Teachers’ structuring of mathematical inquiry lessons: Shifting from ‘task-first’ to ‘scaffolded inquiry’

Abstract

A three-phase ‘task-first’ lesson structure is frequently suggested when teaching mathematics through inquiry. We investigate how secondary school teachers of mathematics structure their inquiry lessons and examine how and why they deviate from a ‘task-first’ structure. We present detailed lesson observation data from three teachers participating in a year-long professional development programme focused on inquiry teaching. We track the developing structure of these teachers’ inquiry lessons through minute-by-minute lesson analysis, describe how their lesson structures altered over time and explore why. Our data show that contextual constraints may explain why teachers departed from the ‘task-first’ lesson structure. In their inquiry teaching, two of the teachers adopted more scaffolded approaches, including the use of a sequence of smaller sub-tasks and teacher interventions. We argue that these modifications to a ‘task-first’ lesson structure are legitimate ways to support student learning through inquiry; indeed, that they may offer some advantages for inquiry teaching.

Keywords

Inquiry-based learning; lesson structure; mathematical inquiry; professional development; scaffolded inquiry

Introduction

Over the past three decades, researchers have examined a variety of student-centred approaches to teaching mathematics, in which teachers engage students in exploring challenging tasks (e.g., Leikin & Rota, 2006; Schoenfeld, 1985; Stein et al., 2008; Wood, 1999). One of these approaches is inquiry-based learning, which involves students working on a challenging task to explore, question, reflect on, explain and communicate their mathematical understandings (Artigue & Blomhøj, 2013). The term ‘challenging task’ refers to a novel question, problem, statement or mathematical situation that requires students to exert a mental effort (see Stein et al., 2000). Mathematical inquiry, hence, serves as a tool for problem solving. Within an inquiry classroom, learning is active and requires students to reason, be creative, connect and communicate their ideas. During inquiry, the teacher works alongside students to facilitate learning through interventions that stimulate their thinking and help them engage with tasks which would otherwise be too demanding.

Inquiry has its roots in cognitive and social constructivism (Piaget, 1985; Vygotsky, 1978) and is a multifaceted activity based on student-centred and collaborative learning for the development of higher-order reasoning. Inquiry is a contested term (see Kirschner et al., 2006) and is often used interchangeably with active, problem-based and investigative approaches to learning mathematics. Consequently, the spectrum of inquiry teaching is broad and encompasses teacher-directed as well as student-centred inquiry. Tafoya et al. (1980) classified inquiry lessons as *structured, guided* and *open*. Within the *structured* type, students are provided with the problem, the method and the materials. In a *guided* inquiry approach, students are given the problem and the necessary materials, but they have to find the appropriate problem-solving strategies and methods to use. *Open* inquiry involves students in deciding about the problem, the methods and the materials that they would like to use. Some research suggests that *open* inquiry offers better opportunities for cognitive development, because students are more likely to work independently of the teacher and take ownership of their inquiries (Rocard et al., 2007). However, *structured* and *guided* inquiry have been found to produce better results when the teacher needs to teach content knowledge, and hence learning is more oriented towards conceptual development (Alfieri et al., 2011).

An essential feature of student-centred approaches to mathematics teaching is a cognitively demanding task (Doyle, 1988; Russo & Hopkins, 2018; Sullivan et al., 2015). Models of student-centred teaching often involve the teacher presenting the task at the *beginning* of the lesson. This often sits within a three-phase lesson structure (see Baxter & Williams, 2010; Stein, Engle, Smith, & Hughes, 2008), incorporating the teacher's task presentation, followed by some student exploration of the task, and ending with a whole-class discussion of student-generated approaches and comparing different solutions to the task (for ease of reference, we will refer to these phases as *presentation*, *exploration* and *summary*). Research indicates that teachers choose how to structure their inquiry lessons according to a number of factors, including their own perceived competence in inquiry teaching (Leikin & Rota, 2006), perceived student readiness for inquiry and the particular intended pedagogical goals (Russo & Hopkins, 2018). However, it is unclear how feasible this structure is for teachers who are learning to teach mathematics through inquiry, and what other options might be available, and why they might be used. Research (see, for example, Sullivan et al., 2015; Russo & Hopkins, 2017) has not yet explored how closely secondary school teachers' inquiry lesson structures follow or deviate from the 'task-first' model proposed in the literature. Hence, in this paper, we investigate how teachers operationalise the 'task-first' structure for inquiry teaching and examine how and why they deviate from it. Gaining an understanding of this process can help to better inform the design of continuing professional development (CPD) programmes that aim to support teachers of mathematics in learning to teach through inquiry.

We begin by reviewing the literature on lesson models across a wide range of student-centred teaching approaches in mathematics. We outline the five broad structures that we found, and explore the similarities and differences among them. All of these structures begin with a 'presentation' phase, in which a challenging task is presented to students, and we categorise them as 'task-first'. We consider possible reasons for this lesson structure and, by drawing on what has been discussed in the literature, we examine *alternatives* to starting lessons with a challenging task.

We then present a case study of six lesson observations of three teachers taking part in a year-long CPD programme of inquiry-based teaching. In our minute-by-minute analysis, we use the classification of Tafoya et al. (1980) to examine how teachers structure their lessons to enact inquiry in their classrooms. By tracking the structure of these teachers' inquiry lessons over the course of the year, we find that, although all of these teachers initially adopted a 'task-first' approach, two of them subsequently departed from this, and adopted what we term 'scaffolded inquiry'. Through detailed analysis, we explore possible reasons for this and the extent to which the lessons taught remained true to the inquiry-based intentions.

Lesson structure within student-centred approaches to teaching mathematics

Teachers opt to use certain types of tasks, based on their learning goals for students, their knowledge of the students and the content knowledge they plan to teach (Williams & Clarke, 1997). Using their knowledge of what students can do and the mathematics that they are required to teach, teachers select tasks and accommodate these within particular lesson structures. However, the lessons teachers actually teach may differ considerably in structure from the planned lessons. The way in which teachers structure their lessons, for example, may depend on the anticipated, perceived and/or emerging struggles that students encounter. Various lesson structures have been advocated, particularly for inquiry-based learning, and we carried out a survey of the literature to discover these various structures and explore their similarities and differences.

We conducted a systematic search using Google Scholar, the Educational Research Abstracts of Taylor & Francis online and Springer, along with a hand search of other key journals, such as the Journal for Research in Mathematics Education, for peer-reviewed mathematics education articles, books and conference proceedings, and mathematics teacher professional development literature, using search terms including "student-centred", "inquiry", "investigations", "problem solving", and "cognitively demanding tasks" (a full list of search terms and results is given in Appendix 1). We further narrowed the search to literature that specifically reported on lesson structure and lesson phases, both as promoted in mathematics professional development programmes, and as enacted by mathematics teachers. We imposed no limit on publication date, but excluded text not written in English.

We carried out the searches during the last week of July 2020 and obtained 2765 matches for the terms "mathematics" and "inquiry"; 5940 matches for the terms "lesson" and "structure" and 420 matches for the terms "lesson" and "phases". Across these searches, we found duplicates and, hence, narrowed down the search, which resulted in 847 remaining items. Our next step was to eliminate articles that did not focus on the teaching of mathematics. When we eliminated these, and narrowed our search further, we were left with 92 items, which we downloaded and read (see Appendix 1). Of these, we found 15 articles specifically investigating teachers' structure of student-centred approaches to teaching mathematics (see Table 1): 4 of these were books, 5 were book chapters, and 6 were journal articles.

Student-centred approaches encompassed five main categories, according to the terms used in the different studies: problem solving, Japanese problem solving, investigative and exploratory, inquiry-based and reform-oriented. While the first two approaches both focus on problem solving, we distinguished them due to the distinctiveness of the Japanese problem-solving lesson structure (see Stigler & Hiebert, 1999). We also note that, across the different categories, most studies dealt with lesson observations of individual teachers and analysed their use of tasks (e.g., Leikin & Rota, 2006; Menezes et al., 2015; Ponte et al., 2014; Shimizu, 1999). Some studies involved teachers working alongside teacher educators (e.g., Ponte et al., 2003), while others involved teachers participating in a CPD programme promoting student-centred approaches to teaching mathematics (e.g., Marshall & Horton, 2011; Wood, 1999). Among these latter studies, there was a consensus that there needs to be a well-defined structure for lessons involving cognitively-demanding tasks, and so it seems important to identify what may be the same and different among the various approaches.

Student- centred Approaches		Lesson Phases 1	2	3	4
Problem Solving	Ingram et al. (2019)	*Launch*	*Explore*	*Summarise*	
	Lappan et al. (1998)	*Launch*	*Explore*	*Summarise*	
	Van de Walle (2004)	*Before*	*During*	*After*	
Japanese Problem Solving	Shimizu (1999)	*Hatsumon*	*Kikan-shido*	*Neriage*	*Matome*
	Stigler & Hiebert (1999)	*Task Presentation*	*Student work on task*	*Discussion*	*Summary*
Investigative and Exploratory	Christiansen & Walther (1986)	*Start*	*Development*	*Discussion & summing up*	
	Ponte et al. (2003)	*Task presentation*	*Development*	*Presentations & discussion*	
	Ponte et al. (2014)	*Task presentation*	*Development*	*Discussion & final synthesis*	
Inquiry-based Learning	Leikin & Rota (2006)	*Introduction*	*Inquiry*	*Summary Discussion*	
	Marshall & Horton (2011)	*Engage*	*Explore*	*Explain*	*Extend*
	Menezes et al. (2015)	*Task launch*	*Exploration*	*Discussion*	*Systematization of learning*
	Ponte (2011)	*Introduction*	*Student work on task*	*Presentations & discussion*	
	Stein et al. (2008)	*Launch*	*Explore*	*Discussion & summary*	
Reform-oriented	Lampert (2001)	*Task launch*	*Exploration*	*Discussion & summary*	
	Wood (1999)	*Introduction*	*Student work on task*	*Discussion*	

Table 1**:** Lesson phases within student-centred approaches

Across the different approaches, lessons tended to incorporate three or four phases. The *presentation* phase involved a sequence of teacher actions aimed at explaining the task, while establishing students' readiness for undertaking it. This was often followed by the *exploration* stage, during which students tackled the task or problem posed, individually or in small groups. The *discussion* and *summary* phases, which researchers considered as either one phase or two successive distinct phases, usually involved a whole-class discussion, during which students presented their ideas and their work on the task. During the final phase, the teacher often also summarised the main teaching points arising from students' work. This body of research shows a clear preference for a 'task-first' lesson structure. In other words, it appears that the assumption is that student-centred approaches to teaching mathematics need to start by presenting students with a challenging task.

In this section, we have summarised the literature on student-centred teaching approaches to mathematics using cognitively-demanding tasks. We have seen that the way teaching approaches are presented suggests that doing mathematics using a cognitively-demanding task requires teachers to follow a particular sequence of phases, beginning with the presentation of a task. While, in the first phase, teachers may introduce students to the task, the resources available, and the nature of what they are expected to produce, the second and third phases require a facilitating rather than directing role by the teacher.

Alternatives to beginning an inquiry lesson with a challenging task

Sullivan et al. (2015) reported that in learning to introduce challenging mathematics tasks teachers found the three-phase lesson structure helpful. In a more recent study with seven- and eight-year old students, Russo and Hopkins (2017) investigated the respective benefits of a 'task-first' and a 'teach-first' approach. As explained above, a 'task-first' approach generally follows the launch-explore-discuss lesson phases (see, for example, Stein et al., 2008), with a challenging task being presented to students at the beginning of the lesson. In contrast, a 'teach-first' approach is intended to reduce the cognitive overload that a 'task-first' approach may entail. In a quasi-experimental design with groups of primary-school students receiving either 'task-first' and 'teach-first' approaches, Russo and Hopkins (2018) found that, while both approaches improved problem-solving performance, the 'teach-first' approach was better at improving fluency. As a result, they suggested that "the decision to endorse a particular instructional approach may depend on the learning objectives and associated learning outcomes valued by educators" (Russo & Hopkins, 2018, p. 11). This conclusion challenges the idea that it is essential to begin an inquiry-oriented lesson with a cognitively-demanding task.

By starting with a teacher-facilitated discussion of key mathematical ideas and content, a 'teach-first' approach is more scaffolded. Following this, students are assigned routine tasks as consolidation work, an approach that prepares them to grapple with the mathematical ideas inherent within the lesson's main, more challenging, task. Consequently, the cognitively-demanding task is more likely to be offered towards the middle or end of the lesson. This approach seems more closely aligned to lesson structures involving an *exposition-practice-consolidation* teaching approach (see Buhagiar & Murphy, 2008; Burkhardt, 1988). In supporting student learning, teachers need to scaffold the *problem solving* rather than the problem (see Foster, 2019). This means that, rather than narrowly helping students solve the particular problem, teachers focus on the processes of mathematical problem solving and broader learning about problem solving. Proponents of 'teach-first' approaches to teaching (see Kirschner et al., 2006) argue that this lesson structure is more guided. Hence, it provides students with more carefully scaffolded teacher instruction that is more beneficial to their learning than 'task-first' approaches. However, it may be problematic to ensure that scaffolding can support students' inquiry learning without undermining the inquiry aspect altogether (Foster, 2019).

Although there are different definitions of the term (Tanner & Jones, 2000), 'scaffolding' is often regarded as the guidance provided by adults or more knowledgeable others to support students in a task that they would not be able to complete on their own (Stone, 1998; Wood et al., 1976). While scaffolding might appear as a strategy used to *simplify* a task, teacher guidance is rather intended to reduce the cognitive load on the students, thus assisting them to focus on and complete the *given* task (Wood et al., 1976). This support is usually provided 'just-in-time', and in a variety of ways – from modelling and giving hints, to posing questions and engaging in discussions with students. For example, Charalambous et al. (2022) provided evidence about how mathematics teachers can use *enablers* (to support struggling students to start or make progress on a task) and *extenders* (to extend the challenge for those who successfully completed a task) as supplementary tools for challenging tasks that can support teachers to adjust the challenge for different students. Drawing on the work of Wood (1994), Bliss, Askew and Macrae (1996) referred to two forms of interaction – *funnelling* and *focusing*. Whereas in *funnelling* teachers select the thinking strategies and guide students to a predetermined solution, in *focusing* teachers draw attention to key features within the task which might still be unclear to the students. The *focusing* approach to interaction assumes an interpersonal and interactive learning process in which teacher and students are active participants (Bliss, Askew & Macrae, 1996; Van de Pol, Volman & Beishuizen, 2010), so that ultimately it is the students who take responsibility for identifying strategies and making decisions. Scaffolding is, thus, "a dynamic process finely tuned to the learner's ongoing progress", where the teacher's support will depend on the type of task that the students engage with and their response to it during different phases of the lesson (Van de Pol, Volman & Beishuizen, 2010, p. 272). According to Van de Pol, Volman and Beishuizen (2010), scaffolding occurs when teachers apply three closely connected strategies: (1) *contingency* – which refers to just-in-time and responsive support that is adapted to students' current performance on the task; (2) *fading* – which refers to the gradual decrease or stepping out by teachers of the scaffolding offered; and (3) *transfer of responsibility* – which refers to how responsibility for task completion is transferred to students, who then take more control of their learning (see Van de Pol, Volman & Beishuizen, 2010).

Designing CPD to support teaching mathematics through inquiry

Inquiry-oriented lesson structures place demands on teachers, because enacting inquiry is complex (Russo & Hopkins, 2017), and encouraging mathematical thinking and reasoning and applying scaffolding strategies is difficult (Bliss, Askew & Macrae, 1996; Stein et al., 2008). To support the demands that inquiry teaching places on teachers, CPD programmes need to model effective inquiry pedagogies (Luft, 2001).

From the vast literature of studies on mathematics teachers' CPD, it is clear that there has been a shift towards programmes that model inquiry pedagogies (Back, Hirst, De Geest, Joubert, & Sutherland, 2009; Loucks-Horsley et al., 2010), promoting a 'task-first' approach to lesson structure (see Sullivan et al., 2015) and including activities that are similar to those that teachers could be using in their classrooms (see Brown, Collins, & Duguid, 1989). For example, the CPD programme offered by Sullivan et al. (2015) involved a two-day induction programme during which teachers were given specific lessons to use, with tasks that matched the mathematical content that teachers intended to teach, along with a suggested lesson structure. The prescriptive and short-term nature of a CPD programme may leave little flexibility for teachers to experiment with activities in the classroom. This lack of teacher ownership and space for reflection on their teaching may account for the teachers in the study by Sullivan et al. (2015) reporting that they found the 'task-first' lesson structure workable. The CPD intervention used in the present study, which was intended to support teachers who were unfamiliar with inquiry teaching, is similar to Sullivan et al.'s (2015), as it involved the provision of tasks, the promotion of inquiry teaching and a suggested 'task-first' lesson structure. However, there were important differences related to freedom for teachers to select and/or design tasks, work collaboratively to plan lessons and experiment to discover what works for them.

According to Jaworski (2007, p. 1693), CPD activities need to bring in "a critically questioning attitude towards practice and knowledge in practice that allows critical reflection on the practice of teaching". From this perspective, to which we adhered, CPD did not only engage teachers to simply align themselves with inquiry, but instead to look critically at inquiry at the same time. Hence, the CPD programme examined in the present study sought to give teachers the freedom to experiment, negotiate, transform and develop their own understandings. By engaging in this process, they were supported to take ownership and responsibility to transform their teaching and to potentially adapt their lesson structure according to their needs.

The present study seeks to answer the following research questions:

1. *What changes are there in how secondary school mathematics teachers structure their inquiry lessons over the course of a year-long inquiry-based CPD programme?*
2. *How and why do these teachers change the structure of their inquiry lessons during this period?*

While teacher proficiency in structuring lessons may improve with experience in managing inquiry-based lessons, lesson structure may also depend on the kinds of tasks that teachers use. Students' familiarity with the mathematical content inherent within a task is also likely to shape the lesson structure. Hence, it seems reasonable for teachers, particularly those starting from practice that is oriented towards *transmission* approaches (see Askew et al., 1997), to encounter challenges when adopting a 'task-first' approach. We focus here on what might happen to the proposed three-phase lesson structure when teachers incorporate inquiry into their teaching. In other words, we seek to understand when teachers depart from a 'task-first' approach, how they do so, and why. Investigating this is important for achieving a better understanding of how teachers develop lesson structure and, as a result, learn to enact inquiry practices. Understanding this process has the potential to help improve the design of CPD programmes supporting teachers in learning to teach through inquiry.

The learning to teach mathematics through inquiry (LTMI) CPD programme

The empirical research reported in this paper was carried out in Malta. Reports (see OECD, 2019) show that, while teachers of mathematics in Malta are supportive of student-centred teaching and learning, they describe teacher-centred practices, as in the case in many parts of the world. In the Maltese context, this practice is accentuated by a focus on content-based mathematics syllabi and the dominant role of high-stakes examinations (Buhagiar & Murphy, 2008). Hence, rather than unwilling, Maltese teachers appear unprepared to teach mathematics through inquiry. The 'Learning to Teach Mathematics through Inquiry' (LTMI) CPD programme was, hence, an intervention programme offering teachers opportunities, over one scholastic year, to experience, integrate, reflect upon and develop their inquiry teaching practices. Designed and implemented in Malta by the first author during 2015-2016, LTMI was offered to secondary-school teachers of mathematics as a voluntary course (see Calleja, 2016; 2018). While the first author was a non-participant observer, gathering field notes and other data about teachers' learning journeys, teachers and teacher educators, with experience in inquiry practices, facilitated the sessions with participants.

Research studies suggest that CPD programmes for mathematics teachers need to model inquiry pedagogies through a 'task-first' approach (Silver, Clark & Ghousseini, 2007). Consequently, in LTMI, teachers were introduced to inquiry by working on mathematical tasks. LTMI promoted a three-phase model to lesson structure, drawing on research-based principles (see Loucks-Horsely et al., 2010) to involve teachers in an ongoing collaborative engagement to design, implement, share, discuss and reflect on their classroom practices.

LTMI included four half-day summer workshops, followed by participation in monthly meetings held during the scholastic year (see Calleja, 2016; 2018). Sessions were led by teachers with experience in inquiry teaching, and the session themes, informed by the work of Schoenfeld (2013), focused on four inquiry features: *mathematical tasks*, *collaborative learning*, *purposeful questioning* and *student agency and responsibility*. All of the LTMI materials are available online (see www.iblmaths.com).

However, importantly, in its design, LTMI endorsed teacher flexibility to select tasks and develop their own inquiry teaching approaches. Hence, teachers were given autonomy to take control over their learning and offered the space to engage in practice-based discussions with colleagues over time. As an intervention programme, LTMI provided readily available online materials, opportunities for reflection, collaborative support and feedback.

Method

The study of teachers' inquiry lesson structure, described in this paper, is part of a larger study investigating teachers' experiences in learning to teach mathematics through inquiry (see Calleja, 2019; 2021; 2022). To investigate teachers' inquiry practices and their lesson structure, we adopted a case-study approach (Yin, 2003). Our study focuses on the lesson structure of three Maltese secondary-school mathematics teachers as they learned to enact inquiry while participating in the one-year-long CPD programme.

A data-driven, inductive approach (see Boyatzis, 1998) was used, which allows for the identification of patterns, represented by observations grounded in the data. Hence, our understanding of teacher lesson structure emerges from the realities happening in the participants' classrooms through data from recordings of lesson observations and the tasks that the teachers used.

The participants

In this paper, we focus on three teachers (2 females and 1 male) purposefully selected from seven teachers (5 females and 2 males) who participated in the LTMI programme. We selected teachers using Patton's (1990) purposeful sampling strategies of *intensity* (information-rich) and *maximum variation* (differing cases). While all three teachers provided access to extensive data about their lessons, they differed in the type of school, their prior knowledge of inquiry, teaching experience, year groups taught and lesson structure. Data about these participants – Chris, Greta and Sarah (pseudonyms) – and how their selection allows spanning across various relevant variables, is provided in Table 2.

Teacher	School	Prior knowledge of inquiry	Teaching experience (Years)	Year group taught	Lesson structure
Chris	Church	Initial teacher education	1 – 5	7	Multi-phase
Greta	Church	Postgraduate course	16 – 20	8	Multi-phase
Sarah	State	18-month inquiry course	16 – 20	9	Three-phase

Table 2: Data about the participants

These three teachers came into the LTMI programme with different prior knowledge of inquiry. Chris had participated in a 10-hour study unit in his Initial Teacher Education programme. Sarah had participated in PRIMAS (Promoting Inquiry in Mathematics and Science Education across Europe) – a three-year (2010-2013) EU-funded project for secondary-school teachers (see https://primas-project.eu/). Greta had learned about inquiry teaching during her Master's course.

Ethical approval was granted by the University of Nottingham and all participating schools, and informed consent was then obtained from all teacher participants and heads of school prior to conducting the research. This study adhered to the ethical principles of informed consent, confidentiality, anonymity and the right to withdraw at any point (see BERA, 2018).

Lesson observations

Over the course of the study, each teacher was observed teaching on six occasions – twice prior and four times during the CPD programme. The two pre-CPD lesson observations – a 'typical' and an 'inquiry' lesson – were requested, and these lasted between 35 and 45 minutes. All lessons were audio recorded, and field notes were taken. A 'typical' lesson denotes a lesson that teachers offered for observation that they felt exemplified their usual style of teaching. This lesson was intended to demonstrate the ways in which the teachers normally structured and taught their mathematics lessons. An 'inquiry' lesson denotes a lesson that teachers felt resembled inquiry teaching, and this was intended to capture teachers' perception and enactment of inquiry at that particular point in time. Field notes included records of timing and content of the main phases of the lesson, the resources used and anything that the teacher or the students wrote on the whiteboard. The field notes and the audio recordings were combined to produce detailed transcripts and accounts of each lesson (for a complete transcript of one of the lessons see https://doi.org/10.6084/m9.figshare.12593390.v3).

Over the scholastic year, each teacher offered four lesson observations during which they implemented inquiry, lasting between 35 and 45 minutes (see Table 3). Over the course of LTMI, teachers had established a trusting relationship with the first author, and so at this point they provided consent for video recording of all lessons. A fixed mini wide-angle video camera with rotatable lens, placed at height of 2 metres, usually at the top of a cupboard, captured the whole-class activity, while a portable tablet camera was used to focus on small-group interactions. The video recordings were viewed several times and the lessons were then transcribed in full. To complement this observational analysis, we used data from teachers' lesson journals (see Appendix 2 for an exemplar), lesson plans and the task sheets provided to students.

Lesson Observations			**Period**
Typical lesson Inquiry lesson	Audio recorded Field notes	Pre-CPD	May 2015
Inquiry lesson 1 Inquiry lesson 2 Inquiry lesson 3 Inquiry lesson 4	Video recorded	During CPD	November to December 2015 January 2016 April 2016 May 2016

Table 3: Conducting lesson observations

Lesson analysis

For lesson analysis, the time-line method, used in problem-solving research by Schoenfeld (1985), was applied. Figures 1, 2 and 3 show the lesson structure analysis, with lessons divided into 1-minute intervals. The four phases (teacher presentation, individual work, small-group activity and whole-class discussion) are presented on the vertical axis and their occurrence during the lesson is shown along the time-line (with 1 minute as a time unit on the horizontal axis).

Video recordings were first viewed in their entirety to get a sense of the content, with each lesson transcribed by writing short, time-coded descriptions of the content of the recordings. Emerging issues from this preliminary analysis were then investigated further for a more detailed account of lesson structure, inquiry features seen, teacher questioning and activities taking place. Using the framework of Stein, Smith, Henningsen and Silver (2000), we also classified teachers' inquiry tasks (see https://doi.org/10.6084/m9.figshare.21973103.v1).

This classification exercise, carried out in discussion with the second and third authors, involved an analysis of lesson videos, with a focus on how the nature of the task and the cognitive demands changed during instruction. A task involving only factual recall and basic procedures, without explicit emphasis on connections, was classified as requiring a low cognitive demand, whereas one involving more complex procedures, and requiring students to make connections between and among mathematical ideas in new ways, as a high cognitive demand task. Hence, a task was classified as having a high cognitive demand when it required students to make decisions about how to use mathematical knowledge and skills in particular circumstances in order to interpret, analyse and solve problems (Doyle, 1988; Stein et al., 2000).

We also coded mathematical content knowledge, classifying a task as *familiar* when it incorporated mathematical content that students had already studied in class (based on what teachers reported in their lesson plan). The term *familiar* does not imply that the task was easy for students to do, as this also depended on the way in which mathematical content was used (i.e., the cognitive demand). For teachers, the mathematical content inherent within a task is usually cross-referenced to the learning objectives listed in the examination syllabus. Hence, of particular interest were the decisions teachers made in relation to their choice of tasks for their inquiry lessons and the ensuing lesson structure.

Lesson journal and data analysis

Journal writing has been used as a strategy for developing critically reflective teachers (see Nichols et al., 1997), and a 'lesson journal' was provided to teachers, comprising a booklet in which they were invited to write post-lesson reflections on their inquiry practices. Teachers were free to reveal any aspects of their thinking that they wished to, but the lesson journal contained prompts inviting them to comment on the lesson and describe what they wanted to achieve, what challenges they encountered in planning for and teaching their lessons, what they had learned, and what they would like to improve on or do differently in the future (see Appendix 2 for an exemplar). In addition, teachers were also asked to provide the approximate time dedicated to each lesson phase (see Appendix 3). Over the period of the study, journal entries varied in both number and the amount of detail that the teachers included. On average, teachers described between eight and 10 learning experiences each, with a total of 27 collected from the three case-study teachers. For analysis, the data from teachers' lesson journals was carefully read several times and cross-referenced to the task sheets that the teachers provided, as well as the lesson observation data. This allowed tentative findings to be triangulated and, in particular, the lesson journals provided an important insight into teachers' expressed motivations and reasons for their actions, which considerably enhanced the detail of the analysis. Table 4 below illustrates how data was collected and analysed to produce codes that together addressed the research questions.

<table>
<tr><th>Research question</th><th>Data collection</th><th>Analysis of</th><th>Codes</th><th></th></tr>
<tr><td rowspan="2">What changes are there in how secondary-school mathematics teachers structure their inquiry lessons over the course of a year-long inquiry-based CPD programme?</td><td>Lesson observations</td><td>• Video recordings
• Lesson structure time-line</td><td rowspan="2">• Teacher's inquiry approach
• Lesson phases
• Time dedicated to each lesson phase
• Choice of task
• Teacher's role
• Students' role</td><td rowspan="2">What</td></tr>
<tr><td>Reflective journal</td><td>• Lesson goals
• Teaching strategies</td></tr>
<tr><td>How and why do these teachers change the structure of their inquiry lessons during this period?</td><td>Lesson observations</td><td>• Video recordings
• Lesson plans
• Task sheets</td><td>• Choice of task
• Lesson phases
• Teacher's role
• Students' role
• Intervention strategies</td><td>How</td></tr>
</table>

	Reflective journal	• Post-lesson reflections • Task sheets	• Conceptions of inquiry teaching • Constraints due to syllabus and exams • Inquiry as a new pedagogy	Why

Table 4: Data collection and analysis cross-referenced to research questions

Findings

Lesson observations pre-CPD

Figures 1, 2 and 3 illustrate how the teachers structured their two pre-CPD lessons, and enable comparison between the typical and inquiry lessons of each teacher. Typical lessons seemed to be dominated by teacher presentation. Chris, for example, devoted 25 minutes to teacher presentation, Greta 22 minutes and Sarah 25 minutes. This phase usually included the teacher explaining mathematics to the whole class: explaining concepts, showing examples, giving notes, correcting homework and clarifying student difficulties. In this phase, the teacher usually dominated the talk, with little student discussion taking place. In the first part of the lesson, both Chris and Sarah included a single stretch of teacher presentation. Greta followed a similar structure, yet engaged the class in two shorter discussions, followed by a longer discussion based on a problem that students worked on individually. Such lessons were dominated by the teacher explaining things at the beginning and then, usually half-way through the lesson, shifting to an activity in which students did some practice on what had been presented and explained.

A pattern also emerged from the structure of teachers' inquiry lessons. All of the teachers adopted a three-phase lesson structure, consisting of a task presentation, collaborative work and a whole-class discussion (see Table 5). Chris devoted extensive time to collaborative work, with Sarah allowing practically the same amount of time for small-group work and the final whole-class presentation and discussion. Unlike Chris and Sarah, Greta intervened twice (17th and 37th minute), addressing the whole class about an aspect that seemed unclear. It appeared that, prior to their engagement in the LTMI CPD, these three teachers already conceived of inquiry lessons as consisting of three main phases, similar to the structure eventually presented and modelled in LTMI. Indeed, teachers were exposed to this same structure during the summer workshops.

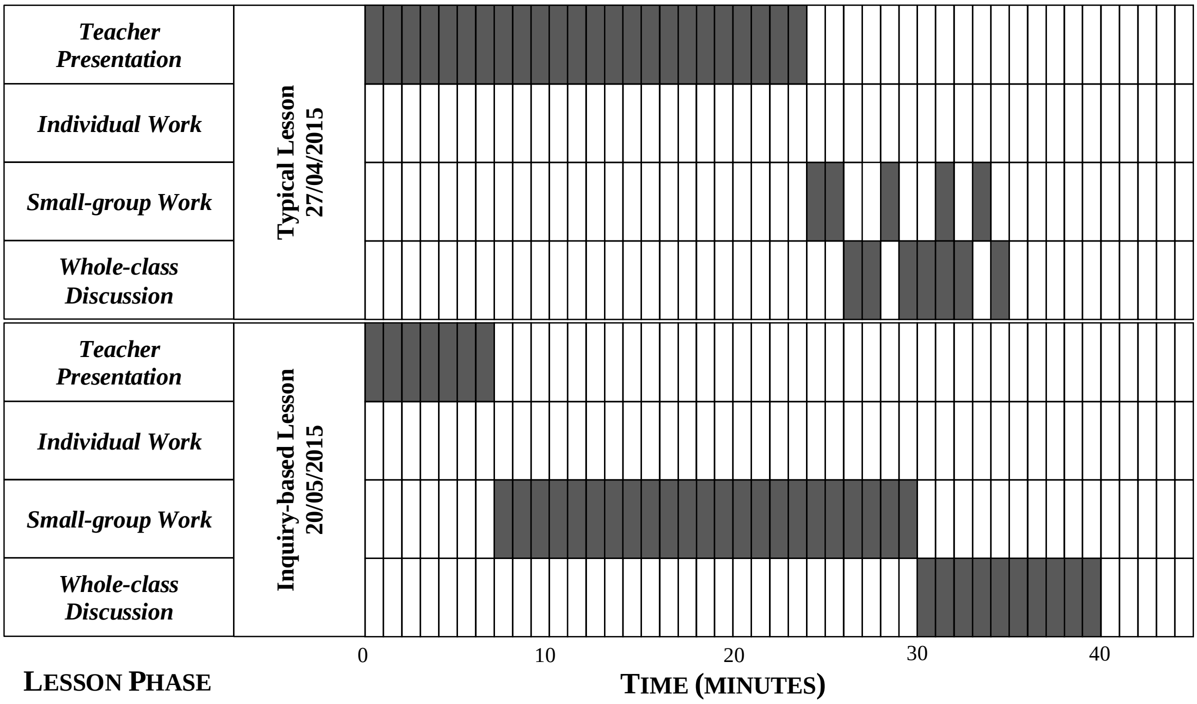

Figure 1: Chris' structure of lessons pre-CPD

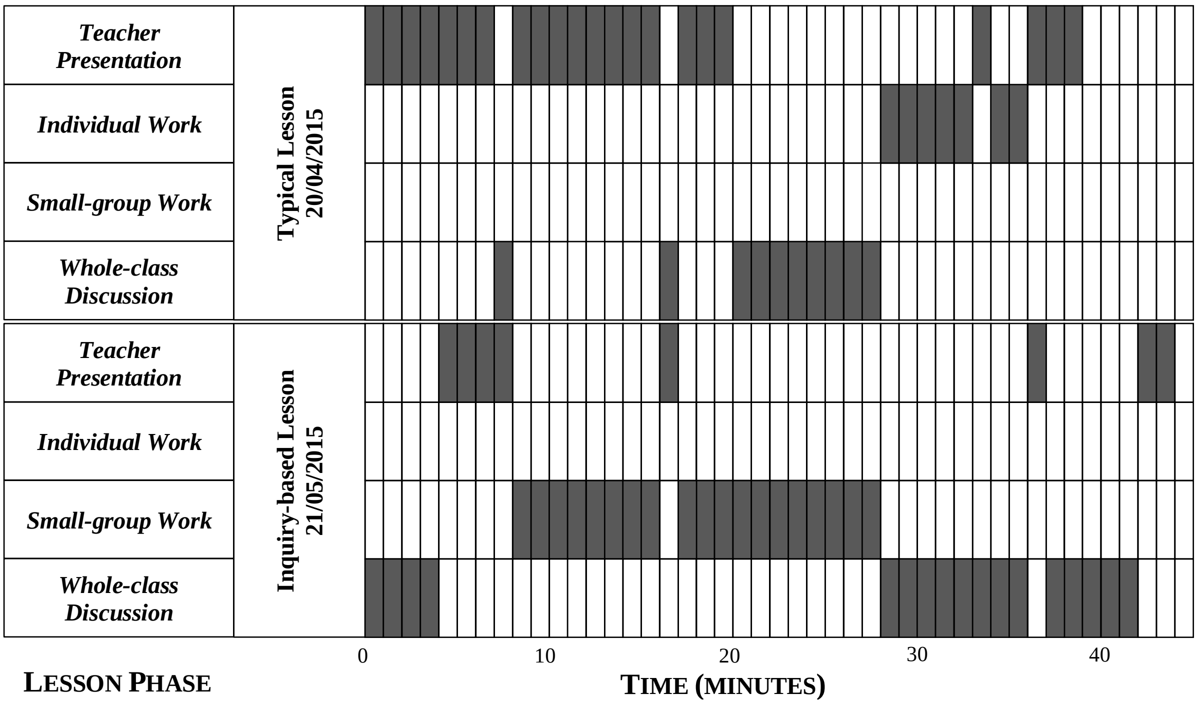

Figure 2: Greta’s structure of lessons pre-CPD

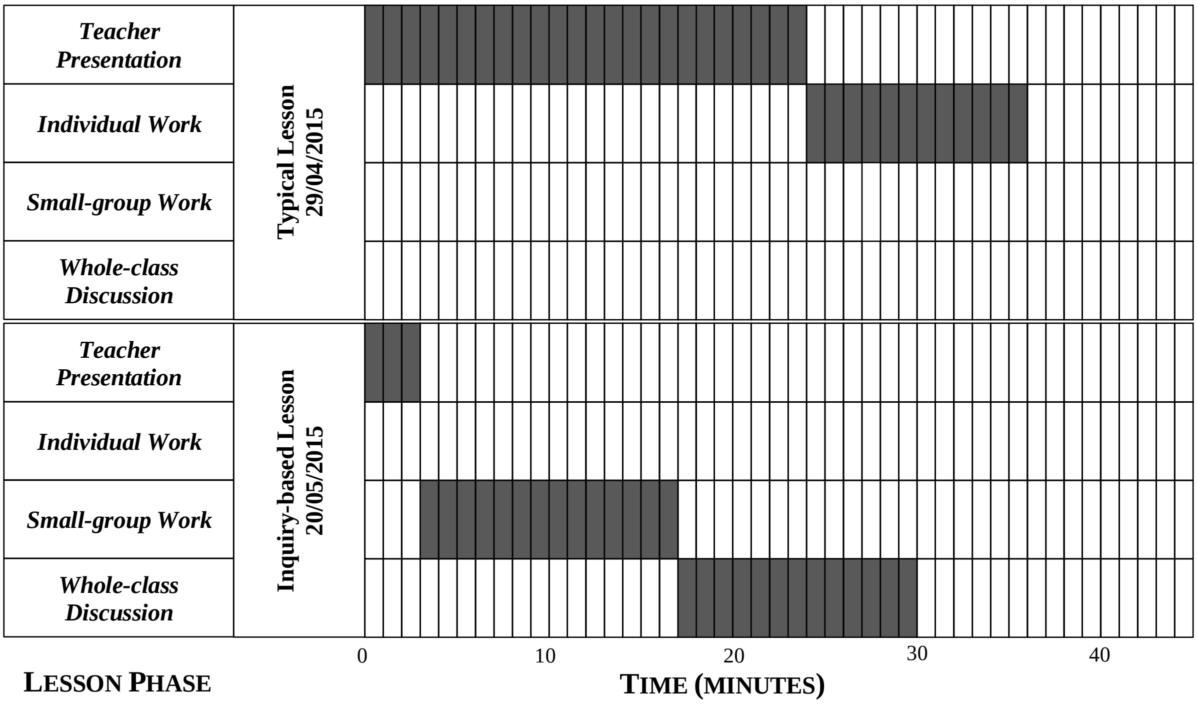

Figure 3: Sarah's structure of lessons pre-CPD

For their inquiry lessons, all three teachers used tasks that they had found on the internet, as revealed in their lesson plans. The task offered by Chris presented students with the question: "*My shoe size is 28 cm. How tall am I?*". Greta presented students with a task entitled "Odds and Evens", obtained from the NRICH website[1]. The task offered by Sarah presented students with a set of probability statements for discussion (see Swan, 2005).

Teachers	**Lesson phases (minutes)**		
	Task presentation	**Group exploration**	**Whole-class discussion**
Chris	7	23	10
Greta	8	20	14
Sarah	3	14	13

Table 5: Time dedicated to each lesson phase pre-CPD

Teachers	**Actions taken to enact inquiry**			**Type of inquiry**
	Task presentation	**Group exploration**	**Whole-class discussion**	

[1] A University of Cambridge affiliated website providing mathematical thinking tasks (see https://nrich.maths.org/4308 for the task).

Chris	Explains task and expectations of how students need to work	Observes without intervening unless prompted by students	Asks each group to describe their solution strategies	Guided
Greta	Revises content through a whole-class discussion and then presents task	Observes and asks students to explain their work	Challenges students' answers, asking questions to uncover their thinking	Structured
Sarah	Explains task and how students need to work	Observes students and encourages discussion	Requests reasons, asks students to comment and then summarises main points	Structured

Table 6: Data from lesson observations on how teachers enacted inquiry pre-CPD

Data from lesson observations (see Table 6) shows that, in the task presentation phase, Chris and Sarah explained the task, whereas Greta required students to recall content from previous lessons before presenting the task. During the final whole-class discussion, all three teachers invited students to share their work with the rest of the class. Chris employed a 'show and tell' activity (see Ball, 2001), asking each group of four students to report on their methods. Each group took turns and the ensuing discussion involved interaction between the teacher and a student from each group. In his *guided* inquiry approach (see Tafoya et al., 1980), Chris provided his students with the problem and materials, but not the method. In contrast, the inquiry approaches of Greta and Sarah were *structured*, as they provided their students with the problem, the materials *and* the methods to use. During group work, both teachers observed students and encouraged thinking and discussion. Greta's students struggled to explain their work and communicate their thoughts during the whole-class discussion. Greta usually intervened to support students by describing what they had done. Sarah also attempted to model inquiry by asking students to explain their reasons. Whenever a group contributed an idea, she involved others, asking them to comment and say whether they agreed or not.

Lesson observations during CPD

Analysis of the inquiry lessons observed during the CPD period revealed more differences for each teacher between their four inquiry lessons during the CPD period (see Figures 4, 5 and 6) than between their 'typical' and 'inquiry' lessons before CPD. Given that LTMI was encouraging teachers to experiment and try things out, teachers' inquiry lessons and their lesson structures were expected to be more in flux. During the CPD period, Sarah maintained the three-phase lesson structure in all four lessons observed, while Chris and Greta seemed to adopt variations of it, for reasons that we explore below.

Chris incorporated a sequence of two or three tasks into each of the first three lessons that were observed (see https://doi.org/10.6084/m9.figshare.21973103.v1). With each task that he introduced (see Figure 4 for the instances in which Chris presented these), he applied iterations of a three-phase structure – presenting the task, allowing time for students to work collaboratively in groups and finally doing a short whole-class discussion. Chris always presented tasks through a whole-class discussion and managed group work by initially supervising student work (see Table 6, lessons 1 and 2), in which he observed and encouraged student inquiry, adopting a facilitator role, promoting mathematical thinking and reasoning (lessons 3 and 4). Chris initially adopted an open and unguided approach, where he allowed students to discover things on their own. Over time, however, he shifted towards adopting a more scaffolded and collaborative approach, developing a more active role with students through the use of questioning and probing.

Like Chris, Greta also chose to start her lessons with whole-class discussions, usually to review work done in the previous lesson, before introducing the main task. She wished to ensure that content was ‘incorporated and visible to students’ (Greta, lesson journal). In the task presentation phase, Greta usually asked students to recall the work that they had covered in class and, in three of the four lesson observations, she told students that the task required them to apply content already known (lessons 1, 3 and 4). Greta also made a notable change in her role, particularly in lessons 3 and 4 (see Table 7). In lessons 1 and 2, she intervened more often, and her lessons were aligned to structured inquiry; whenever students appeared to struggle, she would intervene to help them resolve their issues. For example, in observing students during group work, she identified their mistakes and spoke about them. However, in lessons 3 and 4 a shift occurred. Greta initially observed students and did not interact with them as quickly as she had done in previous lessons. She appeared to be attempting to understand their thinking. Once a group *reached a conclusion*, she questioned it and challenged them about it.

Unlike Chris and Greta, Sarah structured her inquiry lessons in three distinct phases. In the task presentation phase, she alternated between initiating a whole-class discussion and starting off with an explanation of the task. However, her role during group work seemed unchanged (see Table 7). During three of the four lessons, Sarah observed students and made sure that they were on task, clarified issues and encouraged discussion. She did not intervene but seemed to take the role of clarifying any difficulties that students had, related to what they were required to do, while also encouraging further discussion and thinking. During the whole-class discussion, she then challenged students to explain their thinking. It was then through the whole-class discussion that she challenged students' thinking and eventually expected the class to reach agreed-upon solutions. Like Chris and Greta, over time Sarah adopted an approach that was aligned to guided inquiry.

Over the scholastic year, for Sarah the three-phase lesson structure seemed to be a standard practice. On the other hand, Greta and Chris experimented with different lesson structures. This is how the three teachers described their approaches to lesson structure at the end of the scholastic year:

> *I divide the lesson into three parts because that is how inquiry lessons are structured. This structure works well for me as it allows time for students to work collaboratively and independently.*
>
> (Sarah, lesson journal)

> *Sometimes I felt students struggled with group work and [I] decided to change the lesson structure to support their work and learning.*
>
> (Greta, lesson journal)

> *I do not think there is just one way to do an inquiry lesson. I usually experiment with different ways sometimes starting with a discussion, presenting a question or a series of small tasks.*
>
> (Chris, lesson journal)

As part of the CPD programme, these teachers met monthly to share tasks and classroom experiences, and discuss possibilities for teaching through inquiry. Since one of the main features of LTMI was the exploration of inquiry tasks, the tasks that these teachers used to engage students in inquiry were a prominent aspect that surfaced in their discussions, their journal reflections and the lesson observations. To analyse this, we classified tasks using the descriptions provided by teachers in their lesson journals and the task sheets teachers offered during lesson observations. Mathematical tasks were classified according to two dimensions: the *mathematical content* of the task and the *cognitive challenge*.

Lesson phases	CHRIS			
	Lesson 1	**Lesson 2**	**Lesson 3**	**Lesson 4**
Task presentation	Explains task and roles during group work	Initiates a whole-class discussion to revise work done	Short whole-class discussion to introduce task	Whole-class discussion through a problem related to task
Group work	Observes students	Observes and encourages students	Observes, asks and request reasons	Observes, asks and requests reasons
Whole-class discussion	Requests answers	Requests answers and reasons	Requests reasons	Requests reasons
Type of inquiry	Open	Open	Guided	Guided
	GRETA			
	Lesson 1	**Lesson 2**	**Lesson 3**	**Lesson 4**
Task presentation	Whole-class discussion to revise work done and introduce task	Whole-class discussion followed by task presentation	Explains task and reads set of probability statements presented	Whole-class discussion to revise work done and present task
Group work	Observes students, asks questions and resolves issues	Observes students, provides suggestions and answers questions	Observes students and asks questions	Observes students
Whole-class discussion	Requests answers and guides students to reasons	Requests answers and guides students to reasons	Requests reasons and shifts decisions onto students	No whole-class discussion
Type of inquiry	Structured	Structured	Guided	Open
	SARAH			
	Lesson 1	**Lesson 2**	**Lesson 3**	**Lesson 4**
Task presentation	Initiates a whole-class discussion to revise work done and then presents task	Explains task	Initiates a whole-class discussion to revise work done and then presents task	Explains task
Group work	Observes students and answers questions	Observes students and allows students to resolve uncertainties	Observes students and encourages discussion	Observes students and encourages discussion
Whole-class discussion	Requests answers and guides students to reasons	Requests answers and guides students to reasons	Requests reasons and shifts decisions onto students	Requests reasons and shifts decisions onto students
Type of inquiry	Structured	Open	Guided	Guided

Table 7: Data from lesson observations on how teachers enacted inquiry during CPD

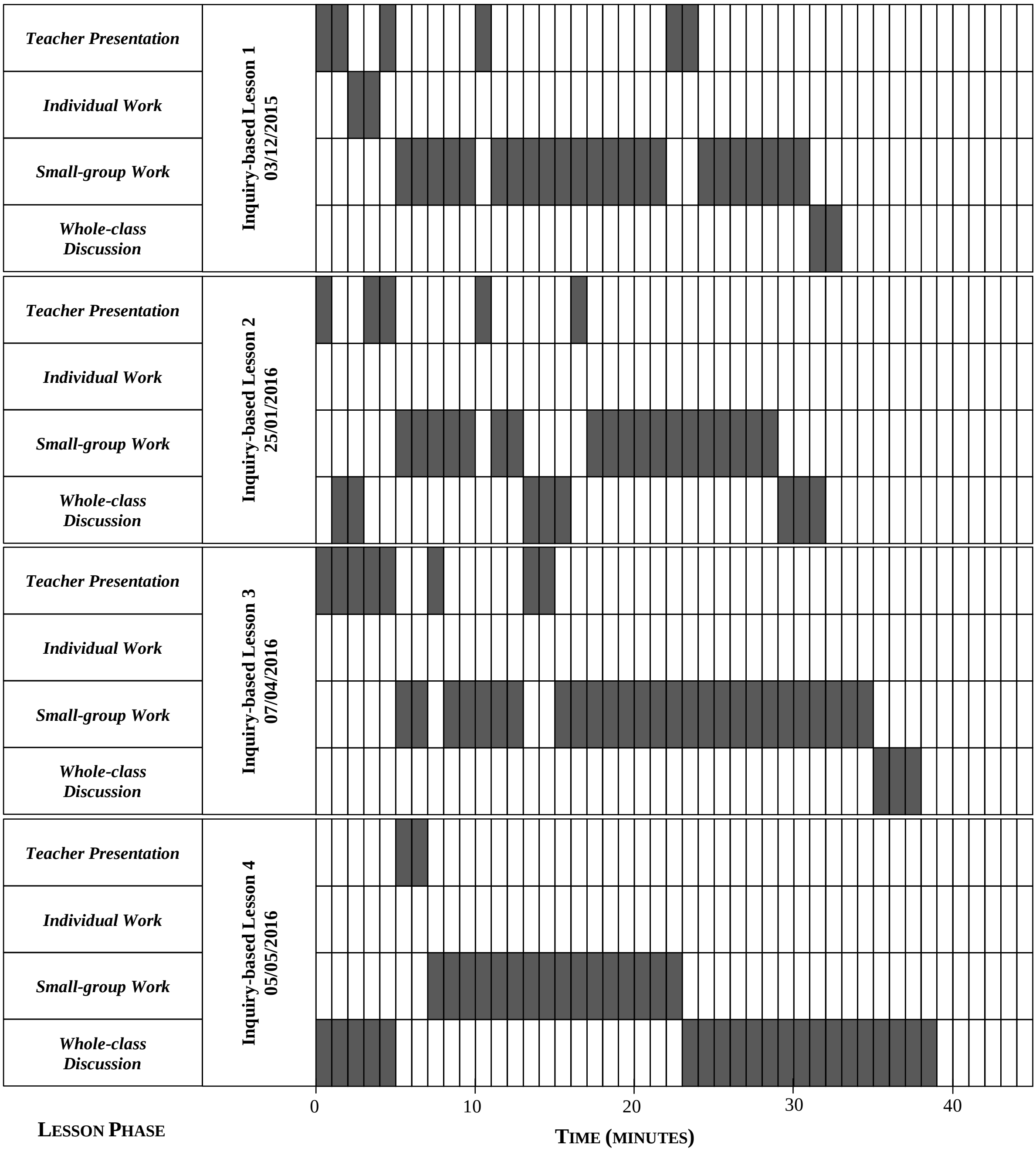

Figure 4: Chris' inquiry lesson structure during the CPD

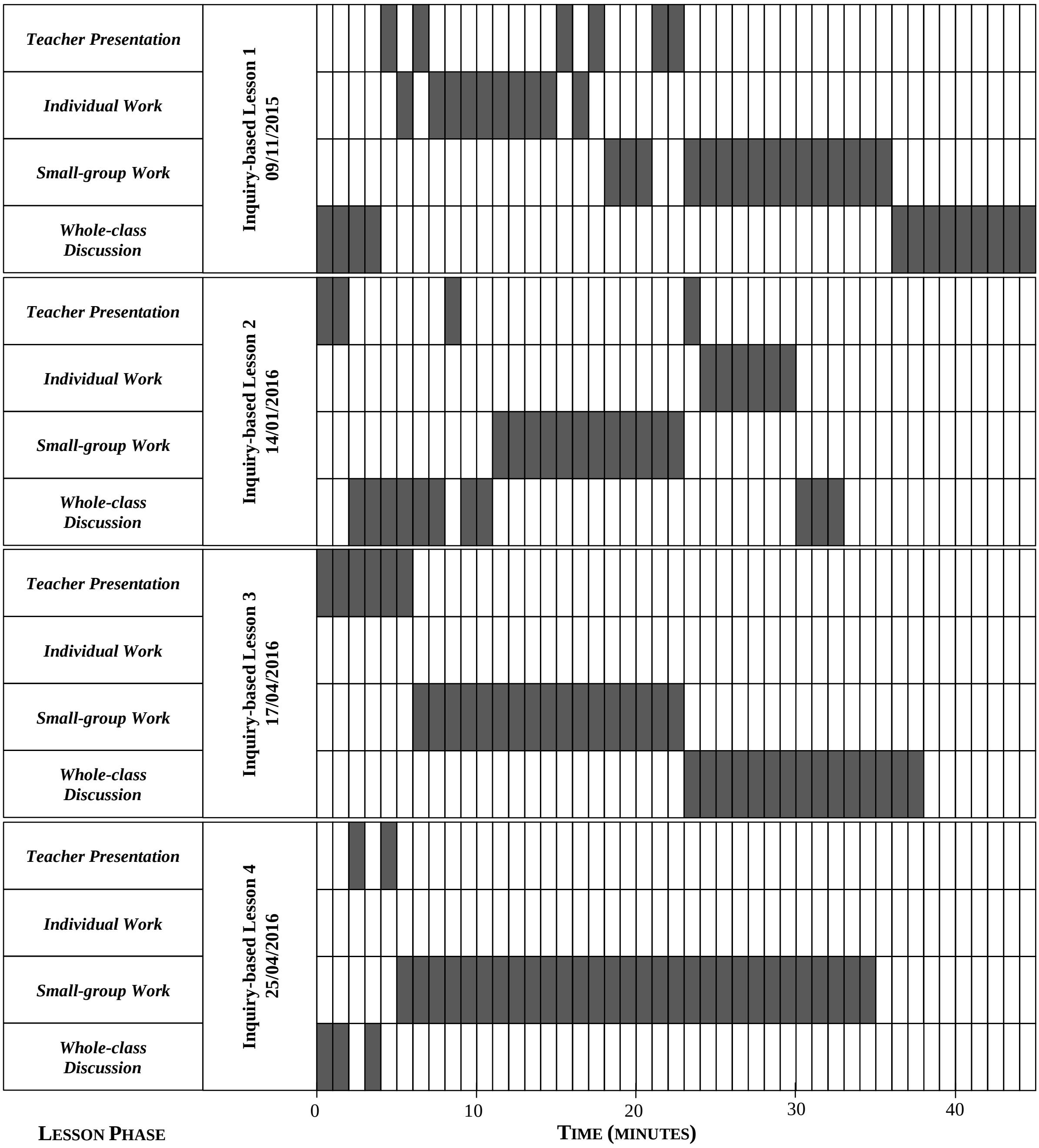

Figure 5: Greta’s inquiry lesson structure during the CPD

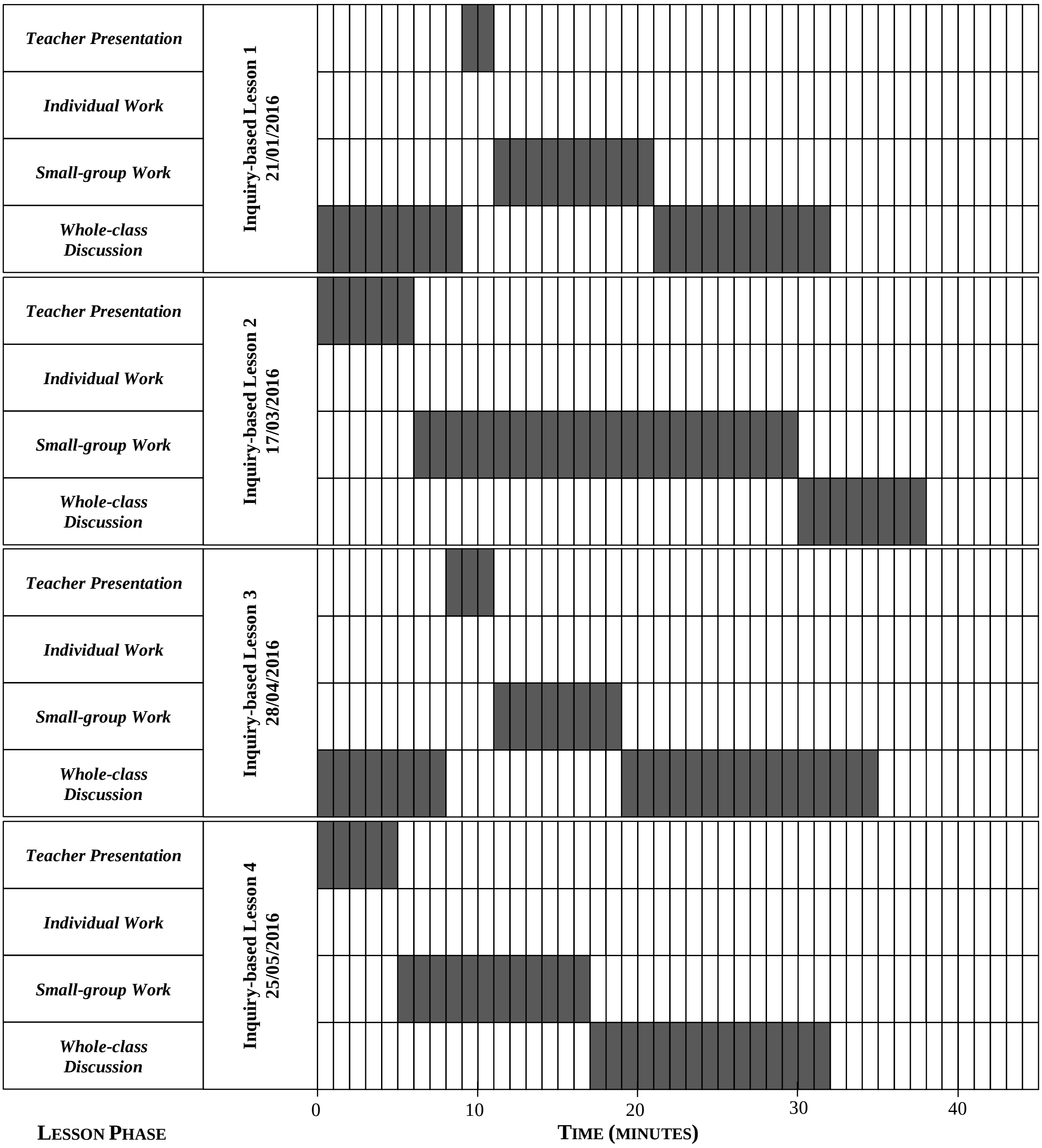

Figure 6: Sarah's inquiry lesson structure during the CPD

Teachers' use of inquiry tasks and lesson structure

We now move on to analyse teachers' use of inquiry tasks and explore reasons for adapting the lesson structure promoted in LTMI. We use data generated from the reflective journals, particularly the post-lesson reflections, and the submitted task sheets. In their inquiry lessons, teachers generally chose to use challenging tasks, because they interpreted these as lending themselves more to inquiry. However, analysis of their lesson journals indicated that mathematical content appeared to be an important factor driving the choice of instructional tasks when implementing inquiry. One reason for this was to address the time constraint that inquiry posed in teaching the syllabus:

> *My inquiry lessons are directly related to the content because there is so much content I need to teach in a very limited time.*
>
> (Greta, lesson journal)

> *I feel I want to do lessons differently but I feel restricted by a system which prioritises content delivery and is exam-driven.*
>
> (Chris, lesson journal)

Analysis of the teachers' tasks (see Table 8) suggests that, in three of their four inquiry lessons, these teachers used tasks that incorporated a high cognitive challenge and familiar content. However, in their lesson implementation, Chris and Greta generally adopted a multiple-phase structure, while Sarah retained a three-phase lesson structure. One explanation for the use of a multiphase structure is that Chris and Greta used different intervention strategies to guide students towards task completion. Chris structured his lessons 'to facilitate students' uptake and progression with inquiry' (Chris, lesson journal) by gradually increasing the cognitive demand. He did this by planning lessons incorporating a sequence of short tasks (in his first three lessons – see https://doi.org/10.6084/m9.figshare.21973103.v1). Greta, on the other hand, addressed students' challenges as they arose during lessons. She intervened to 'minimise students' struggles and prevent them from giving up' (Greta, lesson journal). This could imply that teachers modified lesson structure to provide support to students:

> *Students are not used to inquiry and providing a sequence of tasks worked well as they could see that the challenges that I offered were achievable.*
>
> (Chris, lesson journal)

In the case of Chris, this help appeared aimed at facilitating the process of engaging students with inquiry, while, for Greta, help appeared to be targeted at making tasks more accessible to students. Both teachers intervened and stopped group activity. In the case of Chris, this was to introduce a new task (see Figure 4 and his interventions in lessons 1, 2 and 3), while Greta stopped group activity and conducted a short whole-class discussion to address student difficulties (see Figure 5 and her interventions during lessons 1 and 2).

	INQUIRY TASKS							
	LESSON 1		LESSON 2		LESSON 3		LESSON 4	
	High Cognitive Demand	**Familiar**	**High Cognitive Demand**	**Familiar**	**High Cognitive Demand**	**Familiar**	**High Cognitive Demand**	**Familiar**
Chris	✓	✓	✓	✓	✓	×	✓	✓
Greta	✓	✓	×	×	✓	✓	✓	✓
Sarah	×	×	✓	✓	✓	✓	✓	✓

Table 8: Analysis of inquiry tasks used by teachers

Unlike Chris and Greta, Sarah used a three-phase structure in all of her lessons. She encountered similar challenges, but adopted a different intervention strategy – addressing these by modifying the time and the amount of work allocated to each group.

> *When a task included a number of problems or questions, rather than asking each group to work on all of them, I preferred to have them focus on one or two. In that way, students spent more time discussing each one during the group activity.*
>
> (Sarah, lesson journal)

Moreover, when Sarah recognised that the final whole-class discussion required more time, she amended her tasks in such a way that the small-group activity involved thinking about one or two situations rather than say six different ones. Hence, in lessons 3 and 4, each group worked on different tasks dealing with just one or two problems (or statements), and she could then dedicate more time to the whole-class discussion (see Table 9).

The 40-minute lesson time-frame was a constraint to inquiry implementation. For Greta and Chris, tasks with a high cognitive challenge seemed difficult to implement within a three-phase lesson structure, because the teachers considered that students needed extensive time to work on solving the task within the group activity. As a result, group work tended to take a large proportion of the lesson time (see Table 9), in some cases consequently allowing limited time for the final whole-class discussion. For this reason, it seemed that teachers were also less likely to select tasks that presented students with novel mathematical content. Whenever they did – Chris in lesson 3 and Greta in lesson 2 – the teachers guided students at an early stage during the lesson (see also Figures 4 and 5). Chris intervened, addressing the whole class, after two minutes of group activity, while Greta adopted a different approach, providing hints to individual students as they engaged with the task. It seems likely that, in both cases, the teachers wanted the students to reach the intended learning outcomes within the limited lesson time available.

	Lesson 1		Lesson 2		Lesson 3		Lesson 4	
	Phase	Time (min)	Phase	Time (min)	Phase	Time (min)	Phase	Time (min)
Chris	Presentation	6	Presentation	7	Presentation	8	Presentation	7
	Individual work	2	Individual work	0	Individual work	0	Individual work	0
	Group-work	23	Group-work	19	Group-work	27	Group-work	16
	Class discussion	2	Class discussion	6	Class discussion	3	Class discussion	16
Greta	Presentation	10	Presentation	11	Presentation	6	Presentation	5
	Individual work	10	Individual work	6	Individual work	0	Individual work	0
	Group-work	16	Group-work	12	Group-work	17	Group-work	31
	Class discussion	9	Class discussion	3	Class discussion	14	Class discussion	0
Sarah	Presentation	11	Presentation	6	Presentation	11	Presentation	5
	Individual work	0	Individual work	0	Individual work	0	Individual work	0
	Group-work	10	Group-work	24	Group-work	8	Group-work	12
	Class discussion	11	Class discussion	8	Class discussion	16	Class discussion	15

Table 9: Time allocated for each lesson phase

Sarah, who held to a three-phase lesson structure throughout, appeared more consistent in managing the time allocated to each phase. She did not dedicate any time to individual activity – a strategy that Chris and Greta used inconsistently over time (data cross-referenced to teachers' lesson journals – see Appendix 3). As already mentioned, unlike Chris and Greta, Sarah managed to allocate a higher proportion of time to the final class discussion.

Discussion

In this paper, we have focused on the changes to the standard three-phase inquiry lesson structure made by three teachers who participated in a year-long inquiry-based CPD programme. We tracked the developing structure of their inquiry lessons through minute-by-minute analysis, before and during the CPD, to understand whether, how and why teachers might depart from a 'task-first' approach. In the following sections, we discuss findings involving teachers' lesson structure related to their inquiry task selection, contextual constraints and the observed shift from a 'task-first' approach to what we call 'scaffolded inquiry'.

Teachers deviate from the 'task-first' lesson structure in order to address contextual constraints and offer increased support to students

Our analysis shows that while one teacher (Sarah) followed the three-phase lesson structure advocated by the LTMI CPD programme through the entire year, the other two teachers (Chris and Greta) did not. This finding resonates with that of Russo and Hopkins (2018), who found that primary-school teachers generally preferred a 'teach-first' approach, as they felt that it offered more opportunities to scaffold student learning. In their study, primary-school students first worked on routine tasks, with the aim of mastering strategies for problem solving, before finally working on a challenging task. This approach may appear similar to that adopted by Chris. However, there would seem to be some differences. The initial tasks offered by Chris were not routine (see https://doi.org/10.6084/m9.figshare.21973103.v1), and required students to think deeply (for example: asking students to write down everything that they could think of concerning the number 24, and then explain their ideas to the class). Additionally, following each task, Chris led a short whole-class discussion before moving to the next task. In the case of Greta, her interventions (through prompts and questions) appeared more focused on enabling students to complete the task successfully. These interventions can be considered as *enablers* (see Charalambous et al., 2022) and operate within what Williams and Baxter (1996) called 'discourse-oriented teaching'; the enablers serve as scaffolding. Hence, in their interventions and actions, teachers either broke down the main task into smaller sub-tasks (as in the case of Chris) or intervened to direct students to the outcome of the activity (as in the case of Greta). A critical aspect emerging from teachers' lesson structures was the way in which they managed classroom activity – in particular, how they used facilitation strategies to support students in engaging with mathematical tasks.

It appears that modifications to lesson structure were made to facilitate students' work on tasks and, hence, make them more accessible, so that they could engage in inquiry. For two of the three teachers, adopting a 'task-first' lesson structure did not seem feasible. To facilitate learning and subsequent inquiry, Ward and Sweller (1990) suggested the use of appropriately structured worked examples. Similar to the sequence of tasks (Chris) and enablers (Greta), worked examples are purposefully introduced to serve as a form of 'telling' (see Williams & Baxter, 1996) in supporting students' task completion. Drawing on our case studies, we interpret 'telling' as a scaffolding approach that teachers use to support students in persisting with challenging tasks. Teacher 'telling' took different forms that included explaining the task and what needs to be done, providing feedback on students' work, giving suggestions to guide students forward in their inquiry, and questioning to engage in interactions that probe their thinking.

In structuring their inquiry lessons, the teachers in the present study also sought to address contextual constraints. Contextual concerns arose as a result of the limited lesson time available, the amount of content they were required to teach and the necessity of using cognitively demanding tasks for inquiry. Initially, teachers seemed to believe that students needed extensive time to work on solving the task, with small-group activity taking a large proportion of the lesson time (sometimes 20 minutes or more). Similar to what Priestley (2010) found in Scottish secondary schools, the 40-minute lesson time-frame available was a constraint to the implementation of collaborative, inquiry-based methods. As a result, very little time was usually left for the final whole-class discussion. For the same reason, teachers were also less likely to select tasks that presented students with novel mathematical content.

Sarah, on the other hand, addressed the lesson time issue differently. While following the three-phase lesson structure, she reduced the amount of work she assigned to students during group activity. This resulted in group exploration that focused on fewer tasks, with more time consequently available for the whole-class discussion part of the lesson. Indeed, the split between group work and the whole-class discussion part for the first two lessons (10 min to 11 min and 24 min to 8 min) and lessons 3 and 4 (8 min to 16 min and 12 min to 15 min) shows that Sarah dedicated more time to the final whole-class discussion in her later lessons. When cross-referenced to her roles and actions during these two phases (see Table 6), we find that while Sarah offered students some time (about 10 minutes) for unassisted discovery during group work, she then adopted a more active role in leading, intervening and summarising the main learning points emerging from students' presentations during the final whole-class discussion. Her decision to dedicate more time to assisted inquiry is supported by research (see, for example, Alfieri et al., 2011) that suggests that it may be more useful for learners than presenting them with either totally unguided or explicitly-taught tasks. It can be argued, hence, that while Sarah first offered an opportunity for unguided collaborative learning, she then followed this up with a more guided whole-class discussion.

Teachers shifted from a 'task-first' approach to 'scaffolded inquiry'

The teachers in this study did not find that using a 'task-first' approach necessarily facilitated the implementation of inquiry. Leikin and Rota (2006) attributed adopting a three-phase structure to a teacher's growth in expertise. The present study challenges this assumption and we think this may be related to contextual differences. Whether teachers adopt a three-phase lesson structure may also depend on their preconceptions of inquiry lessons and the tasks that they use. For instance, prior to their participation in LTMI, the three teachers in the present study adopted a three-phase structure for the inquiry lesson that they offered for observation. Sarah, who was already familiar with this structure prior to her participation in CPD, maintained it during LTMI. However, Chris and Greta, for whom LTMI was the first inquiry-related CPD that they had participated in, structured their inquiry lessons differently.

In planning for inquiry teaching, Chris structured his lessons using a series of tasks that gradually introduced students to more challenging work. Chris introduced inquiry by adopting a scaffolded approach usually consisting of offering two or three tasks with increasing cognitive challenge. Similarly, Greta's lesson structure addressed the emerging struggles that students encountered. By using a 'teach-first' approach (see Russo & Hopkins, 2018), Greta presented the challenging task through a whole-class discussion reviewing prior mathematical content that was relevant to the task. As students worked collaboratively on the task, Greta would then offer support generally through direct interventions or 'telling actions' (see Lobato, Clarke & Ellis, 2005); that is, by spotting and indicating mistakes, questioning methods used in arriving at an answer, or stopping the activity and conducting a short, focused whole-class discussion that addressed emerging difficulties. From midway through the scholastic year, Greta began to move away from such scaffolding and became more likely to listen to and challenge student thinking in one-to-one conversations. Both teachers appeared mindful of not presenting tasks with too much cognitive demand, as characterised by the 'task-first' approach, but adopted what we term here a 'scaffolded inquiry' approach, by offering initial support through increased challenges (Chris), discussion and direct feedback (Greta).

Sarah, on the other hand, was consistent in using a three-phase lesson structure in enacting inquiry. Moreover, when challenges appeared, she learned to make adaptations to the tasks she presented, without changing her lesson structure. Over time, she reduced the amount of work assigned (for example, the number of statements to discuss or cards to sort out) and limited collaborative tasks to about 10 minutes, which, in turn, allowed more time for the concluding whole-class discussion. Sarah found that, in reducing the quantity of work assigned, students eventually engaged in more focused, in-depth discussions, and so she found herself with more time to challenge students' reasoning during the final discussion.

Conclusion

The present study builds on the results reported in the study by Leikin and Rota (2006). It suggests that a teacher's ability in using a 'task-first' lesson structure does not necessarily indicate proficiency in managing an inquiry-based classroom. The study by Leikin and Rota (2006), which describes a mathematics teacher's development in using a 'task-first', inquiry-based lesson structure, focused on a case study of a teacher-researcher. This teacher had two years' teaching experience before engaging in a further 15-month period implementing inquiry with her second-grade students. This case-study participant has similarities with the experiences of Chris in the present study, who had a short teaching experience before joining LTMI. Different from the findings reported by Leikin and Rota (2006), our data does not indicate that Chris developed increased confidence and growing expertise over time. However, he did gradually adapt the three-phase lesson structure. The present study indicates that, irrespective of the lesson structure that teachers use during inquiry teaching, they may be likely to adopt a structure that facilitates and scaffolds student engagement and progression with inquiry. We think that this move to a more scaffolded inquiry was enabled by LTMI, which required teachers to take ownership of their learning to make decisions and to structure their lessons according to what they perceived as practicable in their classroom context.

While concurring with Leikin and Rota (2006) that positive changes towards inquiry in both the teachers' and their students' actions and behaviours occurred over time, the present study shows that modifications to lesson structure may be expected, because they depend on the kind of tasks that teachers decide to use. The teachers in this study did not simply implement the tasks that LTMI offered; they adapted and designed their own, playing an active part in task design, planning and enactment. They scaffolded students' thinking "by providing a predictable structure for inquiry" (Goos, 2004, p. 282). For example, Chris favoured a sequence of tasks, rather than just one cognitively demanding task, and he adopted a different kind of lesson structure. Chris's decision to provide a sequence of tasks that required students to focus their attention on solving a number of separate tasks could have implications for how teachers may learn to teach through inquiry. Rather than promoting the 'task-first' lesson structure as the main approach, CPD could expose teachers to more scaffolded forms of lesson structure. Since these might be more practical and achievable to implement than the 'task-first' structure, teachers might then be in a better position to recognise the use, purposes and benefits of each in promoting student inquiry.

Scaffolded forms of lesson structure are those that deliberately require teachers to provide students with support at different phases of the lesson. In line with an inquiry teaching approach, scaffolded forms of lesson structure intentionally involve a dynamic and reflective process (Tanner & Jones, 2000), based on teacher-student interactions (see Bliss, Askew & Macrae, 1996; Van de Pol, Volman & Beishuizen, 2010). Such structures support Vygotsky's (1978) emphasis on the social dimension of learning and the extent to which a student can take up the challenge offered within tasks, provided that there is support to do this. Students can benefit from socially constructed mathematics knowledge when they engage in just-in-time interactions with teachers (as more knowledgeable others) at specific instances in the lesson. For this to happen, teachers need to hone their skills to scaffold student learning through modelling and coaching (Hmelo-Silver et al., 2007). Teachers, hence, need CPD opportunities through which they learn to use a variety of scaffolding tools – for example, the use of enablers and extenders (see Charalambous et al., 2022). However, besides understanding the *means* (*how* to scaffold), teachers need to be aware of the *purpose* (*what* to scaffold), so that their interventions can bring about the desired behaviour in students as they engage in inquiry (Van de Pol, Volman & Beishuizen, 2010). For example, teachers can learn when to ask questions that assess students' progress, and how to use questions that are intended to assist students in developing their thinking and decision making.

A scaffolded approach to inquiry teaching that emerged from the study reported in this paper involved the breaking down of tasks into a sequence. This approach, we think, may initially support students to learn how to engage in mathematical inquiry, and eventually also lead them to become better equipped to undertake more challenging tasks over time. In turn, with a more scaffolded approach, teachers may perceive the challenges arising within their inquiry classroom as more attainable, and their learning to teach mathematics through inquiry could also take place more gradually.

Our study suggests that the 'task-first' lesson structure may not necessarily be optimal for teachers in enacting inquiry teaching. Only one of our teachers (Sarah) retained the three-phase lesson structure promoted by LTMI. Like the teachers in the study by Sullivan et al. (2015), Sarah found the proposed lesson structure practicable and, hence, did not deviate from it. On the other hand, Chris and Greta gradually moved away from this structure. In the process of learning to teach through inquiry, teachers sought a balance between integrating inquiry and, at the same time, addressing contextual challenges. While serving as a useful guideline, the 'task-first' approach may offer limitations in particular teaching situations. Within a context influenced by numerous constraints (restricted lesson time, content coverage and an examination-oriented system), a 'task-first' approach may be too rigid a lesson structure, and teachers may find it difficult to adhere to. In addressing this, the teachers in this study used their "sense of practicality" (see Hargreaves, 1994, p. 12) to adapt the lesson structure according to what appeared to work for them in practice.

For two of the teachers in this study, an alternative lesson structure emerged – which we term the 'scaffolded inquiry' approach. While different from 'teach-first', as it intentionally incorporates scaffolding strategies to support students' learning through inquiry, a 'scaffolded inquiry' approach may still start off with a discussion of key mathematical ideas and include appropriately selected teacher actions and interventions, such as the use of worked examples (see Ward & Sweller, 1990). Alternatively, the 'scaffolded inquiry' approach may incorporate a sequence of smaller tasks, leading up to the main inquiry task. For teachers with limited prior knowledge of inquiry teaching, this approach may be more feasible. 'Scaffolded inquiry' appears to offer a plausible avenue for teachers, because it allows them to shift gradually towards more inquiry-oriented teaching, enabling them in the process to transfer and exploit their more transmission-oriented intervention strategies.

We argue that inquiry may not be an unguided activity but may *include* teachers' supportive interventions. However, inquiry is not a highly-guided activity, and teachers need to be sensitive *not* to scaffold instruction when students can already access the task on their own. A scaffolded inquiry approach, which aligns most closely with guided forms of inquiry – rather than more structured and open inquiry (see Tafoya et al., 1980) – provides teachers with opportunities to support students with finding the appropriate problem-solving strategies and methods to use. By 'scaffolded inquiry' we mean teachers structuring their lessons and intervening to help their students engage with tasks that would otherwise be too demanding. Unlike structured inquiry (see Tafoya et al., 1980), in scaffolded inquiry the provision of student support is less explicit to students, and is provided by teachers when they recognise that students can use it to progress with their exploration of strategies and methods for solving the task. While scaffolding strategies may appear to reduce the cognitive load and, hence, the challenge of inquiry tasks, we draw on Hmelo-Silver et al. (2007) to argue that student inquiry benefits from thoughtful support by teachers at different stages of the inquiry process. In implementing inquiry, teachers provide scaffolding when learning outcomes are comprehensible to students but not easily achievable without support. Through scaffolding, the teacher draws attention to significant ideas by providing "content knowledge on a just-in-time basis" (Hmelo-Silver et al., 2007, p. 100); that is, when they judge that students need it and can make use of it.

In inquiry teaching, teachers should avoid relying on telling as the main form of interaction to support students. 'Scaffolded inquiry' can involve using a range of teacher intervention strategies (e.g., discussion, collaboration, structured tasks, questioning and telling) to enhance student learning. We see 'scaffolded inquiry' as having the potential to support students in the transformation of their roles towards becoming more independent learners. The challenge for teachers, which inquiry-based CPD initiatives need to address, is how to integrate this approach within their inquiry teaching; learning when, how and why they may step in and out of student inquiry. There is a need for CPD programmes that offer teachers different approaches to lesson structure, identifying and discussing differences while supporting them to make informed decisions about which approach to adopt and adapt when, and how such choices might evolve as the teachers' experience with inquiry-based teaching develops. Such decisions also need to be based on the learning objectives and the associated learning outcomes that teachers intend to achieve through inquiry. Hence, in CPD, rather than just teaching about inquiry, teacher educators need to be sensitive to the challenges that teachers might encounter and, hence, support them through a 'scaffolded inquiry' approach that mirrors strategies that teachers can potentially use with their students.

Appendix 1: Searches for literature on lesson structure within student-centred approaches

Google Scholar advanced online search

Articles			
With all words	*With the exact phrase*	*Anywhere in the article (and in title)*	*In the title of the article*
mathematics classroom teaching	lesson structure	5940	1*
mathematics classroom teaching	lesson phases	420	0
mathematics classroom teaching	inquiry lesson structure	7*	0
mathematics classroom teaching	inquiry lesson phases	0	0
classroom teaching	mathematics lesson structure	51*	1*
classroom teaching	mathematics lesson phases	1*	0

* *These texts were downloaded and read*

Taylor & Francis through the Educational Research Abstracts online search

Anywhere in document	*Title*	*Keywords*	*Number of articles found*
mathematics / maths / math; inquiry			2765
mathematics / maths / math; inquiry-based			1872
mathematics / maths / math; lesson phases			854
mathematics / math / math; lesson phases; inquiry / inquiry-based			180
mathematics / math / math; lesson phases	cognitively demanding tasks		4*
	problem solving / problem-solving		18*
	inquiry / inquiry-based		10*
	student-centred / student-centred		5*
	investigation(s)		18*
		cognitively demanding tasks	2*
		problem solving/ problem-solving	13*
		inquiry / inquiry-based	8*
		student-centred / student-centred	6*
		investigation(s)	1*
mathematics / math / math; lesson structure			1553
mathematics / math / math; lesson structure; inquiry			481
mathematics / math / math; lesson structure	cognitively demanding tasks		4*
	problem solving / problem-solving		41*
	inquiry / inquiry-based		26*
	student-centred / student-centred		11*
	investigation(s)		33*
		cognitively demanding tasks	2*
		problem solving / problem-solving	33*
		inquiry / inquiry-based	33*
		student-centred / student-centred	10*
		investigation(s)	3*

** These texts were downloaded and read*

Springer online hand search

Topic	*Keywords*	*Number of book chapters*
Mathematics education	lesson phases / structure; inquiry / inquiry-based; cognitively demanding tasks	8*
Journal	*Keywords*	*Number of journal articles*
Journal of Mathematics Teacher Education	lesson phases / structure	600
	lesson phases / structure; inquiry / inquiry-based	341
	lesson phases / structure; inquiry / inquiry-based; cognitively demanding tasks	19*
Educational Studies in Mathematics	lesson phases / structure	301
	lesson phases / structure; inquiry / inquiry-based	145
	lesson phases / structure; inquiry / inquiry-based; cognitively demanding tasks	9
International Journal of Science and Mathematics Education	lesson phases / structure	265
	lesson phases / structure; inquiry / inquiry-based; cognitively demanding tasks	164
	lesson phases / structure; inquiry / inquiry-based; cognitively demanding tasks	7*
Mathematics Education Research Journal	lesson phases / structure	146
	lesson phases / structure; inquiry / inquiry-based	56
	lesson phases / structure; inquiry / inquiry-based; cognitively demanding tasks	6*
ZDM Mathematics Education	lesson phases / structure	370
	lesson phases / structure; inquiry / inquiry-based	130
	lesson phases / structure; inquiry / inquiry-based; cognitively demanding tasks	12*

* *These texts were downloaded and read*

List of key sources for the literature review

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Appendix 2: Exemplar from a lesson journal entry

Date: 16/10/2015 Class: form 1 Time: 40 min

Topic: Directed Numbers (Addition + Subtraction)

Approximate time spent

Teacher explanation to the whole class	5 min
Students working individually	5 min
Students working in small-groups	10 min
Whole-class presentation of work	10 min
Other (please specify) Reinforcement from textbook	5 min

Reflections on the lesson

Briefly describe the lesson plan.

This lesson was done to reinforce addition and subtraction of directed numbers. The letters of the alphabet were assigned a numeric value (+ve or -ve). Students were to find the 'numeric value' of their names.

Appendix 3: Data analysis of other inquiry lessons reported in teachers' lesson journals

Teacher		Reported time dedicated to each lesson phase (minutes)				
		Teacher explanation	*Individual work*	*Small-group work*	*Whole-class presentation*	*Textbook practice*
Chris	Lesson 1	5	5	10	10	5
	Lesson 2	5	0	20	10	0
	Lesson 3	5	5	20	5	0
	Lesson 4	5	0	25	10	0
	Lesson 5	5	10	0	0	20
	Lesson 6	5	0	30	5	0
	Lesson 7	5	5	15	15	0
Greta	Lesson 1	5	0	10	5	0
	Lesson 2	2	0	15	5	18
	Lesson 3	5	5	10	10	0
	Lesson 4	2	0	15	3	0
	Lesson 5	4	0	20	5	0
	Lesson 6	4	0	15	5	0
	Lesson 7	5	5	20	10	0
	Lesson 8	2	3	20	15	0
	Lesson 9	2	5	15	20	0
	Lesson 10	5	0	20	5	0
	Lesson 11	5	0	35	0	0
Sarah	Lesson 1	5	0	10	20	0
	Lesson 2	5	0	10	15	0
	Lesson 3	5	0	10	20	0
	Lesson 4	5	0	15	10	0
	Lesson 5	5	0	10	15	0
	Lesson 6	5	0	10	15	0
	Lesson 7	10	0	15	10	0
	Lesson 8	5	0	10	20	0
	Lesson 9	10	0	10	15	0

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