# A relative trace formula and counting geodesic segments in the hyperbolic plane

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I, Dimitrios Lekkas, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

# Abstract

In this work we study a modification of the hyperbolic circle problem, which is one of the problems originally studied by A. Good. We consider the orbit of double cosets of a Fuchsian group $\Gamma$ by two hyperbolic subgroups $H_1, H_2$ in the hyperbolic plane.

We use a relative trace formula with suitable test functions for the counting of lengths of geodesic segments perpendicular to the closed geodesics corresponding to $H_1$ and $H_2$. We present an elementary proof providing the main term in the asymptotics and an error term of order $O(X^{2/3})$. We study the mean square of the error term and prove that it is consistent with the conjectural optimal error term $O(X^{1/2+\epsilon})$. To apply the relative trace formula we develop a large sieve inequality for periods of Maass forms. This requires a more subtle understanding of Huber's transform, which is a special case of the Jacobi transform studied by Flensted-Jensen and Koornwinder. Our counting problem is a special case of counting in the orthospectrum. We are motivated by previous work on geodesic segments between a point and a closed geodesic, studied by Huber and Chatzakos–Petridis.

# Impact Statement

The hyperbolic circle problem, analogue of the Gauss circle problem in euclidean space, is at the centre of investigations on the interaction of groups, geometry, and number theory. We study a modified problem about affine symmetric spaces: counting in the orbit of double cosets of a Fuchsian group $\Gamma$ by two hyperbolic subgroups $H_1$ and $H_2$ in the hyperbolic plane.

In this thesis we examine some important questions in number theory, the branch of mathematics that underlies digital communication and internet security. It supports the UK to keep its privileged status in fundamental research in number theory. This thesis aligns with the EPSRC's strategic focus for Number Theory in its Mathematical Sciences theme. This research makes connections to neighbouring fields such as the Mathematical Analysis, via spectral theory, and ergodic theory and dynamical systems.

The beneficiaries of this research are other researchers in number theory, automorphic forms, and dynamical systems. It is expected that the results will have high long term impact on these disciplines.

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# Chapter 1

# Introduction

## 1.1 The hyperbolic circle problem

Hyperbolic lattice counting problems concern the counting of the number of points in the orbit of a discrete group $\Gamma$ that lie in a subset of the upper half-plane $\mathbb{H}$. The discrete subgroup $\Gamma$ is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ thought as the orientation preserving isometries on $\mathbb{H}$, on which it acts by linear fractional transformations. One of the classical problems in the field is the hyperbolic analogue of the Gauss circle problem of counting on average

$$r(k) = \#\left\{(a,b) \in \mathbb{Z}^2 | a^2 + b^2 = k\right\}.$$

Gauss used a geometric argument to show for $r(k)$ that:

$$\sum_{k \leq X} r(k) = \pi X + O\big(X^{1/2}\big).$$

Counting the average representations of $k$ as a sum of two squares, translates geometrically into counting the number of integer points inside the disk of radius $\sqrt{X}$ centered at the origin $(0,0)$. In the hyperbolic setting, the integer points $(x,y)$ are substituted by points in the orbit $\Gamma z$ of some point $z \in \mathbb{H}$. The question in this case is, whether these points lie in the hyperbolic circle of radius $X$ centered at a point $w \in \mathbb{C}$, namely the estimating of the number

$$N(X,z,w) = \#\left\{\gamma \in \Gamma | 2\cosh(\mathrm{dist}(\gamma \mathrm{z}, \mathrm{w})) \le \mathrm{X}\right\},$$

where the hyperbolic distance dist is defined in (2.4).

## 1.2 Counting cosets on Fuchsian groups

The classification of elements of $\mathrm{SL}(2,\mathbb{R})$ by means of their trace into elliptic, parabolic and hyperbolic elements describe all different kinds of motions that could occur on $\mathbb{H}$, when we act by $\Gamma$. Let $\Gamma_1$ and $\Gamma_2$, be each a stabilizer of a point, cusp or geodesic. Considering the elements of the double coset $\Gamma_1 \backslash \Gamma / \Gamma_2$ of $\Gamma$, instead of the full group $\Gamma$, leads to nine different counting problems. This set of problems was studied by Good [11], who achieved the same asymptotic (letting the 'radius '$X \to \infty$) for all of the problems with an error term $O\left(X^{2/3}\right)$. Although this work dates back to the 1983, it is the best result so far. Unfortunately, Good's book is quite hard to understand and suffers from peculiar notation, something that discourages the reader. His method was to define a certain type of generalized Poincaré series for each different category of elements (elliptic, parabolic, hyperbolic) that can be used in the study of the aforementioned counting problems. Our goal is to use a different, more flexible technique for the solution of the problem, in the case that both subgroups in the double coset are stabilizers of closed geodesics. We use a relative trace formula with fairly general test functions.

For the classical hyperbolic circle problem there is more extensive literature, see [15],[16],[17, Ch. 12]. Already in [14] Huber was interested in the case where the coset is formed by the stabilizer $H_1$ of a closed geodesic and the other stabilizer is trivial. Some results on this problem can be found in further work of Huber [16], as well as in Chatzakos' thesis [5]. We apply Huber's method to counting double

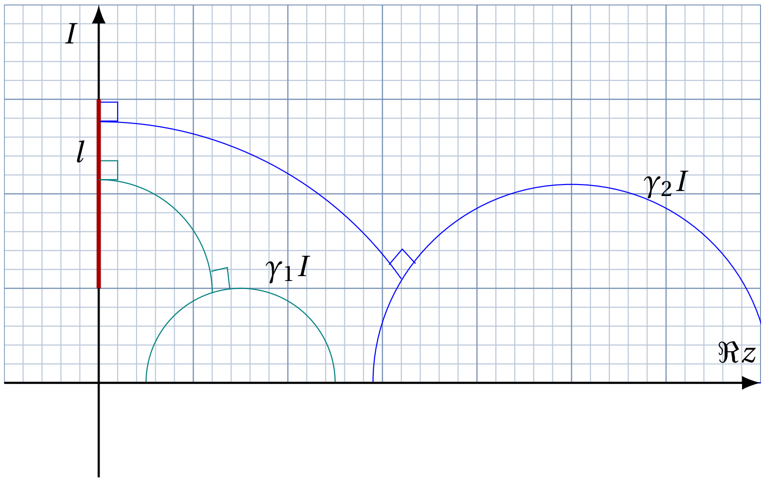


Figure 1.1: The orbit of $\gamma l$ for $\gamma \in H_1 \backslash \Gamma / H_1$.

cosets of a Fuchsian group $\Gamma$ by two hyperbolic subgroups $H_1, H_2$.

More specifically, this problem concerns the estimation of

$$\widetilde{N}(X, l_1, l_2) = \#\left\{\gamma \in H_1 \backslash \Gamma / H_2 | \inf_{\substack{z \in l_1 \\ w \in l_2}} \cosh(\text{dist}(\gamma \text{z}, \text{w})) \leq \text{X}\right\},$$

where the group $\Gamma$ is a Fuchsian group of the first kind and $H_1 \backslash \Gamma / H_2$ is a double coset of $\Gamma$ by hyperbolic subgroups $H_1, H_2$, that correspond to closed geodesics $l_1$ and $l_2$ (see chapter 4 for the relation). In comparison to the case studied on [16] and [6], this problem appears to be more complicated as we consider two hyperbolic subgroups of $\Gamma$ that make the double coset, instead of the one stabilizer being trivial. As it was explained in [23], the problem concerns the counting of the number of $\gamma \in H_1 \backslash \Gamma / H_2$ such that $\gamma \cdot l_1$ and $l_2$ have distance less than $X$. We can assume that $l_1$ lies on $I$, where $I$ is the imaginary axis. The distance between $l_1$ and $l_2$ is given by the length of the line segment, that is orthogonal to both geodesics, see figure 1.1. Moreover we take $l_1 = l_2$. In [23] the distance between $l_1$ and $\gamma l_1$ is related to $\delta(\gamma) := 2|ad + bc|$, for a given $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H_1 \backslash \Gamma / H_1$. More specifically it is shown in [23, Lemma 1] that:

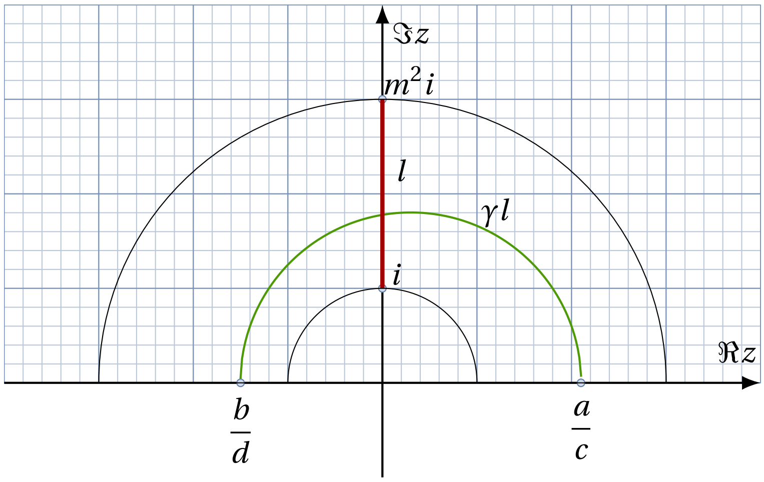


Figure 1.2: A closed geodesic $l$ with norm $m^2$ corresponding to the action of an exceptional point $\gamma$.

**Lemma 1.1.** *For* $\gamma \in PSL(2,\mathbb{R})$ *and such that* $abcd \neq 0$ *and* $u(z,w)$ *the point-pair invariant defined in* (2.3) *we have*

$$\inf(u(\gamma \cdot ix, iy) \,|\, x, y \in \mathbb{R}^+) = \begin{cases} \delta(\gamma) - 2, & abcd > 0\,, \\ 0, & abcd < 0\,. \end{cases}$$

*Thus*

$$\max(\delta(\gamma), 2) = 2\cosh(\mathrm{dist}(\gamma \mathrm{I}, \mathrm{I}))\,,$$

*where* $\mathrm{dist}(\gamma\mathrm{I},\mathrm{I}) = \inf_{\mathrm{z},\mathrm{w}\in\mathrm{I}} \rho(\gamma\mathrm{z},\mathrm{w})$ *and* $\rho(z_1, z_2)$ *is the hyperbolic distance between* $z_1$ *and* $z_2$.

Similarly to [23, see p.11] and [32, Lem. 8] we may assume that $\gamma \in \Gamma - H_1$ is such that $abcd \neq 0 \implies \delta(\gamma) \neq 2$. Those elements $\gamma$ are called regular. From [23, p. 11–12] we know that there finitely many double coset representatives $\gamma$ in $H_1 \backslash \Gamma / H_1$ for which $\delta(\gamma) < 2$ or equivalently $abcd < 0$. Those elements $\gamma$ are called exceptional, see figure 1.2.

The normalization $\inf_{\substack{z\in l_1\\ w\in l_1}} \cosh(\mathrm{dist}(\gamma \mathrm{z}, \mathrm{w})) \leq \mathrm{X}$ is equivalent to $B(\gamma) := \delta(\gamma)/2 < X$ for $\delta(\gamma) > 2$. Since the exceptional double cosets are finite, our problem concerns the estimation of

$$N(X, l_1, l_1) = \#\Big\{\gamma \in H_1\backslash\Gamma/H_1 | B(\gamma) < X\Big\},$$

since $N(X, l_1, l_1) = \widetilde{N}(X, l_1, l_1) + O(1)$.

Similarly, see [23, Chapter 6],

$$N(X, l_1, l_2) = \#\Big\{\gamma \in H_1\backslash\Gamma/H_2 | B(\tau^{-1}\gamma) < X\Big\},$$

where $\tau \in \mathrm{PSL}_2(\mathbb{R})$ is such that $\tau^{-1} \cdot l_2$ lies on the imaginary axis.

However [23] does not give us a result regarding the main or error term of estimating the number $N(X, l_1, l_2)$ or $N(X, l_1, l_1)$. The authors develop a relative trace formula that relates sums of integrals of a test function $\Phi$ to the Selberg–Harish-Chandra transform of $\Phi$ and periods of Maass forms $u_j$ (see Proposition 1 in [23]). Further, replacing the test function with a Bessel function they are able to give results about the periods of Maass forms. The main result ([23, Theorem 2]) achieves an asymptotic formula for the sum of the period squares.

The problem when the two geodesics are the same $l_1 = l_2$ was also studied by Tsuzuki [31], but unfortunately his result with error term $O\big(X^{5/6}\big)$ is worse than the general result of Good, see [11, Th. 4], who obtained $O\big(X^{2/3}\big)$. We are going to state Good's theorem [11, Th. 4] in Chapter 3.

## 1.3 Statements of our results

Set $N(X, l) := N(X, l, l)$. Suppose that $\Gamma$ is cocompact and torsion-free. Let $\{u_j\}_{j=0}^{\infty}$ be a complete orthonormal system of real-valued, normalized eigenfunctions for the discrete spectrum of the hyperbolic Laplacian with eigenvalues $\lambda_j = s_j(1 - s_j)$. We call the eigenfunctions $u_j$ Maass forms. The eigenvalues $\lambda_j$ such that $\lambda_j <$

$1/4 \Leftrightarrow 1/2 < s_j \leq 1$ are called small eigenvalues. We give an elementary proof of Good's theorem for $H_1 \backslash \Gamma / H_1$ with $H_1$ a hyperbolic subgroup:

**Theorem 1.2.** *We have*

$$N(X,l) = \sum_{1/2 < s_j \leq 1} \frac{2}{\pi} \gamma_1(s_j) \widehat{u_j}^2 X^{s_j} + O\big(X^{2/3}\big),$$

*where*

$$\gamma_1(s) = \frac{\pi}{2} \frac{\Gamma(s+1/2)\Gamma((s+1)/2)}{\Gamma(s/2)^3} \frac{4}{s(2s-1)}$$

*and*

$$\widehat{u_j} = \int_l u_j ds$$

*for the geodesic $l$. The big-O estimate depends on the geodesic $l$.*

We call $\widehat{u_j}$ the periods of the Maass forms $u_j$ along the geodesic $l$. We will prove the following large sieve inequality for the periods $\widehat{u_j}$:

**Theorem 1.3.** *Let $T, X > 1$ and $x_1, \ldots, x_R \in [X, 2X]$. If $|x_\nu - x_\mu| > \delta > 0$ for $\nu \neq \mu$, then*

$$\sum_{\nu=1}^{R} \Big| \sum_{|t_j| \leq T} a_j x_\nu^{it_j} \widehat{u_j} \Big|^2 \ll \big(T + X \log T\delta^{-1}\big) ||\mathbf{a}||_*^2,$$

*where*

$$||\mathbf{a}||_* = \Big( \sum_{|t_j| \leq T} |a_j|^2 \Big)^{1/2}.$$

Note that this is analogous to Chamizo's large sieve inequality [3, Theorem 2.2], who worked with sums of Maass forms $u_j(z)$ instead of periods $\widehat{u_j}$. We use similar methods for our proof.

We define the main term of our hyperbolic lattice counting problem as

$$M(X,l) = \sum_{1/2 < s_j \leq 1} \frac{2}{\pi} \gamma_1(s_j) \widehat{u_j}^2 X^{s_j}$$

and the error term as

$$E(X,l) = N(X,l) - M(X,l)\,.$$

Similarly to [6] (see Chapter 2, Proposition 2.12 and Theorem 2.13) we want to show that on average $E(X,l)$ is $O\big(X^{1/2}\big)$. We will use Theorem 1.3 to prove the following result about the mean square of the error term $E(X,l)$:

**Proposition 1.4.** *Let $X > 2$ and $\delta > 0$. Let $X_1, X_2, \ldots, X_R \in [X, 2X]$ such that $|X_i - X_j| > \delta$, when $i \neq j$. Then the following bound holds*

$$\sum_{m=1}^{R} |E(X_m, l)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^3 X\,.$$

Using Proposition 1.4 we will prove the following bounds for the second moment of the error term $E(X,l)$:

**Theorem 1.5.** *If $R\delta \gg X$ and $R > X^{1/2}$, then*

$$\frac{1}{R} \sum_{m=1}^{R} |E(X_m, l)|^2 \ll X \log^3 X \tag{1.1}$$

*As $R \to \infty$, we have*

$$\frac{1}{X} \int_X^{2X} |E(x,l)|^2 dx \ll X \log^3 X\,. \tag{1.2}$$

From Theorems 1.2 and 1.5 we are able to formulate the analogous conjecture to [6, Conj. 5.7]:

**Conjecture 1.6.** *Let $\epsilon > 0$, then the error term $E(X,l)$ satisfies the bound*

$$E(X,l) = O\big(X^{1/2+\epsilon}\big)\,,$$

*where the estimate depends on $l$ and $\epsilon$.*

# Chapter 2

# Hyperbolic lattice counting problems

## 2.1 Preliminaries

In this section we review the core parts of the theory of automorphic forms that will be used in this work. Our main reference is [17].

Let $\mathbb{H} = \{x + iy | x \in \mathbb{R}, y > 0\}$ denote the hyperbolic upper half-plane and $G$ be the group $\mathrm{PSL}(2,\mathbb{R})$. The upper half-plane is equipped by the hyperbolic metric

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}\,.$$

The measure element is given by

$$d\mu(z) = \frac{dxdy}{y^2}\,.$$

By $\Gamma$ we denote a Fuchsian group of the first kind ($\Gamma \leq G$) such that the space $\Gamma\backslash\mathbb{H}$ has finite volume with respect to the hyperbolic measure. More generally, a group $\Gamma_0$ such that $\Gamma_0\backslash\mathbb{H}$ has finite volume is called cofinite.

The group $G$ (and its subgroups) act on $\mathbb{H}$ by fractional linear transformations. Those are defined by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d}\,.$$

An element $g$ of $G$ can be characterized by the value of its trace. It is called:

- elliptic if $|\operatorname{Tr}(g)| < 2$ ,
- hyperbolic if $|\operatorname{Tr}(g)| > 2$ ,
- parabolic if $|\operatorname{Tr}(g)| = 2$ .

Elliptic elements can be shown to be conjugate to a rotation $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. On the other hand, hyperbolic elements are conjugate to a magnification $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, $\lambda > 1$ and parabolic elements are conjugate to translation matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, that shift an element $z \in \mathbb{H}$ horizontally by $x$.
We are interested in studying functions $f : \mathbb{H} \to \mathbb{C}$ which satisfy the periodicity condition

$$f(\gamma z) = f(z), \quad \gamma \in \Gamma, \quad z \in \mathbb{H}\,, \tag{2.1}$$

hence they are invariant under this action of $\Gamma$. We call those functions automorphic. If $\Gamma$ contains a parabolic element of the form $\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , then the action of $\gamma$ on $z \in \mathbb{H}$, gives us $f(z+1) = f(z)$ by the automorphic condition for $f$. This suggests that we can apply concepts of Fourier Analysis for such $f$. Actually, this is possible in the case that $\Gamma$ is cocompact, i.e. $\Gamma\backslash\mathbb{H}$ is compact, where we decompose $f$ in terms of eigenfunctions of the Laplace operator.

The inner product of two functions $f, g \in L^2(\Gamma\backslash\mathbb{H})$ is defined by

$$\langle f, g\rangle := \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}\,d\mu(z)\,.$$

We define the hyperbolic Laplace operator on $\mathbb{H}$ by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

**Definition 2.1.** A smooth function $f \in L^2(\Gamma\backslash\mathbb{H})$ that satisfies the automorphic condition (2.1) and is also an eigenfunction of the Laplace operator is called a Maass form. We denote those by $u_j$, where $\lambda_j$ stands for the corresponding eigenvalue.

Another example of automorphic functions are the Eisenstein series:

**Definition 2.2.** We define the Eisenstein series on $\mathbb{H}$ around a cusp $\alpha$ as

$$E_\alpha(z,s) = \sum_{\gamma\in\Gamma_\alpha\backslash\Gamma} \Im(\sigma_\alpha^{-1}\gamma z)^s\,,$$

where the sum runs over the coset of $\Gamma$ by the stabiliser $\Gamma_\alpha$ of $\alpha$ and $\sigma_\alpha$ is a scaling matrix.

Let $D(\Gamma\backslash\mathbb{H})$ be the space of functions $f$ on $L^2(\Gamma\backslash\mathbb{H})$ such that $f$ and $\Delta f$ are smooth and bounded. Every automorphic function can be written as a sum of Maass forms and Eisenstein series, which is known as the spectral theorem in the theory of automorphic forms. We recall [17, Theorem 7.3]:

**Theorem 2.3.** *Let $f \in L^2(\Gamma\backslash\mathbb{H})$ be an automorphic function. Then $f$ can be expressed in terms of Maass forms $u_j$ and Eisenstein series $E_\alpha$ as follows:*

$$\begin{aligned} f(z) &= \sum_j \langle f, u_j\rangle u_j(z) \\ &\quad + \sum_\alpha \frac{1}{4\pi}\int_{-\infty}^{+\infty} \langle f, E_\alpha(\cdot, \tfrac{1}{2}+it)\rangle E_\alpha(z, \tfrac{1}{2}+it)\,dt\,, \end{aligned} \tag{2.2}$$

*where the $u_j$ have eigenvalues $\lambda_j$, a finite number of which satisfy $0 \le \lambda_j < 1/4$. The expansion converges in the norm topology. If $f(z) \in D(\Gamma\backslash\mathbb{H})$ then the expansion* (2.2) *converges pointwise absolutely and uniformly on compact sets.*

The question now becomes: What automorphic functions are useful in our lattice point problem and what information can we extract from the spectral decomposition? All of the hyperbolic lattice counting problems mentioned in the beginning involve the notion of distance in the hyperbolic plane. This distance $\rho$ can be expressed in terms of the point-pair invariant function $u$.

**Definition 2.4.** For two points $z, w \in \mathbb{H}$ we define the point-pair invariant $u$ by

$$u(z,w) = \frac{|z-w|^2}{4\Im z \Im w}. \tag{2.3}$$

We see immediately that it satisfies $u(gz, gw) = u(z,w), g \in G$. Then the hyperbolic distance $\rho$ is given by

$$\cosh\rho(z,w) = 1 + 2u(z,w). \tag{2.4}$$

We notice that the distance $\cosh\rho(z,w)$ would change if we acted on either $z$ or $w$ by some element of $\Gamma$. Hence the function cannot be automorphic on any of the two variables and the spectral theorem cannot be applied. We solve that issue by summing over all elements of the group.

**Definition 2.5.** Let $k(z,w)$ be a point-pair invariant on $\mathbb{H}$, that is $k(z,w) = k(u(z,w))$ and $k(t)$ is a function on $\mathbb{R}^+$ of compact support. We define its automorphization by

$$K(z,w) = \sum_{\gamma\in\Gamma} k(u(\gamma z, w)).$$

The sum converges because it has finite support for any fixed $z, w$.

**Definition 2.6** (Eq. 1.62 [17])**.** For a function $k = k(u)$ that depends only on the distance $u$, i.e.

$$k(z,w) = k(u(z,w)),$$

we define the Selberg–Harish-Chandra transform through the equations

$$q(v) = \int_v^\infty k(u)(u-v)^{-1/2}du\,,$$
$$g(r) = 2q\left(\left(\sinh\frac{r}{2}\right)^2\right),$$
$$h(t) = \int_{-\infty}^\infty e^{irt}g(r)dr\,.$$

For the automorphization $K$ one can prove its spectral expansion, see [17, Th. 7.4]:

**Theorem 2.7.** *Let $K(z,w)$ be the automorphization of a smooth test function $k$ satisfying $k(z,w) = k(u(z,w))$ and whose Selberg–Harish-Chandra transform $h(t)$ satisfies the following conditions for $\epsilon > 0$:*

$$h(t) \quad \textit{even,}$$
$$h(t) \quad \textit{is holomorphic in the strip} \quad |\Im t| \le \frac{1}{2}+\epsilon,$$
$$h(t) \ll (|t|+1)^{-2-\epsilon} \quad \textit{in the strip}\,.$$

*Then $K(z,w)$ has the spectral expansion*

$$K(z,w) = \sum_j h(t_j)u_j(z)\overline{u_j(w)}$$
$$+\sum_\alpha \frac{1}{4\pi}\int_{-\infty}^\infty h(t)E_\alpha(z,\frac{1}{2}+it)\overline{E_\alpha}(w,\frac{1}{2}+it)dt\,,$$

*where the $t_j$'s parametrize the discrete spectrum i.e. $s_j = \frac{1}{2}+it_j$, with $\lambda_j = s_j(1-s_j)$.*

## 2.2 Results on hyperbolic lattice counting problems

### 2.2.1 Results on the hyperbolic circle problem

The most studied problem among hyperbolic lattice counting problems in the hyperbolic space was the hyperbolic circle problem. We aim to estimate

$$N(X) = \#\{\gamma \in \Gamma | 4u(\gamma z, w) + 2 \leq X\} .$$

The first to consider was Delsartre [8]. Huber began his investigation on the hyperbolic circle problem in 1959 with his paper [15]. He used a Dirichlet series given by

$$G(s, z, w) = \sum_{\gamma \in \Gamma} \left( \frac{1}{\cosh \rho(\gamma z, w)} \right)^s ,$$

which converges absolutely and uniformly for $z$ and $w$ in compact sets of $\mathbb{H}$ and $\Re s > 1$.

There is the following main result about $N(X)$:

**Theorem 2.8 (Selberg[29], Günther[13], Good[11]).** *Let $\Gamma$ be cocompact or cofinite and $z, w$ be two points in $\mathbb{H}$, then*

$$N(X) = \sum_{1/2 < s_j \leq 1} \sqrt{\pi} \frac{\Gamma\left(s_j - \frac{1}{2}\right)}{\Gamma(s_j + 1)} u_j(z) \overline{u_j(w)} X^{s_j} + O\left(X^{2/3}\right) .$$

Selberg's work is unpublished but available online. Günter's work is for general rank one symmetric spaces. Good's book covers all nine cases. One method of proof, see also the proof of [17, Thm. 12.1], is to apply Theorem 2.7 to the functions

$$k^+(u) = \begin{cases} 1, & u \leq \frac{X-2}{4} , \\ \frac{-4u}{Y} + \frac{X+Y-2}{Y}, & \frac{X-2}{4} \leq u \leq \frac{X+Y-2}{4} , \\ 0, & \frac{X+Y-2}{4} \leq u , \end{cases}$$

and

$$k^{-}(u) = \begin{cases} 1, & u \leq \frac{X-Y-2}{4}\,, \\ \frac{-4u}{Y} + \frac{X-2}{Y}, & \frac{X-Y-2}{4} \leq u \leq \frac{X-2}{4}\,, \\ 0, & \frac{X-2}{4} \leq u\,, \end{cases}$$

which are smoothings of the characteristic function on the interval $[0,(X-2)/4]$ and compute suitable bounds for the corresponding Selberg–Harish-Chandra transforms $h^{\pm}(t)$. Here $Y$ is a large parameter such that $Y \ll X$ to be determined for optimizing error terms.

### 2.2.2 Huber's work on counting in conjugacy classes

Assume, unless stated otherwise, that $\Gamma$ is cocompact. Let $\mathfrak{T}$ be a hyperbolic conjugacy class, i.e. $\mathfrak{T} = \{a^{-1}P^{\nu}a, a \in \Gamma\}$ for a primitive hyperbolic element $P$. Here $\nu$ is the number of times the invariant geodesic of $P$ wraps around itself. For $\gamma \in \Gamma$, let

$$\mu = \mu(\gamma) = \inf_{z \in \mathbb{H}} \rho(\gamma z, z)\,.$$

When $\gamma \in \mathfrak{T}$, we see that $\mu(\gamma) = \mu(P^{\nu})$, hence $\mu(\gamma)$ is constant in conjugacy classes. We write $\mu := \mu(\mathfrak{T}) = \mu(P^{\nu})$, which is the length of the closed geodesic corresponding to the hyperbolic conjugacy class $\mathfrak{T}$. The hyperbolic lattice counting problem in conjugacy classes counts

$$N(t,\mathfrak{T},z) = \#\Big\{\gamma \in \mathfrak{T} \mid \rho(\gamma z, z) \leq t\Big\}\,. \tag{2.5}$$

Huber in 1954 used the Dirichlet series

$$G(z,s) = \sum_{\gamma \in \mathfrak{T}} \left(\frac{1}{\cosh\rho(\gamma z, z) - 1}\right)^{s}$$

to prove (see [14, Satz B])

$$N(t,\mathfrak{T},z) \sim \frac{1}{p-1}\frac{1}{v}\frac{\mu}{\sinh\left(\frac{\mu}{2}\right)}e^{t/2}$$

as $t \to \infty$, where $p$ is the genus of $\Gamma\backslash\mathbb{H}$. We notice that the asymptotic growth is exponential for $t$, because Huber counts the number of $\gamma \in \mathfrak{T}$ such that $\rho(\gamma z, z) \leq t$. This is different to the distance used in the hyperbolic circle problem, where the counting is performed in terms of $\cosh\rho(\gamma z, z)$ for $\gamma \in \Gamma$.

When Huber revisited the problem in 1998, in [16], he increased his flexibility by considering a bigger family of automorphic functions and their spectral expansion.

We now explain his work. We introduce a new system of coordinates $(u, v)$ on the upper half-plane: let $z = x + iy$ and define

$$u(z) = \log|z| \quad \text{and} \quad v(z) = -\arctan\left(\frac{x}{y}\right). \tag{2.6}$$

For these variables, we can verify that the ranges are

$$-\infty < u(z) < +\infty \quad \text{and} \quad -\frac{\pi}{2} < v(z) < \frac{\pi}{2}.$$

From the definitions of $u$ and $v$ it also follows that

$$\cos v(z) = \frac{y}{|z|} \quad \text{and} \quad \sin v(z) = -\frac{x}{|z|}.$$

Moreover, for a diagonal hyperbolic element $P = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ we have (see equations

[16, p. 19 eq.23])

$$u(Pz) = u(z) + 2\log\lambda \quad \text{and} \quad v(Pz) = v(z)\,,$$

where $\lambda^2$ is the norm of $P$. Hence, acting by diagonal hyperbolic elements does not change the coordinate $v$. The hyperbolic metric $ds^2 = \left(dx^2 + dy^2\right)/y^2$ on the hyperbolic upper half-plane $\mathbb{H}$ now becomes $ds^2 = \left(du(z)^2 + dv(z)^2\right)/\cos^2 v(z)$. This approach allowed him to obtain a much stronger result for $N(t,\mathfrak{T},z)$, namely the inequality (see [16, p. 11]) for $X \geq 4$:

$$\left|N(t,\mathfrak{T},z) - \frac{1}{2\pi(p-1)}\frac{\mu}{v}X\right| \leq \frac{\mu}{v}(2.56X^{3/4} + 6.75X^{1/2})\,, \tag{2.7}$$

where

$$X = \frac{\sinh(t/2)}{\sinh(\mu/2)}\,. \tag{2.8}$$

The problem was solved by studying the spectral expansion of the series (see [16, p. 16, eq. 4])

$$A(f)(z) = \sum_{\gamma\in\mathfrak{T}} f\left(\frac{\cosh(z,\gamma z) - 1}{\cosh\mu - 1}\right),$$

for $f \in C_0^*[1,\infty)$: the space of real functions of compact support that are bounded in $[1,\infty)$ and have at most finitely many discontinuities. Since $\Gamma$ is cocompact and $f$ has compact support, $A(f)(z)$ is finite. Using equation [16, p. 20, eq. 26] Huber rewrote this series as

$$A(f)(z) = \sum_{\gamma\in\langle P\rangle\backslash\Gamma} f\left(\frac{1}{\cos^2 v(\gamma z)}\right),$$

in the coordinates $u, v$. Huber explained that $A(f)(z)$ is an automorphic function that captures the analytic information needed to attack the problem of counting $N(t,\mathfrak{T},z)$. As an analogue to Theorem 2.7 he proved the following spectral expansion about $A(f)(z)$ (see [16, p. 17]):

**Theorem 2.9.** *The automophic function $A(f)(z)$ has the following spectral expansion*

$$A(f)(z) = \sum_j c_j(f) u_j(z)\,,$$

*where $c_j(f) = 2\widehat{u_j} d_{t_j}(f)$. Here*

$$\widehat{u_j} = \int_l u_j ds$$

*is the period of the eigenfunction $u_j$ along a segment $l$ of length $\int_l ds = \mu/\nu$ on the invariant geodesic of $P$. The transform $d_t$, called Huber transform, is given by*

$$d_t(f) = \int_0^{\pi/2} f\Big(\frac{1}{\cos^2 v}\Big)\frac{\xi_\lambda(v)}{\cos^2 v} dv\,,$$

*with $\lambda = \frac{1}{4} + t^2$. The function $\xi_\lambda$ is the solution to the differential equation*

$$\xi''_\lambda(v) + \frac{\lambda}{\cos^2 v}\xi_\lambda(v) = 0\,, v \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\,,$$

*with initial conditions $\xi_\lambda(0) = 1$ and $\xi'_\lambda(0) = 0$.*

We can write $\xi_\lambda(v)$ in terms of a sum of Legendre functions (see [6, page 5]):

$$\xi_\lambda(v) = \frac{1}{2\sqrt{\pi}}\Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(1-\frac{s}{2}\Big)(P_{s-1}(i\tan v) + P_{s-1}(-i\tan v))$$

and after substituting $x = \tan v$ , the transform takes the form

$$d_t(f) = \frac{1}{2\sqrt{\pi}}\Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(1-\frac{s}{2}\Big)\int_0^\infty f(x^2+1)(P_{s-1}(ix) + P_{s-1}(-ix))dx\,. \tag{2.9}$$

Huber, also proved the following important Lemma about the periods $\widehat{u_j}$:

**Lemma 2.10.** *For the sequence of period integrals $\{\widehat{u_j}\}_{j=0}^\infty$, the following bound is true:*

$$\sum_{t_j \le T} |\widehat{u_j}|^2 \ll T\,.$$

We have better results than Lemma 2.10, for example Tsuzuki proved the following asymptotic, see [32, Th. 1, p. 2388]

$$\sum_{\lambda_j \le x} |\widehat{u_j}|^2 \sim \frac{\mathrm{len}(l)}{\pi} x^{1/2}, \quad x \to \infty .$$

In this problem the Huber transform $d_t(f)$, is an analogue to the Selberg–Harish-Chandra transform. With the right choice of test function $f$ the series $A(f)(z)$ can be used to count $N(t, \mathfrak{T}, z)$. Huber showed that by choosing $f$ to be the characteristic function on the interval $\left[0, \sqrt{X^2-1}\right]$ one gets $A(f)(z) = N(t, \mathfrak{T}, z)$. As is done in the hyperbolic circle problem, we smooth the characteristic functions in order to get good results for the error term. Chatzakos and Petridis in [6] defined the functions for $x > 0$:

$$f^+(x^2+1) = \begin{cases} 1, & x \le U, \\ \frac{V-x}{V-U}, & U \le x \le V, \\ 0, & V \le x, \end{cases}$$

and

$$f^-(x^2+1) = \begin{cases} 1, & x \le T, \\ \frac{U-x}{U-T}, & T \le x \le U, \\ 0, & U \le x, \end{cases}$$

for $U = \sqrt{X^2-1}$ and $T, V$ satisfying the condition $0 < U/2 < T < U < V < 2U$. With this choice of test functions they provided another proof of Good's theorem on $N(t, \mathfrak{T}, z)$ for $\Gamma$ cocompact or cofinite :

**Theorem 2.11 (Chatzakos–Petridis [6]).** *Let $X = \sinh(t/2)\big/\sinh(\mu/2)$, then we have*

$$N(t, \mathfrak{T}, z) = \sum_{1/2 < s_j \le 1} A(s_j) \widehat{u}_j u_j(z) X^{s_j} + O\left(X^{2/3}\right),$$

*where $A(s)$ is given by*

$$A(s) = 2^s \cos\left(\frac{\pi(s-1)}{2}\right)\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\pi\Gamma(s+1)}\,.$$

Let

$$E(X,z) := N(t,\mathfrak{T},z) - \sum_{1/2<s_j\le 1} A(s_j)\hat{u}_j u_j(z) X^{s_j}\,,$$

be the error term in the hyperbolic conjugacy class problem. Chatzakos–Petridis also showed the following results about the second moment of the error term $E(X,z)$ for $\Gamma$ cocompact or cofinite:

**Proposition 2.12 (Chatzakos–Petridis [6]).** *Let $X>2$ and $X_1, X_2, \dots, X_R \in [X, 2X]$, such that $|X_i - X_j| > \delta$ for some $\delta > 0$, when $i \neq j$. Then the following bound holds*

$$\sum_{m=1}^{R} |E(X_m,z)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^2 X\,.$$

**Theorem 2.13 (Chatzakos–Petridis [6]).** *If $R\delta \gg X$ and $R > X^{1/2}$, then*

$$\frac{1}{R}\sum_{m=1}^{R} |E(X_m,z)|^2 \ll X \log^2 X\,.$$

*As $R\to\infty$, we have*

$$\frac{1}{X}\int_X^{2X} |E(x,z)|^2 dx \ll X \log^2 X\,.$$

This theorem suggests that the correct order of growth for the error term $E(X,z)$ is $O\left(X^{1/2+\epsilon}\right)$.

Good in his book [11] considers all counting problems for $\Gamma$ cocompact or cofinite. For the conjugacy class problem instead of using the series $A(f)(z)$, used by Huber and Chatzakos–Petridis, he considers Poincaré series $P_\xi(z,s,m)$[11, p. 73, Eq. 7.1] over single cosets $\Gamma_1\backslash\Gamma$ and its automorphic expansion. His result is the following formula, (see [11, Th. 4]):

**Theorem 2.14 (Good [11]).** *Let $\mathfrak{T}$ be a hyperbolic conjugacy class, $\lambda_{\mathfrak{T}}, \lambda_z$ be specific constants and $a_j(\mathfrak{T}, z)$ be the product of $u_j(z), \widehat{u}_j, g(s_j)$, where $g(s_j)$ is an explicit product of Gamma functions. Then*

$$N(t, \mathfrak{T}, z) = \frac{2}{vol(\Gamma\backslash\mathbb{H})}\frac{\mu}{\nu}X + 2\lambda_{\mathfrak{T}}\lambda_z \sum_{1/2<s_j<1} a_j(\mathfrak{T}, z)X^{s_j} + O\big(X^{2/3}\big).$$

Unfortunately we cannot match the Gamma functions $g(s)$ with $A(s)$.

Parkkonen and Paulin used ergodic methods and more specifically the geodesic flow to study the hyperbolic lattice counting problem in conjugacy classes in [25] and gave an asymptotic for the counting of common perpendicular arcs in negative curvature, see [26, Th. 1, p. 901].

## Chapter 3

# A proof of Good's theorem for counting geodesic segments

## 3.1 A relative trace formula

We let $\Gamma$ be cocompact. In this section we investigate a relative trace formula suitable for the problem of counting in the double coset $H_1 \backslash \Gamma / H_1$.

For simplicity we assume that $L$ is a primitive closed geodesic, namely $L$ is traversed only once. By conjugation we can assume that the axis of the closed geodesic $L$ is the imaginary axis $I$, so that $L = H_1 \backslash I$, where $H_1$ is a hyperbolic subgroup of $\Gamma$ of the form

$$H_1 = \left\langle \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix} \right\rangle \quad , m > 1 \, .$$

Let $l$ represent the closed geodesic $L$ on the imaginary axis. Here $m^2$ is the norm of the primitive closed geodesic $L$. For a path $\sigma : [a, b] \to \mathbb{H}$, we define its length by

$$\operatorname{len}(\sigma) = \int_a^b \frac{|\sigma'(t)|}{\Im(\sigma(t))} dt \, .$$

It follows that $\text{len}(l) = 2\log m$. Suppose $\Gamma\backslash\mathbb{H}$ is compact and let $k : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ be a test function that depends only on $v$, the angle defined in (2.6). In practice we also assume that $k$ is even. We consider its automorphization with respect to $\Gamma$:

$$K(z) := \sum_{\gamma \in H_1\backslash\Gamma} k(v(\gamma z)) \,.$$

Firstly, we develop the geometric side of the relative trace formula by integrating $K$ over the geodesic segment $l$.

We split the sum into the identity coset $H_1$ and the rest of the cosets. For the coset $H_1$ and $z = iy \in l$ we have that $k(v(\gamma z)) = k(v(z)) = k(0)$, since an element $\gamma \in H_1$ is a magnification, that preserves the angle $v(z)$ to the imaginary axis. Hence, we compute

$$\int_l K(z)ds = k(0)\text{len}(l) + \int_l \sum_{\gamma \in H_1\backslash\Gamma - H_1} k(v(\gamma z))ds. \tag{3.1}$$

For the second sum we have:

$$\begin{aligned} \int_l \sum_{\gamma \in H_1\backslash\Gamma - H_1} k(v(\gamma z))ds &= \sum_{\gamma \in H_1\backslash\Gamma - H_1/H_1} \sum_{\gamma_0 \in H_1} \int_l k(v(\gamma\gamma_0 z))ds \\ &= \sum_{\gamma \in H_1\backslash\Gamma - H_1/H_1} \int_I k(v(\gamma z))ds\,, \end{aligned} \tag{3.2}$$

because $H_1$ is the stabilizer of $l$ and we notice that action with the elements of $H_1$ on $l$ will cover the whole imaginary axis $I$.

The function $K(z)$ is automorphic with respect to $\Gamma$. Since $k(v)$ is a function of the angle $v$ to the imaginary axis only, we wish to apply Theorem 2.9 for

$$k(v(z)) := f\left(\frac{1}{\cos^2(v(z))}\right).$$

In order to ensure convergence of the series $K(z) = A(f)(z)$ we assume that either $f \in C_0^*[1,\infty)$ or $f$ is in the Schwartz class. In the first case the series is finite, while in the latter case we can write $K(z)$ as a Stiltjes integral:

$$\sum_{\gamma\in H_1\backslash\Gamma} f\left(\frac{1}{\cos^2 v(\gamma z)}\right) = \int_0^{\pi/2} f\left(\frac{1}{\cos^2 v}\right) d\big(\widetilde{N}(v,z)\big)\,, \tag{3.3}$$

where

$$\widetilde{N}(V,z) := \#\big\{\gamma\in H_1\backslash\Gamma \,\big|\, v(\gamma z)\le V\big\}\,.$$

For $X = 1/\cos v$ we have that

$$\widetilde{N}(v,z) = N(t,\mathfrak{T},z) \ll X$$

by (2.5), (2.8) and Huber's bound (2.7). Applying integration by parts on the integral from (3.3) shows that $K(z)$ converges if $f$ and its derivatives are rapidly decreasing.

By Theorem 2.9 we get:

$$K(z) = \sum_j 2d_{t_j}(f)\widehat{u_j}u_j(z)\,. \tag{3.4}$$

The spectral side of the relative trace formula comes from integrating the above over $l$. Equating (3.1) and (3.2) with (3.4) gives:

$$f(1)\cdot\mathrm{len}(l) + \sum_{\gamma\in H_1\backslash\Gamma - H_1/H_1}\int_I f\left(\frac{1}{\cos^2 v(\gamma z)}\right)ds = \sum_j 2d_{t_j}(f)\widehat{u_j}^2\,.$$

In order to evaluate

$$N(X,l) = \sum_{\substack{\gamma\in H_1\backslash\Gamma/H_1\\ B(\gamma)<X}} 1\,,$$

where $B(\gamma) = |ad+bc|$, we analyse further the geometric side, specifically the integral

$$J_\gamma^I(f) = \int_I f\left(\frac{1}{\cos^2 v(\gamma z)}\right) ds = \int_0^\infty f\left(\frac{1}{\cos^2 v(\gamma \cdot iy)}\right) \frac{dy}{y},$$

where we used that the hyperbolic metric $ds$ is given by $ds^2 = (dx^2 + dy^2)/y^2$. We would like to express $J_\gamma^I(f)$ in terms of the matrix entries of $\gamma$. We show the following result.

**Lemma 3.1.** *Let $z = iy$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then*

$$\frac{1}{\cos^2 v(\gamma z)} = \frac{(a^2y^2 + b^2)(c^2y^2 + d^2)}{y^2}.$$

*Proof.* We notice that

$$\cos v(\gamma z) = \frac{\Im(\gamma z)}{|\gamma z|}.$$

Since $z = iy$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we compute:

$$\gamma z = \frac{aiy + b}{ciy + d} = \frac{(aiy + b)(-ciy + d)}{|ciy + d|^2} = \frac{acy^2 + bd + (ayd - bcy)i}{|ciy + d|^2} = \frac{acy^2 + bcd + yi}{|ciy + d|^2},$$

hence $\Im(\gamma z) = \frac{y}{|ciy+d|^2}$. Therefore,

$$\begin{aligned}\cos v(\gamma z) = \frac{\Im(\gamma z)}{|\gamma z|} &= \frac{y}{|ciy + d|^2}\left(\frac{|aiy + b|}{|ciy + d|}\right)^{-1} \\ &= \frac{y}{|aiy + b||ciy + d|} \\ &= \frac{y}{\sqrt{a^2y^2 + b^2}\sqrt{c^2y^2 + d^2}},\end{aligned}$$

from which we conclude that

$$\frac{1}{\cos^2 v(\gamma z)} = \frac{(a^2y^2 + b^2)(c^2y^2 + d^2)}{y^2}.$$

□

The result of Lemma 3.1 is an analogue to Lemma 1.1, which was proved in [23]. We now present its proof:

*Proof of Lemma 1.1.* From [23, p. 9, eq. 9] we have that

$$4u(\gamma \cdot ix, iy) = \frac{a^2 x}{y} + \frac{b^2}{xy} + c^2 xy + d^2 \frac{y}{x} - 2 .$$

Let $h(x, y) = 4u(\gamma \cdot ix, iy)$. We want to find the minimum value of $h$, hence we compute its gradient:

$$h_x(x, y) = 0 \implies y^2 = \frac{a^2 x^2 - b^2}{d^2 - c^2 x^2} .$$

Using that $y$ in the equation of $h_y(x, y) = 0$ we get the solutions

$$x^4 = \frac{b^2 d^2}{a^2 c^2} .$$

Working in a similar way we find that the solutions to the equation $h_x(x, y)$ are

$$y^4 = \frac{a^2 b^2}{c^2 d^2} .$$

Let $x_{\min}^2 = |(bd)/(ac)|$ and $y_{\min}^2 = |(ab)/(cd)|$. Then we compute

$$h(x_{\min}, y_{\min}) = 2|ad| + 2|bc| - 2 .$$

Now, suppose that $abcd < 0$ then there are two cases: either i) $ad > 0$ and $bc < 0$ or ii) $ad < 0$ and $bc > 0$. The second case is not possible because $ad - bc = 1$. In the first case we have $h(x_{\min}, y_{\min}) = 0$.

On the other hand, if $abcd > 0$ then either i) $ad > 0$ and $bc > 0$ or ii) $ad < 0$

and $bc < 0$. In both cases we have

$$h(x_{\min}, y_{\min}) = 2|ad + bc| - 2\,.$$

Moreover the point $(x_{\min}, y_{\min})$ minimizes h because $h(x, y)$ goes to $\infty$ as $x$ or $y$ approaches either 0 from above or $\infty$. More precisely, there exist constants $C, M \in \mathbb{R}_{>0}$ such that for all $N \geq M$ and for all $(x, y) \notin [1/N, N]^2$, $h(x, y) \geq C \cdot N$. □

Using Lemma 3.1, we see that the integral $J_\gamma^I(f)$ takes the form

$$J_\gamma^I(f) = \int_0^\infty f\left(a^2c^2y^2 + \frac{b^2d^2}{y^2} + a^2d^2 + b^2c^2\right)\frac{dy}{y}\,.$$

Now, similarly to [23], we introduce the change of variables $y = e^t$, hence $dy = e^t dt \implies dy/y = dt$. So we have:

$$J_\gamma^I(f) = \int_{-\infty}^\infty f\left(a^2c^2e^{2t} + b^2d^2e^{-2t} + a^2d^2 + b^2c^2\right)dt\,.$$

Let $p = a'e^{2t} + b'e^{-2t}$, with $a' = a^2c^2$ and $b' = b^2d^2$, so that $dp = \left(2a'e^{2t} - 2b'e^{-2t}\right)dt$. We compute:

$$dt = \frac{dp}{2a'e^{2t} - 2b'e^{-2t}} = \frac{dp}{2(a'e^{2t} - b'e^{-2t})} = \frac{dp}{2\sqrt{p^2 - 4a'b'}}$$

because

$$\begin{aligned} p^2 - 4a'b' &= \left(a'\right)^2e^{4t} + \left(b'\right)^2e^{-4t} + 2a'b' - 4a'b' = \left(a'e^{2t} - b'e^{-2t}\right)^2 \\ &= \left(a^2c^2e^{2t} - b^2d^2e^{-2t}\right)^2\,. \end{aligned}$$

In order to find the lower limit of the integral after the change of variables to $p$, we need to compute $\min_{t\in\mathbb{R}}(a^2c^2e^{2t} - b^2d^2e^{-2t})$. Using the first derivative test, we end up with:

$$2a^2c^2e^{2t} - 2b^2d^2e^{-2t} = 0 \Leftrightarrow e^{4t} = \frac{b^2d^2}{a^2c^2} \Leftrightarrow t = \frac{1}{2}\log\left|\frac{bd}{ac}\right|.$$

We conclude that

$$J_\gamma^I(f) = \int_{2|abcd|}^{\infty} f\big(p + a^2d^2 + b^2c^2\big)\frac{dp}{\sqrt{p^2 - 4a^2b^2c^2d^2}}\,.$$

Suppose that the element $\gamma$ is such that $abcd > 0$, namely $B(\gamma) > 1$. Because $ad - bc = 1$, we immediately get that $a^2d^2 + b^2c^2 = 2abcd + 1$. Now let $q = p + 2abcd$, which gives us

$$J_\gamma^I(f) = \int_{4abcd}^{\infty} \frac{f(q+1)}{\sqrt{q(q-4abcd)}}\,dq\,.$$

In order to match our test function with the ones defined in [16] and [6], we finally introduce the change of variables $x^2 + 1 = q + 1 \implies dx = dq/2x = dq/(2\sqrt{q})$ and the integral becomes

$$J_\gamma^I(f) = 2\int_{\sqrt{4abcd}}^{\infty} \frac{f(x^2+1)}{\sqrt{x^2 - 4abcd}}\,dx\,.$$

We recall at this point that

$$\delta(\gamma) = 2B(\gamma) = 2(ad + bc)\,.$$

Since $ad - bc = 1$ implies $a^2d^2 + b^2c^2 - 2abcd = 1$, we can relate the quantity $4abcd$ that appears inside the integral with $B(\gamma)$. We have:

$$B(\gamma)^2 = a^2d^2 + b^2c^2 + 2abcd = 1 + 4abcd$$
$$\implies B(\gamma)^2 - 1 = 4abcd\,,$$

which leads to

$$J_\gamma^I(f) = 2\int_{\sqrt{B(\gamma)^2-1}}^{\infty} \frac{f(x^2+1)}{\sqrt{x^2-(B(\gamma)^2-1)}}dx\,.$$

This is the contribution to the geometric side of our trace formula for the double coset $H_1\backslash\Gamma/H_1$.

Now, suppose that $abcd < 0$, i.e. $B(\gamma) < 1$. Following the same process we find that

$$J_\gamma^I(f) = 2\int_0^{\infty} \frac{f(x^2+1)}{\sqrt{x^2-(B(\gamma)^2-1)}}dx\,.$$

Let

$$q(z) = 2\int_{\sqrt{z^2-1}}^{\infty} \frac{f(x^2+1)}{\sqrt{x^2-(z^2-1)}}dx \tag{3.5}$$

and

$$\widetilde{q}(z) = 2\int_0^{\infty} \frac{f(x^2+1)}{\sqrt{x^2-(z^2-1)}}dx\,, \tag{3.6}$$

then we have proved the following formula:

**Proposition 3.2 (Relative Trace Formula).** *Let $f \in C_0^*[1,\infty)$ or $f : [1,\infty) \to \mathbb{R}$ be in the Schwartz class. For $q$ as in* (3.5) *and $\widetilde{q}$ as in* (3.6)*, we have:*

$$f(1)\,len(l) + \sum_{\substack{\gamma\in H_1\backslash\Gamma - H_1/H_1\\ B(\gamma)<1}} \widetilde{q}(B(\gamma)) + \sum_{\substack{\gamma\in H_1\backslash\Gamma - H_1/H_1\\ B(\gamma)>1}} q(B(\gamma)) = \sum_j 2d_{t_j}(f)\widehat{u_j}^2\,.$$

## 3.2 Applications of the relative trace formula

### 3.2.1 The geometric side of the trace formula

To apply the relative trace formua we want to choose a test function $f$ so that $q(B(\gamma)) = 1$, whenever $B(\gamma) < X$, namely $q$ is the characteristic function on the interval $[0, X]$. However, this is not continuous and we will need to use smoothings of the characteristic function instead, following ideas from [6]. Motivated by the Selberg–Harish-Chandra transform, we want to write $q$ as a Weyl integral (see [9, Chapter XIII] for the definition of such integrals, more information will be given in Chapter 4). We rewrite (3.5) using the substitution $u = x^2 + 1 \implies dx = du/\left(2\sqrt{u-1}\right)$, as

$$q(z) = \int_{z^2}^{+\infty} \frac{f(u)}{\sqrt{u-1}} \frac{du}{\sqrt{u-z^2}} .$$

We therefore set

$$F(u) = \frac{f(u)}{\sqrt{u-1}} .$$

We now let

$$g(v) = \int_{v}^{\infty} \frac{F(u)}{\sqrt{u-v}} du .$$

From [17, equations 1.64] we can recover $F$ (and consequently $f$). We get:

$$F(u) = -\frac{1}{\pi} \int_{u}^{\infty} \frac{1}{\sqrt{v-u}} dg(v) . \tag{3.7}$$

The outcome is

$$q(z) = g(z^2) .$$

Now, we choose $g$ to be piecewise linear so that $q$ is a smoothing of the characteristic function. Let $H = Y^2 + 2YX$. We choose

$$g(y) = \begin{cases} 1, & y \le X^2, \\ \dfrac{(X+Y)^2 - y}{H}, & X^2 \le y \le (X+Y)^2, \\ 0, & y \ge (X+Y)^2 . \end{cases} \tag{3.8}$$

We now compute $F$ using (3.7).

- When $u > (X+Y)^2$ then $F(u) = 0$.

- For $u < X^2$ we get

$$F(u) = -\frac{1}{\pi} \int_{X^2}^{(X+Y)^2} \frac{1}{\sqrt{v-u}} g'(v) dv ,$$

since the derivative of $g$ is 0 for $u < X^2$. We compute:

$$F(u) = \frac{1}{\pi H}\int_{X^2}^{(X+Y)^2} \frac{1}{\sqrt{v-u}}dv = \frac{2}{\pi H}\left[\sqrt{v-u}\right]_{X^2}^{(X+Y)^2}$$
$$= \frac{2}{\pi H}\left(\sqrt{(X+Y)^2-u} - \sqrt{X^2-u}\right).$$

Hence, for $u < X^2$, we get

$$f(u) = \frac{2}{\pi H}\sqrt{u-1}\left(\sqrt{(X+Y)^2-u} - \sqrt{X^2-u}\right).$$

Since in (3.5) $f$ appears as $f(x^2+1)$ we write

$$f(x^2+1) = \frac{2}{\pi H}x\left(\sqrt{(X+Y)^2-x^2-1} - \sqrt{X^2-x^2-1}\right).$$

After setting $a = \sqrt{X^2-1}$ and $A = \sqrt{(X+Y)^2-1}$ the previous expression becomes

$$f(x^2+1) = \frac{2}{\pi H}x\left(\sqrt{A^2-x^2} - \sqrt{a^2-x^2}\right).$$

- When $X^2 \le u \le (X+Y)^2$ or equivalently $a \le x \le A$ for $u = x^2+1$ we get:

$$F(u) = -\frac{1}{\pi}\int_{u}^{(X+Y)^2} \frac{1}{\sqrt{v-u}}g'(v)dv$$

and with a similar computation:

$$f(x^2+1) = \frac{2}{\pi H}x\sqrt{A^2-x^2}.$$

We conclude that

$$f(x^2+1) = \begin{cases} \dfrac{2}{\pi H} x\Big(\sqrt{A^2-x^2} - \sqrt{a^2-x^2}\Big), & x \le a, \\ \dfrac{2}{\pi H} x\sqrt{A^2-x^2}, & a \le x \le A, \\ 0, & \text{else}\,. \end{cases} \tag{3.9}$$

Since the corresponding $q$ is an overestimate of the characteristic function of the interval $[0, X]$ we denote this function as $q^+$ and similarly we denote $f$ by $f^+$ and $g$ by $g^+$. After defining

$$g^-(y) = \begin{cases} 1, & y \le X^2 - H, \\ \dfrac{X^2-y}{H}, & X^2 - H \le y \le X^2, \\ 0, & y \ge X^2, \end{cases} \tag{3.10}$$

and repeating the same process, we obtain the test function

$$f^-(x^2+1) = \begin{cases} \dfrac{2}{\pi H} x\Big(\sqrt{a^2-x^2} - \sqrt{T^2-x^2}\Big), & x \le T, \\ \dfrac{2}{\pi H} x\sqrt{a^2-x^2}, & T \le x \le a, \\ 0, & \text{else}\,, \end{cases} \tag{3.11}$$

where $T = \sqrt{X^2 - H - 1}$.

Given these choices of $f^-, f^+$, let $\widetilde{g}^-(z^2) = \widetilde{q}^-(z)$ and $\widetilde{g}^+(z^2) = \widetilde{q}^+(z)$ be the corresponding functions for the exceptional terms, namely

$$\widetilde{g}^+(z^2) = 2\int_0^\infty \frac{f^+(x^2+1)}{\sqrt{x^2-(z^2-1)}}dx$$

and

$$\widetilde{g}^-(z^2) = 2\int_0^\infty \frac{f^-(x^2+1)}{\sqrt{x^2-(z^2-1)}}dx\,.$$

For $B(\gamma) < 1$ we notice that

$$\widetilde{g}^{+}\left(B(\gamma)^2\right) \leq \int_0^{\infty} \frac{f^{+}(x^2+1)}{x} dx = \frac{2}{\pi H}\left(\int_0^{A} \sqrt{A^2 - x^2} dx - \int_0^{a} \sqrt{a^2 - x^2} dx\right).$$

After the change of variables $x = A \sin t$ for the first integral and $x = a \sin t$ for the second integral we see that

$$\widetilde{g}^{+}\left(B(\gamma)^2\right) \leq \frac{2}{\pi H} \frac{(A^2 - a^2)\pi}{4} \leq 1 . \tag{3.12}$$

A similar calculation gives that $\widetilde{g}^{-}\left(B(\gamma)^2\right) \leq 1$. We note that by discretness and inequality (3.12) the sums for the exceptional terms are $O(1)$. Also note that $f^{+}(1) = f^{-}(1) = 0$.

From (3.8) and (3.10), since $0 \leq g^{-}(y) \leq 1$ in $\left[0, X^2\right]$ and $g^{-}(y) = 0$ for $y > X^2$, we clearly have that

$$\sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) > 1}} g^{-}\left(B(\gamma)^2\right) \leq \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ 1 < B(\gamma) < X}} 1 \leq \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) > 1}} g^{+}\left(B(\gamma)^2\right). \tag{3.13}$$

By its definition $N(X, l)$ can be written as

$$N(X, l) = \sum_{\substack{\gamma \in H_1 \backslash \Gamma / H_1 \\ B(\gamma) \leq 1}} 1 + \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ 1 < B(\gamma) < X}} 1 ,$$

so from (3.13) we have

$$\sum_{\substack{\gamma \in H_1 \backslash \Gamma / H_1 \\ B(\gamma) \leq 1}} 1 + \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) > 1}} g^{-}\left(B(\gamma)^2\right) \leq N(X, l) \leq \sum_{\substack{\gamma \in H_1 \backslash \Gamma / H_1 \\ B(\gamma) \leq 1}} 1 + \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) > 1}} g^{+}\left(B(\gamma)^2\right). \tag{3.14}$$

### 3.2.2 The spectral side of the relative trace formula

We proceed with the analysis of the spectral side of Proposition 3.2 for $f^+$. The computations regarding $f^-$ follow in the same way. The integral transform that appears in the Huber transform in (2.9) is

$$\int_0^\infty f^+(x^2+1)(P_{s-1}(ix)+P_{s-1}(-ix))dx$$

and for the specific test function $f^+$ that we chose in (3.9), it becomes

$$\begin{aligned}&\int_0^a \frac{2}{\pi H}x\Big(\sqrt{A^2-x^2}-\sqrt{a^2-x^2}\Big)(P_{s-1}(ix)+P_{s-1}(-ix))\,dx\\ &+\int_a^A \frac{2}{\pi H}x\sqrt{A^2-x^2}(P_{s-1}(ix)+P_{s-1}(-ix))dx\,.\end{aligned}$$

For $y>0$, let

$$J_s(y)=\int_0^{\sqrt{y}} x\sqrt{y-x^2}\big(P_{s-1}(ix)+P_{s-1}(-ix)\big)dx\,,$$

then the Huber transform of $f^+$ can be written as

$$d_t(f^+)=\pi^{-3/2}\Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(1-\frac{s}{2}\Big)\frac{J_s(A^2)-J_s(a^2)}{H}\,. \tag{3.15}$$

We prove the following Lemma for $d_t(f^+)$:

**Lemma 3.3.** *For any $s=\frac{1}{2}+it$ with $t\in\mathbb{R}$ or $1/2<s<1$ and $|s|\ll A^2$ we have:*

$$\begin{aligned}d_t(f^+)=&\frac{\gamma_1(s)}{\pi}\frac{2}{s+2}\frac{A^{s+2}-a^{s+2}}{A^2-a^2}(1+O(|t|X^{-2}))\\ &+\frac{\gamma_2(s)}{\pi}\frac{A^{3-s}-a^{3-s}}{A^2-a^2}(1+O(|t|X^{-2}))\\ &+\frac{\gamma_3(s)}{\pi}\frac{A-a}{A^2-a^2}(1+O(|t|X^{-2}))\,,\end{aligned} \tag{3.16}$$

*where*

$$\gamma_1(s) = \frac{\pi}{2}\frac{\Gamma(s+1/2)\Gamma((s+1)/2)}{\Gamma(s/2)^3}\frac{4}{s(2s-1)} \quad , \quad \gamma_2(s) = \frac{\pi}{2}\frac{\Gamma((1-2s)/2)\Gamma((2-s)/2)}{\big(\Gamma((1-s)/2)\big)^2\Gamma((5-s)/2)}$$

$$\text{and} \quad \gamma_3(s) = -\frac{1}{2}\frac{\Gamma((-1-s)/2)\Gamma((s-2)/2)}{\Gamma((1-s)/2)\Gamma(s/2)}\,.$$

*Proof.* By the definition of the Legendre function $P_{s-1}$ we have that

$$\begin{aligned} J_s\big(A^2\big) &= \int_0^A x\sqrt{A^2-x^2}\big(P_{s-1}(ix)+P_{s-1}(-ix)\big)dx \\ &= \int_0^A x\sqrt{A^2-x^2}\Big({}_2F_1\Big(1-s,s;1;\frac{1-ix}{2}\Big)+{}_2F_1(1-s,s;1;\frac{1+ix}{2}\Big)\Big)dx\,. \end{aligned}$$

Applying formulas [12, 9.136.2 and 9.136.3] for $\alpha = (1-s)/2, \beta = s/2, z = -x^2$ to both hypergeometric functions inside the integral, we get

$$J_s\big(A^2\big) = 2D(s)\int_0^A x\sqrt{A^2-x^2}\,{}_2F_1\Big(\frac{1-s}{2},\frac{s}{2};\frac{1}{2};-x^2\Big)dx\,, \tag{3.17}$$

where

$$D(s) = \frac{\sqrt{\pi}}{\Gamma\big(\frac{2-s}{2}\big)\Gamma\big(\frac{s+1}{2}\big)}\,.$$

Let $x^2 = u \implies 2xdx = du$, then

$$J_s\big(A^2\big) = D(s)\int_0^{A^2}\sqrt{A^2-u}\,{}_2F_1\Big(\frac{1-s}{2},\frac{s}{2};\frac{1}{2};-u\Big)du$$

and for $u = vA^2 \implies du = dvA^2$, the resulting integral is

$$A^3D(s)\int_0^1\sqrt{1-v}\,{}_2F_1\Big(\frac{1-s}{2},\frac{s}{2};\frac{1}{2};-vA^2\Big)dv\,.$$

By [12, Eq. 7.512.12] we have

$$J_s(A^2) = A^3D(s)\frac{\Gamma(3/2)}{\Gamma(5/2)}\,{}_3F_2\Big(\frac{1-s}{2},\frac{s}{2},1;\frac{1}{2},\frac{5}{2};-A^2\Big)\,.$$

By [24, Eq. 16.8.8] for $q = 2, z = -A^2$ we have for $s \neq 1/2, 1$

$$\begin{aligned}
J_s(A^2) =& A^3 D(s) \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma((1-s)/2)\Gamma(s/2)} \Bigg( \frac{\Gamma((1-s)/2)\Gamma((2s-1)/2)\Gamma((1+s)/2)}{\Gamma(s/2)\Gamma((4+s)/2)} A^{s-1} \times \\
& \times {}_3F_2\Big(\frac{1-s}{2}, 1-\frac{s}{2}, -1-\frac{s}{2}; \frac{3-2s}{2}, \frac{1-s}{2}; -A^{-2}\Big) \\
& + \frac{\Gamma(s/2)\Gamma((1-2s)/2)\Gamma(1-s/2)}{\Gamma((1-s)/2)\Gamma((5-s)/2)} A^{-s} {}_3F_2\Big(\frac{s}{2}, \frac{s+1}{2}, \frac{s-3}{2}; \frac{2s+1}{2}, \frac{s}{2}; -A^{-2}\Big) \\
& + \frac{\Gamma((-1-s)/2)\Gamma((s-2)/2)}{\Gamma(-1/2)\Gamma(3/2)} A^{-2} {}_3F_2\Big(1, \frac{3}{2}, -\frac{1}{2}; \frac{s+3}{2}, \frac{4-s}{2}; -A^{-2}\Big)\Bigg) \\
=& A^3 D(s) \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma((1-s)/2)\Gamma(s/2)} \Bigg( \frac{\Gamma((1-s)/2)\Gamma((2s-1)/2)\Gamma((1+s)/2)}{\Gamma(s/2)\Gamma((4+s)/2)} A^{s-1} \times \\
& \times {}_2F_1\Big(1-\frac{s}{2}, -1-\frac{s}{2}, \frac{3-2s}{2}, -A^{-2}\Big) \\
& + \frac{\Gamma(s/2)\Gamma((1-2s)/2)\Gamma(1-s/2)}{\Gamma((1-s)/2)\Gamma((5-s)/2)} A^{-s} {}_2F_1\Big(\frac{s+1}{2}, \frac{s-3}{2}, \frac{2s+1}{2}; -A^{-2}\Big) \\
& + \frac{\Gamma((-1-s)/2)\Gamma((s-2)/2)}{\Gamma(-1/2)\Gamma(3/2)} A^{-2} {}_3F_2\Big(1, \frac{3}{2}, -\frac{1}{2}; \frac{s+3}{2}, \frac{4-s}{2}; -A^{-2}\Big)\Bigg).
\end{aligned} \tag{3.18}$$

Excluding $D(s)$, we compute the Gamma factors for the first summand to be $\gamma_1(s)(2/(s+2))$, where

$$\gamma_1(s) := \frac{\pi}{2} \frac{\Gamma(s+1/2)\Gamma((s+1)/2)}{\Gamma(s/2)^3} \frac{4}{s(2s-1)}$$

are the Gamma functions appearing in [32, p.16, Th.25]. We work similarly for the other summands to get

$$\gamma_2(s) = \frac{\pi}{2} \frac{\Gamma((1-2s)/2)\Gamma((2-s)/2)}{\big(\Gamma((1-s)/2)\big)^2 \Gamma((5-s)/2)}$$

and

$$\gamma_3(s) = -\frac{1}{2} \frac{\Gamma((-1-s)/2)\Gamma((s-2)/2)}{\Gamma((1-s)/2)\Gamma(s/2)}.$$

If $|s| \ll A^2$, we use the series expansion of the hypergeometric functions that appear in the expansion of $J_s(A^2)$ to compute that

$${}_2F_1\Big(1-\frac{s}{2}, -1-\frac{s}{2}, \frac{3-2s}{2}, -A^{-2}\Big) = 1 + O\big(|t|A^{-2}\big),$$

$$ {}_2F_1\Big(\frac{s+1}{2},\frac{s-3}{2},\frac{2s+1}{2};-A^{-2}\Big)=1+O\big(|t|A^{-2}\big)\,, $$

$$ {}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-A^{-2}\Big)=1+O\big(|t|^{-2}A^{-2}\big)\,. $$

We use the same methods to compute $J_s(a^2)$. Since $a, A \sim X$ we have

$$ \begin{aligned} d_t(f^+) &= \frac{1}{2\sqrt{\pi}}\Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(1-\frac{s}{2}\Big)\frac{2}{\pi H}\big(J_s\big(A^2\big)-J_s\big(a^2\big)\big) \\ &= \frac{\gamma_1(s)}{\pi}\frac{2}{s+2}\frac{A^{s+2}-a^{s+2}}{A^2-a^2}\big(1+O\big(|t|X^{-2}\big)\big) \\ &\quad+\frac{\gamma_2(s)}{\pi}\frac{A^{3-s}-a^{3-s}}{A^2-a^2}\big(1+O\big(|t|X^{-2}\big)\big) \\ &\quad+\frac{\gamma_3(s)}{\pi}\frac{A-a}{A^2-a^2}\big(1+O\big(|t|^{-2}X^{-2}\big)\big)\,. \end{aligned} $$

□

We note that using Stirling's approximation (A.1) the gamma functions appearing can be bounded as follows:

$$ \gamma_1(s) \ll |t|^{-1/2}\,, \quad \gamma_2(s) \ll |t|^{-3/2} \quad \text{and} \quad \gamma_3(s) \ll |t|^{-2}\,. \tag{3.19} $$

More generally we can prove the following result about $d_t(f^+)$:

**Lemma 3.4.** *Let $f^+$ be the function from* (3.9)*, then there exists $\xi \in [a^2, A^2]$ such that*

$$ d_t(f^+) = \frac{\Gamma\big(\frac{s+1}{2}\big)\Gamma\big(1-\frac{s}{2}\big)}{\pi^{3/2}}\int_0^{\sqrt{\xi}}\frac{x}{\sqrt{\xi-x^2}}\big(P_{s-1}(ix)+P_{s-1}(-ix)\big)dx\,. $$

*Proof.* We define

$$ G_s(y,x) = x\sqrt{y-x^2}\big(P_{s-1}(ix)+P_{s-1}(-ix)\big)\,, $$

so that the Huber transform of $f$ can be written as

$$d_t(f^+) = \frac{2\Gamma\big(\frac{s+1}{2}\big)\Gamma\big(1-\frac{s}{2}\big)}{\pi^{3/2}} \frac{J_s\big(A^2\big)-J_s\big(a^2\big)}{H}\,.$$

Since $H = A^2 - a^2$, we can apply the Mean Value Theorem for $J_s$ on the interval $[a^2, A^2]$. In order to do so, first we have to compute the derivative of $J_s$. By the Leibniz Rule, we have

$$\begin{aligned} J_s'(y) &= G_s(y,\sqrt{y})(\sqrt{y})' - G_s(y,0)0' + \int_0^{\sqrt{y}} \frac{\partial}{\partial y} G_s(y,x)\,dx \\ &= \frac{1}{2}\int_0^{\sqrt{y}} \frac{x}{\sqrt{y-x^2}}\big(P_{s-1}(ix)+P_{s-1}(-ix)\big)\,dx\,. \end{aligned}$$

Using the Mean Value Theorem we find $\xi \in [a^2, A^2] \implies X^2 - 1 < \xi < (X+Y)^2 - 1$, such that

$$\begin{aligned} d_t(f^+) &= \frac{2\Gamma\big(\frac{s+1}{2}\big)\Gamma\big(1-\frac{s}{2}\big)}{\pi^{3/2}} J_s'(\xi) \\ &= \frac{\Gamma\big(\frac{s+1}{2}\big)\Gamma\big(1-\frac{s}{2}\big)}{\pi^{3/2}} \int_0^{\sqrt{\xi}} \frac{x}{\sqrt{\xi-x^2}}\big(P_{s-1}(ix)+P_{s-1}(-ix)\big)\,dx\,. \end{aligned}$$

□

Using Lemma 3.3 we prove the main estimate for the Huber transform of $f^+$ as follows:

**Proposition 3.5.** *i) For any $s = \frac{1}{2} + it$ with $t \in \mathbb{R}$ or $1/2 < s < 1$ and $|s| \ll A^2$, we have*

$$\begin{aligned} d_t(f^+) = &\frac{\gamma_1\big(\frac{1}{2}+it\big)}{\pi} X^{\frac{1}{2}+it} + \frac{\gamma_2\big(\frac{1}{2}+it\big)}{\pi}\Big(\frac{5}{2}-it\Big)X^{\frac{1}{2}-it} \\ &+ O\Big(\Big|\gamma_1\Big(\frac{1}{2}+it\Big)\Big||t|X^{-\frac{1}{2}}Y + \Big|\gamma_2\Big(\frac{1}{2}+it\Big)\Big||t|^2X^{-\frac{1}{2}}Y\Big)\,. \end{aligned}$$

*ii) Let $t \in \mathbb{R}$ and $t \neq 0$. Then $d_t(f^+)$ can be written in the form*

$$d_t(f^+) = a(t,X,Y)X^{\frac{1}{2}+it} + b(t,X,Y)X^{\frac{1}{2}-it},$$

*where*

$$a(t,X,Y), b(t,X,Y) = O\big(\min\{|t|^{-1/2}, |t|^{-3/2}X/Y\}\big).$$

*iii) Let $t \notin \mathbb{R}$, i.e. $s \in \left(\frac{1}{2}, 1\right]$, then the Huber transform can be written as*

$$d_t(f^+) = \frac{\gamma_1(s)}{\pi}X^s + \frac{\gamma_2(s)}{\pi}\Big(\frac{3-s}{2}\Big)X^{1-s} + O\Big(Y + \Big|\Gamma\Big(\frac{1}{2}-s\Big)\Big|X^{1/2}\Big).$$

*iv) For $t = 0$, we have*

$$d_0(f^+) \ll X^{1/2}\log X.$$

*Proof.* i) We use the Mean Value Theorem for the function $x^{(s+2)/2}$ to find $\xi \in [a^2, A^2]$ such that

$$\frac{A^{s+2} - a^{s+2}}{A^2 - a^2} = \frac{s+2}{2}\xi^{s/2} \tag{3.20}$$

and notice that $\xi^{s/2} = X^s + O\big(|s|\big|X^{s-1}\big|Y\big)$. Also by the Mean Value Theorem for the function $x^{(3-s)/2}$, we find $\zeta \in [a^2, A^2]$ such that

$$\frac{A^{3-s} - a^{3-s}}{A^2 - a^2} = \frac{3-s}{2}\zeta^{(1-s)/2}. \tag{3.21}$$

We see that $\zeta^{(1-s)/2} = X^{1-s} + O\big(|1-s|\big|X^{-s}\big|Y\big)$. For the last summand in (3.16) we use Stirling's approximation to conclude that it is $O(1)$. Hence, using (3.20) and (3.21), for $|s| \ll A^2$ we write (3.16) as:

$$d_t(f^+) = \frac{\gamma_1(s)}{\pi}X^s + \frac{\gamma_2(s)}{\pi}\frac{3-s}{2}X^{1-s} + O\Big(|\gamma_1(s)||t|\big|X^{s-1}\big|Y + |\gamma_2(s)||t|^2\big|X^{-s}\big|Y\Big).$$

After plugging $s = \frac{1}{2} + it$ to the last formula the result follows.

ii) Suppose that $t \in \mathbb{R}, t \neq 0$. From (3.15) and (3.18) we have that

$$\begin{aligned} d_t(f^+) =& \frac{\gamma_1(s)}{\pi H}\frac{2}{s+2}\Big(A^{s+2}\,{}_2F_1\Big(1-\frac{s}{2},-1-\frac{s}{2},\frac{3-2s}{2},-A^{-2}\Big) \\ &- a^{s+2}\,{}_2F_1\Big(1-\frac{s}{2},-1-\frac{s}{2},\frac{3-2s}{2},-a^{-2}\Big)\Big) \\ &+ \frac{\gamma_2(s)}{\pi H}\Big(A^{3-s}\,{}_2F_1\Big(\frac{s+1}{2},\frac{s-3}{2},\frac{2s+1}{2};-A^{-2}\Big) \\ &- a^{3-s}\,{}_2F_1\Big(\frac{s+1}{2},\frac{s-3}{2},\frac{2s+1}{2};-a^{-2}\Big)\Big) \\ &+ \frac{\gamma_3(s)}{\pi H}\Big(A\,{}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-A^{-2}\Big) \\ &- a\,{}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-a^{-2}\Big)\Big). \end{aligned}$$

We know from part $i)$ that the terms containing $X^{1/2+it}$ come from the terms containing the function $G_s(x)$, where

$$G_s(x) = x^{(s+2)/2}\,{}_2F_1\Big(1-\frac{s}{2},-1-\frac{s}{2},\frac{3-2s}{2},-x^{-1}\Big).$$

Using $G_s(x)$ we can write

$$\begin{aligned} d_t(f^+) =& \frac{\gamma_1(s)}{\pi}\frac{2}{s+2}\frac{G_s(A^2)-G_s(a^2)}{H} \\ &+ \frac{\gamma_2(s)}{\pi}\frac{G_{1-s}(A^2)-G_{1-s}(a^2)}{H} \\ &+ \frac{\gamma_3(s)}{\pi H}\Big(A\,{}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-A^{-2}\Big) \\ &- a\,{}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-a^{-2}\Big)\Big). \end{aligned}$$

We use formula (A.3) for the two hypergeometric functions appearing in $G_s(A^2)$ and $G_s(a^2)$ for parameters $\alpha = 1$, $\beta = 1/2$, $\gamma = 3$, $\lambda = -s/2$, $z = -A^{-2}$ or $z = -a^{-2}$ and $e^{\pm\phi} = \big(2 - z \pm 2(1-z)^{1/2}\big)/z$. We also use Stirling's approximation for the Gamma functions appearing to get the bound

$$\frac{2\gamma_1(s)}{\pi}\frac{2}{s+2}\frac{G_s(A^2)-G_s(a^2)}{H} \ll |t|^{-3/2}\frac{A^{5/2}+a^{5/2}}{H} \ll |t|^{-3/2}X^{3/2}Y^{-1}$$

By the Mean Value Theorem, we find $\xi \in [a^2, A^2]$ such that

$$\frac{G_s(A^2)-G_s(a^2)}{H} = G_s'(\xi)\,.$$

But we have

$$\begin{aligned} G_s'(\xi) = &\frac{s+2}{2}\xi^{s/2}\,{}_2F_1\Big(1-\frac{s}{2},-1-\frac{s}{2},\frac{3-2s}{2},-\xi^{-1}\Big) \\ &+\xi^{s/2}\frac{(1-s/2)(-1-s)/2}{(3-2s)/2}\,{}_2F_1\Big(2-\frac{s}{2},-\frac{s}{2},\frac{5-2s}{2},-\xi^{-1}\Big)\,, \end{aligned}$$

using (A.2) for the derivative of ${}_2F_1$. We again apply Stirling's approximation for the gamma functions $\gamma_1$ and formula (A.3) for the two hypergeometric functions in $G_s'(\xi)$ with parameters $\alpha = 1$, $\beta = 1/2$, $\gamma = 3$, $\lambda = -s/2$, $z = -\xi^{-1}$ for first one and $\alpha = 2$, $\beta = 1/2$, $\gamma = 3$, $\lambda = -s/2$, $z = -\xi^{-1}$ for the second one. The resulting bound is given by

$$\frac{\gamma_1(s)}{\pi}\frac{2}{s+2}\frac{G_s(A^2)-G_s(a^2)}{H} \ll |t|^{-1/2}X^{1/2}\,.$$

Hence, if we set

$$a(t,X,Y) = \frac{\gamma_1(s)}{\pi}\frac{2}{s+2}\frac{G_s(A^2)-G_s(a^2)}{H}X^{-(1/2+it)}$$

and use the Stirling asymptotic for the Gamma functions appearing in the expression we get

$$a(t,X,Y) = O\Big(\min\big\{|t|^{-1/2},|t|^{-3/2}X/Y\big\}\Big)\,.$$

We do the same for $G_{1-s}(x)$ as above and the coefficient $b(t,X,Y)$ defined as

$$\begin{aligned} b(t,X,Y) = \Bigg(&\frac{\gamma_2(s)}{\pi}\frac{G_{1-s}(A^2)-G_{1-s}(a^2)}{H} \\ &+ \frac{\gamma_3(s)}{\pi H}\Bigg(A\,{}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-A^{-2}\Big) \\ &- a\,{}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-a^{-2}\Big)\Bigg)\Bigg)X^{-(1/2-it)}\,. \end{aligned}$$

Note that here we also bound the summands involving the functions ${}_3F_2$ by using the hypergeometric series expansion (notice that the factors involving $s$ appear on the denominator) and Stirling's approximation for $\gamma_3$, see (3.19). For $z=A$ or $a$, we have that

$$\frac{\gamma_3(s)}{\pi H} z\,{}_3F_2\Big(1,\frac{3}{2},-\frac{1}{2};\frac{s+3}{2},\frac{4-s}{2};-z^{-2}\Big) \ll |t|^{-2}Y^{-1}\,,$$

hence those terms are negligible and will not affect the overall bound that we get from the terms involving $G_{1-s}(x)$. We conclude the same bound for $b(t,X,Y)$ that we got for $a(t,X,Y)$ and the result follows.

iii) It follows from i). If $s=1$, i.e. $t=-i/2$, from [12, Eq.8.711.1] we have that $P_0(\pm ix)=1$. We see that $d_{-i/2}(f)=(2/\pi)X+O(Y)$. If $s\neq 1$ we estimate the Gamma factors in i) using Stirling's approximation and keep the factor $\Gamma\big(\frac{1}{2}-s\big)$.

iv) We first need a bound on the sum $P_{-1/2}(ix)+P_{-1/2}(-ix)$. The transformation formula [24, Eq. 16.8.8] for the hypergeometric function used above, is not valid for $s=1/2$, hence we are going to use the integral representation [12, 8.713.3]. Firstly, assume that $x < 1$. Then we have

$$\begin{aligned} P_{-1/2}(ix) + P_{-1/2}(-ix) &\ll \int_0^\infty (\cosh^2 t + x^2)^{-1/4} dt \\ &\ll \int_0^\infty (\cosh t)^{-1/2} dt \\ &\ll 1\,. \end{aligned}$$

Secondly, assume that $x \geq 1$, then we have

$$\begin{aligned} P_{-1/2}(ix) + P_{-1/2}(-ix) &\ll \int_0^\infty (\cosh^2 t + x^2)^{-1/4} dt \\ &\ll x^{-1/2} \int_0^\infty \left( \left( \frac{\cosh t}{x} \right)^2 + 1 \right)^{-1/4} dt\,. \end{aligned}$$

Now we set $u = \cosh t / x$ and compute:

$$\begin{aligned} \int_0^\infty \left( \left( \frac{\cosh t}{x} \right)^2 + 1 \right)^{-1/4} dt &= \int_{1/x}^\infty (u^2+1)^{-1/4} \frac{x}{\sqrt{x^2u^2-1}} du \\ &= \int_{1/x}^2 (u^2+1)^{-1/4} \frac{x}{\sqrt{x^2u^2-1}} du \\ &\quad + \int_2^\infty (u^2+1)^{-1/4} \frac{x}{\sqrt{x^2u^2-1}} du\,. \end{aligned}$$

For $u \geq 2$ we notice that

$$\frac{x}{\sqrt{x^2-u^2-1}} \ll \frac{1}{u}\,,$$

hence we can bound the second integral as follows:

$$\int_2^\infty (u^2+1)^{-1/4} \frac{x}{\sqrt{x^2u^2-1}} du \ll \int_1^\infty (u^2+1)^{-1/4} \frac{1}{u} \ll 1\,.$$

As for the first integral we compute

$$\int_{1/x}^2 (u^2+1)^{-1/4} \frac{x}{\sqrt{x^2u^2-1}} du \ll \int_{1/x}^2 \frac{x}{\sqrt{x^2u^2-1}} du\,.$$

Let $xu = \cosh r$, then we have

$$\int_{1/x}^{2} \frac{x}{\sqrt{x^2u^2-1}}du = \int_0^{\cosh^{-1}(2x)} dr = \cosh^{-1}(2x)\,.$$

We conclude that

$$\int_{1/x}^{2} (u^2+1)^{-1/4}\frac{x}{\sqrt{x^2u^2-1}}du \ll \log x\,.$$

Now, we have shown in Lemma 3.4 that $d_t(f^+) \ll \int_0^{\sqrt{\xi}} \frac{x}{\sqrt{\xi-x^2}}\big|P_{s-1}(ix) + P_{s-1}(-ix)\big|dx$ for every $t$ and for some $\xi \in [a^2, A^2]$, hence we have

$$\begin{aligned}
d_0(f^+) &\ll \int_0^1 \frac{x}{\sqrt{\xi-x^2}}\big|P_{-1/2}(ix)+P_{-1/2}(-ix)\big|dx \\
&\quad + \int_1^{\sqrt{\xi}} \frac{x}{\sqrt{\xi-x^2}}\big|P_{-1/2}(ix)+P_{-1/2}(-ix)\big|dx \\
&\ll \int_0^1 \frac{x}{\sqrt{\xi-x^2}}dx + \int_1^{\sqrt{\xi}} \sqrt{\frac{x}{\xi-x^2}}\log x dx \\
&\ll 1 + \int_1^{\sqrt{\xi}} \frac{\xi^{1/4}}{\sqrt{\xi}\sqrt{1-\frac{x^2}{\xi}}}\log\sqrt{\xi}dx \\
&\ll 1 + \xi^{-1/4}\log\xi \int_1^{\sqrt{\xi}} \frac{1}{\sqrt{1-\frac{x^2}{\xi}}}dx \\
&\ll \xi^{1/4}\log\xi \ll X^{1/2}\log X\,.
\end{aligned}$$

□

### 3.2.3 Proof of Theorem 1.2

We will now use our results about the Huber transform $d_t(f^+)$ to analyse the spectral contribution to the relative trace formula, Proposition 3.2, and finally prove Theorem 1.2.

*Proof of Theorem 1.2.* We have by Proposition 3.5:

$$\sum_j 2d_t(f^+)\widehat{u_j}^2 = \sum_{1/2<s_j\le 1} \frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2 X^{s_j} + \frac{1}{\pi}(3-s_j)\gamma_2(s_j)\widehat{u_j}^2 X^{1-s_j}$$
$$+ O\Big(\sum_{1/2<s_j\le 1} \Big(\widehat{u_j}^2 Y + \Big|\Gamma\Big(\frac{1}{2}-s_j\Big)\Big|\widehat{u_j}^2 X^{1/2}\Big)\Big)$$
$$+ \sum_{0\neq t_j\in\mathbb{R}} 2d_t(f^+)\widehat{u_j}^2 + O\big(X^{1/2}\log X\big),$$

where the last term is due to the estimate for $d_0(f^+)$ (see Proposition 3.5(iv)). Since the spectrum is discrete, for $s_j$ corresponding to a small eigenvalue, $s_j - \frac{1}{2}$ is bounded away from zero. As the number of small eigenvalues is finite, we get

$$\sum_{1/2<s_j\le 1} \Big(\widehat{u_j}^2 Y + \Big|\Gamma\Big(\frac{1}{2}-s_j\Big)\Big|\widehat{u_j}^2 X^{1/2}\Big) = O\big(Y + X^{1/2}\big).$$

For the same reason,

$$\sum_{1/2<s_j\le 1} \frac{1}{\pi}\gamma_2(s_j)(3-s_j)\widehat{u_j}^2 X^{1-s_j} = O\big(X^{1/2}\big).$$

Let

$$S(f^+) = \sum_{0\neq t_j\in\mathbb{R}} 2d_t(f^+)\widehat{u_j}^2,$$

then

$$\sum_j 2d_t(f^+)\widehat{u_j}^2 = \sum_{1/2<s_j\le 1} \frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2 X^{s_j} + S(f^+) + O\big(Y + X^{1/2}\log X\big). \tag{3.22}$$

Using Proposition 3.5 and the discreteness of the spectrum, we get

$$S(f^+) = \sum_{|t_j|\ge 1} 2d_t(f^+)\widehat{u_j}^2 + \sum_{|t_j|<1} 2d_t(f^+)\widehat{u_j}^2$$
$$= \sum_{|t_j|\ge 1} 2d_t(f^+)\widehat{u_j}^2 + O\big(X^{1/2}\big).$$

Since $d_t(f^+)$ is an even function of $t$, see e.g. (4.1), after using dyadic decomposition we get the bound

$$\begin{aligned}\sum_{|t_j|\geq 1} 2d_t(f^+)\widehat{u_j}^2 &\ll \sum_{t_j\geq 1} 2\left|d_t(f^+)\right|\widehat{u_j}^2 \\ &= \sum_{n=0}^{\infty}\left(\sum_{2^n\leq t_j<2^{n+1}} 2\left|d_t(f^+)\right|\widehat{u_j}^2\right) \\ &\ll \sum_{n=0}^{\infty}\sup_{2^n\leq t_j<2^{n+1}}\left|d_{t_j}(f^+)\right|\left(\sum_{2^n\leq t_j<2^{n+1}}\widehat{u_j}^2\right).\end{aligned} \tag{3.23}$$

From Proposition 3.5 and Lemma 2.10, we compute

$$S(f^+) \ll X^{1/2}\sum_{n=0}^{\infty} 2^{-n/2}\min\left\{2^n, XY^{-1}\right\} + X^{1/2}.$$

We split the sum according to $n < \log_2(X/Y)$ and $n > \log_2(X/Y)$. We get

$$\begin{aligned}S(f^+) &\ll X^{1/2}\sum_{n<\log_2(X/Y)} 2^{-n/2}\min\left\{2^n, XY^{-1}\right\} \\ &\quad + X^{1/2}\sum_{n\geq\log_2(X/Y)} 2^{-n/2}\min\left\{2^n, XY^{-1}\right\} + X^{1/2} \\ &\ll X^{\frac{1}{2}}\sum_{n<\log_2(X/Y)} 2^{n/2} + X^{3/2}Y^{-1}\sum_{n\geq\log_2(X/Y)} 2^{-n/2} + X^{1/2} \\ &\ll XY^{-1/2} + X^{1/2}.\end{aligned} \tag{3.24}$$

With this result for the spectral side and the analysis for the geometric side of our trace formula in Proposition 3.2, we can finally show Theorem 1.2. From the above we have that

$$\begin{aligned}\sum_{\substack{\gamma\in H_1\backslash\Gamma-H_1/H_1\\ B(\gamma)<1}}\widetilde{g}^+\left(B(\gamma)^2\right) + \sum_{\substack{\gamma\in H_1\backslash\Gamma-H_1/H_1\\ B(\gamma)>1}} g^+\left(B(\gamma)^2\right) &= \sum_{1/2<s_j\leq 1}\frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2X^{s_j} \\ &\quad + O\left(XY^{-1/2} + Y + X^{1/2}\log X\right).\end{aligned}$$

But the sum for the exceptional terms $\gamma$, i.e. $B(\gamma) < 1$, is $O(1)$ because there are finitely many such terms and by equation (3.12) we know that $\widetilde{g}^{+}\big(B(\gamma)^2\big) = O(1)$ for $B(\gamma) < 1$. Hence we have that

$$\sum_{\substack{\gamma\in H_1\backslash\Gamma-H_1/H_1 \\ B(\gamma)>1}} g^{+}\big(B(\gamma)^2\big) = \sum_{1/2<s_j\le 1} \frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2 X^{s_j} + O\Big(XY^{-1/2} + Y + X^{1/2}\log X\Big).$$

We notice that the choice $XY^{-\frac{1}{2}} = Y \iff Y = X^{2/3}$ balances the two error terms and makes them equal to $O\big(X^{2/3}\big)$. It follows that

$$\sum_{\substack{\gamma\in H_1\backslash\Gamma-H_1/H_1 \\ B(\gamma)>1}} g^{+}\big(B(\gamma)^2\big) = \sum_{1/2<s_j\le 1} \frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2 X^{s_j} + O\big(X^{2/3}\big).$$

We work similarly for the sums of $g^{-}$, $\widetilde{g}^{-}$ and use equation (3.14) to get that

$$N(X,l) = \sum_{\substack{\gamma\in H_1\backslash\Gamma/H_1 \\ B(\gamma)\le 1}} 1 + \sum_{1/2<s_j\le 1} \frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2 X^{s_j} + O\big(X^{2/3}\big).$$

Since the first sum is of order $O(1)$ we conclude that

$$N(X,l) = \sum_{1/2<s_j\le 1} \frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2 X^{s_j} + O\big(X^{2/3}\big).$$

□

**Remark 3.6.** For $s = 1$, using the fact that

$$\widehat{u_0} = \frac{\mu}{\sqrt{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\,\nu}$$

the contribution to the main term of $N(X,l)$ is

$$2\frac{\gamma_1(1)}{\pi}\widehat{u_0}^2 X = \frac{2}{\pi}\frac{\mu^2}{\mathrm{vol}(\Gamma\backslash\mathbb{H})\nu^2}X.$$

**Remark 3.7.** Let $l_1, l_2$ be representatives of two closed geodesics and $H_1$ a hyperbolic subgroup of $\Gamma$ that corresponds to $l_1$ and $H_2$ a hyperbolic subgroup that corresponds to $l_2$. We assume that $l_1$ and $\tau^{-1} \cdot l_2$ lie on the imaginary axis $I$, for some $\tau \in \mathrm{PSL}_2(\mathbb{R})$. Then we consider the series

$$\widetilde{K}(z) := \sum_{\gamma \in H_2 \backslash \Gamma} k(v(\tau^{-1}\gamma z)) .$$

Similarly to the proof of Proposition 3.2 we have

$$\begin{aligned}\int_{l_1} \widetilde{K}(z) ds &= \sum_{\gamma \in H_2 \backslash \Gamma} \sum_{\gamma_0 \in H_1} \int_{l_1} k(v(\tau^{-1}\gamma\gamma_0 z)) ds \\ &= \sum_{\gamma \in H_2 \backslash \Gamma / H_1} \int_I k(v(\tau^{-1}\gamma z)) ds .\end{aligned}$$

The geometric side of Proposition 3.2 is the same for $B(\tau^{-1}\gamma)$ in place of $B(\gamma)$. The difference in the spectral side case comes from the periods $\widehat{u_j}$. The sum appearing there becomes

$$\sum_{t_j} 2 d_{t_j}(f) \int_{l_1} u_j(z) ds \int_{l_2} u_j(z) ds .$$

For the sum $\sum_{t_j \le T} \int_{l_1} u_j(z) ds \int_{l_2} u_j(z) ds$, we use Lemma 2.10 and the Cauchy–Schwarz inequality to obtain the same bound as the one in the proof of the main theorem. We have

$$\left| \sum_{0 \le t_j \le T} \int_{l_1} u_j(z) ds \int_{l_2} u_j(z) ds \right| \le \left( \sum_{0 \le t_j \le T} \left| \int_{l_1} u_j(z) ds \right|^2 \sum_{0 \le t_j \le T} \left| \int_{l_2} u_j(z) ds \right|^2 \right)^{1/2} \ll T ,$$

which can be used to show equations (3.23) and (3.24) in this case and the same error term as in Theorem 1.2 follows for $N(X, l_1, l_2)$.

# Chapter 4

# Average results for the error term

## 4.1 The large sieve inequality

### 4.1.1 The Jacobi transform

In this chapter we prove Theorem 1.5, namely that on average the error term $E(X,l)$ is $O\big(X^{1/2}\log^{3/2}X\big)$. In order to do so, we relate the integral transform $d_t(f)$, which we call Huber's transform, to the Jacobi transform studied by Flensted-Jensen [10] and Koornwinder [20],[21]. Then, we develop a large sieve inequality for periods of Maass forms (see Theorem 1.3). To prove the inequality, we choose the integral transform to be a Gaussian and study its inverse transform. We use it in the relative trace formula (Proposition 3.2) and follow Chamizo's proof of a similar sieve inequality for values of Maass forms (see [3, Th. 2.2, p. 306]). For the proof of Theorem 1.5, we follow the ideas of Chatzakos and Petridis, where we use our new sieve inequality (Theorem 1.3) in place of Chamizo's inequality [3, Th. 2.2, p. 306].

From the definition of the Huber transform (2.9) and working similarly to the proof of Lemma 3.3 (see (3.17)) we can see that given a test function $f$, its Huber transform can be written as:

$$d_t(f) = \int_0^{\infty} f(x^2+1)\,{}_2F_1\Big(\frac{1-s}{2},\frac{s}{2},\frac{1}{2},-x^2\Big)dx\,, \tag{4.1}$$

where $s = \frac{1}{2} + it$, as usual $t \in \mathbb{R}$ or $1/2 \le s \le 1$. Now, let us make the change of variables $x = \sinh w \implies dx = \cosh w dw$. Hence, we can rewrite the Huber transform as

$$d_t(f) = \int_0^\infty f(\sinh^2 w + 1)\, {}_2F_1\Big(\frac{1-s}{2}, \frac{s}{2}, \frac{1}{2}, -\sinh^2 w\Big) \cosh w dw\,.$$

We now introduce the Jacobi functions. For the discussion and formulas below, see [20, Ch. 2]. Consider for $\alpha, \beta, \mu \in \mathbb{C}$ and $w > 0$ the differential equation

$$\Big(\Delta_{(\alpha,\beta)}(w)\Big)^{-1} \frac{d}{dw}\Big(\Delta_{(\alpha,\beta)}(w) \frac{dU(w)}{dw}\Big) = -(\mu^2 + \rho^2) U(w)\,,$$

where $\rho = \alpha + \beta + 1$ and

$$\Delta_{(\alpha,\beta)}(w) := \Delta(w) := (e^t - e^{-t})^{2\alpha+1}(e^t + e^{-t})^{2\beta+1}\,.$$

Then the function

$$\phi_\mu(w) := \phi_\mu^{(\alpha,\beta)}(w) := {}_2F_1\Big(\frac{1}{2}(\rho + i\mu), \frac{1}{2}(\rho - i\mu), \alpha + 1; -\sinh^2 w\Big)\,,$$

is a solution to the differential equation above with $\phi(0) = 1$ and $\phi'(0) = 0$. This is called the Jacobi function of the first kind. A second linearly independent solution (Jacobi function of the second kind) to the same differential equation for $\mu \ne -i, -2i, -3i, \ldots$, is the function:

$$\Phi_\mu(w) = (e^t - e^{-t})^{i\mu-\rho}\, {}_2F_1\Big(\frac{1}{2}(\beta - \alpha + 1 - i\mu), \frac{1}{2}(\rho - i\mu), 1 - i\mu, -\frac{1}{\sinh^2 w}\Big)\,.$$

The two solutions $\phi$ and $\Phi$ are related via the formula

$$\sqrt{\pi}(\Gamma(\alpha+1))^{-1}\phi_\mu(w) = \frac{1}{2}c(\mu)\Phi_\mu(w) + \frac{1}{2}c(-\mu)\Phi_{-\mu}(w)\,,$$

for

$$c(\mu) := c_{(\alpha,\beta)}(\mu) = \frac{2^{\rho}\Gamma\left(\frac{1}{2}i\mu\right)\Gamma\left(\frac{1}{2}(1+i\mu)\right)}{\Gamma(1/2(\rho+i\mu))\Gamma(1/2(\alpha-\beta+1+i\mu))}\,.$$

The Jacobi transform of a function $f$ is defined as

$$\hat{f}(\mu) := \hat{f}_{(\alpha,\beta)}(\mu) := \left(\sqrt{2}/\Gamma(a+1)\right)\int_0^{\infty} f(w)\phi_\mu(w)\Delta(w)dw\,.$$

We notice that the Huber transform is related to the Jacobi tranform as follows

$$d_t(f) = \left(\sqrt{\pi}/2^{3/2}\right)\hat{h}(\mu)\,,$$

for $\alpha = -\frac{1}{2}, \beta = 0, \rho = \frac{1}{2}, \mu = t$ and $h(w) := f(\sinh^2 w + 1)$. The Jacobi transform has been studied extensively in [10],[20] and [21], where many properties for $\phi_\mu(w)$ and $c(\mu)$ have been shown. For $\Re\mu > 0, s' \geq 0, \sigma > 0$, following [20, eq. 3.10,3.11] and [9, Chapter XIII] we define the Weyl fractional integral transform as $\left(W_\mu^\sigma(f)\right)(s')$ of a function $f$ as

$$\left(W_\mu^\sigma(f)\right)(s') = \frac{1}{\Gamma(\mu)}\int_{s'}^{\infty} f(v)(\cosh(\sigma v) - \cosh(\sigma s'))^{\mu-1} d(\cosh\sigma v)\,,$$

where $d(\cosh v) = \sinh v dv$ and $f$ is taken such that the integral converges (e.g. $f$ is rapidly decreasing). By analytic continuation $W_\mu^\sigma$ extends to an entire function of $\mu$ by

$$\left(W_\mu^\sigma(f)\right)(s') = \frac{(-1)^n}{\Gamma(\mu+n)}\int_{s'}^{\infty}\left(\frac{d^n}{d(\cosh\sigma v)^n} f(v)\right)(\cosh(\sigma v) - \cosh(\sigma s'))^{\mu+n-1} d(\cosh\sigma v)\,,$$

where

$$n = 0, 1, 2, \ldots,\ \Re\mu > -n\,.$$

Here, by $d^n f(v)/d(\cosh\sigma v)^n$ we mean the application of chain rule to the function $r(x) = f(v)$, for $x = \cosh \sigma v$, hence

$$\frac{d^n}{dx^n} r(x) = \frac{d^n f(v)}{d(\cosh \sigma v)^n} .$$

It can easily be seen (see [19, page 4],[20, page 153]) that

$$W^{\sigma}_{\mu+\nu} = W^{\sigma}_{\mu} \circ W^{\sigma}_{\nu} \tag{4.2}$$

and in particular

$$W^{\sigma}_{\mu} \circ W^{\sigma}_{-\mu} = W^{\sigma}_{-\mu} \circ W^{\sigma}_{\mu} = id ,$$

which shows that the Weyl transform can be inverted. One very important property of the Jacobi transform, proved in [20, eq. 3.7, Cor. 3.3] (see also [19, eq. 2.10]) is that

$$\hat{f}(\mu) = \mathscr{F}_c \circ F_{\alpha,\beta}(f)(\mu) , \tag{4.3}$$

where

$$F_{\alpha,\beta}(f)(s) = 2^{3\alpha+\frac{3}{2}} W^1_{\alpha-\beta} \circ W^2_{\beta+\frac{1}{2}}(f)(s) \tag{4.4}$$

and

$$\mathscr{F}_c(f)(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(st) dt$$

denotes the Fourier cosine transform (see also [12, p. 1121]). This was proved in [20] for functions $f$ of compact support, but the proof remains the same for $f$ such that the integral defining the Jacobi transform converges. We can use equations (4.3) and the properties of the Weyl fractional integrals and Fourier cosine transform to invert the Jacobi transform. We have

$$f(s) = 2^{-3\alpha-\frac{3}{2}} W^2_{-\beta-\frac{1}{2}} \circ W^1_{\beta-\alpha} \circ \mathscr{F}_c^{-1}(\hat{f})(s) , \tag{4.5}$$

where $\mathscr{F}_c^{-1}$ is the inverse cosine transform of a function. For simplicity, from now on we assume that $f$ is such that $f(\sinh^2 s+1) \in \cosh^{-1} s \cdot \mathscr{S}$, where $\mathscr{S}$ is the space of even rapidly decreasing functions on $\mathbb{R}$ (see [19, p. 3]). From [12, p. 1092, eq. 4.(iv)] even functions satisfy $\mathscr{F}_c(f)(s) = \mathscr{F}(f)(s)$, where $\mathscr{F}(f)(s)$ is the classical Fourier Transform:

$$\mathscr{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt .$$

Hence, in equation (4.3) we can substitute the Fourier cosine transform $\mathscr{F}_c$ with the classical Fourier transform $\mathscr{F}$. We also define the convolution of two functions $f, g$ (see [12, p. 1088]) as

$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau .$$

The convolution satisfies the properties $f * g = g * f$ and $f * (g * h) = (f * g) * h$. For the Fourier transform of a convolution of two functions the following property, also known as convolution theorem, holds

$$\mathscr{F}(f * g)(s) = \mathscr{F}(f)(s) \mathscr{F}(g)(s) .$$

### 4.1.2 Proof of the large sieve inequality

In order to prove our large sieve inequality (Theorem 1.3) we need to apply the relative trace formula (3.2) for a suitable choice of test function $f$. Here we choose $d_t(f)$ and estimate $f$.

**Lemma 4.1.** *Let $T, C_0, c, u, r$ be positive with $T \geq 1$ and $r \ll 1$. Suppose that $u \geq u_0$ for some fixed $u_0 > 0$. Let $d_t(f) = e^{-t^2/4T^2} \cos(rt)$, where $f$ is defined through* (4.5), *then the following inequalities hold:*

*a) For $f$ we have that:*

$$f(1) \ll \min\{T, r^{-1}\} \quad \text{and} \quad f(\sinh^2 s + 1) \ll T e^{-cT^2(s-r)^2} \quad \textit{for} \quad s \geq 2r .$$

*b) Moreover, we can show for* $\sinh^{-1} u \geq 2r$

$$\int_u^\infty \frac{f(x^2+1)}{\sqrt{x^2-u^2}}dx = \int_{\sinh^{-1} u}^\infty \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s - u^2}}ds \ll Te^{-CT^2(\sinh^{-1} u - r)^2}. \tag{4.6}$$

*c) For* $\sinh^{-1} u < 2r$ *we have*

$$\int_{\sinh^{-1} u}^\infty \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s - u^2}}ds \ll Te^{-CT^2 r^2}.$$

*d) Also we have*

$$\int_0^\infty \frac{f(x^2+1)}{\sqrt{x^2+u^2}}dx \ll Te^{-CT^2 r^2}.$$

*Proof.* a) Using that the Huber transform is a Jacobi transform with $\alpha = -1/2$ and $\beta = 0$, we can use that $d_t(f) = e^{-t^2/4T^2}\cos(rt)$ to recover $f$. By linearity of the transforms $\mathscr{F}$ and $W_\mu^\sigma$ we write the inversion formula (4.5) as

$$f(\sinh^2 s+1) = \frac{2^{3/2}}{\sqrt{\pi}} W^2_{-\frac{1}{2}} \circ W^1_{\frac{1}{2}} \circ \mathscr{F}^{-1}(d_t(f))(s). \tag{4.7}$$

We use the convolution theorem for ordinary Fourier transform to compute the inverse Fourier transform of $d_t(f)$. Informally, let $h_1(x) = \sqrt{2}Te^{-x^2T^2}$ and $h_2(x) = \left(\sqrt{\pi}/\sqrt{2}\right)\left(\delta(x-r)+\delta(x+r)\right)$, where $\delta$ is the delta function. Then from [12, Page 1119, eq.4] and [12, Page 1119, eq.13] we find that $\mathscr{F}(h_1)(t) = e^{-t^2/4T^2}$ and $\mathscr{F}(h_2)(t) = \cos(rt)$. Since

$$\begin{aligned}\mathscr{F}^{-1}\big(d_t(f)\big)(s) &= \mathscr{F}^{-1}\big(e^{-s^2/4T^2}\cos(rs)\big) = \mathscr{F}^{-1}\big(\mathscr{F}(h_1)\mathscr{F}(h_2)\big)(s)\\ &= \mathscr{F}^{-1}\big(\mathscr{F}(h_1 * h_2)\big)(s) = (h_1 * h_2)(s).\end{aligned}$$

We compute the convolution:

$$(h_1 * h_2)(s) = \sqrt{\pi} T \int_{-\infty}^{\infty} e^{-\tau^2 T^2} \big(\delta(s-r-\tau) + \delta(s+r-\tau)\big) d\tau$$
$$= \sqrt{\pi} T \big(e^{-T^2(s-r)^2} + e^{-T^2(s+r)^2}\big) .$$

We now proceed with the Weyl fractional integrals (equations (4.3) and (4.4)). From the properties of the Weyl fractional integral [19, Eq. 2.8] and the proof of [19, Lemma 3.4] we can see that

$$W^2_{-1} = \frac{1}{2\cosh(\cdot)} W^1_{-1} .$$

By equation (4.2) we can rewrite the Weyl fractional integrals as

$$W^2_{-\frac{1}{2}} \circ W^1_{\frac{1}{2}} = W^2_{\frac{1}{2}} \circ W^2_{-1} \circ W^1_{\frac{1}{2}} = W^2_{\frac{1}{2}} \circ \Big(\frac{1}{2\cosh(\cdot)} W^1_{-\frac{1}{2}}\Big) .$$

Let $g(s) := \sqrt{2} T \big(e^{-T^2(s-r)^2} + e^{-T^2(s+r)^2}\big)$. First we calculate $W^1_{-\frac{1}{2}}(g)(s)$:

$$W^1_{-\frac{1}{2}}(g)(s) = \frac{1}{\sqrt{2}} \int_s^{\infty} \frac{dg(v)}{d\cosh v} \frac{1}{\sqrt{\cosh v - \cosh s}} d\cosh v$$
$$= 2T^3 \int_s^{\infty} \frac{(v-r)e^{-T^2(v-r)^2} + (v+r)e^{-T^2(v+r)^2}}{\sqrt{\cosh v - \cosh s}} dv .$$

We turn to the study of $f(1)$, i.e. $s = 0$. Using (4.5) we compute

$$f(1) = \frac{2^{3/2}}{\sqrt{\pi}} \int_0^{\infty} \frac{W^1_{-\frac{1}{2}}(g)(v)}{\cosh v} (\cosh 2v - 1)^{-1/2} d\cosh 2v = \frac{2^{7/2}}{\sqrt{\pi}} \int_0^{\infty} W^1_{-\frac{1}{2}}(g)(v) dv .$$

Applying the Mean Value Theorem for the denominator, we see that

$$\begin{aligned} f(1) &\ll T^3 \int_0^\infty \int_v^\infty \frac{|w-r|e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}}{\sqrt{\cosh w - \cosh v}} dw dv \\ &\ll T^3 \int_0^\infty \int_v^\infty \frac{(\sinh v)^{-1/2}}{\sqrt{w-v}} \Big(|w-r|e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}\Big) dw dv\,. \end{aligned}$$

Rearranging the order of integration in the double integral, we see that

$$\begin{aligned} f(1) &\ll T^3 \int_0^\infty \int_v^\infty \frac{(\sinh v)^{-1/2}}{\sqrt{w-v}} \Big(|w-r|e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}\Big) dw dv \\ &= T^3 \int_0^\infty \left( \int_0^w \frac{(\sinh v)^{-1/2}}{\sqrt{w-v}} dv \right) \Big(|w-r|e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}\Big) dw\,. \end{aligned}$$

We have

$$\int_0^w \frac{(\sinh v)^{-1/2}}{\sqrt{w-v}} dv \ll \int_0^w \frac{1}{\sqrt{v}} \frac{1}{\sqrt{w-v}} dv = \pi\,.$$

Hence, we compute

$$\begin{aligned} f(1) &\ll \int_0^\infty W^2_{-\frac{1}{2}}(g)(v) dv \ll T^3 \int_0^\infty \Big(|w-r|e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}\Big) dw \\ &\ll T^3 \frac{e^{-r^2T^2}}{2T^2} \ll Te^{-r^2T^2}\,. \end{aligned}$$

If $rT < 1 \iff T < r^{-1}$ then $Te^{-r^2T^2} \le T$, while if $rT \ge 1$ we have $rTe^{-r^2T^2} \le 1 \implies Te^{-r^2T^2} \ll r^{-1}$. We conclude that

$$f(1) \ll \min\left\{T, r^{-1}\right\}.$$

Assume now that $s \ge 2r$. Up to a constant the integral $W^1_{-\frac{1}{2}}(g)(s)$ appeared in the proof of [3, Lemma 3.1(b), p.312]. He proved that for $s \ge 2r$ and some

$C > 0$

$$W^1_{-\frac{1}{2}}(g)(s) \ll T^2 e^{-CT^2(s-r)^2}.$$

Since $W^2_{\frac{1}{2}}$ is a positive operator, i.e. $g \geq 0$ implies $W^2_{\frac{1}{2}}(g) \geq 0$, we can bound the next Weyl fractional integral in (4.7) as follows

$$\begin{aligned} W^2_{\frac{1}{2}} \circ \left(\frac{1}{2\cosh(\cdot)} W^1_{-\frac{1}{2}}\right)(g)(s) &\ll W^2_{\frac{1}{2}}\left(\frac{T^2 e^{-CT^2(s-r)^2}}{\cosh s}\right) \\ &\ll T^2 \int_s^\infty \frac{e^{-CT^2(v-r)^2}}{\cosh v} (\cosh 2v - \cosh 2s)^{-1/2} \sinh 2v\, dv \\ &\ll T^2 \int_s^\infty \frac{e^{-CT^2(v-r)^2}}{\sqrt{\cosh 2v - \cosh 2s}} \sinh v\, dv. \end{aligned} \tag{4.8}$$

Suppose now that $rT \geq 1$. Let $C_1 = C/2$, then from (4.8) we see that

$$f(\sinh^2 s + 1) \ll T^2 e^{-C_1 T^2(s-r)^2} \int_s^\infty \frac{e^{-C_1 T^2(v-r)^2}}{\sqrt{\cosh 2v - \cosh 2s}} \sinh v\, dv.$$

By the Mean Value Theorem on the interval $[2s, 2v]$ we get the bound

$$f(\sinh^2 s + 1) \ll T^2 \frac{e^{-C_1 T^2(s-r)^2}}{\sqrt{\sinh 2s}} \int_s^\infty \frac{e^{-C_1 T^2(v-r)^2}}{\sqrt{v-s}} \sinh v\, dv.$$

We apply integration by parts to our integral:

$$\begin{aligned} \int_s^\infty \frac{e^{-C_1 T^2(v-r)^2}}{\sqrt{v-s}} \sinh v\, dv &\ll \int_s^\infty e^{-C_1 T^2(v-r)^2} \sqrt{v-s} \cosh v\, dv \\ &\quad + T^2 \int_s^\infty e^{-C_1 T^2(v-r)^2} (v-r)\sqrt{v-s} \sinh v\, dv. \end{aligned}$$

Let

$$K_1 = \int_s^\infty e^{-C_1 T^2(v-r)^2} \sqrt{v-s} \cosh v\, dv$$

and

$$K_2 = T^2 \int_s^\infty e^{-C_1 T^2(v-r)^2} (v-r)\sqrt{v-s} \sinh v\, dv.$$

To bound $K_2$, as in the proof of [19, Lemma 3.2,page 5] we use the simple bound $\sinh v \ll ve^v/(1+v)$ to conclude for $s \geq 2r$:

$$K_2 \ll T^2 \int_s^\infty e^{-C_1 T^2 (v-r)^2} (v-r)^{3/2} \sinh v dv \\ \ll T^2 \int_s^\infty e^{-C_1 T^2 (v-r)^2 + v} (v-r)^{3/2} v dv .$$

With the substitution $w = v - r$, then

$$K_2 \ll T^2 \int_{s-r}^\infty e^{-C_1 T^2 w^2 + w} w^{3/2} (w+r) dw \\ \ll T^2 \int_{s-r}^\infty e^{-C_1 T^2 w^2 + w} w^{5/2} dw \\ \ll T^2 \int_{s-r}^\infty e^{-C_1 T^2 \left(w - \frac{1}{2C_1 T^2}\right)^2} w^{5/2} dw .$$

Next, we substitute $m = w - 1/(2C_1 T^2)$ to get

$$K_2 \ll T^2 \int_{s-r-\frac{1}{2C_1 T^2}}^\infty e^{-C_1 T^2 m^2} m^{5/2} dm .$$

Let $N = s - r - 1/(2C_1 T^2)$ and $l = C_1 T^2 m^2$, then we have

$$K_2 \ll T^2 \int_N^\infty e^{-C_1 T^2 m^2} m^{5/2} dm \ll T^2 \int_{C_1 T^2 N^2}^\infty e^{-l} \left(\frac{l}{C_1 T^2}\right)^{5/4} \frac{1}{C_1 T^2 \sqrt{l/C_1 T^2}} dl \\ \ll \frac{T^2}{T^{7/2}} \int_{C_1 T^2 N^2}^\infty e^{-l} l^{3/4} dl = T^{-3/2} \Gamma\left(\frac{7}{4}, C_1 N^2 T^2\right),$$

where $\Gamma(a,x)$ denotes the incomplete Gamma function. Bounds for the incomplete Gamma function are well-known (see [12, Eq. 8.350]), we just need that $\Gamma(a,x) \ll \Gamma(a) \ll 1$ for $a > 0$ and $x > 0$, to conclude that $K_2 \ll T^{-3/2}$.

For the integral $K_1$, we use the simple bound $\cosh v \ll e^v$ and the same method as before to show that $K_1 \ll T^{-3/2} \Gamma\left(\frac{1}{2}, C_1 N^2 T^2\right) \ll T^{-3/2}$. If $rT > 1$, then

$$\frac{1}{\sqrt{\sinh 2s}} \ll \frac{1}{\sqrt{\sinh 2r}} \ll \frac{1}{\sqrt{r}} \ll \sqrt{T}$$

and it follows that $f(\sinh^2 s+1) \ll Te^{-C_1T^2(s-r)^2}$.

Suppose that $rT \leq 1$. We know that there is a constant $C_1 > 0$ such that

$$f(\sinh^2 s+1) \ll T^2 e^{-C_1T^2(s-r)^2} \int_s^\infty \frac{e^{-C_1T^2(v-r)^2}}{\sqrt{\cosh 2v - \cosh 2s}} \sinh v\, dv\,.$$

From the bound $e^{-C_1T^2(s-r)^2} \ll e^{-C_2T^2s^2}$ used in [3, p. 312] when $rT \leq 1$, we find $C_2 > 0$ such that

$$f(\sinh^2 s+1) \ll T^2 e^{-C_2T^2(s-r)^2} \int_s^\infty \frac{e^{-C_2T^2v^2}}{\sqrt{\cosh 2v - \cosh 2s}} \sinh v\, dv\,.$$

We apply integration by parts to get

$$\begin{aligned} f(\sinh^2 s+1) \ll T^2 e^{-C_2T^2(s-r)^2} \Bigg( &\left[\sqrt{\cosh 2v - \cosh 2s}\,\frac{e^{-C_2T^2v^2}}{\cosh v}\right]_s^\infty \\ &- T^2 \int_s^\infty \frac{e^{-C_2T^2v^2}}{\cosh v} v\sqrt{\cosh 2v - \cosh 2s}\, dv \\ &- \int_s^\infty \frac{e^{-C_2T^2v^2}\sinh v}{\cosh^2 v}\sqrt{\cosh 2v - \cosh 2s}\, dv \Bigg) \end{aligned}$$

By the Mean Value Theorem on the interval $[2s, 2v]$ we find the bound

$$\begin{aligned} f(\sinh^2 s+1) &\ll T^2 e^{-C_2T^2(s-r)^2} \Bigg( T^2 \int_s^\infty \frac{e^{-C_2T^2v^2}}{\cosh v} v\sqrt{\sinh 2v}\sqrt{v-s}\, dv \\ &\quad + \int_s^\infty \frac{e^{-CT^2v^2}\sinh v}{\cosh^2 v}\sqrt{\sinh 2v}\sqrt{v-s}\, dv \Bigg) \\ &\ll T^2 e^{-C_2T^2(s-r)^2} \Bigg( T^2 \int_s^\infty e^{-C_2T^2v^2} v\sqrt{\frac{\sinh v}{\cosh v}}\sqrt{v-s}\, dv \\ &\quad + \int_s^\infty e^{-C_2T^2v^2} \left(\frac{\sinh v}{\cosh v}\right)^{3/2} \sqrt{v-s}\, dv \Bigg). \end{aligned}$$

Let

$$L_1 = T^2 \int_s^\infty e^{-C_2 T^2 v^2} v\sqrt{\tanh v}\sqrt{v-s}\, dv\,.$$

and

$$L_2 = \int_s^\infty e^{-C_2 T^2 v^2} \Big(\tanh v\Big)^{3/2} \sqrt{v-s}\, dv\,.$$

Using the bound $\tanh v \ll v$, we see that

$$L_1 \ll T^2 \int_s^\infty e^{-C_2 T^2 v^2} \sqrt{v} v \sqrt{v-s}\, dv \ll T^2 \int_s^\infty e^{-C_2 T^2 v^2} v^2 dv$$

and

$$L_2 \ll \int_s^\infty e^{-C_2 T^2 v^2} v^{3/2} \sqrt{v-s}\, dv \ll \int_s^\infty e^{-C_2 T^2 v^2} v^2\, dv\,.$$

Similarly to $K_1$ and $K_2$ we use the change of variables $l = C_2 T^2 v^2$. We get

$$\int_s^\infty e^{-C_2 T^2 v^2} v^2\, dv \ll \int_{C_2 T^2 s^2}^\infty e^{-l} \frac{l}{C_2 T^2} \frac{\sqrt{C_2} T}{2 C_2 T^2 \sqrt{l}}\, dl \ll T^{-3} \int_{C_2 T^2 s^2}^\infty e^{-l} \sqrt{l}\, dl\,.$$

Hence

$$\int_s^\infty e^{-C_2 T^2 v^2} v^2\, dv \ll T^{-3} \Gamma\Big(\frac{3}{2}, C_2 T^2 s^2\Big) \ll T^{-3}\,.$$

We see that $L_1 \ll T^{-1}$ and $L_2 \ll T^{-3}$, hence we conclude that

$$f(\sinh^2 s + 1) \ll T e^{-C_2 T^2 (s-r)^2}\,.$$

b) We now prove the estimate in b). Suppose that $\sinh^{-1} u \geq 2r$. Using a) we have

$$\begin{aligned}\int_u^\infty \frac{f(x^2+1)}{\sqrt{x^2-u^2}} dx &= \int_{\sinh^{-1} u}^\infty \frac{f(\sinh^2 s + 1)\cosh s}{\sqrt{\sinh^2 s - u^2}} ds \\ &\ll T \int_{\sinh^{-1} u}^\infty \frac{e^{-C_0 T^2 (s-r)^2} \cosh s}{\sqrt{\sinh^2 s - u^2}} ds\,.\end{aligned}$$

Let $k = \sinh^{-1} u$ and $c = C_0/2$. Suppose that $k > 2r$ then we have

$$\int_u^\infty \frac{f(x^2+1)}{\sqrt{x^2-u^2}}dx \ll Te^{-cT^2(k-r)^2}\int_k^\infty \frac{e^{-cT^2(s-r)^2}\cosh s}{\sqrt{\sinh^2 s - \sinh^2 k}}ds\,.$$

We apply integration by parts to the last integral and the Mean Value Theorem to get:

$$\begin{aligned}\int_k^\infty \frac{e^{-cT^2(s-r)^2}\cosh s}{\sqrt{\sinh^2 s-\sinh^2 k}}ds &= \frac{1}{2}\int_k^\infty \frac{-2cT^2(s-r)e^{-cT^2(s-r)^2}}{\sinh s}\sqrt{\sinh^2 s-\sinh^2 k}ds\\ &\quad -\frac{1}{2}\int_k^\infty \frac{e^{-cT^2(s-r)^2}\cosh s}{\sinh^2 s}\sqrt{\sinh^2 s-\sinh^2 k}ds\\ &\ll T^2\int_k^\infty e^{-cT^2(s-r)^2}(s-r)\sqrt{\frac{\cosh s}{\sinh s}}\sqrt{s-k}ds\\ &\quad +\frac{1}{2}\int_k^\infty e^{-cT^2(s-r)^2}\Big(\frac{\cosh s}{\sinh s}\Big)^{3/2}\sqrt{s-k}ds\,.\end{aligned}$$

These integrals can be bounded the same way as the integrals $K_1$ and $K_2$ that appeared before in both cases $rT \geq 1$ and $rT < 1$.

We use bounds for $\cosh s$ and $\sinh s$ as before to get

$$\begin{aligned}\int_k^\infty \frac{e^{-cT^2(s-r)^2}\cosh s}{\sqrt{\sinh^2 s-\sinh^2 k}}ds &\ll T^2\int_k^\infty e^{-cT^2(s-r)^2}(s-r)^{3/2}\frac{1}{\sqrt{s}}ds\\ &\quad +\int_k^\infty e^{-cT^2(s-r)^2}\sqrt{s-r}s^{-3/2}ds\,.\end{aligned}$$

Since $s \geq k > 0$ we have that $s^{-3/2} \ll 1$. After setting $w = s-r$ we get

$$\begin{aligned}\int_k^\infty \frac{e^{-cT^2(s-r)^2}\cosh s}{\sqrt{\sinh^2 s-\sinh^2 k}}ds &\ll T^2\int_{k-r}^\infty e^{-cT^2w^2}w\,dw+\int_{k-r}^\infty e^{-cT^2w^2}\sqrt{w}dw\\ &\ll e^{-c(k-r)^2T^2}+\int_{cT^2(k-r)^2}^\infty e^{-l}\Big(\frac{l}{cT^2}\Big)^{1/4}\frac{dl}{2cT^2\sqrt{l/(cT^2)}}\\ &\ll e^{-c(k-r)^2T^2}+T^{-3/2}\Gamma\Big(\frac{3}{4},cT^2(k-r)^2\Big)\\ &\ll e^{-c(k-r)^2T^2}\,.\end{aligned}$$

We conclude that

$$\int_u^\infty \frac{f(x^2+1)}{\sqrt{x^2-u^2}}dx \ll Te^{-cT^2(k-r)^2}.$$

c) Suppose that $k < 2r$. Then from (4.7) and the definition of $g$ we have that

$$\begin{aligned}
f(\sinh^2 s+1) &\ll \int_s^\infty \frac{W^1_{-\frac{1}{2}}(g)(v)}{\cosh v}(\cosh 2v - \cosh 2s)^{-1/2} d(\cosh 2v) \\
&\ll \int_s^\infty \frac{W^1_{-\frac{1}{2}}(g)(v)\sinh v}{\sqrt{\cosh 2v - \cosh 2s}} dv \\
&\ll T^3 \int_s^\infty \int_v^\infty \frac{\left((w-r)e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}\right)\sinh v}{\sqrt{\cosh 2v - \cosh 2s}\sqrt{\cosh w - \cosh v}} dw dv \\
&\ll \frac{T^3}{\sqrt{\sinh 2s}} \int_s^\infty \int_v^\infty \frac{\left((w-r)e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}\right)\sqrt{\sinh v}}{\sqrt{v-s}\sqrt{w-v}} dw dv \\
&\ll \frac{T^3}{\sqrt{\sinh 2s}} \int_0^\infty \left(\int_s^w \frac{\sqrt{\sinh v}}{\sqrt{w-v}\sqrt{v-s}} dv\right)\left((w-r)e^{-(w-r)^2T^2} + (w+r)e^{-(w+r)^2T^2}\right) dw.
\end{aligned}$$

We bound the inside integral:

$$\int_s^w \frac{\sqrt{\sinh v}}{\sqrt{w-v}\sqrt{v-s}} dv \ll e^{w/2}\sqrt{w} \int_s^w \frac{1}{\sqrt{w-v}\sqrt{v-s}} dv \ll e^{w/2}\sqrt{w}.$$

Hence, we have that

$$f(\sinh^2 s+1) \ll \frac{T^3}{\sqrt{\sinh 2s}} \int_0^\infty ((w-r)e^{-(w-r)^2T^2+w/2}\sqrt{w} + (w+r)e^{-(w+r)^2T^2+w/2}\sqrt{w}) dw.$$

When $w \le 2r$ we have

$$\begin{aligned}
\int_0^{2r} (w-r)e^{-(w-r)^2T^2+w/2}\sqrt{w}\, dw \ll \sqrt{r}\int_0^{2r} e^{-(w-r)^2T^2}(w-r)\, dw \ll \frac{\sqrt{r}}{T^2}\left[e^{-(w-r)^2T^2}\right]_0^{2r} \\
\ll \frac{\sqrt{r}}{T^2} e^{-Cr^2T^2}.
\end{aligned}$$

Similarly

$$\int_0^{2r} (w+r)e^{-(w+r)^2T^2+w/2}\sqrt{w}\,dw \ll \frac{\sqrt{r}}{T^2}e^{-Cr^2T^2}\,.$$

For the integral

$$\int_{2r}^{\infty} \left((w-r)e^{-(w-r)^2T^2+w/2}\sqrt{w}+(w+r)e^{-(w+r)^2T^2+w/2}\sqrt{w}\right)dw$$

we consider again the cases $rT>1$ and $rT\le 1$ separately and work as in the previous step to conclude that

$$\int_{2r}^{\infty} (w-r)e^{-(w-r)^2T^2+w/2}\sqrt{w}+(w+r)e^{-(w+r)^2T^2+w/2}\sqrt{w}dw \ll T^{-3/2}e^{-Cr^2T^2}.$$

Overall we see that

$$f(\sinh^2 s+1) \ll T\frac{\sqrt{r}}{\sqrt{\sinh 2s}}e^{-Cr^2T^2}\,.$$

For $0<k=\sinh^{-1}u<2r$ we split the integral in (4.6) as

$$\int_k^{\infty} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s-u^2}}ds = \int_k^{2r} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s-u^2}}ds+\int_{2r}^{\infty} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s-u^2}}ds.$$

For the second integral we have $s\ge 2r$ and we use the bound from a)

$$f(\sinh^2 s+1) \ll Te^{-C(s-r)^2T^2}$$

to conclude again that

$$\int_{2r}^{\infty} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s-u^2}}ds \ll Te^{-Cr^2T^2}\,.$$

For the first integral we use the bound $f(\sinh^2 s+1) \ll T\frac{\sqrt{r}}{\sqrt{\sinh 2s}}e^{-Cr^2T^2}$ to get

$$\begin{aligned}\int_k^{2r} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s-u^2}}ds &\ll Te^{-Cr^2T^2}\int_k^{2r} \frac{\cosh s}{\sqrt{\sinh 2s}\sqrt{\sinh^2 s-\sinh^2 k}}ds \\ &\ll Te^{-Cr^2T^2}\int_k^{2r} \frac{1}{\sqrt{\sinh 2k}\sqrt{s-k}}ds \\ &\ll Te^{-Cr^2T^2}\left[\sqrt{s-k}\right]_k^{2r} \\ &\ll Te^{-Cr^2T^2}.\end{aligned}$$

d) Lastly we consider the integral

$$\int_0^{\infty} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s+u^2}}ds = \int_0^{2r} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s+u^2}}ds + \int_{2r}^{\infty} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s+u^2}}ds.$$

For the second integral we use again the bound $f(\sinh^2 s+1) \ll Te^{-C(s-r)^2T^2}$ and work as before to get that

$$\int_{2r}^{\infty} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s+u^2}}ds \ll Te^{-Cr^2T^2}.$$

For the first integral from the bound $f(\sinh^2 s+1) \ll T\frac{\sqrt{r}}{\sqrt{\sinh 2s}}e^{-Cr^2T^2}$ we have

$$\begin{aligned}\int_0^{2r} \frac{f(\sinh^2 s+1)\cosh s}{\sqrt{\sinh^2 s+u^2}}ds &\ll T\sqrt{r}e^{-Cr^2T^2}\int_0^{2r} \frac{\cosh s}{\sqrt{\sinh 2s}\sqrt{\sinh^2 s+u^2}} \\ &\ll T\sqrt{r}e^{-Cr^2T^2}\int_0^{2r} \frac{1}{\sqrt{s}}ds \ll Te^{-Cr^2T^2}.\end{aligned}$$

□

We will also use [3, Lemma 3.2], which states:

**Lemma 4.2.** *Let $R \in \mathbb{N}$, $\mathbf{b} = (b_1, \ldots, b_R)$ be a unit complex vector and let $M = (m_{\nu\mu})$*

*be an $R\times R$ complex matrix such that* $|m_{\nu\mu}|=|m_{\mu\nu}|$. *Then*

$$|\mathbf{b}\cdot M\mathbf{b}|=\Big|\sum_{\nu,\mu=1}^{R} b_\nu \bar{b}_\mu m_{\nu\mu}\Big|\le \max_{\nu}\sum_{\mu=1}^{R}|m_{\nu\mu}|\,.$$

Using Lemmata 4.1, 4.2 and ideas from [3] we can now prove Theorem 1.3:

*Proof of Theorem 1.3.* We follow the proof of [3, Theorem 2.2] replacing $u_j(z)$ by $\widehat{u_j}$:

Let

$$S=\sum_{\nu=1}^{R}\Big|\sum_{|t_j|\le T} a_j x_\nu^{it_j}\widehat{u_j}\Big|^2 .$$

By duality there exists a unit complex vector $\mathbf{b}=(b_1,b_2,\dots,b_R)$, such that

$$S=\left(\sum_{\nu=1}^{R} b_\nu \sum_{|t_j|\le T} a_j x_\nu^{it_j}\widehat{u_j}\right)^2 .$$

After changing the order of summation and applying the Cauchy–Schwarz inequality on the space $\mathbb{C}^R$ we get

$$S\ll ||\mathbf{a}||_*^2\,\tilde{S}\,,$$

where $\tilde{S}$ is defined as

$$\tilde{S}=\sum_{|t_j|\le T}\Big|\sum_{\nu=1}^{R} b_\nu x_\nu^{it_j}\widehat{u_j}\Big|^2 .$$

We can extend the spectrum and achieve the following bound in order to smooth our sum:

$$\tilde{S}\ll\sum_j e^{-t_j^2/(4T^2)}\Big|\sum_{\nu=1}^{R} b_\nu x_\nu^{it_j}\widehat{u_j}\Big|^2 .$$

After opening the squares and changing the order of summation we get

$$S\ll ||\mathbf{a}||_*^2\max_{\mu\in\{1,2,\dots,R\}}\sum_{\nu=1}^{R}\left|S_{\nu\mu}\right|,$$

where we define

$$S_{\nu\mu} = \sum_j e^{-t_j^2/(4T^2)} \cos(r_{\nu\mu} t_j) \widehat{u_j}^2$$

and

$$r_{\nu\mu} = |\log(x_\nu / x_\mu)| \, .$$

Since $|x_\nu - x_\mu| \ge \delta$ by the spacing condition, we have

$$\left| \frac{x_\nu}{x_\mu} - 1 \right| \ge \frac{\delta}{|x_\mu|} \ge \frac{\delta}{X} \, .$$

Since $X \le x_\nu, x_\mu \le 2X$ we get

$$r_{\nu\mu} = |\log(x_\nu / x_\mu)| \le \log 2 \, .$$

We recall that there are finitely many double coset representatives $\gamma \in H_1 \backslash \Gamma / H_1$ such that $B(\gamma) < 1$. By Proposition 3.2 and Lemma 4.1(a) it follows that

$$S_{\nu\mu} \ll \min\left(T, r_{\nu\mu}^{-1}\right) + \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) < 1}} \widetilde{q}(B(\gamma)) + \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ 1 < B(\gamma) < \cosh(2 r_{\nu\mu})}} q(B(\gamma)) + \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) \ge \cosh(2 r_{\nu\mu})}} q(B(\gamma)) \, .$$

By Lemma 4.1(c), (d) for the first two sums and the discreteness of the terms $B(\gamma)$ we have

$$S_{\nu\mu} \ll \min\left(T, r_{\nu\mu}^{-1}\right) + \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) \ge \cosh(2 r_{\nu\mu})}} q(B(\gamma)) \, .$$

Lastly, by Lemma 4.1(b) we conclude that

$$S_{\nu\mu} \ll \min\left(T, r_{\nu\mu}^{-1}\right) + T \sum_{\substack{\gamma \in H_1 \backslash \Gamma - H_1 / H_1 \\ B(\gamma) \ge \cosh(2 r_{\nu\mu})}} e^{-cT^2 \left( \sinh^{-1}\left( \sqrt{B(\gamma)^2 - 1} \right) - r_{\nu\mu} \right)^2} ,$$

where $c$ is a positive constant. By a trivial estimate (see bound for $\pi_\delta(x) = N(X/2, l)$ in [23, p. 14]) we can prove that the above series converges quickly and is smaller than the first term as follows:

Since the terms $B(\gamma)$ are discrete we can assume that $B(\gamma) > M$, where $M > 1$. Using that $\sinh^{-1}\left(\sqrt{x^2-1}\right) = \cosh^{-1} x$ we write

$$\sum_{\substack{\gamma\in H_1\backslash\Gamma - H_1/H_1 \\ B(\gamma)>M}} e^{-cT^2\left(\sinh^{-1}\left(\sqrt{B(\gamma)^2-1}\right)-r_{\nu\mu}\right)^2} = \int_M^\infty e^{-cT^2\left(\cosh^{-1}x-r_{\nu\mu}\right)^2}\, d(N(x,l))\,.$$

We apply integration by parts to get

$$\begin{aligned}
\int_M^\infty e^{-cT^2\left(\cosh^{-1}x-r_{\nu\mu}\right)^2}\, d(N(x,l)) &= \left[N(x,l)e^{-cT^2\left(\cosh^{-1}x-r_{\nu\mu}\right)^2}\right]_M^\infty \\
&\quad -2cT^2\int_M^\infty N(x,l)e^{-cT^2\left(\cosh^{-1}x-r_{\nu\mu}\right)^2}\left(\cosh^{-1}x-r_{\nu\mu}\right)\frac{1}{\sqrt{x^2-1}}\,dx \\
&\ll e^{-cr_{\nu\mu}^2T^2} + T^2\int_M^\infty e^{-cT^2\left(\cosh^{-1}x-r_{\nu\mu}\right)^2}\left(\cosh^{-1}x-r_{\nu\mu}\right)dx \\
&\ll e^{-cr_{\nu\mu}^2T^2} + T^2\int_{\cosh^{-1}M}^\infty e^{-cT^2(w-r_{\nu\mu})^2}\frac{w-r_{\nu\mu}}{\sinh w}\,dw \\
&\ll e^{-cr_{\nu\mu}^2T^2} + T^2e^{-cr_{\nu\mu}^2T^2}\int_{\cosh^{-1}M}^\infty e^{-cT^2(w-r_{\nu\mu})^2}(w-r_{\nu\mu})\,dw \\
&\ll e^{-cr_{\nu\mu}^2T^2}\,.
\end{aligned}$$

Hence we have that

$$S_{\nu\mu} \ll \min\left(T, \frac{1}{r_{\nu\mu}}\right).$$

By the definition of $r_{\nu\mu}$ we conclude that

$$\sum_{\mu=1}^R \left|S_{\nu\mu}\right| \ll \sum_{\mu=1}^R \min\left(T, X|x_\nu - x_\mu|^{-1}\right).$$

By the spacing condition we have

$$\sum_{\mu=1}^R \min\left(T, X|x_\nu - x_\mu|^{-1}\right) \ll \sum_{j=1}^R \min\left(T, \frac{X}{j\delta}\right).$$

Then we have

$$\sum_{\mu=1}^{R}\min(T, X|x_\nu - x_\mu|^{-1}) \ll T + \sum_{1\le j\le \frac{X}{T\delta}} T + \sum_{\frac{X}{T\delta}<j\le R} \frac{X}{j\delta} \ll T + X\delta^{-1}\log T\,,$$

which concludes the estimate for $S$. □

## 4.2 Second moment of the error term

In this section we give a bound for the second moment of the averaging of the error term in counting $N(X,l)$.

We recall that the main term of our hyperbolic lattice counting problem is

$$M(X,l) = \sum_{1/2<s_j\le 1} \frac{2}{\pi}\gamma_1(s_j)\widehat{u_j}^2 X^{s_j}$$

and the error term is

$$E(X,l) = N(X,l) - M(X,l)\,.$$

Also we define the error term related to a test function $f$ as

$$E_f(X,l) = f(1)\mathrm{len}(l) + \sum_{\gamma\in H_1\backslash\Gamma - H_1/H_1} Q(B(\gamma)) - 2\sum_{1/2<s_j\le 1} d_{t_j}(f)\widehat{u_j}^2\,,$$

where

$$Q(B(\gamma)) = \begin{cases} q(B(\gamma)), & B(\gamma)>1,\\ \widetilde{q}(B(\gamma)), & B(\gamma)<1,\end{cases}$$

and $q$, $\widetilde{q}$ are given by (3.5) and (3.6). In the proof of Theorem 1.2 (see (3.22),(3.24)) we showed that

$$E_{f^+}(X,l) = O(XY^{-1/2} + X^{1/2})\,,$$

$$E_{f^-}(X,l) = O(XY^{-1/2} + X^{1/2})\,,$$

$$E_{f^-}(X,l) < E(X,l) + O\Big(Y + X^{1/2}\log X\Big) < E_{f^+}(X,l)\,.$$

We choose $Y$ such that $X^{1/2}\log X < Y < X$, then

$$E_{f^+}(X,l) < E(X,l) + O(Y) < E_{f^-}(X,l)\,.$$

We can now use Theorem 1.3 to prove Proposition 1.4:

*Proof.* We follow the proof of [6, Proposition 5.3]. We start by choosing $Y$ such that $X^{1/2}\log X \ll Y \ll X$, then

$$E_{f^+}(X,l) < E(X,l) + O(Y) < E_{f^-}(X,l)\,.$$

In the following we write $f$ for either $f^-$ or $f^+$ defined in (3.9) and (3.11). For the average sum of $E(X,l)$ over points $X_1, X_2, \ldots, X_R$ we have

$$\sum_{m=1}^{R} |E(X_m,l)|^2 \ll \sum_{m=1}^{R} |E_f(X_m,l)|^2 + RY^2\,.$$

We define the sum

$$S(X,T) = 2\sum_{T<|t_j|\le 2T} d_{t_j}(f)\widehat{u_j}^2\,.$$

and split the $t_j$ in the following intervals:

$$A_1 = \{t_j : 0 < |t_j| \le 1\},$$

$$A_2 = \{t_j : 1 \le |t_j| \le X^2 Y^{-2}\},$$

$$A_3 = \{t_j : |t_j| > X^2 Y^{-2}\}\,.$$

Let also

$$S_i = 2\sum_{t_j\in A_i} d_{t_j}(f)\widehat{u_j}^2\,,$$

so that the error term with regard to $f$ can be written as

$$E_f(X,l) = S_1 + S_2 + S_3\,.$$

We start by bounding $S_1$: using the estimates for $d_{t_j}(f)$ from Proposition 3.5 we have

$$\sum_{t_j\in A_1} 2d_{t_j}(f)\widehat{u_j}^2 \ll X^{1/2}\sum_{|t_j|<1} t_j^{-3/2}\min\{t_j, X/Y\}\widehat{u_j}^2 \ll X^{1/2} \ll Y\,,$$

since there exist finitely many eigenvalues $\lambda_j$ with spectral parameter $|t_j| \le 1$. For $S_3$ we compute

$$\begin{aligned}\sum_{t_j\in A_3} 2d_{t_j}(f)\widehat{u_j}^2 &\ll \sum_{|t_j|>X^2Y^{-2}} |t_j|^{-3/2}\min\{t_j, X/Y\}X^{1/2}\widehat{u_j}^2\\ &\ll \sum_{t_j>X^2Y^{-2}} t_j^{-3/2}X^{3/2}Y^{-1}\widehat{u_j}^2\,.\end{aligned}$$

By partial summation and Lemma 2.10 we get

$$S_3 \ll X^{1/2} \ll Y\,.$$

Hence we have shown that

$$E_f(X,l) = \sum_{t_j\in A_2} 2d_{t_j}(f)\widehat{u_j}^2 + O(Y)\,.$$

We will consider values $T = 2^k, k = 0,1,\ldots,\log_2(X^2Y^{-2})$ and we sum over only those $T's$ to get

$$E_f(X,l) \ll \sum_{1\le T=2^k\le X^2Y^{-2}} S(X,T) + Y$$

and we also add for $X_1, X_2, \dots X_R$ to get

$$\sum_{m=1}^{R} |E_f(X_m, l)|^2 \ll \sum_{m=1}^{R} \Big| \sum_{1\le T=2^k\le X^2Y^{-2}} S(X_m, T)\Big|^2 + RY^2 \,.$$

At this point we use the Cauchy–Schwarz inequality for the inner sum

$$\Big| \sum_{1\le T=2^k\le X^2Y^{-2}} S(X_m, T)\Big|^2 \ll \log X \sum_{1\le T=2^k\le X^2Y^{-2}} |S(X_m, T)|^2 \,.$$

Combining the last two inequalities we show

$$\sum_{m=1}^{R} |E_f(X_m, l)|^2 \ll \log X \sum_{1\le T=2^k\le X^2Y^{-2}} \Big( \sum_{m=1}^{R} |S(X_m, T)|^2\Big) + RY^2 \,.$$

In Proposition 3.5 we have written the Huber transform of f as

$$d_{t_j}(f) = X^{1/2}(a(t, X, Y)X^{it} + b(t, X, Y)X^{-it}) \,,$$

where $a(t, X, Y)$ and $b(t, X, Y)$ are functions satisfying

$$a(t, X, Y), b(t, X, Y) \ll |t|^{-3/2} \min\{|t|, X/Y\} \,.$$

Now using Theorem 1.3 for $a_j x_\nu^{it_j} = d_{t_j}(f)\widehat{u_j}$, we have

$$\sum_{m=1}^{R} \Big| \sum_{T<|t_j|\le 2T} d_{t_j}(f)\widehat{u_j}^2 \Big|^2 \ll (T + X \log T\delta^{-1})||a||_*^2$$

or equivalently

$$\sum_{m=1}^{R} |S(X_m, T)|^2 \ll (T + X \log T\delta^{-1})||a||_*^2 \,.$$

Let us now bound the norm $||a||_*^2$:

$$||a||_*^2 \ll \sum_{T<|t_j|\le 2T} \left| |t_j|^{-3/2} \min\{t_j, X/Y\} X^{1/2} \widehat{u_j} \right|^2$$
$$\ll XT^{-3} \min\{T^2, X^2Y^{-2}\} \sum_{T<|t_j|\le 2T} |\widehat{u_j}|^2 .$$

By Lemma 2.10 we can see that

$$||a||_*^2 \ll XT^{-2} \min\{T^2, X^2Y^{-2}\} ,$$

hence overall we have shown:

$$\sum_{m=1}^{R} |S(X_m, T)|^2 \ll (T + X \log T\delta^{-1}) XT^{-2} \min\{T^2, X^2Y^{-2}\} .$$

The last part of the proof follows from [6, Page 20]: using the last bound we have

$$\sum_{m=1}^{R} |E(X_m, l)|^2 \ll \log X \sum_{1\le T=2^k\le X^2Y^{-2}} \left( \sum_{m=1}^{R} |S(X_m, T)|^2 \right) + RY^2$$
$$\ll \log X \sum_{1\le T=2^k\le X^2Y^{-2}} (T + X \log T\delta^{-1}) XT^{-2} \min\{T^2, X^2Y^{-2}\} + RY^2 .$$

We get the bound

$$\sum_{m=1}^{R} |E(X_m, l)|^2 \ll X \log X \sum_{1\le T=2^k<XY^{-1}} T + X^2\delta^{-1} \log^2 X \sum_{1\le T=2^k<XY^{-1}} 1$$
$$+ X^3Y^{-2} \log X \sum_{XY^{-1}\le T=2^k<X^2Y^{-2}} T^{-1}$$
$$+ X^4\delta^{-1}Y^{-2} \log^2 X \sum_{XY^{-1}\le T=2^k<X^2Y^{-2}} T^{-2} + RY^2 .$$

By trivial bounds for each term individually we can show that

$$\sum_{m=1}^{R} |E(X_m,l)|^2 \ll X^2 Y^{-1} \log X + \delta^{-1} X^2 \log^3 X + RY^2 \,.$$

We notice that the optimal choice for $Y$ is $Y = R^{-1/3} X^{2/3}$ and that choice gives us the bound

$$\sum_{m=1}^{R} |E(X_m,l)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^3 X \,,$$

which concluded the proof of Proposition 1.4. □

The proof of Theorem 1.5 also follows from the proof of [6, Theorem 5.4]:

*Proof.* To prove equation (1.1) we choose $\delta^{-1} \ll RX^{-1}$ and $R > X^{1/2}$ in the bound

$$\sum_{m=1}^{R} |E(X_m,l)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^3 X \,.$$

We get

$$\sum_{m=1}^{R} |E(X_m,l)|^2 \ll R^{1/3} X^{4/3} \log X + RX \log^3 X \ll RX \log^3 X \,.$$

To prove equation (1.2) we choose the points $X_i$ to be equally spaced in $[X, 2X]$ with $\delta = R^{-1}X$. By taking the mesh $X/R$ to tend to 0 and, since the function $|E(x,l)|^2$ is integrable, because it has finitely many discontinuities as a function of $x$, we have

$$\sum_{m=1}^{R} |E(X_m,l)|^2 \frac{X}{R} \to \int_X^{2X} |E(x,l)|^2 dx \,.$$

This implies

$$\frac{1}{X} \int_X^{2X} |E(x,l)|^2 dx \ll X \log^3 X \,.$$

□

## Appendix A

# Special functions

We use the following special functions, see [12]. We list the main properties that we use.

**Definition A.1.** For $\Re z > 0$ the Gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt .$$

By [12, eq. 8.328.1] we have the following Stirling's approximation

$$\lim_{|y|\to\infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} \sim \sqrt{2\pi} . \tag{A.1}$$

**Definition A.2.** The lower incomplete Gamma function is defined as

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt ,$$

whereas the upper incomplete Gamma function is defined as

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt .$$

**Definition A.3.** Let $\Re x > 0$ and $\Re y > 0$, then the Beta function is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt\,.$$

By [12, Eq. 8.384] the Beta function satisfies

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y)\,.$$

**Definition A.4.** For $|z| < 1$ the Gauss hypergeometric function is defined by the power series

$${}_2F_1(a,b,c;z) = 1+\frac{a\cdot b}{c\cdot 1}z+\frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)\cdot 1\cdot 2\cdot 3}z^3+\ldots\,.$$

It follows that ${}_2F_1(a,b,c;0) = 1$. For $\Re b > \Re c > 0$ the Gauss hypergeometric function has the integral representation [12, 9.111]

$${}_2F_1(a,b,c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt\,.$$

The derivative of ${}_2F_1(a,b,c;z)$ with respect to $z$ is given by

$$\frac{d}{dz}\big({}_2F_1(a,b,c;z)\big) = \frac{ab}{c}{}_2F_1(a+1,b+1,c+1;z)\,. \tag{A.2}$$

We also need the transformation formula [12, 9.132.2]

$$\begin{aligned}{}_2F_1(a,b,c;z) = &\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\gamma(c-a)}(-z)^{-a}{}_2F_1\Big(a,a+1-c,a+1-b;\frac{1}{z}\Big)\\ &+\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1\Big(b,b+1-c,b+1-a;\frac{1}{z}\Big)\,,\end{aligned}$$

where $\arg z < \pi,\ a-b \neq \pm m,\quad m = 0,1,2,\ldots$ and formulas [12, 9.136.1–9.136.2]:

Let

$$A = \frac{\Gamma\big(a+b+\frac{1}{2}\big)\sqrt{\pi}}{\Gamma\big(a+\frac{1}{2}\big)\Gamma\big(b+\frac{1}{2}\big)}, \quad B = \frac{-\Gamma\big(a+b+\frac{1}{2}\big)2\sqrt{\pi}}{\Gamma(a)\Gamma(b)}\,;$$

then we have

$${}_2F_1\left(2a, 2b, a+b+\frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) = A\,{}_2F_1\big(a, b, \frac{1}{2}; z\big) + B\sqrt{z}\,{}_2F_1\left(a+\frac{1}{2}, b+\frac{1}{2}, \frac{3}{2}; z\right)$$

and

$${}_2F_1\left(2a, 2b, a+b+\frac{1}{2}; \frac{1+\sqrt{z}}{2}\right) = A\,{}_2F_1\big(a, b, \frac{1}{2}; z\big) - B\sqrt{z}\,{}_2F_1\left(a+\frac{1}{2}, b+\frac{1}{2}, \frac{3}{2}; z\right).$$

The series

$${}_pF_q(\alpha_1, \alpha_2, \ldots \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_p)_k} \frac{z^k}{k!}$$

is called a generalized hypergeometric series.

For the series ${}_pF_q$ we use the integral formula [12, 7.512.12]:

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-\gamma-1}\,{}_pF_q(a_1, \ldots, a_p, b_1, \ldots, b_q; ax)\,dx$$
$$= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\,{}_{p+1}F_{q+1}(\nu, a_1, \ldots, a_p, \mu+\nu, b_1, \ldots, b_q; a)\,,$$

where $\Re\mu > 0$, $\Re\nu > 0$, $p \le q+1$, if $p = q+1$ then $|a| < 1$. We also write the transformation formula [24, 16.8.8] which reads for $z \in \mathbb{R}$ with $|\arg(-z)| < \pi$ :

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}, b_1, \ldots, b_q; z) = \sum_{j=1}^{q+1} \left( \prod_{\substack{k=1\\k\neq j}}^{q+1} \frac{\Gamma(a_k - a_j)}{\Gamma(a_k)} \Big/ \prod_{k=1}^{q} \frac{\Gamma(b_k - a_j)}{\Gamma(b_k)} \right) \widetilde{w}_j(z)\,,$$

with

$$\widetilde{w}_j(z) = (-z)^{a_j}\,{}_{q+1}F_q\left(a_j, 1-b_1+a_j, \ldots, 1-b_q+a_j, 1-a_1+a_j, \cdots * \ldots, 1-a_{q+1}+a_j; \frac{1}{z}\right),$$

for $j = 1, \dots, q+1$, where $*$ indicates that the entry $1 - a_j + a_j$ is omitted. Lastly we use the asymptotic of Watson [33], in the form found in [22, p. 237, eq. 11]

$$
\begin{aligned}
&{}_2F_1(\alpha+\lambda, 1+\alpha-\gamma+\lambda, 1+\alpha-\beta+2\lambda; z) \\
&\qquad = \frac{2^{\alpha+\beta}\Gamma(1+\alpha-\beta+2\lambda)(\pi/\lambda)^{1/2} e^{-(\alpha+\lambda)\phi}(1-e^{-\phi})^{\gamma-\alpha-\beta-1/2}\big(1+O(\lambda^{-1})\big)}{\Gamma(1+\alpha-\gamma+\lambda)\Gamma(\gamma-\beta+\lambda) z^{\alpha+\lambda}(1+e^{-\phi})^{\gamma-1/2}},
\end{aligned}
\tag{A.3}
$$

where

$$
e^{\pm\phi} = \frac{2-z\pm 2(1-z)^{1/2}}{z}.
$$

**Definition A.5.** The associate Legendre function of the first kind is defined as (see [12, 8.704])

$$
P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(-\nu, \nu+1, 1-\mu; \frac{1-z}{2}\right).
$$

The associate Legendre function of the first kind is defined as (see [12, 8.705])

$$
Q_\nu^\mu(z) = \frac{\pi}{2\sin\mu\pi}\left(P_\nu^\mu(z)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(x)\right).
$$

We use the integral representation for $P_\nu^{-\mu}(z)$ [12, 8.713.3]

$$
P_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\mu+\frac{1}{2}\right)\left(z^2-1\right)^{\mu/2}}{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)} \int_0^\infty \frac{\cosh\left(\nu+\frac{1}{2}\right)t}{(z+\cosh t)^{\mu+\frac{1}{2}}} dt,
$$

where $\Re z > -1$, $|\arg(z\pm 1)| < \pi$, $\Re(\nu+\mu) > -1$ and $\Re(\mu-\nu) > 0$.

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