Portfolio Selection, Periodic Evaluations and Risk Taking*

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Abstract

We present a continuous-time portfolio selection problem faced by an agent with S-shaped preference who maximizes the utilities derived from the portfolio's periodic performance over an infinite horizon. The periodic reward structure creates subtle incentive distortion. In some cases, local risk aversion is induced which discourages the agent from risk taking in the extreme bad states of the world. In some other cases, eventual ruin of the portfolio is inevitable and the agent underinvests in the good states of the world to manipulate the basis of subsequent performance evaluations. We outline several important elements of incentive design to contain the long-term portfolio risk.

Keywords: Portfolio selection, S-shaped utility, periodic evaluation, agency, incentive.

JEL Classification: G11, G19, G40, D81

1 Introduction

Since the seminal work of Merton (1969, 1971), two popular problem formulations have emerged to understand dynamic portfolio decision made by individuals: an infinite horizon intertemporal consumption problem where one jointly chooses investment and consumption strategy to maximize the discounted expected utility generated by a perpetual consumption stream; or a finite

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horizon problem where one finds an investment strategy to maximize the expected utility of the terminal portfolio value on a given maturity date. Despite the prevalence of such objective functions in the literature, they are not necessarily realistic to all types of economic agents.

Consider a scenario where an agent is making investment decision on behalf of a principal. Examples include an asset manager overseeing a client's portfolio, a trader working for a hedge fund, or an investment officer deciding how a firm's capital should be allocated across various risky projects. There may not exist a realistic way for those agents to directly "consume" the underlying portfolio because it is not their personal wealth. Their economic incentives are also not tied to the portfolio value in the distant future, but rather its periodic performance driven by many corporate practices in real life. For instance, a firm typically announces its financial results quarterly with extensive media coverage. Staff appraisal happens regularly within many companies to review how well an employee has been doing in a given period, and the outcome plays a huge role for the decision of one's salary increment and career progression. Interim periodic performance can be economically crucial and it is important to take this into the account when developing a portfolio selection model.

Behavioral economics literature also supports the idea that investors do not necessarily evaluate an investment prospect solely based on their final wealth at a single time point. Prospect Theory of Kahneman and Tversky (1979) and Tversky and Kahneman (1992), arguably the most prominent alternative to the expected utility paradigm, suggests that individuals care about changes in their wealth level relative to some reference point. Kahneman and Sugden (2005) advocate the relevance of "experienced utility" in policy evaluation tasks where the entire portfolio path matters (Loosely speaking, experienced utility can be viewed as an integral of instantaneous satisfaction over the entire period of the experience. See Read (2007) and Kahneman and Sugden (2005) for a detailed discussion). Periodic evaluations at an individual investor level can arise as a manifestation of the above behavioral notions in conjunction with mental accounting (Thaler (1999) and Benartzi and Thaler (1995)). For example, filing taxes can prompt an investor to review the portfolio annual performance and hence one year can be a natural cycle of his mental account. The mental evaluation then results in a burst of experienced utility linked to the portfolio paper gain-and-loss during each tax season.

The first contribution of this paper is that we consider a portfolio selection problem which objective function has not received much attention to date. In the model, trading decision can be adjusted continuously while the portfolio performance is evaluated on a sequence of deterministic dates. The agent derives utility on each evaluation date based on the change in the portfolio value in the prevailing period relative to a performance benchmark parameter. The goal of the agent is to maximize the expected total discounted utilities over an infinite horizon. The utilities can be interpreted in many different ways. They can include monetary remunerations (salary and bonus), intangible payoffs (gain/loss of professional reputation and client base) and mental benefits (pleasure/disappointment from public recognition). The utility function of our choice is S-shaped. On the one hand, this specification can capture the robust behavioral biases reported by Kahneman and Tversky (1979) under which an individual tends to be risk averse over positive outcomes but risk seeking over negative outcomes. On the other hand, local convexity encoded within an S-shaped utility function can also reflect distortion brought by convex incentive schemes such as option-based compensation and limited-liability protection.

In terms of economic contributions, consideration of periodic reward structure has meaningful consequences and it unravels new phenomena that have not been documented in the literature. In a standard finite horizon portfolio optimization problem with S-shaped utility function, the optimal portfolio delivers a random payoff of unbounded upside in the good states of the world but suffers from a total loss in the bad states of the world. This is not necessarily true under a periodic reward structure. If the trading opportunity is favorable, the agent is not willing to take extreme downside risk anymore because otherwise a bad realization of the economy in a given period will then wipe out the entire portfolio, leaving the agent nothing to be earned in the subsequent periods. In this case, it is optimal for the agent to deleverage under a distressed market to ensure the portfolio is strictly solvent at all time, and we will show that the optimal proportion of wealth invested in the risky asset indeed converges to the Merton ratio in both very good and very bad states of the world.

However, the trading opportunity may be less valuable under some setups (e.g. there is a demanding performance target or the agent is highly loss averse). The agent may thus put more weight on the very short-term reward at the end of the current period over the potential benefits in the future. He is happy with taking extreme downside risk again which will result in eventual portfolio ruin with certainty in the long run. Moreover, the agent may also intentionally limit the portfolio growth by a gain-exit strategy to avoid setting up a high basis for performance evaluation in the future periods since he anticipates it is difficult to consistently outperform the benchmark under a tougher trading environment.

The theoretical predictions of our model carry important policy implications in relationship to incentive design, agency problems and portfolio risk. From a delegated portfolio management perspective, excessive risk taking and in particular the possibility of inevitable ruin in the long run under some cases reveal a devastating misalignment of the agent's short-term, periodic incentives against the principal's long-term investment goal. Our results shed lights on how this agency problem can be circumvented via proper design of incentive or "nudging". Possible measures include adoption of cliquet-style option in place of traditional long-dated vanilla option as an incentive scheme, imposing a more lenient performance target and moderation of the agent's effective loss aversion level via reducing the emphasis on penalizing an underperforming agent for example.

On the technical front, our dynamic optimization problem is interesting because it cannot be directly tackled by the Hamilton-Jacobi-Bellman (HJB) equation nor martingale duality approach. The elements of path-dependency, periodicity as well as non-concave utility function lead to a non-standard HJB equation with periodic boundary conditions depending on the solution itself, and the optimal investment strategy cannot be easily pinned down by a simple first-order condition due to the non-concave utility function. Meanwhile, the martingale duality method is not suitable to handle an infinite horizon, path-dependent problem. In view of the above difficulties, our technical contribution is to provide a complete solution to the problem under the specialization to a Black-Scholes economy with the Tversky and Kahneman (1992) piecewise power utility function. The main idea is to consider a family of finite horizon problems where each of them can be individually solved by martingale duality. Then the required solution can be identified by the fixed-point of a "Bellman-style" operator associated with this family of optimization problems.

We close the introduction by discussing how our work is related to the literature of behavioral portfolio selection, managerial risk taking incentives and more generally non-concave utility maximization. There are a few papers incorporating one or several features of the Prospect Theory framework of Tversky and Kahneman (1992) within a continuous-time portfolio selection model. Berkelaar et al. (2004) study a finite horizon problem with a piecewise power S-shaped utility function. Their work is extended in multiple directions such as inclusion of probability weighting, incorporation of risk and trading constraints, applications to insurance and pension management, etc (see, for example, Jin and Zhou (2008), Dong and Zheng (2020, 2019), Chen et al. (2017) and the references therein). All of these models consider a finite horizon objective function and omit interim rewards. In parallel, there is also a vast literature on discrete-time portfolio selection with behavioral preferences. See Benartzi and Thaler (1995), Levy and Levy (2004), He and Zhou (2011), De Giorgi and Legg (2012) and Shi et al. (2015), among others. Some of these models are static in nature and are able to include other important behavioral elements like probability weighting. Since the main interest of our paper is to investigate the impact of periodic evaluations, a continuous-time model is a more convenient framework for our purpose which captures a key feature that evaluations can occur much less frequently than the trading activities.

Our work is conceptually close to the realization utility models studied by Barberis and Xiong (2012), Ingersoll and Jin (2013) and He and Yang (2019). In this type of models, an agent repeatedly purchases and sells a series of statistically identical assets and a utility burst (generated by an S-shaped function over gain/loss) is realized upon completion of each round-trip transaction. Although both their models and ours feature episodicity, there are several important differences. First, the periodic structure of our model is exogenously given (e.g. it depends on the given corporate or mental accounting cycle) while the trading episodes are created via the agent's choice in a realization utility model. Second, utilities in our model can be brought by paper gains-and-losses in form of experienced utility, whereas those in a realization utility model are originating from realized gains-and-losses. Finally, models based on realization utility focus on the purchase/liquidation decision of indivisible asset while our model considers portfolio effect.

Away from behavioral considerations, our work is also broadly related to dynamic models of managerial risk taking under convex incentives such as Carpenter (2000), Kouwenberg and Ziemba (2007), Basak et al. (2007) and Buraschi et al. (2014). We do not explicitly model the contractual payoff to the agent but rather take the utility function as exogenously given. But, as we will show in Section 2, one special case of our setup reflects option-based compensation. Theoretical literature in this area mostly focuses on how managers' risk seeking incentives are influenced by the compensation structure itself evaluated at a single terminal time. Papers that emphasize repeated payouts/evaluations include Panageas and Westerfield (2009) and Hodder and Jackwerth (2007). In a model with continuously updated high-water mark, Panageas and Westerfield (2009) highlight that excessive risk taking due to convex incentive schemes relies on the setting of a finite number of performance fee payment dates. Our work offers an alternative perspective on how the periodic payout structure may drastically change one's risk taking behaviors relative to a finite horizon benchmark. Part of our findings also echo Hodder and Jackwerth (2007), who give numerical evidence that extreme risk taking can be dampened when performance is evaluated annually instead of at a single terminal time. However, our analysis shows that the risk seeking moderation effect depends subtly on the parameters of the reward structure, and this yields important implications to managerial incentive design in terms of how a suitably chosen performance target together with periodic payouts can avoid excessive risk taking and underinvestment.

Martingale duality is the primary solution method to study a finite horizon non-concave utility maximization problem. The idea can date back to Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989). The problem is analyzed under a very general setup for concave utility function by Kramkov and Schachermayer (1999), and is extended to non-concave utility functions by Kouwenberg and Ziemba (2007), Reichlin (2013) and Bichuch and Sturm (2014), etc. Nonetheless, there has not been unified result yet on solving a non-concave utility maximization problem with infinite horizon and path-dependency.

The rest of the paper is organized as follows. Section 2 introduces our modeling framework. We heuristically outline in Section 3 how the periodic portfolio selection problem shall be solved. The main results of this paper are collected and discussed in Section 4. Several empirical and policy implications of our results are further explored in Section 5. Section 6 concludes. Technical materials, proofs and extensions are deferred to the internet appendix (e-companion).

2 Modeling setup

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a standard filtered probability space supporting a one-dimensional Brownian motion $B = (B_t)_{t\geq 0}$. Consider a Black-Scholes economy with one risky asset and one riskfree money market instrument. The price process of the risky asset $S = (S_t)_{t\geq 0}$ is a geometric Brownian motion such that

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

with μ and $\sigma > 0$ being the drift and volatility of the asset respectively. The money market instrument has a constant interest rate r and its price process $D = (D_t)_{t\geq 0}$ is given by $D_t = e^{rt}$.

Write $\theta := \frac{\mu - r}{\sigma}$ as the Sharpe ratio of the asset. The Black-Scholes economy is complete with a unique pricing kernel $Z = (Z_t)_{t \geq 0}$ given by

$$Z_t = \exp\left(-\theta B_t - \left(r + \frac{\theta^2}{2}\right) t\right). \tag{1}$$

We assume $\theta \neq 0$ throughout this paper to ensure Z is a non-degenerate stochastic process. The corner case of $\theta = 0$ can still be analyzed thoroughly, but this would require us to interpret our main results somewhat differently. For a discussion of the special features of a non-concave utility maximization problem when $\theta = 0$, see Section 3.2 of Bichuch and Sturm (2014).

A trading strategy $\phi = (\phi_t^{(S)}, \phi_t^{(D)})_{t \geq 0}$ is a predictable process representing the holding of $\phi_t^{(S)}$ units of the risky asset and $\phi_t^{(D)}$ units of the money market instrument at time t. ϕ and the resulting portfolio value process $X = (X_t)_{t \geq 0}$ is said to be self-financing if

$$X_t := \phi_t^{(S)} S_t + \phi_t^{(D)} D_t = X_0 + \int_0^t \phi_u^{(S)} dS_u + \int_0^t \phi_u^{(D)} dD_u$$

for all $t \geq 0$. Define the set of admissible portfolio processes as

$$\mathcal{A}_t(x) := \left\{ X : X_s = x + \int_t^s \phi_u^{(S)} dS_u + \int_t^s \phi_u^{(D)} dD_u \geq 0 \quad \forall s \geq t \text{ for predictable and self-financing } \phi \right\}$$

for $x > 0$ and $t \geq 0$, which refers to the collection of all non-negative self-financing portfolios that can be generated by a starting capital of x at time t.

The portfolio value is observed periodically on a sequence of deterministic dates $\{T_i\}_{i=0,1,2,\ldots}$ with $T_0 := 0$. For simplicity, we assume $T_i = i\tau$ for $i \in \{0, 1, 2...\}$ with some $\tau > 0$ such that the portfolio is evaluated every τ unit of time (e.g. monthly or annually). On the i^{th} evaluation date, the portfolio performance in the period $(T_{i-1}, T_i]$ is computed as

$$X_{T_i} - \gamma X_{T_{i-1}} \tag{2}$$

for some parameter $\gamma \geq 0$ which we will refer to as the "performance target". If $\gamma = 1$, the expression in (2) is simply the profit-and-loss of the portfolio in the i^{th} period. The parameter γ captures the required gross return per period as the agent's performance target. If (2) is positive (negative), then the agent is overperforming (underperforming) relative to the target benchmark in the i^{th} period. We assume the agent derives utility from the portfolio performance in each accounting period, but not from the portfolio value at any arbitrary terminal time point. As discussed in the introduction, this criterion is reasonable especially in the context of delegated portfolio management. The agent is not concerned about the long-term prospect of the underlying portfolio, but instead their incentives are tied to short-term economic rewards based on how good or bad they have been doing over different accounting periods.

We adopt the Tversky and Kahneman (1992) Prospect Theory piecewise power utility function

$$U(x) := \begin{cases} x^{\alpha}, & x \geq 0; \\ -k|x|^{\alpha}, & x < 0. \end{cases} \tag{3}$$

Here $\alpha \in (0,1)$ is the coefficient of risk aversion/seeking over the domain of gains/losses respectively, and $k \geq 0$ is an asymmetry parameter. Tversky and Kahneman (1992) report the estimates of $\alpha = 0.88$ and $k = 2.25$. The function U is concave on the positive domain and convex on the negative domain, and thus the agent is risk averse over gains but risk seeking over losses. If $k > 1$, then U is asymmetric near the origin in that $|U(-1)| > |U(1)|$. The phenomenon is known as loss aversion reflecting the larger psychological impact from a loss in comparison to a gain of the same size.

The goal of the agent with an initial capital $x > 0$ is to maximize the total discounted expected utilities derived from his trading performance over an infinite number of periods, i.e. to find the optimal portfolio value process which solves

$$V(x) := \sup_{X \in \mathcal{A}_0(x)} \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}})\right], \tag{4}$$

where $\beta > 0$ is the agent's subjective discount factor. In Appendix E, we also consider an extension of the model featuring a random terminal horizon and utility from terminal wealth.

The formulation in (4) is similar to that of Shi et al. (2015), who study a finite horizon discrete-time portfolio selection problem with periodic utilities criterion. They consider a more general adaptive reference point structure but focus on a piecewise linear utility function, and they implicitly assume that the performance evaluation frequency is the same as the agent's trading frequency.

It is important to highlight that (4) is a criterion of "broad framing" or "portfolio-level framing" where utilities are derived from the performance of the aggregate portfolio (see Section 3.5 of Guo and He (2021), Section 8 of He and Zhou (2014) and Section IIIG of Barberis and Huang (2008b)). In contrast, behavioral economics literature also commonly considers models of "narrow framing" where utilities come from the gains-and-losses of an individual risky asset (Barberis et al. (2006), Barberis and Huang (2008a), De Giorgi and Legg (2012), among others). A model featuring broad framing is perhaps better suited to describe the trading behaviors of a more sophisticated agent who exhibits weaker narrow framing (Liu et al. (2010)). Moreover, adoption of a broad framing perspective also enables a meaningful and non-behavioral interpretation of (4) from a delegated portfolio management viewpoint.

As an example, consider a risk-neutral agent whose periodic remuneration is a combination of a base salary $b > 0$ plus a bonus depending on the positive part of the portfolio's after-tax profit in the period. The total salary he receives in period i is

$$W_i := b + \Lambda(a(X_{T_i} - X_{T_{i-1}})^+),$$

where $a > 0$ is a profit-sharing parameter and $\Lambda(\cdot)$ is a concave function reflecting the convex tax schedule. We assume that W_i is not directly paid out from the underlying portfolio such that X_t will not be reduced by an amount of W_i at $t = T_i$. This assumption makes sense for example when the agent is one of the many traders in a large investment firm with his own portfolio, and salaries are paid out centrally from a separate pool of corporate resources. Due to the affine structure of the payoff, optimizing the net present value of the salaries can be described by problem (4) upon the approximation of $\Lambda(x) \propto x^\alpha$ and letting $k = 0$ and $\gamma = 1$ (It is a common practice in the tax literature to use a concave power function to model after-tax income. See, for example, Wen and Gordon (2014) and Benabou (2000)). In general, the function U can be understood as the composite of the agent's individual utility function, the payout function of the incentive scheme and other various market frictions. We opt to omit to the precise modeling details and simply approximate U by a piecewise power utility function. This specification nonetheless is still flexible enough to qualitatively capture some important features of an incentive scheme. In particular, the case of $k = 0$ can describe limited-liability protection or option-based compensation, and the flexibility in choosing γ in (4) allows us to consider convex incentive schemes of different aggressiveness levels.

An interesting special case is $\gamma = 0$ such that the objective function in (4) reflects the expected experienced utility criterion where the agent derives utilities from his wealth level at different time points. In the context of delegated portfolio management, $\gamma = 0$ corresponds to a management fee scheme where the manager periodically receives a fraction of the asset under management (AUM). As we will see in Corollary 1, specialization of $\gamma = 0$ greatly simplifies the mathematical analysis since the convex regime of U is no longer relevant.

We briefly summarize how the various model interpretations above can translate to the choices of the parameters. From a behavioral portfolio selection viewpoint, we expect k to be a value in excess of one to capture loss aversion and γ should be somewhat close to one since the starting wealth (of a period) is a natural choice of the reference point. In a delegated portfolio management application, k can be much smaller than one or even zero if the utilities entirely represent monetary outcomes and the manager has limited-liability protection, while γ can vary a lot from zero (a management fee system linked to AUM) to a high value (an aggressive incentive scheme with high performance benchmark). Under either model interpretation, α is the most difficult to be estimated. It depends heavily on the agent's diminishing sensitivity to gains/losses and the experimental literature reports a wide range of values from 0.22 to 1 (Booij et al. (2010)).

To close this section, we present below the standing assumption of this paper which is a sufficient condition to ensure problem (4) is well-posed.

Assumption 1. *The model parameters are such that*

$$\beta > h := r\alpha + \frac{\alpha\theta^2}{2(1-\alpha)}. \tag{5}$$

An estimate of the value function is given in the proposition below.

Proposition 1. *If* (5) *holds, then*

$$\frac{e^{-\beta\tau}U(e^{r\tau}-\gamma)}{1-e^{-(\beta-r\alpha)\tau}}x^\alpha \leq V(x) \leq \frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}}x^\alpha, \tag{6}$$

where $V(x)$ is the value function of problem (4)*. In particular, problem* (4) *is well-posed such that $V(x)$ is finite for any $x \geq 0$.*

Note that (5) is a sufficient (but not necessary in general) condition of well-posedness, which does not depend on k or γ. Nonetheless, this sufficient condition is not restrictive, and it is precisely the condition under which an infinite horizon Merton consumption-investment problem with utility function $u(c) = c^\alpha$ is well-posed (p.20, Rogers (2013)). Hence the assumption required in our setup is exactly the same as the one required in the standard literature of portfolio selection: we only need the discount factor to be sufficiently strong relative to the preference parameter and the quality of the risky asset measured in terms of the Sharpe ratio. Indeed, the upper bound of $V(x)$ in (6) is precisely the value function of the problem when $\gamma = 0$. See Corollary 1.

3 The solution approach: a preliminary discussion

When solving a dynamic optimization problem, there are two main methods in general: the primal HJB equation formulation and the martingale duality approach. Our general problem (4) has a few specific features which make neither of these two methods directly applicable.

The HJB approach involves solving for the optimal trading strategy and value function directly via the dynamic programming principle. If we write $\pi_t := \frac{\phi_t^{(s)} S_t}{X_t}$ which represents the proportion of capital invested in the risky asset, then the dynamics of $X = X^\pi$ can be rewritten as

$$dX_t = \pi_t X_t \frac{dS_t}{S_t} + (1-\pi_t) X_t \frac{dD_t}{D_t} = [r + (\mu - r)\pi_t] X_t dt + \sigma \pi_t X_t dB_t. \tag{7}$$

The time-t version of the value function in (4) can be defined as

$$V(t,x,\ell) := \sup_{\pi : X^\pi \geq 0} \mathbb{E}\left[\sum_{i=\lfloor t/\tau \rfloor + 1}^{\infty} e^{-\beta(T_i - t)} U(X_{T_i} - \gamma L_{T_i}) \Big| X_t = x, L_t = \ell \right],$$

where the process L defined via $L_0 := 0$ and $L_t := X_{T_{i-1}}$ on $T_{i-1} < t \leq T_i$ for each i represents the lagged portfolio value as the basis of performance valuation. The HJB equation associated with this problem is then

$$\begin{cases} \sup_\pi \left\{ \frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + (\mu - r)\pi x \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} x^2 \pi^2 \frac{\partial^2 V}{\partial x^2} - \beta V \right\} = 0, & \forall t \notin \{T_i : i \in \mathbb{N}\}; \\ V(T_i -, x, \ell) = V(T_i, x, x) + U(x - \gamma \ell), & i \in \mathbb{N}. \end{cases} \tag{8}$$

It is hard to make analytical progress with equation (8) because of a few reasons. First, the state space of the problem involves time, current wealth and the starting wealth of the current period, which is more complicated than that of a standard infinite horizon portfolio selection problem. Second, the utility function U is S-shaped and hence the value function is unlikely to be a concave function. Then it is not straightforward to characterize the optimal strategy π without a simple first-order condition. Third, the periodic boundary condition at $t = T_i-$ not only depends on the utility function U but also the value function itself at $t = T_i$ which has to be solved as a part of the problem. The lack of an explicit boundary condition makes the problem hard to be solved even numerically. Finally, to show that the solution to the HJB equation is coinciding with the value function of the optimization problem, one typically needs to rigorously establish a verification theorem and/or invoke the machineries of viscosity solutions.

These often require careful study of the regularities of the candidate value function which can be mathematically difficult.

The second key approach is the martingale duality. For a finite horizon problem in form of $\sup_{X\in\mathcal{A}_0(x)}\mathbb{E}[G(X_T)]$ in a complete market, the insight is that one can directly take the random variable representing the terminal portfolio value X_T as a decision variable and convert the problem into a static optimization problem

$$\sup_{X\in\mathcal{F}_T^+}\mathbb{E}[G(X)] \qquad \text{subject to} \qquad \mathbb{E}[Z_T X]\leq X_0=x, \tag{9}$$

where $\mathcal{F}_T^+$ is the set of non-negative $\mathcal{F}_T$-measurable random variables, and Z_T is the time-T value of the pricing kernel. With the optimal terminal random variable X^*, the whole optimal portfolio value process can be recovered via $X_t=Z_t^{-1}\mathbb{E}[Z_T X^*|\mathcal{F}_t]$, and, if required, the associated trading strategies can be characterized by the martingale representation theorem. The main advantage of this duality approach is that it can work well with non-concave payoff function G. Moreover, one does not have to rely on dynamic programming principle which in turn bypasses the need of establishing a verification theorem. Indeed, most finite horizon problems (with non-concave G) in the literature are solved by such an argument. However, our general problem (4) is an infinite horizon problem which depends on infinite number of random variables $(X_{T_i})_{i\in\mathbb{N}}$ and the objective function exhibits path-dependency. Thus it is not immediately obvious how martingale duality can be applied here.

Our solution strategy relies on a novel hybrid approach which combines the primal dynamic programming principle and the martingale duality. Starting from (4), using the dynamic programming principle and the recursive structure of the problem, we heuristically expect

$$V(x)=\sup_{X\in\mathcal{A}_0(x)}\mathbb{E}\left[e^{-\beta T_1}U(X_{T_1}-\gamma x)+e^{-\beta T_1}V(X_{T_1})\right]. \tag{10}$$

The right hand side of (10) has the form of a finite horizon portfolio optimization problem with maturity $T_1=\tau$. However, the difficulty is that the "payoff function" of this problem involves V which is the solution that we want to determine in the first place. To proceed, (10) can be thought as a fixed-point iteration problem. Define a "Bellman-style" operator $\mathcal{G}$ via

$$\mathcal{G}f(x)=\sup_{X\in\mathcal{A}_0(x)}\mathbb{E}\left[e^{-\beta\tau}U(X_\tau-\gamma x)+e^{-\beta\tau}f(X_\tau)\right]. \tag{11}$$

If we can show that the map $\mathcal{G}$ is a contraction (on some suitably chosen complete metric space), then the Banach contraction theorem can be invoked such that V can be characterized by the unique fixed-point of $\mathcal{G}$.

Using the scaling property of the utility function $U(y) = y^\alpha U(1)$ (on $y > 0$), we conjecture that our value function should have the form $V(x) = A^* x^\alpha$ for all $x > 0$ and some constant A^*. If we substitute this form of V in (10) and divide both side by x^α, we obtain

$$\begin{aligned} A^* &= \sup_{X \in \mathcal{A}_0(x)} \mathbb{E}\left[e^{-\beta T_1} U\left(\frac{X_{T_1}}{x} - \gamma\right) + e^{-\beta T_1} A^* \left(\frac{X_{T_1}}{x}\right)^\alpha \right] \\ &= \sup_{Y \in \mathcal{A}_0(1)} \mathbb{E}\left[e^{-\beta\tau} U\left(Y_\tau - \gamma\right) + e^{-\beta\tau} A^* Y_\tau^\alpha \right]. \end{aligned} \tag{12}$$

Hence the characterization of V now simplifies to the characterization of the unknown constant A^*, and it is given by the fixed-point of a simpler operator defined on a one-dimensional Euclidean space. Define a function $G_a : \mathbb{R}_+ \to \mathbb{R}$ via

$$G_a(y) := U(y - \gamma) + a y^\alpha, \tag{13}$$

where $a \in \mathbb{R}$ is a parameter of G_a. Consider a family of optimization problems

$$F(a) := \sup_{Y \in \mathcal{F}_\tau^+} \mathbb{E}\left[G_a(Y)\right] \qquad \text{subject to} \qquad \mathbb{E}(Z_\tau Y) \leq 1, \tag{14}$$

where Z is defined in (1) and $\mathcal{F}_\tau^+$ represents the set of non-negative $\mathcal{F}_\tau$-measurable random variables. Using the idea of martingale duality, (12) can now be restated as

$$A^* = e^{-\beta\tau} F(A^*). \tag{15}$$

The key solution idea is to show that the map $a \to e^{-\beta\tau} F(a)$ is a contraction such that there indeed exists a unique A^* solving equation (15). Then one can formally prove that $A^* x^\alpha$ coincides with the value function via a verification theorem. See Appendix A and B for the full theoretical details.

4 The solution to the periodic portfolio selection problem

4.1 The main theoretical results

Theorem 1. *The value function of problem* (4) *is given by* $V(x) = A^* x^\alpha$*, where* A^* *is the unique fixed-point of the map* $a \to e^{-\beta\tau} F(a)$*. The optimal portfolio value process* X^* *at time* T_i *is given by*

$$X^*_{T_i} = X^*_{T_{i-1}} y^*_{A^*}\left(\lambda^* \frac{Z_{T_i}}{Z_{T_{i-1}}}\right), \qquad i = 1, 2, ..., \tag{16}$$

with $X^*_{T_0} = x$. $F(\cdot)$ *is defined in* (14) *and the function* $y^*_{A^*}(\cdot)$ *is defined in Proposition 3 of Appendix A. The support of the random variable* $y^*_{A^*}\left(\lambda^* \frac{Z_{T_i}}{Z_{T_{i-1}}}\right)$ *is*

$$
supp\left(y^*_{A^*}\left(\lambda^* \frac{Z_{T_i}}{Z_{T_{i-1}}}\right)\right) = \begin{cases} (0, c_1\gamma) \cup (c_2\gamma, \infty), & A^* > 0 \quad \textit{(Case 1)}; \\ \{0\} \cup (c_3\gamma, \infty), & -1 \leq A^* \leq 0 \quad \textit{(Case 2a)}; \\ \{0\} \cup (c_3\gamma, \frac{\gamma}{1-|A^*|^{-1/(1-\alpha)}}), & -(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < A^* < -1 \quad \textit{(Case 2b)}; \\ \{0\}, & A^* \leq -(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} \quad \textit{(Case 3)}, \end{cases}
$$

for some constants $0 < c_1 < 1 < c_2$ *and* $c_3 > 1$ *depending on* A^*. *In Case 1, 2a and 2b,* $\lambda^* > 0$ *is the unique solution to the equation* $\mathbb{E}[Z_\tau y^*_{A^*}(\lambda^* Z_\tau)] = 1$.

$V(x) = A^* x^\alpha$ is the value function at time zero which has a very simple form (concave increasing if $A^* \geq 0$ or convex decreasing if $A^* \leq 0$). Note that Theorem 1 does not directly tell us anything about the form of the value function at an arbitrary time point, which we expect will be more complicated with completely different analytical behaviors (see the discussion in Section 3). Nonetheless, the characterization of the optimal portfolio under our current martingale duality approach does not require us to solve for the value function at all time points. At the first sight, it is perhaps surprising to see that the value function can be decreasing. The rationale is that the argument x of the time-zero value function defined in (4) reflects both the starting wealth as well as the benchmark to be used in the first evaluation period. They have opposite effects on the agent's welfare and hence the net impact on $V(x)$ is non-trivial. The roles of the sign of A^* will be further discussed in Section 4.2.

Theorem 1 suggests the agent will trade in a way such that the gross portfolio return in each period, $\frac{X^*_{T_{n+1}}}{X^*_{T_n}}$, is given by a random variable $y^*_{A^*}\left(\lambda^* \frac{Z_{T_{n+1}}}{Z_{T_n}}\right)$. The target returns from different periods are independent and identically distributed since Z is a stationary process. The infinite horizon problem (relative to the one-period version such as Berkelaar et al. (2004)) has a rich and complicated solution structure. The optimal portfolio and the associated trading strategies crucially depend on the value of A^*, which can be interpreted as the certainty equivalent of the trading opportunity per unit capital under management.

Corollary 1. *Suppose* $\gamma = 0$. *Then* $A^* = \frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}} > 0$. *The optimal* $X^* = (X^*_t)_{t\geq 0}$ *is given* *by the Merton portfolio*

$$X_t^* = x \exp\left[-\frac{\alpha}{1-\alpha}\left(r + \frac{\theta^2}{2(1-\alpha)}\right)t\right] Z_t^{-\frac{1}{1-\alpha}} = x \exp\left[\left(r + (\mu - r)\pi^* - \frac{\sigma^2(\pi^*)^2}{2}\right)t + \sigma\pi^* B_t\right], \tag{17}$$

where $\pi^* := \frac{\mu - r}{(1-\alpha)\sigma^2}$ *is the Merton ratio.* X^* *can be constructed by investing a constant fraction* π^* *of the current wealth in the risky asset.*

As discussed in Section 2, $\gamma = 0$ corresponds to an expected experienced utility criterion or an AUM-based compensation structure, where the periodic utilities/rewards received are simply linked to the portfolio values (rather than the profits-and-losses) on the evaluation dates. The optimal investment level in the risk asset is precisely given by the constant Merton ratio π^*. The loss aversion parameter k does not enter the picture since all outcomes are framed as gains under $\gamma = 0$. Notice that the optimal portfolio (17) is always strictly solvent, i.e. $X_t^* > 0$ for all t almost surely.

The form of the optimal portfolio is more complicated away from the special case of $\gamma = 0$. Theorem 1 shows that in general there are four possible cases with the optimal solution as A^* varies. We do not know the value of A^* ex-ante even though it can be conveniently computed by iterative method. Then a natural question is whether all the four solution regimes can indeed arise under some combinations of the underlying model parameters. The following proposition gives a positive answer to this question.

Proposition 2. *Let* $A^*(\gamma, k)$ *be the value of* A^* *as a function of the model parameters* γ *and* k *(while all other parameters are held fixed). Then:*

1. *Suppose* k *is fixed.* $A^*(\gamma, k)$ *is decreasing in* γ*, and* $A^*(\gamma, k) > 0$ *on* $\gamma \leq e^{r\tau}$*. If* $k > 0$*, then* $\lim_{\gamma \to +\infty} A^*(\gamma, k) = -\infty$*.*

2. *Suppose* γ *is fixed.* $A^*(\gamma, k)$ *is decreasing in* k *and* $A^*(\gamma, k = 0) > 0$*. If* $\gamma > e^{r\tau}$*, then* $\lim_{k \to +\infty} A^*(\gamma, k) = -\infty$*.*

Proposition 2 shows that changes in the underlying model parameters lead to variation in A^*. In Figure 1, we plot the values of A^* as γ changes and mark the corresponding critical levels that differentiate the four regimes of the solution structure. When γ increases, A^* decreases and cuts through all the three critical levels. The transition of solution behavior happens in the order of

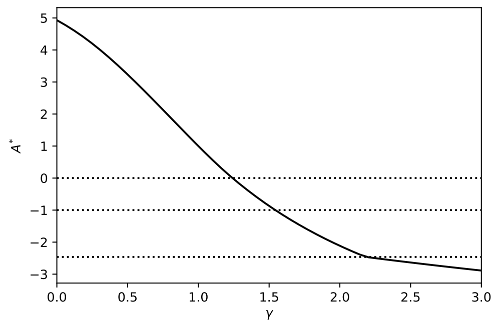


Figure 1: The value of A^* as γ varies while all other model parameters are fixed. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $\mu = 0.1$, $\sigma = 0.2$, $r = 0.005$, $\beta = 0.3$ and $\tau = 1$. The three horizontal dotted lines indicate the level of 0, -1 and $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha}$ respectively.

Case 1, 2a, 2b and 3 as described by Theorem 1. Further numerical experiments reveal that Case 1 and Case 2a tend to occur under reasonable choices of economic parameters. If γ is interpreted as a performance benchmark to beat and $\tau = 1$ (i.e. annual evaluation), then we expect it should not exceed two in practice such that the order of magnitude of $\gamma - 1$ is comparable to that of μ.

The gross riskfree rate $e^{r\tau}$ is an important benchmark of γ. From part (1) of Proposition 2, any value of γ below or equal to $e^{r\tau}$ will ensure $A^* > 0$. In contrary, part (2) of Proposition 2 suggests if γ is strictly above $e^{r\tau}$, then there must exist some model parameters (e.g. large k) such that $A^* < 0$. In summary, the gross riskfree rate is the highest performance benchmark one can impose on an agent if $A^* > 0$ is required. The strict positivity of A^* has some important implications related to risk taking behaviors, as we will elaborate in Section 4.2 and 5.3.

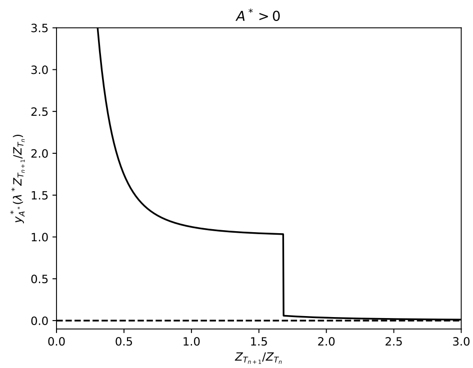
4.2 Discussion of the optimal strategies under different cases

Now we proceed to discuss the economic intuitions of the optimal strategies in each case. The optimal target periodic gross return and the optimal proportion of wealth invested in the risky asset (the latter can be computed using Proposition 5 in Appendix C) are numerically illustrated in Figure 2 and 3.

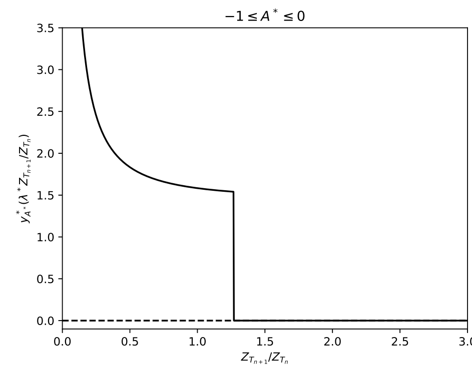
It is useful to recall the dynamic programming equation again:

$$V(x) = \sup_{X \in \mathcal{A}_0(x)} \mathbb{E}\left[e^{-\beta T_1} U(X_{T_1} - \gamma x) + e^{-\beta T_1} V(X_{T_1})\right]. \tag{18}$$

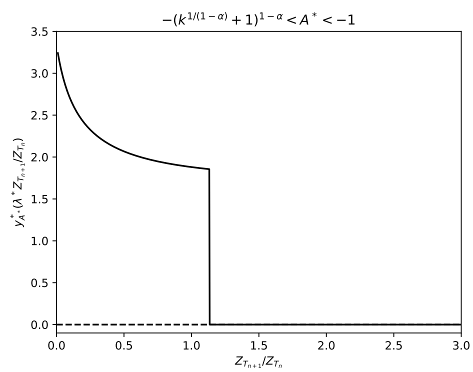
In the one-period problem of Berkelaar et al. (2004), the agent does not care whether the portfolio



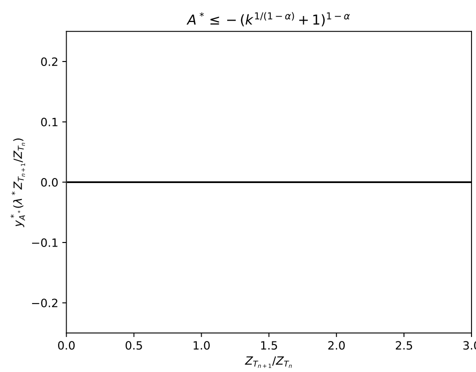
(a) Case 1.



(b) Case 2a.

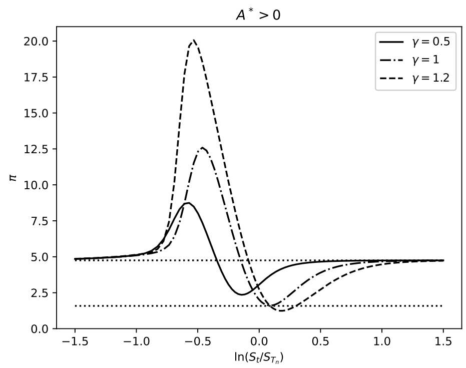


(c) Case 2b.

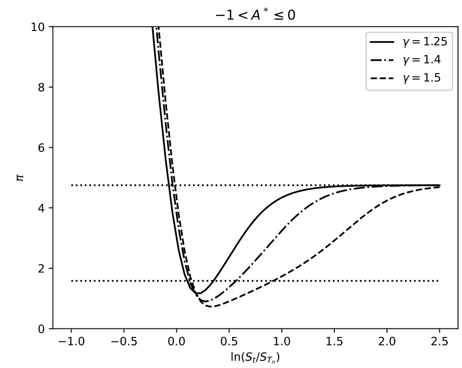


(d) Case 3.

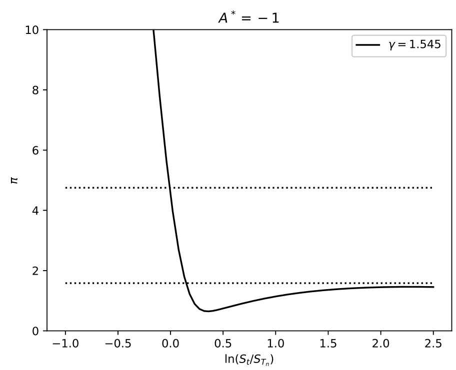
Figure 2: The optimal target gross return random variable as a function of $Z_{T_{n+1}}/Z_{T_n}$ under all the four possible cases described by Theorem 1. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $\mu = 0.1$, $\sigma = 0.2$, $r = 0.005$, $\beta = 0.3$ and $\tau = 1$. The four cases are generated by $\gamma = 1$, $\gamma = 1.45$, $\gamma = 1.7$ and $\gamma = 2.5$ respectively.



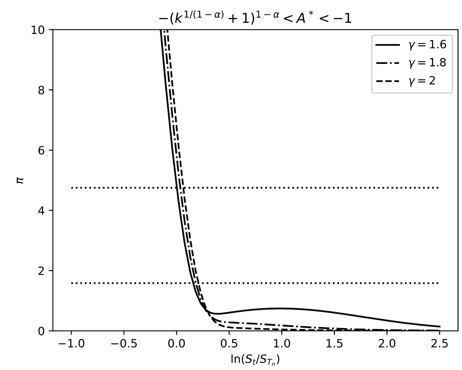
(a) Case 1.



(b) Case 2a with $A^* \neq -1$.



(c) Case 2a with $A^* = -1$.



(d) Case 2b.

Figure 3: The optimal proportion of wealth invested in the risky asset as a function of $\ln(S_t/S_{T_n})$ at a fixed time $t = T_{n+1} - 0.5\tau$ under various values of γ. The results are collected in different sub-figures according to the value of A^*. The two horizontal dotted lines mark the Merton ratio $(\mu - r)/((1 - \alpha)\sigma^2)$ and the constant $(\mu - r)/((2 - \alpha)\sigma^2)$ respectively. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $\mu = 0.1$, $\sigma = 0.2$, $r = 0.005$, $\beta = 0.3$ and $\tau = 1$. Under the choice of $\gamma = 1.545$, the numerical value of A^* is approximately -1 (accurate up to three decimal places).

goes bust or not at the terminal time because he does not have any future incomes or utilities tied to the residual portfolio value, i.e. the term $V(X_{T_1})$ is absent in (18) for a one-period problem. In a multi-period setup, however, the agent must take into the account how the residual portfolio value in the current period affects his position in the next trading round. For example, a very aggressive trading strategy like the one in the one-period problem will lead to zero portfolio value at the end of the first period if the bet goes wrong. This will leave no money on the table for the agent to invest in the subsequent periods, and the utilities will become zero thereafter since no more trading is possible.

Mathematically, (18) is demonstrating the trade-off between maximizing the reward from the current period $U(X_{T_1} - \gamma x)$ and the continuation value at the beginning of the next period summarized by $V(X_{T_1})$. It is known in our setup that $V(X_{T_1}) = A^* X_{T_1}^{\alpha}$. Larger the value of A^*, more valuable it is for the agent to stay in the trading business. Then we expect that the agent should prioritize preserving the portfolio value at a high level when A^* is large, and vice versa.

Suppose $A^* > 0$, which for example occurs under a delegated portfolio management application with the utilties entirely representing monetary compensation and managers are protected under limited-liability such that $k = 0$. Then Case 1 of Theorem 1 suggests that the support of the optimal gross return random variable $\frac{X^*_{T_{n+1}}}{X^*_{T_n}}$ is $(0, c_1\gamma) \cup (c_2\gamma, \infty)$ which does not include $\{0\}$. See Figure 2a. Hence the optimal portfolio is strictly solvent almost surely, which is drastically different from the one-period optimal strategy such as Berkelaar et al. (2004) where the ruin probability is strictly positive. Indeed, if we view the right hand side of (18) as a payoff function in X_{T_1}, then we can see that $V(X_{T_1}) = A^* X_{T_1}^{\alpha}$ is introducing a local concavity near $X_{T_1} = 0$ under $A^* > 0$. Such moderation effect is graphically illustrated in Figure 6a of Appendix A, where the function $G_{A^*}(y) := U(y - \gamma) + A^* y^{\alpha}$ can be interpreted as the "effective payoff function". This reveals a very interesting implication regarding the agent's preference: although his utility function U is S-shaped, the repeated-game nature of the problem creates incentive distortion. For the case of $A^* > 0$, local risk aversion is introduced near zero and this discourages the agent from exposing the portfolio to potential maximal loss. Proposition 5 in Appendix C further tells us that the optimal proportion of wealth invested in the risky asset is approaching the Merton ratio in both very good and very bad states of the world, as shown in Figure 3a.

However, a caveat is that the trading experience is not necessarily a favorable one in that A^* can be negative. As in Proposition 2, this can occur for example when either k or γ is too high. These conditions for instance are relevant in a behavioral context where the agent has high aspiration (reference point) and loss aversion level, or a delegated portfolio management context where the utilities also reflect other intangible losses of underperformance (such as reputational damage) in a competitive industry with a high performance benchmark. If the continuation value is negative, then there is no longer local risk aversion introduced near zero and risk aggression dominates again due to the shape of U. When A^* is mildly negative as described by Case 2a of Theorem 1, the gross return random variable for each period has an atom with strictly positive probability measure at zero and a continuous distribution on $(c_3\gamma, \infty)$ as illustrated in Figure 2b. This is qualitatively very similar to the one-period optimal solution where the agent is taking excessive downside risk and there is a strictly positive chance that the portfolio value will hit zero at the end of the period. Although Figure 3a and 3b may look indistinguishable in the bad states where the optimal portfolio gross return is very close to zero in Case 1 and is exactly zero in Case 2a, the risk taking behaviors are drastically different. In Case 2a, the portfolio weight in the risky asset still approaches the Merton ratio in the good states but explodes in the bad states. Such excessive risk taking results in a sizable chance of bankruptcy which is around 20% in the numerical example of Figure 3b. Meanwhile, the optimal portfolio in Case 1 is always strictly solvent even though it can experience a massive drawdown.

$A^* = -1$ is an interesting corner case from a theoretical perspective. The optimal portfolio weight in the risky asset now converges to another steady positive value $(\mu - r)/((2-\alpha)\sigma^2)$ in the extremely good states of the world. This number is strictly below the Merton ratio in magnitude. See Figure 3c. The agent now exhibits behaviors of underinvestment when his portfolio is performing well. This corner case arises due to the fact that, when $A^* = -1$, the effective payoff function $G_{A^*}(y)$ is saturating for large value of y and in turn the optimal target return function $y_{A^*}(q)$ in Theorem 1 has different asymptotic behaviors near $q = 0$.

As A^* becomes further negative such that we are in Case 2b of Theorem 1 (e.g. due to an even higher loss aversion level of a behavioral agent, or a more demanding performance target imposed on a portfolio manager), the support of the gross return random variable now becomes $\{0\} \cup (c_3\gamma, \frac{\gamma}{1-|A^*|^{-1/(1-\alpha)}})$. See Figure 2c. Similar to Case 2a, the portfolio may suffer from total loss. If we look at the threshold of $Z_{T_{n+1}}/Z_{T_n}$ above which the portfolio gross return becomes zero, the level of Case 2b is smaller than that of Case 2a. It suggests the ruin probability in Case 2b is larger than that of Case 2a, i.e. a larger downside risk in Case 2b. The agent now also severely underinvests in the good states of the economy such that the gross return variable has an upper bound of $\frac{\gamma}{1-|A^*|^{-1/(1-\alpha)}}$. To see why this upper bound exists, recall that the utilities of the agent are derived from the periodic performance $X_{T_i} - \gamma X_{T_{i-1}}$ which scales with the portfolio size. Hence if the performance target or the loss aversion level is too high (which results in large negative value of A^* due to Proposition 2), the agent will cautiously limit the growth of the portfolio value to avoid setting up a high basis of performance evaluation in the next period. Otherwise, if the bar is set too high then underperformance is likely to occur in the future under a demanding profit target which will be emotionally or economically painful to a loss-averse agent. This idea is also mathematically demonstrated in Figure 6c that the effective payoff function $G_{A^*}(y)$ is actually decreasing in the regime of large portfolio return, and hence the agent has an incentive to suppress the portfolio growth to avoid entering the regime on which $G_{A^*}(y)$ is decreasing. The underinvestment behavior is more intuitively shown in Figure 3d, where, away from the corner case of $A^* = -1$, the agent will adopt a gain-exit style strategy and offload his entire risky asset holding as its price increases.

Finally, when A^* becomes extremely negative as in Case 3 of Theorem 1, the optimal portfolio value process has an atom of unity at zero. In this case, the value of the trading opportunity is very unfavorable to the agent (i.e. low level of agent's value function). This can be due to an unprofitable trading environment, extreme preference parameters of the agent or an unrealistic performance goal. The agent hence just wants to walk away as soon as possible by intentionally depleting all the available capital (the portfolio budget constraint $\mathbb{E}[Z_{T_n} X^*_{T_n}] \leq x$ is not binding in, and only in, this case). Then there is no more portfolio to be managed in the future and the agent does not need to face any possible disappointment or penalty from poor trading performance. This prediction, while mathematically possible, is rather extreme. Numerically, it requires a very high performance target parameter γ for this case to occur (e.g. more than 200% per year in the example we gave in Figure 2). It arises in our model because we implicitly assume the agent does not have any exit option and the only way for the agent to get out of the trading business is to purposely get fired by losing everything. In Appendix D, we consider a variation of our model where an agent can choose to early retire at the beginning of each period.

5 Empirical and policy implications

5.1 Risk taking behaviors and performance target

Proposition 2 shows that A^* is decreasing in the performance target γ. If an increase in γ reduces A^* which in turn triggers a change of the solution regime, we know from the discussion in Section 4.2 that risk taking intensifies in the bad states of the world (when Case 1 transits to Case 2a) and underinvestment occurs in the good states of the world (when Case 2a transits to Case 2b). How do risk taking behaviors change if there is a small increase in γ which does not result in a shift of the solution regime? While there is no clear monotonicity, Figure 3a, 3b and 3d demonstrate numerically that an increase in γ tends to increase (decrease) investment in the risky asset in the bad (good) states. Combining all the theoretical and numerical findings, we conclude that an increased performance target results in more (less) risk taking in the bad (good) states.

The empirical findings of Kouwenberg and Ziemba (2007) show a positive relationship between hedge funds downside risk and incentive fee level. Our model offers a complementary empirical hypothesis, where we expect investment funds downside (upside) risk to be positively (negatively) related to the portfolio managers' performance target.

5.2 Risk taking behaviors under option-based compensation

As discussed in Section 2, the portfolio selection problem faced by an agent with limited-liability protection or option-based compensation can be described by the special case of zero loss asymmetry parameter $k = 0$. Then the periodic reward in period i, $U(X_{T_i} - \gamma X_{T_{i-1}})$, is always non-negative. From Proposition 2, $k = 0$ implies A^* is strictly positive. The optimal portfolio strategy pursued by such an agent is hence always described by Case 1 of Theorem 1, as shown in Figure 2a and 3a.

It is useful to compare this finding against standard results in the literature of risk taking incentive under option-based compensation such as Carpenter (2000). In a model featuring a manager endowed with a call option with fixed maturity on the managed portfolio, Carpenter (2000) shows that the optimal portfolio weight in the risky asset grows to infinity when the portfolio value declines and approaches the Merton ratio as the value goes up. Figure 3a reveals a new phenomenon that under periodic evaluation, the risk taking behavior changes significantly where the optimal portfolio weight now also converges to the Merton ratio when the portfolio is performing poorly. The agent avoids taking excessive risk during market downturns to ensure strict solvency, which in turn enable him to stay in the trading business and harvest future rewards beyond the current period. While the ideas about the relevance of multi-period evaluation relative to risk taking behavior are not entirely new (see, for example, Hodder and Jackwerth (2007) and Footnote 1 of Carpenter (2000)), we have provided analytical evidence to support these intuitions. In particular, the agent is taking risk in accordance to the Merton ratio as well even when the option granted for the current period is deeply out-of-money.

One natural implementation of option-based compensation with periodic payout is the cliquet option, under which the agent will receive a series of forward-start options with payoff $(X_{T_i} - \gamma X_{T_{i-1}})^+$ on each fixing date T_i. Our performance benchmark parameter γ can then be interpreted as the forward strike level of this product. Our result yields an important policy implication which favors the use of cliquet option over the traditional long-dated employee call option expiring on a fixed terminal date from the perspective of managerial risk taking (A related idea can be found in Ruß and Schelling (2018), who demonstrate via Monte Carlo simulation that the demand for cliquet-style products can be explained by a multi-period extension of Cumulative Prospect Theory (CPT) evaluation framework but not the static expected utility nor CPT criterion). Panageas and Westerfield (2009) show that excessive risk taking can also be moderated by a contract which pays out a proportional fee whenever a high-water mark has been reached. Nonetheless, cliquet option does not require continuous monitoring for payoff computation and thus it should be easier to implement such compensation structure in practice. This idea also yields an interesting and potentially testable empirical hypothesis, where we conjecture that managers who are granted cliquet options as compensation will take less risk (e.g. reflected by leverage ratio or other risk measures) during market downturns compared to those who are given long-dated call options.

5.3 Long-term portfolio risk and incentive design

From a risk management point of view, an important feature of the optimal trading strategy is the potential maximum portfolio loss. From Theorem 1, if $A^* \leq 0$ then the periodic gross return random variable has a strictly positive probability to be zero which represents a total loss. Since the gross returns from different periods are independent and identically distributed, in the long run the portfolio value must reach zero in finite time whenever $A^* \leq 0$ as a simple consequence of Borel-Cantelli lemma. We hence arrive at the following corollary.

Corollary 2. *The optimal portfolio is ruined in finite time almost surely, i.e.*

$$\mathbb{P}\left(\textit{There exists } s \textit{ such that } X_t^* = 0 \textit{ for all } t \geq s\right) = 1,$$

if and only if $A^* \leq 0$.

Our model focuses on the portfolio decision made by an agent who derives utilities based on the periodic performance in form of monetary incomes, changes in career prospects like promotion and demotion, gain/loss of professional reputation, subjective well-being from public recognition at the end of a financial quarter, etc. Their economic objectives are relatively short-sighted, which can be very different from the goal of his principal who might be looking for a long-term portfolio growth. Corollary 2 reveals a severe agency problem if the agent's incentive is not properly managed: if the model parameters are such that $A^* \leq 0$, then the agent takes excessive risk in a way that the portfolio will eventually suffer a 100% loss.

If the utility function U purely represents monetary payouts, then limited-liability protection (i.e. $k = 0$) alone is sufficient to ensure $A^* > 0$ as per Proposition 2 to avoid the long-term ruin risk, regardless of the value of γ. Otherwise if $k > 0$ (e.g. under the behavioral context with loss aversion or the interpretation that U captures other intangible negative payouts after underperformance), then γ should be determined carefully to avoid the inevitable ruin. By Proposition 2, we can find $\hat{\gamma}$ such that $A^* = A^*(\hat{\gamma}, k) = 0$. Then $\hat{\gamma}$ represents the critical value of the performance target below which the long-term solvency of the portfolio is guaranteed. We plot the values of $\hat{\gamma} = \hat{\gamma}(k)$ in Figure 4 for different values of k. The critical performance target levels are somewhat reasonable for moderately loss-averse agents (e.g a performance target below 1.25 can ensure $A^* > 0$ for agents with k below 2.25). $\hat{\gamma}(k)$ is decreasing in k, which suggests that avoidance of an overly ambitious performance target is especially important when k is large.

In parallel, it is also useful to consider measures that could help reduce the value of k if $A^* > 0$ is desired. Under a delegated portfolio management interpretation of the model, an underperforming trader may be punished which creates negative utilities (say in form of emotional impact or worsened career prospect). It is thus reasonable to expect that k is linked to the severity of punishment, and we conjecture that avoiding excessive punishment against underperformance may help reduce k. Under a behavioral interpretation that k is the agent's loss aversion level,

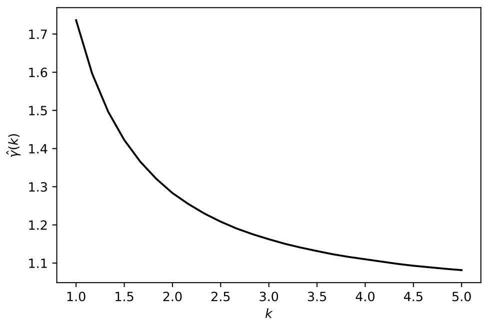


Figure 4: The value of $\hat{\gamma}$ as k varies. Each $\hat{\gamma} = \hat{\gamma}(k)$ represents the value of γ such that $A^* = A^*(\gamma, k) = 0$. Base parameters used are $\alpha = 0.5$, $\mu = 0.1$, $\sigma = 0.2$, $r = 0.005$, $\beta = 0.3$ and $\tau = 1$.

psychology literature also hints a few possible behavioral engineering ("nudging") approaches to moderate one's loss aversion. Examples include enforcing the accountability of the agents (Vieider (2009)), promoting reappraisal of the investment process (Sokol-Hessner et al. (2009)), assisting the agents to acquire experience and knowledge (List (2003), Mrkva et al. (2020)), etc.

5.4 Evaluation horizon, demand for risky assets and equity premium

In their seminal work, Benartzi and Thaler (1995) consider a portfolio selection problem faced by a myopic, loss-averse agent who evaluates his portfolio regularly according to a mental accounting schedule. They find that a shorter evaluation horizon (i.e. more frequent evaluations) results in a lower wealth allocation to risky stocks (see also He and Zhou (2014)). Benartzi and Thaler (1995) conclude this feature offers an explanation to the equity premium puzzle because a myopic agent with short evaluation horizon finds it unattractive to hold stocks and hence demands a higher equity return in equilibrium. Our model is quite different from that of Benartzi and Thaler (1995), as we do not assume the agent is myopic but instead he is able to take the subsequent rewards beyond the first evaluation period into account. Since the evaluation horizon is a crucial quantity which forms the basis of Benartzi and Thaler's argument, it is useful to examine how the optimal investment is influenced by τ in our model. Figure 5 shows the optimal fraction of wealth invested in the risky asset at the beginning of each period under several values of τ. We fix $\gamma = 1$, which is consistent with the definition of gain-and-loss in Benartzi and Thaler (1995). An increase in τ generally leads to higher leverage (i.e. stock becomes more attractive) in the

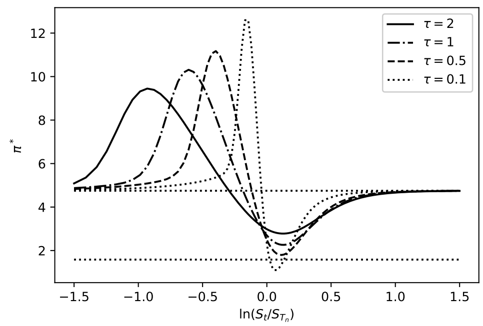


Figure 5: The optimal proportion of wealth invested in the risky asset as a function of $\ln(S_t/S_{T_n})$ at $t = T_n+$ under several values of τ. The two horizontal dotted lines mark the Merton ratio $(\mu - r)/((1-\alpha)\sigma^2)$ and the constant $(\mu - r)/((2-\alpha)\sigma^2)$ respectively. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $\mu = 0.1$, $\sigma = 0.2$, $r = 0.005$ and $\beta = 0.3$.

mildly good states (when $\ln \frac{S_t}{S_{T_n}}$ is between 0% and 15%) or extremely bad states (when $\ln \frac{S_t}{S_{T_n}}$ is less than -70%). But the relationship is reversed in the moderately bad states. The results are perhaps not too surprising: if the agent begins in the bad states of the world and the evaluation date is fast approaching (τ decreases), the convexity of U over losses encourages the agent to gamble more aggressively to get out from the bad states. However, as the market reaches an extremely poor state, the agent will be reluctant to take more risk even when τ drops because he has to ensure the long-term solvency of the portfolio as per Case 1 of our solution regime.

While Benartzi and Thaler (1995) suggest that increasing the evaluation horizon makes risky investment more appealing (and vice versa), our result highlights that the explanatory power of evaluation horizon on demand for stocks interacts with the states of the world in a subtle way. It is highly non-trivial to conclude how the asset allocation in our model changes on average across the realized stock price paths when the evaluation horizon τ changes. Our findings may have potential implications which connect τ to the variability of the equity premium with the business cycle, although this can only be examined in a more carefully crafted equilibrium model which is beyond the scope of the current paper.

6 Concluding remarks

We have solved an infinite horizon portfolio selection problem under S-shaped utility function and periodic evaluation criterion. The non-concave utility function in conjunction with the periodic, infinite horizon nature make the problem economically and technically interesting. The optimal trading strategies have a number of possible forms depending on the model parameters, and such behaviors cannot be observed in a simple one-period, finite horizon model. Risk taking behaviors in the bad states of the world can be moderated by a favorable periodic reward structure because the agent wants to ensure solvency of the portfolio to sustain a perpetual income flow. If the trading environment is challenging, underinvestment might occur in the good states of the world to alleviate the pressure on the agent to outperform his yesterday's self. Under loss aversion and/or the lack of limited-liability protection, the optimal portfolio might be ruined in the long run. Consideration of cliquet-style compensation, imposing a realistic performance target and reducing the agent's loss aversion are examples of crucial features within incentive design and behavioral engineering to curb the excessive risk taking behaviors of an agent with S-shaped preference.

Our model focuses on the Tversky-Kahneman utility function with identical risk aversion and risk seeking parameters. The scaling property of the power function allows us to offer an economically transparent characterization of the optimal portfolio and the trading strategies. One tempting extension is to consider a more generic class of utility (payoff) functions such that the model can be generalized to agents under different preferences or compensation structures, or the combination of both. The issue, however, is that establishing the existence and uniqueness of the fixed-point of the Bellman operator as well as identifying the general form of the optimizer to the auxiliary problem can be very challenging without any restriction on the underlying utility function.

The modeling framework in this paper can also serve as the foundation of a more sophisticated principal-agent problem. Our results reveal that under certain model parameters the optimal strategy pursued by the agent will result in eventual collapse of the portfolio and this is clearly undesirable to a principal seeking long-term performance. Then a natural follow-up question is how the periodic contractual payoff (i.e. the utility function) can be optimally structured to align the short-term periodic interest of the agent and the long-term investment goal of a principal. Such consideration should prove to be an interesting proposal for future research.

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Appendix

A Solution to the auxiliary problem

In this section, we analyze in details the auxiliary problem (14). It turns out that the function G_a in (13) can have vastly different shapes depending on the value of a. It is not concave nor even monotonic in general. As a result, problem (14) is not directly covered by standard results in the literature for finite horizon non-concave utility maximization such as Reichlin (2013) and Bichuch and Sturm (2014), where the utility function is required to be strictly increasing. To solve the auxiliary problem (14), we will adopt the concavification technique where the idea is to solve a different version of problem (14) upon replacing G_a by $\bar{G}_a$ which is defined as the smallest concave majorant of G_a. This leads to an upper bound of the value function in (14), and we will show that the value function of the original and the concavified version of the problem coincide.

The precise mathematical expressions of G_a and $\bar{G}_a$ under different cases are summarized by the following two lemmas.

Lemma 1. *$G_a(y)$ has the following properties:*

1. *For $a > 0$, $G_a(y)$ is concave increasing on $0 \leq y \leq \frac{\gamma}{1+(k/a)^{1/(2-\alpha)}}$, convex increasing on $\frac{\gamma}{1+(k/a)^{1/(2-\alpha)}} < y \leq \gamma$ and concave increasing on $y > \gamma$.*
2. *For $a = 0$, $G_a(y)$ is convex increasing on $0 \leq y \leq \gamma$ and concave increasing on $y > \gamma$.*
3. *For $-1 \leq a < 0$, $G_a(y)$ is convex decreasing on $0 \leq y \leq \frac{\gamma}{1+(k/|a|)^{1/(1-\alpha)}}$, convex increasing on $\frac{\gamma}{1+(k/|a|)^{1/(1-\alpha)}} < y \leq \gamma$, and concave increasing on $y > \gamma$.*
4. *For $a < -1$, $G_a(y)$ is convex decreasing on $0 \leq y \leq \frac{\gamma}{1+(k/|a|)^{1/(1-\alpha)}}$, convex increasing on $\frac{\gamma}{1+(k/|a|)^{1/(1-\alpha)}} < y \leq \gamma$, concave increasing on $\gamma < y \leq \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$, concave decreasing on $\frac{\gamma}{1-|a|^{-1/(1-\alpha)}} < y \leq \frac{\gamma}{1-|a|^{-1/(2-\alpha)}}$ and convex decreasing on $y > \frac{\gamma}{1-|a|^{-1/(2-\alpha)}}$. Moreover,*
   (a) *If $-(k^{\frac{1}{1-\alpha}}+1)^{1-\alpha} < a < -1$, $G_a(y)$ attains its global maximum at $y = \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$.*
   (b) *If $a < -(k^{\frac{1}{1-\alpha}}+1)^{1-\alpha}$, $G_a(y)$ attains its global maximum at $y = 0$.*

Proof. Proof of Lemma 1. The results follow from elementary calculus. Differentiation gives

$$G'_a(y) = \begin{cases} \alpha[(y-\gamma)^{\alpha-1} + ay^{\alpha-1}], & y \geq \gamma; \\ \alpha[k(\gamma-y)^{\alpha-1} + ay^{\alpha-1}], & 0 \leq y < \gamma, \end{cases}$$

and

$$G_a''(y)=\begin{cases}-\alpha(1-\alpha)[(y-\gamma)^{\alpha-2}+ay^{\alpha-2}], & y\geq\gamma;\\ \alpha(1-\alpha)[k(\gamma-y)^{\alpha-2}-ay^{\alpha-2}], & 0\leq y<\gamma.\end{cases}$$

If $a>0$, then trivially $G_a'(y)\geq 0$ for all y. Furthermore, it is easy to verify that $G_a''(y)\leq 0$ on $y\geq\gamma$ and $0\leq y\leq\frac{\gamma}{1+(k/a)^{1/(2-\alpha)}}$, and $G_a''(y)\geq 0$ on $\frac{\gamma}{1+(k/a)^{1/(2-\alpha)}}\leq y\leq\gamma$. Thus G_a has the shape described in Case 1.

Case 2 with $a=0$ is trivial.

For $a<0$ in Case 3, on $0\leq y\leq\gamma$ we have $G_a'(y)=\alpha[k(\gamma-y)^{\alpha-1}-|a|y^{\alpha-1}]$ which is increasing in y with $G_a'(0+)=-\infty$, $G_a'(\gamma-)=+\infty$ and $G_a'(\frac{\gamma}{1+(k/|a|)^{1/(1-\alpha)}})=0$ where $\frac{\gamma}{1+(k/|a|)^{1/(1-\alpha)}}<\gamma$. Moreover, on $y\leq\gamma$ we also have $G_a''(y)=\alpha(1-\alpha)[k(\gamma-y)^{\alpha-2}+|a|y^{\alpha-2}]\geq 0$ and the shape of $G_a(y)$ on $0\leq y\leq\gamma$ follows immediately. Furthermore, with $-1\leq a\leq 0$, on $y\geq\gamma$ one can see that $G_a'(y)=\alpha[(y-\gamma)^{\alpha-1}-|a|y^{\alpha-1}]\geq\alpha[(y-\gamma)^{\alpha-1}-y^{\alpha-1}]\geq 0$ and $G_a''(y)=-\alpha(1-\alpha)[(y-\gamma)^{\alpha-2}-|a|y^{\alpha-2}]\leq-\alpha(1-\alpha)[(y-\gamma)^{\alpha-2}-y^{\alpha-2}]\leq 0$ so $G_a(y)$ is increasing and concave on $y\geq\gamma$.

Finally, when $a<-1$ as in Case 4, on $y\geq\gamma$ check that $G_a'(y)=\alpha[(y-\gamma)^{\alpha-1}-|a|y^{\alpha-1}]=\alpha y^{\alpha-1}[(1-\gamma/y)^{\alpha-1}-|a|]$ such that $G_a'(y)\geq 0$ on $\gamma\leq y\leq\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$ and $G_a'(y)\leq 0$ on $y>\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$ with $\gamma<\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}<\infty$, and $G_a''(y)=-\alpha(1-\alpha)[(y-\gamma)^{\alpha-2}-|a|y^{\alpha-2}]=-\alpha(1-\alpha)y^{\gamma-2}[(1-\gamma/y)^{\alpha-2}-|a|]$ whence $G_a''(a)\leq 0$ on $\gamma\leq y\leq\frac{\gamma}{1-|a|^{-1/(2-\alpha)}}$ and $G_a''(y)\geq 0$ on $y>\frac{\gamma}{1-|a|^{-1/(2-\alpha)}}$ with $\gamma<\frac{\gamma}{1-|a|^{-1/(2-\alpha)}}<\infty$. Since $y=\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$ is a local maxima, the global maximum of $G_a(y)$ is given by

$$\max_{y\geq 0}G_a(y)=\max\left(G_a(0),G_a\left(\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}\right)\right)=\max(-k\gamma^\alpha,-|a|(1-|a|^{-\frac{1}{1-\alpha}})^{1-\alpha}\gamma^\alpha)$$

where the ordering is determined by the ordering of $|a|$ versus $(k^{\frac{1}{1-\alpha}}+1)^{1-\alpha}$. Then the results in Case 4a and 4b follow. □ □

Lemma 2. *$\bar{G}_a(y)$, the smallest concave majorant of $G_a(y)$, has the following expression:*

1. *For $a>0$,*

$$\bar{G}_a(y)=\begin{cases}-k(\gamma-y)^\alpha+ay^\alpha, & 0\leq y<c_1\gamma;\\ -k\gamma^\alpha(1-c_1)^\alpha+ac_1^\alpha\gamma^\alpha+m_1(y-c_1\gamma), & c_1\gamma\leq y\leq c_2\gamma;\\ (y-\gamma)^\alpha+ay^\alpha, & y>c_2\gamma,\end{cases}$$

where

$$m_1 := \gamma^{\alpha-1} \frac{(c_2-1)^\alpha + ac_2^\alpha + k(1-c_1)^\alpha - ac_1^\alpha}{c_2 - c_1}, \tag{19}$$

and c_1, c_2 *with* $0 < c_1 < \frac{1}{1+(k/a)^{1/(2-\alpha)}} < 1 < c_2 < \infty$ *are the solutions to the system of equations*

$$\frac{(c_2-1)^\alpha + ac_2^\alpha + k(1-c_1)^\alpha - ac_1^\alpha}{c_2 - c_1} = \alpha[(c_2-1)^{\alpha-1} + ac_2^{\alpha-1}] = \alpha[k(1-c_1)^{\alpha-1} + ac_1^{\alpha-1}]. \tag{20}$$

2. *For* $-1 \le a \le 0$,

$$\bar{G}_a(y) = \begin{cases} -k\gamma^\alpha + m_2 y, & 0 \le y \le c_3\gamma; \\ (y-\gamma)^\alpha + ay^\alpha, & y > c_3\gamma, \end{cases}$$

where

$$m_2 := \gamma^{\alpha-1} \frac{(c_3-1)^\alpha + ac_3^\alpha + k}{c_3}, \tag{21}$$

and $c_3 > 1$ *is the solution to the equation*

$$\frac{(c_3-1)^\alpha + ac_3^\alpha + k}{c_3} = \alpha[(c_3-1)^{\alpha-1} + ac_3^{\alpha-1}]. \tag{22}$$

3. *For* $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < a < -1$,

$$\bar{G}_a(y) = \begin{cases} -k\gamma^\alpha + m_2 y, & 0 \le y \le c_3\gamma; \\ (y-\gamma)^\alpha + ay^\alpha, & c_3\gamma < y < \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}; \\ -\gamma^\alpha |a| (1-|a|^{-\frac{1}{1-\alpha}})^{1-\alpha}, & y \ge \frac{\gamma}{1-|a|^{-1/(1-\alpha)}} \end{cases}$$

where again m_2 *is defined in* (21) *and* c_3 *is the solution to equation* (22).

4. *For* $a \le -(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha}$, $\bar{G}_a(y) = -k\gamma^\alpha$.

Proof. Proof of Lemma 2. The results follow from the various shapes of G_a derived in Lemma 1.

In Case 1, $\bar{G}_a$ coincides with G_a for small and large value of y, and for intermediate value of y it is a straight tangent line touching G_a at some y_1 and y_2 where $y_1 < \frac{\gamma}{1+(k/a)^{1/(2-\alpha)}} < \gamma < y_2$.

See Figure 6a. By equating the slope of this tangent and the first order derivatives of G_a at y_1 and y_2, we have

$$\frac{G_a(y_2) - G_a(y_1)}{y_2 - y_1} = G_a'(y_1) = G_a'(y_2)$$

and we arrive at (20) upon setting $y_1 = c_1\gamma$ and $y_2 = c_2\gamma$ for some $0 < c_1 < 1 < c_2$.

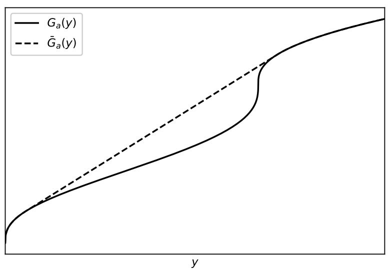
In Case 2, $\bar{G}_a$ coincides with G_a for large value of y only, and it is a straight tangent line passing through $(0, G_a(0))$ which touches $G_a(y)$ at some $y_3 > \gamma$. The point of contact y_3 satisfies the condition $\frac{G_a(y_3)-G_a(0)}{y_3} = G_a'(y_3)$ leading to (22) after setting $y_3 = c_3\gamma$ for some $c_3 > 1$. See Figure 6b.

In Case 3, $G_a(y)$ attains its global maximum at $y = \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$ and is monotonically decreasing afterwards with $G_a''(y) > 0$ for large y and $\lim_{y\to\infty} G_a'(y) = 0$. Hence $\bar{G}_a(y)$ must be a flat horizon line with level $G_a\left(\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}\right) = -\gamma^\alpha|a|(1-|a|^{-\frac{1}{1-\alpha}})^{1-\alpha}$ for $y \geq \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$. Meanwhile, for small value of y, $\bar{G}_a$ is a tangent line passing through $(0, G_a(0))$ which touches $G_a(y)$ at some $y_3 > \gamma$, and for $y_3 \leq y \leq \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$ $\bar{G}_a$ is coinciding with G_a. Here y_3 again satisfies $\frac{G_a(y_3)-G_a(0)}{y_3} = G_a'(y_3)$ which is identical to the condition in Case 2. See Figure 6c.

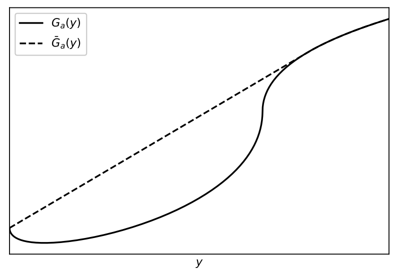
Finally in Case 4 where the global maximum is attained at $y = 0$, $\bar{G}_a(y)$ must simply be a flat horizon line with level $G_a(0) = -k\gamma^\alpha$ since G_a is convex for large value of y with $\lim_{y\to\infty} G_a'(y) = 0$. See Figure 6d. □ □

The possible shapes of G_a and $\bar{G}_a$ are displayed in Figure 6. When $a > 0$, $G_a(y)$ is increasing and locally concave for small and large value of y, and is convex elsewhere. The function is hence concavified by replacing the intermediate region of the function by a linear chord touching the function itself at two different points (Figure 6a). If $-1 \leq a \leq 0$, G_a is not monotonic in general except when $a = 0$. It is convex (resp. concave) for small (resp. large) value of y, and thus the concavification is given by a straight line passing through $(0, G_a(0))$ which touches G_a at some critical point (Figure 6b). For $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < a < -1$, $G_a(y)$ attains an interior global maximum at some y_{max} and the function is now convex decreasing for very large y with vanishing slope. Hence another concavification has to be applied by drawing a flat horizon line truncating $G_a(y)$ on $y > y_{max}$ (Figure 6c). Finally, when $a \leq -(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha}$, the function attains its global maximum at $y = 0$. The concavification turns out to simply be a flat horizontal line with level $G_a(0)$ for all $y \geq 0$.

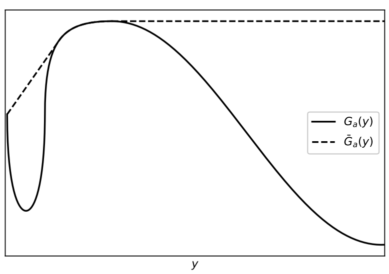
The solution to the auxiliary problem (14) is as follows.



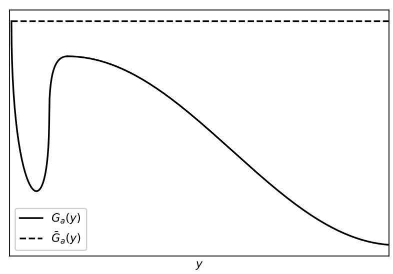
(a) $a > 0$.



(b) $-1 \leq a \leq 0$.



(c) $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < a < -1$.



(d) $a \leq -(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha}$.

Figure 6: Stylized plots of the objective function of the auxiliary problem $G_a(y)$ and its smallest concave majorant $\bar{G}_a(y)$ under different values of a.

Proposition 3. *The optimizer to problem* (14) *is given by* $Y^* = y_a^*(\lambda^* Z_\tau)$, *where:*

1. *If* $a > 0$, *then*

$$y_a^*(q) = \begin{cases} I_2(q), & 0 < q \leq m_1; \\ I_1(q), & q > m_1, \end{cases}$$

where $I_1 = I_1(q) : (m_1, \infty) \to (0, c_1\gamma)$ *is the solution to the equation* $\alpha[k(\gamma - y)^{\alpha-1} + ay^{\alpha-1}] = q$ *on* $y \in (0, c_1\gamma)$ *and* $I_2 = I_2(q) : (0, m_1] \to [c_2\gamma, \infty)$ *is the solution to the equation* $\alpha[(y-\gamma)^{\alpha-1} + ay^{\alpha-1}] = q$ *on* $y \in [c_2\gamma, \infty)$.

2. *If* $-1 \leq a \leq 0$, *or* $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < a < -1$ *and* $|a|^{-\frac{1}{1-\alpha}} > 1 - \gamma e^{-r\tau}$, *then*

$$y_a^*(q) = \begin{cases} I_2(q), & 0 < q \leq m_2; \\ 0, & q > m_2, \end{cases}$$

where $I_2 = I_2(q)$ *is the solution to the equation* $\alpha[(y-\gamma)^{\alpha-1} + ay^{\alpha-1}] = q$ *on* $y \geq c_3\gamma$ *for* $q \in (0, m_2]$. *Furthermore:*

   (a) *If* $-1 \leq a \leq 0$, $I_2(q) : (0, m_2] \to [c_3\gamma, \infty)$.

   (b) *If* $-(k^{\frac{1}{1-\alpha}}+1)^{1-\alpha} < a < -1$ *and* $|a|^{-\frac{1}{1-\alpha}} > 1-\gamma e^{-r\tau}$, $I_2(q) : (0, m_2] \to [c_3\gamma, \frac{\gamma}{1-|a|^{-1/(1-\alpha)}})$.

3. *If* $a \leq -(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha}$, *then* $y_a^*(\cdot)$ *is a constant function equal to* 0.

4. *If* $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < a < -1$ *and* $|a|^{-\frac{1}{1-\alpha}} \leq 1 - \gamma e^{-r\tau}$, *then* $y_a^*(\cdot)$ *is a constant function equal to* $\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$.

In the above definitions, $m_1 > 0$, $m_2 > 0$, $0 < c_1 < 1 < c_2$ *and* $c_3 > 1$ *are constants depending on* a *which are defined in Lemma 2. In Case 1, 2a and 2b,* λ^* *is given by the unique* $\lambda > 0$ *such that* $\mathbb{E}[Z_\tau y_a^*(\lambda Z_\tau)] = 1$.

Proof. Proof of Proposition 3. We first prove the results for Case 3 and 4. In Case 4 where $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < a < -1$, $G_a(y)$ attains its global maximum at $\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$ and as such $\mathbb{E}[G(Y)] \leq G\left(\frac{\gamma}{1-|a|^{-1/(1-\alpha)}}\right)$ with

$$\mathbb{E}\left[Z_\tau \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}\right] = \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}\mathbb{E}[Z_\tau] = \frac{\gamma e^{-r\tau}}{1-|a|^{-1/(1-\alpha)}} \leq 1$$

under the additional condition $|a|^{-\frac{1}{1-\alpha}} \leq 1 - \gamma e^{-r\tau}$. Hence $Y^* = \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}$ is feasible to problem (14) and in turn optimal. Case 3 can be handled similarly.

Now we consider Case 1, 2a and 2b. The Legendre-Fenchel transformation of a function f is defined as

$$J_f(q) := \sup_{y \geq 0}(f(y) - qy), \qquad q > 0.$$

Provided that f is continuous and concave with $f'(\infty) = 0$, the maximizer to the above supremum always exists (although not necessarily unique) and we write $y_f^*(q) := \arg\max_{y \geq 0}(f(y) - qy)$ such that

$$J_f(q) = f(y_f^*(q)) - qy_f^*(q).$$

Then for any $\lambda > 0$ and $Y \in \mathcal{F}_\tau^+$ with $\mathbb{E}[Z_\tau Y] \leq 1$,

$$\mathbb{E}[G_a(Y) - \lambda(Z_\tau Y)] \leq \mathbb{E}[\bar{G}_a(Y) - \lambda(Z_\tau Y)] \leq \mathbb{E}[J_{\bar{G}_a}(\lambda Z_\tau)] = \mathbb{E}[\bar{G}_a(y^*_{\bar{G}_a}(\lambda Z_\tau)) - \lambda Z_\tau y^*_{\bar{G}_a}(\lambda Z_\tau)]$$

and in turn

$$\begin{aligned}\mathbb{E}[G_a(Y)] &\leq \mathbb{E}[\bar{G}_a(y^*_{\bar{G}_a}(\lambda Z_\tau)) - \lambda Z_\tau y^*_{\bar{G}_a}(\lambda Z_\tau)] + \lambda \mathbb{E}(Z_\tau Y) \\ &\leq \mathbb{E}[\bar{G}_a(y^*_{\bar{G}_a}(\lambda Z_\tau))] - \lambda(\mathbb{E}(Z_\tau y^*_{\bar{G}_a}(\lambda Z_\tau)) - 1).\end{aligned}$$

If there exists $\lambda^* > 0$ such that $\mathbb{E}(Z_\tau y^*_{\bar{G}_a}(\lambda^* Z_\tau)) = 1$, then we have $\mathbb{E}[G_a(Y)] \leq \mathbb{E}[\bar{G}_a(y^*_{\bar{G}_a}(\lambda^* Z_\tau))]$ and hence

$$F(a) := \sup_{Y \in \mathcal{L}_{0,\tau}(1)} \mathbb{E}[G_a(Y)] \leq \mathbb{E}[\bar{G}_a(y^*_{\bar{G}_a}(\lambda^* Z_\tau))].$$

where we write

$$\mathcal{L}_{s,t}(x) := \left\{ Y \in \mathcal{F}_t^+ : \mathbb{E}\left[\frac{Z_t}{Z_s} Y \middle| \mathcal{F}_s\right] \leq x \right\}, \qquad s \leq t. \tag{23}$$

Moreover, if we can show that the support of the random variable $y^*_{\bar{G}_a}(\lambda^* Z_\tau))$ is a subset of $\{y \geq 0 : G_a(y) = \bar{G}_a(y)\}$. Then we can conclude $\mathbb{E}[G_a(y^*_{\bar{G}_a}(\lambda^* Z_\tau))] = \mathbb{E}[\bar{G}_a(y^*_{\bar{G}_a}(\lambda^* Z_\tau))]$ and hence $Y^* := y^*_{\bar{G}_a}(\lambda^* Z_\tau)$ must be an optimizer to problem (14).

Write $y_a^*(\cdot) = y^*_{\bar{G}_a}(\cdot)$ for brevity. We now identify the form of $y_a^*(\cdot)$ for each case. If $a > 0$ (Case 1), we have

$$\bar{G}_a'(y) = \begin{cases} \alpha[k(\gamma - y)^{\alpha-1} + ay^{\alpha-1}] =: q_1(y), & 0 \leq y < c_1\gamma; \\ \gamma^{\alpha-1} \frac{(c_2-1)^\alpha + ac_2^\alpha + k(1-c_1)^\alpha - ac_1^\alpha}{c_2 - c_1} =: m_1, & c_1\gamma \leq y \leq c_2\gamma; \\ \alpha[(y-\gamma)^{\alpha-1} + ay^{\alpha-1}] =: q_2(y), & y > c_2\gamma. \end{cases}$$

By considering the first order condition, $y_a^*(q)$ is the solution to $q_1(y) = q$ on $y \in (0, c_1\gamma)$ when $q > m_1$, the solution to $q_2(y) = q$ on $y \in (c_2\gamma, \infty)$ when $q < m_1$, and any value in $[c_1\gamma, c_2\gamma]$ will attain the maximum when $q = m_1$. Hence an optimizer is

$$y_a^*(q) = \begin{cases} I_2(q), & 0 < q \le m_1; \\ I_1(q), & q > m_1. \end{cases}$$

From the above it is also easy to see that $y_a^*(q)$ is decreasing, $y_a^*(0+) = +\infty$ and $y_a^*(+\infty) = 0$. Since Z_τ is atomless, $\zeta(\lambda) := \mathbb{E}(Z_\tau y_a^*(\lambda Z_\tau))$ is a continuous decreasing function with $\zeta(0+) = +\infty$ and $\zeta(+\infty) = 0$ by monotone convergence theorem. There must exist a unique $\lambda^* > 0$ such that $\zeta(\lambda^*) = 1$. The optimal random variable is $Y_a^* = y_a^*(\lambda^* Z_\tau)$ with support $(0, c_1\gamma) \cup (c_2\gamma, \infty)$, which is exactly the set on which $G_a(y) = \bar{G}_a(y)$.

Case 2a is omitted since the analysis is largely similar to that of Case 1. For Case 2b where $-(k^{\frac{1}{1-\alpha}} + 1)^{1-\alpha} < a < -1$, we have

$$\bar{G}_a'(y) = \begin{cases} \gamma^{\alpha-1}\frac{(c_3-1)^\alpha + ac_3^\alpha + k}{c_3} =: m_2, & 0 \le y \le c_3\gamma; \\ \alpha[(y-\gamma)^{\alpha-1} + ay^{\alpha-1}] = q_2(y), & c_3\gamma < y < \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}; \\ 0, & y \ge \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}, \end{cases}$$

from which we can deduce from the first order condition that

$$y_a^*(q) = \begin{cases} I_2(q), & 0 < q \le m_2; \\ 0, & q > m_2, \end{cases}$$

with $I_2(q)$ being the solution to $q_2(y) = q$ on $y \in [c_3\gamma, \frac{\gamma}{1-|a|^{-1/(1-\alpha)}})$ when $q \in (0, m_2]$. Thus $\zeta(\lambda) = \mathbb{E}(Z_\tau y_a^*(\lambda Z_\tau))$ is a continuous, bounded and decreasing function with $\zeta(0+) = \frac{\gamma}{1-|a|^{-1/(1-\alpha)}}\mathbb{E}(Z_\tau) = \frac{\gamma e^{-r\tau}}{1-|a|^{-1/(1-\alpha)}}$ and $\zeta(+\infty) = 0$. Hence provided that $\frac{\gamma e^{-r\tau}}{1-|a|^{-1/(1-\alpha)}} > 1 \iff |a|^{-\frac{1}{1-\alpha}} > 1 - \gamma e^{-r\tau}$, there exists $\lambda^* > 0$ such that $\zeta(\lambda^*) = 1$. The optimal random variable $Y^* = y_a^*(\lambda^* Z_\tau)$ has support $\{0\} \cup (c_3\gamma, \frac{\gamma}{1-|a|^{-1/(1-\alpha)}})$ coinciding $\{y \ge 0 : G_a(y) = \bar{G}_a(y)\}$. □ □

An important technical result is given below which will help characterize our value function as a fixed-point.

Proposition 4. *Define the function*

$$\Phi(a) := e^{-\beta\tau} F(a) \tag{24}$$

where $F(a)$ is the solution to problem (14). Then Φ is a contraction on the metric space $(\mathbb{R}, |\cdot|)$ where $|\cdot|$ is the Euclidean norm. In turn, Φ admits a unique fixed-point A^* such that $A^* = \Phi(A^*)$.

Proof. Proof of Proposition 4. Consider any arbitrary $a_1, a_2 \in \mathbb{R}$. From Proposition 3 there exists an optimizer Y_1^* such that

$$\Phi(a_1) = e^{-\beta\tau}F(a_1) = \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[e^{-\beta\tau}U(Y-\gamma) + a_1 e^{-\beta\tau}Y^\alpha\right] = \mathbb{E}\left[e^{-\beta\tau}U(Y_1^*-\gamma) + a_1 e^{-\beta\tau}(Y_1^*)^\alpha\right]$$

and hence

$$\begin{aligned}
\Phi(a_1) - \Phi(a_2) &= \mathbb{E}\left[e^{-\beta\tau}U(Y_1^*-\gamma) + a_1 e^{-\beta\tau}(Y_1^*)^\alpha\right] - \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[e^{-\beta T_1}U(Y-\gamma) + a_2 e^{-\beta\tau}Y^\alpha\right] \\
&\leq \mathbb{E}\left[e^{-\beta\tau}U(Y_1^*-\gamma) + a_1 e^{-\beta\tau}(Y_1^*)^\alpha\right] - \mathbb{E}\left[e^{-\beta\tau}U(Y_1^*-\gamma) + a_2 e^{-\beta\tau}(Y_1^*)^\alpha\right] \\
&= (a_1 - a_2)e^{-\beta\tau}\mathbb{E}\left[(Y_1^*)^\alpha\right].
\end{aligned}$$

Then

$$|\Phi(a_1) - \Phi(a_2)| \leq |a_1 - a_2| e^{-\beta\tau}\left[\sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[Y^\alpha\right]\right] = |a_1 - a_2| e^{-(\beta-h)\tau} < |a_1 - a_2|$$

using the fact that the supremum above is equal to $e^{h\tau}$ since it is simply the value function of a standard finite horizon Merton problem with utility function $u(w) = w^\alpha$, maturity τ and unit initial wealth. The last inequality is due to our standing assumption that $\beta > h$. The existence and uniqueness of the fixed-point A^* of Φ immediately follow from Banach contraction theorem. □ □

As a by-product of the proposition, A^* can be computed numerically by an iterative method in form of $A_{n+1} = \Phi(A_n)$ for any initial guess $A_0 \in \mathbb{R}$. Each step of iteration can be performed by solving the auxiliary problem (14) using the results in Proposition 3.

Finally, we offer few additional technical results related to $\Phi(a)$ and A^* which will be used in some of the subsequent proofs.

Lemma 3. *Recall the definition of Φ in (24).*

1. *For all $a \in \mathbb{R}$,*

$$\max(e^{-\beta\tau}U(e^{r\tau} - \gamma) + ae^{-(\beta-\alpha r)\tau}, -k\gamma^\alpha e^{-\beta\tau}) \leq \Phi(a) \leq e^{-(\beta-h)\tau}(a^+ + 1).$$

2. *A^* the unique fixed-point of $\Phi(\cdot)$ satisfies*

$$\frac{e^{-\beta\tau}U(e^{r\tau}-\gamma)}{1-e^{-(\beta-r\alpha)\tau}} \leq A^* \leq \frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}}.$$

3. *$A^* \leq 0$ if and only if $\Phi(0) \leq 0$.*

Proof. Proof of Lemma 3. For part (1), the lower bound can be derived by noticing that both $Y = e^{r\tau}$ and $Y = 0$ are admissible to problem (14). Meanwhile,

$$\begin{aligned}\Phi(a) &= \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[e^{-\beta\tau}U(Y-\gamma) + ae^{-\beta\tau}Y^\alpha\right] \\ &\leq \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[e^{-\beta\tau}U(Y-\gamma)\right] + a^+ \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[e^{-\beta\tau}Y^\alpha\right] \\ &\leq \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[e^{-\beta\tau}U(Y)\right] + a^+ \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[e^{-\beta\tau}Y^\alpha\right] = (a^+ + 1)e^{-\beta\tau} \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[Y^\alpha\right]\end{aligned}$$

where we have used the fact that U is increasing and $\gamma > 0$. But again the supremum in the last term is $e^{h\tau}$ the value function of a standard finite horizon Merton problem and hence $\Phi(a) \leq (a^+ + 1)e^{-(\beta-h)\tau}$.

For part (2), the lower bound of A^* can be obtained using part (1) of the lemma that $A^* = \Phi(A^*) \geq e^{-\beta\tau}U(e^{r\tau}-\gamma) + A^* e^{-(\beta-\alpha r)\tau}$. For the upper bound, notice that the map $a \to e^{-(\beta-h)\tau}(a^+ + 1)$ crosses the identity function $a \to a$ exactly once on $a > 0$ at $\hat{a} := \frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}}$. Then we must have $A^* \leq \hat{a}$ since $\Phi(\cdot)$ is dominated by $e^{-(\beta-h)\tau}(a^+ + 1)$ and A^* is given by the unique crossing point between Φ and the identity function.

Finally, since Φ is continuous and admits a unique fixed-point, the function $f(a) := \Phi(a) - a$ should change sign exactly once at $a = A^*$. Using part (1) of the lemma, Φ is bounded from below by a constant and hence $\lim_{a\to-\infty} f(a) = +\infty$. Thus if $\Phi(0) \leq 0$ then $f(0) = \Phi(0) - 0 \leq 0$ and we must have $A^* \leq 0$ since f is continuous. Similarly, if $A^* \leq 0$ then we must have $f(a) \leq 0$ for $a \geq 0 \geq A^*$. Then $\Phi(0) - 0 = f(0) \leq 0$. This establishes the claim in part (3). □ □

B Proofs of the main results

Proof. Proof of Proposition 1. The lower bound of $V(x)$ can be obtained by the fact that $X_t = xe^{rt}$ is an admissible portfolio in $\mathcal{A}_0(x)$.

To establish the upper bound, since U is increasing and $\gamma > 0$, we have for any $X \in \mathcal{A}_0(x)$ and $n \geq 1$ that

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}\left[e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}})\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[e^{-\beta T_i} U(X_{T_i})\right] &\leq \sum_{i=1}^{n} e^{-\beta T_i} \sup_{X\in\mathcal{A}_0(x)} \mathbb{E}\left[U(X_{T_i})\right] \\ &= \sum_{i=1}^{n} e^{-\beta T_i} \sup_{X\in\mathcal{A}_0(x)} \mathbb{E}\left[X^{\alpha}_{T_i}\right].\end{aligned}$$

But $\sup_{X\in\mathcal{A}_0(x)} \mathbb{E}\left[X^{\alpha}_{T_i}\right]$ is simply the value function of a finite horizon Merton investment problem with power utility function, maturity T_i and initial wealth of x. The solution is known as

$$\sup_{X\in\mathcal{A}_0(x)} \mathbb{E}\left[X^{\alpha}_{T_i}\right] = x^{\alpha} \exp\left[\left(r\alpha + \frac{(\mu-r)^2}{2\sigma^2}\frac{\alpha}{1-\alpha}\right) T_i\right] = x^{\alpha} e^{i\tau h}.$$

Hence

$$\sum_{i=1}^{n} \mathbb{E}\left[e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}})\right] \leq x^{\alpha} \sum_{i=1}^{n} e^{-(\beta-h)i\tau} \leq \frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}} x^{\alpha}$$

provided that $\beta > h$. We deduce

$$\frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}} x^{\alpha} \geq \sum_{i=1}^{\infty} \mathbb{E}\left[e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}})\right] = \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}})\right]. \tag{25}$$

Here the last equality is justified by Fubini's theorem where it is easy to see that both $\sum_{i=1}^{\infty} \mathbb{E}[e^{-\beta T_i}(U(X_{T_i} - \gamma X_{T_{i-1}}))^{+}]$ and $\sum_{i=1}^{\infty} \mathbb{E}[e^{-\beta T_i}(U(X_{T_i} - \gamma X_{T_{i-1}}))^{-}]$ are convergent. The result follows after taking supremum over $X \in \mathcal{A}_0(x)$ in (25). □ □

Proof. Proof of Theorem 1. For any admissible process $X \in \mathcal{A}_0(x)$, define a discrete-time stochastic process $M = (M_n)_{n=0,1,2,\ldots}$ via

$$M_n := \sum_{i=1}^{n} e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}}) + A^{*} e^{-\beta T_n} X^{\alpha}_{T_n}.$$

Then

$$\begin{aligned}M_{n+1} &= M_n + e^{-\beta T_{n+1}} U(X_{T_{n+1}} - \gamma X_{T_n}) - A^{*} e^{-\beta T_n} X^{\alpha}_{T_n} + A^{*} e^{-\beta T_{n+1}} X^{\alpha}_{T_{n+1}} \\ &= M_n + e^{-\beta T_n}\left[e^{-\beta\tau}\left(U(X_{T_{n+1}} - \gamma X_{T_n}) + A^{*} X^{\alpha}_{T_{n+1}}\right) - A^{*} X^{\alpha}_{T_n}\right]\end{aligned}$$

and hence

$$\mathbb{E}(M_{n+1}|\mathcal{F}_{T_n}) = M_n + e^{-\beta T_n} X^{\alpha}_{T_n}\left[e^{-\beta\tau}\mathbb{E}\left[U\left(\frac{X_{T_{n+1}}}{X_{T_n}} - \gamma\right) + A^{*}\left(\frac{X_{T_{n+1}}}{X_{T_n}}\right)^{\alpha} \middle| \mathcal{F}_{T_n}\right] - A^{*}\right].$$

Since $X \in \mathcal{A}_0(x)$, ZX is a supermartingale and hence $\mathbb{E}\left[Z_{T_{n+1}}X_{T_{n+1}}\middle|\mathcal{F}_{T_n}\right] \leq Z_{T_n}X_{T_n}$. Then $\mathbb{E}\left[\frac{Z_{T_{n+1}}}{Z_{T_n}}\frac{X_{T_{n+1}}}{X_{T_n}}\middle|\mathcal{F}_{T_n}\right] \leq 1$ and

$$\mathbb{E}\left[U\left(\frac{X_{T_{n+1}}}{X_{T_n}} - \gamma\right) + A^*\left(\frac{X_{T_{n+1}}}{X_{T_n}}\right)^\alpha \middle|\mathcal{F}_{T_n}\right] \leq \sup_{Y \in \mathcal{L}_{T_n,T_{n+1}}(1)} \mathbb{E}\left[U\left(Y-\gamma\right) + A^*Y^\alpha \middle|\mathcal{F}_{T_n}\right] = F(A^*). \tag{26}$$

Using the fact that A^* is the fixed-point of the function $\Phi(A) = e^{-\beta\tau}F(a)$ such that $e^{-\beta\tau}F(A^*) = A^*$, we have

$$\mathbb{E}(M_{n+1}|\mathcal{F}_{T_n}) \leq M_n + e^{-\beta T_n}X_{T_n}^\alpha[e^{-\beta\tau}F(A^*) - A^*] = M_n. \tag{27}$$

Thus M is a $\{\mathcal{G}_n\}$-supermartingale where $\mathcal{G}_n := \mathcal{F}_{T_n}$. Then

$$A^*x^\alpha = M_0 \geq \mathbb{E}[M_{T_n}] = \mathbb{E}\left[\sum_{i=1}^n e^{-\beta T_i}U(X_{T_i} - \gamma X_{T_{i-1}}) + A^*e^{-\beta T_n}X_{T_n}^\alpha\right]$$

and hence

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^n e^{-\beta T_i}U(X_{T_i} - \gamma X_{T_{i-1}})\right] &\leq A^*x^\alpha - A^*\mathbb{E}\left[e^{-\beta T_n}X_{T_n}^\alpha\right] \\ &\leq A^*x^\alpha + e^{-\beta T_n}|A^*|\sup_{X\in\mathcal{A}_0(x)}\mathbb{E}\left[X_{T_n}^\alpha\right] = A^*x^\alpha + e^{-(\beta-h)T_n}|A^*|x^\alpha.\end{aligned}$$

Here we have used again the fact that $\sup_{X\in\mathcal{A}_0(x)}\mathbb{E}\left[X_{T_n}^\alpha\right] = x^\alpha e^{hT_n}$ is the solution to a finite horizon Merton problem. With $\beta > h$, taking limit $n \to \infty$ in conjunction with Fubini's theorem leads to

$$\mathbb{E}\left[\sum_{i=1}^\infty e^{-\beta T_i}U(X_{T_i} - \gamma X_{T_{i-1}})\right] \leq A^*x^\alpha$$

and therefore

$$V(x) = \sup_{X\in\mathcal{A}_0(x)}\mathbb{E}\left[\sum_{i=1}^\infty e^{-\beta T_i}U(X_{T_i} - \gamma X_{T_{i-1}})\right] \leq A^*x^\alpha.$$

To show the reverse inequality, it is sufficient to demonstrate the existence of some admissible process $\hat{X}$ such that $\mathbb{E}\left[\sum_{i=1}^\infty e^{-\beta T_i}U(\hat{X}_{T_i} - \gamma\hat{X}_{T_{i-1}})\right] = A^*x^\alpha$. Note that equality holds in (26) and in turn (27) under the choice of $\frac{X_{Tn+1}}{X_{Tn}} = y^*_{A^*}\left(\lambda_n^*\frac{Z_{Tn+1}}{Z_{Tn}}\right)$ due to Proposition 3. If A^* has value given by Case 1, 2a or 2b in Proposition 3, then λ_n^* is given by the unique solution to the equation

$$\mathbb{E}\left[\frac{Z_{Tn+1}}{Z_{Tn}}y^*_{A^*}\left(\lambda_n^*\frac{Z_{Tn+1}}{Z_{Tn}}\right)\middle|\mathcal{F}_{T_n}\right] = 1.$$

But since Z has the form given by (1), the stationary increment property of Brownian motion implies that

$$\mathbb{E}\left[\frac{Z_{Tn+1}}{Z_{Tn}}y^*_{A^*}\left(\lambda^*_n\frac{Z_{Tn+1}}{Z_{Tn}}\right)\Big|\mathcal{F}_{T_n}\right]=\mathbb{E}\left[\frac{Z_{T_1}}{Z_{T_0}}y^*_{A^*}\left(\lambda^*_n\frac{Z_{T_1}}{Z_{T_0}}\right)\Big|\mathcal{F}_{T_0}\right]=\mathbb{E}[Z_\tau y^*_{A^*}(\lambda^* Z_\tau)]$$

for all n and $\lambda^*_n=\lambda^*_1=:\lambda^*$ where $\mathbb{E}[Z_\tau y^*_{A^*}(\lambda^* Z_\tau)]=1$.

Now, define a sequence of random variables recursively as follows:

$$H_n:=H_{n-1}y^*_{A^*}\left(\lambda^*\frac{Z_{Tn}}{Z_{Tn-1}}\right),\qquad n=1,2,3,...,$$

with $H_0:=x$. By construction of $y^*_{A^*}(\cdot)$ and λ^*, $H_n\in\mathcal{F}^+_{T_n}$ and $\mathbb{E}[Z_{T_{n+1}}H_{n+1}|\mathcal{F}_{T_n}]\le Z_{T_n}H_n$. Using standard arguments of martingale representation theorem, there exists $\hat{X}\in\mathcal{A}_0(x)$ such that $\hat{X}_{T_n}=H_n$ for all n (Note that in Case 3 where $y^*_{A^*}(\cdot)\equiv 0$, the optimal portfolio value process is a strict supermartingale which hits zero at time T_1 almost surely. Such portfolio can be replicated by a "doubling down" style strategy where the notional invested in the risky asset at time t scales inversely with $\sqrt{T_1-t}$. See p.103-104 of Duffie (2001)). Let

$$\hat{M}_n:=\sum_{i=1}^n e^{-\beta T_i}U(\hat{X}_{T_i}-\gamma\hat{X}_{T_{i-1}})+A^*e^{-\beta T_n}\hat{X}^\alpha_{T_n}.$$

Using the same arguments leading to (27), we can conclude that

$$\mathbb{E}(\hat{M}_{n+1}|\mathcal{F}_{T_n})=\hat{M}_n+e^{-\beta T_n}\hat{X}^\alpha_{T_n}[e^{-\beta\tau}F(A^*)-A^*]=\hat{M}_n$$

and hence $\hat{M}$ is a $\{\mathcal{G}_n\}$-martingale. Then

$$\mathbb{E}\left[\sum_{i=1}^n e^{-\beta T_i}U(\hat{X}_{T_i}-\gamma\hat{X}_{T_{i-1}})\right]=A^*x^\alpha-A^*\mathbb{E}\left[e^{-\beta T_n}\hat{X}^\alpha_{T_n}\right]$$

from which we can eventually conclude

$$\mathbb{E}\left[\sum_{i=1}^\infty e^{-\beta T_i}U(\hat{X}_{T_i}-\gamma\hat{X}_{T_{i-1}})\right]=A^*x^\alpha.$$

Finally, we show that it is impossible to observe $-(k^{\frac{1}{1-\alpha}}+1)^{1-\alpha}<A^*<-1$ and $|A^*|^{-\frac{1}{1-\alpha}}\le 1-\gamma e^{-r\tau}$ such that $y^*_{A^*}(\cdot)$ cannot have the form given by Case 4 of Proposition 3. Obviously when $\gamma\ge e^{r\tau}$ there does not exist A^* compatible with the second condition. But if $\gamma<e^{r\tau}$, then part (2) of Lemma 3 suggests that $A^*\ge\frac{e^{-\beta\tau}U(e^{r\tau}-\gamma)}{1-e^{-(\beta-r\alpha)\tau}}>0$ contradicting the first condition. □ □

Proof. Proof of Corollary 1. If $\gamma = 0$, equation (12) simplifies to

$$A^* = e^{-\beta\tau} \sup_{Y \in \mathcal{A}_0(1)} \mathbb{E}\left[(A^*+1)Y_\tau^\alpha\right].$$

The right hand side of the above expression can be written as $e^{-(\beta-h)\tau}(A^*+1)^+$ since $\sup_{Y\in\mathcal{A}_0(1)} \mathbb{E}\left[Y_\tau^\alpha\right] = e^{h\tau}$ is the solution to a finite horizon Merton problem. Hence $A^* = \frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}} > 0$ is the unique solution to (12) and the corresponding optimizer (the optimal periodic gross return random variable) Y^* has its distribution coinciding with the time-τ value of a Merton portfolio with unit initial wealth. Using similar arguments in the proof of Theorem 1, one can verify that the candidate value function $V^C(x) := \frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}} x^\alpha$ is indeed the value function and this can be attained by the Merton portfolio which is given by (17). □ □

Proof. Proof of Proposition 2. For brevity, in this proof we will write $A^*(\gamma) = A^*(\gamma, k)$ if k is considered as fixed, and $A^*(k) = A^*(\gamma, k)$ if γ is considered as fixed. For the first part of the proposition, from Theorem 1,

$$A^*(\gamma) = V(1;\gamma) = \sup_{X\in\mathcal{A}_0(1)} \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}})\right]$$

is decreasing in γ since U is increasing. From part (2) of Lemma 3, if $\gamma \le e^{r\tau}$ then $A^*(\gamma) \ge \frac{e^{-\beta\tau}U(e^{r\tau}-\gamma)}{1-e^{-(\beta-r\alpha)\tau}} \ge 0$ where the last inequality is strict when $\gamma < e^{r\tau}$. To conclude $A^*(\gamma) > 0$ for all $\gamma \le e^{r\tau}$, it is sufficient to show $A^*(\gamma = e^{r\tau}) > 0$. Suppose on contrary that $A^*(\gamma = e^{r\tau}) = 0$ instead, then by (12) we must have

$$\sup_{Y\in\mathcal{A}_0(1)} \mathbb{E}\left[U(Y_\tau - e^{r\tau})\right] = 0.$$

The left hand side of the above expression is a finite horizon portfolio optimization problem with S-shaped utility function which is studied in Berkelaar et al. (2004). Write $f(y) := U(y - e^{r\tau})$ and let $\bar{f}(y)$ be the concave majorant of $f(y)$ on $y \ge 0$. Then

$$0 = \sup_{Y\in\mathcal{A}_0(1)} \mathbb{E}\left[f(Y_\tau)\right] = \sup_{Y\in\mathcal{A}_0(1)} \mathbb{E}\left[\bar{f}(Y_\tau)\right] \ge \bar{f}(e^{r\tau}) > f(e^{r\tau}) = 0,$$

resulting in a contradiction. In the above, the second equality is due to the standard concavification argument for S-shaped utility maximization problem, the first inequality is due to the fact that $Y = (e^{rt})_{t\ge 0}$ is an admissible portfolio, and the last inequality can be verified easily using the form of $\bar{f}$. Hence $A^*(\gamma = e^{r\tau}) > 0$ and in turn $A^*(\gamma) > 0$ for all $\gamma \le e^{r\tau}$.

It remains to show that $A^*(\gamma) \to -\infty$ as $\gamma \to +\infty$ under $k > 0$. Using simple calculus, one can show that for any ξ satisfying $\xi > 1$ and $\xi(1-\xi^{-\frac{1}{1-\alpha}})^{1-\alpha} > k$, we have $\xi y^\alpha - k\gamma^\alpha \ge U(y-\gamma)$ for all $y \ge 0$. Using the definition that A^* is the fixed-point of $\Phi(a) = e^{-\beta\tau}F(a)$ where F is defined in (14), we have

$$
\begin{aligned}
A^*(\gamma) = \Phi(A^*(\gamma)) &= \sup_{Y\in\mathcal{L}_{0,\tau}(1)} e^{-\beta\tau}\mathbb{E}[U(Y-\gamma) + A^*(\gamma)Y^\alpha] \\
&\le e^{-\beta\tau}\left[\sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}[U(Y-\gamma)] + \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}[A^*(\gamma)Y^\alpha]\right] \\
&\le e^{-\beta\tau}\left[\sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}[\xi Y^\alpha - k\gamma^\alpha] + |A^*(\gamma)| \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}[Y^\alpha]\right] \\
&= e^{-\beta\tau}(|A^*(\gamma)| + \xi) \sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[Y^\alpha\right] - e^{-\beta\tau}k\gamma^\alpha \\
&\le e^{-\beta\tau}\left(\frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}} + \xi\right)e^{h\tau} - e^{-\beta\tau}k\gamma^\alpha \to -\infty
\end{aligned}
$$

as $\gamma \uparrow +\infty$ if $k > 0$, where in the last line we have used part (1) of Lemma 3 and the fact that $\sup_{Y\in\mathcal{L}_{0,\tau}(1)} \mathbb{E}\left[Y^\alpha\right] = e^{h\tau}$ is the solution of a finite horizon Merton problem.

Now we prove the second part of the proposition. Similar to part (1), U is decreasing in k and thus $A^*(k) = V(1;k)$ is decreasing in k. If $k = 0$, then U is non-negative and as such $A^*(k=0) \ge 0$. The inequality is indeed straight since $\hat{X} \in \mathcal{A}_0(1)$ for

$$
\hat{X}_t := \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right]
$$

and $\mathbb{P}(U(\hat{X}_{T_n} - \gamma\hat{X}_{T_{n-1}}) > 0) = \mathbb{P}(\hat{X}_{T_n} - \gamma\hat{X}_{T_{n-1}} > 0) = N\left(-\frac{\ln\gamma - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) > 0$ where $N(\cdot)$ is the cumulative distribution function of a normal random variable.

Suppose $\gamma > e^{r\tau}$. Consider a new utility function $\bar{U}$ defined via

$$
\bar{U}(x) := \mathbb{1}_{(x\ge 0)}x^\alpha - k|x|^\alpha\mathbb{1}_{(-\epsilon\le x<0)} - k\epsilon^\alpha\mathbb{1}_{(x<-\epsilon)} = U(x)\mathbb{1}_{(x\ge-\epsilon)} - k\epsilon^\alpha\mathbb{1}_{(x<-\epsilon)}
$$

for some $0 < \epsilon < \gamma - e^{r\tau}$. Then $\bar{U} \ge U$. Hence

$$
\begin{aligned}
A^*(k) = \Phi(A^*(k)) &= \sup_{Y\in\mathcal{L}_{0,\tau}(1)} e^{-\beta\tau}\mathbb{E}[U(Y-\gamma) + A^*(k)Y^\alpha] \qquad (28) \\
&\le \sup_{Y\in\mathcal{L}_{0,\tau}(1)} e^{-\beta\tau}\mathbb{E}[\bar{U}(Y-\gamma) + A^*(k)Y^\alpha] \\
&= \sup_{Y\in\mathcal{L}_{0,\tau}(1)} e^{-\beta\tau}\left(\mathbb{E}[U(Y-\gamma)\mathbb{1}_{(Y\ge\gamma-\epsilon)} + A^*(k)Y^\alpha] - k\epsilon^\alpha\mathbb{P}(Y < \gamma-\epsilon)\right)
\end{aligned}
$$

$$\leq e^{-\beta\tau}(1+|A^*(k)|)\sup_{Y\in\mathcal{L}_{0,\tau}(1)}\mathbb{E}\left[Y^\alpha\right]-ke^{-\beta\tau}\epsilon^\alpha\inf_{Y\in\mathcal{L}_{0,\tau}(1)}\mathbb{P}(Y<\gamma-\epsilon)$$

$$=e^{-(\beta-h)\tau}\left(1+\frac{e^{-(\beta-h)\tau}}{1-e^{-(\beta-h)\tau}}\right)-ke^{-\beta\tau}\epsilon^\alpha\inf_{Y\in\mathcal{L}_{0,\tau}(1)}\mathbb{P}(Y<\gamma-\epsilon) \tag{29}$$

where we have used part (2) of Lemma 3 and the fact that $\sup_{Y\in\mathcal{L}_{0,\tau}(1)}\mathbb{E}\left[Y^\alpha\right]=e^{h\tau}$. Finally, the infimum in (29) is equal to $1-\sup_{Y\in\mathcal{L}_{0,\tau}(1)}\mathbb{P}(Y\geq\gamma-\epsilon)=1-\sup_{Y\in\mathcal{L}_{0,\tau}(1)}\mathbb{E}[\mathbb{1}_{(Y\geq\gamma-\epsilon)}]$. It can be computed explicitly via martingale duality which involves solving the static optimization problem $\sup_{y\geq 0}\left(\mathbb{1}_{(y\geq\gamma-\epsilon)}-\lambda Z_\tau y\right)$. The optimizer is given by $Y^*=(\gamma-\epsilon)\mathbb{1}_{(Z_\tau<1/(\lambda(\gamma-\epsilon)))}$ for some Lagrangian multiplier $\lambda\geq 0$. Note that

$$g(\lambda):=\mathbb{E}[Z_\tau Y^*]=(\gamma-\epsilon)\mathbb{E}[Z_\tau\mathbb{1}_{(Z_\tau<1/(\lambda^*(\gamma-\epsilon)))}]=(\gamma-\epsilon)\int_0^{\frac{1}{\lambda(\gamma-\epsilon)}}\frac{1}{|\theta|\sqrt{\tau}}\phi\left(\frac{\ln v+\left(r+\frac{\theta^2}{2}\right)\tau}{|\theta|\sqrt{\tau}}\right)dv,$$

where $\phi(\cdot)$ is the probability density function of a standard normal random variable. Thus $g(\lambda)$ is decreasing with $g(\infty)=0$ and $g(0)=(y-\epsilon)e^{-r\tau}>1$. Hence there exists a unique $\lambda^*\in(0,\infty)$ such that the budget constraint $1=\mathbb{E}[Z_\tau Y^*]=g(\lambda^*)$ is satisfied. Then

$$\inf_{Y\in\mathcal{L}_{0,\tau}(1)}\mathbb{P}(Y<\gamma-\epsilon)=\mathbb{P}(Y^*<\gamma-\epsilon)=\mathbb{P}\left(Z_\tau\geq\frac{1}{\lambda^*(\gamma-\epsilon)}\right)>0$$

and the value does not depend on k. The result follows after sending $k\to\infty$ in (29). □ □

C Optimal portfolio strategy as fraction of wealth invested in risky asset

In this section, we characterize the optimal portfolio via the fraction of wealth invested in the risky asset.

Proposition 5. *Suppose the optimal solution to problem* (4) *has the form given by Case 1, 2a or 2b of Theorem 1, then the optimal proportion of wealth invested in the risky asset at time t, denoted by π_t^*, is given by*

$$\pi_t^*=\pi^*\left(T_{n+1}-t,\frac{Z_t}{Z_{T_n}}\right):=-\frac{\mu-r}{\sigma^2}\frac{Z_t}{Z_{T_n}}\frac{\vartheta_z\left(T_{n+1}-t,\frac{Z_t}{Z_{T_n}}\right)}{\vartheta\left(T_{n+1}-t,\frac{Z_t}{Z_{T_n}}\right)},\qquad T_n\leq t<T_{n+1}, \tag{30}$$

with

$$\vartheta(s,z):=\int_0^\infty y_{A^*}^*\left(\lambda^* zv\right)\frac{1}{|\theta|\sqrt{s}}\phi\left(\frac{\ln v+\left(r+\frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}}\right)dv,$$

where $y^*_{A^*}(\cdot)$ and λ^* are defined in Theorem 1, $\vartheta_z(s,z) := \frac{\partial}{\partial z}\vartheta(s,z)$ and $\phi(\cdot)$ is the probability density function of a standard normal random variable. Moreover:

1. *If $A^* > 0$, $\pi^*(s,z) \to \frac{\mu-r}{(1-\alpha)\sigma^2}$ as $z \to 0$ or $z \to \infty$.*

2. *If $-1 < A^* \leq 0$, $\pi^*(s,z) \to \frac{\mu-r}{(1-\alpha)\sigma^2}$ as $z \to 0$ and $|\pi^*(s,z)| \to +\infty$ as $z \to \infty$.*

3. *If $A^* = -1$, $\pi^*(s,z) \to \frac{\mu-r}{(2-\alpha)\sigma^2}$ as $z \to 0$ and $|\pi^*(s,z)| \to +\infty$ as $z \to \infty$.*

4. *$-(k^{\frac{1}{1-\alpha}}+1)^{1-\alpha} < A^* < -1$, $\pi^*(s,z) \to 0$ as $z \to 0$ and $|\pi^*(s,z)| \to +\infty$ as $z \to \infty$.*

Proof. Proof of Proposition 5. From Theorem 1, the optimal portfolio value process in Case 1, 2a or 2b is given by

$$X^*_t = Z_t^{-1}\mathbb{E}\left[Z_{T_{n+1}}X^*_{T_n}y^*_{A^*}\left(\lambda^*\frac{Z_{T_{n+1}}}{Z_{T_n}}\right)\Big|\mathcal{F}_t\right], \qquad T_n \leq t < T_{n+1}. \tag{31}$$

Then since Z is an exponential Brownian motion, $\frac{Z_{T_{n+1}}}{Z_t} \overset{dist.}{=} Z_{T_{n+1}-t}$ which is a log-normal random variable independent of $\mathcal{F}_t$ with probability density function $f(v) = \frac{1}{|\theta|v\sqrt{T_{n+1}-t}}\phi\left(\frac{\ln v + \left(r+\frac{\theta^2}{2}\right)(T_{n+1}-t)}{|\theta|\sqrt{T_{n+1}-t}}\right)$. In turn,

$$\begin{aligned} X^*_t = X^*_{T_n}\mathbb{E}\left[\frac{Z_{T_{n+1}}}{Z_t}y^*_{A^*}\left(\lambda^*\frac{Z_t}{Z_{T_n}}\frac{Z_{T_{n+1}}}{Z_t}\right)\Big|\mathcal{F}_t\right] &= X^*_{T_n}\mathbb{E}\left[Z_{T_{n+1}-t}y^*_{A^*}\left(\lambda^*\frac{Z_t}{Z_{T_n}}Z_{T_{n+1}-t}\right)\right] \\ &= X^*_{T_n}\vartheta\left(T_{n+1}-t, \frac{Z_t}{Z_{T_n}}\right). \end{aligned}$$

Write $\dot{\vartheta}(s,z) := \frac{\partial}{\partial s}\vartheta(s,z)$, $\vartheta_z(s,z) := \frac{\partial}{\partial z}\vartheta(s,z)$ and $\vartheta_{zz}(s,z) := \frac{\partial^2}{\partial z^2}\vartheta(s,z)$. Application of Ito's lemma to $X^*_{T_n}\vartheta(T_{n+1}-t, \frac{Z_t}{Z_{T_n}})$ on $T_n \leq t < T_{n+1}$ gives

$$\begin{aligned} X^*_t = X^*_{T_n} + X^*_{T_n}\int_{T_n}^t &\left[-\dot{\vartheta}\left(T_{n+1}-u, \frac{Z_u}{Z_{T_n}}\right) - r\frac{Z_u}{Z_{T_n}}\vartheta_z\left(T_{n+1}-u, \frac{Z_u}{Z_{T_n}}\right)\right. \\ &\left.+ \frac{\theta^2}{2}\left(\frac{Z_u}{Z_{T_n}}\right)^2\vartheta_{zz}\left(T_{n+1}-u, \frac{Z_u}{Z_{T_n}}\right)\right]du - X^*_{T_n}\int_{T_n}^t\left[\theta\frac{Z_u}{Z_{T_n}}\vartheta_z\left(T_{n+1}-u, \frac{Z_u}{Z_{T_n}}\right)\right]dB_u. \end{aligned} \tag{32}$$

On the other hand, we can parameterize the self-financing strategy ϕ associated with X^* via $\phi^{(S)}_t = \pi_t X^*_t/S_t$ and $\phi^{(D)}_t = (1-\pi_t)X^*_t/D_t$ for some process $\pi = (\pi_t)_{t\geq 0}$. This leads to the dynamics

$$X^*_t = X^*_{T_n} + \int_{T_n}^t (r+(\mu-r)\pi_u)X^*_u du + \int_{T_n}^t \sigma\pi_u X^*_u dB_u. \tag{33}$$

To equate the Brownian motion terms in (33) and (32), we choose

$$\pi_t = -X^*_{T_n}\frac{\theta}{\sigma}\frac{Z_t}{Z_{T_n}}\frac{\vartheta_z\left(T_{n+1}-t,\frac{Z_t}{Z_{T_n}}\right)}{X^*_t} = -\frac{\mu-r}{\sigma^2}\frac{Z_t}{Z_{T_n}}\frac{\vartheta_z\left(T_{n+1}-t,\frac{Z_t}{Z_{T_n}}\right)}{\vartheta\left(T_{n+1}-t,\frac{Z_t}{Z_{T_n}}\right)}$$

on $T_n \le t < T_{n+1}$.

By definition,

$$\mathbb{E}\left[Z_{T_{n+1}}y^*_{A^*}\left(\lambda^*\frac{Z_{T_{n+1}}}{Z_{T_n}}\right)\Big|\mathcal{F}_t\right] = Z_t\vartheta\left(T_{n+1}-t,\frac{Z_t}{Z_{T_n}}\right) =: \Psi(t,Z_t)$$

and hence the process on the right hand side must be a martingale. Apply Ito's lemma to $\Psi(t,Z_t)$ and set the drift term to zero, we get (where the arguments in Ψ and ϑ are suppressed for brevity)

$$\begin{aligned}0 = \dot\Psi - rZ_t\Psi_z + \frac{\theta^2}{2}Z_t^2\Psi_{zz} &= -Z_t\dot\vartheta - rZ_t\left(\vartheta + \frac{Z_t}{Z_{T_n}}\vartheta_z\right) + \frac{\theta^2}{2}Z_t^2\left(\frac{2}{Z_{T_n}}\vartheta_z + \frac{Z_t}{Z_{T_n}^2}\vartheta_{zz}\right)\\ &= Z_t\left[-\dot\vartheta - r\vartheta + \left(\theta^2\frac{Z_t}{Z_{T_n}} - r\frac{Z_t}{Z_{T_n}}\right)\vartheta_z + \frac{\theta^2}{2}\frac{Z_t^2}{Z_{T_n}^2}\vartheta_{zz}\right]\end{aligned}$$

and therefore

$$-\dot\vartheta - r\frac{Z_t}{Z_{T_n}}\vartheta_z + \frac{\theta^2}{2}\left(\frac{Z_t}{Z_{T_n}}\right)^2\vartheta_{zz} = r\vartheta - \theta^2\frac{Z_t}{Z_{T_n}}\vartheta_z = (r+(\mu-r)\pi_t)\frac{X^*_t}{X^*_{T_n}}.$$

The above expression allows us to conclude that the drift terms of (33) and (32) are identical under our choice of π_t.

Now we prove the asymptotic results of $\pi(s,z)$. Suppose we are in Case 1 ($A^*>0$) such that

$$y^*_{A^*}(q) = \begin{cases} I_2(q), & 0<q\le m_1;\\ I_1(q), & q>m_1,\end{cases}$$

as per Theorem 1. Recall that $I_2 = I_2(q) : (0,m_1] \to [c_2\gamma,\infty)$ is the inverse to $G'_{A^*}(y) = \alpha[(y-\gamma)^{\alpha-1} + A^*y^{\alpha-1}]$ on $y\in[c_2\gamma,\infty)$, and $I_1 = I_1(q) : (m_1,\infty)\to(0,c_1\gamma)$ is the inverse to $G'_{A^*}(y) = \alpha[k(\gamma-y)^{\alpha-1} + ay^{\alpha-1}] = q$ on $y\in(0,c_1\gamma)$. It is easy to verify that

$$\lim_{q\to0}\left[G'_{A^*}(I_2(q)) - G'_{A^*}([\alpha(A^*+1)]^{\frac{1}{1-\alpha}}q^{-\frac{1}{1-\alpha}})\right] = \lim_{q\to0}\left[q - G'_{A^*}([\alpha(A^*+1)]^{\frac{1}{1-\alpha}}q^{-\frac{1}{1-\alpha}})\right] = 0$$

from which we deduce $\lim_{q\to0} q^{\frac{1}{1-\alpha}}I_2(q) = [\alpha(A^*+1)]^{\frac{1}{1-\alpha}}$ using the continuity of $G'_{A^*}(y)$ on $y\ge c_2\gamma$. This in turn gives

$$\lim_{q\to0}\frac{G''_{A^*}(I_2(q))}{q^{\frac{2-\alpha}{1-\alpha}}} = \lim_{q\to0}\frac{\alpha(\alpha-1)[(I_2(q)-\gamma)^{\alpha-2} + A^*I_2(q)^{\alpha-2}]}{q^{\frac{2-\alpha}{1-\alpha}}}$$

$$= \lim_{q\to 0} \alpha(\alpha-1)[(q^{\frac{1}{1-\alpha}} I_2(q) - \gamma q^{\frac{1}{1-\alpha}})^{\alpha-2} + A^*(q^{\frac{1}{1-\alpha}} I_2(q))^{\alpha-2}]$$
$$= (\alpha-1)[\alpha(A^*+1)]^{-\frac{1}{1-\alpha}}$$

such that

$$\lim_{q\to 0} q^{\frac{2-\alpha}{1-\alpha}} I_2'(q) = \lim_{q\to 0} \left(\frac{G''_{A^*}(I_2(q))}{q^{\frac{2-\alpha}{1-\alpha}}} \right)^{-1} = \frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha-1}.$$

Likewise, we can deduce

$$\lim_{q\to\infty} q^{\frac{1}{1-\alpha}} I_1(q) = (\alpha A^*)^{\frac{1}{1-\alpha}}, \quad \lim_{q\to\infty} q^{\frac{2-\alpha}{1-\alpha}} I_1'(q) = \frac{(\alpha A^*)^{\frac{1}{1-\alpha}}}{\alpha-1}.$$

Now,

$$\begin{aligned}
\lim_{z\to 0}(\lambda^* z)^{\frac{1}{1-\alpha}} \vartheta(s,z) &= \lim_{z\to 0}(\lambda^* z)^{\frac{1}{1-\alpha}} \mathbb{E}[Z_s \left(I_2\left(\lambda^* Z_s z\right) \mathbb{1}_{(\lambda^* Z_s z \le m_1)} + I_1\left(\lambda^* Z_s z\right) \mathbb{1}_{(\lambda^* Z_s z > m_1)}\right)] \\
&= \lim_{z\to 0} \mathbb{E}[Z_s^{-\frac{\alpha}{1-\alpha}} (\lambda^* Z_s z)^{\frac{1}{1-\alpha}} I_2\left(\lambda^* Z_s z\right)) \mathbb{1}_{(\lambda^* Z_s z \le m_1)}] \\
&\quad + \lim_{z\to 0} \mathbb{E}[Z_s^{-\frac{\alpha}{1-\alpha}} (\lambda^* Z_s z)^{\frac{1}{1-\alpha}} I_1\left(\lambda^* Z_s z\right)) \mathbb{1}_{(\lambda^* Z_s z > m_1)}] \\
&= \mathbb{E}[Z_s^{-\frac{\alpha}{1-\alpha}} \lim_{z\to 0} (\lambda^* Z_s z)^{\frac{1}{1-\alpha}} I_2\left(\lambda^* Z_s z\right)] + 0 \\
&= [\alpha(A^*+1)]^{\frac{1}{1-\alpha}} \mathbb{E}[Z_s^{-\frac{\alpha}{1-\alpha}}]
\end{aligned}$$

by dominated convergence theorem. Meanwhile,

$$\vartheta_z(s,z) = \frac{\partial}{\partial z}\left\{ \int_0^{\frac{m_1}{\lambda^* z}} v I_2(\lambda^* z v) f(v) dv + \int_{\frac{m_1}{\lambda^* z}}^{\infty} v I_1(\lambda^* z v) f(v) dv \right\}$$

with $f(v) := \frac{1}{|\theta| v \sqrt{s}} \phi\left(\frac{\ln v + \left(r + \frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}} \right)$ being the probability density function of Z_s. Note that

$$\int_0^{\frac{m_1}{\lambda^* z}} v I_2(\lambda^* z v) f(v) dv = \int_0^{m_1} \frac{q}{(\lambda^* z)^2} I_2(q) f\left(\frac{q}{\lambda^* z}\right) dq$$

and hence Fubini's theorem gives

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\int_0^{\frac{m_1}{\lambda^* z}} v I_2(\lambda^* z v) f(v) dv \right] \\
&= -\frac{1}{z(\lambda^* z)^2}\left\{ 2\int_0^{m_1} q I_2(q) f\left(\frac{q}{\lambda^* z}\right) dq + \int_0^{m_1} \frac{q^2 I_2(q)}{\lambda^* z} f'\left(\frac{q}{\lambda^* z}\right) dq \right\} \\
&= -\frac{1}{z(\lambda^* z)^2}\left\{ m_1^2 I_2(m_1) f\left(\frac{m_1}{\lambda^* z}\right) - \lim_{q\to 0} q^2 I_2(q) f\left(\frac{q}{\lambda^* z}\right) - \int_0^{m_1} q^2 I_2'(q) f\left(\frac{q}{\lambda^* z}\right) dq \right\}
\end{aligned}$$

where we have used integration by part, and the second term vanishes using the fact that $I_2(q) \propto q^{-1/(1-\alpha)}$ for small q. Then

$$
\begin{aligned}
&(\lambda^* z)^{\frac{1}{1-\alpha}} z \frac{\partial}{\partial z} \left[\int_0^{\frac{m_1}{\lambda^* z}} v I_2(\lambda^* z v) f(v) dv \right] \\
&\qquad = -m_1 (\lambda^* z)^{\frac{\alpha}{1-\alpha}} \frac{m_1}{\lambda^* z} f\left(\frac{m_1}{\lambda^* z}\right) I_2(m_1) + (\lambda^* z)^{\frac{1}{1-\alpha}-2} \int_0^{m_1} q^2 I_2'(q) f\left(\frac{q}{\lambda^* z}\right) dq \\
&\qquad = -m_1 (\lambda^* z)^{\frac{\alpha}{1-\alpha}} \frac{m_1}{\lambda^* z} f\left(\frac{m_1}{\lambda^* z}\right) I_2(m_1) + \int_0^{\frac{m_1}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{2-\alpha}{1-\alpha}} I_2'(\lambda^* z v) f(v) dv.
\end{aligned}
$$

The first term converges to zero as $z \to 0$. Recall $\lim_{q\to 0} q^{\frac{2-\alpha}{1-\alpha}} I_2'(q) = \frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha-1}$. For any $\epsilon > 0$, there exits $\delta > 0$ arbitrarily small such that $|q^{\frac{2-\alpha}{1-\alpha}} I_2'(q) - \frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha-1}| < \epsilon$ for $q < \delta$. If we choose $\delta < m_1$, then

$$
\begin{aligned}
&\int_0^{\frac{m_1}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{2-\alpha}{1-\alpha}} I_2'(\lambda^* z v) f(v) dv \\
&\quad \le \left(\frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha - 1} + \epsilon \right) \int_0^{\frac{\delta}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} f(v) dv + (\lambda^* z)^{\frac{1}{1-\alpha}-2} \int_\delta^{m_1} q^2 I_2'(q) f\left(\frac{q}{\lambda^* z}\right) dq \\
&\quad = \left(\frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha - 1} + \epsilon \right) \int_0^{\frac{\delta}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} f(v) dv + (\lambda^* z)^{\frac{\alpha}{1-\alpha}} \int_\delta^{m_1} \frac{q I_2'(q)}{|\theta|\sqrt{s}} \phi \left(\frac{\ln \frac{q}{\lambda^* z} + \left(r + \frac{\theta^2}{2}\right) s}{|\theta|\sqrt{s}} \right) dq \\
&\quad \to \left(\frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha - 1} + \epsilon \right) \int_0^{\infty} v^{-\frac{\alpha}{1-\alpha}} f(v) dv
\end{aligned}
$$

as $z \to 0$ since the second term vanishes due to dominated convergence theorem. Following the same derivation, we can also show that

$$
\lim_{z\to 0} \int_0^{\frac{m_1}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{2-\alpha}{1-\alpha}} I_2'(\lambda^* z v) f(v) dv \ge \left(\frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha - 1} - \epsilon \right) \int_0^{\infty} v^{-\frac{\alpha}{1-\alpha}} f(v) dv.
$$

As ϵ is arbitrary, we conclude

$$
\begin{aligned}
\lim_{z\to 0} (\lambda^* z)^{\frac{1}{1-\alpha}} z \frac{\partial}{\partial z} \left[\int_0^{\frac{m_1}{\lambda^* z}} v I_2(\lambda^* z v) f(v) dv \right] &= \frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha - 1} \int_0^{\infty} v^{-\frac{\alpha}{1-\alpha}} f(v) dv \\
&= \frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha - 1} \mathbb{E}[Z_s^{-\frac{\alpha}{1-\alpha}}].
\end{aligned}
$$

Similarly, we can deduce

$$
\begin{aligned}
&(\lambda^* z)^{\frac{1}{1-\alpha}} z \frac{\partial}{\partial z} \left[\int_{\frac{m_1}{\lambda^* z}}^{\infty} v I_1(\lambda^* z v) f(v) dv \right] \\
&\qquad = m_1 (\lambda^* z)^{\frac{\alpha}{1-\alpha}} \frac{m_1}{\lambda^* z} f\left(\frac{m_1}{\lambda^* z}\right) I_1(m_1) + \int_{\frac{m_1}{\lambda^* z}}^{\infty} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{2-\alpha}{1-\alpha}} I_1'(\lambda^* z v) f(v) dv \to 0
\end{aligned}
$$

as $z \to 0$ using the fact that $\lim_{q\to\infty} q^{\frac{2-\alpha}{1-\alpha}} I_1'(q) = \frac{(\alpha A^*)^{\frac{1}{1-\alpha}}}{\alpha-1}$. Therefore,

$$\pi^*(s,z) = -\frac{\mu-r}{\sigma^2}\frac{z\vartheta_z(s,z)}{\vartheta(s,z)} = -\frac{\mu-r}{\sigma^2}\frac{(\lambda^* z)^{\frac{1}{1-\alpha}} z\vartheta_z(s,z)}{(\lambda^* z)^{\frac{1}{1-\alpha}}\vartheta(s,z)} \to \frac{\mu-r}{(1-\alpha)\sigma^2}$$

as $z \to 0$. The result for $z \to \infty$ can be obtained similarly.

For Case 2a, suppose for now $A^* \neq -1$. Then the limiting result for $z \to 0$ follows from the same argument in Case 1. To show the result for $z \to \infty$, by following a similar derivation in Case 1 we can write

$$\begin{aligned}
\frac{z\vartheta_z(s,z)}{\vartheta(s,z)} &= \frac{(\lambda^* z)^{\frac{1}{1-\alpha}} z\vartheta_z(s,z)}{(\lambda^* z)^{\frac{1}{1-\alpha}}\vartheta(s,z)} \\
&= \frac{-m_2(\lambda^* z)^{\frac{\alpha}{1-\alpha}} \frac{m_2}{\lambda^* z} f\left(\frac{m_2}{\lambda^* z}\right) I_2(m_2) + \int_0^{\frac{m_2}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{2-\alpha}{1-\alpha}} I_2'(\lambda^* z v) f(v)dv}{\int_0^{\frac{m_2}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{1}{1-\alpha}} I_2(\lambda^* z v) f(v)dv} \\
&= -\frac{m_2(\lambda^* z)^{\frac{\alpha}{1-\alpha}} \frac{m_2}{\lambda^* z} f\left(\frac{m_2}{\lambda^* z}\right) I_2(m_2)}{\int_0^{\frac{m_2}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{1}{1-\alpha}} I_2(\lambda^* z v) f(v)dv} + \frac{\int_0^{\frac{m_2}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{2-\alpha}{1-\alpha}} I_2'(\lambda^* z v) f(v)dv}{\int_0^{\frac{m_2}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} (\lambda^* z v)^{\frac{1}{1-\alpha}} I_2(\lambda^* z v) f(v)dv}.
\end{aligned}$$

The second term converges to $1/(\alpha-1)$ as $z \to \infty$ due to the fact that $q^{1/(1-\alpha)} I_2(q) \to \alpha(A^* + 1)^{\frac{1}{1-\alpha}}$ and $q^{\frac{2-\alpha}{1-\alpha}} I_2'(q) \to \frac{[\alpha(A^*+1)]^{\frac{1}{1-\alpha}}}{\alpha-1}$ for small q. To show that the first term is diverging to $-\infty$ when $z \to \infty$, it is sufficient to show

$$\frac{(\lambda^* z)^{\frac{\alpha}{1-\alpha}} \frac{m_2}{\lambda^* z} f\left(\frac{m_2}{\lambda^* z}\right)}{\int_0^{\frac{m_2}{\lambda^* z}} v^{-\frac{\alpha}{1-\alpha}} f(v)dv} = \frac{(\lambda^* z)^{\frac{\alpha}{1-\alpha}} \phi\left(\frac{\ln\frac{m_2}{\lambda^* z} + \left(r+\frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}}\right)}{\int_0^{\frac{m_2}{\lambda^* z}} v^{-\frac{1}{1-\alpha}} \phi\left(\frac{\ln v + \left(r+\frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}}\right) dv}$$

diverges to $+\infty$. This can be verified by an application of L'Hôpital's rule. In the corner case of $A^* = -1$, $I_2(q)$ and $I_2'(q)$ have slightly different asymptotic behaviors near $q = 0$ given by

$$\lim_{q\to 0} q^{\frac{1}{2-\alpha}} I_2(q) = [\alpha\gamma(1-\alpha)]^{\frac{1}{2-\alpha}}, \qquad \lim_{q\to 0} q^{\frac{3-\alpha}{2-\alpha}} I_2'(q) = \frac{[\alpha\gamma(1-\alpha)]^{\frac{1}{2-\alpha}}}{\alpha-2}.$$

The limiting result for $z \to 0$ can be computed by following the same procedure in Case 1, but the answer is different now since the ratio of the above two values becomes $\alpha - 2$ instead of $\alpha - 1$.

Finally, in Case 2b we have

$$y_a^*(q) = \begin{cases} I_2(q), & 0 < q \le m_2; \\ 0, & q > m_2, \end{cases}$$

where $I_2 = I_2(q) : (0, m_2] \to [c_3\gamma, \frac{\gamma}{1-|A^*|^{-1/(1-\alpha)}})$ is the inverse to $G'_{A^*}(y) = \alpha[(y-\gamma)^{\alpha-1} + A^* y^{\alpha-1}] = q$ on $y \in [c_3\gamma, \frac{\gamma}{1-|A^*|^{-1/(1-\alpha)}})$. Then both $I_2(q)$ and $I_2'(q) = 1/G''_{A^*}(I_2(q))$ are bounded on $0 < q \le m_2$, and in particular I_2 is bounded away from zero. We thus have

$$\vartheta(s,z) = \mathbb{E}[Z_s I_2(\lambda^* Z_s z) \mathbb{1}_{(\lambda^* Z_s z \le m_2)}] \to \frac{\gamma \mathbb{E}(Z_s)}{1 - |A^*|^{-1/(1-\alpha)}} > 0$$

and

$$\begin{aligned} z\vartheta_z(s,z) &= -\frac{1}{(\lambda^* z)^2} \left\{ m_2^2 I_2(m_2) f\left(\frac{m_2}{\lambda^* z}\right) - \int_0^{m_2} q^2 I_2'(q) f\left(\frac{q}{\lambda^* z}\right) dq \right\} \\ &= -\frac{1}{(\lambda^* z)^2} m_2^2 I_2(m_2) f\left(\frac{m_2}{\lambda^* z}\right) + \lambda^* z \int_0^{\frac{m_2}{\lambda^* z}} v^2 I_2'(\lambda^* z v) f(v) dv \to 0 \end{aligned}$$

as $z \to 0$ using dominated convergence theorem, and hence $\lim_{z\to 0} \pi(s,z) = 0$. For the diverging result with $z \to \infty$, note that

$$\frac{z\vartheta_z(s,z)}{\vartheta(s,z)} = \frac{-\frac{1}{(\lambda^* z)^2} m_2^2 I_2(m_2) f\left(\frac{m_2}{\lambda^* z}\right) + \lambda^* z \int_0^{\frac{m_2}{\lambda^* z}} v^2 I_2'(\lambda^* z v) f(v) dv}{\int_0^{\frac{m_2}{\lambda^* z}} v I_2(\lambda^* z v) f(v) dv}.$$

The boundedness of I_2 and I_2' implies

$$\begin{aligned} \frac{z\vartheta_z(s,z)}{\vartheta(s,z)} &\ge -K_1 \frac{\left(\frac{m_2}{\lambda^* z}\right)^2 f\left(\frac{m_2}{\lambda^* z}\right)}{\int_0^{\frac{m_2}{\lambda^* z}} v f(v) dv} + K_2 \lambda^* z \frac{\int_0^{\frac{m_2}{\lambda^* z}} v^2 f(v) dv}{\int_0^{\frac{m_2}{\lambda^* z}} v f(v) dv} \\ &= -K_1 \frac{\left(\frac{m_2}{\lambda^* z}\right) \phi\left(\frac{\ln\frac{m_2}{\lambda^* z} + \left(r + \frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}}\right)}{\int_0^{\frac{m_2}{\lambda^* z}} \phi\left(\frac{\ln v + \left(r + \frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}}\right) dv} + K_2 \lambda^* z \frac{\int_0^{\frac{m_2}{\lambda^* z}} v \phi\left(\frac{\ln v + \left(r + \frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}}\right) dv}{\int_0^{\frac{m_2}{\lambda^* z}} \phi\left(\frac{\ln v + \left(r + \frac{\theta^2}{2}\right)s}{|\theta|\sqrt{s}}\right) dv} \end{aligned}$$

for some positive constants K_1 and K_2 independent of z. The result follows from L'Hôpital's rule that the first term diverges to $-\infty$ and the second term converges to some constant as $z \to \infty$. □ □

Under the Black-Scholes economy, there is a one-to-one correspondence between the price of the risky asset and the pricing kernel Z because both variables depend on the underlying Brownian motion B in a one-to-one manner. It is therefore also possible to express the optimal portfolio weight in Proposition 5 as a function of time and $\ln(S_t/S_{T_n})$ the running log-return of the risky asset measured in the current the evaluation period, as we did in Figure 3 and 5.

D Variant of the model with an option to early retire

To capture the possibility of early retirement, we now consider a variation of the model that the agent can choose to quit trading at the beginning of each accounting period after receiving the reward from the previous period. The flow of utilities will stop after the declaration of retirement.

Let $\mathcal{T}$ be the set of $\mathcal{F}_{T_n}$-stopping times valued in $\{0, 1, 2, ...\} \cup \{+\infty\}$. A choice of $O \in \mathcal{T}$ where $\{O \leq n\} \in \mathcal{F}_{T_n}$ means that the agent opts to retire at the end of the O^{th} period. The special case of $O = 0$ refers to the decision that the agent does not take the trading job from the outset. The optimization problem with the retirement option can be stated as

$$V^R(x) := \sup_{X \in \mathcal{A}_0(x), O \in \mathcal{T}} \mathbb{E}\left[\sum_{i=1}^{O} e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}})\right]. \tag{34}$$

Theorem 2. *For problem* (34)*, the value function is given by*

$$V^R(x) = \max(A^* x^\alpha, 0),$$

where A^* is the fixed-point of $\Phi(\cdot)$. Furthermore:

1. *If $A^* \leq 0$, then $O = 0$ is optimal, i.e. the agent will not take the trading job right from the beginning.*

2. *If $A^* \geq 0$, then $O = \infty$ is optimal where the agent will take the job and never retire. The optimal portfolio value process is given by either Case 1 (when $A^* > 0$) or Case 2a (when $A^* = 0$) of Theorem 1.*

In the corner case of $A^* = 0$, the agent is indifferent between the above two strategies.

The proof is presented at the end of this section. If the agent strictly prefers participation in trading from the beginning (i.e. $A^* > 0$), then he must be trading according to Case 1 of Theorem 1 under which the periodic gross return random variable is strictly positive. This in turn completely eliminates the long-term portfolio ruin risk described by Corollary 2. The retirement option serves as a self-screening tool where an agent who finds himself unsuitable for trading (for example, due to his loss aversion or lack of confidence to meet the performance target) can opt to quit the job, rather than forcing himself to take excessive risk to meet the investment goal which eventually is detrimental to both the agent and the principal.

One further extension to the model with early retirement is to assume the agent will receive some outside reservation value $\mathcal{R}$ at the time of retirement which can be positive (e.g. the present value of jobseeker's allowance) or negative (e.g. the loss of employee's perks or the psychological displeasure from leaving a professional sector). The optimization problem can then be stated as

$$\sup_{X\in\mathcal{A}_0(x),O\in\mathcal{T}} \mathbb{E}\left[\sum_{i=1}^{O} e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})+\mathcal{R}e^{-\beta T_O}\right]. \tag{35}$$

It is challenging to solve (35) completely and the optimal retirement and portfolio strategy can no longer be inferred from Theorem 1 since the retirement decision should now depend on the current portfolio value. Nonetheless, it is clear that the degenerate strategy of intentionally depleting the portfolio value to zero in the first period (Case 3 of Theorem 1) will not occur provided that $\mathcal{R}\geq 0$ since it is strictly dominated by the strategy of retiring at time zero.

Proof. Proof of Theorem 2. Denote by $V^C(x):=\max(A^*x^\alpha,0)$ the candidate value function of problem (34). Our goal is to show that $V^R(x)=V^C(x)$.

For any admissible process $X\in\mathcal{A}_0(x)$, define $M=(M_n)_{n=0,1,2,\dots}$ as

$$M_n:=\sum_{i=1}^{n} e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})+e^{-\beta T_n}V^C(X_{T_n}),$$

and for any $O\in\mathcal{T}$ define $Q=(Q_n)_{n=0,1,2}$ via $Q_n:=M_{\min(n,O)}$. Here Q is referring the process M stopped at the stopping time O.

Now,

$$\begin{aligned}
Q_{n+1}-Q_n &= \mathbb{1}_{(O\geq n+1)}[M_{n+1}-M_n]\\
&= \mathbb{1}_{(O\geq n+1)}\left[e^{-\beta T_{n+1}}U(X_{T_{n+1}}-\gamma X_{T_n})-e^{-\beta T_n}V^C(X_{T_n})+e^{-\beta T_{n+1}}V^C(X_{T_{n+1}})\right]\\
&= \mathbb{1}_{(O\geq n+1)}\left\{e^{-\beta T_n}\left[e^{-\beta\tau}\left(U(X_{T_{n+1}}-\gamma X_{T_n})+V^C(X_{T_{n+1}})\right)-V^C(X_{T_n})\right]\right\}
\end{aligned}$$

and, by using similar arguments in the proof of Theorem 1, we have

$$\begin{aligned}
&\mathbb{E}[Q_{n+1}|\mathcal{F}_{T_n}]-Q_n\\
&= \mathbb{1}_{(O\geq n+1)}\mathbb{E}\left\{e^{-\beta T_n}\left[e^{-\beta\tau}\left(U(X_{T_{n+1}}-\gamma X_{T_n})+\max(A^*,0)X^\alpha_{T_{n+1}}\right)-\max(A^*,0)X^\alpha_{T_n}\right]\middle|\mathcal{F}_{T_n}\right\}\\
&= \mathbb{1}_{(O\geq n+1)}\left\{e^{-\beta T_n}X^\alpha_{T_n}\left[e^{-\beta\tau}\mathbb{E}\left[U\left(\frac{X_{T_{n+1}}}{X_{T_n}}-\gamma\right)+\max(A^*,0)\left(\frac{X_{T_{n+1}}}{X_{T_n}}\right)^\alpha\middle|\mathcal{F}_{T_n}\right]-\max(A^*,0)\right]\right\}\\
&\leq \mathbb{1}_{(O\geq n+1)}\left\{e^{-\beta T_n}X^\alpha_{T_n}\left[e^{-\beta\tau}\sup_{Y\in\mathcal{L}_{T_n,T_{n+1}}(1)}\mathbb{E}\left[U(Y-\gamma)+\max(A^*,0)Y^\alpha\middle|\mathcal{F}_{T_n}\right]-\max(A^*,0)\right]\right\}\\
&= \mathbb{1}_{(O\geq n+1)}e^{-\beta T_n}X^\alpha_{T_n}[\Phi(\max(A^*,0))-\max(A^*,0)].
\end{aligned}$$

If $A^*>0$, then $\Phi(\max(A^*,0))-\max(A^*,0)=A^*-A^*=0$. If $A^*\leq 0$, then $\Phi(\max(A^*,0))-\max(A^*,0)=\Phi(0)\leq 0$ using part (3) of Lemma 3. We conclude $\Phi(\max(A^*,0))-\max(A^*,0)\leq 0$ and as such $\mathbb{E}[Q_{n+1}|\mathcal{F}_{T_n}]-Q_n\leq 0$. Hence Q is a $\{\mathcal{G}_n\}$-supermartingale where $\mathcal{G}_n:=\mathcal{F}_{T_n}$. Then

$$V^C(x)=Q_0\geq\mathbb{E}[Q_n]=\mathbb{E}\left[\sum_{i=1}^{\min(n,O)}e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})+e^{-\beta T_{\min(n,O)}}V^C(X_{T_{\min(n,O)}})\right].$$

We deduce

$$\mathbb{E}\left[\sum_{i=1}^{\min(n,O)}e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})\right]\leq V^C(x)-\mathbb{E}\left[e^{-\beta T_{\min(n,O)}}V^C(X_{T_{\min(n,O)}})\right]\leq V^C(x)\quad(36)$$

since V^C is non-negative. The random variable $|\sum_{i=1}^{\min(n,O)}e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})|$ is dominated by $\sum_{i=1}^{\infty}e^{-\beta T_i}|U(X_{T_i}-\gamma X_{T_{i-1}})|$ and it is easy to show that the latter is integrable using similar arguments in the proof of Proposition 1. Sending $n\uparrow\infty$ under dominated convergence theorem and then taking supremum over $X\in\mathcal{A}_0(x)$ and $O\in\mathcal{T}$ in (36) gives

$$V^R(x)=\sup_{X\in\mathcal{A}(x),O\in\mathcal{T}}\mathbb{E}\left[\sum_{i=1}^{O}e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})\right]\leq V^C(x).$$

$V^R(x)\geq V^C(x)$ will hold if we can demonstrate the existence of admissible O and X such that $\mathbb{E}\left[\sum_{i=1}^{O}e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})\right]=\max(A^*,0)x^\alpha$. If $A^*\leq 0$, then the choice of $O^*=0$ clearly attains the above equality. Else if $A^*\geq 0$, we can choose $O^*=\infty$ and X^* according to the strategy in Theorem 1 where the attained value function is A^*x^α. □ □

E Extended model with random horizon and terminal utility

There is a strand of behavioral models which assume the agent derives utilities not only from the gains-and-losses but also from the terminal wealth (e.g. Kőszegi and Rabin (2006)). This can be incorporated within our model by modifying (4) as

$$\tilde{V}(x):=\sup_{X\in\mathcal{A}_0(x)}\mathbb{E}\left[\sum_{i=1}^{N}e^{-\beta T_i}U(X_{T_i}-\gamma X_{T_{i-1}})+e^{-\beta T_N}U_B(X_{T_N})\right],\quad(37)$$

where $N>0$ represents the terminal horizon (in terms of number of periods) and $U_B(\cdot)$ is a utility function associated with the terminal wealth. Practically, the bequest term in (37) can capture the severance pay to the agent upon termination of his job. Our baseline framework (4) is recovered upon setting $N\to+\infty$ and $U_B(\cdot)=0$. Although it is more challenging to study

(37) due to the finite horizon nature of the problem, analytical progress can still be made under a particular specification.

Suppose $U_B(y) = \omega U(y) = \omega y^\alpha$ on $y \geq 0$, where $\omega \geq 0$ is a constant representing the weight attached to the bequest term. We further assume N is an independent geometric random variable with parameter $p \in (0,1)$ such that $\mathbb{P}(N = i) = p(1-p)^i$ and $\mathbb{P}(N \geq i) = (1-p)^i$ for $i \in \{0, 1, 2, ...\}$. p reflects the probability that the agent is dismissed at the beginning of each period. The objective function can be written as

$$\begin{aligned}
\tilde{J}(x) &:= \mathbb{E}\left[\sum_{i=1}^{N} e^{-\beta T_i} U(X_{T_i} - \gamma X_{T_{i-1}}) + \omega e^{-\beta T_N} U(X_{T_N})\right] \\
&= \mathbb{E}\left\{\sum_{i=1}^{\infty} e^{-\beta T_i} \left[(U(X_{T_i} - \gamma X_{T_{i-1}})\mathbb{1}_{(N\geq i)} + \omega U(X_{T_i})\mathbb{1}_{(N=i)}\right]\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{\infty} e^{-\beta i\tau} \left[(U(X_{T_i} - \gamma X_{T_{i-1}})(1-p)^i + \omega U(X_{T_i})p(1-p)^i\right]\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{\infty} e^{-(\beta+\nu)i\tau} \left[(U(X_{T_i} - \gamma X_{T_{i-1}}) + p\omega U(X_{T_i})\right]\right\},
\end{aligned}$$

where we have used the assumption that N is independent of the underlying Brownian motion and we have set $\nu := \frac{1}{\tau}\ln\frac{1}{1-p}$. Following the same intuitions in Section 3, the dynamic programming equation (10) now becomes

$$\tilde{V}(x) = \sup_{X\in\mathcal{A}_0(x)} \mathbb{E}\left[e^{-(\beta+\nu)T_1}[U(X_{T_1} - \gamma x) + p\omega U(X_{T_1})] + e^{-(\beta+\nu)T_1}\tilde{V}(X_{T_1})\right]. \tag{38}$$

We expect the scaling property still holds such that $\tilde{V}(x) = \tilde{A}x^\alpha$ for some constant $\tilde{A}$ to be identified. Then the equation for $\tilde{A}$ becomes

$$\begin{aligned}
\tilde{A} &= \sup_{Y\in\mathcal{A}_0(1)} \mathbb{E}\left[e^{-(\beta+\nu)\tau}[U(Y_\tau - \gamma) + p\omega Y_\tau^\alpha] + \tilde{A}Y_\tau^\alpha\right] \\
&= e^{-(\beta+\nu)\tau} \sup_{Y\in\mathcal{A}_0(1)} \mathbb{E}\left[U(Y_\tau - \gamma) + (p\omega + \tilde{A})Y_\tau^\alpha\right].
\end{aligned} \tag{39}$$

The unknown $\tilde{A}$ can now be solved from (39) by using a similar idea of fixed-point iteration as in our baseline problem. Consider a family of optimization problems

$$\tilde{F}(a) := F(a + p\omega) = \sup_{Y\in\mathcal{F}_\tau^+} \mathbb{E}\left[G_{a+p\omega}(Y)\right] \qquad \text{subject to} \qquad \mathbb{E}(Z_\tau Y) \leq 1, \tag{40}$$

where G and F are defined in (13) and (14) respectively. We can show that the map $a \to e^{-(\beta+\nu)\tau}\tilde{F}(a) =: \tilde{\Phi}(a)$ is a contraction based on a trivial extension of Proposition 4. Then $\tilde{\Phi}$

admits a unique fixed-point which characterizes $\tilde{A}$. The optimality of the value function can be justified by a verification theorem using similar arguments in the proof of Theorem 1.

Once $\tilde{A}$ is characterized, the optimal portfolio gross return random variable $\tilde{Y}^*$ can be identified by studying the maximizer in (39). The range of $\tilde{A}+p\omega$ distinguishes the form of $\tilde{Y}^*$ as in Theorem 1 and Proposition 3. The results are summarized by the following theorem where the full proof is omitted.

Theorem 3. *Consider problem* (37) *under* $U_B(y) = \omega y^\alpha$ *for* $\omega \geq 0$*, and* N *has a probability mass function of* $\mathbb{P}(N = i) = p(1-p)^i$ *with* $p \in (0,1)$ *for* $i \in \{0, 1, 2, ...\}$ *which is independent of the underlying Brownian motion. The corresponding value function is* $\tilde{V}(x) = \tilde{A}x^\alpha$ *where* $\tilde{A}$ *is the unique fixed-point of* $a \to e^{-(\beta+\nu)\tau}\tilde{F}(a)$ *with* $\nu := \frac{1}{\tau}\ln\frac{1}{1-p}$ *and* $\tilde{F}(\cdot)$ *is defined in* (40)*. The optimal portfolio value process* $\tilde{X}^*$ *at time* T_i *is given by*

$$\tilde{X}^*_{T_i} = \tilde{X}^*_{T_{i-1}} y^*_{\tilde{A}+p\omega}\left(\tilde{\lambda}^* \frac{Z_{T_i}}{Z_{T_{i-1}}}\right), \qquad i = 1, 2, ...,$$

with $\tilde{X}^*_{T_0} = x$*, and the function* $y^*_a(\cdot)$ *is given by Proposition 3.* $y^*_{\tilde{A}+p\omega}(\cdot)$ *has four possible forms depending on the value of* $\tilde{A}+p\omega$ *as per Case 1, 2a, 2b and 3 of Proposition 3 (Case 4 cannot occur). In Case 1, 2a and 2b,* $\tilde{\lambda}^* > 0$ *is the unique solution to the equation* $\mathbb{E}[Z_\tau y^*_{\tilde{A}+p\omega}(\tilde{\lambda}^* Z_\tau)] = 1$.

An interesting question is how the presence of the terminal utility term affects the optimal portfolio. The following lemma sheds some lights in this direction.

Lemma 4. *Let* $\tilde{A}(\omega)$ *be the value of* $\tilde{A}$ *in Theorem 3 as a function of* ω *while all other model parameters are fixed. Then* $\tilde{A}(\omega)$ *is increasing in* ω*.*

Proof. Proof of Lemma 4. It is not hard to see that $\tilde{\Phi}(a)$ is bounded from below by a constant. Thus $\tilde{\Phi}(a)$ must cross the identity function only once from the above at $a = \tilde{A}$. As $p > 0$, $\tilde{\Phi}(a) = F(a + p\omega)$ is increasing in ω and hence $\tilde{A} = \tilde{A}(\omega)$ is increasing in ω as well. □ □

For $\omega_2 > \omega_1 > 0$, Lemma 4 suggests that $\tilde{A}(\omega_2) \geq \tilde{A}(\omega_1) \geq \tilde{A}(0) = A^*$ and in turn $\tilde{A}(\omega_2) + p\omega_2 > \tilde{A}(\omega_1) + p\omega_1 > A^*$. As in the discussion of Section 4.2, the solution behavior transits in the order of Case 1, 2a, 2b and 3 as $\tilde{A}+p\omega$ gradually decreases. Hence an increase in ω will push the solution regime towards Case 1. Informally speaking, the inclusion of a bequest term helps reduce excessive risk taking in the bad states of the world and discourage underinvestment in the good states of the world. This is not too surprising as the presence of the terminal utility makes the agent behave more in line with the predictions of a neoclassical model.