Einstein-Maxwell theory and the Weyl double copy

Damien A. Easson[*]

Department of Physics, Arizona State University, Tempe, Arizona 85287-1504, USA

Tucker Manton[†]

Department of Physics, Brown University, Providence, Rhode Island 02912, USA

Andrew Svesko[‡]

*Department of Physics and Astronomy, University College London,
London WC1E 6BT, United Kingdom*

The Weyl double copy relates vacuum solutions in general relativity to Abelian gauge fields in Minkowski spacetime. In a previous work, we showed how the Weyl double copy can be extended to provide a treatment of external gravitational sources consistent with the classical Kerr-Schild double copy. Using this generalization, here we provide a complete double-copy analysis of electrovacuum Petrov type-D spacetimes. This includes the first analysis of the charged C metric, whose single-copy interpretation invokes the two-potential formalism of electrodynamics. We also present the first double-copy prescription for the Ricci spinor, which for nonaccelerating spacetimes, takes a form similar to the original double-copy relation for the Weyl spinor.

I. INTRODUCTION

Gravity and non-Abelian gauge theories share several formal similarities. For example, general relativity and Yang-Mills are each characterized by nonlinear equations of motion whose solution yields the (gauge) curvature of the background. Color-kinematics duality [1] realizes an equivalence between gravity and gauge theory at the level of perturbative scattering amplitudes. Namely, amplitudes in gravity are expressed as a double copy of Yang-Mills gauge theory amplitudes, i.e., a multiloop, multipoint gravity amplitude is given by the product of two gauge theory amplitudes [2,3]. This result was anticipated by the Kawai-Lewellen-Tye relation demonstrating any closed-string amplitude may be cast as a linear sum of factors, each of which is a product of two open-string amplitudes, at tree level [4]. The discovery of the double copy resulted in powerful new techniques for computing amplitudes in (super)gravity at tree level and beyond [3,5–8], offering new insights into the fundamental nature of perturbative quantum gravity, and has numerous applications, particularly in gravitational wave physics [9–12] (see [13–15] for comprehensive reviews).

The success of the perturbative double copy motivated the development of a *classical* double copy, where exact solutions in gravity have a correspondence to exact (linearized) Yang-Mills solutions [16–19] (see also, [20–24]). For example, the Kerr-Schild double copy is a copying relation at the level of the graviton and gauge field [17]. Specifically, consider a spacetime $g_{\mu\nu}$ in Kerr-Schild form,

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu, \qquad g^{\mu\nu} = \eta^{\mu\nu} - \phi k^\mu k^\nu,$$

$$g^{\mu\nu}k_\mu k_\nu = \eta^{\mu\nu}k_\mu k_\nu = 0, \tag{1.1}$$

where ϕ is a scalar-field function of the background coordinates and k_μ is a null geodesic with respect to both $\eta_{\mu\nu}$ and $g_{\mu\nu}$. The Kerr-Schild ansatz (1.1) linearizes Einstein's field equations, and when $g_{\mu\nu}$ solves Einstein's equations, the four-vector $A_\mu \equiv \phi k_\mu$ is a solution to Maxwell's equations. Indeed, choosing $k_0 = 1$ and introducing the tensor $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, from the trace-reversed Einstein equations we identify the sourced Maxwell equations (working in units of $8\pi G = 1$) [25]:

$$R^0_{\ \mu} = T^0_{\ \mu} - \frac{T}{(d-2)}\delta^0_{\ \mu} \leftrightarrow \frac{1}{2}\partial^\nu F_{\nu\mu} = -\frac{1}{2}J_\mu, \tag{1.2}$$

where the (external) sources of the Einstein field equations correspond to (external) sources of the Maxwell equations,

[*]deasson@asu.edu
[†]tucker_manton@brown.edu
[‡]a.svesko@ucl.ac.uk

$J^\mu = -2(T^\mu_{\ 0} - T\delta^\mu_0/(d-2))$, where we used the relation $R^\mu_{\ 0} = -\frac{1}{2}\partial_\nu F^{\mu\nu}$.[1]

The simplest example is the four-dimensional (exterior) Schwarzschild black hole, where, under an elementary gauge transformation, A_μ is recognized as the Coulomb solution for a static point charge located at the origin [17][2]; the Schwarzschild solution is the classical double copy of a static point charge.[3] This version of the classical double copy holds for a variety of spacetimes in higher- and lower dimensions and has been generalized such that $\eta_{\mu\nu}$ in the Kerr-Schild ansatz (1.1) may be replaced by a curved background $\bar{g}_{\mu\nu}$, e.g., [30–35]. Notably, the Kerr-Schild classical double copy accounts for gravity theories with sources, such as Einstein-Maxwell theory, and is thus capable of describing, for example, charged black holes. Stronger links between the perturbative double copy and Kerr-Schild double copy are presented in [36,37].

Another version of the classical double copy was explored in [19], known as the Weyl double copy, and is defined as follows. Let Ψ_{ABCD} denote the completely symmetric Weyl spinor satisfying the four-dimensional vacuum Einstein equations for Petrov type-D or type-N spacetimes. Then the Weyl double copy relates a single copy Maxwell field-strength spinor f_{AB} and a (complex) scalar field S to the gravity solution that constructs Ψ_{ABCD},

$$\Psi_{ABCD} = \frac{1}{S} f_{(AB} f_{CD)}. \qquad (1.3)$$

In this context, S is referred to as the zeroth copy since in the case of single Kerr-Schild spacetimes the real part of S is equal to ϕ up to a constant. The spinors f_{AB} and Ψ_{ABCD} are related to their tensorial counterparts, $F_{\mu\nu}$ and $W_{\mu\nu\lambda\gamma}$, via Infeld–van der Waerden symbols built from Pauli matrices and spacetime vierbeins. When Ψ_{ABCD} is constructed from a vacuum type-D spacetime, then [19]

$$\Box^{(0)}S = 0, \qquad \nabla^{(0)}_\mu F^{\nu\mu} = \nabla^{(0)}_\mu \tilde{F}^{\nu\mu} = 0, \qquad (1.4)$$

where $\tilde{F}_{\mu\nu} = \frac{\sqrt{-g}}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is the dual single-copy field-strength tensor. The (0) superscript indicates that these

derivatives are taken over the flat background metric, which is obtained by taking the appropriate limit of the full spacetime metric associated with Ψ_{ABCD}. The Weyl double copy is consistent with the Kerr-Schild double copy and furthermore, resolves some ambiguities of the latter [19]. The Weyl double copy has since been shown to hold for vacuum spacetimes of arbitrary Petrov type using methods from twistor theory [21,22,38].

Despite its fundamental underpinnings, a weakness of the Weyl double copy (1.3), compared to the Kerr-Schild double copy, is that it cannot describe classical spacetimes with external sources. More bluntly, the Weyl double copy cannot be used to construct a four-dimensional charged black hole. This is certainly a drawback to the Weyl double copy as realistic spacetimes typically involve external matter sources. Thus, to have any empirical merit, the Weyl double copy requires an extension to describe gravity theories with external sources.

In a recent paper [39], we proposed a simple generalization of the Weyl double copy (1.3) to include sources on the gravity side of the duality. Rather than Ψ_{ABCD} being constructed from a single scalar-gauge theory, we promote it to a *sum* over m scalar-gauge theories,

$$\Psi_{ABCD} = \sum_{n=1}^{m} \frac{1}{S_{(n)}} f^{(n)}_{(AB} f^{(n)}_{CD)}, \qquad (1.5)$$

where for $n > 1$, each $S_{(n)}$ and $f^{(n)}_{AB}$ satisfy a particular sourced wave equation and sourced Maxwell equations, respectively,

$$\nabla^{(0)}_\mu F^{\nu\mu}_{(n)} = J^\nu_{(n)}, \qquad \Box^{(0)}S_{(n)} = \rho^S_{(n)}. \qquad (1.6)$$

We thus refer to (1.5) as the *sourced Weyl double copy*. Here $\rho^S_{(n)}$ and $J^\nu_{(n)}$ are external sources for the zeroth-[4] and single copies, respectively, with the $n = 1$ term corresponding to (1.3), or what we may consider the vacuum part of the metric. Terms associated with $n = 2, 3, \ldots, m$ correspond to each external source. That is, all parameters that contribute nontrivially to the Ricci curvature pair with a gauge theory labeled by n for each $n > 1$.

The purpose of this article is to explore the sourced Weyl double copy for exact solutions to Einstein-Maxwell theory, focusing on stationary spacetimes. Our work not only provides the first complete double-copy analysis of canonical electrovacuum type-D spacetimes, including the charged C-metric and Kerr-Newman-Taub–Newman-Unti-Tamburino (NUT) black hole, but also presents the first double-copy prescription for the Ricci spinor.

---

[1]It is generally less clear how one should interpret the spatial components $R^i_{\ j}$ in the double copy. For vacuum spacetimes $R^i_{\ j} = 0$ acts as a constraint on the Maxwell theory akin to color-kinematics duality [19], while $R^i_{\ j} \neq 0$ is fixed by the charge density and the spatial component of the current parallel to k_i [25].

[2]"Vacuum" Schwarzschild is already an example of Einstein's equations with sources: $T^{\mu\nu} = M v^\mu v^\nu \delta^{(3)}(\vec{x})$ with $v^\mu = (1, 0, 0, 0)$. Correspondingly, identifying $k^0 = +1$, the Maxwell current is $J^\mu = -\frac{M}{2} v^\nu \delta^{(3)}(\vec{x})$ [17].

[3]There is actually a bit more to this story. The Coulomb charge has been shown to double copy into a more general static, spherically symmetric configuration known as the JNW solution [26], which includes both the metric and a dilaton field [18,27–29].

[4]Note that here S will always turn out to be time independent. This is related to the fact we are considering stationary KS spacetimes, for which ϕ is time-independent. Thus, $\rho^S_{(n)}$ is related to the charge density J^0 of the nth term of the expansion (1.5).

The article is outlined as follows. We begin in Sec. II by offering a Lagrangian perspective of the double-, single-, and zeroth-copy fields, together with their respective sources. Although not often discussed in the Weyl double-copy literature, the Lagrangian perspective provides a useful way to organize each field and the spacetimes on which they live. This will be crucial for our purposes, since we must distinguish a Maxwell source on the double-copy side from the Maxwell fields on the single-copy side. We proceed by describing the relationship between the Maxwell source on the double-copy side to the appropriate single-copy quantity in Sec. II A, before presenting new relations for the spinorial counterpart of the Ricci tensor in Sec. II B.

In the remainder of the article, we apply our general prescriptions to specific electrovacuum type-D spacetimes. Section III is devoted to studying the Kerr-Newman black hole, which was given a preliminary treatment in [39]; however, here we include an analysis of its associated Ricci spinor. In Sec. IV, we provide the first complete analysis of the charged C metric. Here we encounter a peculiar feature of the sourced Weyl double copy for spacetimes that are not (Riemann) asymptotically flat.[5] The single-copy theory associated with the black hole charge parameter exhibits a nontrivial divergence of the dual field-strength tensor, corresponding to a magnetic current vector. A covariant description of such a solution can be obtained by introducing a so-called two-potential electrodynamic theory [40], from which we provide explicit formulas for the two necessary vector potentials associated with the charged C metric. In Sec. V, after we review the most general Petrov type-D electrovac solution obtained by Plebański and Demiański [41], we apply the sourced Weyl double copy to the Kerr-Newman-Taub-NUT spacetime for the first time.

Finally, we conclude in Sec. VI, outlining future research directions. To keep this article streamlined yet self-contained, we relegate a review of the spinor formalism to the Appendix.

II. LAGRANGIAN PERSPECTIVE AND NEW SPINOR RELATIONS

The Kerr-Schild double copy is often explained at the level of the equations of motion, e.g., (1.2). It is illuminating to see how this translates to the actions characterizing the double-, single-, and zeroth-copy fields. This is xparticularly useful for our purposes as we must keep track of three different Maxwell fields. Beginning with the double-copy action (standard Einstein-Maxwell theory), we have a $U(1)$ gauge field minimally coupled to general relativity,

$$I_{dc} = \int d^4x \sqrt{-g} \left(\frac{1}{2} R - \frac{1}{4} F_s^2 \right), \quad (2.1)$$

where $F_s = dA_s$ is the field-strength tensor of the Maxwell source. The single- and zeroth-copy theories are described by

$$I_{sc} = \int d^4x \sqrt{-g^{(0)}} \left(-\frac{1}{4} F_{(1)}^2 - \frac{1}{4} F_{(2)}^2 + A_\mu^{(2)} J^\mu \right), \quad (2.2)$$

$$I_{zc} = \int d^4x \sqrt{-g^{(0)}} \left(-\frac{1}{2} (\nabla^{(0)} S_{(1)})^2 - \frac{1}{2} (\nabla^{(0)} S_{(2)})^2 + S_{(2)} \rho^S \right). \quad (2.3)$$

Here $F_{(1)}$ and $S_{(1)}$ are the single-copy field strength and zeroth-copy scalar that map to the "vacuum" term of the Weyl tensor, $F_{(2)}$ and $S_{(2)}$ are the single- and zeroth-copy fields that map to the $U(1)$ charge on the gravity side, produced by F_s. We have explicitly introduced an electric current density J^μ directly coupling to the single-copy gauge field $A_{(2)}^\mu$ mapped to the $U(1)$ gravity source, and, similarly, ρ^S is the gravity source for the zeroth-copy $S_{(2)}$. Further, $g_{\mu\nu}$ denotes the full spacetime metric while $g_{\mu\nu}^{(0)}$ is the appropriate "flat-space" limit of $g_{\mu\nu}$, where all parameters responsible for generating curvature are tuned to zero (we will see explicit examples of this in subsequent sections).

Varying the actions (2.1)–(2.3) with respect to the fields $\{g^{\mu\nu}, A_s^\mu, A_{(1)}^\mu, A_{(2)}^\mu, S_{(1)}, S_{(2)}\}$ yields the field equations

$$G_{\mu\nu} = F_{s,\mu}{}^\alpha F_{s,\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_s^2, \qquad \nabla_\nu F_s^{\mu\nu} = 0, \quad (2.4)$$

$$\nabla_\nu^{(0)} F_{(1)}^{\mu\nu} = 0, \qquad \nabla_\nu^{(0)} F_{(2)}^{\mu\nu} = J^\mu, \quad (2.5)$$

$$\Box^{(0)} S_{(1)} = 0, \qquad \Box^{(0)} S_{(2)} = \rho^S, \quad (2.6)$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$ and $\nabla_\mu^{(0)}$ is the covariant derivative with respect to $g_{\mu\nu}^{(0)}$, with $\Box^{(0)} \equiv g_{(0)}^{\mu\nu} \nabla_\mu^{(0)} \nabla_\nu^{(0)}$. In principle, the scalar fields $S_{(1)}$ and $S_{(2)}$ could both be complex; however, in what follows we will see for specific examples that $S_{(1)}$ is typically complex, while $S_{(2)}$ is typically real.[6]

The Kerr-Schild double copy [17] provides the prescription for the single-copy charge density and the zeroth-copy scalar charge as

---

[5] By Riemann asymptotically flat we mean that in Boyer-Lindquist coordinates, the Riemann curvature tensor vanishes in the limit of large radial coordinate. We will discuss this further in Sec. II B.

[6] For complex scalar fields, the action I_{sc} should be modified by, e.g., replacing the real quadratic terms with their modulus, and so forth, to ensure a real action.

$$R^\mu{}_0 = -\frac{1}{2} J^\mu, \qquad R^0{}_0 = 2\rho^S. \tag{2.7}$$

Further, the actions (2.5) and (2.6) with the sourced Weyl double copy (1.5) imply the Weyl spinor for Einstein-Maxwell theories is cast as the following sum [39]:

$$\Psi_{ABCD} = \frac{1}{S_{(1)}} f^{(1)}_{(AB} f^{(1)}_{CD)} + \frac{1}{S_{(2)}} f^{(2)}_{(AB} f^{(2)}_{CD)}. \tag{2.8}$$

The spinors Ψ_{ABCD}, $f^{(1)}_{AB}$, and $f^{(2)}_{AB}$ are related to their tensorial counterparts through contractions with the Infeld–van der Waerden symbols $\sigma^{\mu\nu}_{AB}$, constructed from Pauli matrices and spacetime vierbeins (see the Appendix). The Weyl tensor $W_{\mu\nu\alpha\beta}$ is related to the (completely symmetric) Weyl spinor Ψ_{ABCD} by

$$\Psi_{ABCD} = \frac{1}{4} W_{\mu\nu\alpha\beta} \sigma^{\mu\nu}_{AB} \sigma^{\alpha\beta}_{CD}. \tag{2.9}$$

Similarly, the source field-strength spinor is

$$f^s_{AB} = \frac{1}{2} F^s_{\mu\nu} \sigma^{\mu\nu}_{AB}, \tag{2.10}$$

while the single-copy field strengths are

$$f^{(n)}_{AB} = \frac{1}{2} F^{(n)}_{\mu\nu} \sigma^{(0),\mu\nu}_{AB}, \quad n = 1, 2. \tag{2.11}$$

We write $\sigma^{(0),\mu\nu}_{AB}$ to make explicit that the vierbeins required to construct (2.11) are those which build the appropriate flat limit of the full spacetime metric.

Due to the presence of $F^s_{\mu\nu}$, the solution necessarily has a nonzero Ricci tensor and associated Ricci spinor. The key result from [39] is that by writing the Weyl spinor as the sum of two terms as (2.8), then the field equations (2.5) and (2.6) are satisfied and the sources are given by (2.7). We therefore have three quantities which require a single- and zeroth-copy prescription, namely,

$$\Psi_{ABCD}, \qquad \Phi_{ABC'D'}, \qquad f^s_{AB}, \tag{2.12}$$

where $\Phi_{ABC'D'}$ is related to the Ricci tensor and will be defined shortly (see the Appendix for a detailed discussion). Interestingly, for all of the electrovacuum spacetimes considered in this article we find

$$\frac{1}{S_{(1)}} f^{(1)}_{AB} \propto \frac{1}{S_{(2)}} f^{(2)}_{AB}. \tag{2.13}$$

Here the proportionality symbol denotes equality up to a constant ratio depending on the dynamical parameters of the full spacetime. As noted, the sourced Weyl double copy (2.8) provides the prescription for Ψ_{ABCD}. We now provide copying prescriptions for the double-copy field strength and Ricci spinors, f^s_{AB} and $\Phi_{ABC'D'}$, respectively.

A. Prescription for the $U(1)$ source

The field equation for $F^s_{\mu\nu}$ (2.4) is that of a free Maxwell gauge field over the full metric $g_{\mu\nu}$, while the field equation for $F^{(1)}_{\mu\nu}$ in (2.5) is that of a free Maxwell field over $g^{(0)}_{\mu\nu}$. Since they both satisfy a vacuum equation, the form of their spinor field strengths are in fact equivalent, which can be seen as follows. Define the frame field strength associated with $F^{(1)}_{\mu\nu}$ as $F^{(1)}_{ab} = F^{(1)}_{\mu\nu} (e^{(0)})^\mu_a (e^{(0)})^\nu_b$. It is straightforward to show when $F^{(1)}_{\mu\nu}$ satisfies a vacuum equation over $g^{(0)}_{\mu\nu}$, then $F^{(1)}_{ab}$ satisfies the vacuum equations over $\eta_{ab} = (-1, 1, 1, 1)$. Explicitly, in terms of vierbeins and Infeld–van der Waerden symbols (A10),

$$f^{(1)}_{AB} = \frac{1}{2} F^{(1)}_{\mu\nu} (e^{(0)})^\mu_a (e^{(0)})^\nu_b \sigma^{[a}_{AC'} \bar{\sigma}^{b]C'}{}_B = \frac{1}{2} F^{(1)}_{ab} \sigma^{[a}_{AC'} \bar{\sigma}^{b]C'}{}_B, \tag{2.14}$$

where we used $(e^{(0)})^a_\mu (e^{(0)})^\mu_b = \delta^a_b$. For the source field strength, we have

$$f^s_{AB} = \frac{1}{2} F^s_{\mu\nu} e^\mu_a e^\nu_b \sigma^{[a}_{AC'} \bar{\sigma}^{b]C'}{}_B = \frac{1}{2} F^s_{ab} \sigma^{[a}_{AC'} \bar{\sigma}^{b]C'}{}_B, \tag{2.15}$$

where $F^s_{ab} = F^s_{\mu\nu} e^\mu_a e^\nu_b$ also satisfies the vacuum Maxwell equation over η_{ab},[7] and we used $e^a_\mu e^\mu_b = \delta^a_b$. The key point is that although $F^s_{\mu\nu}$ and $F^{(1)}_{\mu\nu}$ are constructed using different vierbeins, their frame field strengths F^s_{ab} and $F^{(1)}_{ab}$ both satisfy the vacuum Maxwell equations over η_{ab}. Consequently,

$$F^s_{ab} \propto F^{(1)}_{ab} \Leftrightarrow f^s_{AB} \propto f^{(1)}_{AB}, \tag{2.16}$$

which is clear from the rightmost equality in (2.14) and (2.15).

B. Prescription for the Ricci spinor

We describe the essential relations between the Ricci tensor $R_{\mu\nu}$ and the Ricci spinor $\Phi_{ABC'D'} = \Phi_{(AB)(C'D')}$ in the Appendix. For Einstein-Maxwell theory, the only nonzero contraction[8] from (A26) is the component $\Phi_{010'1'} \equiv \Phi_{11} = -\frac{1}{2} R_{\mu\nu} n^\mu l^\nu + 3\Pi$, where $\Pi = \frac{1}{12} R$. The Ricci spinor thus dramatically simplifies to

$$\Phi^{\text{EM}}_{ABC'D'} = 4\Phi_{11} o_{(A} l_{B)} \bar{o}_{(C'} \bar{l}_{D')}. \tag{2.17}$$

---

[7]To see this, note that $\nabla_a F^{ab} = e^\lambda_a (\nabla_\lambda F^{\mu\nu}) e^a_\mu e^b_\nu = \delta^\lambda_\mu (\nabla_\lambda F^{\mu\nu}) e^b_\nu = (\nabla_\mu F^{\mu\nu}) e^b_\nu = 0$, where we have used the no-torsion constraint $\nabla_\lambda e^a_\nu = 0$.

[8]More specifically, we work with the principle null tetrad that results in the Weyl scalar $\Psi_2 \neq 0$, while all others (A23) vanish. The same tetrad is used to compute $\Phi_{ABC'D'}$, which is such that only $\Phi_{11} \neq 0$. Choosing a different tetrad generically will result in additional Ricci scalars being nonzero.

Now, the fields $S_{(2)}$ and $f_{AB}^{(2)}$ are nonzero due to the presence of $F_{\mu\nu}^s$, which in turn results in $\Phi_{ABC'D'}$ being nonzero. Therefore we expect $\Phi_{ABC'D'}$ to be related to the $S_{(2)}$ and $f_{AB}^{(2)}$. The spinor $\Phi_{ABC'D'}$ clearly carries two conjugate indices, and itself is real for all spacetimes we consider. In general, $f_{AB}^{(2)}$ is complex; therefore, its conjugate must be involved in the copying prescription for $\Phi_{ABC'D'}$. In fact, the desired relationship is analogous to the prescription for the Weyl spinor,

$$\Phi_{ABC'D'} = \frac{1}{3}\frac{1}{S_{(2)}} f_{AB}^{(2)} \bar{f}_{C'D'}^{(2)}, \qquad (2.18)$$

where $\bar{f}_{A'B'}^{(2)} \propto \bar{o}_{(A'}\bar{\iota}_{B')}$ is the complex conjugate of $f_{AB}^{(2)}$.

As we stated previously, the scalar- and spinor field strength for the two terms in the sourced Weyl double copy (2.8) are proportional up to parameters characterizing the spacetime when we take the combination $\frac{1}{S}f_{AB}$, (2.13). Further, when the scalar fields $S_{(1)}$ and $S_{(2)}$ obey

$$S_{(2)} \propto |S_{(1)}|^2. \qquad (2.19)$$

Then, combined with (2.13) it follows

$$\frac{1}{S_{(2)}} f_{AB}^{(2)} \bar{f}_{C'D'}^{(2)} \propto \left(\frac{1}{S_{(1)}} f_{AB}^{(1)}\right)\left(\frac{S_{(2)}}{S_{(1)}} f_{CD}^{(1)}\right)^*$$
$$= \frac{S_{(2)}}{|S_{(1)}|^2} f_{AB}^{(1)} \bar{f}_{C'D'}^{(1)} \propto f_{AB}^{(1)} \bar{f}_{C'D'}^{(1)}, \qquad (2.20)$$

where the $*$ refers to complex conjugation. Subsequently,

$$\Phi_{ABC'D'} \propto f_{AB}^{(1)} \bar{f}_{C'D'}^{(1)}. \qquad (2.21)$$

Despite relating the Ricci spinor—which vanishes in pure vacuum—directly to the product of the single-copy field-strength spinors associated with the vacuum solution, this relationship in fact should not come as a surprise. In Einstein-Maxwell theory, the trace-reversed Einstein equations yield $R_{\mu\nu} = T_{\mu\nu}$. In the language of spinors, this is [42]

$$\Phi_{ABC'D'} \propto f_{AB}^s \bar{f}_{C'D'}^s. \qquad (2.22)$$

As we showed in Sec. II A, $f_{AB}^{(1)} \propto f_{AB}^s$; therefore (2.22) implies (2.21).

C. Comment on asymptotic flatness

The single-copy fields building the Ricci spinor (2.17) will be different depending on the behavior of the curvature of the double-copy spacetime at asymptotic spatial infinity. In particular and as we will show explicitly in subsequent sections, *all* spacetimes examined in this article, including

the most general type-D spacetime, naturally have Ricci spinor obeying (2.22) and therefore (2.21); however, not all electrovacuum spacetimes satisfy the relations (2.18) or (2.19).

To appreciate this point, we briefly comment on the notion of asymptotic flatness. A traditional and practical definition is that a spacetime is said to be asymptotically flat if at spatial infinity the spacetime metric takes a Minkowski form. For example, the Schwarzschild black hole is asymptotically flat while the Taub-NUT solution is not, the latter having a nonzero $g_{t\phi}$ component in the large radial-r limit in Boyer-Lindquist coordinates. With this notion, any metric which is asymptotically flat will necessarily have vanishing curvature at spatial infinity. However, the reverse need not be true: asymptotically, all components of the Riemann curvature of the Taub-NUT solution vanish. In what follows, with a slight abuse of terminology, we will work with a refined notion of asymptotically flat, dubbed "Riemann asymptotically flat,"[9] as being a spacetime whose curvature tensor vanishes at infinity. By this notion, the (charged) Taub-NUT solution is Riemann asymptotically flat (counter to the traditional sense of asymptotic flatness of the metric).

Intriguingly, when the Riemann tensor is nonvanishing at infinity, we will observe the gauge field strengths $f_{AB}^{(2)}$ do not satisfy the relations (2.18) or (2.19). In particular, this is the case for the charged C metric and general Plebanski and Demianski solution [41], for which the charged C metric is a special limit. However, the Kerr-Newman-Taub-NUT does obey these relations, and, subsequently, the Ricci spinor may be cast as (2.18).

Further, as we will see, for the two non-Riemann asymptotically flat spacetimes we analyze, the dual field-strength tensor $\tilde{F}_{\mu\nu}^{(2)}$ associated with $f_{AB}^{(2)}$ has a nontrivial divergence, signaling the presence of a magnetic four-current:

$$\nabla_\nu^{(0)} \tilde{F}_{(2)}^{\mu\nu} = J_m^\mu. \qquad (2.23)$$

This is equivalent to the field-strength tensor failing to satisfy the Bianchi identity $\nabla_{[\mu}^{(0)} F_{\rho\sigma]} \neq 0$. Traditional Maxwell theory cannot accommodate a nontrivial divergence of the dual field strength. However, a magnetic current can be accounted for if Maxwell theory is extended to include a second vector potential [40,43], which we will describe in Sec. IV.

Having established a general prescription relating the double-copy spinor fields to single- and zeroth copies, let us survey several electrovacuum type-D spacetimes. We will find the sum over products of gauge field strengths

---

[9]In other words, asymptotic flatness implies Riemann asymptotic flatness, but the converse need not be true.

(2.8) leads to a consistent Weyl double copy that incorporates external Maxwell sources.

III. KERR-NEWMAN BLACK HOLE

The Kerr-Newman solution describes a rotating black hole with rotation parameter a that carries an electric charge Q, and is an exact solution to the Einstein-Maxwell equations. The line element in Boyer-Lindquist coordinates is

$$
ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}((r^2 + a^2)d\phi - adt)^2
$$
$$
+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2,
$$
$$
\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2 + Q^2. \quad (3.1)
$$

The flat-space limit of the line element is Minkowski space in oblate spheroidal coordinates,

$$
ds^2_{(0)} = -dt^2 + \frac{\rho^2}{r^2 + a^2}dr^2 + \rho^2 d\theta^2 + (a^2 + r^2)\sin^2\theta d\phi^2,
$$
$$
(3.2)
$$

which is obtained by tuning the "dynamical parameters" $\{M \to 0, Q \to 0\}$, as this is the limit in which the Weyl curvature is vanishing.

The Weyl spinor associated with the Kerr-Newman solution is given by [44]

$$
\Psi_{ABCD} = \Psi^{(1)}_{ABCD} + \Psi^{(2)}_{ABCD}
$$
$$
= 6\left(-\frac{M}{(r + ia\cos\theta)^3}\right.
$$
$$
\left. + \frac{Q^2}{(r + ia\cos\theta)^3(r - ia\cos\theta)}\right)o_{(A}o_B\iota_C\iota_{D)}.
$$
$$
(3.3)
$$

The contribution proportional to parameter M is present in the "pure" Weyl double copy and appears in the first term in the sourced Weyl double-copy sum (2.8),

$$
\frac{1}{S_{(1)}}f^{(1)}_{(AB}f^{(1)}_{CD)} = -\frac{6M}{(r + ia\cos\theta)^3}o_{(A}o_B\iota_C\iota_{D)}. \quad (3.4)
$$

The single- and zeroth-copy interpretation of this quantity was given in [19], where the scalar and gauge fields were identified to be

$$
S_{(1)} = -\frac{q_1^2}{6M}\frac{1}{r + ia\cos\theta}, \qquad f^{(1)}_{AB} = \frac{q_1}{(r + ia\cos\theta)^2}o_{(A}\iota_{B)},
$$
$$
(3.5)
$$

both of which satisfy the vacuum equations over the flat-space metric (3.2). Here one has introduced the free parameter q_1 associated with the mass parameter M [17].[10]

A. Sourced Weyl double copy

The Q^2 contribution to the Weyl spinor (3.3) implies the second term in the sum (2.8) is

$$
\frac{1}{S_{(2)}}f^{(2)}_{(AB}f^{(2)}_{CD)} = \left(\frac{6Q^2}{(r + ia\cos\theta)^3(r - ia\cos\theta)}\right)o_{(A}o_B\iota_C\iota_{D)}.
$$
$$
(3.6)
$$

Following [39], we choose the zeroth-copy scalar field and single-copy spinor field strength corresponding to the Maxwell source of the full space to be

$$
S_{(2)} = \frac{q_2^2}{6Q^2}\frac{1}{(r + ia\cos\theta)(r - ia\cos\theta)},
$$
$$
f^{(2)}_{AB} = \frac{q_2}{(r + ia\cos\theta)^2(r - ia\cos\theta)}o_{(A}\iota_{B)}, \quad (3.7)
$$

where we have introduced a second free parameter q_2 associated with the black hole charge Q. With respect to the flat-space metric (3.2), it is straightforward to show the zeroth-copy scalar field satisfies

$$
\Box^{(0)}S_{(2)} = -\frac{2q_2^2}{Q^2}\frac{(r^2 + a^2) + a^2\sin\theta^2}{\rho^6}, \quad (3.8)
$$

while the field strength obeys

$$
\nabla^{(0)}_\mu F^{\nu\mu}_{(2)} = -q_2\begin{pmatrix} \frac{(r^2+a^2)+a^2\sin\theta^2}{2\rho^6} \\ 0 \\ 0 \\ \frac{a}{\rho^6} \end{pmatrix}, \quad \nabla^{(0)}_\mu \tilde{F}^{\nu\mu}_{(2)} = 0. \quad (3.9)
$$

The presence of the rotation parameter introduces a current density and therefore a magnetic field, as expected. Another new element not present in the pure Weyl double copy is the gravitational energy density ρ^{grav},

$$
\rho^{\text{grav}} = 8Q^2\frac{(r^2 + a^2) + a^2\sin\theta^2}{\rho^6}. \quad (3.10)
$$

Comparing to the scalar wave equations (3.8), the gravitational energy density is proportional to the single-copy electric charge density $\rho^e \equiv J^0$, and the scalar charge density ρ^S.

---

[10]Note the real scalar $S_{(1)} + \bar{S}_{(1)}$ is identified as the Kerr-Schild function ϕ in the metric decomposition (1.1).

Computing the (1,1) Ricci tensor, we find

$$R_0^\mu = -16Q^2 \begin{pmatrix} \frac{(r^2+a^2)+a^2\sin^2\theta}{2\rho^6} \\ 0 \\ 0 \\ \frac{a}{\rho^6} \end{pmatrix} \propto J_{(2)}^\mu, \quad (3.11)$$

in agreement with our expectations from the sourced Kerr-Schild double copy. By inspection of (3.5) and (3.7), we see

$$\frac{1}{S_{(1)}} f_{AB}^{(1)} \propto \frac{1}{S_{(2)}} f_{AB}^{(2)}, \quad \text{and} \quad S_{(2)} \propto |S_{(1)}|^2, \quad (3.12)$$

as claimed above for asymptotically flat spacetimes. We will return to this point momentarily.

As discussed in [39], the gauge field required to supplement the Kerr-Newman solution is

$$A_\mu = \frac{Q^2 r}{\rho^2} (1, \rho^2/(r^2+a^2), 0, a\sin^2\theta), \quad (3.13)$$

which we note is (gauge) equivalent to the single-copy gauge field associated with the Kerr solution, i.e., the solution associated with $f_{AB}^{(1)}$. More precisely, when the Kerr-Newman black hole is expressed in Kerr-Schild form (1.1), one may decompose the Kerr-Schild scalar ϕ as

$$\phi = \phi^K + \phi^N, \qquad \phi^K = \frac{2Mr}{\rho^2}, \qquad \phi^N = -\frac{Q^2}{\rho^2}. \quad (3.14)$$

In this case, the single-copy gauge field $A_\mu \equiv \phi k_\mu$, for null vector $k_\mu = (1, \rho^2/(r^2+a^2), 0, a\sin^2\theta)$, takes a similar decomposition, $A_\mu = A_\mu^K + A_\mu^N$. The gauge field A_μ^K matches the one found in [19] (up to a parameter rescaling), and whose corresponding gauge field strength $F_{\mu\nu}^{(1)}$ is divergenceless. Meanwhile, the gauge field A_μ^N produces a field strength $F_{\mu\nu}^{(2)}$ whose divergence is proportional to (3.11). This demonstrates a consistency between the Kerr-Schild and sourced Weyl double copy.

B. Ricci spinor

We now analyze the Ricci spinor for the Kerr-Newman solution and provide its double-copy analysis. The only nonzero component is $\Phi_{010'1'}$ such that [45]

$$\Phi_{ABC'D'} = \frac{2Q^2}{(r^2 + a^2\cos^2\theta)^2} o_{(A}\iota_{B)}\bar{o}_{(C'}\bar{\iota}_{D')}. \quad (3.15)$$

Comparing to the single-copy field-strength spinor $f_{AB}^{(1)}$ (3.5), we see the Ricci spinor is proportional to the complex square of $f_{AB}^{(1)}$,

$$f_{AB}^{(1)} \bar{f}_{C'D'}^{(1)} = \frac{q_1^2}{(r^2 + a^2\cos^2\theta^2)^2} o_{(A}\iota_{B)}\bar{o}_{(C'}\bar{\iota}_{D')} = \frac{q_1^2}{2Q^2}\Phi_{ABC'D'}. \quad (3.16)$$

Alternately, via the zeroth- and single-copy fields $S^{(2)}$ and $f_{AB}^{(2)}$ (3.7), we observe

$$\frac{1}{S_{(2)}} f_{AB}^{(2)} \bar{f}_{C'D'}^{(2)} = \frac{6Q^2}{(r^2 + a^2\cos^2\theta)^2} o_{(A}\iota_{B)}\bar{o}_{(C'}\bar{\iota}_{D')} = 3\Phi_{ABC'D'}. \quad (3.17)$$

Hence, the Ricci spinor for the Kerr-Newman black hole may be characterized by the product of a single-copy field-strength spinor and its complex conjugate, reminiscent of the double-copy relation of the Weyl spinor. The fact the Ricci spinor can be cast as a product of $f_{AB}^{(2)}$ is the relation which is only enjoyed by Riemann asymptotically flat spacetimes, as we now explore.[11]

IV. CHARGED C METRIC

The charged C metric will serve as a simple example of the rich new properties of non-Riemann asymptotically flat, nonvacuum spacetimes in the framework of the Weyl double copy. In spherical-like coordinates, its metric takes the form [45]

$$ds^2 = \frac{1}{\Omega^2}\left[-\frac{A}{r^2}dt^2 + \frac{r^2}{A}dr^2 + \frac{r^2}{B}d\theta^2 + Br^2\sin^2\theta d\phi^2\right], \quad (4.1)$$

with

$$\begin{aligned}\Omega &= 1 - \alpha r\cos\theta, \\ A &= (Q^2 - 2Mr + r^2)(1 - \alpha^2 r^2), \\ B &= 1 - 2\alpha M\cos\theta + Q^2\alpha^2\cos^2\theta. \end{aligned} \quad (4.2)$$

This solution describes a (charged Q) black hole of mass M that is uniformly accelerating with acceleration parameter α, generated by a cosmic string pulling it outward. Alternatively, the solution represents a pair of charged, uniformly accelerating black holes moving in opposite directions, either due to a connecting cosmic string between pushing them apart or as two semi-infinite cosmic strings pulling them away from each other (cf. [46]). In the $\alpha \to 0$ limit, the solution describes a single Reissner-Nordström black hole. Importantly, the C metric is not asymptotically flat, in the sense of the metric not being asymptotically

---

[11]The double-copy relations of the Reissner-Nordström black hole follow from the above analysis of the Kerr-Newman black hole in the $a \to 0$ limit, or equivalently in the limit that $\alpha \to 0$ of the charged C metric of Sec. IV.

Minkowskian and, correspondingly, has nonvanishing Riemann curvature at infinity.

With respect to these coordinates, the Weyl spinor Ψ_{ABCD} naturally decomposes into a term proportional to M and Q^2:

$$
\begin{aligned}
\Psi_{ABCD} &= \Psi^{(1)}_{ABCD} + \Psi^{(2)}_{ABCD} \\
&= -6M \frac{(1-\alpha r \cos\theta)^3}{r^3} o_{(A} o_B l_C l_{D)} \\
&\quad + 6Q^2 \frac{(1+\alpha r \cos\theta)(1-\alpha r \cos\theta)^3}{r^4} o_{(A} o_B l_C l_{D)},
\end{aligned}
\tag{4.3}
$$

while the Ricci spinor and Ricci tensor are, respectively,

$$
\Phi_{ABC'D'} = 2Q^2 \frac{(1-\alpha r \cos\theta)^4}{r^4} o_{(A} l_{B)} \bar{o}_{(C'} \bar{l}_{D')},
\tag{4.4}
$$

$$
R^\mu_\nu = Q^2 \frac{(1-\alpha r \cos\theta)^4}{r^4} \operatorname{diag}(-1,-1,1,1).
\tag{4.5}
$$

The pure Weyl double copy of the neutral C metric, with Weyl spinor $\Psi^{(1)}_{ABCD}$, was examined in [19], where it was recognized that the gauge potential A_μ associated with the single-copy field strength $F^{(1)}_{\mu\nu}$ corresponds to the Liénard-Wiechert potential describing a pair of causally disconnected charges, uniformly accelerating in opposite directions with acceleration α. The zeroth- and single-copy fields $S_{(1)}$ and $f^{(1)}_{AB}$ associated with $\Psi^{(1)}_{ABCD}$ in (4.3) are

$$
S_{(1)} = \frac{q_1^2}{M} \frac{1-\alpha r \cos\theta}{r}, \qquad f^{(1)}_{AB} = q_1 \frac{(1-\alpha r \cos\theta)^2}{r^2} o_{(A} l_{B)},
\tag{4.6}
$$

where we have introduced the real, free parameter q_1, understood as the charge of the particle in the single-copy picture. These fields construct the product $\Psi^{(1)}_{ABCD} = \frac{1}{S_{(1)}} f^{(1)}_{(AB} f^{(1)}_{CD)}$ and live over the flat background obtained by sending $\{M, Q\} \to \{0, 0\}$ in (4.1),

$$
ds^2_{(0)} = \frac{1}{\Omega^2} \left[-(1-\alpha^2 r^2) dt^2 + \frac{dr^2}{1-\alpha^2 r^2} + r^2 d\Omega^2 \right],
\tag{4.7}
$$

which is Minkowski space in accelerated coordinates. Additionally, the zeroth- and single-copy field-strength tensor associated with $f^{(1)}_{AB}$ satisfy the vacuum equations

$$
\Box^{(0)} S_{(1)} = 0, \qquad \nabla^{(0)}_\nu F^{\mu\nu}_{(1)} = \nabla^{(0)}_\nu \tilde{F}^{\mu\nu}_{(1)} = 0.
\tag{4.8}
$$

From here, it is straightforward to show the gauge potential corresponding to $F^{(1)}_{\mu\nu}$ is

$$
A_{(1)} = \frac{q_1}{r} dt,
\tag{4.9}
$$

which (functionally) agrees with the potential required to solve the Einstein-Maxwell equations on the double-copy side, $A^s = \frac{Q}{r} dt$. To express the gauge potential in a more standard Liénard-Wiechert form, one may transform the accelerated coordinate system to Minkowski space in Cartesian coordinates [19].

A. Sourced Weyl double copy

Let us now determine the fields required to construct the second term $\Psi^{(2)}_{ABCD}$ of the complete Weyl spinor (4.3) such that it may be cast as

$$
\Psi^{(2)}_{ABCD} = \frac{1}{S_{(2)}} f^{(2)}_{(AB} f^{(2)}_{CD)}.
\tag{4.10}
$$

The scalar field is required to satisfy $\Box^{(0)} S_{(2)} \propto R^t_t \sim (1-\alpha r \cos\theta)^4 / r^4$, which has the solution

$$
S_{(2)} = \frac{q_2^2}{6Q^2} \frac{(1+\alpha r \cos\theta)(1-\alpha r \cos\theta)}{r^2},
\tag{4.11}
$$

where we have introduced another free parameter q_2. Thus, the single-copy field strength is

$$
f^{(2)}_{AB} = q_2 \frac{(1+\alpha r \cos\theta)(1-\alpha r \cos\theta)^2}{r^3} o_{(A} l_{B)}.
\tag{4.12}
$$

Correspondingly, the tensor field strength associated with $f^{(2)}_{AB}$ is

$$
F^{(2)} = -q_2 \frac{1+\alpha r \cos\theta}{2r^3} dt \wedge dr,
\tag{4.13}
$$

whose divergence corresponds to the expectation from the Kerr-Schild double copy,

$$
\nabla^{(0)}_\nu F^{\mu\nu}_{(2)} = -\frac{q_2}{2} \frac{(1-\alpha r \cos\theta)^4}{r^4} \delta^\mu_t \equiv J^\mu_e \propto R^\mu{}_t.
\tag{4.14}
$$

A new feature of the C metric is that we also have a nontrivial divergence of the dual tensor $\tilde{F}^{\mu\nu}_{(2)}$,

$$
\nabla^{(0)}_\nu \tilde{F}^{\mu\nu}_{(2)} = -\frac{q_2 \alpha}{2} \frac{(1-\alpha r \cos\theta)^4}{r^4} \delta^t_\phi \equiv J^\mu_m.
\tag{4.15}
$$

A nonzero divergence of the dual tensor typically implies the presence of a magnetic monopole with its own magnetic

current, J_m^μ. We emphasize, this magnetic current is not present in the neutral C metric. To better interpret the corresponding single-copy field strength $F_{\mu\nu}^{(2)}$, let us briefly digress to discuss how two-potential electrodynamics can account for the new magnetic current.

B. Two-potential electrodynamics

In the presence of magnetic charges, Maxwell's equations may be modified while maintaining the traditional definition of the Maxwell potential A_μ. Doing so, however, comes at the cost of placing magnetic charges at the end of a string on which A_μ develops a singularity [43,47]. Alternatively, one may avoid introducing these singular and nonlocal Dirac strings by way of two-potential electrodynamics, where one introduces a pseudo-4-vector potential, C_μ, in addition to the standard potential A_μ [40] (see also [48–50]). While this alternative formulation of classical electrodynamics is not entirely necessary for understanding the double copy of type-D spacetimes, it is nonetheless useful, particularly for the single-copy interpretation of the charged C metric.

The role of the potential C_μ is to modify the Maxwell equations such that they are symmetric in electric and magnetic fields in the presence of magnetic charges and currents. Doing so, however, requires an extension of the field-strength tensor $F_{\mu\nu}$ to incorporate both four-vector potentials A_μ and C_μ,

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + \varepsilon_{\mu\nu}{}^{\rho\sigma}\partial_\rho C_\sigma. \qquad (4.16)$$

This field strength and its dual satisfy

$$\nabla_\nu F^{\mu\nu} = J_e^\mu, \qquad \nabla_\nu \tilde{F}^{\mu\nu} = J_m^\mu, \qquad (4.17)$$

as desired, where the new potential C_μ is solely responsible for the magnetic current J_m^μ. The components of (4.17) are

$$\begin{aligned}\nabla\cdot\vec{E} = \rho_e, & \qquad \nabla\times\vec{B} = \vec{J}_e + \partial_t\vec{E}, \\ \nabla\cdot\vec{B} = \rho_m, & \qquad -\nabla\times\vec{E} = \vec{J}_m + \partial_t\vec{B}.\end{aligned} \qquad (4.18)$$

Let us now apply this two-potential formalism to the charged C metric, where we can explicitly construct the gauge potentials A_μ and C_μ required to produce the field equations (4.14) and (4.15). The full field-strength tensor $F_{\mu\nu}^{(2)}$ is

$$F_{\mu\nu}^{(2)} = \partial_{[\mu}A_{\nu]}^{(2)} + \sqrt{-g^{(0)}}\, g_{(0)}^{\rho\lambda} g_{(0)}^{\sigma\gamma}\epsilon_{\mu\nu\lambda\gamma}\partial_\rho C_\sigma, \qquad (4.19)$$

with $g_{\mu\nu}^{(0)}$ being the accelerated Minkowski metric (4.7). Using that $F_{\mu\nu}^{(2)}$ takes the form in (4.13), we find that the two vector potentials are

$$A_\mu^{(2)} = -\frac{q_2}{4r^2}\delta_\mu^t, \qquad C_\mu = -\frac{q_2\alpha}{4}\sin^2\theta\,\delta_\mu^\phi. \qquad (4.20)$$

The $A_\mu^{(2)}$ leads to a correction to the Liénard-Wiechert potential (4.9),[12] while the C_μ accounts for the magnetic monopole-like behavior. Moreover, we obtain the Reissner-Nordström solution from the charged C metric in the limit that $\alpha \to 0$. Clearly, C_μ in (4.20) vanishes in that limit, and we recover traditional Maxwell theory.

C. Ricci spinor

From the zeroth- and single-copy fields (4.3), (4.11), and (4.12), it is clear

$$\frac{1}{S_{(1)}}f_{AB}^{(1)} \propto \frac{1}{S_{(2)}}f_{AB}^{(2)}, \qquad S_{(2)} \not\propto |S_1|^2. \qquad (4.21)$$

As expected, the Ricci spinor (4.4) is proportional to the modulus of the single-copy spinors $f_{AB}^{(1)}$,

$$f_{AB}^{(1)}\bar{f}_{C'D'}^{(1)} = q_1^2\frac{(1-\alpha r\cos\theta)^4}{r^4}o_{(A}\iota_{B)}\bar{o}_{(C'}\bar{\iota}_{D')} = \frac{q_1^2}{2Q^2}\Phi_{ABC'D'}. \qquad (4.22)$$

However,

$$\frac{1}{S_{(2)}}f_{AB}^{(2)}\bar{f}_{C'D'}^{(2)} = 6Q^2\frac{(1+\alpha r\cos\theta)(1-\alpha r\cos\theta)^3}{r^4}$$
$$\times o_{(A}\iota_{B)}\bar{o}_{(C'}\bar{\iota}_{D')} \not\propto \Phi_{ABC'D'}, \qquad (4.23)$$

as alluded to above, although the two expressions agree in the $\alpha \to 0$ limit.

V. GENERAL ELECTROVACUUM TYPE-D SPACETIMES

The most general Petrov type-D solution of the Einstein-Maxwell field equations with a nonzero cosmological constant Λ is described by the Plebanski-Demianski (PD) family of metrics [41]. A useful form of the line element[13] is

---

[12]The Liénard-Wiechert potential in the accelerated coordinates falls off as $1/r$, while $A^{(2)}$ falls off as $1/r^2$. Transforming out of the accelerated coordinates into standard Minkowski space in spherical coordinates (T, R, Θ, Φ) centered on the point charge, it follows that the sum of $A^{(1)}$ and $A^{(2)}$ can be written as

$$A^{(1)} + A^{(2)} = \left[1 + \frac{q_2}{q_1}\left(\alpha + \frac{2\cos^2\Theta}{\alpha R^2} + O(R^{-4})\right)\right]A^{(LW)},$$

where $A^{(LW)}$ is the Liénard-Wiechert potential. Note that the $\alpha \to 0$ is not well defined because the diffeomorphism between the accelerated coordinates to standard Minkowski space is singular in that limit [19].

[13]We use the form of the metric presented in [51] with a "mostly plus" signature.

$$ds^2 = \frac{1}{(1-pq)^2} \left[\frac{p^2+q^2}{P(p)} dp^2 + \frac{p^2+q^2}{Q(q)} dq^2 \right.$$
$$\left. + \frac{P(p)}{p^2+q^2} (d\tau + q^2 d\sigma)^2 - \frac{Q(q)}{p^2+q^2} (d\tau - p^2 d\sigma)^2 \right],$$

(5.1)

with

$$Q(q) = k + e^2 + g^2 - 2mq + \epsilon q^2 - 2nq^3 - (k + \Lambda/3)q^4,$$
$$P(p) = k + 2np - \epsilon p^2 + 2mp^3 - (e^2 + g^2 + k + \Lambda/3)p^4.$$

(5.2)

The real parameters $\{m, n, e, g, \epsilon, \Lambda\}$ are taken to be arbitrary; however, limiting cases of the PD metric, e.g., the Kerr-Newman black hole, provide a physical interpretation for these parameters; e.g., m and n correspond to mass and NUT parameters, respectively, and e and g correspond to electric and magnetic monopole charges. While the classical Kerr-Schild double copy has been extended to describe maximally symmetric spacetimes [32], we will set $\Lambda = 0$. For $e = g = \Lambda = 0$ and $k = \gamma$ one recovers the PD metric analyzed in the pure Weyl double copy [19].

Notably, the PD metric cannot generally be placed in Kerr-Schild form (1.1); however, it may be put into double Kerr-Schild form via the complex diffeomorphism [41]

$$\tau = u + \int \frac{q^2 dq}{Q(q)} + i \int \frac{p^2 dp}{P(p)},$$
$$\sigma = v - \int \frac{dq}{Q(q)} + i \int \frac{dp}{P(p)},$$

(5.3)

such that the metric (5.1) takes the form

$$ds^2 = \frac{1}{(1-pq)^2} \left[2(iKdp - Ldq) + \frac{P(p)}{p^2+q^2} K^2 \right.$$
$$\left. - \frac{Q(q)}{p^2+q^2} L^2 \right],$$

(5.4)

where one has introduced null, geodesic, and mutually orthogonal covectors K and L:

$$K = du + q^2 dv, \qquad L = du - p^2 dv. \qquad (5.5)$$

The flat-space limit is identified by simultaneously setting all dynamical, curvature-producing parameters to zero, namely, $m = n = g = e = 0$. The remaining parameters, k and ϵ, are said to be *kinematical* as they do not generate curvature. Following [19], we include the kinematical parameters in the definition of the flat-space metric $ds_{(0)}^2$ which is identified as

$$ds_{(0)}^2 = \frac{1}{(1-pq)^2} \left[2(iKdp - Ldq) + \frac{k(1-p^4) - \epsilon p^2}{(p^2+q^2)} K^2 \right.$$
$$\left. - \frac{k(1-q^4) + \epsilon q^2}{(p^2+q^2)} L^2 \right].$$

(5.6)

From here it is easy to see the PD metric in these coordinates is of double Kerr-Schild form with Kerr-Schild functions ϕ_K and ϕ_L:

$$\phi_K = \frac{2np + 2mp^3 - (e^2+g^2)p^4}{(p^2+q^2)(1-pq)^2},$$
$$\phi_L = \frac{2mq + 2nq^3 - e^2 - g^2}{(p^2+q^2)(1-pq)^2},$$

(5.7)

and hence

$$ds^2 = ds_{(0)}^2 + \phi_K K^2 + \phi_L L^2. \qquad (5.8)$$

Keeping the kinematical parameters in the flat-space portion of the double Kerr-Schild decomposition is a choice, and thus far, ambiguous. We will return to this ambiguity in a moment and see how it affects the source contributions to the Weyl double copy.

While the Ricci tensor $R^\mu{}_\nu$ is linear in Kerr-Schild form, this property is generally not the case for metrics in double Kerr-Schild form. Nonetheless, as shown in [19], the PD metric (5.4) has a linear Ricci tensor, given by

$$R^\mu{}_\nu = (e^2 + g^2) \begin{pmatrix} \frac{(pq-1)^4(p^2-q^2)}{(p^2+q^2)^3} & \frac{2p^2q^2(pq-1)^4}{(p^2+q^2)^3} & 0 & 0 \\ \frac{2(pq-1)^4}{(p^2+q^2)^3} & -\frac{(pq-1)^4(p^2-q^2)}{(p^2+q^2)^3} & 0 & 0 \\ 0 & 0 & \frac{(pq-1)^4}{(p^2+q^2)^2} & 0 \\ 0 & 0 & 0 & -\frac{(pq-1)^4}{(p^2+q^2)^2} \end{pmatrix},$$

(5.9)

from which we obtain the energy density ρ^{grav}, i.e., the negative of the first entry of (5.9),

$$\rho^{\text{grav}} = \frac{(e^2 + g^2)(pq-1)^4(q^2-p^2)}{(p^2+q^2)^3}. \quad (5.10)$$

With the Ricci tensor we can build its spinorial counterpart [51],

$$\Phi_{ABC'D'} = 2(e^2+g^2)\frac{(1-pq)^4}{(q^2+p^2)^2}o_{(A}l_{B)}\bar{o}_{(C'}\bar{l}_{D')}. \quad (5.11)$$

Lastly, the Weyl spinor corresponding to the metric (5.4) is [51]

$$\Psi_{ABCD} = \Psi^{(1)}_{ABCD} + \Psi^{(2)}_{ABCD}$$
$$= -6(m+in)\left(\frac{1-pq}{q+ip}\right)^3 o_{(A}o_B l_C l_{D)}$$
$$+ 6(e^2+g^2)\left(\frac{1-pq}{q+ip}\right)^3\frac{1+pq}{q-ip}o_{(A}o_B l_C l_{D)}. \quad (5.12)$$

The first term $\Psi^{(1)}_{ABCD} = \frac{1}{S_{(1)}}f^{(1)}_{(AB}f^{(1)}_{CD)}$ was previously determined in [19], explicitly given by

$$S_{(1)} = \frac{i}{6}\frac{(\tilde{m}+i\tilde{n})^2(1-pq)}{(m+in)(p-iq)},$$
$$f^{(1)}_{AB} = \frac{(\tilde{m}+i\tilde{n})(1-pq)^2}{(p-iq)^2}o_{(A}l_{B)}, \quad (5.13)$$

and satisfy the vacuum equations over the flat background (5.6). The constants $\tilde{m}$ and $\tilde{n}$ are introduced in the spirit

of having gauge parameters separate from the gravity parameters.

A. Sourced Weyl double copy

We analyzed the second term (5.12) in [39]:

$$S_{(2)} = \frac{(\tilde{e}^2+\tilde{g}^2)^2}{6(e^2+g^2)}\left(\frac{1-pq}{q+ip}\right)\frac{1+pq}{q-ip},$$
$$f^{(2)}_{AB} = (\tilde{e}^2+\tilde{g}^2)\left(\frac{1-pq}{q+ip}\right)^2\frac{1+pq}{q-ip}o_{(A}l_{B)}, \quad (5.14)$$

introducing gauge parameters $\tilde{e}$ and $\tilde{g}$. Notice the wave equation of the zeroth-copy scalar field obeys

$$\Box^{(0)}S_{(2)} = \frac{(\tilde{e}^2+\tilde{g}^2)^2}{6(e^2+g^2)}\left(-2\epsilon\frac{(1-pq)^4(q^2-p^2)}{(p^2+q^2)^3}\right.$$
$$\left. + 4k\frac{(1-pq)^4(1+p^2q^2)}{(p^2+q^2)^3}\right)$$
$$\propto \epsilon\rho^{\text{grav}} + k\Delta(p,q). \quad (5.15)$$

This shows $\Box^{(0)}S_{(2)} \propto \rho^{\text{grav}}$ is met *only* if the kinematical parameter vanishes, $k \to 0$. Due to the ambiguity in choosing the flat background, this is no issue. Indeed, the vacuum equations for $S_{(1)}$ and $f^{(1)}_{AB}$ are immune to the choice of k. This observation suggests the sourced Weyl double copy may specify an appropriate flat background where the single- and zeroth-copy fields live.

The Maxwell tensor associated with the spinor field strength $f^{(2)}_{AB}$ is

$$F^{(2)}_{\mu\nu} = (\tilde{e}^2+\tilde{g}^2)\begin{pmatrix} 0 & 0 & \frac{p(pq+1)}{2(p^2+q^2)^2} & \frac{q(pq+1)}{2(p^2+q^2)^2} \\ 0 & 0 & \frac{pq^2(pq+1)}{2(p^2+q^2)^2} & -\frac{p^2q(pq+1)}{2(p^2+q^2)^2} \\ -\frac{p(pq+1)}{2(p^2+q^2)^2} & -\frac{pq^2(pq+1)}{2(p^2+q^2)^2} & 0 & 0 \\ -\frac{q(pq+1)}{2(p^2+q^2)^2} & \frac{p^2q(pq+1)}{2(p^2+q^2)^2} & 0 & 0 \end{pmatrix}, \quad (5.16)$$

and obeys

$$\nabla^{(0)}_\mu F^{\nu\mu}_{(2)} = (\tilde{e}^2+\tilde{g}^2)\begin{pmatrix} -\frac{(q^2-p^2)(pq-1)^4}{2(p^2+q^2)^3} \\ \frac{(pq-1)^4}{(p^2+q^2)^3} \\ 0 \\ 0 \end{pmatrix}. \quad (5.17)$$

Thus, (5.17) shows the single-copy charge density is proportional to the gravitational energy density, and the current density is proportional to an angular-momentum term, i.e., $R^\mu_0 \propto J^\mu_{(2)}$, as anticipated from the Kerr-Schild

relation (1.2). We also find that the Jacobi identity for (5.16) fails, $\nabla^{(0)}_{[\mu}F^{(2)}_{\lambda\gamma]} \neq 0$. Or, in terms of the divergence of $\tilde{F}^{(2)}_{\mu\nu}$,

$$\nabla^{(0)}_\nu \tilde{F}^{\mu\nu}_{(2)} = (\tilde{e}^2+\tilde{g}^2)\begin{pmatrix} -\frac{p^2q^2(pq-1)^4}{(p^2+q^2)^3} \\ \frac{(p^2-q^2)(pq-1)^4}{2(p^2+q^2)^3} \\ 0 \\ 0 \end{pmatrix}. \quad (5.18)$$

A nonzero divergence of $\tilde{F}_{(2)}^{\mu\nu}$ suggests the presence of a magnetic charge and magnetic current. This is consistent with what we found for the charged C metric, which is a limiting case of the general PD metric. That is, since the PD metric is not asymptotically flat, the single-copy gauge fields are sourced by both an electric and magnetic current density.[14]

B. Ricci spinor

Using single-copy spinor field strength $f_{AB}^{(1)}$ (5.13), we have

$$f_{AB}^{(1)} \bar{f}_{C'D'}^{(1)} = \frac{(\tilde{m}^2 + \tilde{n}^2)}{2(e^2 + g^2)} \Phi_{ABC'D'}, \qquad (5.19)$$

which is clear by inspection to (5.11). Calling $(\tilde{m}^2 + \tilde{n}^2) \equiv q_1^2$ and $(e^2 + g^2) \equiv Q^2$, we uncover the same relation as in (3.16) and (4.22). Meanwhile, the nonvacuum single-copy fields (5.14) are not directly proportional to the Ricci spinor. This is consistent with the our Ricci spinor analysis of the charged C metric (4.23), and is a by-product of the fact $S_{(2)} \not\propto |S_{(1)}|^2$ in the general PD metric.

C. Kerr-Newman-Taub–NUT

As our final example, let us consider a the Kerr-Newman-Taub–NUT solution, a special limiting case of the general PD metric. One can acquire the line element of this solution via a combination of coordinate and parameter rescalings of the general PD metric (5.4). Specifically, perform the coordinate rescalings

$$u \to \ell u, \quad v \to \ell^3 v, \quad p \to \ell^{-1} p, \quad q \to \ell^{-1} q, \qquad (5.20)$$

along with the parameter rescalings

$$m \to \ell^{-3} m, \quad n \to \ell^{-3} n, \quad e \to \ell^{-2} e, \quad g \to \ell^{-2} g,$$
$$\epsilon \to \ell^{-2} \epsilon, \quad k \to \ell^{-4} k. \qquad (5.21)$$

Then, taking the $\ell \to \infty$ limit yields the line element

$$ds^2 = 2(iK dp - L dq) + k(2 du + dv(q^2 - p^2)) dv$$
$$- \epsilon (du^2 + p^2 q^2 dv^2) + \phi_K K^2 + \phi_L L^2, \qquad (5.22)$$

where the first three terms correspond to the flat-space line element and now the Kerr-Schild functions (5.7) ϕ_K and ϕ_L are

$$\phi_K = \frac{2np}{p^2 + q^2}, \qquad \phi_L = \frac{2mq - (e^2 + g^2)}{p^2 + q^2}. \qquad (5.23)$$

Technically, the line element (5.22) is the Kerr-Newman-Taub–NUT solution, including rotation, electric, magnetic, and NUT charges (when $e = g = 0$ we recover the form of the neutral Kerr-Taub–NUT solution considered in [19]). Additionally, while this metric does not approach Minkowski space in the asymptotic limit, it nonetheless has vanishing Riemann curvature at spatial infinity.[15]

Analogous to [19], the zeroth- and single-copy fields for the Kerr-Newman-Taub–NUT solution readily follow from applying the rescalings (5.20) and (5.21), together with rescaling $(\tilde{e}, \tilde{g}) \to \ell^{-2}(\tilde{e}, \tilde{g})$, to the gauge fields of the general PD solution (5.13) and (5.14). Precisely, upon rescaling $S_{(1)}$ and $f_{AB}^{(1)}$, take the $\mathcal{O}(\ell^{-2})$ and $\mathcal{O}(\ell^{-1})$ coefficients, respectively, in a large ℓ limit,

$$S_{(1)} = \frac{i}{6} \frac{(\tilde{m} + i\tilde{n})^2}{(m + in)(p - iq)}, \qquad f_{AB}^{(1)} = \frac{\tilde{m} + i\tilde{n}}{(p - iq)^2} o_{(A} \iota_{B)}. \qquad (5.24)$$

Likewise, we find

$$S_{(2)} = \frac{(\tilde{e}^2 + \tilde{g}^2)^2}{6(e^2 + g^2)(p^2 + q^2)},$$
$$f_{AB}^{(2)} = -i \frac{(\tilde{e}^2 + \tilde{g}^2)}{(p - iq)^2 (p + iq)} o_{(A} \iota_{B)}. \qquad (5.25)$$

It is worth pointing out that for real parameters $e, g, \tilde{e}$ and $\tilde{g}$, the function $S_{(2)}$ is a real function, unlike the zeroth copy $S_{(1)}$. Additionally,

$$\frac{1}{S_{(1)}} f_{AB}^{(1)} \propto \frac{1}{S_{(2)}} f_{AB}^{(2)}, \qquad S_{(2)} \propto |S_{(1)}|^2, \qquad (5.26)$$

unlike the relation (4.21) for the charged C metric and general PD solution.

In the same vein, upon the rescalings (5.20) and (5.21) and taking the $\ell \to \infty$ limit of (5.11) and (5.12), the Ricci spinor and Weyl spinor are, respectively,

$$\Phi_{ABC'D'} = 2 \frac{(e^2 + g^2)}{(p^2 + q^2)^2} o_{(A} \iota_{B)} \bar{o}_{(C'} \bar{\iota}_{D')}, \qquad (5.27)$$

$$\Psi_{ABCD} = -6i \frac{(m + in)}{(p - iq)^3} o_{(A} o_B \iota_C \iota_{D)}$$
$$- \frac{6(e^2 + g^2)}{(p - iq)^3 (p + iq)} o_{(A} o_B \iota_C \iota_{D)}. \qquad (5.28)$$

---

[14]In [39] we imprecisely remarked the appearance of the magnetic monopole arises due to the moving NUT charge. Now it is clear the reason for the magnetic monopole-like behavior is primarily due to the acceleration.

[15]This is mostly easily verified by switching to Boyer-Lindquist form coordinates; see, e.g., [51].

And, by virtue of (5.26), we see

$$\frac{1}{S_{(2)}} f_{AB}^{(2)} \bar{f}_{C'D'}^{(2)} = \frac{6(e^2 + g^2)}{(p^2 + q^2)^2} o_{(A} \iota_{B)} \bar{o}_{(C'} \bar{\iota}_{D')} = 3\Phi_{ABC'D'},$$

(5.29)

which is the same form as the Kerr-Newman solution (3.17) and is proportional to $f_{AB}^{(1)} \bar{f}_{C'D'}^{(1)}$.

VI. DISCUSSION

In this work, we explored the sourced Weyl double +copy [39] for exact solutions to Einstein-Maxwell theory. In doing so we provided the first double-copy analysis of the charged C metric and Kerr-Newman-Taub–NUT black hole, and developed a double prescription for the Ricci spinor for the type-D solutions considered. Paramount in our interpretation of the zeroth- and single-copy fields was the role played by Riemann asymptotic flatness, i.e., whether the Riemann curvature tensor was nonvanishing at spatial infinity. Spacetimes whose Riemann curvature tensor was nonvanishing asymptotically, namely, the charged C metric and general Plebanski-Demianski solution, had a corresponding single-copy dual field-strength tensor with nonvanishing divergence. This indicates the existence of a nontrivial magnetic current. In the case of the charged C metric, we used a two-potential formulation of electrodynamics to provide a consistent description of the single-copy field strength, $F_{\mu\nu}^{(2)}$, leading to a correction of the Liénard-Wiechert potential (owed to $A_\mu^{(2)}$) and a second potential C_μ responsible for the magnetic current density. Additionally, we showed all Riemann asymptotically flat electrovacuum spacetimes, including the Kerr-Newman-Taub–NUT solution, have a Ricci spinor whose double-copy form is reminiscent of the Weyl double copy. Summarily,

$$\Phi_{ABC'D'} = \frac{2q_s^2}{q_1^2} f_{AB}^{(1)} \bar{f}_{C'D'}^{(1)}, \quad \Phi_{ABC'D'} = \frac{1}{3S_{(2)}} f_{AB}^{(2)} \bar{f}_{C'D'}^{(2)}, \quad (6.1)$$

where the first equality holds for all electrovacuum solutions with external source charge q_s, while the second equality only holds for asymptotically flat electrovacuum spacetimes.

There are multiple potential avenues of future research. Firstly, it would be worthwhile to examine the sourced Weyl double copy including matter field sources beyond Einstein-Maxwell theory. More generally one may include additional gauge fields, scalar fields, etc., motivated by, e.g., string theory or supergravity. Such extensions have been considered in the context of the perturbative double copy; however, the exact Kerr-Schild double copy for pure Einstein gravity has difficulty describing an additional massless NS-NS field like a Kalb-Ramond field or a dilaton using the single-null congruence in the conventional Kerr-Schild ansatz. To understand the double copy of such fields in a nonperturbative way, the Kerr-Schild framework was generalized to describe double-field theory and supergravities in [52,53] (see also the extensions [54–56]). It would be interesting to try to incorporate such fields using the sourced Weyl double copy, and compare to the spinorial representation of the torsionful Riemann curvature introduced in [37] to study solutions of the universal massless sector of supergravity. Doing so may also lead to a better understanding how the sourced Weyl double copy relates to the scattering amplitudes of the corresponding single-copy gauge-field theory. Alternately, imposing the (sourced) Weyl double copy in double-field theory could be provide a mechanism to study the relation between T duality and double copy for type-D solutions.[16]

Further, it would be interesting to extend our current analysis to other types of black hole spacetimes, particularly those including a cosmological constant (whose Weyl double copy was presented in [57]), additional Abelian gauge fields [58], and massless or higher-spin fields [59,60]. It would also be interesting to adapt our formalism to three-dimensional spacetimes, such as the Bañados-Teitelboim-Zanelli (BTZ) black hole [61], its quantum and de Sitter generalizations [62,63], or generalized three-dimensional solutions described in [64,65]. To this end, it would be worthwhile to also attempt to extend the Cotton double copy [66] to include external sources.

Finally, we have studied an elementary extension of the original Weyl double copy, and begun to gather evidence of a double-copy prescription for the Ricci spinor. To better understand both double-copy structures on a more fundamental level, it would be interesting to explore the twistorial foundations of the sourced Weyl double copy, utilizing and extending [21,22,36,38,67–69].

APPENDIX: SPINOR FORMALISM

Here we briefly illustrate how to obtain various spinorial quantities of interest from a four-dimensional curved spacetime metric $g_{\mu\nu}$. In our notation, spacetime indices are given by $\{\mu, \nu, \ldots\}$, frame indices by $\{a, b, \ldots\}$, while the spinor indices are $\{A, B, \ldots\}$ and their conjugates $\{A', B', \ldots\}$. Our remaining conventions primarily follow [42,70].

---

[16]We thank Eric Lescano for suggesting this to us.

1. Tetrads and the spinor basis

We introduce a complex null tetrad $\{l, n, m, \bar{m}\}$ constructing the metric $g_{\mu\nu}$ and it satisfies

$$
\begin{aligned}
g_{\mu\nu} &= -2l_{(\mu}n_{\nu)} + 2m_{(\mu}\bar{m}_{\nu)}, \\
l_\mu l^\mu &= n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = 0, \\
n^\mu l_\mu &= -1, \qquad m^\mu \bar{m}_\mu = +1,
\end{aligned}
\tag{A1}
$$

for which null vectors n^μ and l^μ are real and m^μ is generally complex with $\bar{m}^\mu$ its conjugate. The tetrad vectors have an associated frame tetrad that can be obtained using the vierbein as, e.g., $l^a = e^\mu{}_\mu l^\mu$. Explicitly, we work with the tetrad set

$$
l_a = \frac{1}{\sqrt{2}}(1, -1, 0, 0), \qquad n_a = \frac{1}{\sqrt{2}}(1, 1, 0, 0)
$$

$$
m_a = \frac{1}{\sqrt{2}}(0, 0, i, 1), \qquad \bar{m}_a = \frac{1}{\sqrt{2}}(0, 0, -i, 1),
\tag{A2}
$$

which constructs Minkowski space analogously to (A1) as $\eta_{ab} = -2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)}$. All frame indices are raised with η^{ab}.

To translate between tensors and spinors, we use the Pauli four-vectors:

$$
\sigma^a_{AA'} = \frac{1}{\sqrt{2}}(1, \vec{\sigma})_{AA'},
\tag{A3}
$$

where the σ_i are the standard $SU(2)$ generators:

$$
\sigma_1 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{A4}
$$

The Pauli four-vectors satisfy

$$
\sigma^a_{AA'}\sigma_a^{BB'} = \delta^A_B \delta^{A'}_{B'}, \qquad \sigma^a_{AA'}\sigma_b^{BB'} = \delta^a_b.
\tag{A5}
$$

Then, any spacetime (frame) vector has a spinor analog,

$$
V_a \to V_{AA'} = V_a \sigma^a_{AA'},
\tag{A6}
$$

which has its associated spacetime vector $V_\mu = e^a_\mu V_a$.

We next identify a spinor basis $\{o_A, \iota_A\}$ (and the conjugate basis $\{\bar{o}_{A'}\bar{\iota}_{A'}\}$), whose indices are raised and lowered by the two-dimensional Levi-Civita symbol,

$$
\epsilon^{AB} \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{AB}
\tag{A7}
$$

and its conjugate $\epsilon^{A'B'}$. The basis spinors are related to the frame tetrad by

$$
\begin{aligned}
o_A \bar{o}_{A'} &= l_a \sigma^a_{AA'}, & \iota_A \bar{\iota}_{A'} &= n_a \sigma^a_{AA'}, \\
\iota_A \bar{o}_{A'} &= m_a \sigma^a_{AA'}, & o_A \bar{\iota}_{A'} &= \bar{m}_a \sigma^a_{AA'}.
\end{aligned}
\tag{A8}
$$

The spinor basis satisfies $\epsilon^{AB}o_B o_A = \epsilon^{AB}\iota_B \iota_A = 0$, $\epsilon^{AB}\iota_B o_A = \iota^A o_A = 1$, with $\epsilon^{AB}o_B \iota_A = o^A \iota_A = -1$. Using (A8) along with our choice of tetrads (A2), we can deduce that the (normalized) spinor basis vectors are given by

$$
o_A = \frac{1}{\sqrt{2}}(1, 1), \qquad \iota_A = \frac{1}{\sqrt{2}}(1, -1).
\tag{A9}
$$

We next introduce the Infeld–van der Waerden symbols:

$$
\sigma^{ab}_{AB} = \sigma^{[a}_{AA'}\bar{\sigma}^{b]A'C}\epsilon_{CB}, \qquad \bar{\sigma} = \frac{1}{\sqrt{2}}(1, -\vec{\sigma}),
\tag{A10}
$$

which, along with the spacetime vierbeins, allow us to obtain the spinorial forms of any (even) rank-2 and higher tensor. For example, defining $\sigma^{\mu\nu}_{AB} = e^\mu_a e^\nu_b \sigma^{ab}_{AB}$, the Weyl spinor and spinor field strength are given by

$$
\Psi_{ABCD} = \frac{1}{4}W_{\mu\nu\alpha\beta}\sigma^{\mu\nu}_{AB}\sigma^{\alpha\beta}_{CD},
\tag{A11}
$$

$$
f_{AB} = \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}_{AB},
\tag{A12}
$$

where $W_{\mu\nu\alpha\beta}$ is the Weyl tensor and $F_{\mu\nu}$ is the standard field-strength tensor. Both Ψ_{ABCD} and f_{AB} are completely symmetric in their indices.

2. Field-strength spinor

In the Weyl double copy, the spinor field strength lives in an appropriate flat limit of the full spacetime metric, which we relate to the "Minkowski vierbeins" $e^{(0)}$ as $g^{(0)}_{\mu\nu} = e^{(0),a}_\mu e^{(0),b}_\nu \eta_{ab}$. We then define the "frame field strength" F_{ab}, related to the spacetime field strength by

$$
F_{\mu\nu} = e^{(0),a}_\mu e^{(0),b}_\nu F_{ab}.
\tag{A13}
$$

Importantly, F_{ab} lives on Minkowski space defined by $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, while $F_{\mu\nu}$ lives on a different form of flat space described by $g^{(0)}_{\mu\nu}$.

The spinor field strength associated with the frame field strength F_{ab} is analogous to (A12),

$$
f_{AB} = \frac{1}{2}F_{ab}\sigma^{ab}_{AB}.
\tag{A14}
$$

Componentwise, f_{AB} is

$$
\begin{aligned}
f_{00} &= -F_{01} + F_{13} + i(F_{02} - F_{23}), \\
f_{01} &= F_{03} + iF_{12,} \\
f_{11} &= F_{01} + F_{13} + i(F_{02} + F_{23}),
\end{aligned}
\tag{A15}
$$

We can invert (A15) together with their complex conjugate $\bar{f}_{A'B'}$ to solve for the F_{ab}'s:

$$F_{01} = \frac{1}{4}[f_{11} - f_{00} + \bar{f}_{1'1'} - \bar{f}_{0'0'}], \qquad F_{02} = \frac{1}{4i}[\bar{f}_{1'1'} + \bar{f}_{0'0'} - (f_{11} + f_{00})], \qquad F_{03} = \frac{1}{2}[f_{01} + \bar{f}_{0'1'}]$$

$$F_{13} = \frac{1}{4}[f_{00} + f_{11} + \bar{f}_{0'0'} + \bar{f}_{1'1'}], \qquad F_{23} = \frac{1}{4i}[f_{11} - f_{00} - (\bar{f}_{1'1'} - \bar{f}_{0'0'})], \qquad F_{12} = \frac{1}{2i}[f_{01} - \bar{f}_{0'1'}]. \qquad \text{(A16)}$$

Note that for each of the type-D metrics we consider, the matrix structure of f_{AB} is the Pauli matrix σ_3, i.e.,

$$f_{AB} = Z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{AB}, \qquad \text{(A17)}$$

where in general, $Z \in \mathbb{C}$. Thus, $f_{00} = -f_{11} = Z$, $f_{0'0'} = -f_{1'1'} = Z^*$, and $f_{10} = f_{01} = 0$. This allows us to write the two nonzero components of F_{ab} solely in terms of Z as

$$F_{01} = -\frac{1}{2}\text{Re}(Z), \qquad F_{23} = -\frac{1}{2}\text{Im}(Z). \qquad \text{(A18)}$$

Finally, to obtain $F_{\mu\nu}$, we now contract F_{ab} terms using the flat-space vierbeins $e_\mu^{(0),a}$. Dropping the (0) superscript on the vierbeins for conciseness, this reads

$$F_{\mu\nu} = F_{ab}e_\mu^a e_\nu^b = -\frac{1}{2}(\text{Re}(Z)e^0_{[\mu}e^1_{\nu]} + \text{Im}(Z)e^2_{[\mu}e^3_{\nu]}). \qquad \text{(A19)}$$

We point out that when vierbeins are diagonal (as would be the case if $g^{(0)}_{\mu\nu}$ is the metric in spherical polar or oblate spheroidal coordinates), then, from (A19), if Z has an imaginary component, then $F_{\mu\nu}$ has a magnetic field component. This is in line with standard results for the Schwarzschild solution contrasted to the Kerr solution: the presence of the rotation parameter resulted in Z being complex in the latter case, which corresponds to the magnetic field present as a result of the angular momentum of the black hole. For Schwarzschild, $Z \in \mathbb{R}$ and the single-copy gauge field strength contains a (radial) electric field only.

Finally, as described in Sec. II A, let us explicitly show the form of the spinor field strengths of the single-copy source $f^{(1)}_{AB}$ and double-copy source f^s_{AB} are equivalent. For frame field strength $F^{(1)}_{ab} = F^{(1)}_{\mu\nu}(e^{(0)})_a^\mu(e^{(0)})_b^\nu$, then

$$\begin{aligned}
f^{(1)}_{AB} &= \frac{1}{2}F^{(1)}_{\mu\nu}(e^{(0)})_a^\mu(e^{(0)})_b^\nu \sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B \\
&= \frac{1}{2}F^{(1)}_{cd}(e^{(0)})_\mu^c(e^{(0)})_\nu^d(e^{(0)})_a^\mu(e^{(0)})_b^\nu \sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B \\
&= \frac{1}{2}F^{(1)}_{cd}\delta_a^c\delta_b^d \sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B \\
&= \frac{1}{2}F^{(1)}_{ab}\sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B, \qquad \text{(A20)}
\end{aligned}$$

where we used $(e^{(0)})_\mu^a(e^{(0)})_b^\mu = \delta_b^a$ in the third line. Likewise,

$$\begin{aligned}
f^s_{AB} &= \frac{1}{2}F^s_{\mu\nu}e_a^\mu e_b^\nu \sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B \\
&= \frac{1}{2}F^s_{cd}e_\mu^c e_\nu^d e_a^\mu e_b^\nu \sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B \\
&= \frac{1}{2}F^s_{cd}\delta_a^c\delta_b^d \sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B \\
&= \frac{1}{2}F^s_{ab}\sigma^{[a}_{AC'}\bar{\sigma}^{b]C'}{}_B, \qquad \text{(A21)}
\end{aligned}$$

where we used $e_\mu^a e_b^\mu = \delta_b^a$.

3. Weyl spinor

The Weyl spinor Ψ_{ABCD} may be expanded using the spinor basis $\{o_A, \iota_B\}$ as [42,71]

$$\begin{aligned}
\Psi_{ABCD} &= \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A}\iota_B \iota_C \iota_{C)} + 6\Psi_2 o_{(A}o_B \iota_C \iota_{D)} \\
&\quad - 4\Psi_3 o_{(A}o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D. \qquad \text{(A22)}
\end{aligned}$$

Here parentheses represent symmeterization, with the convention, $H_{(AB)} = \frac{1}{2}(H_{AB} + H_{BA})$. The $\Psi_I \in \mathbb{C}$ denote the Weyl scalars involved in the Petrov classification of the spacetime (see, e.g., [70] for a thorough description of the classification scheme), and are defined using the Weyl tensor $W_{\mu\nu\lambda\gamma}$ and the complex null tetrad (A1):

$$\begin{aligned}
\Psi_0 &= W_{\mu\nu\rho\lambda}n^\mu m^\nu n^\rho m^\lambda, \\
\Psi_1 &= W_{\mu\nu\rho\lambda}n^\mu l^\nu n^\rho m^\lambda, \\
\Psi_2 &= W_{\mu\nu\rho\lambda}n^\mu m^\nu \bar{m}^\rho l^\lambda, \\
\Psi_3 &= W_{\mu\nu\rho\lambda}n^\mu l^\nu \bar{m}^\rho l^\lambda, \\
\Psi_4 &= W_{\mu\nu\rho\lambda}\bar{m}^\mu l^\nu \bar{m}^\rho l^\lambda. \qquad \text{(A23)}
\end{aligned}$$

For all Petrov type-D spacetimes, one can always choose a coordinate system such that $\Psi_2 \neq 0$ while all other Ψ_I vanish, such that the Weyl spinor becomes

$$\Psi^{\text{type D}}_{ABCD} = 6\Psi_2 o_{(A}o_B \iota_C \iota_{D)}. \qquad \text{(A24)}$$

4. Ricci spinor

The Riemann tensor decomposes into the Weyl tensor, Ricci tensor, and Ricci scalar schematically as $\text{Riem} = \text{Weyl} + \text{Ric} \otimes g + R(g \otimes g)$. Similarly, the Riemann spinor decomposes as

$$R_{AA'BB'CC'DD'} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}$$
$$+ 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'}), \tag{A25}$$

where Λ and $\Phi_{ABC'D'} = \Phi_{(AB)(C'D')}$ are related to the Ricci scalar and Ricci tensor, respectively, and $\bar{\Phi}_{A'B'CD}$ is the complex conjugate of $\Phi_{ABC'D'}$. We refer to $\Phi_{ABC'D'}$ as the Ricci spinor. The components of the Ricci spinor can be calculated using the Ricci tensor and the complex null tetrad $\{l, n, m, \bar{m}\}$. There are nine individual components of $\Phi_{ABC'D'}$ [42],

$$\Phi_{000'0'} = -\frac{1}{2}R_{\mu\nu}n^\mu n^\nu, \qquad \Phi_{000'1'} = -\frac{1}{2}R_{\mu\nu}n^\mu m^\nu, \qquad \Phi_{001'1'} = -\frac{1}{2}R_{\mu\nu}m^\mu m^\nu,$$

$$\Phi_{010'0'} = -\frac{1}{2}R_{\mu\nu}n^\mu \bar{m}^\nu, \qquad \Phi_{010'1'} = -\frac{1}{2}R_{\mu\nu}n^\mu l^\nu + 3\Pi, \qquad \Phi_{011'1'} = -\frac{1}{2}R_{\mu\nu}m^\mu l^\nu,$$

$$\Phi_{110'0'} = -\frac{1}{2}R_{\mu\nu}\bar{m}^\mu \bar{m}^\nu, \qquad \Phi_{110'1'} = -\frac{1}{2}R_{\mu\nu}\bar{m}^\mu l^\nu, \qquad \Phi_{111'1'} = -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu. \tag{A26}$$

Here, Π is related to the Ricci scalar via $\Pi = \frac{1}{12}R$. The scalar functions $\{\Phi_{000'0'}, ..., \Phi_{111'1'}\}$ can also be obtained via contractions with $\{o^A, \iota^A\}$, for example (cf. page 22 of [71]),

$$\Phi_{000'0'} = \Phi_{ABC'D'}o^A o^B \bar{o}^{C'} \bar{o}^{D'}, \quad \Phi_{110'1'} = \Phi_{ABC'D'}\iota^A \iota^B \bar{o}^{C'} \bar{\iota}^{D'}, \tag{A27}$$

and so forth. Following the notation of [42], the Ricci scalars are conveniently written as

$$\Phi_{00} \equiv \Phi_{000'0'}, \qquad \Phi_{01} \equiv \Phi_{000'1'}, \qquad \Phi_{02} \equiv \Phi_{001'1'},$$

$$\Phi_{10} \equiv \Phi_{010'0'}, \qquad \Phi_{11} \equiv \Phi_{010'1'}, \qquad \Phi_{12} \equiv \Phi_{011'1'},$$

$$\Phi_{20} \equiv \Phi_{110'0'}, \qquad \Phi_{21} \equiv \Phi_{110'1'}, \qquad \Phi_{22} \equiv \Phi_{111'1'}. \tag{A28}$$

Note $\Phi_{ABC'D'}$ is generally *Hermitian*, since $\Phi_{ab} = \bar{\Phi}_{ba}$.

Using the normalized spinor basis, we can deduce an expansion for the Ricci spinor:

$$\Phi_{ABC'D'} = \Phi_{22}o_A o_B \bar{o}_{C'}\bar{o}_{D'} - 2\Phi_{12}o_{(A}\iota_{B)}\bar{o}_{C'}\bar{o}_{D'}$$
$$+ \Phi_{02}\iota_A \iota_B \bar{o}_{C'}\bar{o}_{D'} - 2\Phi_{21}o_A o_B \bar{o}_{(C'}\bar{\iota}_{D')}$$
$$+ 4\Phi_{11}o_{(A}\iota_{B)}\bar{o}_{(C'}\bar{\iota}_{D')} - 2\Phi_{01}\iota_A \iota_B \bar{o}_{(C'}\bar{\iota}_{D')}$$
$$+ \Phi_{20}o_A o_B \bar{\iota}_{C'}\bar{\iota}_{D'} - 2\Phi_{10}\iota_{(A}o_{B)}\bar{\iota}_{C'}\bar{\iota}_{D'}$$
$$+ \Phi_{00}\iota_A \iota_B \bar{\iota}_{C'}\bar{\iota}_{D'}. \tag{A29}$$

As was discussed in the text, the only nonzero Ricci coefficient in Einstein-Maxwell theory is Φ_{11}; thus, the Ricci spinor is

$$\Phi_{ABC'D'} = 4\Phi_{11}o_{(A}\iota_{B)}\bar{o}_{(C'}\bar{\iota}_{D')}. \tag{A30}$$

[1] Z. Bern, J. J. M. Carrasco, and H. Johansson, New relations for gauge-theory amplitudes, Phys. Rev. D **78,** 085011 (2008).

[2] Z. Bern, T. Dennen, Y.-t. Huang, and M. Kiermaier, Gravity as the square of gauge theory, Phys. Rev. D **82,** 065003 (2010).

[3] Z. Bern, J. J. M. Carrasco, and H. Johansson, Perturbative Quantum Gravity as a Double Copy of Gauge Theory, Phys. Rev. Lett. **105,** 061602 (2010).

[4] H. Kawai, D. C. Lewellen, and S. H. H. Tye, A relation between tree amplitudes of closed and open strings, Nucl. Phys. **B269,** 1 (1986).

[5] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, The complete four-loop four-point amplitude in $N = 4$ Super-Yang-Mills theory, Phys. Rev. D **82,** 125040 (2010).

[6] J. J. M. Carrasco and H. Johansson, Five-point amplitudes in $N = 4$ Super-Yang-Mills theory and $N = 8$ supergravity, Phys. Rev. D **85,** 025006 (2012).

[7] S. Oxburgh and C. D. White, BCJ duality and the double copy in the soft limit, J. High Energy Phys. 02 (2013) 127.

[8] Z. Bern, S. Davies, T. Dennen, Y.-t. Huang, and J. Nohle, Color-kinematics duality for pure Yang-Mills and gravity at one and two loops, Phys. Rev. D **92,** 045041 (2015).

[9] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, Phys. Rev. Lett. **122,** 201603 (2019).

[10] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, J. High Energy Phys. 10 (2019) 206.

[11] H. Godazgar, M. Godazgar, R. Monteiro, D. Peinador Veiga, and C. N. Pope, Weyl Double Copy for Gravitational Waves, Phys. Rev. Lett. **126**, 101103 (2021).

[12] M. Carrillo González, A. Momeni, and J. Rumbutis, Cotton double copy for gravitational waves, Phys. Rev. D **106**, 025006 (2022).

[13] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, The duality between color and kinematics and its applications, arXiv:1909.01358.

[14] D. A. Kosower, R. Monteiro, and D. O'Connell, The SAGEX review on scattering amplitudes, Chapter 14: Classical gravity from scattering amplitudes, J. Phys. A **55**, 443015 (2022).

[15] T. Adamo, J. J. M. Carrasco, M. Carrillo-González, M. Chiodaroli, H. Elvang, H. Johansson, D. O'Connell, R. Roiban, and O. Schlotterer, Snowmass white paper: The double copy and its applications, arXiv:2204.06547.

[16] D. Neill and I. Z. Rothstein, Classical space-times from the S matrix, Nucl. Phys. **B877**, 177 (2013).

[17] R. Monteiro, D. O'Connell, and C. D. White, Black holes and the double copy, J. High Energy Phys. 12 (2014) 056.

[18] A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O'Connell, N. Westerberg, and C. D. White, Perturbative spacetimes from Yang-Mills theory, J. High Energy Phys. 04 (2017) 069.

[19] A. Luna, R. Monteiro, I. Nicholson, and D. O'Connell, Type D spacetimes and the Weyl double copy, Classical Quantum Gravity **36**, 065003 (2019).

[20] G. Elor, K. Farnsworth, M. L. Graesser, and G. Herczeg, The Newman-Penrose map and the classical double copy, J. High Energy Phys. 12 (2020) 121.

[21] C. D. White, Twistorial Foundation for the Classical Double Copy, Phys. Rev. Lett. **126**, 061602 (2021).

[22] E. Chacón, S. Nagy, and C. D. White, Alternative formulations of the twistor double copy, J. High Energy Phys. 03 (2022) 180.

[23] M. Campiglia and S. Nagy, A double copy for asymptotic symmetries in the self-dual sector, J. High Energy Phys. 03 (2021) 262.

[24] T. Adamo and U. Kol, Classical double copy at null infinity, Classical Quantum Gravity **39**, 105007 (2022).

[25] A. K. Ridgway and M. B. Wise, Static spherically symmetric Kerr-Schild metrics and implications for the classical double copy, Phys. Rev. D **94**, 044023 (2016).

[26] A. I. Janis, E. T. Newman, and J. Winicour, Reality of the Schwarzschild Singularity, Phys. Rev. Lett. **20**, 878 (1968).

[27] W. D. Goldberger and A. K. Ridgway, Radiation and the classical double copy for color charges, Phys. Rev. D **95**, 125010 (2017).

[28] K. Kim, K. Lee, R. Monteiro, I. Nicholson, and D. Peinador Veiga, The classical double copy of a point charge, J. High Energy Phys. 02 (2020) 046.

[29] A. Luna, S. Nagy, and C. White, The convolutional double copy: A case study with a point, J. High Energy Phys. 09 (2020) 062.

[30] A. Luna, R. Monteiro, D. O'Connell, and C. D. White, The classical double copy for Taub–NUT spacetime, Phys. Lett. B **750**, 272 (2015).

[31] N. Bahjat-Abbas, A. Luna, and C. D. White, The Kerr-Schild double copy in curved spacetime, J. High Energy Phys. 12 (2017) 004.

[32] M. Carrillo-González, R. Penco, and M. Trodden, The classical double copy in maximally symmetric spacetimes, J. High Energy Phys. 04 (2018) 028.

[33] M. Carrillo González, B. Melcher, K. Ratliff, S. Watson, and C. D. White, The classical double copy in three spacetime dimensions, J. High Energy Phys. 07 (2019) 167.

[34] C. Keeler, T. Manton, and N. Monga, From Navier-Stokes to Maxwell via Einstein, J. High Energy Phys. 08 (2020) 147.

[35] D. A. Easson, C. Keeler, and T. Manton, Classical double copy of nonsingular black holes, Phys. Rev. D **102**, 086015 (2020).

[36] A. Guevara, Reconstructing classical spacetimes from the S-matrix in twistor space, arXiv:2112.05111.

[37] R. Monteiro, S. Nagy, D. O'Connell, D. Peinador Veiga, and M. Sergola, NS-NS spacetimes from amplitudes, J. High Energy Phys. 06 (2022) 021.

[38] E. Chacón, S. Nagy, and C. D. White, The Weyl double copy from twistor space, J. High Energy Phys. 05 (2021) 2239.

[39] D. A. Easson, T. Manton, and A. Svesko, Sources in the Weyl Double Copy, Phys. Rev. Lett. **127**, 271101 (2021).

[40] N. Cabibbo and E. Ferrari, Quantum electrodynamics with Dirac monopoles, Nuovo Cimento **23**, 1147 (1962).

[41] J. F. Plebanski and M. Demianski, Rotating, charged, and uniformly accelerating mass in general relativity, Ann. Phys. (N.Y.) **98**, 98 (1976).

[42] R. Penrose and W. Rindler, *Spinors and Space-Time*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2011).

[43] P. A. M. Dirac, The theory of magnetic poles, Phys. Rev. **74**, 817 (1948).

[44] T. Adamo and E. T. Newman, The Kerr-Newman metric: A review, Scholarpedia **9**, 31791 (2014).

[45] J. Podolsky and J. B. Griffiths, Accelerating Kerr-Newman black holes in (anti-)de Sitter space-time, Phys. Rev. D **73**, 044018 (2006).

[46] J. B. Griffiths, P. Krtous, and J. Podolsky, Interpreting the C-metric, Classical Quantum Gravity **23**, 6745 (2006).

[47] P. A. M. Dirac, Quantised singularities in the electromagnetic field, Proc. R. Soc. A **133**, 60 (1931).

[48] D. Zwanziger, Local Lagrangian quantum field theory of electric and magnetic charges, Phys. Rev. D **3**, 880 (1971).

[49] W. A. Barker and F. Graziani, Quantum mechanical formulation of the electron-monopole interaction without Dirac strings, Phys. Rev. D **18**, 3849 (1978).

[50] D. Singleton, Electromagnetism with magnetic charge and two photons, Am. J. Phys. **64**, 452 (1996).

[51] J. B. Griffiths and J. Podolsky, A new look at the Plebanski-Demianski family of solutions, Int. J. Mod. Phys. D **15**, 335 (2006).

[52] K. Lee, Kerr-Schild double field theory and classical double copy, J. High Energy Phys. 10 (2018) 027.

[53] W. Cho and K. Lee, Heterotic Kerr-Schild double field theory and classical double copy, J. High Energy Phys. 07 (2019) 030.

[54] E. Lescano and J. A. Rodríguez, $\mathcal{N} = 1$ supersymmetric double field theory and the generalized Kerr-Schild ansatz, J. High Energy Phys. 10 (2020) 148.

[55] E. Lescano and N. Mirón-Granese, Double field theory with matter and its cosmological application, arXiv:2111.03682.

[56] E. Lescano and S. Roychowdhury, Heterotic Kerr-Schild double field theory and its double Yang-Mills formulation, J. High Energy Phys. 04 (2022) 090.

[57] S. Han, The Weyl double copy in vacuum spacetimes with a cosmological constant, J. High Energy Phys. 09 (2022) 238.

[58] S. Chawla and C. Keeler, Aligned fields double copy to Kerr-NUT-(A)dS, arXiv:2209.09275.

[59] S. Han, Weyl double copy and massless free fields in curved spacetimes, Classical Quantum Gravity **39**, 225009 (2022).

[60] V. E. Didenko and N. K. Dosmanbetov, Classical double copy and higher-spin fields, arXiv:2210.04704.

[61] M. Banados, C. Teitelboim, and J. Zanelli, The Black Hole in Three-Dimensional Space-Time, Phys. Rev. Lett. **69,** 1849 (1992).

[62] R. Emparan, A. M. Frassino, and B. Way, Quantum BTZ black hole, J. High Energy Phys. 11 (2020) 137.

[63] R. Emparan, J. F. Pedraza, A. Svesko, M. Tomašević, and M. R. Visser, Black holes in dS$_3$, J. High Energy Phys. 11 (2022) 073.

[64] G. Alkac, M. K. Gumus, and M. A. Olpak, Generalized black holes in 3D Kerr-Schild double copy, Phys. Rev. D **106,** 026013 (2022).

[65] P. Bueno, P. A. Cano, J. Moreno, and G. van der Velde, Regular black holes in three dimensions, Phys. Rev. D **104,** L021501 (2021).

[66] W. T. Emond and N. Moynihan, Scattering amplitudes and the cotton double copy, arXiv:2202.10499.

[67] E. Chacón, A. Luna, and C. D. White, The double copy of the multipole expansion, Phys. Rev. D **106,** 086020 (2022).

[68] K. Farnsworth, M. L. Graesser, and G. Herczeg, Twistor space origins of the Newman-Penrose map, SciPost Phys. **13,** 099 (2022).

[69] A. Luna, N. Moynihan, and C. D. White, Why is the Weyl double copy local in position space?, J. High Energy Phys. 12 (2022) 046.

[70] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2003).

[71] R. Penrose and W. Rindler, *Spinors and Space-Time Vol. 2: Spinor and Twistor Methods in Space-Time Geometry*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1988).