DEPARTMENT OF MATHEMATICS

Dynamics of an ice particle submerged in water

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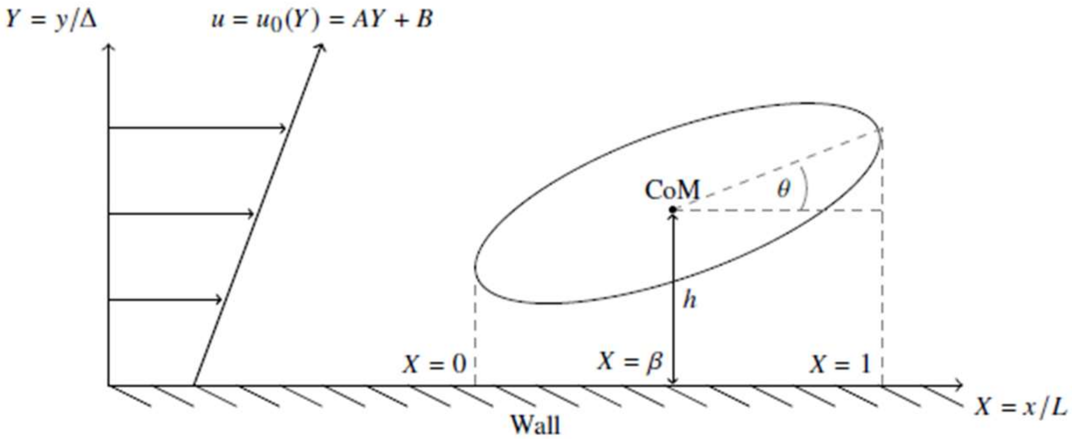
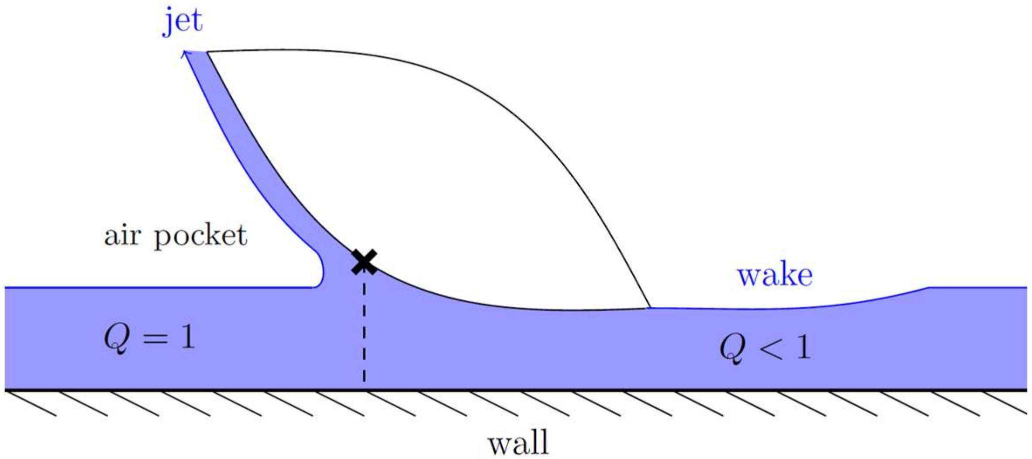
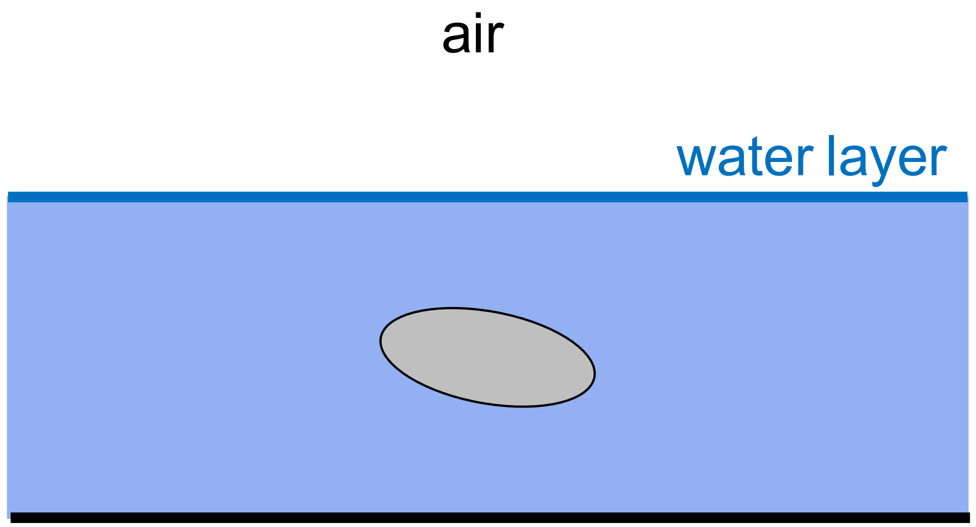
Modelling aircraft icing

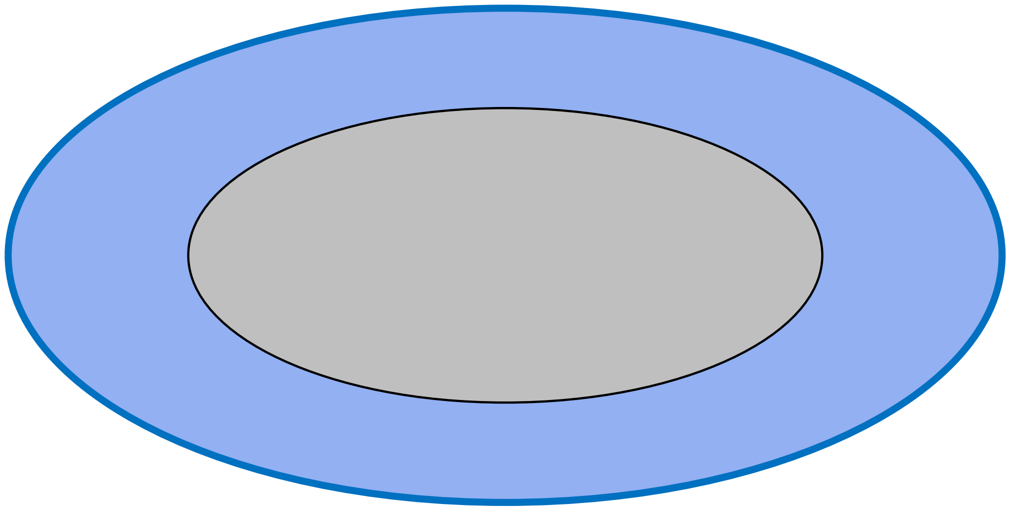




Ice particle in water



Ice particle in a water layer



Melting particle coated in a layer of water

Governing equations

underbody curve

$$F = F_u(X) + h(T) + (X - \beta)\theta(T),$$

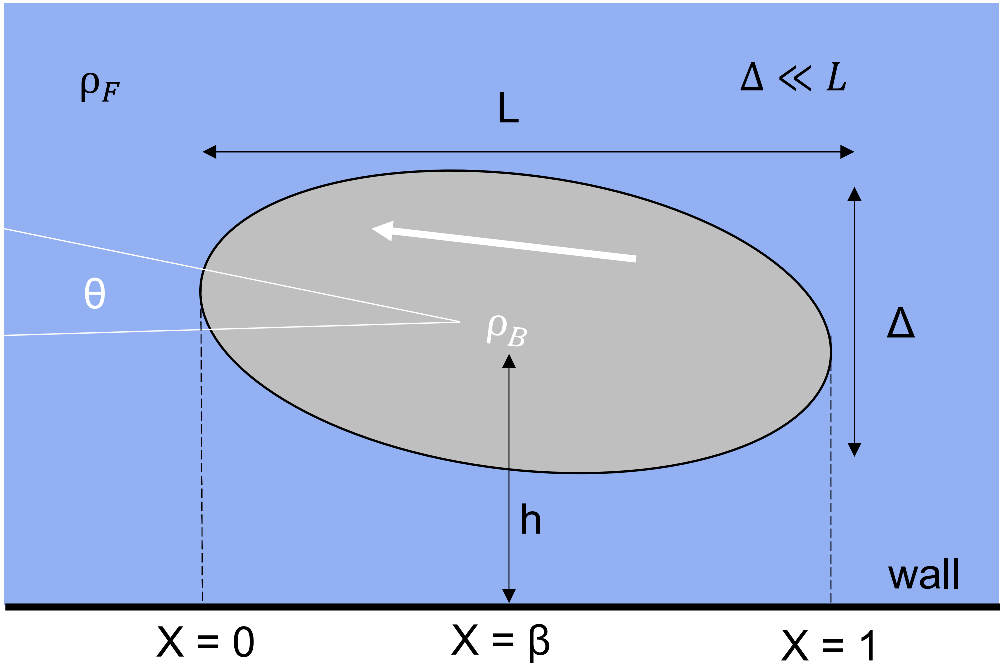
conservation of mass and momentum

$$F_T + (uF)_X = 0,$$

$$u_T + uu_X = -p_X, \quad 0 = -p_Y,$$

body motion equations

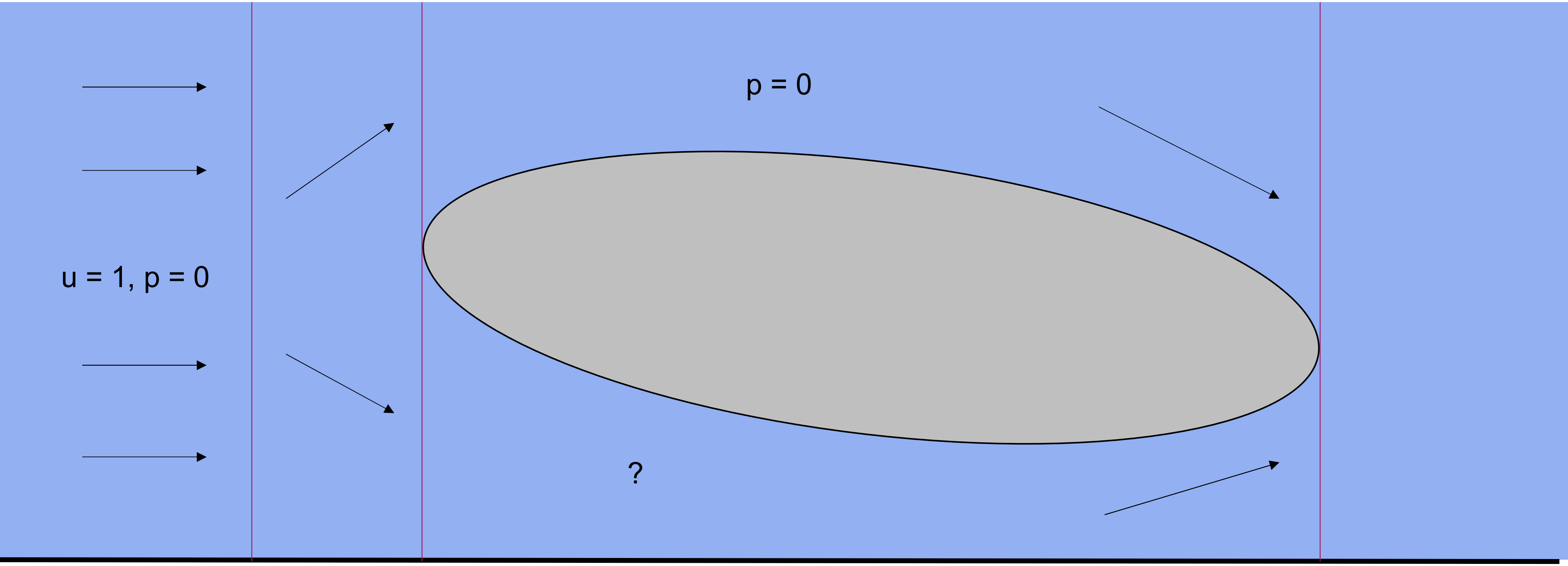
$$Mh_{TT} = \int_0^1 p \, dx, \quad I\theta_{TT} = \int_0^1 (x - \beta)p \, dx.$$



density ratio

$$\frac{\rho_B}{\rho_F} = O\left(\frac{L}{\Delta}\right)^2 \Rightarrow M, I \ll 1$$

Boundary conditions



$$\frac{1}{2}u^2 + p = \frac{1}{2} \text{ at } X = 0$$

$$p = 0 \text{ at } X = 1$$

Governing equations

To approach an O(1) density ratio, assume:

$$M, I \ll 1$$

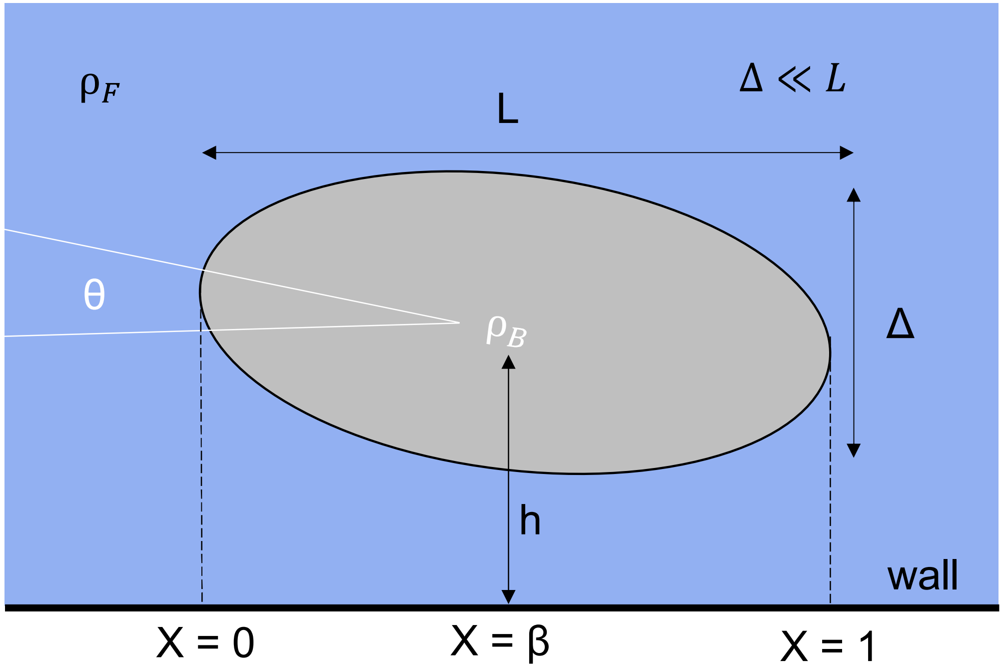
New governing equations:

$$F = F_u(X) + h(T) + (X - \beta)\theta(T),$$

$$F_T + (uF)_X = 0,$$

$$u_T + uu_X = -p_X, 0 = -p_Y$$

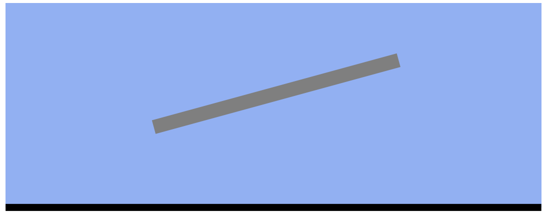
$$\int_0^1 p\, dx = \int_0^1 (x - \beta) p\, dx = 0.$$



Boundary conditions

$$\frac{1}{2}u^2 + p = \frac{1}{2} \text{ at } X = 0. \qquad p = 0 \text{ at } X = 1.$$

Linear analysis for flat plate



$$(F, h_C, \theta, u, p) = (1 + \delta F_1, 1 + \delta h_1, \delta\theta_1, 1 + \delta u_1, \delta p_1) + \dots,$$

$(h_C = h - \beta\theta)$

Linearised equations:

$$F_1 = h_1(T) + X\theta_1(T),$$

$$F_{1T} + F_{1X} + u_{1X} = 0,$$

$$u_{1T} + u_{1X} = -p_{1X},$$

$$\int_0^1 p_1 \, dx = \int_0^1 x p_1 \, dx = 0,$$

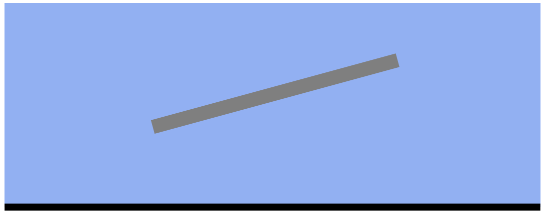
$$u_1 = -p_1 \text{ at } X = 0,$$

$$p_1 = 0 \text{ at } X = 1.$$

$$u_1 = -(h_1' + \theta_1)X - \theta_1' X^2/2 - A_1(T),$$

$$p_1 = (h_1'' + 2\theta_1')X^2/2 + \theta_1'' X^3/6 + (A_1' + h_1' + \theta_1)X + A_1(T),$$

Linear analysis for a flat plate



$$(F, h_C, \theta, u, p) = (1 + \delta F_1, 1 + \delta h_1, \delta\theta_1, 1 + \delta u_1, \delta p_1) + \ldots,$$

Linearised equations:

$$F_1 = h_1(T) + X\theta_1(T),$$

$$F_{1T} + F_{1X} + u_{1X} = 0,$$

$$u_{1T} + u_{1X} = -p_{1X},$$

$$\int_0^1 p_1 \, dx = \int_0^1 x p_1 \, dx = 0,$$

$$u_1 = -p_1 \text{ at } X = 0,$$

$$p_1 = 0 \text{ at } X = 1.$$

$$h_1''/2 + \theta_1''/6 + A_1' = \mathcal{R},$$

$$h_1''/6 + \theta_1''/24 + A_1'/2 = \mathcal{S},$$

$$h_1''/8 + \theta_1''/30 + A_1'/3 = \mathcal{T}.$$

$$\mathcal{R} = -\theta_1' - h_1' - \theta_1 - A_1,$$

$$\mathcal{S} = -\theta_1'/3 - h_1'/2 - \theta_1/2 - A_1,$$

$$\mathcal{T} = -\theta_1'/4 - h_1'/3 - \theta_1/3 - A_1/2.$$

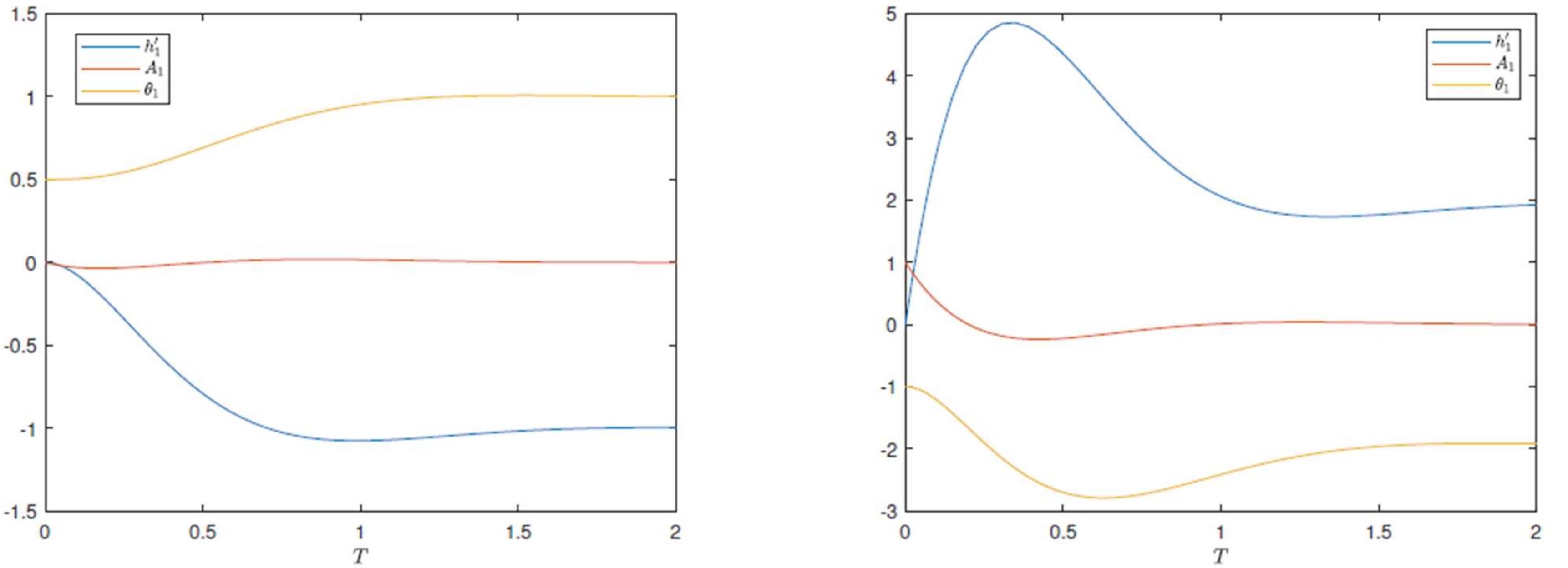
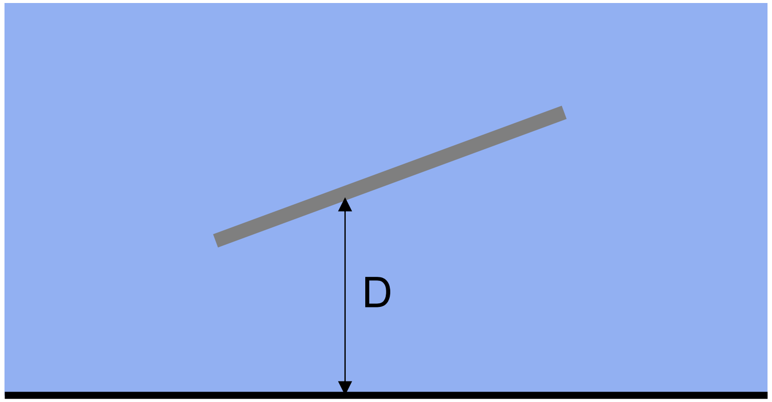


Figure 1: Results from solving the linearized system, showing convergence to two different steady states for the two different initial conditions, each with $h_1' = -\theta_1 =$ const and $A_1 = 0$. The body may have either a negative velocity (left) or a positive velocity (right).

Full nonlinear system



$$F_T + (uF)_X = 0,$$

$$u_T + uu_X = -p_X, 0 = -p_Y$$

$$\int_0^1 p\, dx = \int_0^1 (x-\beta)p\, dx = 0.$$

$$\frac{1}{2}u^2 + p = \frac{1}{2} \text{ at } X = 0.$$

$$p = 0 \text{ at } X = 1.$$

$$u = -\frac{[h'_C X + \frac{1}{2}\theta' X^2 + A(T)]}{D}, \text{ with } D = h_C + X\theta.$$

$$p = -\int_0^X u_T\, dx - \frac{1}{2}u^2 + \frac{1}{2}.$$

Full nonlinear system

$$\alpha_0 A' + \alpha_1 h_C'' + \tfrac{1}{2}\alpha_2\theta'' = e_4 - \tfrac{1}{2} + \beta_0,$$
$$\alpha_1 A' + \alpha_2 h_C'' + \tfrac{1}{2}\alpha_3\theta'' = e_4 - \tfrac{1}{2}b_4 + \beta_1,$$
$$\alpha_2 A' + \alpha_3 h_C'' + \tfrac{1}{2}\alpha_4\theta'' = e_4 - d_4 + \beta_2,$$

$$\alpha_i = \sum_{j=0}^{i-1} \frac{h_C^j(-1)^j}{(i-j)\theta^{j+1}} + \frac{h_C^i(-1)^i}{\theta^i}\alpha_0, \quad i \geq 1,$$

$$\alpha_0 = \frac{1}{\theta}\log\left(1 + \frac{\theta}{h_C}\right).$$

$$\alpha_i = \int_0^1 \frac{x^i}{D}\,dx. \quad i = 0, 1, 2, 3, \ldots$$

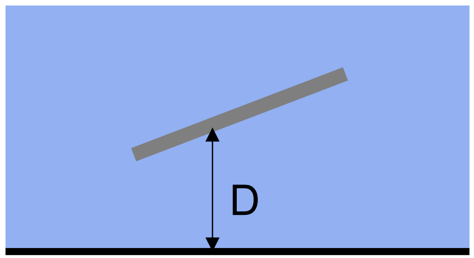
$$\beta_i = \int_0^1 \frac{x^i(h_C'x + \theta'x^2/2 + A)D'}{D^2}\,dx.$$

$$b_4 = \int_0^1 \frac{(h_C'x + \theta'x^2/2 + A)^2}{D^2}\,dx,$$

$$d_4 = \int_0^1 \frac{x(h_C'x + \theta'x^2/2 + A)^2}{D^2}\,dx,$$

$$e_4 = \frac{1}{2}\frac{(h_C' + \theta'/2 + A)^2}{D_1^2},$$

$$D = h_C + X\theta$$
$$D_1 = h_C + \theta$$



Leading edge collisions

In the limit of collision at the leading edge, i.e. $h_C \to 0$, the following asymptotic expansions apply:

$$h_C = (t_0 - t)h_1 + (t_0 - t)\Lambda h_\lambda + \dots,$$
$$\theta = \theta_0 + (t_0 - t)\theta_1 + (t_0 - t)\Lambda\theta_\lambda + \dots,$$
$$A = \Lambda A_\lambda + (t_0 - t)A_1 + \dots,$$

$$\Lambda = \frac{1}{\log(t_0 - t)},$$

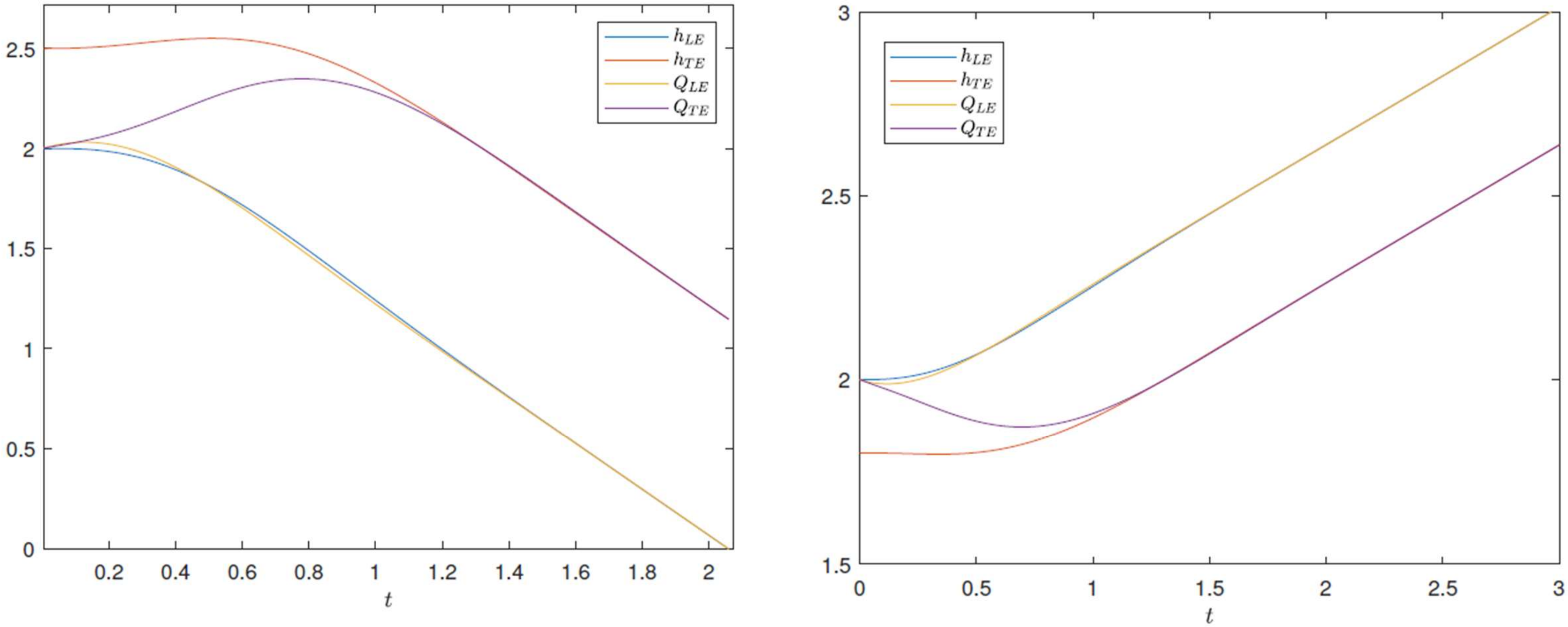
$$\alpha_0 A' + \alpha_1 h_C'' + \tfrac{1}{2}\alpha_2\theta'' = e_4 - \tfrac{1}{2} + \beta_0,$$
$$\alpha_1 A' + \alpha_2 h_C'' + \tfrac{1}{2}\alpha_3\theta'' = e_4 - \tfrac{1}{2}b_4 + \beta_1,$$
$$\alpha_2 A' + \alpha_3 h_C'' + \tfrac{1}{2}\alpha_4\theta'' = e_4 - d_4 + \beta_2,$$

$$\alpha_i = \sum_{j=0}^{i-1} \frac{h_C^j(-1)^j}{(i-j)\theta^{j+1}} + \frac{h_C^i(-1)^i}{\theta^i}\alpha_0, \quad i \geq 1,$$

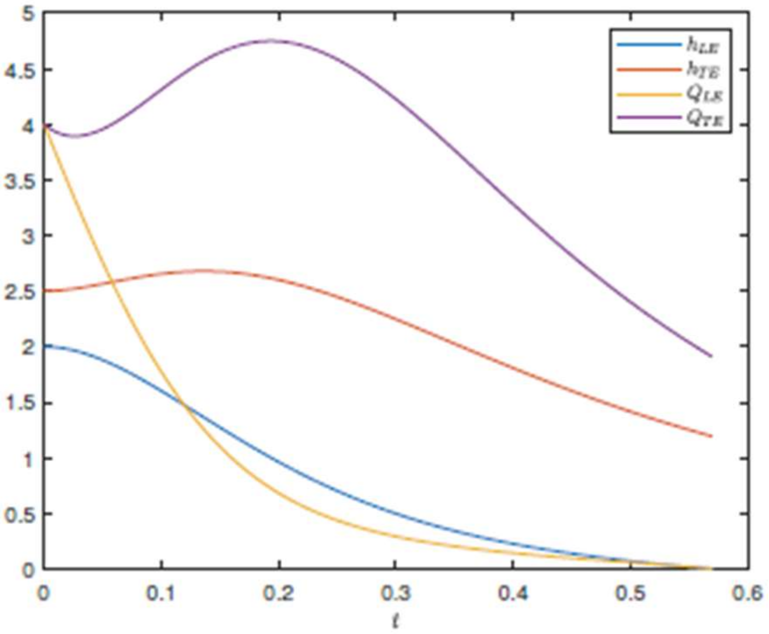
$$\alpha_0 = \frac{1}{\theta}\log\left(1 + \frac{\theta}{h_C}\right).$$

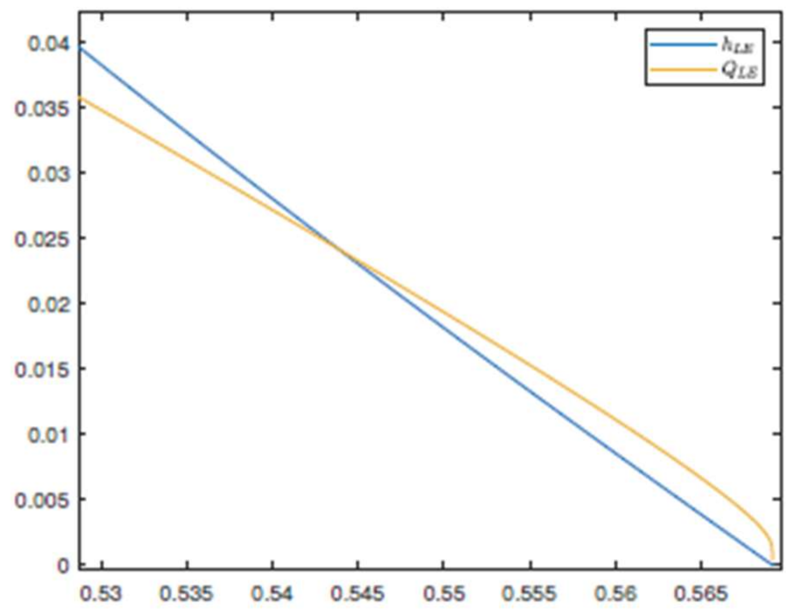
$$A_\lambda\left[A_\lambda + \theta_0 h_1\left(1 + \frac{1}{6}\log\left(\frac{\theta_0}{h_1}\right)\right)\right] = 0,$$

$$A_1 = -\frac{h_1^2}{\theta_0}.$$


h_{LE}
h_{TE}
Q_{LE}
Q_{TE}
2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t
3
2.5
2
1.5
0
0.5
1
1.5
2
2.5
3
t

$$A_\lambda = 0$$





$$A_\lambda \neq 0$$

Summary

- Adapted model of ice particles in air to address ice particles in water
- Found solutions to linearized problem for flat plate
- Found solutions to full non-linear problem for flat plate
- Asymptotically described leading edge collisions

Thank you!