LOW REGULARITY ESTIMATES FOR CUTFEM APPROXIMATIONS OF AN ELLIPTIC PROBLEM WITH MIXED BOUNDARY CONDITIONS

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Abstract. We show error estimates for a cut finite element approximation of a second order elliptic problem with mixed boundary conditions. The error estimates are of low regularity type where we consider the case when the exact solution $u \in H^s$ with $s \in (1, 3/2]$. For Nitsche type methods this case requires special handling of the terms involving the normal flux of the exact solution at the the boundary. For Dirichlet boundary conditions the estimates are optimal, whereas in the case of mixed Dirichlet-Neumann boundary conditions they are suboptimal by a logarithmic factor.

1. Introduction

In this paper we will consider the finite element approximation of the Poisson problem with mixed boundary conditions under minimal regularity assumptions. Let Ω be a domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, which is decomposed into two subdomains $\partial\Omega_D$ and $\partial\Omega_N$ such that $\partial\Omega = \overline{\partial\Omega_D} \cup \partial\Omega_N = \partial\Omega_D \cup \overline{\partial\Omega_N}$ and $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Consider the problem: find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \qquad \text{in } \Omega \tag{1.1}$$

$$u = g_D \qquad \text{on } \partial\Omega_D \tag{1.2}$$

$$\nabla_n u = g_N \qquad \text{on } \partial\Omega_N \tag{1.3}$$

where $f : \Omega \to \mathbb{R}$, $g_D : \Gamma_D \to \mathbb{R}$ and $g_N : \Gamma_N \to \mathbb{R}$ satisfy the following bound for $s > 1$,

$$\|f\|_{H^{s-2}(\Omega)} + \|g_D\|_{H^{s-1/2}(\partial\Omega_D)} + \|g_N\|_{H^{s-3/2}(\partial\Omega_N)} \lesssim 1 \tag{1.4}$$

Here and below we used the notation $a \lesssim b$ for $a \le Cb$, with C a positive constant.

For the approximation of the problem we apply a Cut Finite Element Method (CutFEM). In CutFEM the boundary is allowed to cut through the computational cells in an (almost) arbitrary way and stabilization terms are added in the vicinity of the boundary to ensure that the method is coercive and that the resulting linear system of equations is invertible.

In previous work on fictitious domain finite element methods see [2, 1], error estimate were shown under the assumption that $u \in H^s(\Omega)$ with $s > 3/2$. The objective of the present work is to relax this regularity requirement. Indeed, we show an a priori error estimate in the energy norm, requiring only that $u \in H^s(\Omega)$, where $s > 1$, and Δu is in $L^2(U_{\delta_0})$ on some arbitrarily thin neighborhood U_{δ_0} of the Dirichlet boundary $\partial\Omega_D$. Since the test functions in the Nitsche formulation of the Dirichlet condition are not zero on $\partial\Omega_D$, we will also have to choose the Neumann

Date: June 9, 2023.

data g_N in a slightly smaller space than $H^{-1/2}(\partial\Omega_N)$. We focus our attention on the effects of rough data in CutFEM. We assume that the boundary $\partial\Omega$ of the domain Ω is smooth and that we can evaluate integrals on the intersection of simplices and the domain and its boundary, exactly. Estimation of the error resulting from approximation of the domain can be handled using the techniques in [4].

The study of the convergence of nonconforming methods for the approximation of solution with low regularity has received increasing interest since the seminal paper by Gudi [9]. In that work optimal convergence for low regularity solutions were obtained using ideas from a posteriori error analysis, where the error is upper bounded by certain residuals of the discrete solution. These residuals are then shown to lead to optimal upper bounds using the discrete local efficiency bounds. A similar approach was used by Lüthen et al. [10] for a generalised Nitsche's method on fitted meshes. This approach does however not seem to be suitable for the case of cut finite element method since for cut elements the local efficiency bounds are not robust with respect to the mesh boundary intersection. Instead, in the spirit of [7], we use a version of duality pairing to handle the term involving the normal flux of the interpolation error. This is made more delicate by the presence of mixed boundary conditions. Indeed to include this case in the analysis we introduce a regularized bilinear form and use the solution to the regularized problem as pivot in the error estimate. The regularization gives rise to a logarithmic factor. Observe that this is due to the mixed boundary conditions. For pure Dirichlet conditions or pure Neumann conditions the analysis results in optimal error bounds for $s \geq 1$.

The paper is organized as follows: In Section 2 we introduce the functional framework for the model problem and formulate the finite element method and in Section 3 we derive the error estimates.

2. Weak Formulation and the Finite Element Method

Since we consider low regularity solutions of a problem with mixed boundary condition we must be careful with the fractional Sobolev spaces for the traces of the functions. In this section we first introduce the notations and definitions for the functional analytical framework, leading to the weak formulation of (1.1)–(1.3). Then we introduce the cut finite element method for the approximation of the weak solutions.

2.1. **Function Spaces.**

Let $\omega \subset \mathbb{R}^n$ be an open set and let $H^s(\omega)$ denote the usual Sobolev spaces on ω. Define

$$H^{1/2}(\partial\Omega) = \mathrm{tr}_{\partial\Omega}(H^1(\Omega)) \tag{2.1}$$

$$\|v\|_{H^{1/2}(\partial\Omega)} = \inf_{w \in H^1(\Omega), w|_{\partial\Omega} = v} \|w\|_{H^1(\Omega)} \tag{2.2}$$

where $\mathrm{tr}_{\partial\Omega}$ is the trace operator obtained by extending the restriction operator $|_{\partial\Omega}$ from $C^\infty(\overline{\Omega})$ to $H^1(\Omega)$, see [8]. For $\Gamma \subset \partial\Omega$, Γ open in $\partial\Omega$, define

$$H^{1/2}(\Gamma) = H^{1/2}(\partial\Omega)|_\Gamma \tag{2.3}$$

$$\|v\|_{H^{1/2}(\Gamma)} = \inf_{w \in H^1(\Omega), w|_\Gamma = v} \|w\|_{H^1(\Omega)} \tag{2.4}$$

and

$$\widetilde{H}^{1/2}(\Gamma) = \{v \in H^{1/2}(\partial\Omega) : \mathrm{supp}(v) \subset \overline{\Gamma}\} \tag{2.5}$$

We note for each $v \in \widetilde{H}^{1/2}(\Gamma)$ we can define an extension $v^e \in H^{1/2}(\partial\Omega)$ such that $v^e = 0$ on $\partial\Omega \setminus \Gamma$ and $v^e = v$ on Γ. Letting $[H^{1/2}(\Gamma)]^e \subset H^{1/2}(\partial\Omega)$ be the space of such extended functions and observing that $[H^{1/2}(\Gamma)]^e$ is precisely the kernel of the restriction operator $|_{\partial\Omega\setminus\overline{\Gamma}} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega \setminus \overline{\Gamma})$, we obtain $H^{1/2}(\partial\Omega \setminus \overline{\Gamma}) \cong H^{1/2}(\partial\Omega)/[\widetilde{H}^{1/2}(\Gamma)]^e$.

Next define the dual spaces

$$H^{-1/2}(\partial\Omega) = [H^{1/2}(\partial\Omega)]^* \tag{2.6}$$

$$H^{-1/2}(\Gamma) = [\widetilde{H}^{1/2}(\Gamma)]^* \tag{2.7}$$

$$\widetilde{H}^{-1/2}(\Gamma) = [H^{1/2}(\Gamma)]^* \tag{2.8}$$

consisting of functionals $g : X \to \mathbb{R}$ with duality pairing $\langle g, v\rangle_{X^*\times X} = g(v)$ and norm

$$\|g\|_{X^*} = \sup_{v\in X\setminus\{0\}} \frac{g(v)}{\|v\|_X} \tag{2.9}$$

with $X \in \{H^{1/2}(\partial\Omega), H^{1/2}(\Gamma), \widetilde{H}^{1/2}(\Gamma)\}$. For $X \in \{H^{1/2}(\Gamma), \widetilde{H}^{1/2}(\Gamma)\}$ we will use the simplified notation $(g, v)_\Gamma = \langle g, v\rangle_{X^*\times X} = g(v)$ for the duality pairing with $g \in \widetilde{H}^{-1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$, and $g \in H^{-1/2}(\Gamma)$ and $v \in \widetilde{H}^{1/2}(\Gamma)$, respectively.

Note that for each $g \in \widetilde{H}^{-1/2}(\Gamma)$ we may define $g^e \in H^{-1/2}(\partial\Omega)$ by $g^e(v) = g(v|_\Gamma)$ and thus $H^{-1/2}(\Gamma) \hookrightarrow H^{-1/2}(\partial\Omega)$.

2.2. **Weak Formulation.** The problem (1.1)–(1.3) can be cast on weak form: find $u \in V_{g_D}$ such that

$$a(u, v) = l(v) \qquad v \in V_0 \tag{2.10}$$

where

$$a(v, w) = (\nabla v, \nabla w)_\Omega, \qquad l(v) = (f, v)_\Omega + (g_N, v)_{\partial\Omega_N} \tag{2.11}$$

and, for each $g_D \in H^{1/2}(\partial\Omega_D)$,

$$V_{g_D} = \{v \in H^1(\Omega) : v|_{\partial\Omega_D} = g_D\} \tag{2.12}$$

For $f \in H^{-1}(\Omega)$, $g_D \in H^{1/2}(\partial\Omega_D)$, and $g_N \in H^{-1/2}(\partial\Omega_N)$, there exists a unique weak solution to (2.10) and the following elliptic regularity estimate holds, $1 \leq s < 3/2$,

$$\|u\|_{H^s(\Omega)} \lesssim \|f\|_{H^{s-2}(\Omega)} + \|g_D\|_{H^{s-1/2}(\partial\Omega_D)} + \|g_N\|_{H^{s-3/2}(\partial\Omega_N)} \tag{2.13}$$

see Savaré [11], Theorem 2, page 876.

2.3. **The Normal Flux.** The normal flux $\nabla_n u = n \cdot \nabla u \in H^{-1/2}(\partial\Omega)$, where n is the exterior unit normal, plays an important role in what follows. For $u \in H^1(\Omega)$, with $\Delta u \in L^2(\Omega)$, it can be defined by the identity

$$(\nabla_n u, v)_{\partial\Omega} = (\Delta u, v)_\Omega + (\nabla u, \nabla v)_\Omega \qquad \forall v \in H^1(\Omega) \tag{2.14}$$

Observe that in the finite element method we work with weakly enforced boundary conditions and therefore we will not have test functions that vanish on $\partial\Omega_D$, i.e. the test functions are not in V_0, and herefore we will consider boundary data such that

$$g_D \in H^{1/2}(\partial\Omega_D), \qquad g_N \in \widetilde{H}^{-1/2}(\partial\Omega_N) \tag{2.15}$$

where the Neumann data g_N is chosen in the smaller space $\widetilde{H}^{-1/2}(\partial\Omega_N) \subset H^{-1/2}(\partial\Omega_N)$, compared to the strong formulation and corresponding weak form (2.10). We will also assume that the source term f is square integrable over some (arbitrary thin) neighbourhood of the boundary $\partial\Omega$, see (3.38) below.

2.4. **Finite Element Method.** To define the cut finite element method let Ω_0 be a polygonal domain such that $\Omega \subset \Omega_0$ and let $\{\mathcal{T}_{h,0} : h \in (0, h_0]\}$ be a family of quasiuniform meshes covering Ω_0 with mesh parameter $h := \max_{T\in\mathcal{T}_{h,0}} \mathrm{diam}(T)$. For a subset $\omega \subset \Omega_0$, define the submesh of elements intersecting ω, by $\mathcal{T}_h(\omega) := \{T \in \mathcal{T}_{h,0} : T \cap \omega \neq \emptyset\}$, and let $\mathcal{T}_h := \mathcal{T}_h(\Omega)$ be the so called active mesh. Let $V_{h,0}$ be the conforming finite element space defined on $\mathcal{T}_{h,0}$ consisting of piecewise affine functions and define $V_h = V_{h,0}|_{\mathcal{T}_h}$. Define the bilinear forms

$$A_h(v, w) := a(v, w) - (\nabla_n v, w)_{\partial\Omega_D} - (v, \nabla_n w)_{\partial\Omega_D} + \beta h^{-1}(v, w)_{\partial\Omega_D} \tag{2.16}$$

$$s_h(v, w) := \sigma h([\nabla_n v], [\nabla_n w])_{\mathcal{F}_h(\partial\Omega)} \tag{2.17}$$

$$L_h(v) := (f, v)_\Omega + (g_N, v)_{\partial\Omega_N} - (g_D, \nabla_n v)_{\partial\Omega_D} + \beta h^{-1}(g_D, v)_{\partial\Omega_D} \tag{2.18}$$

with positive parameters β and σ, $\mathcal{F}_h(\partial\Omega)$ the set of interior faces in $\mathcal{T}_h$, that is faces belonging to two different elements $\mathcal{T}_h$, associated with an element $T \in \mathcal{T}_h(\partial\Omega) = \{T \in \mathcal{T}_h : T \cap \partial\Omega \neq \emptyset\}$ that intersects the boundary, and the jump in the normal flux at face F shared by elements T_1 and T_2 is defined by

$$[\nabla_n v] = \nabla_{n_1} v_1 + \nabla_{n_2} v_2 \qquad \text{on } F \tag{2.19}$$

where $v_i = v|_{T_i}$ and n_i is the unit exterior normal.

Define the finite element method: find $u_h \in V_h$ such that

$$A_h(u_h, v) + s_h(u_h, v) = L_h(v) \qquad \forall v \in V_h \tag{2.20}$$

Remark 2.1. *We have assumed quasiuniformity in order to simplify the notation but it is sufficent to assume that the elements are shape regular and that local quasiuniformity holds, see Definition 2.2 in* [5].

3. Error Analysis

In this section we will derive the error estimates, here as usual the consistency of the method is of essence. However, for solutions with low regularity this is delicate in the case of mixed boundary conditions. Indeed, in the low regularity case, (2.14) is not sufficient to make sense of the term $(\nabla_n u, w)_{\partial\Omega_D}$ for approximation purposes, since the division on $\partial\Omega_D$ and $\partial\Omega_N$ necessarily results in a boundary integral over one of the subdomains that has to be lifted in some other fashion. This is problematic since the solution is not regular enough to allow for the usual trace inequality arguments. To handle this difficulty we introduce a regularized finite element formulation (for analysis purposes only), where a smooth weight function χ is introduced and the problematic term is replaced by

$$(\nabla_n u, w)_{\chi,\partial\Omega} := (\chi \nabla_n u, w)_{\partial\Omega} \tag{3.1}$$

The regularized method has a consistency error that can be controlled by sharpening the cut off function χ.

3.1. **Outline.** We shall prove low regularity energy norm error estimates using the following approach:

- Similarly to [9] we estimate the error in a norm which does not involve the L^2 norm of the normal trace of the gradient.
- For the case of mixed boundary conditions, we introduce a regularized bilinear form and the corresponding (nonconsistent) finite element method. The regularization takes the form of a weight function smoothing the transition from the Dirichlet to the Neumann boundary condition in the first boundary integral of the form A_h, see equation (2.16). In the regularized norm we can use a version of $H^{-1/2} - H^{1/2}$ duality in an ϵ neighborhood of $\partial\Omega_D$.
- The total error is estimated using a Strang type argument. The error is divided into the approximation error, the discrete error between an interpolant and the finite element solution of the regularized formulation and finally the regularization error between the regularized and standard finite element solutions.

3.2. **The Cut Off Function.** Key to the regularized problem is the design of the weight function, $\chi : \Omega \to \mathbb{R}$ with support in a neighbourhood of $\partial\Omega_D$. This function takes the value 1 on $\partial\Omega_D$ and decays smoothly to zero in an ϵ neighbourhood of $\overline{\partial\Omega}_D \cap \overline{\partial\Omega}_N$ and into the domain away from the boundary. This way it plays the role of a cut off, that localizes the boundary integral to $\partial\Omega_D$, while the form remains well defined for low regularity solutions. In order to define the cut off function we introduce some notation.
Notation. For $x \in \mathbb{R}^d$, $\omega \subset \mathbb{R}^d$, let $\rho_\omega(x) \geq 0$ be the distance function $\rho_\omega(x) = \text{dist}(x, \omega)$ and let $p_\omega : \mathbb{R}^d \to \omega$ be the closest point mapping. In the case $\omega \equiv \partial\Omega$ we drop the subscript. For $\delta \in (0, \delta_0]$, define the δ-neighbourhood of $\partial\Omega$,

$$U_\delta(\partial\Omega) = \{x \in \Omega : \rho(x) < \delta\} \tag{3.2}$$

Then there is $\delta_0 > 0$ such that the closest point mapping $p : U_{\delta_0}(\partial\Omega) \to \partial\Omega$ maps every x to precisely one point at $\partial\Omega$. We also define δ-neighbourhood of $\partial\Omega_D$ and $\partial\Omega_N$ as follows

$$U_\delta(\partial\Omega_D) = \{x \in U_\delta(\partial\Omega) : p(x) \in \partial\Omega_D\}, \qquad U_\delta(\partial\Omega_N) = U_\delta \setminus U_\delta(\partial\Omega_D) \tag{3.3}$$

Let $\Sigma = \partial(\partial\Omega_D) = \partial(\partial\Omega_N)$ be the smooth interface separating $\partial\Omega_D$ and $\partial\Omega_N$ and let ν be the unit conormal to Σ exterior to $\partial\Omega_N$ and tangent to $\partial\Omega$. See Figure 1. For $t \in [0, \delta_0]$ let

$$\partial\Omega_t = \{x \in \Omega : \rho(x) = t\} \tag{3.4}$$

$$\partial\Omega_{N,t} = \{x \in \partial\Omega_t : p(x) \in \partial\Omega_N\} \tag{3.5}$$

$$\Sigma_t = \{x \in \partial\Omega_t : p(x) \in \Sigma\} \tag{3.6}$$

Note that $p : \partial\Omega_t \to \partial\Omega$ is a bijection for all $t \in [0, \delta_0]$. Let

$$U_{t,\gamma}(\Sigma_t) = \{x \in \partial\Omega_{N,t} : \rho_{\Sigma_t}(x) < \gamma\} \subset \partial\Omega_{N,t} \tag{3.7}$$

be the γ tubular neighborhood of Σ_t in $\partial\Omega_{N,t}$, and assume that $\gamma \in (0, \gamma_0]$ with γ_0 small enough to guarantee that the closest point mappings p_{Σ_t} are well defined for all $t \in [0, \delta_0]$, and let

$$U_\gamma(\Sigma) = U_{0,\gamma}(\Sigma_0) \subset \partial\Omega_N \tag{3.8}$$

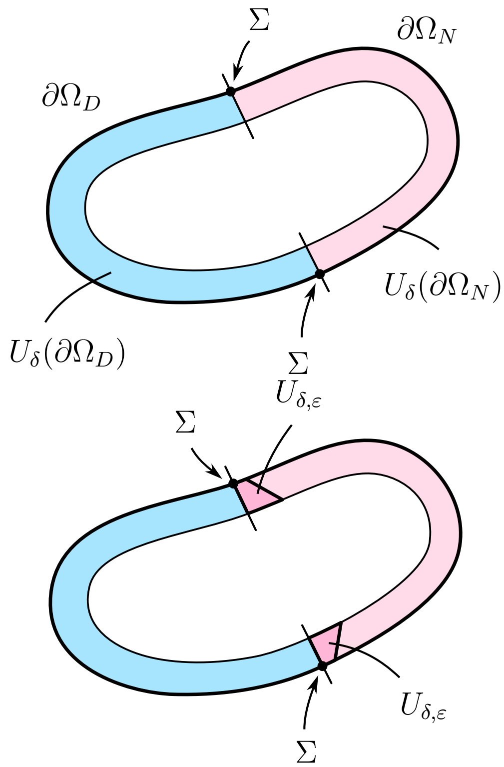

FIGURE 1. Left: the Dirichlet boundary $\partial\Omega_D$, the Neumann boundary $\partial\Omega_N$, the interface Σ, and the tubular neighborhood $U_\delta(\partial\Omega) = U_\delta(\partial\Omega_D) \cup U_\delta(\partial\Omega_N)$. Right: the set $U_{\delta,\epsilon} \subset U_\delta(\partial\Omega_N)$.

Define

$$U_{\delta,\epsilon} = \cup_{t\in[0,\delta]} U_{t,\gamma(t)}(\Sigma_t) \tag{3.9}$$

with $\gamma(t) = t + \epsilon$ for $\epsilon \in (0, \epsilon_0]$ and $\epsilon << \delta$, see Figure 2. Defining, for $z \in \Sigma$,

$$U_{\delta,\epsilon}(z) = \{x \in U_{\delta,\epsilon} : p_\Sigma(x) = z\} \tag{3.10}$$

where p_Σ is the closest point mapping associated with Σ, we have $U_{\delta,\epsilon} = \cup_{z\in\Sigma} U_{\delta,\epsilon}(z)$. Note that $U_{\delta,\epsilon}(z) = U_{\delta,\epsilon} \cap p_\Sigma^{-1}(z) \subset U_{\delta_0}(\Sigma) \cap p_\Sigma^{-1}(z)$, which is a subset of the 2 dimensional normal space $N_\Sigma(z)$ to the $d-2$ dimensional tangent space $T_\Sigma(z)$ of Σ at z. In the case $d = 2$, Σ consists of distinct points and in that case $U_{\delta,\epsilon} \subset U_{\delta_0}(z) \subset p_\Sigma^{-1}(z)$, for δ_0 small enough. Finally, let

$$U_\delta = U_\delta(\partial\Omega_D) \cup U_{\delta,\epsilon} \tag{3.11}$$

The Cut Off Function. We will below take $\delta \sim h$ and $\epsilon \sim h^\alpha$ with $\alpha = d$. Let $\chi : \Omega \to [0, 1]$ be smooth such that

$$\begin{cases} \chi = 1 & \text{on } \partial\Omega_D \\ \chi = 0 & \text{on } \partial\Omega_N \setminus U_\epsilon(\Sigma) \\ \chi = 0 & \text{on } \Omega \setminus U_\delta \end{cases} \qquad \begin{cases} \|\nabla\chi\|_{L^\infty(U_\delta \setminus U_{\delta,\epsilon})} \lesssim \delta^{-1} \\ \|\nabla_n\chi\|_{L^\infty(U_{\delta,\epsilon})} \lesssim \delta^{-1} \\ \|\nabla_\Sigma\chi\|_{L^\infty(U_{\delta,\epsilon})} \lesssim 1 \\ \|\nabla_\nu\chi\|_{L^\infty(U_{t,\gamma(t)})} \lesssim (\gamma(t))^{-1} \quad t \in [0,\delta] \end{cases} \tag{3.12}$$

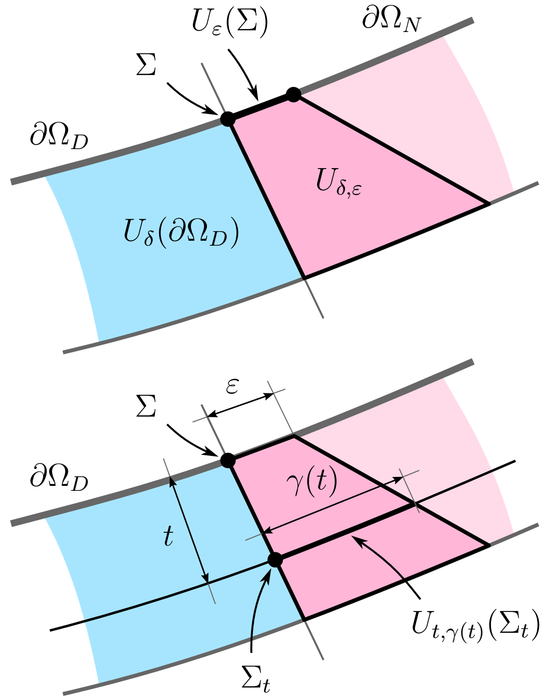

FIGURE 2. Left: Close up of the set $U_{\delta,\epsilon}$ including $U_\epsilon(\Sigma) \subset \partial\Omega_N$. Right: The set Σ_t and $U_{t,\gamma}(t)(\Sigma_t)$.

Observe that in the definition above ∇_Σ denotes the projection of the gradient on the tangent plane of Σ. By the construction of χ, $\|\nabla_\Sigma \chi\|_{L^\infty(U_{\delta,\epsilon})}$ is bounded and depends only on ϵ, δ and the regularity of Σ.

Lemma 3.1. *The cut off function χ satisfies the following estimate*

$$\sup_{z\in\Sigma} \|\nabla_\nu \chi\|^2_{U_{\delta,\epsilon}(z)} \lesssim |\ln(1+\delta/\epsilon)| \tag{3.13}$$

and with

$$\delta \sim h, \qquad \epsilon \sim h^\alpha \tag{3.14}$$

for $1 \leq \alpha \lesssim 1$ we obtain

$$\|\nabla_\nu \chi\|^2_{U_{\delta,\epsilon}} \lesssim 1 + |\ln(h)| \tag{3.15}$$

Proof. Using the bounds for $\nabla_\nu \chi$ we obtain

$$\|\nabla_\nu \chi\|^2_{U_{\delta,\epsilon}(z)} = \int_{U_{\delta,\epsilon}(z)} |\nabla_\nu \chi|^2 \lesssim \int_0^\delta \int_{U_{t,t+\epsilon}(z)} (t+\epsilon)^{-2} \tag{3.16}$$

$$\lesssim \int_0^\delta (t+\epsilon)^{-1} = [\ln(t+\epsilon)]_0^\delta = \ln(1+\delta/\epsilon) \tag{3.17}$$

Estimate (3.15) follows directly from the definition of δ and ϵ. $\square$

3.3. **The Regularized Problem.** For $\epsilon \in (0, \epsilon_0]$ define the regularized form

$$A_{h,\epsilon}(v, w) = (\nabla v, \nabla w)_\Omega - (\nabla_n v, w)_{\chi,\partial\Omega} - (v, \nabla_n w)_{\partial\Omega_D} + \beta h^{-1}(v, w)_{\partial\Omega_D} \tag{3.18}$$

and define $A_{h,0} = A_h$. We will show that the mapping $[0, \epsilon_0] \ni \epsilon \mapsto A_{h,\epsilon}$ is continuous for ϵ_0 small enough, see Lemma 3.3 below for details.

For $\epsilon \in [0, \epsilon_0]$, define the regularized finite element method: find $u_{h,\epsilon} \in V_h$ such that

$$A_{h,\epsilon}(u_{h,\epsilon}, v) + s_h(u_h, v) = L_h(v) \qquad \forall v \in V_h \tag{3.19}$$

This method is not consistent, but we have the identity

$$A_{h,\epsilon}(u - u_{h,\epsilon}, v) = A_{h,\epsilon}(u, v) - L_h(v) + s_h(u_h, v) \tag{3.20}$$

$$= s_h(u_h, v) - (g_N, v)_{\chi,\partial\Omega_N} \qquad \forall v \in V_h \tag{3.21}$$

since using Green's formula gives

$$A_{h,\epsilon}(u, v) = (\nabla u, \nabla v)_\Omega - (\nabla_n u, v)_{\chi,\partial\Omega} - (u, \nabla_n v)_{\partial\Omega_D} + \beta h^{-1}(u, v)_{\partial\Omega_D} \tag{3.22}$$

$$= -(\Delta u, v)_\Omega + (\nabla_n u, v)_{\partial\Omega} - (\nabla_n u, v)_{\chi,\partial\Omega} \tag{3.23}$$

$$- (u, \nabla_n v)_{\partial\Omega_D} + \beta h^{-1}(u, v)_{\partial\Omega_D} \tag{3.24}$$

$$= (f, v)_\Omega - (g_D, \nabla_n v)_{\partial\Omega_D} + \beta h^{-1}(g_D, v)_{\partial\Omega_D} + (g_N, v)_{\partial\Omega_N} - (g_N, v)_{\chi,\partial\Omega_N} \tag{3.25}$$

$$= L_h(v) - (g_N, v)_{\chi,\partial\Omega_N} \tag{3.26}$$

where we used the fact that $\chi = 1$ on $\partial\Omega_D$ to conclude that

$$(\nabla_n u, v)_{\partial\Omega} - (\nabla_n u, v)_{\chi,\partial\Omega} = (\nabla_n u, v)_{\partial\Omega_N} - (\nabla_n u, v)_{\chi,\partial\Omega_N} \tag{3.27}$$

$$= (g_N, v)_{\partial\Omega_N} - (g_N, v)_{\chi,\partial\Omega_N} \tag{3.28}$$

3.4. **Properties of the Bilinear Forms.** We here summarize the basic results on the bilinear forms and conclude with a proof of existence, uniqueness, and stability of the finite element solutions.

Inverse and Trace Inequalities. Let us recall some inverse and trace inequalities. Here $\mathbb{P}_1(T)$ denotes the set of polynomials of degree less than or equal to 1 on the simplex T.

- Inverse inequalities (see [6, Section 1.4.3]),

$$\|\nabla v\|_{H^1(T)} \lesssim h_T^{-1}\|v\|_{L^2(T)} \quad \forall v \in \mathbb{P}_1(T) \tag{3.29}$$

  and

$$\|v\|_{L^\infty(T)} \lesssim h^{-\frac{d}{2}}\|v\|_{L^2(T)} \quad \forall v \in \mathbb{P}_1(T) \tag{3.30}$$

- Trace inequalities (see [6, Section 1.4.3]),

$$\|v\|_{L^2(\partial T)} \leq C_T \left(h_T^{-1/2}\|v\|_{L^2(T)} + h_T^{1/2}\|\nabla v\|_T \right) \quad \forall v \in H^1(T) \tag{3.31}$$

  and

$$\|v\|_{L^2(\partial T)} \leq C_t h_T^{-1/2}\|v\|_{L^2(T)} \quad \forall v \in \mathbb{P}_1(T) \tag{3.32}$$

- Inverse trace inequality on cut elements. For a simplex T such that $T \cap \partial\Omega \neq \emptyset$, there holds

$$\|v\|_{L^2(T\cap\partial\Omega)} \lesssim h_T^{-1/2}\|v\|_{L^2(T)} \quad \forall v \in \mathbb{P}_1(T) \tag{3.33}$$

Stabilization Estimates. For any two elements T_1 and T_2 in $\mathcal{T}_h$, sharing a face F, we have the estimate

$$\|\nabla^m v\|^2_{T_1} \lesssim \|\nabla^m v\|^2_{T_2} + h^{3-2m}\|[\nabla_n v]\|^2_F \qquad m = 0, 1, \quad v \in V_h \tag{3.34}$$

Repeated use of (3.34) leads to

$$\|\nabla v\|^2_{\mathcal{T}_h} \lesssim \|\nabla v\|^2_\Omega + \|v\|^2_{s_h} \qquad v \in V_h \tag{3.35}$$

For sets $\omega_0 \subset \omega_1 \subset \Omega$ such that $\mathrm{diam}(\omega_1 \setminus \omega_0) \lesssim h$, we may also derive the estimate

$$\|\nabla^m v\|^2_{\mathcal{T}_h(\omega_1)} \lesssim \|\nabla^m v\|^2_{\mathcal{T}_h(\omega_0)} + h^{3-2m}\|[\nabla_n v]\|^2_{\mathcal{F}_h(\omega_1)} \qquad m = 0, 1, \quad v \in V_h \tag{3.36}$$

where $\mathcal{F}_h(\omega_1)$ denotes the interior faces of $\mathcal{T}_h(\omega_1)$.

The Energy Norm. We equip the finite element space V_h with the energy norm

$$|||v|||^2_h = \|\nabla v\|^2_\Omega + \|v\|^2_{s_h} + h^{-1}\|v\|^2_{\partial\Omega_D} \tag{3.37}$$

where $\|v\|^2_{s_h} := s_h(v, v)$. In order to have the normal flux well defined on the Dirichlet boundary we assume that

$$v \in V = \{v \in H^1(\Omega) : \Delta v|_{U_{\delta_0}} \in L^2(U_{\delta_0})\} \tag{3.38}$$

where we recall, see (3.11), that $\mathrm{supp}(\chi) \subset U_{\delta_0} = U_{\delta_0}(\partial\Omega_D) \cup U_{\delta_0,\epsilon_0}$ for all regularization parameters $\epsilon \in [0, \epsilon_0]$. The stabilization form s_h is not defined on V, due to the low regularity, and therefore we equip V with the weaker energy norm

$$|||v|||^2 = \|\nabla v\|^2_\Omega + h^{-1}\|v\|^2_{\partial\Omega_D} \tag{3.39}$$

Lemma 3.2. *There is constant such that for all $v \in V + V_h$, $w \in V_h$, and $\epsilon \in [0, \epsilon_0]$,*

$$\boxed{A_{h,\epsilon}(v, w) \lesssim |||v|||\,|||w|||_h + |(\nabla_n v, w)_{\chi,\partial\Omega}|} \tag{3.40}$$

where we use the norm $|||\cdot|||$, which does not include the stabilization, on $V + V_h$.

Proof. To verify this estimate we start from the definition (3.18) of the regularized form and using the Cauchy Schwarz inequality we get

$$A_{h,\epsilon}(v, w) \lesssim \|\nabla v\|_\Omega \|\nabla w\|_\Omega + |(\nabla_n v, w)_{\chi,\partial\Omega}| \tag{3.41}$$

$$\qquad + h^{-1/2}\|v\|_{\partial\Omega_D} h^{1/2}\|\nabla_n w\|_{\partial\Omega_D} + \beta h^{-1}\|v\|_{\partial\Omega_D}\|w\|_{\partial\Omega_D} \tag{3.42}$$

$$\lesssim |||v|||\,|||w|||_h + |(\nabla_n v, w)_{\chi,\partial\Omega}| \tag{3.43}$$

We estimated $h^{1/2}\|\nabla_n w\|_{\partial\Omega_D}$, with $w \in V_h$, using the inverse inequality

$$h\|\nabla_n w\|^2_{\partial\Omega_D} \lesssim \|\nabla v\|^2_{\mathcal{T}_h(\partial\Omega_D)} \lesssim \|\nabla w\|^2_{\mathcal{T}_h} \lesssim \|\nabla w\|^2_\Omega + \|w\|^2_{s_h} \lesssim |||w|||^2_h \tag{3.44}$$

where we first used the inverse trace inequality (3.33) and then the stabilization estimate (3.35). □

We will now prove a bound on the error introduced by replacing A_h by its regularized counterpart $A_{h,\epsilon}$.

Lemma 3.3. *There is a constant such that for all $v, w \in V_h$, and $\epsilon \in [0, \epsilon_0]$ with $\epsilon_0 \sim h$,*

$$\boxed{|A_{h,\epsilon}(v, w) - A_h(v, w)| \lesssim \epsilon h^{1-d}|||v|||_h|||w|||_h} \tag{3.45}$$

Proof. Using the definitions (2.16) and (3.18) of the forms A_h and $A_{h,\epsilon}$ we obtain

$$|A_h(v,w) - A_{h,\epsilon}(v,w)| = |(\nabla_n v, \chi w)_{\partial\Omega_N}| \tag{3.46}$$

$$\lesssim h^{1/2}\|\nabla_n v\|_{U_\epsilon(\Sigma)} h^{-1/2}\|w\|_{U_\epsilon(\Sigma)} \tag{3.47}$$

$$\lesssim \epsilon h^{1-d} |||v|||_h |||w|||_h \tag{3.48}$$

where we used the fact that $\text{supp}(\chi) \cap \partial\Omega_N \subset U_\epsilon(\Sigma)$, see (3.8). To estimate $h\|\nabla_n v\|^2_{U_\epsilon(\Sigma)}$ we proceed in the same way as in (3.44), we first use an inverse estimate and then the stablization (3.35),

$$h\|\nabla_n v\|^2_{U_\epsilon(\Sigma)} \lesssim h\|\nabla_n v\|^2_{\mathcal{T}_h(U_\epsilon(\Sigma))\cap\partial\Omega_N} \lesssim \|\nabla v\|^2_{\mathcal{T}_h(U_\epsilon(\Sigma))} \lesssim \|\nabla v\|^2_{\mathcal{T}_h} \lesssim |||v|||^2_{1,h} \tag{3.49}$$

Next to estimate $h^{-1}\|v\|^2_{U_\epsilon(\Sigma)}$ we pass over to the L^∞ norm in order to extract an ϵ factor and then we use suitable inverse bounds to pass to the energy norm.

$$h^{-1}\|v\|^2_{U_\epsilon(\Sigma)} \lesssim h^{-1}\epsilon\|v\|^2_{L^\infty(U_\epsilon(\Sigma))} \tag{3.50}$$

$$\lesssim h^{-1}\epsilon\|v\|^2_{L^\infty(\mathcal{T}_h(U_\epsilon(\Sigma)))} \tag{3.51}$$

$$\lesssim h^{-1}\epsilon h^{-d}\|v\|^2_{\mathcal{T}_h(U_\epsilon(\Sigma))} \tag{3.52}$$

$$\lesssim h^{-1}\epsilon h^{-d}\Big(h\|v\|^2_{\partial\Omega_D\cap\widetilde{\mathcal{T}}_h(U_\epsilon(\Sigma))} + h^2\|\nabla v\|^2_{\widetilde{\mathcal{T}}_h(U_\epsilon(\Sigma))}\Big) \tag{3.53}$$

$$\lesssim \epsilon h^{1-d}\Big(h^{-1}\|v\|^2_{\partial\Omega_D} + \|\nabla v\|^2_{\mathcal{T}_h}\Big) \tag{3.54}$$

$$\lesssim \epsilon h^{1-d}|||v|||^2_h \tag{3.55}$$

Here $\widetilde{\mathcal{T}}_h(U_\epsilon(\Sigma))$ is a slightly larger patch of elements such that the $d-1$ dimensional measure of its intersection with the Dirichlet boundary satisfies $|\widetilde{\mathcal{T}}_h(U_\epsilon(\Sigma))\cap\partial\Omega_D| \sim h^{d-1}$,which allows us to utilize the control available in $|||v|||_h$ at the Dirichlet boundary and to employ a Poincaré inequality in (3.53), see the appendix in [3]. The patch $\mathcal{T}_h(U_\epsilon(\Sigma))$ does not in general satisfy $\mathcal{T}_h(U_\epsilon(\Sigma)) \cap \partial\Omega_D \sim h^{d-1}$ and therefore it is enlarged by adding a suitable number of face neighboring elements in $\mathcal{T}_h(\partial\Omega_D)$. In the last step (3.55) we also used the stabilization (3.35). Note that due to the assumption that $\epsilon \in [0,\epsilon_0]$ with $\epsilon_0 \sim h$ it follows from shape regularity that there is a uniform bound on the number of elements in $\widetilde{\mathcal{T}}_h(U_\epsilon(\Sigma))$. □

Lemma 3.3 is instrumental for the coercivity that we prove next.

Lemma 3.4. *For β large enough and $\sigma > 0$, the forms $A_{h,\epsilon} + s_h$, $h \in (h, h_0]$, $\epsilon \in [0, ch^d]$ with c small enough, are coercive*

$$\boxed{|||v|||^2_h \lesssim A_{h,\epsilon}(v,v) + s_h(v,v) \qquad v \in V_h} \tag{3.56}$$

Proof. First we note that $A_{h,0}$ is coercive using standard techniques together with the inverse estimate (3.44). Next using the bound (3.45) of Lemma 3.3, we obtain

$$\begin{aligned}
A_{h,\epsilon}(v,v) &= A_{h,0}(v,v) + A_{h,\epsilon}(v,v) - A_{h,0}(v,v) \\
&\geq C_1|||v|||^2_h - |A_{h,\epsilon}(v,v) - A_{h,0}(v,v)| \\
&\geq (C_1 - C_2\epsilon h^{1-d})|||v|||^2_h \\
&\gtrsim |||v|||^2_h
\end{aligned}$$

where in the last step we choose $\epsilon \leq ch^d$ with $h \in (0, h_0]$ and c small enough. □

Using Lax-Milgram we conclude that for each $\epsilon \in [0, ch^d]$, there is a unique solution $u_{h,\epsilon} \in V_h$ to the regularized problem (3.19) such that

$$|||u_{h,\epsilon}|||_h \lesssim \sup_{v \in V_h \setminus \{0\}} L_h(v) \lesssim \|f\|_{H^{-1}(\Omega)} + \|g_N\|_{\widetilde{H}^{-1/2}(\partial\Omega_N)} + h^{-1/2}\|g_D\|_{\partial\Omega_D} \tag{3.57}$$

3.5. **Technical Lemmas.** In this section we collect some technical results that will be useful in the analysis. More precisely we start with four technical lemmas before proving Lemma 3.8 which is used to estimate the problematic term $(\nabla_n v, w)_{\chi,\partial\Omega}$ in the regularized problem.

Lemma 3.5. *There is a constant such that for all $v \in V_h$,*

$$\int_\Sigma \|v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \lesssim (1 + |\ln(h)|)\, |||v|||_h^2 \tag{3.58}$$

Proof. 1. Recall that for $z \in \Sigma$, $U_{\delta,\epsilon}(z) = \{x \in U_{\delta,\epsilon} : p_\Sigma(x) = z\}$, see (3.10), and we have $U_{\delta,\epsilon} = \cup_{z\in\Sigma} U_{\delta,\epsilon}(z)$. There are $\delta_0 \sim \epsilon_0 \sim 1$ such that $\delta \in (0, \delta_0]$, $\epsilon \in (0, \epsilon_0]$ and

$$U_{\delta,\epsilon}(z) \subset U_{\delta_0,\epsilon_0}(z) \tag{3.59}$$

We shall first show that there is a constant such that for all $z \in \Sigma$,

$$\|v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \lesssim (1 + |\ln(h)|)\|v\|^2_{H^1(U_{\delta_0,\epsilon_0}(z))} + h^2\|\nabla v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \tag{3.60}$$

To that end note that U_{δ_0,ϵ_0} has the following cone property: for each $x \in U_{\delta_0,\epsilon_0}(z)$ there is a cone (or sector since U_{δ_0,ϵ_0} is two dimensional) $\Lambda_{r_0}(x) \subset U_{\delta_0,\epsilon_0}(z)$, with vertex x, radius $r_0 \sim \delta_0 \sim 1$, and opening angle $\theta_0 \sim 1$. For $x \in U_{\delta_0,\epsilon_0}(z)$ and $r, \theta \in \Lambda_{r_0}(x)$ we have the identity

$$v(x) = v(r,\theta) - \int_0^r \partial_r v(s,\theta)ds \tag{3.61}$$

and the estimate

$$v^2(x) \lesssim v^2(r,\theta) + \left(\int_0^{r_0} \partial_r v(s,\theta)ds\right)^2 \tag{3.62}$$

We estimate the integral on the right hand side as follows

$$\left(\int_0^{r_0} \partial_r v(s,\theta)ds\right)^2 \lesssim \left(\int_0^{\eta h} \partial_r v(s,\theta)ds\right)^2 + \left(\int_{\eta h}^{r_0} \partial_r v(s,\theta)ds\right)^2 \tag{3.63}$$

$$\lesssim (\eta h)^2\|\nabla v\|^2_{L^\infty(\Lambda_{\eta h})} + |\ln(d/\eta h)| \int_{\eta h}^{r_0} (\partial_r v(s,\theta))^2 s ds \tag{3.64}$$

where for the second term on the right hand side we used the estimate

$$\left(\int_{\eta h}^{r_0} \partial_r v(s,\theta)ds\right)^2 \lesssim \int_{\eta h}^{r_0} s^{-1}ds \int_{\eta h}^{r_0} (\partial_r v(s,\theta))^2 s ds \tag{3.65}$$

$$\lesssim |\ln(d/\eta h)| \int_{\eta h}^{r_0} (\partial_r v(s,\theta))^2 s ds \tag{3.66}$$

Combining (3.62) and (3.64), we get

$$v^2(x) \lesssim v^2(r,\theta) + (\eta h)^2\|\nabla v\|^2_{L^\infty(\Lambda_{\eta h})} + |\ln(r_0/\eta h)| \int_{\eta h}^{r_0} (\partial_r v(s,\theta))^2 s ds \tag{3.67}$$

and integrating over $\Lambda_{r_0}(x)$ gives

$$|\Lambda_{r_0}|v^2(x) \lesssim \int_0^{r_0}\int_0^{\theta_0} v^2(r,\theta) r d\theta dr + |\Lambda_{r_0}|\,(\eta h)^2 \|\nabla v\|^2_{L^\infty(\Lambda_{\eta h}(x))} \tag{3.68}$$

$$+ |\ln(d/\eta h)| \int_0^{r_0}\int_0^{\theta_0} \left(\int_{\eta h}^{r_0} (\partial_r v(s,\theta))^2 s ds \right) r d\theta dr \tag{3.69}$$

$$\lesssim \|v\|^2_{\Lambda_{r_0}(x)} + |\Lambda_{r_0}|\,(\eta h)^2 \|\nabla v\|^2_{L^\infty(\Lambda_{\eta h}(x))} + d^2 |\ln(d/\eta h)|\, \|\nabla v\|^2_{\Lambda_{r_0}(x)} \tag{3.70}$$

Here $r_0 \sim 1$, and $|\Lambda_{r_0}| \sim r_0^2 \sim 1$ is independent of x, and thus we obtain

$$v^2(x) \lesssim \|v\|^2_{\Lambda_{r_0}(x)} + |\ln(d/\eta h)| \|\nabla v\|^2_{\Lambda_{r_0}(x)} + (\eta h)^2 \|\nabla v\|^2_{L^\infty(\Lambda_{\eta h}(x))} \tag{3.71}$$

which leads to

$$\|v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \lesssim (1 + |\ln(h)|) \|v\|^2_{H^1(U_{\delta_0,\epsilon_0}(z))} + h^2 \|\nabla v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \tag{3.72}$$

and thus (3.60) holds.

2. $d = 2$. In the two dimensional case $d = 2$, the interface Σ consist of a set of isolated points and we may cover the two dimensional set $U_{\delta_0,\epsilon_0}(z)$ by a patch of elements $\mathcal{T}_h(U_{\delta_0,\epsilon_0})$, and then apply the element wise inverse inequality (3.30),

$$\|v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \lesssim (1 + |\ln(h)|) \|v\|^2_{H^1(U_{\delta_0,\epsilon_0}(z))} + h^2 \|\nabla v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \tag{3.73}$$

$$\lesssim (1 + |\ln(h)|) \|v\|^2_{H^1(\mathcal{T}_h(U_{\delta_0,\epsilon_0}(z)))} + h^2 \|\nabla v\|^2_{L^\infty(\mathcal{T}_h(U_{\delta_0,\epsilon_0}(z)))} \tag{3.74}$$

$$\lesssim (1 + |\ln(h)|) \|v\|^2_{H^1(\mathcal{T}_h(U_{\delta_0,\epsilon_0}(z)))} + \|\nabla v\|^2_{\mathcal{T}_h(U_{\delta_0,\epsilon_0}(z)))} \tag{3.75}$$

$$\lesssim (1 + |\ln(h)|) |||v|||^2_h \tag{3.76}$$

where we finally used the stabilization estimate (3.35). This completes the proof in the case $d = 2$.

3. $d \geq 3$. Here, the set $U_{\delta_0,\epsilon_0}(z)$, for a given $z \in \Sigma$, is a subset of a two dimensional plane, that cuts through the d dimensional elements in a general way, which requires a more refined argument since an element wise trace inequality can not be applied due to the presence of cut elements. We start by integrating (3.60) over Σ,

$$\int_\Sigma \|v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \lesssim (1 + |\ln(h)|) \int_\Sigma \|v\|^2_{H^1(U_{\delta_0,\epsilon_0}(z))} + h^2 \int_\Sigma \|\nabla v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \tag{3.77}$$

$$\lesssim (1 + |\ln(h)|) \|v\|^2_{H^1(\mathcal{T}_h(U_{\delta_0,\epsilon_0}))} + \|\nabla v\|^2_{\mathcal{T}_h(U_{\delta_0,\epsilon_0})} \tag{3.78}$$

$$\lesssim (1 + |\ln(h)|) \|v\|^2_{H^1(\mathcal{T}_h(U_{\delta_0,\epsilon_0}))} \tag{3.79}$$

$$\lesssim (1 + |\ln(h)|) |||v|||^2_h \tag{3.80}$$

Here we used the inverse estimate

$$h^2 \int_\Sigma \|\nabla v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \lesssim \|\nabla v\|^2_{\mathcal{T}_h(U_{\delta_0,\epsilon_0})} \tag{3.81}$$

To verify (3.81) we first note that, with $w = \nabla v$, we have for each $z \in \Sigma$,

$$\|w\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} = \max_{T \in \mathcal{T}_h(U_{\delta_0,\epsilon_0}(z))} \|w\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z)\cap T)} \tag{3.82}$$

$$\lesssim \sum_{T \in \mathcal{T}_h(U_{\delta_0,\epsilon_0}(z))} \|w\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z)\cap T)} \tag{3.83}$$

$$\lesssim \sum_{T \in \mathcal{T}_h(U_{\delta_0,\epsilon_0})} \|w\|^2_{L^\infty(T)} 1_T(z) \tag{3.84}$$

$$\lesssim \sum_{T \in \mathcal{T}_h(U_{\delta_0,\epsilon_0})} h^{-d} \|w\|^2_T 1_T(z) \tag{3.85}$$

where $1_T(z) = 1$ if $U_{\delta_0,\epsilon_0}(z) \cap T \neq \emptyset$ and 0 otherwise, and we employed an inverse inequality in the last step. We next note that $1_T : \Sigma \to \{0,1\}$ is the characteristic function of the closest point projection $p_\Sigma(T)$ of T on Σ, and therefore

$$\int_\Sigma 1_T \lesssim h^{d-2} \tag{3.86}$$

Integrating, (3.85) over Σ we get

$$\int_\Sigma \|w\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \lesssim \int_\Sigma \sum_{T \in \mathcal{T}_h(U_{\delta_0,\epsilon_0})} h^{-d} \|w\|^2_T 1_T(z) \tag{3.87}$$

$$\lesssim \sum_{T \in \mathcal{T}_h(U_{\delta_0,\epsilon_0})} h^{-d} \|w\|^2_T \int_\Sigma 1_T(z) \tag{3.88}$$

$$= h^{-2} \|w\|^2_{\mathcal{T}_h(U_{\delta_0,\epsilon_0})} \tag{3.89}$$

where we used (3.86). This completes the verification of (3.81).

∎

Lemma 3.6. *Let χ be defined by (3.12), then there is a constant such that for all $v \in V_h$,*

$$\boxed{\|(\nabla \chi) v\|_{U_{\delta,\epsilon}} \lesssim (1 + |\ln(h)|) |||v|||_h} \tag{3.90}$$

Proof. Splitting $\|(\nabla \chi) v\|^2_{U_{\delta,\epsilon}}$ into three contributions corresponding to the directions of the derivative relative to the interface Σ we obtain

$$\|(\nabla \chi) v\|^2_{U_{\delta,\epsilon}} \lesssim \|(\nabla_\Sigma \chi) v\|^2_{U_{\delta,\epsilon}} + \|(\nabla_n \chi) v\|^2_{U_{\delta,\epsilon}} + \|(\nabla_\nu \chi) v\|^2_{U_{\delta,\epsilon}} \tag{3.91}$$

$$\lesssim \|v\|^2_{U_{\delta,\epsilon}} + \delta^{-2} \|v\|^2_{U_{\delta,\epsilon}} + \|(\nabla_\nu \chi) v\|^2_{U_{\delta,\epsilon}} \tag{3.92}$$

$$\lesssim \|v\|^2_{U_{\delta,\epsilon}} + \int_\Sigma \|v\|^2_{L^\infty(U_{\delta,\epsilon}(z))} + (1 + |\ln(h)|)^2 |||v|||^2_h \tag{3.93}$$

$$\lesssim (1 + |\ln(h)|^2)\, |||v|||^2_h \tag{3.94}$$

where we for the second term (3.92) used the facts $|U_{\delta,\epsilon}(z))| \lesssim \delta^2 \lesssim h^2$, $\|v\|_{L^\infty(U_{\delta,\epsilon}(z))} \leq \|v\|_{L^\infty(U_{\delta_0,\epsilon_0}(z))}$ followed by (3.58), and for the third term we used the estimate

$$\|(\nabla_\nu \chi) v\|_{U_{\delta,\epsilon}} \lesssim (1 + |\ln(h)|) |||v|||_h \tag{3.95}$$

which we verify next. This argument completes the proof of (3.90).

To verify (3.95) we use Hölder's inequality twice, first on $U_{\delta,\epsilon}(z)$ and then on Σ, employ (3.15), and finally (3.58),

$$\|(\nabla_\nu \chi) v\|^2_{U_{\delta,\epsilon}} = \int_\Sigma \|(\nabla_\nu \chi) v\|^2_{U_{\delta,\epsilon}(z)} \tag{3.96}$$

$$\lesssim \int_\Sigma \|\nabla_\nu \chi\|^2_{U_{\delta,\epsilon}(z)} \|v\|^2_{L^\infty(U_{\delta,\epsilon}(z))} \tag{3.97}$$

$$\lesssim \Big(\sup_{z\in\Sigma} \|\nabla_\nu \chi\|^2_{U_{\delta,\epsilon}(z)} \Big) \int_\Sigma \|v\|^2_{L^\infty(U_{\delta,\epsilon}(z))} \tag{3.98}$$

$$\lesssim (1+|\ln(h)|) \int_\Sigma \|v\|^2_{L^\infty(U_{\delta,\epsilon}(z))} \tag{3.99}$$

$$\lesssim (1+|\ln(h)|) \int_\Sigma \|v\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \tag{3.100}$$

$$\lesssim (1+|\ln(h)|)^2 |||v|||^2_h \tag{3.101}$$

Thus (3.95) holds. □

Lemma 3.7. *There is a constant such that for all $w \in V_h$,*

$$h^{-2}\|w\|^2_{\mathcal{T}_h(U_\delta)} + \|\nabla w\|^2_{\mathcal{T}_h(U_\delta)} \lesssim |||w|||^2_h \tag{3.102}$$

which holds for $\delta = \eta h$ with η a sufficiently small constant.

Proof. First observe that by construction no point in U_δ is further than $O(\delta)$ from $\partial\Omega_D$. Using estimate (3.36) followed by the Poincaré inequality

$$\|w\|^2_{\mathcal{T}_h(U_\delta(\partial\Omega_D))} \lesssim \delta \|w\|_{\partial\Omega_D} + \delta^2 \|\nabla w\|_{\mathcal{T}_h(U_\delta(\partial\Omega_D))} \tag{3.103}$$

see appendix [3], we obtain

$$\|w\|^2_{\mathcal{T}_h(U_\delta)} \lesssim \|w\|^2_{\mathcal{T}_h(U_\delta(\partial\Omega_D))} + \|w\|^2_{\mathcal{T}_h(U_{\delta,\epsilon})} \tag{3.104}$$

$$\lesssim \|w\|^2_{\mathcal{T}_h(U_\delta(\partial\Omega_D))} + h^3 \|[\nabla_n w]\|^2_{\mathcal{F}_h(\partial\Omega\cap U_\delta)} \tag{3.105}$$

$$\lesssim \delta \|w\|_{\partial\Omega_D} + \delta^2 \|\nabla w\|_{\mathcal{T}_h(U_\delta(\partial\Omega_D))} + h^2 \|\nabla w\|^2_{\mathcal{T}_h(\partial\Omega\cap U_\delta)} \tag{3.106}$$

where we used the estimate

$$h\|[\nabla_n w]\|^2_{\mathcal{F}_h(\partial\Omega\cap U_\delta)} \lesssim \|\nabla v\|^2_{\mathcal{T}_h} \tag{3.107}$$

Applying now (3.35) and using $\delta \sim h$ we conclude that

$$h^{-2}\|w\|^2_{\mathcal{T}_h(U_\delta)} + \|\nabla w\|^2_{\mathcal{T}_h(U_\delta)} \lesssim h^{-1}\|w\|^2_{\partial\Omega_D} + \|\nabla w\|^2_\Omega + \|w\|^2_{s_h} \lesssim |||w|||^2_h \tag{3.108}$$

□

Lemma 3.8. *There is a constant such that for all $v \in V$, $v_h \in V_h$, and $w \in V_h$,*

$$\boxed{\begin{aligned}(\nabla_n(v-v_h), w)_{\chi,\partial\Omega} \lesssim \Big(&(1+|\ln(h)|)\|\nabla(v-v_h)\|_{U_\delta} \\ &+ h\|\Delta v\|_{U_\delta} + h^{1/2}\|[\nabla_n v_h]\|_{\mathcal{F}_h\cap U_\delta}\Big)|||w|||_h\end{aligned}} \tag{3.109}$$

Proof. For $v \in V$, see (3.38), we have $\Delta v \in L^2(\mathrm{supp}(\chi)) \subset L^2(U_{\delta_0})$ and using Green's formula

$$(\Delta v, \chi w)_\Omega = (\nabla_n v, \chi w)_{\partial\Omega} - (\nabla v, (\nabla\chi) w)_\Omega - (\nabla v, \chi \nabla w)_\Omega \tag{3.110}$$

For $v_h \in V_h$ we use Green's formula element wise

$$(\nabla v_h, \chi \nabla w)_\Omega = (\nabla_n v_h, \chi w)_{\partial\Omega} + ([\nabla_n v_h], \chi w)_{\mathcal{F}_h \cap \Omega} \tag{3.111}$$

$$- (\Delta v_h, \chi w)_{\mathcal{T}_h \cap \Omega} - (\nabla v_h, (\nabla \chi) w)_\Omega \tag{3.112}$$

Combining the formulas and rearranging the terms we obtain

$$(\nabla_n (v - v_h), w)_{\chi, \partial\Omega} = (\nabla (v - v_h), \chi \nabla w)_\Omega + (\nabla (v - v_h), (\nabla \chi) w)_\Omega \tag{3.113}$$

$$+ (\Delta v, \chi w)_\Omega + ([\nabla_n v_h], \chi w)_{\mathcal{F}_h \cap \Omega} \tag{3.114}$$

To estimate the right hand side we may directly estimate the first two terms using the Cauchy Schwarz inequality and (3.102),

$$(\nabla (v - v_h), \chi \nabla w)_\Omega \lesssim \|\nabla (v - v_h)\|_{U_\delta} \|\nabla w\|_{U_\delta} \lesssim \|\nabla (v - v_h)\|_{U_\delta} |||w|||_h \tag{3.115}$$

$$(\Delta v, \chi w)_\Omega \lesssim h \|\Delta v\|_{U_\delta} h^{-1} \|w\|_{U_\delta} \lesssim h \|\Delta v\|_{U_\delta} |||w|||_h \tag{3.116}$$

Next using the Cauchy Schwarz inequality, the element wise trace inequality (3.31),

$$([\nabla_n v_h], \chi w)_{\mathcal{F}_h \cap \Omega} \lesssim h^{1/2} \|[\nabla_n v_h]\|_{\mathcal{F}_h \cap U_\delta} h^{-1/2} (h^{-1} \|w\|^2_{\mathcal{T}_h(U_\delta)} + h \|\nabla w\|^2_{\mathcal{T}_h(U_\delta)})^{1/2} \tag{3.117}$$

$$\lesssim h^{1/2} \|[\nabla_n v_h]\|_{\mathcal{F}_h \cap U_\delta} (h^{-2} \|w\|^2_{\mathcal{T}_h(U_\delta)} + \|\nabla w\|^2_{\mathcal{T}_h(U_\delta)})^{1/2} \tag{3.118}$$

$$\lesssim h^{1/2} \|[\nabla_n v_h]\|_{\mathcal{F}_h \cap U_\delta} |||w|||_h \tag{3.119}$$

where for the last inequality we employed (3.102). For the remaining term we use the Cauchy Schwarz inequality, followed by (3.102) and (3.90),

$$(\nabla (v - v_h), (\nabla \chi) w)_\Omega \lesssim \|\nabla (v - v_h)\|_{U_\delta} \Big(\|(\nabla \chi) w\|_{U_\delta(\partial\Omega_D)} + \|(\nabla \chi) w\|_{U_{\delta,\epsilon}} \Big) \tag{3.120}$$

$$\lesssim \|\nabla (v - v_h)\|_{U_\delta} \Big(\delta^{-1} \|w\|_{U_\delta(\partial\Omega_D)} + \|(\nabla \chi) w\|_{U_{\delta,\epsilon}} \Big) \tag{3.121}$$

$$\lesssim (1 + |\ln(h)|) \|\nabla (v - v_h)\|_{U_\delta} |||w|||_h \tag{3.122}$$

Collecting the bounds we arrive at

$$(\nabla_n (v - v_h), w)_{\partial\Omega} \lesssim \Big((1 + |\ln(h)|) \|\nabla (v - v_h)\|_{U_\delta} \tag{3.123}$$

$$+ h \|\Delta v\|_{U_\delta} + h^{1/2} \|[\nabla_n v_h]\|_{\mathcal{F}_h \cap U_\delta} \Big) |||w|||_h \tag{3.124}$$

which completes the proof of (3.109).

□

3.6. **Interpolation.** Let $E : H^s(\Omega) \to H^s(\mathbb{R}^d)$ be a continuous extension operator. Define the interpolant $\pi_h : H^1(\Omega) \to V_h$ by $\pi_h = \pi_{h,Cl} \circ E$ where $\pi_{h,Cl} : L^2(\Omega_h) \to V_h$ is the Clement interpolant and $\Omega_h = \cup_{T \in \mathcal{T}_h} T$. Using the interpolation results for the Clement interpolation operator and the stability of the extension operator we conclude that

$$\|v - \pi_h v\|_{H^m(\Omega)} \lesssim h^{s-m} \|v\|_{H^s(\Omega)} \qquad 0 \leq m \leq s \leq 2 \tag{3.125}$$

For the energy norm (3.39) it holds

$$\boxed{|||v - \pi_h v||| + \|\pi_h v\|_{s_h} \lesssim h^{s-1} \|v\|_{H^s(\Omega)}} \tag{3.126}$$

Proof. With $\rho = v - \pi_h v$ we have

$$|||\rho|||_{0,h}^2 \lesssim \|\nabla\rho\|_\Omega^2 + h^{-1}\|\rho\|_{\partial\Omega_D}^2 \tag{3.127}$$

Using (3.125) we directly have

$$\|\nabla\rho\|_\Omega^2 \lesssim h^{2(s-1)}\|u\|_{H^s(\Omega)}^2 \tag{3.128}$$

and using the trace inequality

$$\|v\|_{\partial\Omega_D}^2 \lesssim \delta^{-1}\|v\|_{U_\delta(\partial\Omega_D)}^2 + \delta\|\nabla v\|_{U_\delta(\partial\Omega_D)}^2 \tag{3.129}$$

with $\delta \sim h$ we obtain

$$h^{-1}\|\rho\|_{\partial\Omega_D}^2 \lesssim h^{-1}(\delta^{-1}\|\rho\|_{U_\delta(\partial\Omega_D)}^2 + \delta\|\nabla\rho\|_{U_\delta(\partial\Omega_D)}^2) \tag{3.130}$$

$$\lesssim h^{-2}\|\rho\|_{U_\delta(\partial\Omega_D)}^2 + \|\nabla\rho\|_{U_\delta(\partial\Omega_D)}^2 \tag{3.131}$$

$$\lesssim h^{2(s-1)}\|v\|_{H^s(\Omega)}^2 \tag{3.132}$$

Finally, we have with $\pi_{h,Cl}\nabla Ev \in V_h^d$,

$$\|\pi_h v\|_{s_h}^2 \lesssim h\|[\nabla\pi_h v - \pi_{h,Cl}\nabla Ev]\|_{\mathcal{F}_h}^2 \tag{3.133}$$

$$\lesssim \|\nabla\pi_h v - \pi_{h,Cl}\nabla Ev\|_{\mathcal{T}_h}^2 \tag{3.134}$$

$$\lesssim \|\nabla_n(\pi_h v - v)\|_{\mathcal{T}_h}^2 + \|\pi_{h,Cl}\nabla Ev - \nabla Ev\|_{\mathcal{T}_h}^2 \lesssim h^{2(s-1)}\|v\|_{H^s(\Omega)}^2 \tag{3.135}$$

In the first inequality the inverse inequality

$$h\|[\nabla w]\|_F^2 \lesssim \|\nabla w\|_{T_1}^2 + \|\nabla w\|_{T_2}^2, \qquad w \in V_h|_{T_1\cup T_2} \tag{3.136}$$

where T_1 and T_2 are the two elements that share face F. □

3.7. **Error Estimates.**

Theorem 3.9. *Let $u \in H^s(\Omega)$, $s \in [1, 3/2]$, be the solution to (1.1)-(1.2) and u_h the finite element approximation defined by (2.20), then*

$$\begin{aligned}|||u - u_h||| + \|u_h\|_{s_h} \lesssim h^{s-1}\Big((1 + |\ln(h)|)\|u\|_{H^s(\Omega)} + \|g_N\|_{\widetilde{H}^{s-3/2}(\partial\Omega_N)}\Big)\\ + h\Big(\|f\|_{U_\delta} + \|f\|_{H^{-1}(\Omega)} + \|g_N\|_{\widetilde{H}^{-1/2}(\partial\Omega_N)} + \|g_D\|_{H^{1/2}(\partial\Omega_D)}\Big)\end{aligned}$$

The logarithmic factor is present only for the case of mixed Dirichlet-Neumann boundary conditions.

Proof. We split the error as follows

$$\begin{aligned}|||u - u_h||| + \|u_h\|_{s_h} &\lesssim |||u - \pi_h u|||_h + |||\pi_h u - u_h|||_h + \|u_h\|_{s_h}\\ &\lesssim \underbrace{|||u - \pi_h u|||_h}_{\lesssim h^{s-1}\|u\|_{H^s(\Omega)}} + \underbrace{|||\pi_h u - u_{h,\epsilon}|||_h}_{I} + \underbrace{|||u_{h,\epsilon} - u_h|||_h}_{II} + \underbrace{\|u_h\|_{s_h}}_{III}\end{aligned}$$

where $u_{h,\epsilon}$ is the solution to the regularized problem (3.19) and we used the interpolation error estimate (3.126) to estimate the first term on the right hand side.
Term *I*. The following estimate holds

$$|||\pi_h u - u_{h,\epsilon}|||_h \lesssim (1+|\ln(h)|)h^{s-1}\Big(\|u\|_{H^s(\Omega)} + \|g_N\|_{\widetilde{H}^{s-3/2}(\partial\Omega_N)}\Big) + h\|f\|_{U_\delta} \tag{3.137}$$

To verify the estimate let $\rho_h = \pi_h u - u_{h,\epsilon}$. Using coercivity (3.56) we obtain

$$|||\rho_h|||_h^2 \lesssim A_{h,\epsilon}(\rho_h, \rho_h) + s_h(\rho_h, \rho_h)$$

and then employing the definition (3.19) of $u_{h,\epsilon}$ we obtain

(3.138)
$$A_{h,\epsilon}(\pi_h u - u_{h,\epsilon}, \rho_h) + s_h(\pi_h u - u_{h,\epsilon}, \rho_h)$$
(3.139)
$$= A_{h,\epsilon}(\pi_h u, \rho_h) - L_h(\rho_h) + s_h(\pi_h u, \rho_h)$$
(3.140)
$$= A_{h,\epsilon}(\pi_h u - u, \rho_h) + A_{h,\epsilon}(u, \rho_h) - L_h(\rho_h) + s_h(\pi_h u, \rho_h)$$
(3.141)
$$\lesssim (|||\pi_h u - u||| + \|\pi_h u\|_{s_h})|||\rho_h|||_h + |(\nabla_n(\pi_h u - u), \rho_h)_{\chi,\partial\Omega}|$$
(3.142)
$$+ |A_{h,\epsilon}(u, \rho_h) - L_h(\rho_h)|$$
(3.143)
$$\lesssim h^{s-1}\|u\|_{H^s(\Omega)}|||\rho_h|||_h + (1 + |\ln(h)|)h^{s-1}\|u\|_{H^s(\Omega)} + h\|f\|_{U_\delta})|||\rho_h|||_h$$
(3.144)
$$+ (1 + |\ln(h)|)|h^{s-1}\|g_N\|_{\widetilde{H}^{s-3/2}(\partial\Omega_N)}|||\rho_h|||_h$$

where we used the continuity (3.40) in (3.141), and in (3.143) we used the interpolation error estimate (3.126) to estimate the first term and then the following estimates

$$|(\nabla_n(\pi_h u - u), \rho_h)_{\chi,\partial\Omega}| \lesssim \Big((1 + |\ln(h)|)h^{s-1}\|u\|_{H^s(\Omega)} + h\|f\|_{U_\delta}\Big)|||\rho_h|||_h \tag{3.145}$$
$$|A_{h,\epsilon}(u, \rho_h) - L_h(\rho_h)| \lesssim (1 + |\ln(h)|)h^{s-1}\|g_N\|_{\widetilde{H}^{s-3/2}(\partial\Omega_N)}|||\rho_h|||_h \tag{3.146}$$

(3.145). Using (3.109) followed by the interpolation estimate (3.126),

$$|(\nabla_n(\pi_h u - u), \rho_h)_{\chi,\partial\Omega}| \tag{3.147}$$
$$\lesssim \Big((1 + |\ln(h)|)\|\nabla(u - \pi_h u)\|_{U_\delta} \tag{3.148}$$
$$+ h\|\Delta u\|_{U_\delta} + h^{1/2}\|[\nabla_n \pi_h u]\|_{\mathcal{F}_h \cap U_\delta}\Big)|||\rho_h|||_h \tag{3.149}$$
$$\lesssim \Big((1 + |\ln(h)|)h^{s-1}\|u\|_{H^s(\Omega)} + h\|f\|_{U_\delta}\Big)|||\rho_h|||_h \tag{3.150}$$

where we used the fact $\Delta u = -f$. **(3.146).** Starting from the identity (3.21) we get

$$|A_{h,\epsilon}(u, \rho_h) - L_h(\rho_h)| = |(g_N, \chi\rho_h)_{\partial\Omega_N}| \tag{3.151}$$
$$\lesssim \|g_N\|_{\widetilde{H}^{s-3/2}(\partial\Omega_N)}\|\chi\rho_h\|_{H^{3/2-s}(\partial\Omega_N)} \tag{3.152}$$

To estimate $\|\chi\rho_h\|_{H^{3/2-s}(\partial\Omega_N)}$ we use a trace inequality on $U_{\delta_0}(\partial\Omega_N))$,

$$\|\chi\rho_h\|_{H^{3/2-s}(\partial\Omega_N)} \lesssim \|\chi\rho_h\|_{H^{2-s}(U_{\delta_0}(\partial\Omega_N))} \tag{3.153}$$

In order to estimate the right hand side using the available bounds we employ the interpolation between norms estimate

$$\|v\|_{H^\gamma(\omega)} \lesssim \|v\|^{1-t}_{H^{s_1}(\omega)}\|v\|^t_{H^{s_2}(\omega)} \tag{3.154}$$

for $t \in [0, 1]$ and $\gamma = (1 - t)s_1 + ts_2$. In our case $\gamma = 2 - s \in [1/2, 1]$ and we take $s_1 = 0$ and $s_2 = 1$, which gives $t = 2-s$. Observing that $\text{supp}(\chi) \cap U_{\delta_0}(\partial\Omega_N) \subset U_{\delta,\epsilon}$

we get

$$\|\chi\rho_h\|_{H^{2-s}(U_{\delta,\epsilon})} \lesssim \|\chi\rho_h\|^{s-1}_{H^0(U_{\delta,\epsilon})}\|\chi\rho_h\|^{2-s}_{H^1(U_{\delta,\epsilon})} \tag{3.155}$$

$$\lesssim \Big((1+|\ln(h)|)h|||\rho_h|||_h\Big)^{s-1}\Big((1+|\ln(h)|)|||\rho_h|||_h\Big)^{2-s} \tag{3.156}$$

$$\lesssim (1+|\ln(h)|)h^{s-1}|||\rho_h|||_h \tag{3.157}$$

Here we used the following two estimates. First

$$\|\rho_h\|^2_{U_{\delta,\epsilon}} = \int_\Sigma \|\rho_h\|^2_{U_{\delta,\epsilon}(z)} \tag{3.158}$$

$$\lesssim \int_\Sigma h^2\|\rho_h\|^2_{L^\infty(U_{\delta,\epsilon}(z))} \tag{3.159}$$

$$\lesssim \int_\Sigma h^2\|\rho_h\|^2_{L^\infty(U_{\delta_0,\epsilon_0}(z))} \tag{3.160}$$

$$\lesssim h^2(1+|\ln(h)|)|||\rho_h|||^2_{1,h} \tag{3.161}$$

where we at last used (3.58). Second

$$\|\chi\rho_h\|_{H^1(U_{\delta,\epsilon})} \lesssim \|\chi\rho_h\|_{U_{\delta,\epsilon}} + \|(\nabla\chi)\rho_h\|_{U_{\delta,\epsilon}} + \|\chi\nabla\rho_h\|_{U_{\delta,\epsilon}} \tag{3.162}$$

$$\lesssim (1+|\ln(h)|)|||\rho_h|||_h \tag{3.163}$$

where we used (3.90) and (3.102). This completes the bound for Term I.
Term II. For $\epsilon \sim h^\alpha$ with $\alpha = d$, we shall prove the estimate

$$|||u_{h,\epsilon} - u_h|||_h \lesssim h\Big(\|f\|_{H^{-1}(\Omega)} + \|g_N\|_{\widetilde{H}(\partial\Omega_N)} + \|g_D\|_{\partial\Omega_D}\Big) \tag{3.164}$$

We start once again with coercivity, this time of $A_h + s_h$, using the notation $\zeta_h = u_{h,\epsilon} - u_h$ we have

$$|||\zeta_h|||^2_h \lesssim A_h(\zeta_h,\zeta_h) + s_h(\zeta_h,\zeta_h) \tag{3.165}$$

Then using the definition of the method and estimate (3.45) we obtain

$$|||\zeta_h|||^2_h \lesssim A_h(u_{h,\epsilon} - u_h,\zeta_h) + s_h(u_{h,\epsilon} - u_h,\zeta_h) \tag{3.166}$$

$$= A_h(u_{h,\epsilon},\zeta_h) + s_h(u_{h,\epsilon},\zeta_h) - L_h(\zeta_h) \tag{3.167}$$

$$= A_h(u_{h,\epsilon},\zeta_h) - A_{h,\epsilon}(u_{h,\epsilon},\zeta_h) \tag{3.168}$$

$$\lesssim \epsilon h^{1-d}|||u_{h,\epsilon}|||_h|||\zeta_h|||_h \tag{3.169}$$

$$\lesssim h^{\alpha+1-d}\Big(\|f\|_{H^{-1}(\Omega)} + \|g_N\|_{\widetilde{H}^{-1/2}(\partial\Omega_N)} + h^{-1/2}\|g_D\|_{H^{1/2}(\partial\Omega_D)}\Big)|||\zeta_h|||_h \tag{3.170}$$

$$\lesssim h\Big(\|f\|_{H^{-1}(\Omega)} + \|g_N\|_{\widetilde{H}^{-1/2}(\partial\Omega_N)} + \|g_D\|_{H^{1/2}(\partial\Omega_D)}\Big)|||\zeta_h|||_h \tag{3.171}$$

for $\alpha = d$, where we used the stability estimate (3.57).
Term III. We finally have the following estimate for the stabilization term

$$\|u_h\|_{s_h} \leq \|u_h - u_{h,\epsilon}\|_{s_h} + \|\pi_h u - u_{h,\epsilon}\|_{s_h} + \|\pi_h u\|_{s_h} \tag{3.172}$$

$$= |||\zeta_h|||_h + |||\rho_h|||_h + \|\pi_h u\|_{s_h} \tag{3.173}$$

where the first two terms are estimated in (3.164) and (3.137) and the third by the interpolation estimate (3.126).
Conclusion. The theorem now follows by collecting the bounds for the terms I, II, and III. □

Remark 3.1. *Observe that the logarithmic factor can be traced to Lemma 3.5, Lemma 3.6 and* (3.146) *all of which are invoked only for the case of mixed boundary conditions*

Acknowledgements. This research was supported in part by the Swedish Research Council Grants Nos. 2017-03911, 2018-05262, 2021-04925, and the Swedish Research Programme Essence. EB was supported in part by the EPSRC grant EP/P01576X/1.

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References

[1] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing. CutFEM: discretizing geometry and partial differential equations. *Internat. J. Numer. Methods Engrg.*, 104(7):472–501, 2015.

[2] E. Burman and P. Hansbo. Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method. *Appl. Numer. Math.*, 62(4):328–341, 2012.

[3] E. Burman, P. Hansbo, and M. G. Larson. A cut finite element method with boundary value correction. *Math. Comp.*, 87(310):633–657, 2018.

[4] E. Burman, P. Hansbo, M. G. Larson, and S. Zahedi. Cut finite element methods for coupled bulk-surface problems. *Numer. Math.*, 133(2):203–231, 2016.

[5] W. Dahmen, B. Faermann, I. G. Graham, W. Hackbusch, and S. A. Sauter. Inverse inequalities on non-quasi-uniform meshes and application to the mortar element method. *Math. Comp.*, 73(247):1107–1138, 2004.

[6] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Heidelberg, 2012.

[7] A. Ern and J.-L. Guermond. Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions. *Comput. Math. Appl.*, 75(3):918–932, 2018.

[8] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[9] T. Gudi. A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comp.*, 79(272):2169–2189, 2010.

[10] N. Lüthen, M. Juntunen, and R. Stenberg. An improved a priori error analysis of nitsche's method for robin boundary conditions. *Numerische Mathematik*, 2017. https://doi.org/10.1007/s00211-017-0927-1.

[11] G. Savaré. Regularity and perturbation results for mixed second order elliptic problems. *Comm. Partial Differential Equations*, 22(5-6):869–899, 1997.