

# Modelling and Control of Complex Cyber-Physical Ecosystems<sup>\*</sup>

Manuela L. Bujorianu<sup>\*</sup> Tristan Caulfield<sup>\*\*</sup> David Pym<sup>\*\*\*</sup>

<sup>\*</sup> *Department of Computer Science, University College London, UK  
([luminita.bujorianu@ucl.ac.uk](mailto:luminita.bujorianu@ucl.ac.uk)).*

<sup>\*\*</sup> *Department of Computer Science, University College London, UK  
([t.caulfield@ucl.ac.uk](mailto:t.caulfield@ucl.ac.uk)).*

<sup>\*\*\*</sup> *UCL & Institute of Philosophy, University of London, UK  
([d.pym@ucl.ac.uk](mailto:d.pym@ucl.ac.uk)).*

**Abstract:** In this paper, we set up a mathematical framework for the modelling and control of complex cyber-physical ecosystems. In our setting, cyber-physical ecosystems (CPES) are cyber-physical systems of systems that are highly connected. CPES are understood as open and adaptive cyber-physical infrastructures. These networked systems combine cyber-physical systems with an interaction mechanism with other systems and the environment (ecosystem capability). The main focus will be on modelling cyber and physical interfaces that play an important role on the control of the emergent properties like safety and security.

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## 1. INTRODUCTION

The concept of Cyber-Physical System (CPS) was introduced by NSF in 2006 to define a new generation of systems that integrate computation, networking and physical processes at different complexity scales. CPSs go beyond traditional embedded/pervasive and distributed systems. CPES are ecosystems of networked CPS, meaning that they are systems of CPS (CPSoS) provided with an interaction activity between them and with their environment. Alternatively, we may call them cyber-physical infrastructures (CPI). Examples are smart grid, autonomous vehicles and maritime ships, autonomous swarm robotics. From the computer science perspective and systems theory, we can understand cyber-physical infrastructures within the framework of distributed systems. In the computer science field, the concept of a distributed system is the essential assembling piece of the systems theory that supports the technical architecture and operations of the CPSoS. For distributed systems, the key concepts are as follows: architecture (i.e., the location of the system component), resource (the nature of the system components) and process (communication, interoperability, service).

When studying properties of complex infrastructures like security and safety, a chink in the armour of the interacting mechanism is the design and modelling of interfaces. For CPS, these interfaces are complex and can be described in a hierarchical manner: cyber-physical layer, information layer, service layer, and so on. In this paper, we set up a mathematical framework for CPES based on the theoretical work described in Collinson et al. (2012); Caulfield et al. (2022); Anderson and Pym (2016) (and papers cited

therein) that builds on a conceptualization of the essential structural (locations, resources) and dynamic (processes) components of distributed systems and their interfaces, situated in stochastic environments. Formally, we consider the architecture of a CPSoS, the components (or CPS) in the system, and the processes that are executed by the system. Locations are modelled as vertices of a network, resources are modelled implicitly by the coefficients of equations that govern the dynamics of the CPS components and eventually some constraints, and processes are characterized by a set of physical and computational variables and their trajectories. The component behaviours are modelled as trajectories of stochastic hybrid systems.

This work combines the distributed systems approach with the behavioural approach for dynamical systems Willems (2007). The first supports a compositional analysis of the structure CPES. The second makes possible the use of different control techniques for CPI. We assume that the reader is familiar the basic theory of stochastic (hybrid) processes. See Bujorianu (2012) for a detailed presentation. Notations are standard for the Markov process theory.

## 2. CPES MODELLING

### 2.1 Conceptual Modelling

Conceptually, we describe CPES as an infrastructure of cyber-physical systems that are interconnected through some interfaces/ports, change some information, interoperate to achieve some common goals.

*Components* In a nutshell, a CPS component can be viewed as a tuple  $C = \langle Loc, X, I, Beh \rangle$ , where  $Loc$  is its location,  $X$  is a finite set of variables (both computational and physical),  $I$  represents its interface thought of as

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a set of variables that can be observed by the other systems of the CPES. Finally,  $Beh$  denotes the set of the component behaviours that are thought as system trajectories (evolutions of its variables).

Because of the interaction with other components, some of these parameters might be modified. For example,  $Loc$  can be changed according to a Markov chain, or the equation that governs the time evolution of  $X$  can be perturbed by a stochastic disturbance coming from the environment.

In this paper, we provide a modelling framework for both local and global behaviour of a CPES. Any CPS component will change continuously its behaviours evolving in parallel with the other components. Moreover, the interactions through the interface states may change the component internal behaviours. The global evolution of a CPES will be modelled as the asynchronous parallel composition of its (interaction altered) component evolutions. An interleaving semantics will be provided to describe the overall CPES dynamics in different interaction scenarios. For control and coordination purposes, an analytic synchronization of the whole CPES will be assembled.

*Composition Design* In the context of CPS, the fingerprints for composition design have been discussed in Tripathi (2016). In the following, we just present at a conceptual level some of ideas adapted to our setting. The CPI can be thought of as the result of composition of the constituent systems by means of parallel composition or interconnection (interoperability) operators. At a high level, we can think that a conceptual CPI is defined as a SoS of  $N$  CPS components; that is,  $(C_1, C_2, \dots, C_N)$ .

Two CPS components are *compatible* if their locations and variables are disjoint (non-overlapping).

The *synchronous parallel composition* of the compatible CPS components  $C_{i_1}$  and  $C_{i_2}$  is the interaction-free component denoted by  $C_{i_1} \parallel C_{i_2}$ , defined as:  $C_{i_1} \parallel C_{i_2} = \langle (Loc_{i_1}, Loc_{i_2}), X_{i_1} \cup X_{i_2}, I_{i_1} \cup I_{i_2}, Beh_{i_1} \times Beh_{i_2} \rangle$ .

To define the interoperability operator we need first to have an interconnection relation. Let  $I = I_1 \cup I_2 \cup \dots \cup I_N$  be the set of all interface states of the CPSoS. Then an *interconnection relation*  $\mathcal{IR}$  is defined as a relation on  $I$ ; that is,  $\mathcal{IR} \subset I \times I$ , which illustrates the interface coupling.

Using  $\mathcal{IR}$ , an interoperability operator  $\Psi$  is defined as a mapping that alters the structure of a component  $C$  when this interacts with other components. For a component  $C$ , we can define the external set of interface states, which belong to the those systems that interact with  $C$  as follows:

$$Ex_{\mathcal{IR}}^C = \{u \in I \setminus I_C \mid \exists v \in I_C : (u, v) \in \mathcal{IR} \vee (v, u) \in \mathcal{IR}\}.$$

By applying the operator  $\Psi$  to the component  $C$ , we obtain a new system  $\Psi(C)$  whose locations, variables, and behaviours may be altered by the interactions with the other systems:  $\Psi(C) = \langle Loc', X', I', Beh' \rangle$ .

The modification of the location  $Loc$  and of the set of variables  $X$  can be done following a specific protocol using a logical reasoning on interface variables. As well, as a result of interconnections, some interface states can be removed by wiring. The operator  $\Psi$  has to provide a methodology for the amendment of the behaviours.

Interoperability can be also result of an *asynchronous parallel composition* (by interleaving), where the interaction between components is allowed through the interface variables. In this paper, we suppose that all the CPS components are atomic and there is no integration into compound components. So, the components have some autonomy and their interoperability with the other components does not produce the behavioural binding. To this end, here, the CPES behaviour is the result of the asynchronous parallel composition of its components.

## 2.2 Mathematical Modelling

The next step is to set up a mathematical framework where the CPES objects and their relationships are described in a formal language. Our CPES model builds on several well-known formalism like stochastic hybrid systems Bujorianu (2012) and the interface theory Caulfield et al. (2022). Our framework provides support for physical and digital modelling, and for both physical and computational interactions. Formally, we model CPES as a network of stochastic hybrid agents, which captures the autonomy of the CPS components. Each CPS is an agent, modelled as a hybrid dynamical system with probabilistic discrete transitions between operation modes. In each mode, the system evolution is continuous and can be either deterministic or stochastic. The interactions are realized through some interfaces and an interaction protocol. We can specify the mathematical description in the language of distributed systems with locations, resources and processes. Then we combine this language with the mathematically powerful stochastic hybrid systems modelling.

### Stochastic Hybrid Network

We model a CPES as a network of stochastic hybrid agents. The locations of the agents are the network vertices. The edges of the network represent the interactions between agents. Note that the network can evolve in time and interactions between agents are dynamic. We consider agents with complex nonlinear dynamics affected by randomness modelled as stochastic hybrid systems with a specific type of interaction. In these systems, the randomness may be present in both continuous and discrete behaviors. Usually, the discrete behaviour include: (i) spontaneous (event triggered) transitions, which are generated by the appearance of certain events; (ii) forced transitions, which are triggered by specific conditions (guards) that are associated with the internal structure of individual agents.

In this section, we consider stochastic hybrid systems with both forced and spontaneous transitions. In the context of SoS, spontaneous transitions can be also generated by specific events like the inter-systems communication, or by the perturbations coming from the environment. Each system interchanges (incoming/outgoing) messages with the other systems, following a specific communication protocol. An outgoing message is released only when a discrete transition is enabled. The incoming messages are collected only during continuous evolution.

The hybrid agent evolution can be described by two types of activities: (i) interface activity (interaction with other agents); (ii) 'internal working' activity. Every input message generates some new constraints on the working

activity of the agent, which can play the role of new guards/rates that trigger new discrete jumps.

Formally, the overall activity of each agent is described as a stochastic hybrid process:  $\mathbf{H}^i = (q_t^i, x_t^i, u_t^i)$ . Standard assumptions are imposed to ensure the Markov property.

*Hybrid state space* For each  $\mathbf{H}^i$ , we define its hybrid state space  $\mathbf{X}^i$ , where  $\mathbf{X}^i = Q^i \times X^i$ , for  $i = 1, \dots, N$ . We assume that for all  $i$ , the hybrid state spaces  $\mathbf{X}^i$  can be embedded in the Euclidean space  $\mathbb{R}^d$ . The space  $\mathbf{X}^i$  will be equipped with its Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbf{X}^i)$ . The hybrid state  $\mathbf{x}_t^i := (q_t^i, x_t^i)$  will be called the *internal state* of the agent  $i$ , which is not accessible to other agents. The discrete state  $q$  corresponds to the computational part and the continuous state  $x$  describes the physical part of the system.

The pair  $\mathbf{u}_t^i := (q_t^i, u_t^i)$  is called *interface state*, and it represents the observable state of the agent  $i$ , visible to the other agents ( $U^i$  will be defined later on). The space  $U^i$  will be divided into digital and physical components, and will play an essential role in defining the agent interaction.

*Discrete transitions* In the absence of interactions, between the jump times, the process follows the dynamics law given by some stochastic differential equations. The jumping times (when the discrete transitions are fired) are defined as hitting times of the active boundaries (called guards denoted by  $\Gamma^i$ ), when the jumps are forced, or exponentially distributed times (with rates  $\lambda^i : \mathbf{X}^i \rightarrow \mathbb{R}_+$ ) when the jumps are spontaneous. The post jump locations are chosen according to a stochastic kernel  $R^i : \mathbf{X}^i \times \mathcal{B}(\mathbf{X}^i) \rightarrow [0, 1]$ . Therefore, the realization of such a hybrid system will be described by the tuple  $(q_t^i, x_t^i)$ ,  $t \geq 0$ , which is a right continuous stochastic process.

For each agent  $i$ , the jumping times are:  $0 < T_1^i < T_2^i < \dots < T_k^i < \dots$ . The discrete transitions might be:

(i) *spontaneous*, when  $\mathbf{x}_{(T_k^i-)}^i \in \mathbf{X}^i \setminus \Gamma^i$  and the sojourn time is given by the survival function (multiplicative functional):  $\Psi^i(t, \omega) = \exp(-\int_0^t \lambda^i(\mathbf{x}_s^i(\omega)) ds)$ ;

(ii) *forced*, when  $\mathbf{x}_{(T_k^i-)}^i \in \Gamma^i$  and the corresponding multiplicative functional is  $I_{[t < t^{i*}(\omega)]}$ , where  $t^{i*}(\omega)$  is the first exit time from  $\mathbf{X}^i \setminus \Gamma^i$ .

It is assumed that  $\mathbf{x}_t^i \in \mathbf{X}^i \setminus \Gamma^i$ , for all  $t \geq 0$ , except for the forced transition moments of time when  $\mathbf{x}_{(T_k^i-)}^i \in \Gamma^i$ .

The active boundary is a static or a dynamic set that might depend on the environment or on the other agents. In each operation mode, this could be changed according to the communication/interaction with other CPES components.

*Infinitesimal Generator* Let us briefly recall the concept of infinitesimal generator. Intuitively, the generator describes the movement of the process in an infinitesimal time interval. Suppose that  $(X_t)$  is a Markov process with an homogeneous transition probability function  $(p_t)_{t \geq 0}$ . For each  $t \geq 0$ , define conditional expectation operator by:  $\mathbf{P}_t f(x) := \int f(y) p_t(x, dy) = \mathbf{E}_x f(x_t)$ ,  $\forall x \in \mathbf{X}$ ; where  $\mathbf{E}_x$  is the expectation with respect to  $\mathbf{P}_x$ . Here,  $f$  belongs to  $\mathbf{B}(\mathbf{X})$ , which is the lattice of all bounded measurable real functions defined on  $\mathbf{X}$ . The linear operators  $\mathbf{P}_t$  satisfy the semigroup property:  $\mathbf{P}_{t+s} = \mathbf{P}_t \mathbf{P}_s$ , and the operator

semigroup  $\mathcal{P} = (\mathbf{P}_t)_{t > 0}$  represents a parameterization for a Markov process. Associated with the semigroup  $(\mathbf{P}_t)$  is its *infinitesimal generator* which, loosely speaking, is the derivative of  $\mathbf{P}_t$  at  $t = 0$ . Let  $D(\mathcal{L}) \subset \mathbf{B}(\mathbf{X})$  be the set of functions  $f$  for which the following limit exists:  $\lim_{t \searrow 0} \frac{1}{t} (\mathbf{P}_t f - f)$ , and denote this limit  $\mathcal{L}f$ . The limit refers to convergence in the supnorm  $\|\cdot\|$  of the Banach space  $\mathbf{B}(\mathbf{X})$ ; that is, for  $f \in D(\mathcal{L})$  we have:  $\lim_{t \searrow 0} \|\frac{1}{t} (\mathbf{P}_t f - f) - \mathcal{L}f\| = 0$ . The process behaviour is implicitly characterized by its generator.

For a stochastic hybrid agent  $i$ , the infinitesimal generator can be written in a simplified version as follows:

$$\mathcal{L}^i f = \mathcal{L}_c^i f + \mathcal{L}_j^i f, \forall f \in \mathbf{B}(\mathbf{X}^i). \quad (1)$$

We use the notation  $\mathcal{L}_c^i$  for the continuous part of the generator, which corresponds to the continuous dynamics of the process. This can be in the form of the Lie derivative if this dynamics is deterministic (governed by an ordinary differential equation), or in the form of the diffusion generator if this dynamics is stochastic (governed by a stochastic differential equation). The notation  $\mathcal{L}_j^i$  is used to designate the discrete part of the generator, which corresponds to the discrete dynamics of the process. This is expressed as the infinitesimal generator of a jump process:

$$\mathcal{L}_j^i f(\mathbf{x}) = \lambda^i(\mathbf{x}) \int [f(\mathbf{y}) - f(\mathbf{x})] R^i(\mathbf{x}, d\mathbf{y}). \quad (2)$$

For each  $\mathbf{x}$  in the active boundary, the boundary condition should be satisfied:

$$f(\mathbf{x}) = \int_{\mathbf{X}^i} f(\mathbf{y}) R(\mathbf{x}, d\mathbf{y}), \forall f \in D(\mathcal{L}^i) \quad (3)$$

In the previous formulas, some superscripts for  $\mathbf{x}$  have been omitted to improve readability.

### 2.3 Interaction and Interfaces

Each CPS has a physical behaviour which is independent when the system evolves in isolation, or is changed by the interaction with other systems of the CPES for the purpose of inter-operability. The CPS dynamics of is modelled as a stochastic hybrid process. Examples of such CPES are the cars on the highway, or the fleet of ships in port. Advanced models could include the dynamics of an autonomous car and its interaction with other traditional cars.

The CPES behaviour modelling requires the description of an interaction protocol between agents. The interaction of a CPS agent with other through some sort of communication will lead to perturbations of the continuous dynamics or will generate additional discrete transitions.

We can consider interactions at different levels: (i) *Discrete layer*: inter-agent interaction influences the discrete mechanism. In this case, the guards and the rates of discrete transitions are reconfigured. (ii) *Physical layer*: inter-agent interaction may change the laws of the continuous (the differential equations) behaviour.

The interaction at the discrete layer is done changing the rates or the guards of the discrete transitions, by using the interface variables. The interaction leads to modification of

the sojourn times of continuous dynamics, and only then the internal physical behaviour is altered.

At the physical layer, the interaction could be done implicitly, when agents change information that leads to the update of some parameters. As well, the interaction can be done in an explicit way, through some direct actions when one agent is changing its continuous dynamics law.

*Interface parameters* Suppose that the evolution of a CPES component  $i$  can be modelled by a hybrid stochastic system. When such a system is interacting with other components, the discrete or the continuous behaviours suffer changes. Mathematically, we formulate these interactions defining the interface variables. Note these variables are not necessarily state variables, but rather observation parameters or action labels. In the following, we define the cyber and physical interface parameters.

*Intensity rate* Let the influence from the agent  $j$  towards agent  $i$  be modelled by an *intensity rate*:  $\lambda^{ij} : Q^i \times Q^j \rightarrow \mathbb{R}; (q^i, q^j) \mapsto \lambda^{ij}(q^i, q^j) := \lambda_{q^i q^j}^{ij}$ . This quantifies how the actions of  $j$  when its discrete state is  $q^j$  are perceived by the agent  $i$  when it works in the mode  $q^i$ . Convention:  $\lambda_{q^i q^i}^{ii} := 0$ . Possible intensity rates are: location awareness, distance from an obstacle/objective, resource usage index.

*Edge weights* We can define  $w_{ij}$  as some interface parameters between physical layer and digital layer, which may affect various types of interaction. The concept is versatile enough to encapsulate different interface variables. In this paper, we take just some simple examples.

*Sojourn local time* For each agent  $i$ , for  $t \geq 0$ ,  $t \in [T_k^i, T_{k+1}^i)$ , let us define  $S_t^i := t - T_k^i$ .

The time  $S_t^i$  is part of the physical interface state of the agent  $i$  and is thought of output variable (observable by the other agents). This is the time elapsed since the last discrete transition of the  $i$ th agent until the moment  $t$ . Obviously, the discrete state remains constant between discrete transitions; that is,  $q_t^i = q_{T_k^i}^i$  if  $t \in [T_k^i, T_{k+1}^i)$ .

When  $t \in [T_k^i, T_{k+1}^i)$ , the time  $S_t^i(\omega)$  can be thought of as a local time random variable that describes the sojourn time of the trajectory  $\omega$  in the mode  $q_{T_k^i}^i$ .

*Alterations of the (discrete) spontaneous transitions* We use the notation  $N^{q^i}$  for the set of all agents that can influence the agent  $i$  when its operational mode is  $q^i$ ; that is,  $N^{q^i} := \{j : |\lambda_{q^i q^j}^{ij}| \geq \gamma\}$ , where  $\gamma > 0$  is a lower bound that controls the agent interactions. Note that  $N^{q^i}$  is time dependent according to the evolution of the agent population that has to achieve a specific objective.

The interaction with the agents from  $N^{q^i}$  can affect spontaneous jumps: we choose a simple interaction map

$$C_t^i := \sum_{j \in N^{q_t^i}} (\lambda^{ij}(q_t^i, q_t^j) \cdot \lambda_{q_t^i}^j), \quad (4)$$

where  $\lambda_{q_t^i}^j$  is an observation (a measurement) of the transition rate associated to the agent  $j$  when operating in mode  $q_t^i$  provided that  $\lambda_{q_t^i}^j$  does not depend on the continuous

state. The functional  $C_t^i$  is thought of as the total rate at which messages coming from  $N^{q_t^i}$  are collected by the agent  $i$  when its mode is  $q_t^i$ .

The jumps of the agent  $i$ , when its discrete mode is  $q_t^i$ , induced by the communication with its vicinity are triggered by the following ‘‘survivor’’ multiplicative functional:  $M_t^i(\omega) := I_{[t < t^{*i}(\omega)]} F^i(t, \omega) \Delta^i(t, \omega)$ , where  $F^i(t, \omega) = \exp(-\int_0^t \lambda^i(\mathbf{x}_s^i(\omega)) ds)$  is the survivor function of agent  $i$ , and  $\Delta^i(t, \omega) := \exp(-\int_0^t C_s^i(\omega) ds)$  is the *interaction multiplicative functional*.

### 3. CPES REALIZATIONS

The third step is the execution modelling stage, where we study the realization or behaviour of the CPES defined in this paper. We present a composition mechanism that allows us to study the evolution of the CPES in the same mathematical framework of stochastic hybrid systems. We propose a modelling approach for CPES that is capable to illustrate how the local properties of the components are lifted to the global ones. The modelling framework is versatile enough to help us proving that the emergent properties of the whole CPES are just the result of the composition of the constituent properties. The overall behaviour of CPES when its constituents are modelled by stochastic hybrid systems will be modelled also as a stochastic hybrid system. The main analytical tool that will be used in both micro and macro scales is the infinitesimal generator of a stochastic process. Then the characterization of a specific property (liveness, invariance, or reachability) for a single system and for the whole infrastructure will be given using different equations associated to these generators.

This section contribution is to characterize the emergent behaviour of a CPES by its infinitesimal generator. This will be the ‘composition’ of the infinitesimal generators of its constituents. This analytical tool will allow us to link local and global properties for the given CPES.

#### 3.1 CPES hybrid network

The state space (resp. location) of a CPES will be obtained as the tensor product of the state spaces (resp. locations) of its components. The interface of the whole CPES will be again the tensor product of the component interfaces. Only the CPES behaviour will be described by interleaving. Moreover, an *analytic synchronization* concept (see Cao et al. (2005)) can be introduced to study discrete transitions of the whole CPES. This technique will generate a single synchronous process that will be employed to show how the local properties of the (asynchronous) CPES components induce the global properties of the CPES.

The overall CPES behaviour (realization) will be characterized using the infinitesimal generator that can be associated to the tensor product of the modified stochastic hybrid processes that model the CPES components.

The CPES behavior is described by the asynchronous parallel composition of its hybrid components. For simplicity, we will consider two cases: (1) The hybrid components have only spontaneous jumps, and the interactions with other components change the rates of these jumps. (2)

The hybrid components have only forced jumps and the interactions with other components generate some particular spontaneous jumps. In both cases, we use the same techniques to define the CPES realizations.

For (1), we consider the CPS components as hybrid processes together with their interface functionals:  $(q_t^i, x_t^i, \Lambda^i)$ ,  $i = 1, \dots, N$ . For simplicity, we suppose that the discrete transition rates do not depend on the continuous state. We define the interaction intensity matrix  $\Lambda$  as:

$$\Lambda = [\Lambda^1, \Lambda^2, \dots, \Lambda^N]^\perp, \Lambda^i = (C_{q^i}^i)_{q^i \in Q^i}.$$

Here,  $\Lambda^i$  is thought of as the interface of the component  $i$ . The vicinity of  $i$ , denoted by  $N^i = \bigcup_{q^i \in Q^i} N^{q^i}$  is the set of all components that interact with it.

Formally, the CPES hybrid system is denoted by

$$\mathbf{H} = \otimes_{i=1}^N (\mathbf{x}_t^i, \Lambda^i)$$

where  $\mathbf{x}_t^i = (q_t^i, x_t^i)$  is the internal hybrid state associated to the component  $i$ . The hybrid state of  $\mathbf{H}$  is:

$$\mathbf{x} = ((q_1^1, q_1^2, \dots, q_1^N), (x_1^1, x_1^2, \dots, x_1^N)) \in Q \times \mathbb{R}^{d \times N};$$

where  $Q = Q^1 \times Q^2 \times \dots \times Q^N$ .

For (2), we consider the CPS components as hybrid processes together with their interface local times:  $(q_t^i, x_t^i, S_t^i)$ . Moreover, we suppose that for any two agents  $i$  and  $j$ , there exists an intensity  $k^{ij}$  that characterizes the strength of influence of agent  $j$  against agent  $i$ . Denote the vector of such intensities by  $k^i = (k^{i1}, k^{i2}, \dots, k^{iN})$ . The influence of the  $j$ th agent towards agent  $i$  is via the information about the last jumping time of the  $j$ th agent that arrives at the  $i$ th agent in a Poisson style; i.e.,  $\xi_t^{ij} = \exp(-k^{ij} S_t^j)$ .

For each agent  $i$ , consider the overall changes in the dynamics due to all communications with its neighbours triggered by  $\Xi_t^i(\omega) := \prod_{j \in N_t^i} \xi_t^{ij}(\omega)$ , where  $N_t^i$  is given by  $N_t^i := \{j | S_t^j < S_t^i\}$ . Note that the interaction vicinity of the agent  $i$  is time dependent.  $\Xi_t^i$  is a multiplicative functional being the product of multiplicative functionals.

In this case, we define the CPES hybrid system as follows:

$$\mathbf{H} = \otimes_{i=1}^N (\mathbf{x}_t^i, S_t^i, \Xi_t^i)$$

where  $\mathbf{x}_t^i = (q_t^i, x_t^i)$  is the internal hybrid state associated to the component  $i$  and  $\Xi_t^i$  is the communication functional. The hybrid state is defined as in the previous case.

### 3.2 Behaviour Characterization

For the process  $\mathbf{H}$ , the initial probabilities  $\mathbf{P}_x$  and the (survivor) multiplicative functionals  $\mathbf{M}_t$  are obtained by the product of the corresponding component objects. These functionals are obtained by the multiplication of the initial functionals (before interaction) with the interaction functional in (1) and the communication functional in (2).

The transition semigroup associated to  $\mathbf{H}$  until the first jumping time is defined as:

$$\mathbf{P}_t(\prod_{i=1}^N f^i)(\mathbf{x}) = \mathbf{P}_x \left[ \prod_{i=1}^N f^i(\mathbf{x}_t^i) \mathbf{M}_t \right].$$

The analytic synchronization technique is merging all  $N$  event time sequences into a single ordered sequence of

event times. Between two event times, all the components of CPES evolve continuously governed by their local laws. This technique consists of ordering the multi-clock  $\mathbf{S}_t := (S_t^1, S_t^2, \dots, S_t^N)$ . Therefore, this technique provides a global time axis, and each event time is, in fact, a time for a local discrete transition in one of the CPES components. One important fact about the multi-clock  $(S_t)$  is that its realization is a piecewise deterministic Markov process and it can play the role of an abstraction for the CPES  $\mathbf{H}$ .

The analytic synchronization and the fact that the component behaviours are described by stochastic processes obtained by subordination with respect to the multiplicative functional  $\mathbf{M}_t^i$  allow us to write down the following: **Theorem 1.** *The behaviour of the CPES  $\mathbf{H}$  is the realization of a stochastic hybrid process.*

Now, we are ready to write the main result of this section. The infinitesimal generator of  $\mathbf{H}$  will be obtained by the composition of the component generators.

For (1), when there are no forced transitions, we denote the infinitesimal generator of the agent  $i$  after it has interacted with other components, by  $\mathcal{L}^{i*}$ ; that is,

$$\mathcal{L}^{i*} f^i(\cdot) = \mathcal{L}_c^i f^i(\cdot) + \lambda_{(\cdot)}^{i*} \left\{ \int [f^i(\mathbf{y}^i) R^i(\cdot, d\mathbf{y}^i) - f^i(\cdot)] \right\}$$

where  $\lambda_{q^i}^{i*} = \lambda_{q^i}^i + C_{q^i}^i$  is the altered transition rate.

In the first case, the following result holds:

**Theorem 2.** *The generator of  $\mathbf{H}$  (with no forced jumps) has the following formula:*

$$\mathcal{L} = \sum_{i=1}^N \mathcal{L}^{i*} \otimes (\otimes_{j \neq i} Id^j),$$

where  $Id^j$  is the identity operator corresponding to  $j$ .

*Proof.* The generator expression is a consequence of the Trotter formula Bujorianu (2012) applied to  $\mathbf{H}$ , thought of as a superposition of stochastic processes.  $\square$

For (2), when the hybrid component  $i$  has initially only forced jumps, and only after interaction with other components, some spontaneous jumps may emerge, the expression of the infinitesimal generator needs to capture also the dynamics of  $S_t^i$ . Therefore, we consider the extended time-space stochastic process.

To give the generator expression, some notations are necessary: (i) The total communication rate is  $\mathbf{k}^i := \sum_{j \in N^i} k^{ij}$ .

(ii)  $\tilde{f}^i$  represents a bounded measurable function of the time and space for the component  $i$ ; (iii) The time-states will be denoted by Greek letters  $\alpha, \beta$ , then we have  $\alpha^i = (t, \mathbf{x}^i)$ . (iv) The time-space reset kernel for the component  $i$  is denoted by  $\tilde{R}^i$ , and it is equal with the tensor product of  $R^i$  with  $R_{null}$  (corresponding to the resetting to 0 of  $S_t^i$ ).

For each hybrid  $i$ , its infinitesimal generator altered expression after the interaction with other agents is given by:  $\tilde{\mathcal{L}}^{i*} \tilde{f}^i(\alpha^i) = \frac{\partial \tilde{f}^i}{\partial t}(\alpha^i) + \mathcal{L}_c^i \tilde{f}^i(\alpha^i) + \mathbf{k}^i \left[ \int \tilde{f}^i(\beta^i) \tilde{R}^i(\alpha^i, d\beta^i) - \tilde{f}^i(\alpha^i) \right]$ . As well, the boundary condition holds.

For the second case, when the forced jumps are allowed, formally the expression of generator is the same

$$\tilde{\mathcal{L}} = \sum_{i=1}^N \tilde{\mathcal{L}}^{i*} \otimes (\otimes_{j \neq i} \tilde{I}d^j),$$

where similar concepts are defined on the extended time-space. The only difficulty comes into play, when we have to write the boundary condition. Note that the boundary of the whole CPES is, in fact, the superposition of the component boundaries. Therefore, the boundary condition for the CPES generator is the coupling of all boundary conditions for the components. For brevity in this short paper, we elide the details of these conditions.

#### 4. CPES CONTROL: COORDINATION AND EMERGENT BEHAVIOUR

The contribution of this section will be to show that invariant measures for the given CPES are obtained by the composition of local invariant measures of its constituents.

##### 4.1 Invariant Measure

The concepts of invariant measure and stationary distribution are well-known in the context of Markov processes. According to Davis (1993), for stochastic hybrid processes, an invariant measure is, in fact a pair  $(\mu, \sigma)$ , where  $\mu$  is invariant for the interior of the state space, and  $\sigma$  is invariant for the boundary. In this section, we work with an extension of this concept called quasi-stationary distribution. This is more appropriate for hybrid processes because of the existence of jumping times, which are exit times from different process modes.

Let  $\zeta$  be the life time of the process  $(X_t)$  (i.e., the time when the process hits an absorbing/cemetery state  $\delta$ ). Connected to the concept of stationary distribution, we have the concept of *quasi-stationary distribution* (QSD). A probability measure  $\nu$  on  $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$  is called QSD if:

$$\nu(A) = \mathbf{P}^\nu(X_t \in A | t < \zeta), \forall A \in \mathcal{B}(\mathbf{X}). \quad (5)$$

The condition (5) is equivalent with the following:

$$\mathbf{P}^\nu(X_t \in A, t < \zeta) = \nu(A)\mathbf{P}^\nu(t < \zeta), \forall A \in \mathcal{B}(\mathbf{X}). \quad (6)$$

The meaning of the QSD definition is that if the initial state  $x_0$  is distributed according to  $\nu$  then the law of  $X_t$  conditional to not reach  $\delta$  is still  $\nu$  for all time instants that are less than  $\zeta$ . For a stochastic hybrid system, we can refine the concept of QSD asking that in the equation (5), the time  $\zeta$  to be replaced by the jumping times of the underlying process. Therefore, instead to have a global QSD, we will have a family of such measures, one for each mode. We can write  $\nu := \{\nu_q | q \in Q\}$ .

##### 4.2 Invariant Measures for CPES

Suppose that we have a probability measure  $\mu$  on  $\mathbb{R}^{N \times d}$  given by  $\mu = \mu^1 \otimes \mu^2 \otimes \dots \otimes \mu^N$ , where  $\mu^i$ ,  $i = 1, \dots, N$  are probability measures on  $\mathbb{R}^d$ . The natural question to ask is that if  $\mu$  is an invariant measure for the CPES, are its components  $\mu^i$  somehow invariant for the individual agents? Or, in other words, an emergent equilibrium regime for the whole CPES is a resultant of the component equilibria? The interaction mechanism is affecting the discrete transitions and the jumping times of the components of  $\mathbf{H}$ . As we have seen in the previous section, the infinitesimal generators of the component agents are perturbed in the discrete

part. Therefore, the invariant measures of agents are also altered. Then, the synergetic behaviour of the CPES can only be captured by the quasi-invariant distributions. The following theorem can be easily proved.

**Theorem 3.**  $\mu$  is a QSD for  $\mathbf{H}$  iff  $\mu^i$  with  $i = 1, \dots, N$  are quasi-stationary distributions for its components.

The above theorem is basically a natural consequence of the structure of CPES infinitesimal generator. It may be derived, as well, from the expression of transition semigroup of CPES. This is to be expected because both the infinitesimal generator and the transition semigroup are equivalent ways to characterize a Markovian process. To this end, we want to emphasize the importance of this result: The global CPES invariance is the result of its component invariance.

#### 5. CONCLUSIONS

We have defined a unifying mathematical framework for complex cyber-physical ecosystems. Conceptually, the skeleton of this framework is provided by the theory of distributed systems and their parallel composition. Then the mathematics of stochastic hybrid systems is grafted on this skeleton with the purpose of behaviour modelling of CPS components. The emergent behaviour and emergent properties of such complex systems are studied using the global behaviour of CPES, which is again modelled by a stochastic hybrid Markov process. Then the CPES properties are characterized employing the concept of infinitesimal generator associated to this Markov process. The main novelty of the paper is the cross-fertilization between the distributed systems paradigm, which provides insides about the CPES architecture and interface interactions of the CPES components, and the behavioural approach, which provides the main tool to study control problems.

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