theorem 5.3.1 by replacing the factor $(\lambda+k-1)y_1$ with $\widetilde{y}+(k-1)\widetilde{v}$ and removing the factors of λ multiplying $w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}$; this will give us Theorem 4.3.1$''$ which we copy here:

Theorem 5.3.1.$'$ The ordinary generating function of the polynomials (5.40) has the T-type continued fraction

$$\sum_{n=0}^{\infty} \widetilde{P}_n(x_1,x_2,\widetilde{y}_1,y_2,u_1,u_2,\widetilde{v}_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o})\, t^n \;=\; \cfrac{1}{1-z_{\rm e}z_{\rm o}\,t-\cfrac{x_1\widetilde{y}_1\,t}{1-\cfrac{(x_2+w_{\rm e})(y_2+w_{\rm o})\,t}{1-\cfrac{(x_1+u_1)(\widetilde{y}_1+\widetilde{v}_1)\,t}{1-\cfrac{(x_2+u_2+w_{\rm e})(y_2+v_2+w_{\rm o})\,t}{1-\cfrac{(x_1+2u_1)(\widetilde{y}_1+2\widetilde{v}_1)\,t}{1-\cfrac{(x_2+2u_2+w_{\rm e})(y_2+2v_2+w_{\rm o})\,t}{1-\cdots}}}}}}} \qquad (5.41)$$

with coefficients

$$\alpha_{2k-1} = [x_1+(k-1)u_1]\,[\widetilde{y}_1+(k-1)\widetilde{v}_1] \tag{5.42a}$$

$$\alpha_{2k} = [x_2+(k-1)u_2+w_{\rm e}]\,[y_2+(k-1)v_2+w_{\rm o}] \tag{5.42b}$$

$$\delta_1 = z_{\rm e}z_{\rm o} \tag{5.42c}$$

$$\delta_n = 0 \qquad \text{for } n \ge 2 \tag{5.42d}$$

We leave the rephrasings of Theorems 5.3.3 and 5.3.4 to the reader and directly proceed to rephrase Theorem 5.3.5. To do this, we introduce the following polynomial:

$$\begin{aligned}\widetilde{P}'_n(x_1,x_2,\widetilde{y}_1,y_2,u_1,u_2,\widetilde{v}_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}) \;=\; \\ \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\rm ereccpeak'(\sigma)} x_2^{\rm eareccdfall'(\sigma)} \widetilde{y}_1^{\rm minval(\sigma)} y_2^{\rm ereccdrise'(\sigma)} \times \\ u_1^{\rm nrcpeak'(\sigma)} u_2^{\rm nrcdfall'(\sigma)} \widetilde{v}_1^{\rm nminval(\sigma)} v_2^{\rm nrcdrise'(\sigma)} \times \\ w_{\rm e}^{\rm evennrfix(\sigma)} w_{\rm o}^{\rm oddnrfix(\sigma)} z_{\rm e}^{\rm evenrar(\sigma)} z_{\rm o}^{\rm oddrar(\sigma)} \,.\end{aligned} \tag{5.43}$$

Theorem 5.3.5 can now simply be restated as follows:

Theorem 5.3.5′. *The ordinary generating function of the polynomials $\widetilde{P}'_n$ defined in* (5.43) *has the same T-type continued fraction* (5.41)/(5.42) *as the polynomials $\widetilde{P}_n$ defined in* (5.40). *Therefore,*

$$\widetilde{P}'_n(x_1,x_2,\widetilde{y}_1,y_2,u_1,u_2,\widetilde{v}_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}) \;=\; \widetilde{P}_n(x_1,x_2,\widetilde{y}_1,y_2,u_1,u_2,\widetilde{v}_1,v_2,w_{\rm e},w_{\rm o},z_{\rm e},z_{\rm o}) \,. \tag{5.44}$$

Recall that we observed in Section 5.1.3 the equivalence between [RZ96a, Conjecture 12] and Conjecture 5.1.2′. We now obtain Conjecture 5.1.2′ as a corollary of Theorem 5.3.5′.

Corollary 5.3.8 (Conjecture 5.1.2′)**.** *Recall the polynomials G_n, defined in (5.9)/(5.11),*

$$\begin{aligned} G_n(x,y,\bar{x},\bar{y}) &= \sum_{\sigma\in\mathfrak{D}^{\mathrm{o}}_{2n}} x^{\mathrm{comi}(\sigma)} y^{\mathrm{lema}(\sigma)} \bar{x}^{\mathrm{cemi}(\sigma)} \bar{y}^{\mathrm{remi}(\sigma)} \\ &= \sum_{\sigma\in\mathfrak{D}^{\mathrm{o}}_{2n}} x^{\mathrm{minval}(\sigma)} y^{\mathrm{ereccpeak}'(\sigma)} \bar{x}^{\mathrm{evennrfix}(\sigma)} \bar{y}^{\mathrm{eareccdfall}'(\sigma)} . \end{aligned}$$

The ordinary generating functions of G_n has the S-type continued fraction

$$1+\sum_{n=1}^{\infty} G_n(x,y,\bar{x},\bar{y})t^n = \cfrac{1}{1-\cfrac{xyt}{1-\cfrac{1(\bar{x}+\bar{y})t}{1-\cfrac{(x+1)(y+1)t}{1-\cfrac{2(\bar{x}+\bar{y}+1)}{1-\cfrac{(x+2)(y+2)t}{1-\cfrac{3(\bar{x}+\bar{y}+2)t}{\cdots}}}}}}} \tag{5.45}$$

with coefficients

$$\alpha_{2k-1} = (x+k-1)\,(y+k-1) \tag{5.46a}$$

$$\alpha_{2k} = k\,(\bar{x}+\bar{y}+k-1) \tag{5.46b}$$

PROOF. It is evident from (5.11)/(5.43) that G_n can be obtained from $\widetilde{P}'_n$ by specialising $\widetilde{y}_1 = x, x_1 = y, w_{\mathrm{e}} = \bar{x}, x_2 = \bar{y}$ and $w_{\mathrm{o}} = z_{\mathrm{o}} = 0$, and setting all other variables to 1. This along with Theorems 5.3.1′/5.3.5′ proves the result. □

5.4 Permutations: Proof of Theorems 5.2.1, 5.2.2, 5.2.3

Sokal and Zeng [SZ22, Section 6.1] used a variant of the Foata–Zeilberger bijection [FZ90] to prove [SZ22, Theorems 2.1(a), 2.2, 2.5, 2.7 and 2.9] i.e., their "first theorems" for permutations. We will provide a new interpretation to this bijection in terms of Laguerre digraphs and then use this interpretation to prove our theorems for permutations.

We first recall Sokal and Zeng's bijection in Subsection 5.4.1, and then introduce our interpretation in Subsection 5.4.2. We complete our proofs in Subsection 5.4.3.

5.4.1 Sokal–Zeng variant of the Foata–Zeilberger bijection

Sokal and Zeng employed a variant of the Foata–Zeilberger bijection to prove [SZ22, Theorems 2.1(a), 2.2, 2.5, 2.7, and 2.9], i.e., their "first theorems" for permutations. We begin by recalling this bijection which is a correspondence between $\mathfrak{S}_n$ to the set of $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled Motzkin paths of length n, where the labels ξ_i lie in the sets

$$\mathcal{A}_h = \{0,\ldots,h\} \qquad \text{for } h \geq 0 \quad (5.47a)$$

$$\mathcal{B}_h = \{0,\ldots,h-1\} \qquad \text{for } h \geq 1 \quad (5.47b)$$

$$\mathcal{C}_h = \left(\{1\}\times C_h^{(1)}\right) \cup \left(\{2\}\times C_h^{(2)}\right) \cup \left(\{3\}\times C_h^{(3)}\right) \qquad \text{for } h \geq 0 \quad (5.47c)$$

where

$$C_h^{(1)} = \{0,\ldots,h-1\} \qquad \text{for } k \geq 0 \quad (5.48a)$$

$$C_h^{(2)} = \{0,\ldots,h-1\} \qquad \text{for } k \geq 0 \quad (5.48b)$$

$$C_h^{(3)} = \{0\} \qquad \text{for } k \geq 0 \quad (5.48c)$$

Notice that our convention for labels introduced in Section 2.2.5 are slightly different from those in [SZ22]. A key difference is that our label set starts at 0 and not at 1. A level step that has label $\xi_h \in \{i\}\times C_h^{(i)}$ will be called a level step of type i $(i = 1,2,3)$.

We will begin by recalling how the Motzkin path ω is defined; then we will explain how the labels ξ are defined; next we sketch the proof that the mapping is indeed a bijection. There were two more steps in [SZ22, Section 6.1] which we will not recall; these were the translation of the various statistics from $\mathfrak{S}_n$ to labelled Motzkin paths; and summing over labels ξ to obtain the weight $W(\omega)$.

Step 1: Definition of the Motzkin path. Given a permutation $\sigma \in \mathfrak{S}_n$, we classify the indices $i \in [n]$ according to the cycle classification. We then define a path $\omega = (\omega_0, \ldots, \omega_n)$ starting at $\omega_0 = (0,0)$ and ending at $\omega_n = (n,0)$, with steps $s_1, \ldots, s_n$, as follows:

- If i is a cycle valley, then s_i is a rise.
- If i is a cycle peak, then s_i is a fall.
- If i is a cycle double fall, then s_i is a level step of type 1.
- If i is a cycle double rise, then s_i is a level step of type 2.
- If i is a fixed point, then s_i is a level step of type 3.

The fact that the resulting path is indeed a Motzkin path was proved by providing an interpretation of the height h_i which we recall here:

Lemma 5.4.1 ([SZ22, Lemma 6.1])**.** *For $i \in [n+1]$ we have*

$$h_{i-1} = \#\{j < i\colon\ \sigma(j) \geq i\} \tag{5.49a}$$

$$= \#\{j < i\colon\ \sigma^{-1}(j) \geq i\}. \tag{5.49b}$$

We also recall ([SZ22, eq. (6.4)]) which is an equivalent formulation of Equation (5.49):

$$h_i = \#\{j \leq i\colon\ \sigma(j) > i\} \tag{5.50a}$$

$$= \#\{j \leq i\colon\ \sigma^{-1}(j) > i\}. \tag{5.50b}$$

Notice that if i is a fixed point, then by comparing (5.49a)/(5.50a) with (2.68) we see that the height of the Motzkin path after (or before) step i equals the number of pseudo-nestings of the fixed point:

$$h_{i-1} \;=\; h_i \;=\; \mathrm{psnest}(i,\sigma)\,. \tag{5.51}$$

Step 2: Definition of the labels $\boldsymbol{\xi_i}$. We now recall the definition of the labels

$$\xi_i \;=\; \begin{cases} \#\{j\colon\; j<i \text{ and } \sigma(j)>\sigma(i)\} & \text{if } \sigma(i)>i \quad \text{if } i\in \mathrm{Cval}\cup\mathrm{Cdrise} \\ \#\{j\colon\; j>i \text{ and } \sigma(j)<\sigma(i)\} & \text{if } \sigma(i)<i \quad \text{if } i\in \mathrm{Cpeak}\cup\mathrm{Cdfall} \\ 0 & \text{if } \sigma(i)=i \quad \text{if } i\in \mathrm{Fix} \end{cases} \tag{5.52}$$

where recall that Cval is the set of all cycle valleys of σ and likewise for the others. For sake of brevity, we are abusing notation for fixed points, cycle double rises and cycle double falls by dropping the first index of ξ_i.

Compare our definition (5.52) with [SZ22, (6.5)] and notice the shift from 1 to 0. These definitions have a simple interpretation in terms of the nesting statistics defined in (2.67b,d):

$$\xi_i \;=\; \begin{cases} \mathrm{unest}(i,\sigma) & \text{if } \sigma(i)>i \quad \text{if } i\in \mathrm{Cval}\cup\mathrm{Cdrise} \\ \mathrm{lnest}(i,\sigma) & \text{if } \sigma(i)<i \quad \text{if } i\in \mathrm{Cpeak}\cup\mathrm{Cdfall} \\ 0 & \text{if } \sigma(i)=i \quad \text{if } i\in \mathrm{Fix} \end{cases} \tag{5.53}$$

To verify that the inequalities (2.37)/(5.47) are satisfied; to do this, we interpret $h_{i-1}-\xi_i$ in terms of the crossing statistics defined in (2.67a,c):

Lemma 5.4.2 (Crossing statistics)**.** *We have*

$$h_{i-1}-\xi_i \;=\; \mathrm{ucross}(i,\sigma) \quad \text{if } i\in\mathrm{Cval} \tag{5.54}$$

$$h_{i-1}-1-\xi_i \;=\; \mathrm{ucross}(i,\sigma) \quad \text{if } i\in\mathrm{Cdrise} \tag{5.55}$$

$$h_{i-1}-1-\xi_i \;=\; \mathrm{lcross}(i,\sigma) \quad \text{if } i\in\mathrm{Cpeak}\cup\mathrm{Cdfall} \tag{5.56}$$

Again compare Lemma 5.4.2 with [SZ22, Lemma 6.2] to see how the shift of 1 affects these quantities.

Since the quantities (5.54)–(5.56) are manifestly nonnegative, it follows immediately that the inequalities (2.37)/(5.47) are satisfied.

Step 3: Proof of bijection. We recall the description of the inverse map for the mapping $\sigma \mapsto (\omega, \xi)$.

First, some preliminaries: Given a permutation $\sigma \in \mathfrak{S}_n$, we define five subsets of $[n]$:

$$F = \{i\colon\ \sigma(i) > i\} = \text{positions of excedances} \tag{5.57a}$$

$$F' = \{i\colon\ i > \sigma^{-1}(i)\} = \text{values of excedances} \tag{5.57b}$$

$$G = \{i\colon\ \sigma(i) < i\} = \text{positions of anti-excedances} \tag{5.57c}$$

$$G' = \{i\colon\ i < \sigma^{-1}(i)\} = \text{values of anti-excedances} \tag{5.57d}$$

$$H = \{i\colon\ \sigma(i) = i\} = \text{fixed points} \tag{5.57e}$$

Let us observe that

$$F \cap F' = \text{cycle double rises} \tag{5.58a}$$

$$G \cap G' = \text{cycle double falls} \tag{5.58b}$$

$$F \cap G' = \text{cycle valleys} \tag{5.58c}$$

$$F' \cap G = \text{cycle peaks} \tag{5.58d}$$

$$F \cap G = \varnothing \tag{5.58e}$$

$$F' \cap G' = \varnothing \tag{5.58f}$$

and of course H is disjoint from F, F', G, G'.

Let us also recall the notion of an *inversion table*: Let S be a totally ordered set of cardinality k, and let $\boldsymbol{x} = (x_1, \ldots, x_k)$ be an enumeration of S; then the (left-to-right) inversion table corresponding to $\boldsymbol{x}$ is the sequence $\boldsymbol{p} = (p_1, \ldots, p_k)$ of nonnegative integers defined by $p_\alpha = \#\{\beta < \alpha\colon\ x_\beta > x_\alpha\}$. Note that $0 \le p_\alpha \le \alpha - 1$ for all $\alpha \in [k]$, so there are exactly $k!$ possible inversion tables. Given the inversion table $\boldsymbol{p}$, we can reconstruct the sequence $\boldsymbol{x}$ by working from right to left, as follows: There are p_k elements of S larger than x_k, so x_k must be the (p_k+1)th largest element of S. Then there are p_{k-1} elements of $S\setminus\{x_k\}$ larger than x_{k-1}, so x_{k-1} must be the $(p_{k-1}+1)$th largest element of $S\setminus\{x_k\}$. And so forth. [Analogously, the right-to-left inversion table corresponding to $\boldsymbol{x}$ is the sequence $\boldsymbol{p}=(p_1,\ldots,p_k)$ of nonnegative integers defined by $p_\alpha=\#\{\beta>\alpha\colon\ x_\beta<x_\alpha\}$.]

With these preliminaries out of the way, we can now describe the map $(\omega,\xi)\mapsto\sigma$. Given the Motzkin path ω, we read off which indices i correspond to cycle valleys, cycle peaks, cycle double falls, cycle double rises, and fixed points; this allows us to reconstruct the sets F,F',G,G',H. We now use the labels ξ to reconstruct the maps $\sigma\restriction F\colon\ F\to F'$ and $\sigma\restriction G\colon\ G\to G'$, as follows: Let $i_1,\ldots,i_k$ be the elements of F written in increasing order; then the sequence $j_1,\ldots,j_k$ defined by $j_\alpha=\sigma(i_\alpha)$ is a listing of F' whose inversion table is given by $p_\alpha=\xi_{i_\alpha}$: this is the content of (5.52) in the case $\sigma(i)>i$. So we can use $\xi\restriction F$ to reconstruct $\sigma\restriction F$. In a similar way we can use $\xi\restriction G$ to reconstruct $\sigma\restriction G$, but now we must use the right-to-left inversion table because of how (5.52) is written in the case $\sigma(i)<i$.

5.4.2 Combinatorial interpretation using Laguerre digraphs

We begin with a Motzkin path ω and an assignment of labels ξ satisfying (2.37)/(5.47). The inverse bijection (Section 5.4.1 Step 3), gives us a permutation σ. We will now break this process into several intermediate steps and reinterpret it using Laguerre digraphs. Recall that a Laguerre digraph of size n is a directed graph where each vertex has a distinct label from the label set $[n]$ and has indegree 0 or 1 and outdegree 0 or 1. Clearly, any subgraph of a Laguerre digraph is also a Laguerre digraph. The connected components in a Laguerre digraph are either directed paths or directed cycles, where a path with one vertex is called an isolated vertex and a cycle with one vertex is called a loop.

Notice that a Laguerre digraph with no paths is a diagrammatic representation of a permutation in cycle notation (see [Sta09, pp. 22-23]). Let L^σ denote this Laguerre digraph corresponding to a permutation $\sigma\in\mathfrak{S}_n$. The directed edges of

L^σ are precisely $u \to \sigma(u)$. Also for $S \subseteq [n]$, we let $L^\sigma|_S$ denote the subgraph of L^σ, containing the same set of vertices $[n]$, but only the edges $u \to \sigma(u)$, with $u \in S$ (we are allowed to have $\sigma(u) \notin S$). Thus, $L^\sigma|_{[n]} = L^\sigma$, and $L^\sigma|_\varnothing$ is the digraph containing n vertices and no edges. Whenever the permutation σ is understood, we shall drop the superscript and only write this subgraph as $L|_S$.

Recall that the inverse bijection in Step 3 Section 5.4.1, begins by obtaining the sets F, F', G, G', H from the Motzkin path ω. We then construct $\sigma \restriction F\colon\ F \to F'$ and $\sigma \restriction G\colon\ G \to G'$ separately by using the labels $\xi \restriction F$ and $\xi \restriction G$ respectively. (Note that just knowing the set H suffices to reconstruct $\sigma \restriction H$.)

In this interpretation, we start with the digraph $L|_\varnothing$. We then go through the set $[n]$. However, the crucial part of our approach is that we use the following unusual total order on $[n]$ (notice that $[n] = F \cup G \cup H$):

Stage (a): We first go through the set H in increasing order.

Stage (b): We then go through the set G in increasing order.

Stage (c): Finally, we go through the set F but in *decreasing* order.

As F, G and H are entirely determined by the path ω, we call the above order the FZ order on $[n]$ with respect to the Motzkin path ω. Thus, the FZ order corresponding to two different permutations σ and σ' coming from the same Motzkin path ω are the same.

Let $u_1, \dots, u_n$ be a rewriting of $[n]$ as per the FZ order. We now consider the "FZ history" $L|_\varnothing \subset L|_{\{u_1\}} \subset L|_{\{u_1,u_2\}} \subset \dots \subset L|_{\{u_1,\dots,u_n\}} = L$ as a process of building up the permutation σ by successively considering the status of vertices $u_1, u_2, \dots, u_n$. Thus, at step u (where the step number is given by the vertex $u \in [n]$) we use the inversion tables to construct the edge $u \to \sigma(u)$. Thus, at each step we insert a new edge into the digraph, and at the end of this process, the resulting digraph obtained is the permutation σ in cycle notation.

Let us now look at the intermediate Laguerre digraphs obtained during stages (a), (b) and (c) more closely.

Stage (a): Going through H:

For each vertex $u \in H$, we introduce a loop edge $u \to u$ thus creating a new loop at the end of each step. After all steps $u \in H$ have been carried out, the resulting Laguerre digraph $L|_H$ consists of loops at all vertices in H. All other vertices are isolated vertices, i.e., have no adjacent edges.

Stage (b): Going through G:

From (5.58), we know that $G = \mathrm{Cdfall}(\sigma) \cup \mathrm{Cpeak}(\sigma)$ where σ is the resulting permutation obtained at the end of the inverse bijection. Let us recall the construction for this case. We construct $\sigma \restriction G\colon\ G \to G'$ using the right-to-left inversion table. Let $G = \{x_1 < x_2 < \ldots < x_k\}$ be the elements of G arranged in increasing order, and similarly let $G' = \{x'_1 < x'_2 < \ldots < x'_k\}$. Then the $(p_j + 1)$th smallest element of $G' \backslash \{\sigma(x_1), \ldots, \sigma(x_{j-1})\}$ is chosen to be $\sigma(x_j)$ where $p_j = \xi_{x_j}$.

Let us reinterpret this in terms of Laguerre digraphs. At this stage, the vertices in G (resp. G') are our designated starting vertices (ending vertices) arranged in increasing order. We then look through the starting vertices in increasing order. At the end of step x_{j-1}, directed edges $x_1 \to \sigma(x_1), \ldots, x_{j-1} \to \sigma(x_{j-1})$ have been inserted, the available starting vertices are $x_j, \ldots, x_k$ and the available ending vertices belong to $G' \backslash \{\sigma(x_1), \ldots, \sigma(x_{j-1})\}$. We then pick the smallest available starting vertex, which is x_j, and connect it to the $(p_j + 1)$th smallest ending vertex available, i.e., the $(p_j + 1)$th smallest element of the set $G' \backslash \{\sigma(x_1), \ldots, \sigma(x_{j-1})\}$. (This is analogous to the partial interpretation of the labels in terms of bipartite digraph for cycle double falls and cycle peaks on [SZ22, p. 96].)

The vertices in $L|_{H \cup \{x_1, \ldots, x_j\}}$ have out-edges in the following situations:

- if $u \in H$, u is a loop.
- if u is one of the j smallest elements of G, it has an edge going to some vertex $v \neq u$ and $v \in G'$.
- All other vertices have no out-edges.

Also, from definition of G in (5.57c), it follows that for any non-loop edge $u \to v$, we must have $u > v$. Thus, the Laguerre digraph $L|_{H \cup \{x_1, \ldots, x_j\}}$ only consists of decreasing directed paths (including isolated vertices) and loops; there are no non-loop cycles.

Lemma 5.4.3. *The Laguerre digraph $L|_{H \cup G}$ consists of the following connected components:*

- *loops on vertices $u \in H$,*
- *directed paths with at least two vertices, in which the initial vertex of the path is a cycle peak in σ (i.e. contained in the set $F' \cap G$), the final vertex is a cycle valley in σ (i.e. contained in the set $F \cap G'$), and the intermediate vertices (if any) are cycle double falls (i.e. contained in the set $G \cap G'$).*
- *isolated vertices at $u \in F \cap F' = \mathrm{Cdrise}(\sigma)$.*

Furthermore, it contains no directed cycles.

PROOF. It suffices to prove that for a directed path with at least two vertices, the initial vertex is a cycle peak in σ (i.e. contained in the set $F' \cap G$) and the final vertex is a cycle valley in σ (i.e. contained in the set $F \cap G'$).

Notice that the initial vertex u of such a path must already have an out-neighbour and hence must belong to the set G as only the elements of the sets G and H have been assigned out-neighbours. As all vertices in G' have already been assigned in-neighbours, $u \notin G'$ and using the fact that $G = \mathrm{Cdfall} \cup \mathrm{Cpeak}$ (from (5.58)), we get that $u \in \mathrm{Cpeak} = F' \cap G$. The proof for the final vertex is similar and we omit it. □

Stage (c): Going through F:

Similar to both the previous cases, at each step we introduce edges $u \to \sigma(u)$. However, now we go through elements of F in decreasing order.

From (5.58), we know that $F = \mathrm{Cdrise} \cup \mathrm{Cval}$. Let us now recall the construction for this case. We construct $\sigma \restriction F\colon\ F \to F'$ using the left-to-right inversion table. Let $F = \{y_1 < y_2 < \ldots < y_l\}$ be the elements of F arranged in increasing order. Then the

(p_j+1)th largest element of $F'\backslash\{\sigma(y_l),\ldots,\sigma(y_{j+1})\}$ is chosen to be $\sigma(y_j)$ where $p_j = \xi_{y_j}$.

We now reinterpret this in terms of Laguerre digraphs. At this stage, the vertices in F (resp. F') are our designated starting vertices (ending vertices). We look through the starting vertices in decreasing order. At the end of step y_{j+1}, directed edges $y_l \to \sigma(y_l),\ldots,y_{j+1} \to \sigma(y_{j+1})$ have been inserted, the available starting vertices are $y_j,\ldots,y_1$ and the available ending vertices belong to $F'\backslash\{\sigma(y_l),\ldots,\sigma(y_{j+1})\}$. We then pick the largest available starting vertex, which is y_j and connect it to the (p_j+1)th largest ending vertex available, i.e., the (p_j+1)th largest element of the set $F'\backslash\{\sigma(y_l),\ldots,\sigma(y_{j+1})\}$.

Lemma 5.4.4. *Let u be the final vertex of a path with at least two vertices in $L|_{H\cup G\cup\{y_l,\ldots,y_j\}}$. Then $u \in \mathrm{Cval} = F \cap G'$.*

PROOF. Since all vertices in $H \cup G$ were already assigned out-neighbours during stages (a) and (b), it must be that $u \in F$. As $F = \mathrm{Cdrise} \cup \mathrm{Cval}$, u is either a cycle valley or a cycle double rise in σ.

Let us assume that u is a cycle double rise in σ, i.e., $\sigma^{-1}(u) < u < \sigma(u)$. We know that u must have an in-neighbour v as its path has atleast two vertices. Thus, $v = \sigma^{-1}(u) < u$, which implies $v \in F$ (from definition of F in (5.57a)). However, this is a situation where we have two vertices $u, v \in F$ with $u > v$ such that the smaller vertex has an out-neighbour even though the larger vertex does not. This clearly cannot happen as we assign out-neighbours to vertices in F in descending order. This is a contradiction and thus $u \in \mathrm{Cval} = F \cap G'$. $\square$

Let $u_1,\ldots,u_n$ be the elements of $[n]$ rearranged as per the FZ order with respect to Motzkin path ω. We say that $u_j \in [n]\backslash H$ is a *cycle closer* if the edge $u_j \to \sigma(u_j)$ is introduced in $L|_{\{u_1,\ldots,u_{j-1}\}}$ as an edge between the two ends of a path turning the path into a cycle. The following lemma classifies all cycle closers:

Lemma 5.4.5. *(Classifying cycle closers) Given a permutation σ, an element $u \in [n]$ is a cycle closer if and only if it is a cycle valley minimum, i.e., it is the smallest element in its cycle.*

PROOF. Let u_j be a cycle closer. Notice that u_j must have been the final vertex of a path in $L|_{\{u_1,\dots,u_{j-1}\}}$ that contains at least two vertices. From Lemma 5.4.4, $u_j \in F \cap G' = \text{Cval}$.

Any other cycle valley $v \neq u_j$ in the resulting cycle must have already been present in this path and hence must have an out-neighbour. Hence, it must be that $v > u_j$ as $\text{Cval} \subseteq F$ and we assign out-neighbours to vertices belonging to F in descending order. Thus, the cycle closer u_j is the smallest cycle valley in its cycle in σ. $\square$

As each non-singleton cycle has exactly one cycle closer, counting cycle closers will give us the number of non-singleton cycles. This is what we do next. But before doing that, we require a technical lemma. However, before going into the lemma, recall that if $y_j \in F \cap G' = \text{Cval}(\sigma)$, step s_{y_j} must be a rise from height h_{y_j-1} to height h_{y_j} and hence, $h_{y_j-1}+1 = h_{y_j}$.

Lemma 5.4.6. *Given a permutation σ and associated sets F, F', G, G', H with $F = \{y_1 < y_2 < \ldots < y_l\}$, and an index j ($1 \le j \le l$) such that $y_j \in F \cap G'$. Then the following is true:*

$$\big|\{u \in F' \backslash \{\sigma(y_l), \dots, \sigma(y_{j+1})\} \colon \; u > y_j\}\big| = h_{y_j-1}+1 = h_{y_j} \tag{5.59}$$

where h_i denotes the height at position i of the Motzkin path ω associated to σ in Step 1.

PROOF. We first establish the following equality of sets:

$$\{u > y_j \colon \; \sigma^{-1}(u) \le y_j\} \; = \; \{u \in F' \backslash \{\sigma(y_l), \dots, \sigma(y_{j+1})\} \colon \; u > y_j\}. \tag{5.60}$$

Whenever $u \in F'$, we have that $\sigma^{-1}(u) \in F$ (by description of F, F' in (5.57a,b)). Additionally, if $u \notin \{\sigma(y_l), \dots, \sigma(y_{j+1})\}$ then it must be that $\sigma^{-1}(u) \le y_j$. This establishes the containment $\{u > y_j \colon \; \sigma^{-1}(u) \le y_j\} \supseteq \{u \in F' \backslash \{\sigma(y_l), \dots, \sigma(y_{j+1})\} \colon \; u > y_j\}$.

On the other hand, if $u > y_j$ and $\sigma^{-1}(u) \leq y_j$, then $u > \sigma^{-1}(u)$ and hence $u \in F'$. As $\sigma^{-1}(u) \leq y_j < y_{j+1} < \ldots < y_l$, u cannot be one of $\sigma(y_{j+1}), \ldots, \sigma(y_l)$. Therefore, $u \in F' \backslash \{\sigma(y_l), \ldots, \sigma(y_{j+1})\}$. This establishes (5.60).

To obtain Equation (5.59), it suffices to show that the cardinality of the set $\{u > y_j \colon \sigma^{-1}(u) \leq y_j\}$ is h_{y_j}. To do this, recall the interpretation of heights in Equation (5.50a) and observe that

$$\begin{aligned} h_{y_j} &= \#\{u \leq y_j \colon \sigma(u) > y_j\} \\ &= \#\{u > y_j \colon \sigma^{-1}(u) \leq y_j\} \end{aligned} \tag{5.61}$$

where the second equality is obtained by replacing u with $\sigma^{-1}(u)$. □

We are now ready to count the number of cycle closers.

Lemma 5.4.7 (Counting of cycle closers for permutations). *Fix a Motzkin path ω of length n and construct the sets F, F', G, G', H (these are totally determined by ω). Let $F = \{y_1 < y_2 < \ldots < y_l\}$ and fix an index j ($1 \leq j \leq l$) such that $y_j \in F \cap G'$. Also fix labels ξ_u for vertices $u \in H \cup G \cup \{y_l, y_{l-1}, \ldots, y_{j+1}\}$ satisfying* (2.37)/(5.47). *Then*

(a) The value of ξ_{y_j} completely determines if y_j is a cycle closer or not.

(b) There is exactly one value $\xi_{y_j} \in \{0, 1, \ldots, h_{y_j - 1}\}$ that makes y_j a cycle closer, and conversely.

PROOF. As $y_j \in F \cap G'$, it must be that y_j is a cycle valley in σ. As y_j does not have an out-neighbour in the Laguerre digraph $L|_{H \cup G \cup \{y_l, \ldots, y_{j+1}\}}$, it must be the final vertex of a path. Let v be the initial vertex of this path. During step y_j, we choose one of the available ending vertices from the set $F' \backslash \{\sigma(y_l), \ldots, \sigma(y_{j+1})\}$ to be $\sigma(y_j)$.

For y_j to be a cycle closer, it must be that $\sigma(y_j) = v$ (so that inserting the edge $y_j \to v$ turns its path into a cycle). As each value for label $\xi_{y_j} \in \{0, 1, \ldots, h_{y_j - 1}\}$

assigns a different vertex to be $\sigma(y_j)$, there is at most one label for which y_j is a cycle closer.

We begin by observing that $v \notin G'$. This is because v does not have an in-neighbour in $L|_{H \cup G \cup \{y_l, \ldots, y_{j+1}\}}$ whereas all $v \in G'$ had in-neighbours at the end of stage (b). Therefore, $v \in F'$ and $y_j \neq v$ as $y_j \in G'$. To show that there is a $\xi_{y_j} \in \{0, 1, \ldots, h_{y_j - 1}\}$ which can connect y_j to v, we must show that v is among the largest $h_{y_j-1} + 1$ elements of $F' \backslash \{\sigma(y_l), \ldots, \sigma(y_{j+1})\}$. However, from Lemma 5.4.6, we only need to show that $v > y_j$. The remainder of the proof does this.

Let $v = v_0, v_1, \ldots, v_\alpha = y_j$ be the vertices of the path containing y_j and v with edges $v_i \to v_{i+1}$. Let β be the smallest index such that $v_\beta \in F$. Using the description of G in (5.57c), we get $v = v_0 > \ldots > v_\beta$ when $\beta > 0$. Thus,

$$v = v_0 \leq v_\beta \tag{5.62}$$

with equality if and only if $\beta = 0$. On the other hand, if $\beta < \alpha$, then v_β already has an out-neighbour and thus $v_\beta > y_j$ (as we are going through elements of F in descending order). Thus,

$$v_\beta \leq v_\alpha = y_j \tag{5.63}$$

with equality if and only if $\beta = \alpha$. Using (5.62), (5.63) and the fact that $v \neq y_j$, we obtain $v > y_j$. This completes the proof. □

Remark. Notice that one can also construct a variant of this interpretation where stage (c) occurs before stage (b). The role of cycle closer will then be played by cycle peak maxima. ■

5.4.3 Computation of weights

We can now compute the weights associated to the Motzkin path ω in Section 5.4.1 Step 1. As our polynomial $\widehat{Q}$ defined in (5.20) is almost the same as the polynomial introduced in [SZ22, eq. (2.77)] except for the extra factor $\lambda^{\mathrm{cyc}(\sigma)}$ and the index of a; the dependence on cycle peaks, cycle double rises, cycle double falls, and fixed points are same as in [SZ22, eq. (2.77)] but the treatment of cycle valleys is different. As we use the same bijection to obtain the continued fraction ([SZ22, eq. (2.78)/(2.79)]), the computation of weights corresponding to the variables $\mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}$ are going to be exactly the same.

The only thing that remains is to compute the weights for the variables a. These correspond to steps s_i in ω where s_i is a rise starting at height $h_{i-1} = k$ (so that i is a cycle valley). Then from Equations (5.53)/(5.54), we get $k = \mathrm{ucross}(i, \sigma) + \mathrm{unest}(i, \sigma)$. Also among the possible choices of labels $\xi \in [0, k]$ there is exactly one which closes a cycle and the others don't (Lemma 5.4.7). Therefore, we obtain

$$a_k \stackrel{\text{def}}{=} \sum_{\xi} a_{k,\xi} \;=\; (\lambda + k)\mathsf{a}_k. \tag{5.64}$$

This completes the proof of Theorem 5.2.3. □

PROOF OF THEOREM 5.2.2. We recall [SZ22, Lemma 2.10] which was used to separate records and antirecords: Let $\sigma \in \mathfrak{S}_n$ and $i \in [n]$.

(a) If i is a cycle valley or cycle double rise, then i is a record if and only if $\mathrm{unest}(i, \sigma) = 0$; and in this case it is an exclusive record.

(b) If i is a cycle peak or cycle double fall, then i is an antirecord if and only if $\mathrm{lnest}(i, \sigma) = 0$; and in this case it is an exclusive antirecord.

We then specialise Theorem 5.2.3 to

$$\mathsf{a}_\ell = p_{+1}^\ell y_1 \tag{5.65a}$$

$$\mathsf{b}_{\ell,\ell'} = p_{-1}^\ell q_{-1}^{\ell'} \times \begin{cases} x_1 & \text{if } \ell' = 0 \\ u_1 & \text{if } \ell' \geq 1 \end{cases} \tag{5.65b}$$

$$\mathsf{c}_{\ell,\ell'} = p_{-2}^\ell q_{-2}^{\ell'} \times \begin{cases} x_2 & \text{if } \ell' = 0 \\ u_2 & \text{if } \ell' \geq 1 \end{cases} \tag{5.65c}$$

$$\mathsf{d}_{\ell,\ell'} = p_{+2}^\ell q_{+2}^{\ell'} \times \begin{cases} y_2 & \text{if } \ell' = 0 \\ v_2 & \text{if } \ell' \geq 1 \end{cases} \tag{5.65d}$$

$$\mathsf{e}_\ell = s^\ell w_l \tag{5.65e}$$

□

Remark. Notice that the specialisation in the above proof is almost the same as [SZ22, eq. (2.81)], except for the treatment of a. ■

PROOF OF THEOREM 5.2.1. Specialise Theorem 5.2.2 to $p_{+1} = p_{-1} = p_{+2} = p_{-2} = q_{-1} = q_{+2} = q_{-2} = s = 1$. □

5.5 D-permutations: Proofs

Motivated by Randrianarivony's [Ran97] bijection for D-o-semiderangements, we came up with two different bijections for D-permutations in Section 4.5. The first of them was constructed in Sections 4.5.1-4.5.3 and was used to prove Theorems 4.2.3, 4.2.7, 4.2.9, i.e., our "first T-fractions" for D-permutations in Chapter 4. We introduced our second bijection in Section 4.5.5 and used it to prove our "variant forms of the first T-fractions" for D-permutations; these were Theorems 4.2.10 and 4.2.11. We will provide new interpretations to both of these bijections and use them to prove our theorems on D-permutations in this chapter.

We recall that both bijections had the same path but different description of labels. In Section 4.5 we introduced almost-Dyck paths. We then constructed a bijection from $\sigma \in \mathfrak{D}_{2n}$ to labelled 0-Schröder paths of length $2n$. We did this by constructing an almost-Dyck path ω and then transformed it into a 0-Schröder path $\widehat{\omega} = \psi(\omega)$ (this is the obvious bijection in which we replace each down-up pair starting and ending at height 0 with a long level step at height 0). The common definition of the almost-Dyck path was provided in Section 4.5.1.

Our bijections are correspondences between $\mathfrak{D}_{2n}$ to the set of $(\mathcal{A},\mathcal{B},\mathcal{C})$-labelled 0-Schröder paths of length $2n$ (these were defined in Section 2.2.5). Here $\mathcal{A},\mathcal{B},\mathcal{C}$ are the sets of the labels ξ_i and they were described in (4.72) which we copy here:

$$\mathcal{A}_h = \{0,\ldots,\lceil h/2\rceil\} \quad \text{for } h \geq 0 \tag{5.66a}$$

$$\mathcal{B}_h = \{0,\ldots,\lceil (h-1)/2\rceil\} \quad \text{for } h \geq 1 \tag{5.66b}$$

$$\mathcal{C}_0 = \{0\} \tag{5.66c}$$

$$\mathcal{C}_h = \varnothing \quad \text{for } h \geq 1 \tag{5.66d}$$

For the convenience of the reader, we now recall how the almost-Dyck path was constructed.

Step 1: Definition of the almost-Dyck path. Given a D-permutation $\sigma \in \mathfrak{D}_{2n}$, we define a path $\omega = (\omega_0,\ldots,\omega_{2n})$ starting at $\omega_0 = (0,0)$, with steps $s_1,\ldots,s_{2n}$ as follows:

- If $\sigma^{-1}(i)$ is even, then s_i is a rise. (Note that in this case we must have $\sigma^{-1}(i) \geq i$, by definition of D-permutation.) Alternatively, if i is a cycle valley, cycle double fall or even fixed point, then s_i is a rise.
- If $\sigma^{-1}(i)$ is odd, then s_i is a fall. (Note that in this case we must have $\sigma^{-1}(i) \leq i$, by definition of D-permutation.) Alternatively, if i is a cycle peak, cycle double rise or odd fixed point, then s_i is a fall.

The fact that the resulting path is indeed an almost-Dyck path can be done by showing that all the heights h_i are ≥ -1 and that $h_{2n} = 0$. This was done by obtaining a precise interpretation of the height h_i in Lemma 4.5.1; we showed that for $k \in [2n]$ we have

$$h_k \;=\; \begin{cases} 2f_k - 1 & \text{if } k \text{ is odd} \\ 2f_k & \text{if } k \text{ is even} \end{cases} \tag{5.67}$$

where f_k was defined in (4.73)/(4.74)

$$f_k \;=\; \#\{i \le k\colon\; \sigma(i) > k\} \;=\; \#\{i \le k\colon\; \sigma^{-1}(i) > k\}\,. \tag{5.68}$$

Also recall that in Lemma 4.2.2 we showed that $2i-1$ and $2i$ are record-antirecords if and only if σ maps $\{1,\ldots,2i-1\}$ onto itself and this is the only situation when $h_k = -1$ where $k = 2i-1$ and $f_k = 0$. In this situation, $f_{2i-2} = f_{2i-1} = f_{2i} = 0$, so that $(h_{2i-2}, h_{2i-1}, h_{2i}) = (0,-1,0)$.

5.5.1 Continued fractions using record classification: Proof of Theorems 5.3.1, 5.3.3, 5.3.4

We first recall the description of labels and the inverse bijection from Section 4.5.1 in Subsection 5.5.1.1. We then reinterpret the inverse bijection described in Section 4.5.3 in Subsection 5.5.1.2.

5.5.1.1 Description of labels and inverse bijection

Step 2: Definition of the labels $\boldsymbol{\xi_i}$. We defined the labels in equation (4.80) which we recall here:

$$\xi_i \;=\; \begin{cases} \#\{j\colon\; \sigma(j) < \sigma(i) \le i < j\} & \text{if } i \text{ is even} \\ \#\{j\colon\; j < i \le \sigma(i) < \sigma(j)\} & \text{if } i \text{ is odd} \end{cases} \tag{5.69}$$

In (4.81) we showed that the definition (5.69) can be written equivalently as

$$\xi_i \;=\; \begin{cases} \#\{2l > 2k\colon\; \sigma(2l) < \sigma(2k)\} & \text{if } i = 2k \\ \#\{2l-1 < 2k-1\colon\; \sigma(2l-1) > \sigma(2k-1)\} & \text{if } i = 2k-1 \end{cases} \tag{5.70}$$

since $\sigma(j) < j$ implies that j is even, and $j < \sigma(j)$ implies that j is odd.

These definitions have a simple interpretation in terms of the nesting statistics defined in Section 2.5.2

$$\xi_i \;=\; \begin{cases} \mathrm{lnest}(i,\sigma) & \text{if } i \text{ is even and } \neq \sigma(i) \text{ [equivalently, } i > \sigma(i)\text{]} \\ \mathrm{unest}(i,\sigma) & \text{if } i \text{ is odd and } \neq \sigma(i) \text{ [equivalently, } i < \sigma(i)\text{]} \\ \mathrm{psnest}(i,\sigma) & \text{if } i = \sigma(i) \text{ (that is, } i \text{ is a fixed point)} \end{cases} \tag{5.71}$$

We then showed in Lemma 4.5.4 that the quantities $\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i$ when s_i is a rise, and $\left\lceil \frac{h_i}{2} \right\rceil - \xi_i$ when s_i is a fall, have the following interpretation in terms of the crossing statistics (also defined in Section 2.2.5):

(a) If s_i a rise and i (hence also h_i) is odd, then

$$\left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i \;=\; \mathrm{ucross}(i,\sigma)\,. \tag{5.72}$$

(b) If s_i a rise and i (hence also h_i) is even, then

$$\begin{aligned} \left\lceil \frac{h_i - 1}{2} \right\rceil - \xi_i \;&=\; \mathrm{lcross}(i,\sigma) + \mathrm{I}[\sigma(i) \neq i] && \text{(5.73a)} \\ &=\; \mathrm{lcross}(i,\sigma) + \mathrm{I}[i \text{ is a cycle double fall}]\,. && \text{(5.73b)} \end{aligned}$$

(c) If s_i a fall and i (hence also h_i) is odd, then

$$\begin{aligned} \left\lceil \frac{h_i}{2} \right\rceil - \xi_i \;&=\; \mathrm{ucross}(i,\sigma) + \mathrm{I}[\sigma(i) \neq i] && \text{(5.74a)} \\ &=\; \mathrm{ucross}(i,\sigma) + \mathrm{I}[i \text{ is a cycle double rise}]\,. && \text{(5.74b)} \end{aligned}$$

(d) If s_i a fall and i (hence also h_i) is even, then

$$\left\lceil \frac{h_i}{2} \right\rceil - \xi_i \;=\; \mathrm{lcross}(i,\sigma)\,. \tag{5.75}$$

(Here I[*proposition*] $= 1$ if *proposition* is true, and 0 if it is false.)

Since the quantities (5.72)–(5.75) are manifestly nonnegative, we obtain that the labels ξ_i satisfy the following inequalities:

$$0 \le \xi_i \le \left\lceil \frac{h_i - 1}{2} \right\rceil = \left\lceil \frac{h_{i-1}}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is even (i.e., } s_i \text{ is a rise)} \tag{5.76a}$$

$$0 \le \xi_i \le \left\lceil \frac{h_i}{2} \right\rceil = \left\lceil \frac{h_{i-1} - 1}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is odd (i.e., } s_i \text{ is a fall)} \tag{5.76b}$$

This was the content of Lemma 4.5.3.

Step 3: Proof of bijection. We recall the description of the inverse map for the mapping $\sigma \mapsto (\omega, \xi)$.

First, some preliminaries: Given a D-permutation $\sigma \in \mathfrak{D}_{2n}$ we can define four subsets of $[2n]$:

$$F = \{2, 4, \ldots, 2n\} = \text{even positions} \tag{5.77a}$$

$$F' = \{i \colon \sigma^{-1}(i) \text{ is even}\} = \{\sigma(2), \sigma(4), \ldots, \sigma(2n)\} \tag{5.77b}$$

$$G = \{1, 3, \ldots, 2n-1\} = \text{odd positions} \tag{5.77c}$$

$$G' = \{i \colon \sigma^{-1}(i) \text{ is odd}\} = \{\sigma(1), \sigma(3), \ldots, \sigma(2n-1)\} \tag{5.77d}$$

Note that F' (resp. G') are the positions of the rises (resp. falls) in the almost-Dyck path ω.

Let us observe that

$$F \cap F' = \text{cycle double falls and even fixed points} \tag{5.78a}$$

$$G \cap G' = \text{cycle double rises and odd fixed points} \tag{5.78b}$$

$$F \cap G' = \text{cycle peaks} \tag{5.78c}$$

$$F' \cap G = \text{cycle valleys} \tag{5.78d}$$

$$F \cap G = \varnothing \tag{5.78e}$$

$$F' \cap G' = \varnothing \tag{5.78f}$$

We can now describe the map $(\omega, \xi) \mapsto \sigma$. Given the almost-Dyck path ω, we can immediately reconstruct the sets F, F', G, G'. We now use the labels ξ to reconstruct the maps $\sigma \upharpoonright F\colon\ F \to F'$ and $\sigma \upharpoonright G\colon\ G \to G'$ as follows: The even subword $\sigma(2)\sigma(4)\cdots\sigma(2n)$ is a listing of F' whose right-to-left inversion table is given by $q_\alpha = \xi_{2\alpha}$; this is the content of (5.70a). Similarly, the odd subword $\sigma(1)\sigma(3)\cdots\sigma(2n-1)$ is a listing of G' whose left-to-right inversion table is given by $p_\alpha = \xi_{2\alpha-1}$; this is the content of (5.70b).

5.5.1.2 Combinatorial interpretation using Laguerre digraphs

The construction here will almost mirror the construction in Section 5.4.2. We will only include the necessary details and state the necessary lemmas and will omit most of the proofs.

We begin with an almost-Dyck path ω and an assignment of labels ξ satisfying (5.76). The inverse bijection in Section 5.5.1.1 Step 3, gives us a D-permutation σ. We will again break this process into several intermediate steps and provide a reinterpretation using Laguerre digraphs. We will use the same conventions for denoting Laguerre digraphs as in Section 5.4.2.

Recall that the inverse bijection (Section 5.5.1.1 Step 3) begins by obtaining the sets F, G (which are fixed for any given n) and F', G' from the almost-Dyck path ω. We then construct $\sigma \upharpoonright F\colon\ F \to F'$ and $\sigma \upharpoonright G\colon\ G \to G'$ separately by using the labels $\xi \upharpoonright F$ and $\xi \upharpoonright G$ respectively.

We again start with the digraph $L|_\varnothing$ and then go through the set $[2n]$. However, we first go through F and then go through G, i.e., the order of our steps is now $2, 4, \ldots, 2n, 2n-1, \ldots, 3, 1$. In this situation, stage (a) will involve going through the even vertices in increasing order and then stage (b) will involve going through the odd vertices but in decreasing order. Thus, unlike the situation in Section 5.4.2, the FZ order corresponding to two different D-permutations $\sigma, \sigma' \in \mathfrak{D}_{2n}$ are the same, irrespective of the underlying almost-Dyck path.

Let us now look at the intermediate Laguerre digraphs obtained during stages (a) and (b).

Stage (a): Going through $F = \{2, 4, \ldots, 2n\}$:

We go through the even vertices in increasing order. From (5.78), we know that $F = \mathrm{Cdfall}(\sigma) \cup \mathrm{Evenfix}(\sigma) \cup \mathrm{Cpeak}(\sigma)$ and $F' = \mathrm{Cdfall}(\sigma) \cup \mathrm{Evenfix}(\sigma) \cup \mathrm{Cval}(\sigma)$ where σ is the resulting D-permutation obtained at the end of the inverse bijection.

The construction here is similar to the construction in stage (b) Section 5.4.2. The connected components at the end of this stage can be described as follows:

Lemma 5.5.1. *The Laguerre digraph $L|_F$ consists of the following connected components:*

- *loops on vertices $u \in \mathrm{Evenfix}$,*
- *directed paths with at least two vertices, in which the initial vertex of the path is a cycle peak in σ (i.e. contained in the set $F \cap G'$), the final vertex is a cycle valley in σ (i.e. contained in the set $F' \cap G$), and the intermediate vertices (if any) are cycle double falls (which belong to the set $F \cap F'$).*
- *isolated vertices at $u \in G \cap G' = \mathrm{Cdrise}(\sigma) \cup \mathrm{Oddfix}(\sigma)$.*

Furthermore, it contains no directed cycles.

Stage (b): Going through $G = \{2n-1, \ldots, 3, 1\}$:

We now go through the odd vertices in decreasing order. From (5.78), we know that $G = \mathrm{Cdrise}(\sigma) \cup \mathrm{Oddfix}(\sigma) \cup \mathrm{Cval}(\sigma)$, and $G' = \mathrm{Cdrise}(\sigma) \cup \mathrm{Oddfix}(\sigma) \cup \mathrm{Cpeak}(\sigma)$, where σ is the resulting D-permutation obtained at the end of the inverse bijection.

The construction here is similar to stage (c) in Section 5.4.2. The final vertices of a path with at least two vertices have the following description:

Lemma 5.5.2. *Let u be the final vertex of a path with at least two vertices in $L|_{F \cup \{2n-1, \ldots, 2(n-j)+1\}}$ for some index j ($1 \le j \le n$). Then $u \in \mathrm{Cval}$.*

Our definition of cycle closers is again the same as in Section 5.4.2. The following lemma classifies all cycle closers.

Lemma 5.5.3. *(Classifying cycle closers) Given a D-permutation σ, an element $u \in [2n]$ is a cycle closer if and only if it is a cycle valley minimum, i.e., it is the smallest element in its cycle.*

Next, we will count the number of cycle closers. But before doing that, we require a technical lemma similar to Lemma 5.4.6 for the case of permutations. However, first notice that if $i \in G \cap F' = \mathrm{Cval}(\sigma)$, step s_i must be a rise from height h_{i-1} to height h_i and hence, $h_{i-1} + 1 = h_i$. Also, from the interpretation of the heights in equation (5.67) we must have $\lceil h_{i-1}/2 \rceil + 1 = \lceil (h_i+1)/2 \rceil = f_i$.

Lemma 5.5.4. *Given a D-permutation σ and associated sets F', G' and an odd $y \in \{2n-1, \ldots, 3, 1\}$ such that $y \in G \cap F' = \mathrm{Cval}$. Then the following is true:*

$$\begin{aligned} \#\{u \in G' \backslash \{\sigma(2n-1), \sigma(2n-3), \ldots, \sigma(y+2)\} \colon\ u > y\} &= \lceil h_{y-1}/2 \rceil + 1 \\ &= \lceil (h_y+1)/2 \rceil \\ &= f_y \end{aligned} \qquad (5.79)$$

where h_i denotes the height at position i of the almost-Dyck path ω associated to σ in Step 1, and f_i is defined in (5.68).

PROOF. We first establish the following equality of sets:

$$\{u > y \colon\ \sigma^{-1}(u) \le y\} = \{u \in G' \backslash \{\sigma(2n-1), \sigma(2n-3), \ldots, \sigma(y+2)\} \colon\ u > y\}. \qquad (5.80)$$

Whenever $u \in G'$, we have that $\sigma^{-1}(u) \in G$ (by description of G, G' in (5.77)). Additionally, if $u \notin \{\sigma(2n-1), \sigma(2n-3), \ldots, \sigma(y+2)\}$ then it must be that $\sigma^{-1}(u) \le y$. This establishes the containment

$$\{u > y \colon\ \sigma^{-1}(u) \le y\} \supseteq \{u \in G' \backslash \{\sigma(2n-1), \sigma(2n-3), \ldots, \sigma(y+2)\} \colon\ u > y\}.$$

On the other hand, if $u > y$ and $\sigma^{-1}(u) \le y$, then $u > \sigma^{-1}(u)$ and as σ is a D-permutation, $\sigma^{-1}(u)$ must be odd. Therefore, $u \in G'$. As $\sigma^{-1}(u) \le y$, u cannot be one of $\sigma(y+2), \ldots, \sigma(2n-3), \sigma(2n-1)$. Therefore, $u \in G' \backslash \{\sigma(2n-1), \sigma(2n-3), \ldots, \sigma(y+2)\}$. This establishes (5.80).

To obtain Equation (5.79), it suffices to show that the cardinality of the set $\{u > y \colon\ \sigma^{-1}(u) \le y\}$ is f_y. To do this, recall the description of f_y in Equation (5.68) and observe that

$$\begin{aligned} f_y &= \#\{u \le y\colon\ \sigma(u) > y\} \\ &= \#\{u > y\colon\ \sigma^{-1}(u) \le y\} \end{aligned} \tag{5.81}$$

where the second equality is obtained by replacing u with $\sigma^{-1}(u)$. □

We are now ready to state the counting of cycle closers.

Lemma 5.5.5 (Counting of cycle closers for D-permutations)**.** *Fix an almost-Dyck path ω of length $2n$ and construct F', G' (these are completely determined by ω). Also fix labels ξ_u for vertices $u \in \{2,4,\ldots,2n\} \cup \{2n-1,\ldots,2(n-j)+1\}$ satisfying* (5.76)*. Also let $y = 2(n-j)-1 \in G \cap F' = \mathrm{Cval}(\sigma)$. Then*

(a) The value of ξ_y completely determines if y is a cycle closer or not.

(b) There is exactly one value $\xi_y \in \{0,1,\ldots,\lceil h_{y-1}/2\rceil\}$ that makes y a cycle closer, and conversely.

Remark. Notice that one can also construct a variant of this interpretation where stage (b) occurs before stage (a). The role of cycle closer will then be played by cycle peak maximum. ■

5.5.1.3 Computation of weights

We can now compute the weights associated to the 0-Schröder path $\widehat{\omega}$ in Step 1. As our polynomial $\widehat{Q}$ defined in (5.31) is almost the same as the polynomial (4.31) except for the extra factor $\lambda^{\mathrm{cyc}(\sigma)}$ and the index of a, the dependence on cycle peaks, cycle double rises, cycle double falls, and even and odd fixed points are the same in both polynomials but the treatment of cycle valleys is different. As we use the same bijection used to obtain the continued fraction (4.32)/(4.33) the computation of weights corresponding to the variables $\mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}$ are going to be exactly the same.

The only thing that remains is to compute the weights for the variables a. These correspond to steps s_i in $\widehat{\omega}$ where s_i is a rise starting at height $h_{i-1} = 2k$ (so that

i is a cycle valley). Then from Equations (5.71)/(5.72) we get $\lceil (h_{i-1}-1)/2 \rceil = \lceil h_i/2 \rceil = \mathrm{ucross}(i,\sigma) + \mathrm{unest}(i,\sigma)$ $(= k)$. Also among the possible choices of labels $\xi \in [0,k]$ there is exactly one which closes a cycle and the others don't (Lemma 5.5.5). Therefore, we obtain

$$a_{2k} \stackrel{\text{def}}{=} \sum_{\xi} a_{2k,\xi} \;=\; (\lambda + k)\mathsf{a}_{2k}. \tag{5.82}$$

This completes the proof of Theorem 5.3.4. □

PROOF OF THEOREM 5.3.3. Specialise Theorem 5.3.4 to

$$\mathsf{a}_{k-1} \;=\; p_{+1}^{k-1} \times y_1 \tag{5.83}$$

$$\mathsf{b}_{k-1-\xi,\xi} \;=\; p_{-1}^{k-1-\xi} q_{-1}^{\xi} \times \begin{cases} x_1 & \text{if } \xi = 0 \\ u_1 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{5.84}$$

$$\mathsf{c}_{k-1-\xi,\xi} \;=\; p_{-2}^{k-1-\xi} q_{-2}^{\xi} \times \begin{cases} x_2 & \text{if } \xi = 0 \\ u_2 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{5.85}$$

$$\mathsf{d}_{k-1-\xi,\xi} \;=\; p_{+2}^{k-1-\xi} q_{+2}^{\xi} \times \begin{cases} y_2 & \text{if } \xi = 0 \\ v_2 & \text{if } 1 \le \xi \le k-1 \end{cases} \tag{5.86}$$

$$\mathsf{e}_k \;=\; \begin{cases} z_{\mathrm{e}} & \text{if } k = 0 \\ s_{\mathrm{e}}^k w_{\mathrm{e}} & \text{if } k \ge 1 \end{cases} \tag{5.87}$$

$$\mathsf{f}_k \;=\; \begin{cases} z_{\mathrm{o}} & \text{if } k = 0 \\ s_{\mathrm{o}}^k w_{\mathrm{o}} & \text{if } k \ge 1 \end{cases} \tag{5.88}$$

□

Remark. Notice that the specialisation in the above proof is almost the same as our specialisation in equations (4.111)-(4.116) except for the treatment of a. ■

PROOF OF THEOREM 5.3.1. Specialise Theorem 5.3.3 to

$$p_{+1} = p_{-1} = p_{+2} = p_{-2} = q_{-1} = q_{+2} = q_{-2} = s_{\mathrm{e}} = s_{\mathrm{o}} = 1. \tag{5.89}$$

□

5.5.2 Continued fractions using variant record classification: Proof of Theorems 5.3.5, 5.3.6, 5.3.7

We first recall the description of the alternate labels and the inverse bijection from Section 4.5.5 in Subsection 5.5.2.1. We then reinterpret the inverse bijection in Subsection 5.5.2.2.

5.5.2.1 Description of labels and inverse bijection

Step 2: Definition of the labels $\widehat{\xi}_i$.

In equation (4.117) we defined our alternate labels for our second bijection and we recall it here:

$$\widehat{\xi}_i = \begin{cases} \#\{2l > 2k\colon\ \sigma(2l) < \sigma(2k)\} & \text{if } i = \sigma(2k) \\ \#\{2l-1 < 2k-1\colon\ \sigma(2l-1) > \sigma(2k-1)\} & \text{if } i = \sigma(2k-1) \end{cases} \tag{5.90a}$$

$$= \begin{cases} \#\{j\colon\ j < i \le \sigma^{-1}(i) < \sigma^{-1}(j)\} & \text{if } \sigma^{-1}(i) \text{ is even} \\ \#\{j\colon\ \sigma^{-1}(j) < \sigma^{-1}(i) \le i < j\} & \text{if } \sigma^{-1}(i) \text{ is odd} \end{cases} \tag{5.90b}$$

These labels have a simple interpretation in terms of variant nesting statistics defined in Section 2.2.5

$$\widehat{\xi}_i = \begin{cases} \mathrm{lnest}'(i,\sigma) & \text{if } \sigma^{-1}(i) \text{ is even (i.e., } s_i \text{ is a rise) and } \neq i \\ \mathrm{unest}'(i,\sigma) & \text{if } \sigma^{-1}(i) \text{ is odd (i.e., } s_i \text{ is a fall) and } \neq i \\ \mathrm{psnest}(i,\sigma) & \text{if } \sigma^{-1}(i) = i \text{ (i.e., } i \text{ is a fixed point)} \end{cases} \tag{5.91}$$

This was the content of Lemma 4.5.7

We then showed in Lemma 4.5.9 that the quantities $\left\lceil \frac{h_i - 1}{2} \right\rceil - \widehat{\xi}_i$ when s_i is a rise, and $\left\lceil \frac{h_i}{2} \right\rceil - \widehat{\xi}_i$ when s_i a fall, have the following interpretation in terms of the variant crossing statistics (also defined in Section 2.2.5):

(a) If s_i a rise (i.e. $\sigma^{-1}(i)$ is even), then

$$\left\lceil \frac{h_i - 1}{2} \right\rceil - \widehat{\xi}_i \;=\; \mathrm{lcross}'(i,\sigma) + \mathrm{I}[i \text{ is even and } \sigma(i) \neq i] \qquad (5.92a)$$
$$\;=\; \mathrm{lcross}'(i,\sigma) + \mathrm{I}[i \text{ is a cycle double fall}]\,. \qquad (5.92b)$$

(b) If s_i a fall (i.e. $\sigma^{-1}(i)$ is odd), then

$$\left\lceil \frac{h_i}{2} \right\rceil - \widehat{\xi}_i \;=\; \mathrm{ucross}'(i,\sigma) + \mathrm{I}[i \text{ is odd and } \sigma(i) \neq i] \qquad (5.93a)$$
$$\;=\; \mathrm{ucross}'(i,\sigma) + \mathrm{I}[i \text{ is a cycle double rise}]\,. \qquad (5.93b)$$

Since the quantities (5.92), (5.93) are manifestly nonnegative, we obtain that the labels $\widehat{\xi}_i$ satisfy the following inequalities:

$$0 \;\le\; \widehat{\xi}_i \;\le\; \left\lceil \frac{h_i - 1}{2} \right\rceil \;=\; \left\lceil \frac{h_{i-1}}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is even (i.e., } s_i \text{ is a rise)} \qquad (5.94a)$$

$$0 \;\le\; \widehat{\xi}_i \;\le\; \left\lceil \frac{h_i}{2} \right\rceil \;=\; \left\lceil \frac{h_{i-1} - 1}{2} \right\rceil \quad \text{if } \sigma^{-1}(i) \text{ is odd (i.e., } s_i \text{ is a fall)} \qquad (5.94b)$$

This was the content of Lemma 4.5.8.

Step 3: Proof of bijection. The proof is similar to that presented in Step 3 Section 5.5.1.1, but using a value-based rather than position-based notion of inversion table. Recall that if $S = \{s_1 < s_2 < \ldots < s_k\}$ is a totally ordered set of cardinality k, and $\boldsymbol{x} = (x_1, \ldots, x_k)$ is a permutation of S, then the (left-to-right) (position-based) inversion table corresponding to $\boldsymbol{x}$ is the sequence $\boldsymbol{p} = (p_1, \ldots, p_k)$ of nonnegative integers defined by $p_\alpha = \#\{\beta < \alpha \colon\; x_\beta > x_\alpha\}$. We now define the (left-to-right) *value-based* inversion table $\boldsymbol{p}'$ by $p'_{x_i} = p_i$; note that $\boldsymbol{p}'$ is a map from S to $\{0, \ldots, k-1\}$, such that p'_{x_i} is the number of entries to the left of x_i (in the word $\boldsymbol{x}$) that are larger than x_i. In particular, $0 \le p'_{s_i} \le k - i$. Given the value-based inversion table $\boldsymbol{p}'$, we can reconstruct the sequence $\boldsymbol{x}$ by working from largest to smallest value, as follows [Knu98, section 5.1.1]: We start from an empty sequence, and insert s_k. Then we insert s_{k-1} so that the resulting word has $p'_{s_{k-1}}$ entries to its left. Next we insert s_{k-2} so that the resulting word has $p'_{s_{k-2}}$ entries to its left, and so on. [The right-to-left value-based inversion table $\boldsymbol{q}'$ is defined analogously, and the reconstruction proceeds from smallest to largest.]

We now recall the definitions

$$F = \{2,4,\ldots,2n\} = \text{even positions} \tag{5.95a}$$

$$F' = \{i\colon \sigma^{-1}(i) \text{ is even}\} = \{\sigma(2),\sigma(4),\ldots,\sigma(2n)\} \tag{5.95b}$$

$$G = \{1,3,\ldots,2n-1\} = \text{odd positions} \tag{5.95c}$$

$$G' = \{i\colon \sigma^{-1}(i) \text{ is odd}\} = \{\sigma(1),\sigma(3),\ldots,\sigma(2n-1)\} \tag{5.95d}$$

Note that F' (resp. G') are the positions of the rises (resp. falls) in the almost-Dyck path ω.

We can now describe the map $(\omega,\widehat{\xi}) \mapsto \sigma$. Given the almost-Dyck path ω, we can immediately reconstruct the sets F,F',G,G'. We now use the labels $\widehat{\xi}$ to reconstruct the maps $\sigma \restriction F\colon\; F \to F'$ and $\sigma \restriction G\colon\; G \to G'$ as follows: The even subword $\sigma(2)\sigma(4)\cdots\sigma(2n)$ is a listing of F' whose right-to-left value-based inversion table is given by $q'_i = \widehat{\xi}_i$ for all $i \in F'$; this is the content of (5.90a). Similarly, the odd subword $\sigma(1)\sigma(3)\cdots\sigma(2n-1)$ is a listing of G' whose left-to-right value-based inversion table is given by $p'_i = \widehat{\xi}_i$ for all $i \in G'$; this again is the content of (5.90a).

Additionally, we state a lemma which gives an alternate way of constructing a value-based inversion table for D-permutation.

Lemma 5.5.6. *Let $\sigma \in \mathfrak{D}_{2n}$ and let $\left(\omega,\widehat{\xi}\right)$ be its associated almost-Dyck path with an assignment of labels determined by* (5.90). *Also let $G' = \{x_1 < \ldots < x_n\}$ and let $F' = \{y_1 < \ldots < y_n\}$. Then*

(a) For any index j ($1 \le j \le n$), $\sigma^{-1}(x_j)$ is the $(\widehat{\xi}_{x_j}+1)$th smallest element of $G \setminus \{\sigma^{-1}(x_1),\ldots,\sigma^{-1}(x_{j-1})\}$.

(b) For any index j ($1 \le j \le n$), $\sigma^{-1}(y_j)$ is the $(\widehat{\xi}_{y_j}+1)$th largest element of $F \setminus \{\sigma^{-1}(y_n), \ldots, \sigma^{-1}(y_{j+1})\}$.

This lemma can be proved using (5.90a) and we omit the details.

5.5.2.2 Combinatorial interpretation using Laguerre digraphs

The construction here will again be similar to those in Sections 5.4.2 and 5.5.1.2. We will only include the necessary details and state the necessary lemmas, and will omit most of the proofs.

We begin with an almost-Dyck path ω and an assignment of labels $\widehat{\xi}$ satisfying (5.94). The inverse bijection in Section 5.5.2.1 Step 3, gives us a D-permutation σ. We will again break this process into several intermediate steps and provide a reinterpretation using Laguerre digraphs.

This time however, we will use a different convention for denoting Laguerre digraphs than the one used in Sections 5.4.2 and 5.5.1.2. Let $\sigma \in \mathfrak{S}_n$ be a permutation on $[n]$. For $S \subseteq [n]$, we let $L'|_S$ denote the subgraph of L^σ, containing the same set of vertices $[n]$, but only containing the edges $\sigma^{-1}(i) \to i$, whenever $i \in S$ (we are allowed to have $\sigma^{-1}(i) \notin S$). Thus, $L'|_{[n]} = L^\sigma$, and $L'|_\varnothing = L^\sigma|_\varnothing$ is the digraph containing n vertices and no edges. Whenever the permutation σ is understood, we shall drop the superscript and denote it as $L'|_S$.

Now, let $\sigma \in \mathfrak{D}_{2n}$ be a D-permutation. Similar to the construction in Section 5.5.1.1, recall that the inverse bijection begins by obtaining the sets F, G, F', G' (the sets F, G are fixed for any given n and the sets F', G' are obtained from the almost-Dyck path ω). We then construct $\sigma \restriction F\colon\ F \to F'$ and $\sigma \restriction G\colon\ G \to G'$ separately but by using the labels $\widehat{\xi} \restriction F'$ and $\widehat{\xi} \restriction G'$ respectively.

In this interpretation, we start with the digraph $L'|_\varnothing$ and then go through the set $[2n]$. This time, we first go through the elements of G' in increasing order (stage (a)) and then through the elements of F' in decreasing order (stage (b)). We call this the *variant DS order* on the set $[2n]$.

Here however, the history that we consider is different from the one in Section 5.5.1.2. Let $u_1, \ldots, u_{2n}$ be a rewriting of $[2n]$ as per the variant DS order. We now consider the "variant DS history" $L'|_{\varnothing} \subset L'|_{\{u_1\}} \subset L'|_{\{u_1,u_2\}} \subset \ldots \subset L'|_{\{u_1,\ldots,u_{2n}\}} = L$. Thus, at step u as per the variant DS order, we use the (value-based) inversion tables and Lemma 5.5.6 to construct the edge $\sigma^{-1}(u) \to u$. Similar to the previous histories, at each step we insert a new edge into the digraph, and at the end of this process, the resulting digraph obtained is the permutation σ in cycle notation.

Let us now look at the intermediate Laguerre digraphs obtained during stages (a) and (b).

Stage (a): Going through G':

From (5.95), we know that $G = \mathrm{Cdrise}(\sigma) \cup \mathrm{Oddfix}(\sigma) \cup \mathrm{Cval}(\sigma)$ and $G' = \mathrm{Cdrise}(\sigma) \cup \mathrm{Oddfix}(\sigma) \cup \mathrm{Cpeak}(\sigma)$ where σ is the resulting D-permutation obtained at the end of the inverse bijection.

The connected components at the end of this stage can be described as follows:

Lemma 5.5.7. *The Laguerre digraph $L'|_{G'}$ consists of the following connected components:*

- *loops on vertices $u \in \mathrm{Oddfix}$,*
- *directed paths with at least two vertices, in which the initial vertex of the path is a cycle valley in σ (i.e. contained in the set $F' \cap G$) and the final vertex is a cycle peak in σ (i.e. contained in the set $F \cap G'$), and the intermediate vertices (if any) are cycle double rises (which belong to the set $G \cap G'$).*
- *isolated vertices at $u \in F \cap F' = \mathrm{Cdfall}(\sigma) \cup \mathrm{Evenfix}(\sigma)$.*

Furthermore, it contains no directed cycles.

Stage (b): Going through F':

We now go through the elements of F' in decreasing order. From (5.95), we know that $F = \mathrm{Cdfall}(\sigma) \cup \mathrm{Evenfix}(\sigma) \cup \mathrm{Cpeak}(\sigma)$ and $F' = \mathrm{Cdfall}(\sigma) \cup \mathrm{Evenfix}(\sigma) \cup \mathrm{Cval}(\sigma)$ where σ is the resulting D-permutation obtained at the end of the inverse bijection. We let $F' = \{y_1 < \ldots < y_n\}$.

This time, we describe the the *initial vertices* of paths with at least two vertices (and not the final vertices):

Lemma 5.5.8. *Let u be the initial vertex of a path with at least two vertices in $L'|_{G'\cup\{y_n,\ldots,y_{j+1}\}}$ for some index j ($1 \le j \le n$). Then $u \in \mathrm{Cval}$.*

Our definition of cycle closers here is different. Let $u_1, \ldots, u_{2n}$ be the vertices $[2n]$ arranged according to the variant DS order. We say that $u_j \in [2n]$ is a *cycle closer* if the edge $\sigma^{-1}(u_j) \to u_j$ is introduced in $L'|_{\{u_1,\ldots,u_{j-1}\}}$ as an edge between the two ends of a path turning the path to a cycle. The following lemma classifies all cycle closers.

Lemma 5.5.9. *(Classifying cycle closers) Given a D-permutation σ, an element $u \in [2n]$ is a cycle closer if and only if it is a cycle valley minimum, i.e., it is the smallest element in its cycle.*

Next, we will count the number of cycle closers. But before doing that, we require a technical lemma similar to Lemmas 5.4.6, 5.5.4. (As before, recall that if $i \in G \cap F' = \mathrm{Cval}(\sigma)$, step s_i must be a rise from height h_{i-1} to height h_i and hence, $h_{i-1}+1 = h_i$. Also, from the interpretation of the heights in equation (5.67) we must have $\lceil h_{i-1}/2 \rceil + 1 = \lceil (h_i+1)/2 \rceil = f_i$.)

Lemma 5.5.10. *Given a D-permutation σ and associated sets F, G, F', G' where $F' = \{y_1 < \ldots < y_n\}$, and an index j ($1 \le j \le n$) such that $y_j \in G \cap F' = \mathrm{Cval}$. Then the following is true:*

$$\#\{u \in F \backslash \{\sigma^{-1}(y_n), \ldots, \sigma^{-1}(y_{j+1})\} \colon\ u > y_j\} = \lceil h_{y_j-1}/2 \rceil + 1 = \lceil (h_{y_j}+1)/2 \rceil = f_{y_j} \tag{5.96}$$

where h_i denotes the height at position i of the almost-Dyck path ω associated to σ in Step 1 and f_i is defined in (5.68).

PROOF. Notice the equality of the following sets:

$$\{u \in F \backslash \{\sigma^{-1}(y_n), \ldots, \sigma^{-1}(y_{j+1})\} \colon\ u > y_j\} \;=\; \{u > y_j : \sigma(u) \le y_j\}. \tag{5.97}$$

Next use (5.68) to notice that the set on the right hand side of (5.97) has cardinality f_{y_j}. $\square$

We are now ready to count the number of cycle closers.

Lemma 5.5.11 (Counting of cycle closers for D-permutations using variant labels)**.** *Fix an almost-Dyck path ω of length $2n$ and construct F', G' (these are completely determined by ω). Let $y_j \in G \cap F'$. Also fix labels $\widehat{\xi}_u$ for vertices $u \in G' \cup \{y_n, \ldots, y_{j+1}\}$ satisfying* (5.94)*. Then*

(a) The value of $\widehat{\xi}_{y_j}$ completely determines if y_j is a cycle closer or not.

(b) There is exactly one value $\widehat{\xi}_{y_j} \in \{0, 1, \ldots, \lceil h_{y_j - 1}/2 \rceil\}$ that makes y_j a cycle closer, and conversely.

5.5.2.3 Computation of weights

PROOF OF THEOREM 5.3.7. The computation of weights is completely analogous to what was done in Section 5.5.1.3, but using Lemma 5.5.11 in place of Lemma 5.5.5. We leave the details to the reader: the upshot (similar to the proof of Theorem 4.2.10) is that for cycle valleys and cycle peaks, "u" and "l" are interchanged compared to Section 5.5.1.3, and all the statistics are primed. It therefore completes the proof of Theorem 5.3.7. □

PROOF OF THEOREMS 5.3.6 AND 5.3.5. Comparing (5.36) with (5.38) we see that the needed specialisation in (5.38) are the same as given in (5.83)–(5.88). Inserting these into Theorem 5.3.7 gives Theorem 5.3.6.

Similarly, the proof of Theorem 5.3.5 follows by specialising the weights in Theorem 5.3.6 to (5.89). □

5.6 Final remarks

We began this work only hoping to prove [SZ22, Conjecture 2.3]. Our initial guess was that this would involve constructing a new bijection from permutations to labelled Motzkin paths, possibly by tweaking the Biane bijection [Bia93]. However, on discovering our proof, we were surprised to see that not only did we not construct any new bijection, but we used the same variant of the Foata–Zeilberger bijection, which Sokal and Zeng use to prove their "first" continued fractions for permutations. As we remarked in Section 4.7, the proofs of our "first" theorems for D-permutations in Chapter 4 are *parallel* to the proofs of the "first" theorems in [SZ22] for permutations; this led us to prove [RZ96a, Conjecture 12] and also Theorem 4.3.1 which we had only conjectured while working on Chapter 4 and then proved it much later while working on the present chapter.

We had to introduce a new total order on $[n]$ ($[2n]$ for D-permutations) to describe histories for these bijections. The crucial reason why our proof works is that the total order is the same for any given Motzkin path (almost-Dyck path for D-permutations), and also because the associated weights are commutative. Thus, the order in which we multiply them has no effect on the product as long as we stick to the same order for any given path.

On the other hand, Flajolet [Fla80] provides a more general combinatorial interpretation with non-commutative weights, as long as these weights are multiplied using the natural order of the path. With this in mind, we think that it will probably not be too difficult to generalize the "second" theorems for permutations in [SZ22], and for D-permutations in Chapter 4 to obtain continued fractions with non-commutative weights. We predict that this will also be possible for continued fractions obtained using the Françon-Viennot bijection [FV79]. However, why such continued fractions might be of interest is not immediate to us, and at the present moment, we refrain from working out the details.

Chapter 6

Higher-order Stirling cycle and subset triangles: Total positivity, continued fractions and real-rootedness[1]

6.1 Introduction and statement of main results

Let us recall that a (finite or infinite) matrix of real numbers is called totally positive (TP) if all its minors are nonnegative. A matrix of polynomials with real coefficients is called coefficientwise-totally positive is all its minors are polynomials with nonnegative coefficients. Let $A = \big(a_{n,k}\big)_{n,k\geq 0}$ be a lower-triangular matrix of non-negative real numbers. In Sections 1.1 and 2.1.6 we asked the following four questions:

(a) Is A totally positive?

(b) Is the lower-triangular matrix $A^{\mathrm{rev}} \stackrel{\mathrm{def}}{=} (a_{n,n-k})_{n,k\geq 0}$ obtained by reversing the rows of A totally positive? (Here the entry $a_{n,n-k} := 0$ when $n < k$.)

(c) Let $A_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$ denote the row generating polynomial of the n-th row of A. Are the polynomials $A_n(x)$ real-rooted? This is equivalent to asking if the Toeplitz matrix of the n-th row sequence $(a_{n,k})_{k\geq 0}$ is totally positive

[1]The work in this chapter was largely in collaboration with Alan D. Sokal. See pp. 18-19 for details.

[ASW52, Edr52]. We then say that the sequence $(a_{n,k})_{k\geq 0}$ is Toeplitz-totally positive[2].

(d) Is the Hankel matrix of the polynomial sequence $(A_n(x))_{n\geq 0}$ coefficientwise-totally positive in the variable x? We then say that the sequence $(A_n(x))_{n\geq 0}$ is coefficientwise-Hankel-totally positive. One sufficient but far-from-necessary condition for this to hold is for the ordinary generating function $\sum_{n=0}^{\infty} A_n(x)t^n$ to have a Stieltjes-type continued fraction whose coefficients are polynomials with variable x with non-negative coefficients, see Theorem 2.1.3.

We then saw a few examples of triangles for which all four of these questions are either known to be true or are conjectured to be true. Let us recall two of them here:

(a) When $A = \left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n,k\geq 0}$ is the matrix of Stirling cycle numbers, where the (n,k)-th entry counts the number of permutations on $[n]$ with k cycles. The total positivity of the matrix A and its reversal A^{rev} were shown by Brenti in 1995 [Bre95]. The row generating polynomials $A_n(x) = x(x+1)\cdots(x+n-1)$ are clearly real-rooted with non-positive roots. The sequence $(A_n(x))_{n\geq 0}$ has a Stieltjes-type continued fraction discovered by Euler in 1760 [Eul60], also see [SZ22, eq. (2.2), (2.6)].

(b) When $A = \left(\left\{\begin{matrix} n \\ k \end{matrix}\right\}\right)_{n,k\geq 0}$ is the matrix of Stirling subset numbers, which count the number of set partitions on $[n]$ with k blocks. The total positivity of the matrix A is due to Brenti from 1995 [Bre95]. However, the total positivity of the reversal A^{rev} was only recently shown by Chen *et al.* in [CDD$^+$21]. The real-rootedness of the row generating polynomials $A_n(x)$ is due to Harper from 1967 [Har67]. The sequence $(A_n(x))_{n\geq 0}$ has a Stieltjes-type continued fraction [SZ22, eq. (3.2), (3.5)].

In this chapter, we introduce two infinite families of lower-triangular matrices generalising the Stirling cycle and subset triangles; we call these the *higher-order*

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[2] Sequences whose Toeplitz matrices are totally positive are also known as Pólya frequency sequences.

Stirling cycle and *subset triangles*. The central theme of this chapter is to investigate the above four questions for these triangular arrays. We have used an experimental approach for our investigation to obtain our conjectures. Even though we can prove very few of our conjectures, our results suggest various interesting consequences in the form of combinatorial interpretations and also new conjectures.

The rest of the introduction is structured as follows: In Section 6.1.1 we define the r-th order Stirling cycle and subset numbers, we set up basic notation for the rest of the chapter and state some recurrences that these numbers satisfy. In Section 6.1.2, we state our main conjectures and theorems. The outline of the rest of the chapter is then mentioned in Section 6.1.3.

6.1.1 Higher order Stirling cycle and subset numbers

There are several generalisations and extensions of the Stirling cycle and subset numbers. One such family of generalisations for the Stirling cycle numbers are the r-associated Stirling cycle numbers $\left[{n \atop k}\right]_r$, where $\left[{n \atop k}\right]_r$ denotes the number of permutations on $[n]$ with k cycles such that each cycle has at least r elements (see [Com74, pp. 256,257,295]). Thus we have $\left[{n \atop k}\right]_1 = \left[{n \atop k}\right]$. The first few 2-associated Stirling cycle numbers starting at $n = 2$ are given below [OEI19, A008306]:

$$\begin{array}{cccc} 1 & & & \\ 2 & & & \\ 6 & 3 & & \\ 24 & 20 & & \\ 120 & 130 & 15 & \\ 720 & 924 & 210 & \\ 5040 & 7308 & 2380 & 105 \end{array} \tag{6.1}$$

Note that $\left[{n \atop k}\right]_r$ is nonzero only when $n \geq rk$, or equivalently $k \leq \lfloor n/r \rfloor$.

Similarly, the r-associated Stirling subset numbers $\left\{{n \atop k}\right\}_r$ are defined to be the number of set partitions of $[n]$ into k blocks such that each block has at least r elements (see [Com74, pp. 221,222]). Thus we have $\left\{{n \atop k}\right\}_1 = \left\{{n \atop k}\right\}$. The first few

2-associated Stirling subset numbers starting at $n = 2$ are given below [OEI19, A008299]:

$$\begin{array}{llll} 1 & & & \\ 1 & & & \\ 1 & 3 & & \\ 1 & 10 & & \\ 1 & 25 & 15 & \\ 1 & 56 & 105 & \\ 1 & 119 & 490 & 105 \end{array} \tag{6.2}$$

Note that $\left\{ n \atop k \right\}_r$ is nonzero only when $n \geq rk$, or equivalently $k \leq \lfloor n/r \rfloor$.

For $r \geq 2$, the Hankel matrices of the sequence of row-generating polynomials of these triangles are *not* coefficientwise-totally positive. This is because the n-th row-generating polynomial is of degree $\lfloor n/r \rfloor$ — *not* n — and one can show that even the 2×2 minors of the Hankel matrix of the row-generating polynomials must have some negative coefficients.

Instead, we modify these triangles so that the columns are shifted up to make the diagonal entries nonzero, i.e., for a fixed $r \geq 1$, consider the lower-triangular matrices $\left(\left[n \atop k \right]^{(r)} \right)_{n,k \geq 0}$ and $\left(\left\{ n \atop k \right\}^{(r)} \right)_{n,k \geq 0}$ where the (n,k)-entries are defined by:

$$\left[n \atop k \right]^{(r)} \stackrel{\text{def}}{=} \left[n+(r-1)k \atop k \right]_r \tag{6.3}$$

$$\left\{ n \atop k \right\}^{(r)} \stackrel{\text{def}}{=} \left\{ n+(r-1)k \atop k \right\}_r \tag{6.4}$$

and we set $\left[0 \atop 0 \right]^{(r)} = \left\{ 0 \atop 0 \right\}^{(r)} = 1$. We call the numbers $\left[n \atop k \right]^{(r)}$ and $\left\{ n \atop k \right\}^{(r)}$ as the *r-th order Stirling cycle and subset numbers*, respectively. We then clearly have,

$$\left[n \atop k \right]^{(1)} = \left[n \atop k \right]_1 = \left[n \atop k \right] \tag{6.5}$$

$$\left\{ n \atop k \right\}^{(1)} = \left\{ n \atop k \right\}_1 = \left\{ n \atop k \right\} \tag{6.6}$$

and thus, the numbers of order 1 of both kinds are the usual Stirling cycle and subset numbers, respectively. For the convenience of the reader and for our utility in the rest of this chapter, we rewrite (6.3)/(6.4) for $r = 2$:

$$\left[{n \atop k}\right]^{(2)} = \left[{n+k \atop k}\right]_2 \tag{6.7}$$

$$\left\{{n \atop k}\right\}^{(2)} = \left\{{n+k \atop k}\right\}_2 . \tag{6.8}$$

For a fixed $r \geq 1$, let $C^{(r)}$ and $S^{(r)}$ be the lower-triangular matrices defined by

$$C^{(r)} \stackrel{\text{def}}{=} \left(\left[{n \atop k}\right]^{(r)}\right)_{n,k\geq 0} \tag{6.9a}$$

$$S^{(r)} \stackrel{\text{def}}{=} \left(\left\{{n \atop k}\right\}^{(r)}\right)_{n,k\geq 0} . \tag{6.9b}$$

The triangle $C^{(2)}$ is [OEI19, A259456, A269940, A111999] and its row sums are [OEI19, A032188]. The triangle $S^{(2)}$ is [OEI19, A134991] and its row sums are [OEI19, A000311]. The second-order Stirling subset numbers are usually referred to as the *Ward numbers*.

We now provide a recurrence relation for the Stirling cycle and subset numbers of all orders.

Lemma 6.1.1. (a) The numbers $\left[{n \atop k}\right]^{(r)}$ satisfy the recurrence relation:

$$\left[{n \atop 0}\right]^{(r)} = \delta_{n,0} \tag{6.10a}$$

$$\left[{n \atop k}\right]^{(r)} = (r-1)!\binom{n+(r-1)k-1}{r-1}\left[{n-1 \atop k-1}\right]^{(r)} + \big(n+(r-1)k-1\big)\left[{n-1 \atop k}\right]^{(r)} \qquad \text{for } n,k \geq 1. \tag{6.10b}$$

(b) The numbers ${n \brace k}^{(r)}$ satisfy the recurrence relation:

$$
\begin{aligned}
{n \brace 0}^{(r)} &= \delta_{n,0} && (6.11a)\\
{n \brace k}^{(r)} &= \binom{n+(r-1)k-1}{r-1} {n-1 \brace k-1}^{(r)} + k {n-1 \brace k}^{(r)} \\
&\qquad \text{for } n,k \geq 1. && (6.11b)
\end{aligned}
$$

The recurrences (6.10), (6.11) are straightforward consequences of well-known recurrences for the r-associated Stirling cycle [Com74, (7) p. 257] and subset [Com74, p. 222] numbers. However, for the sake of completeness, we provide a combinatorial proof.

PROOF OF LEMMA 6.1.1. *(a)* For a fixed $r \geq 1$, let $\left[{[n] \atop k}\right]_r$ denote the set of all permutations on $[n]$ with k cycles such that each cycle has at least r elements. Now let $\sigma \in \left[{[n+(r-1)k] \atop k}\right]_r$ and consider the status of the cycle containing n: it either has exactly r elements or it has $r+1$ or more elements. In the former case, we remove the cycle containing n and rename the elements in the other cycles to obtain a permutation of $[(n-1)+(r-1)(k-1)]$ into $k-1$ blocks, each containing at least r elements. In the latter case, we remove n from its cycle to obtain a permutation of $[(n-1)+(r-1)k]$ into k blocks, each containing at least r elements.

Conversely, to get a permutation $\sigma \in \left[{[n+(r-1)k] \atop k}\right]_r$ with n in a cycle of size exactly r, we first pick $r-1$ elements from the set $[n+(r-1)k-1]$ and then select a cycle in one of $(r-1)!$ possible ways, the total number of choices in this case is $(r-1)!\binom{n+(r-1)k-1}{r-1}\left[{n-1 \atop k-1}\right]^{(r)}$. On the other hand, to get a permutation $\sigma \in \left[{[n+(r-1)k] \atop k}\right]_r$ with n in a cycle of size at least $r+1$, we pick a permutation $\tau \in \left[{[n-1+(r-1)k] \atop k}\right]_r$ and insert n into one of the cycles by picking $\sigma^{-1}(n)$ in $(n+(r-1)k-1)$ ways, the total number of choices in this case is $(n+(r-1)k-1){n-1 \brace k}^{(r)}$. Adding the number of choices in both cases proves the recurrence (6.10).

(b) We leave this as an exercise for the reader. □

When $r = 2$, the recurrences (6.10b)/(6.11b) are

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(2)} = (n+k-1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{(2)} + (n+k-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}^{(2)} \tag{6.12a}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}^{(2)} = (n+k-1)\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}^{(2)} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}^{(2)} \tag{6.12b}$$

Note that the coefficients here are affine in n and k (this is no longer the case for $r \geq 3$). It is curious that we see here the combination $n+k$, in contrast with the $n-k$ occurring in (3.2)/(3.4)/(3.5).

We use $c_{r,n}(x)$ to denote the row-generating polynomial of the n-th row of the matrix $C^{(r)}$; we call them the *r-th order Stirling cycle polynomials*. Similarly, we use $s_{r,n}(x)$ to denote the row-generating polynomial of the n-th row of the matrix $S^{(r)}$; we call them the *r-th order Stirling subset polynomials*. Thus, we have

$$c_{r,n}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{(r)} x^k, \tag{6.13}$$

$$s_{r,n}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}^{(r)} x^k. \tag{6.14}$$

6.1.2 Statement of main conjecture and results

Having defined the Stirling cycle and subset numbers of all orders r, we are now ready to state our conjectures, which constitute our main contribution in this chapter. We will then state which of these cases are known and which of these we can prove here.

Our conjecture for the Stirling cycle numbers of all orders is the following:

Conjecture 6.1.2 (Conjecture for Stirling cycle numbers of all orders)**.** The following are true:

(a) The triangle $C^{(r)}$ is totally positive for all $r \geq 1$.

(b) The triangle $\check{C}^{(r)} \stackrel{\text{def}}{=} \left(C^{(r)}\right)^{\text{rev}}$ formed by reversing the rows of $C^{(r)}$ is totally positive for $r = 1$ and $r = 2$ and is not totally positive when $r \geq 3$.

(c) The row-generating polynomials $c_{r,n}(x)$ are real-rooted and have non-positive zeroes when $r=1$ and $r=2$, and they have non-real complex zeroes for $r \geq 3$. Also, the row sequences of the matrix $C^{(r)}$ are log-concave for $1 \leq r \leq 5$.

(d) The sequence $(c_{r,n}(x))_{n\geq 0}$ is coefficientwise-Hankel-totally positive in the variable x for all $r \geq 1$.

As mentioned earlier, the $r=1$ cases of Conjecture 6.1.2 are known to be true. In this chapter, we will prove *(c)* and *(d)* for $r=2$.

Our conjecture for the Stirling subset numbers of all orders is the following:

Conjecture 6.1.3 (Conjecture for Stirling subset numbers of all orders)**.** The following are true:

(a) The triangle $S^{(r)}$ is totally positive for all $r \geq 1$.

(b) The triangle $\breve{S}^{(r)} \stackrel{\text{def}}{=} \left(S^{(r)}\right)^{\text{rev}}$ formed by reversing the rows of $S^{(r)}$ is totally positive for $r=1$ and $r=2$ and is not totally positive when $r \geq 3$.

(c) The row-generating polynomials $s_{r,n}(x)$ are real-rooted and have non-positive zeroes when $r=1$ and $r=2$ and they have non-real complex zeroes for $r \geq 3$. Also, the row sequences of the matrix $S^{(r)}$ are log-concave for $1 \leq r \leq 5$.

(d) The sequence $(s_{r,n}(x))_{n\geq 0}$ is coefficientwise-Hankel-totally positive in the variable x for $r=1$ and $r=2$ and is not coefficientwise-Hankel-totally positive for $r \geq 3$.

As mentioned earlier, the $r=1$ cases of Conjecture 6.1.3 are known to be true.

The sequence of polynomials $(s_{2,n}(x))_{n\geq 0}$ were studied by Elvey Price and Sokal in [PS20], they called them the Ward polynomials (compare the recurrence (6.12b) with [PS20, eq. (1.7),(1.8)]). It was shown [PS20, Theorem 1.1] that these polynomials have a Thron-type continued fraction (T-fractions were introduced in Section 2.2.1). It then follows from Theorem 2.3.1 that this sequence of polynomials $(s_{2,n}(x))_{n\geq 0}$ is coefficientwise-Hankel-totally positive in the variable x and thus, *(d)* is true for $r=2$. In this chapter, we will prove *(c)* for the case $r=2$.

Our main result on Hankel-total positivity is the following:

Theorem 6.1.4. The sequence of second-order Stirling cycle polynomials $(c_{2,n}(x))_{n\geq 0}$ is coefficientwise-Hankel-totally positive in the variable x.

We will prove Theorem 6.1.4 by bijectively showing that they are equal to certain multivariate generalisations of the Eulerian polynomials introduced by Pétréolle, Sokal and Zhu [PSZ18]. We will recall the precise definitions in Section 6.2.2 but we state our result here:

Proposition 6.1.5. The second-order Stirling cycle polynomial $c_{2,n}(x)$ is equal to the multivariate Eulerian polynomial $\mathcal{P}_2^{(2)}(x, 1+x, 1+x)$ (defined in [PSZ18, sec 12.2.1]) i.e.,

$$c_{2,n}(x) \;=\; \mathcal{P}_2^{(2)}(x, 1+x, 1+x)\;. \tag{6.15}$$

We will show that Theorem 6.1.4 follows from Proposition 6.1.5 by using [PSZ18, Equation (12.5), Theorems 12.1(a) and 9.8]; we will state this combination in Theorem 6.2.2. Our proof of Proposition 6.1.5 will involve two steps: In Section 6.2.1 we will introduce a second combinatorial interpretation for the numbers $\left[{n \atop k}\right]^{(2)}$, using Stirling permutations, due to Callan [Cal]. Then in Section 6.2.2, we will introduce a third combinatorial interpretation for these numbers in terms of increasing ternary trees, using a well-known bijection to Stirling permutations; Proposition 6.1.5 and then Theorem 6.1.4 will follow from this.

In Section 6.3, we introduce two more combinatorial interpretations for the numbers $\left[{n \atop k}\right]^{(2)}$; thus, we provide a total of five combinatorial interpretations for these numbers in this chapter. Even though these interpretations are not necessary for our proof of Theorem 6.1.4, and we are unable at present to see how to use them to prove any of our other main conjectures, they are still interesting in their own right and provide interesting consequences and also suggest new conjectures. In Section 6.3.1, we introduce our fourth interpretation in terms of certain families of weighted increasing trees with ordered children, and we provide two different proofs: one bijective and one using generating functions. Both proofs suggest new lines of investigation, and also show that the hypothesis in [PS21, Theorem 1.1] due to Pétréolle and Sokal is sufficient but far from necessary. In Section 6.3.2 we introduce our fifth and final interpretation in terms of certain leaf-labelled trees. We then show the relation to multivariate Ward polynomials introduced by Elvey Price and Sokal [PS20, pp. 9-11], and we show that these multivariate Ward polynomials are a common generalisation of both the second-order Stirling cycle polynomials $c_{2,n}(x)$ and also of the second-order Stirling subset polynomials $s_{2,n}(x)$. This observation very naturally leads us to another triangle of numbers which also seems to satisfy our four positivity questions; we state this in Conjecture 6.3.10.

Our second main theorem on real-rootedness is the following:

Theorem 6.1.6. For every $n \geq 1$, the polynomials $c_{2,n}(x)$, $s_{2,n}(x)$ have 0 as a root and the zeroes of the polynomial sequence $\big(c_{2,n}(x)/x\big)_{n\geq 1}$, and also the sequence $\big(s_{2,n}(x)/x\big)_{n\geq 1}$, are simple, interlacing, and they lie in the interval $(-1,0)$.

Thus, in Theorem 6.1.6 we prove that not only are the zeroes non-positive, but in fact, we show that the zeroes lie in an interval.

As mentioned previously, it follows from the work of Aissen, Schoenberg and Whitney [ASW52], and Edrei [Edr52] that for any fixed n and for $r = 1,2$, the real-rootedness of the polynomial $c_{r,n}(x)$, (or of the polynomial $s_{r,n}(x)$) implies that the lower-triangular Toeplitz matrix $\left(\left[{n \atop i-j}\right]^{(r)}\right)_{i,j\geq 0}$ (the matrix $\left(\left\{{n \atop i-j}\right\}^{(r)}\right)_{i,j\geq 0}$, respectively) is totally positive. The positivity of the contiguous 2×2 minors of these Toeplitz matrices gives us the following corollary:

Corollary 6.1.7. The row sequences of the r-th order Stirling cycle and subset matrices $C^{(r)}$ and $S^{(r)}$ are log-concave for $r = 1,2$.

One can show that Theorem 6.1.6 follows very easily from a very general theorem due to Liu and Wang [LW07]. However, for the convenience of the reader, we will provide a direct proof of our theorem in Section 6.4.

Even though the zeroes of the polynomials $c_{r,n}(x)$ and $s_{r,n}(x)$ do not lie on the real line for $r \geq 3$, it might be worthwhile to try and understand how they are distributed on the complex plane. In Section 6.5, we will provide some plots for the distribution of zeroes of the polynomials for orders $r \geq 3$. These plots will suggest that these zeroes, when correctly normalised, have some mysterious structure; they seem to accumulate around some limiting curves. However, at this moment we are unable to provide any proofs describing these distributions.

6.1.3 Structure of chapter

The plan of the rest of this chapter is as follows: In Section 6.2, we prove the coefficientwise-Hankel-total positivity of the second-order Stirling cycle polynomials (Theorem 6.1.4). Then, in Section 6.3, we provide two further combinatorial interpretations for the second-order Stirling cycle numbers; this leads us to some interesting observations and another conjecture. Next, in Section 6.4, we prove Theorem 6.1.6. Finally, in Section 6.5 we provide some plots for the zeroes of r-th order Stirling cycle and subset polynomials for orders $r \in [3, 10]$.

6.2 Coefficientwise Hankel-total positivity of the second-order Stirling cycle polynomials

We will prove Theorem 6.1.4 in this section. The proof consists of the following two steps: In Section 6.2.1 we will first establish our second combinatorial interpretation for the number $\left[{n \atop k}\right]^{(2)}$: it counts Stirling permutations of the multiset $\{1,1,2,2,\ldots,n,n\}$ with optional dots at ascents and containing $n-k$ dots. (The full definitions will be provided in Section 6.2.1.) Then our second step is in Section 6.2.2, where we show that ascent-marked Stirling permutations are in bijection with a certain class of increasing ternary trees on $n+1$ vertices. This will give us our third combinatorial interpretation for the numbers $\left[{n \atop k}\right]^{(2)}$ and will enable us to prove Proposition 6.1.5. Finally, we will see that the theory of branched continued fractions [PSZ18] will immediately show that Proposition 6.1.5 implies Theorem 6.1.4.

6.2.1 Interpretations of the second-order Stirling cycle numbers

Let us recall that the numbers $\left[{n \atop k}\right]^{(2)} = \left[{n+k \atop k}\right]_2$ were defined to be the number of permutations on $[n+k]$ containing k cycles, none of which are singletons. These numbers satisfy the recurrence (6.12a). We will now provide another interpretation for these numbers: namely, as the number of Stirling permutations with optional dots in ascents. This interpretation can be guessed from the entries in [OEI19, A032188], and the ensuing proof is due to David Callan [Cal].

A *Stirling permutation of size n* is a permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ with the *Stirling property* — for any i such that $1 \le i \le n$, all entries between the two occurrences of i exceed i. Equivalently, a Stirling permutation of size n is permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ that avoids the pattern 212. We recall [GS78] that there are $(2n-1)!!$ Stirling permutations of size n. An *ascent-marked Stirling permutation of size n* is a Stirling permutation in which a dot may or may not be inserted at each ascent. We let $\mathcal{V}_{n,k}$ denote the set of ascent-marked Stirling permutations of size n with k dots. Thus,

$$\mathcal{V}_{2,0} \;=\; \{1122, 1221, 2211\} \tag{6.16a}$$

$$\mathcal{V}_{2,1} \;=\; \{11\cdot 22, 1\cdot 221\}. \tag{6.16b}$$

Let $v_{n,k} = |\mathcal{V}_{n,k}|$.

We will prove the following lemma:

Lemma 6.2.1. The numbers $v_{n,k}$ satisfy the following recurrence:

$$v_{n,k} \;=\; (2n-k-1)\big(v_{n-1,k} + v_{n-1,k-1}\big) \tag{6.17}$$

and $v_{0,k} = \delta_{0,k}$, and $v_{n,k} = 0$ when $k < 0$. We then have $\left[{n \atop k}\right]^{(2)} = v_{n,n-k}$.

PROOF. Let $\sigma \in \mathcal{V}_{n,k}$ be an ascent-marked Stirling permutation of size n with k dots. Deleting the two n's (which are necessarily adjacent) together with the dot (if any) preceding them gives an element of $\mathcal{V}_{n-1,k}$ if there was no dot before nn, and an element of $\mathcal{V}_{n-1,k-1}$ if there was a dot before nn. Conversely, to get an element σ of $\mathcal{V}_{n,k}$ with nn not preceded by a dot from a one-size-smaller ascent-marked Stirling permutation, nn can be inserted in any of the $2n-1$ spaces except those occupied by a dot in an element of $\mathcal{V}_{n-1,k}$: thus, there are $(2n-1-k)$ choices. To get a σ with a dot followed by nn, $\cdot nn$ can be inserted in any of the $2n-1$ spaces except the first space and those occupied by a dot in an element of $\mathcal{V}_{n-1,k-1}$: again the number of choices is $(2n-1-k)$. This proves the recurrence (6.17).

From the recurrence (6.17) it follows that the numbers $u_{n,k} = v_{n,n-k}$ satisfy the recurrence

$$u_{n,k} \;=\; (n+k-1)(u_{n-1,k-1}+u_{n-1,k}) \tag{6.18}$$

which is the same as the recurrence (6.12a) satisfied by the numbers $\left[{n \atop k}\right]^{(2)}$. This finishes the proof. □

6.2.2 Proof of Proposition 6.1.5 and Theorem 6.1.4

We will prove Theorem 6.1.4 in this section. We do this by using the theory of branched continued fractions introduced in [PSZ18], see Section 2.3 for a summary of some of the results. Our approach will be as follows: we will first state [PSZ18, Theorem 6.2.2] and introduce the necessary concepts to understand the statement of this theorem. We then provide a bijective proof of Proposition 6.1.5. This proof will involve a bijection between certain increasing ternary trees and ascent-marked Stirling permutations introduced in Section 6.2.1. Finally, we will see that Theorem 6.1.4 is an easy consequence of these two results.

For the convenience of the reader, we now state all the necessary details. For a fixed integer $m \geq 2$, an m-ary tree is a rooted tree with finitely many vertices such that each vertex has 0 or 1 first child, 0 or 1 second child, and so on until 0 or 1 m-th child. An increasing m-ary tree on $n+1$ vertices is an m-ary tree with $n+1$ vertices such that each vertex is given a distinct label from the label set $[0,n]$, such that the label of a vertex is always larger than the label of its parent. For a fixed integer $m \geq 1$ and indeterminates $\mathbf{x} = (x_0, x_1, \ldots, x_m)$ we let $Q_n^{(m)}(\mathbf{x})$ be the generating polynomial for increasing $(m+1)$-ary trees on $n+1$ vertices with labels $[0,n]$ in which each i-edge, an edge joining a vertex with its $(i+1)$-th child, gets a weight x_i. We then define $\mathcal{P}_n^{(m)}(\mathbf{x})$ by

$$\mathcal{P}_0^{(m)}(\mathbf{x}) = 1 \tag{6.19a}$$

$$\mathcal{P}_n^{(m)}(\mathbf{x}) = x_0 \mathcal{Q}_{n-1}^{(m)}(\mathbf{x}) \qquad \text{for } n \geq 1 \tag{6.19b}$$

Therefore, $\mathcal{P}_n^{(m)}(\mathbf{x})$ is the generating polynomial for increasing $(m+1)$-ary trees on on $n+1$ vertices with labels $[0,n]$ such that the only edge (if any) emanating from the root 0 is a 0-edge and in which each i-edge gets a weight x_i. These polynomials were called the multivariate Eulerian polynomials in [PSZ18].

In [PSZ18, Theorem 9.8], Pétréolle, Sokal and Zhu show that the sequence of m-Stieltjes–Rogers polynomials (see Section 2.3 for the definition) is coefficientwise Hankel-totally positive, see Theorem 2.3. Then in [PSZ18, Equation (12.5) and Theorem 12.1(a)] they prove that the multivariate Eulerian polynomials $\mathcal{P}_n^{(m)}(\mathbf{x})$ can be obtained by specialising m-Stieltjes–Rogers polynomials with coefficientwise-positive polynomials. This implies that the sequence $\left(\mathcal{P}_n^{(m)}(\mathbf{x})\right)_{n\geq 0}$ must also be coefficientwise Hankel-totally positive. We now state the combination [PSZ18, Equation (12.5), Theorems 12.1(a) and 9.8]:

Theorem 6.2.2 ([PSZ18, Equation (12.5), Theorems 12.1(a) and 9.8])**.** The polynomials $\mathcal{P}_n^{(m)}(\mathbf{x})$ are equal to the m-Stieltjes–Rogers polynomials $S_n^{(m)}(\boldsymbol{\alpha})$ where the weights $\boldsymbol{\alpha} = \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ are given by

$$\alpha_{m+j+pk} = (k+1)x_j. \tag{6.20}$$

Thus, the sequence of polynomials $\left(\mathcal{P}_n^{(m)}(\mathbf{x})\right)_{n\geq 0}$ is coefficientwise-Hankel-totally positive with respect to the variables $x_0, x_1, \ldots, x_m$.

Now, let $\mathcal{T}_n$ denote the set of all increasing ternary trees (each vertex can have zero-or-one left, middle or right children) on $n+1$ vertices with labels $\{0,\ldots,n\}$ such that the only edge (if any) emanating from the root 0 is a left edge. Thus,

$\mathcal{P}_n^{(2)}(x, 1+x, 1+x)$ is the generating polynomial of trees $T \in \mathcal{T}_n$ with edge-weights x to left edges, and $1+x$ to middle and right edges. We are now ready to prove Proposition 6.1.5 where we show that $c_{2,n}(x) = \mathcal{P}_n^{(2)}(x, 1+x, 1+x)$.

PROOF OF PROPOSITION 6.1.5. It is convenient to assign colours to the edges of our ternary trees in $\mathcal{T}_n$ such that each middle or right edge can be coloured either red or blue but each left edge must always be coloured red. We let $\widehat{\mathcal{T}}_{n,k}$ denote the set of such 2-coloured trees containing k red edges.

Thus, to show that equation (6.15) holds, it suffices to show that the cardinality of the set $\widehat{\mathcal{T}}_{n,k}$ is $\left[n \atop k \right]^{(2)}$. We will do this by showing that the set $\widehat{\mathcal{T}}_{n,k}$ is in bijection with $\mathcal{V}_{n,n-k}$, the set of ascent-marked Stirling permutations of size n with $n-k$ dots; our result will then follow from Lemma 6.2.1.

We must show that ascent-marked Stirling permutations of size n with $n-k$ dots are in bijection with increasing ternary trees on the vertex set $[0,n]$, coloured as explained above, containing k red edges. The first step is to use a classical bijection between Stirling permutations and increasing ternary trees [Par94, JKP11][3]. We now sketch how this bijection is constructed: Let $\sigma = a_1 \cdots a_{2n}$ be any word on the alphabet $\mathbb{Z}_{\geq 1}$ such that each letter, if it occurs in σ, occurs exactly twice. Also assume that σ has the Stirling property, i.e., all entries of σ between the two occurrences of a exceed a. We then recursively define a ternary tree $T(\sigma)$ as follows:

- $T(\sigma)$ is the empty tree if and only if σ is the empty word.
- Otherwise, let a be the smallest letter of σ and also let $\sigma = \sigma_0 \, a \, \sigma_1 \, a \, \sigma_2$. Then $T(\sigma)$ is the ternary tree rooted at a with $T(\sigma_0), T(\sigma_1)$ and $T(\sigma_2)$ as the left, middle and right subtrees of a, respectively.

From the description of this construction, it is clear that $\sigma \mapsto T(\sigma)$ is a bijection from Stirling permutations on $\{1,1,2,2,\ldots,n,n\}$ to increasing ternary trees on $[n]$.

From the description of this bijection, it is clear that the first occurrence of the letter a is the first element of an ascent pair (a pair a_i, a_{i+1} such that $a_i < a_{i+1}$) if and

[3]This bijection was first mentioned by Park in [Par94] who attributed it to Gessel. The details of this bijection were first explicitly stated by Janson, Kuba and Panholzer in [JKP11].

only if σ_1 is not the empty word, which is true if and only if the vertex labelled a has a middle edge in the ternary tree $T(\sigma)$. Similarly, the second a is the first element of an ascent pair if and only if σ_2 is not the empty word, which is true if and only if the vertex labelled a has a right edge in the ternary tree $T(\sigma)$. Thus, the ascents in σ are in correspondence with the middle and right edges of $T(\sigma)$.

Thus, for an ascent-marked Stirling permutation $\tau \in \mathcal{V}_{n,n-k}$ we begin by constructing the ternary tree $T(\tau')$ where τ' is the underlying Stirling permutation of τ. We then put τ' as the left subtree of a root vertex labelled 0. Next, we colour the middle edge emanating from a vertex i as blue if the first occurrence of the letter i in τ is immediately followed by a dot, and we colour the right edge emanating from i blue if the second the vertex labelled i in τ is immediately followed by a dot. The remaining edges are coloured red. Since τ has $n-k$ dots, the resulting tree has $n-k$ blue edges and hence k red edges. Thus, we clearly have a bijection between the sets $\mathcal{V}_{n,k}$ and $\widehat{\mathcal{T}}_{n,n-k}$. This finishes the proof. □

PROOF OF THEOREM 6.1.4. This follows by combining Proposition 6.1.5 and Theorem 6.2.2. □

6.3 Two more combinatorial interpretations of the second-order Stirling cycle numbers

We will provide two more combinatorial interpretations for the second-order Stirling cycle numbers in this section. In Section 6.3.1, we introduce an interpretation involving a class of increasing labelled trees with ordered children. Then in Section 6.3.2, we introduce some leaf-labelled trees with cyclically-ordered children.

6.3.1 Interpretation as increasing ordered trees

We will now provide another interpretation of the second-order Stirling cycle numbers $\left[{n \atop k}\right]^{(2)}$ in terms of a different class of increasing trees.

Let $\mathcal{I}_n$ denote the set of increasing trees on the vertex set $[0,n]$ in which the children of each vertex are linearly ordered. Next, we consider trees $S \in \mathcal{I}_n$ and colour the vertices with colours red or blue, such that the internal vertices are always coloured red. Let $\widehat{\mathcal{I}}_{n,k}$ denote the set of such 2-coloured trees with k red vertices. In this section, we will show that the the cardinality of the set $\widehat{\mathcal{I}}_{n,k}$ is $\left[{n \atop k}\right]^{(2)}$. As preparation for this, recall that $\mathcal{T}_n$ denotes the set of all increasing ternary trees on the vertex set $[0,n]$ such that the only edge (if any) emanating from the root 0 is a left edge. Recall also that $\widehat{\mathcal{T}}_{n,k}$ denotes the set of trees in $\mathcal{T}_n$ along with a colouring of the edges, either blue or red, such that left edges are always coloured red, and which contain k red edges.

We will prove the following result:

Proposition 6.3.1. There exists a bijection $\Phi_n : \mathcal{T}_n \to \mathcal{I}_n$ such that for every tree $T \in \mathcal{T}_n$, the number of left edges of T is the same as the number of internal vertices in $\Phi_n(T)$.

We will provide two proofs of Proposition 6.3.1: we will give a bijective proof in Section 6.3.1.1 and a proof using generating functions in Section 6.3.1.2. Before providing the proofs, let us observe that using the bijection and the correspondence between left edges and internal vertices in Proposition 6.3.1, and we get the following corollary:

Corollary 6.3.2. The sets $\widehat{\mathcal{T}}_{n,k}$ and $\widehat{\mathcal{I}}_{n,k}$ are in bijective correspondence. Thus, the cardinality of $\widehat{\mathcal{I}}_{n,k}$ is given by

$$\left|\widehat{\mathcal{I}}_{n,k}\right| = \begin{bmatrix} n \\ k \end{bmatrix}^{(2)}. \tag{6.21}$$

PROOF OF COROLLARY 6.3.2 ASSUMING PROPOSITION 6.3.1. For a tree $S \in \mathcal{I}_n$, let $\mathrm{intvertex}(S)$ denote the number of internal vertices of S and let $\mathrm{leaf}(S)$ denote the number of leaves of S. Also, for a tree $T \in \mathcal{T}_n$, let $\mathrm{l}(T)$, $\mathrm{m}(T)$, $\mathrm{r}(T)$ denote the number of left, middle and right edges of T, respectively. Then we clearly obtain the following

$$\sum_{k=0}^{n} \left|\widehat{\mathcal{I}}_{n,k}\right| x^k = \sum_{S \in \mathcal{I}_n} x^{\mathrm{intvertex}(S)} (1+x)^{\mathrm{leaf}(S)} \quad (6.22a)$$

$$= \sum_{T \in \mathcal{T}_n} x^{\mathrm{l}(T)} (1+x)^{\mathrm{m}(T)} (1+x)^{\mathrm{r}(T)} \quad (6.22b)$$

$$= \sum_{k=0}^{n} \left|\widehat{\mathcal{T}}_{n,k}\right| x^k \quad (6.22c)$$

where (6.22a) follows from the definition of the sets $\mathcal{I}_n, \widehat{\mathcal{I}}_{n,k}$, (6.22b) follows from Proposition 6.3.1, and (6.22c) follows from the definition of the sets $\mathcal{T}_n, \widehat{\mathcal{T}}_{n,k}$. This completes the proof. $\square$

We will thus have, once Proposition 6.3.1 is proved, our fourth combinatorial interpretation for the numbers $\left[{n \atop k}\right]^{(2)}$.

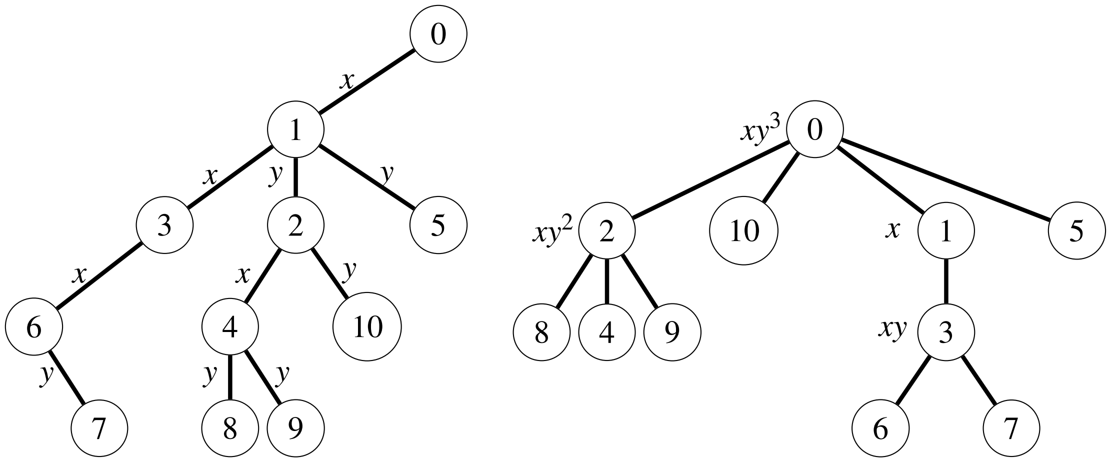
6.3.1.1 Bijective proof of Proposition 6.3.1

We will now construct a bijection $\Phi_n : \mathcal{T}_n \to \mathcal{I}_n$ and provide our first proof of Proposition 6.3.1. We will first construct the map $\Phi_n : \mathcal{T}_n \to \mathcal{I}_n$; we will then construct another map $\Psi_n : \mathcal{I}_n \to \mathcal{T}_n$; finally, we will show that the maps Φ_n and Ψ_n are inverses of each other, thus providing a bijective proof of Proposition 6.3.1. Before doing all of this, we first introduce some terminology on trees $T \in \mathcal{T}_n$ and $S \in \mathcal{I}_n$.

Let $T \in \mathcal{T}_n$ and let $j \in [0,n]$ be a vertex of T. Consider the subtree of T rooted at the vertex j which uses all the middle and right edges but no left edge. We call this tree the *middle-right subtree of j in T*. Next we assume that j is the left child of vertex i. We then use $\mathrm{left}_T(i)$ to denote the middle-right subtree of j in T. If instead, i is a vertex with no left child, we let $\mathrm{left}_T(i)$ be the empty tree. For example, in Figure 6.1(a), the tree $\mathrm{left}_T(1)$ is the subtree consisting of the vertex 3, and the tree $\mathrm{left}_T(2)$ is the subtree induced by the vertices $4, 8$ and 9.

It is easy to see that each vertex of T, other than the root 0, belongs to precisely one of the subtrees $\mathrm{left}_T(i)$; here we have used the fact that the root has no middle or right edge.

Next, let $S \in \mathcal{I}_n$ and let $i \in [0,n]$ be a vertex of S. Then we define the *child word* *of* i, denoted by $\mathrm{ch}_S(i)$, to be the word whose j-th letter is the j-th child of i. Thus, $\mathrm{ch}_S(i)$ is the empty word when i is a leaf. For example, in Figure 6.1(b), we have $\mathrm{ch}_S(1) = 3$ and $\mathrm{ch}_S(2) = 849$.



(a) An increasing ternary tree $T \in \mathcal{T}_{10}$

(b) An increasing ordered tree $S \in \mathcal{I}_{10}$

Figure 6.1: (a) This tree $T \in \mathcal{T}_{10}$ is given edge weights x, y, y for the left, middle and right edges respectively.
(b) This tree $S \in \mathcal{I}_{10}$ has $\phi_0 = 1$ and $\phi_i = xy^{i-1}$ for every $i \geq 1$, where ϕ_i is the weight of a vertex with i children. The leaves get weight 1.
Both trees have weight x^4y^6.

We need one more ingredient before describing the map Φ_n. This is the well-known correspondence between increasing binary trees on a totally-ordered finite vertex set A, and words w on the alphabet A with no repeated letters, see for example [Sta09, pp. 44-45]. However, in our version, instead of using binary trees with left and right edges, we will use binary trees with middle and right edges: If the word $w = \varnothing$, then $\mathrm{Tree}(w) = \varnothing$. Else, $w = u \cdot i \cdot v$ where i is the smallest letter of w. We recursively define $\mathrm{Tree}(w)$ to be the tree with root i whose middle subtree is $\mathrm{Tree}(u)$ and right subtree is $\mathrm{Tree}(v)$.

On the other hand, given an increasing binary tree T on the vertex set A made up of middle and right edges, we define $\mathrm{Word}(T)$ as follows: If $T = \varnothing$ then $\mathrm{Word}(T) = \varnothing$. Else, if T has root i with middle subtree T' and right subtree T'' (both of which may be empty), we define $\mathrm{Word}(T) = \mathrm{Word}(T') \cdot i \cdot \mathrm{Word}(T'')$.

We now state the following lemma which is clear from the construction of the maps $w \mapsto \text{Tree}(w)$ and $T \mapsto \text{Word}(T)$:

Lemma 6.3.3. (a) Let w be a word on the alphabet A with no repeated letters. Then $w = \text{Word}(\text{Tree}(w))$.

(b) Let T be an increasing tree on the vertex set A, made up of middle and right edges. Then $T = \text{Tree}(\text{Word}(T))$.

Construction of the map $\Phi_n : \mathcal{T}_n \to \mathcal{I}_n$.

For a given a tree $T \in \mathcal{T}_n$, we will now construct a tree $\Phi_n(T) \in \mathcal{I}_n$. To do this, we begin with a tree rooted at 0. We then iterate j from 0 through n and insert the children of j in $\Phi_n(T)$ as follows: We first consider the subtree $\text{left}_T(j)$ in T; it only consists of middle and right edges. The children of j in $\Phi_n(T)$ are then the letters of $\text{Word}\big(\text{left}_T(j)\big)$ which are inserted in order from left to right. Thus, we have

$$\text{ch}_{\Phi_n(T)}(j) \;=\; \text{Word}\big(\text{left}_T(j)\big)\,. \tag{6.23}$$

See Figure 6.1 for an example of a tree $T \in \mathcal{T}_{10}$ and $\Phi_{10}(T) \in \mathcal{I}_{10}$.

When $n = 0$, we clearly have that $\Phi_0(T) \in \mathcal{I}_0$. For $n \geq 1$, we will now show that $\Phi_n(T) \in \mathcal{I}_n$ by showing that every vertex $j \in [1,n]$ must have a parent in $\Phi_n(T)$ with label strictly smaller than j. To do this, we identify the parent of j in $\Phi_n(T)$. Consider the unique path from j to the root 0 in the tree T. Clearly, this path must contain at least one left edge as the root must have a left edge. Let $\widetilde{j}$ be the first vertex on this path which is reached from j using a left edge. We refer to $\widetilde{j}$ as *the left-ancestor of j*. Thus, we have $0 \leq \widetilde{j} < j$, and also, j is a vertex of $\text{left}\big(\widetilde{j}\big)$. For example, in Figure 6.1(a), the left-ancestor of 8 is 2, and the left-ancestor of 10 is 0.

We clearly obtain the following lemma:

Lemma 6.3.4. Let $j \in [1,n]$ be a vertex in T and let $\widetilde{j}$ be its left-ancestor. Then $\widetilde{j}$ is the parent of j in $\Phi_n(T)$.

From Lemma 6.3.4 and the fact that $0 \leq \widetilde{j} < j$, we get as a corollary the following:

Lemma 6.3.5. Given $T \in \mathcal{T}_n$, we have $\Phi_n(T) \in \mathcal{I}_n$.

This shows that Φ_n is a map from $\mathcal{T}_n \to \mathcal{I}_n$.

Construction of the map $\Psi_n : \mathcal{I}_n \to \mathcal{T}_n$.

For a given a tree $S \in \mathcal{I}_n$, we will now construct a tree $\Psi_n(S) \in \mathcal{T}_n$ as follows: We start with a tree rooted at 0. We then iterate through $i \in [0,n]$ and look at $\mathrm{ch}_S(i)$, the childword of i in S. If $\mathrm{ch}_S(i) = \varnothing$, then i will have no left edge in $\Psi_n(S)$. Otherwise, we create a left edge emanating out if i in $\Psi_n(S)$ and attach to it the binary tree $\mathrm{Tree}\big(\mathrm{ch}_S(i)\big)$ consisting of only middle and right edges. Thus, we have

$$\mathrm{left}_{\Psi_n(S)}(i) \;=\; \mathrm{Tree}\big(\mathrm{ch}_S(i)\big)\;. \tag{6.24}$$

When $n = 0$, we clearly have $\Psi_0(S) \in \mathcal{T}_0$. For $n \geq 1$, we will now show that $\Psi_n(S) \in \mathcal{T}_n$. In the construction of $\Psi_n(S)$, a vertex i gets a left child j only when $\mathrm{ch}_S(i) \neq \varnothing$ and when j is the root of $\mathrm{Tree}\big(\mathrm{ch}_S(i)\big)$, or in other words j is the smallest letter in $\mathrm{ch}_S(i)$ which is the same as saying that j is the smallest-numbered child of the vertex i in S. Thus, it is clear that every vertex i in $\Psi_n(S)$ can contain at most one left child j and that $j > i$. Next, notice that a vertex i in $\Psi_n(S)$ gets a middle (resp. right) child j only if i and j are both letters of the childword $\mathrm{ch}_S\big(i^\flat\big)$, for some vertex $i^\flat$ in T, and j is the middle (resp. right) child of i in $\mathrm{Tree}\big(\mathrm{ch}_S\big(i^\flat\big)\big)$. Thus, it is clear that every vertex i in $\Psi_n(S)$ can contain at most one middle child j and at most one right child j; furthermore, $j > i$ since $\mathrm{Tree}\big(\mathrm{ch}_S\big(i^\flat\big)\big)$ is an increasing tree. Also, 0 cannot have any middle or right children in $\Psi_n(S)$ as it does not have any siblings in S. This shows that:

Lemma 6.3.6. Given $S \in \mathcal{I}_n$, we have $\Psi_n(T) \in \mathcal{T}_n$.

We are now ready to prove Proposition 6.3.1.

PROOF OF PROPOSITION 6.3.1 USING BIJECTION. We will now show that the maps Φ_n and Ψ_n are inverses of each other. From equations (6.23), (6.24) and Lemma 6.3.3, we observe the following:

(a) For a tree $T \in \mathcal{T}_n$ and a vertex $j \in [0,n]$ of T, we have

$$\begin{aligned} \mathrm{left}_{\Psi_n(\Phi_n(T))}(j) &= \mathrm{Tree}(\mathrm{ch}_{\Phi_n(T)}(j)) && (6.25a) \\ &= \mathrm{Tree}(\mathrm{Word}(\mathrm{left}_T(j))) && (6.25b) \\ &= \mathrm{left}_T(j) \,. && (6.25c) \end{aligned}$$

Recall that every vertex $i \neq 0$ belongs to exactly one of the subtrees $\mathrm{left}_T(j)$ (resp. $\mathrm{left}_{\Psi_n(\Phi_n(T))}(j)$) in tree T (resp. $\Psi_n(\Phi_n(T))$). Thus, from equation (6.25) a vertex i is the middle (resp. right) child of a vertex $i^\flat$ in tree T if and only if i is the middle (resp. right) child of $i^\flat$ in tree $\Psi_n(\Phi_n(T))$. Also, a vertex i is the left child of a vertex $i^\flat$ in tree T (resp. $\Psi_n(\Phi_n(T))$) if and only if it is the root of the tree $\mathrm{left}_T(i^\flat)$ (resp. $\mathrm{left}_{\Psi_n(\Phi_n(T))}(i^\flat)$). Thus, from equation (6.25) we also get that a vertex i is the left child of a vertex $i^\flat$ in tree T if and only if i is the left child of $i^\flat$ in tree $\Psi_n(\Phi_n(T))$. Thus, $\Psi_n(\Phi_n(T)) = T$.

(b) For a tree $S \in \mathcal{I}_n$ and a vertex $i \in [0,n]$ of S, we have

$$\begin{aligned} \mathrm{ch}_{\Phi_n(\Psi_n(S))}(i) &= \mathrm{Word}(\mathrm{left}_{\Psi_n(S)}(i)) && (6.26a) \\ &= \mathrm{Word}(\mathrm{Tree}(\mathrm{ch}_S(i))) && (6.26b) \\ &= \mathrm{ch}_S(i) && (6.26c) \end{aligned}$$

Equation (6.26) gives us $\Phi_n(\Psi_n(S)) = S$.

Thus, Φ_n and Ψ_n are inverses of each other.

Finally, notice that a vertex j has a left edge in $T \in \mathcal{T}_n$ if and only if j has a non-empty childword in the tree $\Phi_n(T)$, which is true if and only if i is an internal vertex of $\Phi_n(T)$. $\square$

6.3.1.2 Generating functions proof of Proposition 6.3.1

We will now use generating functions to provide an alternate proof of Proposition 6.3.1. We begin by introducing a framework which was first introduced by Bergeron, Flajolet and Salvy in [BLL92].

Let $\boldsymbol{\phi} = (\phi_i)_{i \geq 0}$ be indeterminates and let $S \in \mathcal{I}_{n-1}$ be an increasing ordered tree with n labelled vertices with labels in $[0, n-1]$. To every vertex with i children we assign the weight ϕ_i; the weight of the tree, $\mathrm{wt}(T)$, is the product of the weight of its vertices. Let $F(t; \boldsymbol{\phi})$ be the weighted exponential generating function of increasing ordered trees on n vertices, i.e.,

$$F(t; \boldsymbol{\phi}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \left(\sum_{S \in \mathcal{I}_{n-1}} \mathrm{wt}(S) \right) \frac{t^n}{n!} \tag{6.27}$$

and thus,

$$F'(t; \boldsymbol{\phi}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \left(\sum_{S \in \mathcal{I}_{n}} \mathrm{wt}(S) \right) \frac{t^n}{n!} \,. \tag{6.28}$$

Also let $\Phi(u)$ be the ordinary generating function of $\boldsymbol{\phi}$ given by

$$\Phi(u) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \phi_i u^i \,. \tag{6.29}$$

We call $\Phi(u)$ the *degree function.* It is known that $F(t; \boldsymbol{\phi})$ satisfies the differential equation [BLL92, Eq. (8)]

$$\frac{d}{dt} F(t; \boldsymbol{\phi}) \;=\; \Phi(F(t; \boldsymbol{\phi})) \,. \tag{6.30}$$

Notice that even though we defined the generating function $F(t; \boldsymbol{\phi})$ to generate all increasing ordered trees, this set-up is very general and we can use different degree functions to denote different families of weighted trees. For example, when we substitute $\phi_i = \widetilde{\phi}_i / i!$, we may think of $F(t; \boldsymbol{\phi})$ as the exponential generating function enumerating all unordered increasing trees where each vertex with i children gets weight $\widetilde{\phi}_i$.

We will use this framework to prove the following very general result:

Proposition 6.3.7. Let $\widehat{\Phi}(u)$ be a formal power series with zero constant term, and let $\mathcal{X}(s)$ be a formal power series satisfying

$$\widehat{\Phi}'(u) \;=\; \mathcal{X}(\widehat{\Phi}(u)) \,. \tag{6.31}$$

Now define

$$\Phi(u) \stackrel{\text{def}}{=} 1 + x\widehat{\Phi}(u) \quad (6.32a)$$

$$\Psi(u) \stackrel{\text{def}}{=} (1 + xu)\,\mathcal{X}(u) \quad (6.32b)$$

Let $\mathcal{A}(t) \stackrel{\text{def}}{=} F(t;\boldsymbol{\phi})$ with degree function $\Phi(u)$, and let $\mathcal{B}(t) \stackrel{\text{def}}{=} F(t;\boldsymbol{\psi})$ with degree function $\Psi(u)$. Then the following identities hold:

$$\mathcal{B}(t) = \widehat{\Phi}(\mathcal{A}(t)) \quad (6.33a)$$

$$\mathcal{A}'(t) = 1+x\mathcal{B}(t)\,. \quad (6.33b)$$

PROOF. From definition of $\mathcal{B}(t)$, we know that it satisfies the differential equation (using equations (6.30) and (6.32b))

$$\mathcal{B}'(t) = \Psi(\mathcal{B}(t)) = (1 + x\mathcal{B}(t))\,\mathcal{X}(\mathcal{B}(t))\,. \quad (6.34)$$

On the other hand,

$$\frac{d}{dt}\widehat{\Phi}(\mathcal{A}(t)) = \widehat{\Phi}'(\mathcal{A}(t))\,\mathcal{A}'(t) \quad (6.35a)$$

$$= \mathcal{X}(\widehat{\Phi}(\mathcal{A}(t)))\,\Phi(\mathcal{A}(t)) \quad (6.35b)$$

$$= \mathcal{X}(\widehat{\Phi}(\mathcal{A}(t)))\left[1+x\widehat{\Phi}(\mathcal{A}(t))\right] \quad (6.35c)$$

So $\mathcal{B}(t)$ and $\widehat{\Phi}(\mathcal{A}(t))$ satisfy the same differential equation with the same initial condition (namely, vanish at $t=0$), so they are equal. This exhibits (6.33a).

Equation (6.33) follows from combining (6.32b) and (6.33a). □

PROOF OF PROPOSITION 6.3.1 USING GENERATING FUNCTIONS. We will now prove Proposition 6.3.1 by using Proposition 6.3.7. First, consider the exponential generating function $\mathcal{A}(t) \stackrel{\text{def}}{=} F\big(t;(\delta_{i0}+(1-\delta_{i0})xy^{i-1}))_{i\geq 0}\big)$; thus $\mathcal{A}(t)$ denotes the exponential generating function of all increasing ordered trees $T \in \mathcal{I}_n$ where each leaf is given weight $\phi_0 = 1$ and each internal vertex with i (≥ 1) children is given weight $\phi_i = xy^{i-1}$. Thus, the degree function is

$$\Phi(u) \;=\; 1 + \frac{xu}{1-yu} \tag{6.36}$$

and $\mathcal{A}(t)$ satisfies the differential equation

$$\mathcal{A}' \;=\; 1 + \frac{x\mathcal{A}}{1-y\mathcal{A}} \tag{6.37}$$

Thus, we have from (6.27)

$$\sum_{n=1}^{\infty} \left(\sum_{S \in \mathcal{I}_{n-1}} x^{\mathrm{intvertex}(S)} y^{\mathrm{leaf}(S)} \right) \frac{t^n}{n!} \;=\; \mathcal{A}(t) \tag{6.38}$$

where we recall that $\mathrm{intvertex}(S)$ and $\mathrm{leaf}(S)$ denote the number of internal vertices and leaves of of a tree $S \in \mathcal{I}_{n-1}$, respectively.

We use notation from Proposition 6.3.7 to obtain

$$\widehat{\Phi}(u) \;=\; \frac{u}{1-yu} \tag{6.39a}$$

$$\mathcal{X}(s) \;=\; (1+ys)^2 \tag{6.39b}$$

$$\psi(u) \;=\; (1+xu)(1+ys)^2 \tag{6.39c}$$

We then define $\mathcal{B}(t) \stackrel{\mathrm{def}}{=} F(t;\boldsymbol{\psi})$ with degree function $\psi(u) = (1+xu)(1+ys)^2$. Here, we have $\psi_0 = 1$, $\psi_1 = x+2y$, $\psi_2 = 2xy+y^2$, $\psi_3 = xy^2$, and $\psi_i = 0$ for every $i \geq 4$. Thus, we can interpret $\mathcal{B}(t)$ to be the exponential generating function of all ternary trees where each left edge gets a weight x, and each middle or right edge gets a weight y, and the weight of a tree is instead the product of the weights of its edges. Thus, it is clear that the exponential generating function of trees $T \in \mathcal{T}_n$ with weights x for each left edge and weights y for each middle or right edge is simply given by

$$\sum_{n=0}^{\infty} \left(\sum_{T \in \mathcal{T}_n} x^{\mathrm{l}(T)} y^{\mathrm{m}(T)+\mathrm{r}(T)} \right) \frac{t^n}{n!} \;=\; 1 + x\mathcal{B}(t) \tag{6.40}$$

where we recall that $\mathrm{l}(T), \mathrm{m}(T), \mathrm{r}(T)$ denote the number of left, middle and right edges of a tree $T \in \mathcal{T}_n$.

Finally, we use Proposition 6.3.7 to obtain

$$\mathcal{A}'(t) \;=\; 1 + x\mathcal{B}(t) \tag{6.41}$$

Combining equations (6.38), (6.40) and (6.41) finishes the proof. □

We end this section with some interesting remarks.

Final remarks. 1. If we solve equation (6.31) with $\mathcal{X}(v) = (1+yv)$, we get $\widehat{\Phi}(u) \;=\; \big(e^{yu}-1\big)/y$. This shows that unordered increasing trees on $n+1$ vertices are in bijection with increasing binary trees on n vertices. This bijection is well known, see for e.g. [Sta09, pp. 44-46]. Furthermore, this also tells us that the left edges in binary trees are in correspondence with the internal vertices of its corresponding unordered increasing tree.

2. If we solve equation (6.31) with $\mathcal{X}(v) = (1+yv)^k$ with $k \geq 2$, we obtain

$$\widehat{\Phi}(u) \;=\; \frac{[1-(k-1)yu]^{-1/(k-1)} - 1}{y}\;. \tag{6.42}$$

In particular, when $k \geq 3$, the degree function $\Phi(u) = 1 + x\widehat{\Phi}(u)$ obtained from (6.42) describes increasing ordered trees in which each leaf gets a weight 1, each vertex with only one child gets a weight x and for $i \geq 2$, each vertex with i children gets a weight $xy^{i-1}\prod_{j=1}^{i-1}((j-1)k - j + 2)$. Such trees on $n+1$ vertices are in bijection with $(k+1)$-ary trees on $n+1$ vertices in which the root can only have a first child.

3. Finally, and more interestingly, if we solve equation (6.31) with $\mathcal{X}(v) = e^{yv}$, we obtain $\widehat{\Phi}(u) \;=\; \big(e^{yu}-1\big)/y$. The consequence of this is that increasing trees on $n+1$ vertices where the children of every vertex are cyclically ordered, are in bijection with trees on $n+1$ vertices where every vertex has zero or one left child and an arbitrary number of unordered right children but the root can only have a left child.

Problem 6.3.8. Find bijections that explain 2. and 3.

4. On setting $y = (1+x)$ in the degree function (6.36), we get the sequence $\phi_0 = 1$, and for $i \geq 1$, $\phi_i = x(1+x)^{i-1}$. This sequence is not coefficientwise Toeplitz totally positive in the variable x. However, by combining Propositions 6.1.5, 6.3.1 and Theorem 6.1.4, we get that the generating polynomials of weighted trees in $\mathcal{I}_n$ are the polynomials $c_{2,n}(x)$ and that the sequence $c_{2,n}(x)$ is coefficientwise-Hankel-totally positive with respect to the variable x. This has a consequence to the understanding of [PS21, Theorem 1.1]. This shows that even though the sequence $\boldsymbol{\phi}$, in this case, is not Toeplitz totally positive, the resulting sequence in (c) of [PS21, Theorem 1.1] is still coefficientwise-Hankel-totally positive. Thus, the condition of $\boldsymbol{\phi}$ being Toeplitz-totally positive is only a sufficient condition and is far from necessary for [PS21, Theorem 1.1(c)] to hold. In [DDS], $\Phi(u)$ is taken to be a polynomial in u of degrees 3 and 4 and the exact necessary and sufficient conditions under which (c) of [PS21, Theorem 1.1] holds true is studied. ■

6.3.2 Interpretation as cyclically-ordered phylogenetic trees

Here we introduce our fifth and final combinatorial interpretation of the second-order Stirling cycle numbers. This interpretation will lead us to a common generalisation of the second-order Stirling numbers of both kinds and also lead us to another conjecture similar to Conjectures 6.1.2 and 6.1.3.

A *phylogenetic tree* is a leaf-labelled rooted tree in which all non-leaf vertices have at least two children. Unless otherwise mentioned, it is assumed that the children of each vertex of a phylogenetic tree are unordered. However, we will work with various orders. A *cyclically-ordered phylogenetic tree* is a phylogenetic tree in which the children of each vertex are cyclically ordered. These have also been called series-reduced mobiles [AW]. Let $\mathcal{M}_{n,k}$ denote the set of all cyclically-ordered phylogenetic trees with $n+1$ leaves and k internal vertices.

We will now establish the following proposition:

Proposition 6.3.9. The cardinality of the set $\mathcal{M}_{n,k}$ of cyclically-ordered phylogenetic trees with $n+1$ leaves and k internal vertices is given by

$$\left|\mathcal{M}_{n,k}\right| \;=\; \begin{bmatrix} n \\ k \end{bmatrix}^{(2)}. \tag{6.43}$$

We will provide two different proofs of Proposition 6.3.9. Our first proof will be showing that the numbers $\left|\mathcal{M}_{n,k}\right|$ satisfy the recurrence relation (6.12a). Our second proof will use generating functions for multivariate Ward polynomials introduced by Elvey Price and Sokal in [PS20].

PROOF OF PROPOSITION 6.3.9 USING RECURRENCE RELATIONS. We will consider the two cases whether or not the leaf labelled $n+1$ has exactly one sibling.

If the leaf labelled $n+1$ has exactly one sibling, then the tree obtained by removing it is no longer a cyclically-ordered phylogenetic tree as the parent of $n+1$ now has only one child. If the parent of $n+1$ was not the root, we can contract the resulting tree at this internal vertex, to obtain a tree in $\mathcal{M}_{n-1,k-1}$. Thus, to insert $n+1$ back into this tree, we need to select an edge, and elongate it by adding an extra internal vertex. As there are $n+k-2$ edges, we get a contribution of $(n+k-2)|\mathcal{M}_{n-1,k-1}|$. If the parent of $n+1$ was the root, we remove the root as well to obtain a tree rooted at the sibling of $n+1$. This contributes $|\mathcal{M}_{n-1,k-1}|$.

If $n+1$ has at least two siblings, we can remove it and the resulting tree belongs to $\mathcal{M}_{n-1,k}$. Thus, inserting $n+1$ back into this tree, it suffices to choose its sibling that comes immediately before it and we may choose any of the non-root vertices. This contributes the term $(n+k-1)|\mathcal{M}_{n-1,k}|$.

Combining all the contributions, we get

$$|\mathcal{M}_{n,k}| \;=\; (n+k-2)|\mathcal{M}_{n-1,k-1}| + |\mathcal{M}_{n-1,k-1}| + (n+k-1)|\mathcal{M}_{n-1,k}| \tag{6.44}$$

which is exactly the recurrence (6.12a). □

We require the multivariate Ward polynomials introduced in [PS20, pp. 9-11] for our second proof of Proposition 6.3.9. As mentioned earlier in Section 6.1.2, our second-order Stirling subset polynomials $s_{2,n}(x)$ are usually referred to as the Ward polynomials, and the numbers $\left\{{n \atop k}\right\}^{(2)}$ are the Ward numbers. It is known that the Ward number $\left\{{n \atop k}\right\}^{(2)}$ counts the number of phylogenetic trees (with unordered children) on $n+1$ leaves and k internal vertices. Thus, the polynomial $s_{2,n}(x)$ is the generating polynomial of phylogenetic trees on $n+1$ leaves in which each internal vertex gets a weight x and each leaf gets a weight 1. Now let $\mathbf{x} = (x_i)_{i\geq 1}$ be an infinite collection of indeterminates. The multivariate Ward polynomials $\boldsymbol{W}_n(\mathbf{x}) = \boldsymbol{W}_n(x_1, \ldots, x_n)$ are a multivariate generalisation of the polynomials $s_{2,n}(x)$; these are the generating polynomial for phylogenetic trees on $n+1$ labelled leaves in which each internal vertex with i (≥ 2) children gets a weight x_{i-1}. The polynomial $\boldsymbol{W}_n(\mathbf{x})$ is quasi-homogeneous of degree n when x_i is given weight i. Thus $s_{2,n}(x) = \boldsymbol{W}_n(x, \ldots, x)$. The first few $\boldsymbol{W}_n$ are [OEI19, A134685]. The exponential generating function

$$\mathcal{W}(t;\mathbf{x}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \boldsymbol{W}_n(\mathbf{x}) \frac{t^{n+1}}{(n+1)!} = t + \sum_{n=2}^{\infty} \boldsymbol{W}_{n-1}(\mathbf{x}) \frac{t^n}{n!} \tag{6.45}$$

satisfies the functional equation [PS20, eq. (1.18)]

$$\mathcal{W}(t;\mathbf{x}) = t + \sum_{n=2}^{\infty} x_{n-1} \frac{\mathcal{W}(t;\mathbf{x})^n}{n!}, \tag{6.46}$$

where the term n in the sum corresponds to the case in which the root has n children.

PROOF OF PROPOSITION 6.3.9 USING GENERATING FUNCTIONS. It is clear that with the substitution $x_i \mapsto i! \cdot x$, the polynomials $\boldsymbol{W}_n(x, 2x, \ldots, n! \cdot x)$ are the generating polynomials of cyclically-ordered phylogenetic trees with weight x given to each internal vertex and a weight 1 to each leaf. Thus we want to show that $\boldsymbol{W}_n(x, 2x, \ldots, n! \cdot x) \stackrel{?}{=} c_{2,n}(x)$. We call these polynomials $W_n(x) \stackrel{\text{def}}{=} \boldsymbol{W}_n(x, 2x, \ldots, n! \cdot x)$, and the corresponding exponential generating function is denoted by $\mathcal{W}(t;x)$. Thus, the functional equation (6.46) is now

$$\mathcal{W}(t;x) = t + \sum_{n=2}^{\infty} \mathcal{W}(t;x)^n \frac{x}{n}. \tag{6.47}$$

This can be rewritten as

$$\mathcal{W}(t;x) \;=\; t - x\left[\log\left(1-\mathcal{W}(t;x)\right) + \mathcal{W}(t;x)\right] \tag{6.48}$$

and differentiating both sides with respect to t we obtain

$$\mathcal{W}' \;=\; \frac{1}{1 - x\,\dfrac{\mathcal{W}}{1-\mathcal{W}}} \;=\; 1 + \frac{x\,\mathcal{W}}{1-(1+x)\,\mathcal{W}}\,. \tag{6.49}$$

However, this is exactly the same as the differential equation (6.37) with $y = 1+x$ which is satisfied by the exponential generating function for increasing ordered trees on $[0,n]$ where each leaf is given a weight 1 and each vertex with $i \geq 1$ children is given a weight $x(1+x)^{i-1}$. Thus, the result follows by combining Propositions 6.3.1 and 6.1.5. $\square$

Final remark and conjecture. As mentioned earlier, in [PS20] it has been noticed that the number of phylogenetic trees on $n+1$ vertices with unordered children having k internal nodes is the second-order Stirling subset number $\left\{ {n \atop k} \right\}^{(2)}$. In Proposition 6.3.9, we established that the number of phylogenetic trees on $n+1$ vertices with cyclically-ordered children having k internal nodes is the second-order Stirling cycle number $\left[{n \atop k} \right]^{(2)}$. The generating polynomials in both cases can be obtained as specialisations of the multivariate Ward polynomials; the former can be obtained by setting $x_i \mapsto x$, and the latter can be obtained by setting $x_i \mapsto i! \cdot x$. After these two observations, it is very natural to ask what the number of phylogenetic trees on $n+1$ vertices with linearly-ordered children having k internal nodes is.

Let $\widehat{\left\{ {n \atop k} \right\}}$ denote the number of phylogenetic trees with ordered children having $n+1$ vertices and k internal nodes. Their generating polynomials are obtained by setting $x_i \mapsto (i+1)! \cdot x$ in the multivariate Ward polynomials. These numbers seem to match [OEI19, A357367] and it seems that for $k \leq n$, we have $\widehat{\left\{ {n \atop k} \right\}} = \left[(n+k)!/k!\right]\binom{n-1}{k-1}$ and the first few rows are

$$
\begin{array}{cccccc}
1 & & & & & \\
 & 2 & & & & \\
 & 6 & 12 & & & \\
 & 24 & 120 & 120 & & \\
 & 120 & 1080 & 2520 & 1680 & \\
 & 720 & 10080 & 40320 & 60480 & 30240 \\
 & 5040 & 100800 & 604800 & 1512000 & 1663200 & 665280
\end{array}
\tag{6.50}
$$

Let $d_n(x)$ denote the generating polynomials

$$
d_n(x) \stackrel{\text{def}}{=} \sum_{k=0}^{n} \widehat{\left\{ {n \atop k} \right\}} x^k \tag{6.51}
$$

and let D be the lower-triangular matrix $D = \left(\widehat{\left\{ {n \atop k} \right\}} \right)_{n,k \geq 0}$.

Similar to Conjectures 6.1.2 and 6.1.3, we have the following conjecture:

Conjecture 6.3.10 (Conjecture for ordered phylogenetic trees)**.** The following are true:

(a) The triangle D is totally positive.

(b) The triangle D^{rev} formed by reversing the rows of D is totally positive.

(c) The row-generating polynomials $d_n(x)$ are real-rooted and their zeroes lie in the interval $(-1, 0]$.

(d) The sequence $(d_n(x))_{n \geq 0}$ is coefficientwise-Hankel-totally positive in the variable x.

Finally, we leave the further study of the multivariate Ward polynomials as an open problem. ■

6.4 Distribution of zeroes for the second-order Stirling cycle and subset polynomials

We will now prove Theorem 6.1.6 in this section.

For $r, n \geq 1$, it is clear from (6.10a)/(6.11a) that $c_{r,n}(0) = s_{r,n}(0) = 0$. For $n \geq 0$ and $r = 2$, we then define the polynomials $\widehat{c}_n(x)$ and $\widehat{s}_n(x)$ by

$$\widehat{c}_n(x) \stackrel{\text{def}}{=} \frac{c_{2,n+1}(x)}{x} \tag{6.52a}$$

$$\widehat{s}_n(x) \stackrel{\text{def}}{=} \frac{s_{2,n+1}(x)}{x} . \tag{6.52b}$$

We will show that the polynomials $\widehat{c}_n(x)$ and $\widehat{s}_n(x)$ have simple and interlacing zeroes in the interval $(-1,0)$. To prove this, we will first obtain recurrences for the polynomials $\widehat{c}_n(x)$ and $\widehat{s}_n(x)$ and then evaluate them at $x = 0$ and at $x = -1$.

Lemma 6.4.1. (a) The polynomials $\widehat{c}_n(x)$ satisfy the recurrence relation

$$\widehat{c}_n(x) = \big((n+2)x+n+1\big)\,\widehat{c}_{n-1}(x) + x(x+1)\,\widehat{c}\,'_{n-1}(x) \tag{6.53}$$

(b) The polynomials $\widehat{s}_n(x)$ satisfy the recurrence relation

$$\widehat{s}_n(x) = \big((n+2)x+1\big)\,\widehat{s}_{n-1}(x) + x(x+1)\,\widehat{s}\,'_{n-1}(x) \tag{6.54}$$

PROOF. As a consequence of the recurrences (6.12), the polynomials $c_{2,n}(x)$ and $s_{2,n}(x)$ satisfy the recurrences

$$c_{2,n}(x) = \big(nx+(n-1)\big)\,c_{2,n-1}(x) + x(x+1)\,c'_{2,n-1}(x) \tag{6.55a}$$

$$s_{2,n}(x) = nx\,s_{2,n-1}(x) + x(x+1)\,s'_{2,n-1}(x) \tag{6.55b}$$

from which the recurrences (6.53)/(6.54) follow. □

Lemma 6.4.2. For all $n \geq 1$, the polynomials $\widehat{c}_n(x)$ and $\widehat{s}_n(x)$ have the following evaluations at $x=0$ and at $x=-1$:

$$\widehat{c}_n(0) = (n+1)! \tag{6.56a}$$

$$\widehat{s}_n(0) = 1 \tag{6.56b}$$

$$\widehat{c}_n(-1) = (-1)^n \tag{6.56c}$$

$$\widehat{s}_n(-1) = (-1)^n\,(n+1)! \tag{6.56d}$$

PROOF. These follow by substituting $x=0$ and $x=-1$ in the recurrences (6.53) and (6.54) and then using induction. □

We are now ready to prove Theorem 6.1.6.

PROOF OF THEOREM 6.1.6. We prove it for the polynomial sequence $\widehat{c}_{2,n}(x)$. The proof for the polynomial sequence $\widehat{s}_{2,n}(x)$ is mostly the same and is left to the reader.

We first rewrite equation (6.53) as

$$\widehat{c}_n(x) = x\left((n+2)\widehat{c}_{n-1}(x)+(x+1)\widehat{c}'_{n-1}(x)\right) + (n+1)\,\widehat{c}_{n-1}(x) \tag{6.57}$$

Now, we use induction on n. For the case $n=0$, the polynomial $p_0(x)=1$ has no roots and hence the theorem is vacuously true. The case $n=1$ can be checked to be true as well. Let us assume that the zeroes of $\widehat{c}_{n-1}(x)$ are distinct and are contained in the interval $(-1,0)$.

Now let $p_n(x) := \mathrm{GCD}\left(\widehat{c}_{n-1}(x),(x+1)\widehat{c}'_{n-1}(x)\right)$. By Rolle's theorem, the zeroes of $\widehat{c}'_{n-1}(x)$ interlace those of $\widehat{c}_{n-1}(x)$ and thus, lie in the interval $(-1,0)$ and are simple. Thus we get $p_n(x)=1$. Thus, there are no common zeroes between $\widehat{c}_{n-1}(x)$, $\widehat{c}'_{n-1}(x)$ and $\widehat{c}_n(x)$.

Now let $-1<\lambda^{(n-1)}_{n-1}<\ldots<\lambda^{(n-1)}_1<0$ be the zeroes of $\widehat{c}_{n-1}(x)$. We then have $\widehat{c}'_{n-1}\left(\lambda^{(n-1)}_j\right)>0$ if j is odd, and $\widehat{c}'_{n-1}\left(\lambda^{(n-1)}_j\right)<0$ if j is even. Hence,

$$\mathrm{sgn}\left(\widehat{c}_n\left(\lambda^{(n-1)}_j\right)\right) = (-1)^j\,. \tag{6.58}$$

Furthermore, using Lemma 6.4.2, we also have $\widehat{c}_n(-1) = (-1)^n$ and $\widehat{c}_n(0) = (n+1)!$. Thus, the proof of this theorem follows by using the intermediate value theorem. □

Let us finally conclude this section by mentioning that even though we have included a proof of Theorem 6.1.6 for the sake of completeness, one can obtain this result as an immediate consequence of a seminal result due to Liu and Wang [LW07].

6.5 Distribution of zeroes for order 3 and above

We conjectured that the zeroes of any r-th order Stirling cycle or subset polynomials are not real when $r \geq 3$. We already know that $c_{r,n}(0) = s_{r,n}(0) = 0$ for $n \geq 1$. In this section, we first show that the polynomials $c_{r,3}(x)/x$ and $s_{r,3}(x)/x$, which are quadratic polynomials, have a negative discriminant when $r \geq 3$ in the cycle case and $r \geq 4$ in the subset case, thus proving that they have non-real complex zeroes. We will then provide plots for the distribution of these zeroes which suggest that these zeroes accumulate around some mysterious limiting curves.

Using the recurrences in Lemma 6.10, we obtain the following:

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}^{(r)} = (r+1)!, \qquad \begin{bmatrix} 3 \\ 2 \end{bmatrix}^{(r)} = \frac{(2r+1)!}{r(r+1)}, \qquad \begin{bmatrix} 3 \\ 3 \end{bmatrix}^{(r)} = \frac{(3r-1)!}{2r^2}. \tag{6.59}$$

Thus we get that the discriminant of the polynomial $c_{r,3}(x)/x$ is

$$D(r) := \left(\frac{(2r+1)!}{r(r+1)}\right)^2 - 4 \cdot \frac{(r+1)!(3r-1)!}{2r^2}. \tag{6.60}$$

Thus for a fixed $r \geq 3$, showing that $D(r) < 0$ will imply that the polynomial $c_{r,3}(x)$ contains non-real zeroes. We first rewrite $D(r)$ as

$$D(r) = \frac{(r+1)!(2r+1)!}{r^2(r+1)^2} \cdot \left(\frac{(2r+1)!}{(r+1)!} - 2(r+1)^2\frac{(3r-1)!}{(2r+1)!}\right). \tag{6.61}$$

Since $r \geq 3$, we can further rewrite

$$D(r) = \frac{(r+1)!(2r+1)!}{r^2(r+1)^2} \cdot \Big[(r+2)(r+3)\cdot(r+4)\cdots(2r+1) \tag{6.62a}$$

$$- \; 2(r+1)^2\cdot(2r+2)\cdots(3r-1)\Big] \tag{6.62b}$$

from which it is clear that $D(r) < 0$.

One can use a similar argument to show that the discriminant of the quadratic polynomial $s_{r,3}(x)/x$ is negative when $r \geq 4$; we leave the details to the reader. Even though the zeroes of $s_{3,3}(x)$ are real, using computer algebra one can see that the zeroes of the polynomial $s_{3,4}(x)$ are however complex.

Let us first establish some notation before we provide the plots for the zeroes of the polynomials $c_{r,n}(x)$ and $s_{r,n}(x)$. Let $\mathcal{A} \subset \mathbb{C}$ be a subset of complex numbers, let $\alpha \in \mathbb{R}$ be a real number. We then define the set $\alpha\mathcal{A}$ as follows:

$$\alpha\mathcal{A} \stackrel{\text{def}}{=} \{\alpha x : x \in \mathcal{A}\}\,. \tag{6.63}$$

For a fixed $r \geq 1$, we let $\mathcal{RC}_{r,n}$ and $\mathcal{RS}_{r,n}$ denote the set of zeroes of the r-th order Stirling cycle polynomial $c_{r,n}(x)$, and of the r-th order Stirling subset polynomial $s_{r,n}(x)$, respectively. Thus,

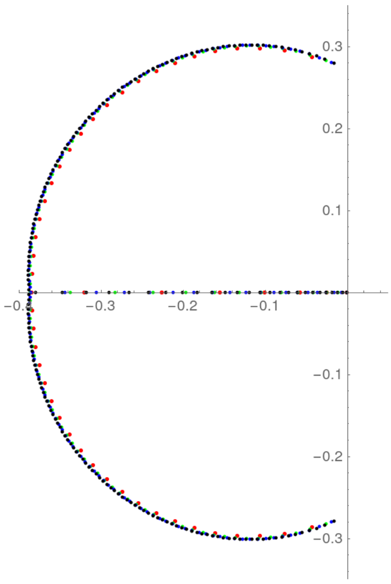
$$\mathcal{RC}_{r,n} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : c_{r,n}(z) = 0\} \tag{6.64a}$$

$$\mathcal{RS}_{r,n} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : s_{r,n}(z) = 0\}\,. \tag{6.64b}$$

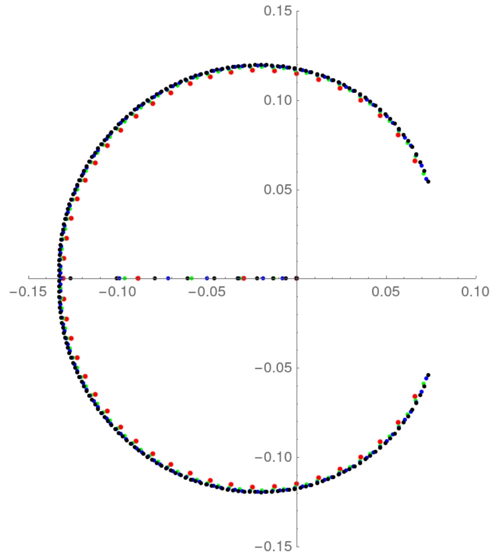
In Section 6.5.1, we provide some normalised plots for $\mathcal{RC}_{r,n}$ for some r, n, and then in Section 6.5.2 we provide normalised plots for $\mathcal{RS}_{r,n}$. These plots will immediately suggest some problems and conjectures which we mention.

6.5.1 Third and higher-order Stirling cycle polynomials

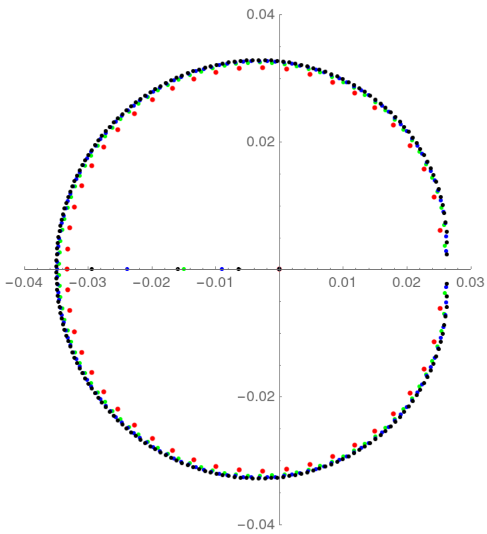
We fix r and plot the set $n^{n-r}\mathcal{RC}_{r,n}$ for $n = 50, 100, 150, 200$ using the different colours red, green, blue and black, respectively. The overlapping plots suggest that the zeroes have some asymptotic distribution around some limiting curves. We first plot the cases $r = 3,4,5,6$ in Figure 6.2 and then $r = 7,8,9,10$ in Figure 6.3.



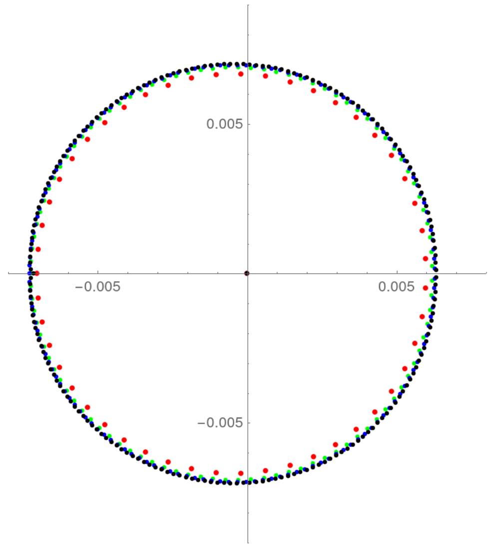
(a) $r =$ 3-rd order Stirling cycle



(b) $r =$ 4-th order Stirling cycle

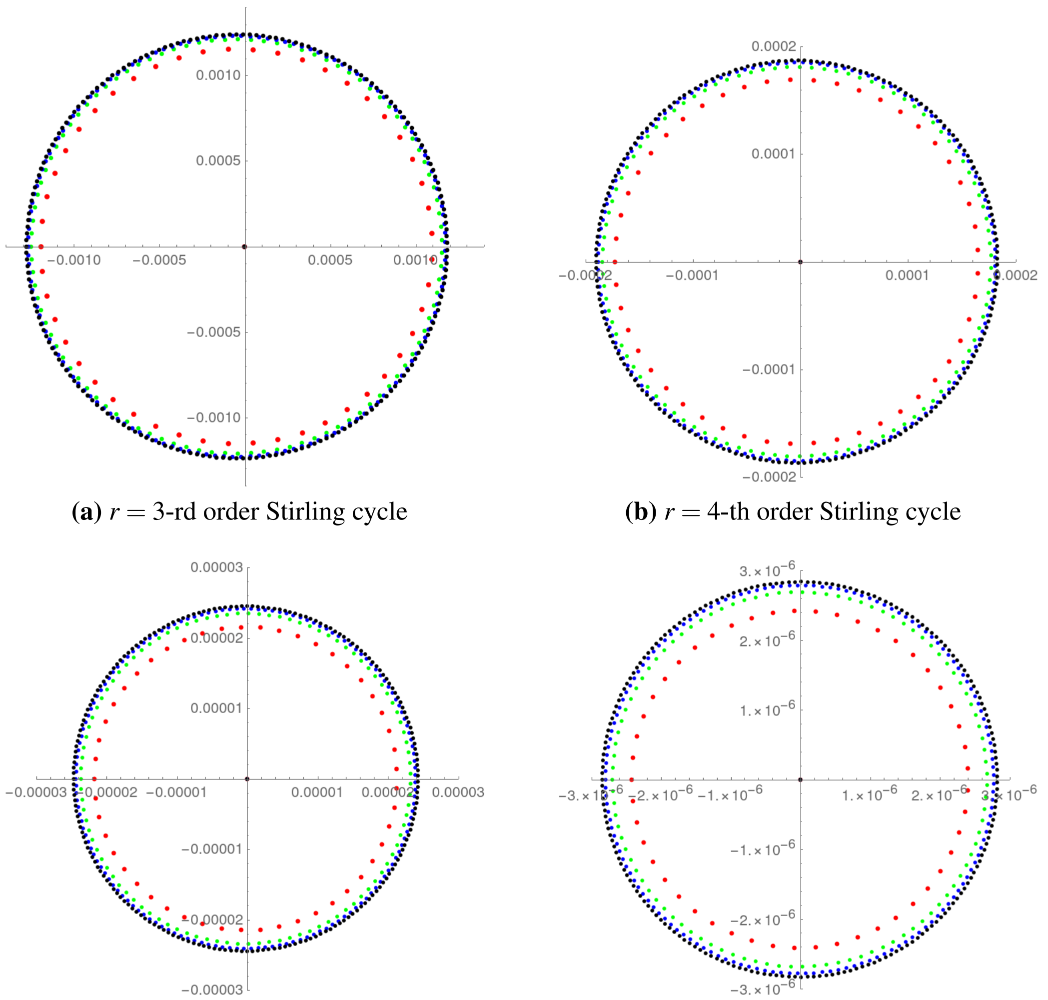


(c) $r =$ 5-th order Stirling cycle



(d) $r =$ 6-th order Stirling cycle

Figure 6.2: We plot $n^{n-r}\mathcal{RC}_{r,n}$ with $r = 3,4,5,6$ in subfigures (a), (b), (c) and (d), respectively, and with $n = 50,100,150,200$ using the different colours red, green, blue and black, respectively.



(a) $r = 3$-rd order Stirling cycle **(b)** $r = 4$-th order Stirling cycle

(c) $r = 5$-th order Stirling cycle **(d)** $r = 6$-th order Stirling cycle

Figure 6.3: We plot $n^{n-r}\mathcal{RC}_{r,n}$ with $r = 7, 8, 9, 10$ in subfigures (a), (b), (c) and (d), respectively, and with $n = 50, 100, 150, 200$ using the different colours red, green, blue and black, respectively.

Our plots suggest the following problem:

Problem 6.5.1. For a fixed $r \geq 1$, study the asymptotic distribution of the set of normalised zeroes $n^{r-2}\mathcal{RC}_{r,n}$. Do they accumulate around a curve?

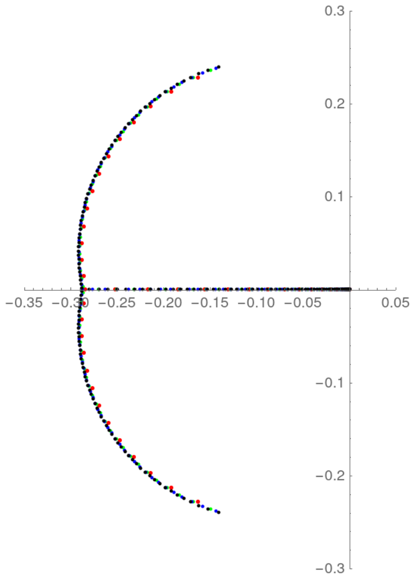
In the case $r = 1$, we clearly get that the normalised zeroes accumulate uniformly on the unit interval $[-1, 0]$.

When $r = 3$, we conjecture the following:

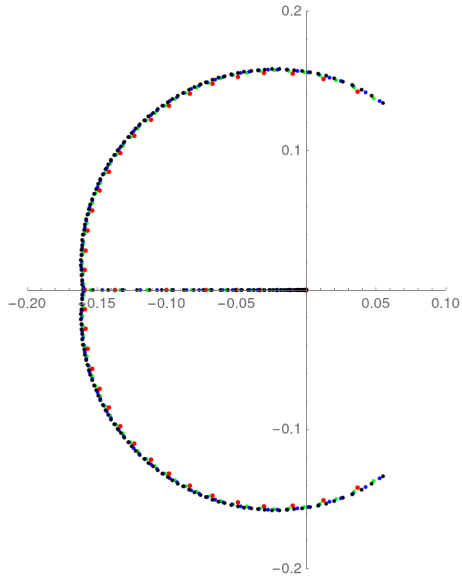
Conjecture 6.5.2. The zeroes of the third-order Stirling cycle polynomials $c_{3,n}(x)$ lie on the left-half plane.

6.5.2 Third and higher-order Stirling subset polynomials

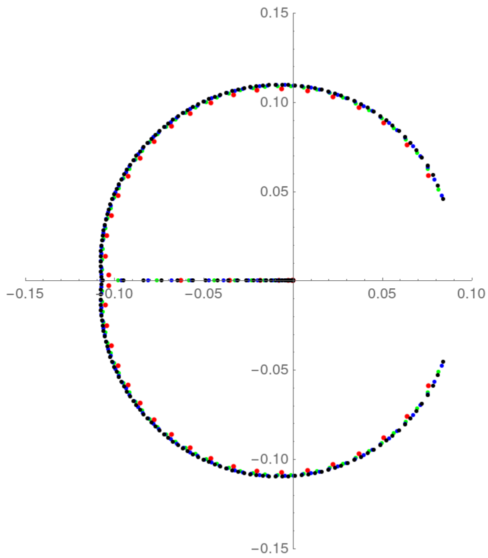
We fix r and plot the set $n^{n-r}\mathcal{RS}_{r,n}$ for $n = 50, 100, 150, 200$ using the different colours red, green, blue and black, respectively. The overlapping plots suggest that the zeroes have some asymptotic distribution around some limiting curves. We first plot the cases $r = 3, 4, 5, 6$ in Figure 6.4 and then $r = 7, 8, 9, 10$ in Figure 6.5.



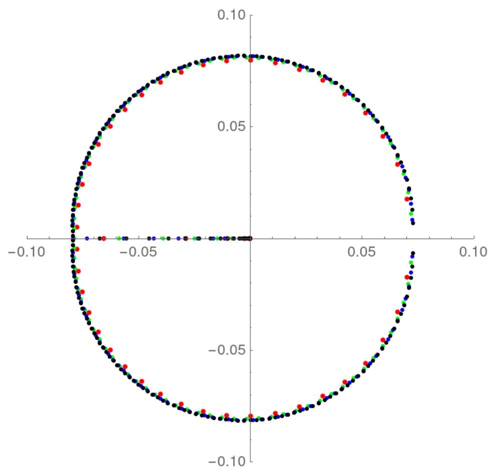
(a) $r = 3$-rd order Stirling subset



(b) $r = 4$-th order Stirling subset

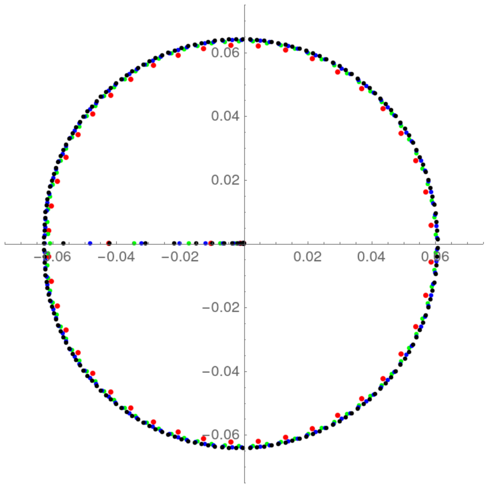


(c) $r = 5$-th order Stirling subset

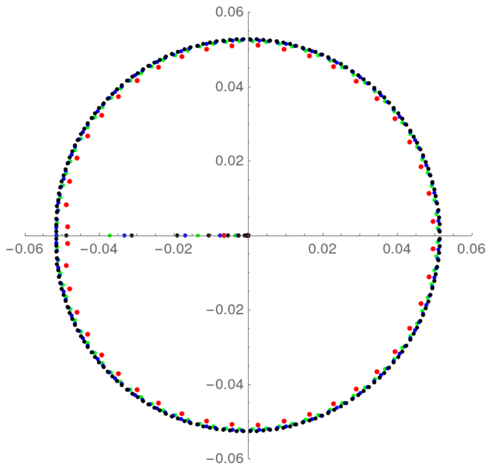


(d) $r = 6$-th order Stirling subset

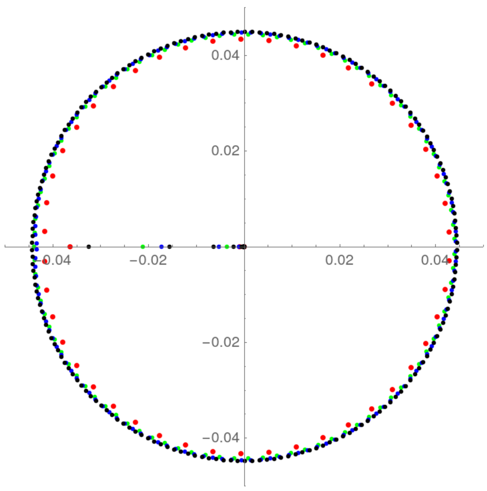
Figure 6.4: We plot $n^{n-r}\mathcal{RS}_{r,n}$ with $r = 3, 4, 5, 6$ in subfigures (a), (b), (c) and (d), respectively, and with $n = 50, 100, 150, 200$ using the different colours red, green, blue and black, respectively.



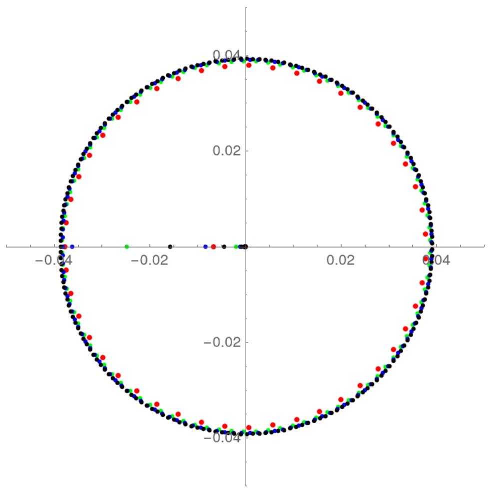
(a) $r = 7$-th order Stirling subset



(b) $r = 8$-th order Stirling subset



(c) $r = 9$-th order Stirling subset



(d) $r = 10$-th order Stirling subset

Figure 6.5: We plot $n^{n-r}\mathcal{RS}_{r,n}$ with $r = 7, 8, 9, 10$ in subfigures (a), (b), (c) and (d), respectively, and with $n = 50, 100, 150, 200$ using the different colours red, green, blue and black, respectively.

Our plots suggest the following problem:

Problem 6.5.3. For a fixed $r \geq 1$, study the asymptotic distribution of the set of normalised zeroes $n^{r-2}\mathcal{RS}_{r,n}$. Do they accumulate around a curve?

In the case $r = 1$, the asymptotic distribution of the normalised zeroes of the Bell polynomials was studied by Elbert [Elb01a, Elb01b].

When $r = 3$, we conjecture the following:

Conjecture 6.5.4. The zeroes of the third-order Stirling subset polynomials $s_{3,n}(x)$ lie on the left-half plane.

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