Essays in Microeconomic Theory

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PhD in Economics

I, Nathan Hancart confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

איבערגעקומענע צורות איז גוט צו דערציילן.

It is good to tell past troubles. (Yiddish proverb)

Abstract

This thesis consists of three independent chapters.

1. A decision-maker must accept or reject a privately informed agent. The agent always wants to be accepted, while the decision-maker wants to accept only a subset of types. The decision-maker has access to a set of feasible tests and, prior to making a decision, requires the agent to choose a test from a menu. I show that the DM does not benefit from commitment in this context. I then show in various environments when the DM benefits from offering a menu. When the domain of feasible tests contains a most informative test, I characterise when only the dominant test is offered and when a dominated test is part of the optimal menu. I also consider settings where types are multidimensional or where tests vary in difficulty.

2. I consider a model of monopoly pricing where a firm makes a price offer to a buyer with reference-dependent preferences. The reference point is the ex-ante probability of trade and the buyer exhibits an attachment effect: the higher his expectations to buy, the higher his willingness-to-pay. When the buyer's valuation is private information, a unique equilibrium exists where the firm plays a mixed strategy and its profits are the same as in the reference-independent benchmark. The equilibrium always entails inefficiencies: the probability of no trade is greater than zero. Finally, I show that when the firm can design a test about the buyer's valuation, it is better off using a noisy test.

3. I provide a sufficient condition under which a principal does not benefit from commitment in economic situations described by a constrained maximisation problem. Commitment has no value when the *marginal* contribution of the constraints is null in the problem with commitment. I use this result to extend multiple results in mechanism design.

Impact Statement

I summarise here how my PhD thesis is most likely to have impact inside and outside of academia. Given the nature of my thesis, I give the impact of each chapter separately.

1. The first chapter is concerned with the design of admission procedure and in particular whether giving a choice of test can be useful for the designer. An important application of this chapter is the design of admission procedure where the possibility of making some tests optional is debated in academia and the general public. For example, some argue that standardised test used in US university admissions like the SAT should be optional (see e.g., Hubler, 2020). This chapter provides arguments under which making some tests optional reveals more information and under which making some tests optional is suboptimal.

2. The second chapter explores theoretically a monopoly pricing model with the following buying behaviour: the more a buyer expects to own a good, the more he is willing to pay for it. I show that the extent to which a monopolist can take advantage depends on the informational environment. This could be important for the regulation of markets where this behaviour is believed to be prevalent.

3. The last chapter develops a theoretical tool to show when commitment has value in economic problems. I apply this tool to a specific mechanism design setting and generalise existing results. It could also be used in other settings where commitment is an important issue other areas of economic theory.

Acknowledgements

In 1975, Primo Levi published his book “The Periodic Table” (*Il sistema periodico* in its original title). The book is a collection of short stories, some autobiographical, others fictional, each named after and linked to a chemical element. In it, he describes some of the most difficult periods of his life in Auschwitz and fascist Italy but also the challenges, joy and troubles of being a chemist. The book, beyond the stories told, is a tribute to research and the human stories happening around it.

Primo Levi decided to open the book with the Yiddish proverb: איבערגעקומענע צורות איז גוט צו דערציילן, transliterated “Ibergekumene tsures iz gut tsu dertseiln”, *It is good to tell past troubles* or *troubles overcome*. At my own scale, these last six years were made of challenges, joy and *tsures* – troubles – overcome. Before telling them, I would like to thank a couple of people that have accompanied me throughout the years.

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Contents

Introduction

This thesis is made of three independent chapters in different areas of microeconomic theory. Each chapter contains a separate introduction that summarises and highlights important aspects of the chapter.

The first chapter asks whether a decision-maker designing an admission procedure can benefit from leaving the choice of test that is conducted to the candidate. The motivation is to understand how a decision-maker limited in his testing technology optimally can improve the admission procedure. One way is to offer a menu of tests to the agent instead of a unique test. By doing so, the decision-maker can use the information from the choice itself to learn about the candidate. I provide technical results regarding the optimal menu of tests and characterise when the decision-maker benefits from giving a choice of tests.

The second chapter is in the field of Behavioural Industrial Organisation. I consider a monopolist selling a good to an agent exhibiting an attachment effect: the agent values the good more if he expects to own it. This type of behaviour is a typical prediction of expectation-based reference-dependent preferences (Kőszegi and Rabin, 2006). Whereas the existing literature explored a setting where the monopolist can commit to a random price distribution, I characterise the equilibrium strategies when the monopolist *cannot* commit. I show that assumptions on the informational environment play a key role for whether the monopolist benefits from facing this type of buyers.

The last chapter provides a tool to test whether commitment has a value in economic problems described by constrained maximisation. The idea is to use the envelope theorem for saddle-point problems to understand how a marginal change in the value function relates to a marginal change in the objective function. I use this tool to generalise several results in the literature on mechanism design with evidence.

While these three chapters are independent, some themes are recurring. In all three chapters the notion of commitment, and the possibility to commit or not, are important concerns. In both the first and the second chapter, I consider how to optimally test an agent and how the test designer can benefit from less precise tests. In both cases, strategic considerations play a key role when choosing which test to conduct or to allow.

Chapter 1

Designing the Optimal Menu of Tests

1.1 Introduction

In many economic settings, decision-makers (DMs) rely on tests to guide their actions. Universities use standardised tests as part of their admission process, firms interview job candidates before they hire them and regulators test products prior to authorisation. In these examples, the DM is deciding whether to accept an agent and his preferences depend on some private information held by the agent: the ability of the student, the productivity of the candidate or the quality of the product. Ideally, the DM would want to use a fully revealing test, but this is often not feasible and thus the ability to learn from the test outcome alone is limited. However, there is an additional channel the DM can learn from: he can offer a menu of tests and let the agent *choose* which test to take. The DM can then use the agent's choice as an additional source of information.

Depending on the setting, constraints on the testing ability can take many different forms. For instance, a hiring firm is constrained by the amount of time and resources it can allocate to the selection process; most universities have to use externally provided tests like the SAT or the GRE for their admission procedures; and medicine regulatory agencies face both technological and ethical constraints when authorising new drugs.

In this paper, I study the DM's optimal design of a menu of tests and provide conditions under which the DM learns from both the test choice and outcome. I first show that the DM does not benefit from committing to a strategy for any arbitrary domains of feasible tests. I then apply this result to natural economic applications and determine which tests are part of the optimal menu and how it depends on their properties. I provide conditions under which it is optimal to include a strictly less informative test in the optimal menu and when it is not. I also characterise the optimal menu when tests vary in their difficulty and when each test can identify only one dimension of the agent's private information.

I study a DM who has to accept or reject a privately informed agent. While the DM wants to accept a subset of types (the A-types) and reject the others (the R-types), the agent always wants to be accepted. The domain of feasible tests is an exogenously given set of Blackwell experiments. The DM designs a menu of tests, a subset of the feasible tests, from which the agent chooses one. For example, a university can let a student decide whether he takes a standardised test in its admission procedure. A regulator can let a pharmaceutical company design the clinical trials when authorising a new drug. After observing, the choice of test and its result, the DM decides whether or not to accept.

I first show there is no value to commitment: regardless of whether the DM is able commit ex-ante to an acceptance rule, the optimal menu and strategies remain the same. This result holds for arbitrary type structures and domain of feasible tests. Moreover, I find that it is without loss to consider menus with as many tests as A-types. This implies that if there is only one type the DM wants to accept, it is sufficient to consider menus with a single test and the only information used is the test outcome. The result on the value of commitment comes from a max-min representation of the DM's problem where the maximiser chooses the DM and A-types' strategy and the minimiser the R-types' strategy.

In Section 1.4, I use Theorem 1 to determine which tests are part of the optimal menu in three natural economic applications. In Section 1.4.1, I consider a domain of feasible tests containing a dominant one, in the sense of Blackwell's (1953) informativeness order. One example of this environment is a university considering whether to allow students to opt out of a test like a standardised test or an interview when applying. This is effectively offering a menu containing the standardised or the interview and an uninformative test.

In Lemma 1, I show that the most informative test is always part of an optimal menu. I then provide conditions under which a dominated test is part of the optimal menu when tests are binary, i.e., pass-fail tests. In this case, types can be ordered by how likely they are to generate the pass signal in the most informative test. I show that the optimal menu always includes a strictly less informative test if, and only if, the DM's payoff is enclosed. This corresponds to the DM wanting to accept at least the worst and the best performer on the test. On the other hand, for any prior an optimal menu contains only the most informative test if, and only if, the DM's payoff is single-peaked with respect to that order. This corresponds to the DM willing to accept either only high types, only low types or only intermediate types, as measured by their performance on the test. Failure of single-peakness can occur for example when the most informative test does not test all relevant dimensions or only tests a proxy of the relevant dimension.

In the case where the domain of feasible tests contains a dominant tests and they generate more than two signals, the results extend as follows. If there exists a subset of signals where single-peakness is violated, there exists a less informative test that is part of the optimal menu for some prior. On the other hand, if the environment is one-dimensional, in the sense that all the tests satisfy the monotone likelihood ratio property and the DM wants to accept any type above a threshold, only the most informative test is offered.

In the first application, I considered a domain of feasible tests where tests can be ordered by their informativeness. In Section 1.4.2, I consider one-dimensional environments where feasible tests are ordered by their difficulty. For example, the DM could be a regulator deciding how hard a compliance test is before authorising a product. The testing technology is a set of pass-fail tests and varying the difficulty of a test changes which types it identifies better. A more difficult test is informative when it is passed, as only high types are likely to produce a high grade but it is less informative when it is failed. In this case, I show again that a singleton menu is optimal.

In previous sections, I show that for natural specifications of one-dimensional environments, a singleton menu is optimal. I then turn to multidimensional environments. For example, a hiring firm could be guided by two considerations, the candidate's technical and managerial skills and specialise the interview on either dimension. More generally, I assume that the agent's type has two components and each test is informative about only one of them.[1] Offering tests for both dimensions allows A-types that perform badly in one dimension to select the test where they perform best. I show that the optimal menu contains both tests for any prior if and only if the DM wants to accept any type that performs well in at least one dimension. This would be the case if the hiring firm would want to hire a candidate with high technical skills but no managerial skills and vice-versa. On the other hand, if the firm cares about both dimension simultaneously, then, for some priors, it uses only one test.

Finally, I consider two extensions to the baseline model. The first one is to allow for cheap-talk communication on top of the test choice. I can show that in this case, each A-type announces his type deterministically and each R-type pretends to be an A-type. I also consider the possibility of going beyond menus and allow for a mechanism where a mechanism can randomly allocate a type to a test. I show that the DM benefits from having access to a randomisation device and provide a characterisation of the optimal mechanism in terms of another max-min problem (Theorem 3). I also show that in this context, the DM does not benefit from committing to a strategy.

Relation to the literature

This paper relates to both the literature on strategic disclosure and mechanism design with evidence and the literature on information design without commitment. The strategic disclosure literature studies information provision by privately informed players. In these papers, information provision is usually modelled with hard evidence (e.g., Grossman, 1981; Milgrom, 1981; Dye, 1985; Milgrom, 2008). Hard evidence is a particular

[1] The results extend easily to more than two dimensions.

kind of test that takes a deterministic form: the agent can provide evidence that he belongs to a certain subset of types. Another difference with modelling information with evidence is that, in my language, not all types can participate in all tests. Instead, I allow arbitrary stochastic tests and all types can participate in any test. I discuss the relation between these two modelling approaches in more details in Section 1.2.2.

Formally, my model is most closely related to Glazer and Rubinstein (2006). They also study a problem where an agent wants to persuade a DM to accept him but in their model, the agent can only present deterministic evidence about his type. They characterise the optimal mechanism that maps evidence to a decision and show that the outcome can be implemented without commitment (see also Hart et al., 2017, for similar results with other payoff structures; Sher, 2011). They also show that with deterministic evidence, the optimal decision rule is deterministic. I extend their analysis in two ways. First, Theorem 1 generalises their result on commitment to arbitrary testing technology and my characterisation result also applies in their setup. I also show that that the optimal decision rule is no longer deterministic when tests are stochastic. Second, I use the characterisation to prove general results on which test is included in the optimal menu depending on the properties of the feasible tests. Glazer and Rubinstein (2004) study a related problem where the agent communicates with the DM who, based on the communication, verifies one dimension of a multidimensional type.

More generally, this paper relates to the mechanism design with evidence literature (e.g., Green and Laffont, 1986; Bull and Watson, 2007; Deneckere and Severinov, 2008; Koessler and Perez-Richet, 2019; Forges and Koessler, 2005; Kartik and Tercieux, 2012; Strausz, 2017). In Section 1.7, I characterise the optimal mechanism and unlike most of that literature, I allow for arbitrary domain of feasible tests that include non-deterministic tests.[2] The payoff structure assumed in this paper is commonly used in this literature, e.g., in Glazer and Rubinstein (2004); Glazer and Rubinstein (2006) and special cases of Ben-Porath et al. (2019); Ben-Porath et al. (2021). Ben-Porath et al. (2021) show a similar result on the no value of commitment.

An important focus of the literature on strategic disclosure is finding conditions under which all information is revealed in equilibrium, see e.g., Grossman (1981), Milgrom (1981), Lipman and Seppi (1995), Giovannoni and Seidmann (2007), Hagenbach et al. (2014) or Carroll and Egorov (2019). In my model, if full information is possible, it is optimal, but I also characterise the optimal choice of test when full information is not attainable. In Proposition 8, I provide the necessary and sufficient conditions for full payoff-relevant information revelation.

The other branch of literature my paper relates to is information design without sender commitment. In these papers, the agent and the DM correspond to the sender and the receiver. In particular, this paper is closer to models characterising receiver-optimal tests where the sender can choose which test to take. Rosar (2017) and Harbaugh and Rasmusen (2018) the receiver designs a test where a privately informed agent can either take the test, possibly at a cost, or take an uninformative test. In these papers, the receiver flexibly designs a test *given*

[2]For an example of mechanism design paper with non-deterministic tests, see Ball and Kattwinkel (2022), Ben-Porath et al. (2021).

that the sender has a choice. In my paper, the receiver designs the choice, i.e., the menu, given the restrictions on the feasible tests.

Other papers consider the receiver-optimal design of tests where the sender's action is partially observed or unobserved, e.g., DeMarzo et al. (2019), Deb and Stewart (2018), Perez-Richet and Skreta (2022) or Ball (2021) (note that Perez-Richet and Skreta (2022) also consider observable action). The design of the optimal test also has to take into account the strategy of the sender, however unobservable actions fundamentally changes the sender's incentives and thus how information is revealed. I discuss in Section 1.2.2 which results would still apply if the outcome of the tests depends on the agent's unobserved effort.[3]

Finally, this paper is related to Ely et al. (2021). They study the optimal allocation of tests from a restricted set to agents with observable characteristics. My paper can be interpreted as a problem of optimal allocation of tests with asymmetric information, thus the allocation must also respect incentive constraints.

1.2 Model

There is a decision-maker (DM) and an agent. The agent has a type $\theta \in \Theta$, $|\Theta| < \infty$, with a common prior $\mu \in \Delta(\Theta)$. The set of types is partitioned in two: $\Theta = A \cup R$, $A \cap R = \emptyset$. The type is private information of the agent. The DM must take an action $a \in \{0, 1\}$, accept or reject. The utilities of the DM and the agent are $v(a, \theta) = a\big(\mathbb{1}[\theta \in A] - \mathbb{1}[\theta \in R]\big)$ and $u(a, \theta) = a$. That is, the DM wants to accept agents in A and reject agents in R. The agent always wants to be accepted. The analysis is virtually unchanged by allowing for DM's utility functions of the form $v(a, \theta) = a\nu(\theta)$ for some $\nu : \Theta \to \mathbb{R}$.

There is a finite exogenous set of test $T \subseteq \Pi \equiv \{\pi : \Theta \to \Delta X\}$, where X is some finite signal space. The conditional probabilities of test t are $\pi_t(\cdot|\theta)$. The set T captures the restriction on the DM's testing capacity. He can only perform one test from that set. For simplicity, I assume that for any $\theta \neq \theta'$, $\pi_t(\cdot|\theta) \neq \pi_t(\cdot|\theta')$. A menu of test is a subset of the feasible tests, $\mathcal{M} \subseteq T$.

The timing of the game is as follows. For a menu $\mathcal{M} \subseteq T$,

1. The agent learns his type θ.

2. The agent chooses a test from the menu, denoted by $\sigma : \Theta \to \Delta\mathcal{M}$.

3. A signal x is drawn according to $\pi_t(\cdot|\theta)$.

[3] There are also papers studying sender-optimal tests when the sender cannot fully commit to reporting the test correctly, e.g., Nguyen and Tan (2021), Lipnowski et al. (2022) or Koessler and Skreta (2022). In Boleslavsky and Kim (2018) and Perez-Richet et al. (2020), the sender can commit but there is a third agent whose effort determines respectively the state of the world or the Blackwell experiment actually performed.

4. The DM chooses an action based on the realised test choice and outcome, the acceptance probability denoted by $\alpha : \mathcal{M} \times X \to [0, 1]$.

Beliefs of the DM are $\tilde{\mu} : \mathcal{M} \times X \to \Delta\Theta$, a probability distribution over types given an observed test and signal realisation.

The solution concept is DM-preferred weak Perfect Bayesian Equilibrium.

I write $(\alpha, \sigma) \in \text{wPBE}(\mathcal{M})$ if there is a belief $\tilde{\mu}$ where $(\alpha, \sigma, \tilde{\mu})$ is a weak PBE when the menu is $\mathcal{M}$.

The optimal design of menu solves

$$V = \max_{\mathcal{M} \subseteq T} \max_{\sigma, \alpha} \sum_{\theta \in A} \mu(\theta) \sum_{t \in \mathcal{M}} \sigma(t|\theta) \sum_{x} \alpha(t, x)\pi_t(x|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_{t \in \mathcal{M}} \sigma(t|\theta) \sum_{x} \alpha(t, x)\pi_t(x|\theta)$$
$$\text{s.t. } (\alpha, \sigma) \in \text{wPBE}(\mathcal{M})$$

The inner maximisation problem selects, for a fixed menu, the DM and agent strategy to maximise the DM's payoff for a fixed menu, under the constraint that they are equilibrium strategies. The outer maximisation problem selects the best possible menu for the DM.

Notation: For any α, denote the probability of type θ to be accepted in test t by $p_t(\alpha; \theta) \equiv \sum_x \alpha(t, x)\pi_t(x|\theta)$.

Off-path beliefs: The results would exactly the same if I would take DM-preferred Sequential Equilibrium (Kreps and Wilson, 1982) as my solution concept. I comment on this in more detail in the discussion of Theorem 3.

Test restriction: The exogenous set of tests T can capture different constraints on DM's testing capacity. It could be a purely technological constraint, e.g., when choosing amongst standardised test, universities can only choose from an exogenously given set of tests (SAT, ACT, GRE, etc.). The constraint can also be on some properties of the tests that can be used, e.g., $T \subset \{\pi : \pi \text{ has the MLRP}\}$. Finally, it could come from a capacity constraint in the information processing/acquisition abilities of the DM, e.g., a limited number of sample sizes a researcher can collect or there could be a cost function associated with each experiment $C : \Pi \to \mathbb{R}$ and a maximum cost the DM can pay $c \in \mathbb{R}$, $T \subset \{\pi : c \geq C(\pi)\}$.

1.2.1 Example: Opting out of an admission test

Suppose a university uses some test for university admission and that there are three types of students: $A = \{A1, A2\}$ and $R = \{R1\}$. Consider the testing set $T = \{t, \emptyset\}$ where $\emptyset$ is an uninformative test. The test t is described by $X = \{x_0, x_1\}$ and

$$\pi_t(x|A1) = \begin{cases} 1/2 & \text{if } x = x_0 \\ 1/2 & \text{if } x = x_1 \end{cases} \qquad \pi_t(x|R1) = \begin{cases} 1/3 & \text{if } x = x_0 \\ 2/3 & \text{if } x = x_1 \end{cases}$$

$$\pi_t(x|A2) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

Furthermore, suppose that $\mu(A1) < \frac{2}{3}\mu(R1) < \mu(A2)$.

This example can be interpreted as follows. The test t is a test a university uses to get information about students, like an interview or the SAT or GRE. The signal x_1 represents a high grade and x_0 a low grade. A common concern about these tests is that they can be too easily gamed or fail to identify good students in some categories of the population (see e.g., Hubler, 2020). The parametrisation of the test t captures this phenomenon. While A_2 and R_1 are naturally ordered, in the sense that A_2 is more likely to have a good grade than R_1, A_1 corresponds to a type of student that the university wants to accept but generates a lower grade than R_1. Adding $\emptyset$ to the menu allows the student to opt out from the standardised test.

When only t is offered: The information structure and prior deliver the following best response when only t is offered,

$$\alpha(x,t) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

The acceptance probabilities of each types are then

$$p_t(\alpha; R1) = 2/3 \qquad p_t(\alpha; A1) = 1/2 \qquad p_t(\alpha; A2) = 1$$

When both t and $\emptyset$ are offered: Consider the equilibrium with the following strategies of the agent:

$$\sigma(\emptyset|R1) = \frac{\mu(A1)}{\mu(R1)} \qquad \sigma(\emptyset|A1) = 1 \qquad \sigma(t|A2) = 1$$

The student $R1$ mixes between the two tests, t and $\emptyset$, whereas $A1$ chooses $\emptyset$ with probability one and $A2$ chooses t with probability one. Note that if all types play a pure strategy, it is not possible to maintain an equilibrium where both tests are chosen. If it is the case, there is a test that is only chosen by an A-type and in equilibrium the DM must accept with probability one after any signal in that test. Thus R_1 mixes in equilibrium to make the menu $\{t, \emptyset\}$ credible.

Given the agent's strategy, the DM's strategy after t remains the same as before. When the DM observes $\emptyset$, he is indifferent between accepting and rejecting. He then mixes in a way that makes $R1$ indifferent between $\emptyset$ and t: $\alpha(x, \emptyset) = 2/3$. The resulting acceptance probabilities are

$$\mathbb{E}[p(\alpha; R1)] = 2/3 \qquad p_\emptyset(\alpha; A1) = 2/3 \qquad p_t(\alpha; A2) = 1$$

Types $R1$ and $A2$ have the same acceptance probabilities as before but $A1$ is accepted with strictly higher probability. Therefore, allowing to opt out strictly increases the DM's payoffs.

1.2.2 Discussion

Effort: The outcome of the test is independent of the agent's action. The model would go unchanged if effort is costless and observable as it could be deterred with off-path beliefs. If the effort is costless but unobservable the results would generally change. However, if signals are ordered and the DM uses a cutoff strategy, as in many natural applications, a reasonable assumption on effort would be that the higher the effort, the likelier a high signal. In this case, the agent would always have an incentive to provide high effort. See Deb and Stewart (2018) and Ball and Kattwinkel (2022) for models that takes into account both asymmetric information and moral hazard in a model of testing.

Relation to models with evidence: The model can be interpreted as a generalisation of models with evidence. The idea of these models is that each type is endowed with a set of messages that only a subset of types can send. Formally, an evidence structure is a correspondence $E : \Theta \rightrightarrows M$ for some finite set of messages M. Thus type θ can only send messages in $E(\theta)$. We can capture these models in the following way. The set of feasible test has $X = \{x_1, x_0\}$ and for all $m \in M$, $\pi_m(x_1|\theta) = 1 \Leftrightarrow \theta \in E^{-1}(m)$. Thus a test m perfectly reveals whether θ is in $E^{-1}(m)$ or in $\Theta \setminus E^{-1}(m)$. In a model with evidence, a type θ can never reveal he is in $\Theta \setminus E^{-1}(m)$ for a message $m \notin E(\theta)$. However, in the testing model, we can always incentivise any type to not choose such a test by setting $\alpha(x_0, m) = 0$ for all m. This strategy could be justified because (x_0, m) would always be off-path. Alternatively, we can set this restriction on α directly and Theorem 3 would still hold.

1.3 Characterisation of the optimal menu

An important step in the characterisation of the optimal menu is to show that commitment to a strategy does not have value for the DM. I also show that in the DM-preferred equilibrium, the A-types play a pure strategy.

Abusing notation, denote by $\sigma^{\Theta'} : \Theta' \to \Delta T$ for any $\Theta' \subseteq \Theta$ the strategies of a subset of types Θ' and by

$$v(\alpha, \sigma^A, \sigma^R) \equiv \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta')$$

the DM's payoffs.

Theorem 1. *The value of the optimal menu is*

$$V = \max_{\alpha, \sigma^A} \min_{\sigma^R} v(\alpha, \sigma^A, \sigma^R). \tag{1.1}$$

For any $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha}, \tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$ *and* $\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$, $(\alpha, \sigma^A, \sigma^R)$ *are the equilibrium strategies in a DM-preferred equilibrium.*

Moreover,

- *the DM does not benefit from commitment,*
- *There exists a DM-preferred equilibrium where* σ^A *is in pure strategies and therefore* $|\mathcal{M}| \leq |A|$.

All proofs are relegated to Section 1.9.

To understand Theorem 1 better first note that we can rewrite (1.1) as

$$\max_{\alpha} \left[\max_{\sigma^A} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta) - \max_{\sigma^R} \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta') \right].$$

That is, it corresponds to the problem of finding the best strategy α when the agent chooses a test to maximise his payoffs given α. Because the interest of the A-types is fully aligned with the DM's, they try to maximise his payoffs whereas the R-types have opposite interests and thus try to minimise the DM's payoffs. Unlike the original DM problem, the DM can commit to α and thus it does not need to be a best-reply. Theorem 1 shows that the optimal α under commitment actually is a best-reply to the strategy chosen by the agent.

Theorem 1 provides two important tools to characterise the optimal menu. The first one is to establish that the DM does not benefit from commitment. This is is a powerful tool to test equilibria. Indeed, it is not necessary to compare equilibria across menus to establish that an equilibrium (α, σ) is not optimal. It is enough to find an alternative DM strategy $\tilde{\alpha}$ such that

$$\max_{\sigma^A} \min_{\sigma^R} v(\alpha, \sigma^A, \sigma^R) < \max_{\sigma^A} \min_{\sigma^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \sigma^R)$$

to show that (α, σ) does not constitute an optimal menu without having to worry whether $\tilde{\alpha}$ is part of an equilibrium strategy.

I interpret this result as a hierarchy over sources of learning. The DM has two sources of information, the "hard information" from the test results and the endogenously created information from the choice of test. When the DM can commit to a strategy, he can "sacrifice" payoffs from the test result by not best replying, in order to create separation of types through the test choice. By showing that the DM always best replies, even when he can commit, I show that he should always prioritise the hard information over creating endogenous information through the test choice.

That commitment has no value in this game comes from the zero-sum structure of the characterisation. Because a minimax theorem holds, this implies that the order of moves do not matter in this game: the DM has the same payoffs if he moves first or last.

The second tool is to establish that the size of the menu is bounded by the number of A-types. This limits the number of tests we need to consider. An immediate corollary is also that if there is only one type the DM would like to accept an optimal menu is to use only one test. In particular, this results shows that in a binary state environment, the optimal mechanism uses only one test, no matter what the available set of test is.

Corollary 1. *Suppose* $|A| = 1$. *Then for any* T*, there is an optimal menu that uses only one test.*

Sequential Equilibrium. Note that if the solution concept is DM-preferred Sequential Equilibrium (SE) (Kreps and Wilson, 1982), Theorem 1 would also hold. If all tests have full support, then all signals are on-path and the PBE and SE coincide. If some tests do not have full support, then I can always assume that the trembling of R-types is more likely than the trembling of A-types. Then, the DM's off-path beliefs after the pair (t, x) are that the type is an A-type if the support of A- and R-types do not coincide and that the type is an R-type otherwise. This guarantees that if an A-type finds it profitable to deviate the problem without commitment then he would also find it profitable in the problem with commitment.

1.4 Applications

1.4.1 Optimal menu with Blackwell dominant test

It is common in applications that the DM has access to a most informative test. This can be because the choice is simply between a test and opting out of the test like in the admission test example. It can also come from the structure of the constraints. For example, the DM could have a time budget to conduct an interview. The more time the interview takes, the more informative it is. Another possibility is that the DM can easily make a test less informative by simply not conducting part of the test. If a test is composed of a series of questions, the DM can ignore some of them.

I will use Blackwell (1953)'s notion of informativeness.

Definition 1 (Blackwell (1953))**.** *A test t is more informative than t', $t \succeq t'$, if there is function $\beta : X \times X \to [0,1]$ such that for all $x' \in X$, $\sum_x \beta(x,x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x,x') = 1$.*

I call a test t a dominant test if $t \succeq t'$ for all $t' \in T$. If a test is more informative than another then in any decision problem, i.e., a pair of utility function and a prior, using the more informative test yields higher expected utility. A first important fact we will record here is that if there is a most informative test, then it is part of an optimal menu.

Lemma 1. *If there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$, then there is an optimal menu that includes t.*

This lemma follows from the commitment result of Theorem 1 and the properties of dominant tests. Indeed, if we find a menu where the dominant test t is not included, we can modify the DM's strategy such that one A-type is accepted with higher probability than the test he is choosing, say t', and all R-types are accepted with lower probability than in t'. Then this A-type has a profitable deviation to t.

As we have seen in the admission test example in Section 1.2.1, it can be optimal to add a strictly less informative in the optimal menu. I first focus on binary signals environment, $X = \{x_0, x_1\}$. Let t be the most informative test in T. When signals are binary, we can order the types by their likelihood of generating signal x_1: $\theta \geq_t \theta' \Leftrightarrow \pi_t(x_1|\theta) \geq \pi_t(x_1|\theta')$.[4] I characterise the optimal menu for different payoff function of the DM.

Definition 2. *The DM's preferences are single-peaked given the order $\geq$ on Θ if there is $\theta_1, \theta_2 \in A$ such that $A = \{\theta : \theta_1 \leq \theta \leq \theta_2\}$.*

Preferences are single-peaked if the DM only wants to either only accept high types, only low types or only intermediate types, where the order is determined by the performance of types on the test. Preferences are not single-peaked whenever it is possible to find $A_1, A_2 \in A$ and $R_1 \in R$ such that $A_1 <_t R_1 <_t A_2$. This was for example the case in the admission test example in Section 1.2.1.

We get the following characterisation.

Proposition 1. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and let $\geq_t$ on Θ be the order implied by t.*

The singleton menu $\{t\}$ is optimal for any $\mu \Leftrightarrow$ the DM's preferences are single-peaked given $\geq_t$.

[4]Note that given that tests are binary, this is equivalent to ordering type by the likelihood ratio, $\frac{\pi(x_1|\theta)}{\pi(x_0|\theta)}$.

From Lemma 1, the most informative test is part of the optimal menu. Whenever the DM's preferences are single-peaked, if the most informative test is included in the menu, the unique resulting equilibrium is one where all types choose the most informative test. The key argument in the analysis is noting that $p_t(\alpha;\theta) - p_{t'}(\alpha;\theta)$ is single-crossing in θ with respect to the order $\geq_t$, for any α. When preferences are single-peaked, we can use the single-crossing condition and properties of tests satisfying the monotone likelihood ratio property to show that there is a unique equilibrium where only t is chosen.

On the other hand, if the preferences are not single-peaked, there is a prior where offering even a completely uninformative test with the most informative test is strictly better for the DM. To illustrate, consider three types $A_1, A_2 \in A$ and $R_1 \in R$ such that $A_1 <_t R_1 <_t A_2$. Suppose the prior is such that if only t is offered, the DM accepts after x_1 and rejects after x_0. The DM can then offer an uninformative test where the probability of being accepted makes R_1 indifferent but is strictly preferred by A_1. This constitutes a deviation in the problem with commitment. This reasoning can be used to show that including a less informative test is always beneficial whenever the DM's payoff is *enclosed* given $\geq$: there is $\theta_1, \theta_2 \in A$ such that $\theta_1 < \theta < \theta_2$ for any $\theta \neq \theta_1, \theta_2$.

Proposition 2. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and let $\geq_t$ on Θ be the order implied by t.*

The DM's preferences are enclosed given $\geq_t \Leftrightarrow$ the DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$ for any μ and $t' \in T$.

The ideas of Proposition 1 and Proposition 2 can be partially extended to more than two signals. First, if all tests satisfy the monotone likelihood ratio property and the DM only wants to accept types above a threshold, the optimal menu is to only offer the most informative test.

Proposition 3. *Suppose $\Theta, X \subset \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ for some $\overline{\theta}$ and all tests in T have full-support and the monotone likelihood ratio property: for $\theta > \theta'$,*

$$\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \text{ is increasing in } x.$$

If there is $t \succeq t'$ for all $t' \in T$, then, the menu $\{t\}$ is optimal.

Again this result holds by showing a single-crossing difference property on the acceptance probability. Intuitively, the reason is that more informative tests send relatively higher signals for higher types. So if a low type chooses the most informative test, the higher types must also choose that one. This prevents any pooling of A-types and R-types on two different tests. Combined with Lemma 1 that guarantees the inclusion of the dominant test, we get our result. Note also that this result would hold using weaker information order like Lehmann (1988) or some weakening of it. The key property delivering the result is the single-crossing condition described above.

If it is possible to find two signals, x, x', two A-types A_1, A_2 and one R-type, R_1 such that $\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}$, then there is a test t' strictly less informative than t and a prior such that offering $\{t, t'\}$ is better for the DM than just offering $\{t\}$.

Proposition 4. *Let t be a test. Suppose there are two signals $x, x' \in X$, types $A_1, A_2 \in A$ and $R_1 \in R$ such that*

$$\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}.$$

There is a prior μ and a test $t' \prec t$ such that the DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$.

Intuitively, if we interpret x as a high signal, the A-type A_1 sends relatively low signals. Suppose that the prior is such that, if only t is offered, x is accepted and x' is not. In a sense, it means that in the test t, type R_1 performing better than A_1 on the signals x, x'. It is then beneficial for the DM to include a test that pools signals x, x' together. In that new test, type A_1 can choose the coarsened test where the superior performance of type R_1 is less important than in the original test.

1.4.2 Optimal menu with tests ordered by their difficulty

In many economic environments, the DM does not necessarily have access to a most informative test but can vary the difficulty to pass a test. This is for example the case for a regulator that can decide how demanding a certification test is. Like in Proposition 1 and Proposition 3, I show that the optimal menu is a singleton.

I first formalise the notion of more difficult test as follows.

Definition 3 (Difficulty environment)**.** *An environment is a Difficulty environment if $\Theta \in \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ for some $\overline{\theta}$, $X = \{x_0, x_1\}$, $T \subset \mathbb{R}$, all tests have full-support, satisfy the monotone likelihood ratio property and for all $t > t'$, and $\theta > \theta'$,*

$$\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} \geq \frac{\pi_{t'}(x_1|\theta)}{\pi_{t'}(x_1|\theta')} \quad \textit{and} \quad \frac{\pi_t(x_0|\theta)}{\pi_t(x_0|\theta')} \geq \frac{\pi_{t'}(x_0|\theta)}{\pi_{t'}(x_0|\theta')}$$

If $t > t'$, I will say that t is harder than t'. To understand the last condition better, let $\mu(\cdot|x, t)$ be a posterior belief after observing signal x in test t. The monotone likelihood ratio property implies $\mu(\cdot|t, x_1) \succeq_{FOSD} \mu(\cdot|t, x_0)$, a higher signal is "good news" about the type (Milgrom, 1981). The last property in the definition further implies $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$. That means that a pass grade shifts beliefs more towards higher type in a harder test and a fail grade shifts more beliefs towards lower types in an easy test. Or put differently, the harder a test the more informative it is about a high type when there is a pass-grade whereas an easier test is informative about the low types when the test is failed. As an example, if $\Theta \subset (0,1)$ and $\pi_t(x_1|\theta) = \theta^t$ we are in a Difficulty environment.

Proposition 5. *In a Difficulty environment, a singleton menu is optimal.*

Like Proposition 1 and Proposition 3, Proposition 5 illustrates how incentive constraints shape the size of the optimal menu. In the case of the single-peaked preferences with dominant test, the equilibrium when the most informative test is offered is unique and only that test is chosen. Here, it is possible to construct an equilibrium where more than one test is chosen in equilibrium. However, the DM strategy needed to sustain that equilibrium is such that he is better off offering only one test.

The proof proceeds in two steps. First, I show that there are at most two tests in the optimal menu and if there are two tests, the harder test must be more lenient that the easy test. In particular, I show that after the hard test, the DM must accept with some probability after a fail signal and in the easy test, reject with positive probability after a pass grade.

This means that to maintain incentives to select both tests, the DM only reacts to the least informative signal from the test: in the hard test after a fail grade, in the easy test after a pass grade. This in turn implies that it would be better for the DM to use only one test and reject after a fail grade and accept after a pass grade.

1.4.3 Bidimensional environment

In this subsection, I apply the tools of Theorem 1 to study environments with bidimensional types. The analysis here can be easily extended to more than two dimensions. I assume that the DM has access to tests that is only informative about one dimension and the preference of the DM have some monotonicity along each dimension.

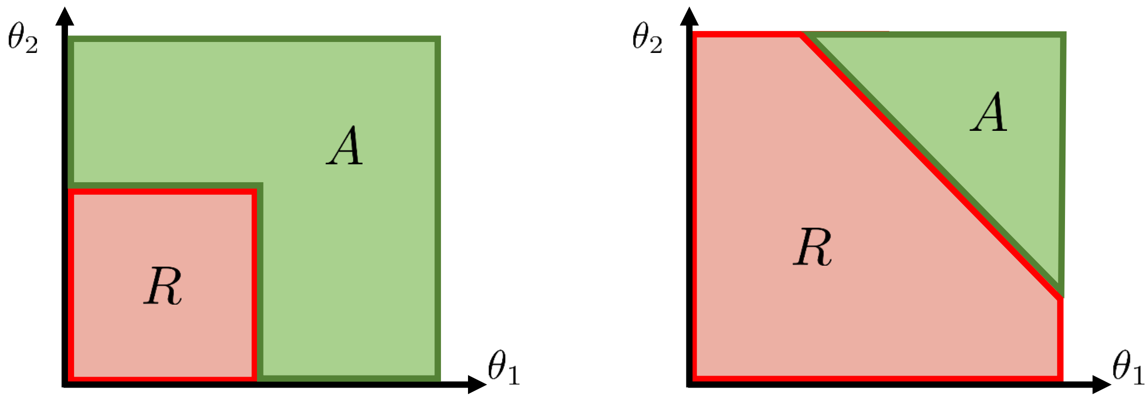
Definition 4. *An environment is bidimensional if* $\Theta = \Theta_1 \times \Theta_2 \subset \mathbb{R}^2$, $X \subset \mathbb{R}$ *and* $T = \{t_1, t_2\}$ *such that for* $i = 1, 2$,

- *if* $\theta \in A$, *then for all* $\theta' \geq \theta$, $\theta' \in A$
- t_i *has full support and for all* $\theta_i > \theta_i'$,

$$\frac{\pi_{t_i}(x|\theta_i, \theta_j)}{\pi_{t_i}(x|\theta_i', \theta_j)} \text{ is strictly increasing in } x \text{ for any } \theta_j \in \Theta_j$$

- $\pi_{t_i}(x|\theta_i, \theta_j) = \pi_{t_i}(x|\theta_i, \theta_j')$ *for all* $\theta_j, \theta_j' \in \Theta_j$ *and* $x \in X$

The first condition captures the idea that a higher type is always better for the DM. The second and third



(a) Full menu always optimal.

(b) Full menu not always optimal.

Figure 1.1: Illustration of DM's preferences for Proposition 6.

condition captures the idea that each test is only informative about one dimension and that a higher signal corresponds to a higher type in that dimension.

In this environment, whether the DM wants to offer a menu depends crucially on his preferences. In particular, I give a necessary and sufficient condition on the preferences such that a menu is optimal for any prior. Let $\overline{\theta}_i = \max \Theta_i$.

Proposition 6. *Suppose we are in a bidimensional environment. Offering a menu $\{t_1, t_2\}$ is strictly optimal for any prior if and only if*

$$\textit{for } i = 1, 2,\ (\overline{\theta}_i, \theta_j) \in A, \textit{ for all } \theta_j \in \Theta_j. \tag{1.2}$$

The proof of Proposition 6 works by showing that a deviation from a single test menu is always profitable when condition (1.2) is satisfied and constructs a prior under which there are no profitable deviations when the condition is not satisfied.

Figure 1.1 illustrates the condition of Proposition 6 with $\Theta \subset [0, 1]^2$. In Figure 1.1a, the DM wants the agent's type to be high enough in at least one dimension. Then the DM always prefers to offer a full menu to the agent. On the other hand, in Figure 1.1b, the DM does not want to accept a type that is high in only one dimension. In this case, for some prior, the DM only wants to offer one test. This happens when after any deviation from the singleton menu any A-type is mimicked by too many R-types that cannot be distinguished from him.

1.5 Sufficient conditions for test inclusion

In this section, I study in more details the notion of efficient allocation of tests to the agent's types. I show that a sufficient condition to include a test in the optimal menu is if it is good at differentiating one A-type from all the R-types. This captures a notion of a test tailored for the A-type.

Definition 5. *Fix $\theta \in A$. Test t θ-dominates t', $t \succeq_\theta t'$, if there is $\beta : X \times X \to [0,1]$ such that for all $x' \in X$*

$$\sum_x \beta(x,x')\pi_t(x|\theta) \leq \pi_{t'}(x'|\theta)$$

$$\textit{for all } \theta' \in R, \quad \sum_x \beta(x,x')\pi_t(x|\theta') \geq \pi_{t'}(x'|\theta')$$

$$\textit{for all } x \in X, \quad \sum_{x'} \beta(x,x') \leq 1$$

To understand this definition better, compare it to Blackwell (1953)'s informativeness order. It requires the existence of a function β such that for all $x' \in X$, $\sum_x \beta(x,x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x,x') = 1$. The key difference is that we restrict attention to one A-type and all the R-types. This captures the idea the test θ-dominant test is tailored to differentiate θ from each R-type. The second difference is that it requires only inequalities whereas the Blackwell order requires equalities. This is because we are fixing the utility function we are interested in, unlike in Blackwell (1953).

If a type $\theta \in A$ has a $\succeq_\theta$-dominant test, then this test is used in an optimal menu. This shows that an important property of tests is not so much how good they are at differentiating types, but how good they are at differentiating one type the DM wants to accept from all the types he wants to reject.

Proposition 7. *Suppose there is $t \in T$ and $\theta \in A$ such that $t \succeq_\theta t'$ for all $t' \in T$, then t is part of an optimal menu.*

The stronger notion of a test able to differentiate some $\theta \in A$ from all R-types is if $\operatorname{supp} \pi_t(\cdot|\theta) \cap \Big(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \Big) = \emptyset$. If each type in A has such a test, then the principal never makes a mistake. This condition is also necessary.

Proposition 8. *The principal's expected payoff is $\sum_{\theta \in A} \mu(\theta)$ if and only if for all $\theta \in A$, there exists $t \in T$ such that*

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \Big(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \Big) = \emptyset$$

Here, the principal just needs for each type he wants to accept a test where he can discriminate between that type and the R-types. Then he can offer a menu of tests where each A-type self selects into the test that discriminates him from the R-types. The actual learning only happens by observing the test selected and the testing technology serves as a detriment to deviations from R-types. The argument is then similar to an unravelling argument à la Milgrom (1981) and Grossman (1981). These are not fully revealing tests but tests that allow to perfectly discriminate *one* A-type from all the R-types. But it could be a very noisy tests for the other A-types.

1.6 Extension: Communication

I consider here the possibility of adding a communication channel on top of the test choice. There is now a finite set C of output messages with $|C| \geq |A|$ and a strategy is a mapping $\sigma : \Theta \to \Delta(T \times C)$. Note that all the results from the previous sections go through as from any finite set T one can create another T' that duplicate each test $|C|$ times. I call this variant of the model the *menu game with communication*.

In line with Theorem 1, each A-type chooses a message-test pair deterministically and each R-type mixes over some A-types message-test pair. Moreover, I show that when communication is added, each type in A announces his type, thus maximally differentiating himself, and each R-type pretends to be an A-type.

Theorem 2. *If communication is allowed, the same construction as Theorem 1 holds. Moreover, there is a DM-preferred equilibrium where each A-type reports his own type.*

Theorem 2 shows that the results extend naturally to an environment where communication is allowed. Because the DM could commit to a strategy, he can always guarantee each A-type at least as much as he would have if he would pool with another A-type. This guarantees that there is an a solution to the problem with commitment where he separates from the other A-types.

Communication can play an important economic role. For example, if the set of feasible tests contains a Blackwell dominant test and communication is allowed, the DM only use the dominant test.

Proposition 9. *Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and communication is allowed. Then an optimal menu is $\{t\}$.*

Example from Section 1.2.1 revisited: Suppose that we allow for communication in the admission test example (Section 1.2.1). In this case, all types choose test t, type $A1$ and $A2$ communicate their type and $R1$ mixed over the two messages:

$$\sigma(A1, t|R1) = \frac{3\mu(A1)}{4\mu(R1)} \qquad \sigma(A1, t|A1) = 1 \qquad \sigma(A2, t|A2) = 1$$

The DM accepts only after the signal x_1 after the message test pair $(A2, t)$ and mixes after x_1 and accepts after

x_1 in the message test pair $(A1, t)$:

$$\alpha(x, A1, t) = \begin{cases} 1 & \text{if } x = x_0 \\ 1/2 & \text{if } x = x_1 \end{cases} \qquad \alpha(x, A1, t) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

The probability of acceptance of each type is now:

$$\mathbb{E}[p(\alpha; R1)] = 2/3 \qquad p_{A1,t}(\alpha; A1) = 3/4 \qquad p_{A2,t}(\alpha; A2) = 1$$

Note that the DM must now use a different decision rule for the same test. In particular, the meaning of the test is different: a "high grade" after message $A1$ is x_0 whereas a "high grade" after message $A2$ is x_1.

1.7 Mechanism and randomisation

So far we have restricted attention to menus of tests. But the DM could potentially use a more elaborate *mechanism* to allocate tests to agents. I define a mechanism $\tau : \Theta \to \Delta T$, a possibly random mapping from type report to distribution over tests. A strategy for the DM remains a mapping from test allocation and signal realisation to an acceptance decision, $\alpha : T \times X \to [0, 1]$, and the agent's strategy is a mapping from type to type report. I assume that the DM cannot observe the type report but we can naturally extend the mechanism τ to allow for output messages in the spirit of Section 1.6. Standard revelation principle arguments show that it is without loss of generality to restrict attention to type reports.

The DM's problem is now to maximise his expected payoff subject to incentive-compatibility constraints. In the baseline model, the DM cannot commit to its strategy α. Let $BR(\tau) := \{\alpha : \mathbb{E}_{\alpha,\tau}[v(a, \theta)] \geq \mathbb{E}_{\alpha',\tau}[v(a, \theta)], \text{ for all } \alpha'\}$, the set of best-reply when the mechanism is τ. The DM's problem is

$$\begin{aligned} \max_{\tau,\alpha} \; & \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha; \theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha; \theta) \\ \text{s.t. } & \sum_t (\tau(t|\theta) - \tau(t|\theta')) p_t(\alpha; \theta) \geq 0 \text{ for all } \theta, \theta' \\ & \alpha \in BR(\tau) \end{aligned}$$

The first constraint is the incentive compatibility constraint of type θ deviating to θ' and the second constraint ensures that the DM best replies to the information revealed by the output of the mechanism.[5]

[5]This definition does not put constraints on off-path optimality but because any strategy is best-reply to some beliefs, this guarantees that satisfying $BR(\tau)$ will lead to a weak PBE.

By using a mechanism, the agent can be randomly allocated to different tests without being indifferent between them. On the other, when restricting attention to menus, the agent has to be indifferent between tests if he randomises over tests. In the following example (inspired by Glazer and Rubinstein, 2004), I show that access to a mechanism can strictly improve the DM payoffs.

Example 1 (Randomised allocation)**.** Suppose there are six types $\Theta = \{(\theta_1, \theta_2, \theta_3) : \theta_i = 0, 1,\ 1 \leq \theta_1 + \theta_2 + \theta_3 \leq 2\}$ and $A = \{(\theta_1, \theta_2, \theta_3) : \theta_1 + \theta_2 + \theta_3 = 2\}$. The DM has access to three tests, each perfectly revealing one dimension: $T = \{1, 2, 3\}$ with $\pi_t(x = \theta_t|\theta) = 1$. The prior μ is uniform. At the optimum, the DM accepts when the signal is equal to 1. The optimal mechanism τ allocates each A-type with probability $1/2$ to each test where their dimension is equal to 1. Each R-type is allocated with probability $1/2$ to the dimension where it has value 1 and $1/4$ in the other dimensions.

This mechanism accepts A-types with probability one and accepts R-types with probability $1/2$. In particular, the mechanism randomises the allocation of R-types over tests they are not indifferent between. If the DM could only use a menu, the R-types would always choose the test that reveals their dimension equal to one. Thus any menu and strategy that accepts R-types with probability $1/2$ must also accept A-types with probability $1/2$.

I now show that the optimal mechanism can also be characterised by a max-min problem and the DM does not benefit from commitment.

To set up the characterisation of the optimal mechanism, let $s : A \to \Delta T$ and $m : R \to \Delta A$ and abusing notation, let $\alpha : T \times X \to [0, 1]$ and

$$v(\alpha, s, m) \equiv \sum_{\theta \in A} \sum_{t \in T} s(t|\theta) \Big[\mu(\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') m(\theta|\theta') p_t(\alpha; \theta')\Big] \tag{1.3}$$
$$= \sum_{\theta \in A} \sum_{t \in T} \mu(\theta) s(t|\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{\theta \in A} m(\theta|\theta') \sum_{t \in T} s(t|\theta) p_t(\alpha; \theta')$$

The function s can be interpreted as A-types choosing a test, m as R-types choosing an A-type to mimic, α as the DM accepting the agent after a test and signal realisation. The function v is then the DM's expected payoffs from a distribution over tests induced by the pair (s, m). I explain these objects in more detail in the discussion of Theorem 3.

Theorem 3. *The value of an optimal mechanism is*

$$V = \max_{\alpha, s} \min_{m} v(\alpha, s, m) \tag{1.4}$$

For any $(\alpha, s) \in \arg\max_{\tilde{\alpha}, \tilde{s}} \min_{\tilde{m}} v(\tilde{\alpha}, \tilde{s}, \tilde{m})$ *and* $m \in \arg\min_{\tilde{m}} \max_{\tilde{\alpha}} v(\tilde{\alpha}, s, \tilde{m})$*, an optimal mechanism is*

- *for* $\theta \in A : \tau(t|\theta) = s(t|\theta)$
- *for* $\theta' \in R : \tau(t|\theta') = \sum_{\theta \in A} m(\theta|\theta')\tau(t|\theta)$
- *the DM's strategy is* α

Moreover, the DM does not benefit from committing to α*.*

Theorem 3 provides another characterisation of the optimal mechanism in terms of a max-min problem. As in Theorem 1, the no value of commitment follows from the max-min structure of the characterisation. To understand the structure of this max-min problem better, consider the objective function v for a fixed α. This can be interpreted as a zero-sum game where where the maximiser, the A-types, chooses $s : A \to \Delta T$ and the minimiser, the R-types, chooses $m : R \to \Delta A$. The payoffs of a given A-type θ choosing test t and a given R-type, θ', choosing an A-type $\tilde{\theta}$ can be expressed as:

$$
\begin{aligned}
&\text{for } \theta \in A \text{ choosing } t,\ \mu(\theta)p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta')m(\theta|\theta')p_t(\alpha;\theta') \\
&\text{for } \theta' \in R \text{ choosing } \tilde{\theta},\ \mu(\theta')\sum_t s(t|\tilde{\theta})p_t(\alpha;\theta') - \sum_{\theta \in A, t} \mu(\theta)s(t|\theta)p_t(\alpha;\theta)
\end{aligned}
$$

Note that in the payoffs of the R-type, his strategy, the choice of $\tilde{\theta}$, only affects the first part of the payoffs. So the R-type is effectively trying to maximise his probability of being accepted. On the other hand, the A-type maximise a modified version of their utility where they maximise their probability of being accepted while being penalised every time a R-type mimics them and is accepted. The A-types' utility is thus modified to align it with the DM's payoffs. The induced distribution over tests determines the optimal mechanism when strategy α is used.

1.8 Conclusion

I study the design of optimal menus of tests. Menus allow the DM to have an additional dimension for information revelation as well as allow for a more efficient allocation of tests to the agent's types. I show that for arbitrary structures on types or tests available, the DM does not benefit from committing to a strategy and the size of the optimal menu is bounded by the number of types the DM wants to accept. In applications, I show that using a menu can be a powerful tool, and even a dominated test, in the Blackwell sense, can be part of the optimal menu. However, this channel also has limits and I show that in some natural economic environments the optimal menu is a singleton. All the results also hold when the DM can commit to an action. I interpreted this result as a hierarchy over information sources: even when the DM can use a suboptimal strategy to "artificially" incentivise the agent to choose different tests, he is better off using a menu only when he can best reply to the information revealed.

An important technical observation throughout the paper is that single-crossing conditions on the acceptance probability play a key role to have a singleton menu. While single-crossing conditions are usually used to maintain separation in signalling and screening models, in this case separation reveals too much information through the choice. This in turn makes it impossible to maintain the incentives to separate in the first place.

Finally, adding the communication highlights the role of tests when there is no communication. Without communication, the tests also serve as a communication channel. When communication is allowed, the test choice does not add any information beyond the test results. This leads to the result that only dominant tests are used when the agent can communicate but a dominant test can be useful when there is no additional communication.

1.9 Omitted proofs of Chapter 1

1.9.1 Proof of Theorem 1

The problem the DM needs to solve when committing to α is

$$\max_{\alpha} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \tilde{\sigma}(t|\theta') p_t(\alpha;\theta')$$
$$\text{s.t. } \tilde{\sigma}(\cdot|\theta) \in \arg\max \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta), \quad \text{for all } \theta \in \Theta$$

Therefore, $\sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta) = \max_{\sigma(\cdot|\theta)} \sigma(t|\theta) p_t(\alpha;\theta)$ for all θ. Abusing notation, denote by $\sigma^{\Theta'} : \Theta' \to \Delta T$. We can plug this is in the DM's maximisation problem to obtain

$$\max_{\alpha} \max_{\sigma^A} \min_{\sigma^R} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta') p_t(\alpha;\theta')$$

where the $\min$ is obtained because of the minus sign. Define

$$v(\alpha, \sigma^A, \sigma^R) \equiv \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta') p_t(\alpha;\theta')$$

Let $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha}, \tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$ and $\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$. I will now show that these strategies are equilibrium strategies.

Because the order of maximisation does not matter, $\sigma^A \in \arg\max\min v(\alpha, \tilde{\sigma}^A, \tilde{\sigma}^R)$. Moreover, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma_1^R) \Leftrightarrow \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) \geq \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta)$ for all $\theta \in A$ and $\tilde{\sigma}^A$. This last expression does not depend σ^R. Therefore, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma^R)$.

Similarly, $\alpha \in \arg\max\min v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$. Because v is linear in both α and σ^R, (α, σ^R) is a saddle-point of $v(\cdot, \sigma^A, \cdot)$ by the minimax theorem. As for σ^A, $\sigma^R \in \arg\min v(\alpha, \sigma^A, \tilde{\sigma}^R) \Leftrightarrow \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) \geq \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta)$ for all $\theta \in R$ and $\tilde{\sigma}^R$. Therefore all strategies are best-reply.

Beliefs on-path are formed using Bayes' rule and off-path beliefs are chosen to justify the off-path actions of α (this is always possible to find as any action is best-reply to some beliefs).

Note also that we can without loss of generality take σ^A to be in pure strategy as it is maximises a linear function.

1.9.2 Proof of Lemma 1

Because $t \succeq t'$ implies $t \succeq_\theta t'$ for some $\theta \in A$, Lemma 1 is a corollary of Proposition 7 proven below.

1.9.3 Proof of Proposition 1 and Proposition 2

Suppose the DM's preferences are single-peaked given $\geq_t$. Suppose there is a menu with both t, t'. Take $A_1, A_2 \in A$ with $A_1 < A_2$ and without loss of generality, suppose A_1 chooses t' and A_2 chooses t in some equilibrium. Let α denote the DM equilibrium strategy in this equilibrium.

Because $t \succeq t'$, there is $\beta : X \times X \to [0,1]$ such that $p_{t'}(\tilde{x}|\theta) = \beta(x,\tilde{x})\pi_t(x|\theta) + \beta(x',\tilde{x})\pi_t(x'|\theta)$ and $\sum_x \beta(\tilde{x}, x) = 1$ for $\tilde{x} = x, x'$. Type $\theta \in \Theta$ prefers test t' over t if

$$\alpha(x_1,t')\Big(\beta(x_1,x_1)\pi_t(x_1|\theta) + \beta(x_0,x_1)\pi_t(x_0|\theta)\Big) \\ + \alpha(x_0,t')\Big(\beta(x_1,x_0)\pi_t(x_1|\theta) + \beta(x_0,x_0)\pi_t(x_0|\theta)\Big) - \alpha(x_1,t)\pi_t(x_1|\theta) - \alpha(x_0,t)\pi_t(x_0|\theta) \geq 0$$

Note that this expression is monotonic in θ. Indeed, if $\pi_t(x_0|\theta) > 0$, then dividing by $\pi_t(x_0|\theta)$ gives

$$\alpha(x_1,t')\Big(\beta(x_1,x_1)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_1)\Big) + \alpha(x_0,t')\Big(\beta(x_1,x_0)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_0)\Big) \\ - \alpha(x_1,t)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} - \alpha(x_0,t)$$

which is linear in $\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)}$, an increasing function of θ. If $\pi_t(x_0|\theta) = 0$, then $\pi_t(x_0|\theta') = 0$ for all $\theta' >_t \theta$ and the expression is constant.

To have A_1 choose t' and A_2 choose t, it must be strictly decreasing[6] in θ, i.e.,

$$\alpha(x_1,t')\beta(x_1,x_1) + \alpha(x_0,t')\beta(x_1,x_0) - \alpha(x_1,t) < 0 \tag{1.5}$$

A necessary condition for (1.5) to hold is that $\alpha(x_1,t) > 0$. Note the strict monotonicity also implies that there is $\overline{\theta} \in A$ such that any $\theta > \overline{\theta}$ prefers t and any $\theta \leq \overline{\theta}$ prefers t'. Let $A^+ = \{\theta \in A : \theta >_t \overline{\theta}\}$ and $R^+ = \{\theta \in R : \theta >_t \theta',$ for all $\theta' \in A\}$. But because only types in $A^+ \cup R^+$ choose t, the likelihood ratios $\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} < \frac{\pi_t(x_0|\theta)}{\pi_t(x_0|\theta')}$ for any $\theta \in A^+, \theta' \in R^+$ and $\alpha(x_1,t) > 0$ imply that $\alpha(x_0,t) = 1$ (Milgrom, 1981).

[6] If all types are indifferent between t and t' then it is also an equilibrium to offer only t and the DM's payoffs are the same.

But then no type ever prefer t' over t. Indeed, the condition to prefer t' over t,

$$\Big(\alpha(x_1,t')\beta(x_1,x_1)+\alpha(x_0,t')\beta(x_1,x_0)-\alpha(x_1,t)\Big)\pi_t(x_1|\theta) \geq \Big(1-\alpha(x_1,t')\beta(x_0,x_1)-\alpha(x_0,t')\beta(x_0,x_0)\Big)\pi_t(x_0|\theta)$$

is never satisfied as the LHS is strictly negative because (1.5) must hold and the RHS is positive because $\beta(x_0,x_1)+\beta(x_0,x_0)=1$ and $\alpha(\tilde{x},t')\leq 1$, $\tilde{x}=x_1,x_0$.

Thus there cannot be an equilibrium where another test than t is chosen.

Suppose the DM's preferences are enclosed given $\geq_t$.

Suppose $(\tilde{\alpha},\tilde{\sigma}^A)\in\arg\max\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R)$ with $\tilde{\sigma}(t|\theta)=1$ for all $\theta\in A$.

Suppose the prior is such that when only t is offered, x_0 is rejected and x_1 is accepted. Let $\underline{\theta}=\min\{\theta\in R\}$ where the $\min$ is taken with respect to $\geq_t$.

Then consider the following deviation: take some $t'\neq t$ and let $\alpha(x,t')=\pi_t(x_1|\underline{\theta})$ for all $x\in X$ and $\alpha=\tilde{\alpha}$ otherwise. Because preferences are enclosed, there is $\theta\in A$ such that $\pi_t(x_1|\theta)<\pi_t(x_1|\underline{\theta})$ and for all $\theta'\in R$, $\pi_t(x_1|\theta')\geq\pi_t(x_1|\underline{\theta})$. Let $\sigma(t'|\theta)=1$ for that type and $\sigma=\tilde{\sigma}$ otherwise. This deviation is strictly profitable, i.e., $\min_{\sigma^R} v(\tilde{\alpha},\tilde{\sigma}^A,\sigma^R)<\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R)$.

Suppose the prior is such that $\tilde{\alpha}(x_1,t)=\tilde{\alpha}(x_0,t)\in\{0,1\}$ when only t is offered. This means that the DM does not react to information. Let $\alpha(x,t')=\tilde{\alpha}(x,t)$ for some $t'\neq t$ and $\sigma(t'|\theta)=1$ for some $\theta\in A$ and $\sigma=\tilde{\sigma}$ otherwise. We get $\min_{\sigma^R} v(\tilde{\alpha},\tilde{\sigma}^A,\sigma^R)=\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R)$, so it is also a solution.

Suppose that the DM's preferences are not single-peaked given $\geq_t$.

In this case, it is possible to find $A_1,A_2\in A$ and $R_1\in R$ such that $A_1<_t R_1<_t A_2$. Let $\mu(\theta)\approx 0$ for $\theta\neq A_1,A_2,R_1$ and be such that x_0 is rejected and and x_1 is accepted when only t is offered. Because t is informative, there is always such prior. Then from the reasoning above the menu $\{t,t'\}$ is strictly better for the DM than $\{t\}$ when only focusing on A_1,A_2,R_1 have positive probability. But because $\mu(\theta)\approx 0$ for $\theta\neq A_1,A_2,R_1$, then the menu $\{t,t'\}$ remains strictly better than $\{t\}$ whatever the behaviour of the other types.

Suppose that the DM's preferences are not enclosed given $\geq_t$.

If the DM's preferences are not enclosed, then suppose without loss of generality that there is $R_1\in R$ such that $R_1\leq_t\theta$ for any $\theta\in\Theta$ (otherwise, simply change the roles of x_1 and x_0).

Suppose it is not the case and take some $A_1\in A$ such that $A_1>_t R_1$. (We have a strict inequality because we assumed that all types generate different distribution over signals.) Suppose that for $\theta\neq A_1,R_1$, $\mu(\theta)\approx 0$. An argument analogue to the proof that single-peakness implies that only t is chosen in equilibrium holds.

1.9.4 Proof of Proposition 3

Proof. Note that in an MLRP environment, the strategy of the DM takes the form of a cutoff strategy. For each test t, there is $x_t \in X$ such that $\alpha(x,t) = 0$ for $x < x_t$, $\alpha(x,t) = 1$ for $x > x_t$ and $\alpha(x_t,t) \in [0,1]$. From Lemma 1, we know that there is an optimal menu containing the Blackwell most informative test. Because all tests are MLRP and the DM's payoffs satisfy single-crossing condition, the Lehmann order is well-defined and the Blackwell order implies the Lehmann order (Lehmann, 1988; Persico, 2000). Let $\succeq^a$ denote the Lehmann order.

The Lehmann order is defined on continuous information structure. But as outlined in Lehmann (1988), we can always make our conditional probabilities continuous by adding independent uniform between each signal. Let's assume, without loss of generality, that $X = \{1, ..., n\}$. The new distribution over signal is $\tilde{y}|\theta = \tilde{x}|\theta - u$ where u ~U$[0,1]$. Denote by F_t the cdf associated with the new information structure.

We have that $t \succeq^a t'$ if $y^*(\theta, y) \equiv F_t(y^*|\theta) = F_{t'}(y|\theta)$ is nondecreasing in θ for all y (Lehmann, 1988). In particular, this condition implies that if $F_t(y|\theta') \leq (<) F_{t'}(y'|\theta')$ then $F_t(y|\theta) \leq (<) F_{t'}(y'|\theta)$ for all $\theta > \theta'$.

Let α be the optimal strategy and x_t be the cutoff signal associated to each test. To each $\big(\alpha(\cdot,t), x_t\big)$ we can associate a $y_t \equiv x_t - \alpha(x_t,t)$.

If t is part of an optimal menu, it must be that there is some $\theta' \in R$ such that $p_t(\alpha;\theta') \geq p_{t'}(\alpha;\theta')$ for all t'. Or put differently, $F_t(y_t|\theta') \leq F_{t'}(y_{t'}|\theta')$ for all t'. But then $F_t(y_t|\theta) \leq F_{t'}(y_{t'}|\theta)$ for all t' and all $\theta > \theta'$, in particular all $\theta \in A$. Therefore all type in A prefer test t as well and there is an solution of the max-min problem where all types in $\theta \in A$ choose t. (If there is an A-type that is indifferent between t and t' then all types in R must be indifferent or prefer t' so choosing t is an equilibrium strategy for such A-type.) $\square$

1.9.5 Proof of Proposition 4

Suppose that t is the only test used in the optimal menu. Define a test t' such that for all $\theta \in \Theta$,

$$\pi_{t'}(x|\theta) = \sum_{\tilde{x}=x,x'} \pi_t(\tilde{x}|\theta)$$

$$\pi_{t'}(\tilde{x}|\theta) = \pi_t(\tilde{x}|\theta), \text{ for all } \tilde{x} \neq x, x'$$

The test t' pools signals x and x' together and is otherwise identical to t. We have $t \succ t'$ as any strategy under

t' can be replicated under t.

Let μ be such that for all $\theta \neq A_1, A_2, R_1$, $\mu(\theta) \approx 0$ and such that when only t is chosen, the DM's best-reply is $\alpha(x,t) = 1$ and $\alpha(x',t) = 0$.

In the problem with commitment, consider the deviation $\tilde{\alpha}$ such that $\tilde{\alpha}(x,t') = \frac{\pi_t(x|A_1)}{\pi_t(x|A_1)+\pi_t(x'|A_1)} + \epsilon$ for some small $\epsilon > 0$ and $\tilde{\alpha} = \alpha$ otherwise.

This implies that $p_t(\alpha; A_1) < p_{t'}(\tilde{\alpha}; A_1)$ but for ϵ small enough $p_t(\alpha;\theta) > p_{t'}(\tilde{\alpha};\theta)$ for $\theta = R_1, A_2$. Therefore A_1 is accepted with strictly higher probability and the other types with the same probability as they choose the same test. If the prior on other types is sufficiently small, the deviation is still strictly profitable for the DM.

1.9.6 Proof of Proposition 5

I first show that if $t > t'$, then $\mu(\cdot|t,x) \succeq_{FOSD} \mu(\cdot|t',x)$ where $\succeq_{FOSD}$ denotes first-order stochastic dominance.

Proof. The proof is similar to the one in Milgrom (1981). Denote by $G_t(\cdot|x)$ the cdf of posterior beliefs after signal x in test t. For all $\theta > \theta'$,

$$\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Take some $\theta^* \geq \theta'$. Summing over θ, we get

$$\sum_{\theta>\theta^*}\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \sum_{\theta>\theta^*}\mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Inverting and summing over θ', we get

$$\frac{\sum_{\theta^*\geq\theta'}\mu(\theta')\pi_t(x|\theta')}{\sum_{\theta>\theta^*}\mu(\theta)\pi_t(x|\theta)} \leq \frac{\sum_{\theta^*\geq\theta'}\mu(\theta')\pi_{t'}(x|\theta')}{\sum_{\theta>\theta^*}\mu(\theta)\pi_{t'}(x|\theta)}$$

which implies

$$\frac{G_t(\theta^*|x)}{1-G_t(\theta^*|x)} \leq \frac{G_{t'}(\theta^*|x)}{1-G_{t'}(\theta^*|x)} \quad \Rightarrow \quad G_t(\theta^*|x) \leq G_{t'}(\theta^*|x)$$

□

The way this proof proceeds is by fixing a menu and dividing tests in two categories: (1) those for which $\alpha(x_0,\tilde{t}) \in (0,1)$ and $\alpha(x_1,\tilde{t}) = 1$ and (2) $\alpha(x_0,\tilde{t}) = 0$ and $\alpha(x_1,\tilde{t}) \in (0,1]$. I exclude the possibility that the DM always accepts or rejects after any signal as it would either be the only test chosen in equilibrium or never chosen. Then, I show that within each category, it is without loss of optimality to have at most one test. It is thus optimal to have at most two tests in the menu. The last part of the proof shows that the resulting menu is dominated by having only one test.

If there are two tests, $t > t'$ such that $\alpha(x_0, \tilde{t}) = 0$ and $\alpha(x_1, \tilde{t}) \in (0, 1]$, I will show that,

$$p_t(\alpha; \theta') \geq p_{t'}(\alpha; \theta') \quad \Rightarrow \quad p_t(\alpha; \theta) \geq p_{t'}(\alpha; \theta) \text{ for all } \theta > \theta'$$

Take two tests such that $\alpha(x_0, \tilde{t}) = 0$, $t > t'$. Let α, α' denote their respective probability of accepting after x_1. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_1|\theta) = 0$. Rearranging, $\alpha(\theta) = \alpha' \frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. From our assumption on the difficulty environment, $\alpha(\theta)$ is decreasing in θ. If $p_t(\alpha; \theta') \geq p_{t'}(\alpha; \theta')$ for some θ' then $\alpha \geq \alpha(\theta')$. Then $\alpha \geq \alpha(\theta)$ for all $\theta > \theta'$.

In equilibrium, we must have that there is one $\theta' \in R$ that chooses t and thus for all $\theta \in A$, $p_t(\alpha; \theta) \geq p_{t'}(\alpha; \theta)$. Then there is an solution of the max-min problem where t' is never chosen.

A similar argument can be made for all tests where $\alpha(x_0, \tilde{t}) > 0$.

Thus we conclude that it is without loss of optimality that the optimal menu has at most two tests.

Suppose the optimal menu uses two tests, $t > t'$. I will now show that it must be that $\alpha(x_0, t) \in (0, 1)$ and $\alpha(x_1, t') \in (0, 1)$, i.e., the DM must accept in the hard test when there is a fail grade and only accept in the easy test if there is a pass grade. Suppose it is not the case and denote by α, α' their respective mixing probabilities. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_0|\theta) - \pi_{t'}(x_1|\theta) = 0$, which is equivalent to $\alpha(\theta) = \alpha' \frac{1}{\pi_t(x_1|\theta)} + (1 - \alpha') \frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. Again from our assumptions, this is decreasing in θ. A type θ chooses t if $\alpha \geq \alpha(\theta)$. Thus if one $\theta \in A$ chooses t all $\theta \in R$ choose t and there is no pooling of A and R-types on t', or it is payoff equivalent to just offering t. Therefore, $\alpha(x_0, t) \in (0, 1)$ and $\alpha(x_1, t') \in (0, 1)$ for $t > t'$.

If the DM mixes, he must be indifferent and thus we have

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta)\pi_t(x_0|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')\pi_t(x_0|\theta') = 0$$
$$\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta') = 0$$

In the easy test, because the DM rejects with positive probability after x_1 and rejects for sure after x_0 (as he uses a cutoff strategy), his payoffs from t' is 0, i.e., he does as well as rejecting for sure.

In the hard test, he accepts with some probability after x_0 and thus his payoffs are

$$\sum_{\theta\in A}\mu(\theta)\sigma(t|\theta)-\sum_{\theta'\in R}\mu(\theta')\sigma(t|\theta')$$

that is the payoffs he would get from accepting all types choosing t. Thus the overall payoffs from the menu is $\sum_{\theta\in A}\mu(\theta)\sigma(t|\theta)-\sum_{\theta'\in R}\mu(\theta')\sigma(t|\theta')$. Offering a menu is better than a singleton menu if this value is strictly greater than offering t and following the signal

$$\begin{aligned}\sum_{\theta\in A}\mu(\theta)\sigma(t|\theta)-\sum_{\theta'\in R}\mu(\theta')\sigma(t|\theta') &> \sum_{\theta\in A}\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta'\in R}\mu(\theta')\pi_t(x_1|\theta')\\ &=\sum_{\theta\in A}\sigma(t|\theta)\mu(\theta)\pi_t(x_1|\theta)+\sum_{\theta\in A}\sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta)\\ &-\sum_{\theta'\in R}\sigma(t|\theta')\mu(\theta')\pi_t(x_1|\theta')-\sum_{\theta'\in R}\sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')\end{aligned}$$

We can rearrange and use the indifference condition at (x_0,t) to get

$$0>\sum_{\theta\in A}\sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta'\in R}\sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')$$

Using the indifference condition at (x_1,t'), we can replace 0 on the LHS and get

$$\begin{aligned}&\sum_{\theta\in A}\mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta)-\sum_{\theta'\in R}\mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta')\\ &\qquad>\sum_{\theta\in A}\sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta'\in R}\sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')\end{aligned}$$

But from the definition of the environment, for all $\theta>\theta'$,

$$\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')}\geq\frac{\pi_{t'}(x_1|\theta)}{\pi_{t'}(x_1|\theta')}$$

which implies that $\mu(\theta|x_1,t)\succeq_{FOSD}\mu(\theta|x_1,t')$. Thus we get a contradiction.

1.9.7 Proof of Proposition 6

Suppose condition (1.2) holds.

Suppose that all types choose the same test testing dimension j. Take $(\tilde{\theta}_i,\tilde{\theta}_j)\in\arg\min_{\theta\in A}p_{t_j}(\alpha;\theta)$. Because $p_{t_j}(\alpha;\theta_i,\theta_j)$ is constant in θ_i, we have $(\overline{\theta}_i,\tilde{\theta}_j)\in\arg\min_{\theta\in\Theta}p_{t_j}(\alpha;\theta)$ as well and from condition (1.2), $(\overline{\theta}_i,\tilde{\theta}_j)\in A$. Consider the deviation in the problem with commitment to $(\tilde{\alpha},\tilde{s})$ such that for t_i,

- $\tilde{\alpha}(\cdot, t_i)$ is set so that it has a cutoff structure and $p_{t_i}(\tilde{\alpha}|\overline{\theta}_i, \tilde{\theta}_j) = p_{t_j}(\alpha; \overline{\theta}_i, \tilde{\theta}_j) + \epsilon$ and $\tilde{\alpha}(\cdot, t_j) = \alpha(\cdot, t_j)$ otherwise.
- $\tilde{\sigma}(t_i|\overline{\theta}_i, \tilde{\theta}_j) = 1$ and $\tilde{\sigma}(\cdot|\theta) = \sigma(\cdot|\theta)$ otherwise.

Because the test t_i has the strict MLRP when restricting attention to dimension i, for all $\theta_i < \overline{\theta}_i$, $\min_{\theta\in\Theta} p_{t_j}(\alpha;\theta) \geq p_{t_i}(\tilde{\alpha}|\overline{\theta}_i, \theta_j) > p_{t_i}(\tilde{\alpha}|\theta_i, \theta_j)$ if ϵ is small enough. This means that no other type has an incentive to choose test i but $(\overline{\theta}_i, \tilde{\theta}_j) \in A$ is accepted with strictly higher probability. Thus the menu with only test j cannot be the optimal menu.

Suppose condition (1.2) does not hold.

If condition (1.2) is not satisfied, then there a dimension, say 1, and $\tilde{\theta}_2 \in \Theta_2$ such that $(\overline{\theta}_1, \tilde{\theta}_2) \in R$. By the monotonicity of payoffs in the bidimensional environment, this implies that $(\theta_1, \tilde{\theta}_2) \in R$ for all $\theta_1 \in \Theta_1$. Moreover, for all $\theta_2 < \tilde{\theta}_2$ and all $\theta_1 \in \Theta_1$, $(\theta_1, \theta_2) \in R$.

Now suppose μ is such that $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2'\neq\theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$. And that $\mu(\theta_1, \theta_2) \approx 0$ for all $(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$.

I am going to show that $\{t_2\}$ is optimal when t_1 fully reveals dimension 1. Because this test can replicate the strategies of any t_1, it is enough to prove our claim.

Suppose there is an optimal menu $\{t_1, t_2\}$. From our assumptions on μ, the DM follows a cutoff strategy after t_2. That's because his payoff is monotone along that dimension, ignoring $(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$ whose prior probability is close to zero. So it does not upset the cutoff structure of the best-response. This implies that $p_{t_2}(\alpha; \theta_1, \theta_2) > p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ because the likelihood ratio is strictly increasing.

Suppose that some $(\theta_1, \tilde{\theta}_2)$ chooses t_1 with probability 1 in equilibrium. Because $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2'\neq\theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$, it must be that the best-response is $\alpha(x = \theta_1, t_1) = 0$ (recall that t_1 fully reveals θ_1). Thus $p_{t_2}(\alpha; \theta_1, \theta_2) = 0$ for all $\theta_2 \in \Theta_2$, otherwise there is a profitable deviation. Either this contradicts the fact that the DM best replies or in equilibrium the DM rejects after all signals in every test. But then he is weakly better off only offering t_2.

Thus to have $\{t_1, t_2\}$ strictly better, it must be that all $(\theta_1, \tilde{\theta}_2)$ choosing t_1 mix in equilibrium. This means that $p_{t_1}(\alpha; \theta_1, \tilde{\theta}_2) = p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$. But by the cutoff structure of $\alpha(\cdot, t_2)$ and the strict MLRP assumption, we have $p_{t_2}(\alpha; \theta_1, \theta_2) > p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ and $p_{t_2}(\alpha; \theta_1, \theta_2) < p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 < \tilde{\theta}_2$. Thus t_2 is strictly preferred for all $(\theta_1, \theta_2) \in A$. Thus choosing only $\{t_2\}$ is an optimal menu.

1.9.8 Proof of Proposition 7

Proof. I will first prove the following lemma. This result already exists in the literature and I provide a proof for completeness.

Lemma 2. *For any $t \succeq t'$ and $\alpha(\cdot, t')$, there is $\alpha(\cdot, t)$ such that*

$$\sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta)$$
$$\textit{for all } \theta' \in R, \quad \sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta')$$

Proof. We can prove this lemma by using a theorem of the alternative (see e.g., Rockafellar (2015) Section 22). Only one of the following statement is true:

- There exists $\alpha(\cdot, t)$ such that

$$\sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta)$$
$$\text{for all } \theta' \in R, \quad \sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta')$$
$$\text{for all } x \in X, \quad \alpha(x,t) \leq 1$$
$$\text{for all } x \in X, \quad \alpha(x,t) \geq 0$$

- There exists $z, y \geq 0$ such that

$$\text{for all } x \in X, \quad -z_\theta \pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_t(x|\theta') + y_x \geq 0 \tag{1.6}$$
$$-z_\theta \sum_{x'} \alpha(x',t')\pi_{t'}(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x',t')\pi_{t'}(x'|\theta') + \sum_{x'} y_{x'} < 0 \tag{1.7}$$

Take inequality (1.6) from the second alternative and multiply by $\beta(x, x')$ as described in Definition 5 and sum over $x \in X$:

$$-z_\theta \sum_x \beta(x,x')\pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_x \beta(x,x')\pi_t(x|\theta') + \sum_x \beta(x,x') y_x \geq 0$$

Because $t \succeq_\theta t'$, we get for all $x' \in X$,

$$-z_\theta \pi_{t'}(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_{t'}(x'|\theta') + \sum_x \beta(x, x') y_x \geq 0$$

We can then multiply by $\alpha(x', t')$ and sum over $x' \in X$:

$$-z_\theta \sum_{x'} \alpha(x', t') \pi_t(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x', t') \pi_{t'}(x'|\theta') + \sum_{x,x'} \alpha(x', t') \beta(x, x') y_x \geq 0 \tag{1.8}$$

Because $\sum_{x'} \beta(x, x') \leq 1$ and $\alpha(x', t') \leq 1$ for all $x' \in X$, we have $\sum_{x,x'} \alpha(x', t') \beta(x, x') y_x \leq \sum_x y_x$. Therefore, the inequality (1.7) cannot hold and the first alternative holds. □

With this result in hand, we can now prove our result. Suppose that t is not part of the optimal menu. Thus we can find an solution of the problem with commitment, (α, σ) with $\sigma(t|\theta) = 0$ for all θ. Take a test t' used in the solution by some $\theta \in A$. Then from Lemma 2, we can construct a $(\tilde{\alpha}, \tilde{\sigma})$ such that

- $p_t(\tilde{\alpha}; \theta) \geq p_{t'}(\alpha; \theta)$
- $p_t(\tilde{\alpha}; \theta') \leq p_{t'}(\alpha; \theta')$ for all $\theta' \in R$
- $\tilde{\sigma}(t|\theta) = 1$
- $\tilde{\sigma} = \sigma$ otherwise

This constitutes a solution to the problem with commitment.

□

1.9.9 Proof of Proposition 8

Proof. ($\Leftarrow$) For each $\theta \in A$, let t_θ such that

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \left(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \right) = \emptyset$$

Then posting a menu $(t_\theta)_{\theta \in A}$ is optimal (eliminating duplicates if there are some). Each $\theta \in A$ chooses t_θ. For any strategy of $\theta' \in R$, the DM accepts after any $(x, t) \in \cup_{\theta: \sigma(t|\theta)=1} \operatorname{supp} \pi_t(\cdot|\theta)$ and rejects otherwise. This gives the DM and the A-types maximal payoffs and the R-types get rejected for any strategy they follow.

($\Rightarrow$) Suppose the DM's payoffs are maximal and there is $\theta \in A$ and for all $t \in T$ there is $\theta' \in R$ and $x \in X$ such that $\pi_t(x|\theta), \pi_t(x|\theta') > 0$. Then when θ chooses t out of the menu of tests, if θ' chooses t as well, at x, either the DM accepts θ' or rejects θ. Therefore, payoffs cannot be maximal. $\square$

1.9.10 Proof of Theorem 2

The only thing we need prove is that it is optimal to have a different message for each type $\theta \in A$, the rest follows from Theorem 1. Suppose it is not the case and take a solution (α, σ) of the problem with commitment where the A-types play a pure strategy.

There is $\theta_1, \theta_2 \in A$ and $(t, c) \in T \times C$ such that $\sigma(t, c|\theta_1) = \sigma(t, c|\theta_2) = 1$ (if they use a different test then we can also change the message and nothing is changed). Then consider the alternative strategy α' where, for some unused (t, c') in the original mechanism, $\alpha'(t, c', x) = \alpha(t, c, x)$ for all $x \in X$ and $\alpha'(t'', c'', x) = \alpha'(t'', c'', x)$ for all other $(t'', c'') \in T \times C$ and all $x \in X$ otherwise. The new strategy α' is thus the same as α but makes sure that if the pair (t, c') is chosen, it uses the same actions as (t, c). Now consider the following strategy $\tilde{\sigma}(\cdot|\theta)$ for $\theta \in A$ in the auxiliary max-min problem, $\tilde{\sigma}(\cdot|\theta) = \sigma(\cdot|\theta)$ for $\theta \neq \theta_1$ and $\tilde{\sigma}(t, c'|\theta_1) = 1$. In the problem with commitment under the strategy α', the payoffs are the same than under (α, σ). Moreover, any deviations under α' gives the same payoff than under α. Therefore, $(\alpha', \tilde{\sigma})$ is an solution to problem with commitment.

1.9.11 Proof of Proposition 9

This follows from Lemma 1. If a Blackwell dominated test is chosen by an A-type, then we can introduce a message test pair (c, t) in the problem with commitment that will make the A-type better off without making any R-type better off. This will thus improve the DM's payoffs.

1.9.12 Proof of Theorem 3

The designer's problem is

$$\begin{aligned} \max_{\tau,\alpha} \; & \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta) \\ \text{s.t. } & \sum_t (\tau(t|\theta) - \tau(t|\theta')) p_t(\alpha;\theta) \geq 0 \text{ for all } \theta, \theta' \\ & \alpha \in BR(\tau) \end{aligned}$$

If the DM could commit over a strategy α, his problem would be

$$\tilde{V}(\alpha) = \max_{\tau} \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta)$$
$$\text{s.t. } \sum_t (\tau(t|\theta) - \tau(t|\theta')) p_t(\alpha;\theta) \geq 0 \text{ for all } \theta, \theta'$$

Step 1: Show that $\tilde{V}(\alpha) = \max_s \min_m v(\alpha, s, m)$ where v is defined in (1.3).

To show this claim, I am going to relax the mechanism design problem by restricting attention to the IC constraints of R-types deviating to reporting an A-type:

$$\tilde{V}(\alpha) = \max_{\tau} \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta)$$
$$\text{s.t. } \sum_t \tau(t|\theta) p_t(\alpha;\theta) \geq \max_{m(\cdot|\theta)} \sum_{\theta' \in A} m(\theta'|\theta) \sum_t \tau(t|\theta') p_t(\alpha;\theta), \text{ for all } \theta \in R$$

The IC constraints are written to express that reporting type θ for $\theta \in R$ is better than any other reporting strategy over the A-types.

Now note that if an IC constraint is slack at the optimum, we could improve the DM's payoff by setting $\tau(\cdot|\theta) = \max_{m(\cdot|\theta)} \sum_{\theta' \in A} m(\theta'|\theta) \sum_t \tau(t|\theta') p_t(\alpha;\theta)$. As that would reduce the probability of type $\theta \in R$ of being accepted and would not change any other constraints in the relaxed problem. Thus at the optimum, $\tau(\cdot|\theta) = \max_{m(\cdot|\theta)} \sum_{\theta' \in A} m(\theta'|\theta) \sum_t \tau(t|\theta') p_t(\alpha;\theta)$. We can plug this expression in the payoffs to get

$$\tilde{V}(\alpha) = \max_{\tau} \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta) \max_{m(\cdot|\theta)} \sum_{\theta' \in A} m(\theta'|\theta) \sum_t \tau(t|\theta') p_t(\alpha;\theta)$$
$$= \max_{\tau} \min_{m} \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta) \sum_{\theta' \in A} m(\theta'|\theta) \sum_t \tau(t|\theta') p_t(\alpha;\theta)$$

Note that we can take out the max of the summation by the linearity of the expression in m and it becomes a min because of the minus sign. This expression also corresponds to v as defined in (1.3).

It remains to show that the solution of this relaxed mechanism is indeed optimal. Take $s \in \arg\max \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$ and $m \in \arg\min v(\alpha, s, \tilde{m})$ and define the optimal mechanism by

- $\tau(t|\theta) = s(t|\theta)$ for $\theta \in A$
- $\tau(t|\theta') = \sum_{\theta \in A} m(\theta|\theta') s(t|\theta)$ for $\theta' \in R$

Note that an outcome of this mechanism gives payoff weakly higher than $\max_{\tilde{s}} \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$ as $v(\alpha, s, m) \geq$

$\min_{\tilde{m}} v(\alpha, s, \tilde{m})$. Thus if it is incentive-compatible, it must be actually equal to the upper bound $\max_{\tilde{s}} \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$.

Note that all allocations are either allocations of A-types or convex combinations of the A-types' allocations thus it is enough to check that no type has any incentive to report any A-type in the mechanism.

By definition of m,

$$v(\alpha, s, m) \geq v(\alpha, s, m'), \text{ for all } m'$$

Therefore, for each R-type, θ',

$$\sum_{\theta \in A} m(\theta|\theta') \sum_t \tau(t|\theta) p_t(\alpha; \theta') \geq \sum_t \tau(t|\tilde{\theta}) p_t(\alpha; \theta'), \text{ for any } \tilde{\theta} \in A$$

For an A-type θ, consider the choice of choosing $\tilde{s}$ such as $\tilde{s}(\cdot|\theta) = s(\cdot|\tilde{\theta})$ for some $\tilde{\theta} \in A$ and the same otherwise. By definition of s,

$$v(\alpha, s, m) = \min_{\tilde{m}} v(\alpha, s, \tilde{m}) \geq \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$$

Rearranging,

$$\mu(\theta) \sum_t \big(s(t|\theta) - s(t|\tilde{\theta})\big) p_t(\alpha; \theta) \geq \max_{\tilde{m}} \sum_{\theta' \in R} \mu(\theta') \sum_{\theta'' \in A} \tilde{m}(\theta''|\theta') \sum_t s(t|\theta) p_t(\alpha; \theta') - \max_{\tilde{m}} \sum_{\theta' \in R} \mu(\theta') \sum_{\theta'' \in A} \tilde{m}(\theta''|\theta') \sum_t \tilde{s}(t|\theta) p_t(\alpha; \theta')$$

where the min is transformed in max because of the negative sign. Note that the LHS is the IC constrain of type θ deviating to type $\tilde{\theta}$. The RHS is the difference payoff is the probability of the R-types of being accepted when they choose a mimicking strategy $\tilde{m}$. Note that the only difference between s and $\tilde{s}$, from their point of view is that there is weakly less choice of allocations to mimic as we have θ choosing the same allocation as $\tilde{\theta}$. Therefore it must be that the RHS is positive which implies that the LHS is as well.

Step 2: Show that the DM does not benefit from commitment in the optimal mechanism.

Take $(\alpha, s) \in \arg\max_{\tilde{\alpha}, \tilde{s}} \min_{\tilde{m}} v(\tilde{\alpha}, \tilde{s}, \tilde{m})$ and $m \in \arg\min_{\tilde{m}} \max_{\tilde{\alpha}} v(\tilde{\alpha}, s, \tilde{m})$. The α selected would be the optimal strategy when the DM can commit.

Note that because the order of maximisation does not matter, we also have $\alpha \in \arg\max_{\tilde{\alpha}} \min_{\tilde{m}} v(\tilde{\alpha}, s, \tilde{m})$.

Note that v is linear in $\tilde{\alpha}$ and $\tilde{m}$ and thus by the minimax theorem,

$$v(\alpha, s, m) \geq v(\alpha', s, m), \text{ for all } \alpha'$$
$$v(\alpha, s, m) \leq v(\alpha, s, m'), \text{ for all } m'$$

Thus α best-replies to the optimal mechanism when the DM can commit and m is also a best reply to (α, s), thus satisfying the condition for characterising the equilibrium in Step 1.

Chapter 2

Managing the expectations of buyers with reference-dependent preferences

2.1 Introduction

The purpose of this paper is to study a monopoly pricing model where the buyer exhibits a specific type of reference-dependent preferences. I consider a buyer that values a good more when he expects to own it. For instance, a buyer expecting to own a specific car or house can get emotionally attached to it and finds it then harder to walk away from an offer. A job applicant, expecting to be employed, can build expectations about the prospects of a different lifestyle or social status, which can reduce his willingness to refuse an offer. More generally, this attachment effect is an "expectation-based endowment effect" and is a prediction of expectation-based reference-dependent preferences (Kőszegi and Rabin, 2006).

When can a firm benefit from facing a buyer with an attachment effect and what is its consequence on a firm's pricing strategy? To study this question, I adapt the monopoly model of Heidhues and Kőszegi (2014). In their model, a firm sells an indivisible good to a buyer by making a take-it-or-leave-it offer. To capture the attachment effect, the buyer's willingness-to-pay (WTP) increases linearly in the ex-ante probability of trading. Following the literature on expectation-based reference-dependent preferences, the buyer plays a Preferred Personal Equilibrium: when setting expectations, he correctly anticipates the firm's strategy and his own action and selects the most favourable plan of action.[1]

[1] See for example Heidhues and Kőszegi (2014), Kőszegi and Rabin (2009), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018) for papers using this selection.

Importantly, in Heidhues and Kőszegi (2014), the firm can commit to a possibly random price offer distribution. They show that the optimal strategy is to randomise over prices. The low prices in the support ensure that the buyer expects to buy with positive probability. This increases the WTP through the attachment effect. The higher prices in the support exploit this higher WTP to increase profits. However, this strategy is not consistent with equilibrium behaviour when the firm cannot commit to a random price strategy.

In this paper, I characterise the firm's pricing strategy when it cannot commit to it. This model has two main features. First, the demand is endogenous: the probability of buying and hence the buyer's WTP depends on the firm's strategy. Second, the firm has a commitment problem. There is a tension between offering low prices to induce expectations to buy and high prices to take advantage of a higher WTP. I characterise the equilibrium strategy under three different information environments: (1) the valuation is the buyer's private information, (2) the buyer's valuation is common knowledge, and (3) the firm can learn about the valuation.

Proposition 2 characterises the unique equilibrium of the game when valuations are private information. In equilibrium, the firm chooses the mixed strategy such that the resulting demand is unit-elastic on the support. That is, it creates the demand that makes it indifferent between any price on the support. This preserves the incentives for the mixed strategy and solves the firm's commitment problem. However, to induce this endogenous unit-elastic demand, different types must trade with different probability. This means that the equilibrium entails inefficiencies: the probability of no trade is always strictly greater than zero.

Moreover, the firm does not benefit from the attachment effect and equilibrium profits are independent of its strength. Indeed, types below the support of the mixed strategy only face prices above their valuation in equilibrium. By the PPE requirement, they cannot expect to buy and behave as if they have no attachment effect. This implies that the profits from the lowest price on the support must be the same as in the reference-independent benchmark[2], pinning down equilibrium profits. The commitment problem created by the attachment effect has thus two effects here: the firm must use random prices to overcome it and it does not benefit from having buyers with reference-dependent preferences.

In Section 2.3.1, I characterise the firm's pricing strategy when it knows the buyer's valuation in two different ways, and get contrasting results. First, I use the incomplete information characterisation to study convergence to complete information. As the distribution over types concentrates on a singleton, the equilibrium strategy converges to a mixed strategy on a non-vanishing support and the limit probability of no trade is bounded away from zero. This follows from the incomplete information characterisation. In order to create a unit-elastic demand, the firm induces a large variation in the trading probability of almost identical types. This results in a positive probability of no trade. This problem is more severe for a stronger attachment effect: the probability of

[2]Throughout the paper, the reference-independent model refers to an equivalent model with no attachment effect, i.e., the WTP is the same as the valuation.

trading converges to zero as the attachment effect grows large.

However, there is a discontinuity at the limit: when the valuation is common knowledge, no equilibrium exists. Indeed, it is impossible for the firm to overcome its commitment problem. Any pricing strategy where the buyer is willing to accept increases his WTP above the price played in equilibrium.[3]

In Heidhues and Kőszegi (2014), the firm can commit to a price distribution and the valuation is common knowledge. In contrast, I consider a firm that cannot commit to a price distribution and the buyer's valuation is private information. I show that inefficiencies are a general feature of this model and that the firm does not benefit from facing a buyer with an attachment effect. Moreover, an equilibrium does not exist when the valuation is common knowledge. This shows that the commitment assumption is key to both existence and benefiting from the attachment effect. On the other hand, my model also predicts random prices like Heidhues and Kőszegi (2014).

The first two sets of results looked at extreme information structures: either incomplete information, where the firm does not benefit from the attachment effect or complete information, where no equilibrium exists. In Section 2.4, I look at intermediate case where the firm is partially informed about the buyer's valuation. Specifically, I allow the firm to design a test to learn about the buyer's valuation. The idea behind this section is to explore whether the firm can use the buyer's expectations about his performance on the test to take advantage of the attachment effect. For example, suppose that a firm designs a screening process before making a wage offer to a candidate.[4] The candidate will use his performance during the screening process to assess what are his chances of getting a high wage. If the candidate believes he did well on the test designed by the firm, he will expect high wages and thus a high probability of accepting the job. If the candidate exhibits an attachment effect, this would weaken his bargaining position: he would be willing to accept lower wage offers to avoid the disappointment of not being employed. This could then be used by the firm to offer lower wages.

In this new environment, the firm first designs a publicly observed test, privately observes a signal realisation then makes an offer. In Proposition 6, I show that the firm can be better off with a noisy test. Intuitively, when the test is noisy, the firm can create random price offers like in Heidhues and Kőszegi (2014). At the same time, given the uncertainty generated by the test, the firm can credibly offer low prices after a low signal and high prices after a high signal. This last section shows that in the presence of an attachment effect, a monopolist has an incentive to design imperfect tests. This allows the buyer to credibly entertain the idea that he will get a low price. The firm can then use these expectations to make higher profits. In Proposition 7, I characterise the firm's optimal testing strategy.

[3]Existence issues with PPE in strategic settings were already pointed out by Dato et al. (2017).

[4]The model is set up as a buyer-seller interaction. It can be easily rewritten as firm making a wage offer to a candidate with unknown outside option.

Relation to the literature

This paper is part of the literature that studies the implication of rational expectations as the reference point in reference-dependent preferences, following Kőszegi and Rabin (2006). The two closest papers in this literature are Heidhues and Kőszegi (2014) and Eliaz and Spiegler (2015).

Eliaz and Spiegler (2015) look at a more abstract model that nests the complete information environment with commitment of this paper as a special case. They show that uniqueness of the PE can be guaranteed through a first-order stochastic dominance property that is useful in this paper. This paper differs from the existing literature by not allowing the firm to commit to a price distribution. I show how to characterise the equilibrium pricing strategy by adapting the result of Eliaz and Spiegler (2015). I also show that an imperfect learning strategy can provide a foundation for the stochastic pricing strategy without commitment. Rosato (2016) also studies a monopoly pricing model where the uncertainty is used to exploit expectation-based reference-dependent preferences. There, the monopolist commits to the limited availability of substitutes to induce the expectations of buying.

The last section is related to the literature on optimal disclosure with a behavioural audiences as it is concerned with the design of the information environment with non-standard preferences, see e.g., Lipnowski and Mathevet (2018); Lipnowski et al. (2020); Levy et al. (2020). In particular, Karle and Schumacher (2017) study a model where a monopolist posts a public price as well as discloses a signal of the valuation of an initially uninformed buyer with expectation-based reference-dependent preferences. The firm benefits from imperfect disclosure when a low valuation is pooled with a high valuation. The buyer then expects to buy at the price posted and thus develops an attachment towards the good. In contrast, I consider a perfectly informed buyer and it is the firm that learns about the valuation. This has two implications. First, the price offered depend on the signal observed, so there is variation in price. Second, inducing expectations to buy is not enough for the firm to benefit from the attachment effect as this happens when prices are relatively low. So the firm must induce both high and low prices to benefit from it.[5]

Finally, there are links to the literature on optimal learning and price discrimination. Bergemann et al. (2015) characterise all the combination of consumers' surplus and monopoly profit after some learning of the firm. I depart from their framework by introducing reference-dependent preferences. Where in their model the optimal learning strategy is to perfectly learn the valuation, introducing reference-dependent preferences incentivises the firm to create a stochastic environment. Roesler and Szentes (2017) and Condorelli and Szentes (2020) look at environments where an agent designs an optimal learning strategy taking into account the effect of information acquisition on the other agent's strategy. Here, the firm designs its optimal learning strategy taking into account

[5] Karle and Schumacher (2017) also show that the monopolist does not benefit from committing to its pricing strategy, unlike this paper. The type of commitment is however different: they consider a setting where the firm can commit to not change the price after the buyer has set expectations but they do not allow commitment to a random price strategy.

its effect on the buyer's preferences.

2.2 The model

There is one firm and one buyer. The firm makes a take-it-or-leave-it offer $p \in \mathbb{R}$ for an indivisible good that the buyer can either accept, $a = 1$, or reject, $a = 0$. The buyer has a reference-point $r \in \{0, 1\}$ and an exogenous valuation v. His payoffs are

$$u(p, v, a|r) = a(v - p) - \lambda \cdot v \cdot r(1 - a),$$

where $r = 1$ stands for "expecting to accept", and $r = 0$ for "expecting to reject". Here, the buyer "pays" a penalty λv whenever he rejects an offer he was expecting to accept. Like in Kőszegi and Rabin (2006), I allow the reference point to be stochastic. The reference point is then $q \in [0, 1]$ which stands for the probability of accepting. The utility of buyer v is written as

$$u(p, v, a|q) = q \cdot \big(a(v - p) - \lambda v \cdot (1 - a)\big) + (1 - q) \cdot a(v - p).$$

The firm's payoff is

$$\pi(p, a) = a \cdot p.$$

The buyer knows v. The firm only knows that $v \sim G$, where G denotes a cdf. It admits a strictly positive density g on the support $V = [\underline{v}, \overline{v}]$, $\underline{v} \geq 0$. I use γ to denote the probability measure associated with G: for any measurable set A, $Pr[v \in A] = \gamma(A)$. I will often refer to a valuation v as the buyer's type. I assume that there is a positive surplus with any type and so the assumption that the firm has no cost is a normalisation.

Buyer's behaviour Given his valuation v and his reference point q, the buyer's payoffs from accepting and refusing at price p are

$$u(p, v, a = 1|q) = v - p,$$
$$u(p, v, a = 0|q) = 0 - \lambda v \cdot q.$$

Therefore, he optimally plays a cutoff strategy: he accepts an offer p if and only if $p \leq v + \lambda v q$.[6] I denote the buyer's optimal strategy by $a^*(p, v|q) = \mathbb{1}[p \leq v + \lambda v q]$.

[6]Here, I assume that, when indifferent, the buyer accepts the price offer. Allowing for different strategies when indifferent could change the PPE outcome. However, one can show that it would not change the equilibrium strategies in this paper. Therefore, to simplify the exposition, I omit this possibility.

Following Kőszegi and Rabin (2006), the buyer forms his reference point based on the correct expectations of trading. I assume that the buyer first learns his type, then forms his expectations based on the price distribution F. The reference point is thus formed after learning his own type but before the price realisation. Therefore, different types can have different expectations of trading. A Personal Equilibrium (PE) is a reference point q such that the probability of trading is consistent with the optimal strategy given the reference point.

Definition 1. *Given a price distribution F, $(Q_v)_v$ is a profile of Personal Equilibria if for each $v \in V$, Q_v satisfies*

$$Q_v = \int_{\mathbb{R}} a^*(p, v|Q_v) dF(p)$$

and $a^*(p, v|Q_v) = \mathbb{1}[p \leq v + \lambda v Q_v] \in \arg\max u(p, v, a|Q_v)$.

In a PE, the buyer with valuation v correctly anticipates how his expectations change his strategy and how his strategy changes his expectations. The PE Q_v depends on the type but also on the distribution over prices. Therefore, the buyer's behaviour will depend directly on the firm's strategy.

The expected utility of type v, for a given PE Q_v and price distribution $F(p)$ is

$$W(v|F, Q_v) = \int_{-\infty}^{v+\lambda v Q_v} (v - p) dF(p) + \int_{v+\lambda v Q_v}^{+\infty} -\lambda v Q_v \, dF(p).$$

For any prices in $(-\infty, v + \lambda v Q_v]$, the buyer accepts the offer and gets a utility $v - p$. For prices larger than $v + \lambda v Q_v$, the buyer rejects the offer and gets a loss of $-\lambda v Q_v$.

Because there can be multiple PEs, I assume the buyer plays his Preferred Personal Equilibrium (PPE). The PPE is the Personal Equilibrium that gives the highest expected utility (Kőszegi and Rabin, 2006, 2007).

Definition 2. *Given a price distribution F, $(Q_v^*)_v$ is a profile of Preferred Personal Equilibria if for each $v \in V$, $Q_v^* \in \arg\max_{Q_v \in PE} W(v|F, Q_v)$.*

This (Personal) equilibrium selection is common in the literature using Personal Equilibria.[7] Its motivation is based on an introspection interpretation of the PE. The buyer can entertain multiple expectations of trading but cannot fool himself: his reference point must be correct given his optimal behaviour. Then, if he can "choose" amongst multiple reference points, he would choose the one with the highest expected utility.

It will be useful to think of the PE or PPE as the cutoff price it generates.

Definition 3. *Given F a distribution over prices and PE Q_v, the PE cutoff price of type $v \in V$ is $\hat{p}(v) = v + \lambda v Q_v$. Given PPE Q_v^*, the PPE cutoff price is $p^*(v) = v + \lambda v Q_v^*$.*

[7]See e.g., Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018).

The PPE cutoff price determines buyer v's willingness-to-pay (WTP). Note also that we have $Q_v^* = F(p^*(v))$, as buyer v accepts any price below $p^*(v)$. In the rest of the paper, the valuation refers to a buyer's type v and his willingness-to-pay to his PPE cutoff price, $p^*(v)$.

Given a profile of PPE $(Q_v^*)_v$, let $V^*(p) = \{v \in V : p \leq p^*(v)\}$ be the set of types accepting price p. Define $v^*(p) = \inf\{v : v \in V^*(p)\}$, the lowest type in $V^*(p)$.

Equilibrium The firm's expected profits given the profile of PPE $(Q_v^*)_v$ are $\mathbb{E}[\pi(p)|(Q_v^*)_v] = p\,\gamma(V^*(p))$. I can now define an equilibrium in this model.

Definition 4. *A profile of strategy and reference points $\big(F(p), (Q_v^*)_v\big)$ is an equilibrium if for each $v \in V$, Q_v^* is type v's PPE given F and for each $p \in \operatorname{supp} F$, $p \in \arg\max_{\tilde{p}} \mathbb{E}[\pi(\tilde{p})|(Q_v^*)_v]$.*

In equilibrium, each buyer v forms his expectations based on on the firm's equilibrium strategy and his type and the firm's strategy is a best response to the buyers' PPEs.

2.2.1 Comments

Utility function The utility function I use allows me to capture an attachment effect in the simplest possible way. With this utility function, the agent with valuation v pays a penalty λv weighted by the probability of accepting q when he does not accept the offer. The original specification of Kőszegi and Rabin (2006) allows for a reference point that depends both on the distribution over consumption and price paid. Here, the utility function is similar to ones used in the literature with loss-aversion in one dimension only.[8] Having loss-aversion in one dimension only allows to cleanly isolate an effect of the reference-dependent preferences, for example aversion to price increases or in this case, the attachment effect.

The commitment assumption Whether commitment to a random pricing strategy is a reasonable assumption depends on the situation considered. When a patient firm post publicly observed price, the commitment assumption can be justified by the incentives of a firm to develop a certain reputation for some price distribution. On the other hand, in many settings, prices are not directly observed. This is the case for example for goods or services that are the outcome of some bargaining or not often traded such as houses, cars or jobs. In this case, the take-it-or-leave-it bargaining structure captures a bargaining process where the firm has all the bargaining power.

[8] For example, section 4.1 in Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Carbajal and Ely (2016), Rosato (2020) or Spiegler (2012).

2.2.2 Characterisation of the PPE

Proposition 1 establishes two properties of the PPE. First, the PPE cutoff is the smallest of the PE cutoffs. Second, it establishes that if $p^*(v)$ is the PPE cutoff then $F(p)$ must lie strictly above $\frac{p-v}{\lambda v}$ on $(-\infty, p^*(v))$. The proof of Proposition 1 also establishes existence of the PPE.

Proposition 1. *For a fixed type v and distribution F, these three statements are equivalent:*

- $p^*(v)$ *is PPE cutoff price*
- $p^*(v) = \min\{p : F(p) = \frac{p-v}{\lambda v}\}$
- $v - p^*(v) = -\lambda v F\big(p^*(v)\big)$ *and for all* $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda v}$

Moreover a PPE exists.

Proof. Fix a type v. First, I show that the PPE cutoff is the lowest PE cutoff. Fix two PE, Q_1, Q_2 and their respective PE cutoffs, p_1, p_2. Then,

$$\begin{aligned} v - p_1 &= -\lambda v F(p_1) \\ \text{and } v - p_2 &= -\lambda v F(p_2). \end{aligned}$$

Note that we have $F(p_1) \neq F(p_2)$, for otherwise $p_1 = p_2$. The expected utility at PE Q_i is

$$W(v|Q_i) = \int_{-\infty}^{p_i} (v-p)dF(p) + \int_{p_i}^{+\infty} -\lambda v F(p_i) dF(p).$$

Using the equality defining the cutoff, p_1 is preferred to p_2 if and only if

$$\begin{aligned} \int_{-\infty}^{p_1} (v-p)dF(p) + (1-F(p_1))(v-p_1) \geq \int_{-\infty}^{p_2} (v-p)dF(p) + (1-F(p_2))(v-p_2) \\ \Leftrightarrow \int_{p_1}^{p_2} p\, dF(p) \geq (1-F(p_1))p_1 - (1-F(p_2))p_2 \\ \Leftrightarrow p_2F(p_2) - p_1F(p_1) - \int_{p_1}^{p_2} F(p)dp \geq (1-F(p_1))p_1 - (1-F(p_2))p_2 \\ \Leftrightarrow p_2 - p_1 \geq \int_{p_1}^{p_2} F(p)dp, \end{aligned}$$

where I obtain the third line by integrating by part. Because $F(p_1) \neq F(p_2)$, this is satisfied if and only if $p_1 < p_2$.

Now let $\tilde{p} = \inf\{p : F(p) \leq \frac{p-v}{\lambda v}\}$. If F is continuous at $\tilde{p}$, then $\inf\{p : F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p :$

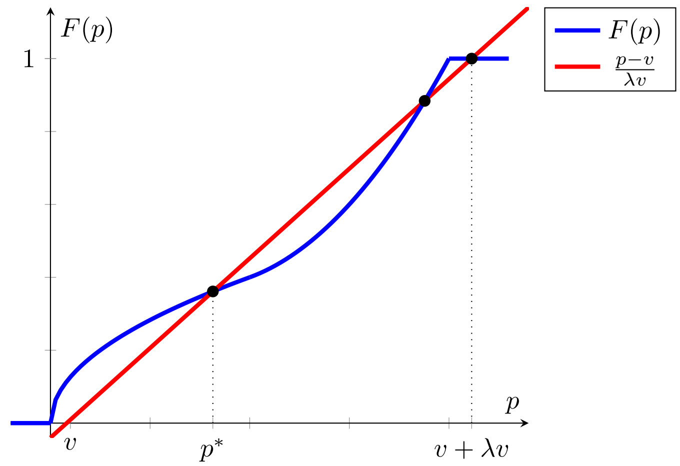


Figure 2.1: Each intersection of the blue and red curve is a PE. The lowest intersection, p^*, is the PPE.

$F(p) \leq \frac{p-v}{\lambda v}\}$. If F is not continuous at $\tilde{p}$, then because it is non-decreasing, $\lim_{p \nearrow \tilde{p}} F(p) < F(\tilde{p})$, which then contradicts that $\tilde{p} = \inf\{p : \; F(p) \leq \frac{p-v}{\lambda v}\}$.

Therefore, F is continuous at $\tilde{p}$ and $\min\{p : \; F(p) \leq \frac{p-v}{\lambda v}\}$ exists. Because F is continuous at $\tilde{p}$, it also implies that $\min\{p : \; F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p : \; F(p) = \frac{p-v}{\lambda v}\}$ which establishes existence of the PPE and the first equivalence.

We can now show $p^*(v) = \min\{p : \; v - p = -\lambda v F(p)\} \Leftrightarrow F(p) > \frac{p-v}{\lambda v}$ for all $p < p^*(v)$ and $F(p^*(v)) = \frac{p^*(v)-v}{\lambda v}$.

($\Rightarrow$) Suppose $p^*(v)$ is a PPE and we have some $\hat{p} < p^*(v)$ with $F(\hat{p}) \leq \frac{\hat{p}-v}{\lambda}$. Because we have established that $\min\{p : \; F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p : \; F(p) = \frac{p-v}{\lambda v}\}$, we get a contradiction.

($\Leftarrow$) If for all $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda v}$, then there are no other PE smaller $p^*(v)$. □

The PPE cutoff price is the smallest of the PE cutoff prices because for any distribution, this cutoff is weakly above the valuation v. Therefore, the lowest PE cutoff minimises trade when $p > v$, i.e., when the buyer has a negative utility. The condition that $F(p) > \frac{p-v}{\lambda v}$ when $p < p^*(v)$ was introduced by Eliaz and Spiegler (2015). This property is similar to the characterisation of first-order stochastic dominance albeit on only part of the support. In particular, it implies that for any F implementing PPE cutoff $p^*(v)$,

$$\int_{-\infty}^{p^*(v)} x \, dF(x) < \int_{v}^{p^*(v)} x \cdot \frac{1}{\lambda v} dx.$$

This observation will be useful in Section 2.4 when we will design the firm's optimal testing strategy. Figure 1 illustrates graphically how to determine the PPE cutoff using Proposition 1.

2.3 Incomplete information

In this section, I first characterise the equilibrium when the valuation is private information. In Section 2.3.1, I study the equilibrium when the set of types converges to a singleton and when the valuation is common knowledge.

To simplify the analysis in this section, I restrict attention to strictly concave reference-independent profits.

Assumption 1. *The function $p(1-G(p))$ is strictly concave.*

Assumption 1 is made to simplify the exposition of the paper. The results of this section extend qualitatively to more general distributions but the exact characterisation of the equilibrium differs.

Proposition 2 characterises the unique equilibrium of the game. It shows that the firm plays a mixed strategy, the equilibrium demand is unit-elastic and the equilibrium profits are the same as in a reference-independent model.

Denote by $\pi^* = \max_p \, p(1-G(p))$ and $p_{ind} = \arg\max_p p(1-G(p))$, the equilibrium profits and prices of the reference-independent benchmark.

Proposition 2. *There is a unique equilibrium $\left(F, (Q_v^*)_v\right)$. In equilibrium,*

- *The firm plays the mixed strategy*
$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)},$$
with $\operatorname{supp} F = [p_{ind}, \overline{p}]$, $\overline{p} := F(\overline{p}) = 1$.
- *If $p \in \operatorname{supp} F$, then $\gamma(V^*(p)) = \frac{\pi^*}{p}$, i.e., the demand is unit-elastic on the support.*
- *The equilibrium profits are $\pi^* = \max_p \, p(1-G(p))$.*

The proof is in Section 2.6.1.

Proposition 2 illustrates how the firm's commitment problem constrains its behaviour and how it can solve it. First, any pure strategy is not credible. To see why, suppose the firm plays a pure strategy p. Then by the PPE requirement, all the types $v \geq p$ accept the offer and all types $v < p$ refuse it. This means that their WTPs are $p^*(v) = v + \lambda v$ for $v \geq p$ and $p^*(v) = v$ for $v < p$. The firm has then a profitable deviation to a higher price as all types that accept p are willing to pay strictly more than p.

The firm can play a mixed strategy only if it is indifferent between any price on the support, i.e., the demand is unit-elastic on the support. Therefore, the firm's mixed strategy induces expectations of trading such that the resulting distribution over WTP is unit-elastic.

Moreover, the firm does not benefit from facing buyers with reference-dependent preferences. The buyers whose valuation is below the support know they will only face prices higher than their valuation. Because they play a PPE, they cannot expect to trade and their WTP is equal to their valuation. Therefore, they behave like players with no attachment effect. When offering the lowest price on the support, only buyers with valuation above that price accept, exactly like in a reference-independent model. By the indifference condition, these must be the equilibrium profits. However, some types do end up buying the good at a price above their valuations with some probability. The exploitation of the buyers' attachment effect is compensated by higher probability of no trade when offering high prices.

2.3.1 (Almost) Complete information

In this subsection, I look at the behaviour of equilibrium objects when the distribution over valuations converges to a singleton and when the valuation is common knowledge.

In what follows, I look at a sequence of games where the only varying primitive is the prior distribution G. Therefore, abusing notation, I will identify a sequence of games with a sequence of prior distributions. Denote by $\xrightarrow{\mathcal{D}}$ convergence in distribution and δ_v the Dirac measure on v.

Proposition 3. *Let $v > 0$. Take a sequence of games $\{G_i\}_{i=0}^{\infty}$ such that $G_i \xrightarrow{\mathcal{D}} \delta_v$. All other primitives of the model are fixed.*

Then, equilibrium profits converge to v and the firm's equilibrium strategy converges in distribution to $F_\infty(p) = \frac{p-v}{\lambda v}$ with $\operatorname{supp} F_\infty = [v, v+\lambda v]$.

Moreover, the limit probability of trade is

$$\frac{1}{\lambda}\log\frac{v+\lambda v}{v}.$$

Proof. **Limit distribution and profits:**

Let F_i be the equilibrium strategy given G_i, $\underline{p}_i = \min \operatorname{supp} F_i$, $\overline{p}_i = \max \operatorname{supp} F_i$ and $\pi_i^* = \underline{p}_i(1 - G_i(\underline{p}_i))$. From Proposition 2, $F_i(p) = \frac{p - G_i^{-1}\left(\frac{p-\pi_i^*}{p}\right)}{\lambda G_i^{-1}\left(\frac{p-\pi_i^*}{p}\right)}$ for all $p \in [\underline{p}_i, \overline{p}_i]$. Using that $G_i^{-1}(x) \to v$ for each $x \in (0,1)$, for each $p \in \mathbb{R}$,

$$F_i(p) \to \begin{cases} 0 & if \quad p < v \\ \frac{p-v}{\lambda v} & if \quad p \in [v, v+\lambda v] \\ 1 & if \quad p > v + \lambda v \end{cases},$$

and thus $F_i \xrightarrow{\mathcal{D}} F_\infty$.

Profits converge to v as $\max_p \, p(1 - G_i(p)) \to v$.

Probability of trade: Denote the probability of trading at price p by $\phi(p)$. This probability is pinned down by the indifference condition:

$$\pi_i^* = p\phi_i(p).$$

The probability of trading is thus $\int_{\mathbb{R}} \phi_i(p) f_i(p)\, dp$ where f_i is the density of F_i. It is easy to verify that $f_i(p) \to \frac{1}{\lambda v}$ for all $p \in [v, v + \lambda v]$ and $\phi_i(p) f_i(p)$ is uniformly bounded. Using the dominated convergence theorem, we get that

$$\int_{\mathbb{R}} \phi_i(p) f_i(p)\, dp \to \int_v^{v+\lambda v} \frac{v}{p} \cdot \frac{1}{\lambda v} dp = \frac{1}{\lambda} \log \frac{v + \lambda v}{v}.$$

□

As the distribution of types converges to the singleton v, the firm's strategy converges to a uniform distribution on $[v, v + \lambda v]$. The profits, on the other hand, are always equal to the reference-independent benchmark, $\pi^* = v$ in the limit. As the mass of types accumulate on v, the support does not converge to a singleton. Even though the interval of valuations could become arbitrarily small, the interval of potential WTP stays large: any $p \in [v, v + \lambda v]$ can be a PPE cutoff. The firm still needs to mix to create the endogenous demand that makes it indifferent on the support. This means that even as we converge to complete information, a large amount of uncertainty is needed to guarantee an equilibrium. This variation leads to a strictly positive probability of no trade in the limit.

Moreover, $\frac{1}{\lambda} \log \frac{v+\lambda v}{v}$, the probability of trade, decreases in the attachment effect λ and converges to 0 as $\lambda \to \infty$. Intuitively, a higher λ makes the buyer more vulnerable to exploitation but also increases the firm's commitment problem. The firm must lower the probability of trading to compensate for the higher demand induced by a higher λ. This can also be seen from the indifference condition: as profits converge to v for any λ, the higher prices must be compensated for by a higher probability of rejection.

Finally, I note that the limit strategy is arbitrarily close to the one the firm would use if it could commit to a price distribution.

Proposition 4 (Heidhues and Kőszegi, 2014)**.** *The solution to the commitment problem:*

$$\sup_{F \in \Delta\mathbb{R}} \int_{-\infty}^{p^*} p\, dF(p)$$

subject to p^* is PPE, is

$$\frac{\lambda + 2}{2} \cdot v.$$

The distribution that attains this profit converges to $F(p) = \frac{p-v}{\lambda v}$.

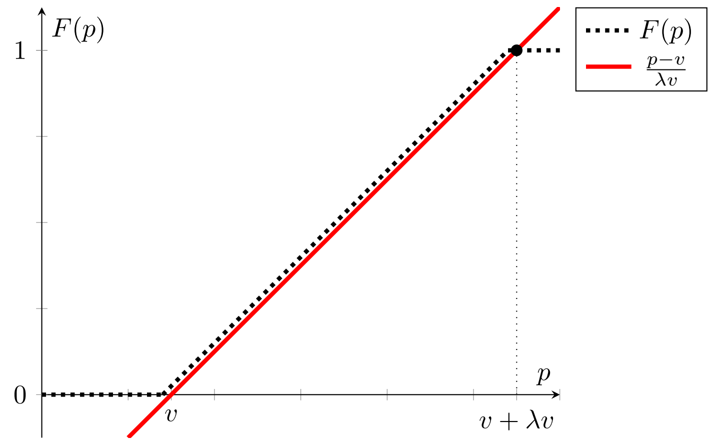


Figure 2.2: Almost optimal distribution over prices.

Proof. See Heidhues and Kőszegi (2014), section 4.1 or Eliaz and Spiegler (2015). □

The firm chooses a price distribution that maximises its profits amongst all the distribution that implement trade with probability one. By FOSD interpretation of the PPE (Proposition 1), this is done by choosing a price distribution as close as possible to $\frac{p-v}{\lambda v}$, see Figure 2.2 for an illustration.

In contrast, if the value of v is common knowledge and there is no commitment, no equilibrium exists.

Proposition 5. *When v is common knowledge and $v > 0$, no equilibrium exists.*

Proof. For any $p^*(v)$, there is a unique best-response of the firm, which is to offer $p^*(v)$. Let p be the equilibrium price, i.e., the equilibrium strategy is $F(\tilde{p}) = \mathbb{1}[\tilde{p} \geq p]$.

If $p \leq v$, then there is a unique PE cutoff $p^*(v) = v + \lambda v F(v + \lambda) = v + \lambda v$. There is a profitable deviation to $p' = v + \lambda v$.

If $p > v$, then $p^*(v) = v + \lambda v F(v) = v$ is a PE cutoff and also the smallest PE cutoff. By Proposition 1, it is the PPE cutoff and thus there is no trade in equilibrium. Then, there is a profitable deviation to any $p' \in (0, v]$. □

The key tension is that the firm wants to take advantage of the attachment effect. However, given the PPE requirement and a deterministic price, the buyer's WTP is only higher than his valuation if the price offered is below his valuation. An equilibrium with no trade is also impossible because any price below the valuation will be accepted for any PPE.

We obtain the non-existence result for the same reason there cannot be a pure strategy equilibrium in the incomplete information environment. Playing a pure strategy shifts the demand if the offer is accepted. Unlike the incomplete information environment, the firm cannot play a mixed strategy because it is facing only one type. Thus, it cannot solve its commitment problem.

Relation to other non-existence results: Dato et al. (2017) have already observed that in games where players are constrained to play a PPE, an equilibrium does not always exist. They note that with binary actions, a PPE strategy never entails mixing. Therefore, if the equilibrium requires mixed strategies, these strategies can never be the players' PPE, even though they could be PEs. Here, the mechanism for non-existence is different as the equilibrium relaxing the PPE constraint would not be in mixed strategies. Instead, it occurs because the buyer's PPE price cutoff is always bounded away from the price offered.

Azevedo and Gottlieb (2012) show that games with prospect theory preferences can suffer from equilibrium existence issues in a game where a risk-neutral firm offers a gamble to an agent. In their case, they observe that for an exogenous reference point and some conditions on the value and probability weighting functions, there always exists a bet with arbitrarily low expected value that an agent with prospect theory preferences is willing to accept. Our analysis differ both in the choice of reference-point as well as the choice of payoff function – they allow for probability weighting as well as restrict attention to gain-loss value function. In the model considered here, the payoffs are always bounded so the mechanism for non-existence is also different.

2.4 Intermediate case: Testing the valuation

We have so far looked at "extreme" information structures, either complete information or complete lack of information. In this section, I allow the firm to collect additional information on the buyer's valuation before setting a price. In Proposition 6, I first establish that having only partial information about the buyer's valuation can be beneficial for the firm. In Proposition 7, I characterise the firm's preferred testing strategy and its profits.

In many bilateral trade settings, one party can gather information about the other before making an offer. A leading example is job applications where an employer designs a screening process before making an offer to a candidate.[9] In the process, he collects information about the candidate's productivity and outside option. These can be explicit tests through an assessment centre or asking to submit certain documents like a CV or recommendation letters. The question being asked in this section is whether a firm can use a combination of the screening process and a candidate's attachment effect to offer lower wages.

This section would also be relevant to settings where a seller can gather information about the buyer's valuation. For example, a car dealer could ask questions to the consumer about his valuation.

[9]The model is cast as a buyer-seller interaction but it can be rewritten as an firm-candidate interaction where the candidate has private information regarding his outside option and the firm makes take-it-or-leave-it wage offer.

I focus on $G \sim U[\underline{v}, \overline{v}]$. This is with loss of generality and I explain after the discussion of Proposition 7 why we need to restrict attention to these distributions. All functions and sets are assumed to be measurable.

Test: Let S be a set of signals and $F : V \to \Delta(S)$ a mapping from types to distributions over signals. A test is a pair (F, S). Denote by $F(s|v)$ the distribution of s conditional on v. Abusing notation, $F(v, s)$ is the joint distribution of (v, s) induced by $F(s|v)$ and $G(v)$.

Players' information: The test is common knowledge but only the firm observes the signal realisation. The valuation v is still privately known.

Players' strategy: A strategy for the firm is $P : S \to \Delta(\mathbb{R})$, a mapping from signals to distributions over prices.

The assumption regarding the players' information and strategies correspond to a setting where the seller can commit to a test or where the test is publicly observable. This would be the case in a job application setting where the candidate can observe the selection process he goes through. An important assumption is also that the candidate does not observe the signal realisation, e.g., he does not know the result of the test, does not observe his recommendation letters or an interview is made opaque to make it hard to interpret. Alternatively, this information could be revealed to him as long as his reference-point is set before learning the outcome of the test. The case of public signals is discussed at the end of the section.

The PE and PPE are defined in the same way as before. I assume that the buyer forms expectations after having observed his type and the test. Given the test (F, S) and strategy P, a reference point Q_v is type v's PE if

$$Q_v = \int_S \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda v Q_v] \, dP(p|s) dF(s|v),$$

and the expected utility of type v given (F, S), P and Q_v is

$$W(v|Q_v) = \int_S \int_{\mathbb{R}} \bigg(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \bigg) dP(p|s) dF(s|v).$$

Then, Q_v^* is a PPE if $Q_v^* \in \arg\max_{Q_v \in PE} W(v|Q_v)$.

The firm's ex-ante payoffs are

$$\mathbb{E}[\pi(P)|(Q_v^*)_v] = \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)] \, p \, dP(p|s) dF(s|v) dG(v).$$

Definition 5. *Fix a test (F, S). An equilibrium is a profile $(P, (Q_v^*)_v)$ such that*

- $P \in \arg\max \mathbb{E}[\pi(\cdot)|(Q_v^*)_v]$,
- *and for all $v \in [\underline{v}, \overline{v}]$, Q_v^* is type v's PPE.*

In equilibrium, the buyer plays according to his PPE based on the test and the firm's strategy. The firm best replies to the buyers' PPE based on its information.

In the reference-independent model, the firm would perfectly learn the buyer's valuation and offer the valuation. This would also be the limit profits of the firm if the test would be arbitrarily close to fully revealing the valuation (Proposition 3).[10] The profits in this case would be $\int_V v\,dG$. A test is completely noisy if for almost all s and v, $Pr_F[s|v] < 1$ and there is no unique v such that $s \in \operatorname{supp} F(\cdot|v)$. That is, a test is completely noisy if no type sends a signal deterministically and any signal realisation does not reveal any type.

Proposition 6. *There is a completely noisy test (F, S) and an equilibrium $(P, (Q_v^*)_v)$ such that firm's profits are strictly greater than $\int_V v\,dG$.*

The proof is in Section 2.6.3.

Proposition 6 shows that the firm benefits from not fully learning the buyer's type. Coming back to the case of screening processes for job candidates, Proposition 6 suggests that firms could benefit from designing noisy or opaque screening procedures. This would let candidates entertain the idea that they would get a good wage offer and thus weaken their bargaining position. However, not all completely noisy tests achieve higher profits. In Proposition 7, I characterise this test and show that it is optimal amongst all tests.

A completely noisy test uses the two types of uncertainty it generates to credibly exploit the buyer's attachment effect. First, the buyer is uncertain about which signal he generated and therefore which types he is pooled with. At low signals, the firm offers low prices, inducing expectations to buy. Then, at higher signals, the firm offers higher prices, taking advantage of the higher WTP. From the buyer's perspective, he is facing random prices like in Heidhues and Kőszegi (2014). Second, the firm uses the uncertainty it has about the type to credibly offer low prices after a low signal, despite facing some buyers willing to accept higher prices.

The next step is to characterise the firm's preferred test. Like in Bergemann and Morris (2016), we can focus on tests that generate action recommendations compatible with the PPE requirement. The distribution $\delta_s(p)$ denotes the Dirac measure.

Lemma 3. *Consider a test (F, S) and an equilibrium $\left(P, (Q_v^*)_v\right)$ in (F, S). Then, there exists a test $(\tilde{F}, \cup_{s\in S} \operatorname{supp} P(\cdot|s))$ and an equilibrium $\left(\delta_s, (Q_v^*)_v\right)$ in that test such that each player gets the same payoffs as in the original equilibrium.*

[10]Remember that if the firm perfectly learns the valuation, then there is no equilibrium in the resulting game.

Proof. See Section 2.6.2 □

Lemma 3 holds because the only thing that matters for the buyer's PPE is the distribution over prices given his type. Therefore, a standard revelation principle argument holds. If after two different signals, the firm offers the same price, we can modify the test to "merge" these two signals. This will not change the distribution over actions and thus all PPEs are preserved.

The firm's problem is to find the supremum profits and the test that attains it:

$$\sup_{(F,S)} \int_{V \times S} \mathbb{1}[v \in V^*(s)]\, s\, dF(v,s) \tag{2.1}$$

$$\text{s.t. } p^*(v) = \min\{p:\ v - p = -\lambda v\, F(p|v)\}, \tag{2.2}$$

$$\text{for all } S' \subseteq S,$$

$$\int_{V \times S'} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V \times S'} \mathbb{1}[v \in V^*(\tilde{P}(x))]\, \tilde{P}(x) dF(v,x) \tag{2.3}$$

$$\text{for all } \tilde{P} : S \to \mathbb{R}$$

We need to solve for the supremum because the set of constraints is not closed. In particular, a sequence of distributions $F_i(\cdot|v) \to F(\cdot|v)$ that induce $p^*(v) = p'$ for each i can have $p' \neq \min\{p:\ v - p = -\lambda v\, F(p|v)\}$.[11] The firm chooses distribution over prices that will determine its profits and the WTP of each type. The constraint (2.2) pins down the WTP of each buyer. The obedience constraint (2.3) ensures that the firm is willing to follow almost all price recommendations. I say that a test is *admissible* if it satisfies the constraints (2.3) where the WTP are pinned down by (2.2). Obedience constraints are required to hold for any subset S' to take into account that some signal realisations might be zero probability events and have no well-defined density.

To fix ideas, I first characterise the firm's first-best solution, i.e., ignoring the obedience constraints. It is equivalent to allow the firm to commit to a distribution over prices, like in Heidhues and Kőszegi (2014). Proposition 4 in the previous section has thus already characterised the first-best solution.

Claim 1. *The first-best solution is to take a sequence* $\{(F_i, S_i)\}$ *such that*

- $F_i(s|v) \to \frac{s-v}{\lambda v}$ *with* $\operatorname{supp} F_i \to [v, v + \lambda v]$ *for all* v *and* $S_i \to [\underline{v}, \overline{v}(1+\lambda)]$.
- *For each* i*, for each* v*,* $p^*(v) = v + \lambda v$ *and there is trade with probability one.*

I call the distribution $\frac{s-v}{\lambda v}$ on $[v, v + \lambda v]$ the commitment distribution. The commitment distribution and its modification defined below will be important for the characterisation of the optimal test.

[11] For example, the distribution $F(s|v) = \frac{1}{2} + \frac{s - v - \frac{\lambda v}{2}}{\lambda v + \epsilon}$ induces a PPE cutoff $p^*(v) = v + \lambda v/2$ for all $\epsilon > 0$ but a PPE cutoff $p^*(v) = v$ for $\epsilon = 0$.

Definition 6 (Censored commitment distribution)**.** *F is the cdf of a censored commitment distribution if there exists a $\tilde{p} \in [v, v+\lambda v]$ and $\tilde{F}(s|v) < \frac{s-v}{\lambda v}$ such that*

$$F(s|v) = \begin{cases} 0 & if \quad s < v \\ \frac{s-v}{\lambda v} & if \quad s \in [v, \tilde{p}] \\ \tilde{F}(s|v) & if \quad s > \tilde{p} \end{cases} .$$

The censored commitment distribution behaves like the commitment distribution for prices below $\tilde{p}$ but stays below $\frac{s-v}{\lambda v}$ for higher prices. The commitment distribution is a censored commitment distribution with $\tilde{p} = v + \lambda v$.

Proposition 7. *Assume $v \sim U[\underline{v}, \overline{v}]$ and let $\hat{v}(s) = \min\{\overline{v}, s\}$. The firm's supremum profits are*

$$\pi^* = \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\min\left\{\hat{v}(s) - \underline{v},\ s\log\frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda \Delta v} ds,$$

and there exists a sequence of admissible tests $\{(F_i, S)\}$ that approximate the firm's supremum profits such that

- *For each v, $F_i(\cdot|v)$ converges to a censored commitment distribution.*
- *The sequence F_i converges to a completely noisy test.*
- *There is downward distortion: the probability of trading is increasing v.*

In the limit, we have $v^*(s) = \max\{\hat{v}(s)\exp\left(\frac{-\hat{v}(s)+\underline{v}}{s}\right), \dfrac{s}{1+\lambda}\}$ and $p^*(\cdot)$ is the inverse of $v^*(\cdot)$.

The proof is in Section 2.6.3.

Proposition 7 characterises the firm's preferred test and supremum profits. As in Proposition 6, the preferred test is a completely noisy test. Moreover, higher types are more likely to trade and also face, and accept, higher prices compared to their valuation. In our job screening example, it means that less productive candidates (the equivalent of high valuation buyers) are more likely to suffer from their attachment towards the job as they are more likely to receive relatively better offers compared to their outside option.

The firm can credibly follow the equilibrium strategy because it uncertain about which type it is facing. However, there are limits to the uncertainty the firm can generate. For example, the firm cannot pool the lowest type in the support with even lower types. Therefore, this type always expects prices higher than his WTP. By the PPE requirement, he must have a WTP equal to his valuation. In turn, this means that he must trade with probability 0. This logic can be extended to more types: relatively low valuations can be pooled with fewer lower types. The firm can take advantage of their attachment effect but not fully. Their probability of trade is then smaller than one.

This argument can also be seen from the action recommendation approach. In the first-best solution, each type trades with probability one and therefore $p^*(v) = v + \lambda v$. In particular, $p^*(v) \geq \underline{v} + \lambda\underline{v}$ for all v. Moreover, the signal space is $S = [\underline{v}, \overline{v}(1+\lambda)]$. This solution does not respect the obedience constraints because for any signal in $[\underline{v}, \underline{v}(1+\lambda))$, there would be a profitable deviation to $\underline{v}(1+\lambda)$. To satisfy the obedience constraints, the firm decreases the WTP of low types to reduce the incentives to increase the price at low signals. But this means that the probability of trading of low types must be less than one. Hence, the firm generates inefficiencies in the form of downward distortion. The optimal way the firm generates the downward distortion is by using censored commitment distributions. These are the distribution over signals that generate the largest profit for a given WTP.

As in the incomplete information environment, the firm generates inefficiencies to maintain the credibility of its own strategy. In both cases, the firm must manage the buyers' expectations to ensure that it is willing to follow its strategy. Unlike the incomplete information environment, the firm can now make more profits than in the reference-independent benchmark. In Proposition 2, the firm can only make the reference-independent profits because types below the support are always expecting prices higher than their valuation. Here, the support of types conditional on the signal is not common knowledge anymore. Therefore, only the lowest type in the prior distribution expects to face prices higher than his valuation.

The case of non-uniform priors The results derived in this section do not hold for all prior distributions over valuations. The proof of Proposition 7 relies on solving for local obedience constraints and then check that global constraints hold as well. This approach works for some distributions, including uniform distributions, but not all.

Public signals Consider a model where the signal realisation is public and the buyer's reference point is set after having observed the signal realisation. In this different environment, the firm's information is common knowledge. Thus, after each signal realisation, we are back to the environment of Section 2.3. The optimal test for the firm is then to take an arbitrarily fine partition of the type space and play the equilibrium of Proposition 2 in each element of the partition.

The profits are the same as in the reference-independent benchmark but like in Section 2.3, there is a positive probability of no-trade despite the near-complete information. Another difference is that the WTP is no longer monotonic in the valuation: in each element of the partition $[v, v+\epsilon)$, the support of the mixed strategy is $\approx [v, v+\lambda v]$ and the WTP vary on an interval $\approx [v, v+\lambda v)$. Therefore, some types might be on higher elements of the partition, but have a lower WTP.

2.5 Conclusion

In this paper, I study a model of monopoly pricing where the buyer has expectation-based reference-dependent preferences, focusing on an attachment effect. The model has two main features. The expectation-based reference point renders the demand an endogenous object. The PPE requirement creates a commitment problem for the firm.

On a theoretical level, this model offers two main lessons. First, uncertainty can help overcome the firm's commitment problem. In all the environments studied, the firm must manage the buyers' expectations, and thus the demand, to maintain a credible strategy. In the incomplete information environment, the firm needs the uncertainty to induce a unit-elastic demand. For its optimal testing strategy, the firm uses the uncertainty to create obedient distributions over prices. While it can deliver equilibrium existence or credible price distributions, using uncertainty necessarily entails inefficiencies. Furthermore, a higher λ, associated with a stronger commitment problem, implies a higher probability of no trade.

The other recurring theme is the impossibility for the firm to exploit the low types. This follows from the buyers' rational expectations as a low type always anticipate prices above his valuation and therefore cannot expect to buy in a PPE. The consequence was particularly stark in the incomplete information model where it made the profits the same as in the reference-independent model. In the optimal testing environment, it generated downward distortions.

One issue put aside in this paper is the possibility for the buyer to experience gain-loss utility in the money dimension as well. This modification would change the characterisation of the PPE. For example, a PE price cutoff would not be determined by the probability of reaching the cutoff but also on the expected loss in price and thus a result like Proposition 1 would not hold.

2.6 Omitted proofs of Chapter 2

2.6.1 Proof of Proposition 2

Proof. **Preliminary lemmas:**

The first lemmas guarantee the good behaviour of two equilibrium objects, F, the firm's strategy, and $p^*(v)$ the WTP of each buyer as a function of his type.

Lemma 4. *In any equilibrium, $p^*(v)$ is strictly increasing.*

Proof. Take $v_1 < v_2$. Let $p^*(v_1) = p_1$ and $p^*(v_2) = p_2$ be their PPE cutoffs. Assume that $p_1 \geq p_2$. Because p_1, p_2 are PE cutoffs,

$$v_1 - p_1 = -\lambda v_1 F(p_1), \tag{2.4}$$

$$v_2 - p_2 = -\lambda v_2 F(p_2). \tag{2.5}$$

Clearly, $p_1 = p_2$ cannot hold. We either have $v_1 - p_2 \geq -\lambda v_2 F(p_2)$ or $v_1 - p_2 < -\lambda v_2 F(p_2)$. In the first case, we have

$$v_2 - p_2 > v_1 - p_2 \geq -\lambda v_2 F(p_2),$$

contradicting equation (2.5). In the second case, $F(p_2) < \frac{p_2 - v_1}{\lambda v_2}$ and $p_1 > p_2$ contradict Proposition 1. □

Lemma 5. *Let F be an equilibrium strategy. If $p \in \operatorname{supp} F$, then there exists $v \in V$ such that $p^*(v) = p$.*

Proof. Assume not: there is a $p \in \operatorname{supp} F$ and no v such that $p^*(v) = p$. First, if $p \in \operatorname{supp} F$, then $p^*(v) \geq p$ for some v, for otherwise the firm makes zero profits. The firm can always make strictly positive profits by offering a price $p < \overline{v}$. This would be accepted by all types $v \in [p, \overline{v}]$ because $p^*(v) \geq v$ in any PPE. Because $p^*(\cdot)$ is strictly increasing, this implies that there is a v such that $p^*(\cdot)$ is not continuous at v and $p \in [\lim_{x \searrow v} p^*(x), \lim_{x \nearrow v} p^*(x)]$. By continuity of G, $\gamma(V^*(p)) = \gamma(V^*(p^*(v)))$. But then both $p^*(v), p \in \operatorname{supp} F$ but they give different profits, a contradiction. □

Lemma 6. *Any equilibrium strategy F is continuous.*

Proof. Assume not. Let $\tilde{p}$ be a point of discontinuity of F. If $\tilde{p}$ is a PE cutoff for some v, then $F(\tilde{p}) = \frac{\tilde{p} - v}{\lambda v}$. Using the upper semicontinuity of F and continuity of $\frac{p-v}{\lambda v}$, there exists $p' < \tilde{p}$ such that $F(p') < \frac{p' - v}{\lambda v}$. By Proposition 1, $\tilde{p}$ cannot be a PPE cutoff of v. This contradicts Lemma 5. □

Lemma 7. *In any equilibrium, $p^*(v)$ is continuous.*

Proof. Assume there exists a point of discontinuity $\tilde{v}$, i.e., $p_1 \equiv \lim_{v \nearrow \tilde{v}} p^*(v) < \lim_{v \searrow \tilde{v}} p^*(v) \equiv p_2$. We have that $F(p^*(v)) = \frac{p^*(v)-v}{\lambda v}$ and F is continuous, therefore,

$$F(p_1) = \lim_{v \nearrow \tilde{v}} \frac{p^*(v) - v}{\lambda v} < \lim_{v \searrow \tilde{v}} \frac{p^*(v) - v}{\lambda v} = F(p_2).$$

We can then find $\tilde{p} \in (p_1, p_2)$ such that $\tilde{p} \in \operatorname{supp} F$ and there exist no v such that $p^*(v) = \tilde{p}$. This contradicts Lemma 5. □

Lemma 6 rules out pure strategies for the firm. It shows that if the firm puts strictly positive mass at one point of the support, it creates a discontinuity in the demand exactly at that point. Then, it wants to take advantage of it.

Lemma 4 and Lemma 7 also imply that we can think of $v^*(p) = \inf\{v : p^*(v) \geq p\}$ as the inverse of $p^*(v)$: for any p in the support, $p^*(v^*(p)) = p$. Furthermore, $V^*(p) = \{v : p^*(v) \geq p\} = [v^*(p), \overline{v}]$ and the demand at any price $\gamma(V^*(p)) = 1 - G(v^*(p))$.

Let $\underline{p} = \min \operatorname{supp} F$ and $\overline{p} = \max \operatorname{supp} F$.

Profits from $\underline{p}$

The profits from $\underline{p}$ are $\underline{p}(1 - G(v^*(\underline{p}))$. Indeed, for any $v \leq p$, $F(v) = 0$. Therefore, $p^*(v) = v + \lambda F(v|v) = v$ is a PE cutoff. This being the smallest PE cutoff possible, it is the PPE cutoff by Proposition 1. Moreover, for any v, $p^*(v) \geq v$. Therefore, all types above $\underline{p}$ accepts it and all types below reject it, i.e., $v^*(\underline{p}) = \underline{p}$. Profits when offering $\underline{p}$ are then $\underline{p}\,(1 - G(\underline{p}))$. These must be the equilibrium profits.

Finding the equilibrium strategy

For any $p \in \operatorname{supp} F$, by indifference on the support,

$$\pi^* \equiv \underline{p}(1 - G(\underline{p})) = p(1 - G(v^*(p))).$$

Therefore,

$$v^*(p) = G^{-1}\left(\frac{p - \pi^*}{p}\right),$$

for all $p \in \operatorname{supp} F$. Since $\frac{p-\pi^*}{p} \in [0, 1)$ for all $p \geq \underline{p}$, the expression above is well-defined. The equilibrium strategy F must guarantee that a PE cutoff of $v^*(p)$ is $p^*(v^*(p)) = p$:

$$v^*(p) - p = -\lambda v^*(p) F(p) \Rightarrow F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)}, \tag{2.6}$$

using that $p^*(v^*(p)) = p$. Note that this discussion also implies that in equilibrium, $p^*(v) = \frac{\pi^*}{1-G(v)}$.

Pinning down $\underline{p}$. For any $p < \underline{p}$, $F(p) = 0$. Therefore, $v^*(p) = p - \lambda v^*(p)F(p) = p$. In equilibrium, we must have

$$\pi^* \geq p(1 - G(p)).$$

For any $p > \underline{p}$, we have:

$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)} \Rightarrow G^{-1}(\frac{p - \pi^*}{p}) = p - \lambda G^{-1}(\frac{p - \pi^*}{p})F(p) < p$$
$$\Leftrightarrow p(1 - G(p)) < \pi^*,$$

using that $F(p) > 0$ for $p > \underline{p}$. Therefore, we have $\underline{p} = \arg\max_p \, p(1 - G(p))$.[12]

F is well-defined on the support

I check here that $\frac{p-G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)}$ is a strictly increasing and positive function. For all $p \geq \underline{p}$,

$$p - G^{-1}(\frac{p - \pi^*}{p}) \geq 0 \Leftrightarrow G(p) \geq \frac{p - \pi^*}{p} \Leftrightarrow \pi^* \geq p(1 - G(p))$$

This is satisfied because $\pi^* = \max p(1 - G(p))$.

I now show that for all $p > \underline{p}$, the derivative of F is strictly positive. This follows from the following fact

1. For $p > \underline{p}$, $\frac{1-G(p)}{p} < g(p)$: by strict concavity of the profit function, the derivative is negative after the maximum.

Taking the derivative of F,

$$F'(p) \propto \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right) - p\frac{\pi^*}{p^2}\frac{1}{g\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)}}{\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)^2} > \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right) - \frac{\pi^*}{p}\frac{G^{-1}\left(\frac{p-\pi^*}{p}\right)}{1-G\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)}}{\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)^2} = 0,$$

using fact 1 to get the inequality and rearranging to get the equality.

Pinning down $\overline{p}$. We have to check that there exists a $\overline{p}$, such that $F(\overline{p}) = 1$. To do that, we will check that there exists p such that $F(p) = 1$. Note that $F(\underline{p}) = 0$ and $F(\overline{v} + \lambda\overline{v}) = \frac{\overline{v}+\lambda\overline{v}-G^{-1}\left(\frac{\overline{v}+\lambda-\pi^*}{\overline{v}+\lambda\overline{v}}\right)}{\lambda G^{-1}\left(\frac{\overline{v}+\lambda\overline{v}-\pi^*}{\overline{v}+\lambda\overline{v}}\right)} > 1$ (rearranging and using that $G(\overline{v}) = 1$). Therefore, by continuity of F, there exists, $\underline{p} < p < \overline{v} + \lambda\overline{v}$ such that $F(p) = 1$.

[12] $\underline{p}$ is well-defined by the strict concavity of $p(1 - G(p))$.

Then, $\overline{p} := F(\overline{p}) = 1$.[13]

Preferred Personal Equilibrium The last step is to check that the PE cutoffs pinned down by equation (2.6) are PPE cutoffs. This follows from the fact that the PE pinned down by equation (2.6) is unique:

$$\frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)} \geq \frac{p-v}{\lambda v}$$
$$\Leftrightarrow G(v) \geq \frac{p-\pi^*}{p}$$
$$\Leftrightarrow p \leq \frac{\pi^*}{1-G(v)} = p^*(v).$$

Hence, it is also a PPE cutoff. □

2.6.2 Proof of Lemma 3

The firm's strategy and test as defined are Markov kernels. For simplicity, for any measurable space $(X, \mathcal{X})$, I simply write X. A mapping $Q : X \times Y \to [0,1]$ is a Markov kernel if (i) for any measurable $A \subseteq Y$, $Q(A|\cdot)$ is measurable and (ii) for any $x \in X$, $Q(\cdot|x)$ is a probability measure. I will make repeated use of composition of Markov kernels. Let $Q : X \times Y \to [0,1]$ and $P : Y \times Z \to [0,1]$ be two Markov kernels. Then the composition of $P \circ Q : X \times Z \to [0,1]$ defined as

$$(P \circ Q)(A|x) = \int_Y P(A|y) dQ(y|x) \text{ for all measurable } A \subseteq Z \text{ and } x \in X$$

is a Markov kernel. Furthermore, for all bounded measurable $f : Z \to \mathbb{R}$,

$$\int f(z) d(P \circ Q)(z|x) = \int \int f(z) dP(z|y) dQ(y|x)$$

See e.g., Bauer (1996), chapter VIII, §36.

Proof. Start with a test (F, S) and an equilibrium $\left(P, (Q_v^*)_v\right)$. We are going to construct a new test $(\tilde{F}, \tilde{S})$ and an equilibrium $\left(\delta_s, (Q_v^*)_v\right)$ such that all players get the same payoffs. We can construct the Markov kernel $\tilde{F} : V \times \tilde{S} \to [0,1]$, where $\tilde{S} = \cup_{s \in S} \operatorname{supp} P(s) \subseteq \mathbb{R}$ as $\tilde{F} = P \circ F$.

Let's first verify that the PPE do not change. Fix a v.

[13]Because F is strictly increasing, $\overline{p}$ is well-defined.

$$
\begin{aligned}
Q_v &= \int_S \int_{\mathbb{R}} \mathbb{1}[p \le v + \lambda v Q_v] dP(p|s) dF(s|v) \\
&= \int_{\tilde{S}} \mathbb{1}[p \le v + \lambda v Q_v] d\tilde{F}(p|v) \\
&= \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[p \le v + \lambda v Q_v] d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v),
\end{aligned}
$$

$$
\begin{aligned}
\text{and } W_{(F,S)}(v|Q_v) &= \int_S \int_{\mathbb{R}} \Big(\mathbb{1}[p \le v + \lambda v Q_v](v-p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \Big) dP(p|s) dF(s|v) \\
&= \int_{\tilde{S}} \Big(\mathbb{1}[p \le v + \lambda v Q_v](v-p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \Big) d\tilde{F}(p|v) \\
&= \int_{\tilde{S}} \int_{\mathbb{R}} \Big(\mathbb{1}[p \le v + \lambda v Q_v](v-p) + \mathbb{1}[p > v + \lambda Q_v](-\lambda v Q_v) \Big) d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v) \\
&= W_{(\tilde{F},\tilde{S})}(v|Q_v).
\end{aligned}
$$

This shows that the set of PE and the payoff from each of them does not change under the new test and equilibrium. Moving to the firm to the firm's payoffs, we get similarly

$$
\begin{aligned}
\mathbb{E}_{(F,S)}[\pi(P)|(Q_v^*)_v] &= \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP(p|s) dF(s|v) dG \\
&= \int_V \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v) dG \\
&= \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q_v^*)_v].
\end{aligned}
$$

Note that the integrand is bounded because $\mathbb{1}[v \in V^*(p)] = 0$ for $p > \overline{v}(1+\lambda)$ and offering a negative is a strictly dominated action. The last step is to check that any deviation from $\delta_{\tilde{s}}(p)$ is suboptimal in the new test. I show that from any strategy in $(\tilde{F},\tilde{S})$, we can construct a strategy in (F,S) that yields the same payoff. Let

$\tilde{P} : \tilde{S} \times \mathbb{R} \to [0,1]$ a strategy in $(\tilde{F}, \tilde{S})$. Define the Markov kernel $P' : S \times \tilde{S} \to [0,1]$ as $P' = \tilde{P} \circ P$. Then,

$$\begin{aligned}
\mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\tilde{P})|(Q_v^*)_v] &= \int_V \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\tilde{P}(p|\tilde{s}) d\tilde{F}(\tilde{s}|v) dG(v) \\
&= \int_V \int_S \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\tilde{P}(p|\tilde{s}) dP(\tilde{s}|s) dF(s|v) dG(v) \\
&= \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP'(p|s) dF(s|v) dG(v) \\
&\leq \mathbb{E}_{(F,S)}[\pi(P)|(Q_v^*)_v] = \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q_v^*)_v].
\end{aligned}$$

□

2.6.3 Proof of Proposition 6 and Proposition 7

Plan of the proof:

1. Relax the problem by requiring that obedience constraint holds on intervals of signals $[\underline{s}, s]$ for all s and only upward deviation, $\tilde{P}(x) = x + \epsilon$ for all $\epsilon > 0$.
2. Use that $F(s|v) > \frac{s-v}{\lambda v}$ for all $s < p^*(v)$ to relax the obedience constraints and make them only depend on a new object $h(x) := \int_{V^*(x)} \frac{1}{v} dG$. If the obedience constraints depend only h, then it is optimal to choose a censored commitment distribution, using the FOSD interpretation of the PPE (Proposition 1). We are left with optimising over h.
3. Look only at local deviations, i.e., $\epsilon \to 0$, to get an integral inequality that pins down h.
   (a) Because h is not necessarily Lipschitz continuous, which is needed for the operation described above, I construct a sequence of relaxed problems with a smaller set of relaxed obedience constraints where deviations are bounded away from 0. For each problem, I show that it is without loss to focus on Lipschitz continuous h.
   (b) Then, I look at the limit of these problems with Lipschitz continuous h and focusing on the smallest possible deviation in each element of the sequence to derive a condition on h.
4. The resulting supremum problem with the condition from local relaxed obedience constraints gives an upper bound on profits.
5. I show that there exists a sequence of tests respecting the obedience constraints converging to the upper bound.
6. Proof of Proposition 6: Compare the characterised payoffs to the full information benchmark to show that it *strictly* dominates it.

Proof. Let $\underline{s} = \min S$ and $\overline{s} = \max S$.

Lemma 8. *For any F, $V^*(\underline{s}) = [\underline{s}, \overline{v}]$.*

Proof. By definition of PPE and $\underline{s}$, for all $v < \underline{s}$, $p^*(v) = v - \lambda v F(v|v) = v < \underline{s}$, using that $F(v|v) = 0$.

On the other hand, $p^*(v) \geq v$, therefore, $p^*(v) \geq \underline{s}$ for all $v \geq \underline{s}$. Thus, $V^*(\underline{s}) = [\underline{v}, \overline{v}]$. □

Define $h(s) := \int_{V^*(s)} \frac{1}{v} dG$. The following lemma states that there exists a relaxation of the original problem where the constraints only depend on h, not on the test.

Lemma 9. *The following problem is a relaxation of the firm's problem:*

$$\sup_{S,\, h \in L^1(S)} \int_S x \frac{h(x) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}}{\lambda} dx$$

s.t. for all $s \in S$ and $\epsilon > 0$,

$$(h(s) - h(s+\epsilon))s\,\epsilon + \int_{\underline{s}}^{s} x(h(x) - h(x+\epsilon))dx \geq \int_{\underline{s}}^{s} \epsilon\,(h(x+\epsilon) - \frac{\log G(x)}{\Delta v})dx \tag{2.7}$$

$$h(s) \in [\frac{1}{\Delta v} \log \frac{\max\{\overline{v}, s\}}{s}, \frac{1}{\Delta v} \log \Big(\frac{\overline{v}}{\max\{\underline{v}, \frac{s}{1+\lambda}\}}\Big)] \text{ for all } s \in S \tag{2.8}$$

$$h(\underline{s}) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}};\ h \text{ non-increasing.} \tag{2.9}$$

Proof. First, focus on the following subset of obedience constraints: for all $s \in S$,

$$\int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x+\epsilon)]\,(x+\epsilon)\, dF(v,x) \text{ for all } \epsilon > 0$$

Noting that $V^*(x+\epsilon) \subseteq V^*(x)$, we can rearrange the relaxed obedience constraint as

$$\int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x) \setminus V^*(x+\epsilon)]\, x\, dF(v,x) \geq \int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x+\epsilon)]\, \epsilon\, dF(v,x)$$

This is equivalent to (see Figure 2.3 for an illustration)

$$\int_{V^*(s) \setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG + \int_{V \setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG$$
$$\geq \int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon\, dF(x|v) dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \epsilon\, dF(x|v) dG$$

I will now use repeatedly the FOSD interpretation of the PPE (Proposition 1): $F(x|v) > \frac{x-v}{\lambda v}$ for $x < p^*(v)$ and $F(p^*(v)|v) = \frac{p^*(v)-v}{\lambda v}$.

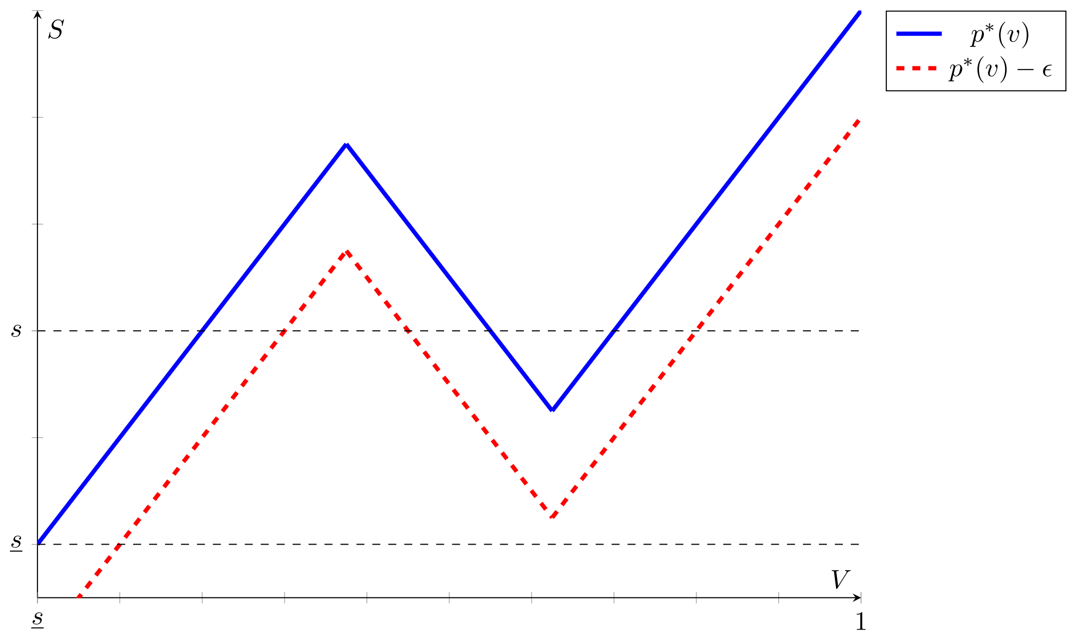


Figure 2.3: Area of integration with $V^*(x) = \{v : p^*(v) \geq x\}$ and $V^*(x+\epsilon) = \{v : p^*(v) - \epsilon \geq x\}$

Take the RHS first.

$$
\begin{aligned}
&\int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon \, dF(x|v) dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \epsilon \, dF(x|v) dG \\
&\geq \int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon \, \mathbb{1}[x \geq v] \frac{1}{\lambda v} dx \, dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \mathbb{1}[x \geq v] \epsilon \, \frac{1}{\lambda v} dx \, dG \\
&= \int_{\underline{s}}^{s} \int_{V^*(x+\epsilon)} \mathbb{1}[x \geq v] \epsilon \frac{1}{\lambda v} dG \, dx \\
&\geq \int_{\underline{s}}^{s} \frac{\epsilon}{\lambda} \Big[\int_{V^*(x+\epsilon)} \frac{1}{v} \, dG - \int_{[x, \max\{x, \overline{v}\}]} \frac{1}{v} \, dG \Big] dx
\end{aligned}
$$

using the FOSD property on the second line, changing the order of integration in the third and using that $1 \geq \gamma(V^*(x+\epsilon)) + \gamma([0,x]) - \gamma\big(V^*(x+\epsilon) \cap [0,x]\big)$ on the last.

Now focusing on the LHS,

$$
\begin{aligned}
&\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG \\
&\leq \underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG}_{I} \\
&+ \underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dx\, dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG}_{II}
\end{aligned}
$$

where the inequality simply follows from adding a positive term on the second line.

$$
\begin{aligned}
I &\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} s\big(F(s|v) - F(p^*(v)-\epsilon|v)\big) dG \\
&\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} s\big(\frac{p^*(v)-v}{\lambda v} - \frac{p^*(v)-\epsilon-v}{\lambda v}\big) dG = \frac{s\epsilon}{\lambda} \int_{V^*(s)\setminus V^*(s+\epsilon)} \frac{1}{v} dG
\end{aligned}
$$

using that $s \geq x$ and the FOSD property.

$$
\begin{aligned}
II &\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dx\, dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dx\, dG \\
&= \int_{\underline{s}}^{s} \int_{V^*(x)\setminus V^*(x+\epsilon)} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dG\, dx \\
&\leq \int_{\underline{s}}^{s} \int_{V^*(x)\setminus V^*(x+\epsilon)} x \frac{1}{\lambda v}\, dG\, dx
\end{aligned}
$$

where I use the FOSD property on the first line, change the order of integration on the second and ignore that we must have $\mathbb{1}[x \geq v]$ on the third.

Using that $h(x) = \int_{V^*(x)} \frac{1}{v} dG$, he resulting, relaxed constraint is

$$
(h(s) - h(s+\epsilon)) s\, \epsilon + \int_{\underline{s}}^{s} x(h(x) - h(x+\epsilon)) dx \geq \int_{\underline{s}}^{s} \epsilon \, (h(x+\epsilon) - \frac{\log \frac{\max\{\overline{v}, x\}}{x}}{\Delta v}) dx
$$

Then, remember that the constraint $p^*(v) = \min\{p : v - p = -\lambda v F(p|v)\}$ is equivalent to $v - p^*(v) = -\lambda v F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) > \frac{p-v}{\lambda v}$ (Proposition 1). Relax it by only requiring $v - p^*(v) = -\lambda v F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) \geq \frac{p-v}{\lambda v}$.

It is now optimal to set $F(s|v) = \frac{s-v}{\lambda v}$ for all $s \leq p^*(v)$, i.e., we choose a censored commitment distribution. This operation does not modify the relaxed PPE requirement nor the relaxed obedience constraints but improves profits. The firm's problem becomes

$$\sup_{S,\, h \in L^1(S)} \int_S x \, \frac{h(x) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}}{\lambda} dx$$

s.t. for all $s \in S$ and $\epsilon > 0$,

$$(h(s) - h(s+\epsilon)) s\, \epsilon + \int_{\underline{s}}^{s} x(h(x) - h(x+\epsilon)) dx \geq \int_{\underline{s}}^{s} \epsilon \, (h(x+\epsilon) - \frac{\log \frac{\max\{\overline{v}, x\}}{x}}{\Delta v}) dx$$

$$h(s) \in [\frac{1}{\Delta v} \log \frac{\max\{\overline{v}, s\}}{s}, \frac{1}{\Delta v} \log \Big(\frac{\overline{v}}{\max\{\underline{v}, \frac{s}{1+\lambda}\}} \Big)] \text{ for all } s \in S$$

$$h(\underline{s}) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}}; \; h \text{ non-increasing.}$$

where $h(s) \in [\frac{1}{\Delta v} \log \frac{\max\{\overline{v}, s\}}{s}, \frac{1}{\Delta v} \log \Big(\frac{\overline{v}}{\max\{\underline{v}, \frac{s}{1+\lambda}\}} \Big)]$ comes from $p^*(v) \in [v, v+\lambda v]$, $h(\underline{s}) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}}$ follows from Lemma 8 and h non-increasing follows from the definition of $V^*(s)$, i.e., increasing the price necessarily decreases the mass of types willing to accept. □

I am going to do a little detour now and focus on a set of obedience constraints and deviations that are bounded away from zero. Specifically, obedience constraints only need to hold for all $s \in S^i = [\underline{s} + \frac{1}{i}, \overline{s}]$ and $\epsilon \in E^i = [\frac{1}{i}, \overline{v}(1+\lambda)]$ for some $i \in \mathbb{N}_0$. Note that $\underline{s} \notin S^i$ and $0 \notin E^i$. Furthermore $S^i \subseteq S^{i+1}$ and $E^i \subseteq E^{i+1}$.

Let K^i be the set of functions satisfying the constraints (2.7), (2.8) and (2.9) for any $s \in S^i$, $\epsilon \in E^i$ and K be the set of functions satisfying these constraints for any $s \in S$ and $\epsilon > 0$. Similarly, define OB^i as the set of functions satisfying the relaxed obedience constraints (2.7) for any $s \in S^i$ and $\epsilon \in E^i$ and define OB for any $s \in S$ and $\epsilon \in E = [0, \overline{v}(1+\lambda)]$. Define

$$\Gamma = \{\phi \in L^1(S) : \text{ satisfying (2.8) and (2.9)}\}$$

where $L^1(S)$ is the set of measurable function from S to $\mathbb{R}$. Note that $K^i = OB^i \cap \Gamma$. Finally, let Lip be the set of Lipschitz continuous functions (not necessarily with the same Lipschitz constant). Endow the spaces defined above with the L^1-norm.

Lemma 10. *Let* $\pi(h) = \int_S x \, \frac{h(x) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}}{\lambda} dx$. *Then,*

$$\sup_{h \in K} \pi(h) \leq \lim_{i \to \infty} \sup_{h \in K^i} \pi(h) = \lim_{i \to \infty} \sup_{h \in K^i \cap Lip} \pi(h)$$

Proof. **1.** $\lim_{i \to \infty} \sup_{h \in K^i} \pi(h)$ exists. This follows from $K^{i+1} \subseteq K^i$, therefore $\sup_{h \in K^{i+1}} \pi(h) \leq \sup_{h \in K^i} \pi(h)$. Moreover, $\sup_{h \in K^i} \pi(h) \geq 0$ as choosing $h(x) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}$ is always possible for any i. Thus, the limit exists.

2. $\lim_{i\to\infty}\sup_{h\in K^i}\pi(h)\geq\sup_{h\in K}\pi(h)$. For each i, $K\subseteq K^i$, therefore, $\sup_{h\in K^i}\pi(h)\geq\sup_{h\in K}\pi(h)$ for each i.

3. $\lim_{i\to\infty}\sup_{h\in K^i}\pi(h)=\lim_{i\to\infty}\sup_{h\in K^i\cap Lip}\pi(h)$

To prove this identity, I will show that $K^i\cap Lip$ is a dense subset of K^i. Because $\pi(h)$ is continuous in h in the L^1-norm, then $\sup_{h\in K^i}\pi(h)=\sup_{h\in K^i\cap Lip}\pi(h)$.

This part is in three steps. Step 1: show that $\Gamma\cap Lip$ is dense in Γ. Step 2: show that int(K^i) is non-empty in Γ. Step 3: Using that Lipschitz continuous functions are dense in int(K^i) because it is open and int$(K^i)\subseteq\Gamma$, and convexity of K^i, show that any function in K^i can be approximated by a function in $K^i\cap Lip$.

Step 1: $\Gamma\cap Lip$ is dense in Γ

Take $\phi\in\Gamma$. Define

$$\phi_n(x)=\begin{cases}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}}+\frac{\phi_n(\underline{s}+1/n)-\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}}}{1/n}(x-\underline{s}) & if\quad x\in[\underline{s},\underline{s}+1/n)\\ n\int_{x-1/n}^{x}\phi(z)dz & if\quad x\geq\underline{s}+1/n\end{cases}$$

ϕ_n is differentiable everywhere but at one point, $\underline{s}+1/n$, and its derivative is bounded by n therefore Lipschitz continuous and $\phi_n\in\Gamma$.

We have to show that

$$\lim_{n\to\infty}\int_{\underline{s}}^{\overline{s}}|\phi_n(x)-\phi(x)|dx=0$$

Focusing on $x\geq\underline{s}+1/n$[14].

$$\begin{aligned}&\int_{\underline{s}+1/n}^{\overline{s}}|n\int_{x-1/n}^{x}\phi(z)dz-\phi(x)|dx\\ &\leq\int_{\underline{s}+1/n}^{\overline{s}}n\int_{x-1/n}^{x}|\phi(z)-\phi(x)|dzdx\\ &=\int_{\underline{s}+1/n}^{\overline{s}}n\int_{-1/n}^{0}|\phi(x+y)-\phi(x)|dydx\\ &=\int_{-1/n}^{0}n\int_{\underline{s}+1/n}^{\overline{s}}|\phi(x+y)-\phi(x)|dxdy\\ &\leq\sup\{\int_{\underline{s}+1/n}^{\overline{s}}|\phi(x+y)-\phi(x)|dx\ :\ y\in[-1/n,0]\}\end{aligned}$$

For simplicity, extend the domain to the real line and set $\phi(x)=0$ when $x\notin[\underline{s},\overline{s}]$. Let $\psi_m\in C_c(\mathbb{R})$, the set of continuous function in $\mathbb{R}$ with compact support, with $\psi_m\to_{L^1}\phi$. By the Heine-Cantor theorem, any ψ_m is

[14] I would like to thank user fourierwho of StackExchange for this proof.

uniformly continuous. We have for all m,

$$\begin{aligned}
&\lim_{y\to 0}\int_{\mathbb{R}}|\phi(x+y)-\phi(x)|dx\\
&\leq \lim_{y\to 0}\int_{\mathbb{R}}|\phi(x+y)-\psi_m(x+y)|dx+\int_{\mathbb{R}}|\psi_m(x+y)-\psi_m(x)|dx+\int_{\mathbb{R}}|\psi_m(x)-\phi(x)|dx\\
&\leq 2\int_{\mathbb{R}}|\psi_m(x)-\phi(x)|dx
\end{aligned}$$

where $\lim_{y\to 0}\int_{\mathbb{R}}|\psi_m(x+y)-\psi_m(x)|dx=0$ holds because ψ_m is uniformly continuous. Therefore, taking $m\to\infty$, $\lim_{y\to 0}\int_{\mathbb{R}}|\phi(x+y)-\phi(x)|dx=0$. In turn, it means that $\sup\{\int_{\underline{s}+1/n}^{\overline{s}}|\phi(x+y)-\phi(x)|dx : y\in[-1/n,0]\}\to 0$ as $n\to\infty$.

Now for $x\in[\underline{s},\underline{s}+1/n)$, because $|\phi_n(x)|$ and $|\phi(x)|$ are bounded as $n\to\infty$, $\lim_{n\to\infty}\int_{\underline{s}}^{\underline{s}+1/n}|\phi_n(x)-\phi(x)|dx=0$.

Therefore, $\Gamma\cap Lip$ is dense in Γ.

Step 2: Non-empty interior of $\Gamma\cap OB^i$ in Γ

Take $h(x)=\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}$. It is easy to check that $h\in K^i=\Gamma\cap OB^i$. Define $z(s,\epsilon)=\int_{\underline{s}}^{s}(x+\epsilon)(h(x)-h(x+\epsilon))dx$ and $\underline{z}=\min_{s,\epsilon}z(s,\epsilon)$. Note that we have $\underline{z}>0$ because h is strictly increasing on parts of its domain and $\underline{s}\notin S^i$ and $0\notin E^i$.

Now take $\phi(x)\in\Gamma$ with $\int_S|\phi(x)-h(x)|dx\leq\eta$, $\eta>0$. I will show that

$$(\phi(s)-\phi(s+\epsilon))s\epsilon+\int_{\underline{s}}^{s}x(\phi(x)-\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx\geq\int_{\underline{s}}^{s}(x+\epsilon)(\phi(x+\epsilon)-\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx$$

for all $\epsilon\in E^i$, $s\in S^i$ for η sufficiently small. Rearranging the obedience constraint,

$$\begin{aligned}
(\phi(s)-\phi(s+\epsilon))s\epsilon+\int_{\underline{s}}^{s}x(\phi(x)-\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx+\int_{\underline{s}}^{s}(x+\epsilon)(h(x)-h(x+\epsilon))\\
\geq\int_{\underline{s}}^{s}(x+\epsilon)(\phi(x+\epsilon)-h(x+\epsilon))dx
\end{aligned}$$

Take the LHS, we have

$$z(s,\epsilon)+(\phi(s)-\phi(s+\epsilon)s\epsilon+\int_{\underline{s}}^{s}x(\phi(x)-h(x))dx\geq\underline{z}-\eta\overline{s}$$

using that ϕ is non-increasing. The RHS gives

$$\int_{\underline{s}}^{s} (x + \epsilon)(\phi(x + \epsilon) - h(x + \epsilon))dx \leq \eta(\overline{s} + \epsilon)$$

Therefore, we need

$$\underline{z} - \eta\overline{s} \geq \eta(\overline{s} + \epsilon)$$
$$\underline{z} \geq (2\overline{s} + \epsilon)\eta$$

which holds for all $s \in S^i$ and $\epsilon \in E^i$ for η small enough.

Step 3: $K^i \cap Lip$ is dense in K^i

First observe that int(K^i) is an open set in Γ in the metric space $(\Gamma, L^1$-norm$)$. Therefore, int$(K^i) \cap Lip$ is dense in int(K^i).

Note that the set K^i is convex. This can be verified by simply summing over the relaxed obedience constraints. The properties of Γ are also maintained when taking convex combinations.

Take some $h \in$ int(K^i). Any function $\phi \in K^i$ can be approximated by a sequence of $\alpha^n h + (1 - \alpha^n)\phi$ with the appropriate sequence of α^n. Moreover, any point in the sequence is in the interior of Γ.[15]

Take any $\phi \in K^i$ and $\epsilon > 0$. Let $\phi^n = \alpha^n\phi + (1 - \alpha^n)h$, such that $|\phi - \phi^n| < \epsilon/2$ for all $n \geq N$ for some $N \in \mathbb{N}$. Define also $\psi^n \in K^i \cap Lip$ such that $|\psi^n - \phi^n| < \epsilon/2$ for all n, using that $\phi^n \in$ int K^i. Therefore,

$$|\phi - \psi^n| \leq |\phi - \phi^n| + |\phi^n - \psi^n| < \epsilon/2 + \epsilon/2 = \epsilon$$

for $n \geq N$.

Now, given that $\pi(h) = \int_S x \frac{h(x) - \frac{\log \frac{\max\{\overline{v}, x\}}{x}}{\Delta v}}{\lambda} dx$ is continuous in the L^1-norm, we have established that $\sup_{h \in K^i} \pi(h) = \sup_{h \in Lip \cap K^i} \pi(h)$. □

This lemma shows that taking the restricted set of constraints provides another upper bound to our problem. Furthermore, in the restricted problem, it is without loss to restrict attention to Lipschitz continuous functions.

Now, let's focus on $\lim_{i\to\infty} \sup_{h \in K^i \cap Lip} \pi(h) = \pi(h^*)$ (for some h^*). This implies that there exist a sequence $\{h^i\}$ with $h^i \in K^i \cap Lip$ such that $h^i \to_{L^1} h^*$. Let $\underline{\epsilon}_i = \min\ E^i$.

[15] To see this note that there exists $\eta > 0$ such that any $B_\eta(h) \subseteq K^i$. Take $\psi = \alpha h + (1 - \alpha)\phi$. I will show that any $w \in B_{\eta\alpha}(\psi)$ is in K^i. First, define $z = h + \frac{w - \psi}{\alpha}$. Then, $|z - h| = |h + \frac{w - \psi}{\alpha} - h| < \alpha\frac{\eta}{\alpha} = \eta$. Therefore $z \in K^i$. Then, choosing $\beta = \alpha$, we have $w = \beta z + (1 - \beta)\phi$ and thus $w \in K^i$.

Because each h^i is bounded and of bounded total variation, by Helly's selection theorem, there exists a subsequence $\{h^{i_k}\}$ such that $h^{i_k}(s) \to h^*(s)$ for all $s \in \text{int } S$. Let's focus on that subsequence and rename its elements: $\{h^k\}_{k=0}^{\infty}$. This implies that for each $s \in \text{int } S$, for all $\eta > 0$, there exists $P(s,\eta) \in \mathbb{N}$ such that $|h^*(s) - h^k(s)| < \eta$ for all $k \geq P(s,\eta)$ and there exists $M(\eta) \in \mathbb{N}$ such that $\int_S |h^*(s) - h^k(s)|ds < \eta$ for all $k \geq M(\eta)$. Note also that $\int_S |h^i(x) - h^k(x)|dx < \eta$ for all $k, i \geq M(\eta/2)$.

Finally, h^* being the limit of monotone function, it is monotone and thus continuous almost everywhere. Therefore, wherever h^* is continuous, there exists $N(s,\eta) \in \mathbb{N}$ such that $|h^*(s) - h^*(s+\underline{\epsilon}_i)| < \eta$ for all $i \geq N(s,\eta)$.

Fix $\eta > 0$ and $s > \underline{s}$ where h^* is continuous. Define $i = \max\{\frac{1}{s}, N(s,\eta/3)\}$. Then, for all $k > k^*(s,\eta) \equiv \max\{i, P(s,\eta/3), P(s+\underline{\epsilon}_i,\eta/3)\}$, we have

$$\begin{aligned}
|h^k(s) - h^k(s+\underline{\epsilon}_k)| &\leq |h^k(s) - h^k(s+\underline{\epsilon}_i)| \\
&\leq |h^k(s) - h^*(s)| + |h^*(s) - h^*(s+\underline{\epsilon}_i)| + |h^*(s+\underline{\epsilon}_i) - h^k(s+\underline{\epsilon}_i)| \\
&< \eta/3 + \eta/3 + \eta/3 = \eta
\end{aligned}$$

using that $\underline{\epsilon}_i > \underline{\epsilon}_k$ on the first line. Therefore, for all $k > \max\{k^*(s,\eta), M(\eta/2)\}$,

$$\begin{aligned}
&(h^k(s) - h^k(s+\underline{\epsilon}_k)s + \int_{\underline{s}}^s x(h^k(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx \\
&\qquad\qquad \geq \int_{\underline{s}}^s (x+\underline{\epsilon}_k)(h^k(x+\underline{\epsilon}_k) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\; s\eta + \int_{\underline{s}}^s x(h^i(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx + s\eta \\
&\qquad\qquad \geq \int_{\underline{s}}^s (x+\underline{\epsilon}_k)(h^i(x+\underline{\epsilon}_k) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx - (s+\underline{\epsilon}_k)\eta
\end{aligned}$$

We can rearrange the constraint and let $k \to \infty$ (which implies $\underline{\epsilon}_k \to 0$),

$$(3s+\underline{\epsilon}_k)\eta + \int_{\underline{s}}^s x\,\frac{h^i(x) - h^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}dx \geq \int_{\underline{s}}^s h^i(x+\underline{\epsilon}_k) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx$$

$$\text{letting } k\to\infty,\quad 3s\eta + \int_{\underline{s}}^s -x\frac{\partial h^i}{\partial x}dx \geq \int_{\underline{s}}^s h^i(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx$$

where we used the dominated convergence theorem, using that $|h^i|$ is bounded and $\frac{h^i(x)-h^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}$ is bounded by Lipschitz continuity. Integrating by part, we get

$$3s\eta - [h^i(x)x]_{\underline{s}}^{s} + \int_{\underline{s}}^{s} h^i(x)dx \geq \int_{\underline{s}}^{s} h^i(x)dx - \int_{\underline{s}}^{s} \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx$$
$$h^i(s) \leq \frac{\int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx + (\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}})\underline{s}}{s} + 3\eta$$

Using that $h^i(\underline{s}) = \frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}}$.

Then, we can take a sequence of $\eta \to 0$, and thus $i \to \infty$, and we get for each s where h^* is continuous

$$h^*(s) = \lim_{\eta\to 0} h^i(s) \leq \lim_{\eta\to 0}\frac{\int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx + (\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}})\underline{s}}{s} + 3\eta$$
$$= \frac{\int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx + (\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}})\underline{s}}{s}$$

This holds for any s where $h^*(s)$ is continuous.

Therefore, we get another upper bound on the firm's problem.

$$\sup_{h\in Lip}\int_{\underline{s}}^{\overline{s}} x\,\frac{h(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}}{\lambda}dx$$
$$\text{s.t. for all } s \in S' :\ h(s) \leq \frac{\int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx + (\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}})\underline{s}}{s}$$
$$h(s) \in [\frac{1}{\Delta v}\log\frac{\max\{\overline{v},s\}}{s}, \frac{1}{\Delta v}\log\Big(\frac{\overline{v}}{\max\{\underline{v},\frac{s}{1+\lambda}\}}\Big)]$$
$$h(\underline{s}) = \frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}}$$

for some $S' \subseteq S$ such that $\mu(S') = \mu(S)$, where $\mu(\cdot)$ is the Lebesgue measure. This is solved by setting $\underline{s} = \underline{v}$, $\overline{s} = \overline{v}(1+\lambda)$ and $h(s) = \min\{\frac{\int_{\underline{v}}^{s}\log\frac{\max\{\overline{v},x\}}{x}dx + \underline{v}\log\frac{\overline{v}}{\underline{v}}}{s\Delta v}, \frac{1}{\Delta v}\log\Big(\frac{\overline{v}}{\max\{\underline{v},\frac{s}{1+\lambda}\}}\Big)\}$. Calculating the integral shows that if $\underline{v} \geq \frac{s}{1+\lambda}$, then $\frac{\int_{\underline{v}}^{s}\log\frac{\max\{\overline{v},x\}}{x}dx + \underline{v}\log\frac{\overline{v}}{\underline{v}}}{s\Delta v} \leq \frac{1}{\Delta v}\log\Big(\frac{\overline{v}}{\underline{v}}\Big)$. We can thus simplify this expression further

$$h(s) = \min\{\frac{\int_{\underline{v}}^{s}\log\frac{\max\{\overline{v},x\}}{x}dx + \underline{v}\log\frac{\overline{v}}{\underline{v}}}{s\Delta v}, \frac{1}{\Delta v}\log\Big(\frac{\overline{v}(1+\lambda)}{s}\Big)\}$$

□

Showing the upper bound is achievable

I will now show that there exists a sequence of tests satisfying the constraints and such that the profits converge to the upper bound.

Lemma 11. *There exists a test F_ϵ that satisfy the obedience constraints for $\epsilon > 0$ small enough and such that the profits converge to the upper bound as $\epsilon \to 0$.*

Proof. **Construct the test** Let $\Lambda = \{p : V \to [\underline{v}, \overline{v}(1+\lambda)] : p(v) \geq v, p(\underline{v}) = \underline{v}$, K-Lipschitz continuous and non-decreasing$\}$ for some $K > 1$. Endow that space with the L^1-norm. Construct the mapping $\Phi : \Lambda \to \Lambda$ as follows. Take $p \in \Lambda$ and define $\underline{p}(v)$ as

$$\frac{1}{(\lambda + \epsilon\,\frac{p(v)-\underline{v}}{\underline{v}})v} = \frac{\frac{p(v)-v}{\lambda v}}{p(v) - \underline{p}(v)} \Leftrightarrow \underline{p}(v) = \underline{v}\,\frac{\lambda v + \epsilon(p(v) - v)}{\lambda \underline{v} + \epsilon(p(v) - v)}$$

Note that if ϵ is small enough then $\underline{p}(v)$ is strictly increasing and it is continuous for any ϵ. Let $\hat{v}(s) = \min\{\underline{p}^{-1}(s), \overline{v}\}$ and $\tilde{v}(s) = \hat{v}(s)\exp\frac{-\hat{v}(s)+\underline{v}}{s}$. One can check that $\tilde{v}$ is continuous with $\tilde{v}(\underline{v}) = \underline{v}$ and has its derivative uniformly bounded away from zero for any p. We can now define $\Phi : p(v) \to \min\{v(1+\lambda), \tilde{v}^{-1}(v)\} \in \Lambda$.

We know want to apply Schauder fixed point theorem: Every continuous self-map on a nonempty compact and convex subset of a normed linear space has a fixed point. (Ok, 2007, p. 469)

The function Φ is a composition of mappings that are continuous in the L^1-norm and is thus continuous. The set Λ is compact by applying Helly's selection theorem. Because Λ is of bounded variation, bounded and closed, by Helly's selection theorem any sequence in Λ admits a convergent subsequence that converges in Λ because it is closed. It is also convex. Finally, $L^1([0,1])$ is a normed linear space.

Let $p^*(v)$ be a fixed point of Φ. Abusing notation, let $\epsilon(s) = \epsilon\,\frac{s-\underline{v}}{\underline{v}}$. The test is

$$F_\epsilon(s|v) = \begin{cases} 0 & \text{if} \quad s < \underline{p}(v) \\ \frac{s-\underline{p}(v)}{(\lambda+\epsilon(s))v} & \text{if} \quad s \in [\underline{p}(v), p^*(v)] \\ \frac{s-v^*(s)}{\lambda v^*(s)} & \text{if} \quad s > p^*(v) \end{cases}$$

Note that $\underline{p}(v) \in [\frac{v(1+\epsilon)}{1+\epsilon v/\underline{v}}, v]$ because $p(v) \in [v, v(1+\lambda)]$ and thus $\hat{v}(s) \in [\min\{s, \overline{v}\}, \min\{\frac{s\underline{v}}{\underline{v}(1+\epsilon)-s\epsilon}, \overline{v}\}]$. Finally, v^* is simply the inverse of p^*.

Show it respects the obedience constraints Let $\pi(s,s') = \int_{v^*(s')}^{\overline{v}(s)} s' f_\epsilon(s|v)dv$. To satisfy the obedience constraints, we must have

$$s \in \arg\max_{s'} \pi(s,s')$$

Let's examine upward and downward deviations separately. First, upward deviation. We can write the profits from offering price s' at signal s as $\pi(s,s') = \frac{1}{\Delta v}\int_{v^*(s')}^{\hat{v}(s)} s' f_\epsilon(s|v)dv$. The derivative with respect s' is proportional to

$$\log \frac{\hat{v}(s)}{v^*(s')} - s' \frac{(v^*(s')'}{v^*(s')}$$

Setting the derivative equal to 0 when $s' = s$, we get $\log \frac{\hat{v}(s)}{v^*(s)} - s\frac{(v^*(s)'}{v^*(s)} = 0$. This is always satisfied when $v^*(s) = \hat{v}(s) \exp \frac{-\hat{v}(s)+\underline{v}}{s}$. When, $v^*(s) = \frac{s}{1+\lambda}$, we need

$$\log \frac{\hat{v}(s)(1+\lambda)}{s} - s\frac{1/(1+\lambda)}{s/(1+\lambda)} \leq 0$$

Note that $v^*(s) = \frac{s}{1+\lambda}$ when $\frac{s}{1+\lambda} \geq \hat{v}(s) \exp \frac{-\hat{v}(s)+\underline{v}}{s}$. Using that $\hat{v}(s) \leq \frac{s\underline{v}}{\underline{v}(1+\epsilon)-s\epsilon}$, we get

$$\log \frac{\hat{v}(s)(1+\lambda)}{s} \leq \frac{\hat{v}(s)-\underline{v}}{s} \leq \frac{\frac{s\underline{v}}{\underline{v}(1+\epsilon)-s\epsilon} - \underline{v}}{s} \leq 1, \text{ for } \epsilon > 0.$$

Then observe that for $s' > s$,

$$\log \frac{\hat{v}(s)}{v^*(s')} - s' \frac{(v^*(s')'}{v^*(s')} \leq \log \frac{\hat{v}(s')}{v^*(s')} - s' \frac{(v^*(s')'}{v^*(s')} \leq 0$$

because $\hat{v}(\cdot)$ is increasing. Therefore, there is no profitable upward deviation.

Now, for downward deviations, assuming $v^*(s) > s/(1+\lambda)$, the payoffs are proportional to

$$\int_{v^*(s)}^{\hat{v}(s)} \frac{s'}{(\lambda+\epsilon(s))v} dv + \int_{v^*(s')}^{v^*(s)} s' \Big(\frac{s-v^*(s)}{\lambda v^*(s)}\Big)' dv$$

Note that $\left(\frac{s-v^*(s)}{\lambda v^*(s)}\right)' = \frac{v^*(s)-(v^*(s))'s}{\lambda v^*(s)^2}$. Note also that $\frac{\lambda+\epsilon(s)}{\lambda} \cdot \frac{v^*(s)-(v^*(s))'s}{v^*(s)} \leq 1$ for ϵ small enough for all s as

$$\begin{aligned}
\frac{\lambda+\epsilon(s)}{\lambda} \cdot \frac{v^*(s)-(v^*(s))'s}{v^*(s)} &= \frac{\lambda+\epsilon(s)}{\lambda} \cdot (1 - \frac{(v^*(s))'s}{v^*(s)}) \\
&= \frac{\lambda+\epsilon(s)}{\lambda} \cdot (1 - \log \hat{v}(s)/v^*(s)) \\
&= \frac{\lambda+\epsilon(s)}{\lambda} \cdot (1 - \frac{\hat{v}(s)-\underline{v}}{s}) \\
&\leq \frac{\lambda+\epsilon\,\frac{s-\underline{v}}{\underline{v}}}{\lambda} \cdot (1 - \frac{\min\{s,\overline{v}\}-\underline{v}}{s}) \leq 1
\end{aligned}$$

where I have use that $\hat{v}(s) \geq \min\{s, \overline{v}\}$. The last inequality is satisfied for all s for ϵ small enough. Taking the derivative and evaluating it at $s' = s$, we have

$$\frac{1}{\lambda+\epsilon(s)}\log\frac{\hat{v}(s)}{v^*(s)} - (v^*(s))' s \frac{v^*(s)-(v^*(s))' s}{\lambda v^*(s)^2} \geq 0$$
$$\Leftrightarrow \log\frac{\hat{v}(s)}{v^*(s)} - \frac{\lambda+\epsilon(s)}{\lambda}\cdot\frac{v^*(s)-(v^*(s))' s}{v^*(s)}\cdot\frac{(v^*(s))' s}{v^*(s)} \geq \log\frac{\hat{v}(s)}{v^*(s)} - \frac{(v^*(s))' s}{v^*(s)} = 0$$

using that $\frac{\lambda+\epsilon(s)}{\lambda}\cdot\frac{v^*(s)-(v^*(s))' s}{v^*(s)} \leq 1$ for ϵ small enough for all s.

We are then left to check that the profit function is concave when $s' < s$. Take the derivative with respect to s' twice, we need to show that

$$-f(s|v)\,(v^*(s'))' - f(s|v)\,(v^*(s'))' - f(s|v)(v^*(s'))''\,s' \leq 0$$
$$\Leftrightarrow -2(v^*(s'))' - (v^*(s'))''\,s' \leq 0$$

To see this is verified, note that $h(s)s = \int_{\underline{v}}^{s}\log\frac{\overline{v}}{\hat{v}(t)}dt + h(\underline{v})\underline{v}$. Taking the derivative twice on both sides, we get, $2h'(s) + sh''(s) = -\frac{\hat{v}'(s)}{\hat{v}(s)} \leq 0$. Given that $h(s) \propto \int_{v^*(s)}^{\overline{v}}\frac{1}{v}dv$, this means that

$$-2\frac{(v^*(s))'}{v^*(s)} + s\frac{-(v^*(s))'' v^*(s) + (v^*(s))'^2}{v^*(s)^2} \leq 0$$

Rearranging, we get

$$-2(v^*(s'))' - (v^*(s'))''\,s' \leq -\frac{s(v^*(s))'^2}{v^*(s)} \leq 0$$

If $v^*(s) = s/(1+\lambda)$, there are no profitable downward deviation as $f(s|v) = 0$ for all $v \leq v^*(s)$.

As $\epsilon \to 0$, profits converge to the upper bound derived in the previous section. In the limit, we also have $\hat{v}(s) = \min\{s, \overline{v}\}$.

□

Simple calculation shows that if $h(s) = \min\{\frac{\int_{\underline{v}}^{s}\log\frac{\max\{\overline{v},x\}}{x}dx + \underline{v}\log\frac{\overline{v}}{\underline{v}}}{s\Delta v}, \frac{1}{\Delta v}\log\frac{\overline{v}(1+\lambda)}{s}\}$, the upper bound is the profits stated in Proposition 7:

$$\int_{\underline{v}}^{\overline{v}(1+\lambda)} s\,\frac{h(s) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},s\}}{s}}{\lambda}ds = \int_{\underline{v}}^{\overline{v}(1+\lambda)}\frac{\min\left\{\hat{v}(s) - \underline{v},\ s\log\frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda\Delta v}ds.$$

Showing that the profits are strictly greater than full information:

The profits for the limit of fully revealing tests is

$$\begin{aligned}\int_{\underline{v}}^{\overline{v}} \frac{v}{\Delta v} dv &= \int_{\underline{v}}^{\overline{v}} \int_{v}^{v(1+\lambda)} \frac{1}{\lambda \Delta v} dv \\ &= \int_{\underline{v}}^{\overline{v}(1+\lambda)} \int_{\max\{\underline{v}, \frac{s}{1+\lambda}\}}^{\hat{v}(s)} \frac{1}{\lambda \Delta v} dv\, ds \\ &= \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\hat{v}(s) - \max\{\underline{v}, \frac{s}{1+\lambda}\}}{\lambda \Delta v} ds.\end{aligned}$$

Clearly, $\hat{v}(s) - \max\{\underline{v}, \frac{s}{1+\lambda}\} \leq \hat{v}(s) - \underline{v}$ and observe that for $s < \overline{v}(1+\lambda)$,

$$\begin{aligned}s \log \frac{(1+\lambda)\hat{v}(s)}{s} &= \int_{\frac{s}{1+\lambda}}^{\hat{v}(s)} \frac{s}{v} dv \\ &> \int_{\frac{s}{1+\lambda}}^{\hat{v}(s)} dv \geq \hat{v}(s) - \max\{\underline{v}, \frac{s}{1+\lambda}\},\end{aligned}$$

where the first inequality follows from $\frac{s}{v} > 1$ when $v \in [\frac{s}{1+\lambda}, \hat{v}(s))$. Note that there must be a strictly positive measure of signals such that $\min\left\{\hat{v}(s) - \underline{v},\, s \log \frac{(1+\lambda)\hat{v}(s)}{s}\right\} = s \log \frac{(1+\lambda)\hat{v}(s)}{s}$ as $\hat{v}(\overline{v}(1+\lambda)) - \underline{v} > 0$ and $\overline{v}(1+\lambda) \log \frac{(1+\lambda)\hat{v}(\overline{v}(1+\lambda))}{\overline{v}(1+\lambda)} = 0$ and all functions are continuous.

Therefore,

$$\int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\min\left\{\hat{v}(s) - \underline{v},\, s \log \frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda \Delta v} ds > \int_{\underline{v}}^{\overline{v}} \frac{v}{\Delta v} dv.$$

Chapter 3

The (No) Value of Commitment

3.1 Introduction

Commitment plays an important role in many economic models. The general insight of economic theory is that the value of commitment is positive: if a principal has commitment, he can replicate any action he would play without commitment. Moreover, commitment plays a key role in many standard tools used in economic theory such as the revelation principle (Myerson, 1982; Bester and Strausz, 2000; Doval and Skreta, 2022). However, commitment is usually a strong assumption and is sometimes hard to justify. Even when it is possible to justify the commitment assumption, it might be an undesirable feature of the model. For example, even if a regulator could commit to a rule, there might be reasons outside the model that require the government to maintain agency over this rule at any point in time.

In this paper, I provide a condition under which commitment has no value for a principal that faces a maximisation problem under constraints. That is I provide a condition under which, even when the principal can commit, he is better off best-replying to the information revealed in the economic problem. The usefulness of this result is twofold. First, as argued above, commitment can be an undesirable feature of economic models. Knowing that the condition provided is satisfied facilitates solving the model. Indeed, models assuming commitment are usually easier to solve as the number of constraints in the problem is smaller. When assuming commitment, the modeller does not need to make sure that the principal best-replies at the optimum. But if the condition holds, we are guaranteed that the omitted best-replies constraints of the principal will hold. Second, in the case commitment is actually assumed, it restricts the set of strategies the modeller has to look at. Even though assuming commitment can simplify the problem, the set of solutions the modeller needs to consider remain quite large. Knowing that the value of commitment is zero restricts the set of potential solutions.

Consider the following maximisation problem. Let α describe the strategy of the principal and σ the strategy of other agents in the economic problem.

$$\begin{aligned} V = \max_{\alpha,\sigma} v(\alpha, \sigma) \\ \text{s.t. } \text{Constraint}(\alpha, \sigma) \\ \alpha \text{ is a best-response to } \sigma \end{aligned}$$

where v denotes the payoffs of the principal. The problem above can represent many economic models. For example, the constraints can be incentive compatibility constraints of some agents in a mechanism design problem. Without commitment, there is an additional constraint guaranteeing that the principal's strategy is a best-reply to the information revealed in the economic interaction. With commitment, the principal can commit to information in an arbitrary way. My approach is to fix the principal's strategy and treat it as a parameter in the maximisation problem.

$$\begin{aligned} V(\alpha) = \max_{\sigma} v(\alpha, \sigma) \\ \text{s.t. Constraint}(\alpha, \sigma) \end{aligned}$$

The problem $V(\alpha)$ is the principal's problem when he commits to the strategy α. One can solve the problem above by finding a saddle-point of a Lagrangian:

$$\mathcal{L}(\sigma, \lambda; \alpha) = v(\alpha, \sigma) + \lambda \cdot \text{Constraint}(\alpha, \sigma),$$

where λ is the Lagrangian multiplier associated with the constraints. For a solution (σ^*, λ^*), we can apply an envelope theorem (Milgrom and Segal, 2002) on the Lagrangian to get that, omitting technical details,

$$\frac{dV(\alpha)}{d\alpha} = \frac{\partial \mathcal{L}(\sigma^*, \lambda^*; \alpha)}{\partial \alpha} = \frac{\partial v(\alpha, \sigma^*)}{\partial \alpha} + \lambda^* \cdot \frac{\partial \, \text{Constraint}(\alpha, \sigma^*)}{\partial \alpha}$$

Now note that if the last term $\lambda^* \cdot \frac{\partial \, \text{Constraint}(\alpha, \sigma^*)}{\partial \alpha} = 0$, then the total derivative of the value function is equal to its partial derivative:

$$\frac{dV(\alpha)}{d\alpha} = \frac{\partial v(\alpha, \sigma^*)}{\partial \alpha}.$$

This is exactly the condition needed to show that commitment does not have any value. Indeed, when the first-order condition is satisfied in the commitment problem, i.e., when taking into account the change it is going to induce in the constraint, it is also satisfied when the principal does not have commitment, i.e., does not take into account the change it induces in the constraints.

Note that if the constraints are slack at the optimum, this condition holds. But it can also hold when they are not. The key condition is not whether the constraints matter, i.e., they are slack, but whether the *marginal* contribution of the constraints to the Lagrangian is null. It is worth noting that to check whether the condition is satisfied, it is not needed to check explicitly whether the principal best-replies to the information revealed in the economic problem.

I apply this result to a mechanism design setting where the principal cannot commit to a mapping from report to an allocation. In Proposition 10, I show that if the allocation is a real number, the agent's payoff is strictly monotonic and the principal's utility is Lipschitz continuous, then he does not benefit from being able to contract over these actions. This result generalises the results of Glazer and Rubinstein (2006), Sher (2011) and Hart et al. (2017) that show under different assumptions on preferences and evidence available that commitment has no value. I discuss in Section 3.4 how this result relates to other results on the value of commitment in mechanism design.

3.2 General setup

A principal must solve the economic problem described as follows. For $n = 1, ..., N$, let Y_n be a finite set and $A = \times_n \Delta(Y_n)$ with typical element α. Let Σ a subset of a convex compact subset of $\mathbb{R}^p$ with typical element σ. Let $v : A \times \Sigma \to \mathbb{R}$, $g : A \times \Sigma \to \mathbb{R}^k$ and $BR(\sigma) = \{\alpha : v(\alpha, \sigma) \geq v(\alpha', \sigma),\ \forall \alpha'\}$. The assumption that A is the product set of simplexes allows for product set of intervals with the right normalisation by taking Y_m to be binary.

Consider the following maximisation problem:

$$
\begin{aligned}
V = &\max_{\alpha,\sigma} v(\alpha, \sigma) \\
&\text{s.t. } g(\alpha, \sigma) \geq 0 \\
&\quad\ \alpha \in BR(\sigma)
\end{aligned}
\tag{$\mathcal{V}$}
$$

The function v denotes the principal's payoff, the function g describes a set of constraint he is facing and $BR(\sigma)$ describes the set of element of A that are a best reply to σ.

If the principal could commit to α, he would solve the following problem:

$$\overline{V} = \max_{\alpha,\sigma} v(\alpha,\sigma) \qquad (\overline{\mathcal{V}})$$
$$\text{s.t. } g(\alpha,\sigma) \geq 0$$

I say that there is no value of commitment if $V = \overline{V}$. The aim of this paper is to find condition under which it is the case.

To do so, I first introduce the following maximisation problem where the principal commits over some α:

$$\overline{V}(\alpha) = \max_{\sigma} v(\alpha,\sigma) \qquad (\overline{\mathcal{V}}(\alpha))$$
$$\text{s.t. } g(\alpha,\sigma) \geq 0$$

and the associated Lagrangian,

$$\mathcal{L}(\sigma,\lambda;\alpha) = v(\alpha,\sigma) + \lambda \cdot g(\alpha,\sigma)$$

We can now state our main theorem. Say that the *first-order conditions are sufficient for* v if for each $\sigma \in \Sigma$, $\alpha^* \in \arg\max_\alpha v(\alpha,\sigma)$ implies that for all n, $y \in \operatorname{supp} \alpha^*(\cdot|n)$ only if $\frac{\partial v(\alpha^*,\sigma)}{\partial \alpha(y|n)} \geq \frac{\partial v(\alpha^*,\sigma)}{\partial \alpha(y'|n)}$ for all y'.

Theorem 4. *Suppose that each element of $\nabla_\alpha v(\alpha,\sigma)$ and $\nabla_\alpha g(\alpha,\sigma)$ is continuous in (α,σ), that first-order conditions are sufficient for v and that the solution of $\overline{\mathcal{V}}(\alpha)$ can be obtained by finding a saddle-point of $\mathcal{L}(\cdot,\cdot;\alpha)$ for all α.*

If there is $\alpha^* \in \arg\max_\alpha \overline{V}(\alpha)$ and saddle-point of $\mathcal{L}(\cdot,\cdot;\alpha^*)$, (σ,λ), such that

$$\lambda \cdot \nabla_\alpha g(\alpha^*,\sigma) = 0 \qquad (3.1)$$

and any saddle-point in a neighbourhood of α^* is bounded and $\overline{V}(\alpha)$ is differentiable at α^*, then $V = \overline{V}$

All proofs are relegated to Section 3.6.

This result tells us that commitment has no value if the *marginal* contribution of the constraints is zero. Note that if all the constraints were slack at the optimum, then $\lambda = 0$ and the condition is satisfied. This is what we would expect: if the principal is not effectively facing any constraint, he is better off best-replying to the information revealed. Theorem 4 tells us that what really matters is not that constraints do not matter but that their marginal contribution is null.

The condition that first-order conditions are sufficient is satisfied whenever $v(\alpha,\sigma)$ is linear in α. This would be the case if $v(\alpha, \sigma)$ is the expected utility over some finite action set $\times_n Y_n$.

What does no commitment mean? To clarify what the notion of best-reply means exactly, let's use the setup of mechanism design without commitment of Bester and Strausz (2001). An agent with private type sends a message with a principal with a strategy represented by σ. In the case of commitment, the principal commits to a mapping from message to allocation, represented by α. In the case of no commitment, the strategy α must best-reply to σ. In this case, the functions $g(\cdot, \cdot)$ represent the agent's IC constraints.

Note that from the definition of $BR(\sigma)$, the equilibrium concept associated is Bayes Nash Equilibrium. But in this case, any Bayes Nash Equilibrium is also a Perfect Bayesian Equilibrium if any allocation is optimal under some beliefs. This reasoning would hold for all the applications considered below.

3.3 Application – Disclosure Mechanism

I now put more structure by considering a mechanism design setup where the principal directly observes the message sent by the agent as in Bester and Strausz (2001) or in disclosure mechanisms Glazer and Rubinstein (2004); Hart et al. (2017); Ben-Porath et al. (2019). There a principal and an agent. The agent has a private type $\theta \in \Theta$ and access to set of messages $M(\theta)$. Let $M = \cup_\theta M(\theta)$. If $M(\theta) = M$ for all θ, we are in a cheap-talk setup. Otherwise, $M(\theta)$ represents a set of evidence the agent can submit. The outcome is denoted by $y \in Y$. The payoffs of the principal are $v(y, \theta)$ and the agents payoff's are $u(y, \theta)$. The sets Θ and M are finite and the set Y is a compact convex subset of $\mathbb{R}$. All utility functions are assumed to be differentiable.

A mechanism maps a message to an outcome y, $\alpha : M \to Y$. A mixed strategy for the agent is a mapping from type to messages: $\sigma : \Theta \to \Delta M$ with the condition $\operatorname{supp} \sigma(\cdot|\theta) \subseteq M(\theta)$.

The optimal mechanism with commitment solves

$$\begin{aligned}
&\max_\alpha \max_\sigma \sum_\theta \sum_m \mu(\theta)\sigma(m|\theta)v(\alpha(m), \theta) \\
&\text{s.t. for all } \theta \text{ and } m' \in M(\theta), \quad \sum_{m \in M(\theta)} \sigma(m|\theta)u(\alpha(m), \theta) \geq u(\alpha(m'), \theta) \\
&\sum_{m \in M(\theta)} \sigma(m|\theta) = 1
\end{aligned}$$

This problem selects for each mechanism the best agent strategy subject to this strategy best replies to the mechanism and then maximises over mechanisms.

The optimal mechanism is implementable without commitment if there is an optimal mechanism α with agent strategy σ such that $\alpha \in BR(\sigma)$.

I say that first-order conditions are sufficient for any expected v if for all $\mu \in \Delta\Theta$, first-order conditions are sufficient for $\sum_\theta \mu(\theta)v(y,\theta)$.

Proposition 10. *If first-order conditions are sufficient for any expected v, $u(\cdot,\theta)$ is strictly monotonic for each θ and*

$$f(\theta, y, y') = \frac{v(y,\theta) - v(y',\theta)}{u(y,\theta) - u(y',\theta)} \text{ is bounded} \tag{3.2}$$

then the optimal mechanism is implementable without commitment.

Given the assumption that $u(\cdot,\theta)$, the condition (3.2) holds whenever $v(\cdot,\theta)$ is differentiable and Lipschitz continuous.

First-order conditions are sufficient for any expected v if v is concave and differentiable in y. Hart et al. (2017) give examples of function that satisfy this condition.

Proposition 10 generalises the results from Hart et al. (2017) that focus on agent's payoffs $u(y,\theta) = y$ and an evidence structure that satisfies a form of transitivity. I show that their result on the no-value of commitment extends to arbitrary evidence structure as well as both increasing and decreasing agent's payoffs.

3.4 Relation to other results in the literature

Many papers in the literature in mechanism design with evidence show that the principal does not benefit from a form of commitment. Ben-Porath et al. (2021) show that when preferences are semi-aligned, i.e. $v(y,\theta) = \nu(\theta)u(y,\theta)$ for some function ν, the principal does not benefit from commitment. In the case y is a mixed strategy over a binary action, they show that any principal's preferences can take this form, generalising the result of Glazer and Rubinstein (2006). Thus, in the case of binary actions, Proposition 10 extends this result by allowing non-expected utility for the principal as $v(y,\theta)$ is not required to be linear in y. Ben-Porath et al. (2019) look at more general preferences where $Y = \Delta^n$ and preferences are $v(y,\theta) = \mathbb{E}_y[\psi(\cdot)] + \nu(\theta)u(y,\theta)$ but focus on evidence structure satisfying the normality condition (Bull and Watson, 2007).

Vohra et al. (2021) show that when the optimal mechanism is partitional, i.e., the mechanism partitions the allocation space and each type strictly prefers his allocation over the other elements of the partition, and $v(y,\theta) = f(u(y,\theta),\theta)$, then the principal does not benefit from commitment. Unfortunately, in many instances the optimal mechanism is not partitional. In particular, it is not when some types have to mix over which message they send as is typical in mechanism design without commitment.

3.5 Conclusion

I have presented a method leveraging the envelope theorem for saddle-point problems to show that commitment has no value in some economic problems. The advantage of this method is that it does not necessitate checking that the principal actually best-replies to the information revealed. Moreover it has a natural economic interpretation in terms of the marginal contribution of the constraints.

I have shown it can be useful in the context of strictly monotonic agent utility. One big limitation of this approach is that it requires that the value function to be differentiable at the optimal α. I conjecture that this condition can be dispensed with. If this conjecture is correct, then many results on the value of commitment can be proved in a unified way using this method, like results from Vohra et al. (2021), Ben-Porath et al. (2021) and Ben-Porath et al. (2019).

3.6 Omitted proofs of Chapter 3

3.6.1 Proof of Theorem 4

First note that $\overline{V} \geq V$. It is also true that $\max_\alpha \overline{V}(\alpha) = \overline{V} \geq V$. The goal of the proof is to show that $\max_\alpha \overline{V}(\alpha) = V$. To do so, I will show that if the condition of the theorem is satisfied for (σ^*, α^*) maximising $\max_\alpha \overline{V}(\alpha)$, then $\alpha^* \in BR(\sigma^*)$.

Let $\mathcal{L}(\sigma, \lambda; \alpha)$ be the Lagrangian associated with $\overline{\mathcal{V}}(\alpha)$.

From the assumptions of Theorem 4,

$$\overline{V}(\alpha) = \max_\sigma \min_\lambda \mathcal{L}(\sigma, \lambda; \alpha)$$

Let $\Sigma^*(\alpha) = \arg\max \min_{\lambda'} \mathcal{L}(\sigma', \lambda'; \alpha)$ and $\Lambda^*(\alpha) = \arg\min \max_{\sigma'} \mathcal{L}(\sigma', \lambda'; \alpha)$. By Milgrom and Segal (2002, Corollary 5), if $\Sigma^*(\alpha)$ and $\Lambda^*(\alpha)$ are bounded in a neighbourhood of α^*, for any $\sigma \in \Sigma^*(\alpha)$ and $\lambda \in \Lambda^*(\alpha)$, for each element of α, α_i

$$\frac{d\overline{V}(\alpha)}{d\alpha_i} = \frac{\partial \mathcal{L}}{\partial \alpha_i} \quad \text{a.e.}$$

Because we assume that $\overline{V}(\alpha)$ is differentiable, this equality holds at α^*.

Furthermore, Milgrom and Segal (2002) show that both the left- and right-derivative exist for any interior α with:

$$\frac{d^+\overline{V}(\alpha)}{d\alpha_i} = \max_{\sigma \in \Sigma^*(\alpha)} \min_{\lambda \in \Lambda^*(\alpha)} \frac{\partial \mathcal{L}}{\partial \alpha_i}$$
$$\frac{d^-\overline{V}(\alpha)}{d\alpha_i} = \min_{\sigma \in \Sigma^*(\alpha)} \max_{\lambda \in \Lambda^*(\alpha)} \frac{\partial \mathcal{L}}{\partial \alpha_i}$$

Observe that from condition (3.1), at α^*, we have $\frac{\partial \mathcal{L}}{\partial \alpha_i} = \frac{\partial v(\alpha, \sigma)}{\partial \alpha_i}$.

For any interior α_i, we have $\frac{d\overline{V}(\alpha)}{d\alpha_i} = \frac{\partial v(\alpha,\sigma)}{\partial \alpha_i} = 0$ and if $\alpha_i = 0, 1$, we must have $\frac{d^+\overline{V}(\alpha)}{d\alpha_i} \leq 0$ or $\frac{d^-\overline{V}(\alpha)}{d\alpha_i} \geq 0$. Because first-order conditions are sufficient for v, then $\alpha^* \in BR(\sigma)$.

3.6.2 Proof of Proposition 10

The problem

$$\max_{\sigma} \sum_{\theta} \sum_{m} \mu(\theta)\sigma(m|\theta)v(\alpha(m), \theta)$$

$$\text{s.t. for all } \theta \text{ and } m' \in M(\theta), \sum_{m \in M(\theta)} \sigma(m|\theta)u(\alpha(m), \theta) \geq u(\alpha(m'), \theta)$$

$$\sum_{m \in M(\theta)} \sigma(m|\theta) = 1$$

is linear in σ and thus solvable by finding a saddle-point of the associated Lagrangian.

Fix a strategy α and a saddle-point (σ, λ). Take $m \in \operatorname{supp} \sigma(\cdot|\alpha)$. For any saddle-point, (σ', λ), if $\lambda(\theta, m') > 0$, then by complementary slackness, $u(\alpha(m'), \theta) = u(\alpha(m), \theta)$. Because $u(\cdot, \theta)$ is injective, this means that $\alpha(m) = \alpha(m')$.

I will now show that for any α, it is possible to find a saddle-point satisfying condition (3.1). Because α is fixed throughout and to simplify the exposition, I will write $v(m, \theta)$ and $u(m, \theta)$ instead of $v(\alpha(m), \theta)$ and $u(\alpha(m), \theta)$. The dual problem is

$$\min_{\lambda, z} \sum_{\theta} \sum_{m' \in M(\theta)} -\lambda(\theta, m')u(m', \theta) + \sum_{\theta} z(\theta)$$

$$\text{s.t. for all } \theta,\ m \in M(\theta),\ z(\theta) - \sum_{m'} \lambda(\theta, m')u(m, \theta) \geq \mu(\theta)v(m, \theta)$$

$$\lambda(\theta, m') \geq 0,\ z(\theta) \in \mathbb{R}$$

Where λ and z are the dual variables associated with the IC constraints and feasibility constraints. Take a solution (σ, λ, z) of the primal and dual problem. First, note that if $\lambda(\theta, m') > 0$, by complementary slackness, we must have

$$\sum_{m} \sigma(m|\theta)u(m, \theta) = u(m'|\theta)$$

and therefore $u(m, \theta) = u(m', \theta)$ for any $m \in \operatorname{supp} \sigma(\cdot|\theta)$. By condition (3.2). Because $u(\cdot, \theta)$ is injective, this means that $\alpha(m) = \alpha(m')$.

Next, I will give a new solution to the primal and dual problem and show that (1) it is indeed a solution and (2) it satisfies condition (3.1).

Let $(\tilde{\sigma}, \lambda, z)$ with $\tilde{\sigma}(m|\theta) = \frac{\lambda(\theta, m)}{\sum_{m'} \lambda(\theta, m')}$ if $\sum_{m'} \lambda(\theta, m') > 0$ and $\tilde{\sigma}(m|\theta) = \sigma(m|\theta)$ otherwise.

(1) $(\tilde{\sigma}, \lambda, z)$ is a solution.

First, let's verify it satisfies all the constraints of the primal and dual problem.

IC constraints: We have established that for any $m' : \lambda(\theta, m') > 0$, $u(m, \theta) = u(m', \theta)$ for any $m \in \operatorname{supp} \sigma(\cdot|\theta)$. Therefore if σ satisfies the IC constraints, so must $\tilde{\sigma}$. The new strategy $\tilde{\sigma}(\cdot|\theta)$ is also a well-defined probability for each θ.

Dual constraints: for any pair θ, m,

$$z(\theta) - \sum_{m'} \lambda(\theta, m') u(m, \theta) \geq \mu(\theta) v(m, \theta)$$

only depends λ and is thus still satisfied.

Let's verify that complementary slackness holds. The payoffs from $\tilde{\sigma}(\cdot|\theta)$ and $\sigma(\cdot|\theta)$ are the same for each θ. Therefore we still have that $\lambda(\theta, m') > 0$ implies $\sum_m \tilde{\sigma}(m|\theta) u(m, \theta) = u(m'|\theta)$.

We are left to check that $\tilde{\sigma}(m|\theta) > 0$ implies

$$z(\theta) - \sum_{m'} \lambda(\theta, m') u(m, \theta) = \mu(\theta) v(m, \theta) \tag{3.3}$$

Now remember that the equality (3.3) would hold for any $m' \in \operatorname{supp} \sigma(\cdot|\theta)$. We have also shown that $u(m, \theta) = u(m', \theta)$ for any $m \in \operatorname{supp} \tilde{\sigma}(\cdot|\theta)$ and by (3.2), we also have $v(m, \theta) = v(m', \theta)$. Therefore, equation (3.3) also holds.

We thus have that $(\tilde{\sigma}, \lambda, z)$ are also a solution to the primal and dual problems.

(2) Show that $(\tilde{\sigma}, \lambda, z)$ satisfies condition (3.1)

Let $u'(\cdot, \theta)$ denote the partial derivative of u with respect to y. The derivative of the constraint part of the Lagrangian with respect to $\alpha(\tilde{m})$ is

$$\begin{aligned}
&\frac{\partial}{\partial \alpha(\tilde{m})} \sum_{\theta} \sum_{m' \in M(\theta)} \lambda(\theta, m') \big[\sum_{m} \tilde{\sigma}(m|\theta) u(\alpha(m), \theta) - u(\alpha(m'), \theta) \big] \\
&= \sum_{\theta} u'(\alpha(\tilde{m}), \theta) \tilde{\sigma}(\tilde{m}|\theta) \sum_{m} \lambda(\theta, m) - u'(\alpha(\tilde{m}), \theta) \lambda(\theta, \tilde{m}) = 0
\end{aligned}$$

Show that the dual variables are bounded

For each α, we can find a saddle-point (σ, λ, z). Now note that σ must maximise the Lagrangian. If $m \in$

$\operatorname{supp}\sigma(\cdot|\theta)$,

$$\frac{\partial \mathcal{L}(\sigma,\lambda,z)}{\partial \sigma(m|\theta)} - \frac{\partial \mathcal{L}(\sigma,\lambda,z)}{\partial \sigma(m'|\theta)} \geq 0, \text{ for all } m'$$

Assuming $u(\alpha(m),\theta) - u(\alpha(m'),\theta) \neq 0$, this conditions boils down

$$\sum_{\tilde{m}} \lambda(\theta,\tilde{m}) \geq \mu(\theta) \frac{v(\alpha(m'),\theta) - v(\alpha(m),\theta)}{u(\alpha(m),\theta) - u(\alpha(m'),\theta)}$$

where I have used that $u(\alpha(m),\theta) - u(\alpha(m'),\theta) \geq 0$ because $m \in \operatorname{supp}\sigma(\cdot|\theta)$.

Because $\lambda \in \arg\min\max \mathcal{L}(\sigma,\lambda,z)$, this constraint must be binding if the RHS is positive for some m', otherwise the minimiser can decrease the Lagrangian, without changing the maximiser's best-reply. Therefore a sufficient condition for $\sum_{\tilde{m}} \lambda(\theta,\tilde{m})$ to be bounded is that $\frac{v(y,\theta)-v(y',\theta)}{u(y,\theta)-u(y',\theta)}$ is bounded for any θ, y, y'.

Any binding dual constraint pins down $z(\theta)$ through $\sum_{\tilde{m}} \lambda(\theta,\tilde{m})$, therefore $z(\theta)$ is bounded if and only if $\sum_{\tilde{m}} \lambda(\theta,\tilde{m})$ is.

Show that the value function is differentiable at the optimum

Take a sequence of strategies (α_n) converging to α and for each n, an optimal mechanism satisfying condition (3.1), $(\sigma_n, \lambda_n, z_n)$. Assume that $\alpha_n(m) - \alpha(m) > 0$. This is a bounded sequence in $\mathbb{R}^n$ and so a converging subsequence exists. Let's focus on this one, with limit $(\sigma^+, \lambda^+, z^+)$. Because the value function is continuous (Milgrom and Segal, 2002, Corollary 5) and the set of constraints satisfy the closed graph property, (σ^+, λ^+, z) is a saddle-point of the Lagrangian at α. By Theorem 5 of (Milgrom and Segal, 2002) $\overline{V}(\alpha)$ is directionally differentiable and by definition of a saddle-point,

$$\frac{\mathcal{L}(\sigma^+,\lambda_n,z_n,\alpha_n) - \mathcal{L}(\sigma^+,\lambda_n,z_n,\alpha)}{\alpha_n(m)-\alpha(m)} \leq \frac{\overline{V}(\alpha_n) - \overline{V}(\alpha)}{\alpha_n(m)-\alpha(m)} \leq \frac{\mathcal{L}(\sigma_n,\lambda^+,z^+,\alpha_n) - \mathcal{L}(\sigma_n,\lambda^+,z^+,\alpha)}{\alpha_n(m)-\alpha(m)}$$

Take the first inequality and add $(\mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha_n) - \mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha_n) + \mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha) - \mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha))$ on the LHS. Define the vector $IC(\sigma,\alpha) = (\sum_{m'} \sigma(m'|\theta) u(\alpha(m'),\theta) - u(\alpha(m),\theta))_{\theta,m}$. Rearranging, we have

$$\frac{\mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha_n) - \mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha)}{\alpha_n(m)-\alpha(m)} + \frac{(\lambda_n - \lambda^+)\cdot(IC(\sigma^+,\alpha_n) - IC(\sigma^+,\alpha))}{\alpha_n(m)-\alpha(m)} \leq \frac{\overline{V}(\alpha_n)-\overline{V}(\alpha)}{\alpha_n(m)-\alpha(m)}$$

Taking $n \to \infty$, we get $\frac{\partial^+ \mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha)}{\partial \alpha(m)} \leq \frac{d^+\overline{V}(\alpha)}{d\alpha(m)}$. A similar reasoning yields $\frac{d^+\overline{V}(\alpha)}{d\alpha(m)} \leq \frac{\partial^+ \mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha)}{\partial \alpha(m)}$ and thus $\frac{d^+\overline{V}(\alpha)}{d\alpha(m)} = \frac{\partial^+ \mathcal{L}(\sigma^+,\lambda^+,z^+,\alpha)}{\partial \alpha(m)}$.

A similar reasoning establishes that $\frac{d^-\overline{V}(\alpha)}{d\alpha(m)} = \frac{\partial^- \mathcal{L}(\sigma^-,\lambda^-,z^-,\alpha)}{\partial \alpha(m)}$ where $(\sigma^-, \lambda^-, z^-)$ is a saddle-point at α and the limit to a sequence of strategies $\alpha_n \to \alpha$ with $\alpha_n < \alpha$ satisfying condition (3.1).

Now note that the set of saddle-points is rectangular and thus $(\sigma^+, \lambda^-, z^-)$ and $(\sigma^-, \lambda^+, z^+)$ are also saddle-points. This means that if $m \in \operatorname{supp} \sigma^+(\cdot|\theta)$, $m' \in \operatorname{supp} \sigma^-(\cdot|\theta)$ and $m \neq m'$, we must have $u(\alpha(m), \theta) = u(\alpha(m'), \theta)$ and by injectivity of $u(\cdot, \theta)$, $\alpha(m) = \alpha(m')$.

Note that for any m, $\frac{d^+\overline{V}(\alpha)}{d\alpha(m)} \leq 0 \leq \frac{d^-\overline{V}(\alpha)}{d\alpha(m)}$, with a strict inequality if it is not differentiable. But if $\sum_\theta \mu(\theta)\sigma^+(m|\theta)v'(\alpha(m), \theta) < 0$ for some m taken with positive probability, then $v'(\alpha(m), \theta) < 0$ for all m' such that $\lambda^-(\theta, m') > 0$. Therefore, we get $\sum_\theta \mu(\theta)\sigma^-(m'|\theta)v'(\alpha(m'), \theta) < 0$ contradicting the optimality of α.

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