Seiberg-Witten Floer Spectra and Contact Structures

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Declaration

I, Bruno Ricieri Souza Roso, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

In this thesis, the author defines an invariant of rational homology 3-spheres equipped with a contact structure as an element of a cohomotopy set of the Seiberg-Witten Floer spectrum as defined in Manolescu (2003). Furthermore, in light of the equivalence established in Lidman & Manolescu (2018a) between the Borel equivariant homology of said spectrum and the Seiberg-Witten Floer homology of Kronheimer & Mrowka (2007), the author shall show that this homotopy theoretic invariant recovers the already well known contact element in the Seiberg-Witten Floer cohomology (vid. e.g. Kronheimer, Mrowka, Ozsváth & Szabó 2007) in a natural fashion. Next, the behaviour of the cohomotopical invariant is considered in the presence of a finite covering. This setting naturally asks for the use of Borel cohomology equivariant with respect to the group of deck transformations. Hence, a new equivariant contact invariant is defined and its properties studied. The invariant is then computed in one concrete example, wherein the author demonstrates that it opens the possibility of considering scenarios hitherto inaccessible.

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Impact Statement

This thesis has impacted the knowledge on a well known and powerful invariant used in numerous results of the past two decades in the field of contact topology. This was achieved by refining it via a homotopy theoretic approach whilst showing that it recovers the classical one in a natural fashion. As an application, the author developed a criterion to solve a concrete class of problems which had proven difficult to study in the past. Moreover, there is great potential for exploring further implications of the work presented in this thesis via use of different topological methods.

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1. Introduction

It is well known from the work of Taubes (1994) that symplectic 4-manifolds with $b_2^+ > 1$ have non-trivial Seiberg-Witten invariants. This is accomplished by perturbing the Seiberg-Witten equations in a manner dictated by the symplectic form multiplied by a large positive real number. The effect of doing so is that the Seiberg-Witten equations gain an obvious canonical solution, which turns out to be unique and non-degenerate.

Perhaps a little less well known is that one can do something similar for contact 3-manifolds. This was pursued in Taubes (2007) and was ultimately crucial for the proof of the Weinstein conjecture. In the 3-dimensional case, one adds, as a perturbation, the contact form multiplied, again, by a positive real number. A canonical solution to the Seiberg-Witten equations immediately becomes apparent, and, if the real number be made large enough, one finds that this solution is automatically non-degenerate. Uniqueness does not hold in the 3-dimensional case in the same form as it does in the 4-dimensional case; if it did, all contact rational homology 3-spheres would be *L-spaces.* Nonetheless, another sort of uniqueness does hold but one which concerns Seiberg-Witten *trajectories.* The distinguished contact monopole does not admit any non trivial Seiberg-Witten trajectories coming into it in the *forward time limit.* This implies that this solution defines a cocycle and therefore a class in *monopole Floer cohomology.* As it turns out, this class is the well known contact invariant studied in Kronheimer, Mrowka, Ozsváth & Szabó (2007) and is equivalent to the contact invariants in Heegaard Floer and embedded contact homologies.

In Manolescu (2003), a Seiberg-Witten Floer *spectrum* was defined, which was later shown to recover the monopole Floer cohomologies through its Borel U(1)-equivariant cohomology. An important detail here is that the construction of the spectrum avoids altogether the use of any *generic* perturbations. Taubes' approach to defining the contact invariant, despite requiring a generic perturbation in order to work in a Morse theoretic setting, has the property that the contact monopole is already non-degenerate *before* the addition of the generic perturbation. In the present thesis, the author applies Taubes' approach to the contact invariant in the context of the Seiberg-Witten Floer spectrum in order to conveniently avoid the use of generic perturbations altogether.

Theorem 1.1: Given a contact rational homology 3-sphere (Y, λ) there exists a *cohomotopical contact invariant,*

$$\mathrm{SWF}(Y, \mathfrak{s}_\lambda) \to \mathcal{T}(\lambda),$$

where $\mathfrak{s}_\lambda$ is the $\mathrm{Spin}^{\mathbf{C}}$ structure on Y defined by the contact form λ and $\mathcal{T}(\lambda)$ is a (de)suspension of a U(1)-equivariant Thom space of a vector bundle over the U(1)-orbit of the contact monopole in global Coulomb gauge. Moreover, the classical *cohomological* contact invariant is recovered by pulling back via this map a class in the Borel U(1)-equivariant cohomology of $\mathcal{T}(\lambda)$.

A similar invariant is constructed in Iida & Taniguchi (2021); however, the work in the present thesis is entirely independent and differs in significant ways. Firstly, the author uses a different set of analytical results to ensure the existence of his invariant and also relies more heavily on certain aspects of *Conley theory* to define it. Secondly, here, U(1)-equivariance is kept manifest throughout, which means one can consider implications in Borel equivariant cohomology; indeed, the author was able to prove, via the techniques developed in Lidman & Manolescu (2018a), how the cohomotopical invariant recovers the well known cohomological invariant by passing to Borel equivariant cohomology. It should also be noted that the construction presented in the present thesis and the one of Iida & Taniguchi (2021) are sufficiently different that it is not clear if the two invariants are indeed equivalent or not. Of course, one would be inclined to think that two such invariants should really be holding the same information. However, proving their equivalence might be a difficult task due to the different analytical foundations used, so this goal is not pursued in the present thesis.

The author's main goal, after arming himself with the cohomotopical invariant, was to study *covering spaces.* The avoidance of generic perturbations is important in this context, as demonstrated in Lidman & Manolescu (2018b), due to the impossibility of producing sufficiently generic *equivariant* perturbations. The author's cohomotopical contact invariant can also be made G-equivariant for G the group of *deck transformations* of a finite regular covering. This allows him to consider Borel G-equivariant cohomology and deduce certain vanishing and non-vanishing results via the use of the *localization theorem.*

The central question one considers is whether the lift of a tight contact structure remains tight or becomes overtwisted in the covering. The results derived via use of the contact invariant shed light on this problem. In the work of Lin & Lipnowski (2022), the term *minimal L-space* is introduced to refer to rational homology 3-spheres having a single solution to the Seiberg-Witten equations for any $\text{Spin}^{\mathbf{C}}$ structure. For such a manifold, the Seiberg-Witten Floer spectrum is always the *sphere spectrum.* Moreover, in the case of a finite covering, the Seiberg-Witten Floer G-spectrum is the *sphere G-spectrum.* In this context, the author shall establish the following theorem.

Theorem 1.2: Let $\pi : Y \to Y/G$ be a *regular prime order covering* of minimal L-spaces and suppose that λ be a tight contact form on Y/G with non-vanishing *cohomological* contact invariant. Then, the lifted contact form, $\pi^*\lambda$, has non-vanishing cohomological contact invariant (and, therefore, is tight) provided that $d_3(\operatorname{Ker}\pi^*\lambda)+1/2 = d(Y, \pi^*\mathfrak{s}_\lambda)$, where d_3 denotes Gompf's three-dimensional invariant of hyperplane fields and d denotes the Ozsváth-Szabó invariant.

Examples of minimal L-spaces include all *sol* rational homology 3-spheres due to the work of Lin (2020). Another example is the *Hantzsche-Wendt* manifold, the unique flat rational homology 3-sphere. A few more examples exist amongst the *hyperbolic* manifolds as shown in Lin & Lipnowski (2022). However, the most evident class of examples of minimal L-spaces is that of the *elliptic manifolds.* In this case, increased knowledge of the contact topology, in particular the fact that the cohomological contact invariant never vanishes for tight contact structures and no two distinct contact structures have the same $\text{Spin}^{\mathbf{C}}$ structure, allows the author to prove the following stronger theorem.

Theorem 1.3: Let $\pi : Y \to Y/G$ be a *prime order regular covering* of elliptic manifolds and suppose that λ be a tight contact form on Y/G. Then, the lifted contact form, $\pi^*\lambda$, is *isotopic* to a tight contact form λ' on Y if and only if it be *homotopic* to λ'.

This result leads to a scheme for determining tightness of the lift of a tight contact structure on an elliptic manifold based purely on the *homotopy theoretic obstruction classes* of the contact structures involved. This reduces significantly the complexity of the problem and can be used in concrete calculations. The rationale is to try to determine the obstruction theoretic invariants of hyperplane fields – that is, the d_3 invariant and the $\text{Spin}^{\mathbf{C}}$ structure – for the *lifted contact structure,* $\pi^*\lambda$, from those of λ. If those be seen to match the values for a known tight contact structure on Y, then one shall know that $\pi^*\lambda$ is isotopic to it. The issue that arises is that it is non trivial to determine the lifting behaviour of the obstruction theoretic invariants, especially of the d_3 invariant.

In order to solve this problem, the author found himself having to develop techniques which seem not be discussed in the literature in a particularly well detailed manner. These shall be detailed in the present thesis and rely mostly on use of the Kirby calculus. As an example, the author shall study the case a certain tight contact structure on the (-8)-surgery on the left-handed trefoil which shall be shown to lift to a virtually overtwisted contact structure on the lens space $L(12, 7)$ via the double covering.

The main ingredient needed to perform these calculations is a form of G-*equivariant almost-complex filling* for the given covering of contact manifolds, which consists of an almost complex 4-manifold-with-boundary extending the given G-action on its contact boundary and potentially having a branching surface in its interior. Such a filling can often be produced by appealing to the notion of *equivariant handle attachments* in the context of the Kirby calculus. With such a filling at hand, one can apply the G-signature theorem, as was done by Khuzam (2012), to deduce the lifting behaviour of d_3 invariants.

The other matter that one must understand carefully is the lifting behaviour of $\mathrm{Spin}^{\mathbf{C}}$ structures. This is more elementary, albeit still difficult in practice, and can be tackled in different manners. The method pursued here shall follow a similar approach to the lifting of d_3 invariants by using Kirby calculus to express *Spin* structures in terms of obstruction theory and then studying the lifting behaviour of Spin structures. The behaviour of $\mathrm{Spin}^{\mathbf{C}}$ structures follows easily thence.

2. Seiberg-Witten Equations and Contact Structures

This section shall introduce the basic definitions and analytical results required from Seiberg-Witten theory. Throughout this thesis, the author shall use a version of the Seiberg-Witten equations adapted to the presence of a contact form which was first introduced in Taubes (2007) and Taubes (2009) and was subsequently used in Taubes' work in the correspondence between monopole Floer homology and embedded contact homology.

Consider an oriented 3-manifold Y satisfying $b_1(Y) = 0$. A contact form λ is a 1-form on Y satisfying $\lambda \wedge \mathrm{d}\lambda > 0$. The subbundle of $\mathrm{T}Y$ given by $\operatorname{Ker}\lambda$ is called a *coorientable* contact structure. In this thesis, all contact structures shall be assumed coorientable. As shall be seen, the version of the Seiberg-Witten equations which shall be used always admits a canonical solution, C_λ, which, provided a certain parameter $r > 0$ be made large enough, is nicely behaved in two fundamental ways. It is non-degenerate irrespective of any genericity requirements, and it is not the forward time limit of any Seiberg-Witten trajectory. These two properties shall be instrumental later in the present thesis. The solution C_λ is essentially defined in a manner that make its spinor component bounded away from zero; a feature which is unique to this solution provided r be made large enough.

In what follows, agree to fix a metric g on Y with the property that $\lambda \wedge \mathrm{d}\lambda = \mathrm{Vol}_g$. Use $\xi := \operatorname{Ker}\lambda$. Fix a complex structure $J \in \operatorname{End}(\xi)$ on the bundle ξ compatible with g in the sense that $g(-,-) = \mathrm{d}\lambda|_\xi(-, J-)$. Use $R \in \Gamma\mathrm{T}Y$ to denote the Reeb vector field; that is, the vector field satisfying $\iota_R \mathrm{d}\lambda = 0$, $\iota_R\lambda = 1$. Write $\xi \otimes \mathbf{C} = \Lambda^{1,0}\xi \oplus \Lambda^{0,1}\xi$, where $\Lambda^{1,0}\xi$ and $\Lambda^{0,1}\xi$ are, respectively, the $(\pm i)$-eigenbundles of J. Likewise, for the dual, write $\xi^* \otimes \mathbf{C} = \Lambda^{1,0}\xi^* \oplus \Lambda^{0,1}\xi^*$. Denote $\Lambda^{p,q}\xi^* := \Lambda^p_{\mathbf{C}}\Lambda^{1,0}\xi^* \otimes \Lambda^q_{\mathbf{C}}\Lambda^{0,1}\xi^*$. There is canonical $\mathrm{Spin}^{\mathbf{C}}$ structure on Y defined via the specification of its spinor representation bundle in the following way.

Definition 2.1: Define the spinor bundle

$$\mathcal{S}_\lambda := \bigoplus_q \Lambda^{0,q}\xi^* = \Lambda^{0,0}\xi^* \oplus \Lambda^{0,1}\xi^*$$

with Clifford multiplication $c\ell : \mathrm{T}Y \to \operatorname{End}_{\mathbf{C}}(\mathcal{S}_\lambda)$ defined so as to satisfy, for $\alpha \in \Lambda^{0,q}\xi^*$ and $X \in \xi$,

$$c\ell(R)\alpha = (-1)^{q+1} i\alpha, \quad c\ell(X)\alpha = \sqrt{2}\left((X^{0,1})^* \wedge \alpha - \iota_{(X^{0,1})}\alpha\right).$$

Remark 2.2: The notation $(X \mapsto X^*) : \mathrm{T}Y \otimes \mathbf{C} \to \mathrm{T}^*Y \otimes \mathbf{C}$ denotes the $\mathbf{C}$-*anti*linear isomorphism induced by the metric.

Remark 2.3: Note that this fully determines the Clifford action due to the fact that $\mathrm{T}Y = \langle R \rangle_{\mathbf{R}} \oplus \xi$. This map can be checked to indeed define an irreducible Clifford module; vid. Petit (2005) for a proof and more details about this matter.

Definition 2.4: Denote the underlying $\mathrm{Spin}^{\mathbf{C}}$ structure, that is, the principal $\mathrm{Spin}^{\mathbf{C}}(3)$-bundle, by $\mathfrak{s}_\lambda \to Y$.

Remark 2.5: Notice that the determinant line bundle is simply $\det \mathfrak{s}_\lambda = \Lambda^{0,1}\xi^* \cong \xi^*$ where ξ^* is equipped with the complex structure induced by J.

Definition 2.6: Use $\mathcal{A}(E)$ to denote the affine space of Hermitian connexions on a Hermitian vector bundle $E \to Y$.

Definition 2.7: Define the *configuration space* by $\mathcal{C}(Y, \mathfrak{s}_\lambda) := \mathcal{A}(\det \mathfrak{s}_\lambda) \times \Gamma(\mathcal{S}_\lambda)$. Use $\mathcal{C}(Y, \mathfrak{s}_\lambda)_k$ to denote its completion in the topology induced by the Sobolev norm L^2_k.

Definition 2.8: The tangent bundle $\mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda) \to \mathcal{C}(Y, \mathfrak{s}_\lambda)$ is the bundle having as fibre over $C \in \mathcal{C}(Y, \mathfrak{s}_\lambda)$ the space $\mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_C = \Gamma(i\mathrm{T}^*Y \oplus \mathcal{S}_\lambda)$. Use $\mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)_k$ for its Sobolev L^2_k completion.

Definition 2.9: Use $\tau : \mathcal{S}_\lambda \to i\mathrm{T}^*Y$ to denote the quadratic map defined by sending

$$\psi \mapsto c\ell^{-1}\left(\psi \otimes \psi^* - \frac{1}{2}|\psi|^2 \mathrm{id}\right).$$

Definition 2.10: Denote $\mathfrak{D}_A : \Gamma\mathcal{S}_\lambda \to \Gamma\mathcal{S}_\lambda$ the *Dirac operator* defined by a connexion A on $\det \mathcal{S}_\lambda$.

Definition 2.11: Define the *contact configuration,* $C_\lambda \equiv (A_\lambda, \psi_\lambda) \in \mathcal{A}(\det \mathfrak{s}_\lambda) \oplus \mathcal{S}_\lambda$ by setting its spinor component to be the constant function $\psi_\lambda := 1 \in \Gamma\xi^*_{0,0} \cong Y \times \mathbf{C}$ and by requiring its connexion component A_λ to solve the Dirac equation $\mathfrak{D}_{A_\lambda}\psi_\lambda = 0$.

Remark 2.12: This canonical configuration shall play a pivotal rôle in this thesis.

Definition 2.13: Let $r > 0$. The *canonically perturbed Seiberg-Witten vector field* of (Y, λ) is the vector field $\mathcal{X}_{\lambda,r} : \mathcal{C}(Y, \mathfrak{s}_\lambda) \to \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)$ given by

$$\mathcal{X}_{\lambda,r}(A, \psi) = \left(*\frac{1}{2}(F_A - F_{A_\lambda}) + r\tau(\psi) - \frac{ir}{2}\lambda,\ \mathfrak{D}_A\psi\right).$$

Remark 2.14: Notice that, for any value of $r > 0$, the contact configuration solves the Seiberg-Witten equation

$$\mathcal{X}_{\lambda,r}(C_\lambda) = 0.$$

That is, C_λ is a *fixed point* of the Seiberg-Witten vector field.

Definition 2.15: Denote the *gauge group* by $\mathcal{G}(Y) := \mathrm{C}^\infty(Y, \mathrm{U}(1))$. Use $\mathcal{G}(Y)_k$ for its completion in the Sobolev L^2_k norm.

Definition 2.16: For $(A, \psi) \in \mathcal{C}(Y, \mathfrak{s}_\lambda)$, define the *linearized gauge action* by

$$\mathfrak{L}_{(A,\psi)} : \mathrm{C}^\infty(Y, i\mathbf{R}) \to \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_{(A,\psi)}, \quad u \mapsto (-\mathrm{d}u, u\psi).$$

Use $\mathfrak{L}^*_{(A,\psi)}$ for its formal L^2-adjoint.

Definition 2.17: The *local Coulomb gauge* is the subspace

$$\mathcal{K}_C := \operatorname{Ker} \mathfrak{L}^*_C \subset \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_C.$$

Denote by $\mathcal{K} \to \mathcal{C}(Y, \mathfrak{s}_\lambda)$ the vector bundle with fibres the local Coulomb gauges. Use $\mathcal{K}_{C,k}$ and $\mathcal{K}_k$ to denote the L^2_k Sobolev completions.

Definition 2.18: Denote by $\Pi^{\mathrm{LC}}_C : \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_C \to \mathcal{K}_C$ the L^2-orthogonal projection.

Definition 2.19: A configuration $(A, \psi) \in \mathcal{C}(Y, \mathfrak{s}_\lambda)$ is said to be *irreducible* if ψ is not identically zero.

Definition 2.20: An irreducible solution $C \in \mathcal{X}^{-1}_{\lambda,r}(0)$ is said to be *non-degenerate* if the derivative

$$\Pi^{\mathrm{LC}}_C \circ \mathrm{D}_C \mathcal{X}_{\lambda,r} : \mathcal{K}_C \to \mathcal{K}_C$$

be surjective.

Remark 2.21: It is customary in Seiberg-Witten theory to ensure non-degeneracy of all solutions through the addition of a generic perturbation; however, often, doing so has its downside. One of the key advantages of the Seiberg-Witten Floer spectrum construction (Manolescu 2003) is that it avoids the need for such a perturbation altogether. This was crucial in the results of Lidman & Manolescu (2018b) concerning the Smith-type inequality of Seiberg-Witten Floer homology. Avoiding the use of a generic perturbation shall also be exploited in the present thesis; however, as will be seen, it is still necessary to make sure that C_λ be non-degenerate. This is ensured by the following theorem of Taubes.

Remark 2.22: For ease of reference, the author shall make use of the following convention. When a proposition state the existence of a certain constant which shall be of use later in the text, that constant shall be labelled with the number of the proposition.

Theorem 2.23: (Taubes 2007) There exists $r_{2.23} > 0$ such that, for $r > r_{2.23}$, the contact configuration is non-degenerate.

Proof: Prior to coming across this result in Taubes (2007), the author wrote an independent proof for it, which turns out to consist of a significantly different argument. This alternative proof can be found in §11 of the present thesis. **QED**

Besides the non-degeneracy property, the configuration C_λ enjoys two *uniqueness* properties that shall prove important.

Theorem 2.24: (Taubes 2009) There exists $r_{2.24} > 0$ and $\delta_{2.24} > 0$ such that, for $r > r_{2.24}$, the only configuration $C = (a, \psi)$, up to a gauge transformation, satisfying

$$\mathcal{X}_{\lambda,r}(C) = 0, \quad |\psi| \geq 1 - \delta_{2.24}$$

is the contact configuration $C = C_\lambda$.

Proof: Vid. Taubes (2009), Proposition 2.8. **QED**

Theorem 2.25: There exists $r_{2.25} > 0$ such that, for $r > r_{2.25}$, any trajectory $\gamma : \mathbf{R} \to \mathcal{C}(Y, \mathfrak{s}_\lambda)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = -\mathcal{X}_{\lambda,r}(\gamma(t)), \quad \lim_{t\to-\infty}\gamma(t) = C, \quad \lim_{t\to\infty}\gamma(t) = C_\lambda,$$

where the limits are with respect to the Sobolev norm L^2_k for any $k \geq 5$, must satisfy $\gamma(t) = C_\lambda$ for all $t \in \mathbf{R}$.

Proof: This is *nearly* what is stated by Taubes (2009), Proposition 5.15, but not quite. Please find an adaptation of Taubes' proof in §12 of the present thesis.

QED

3. Review of Seiberg-Witten Floer Spectra

Armed with the analytic results from the previous section, the first goal of the present thesis shall be to define a homotopy theoretic invariant emerging from the contact configuration. This invariant shall live in an equivariant cohomotopy set of the Seiberg-Witten Floer Spectrum $\mathrm{SWF}(Y, \mathfrak{s}_\lambda)$ defined by Manolescu (2003). This section shall review that construction with a slight adaptation; the Seiberg-Witten flow used shall be the one canonically perturbed by the contact form as was described in the previous section. This shall allow for the definition of the contact invariant in the next section.

Let Y, λ and g be as in the previous section.

Definition 3.1: The *unperturbed Seiberg-Witten vector field* is

$$\mathcal{X} : \mathcal{C}(Y, \mathfrak{s}_\lambda) \to \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda), \quad \mathcal{X}(A, \psi) = \left(\frac{1}{2} {*F_A} - \tau(\psi),\ \mathfrak{D}_A \psi \right).$$

Remark 3.2: The construction in Manolescu (2003) uses $\mathcal{X}$ as the Seiberg-Witten vector field. The version of the Seiberg-Witten vector field used in the present thesis is $\mathcal{X}_{\lambda,r}$ and it differs from $\mathcal{X}$ in two ways. Firstly, the spinor component of $\mathcal{X}_{\lambda,r}$ is scaled by $r^{1/2}$ compared to $\mathcal{X}$; this distinction shall be evidently immaterial in the construction. Secondly, $\mathcal{X}_{\lambda,r}$ contains the constant term $(\frac{ir}{2}\lambda - *\frac{1}{2}F_{A_\lambda}, 0)$ added on. In the language of Lidman & Manolescu (2018a), this amounts to the addition of a "very tame" perturbation (vid. Lidman & Manolescu 2018a, Definition 4.4.2), which, as demonstrated there (vid. Lidman & Manolescu 2018a, Proposition 6.1.6), does not affect the construction of the spectrum. Hence, the spectra defined with $\mathcal{X}_{\lambda,r}$ and $\mathcal{X}$ shall be the same.

Definition 3.3: The *normalized Gauge group* is the subgroup $\mathcal{G}^\circ(Y) \subset \mathcal{G}(Y)$ consisting of those $u \in \mathcal{G}(Y)$ which can be written as $u = e^{if}$ such that $\int_Y *f = 0$.

Definition 3.4: By the *global Coulomb gauge with respect to the connexion A_λ,* one means

$$W := \left(A_\lambda + \mathrm{Ker}(\mathrm{d}^* : i\Omega^1(Y) \to i\Omega^0(Y))\right) \oplus \Gamma(\mathcal{S}_\lambda) \subset \mathcal{C}(Y, \mathfrak{s}_\lambda).$$

Remark 3.5: The affine space W shall be thought of as a vector space with zero being $(A_\lambda, 0)$; that is, connexions shall be thought of as purely imaginary 1-forms by subtracting A_λ.

Remark 3.6: Any $\mathcal{G}^\circ(Y)$-equivalance class $[(A_\lambda + a, \psi)] \in \mathcal{C}(Y, \mathfrak{s}_\lambda)/\mathcal{G}^\circ(Y)$ has a unique representative in global Coulomb gauge; that is, there is a unique $(A_\lambda + a', \psi') \in W$ such that $[(A_\lambda + a', \psi')] = [(A_\lambda + a, \psi)] \in \mathcal{C}(Y, \mathfrak{s}_\lambda)/\mathcal{G}^\circ(Y)$.

Definition 3.7: The *global Coulomb projection*, $\Pi^{\mathrm{GC}} : \mathcal{C}(Y, \mathfrak{s}_\lambda) \to W$, is given by sending a configuration $(A_\lambda + a, \psi) \in \mathcal{C}(Y, \mathfrak{s}_\lambda)$ to its unique $\mathcal{G}^\circ(Y)$-equivalent in global Coulomb gauge.

Remark 3.8: The map Π^{GC} may be computed as follows. Let $G : \mathrm{L}^2_m(Y) \to \mathrm{L}^2_{m+2}(Y)$ denote the Green's operator of the Laplacian $\Delta : \Omega^0(Y) \to \Omega^0(Y)$. One can show that

$$\Pi^{\mathrm{GC}}(A_\lambda + a, \psi) = (A_\lambda + a - \mathrm{d}G\mathrm{d}^* a, e^{G\mathrm{d}^* a}\psi).$$

As a consequence, note that Π^{GC} maps bounded sets to bounded sets.

Definition 3.9: The *enlarged local Coulomb slice* is the subspace

$$\mathcal{K}^{\mathrm{E}}_{(A,\psi)} \subset \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_{(A,\psi)}$$

defined as the L^2-orthogonal complement to the orbits of $\mathcal{G}^\circ(Y)$.

Definition 3.10: Denote by $\mathfrak{G}^\circ(Y)$ the Lie algebra of $\mathcal{G}^\circ(Y)$.

Remark 3.11: Any equivalence class $[(b, \phi)] \in \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_{(A,\psi)}/\mathfrak{G}^\circ(Y)$ has a unique representative in enlarged local Coulomb gauge.

Definition 3.12: By the *enlarged local Coulomb projection*, one means

$$\Pi^{\mathrm{ELC}}_{(A,\psi)} : \mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_{(A,\psi)} \to \mathcal{K}^{\mathrm{E}}_{(A,\psi)}$$

defined by sending a vector to the unique representative in enlarged local Coulomb gauge of its equivalence class in the quotient $\mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)|_{(A,\psi)}/\mathfrak{G}^\circ(Y)$.

Remark 3.13: Note that Π^{LC} and Π^{ELC} are maps defined on the tangent bundle $\mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda)$, whereas Π^{GC} is defined on $\mathcal{C}(Y, \mathfrak{s}_\lambda)$. Of course, Π^{GC} induces a map $\mathrm{T}\mathcal{C}(Y, \mathfrak{s}_\lambda) \to \mathrm{T}W$ via the pushforward Π^{GC}_*.

Definition 3.14: Set $\mathcal{X}^{\mathrm{GC}}_{\lambda,r} := \Pi^{\mathrm{GC}}_* \mathcal{X}_{\lambda,r}$.

Remark 3.15: Fix some integer $k \geq 5$ and consider, henceforth, $\mathcal{X}^{\mathrm{GC}}_{\lambda,r}$ as a map $W_k \to W_{k-1}$ where W_m denotes the completion of W in the Sobolev norm L^2_m.

Definition 3.16: Define the Fredholm linear operator $\ell : W_k \to W_{k-1}$ by the formula

$$\ell(A_\lambda + a, \psi) = \left(A_\lambda + \frac{1}{2}{*}\mathrm{d}a,\ \mathfrak{D}_{A_\lambda}\psi\right).$$

Definition 3.17: Define the (non-linear) operator $c : W_k \to W_{k-1}$ by $c := \mathcal{X}^{\mathrm{GC}}_{\lambda,r} - \ell$.

Remark 3.18: Note that $\mathcal{X}^{\mathrm{GC}}_{\lambda,r} = \ell + c$ where c is a *compact* operator as explained in Manolescu (2003), §4.

Definition 3.19: For $\mu > 1$, denote by $W^\mu \subset W_k$, the subspace consisting of the span of the eigenvectors of ℓ with eigenvalues in the interval $(-\mu, \mu)$. Use $\tilde{p}^\mu : W_k \to W^\mu$ to denote the L^2-orthogonal projection.

The family of operators $\tilde{p}^\mu$ must now be smoothed out in a particular way. For that end, fix a smooth function $\beta : \mathbf{R} \to \mathbf{R}$ satisfying $\operatorname{supp} \beta = [0, 1]$ and $\int_{\mathbf{R}} \beta(x) \mathrm{d}x = 1$. A preliminary version of the smoothed out family is as follows.

Definition 3.20: Define a family of operators $p^\mu_{\mathrm{prel}} : W_k \to W^\mu$ by

$$p^\mu_{\mathrm{prel}} := \int_0^1 \beta(t) \tilde{p}^{\mu - t} \mathrm{d}t$$

Remark 3.21: This preliminary version could well be used to define the Seiberg-Witten Floer spectrum and, indeed, is essentially the operator family which appears in the original definition in Manolescu (2003). However, in Lidman & Manolescu (2018a), the authors use a slightly modified version which turns out to be needed in proving some technical results. Some of those technical results shall be used in the present thesis. To define the final version of the operator family, a few more data need to be fixed. Firstly, choose an unbounded strictly increasing sequence $\{\mu_i\} \subset \mathbf{R}$ such that, for no i, be μ_i an eigenvalue of ℓ. Next, fix a sequence of small real numbers $\{\varepsilon_i\} \subset \mathbf{R}$ such that the intervals $[\mu_i - \varepsilon_i, \mu_i + \varepsilon_i]$ be disjoint and not contain any eigenvalue of ℓ. At last, pick smooth bump functions $\{\beta_i : \mathbf{R} \to [0, 1]\}$ such that $\operatorname{supp} \beta_i \subset [\mu_i - \varepsilon_i, \mu_i + \varepsilon_i]$.

Definition 3.22: Define the family of operators $p^\mu : W_k \to W^\mu$ by

$$p^\mu := \sum_i \beta_i(\mu) \tilde{p}^\mu + \left(1 - \sum_i \beta_i(\mu)\right) p^\mu_{\mathrm{prel}}.$$

Remark 3.23: The family of operators p^μ is smooth in μ but still has the property that, for all i, $p^{\mu_i} = \tilde{p}^{\mu_i}$.

Definition 3.24: By the *canonically perturbed Chern-Simons-Dirac functional*, one means

$$\mathrm{CSD}_{\lambda,r} : \mathcal{C}(Y, \mathfrak{s}_\lambda) \to \mathbf{R},$$
$$\mathrm{CSD}_{\lambda,r}(A_\lambda + a, \psi) := \int_Y *r\langle \psi, \mathfrak{D}_{A_\lambda + a} \psi \rangle - \int_Y a \wedge \mathrm{d}a - \frac{ri}{2} \int_Y \lambda \wedge \mathrm{d}a.$$

Definition 3.25: A *finite type curve* $\gamma : \mathbf{R} \to \mathcal{C}(Y, \mathfrak{s}_\lambda)$, $\gamma = (A, \psi)$, is a curve such that the maps $t \mapsto \mathrm{CSD}_{\lambda,r}(\gamma(t))$ and $t \mapsto \|\psi(t)\|_{\mathrm{C}^0}$ be bounded as functions $\mathbf{R} \to \mathbf{R}$.

Definition 3.26: A curve $\gamma : \mathbf{R} \to W_k$ is said to be a *Seiberg-Witten trajectory in global Coulomb gauge* if

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -\mathcal{X}^{\mathrm{GC}}_{\lambda,r}(\gamma(t)).$$

Definition 3.27: For $R > 0$, and a normed vector space V, use $\mathrm{B}(V, R) \subset \mathrm{D}(V, R) \subset V$ to denote, respectively, the open and closed balls of radius R. Use $\mathrm{S}(V, R) = \mathrm{D}(V, R) \setminus \mathrm{B}(V, R)$ to denote the sphere of radius R.

Theorem 3.28: (cf. Manolescu 2003, Proposition 1) There exists $R > 0$ such that all finite type trajectories of $\mathcal{X}^{\mathrm{GC}}_{\lambda,r}$ are contained in the ball $\mathrm{B}(W_k, R) \subset W_k$.

Proof: Firstly, note that the proof in Manolescu (2003) can be easily adapted to the present case of the *perturbed* Seiberg-Witten flow. That result provides a constant $R' > 0$ such that, *up to a gauge transformation*, all Seiberg-Witten trajectories of finite type sit inside the ball $\mathrm{B}(W_k, R') \subset \mathcal{C}(Y, \mathfrak{s}_\lambda)_k$. Therefore, a Seiberg-Witten trajectory in global Coulomb gauge is, locally, the global Coulomb projection of a Seiberg-Witten trajectory residing in the ball $\mathrm{B}(W_k, R') \subset \mathcal{C}(Y, \mathfrak{s}_\lambda)_k$. But the global Coulomb projection map $\Pi^{\mathrm{GC}} : \mathcal{C}(Y, \mathfrak{s}_\lambda)_k \to W_k$ maps bounded sets to bounded sets. **QED**

Remark 3.29: Henceforth, assume $R > 0$ to be such that all finite type Seiberg-Witten trajectories in Coulomb gauge fit in $B(W_k, R)$.

Remark 3.30: Also, fix a family of U(1)-equivariant bump functions $u^\mu : W^\mu \to \mathbf{R}$ satisfying

$$u^\mu|_{\mathrm{D}(W^\mu, 2R)} = 1, \quad u^\mu|_{W^\mu \setminus \mathrm{B}(W^\mu, 3R)} = 0$$

and, u^μ constant on the sphere $\mathrm{S}(W^\mu, t)) \subset W^\mu$ of radius t for all $t \in [0, \infty)$. Note that the norm on $W^\mu \subset W_k$ is defined by the Sobolev norm L^2_k of W_k.

Definition 3.31: Define the *finite dimensional approximation to the Seiberg-Witten vector field* as

$$\mathcal{X}^\mu_{\lambda,r} = u^\mu \cdot (\ell + p^\mu c).$$

Furthermore, use $\varphi^\mu_{\lambda,r} : W^\mu \times \mathbf{R} \to W^\mu$ to denote the flow given by the O.D.E.

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t) = -\mathcal{X}^\mu_{\lambda,r}(\gamma(t)).$$

The flow lines of $\varphi^{\mu}_{\lambda,r}$ are called *approximate Seiberg-Witten trajectories in global Coulomb gauge.*

Definition 3.32: Use $S^{\mu}_{\lambda,r} \subset \mathrm{B}(W^{\mu}, R)$ to denote the union of all flow lines of $\varphi^{\mu}_{\lambda,r}$ which remain inside of $B(W^{\mu}, R)$ for all time.

Theorem 3.33: (Manolescu 2003, Proposition 3) For $\mu > 0$ sufficiently large compared to R, any flow line of $\varphi^{\mu}_{\lambda,r}$ which be contained in the disk $\mathrm{D}(W^{\mu}, 2R)$ is, in fact, contained in the open ball $\mathrm{B}(W^{\mu}, R)$.

Proof: The proof in Manolescu (2003) of the non-perturbed case of this theorem can be trivially adapted to the case at hand. Alternatively, this result is a special case of Lidman & Manolescu (2018a), Proposition 6.1.2(i) and Proposition 6.1.5, noting that the perturbation used in this thesis is "very tame". **QED**

The author shall now recall the pertinent definitions from Conley theory. For a reference, the reader is directed to Conley (1978), Floer (1987) and Mischaikow (1995). In what follows, suppose that G be a compact Lie group, Γ be a locally compact Hausdorff space with a continuous G-action and $\phi : \Gamma \times \mathbf{R} \to \Gamma$ a continuous and equivariant flow.

Definition 3.34: Let $U \subset \Gamma$ be a G-invariant subset, the *maximal invariant set* of U is

$$\mathrm{Inv}(U) := \{u \in U \mid (\forall t \in \mathbf{R})(\phi(u, t) \in U)\}.$$

Definition 3.35: Let $S \subset \Gamma$ be a G-invariant compact subset. S is called an *isolated invariant set* if there be a compact neighbourhood $U \supset S$ such that $\mathrm{Inv}(U) = S$.

Definition 3.36: Let $S \subset \Gamma$ be an isolated invariant set. A pair of G-invariant compact sets (M, N) satisfying $N \subset M \subset \Gamma$ is called an *index pair* for S when:

(i) $M \setminus N$ be an isolating neighbourhood for S;

(ii) for all $t \geq 0$ and $x \in N$, if $\varphi(\{x\} \times [0, t]) \subset M$, then $\varphi(\{x\} \times [0, t]) \subset N$;

(iii) for all $t \geq 0$ and $x \in M$, if $\varphi(x, t) \notin M$, then $\varphi(\{x\} \times [0, t]) \cap N \neq \emptyset$.

Theorem 3.37: (Conley 1978, non-equivariant; Floer 1987 and Floer & Zehnder 1988, equivariant) For any isolated invariant set $S \subset \Gamma$, there exists an index pair (M, N) and the G-equivariant pointed homotopy type M/N is independent of the choice of (M, N).

Definition 3.38: The G-equivariant homotopy type of M/N where (M, N) is an index pair for an isolated invariant set S is called the *Conley index* of S and denoted $I_G(S, \phi)$.

Theorem 3.33 can now be reinterpreted in this language.

Corollary 3.39: $S^{\mu}_{\lambda,r}$ is a U(1)-invariant isolated invariant set with isolating neighbourhood $\mathrm{D}(W^{\mu}, 2R)$ for the U(1)-equivariant flow $\varphi^{\mu}_{\lambda,r}$.

The author shall now introduce the relevant definitions from *equivariant stable homotopy theory.* Here, the author shall deviate slightly from the route taken in Manolescu (2003); this is done in the interests of later sections that shall deal with Seiberg-Witten Floer spectra equivariant with respect to the deck transformations of a covering. In Manolescu (2003), in order to perform the required *desuspensions,* an *ad hod* version of the Spanier-Whitehead category is used. In the present thesis, instead, the author shall use the, by now, more standard category of spectra. This increases slightly the complexity of the definitions, but nothing new is gained as the Spanier-Whitehead category embeds into the category of spectra in a simple way. For more details, the reader is directed to May & al. (1996). In what follows, let G be a compact Lie group. Whenever the author say *G-space,* he means in fact *pointed G-space.*

Definition 3.40: A *G-universe* $\mathcal{U}$ is an orthogonal G-representation of countable dimension having the following two properties:

(i) For each finite dimensional subrepresentation $V \subset \mathcal{U}$, the direct sum of V with itself countably many times, V^{∞}, also occurs as a subrepresentation in $\mathcal{U}$.

(ii) The trivial representation $\mathbf{R}$, therefore also $\mathbf{R}^{\infty}$, occurs as a subrepresentation in $\mathcal{U}$.

Definition 3.41: For V a G-representation, denote by V^+ its *one-point compactification;* note that V^+ is a G-space and call it a *representation sphere of G.* For X a G-space, the V^{th} *suspension* of X is the smash product $\Sigma^V X := V^+ \wedge X$. The V^{th} *loop space* of X is the G-space $\Omega^V X$ of all maps $V^+ \to X$ with G acting by conjugation.

Remark 3.42: There is an *adjunction* between the suspension functor Σ^V and the loop space functor Ω^V on G-spaces. That is to say that, for G-spaces X_1 and X_2, there is a *natural bijection* between the space of maps $\Sigma^V X_1 \to X_2$ and the space of maps $X_1 \to \Omega^V X_2$.

Definition 3.43: A *G-prespectrum E indexed on the G-universe* $\mathcal{U}$ consists of the following data.

(i) A set of G-spaces, E_V, one for each finite dimensional subrepresentation V in $\mathcal{U}$.

(ii) A set of G-equivariant *structure maps,* $\sigma_{V,W} : \Sigma^{W-V} E_V \to E_W$, one for each pair of nested finite dimensional subrepresentations, $V \subset W \subset \mathcal{U}$, where $W - V$ denotes the orthogonal complement of V in W.

Definition 3.44: A *map of G-prespectra f* from a G-prespectrum E to a G-prespectrum F, both indexed on the same universe $\mathcal{U}$, consists of a set of G-equivariant maps of G-spaces $f_V : E_V \to F_V$, one for each finite dimensional subrepresentation $V \subset \mathcal{U}$, such that the evident diagrams

$$\begin{array}{ccc} \Sigma^{W-V} E_V & \longrightarrow & E_W \\ \Big\downarrow & & \Big\downarrow \\ \Sigma^{W-V} F_V & \longrightarrow & F_W \end{array}$$

all commute for any pair of nested representations $V \subset W \subset \mathcal{U}$. A *homotopy* between two maps of prespectra $f, g : E \to F$ is a map of prespectra $h : E \wedge I_+ \to F$ where I_+ denotes the interval $[0, 1]$ with a disjoint base point added; here, the smash $E \wedge I_+$ between a prespectrum and a space is simply to be interpreted spacewise. A map of prespectra f is called a *weak equivalence* if all of its constituent maps of spaces, f_V, be weak equivalences.

Definition 3.45: The *suspension prespectrum functor,* $\Sigma^\infty_{\mathcal{U}}$, from G-spaces to G-prespectra indexed on the universe $\mathcal{U}$ is defined by assigning to a G-space X the prespectrum $\Sigma^\infty_{\mathcal{U}} X$ consisting of $(\Sigma^\infty_{\mathcal{U}} X)_V := \Sigma^V X$ and structure maps the identity maps.

Definition 3.46: Given a G-representation V in the universe $\mathcal{U}$ and X a G-space, the V^{th} *desuspension of X,* denoted $\Sigma^{-V} \Sigma^\infty_{\mathcal{U}} X$, is a G-prespectrum indexed on $\mathcal{U}$ defined as follows. For $W \subset \mathcal{U}$, the W^{th} space of $\Sigma^{-V} \Sigma^\infty_{\mathcal{U}} X$ is either a single point, in the event that V not be contained in W, or it is the G-space $\Sigma^{W-V} X$, in the event that V be contained in W. Meanwhile, the structure maps $\Sigma^{U-W} \Sigma^{W-V} X \to \Sigma^{U-V} X$ are the evident ones.

Definition 3.47: A G-prespectrum E indexed on the G-universe $\mathcal{U}$ is called a *spectrum* whenever all the adjoints, $E_V \to \Omega^{W-V} E_W$, to the structure maps be *homeomorphisms.* The *category* of G-spectra and maps as defined above for prespectra is denoted $G\mathcal{SU}$. The *homotopy category* of G-spectra is defined as the category with the same objects as $G\mathcal{SU}$ but with morphisms being the homotopy classes of maps of prespectra; this category is denoted $hG\mathcal{SU}$. The *stable homotopy category* of G-spectra consists of the category $hG\mathcal{SU}$ together with formal inverses for all the weak equivalences; this category is denoted $\bar{h}G\mathcal{SU}$.

Theorem 3.48: The forgetful functor from spectra to prespectra has a *right adjoint* called the *spectrification functor.*

Proof: Vid. May & al. (1996), §XII.2. **QED**

Definition 3.49: By composing the *suspension prespectrum functor,* $\Sigma^\infty_{\mathcal{U}}$, with the *spectrification functor,* one obtains a functor from G-spaces to $G\mathcal{SU}$. This functor shall be called the *suspension spectrum functor* and shall also be denoted by $\Sigma^\infty_{\mathcal{U}}$. Likewise, the desuspension of a G-space X, $\Sigma^{-V}\Sigma^\infty_{\mathcal{U}}X$ can be regarded as being in $\bar{h}G\mathcal{SU}$.

Remark 3.50: As the notation suggests, there is a desuspension functor, Σ^{-V}, defined for all spectra, not just for suspension spectra. However, it is somewhat more subtle to define and the author shall not require it in the present thesis.

Remark 3.51: The stronger notion of spectra as opposed to prespectra is not so important in the present thesis because the main desire is simply to be able to perform *desuspensions.* Nonetheless, it has become standard in the literature to work with spectra because of their ability to classify homology and cohomology theories and also the superior properties that the category of spectra enjoys. Therefore, the author decided to phrase everything in terms of spectra for ease of reference.

Remark 3.52: Notice that the Coulomb gauge W is a U(1)-universe isomorphic to $\mathbf{R}^\infty \oplus \mathbf{C}^\infty$. Such an isomorphism can be defined by picking a basis of eigenvectors of ℓ. Note that this universe does *not* contain all representations of U(1). Indeed, if $\mathbf{C}$ denote the standard representation, where $\mathrm{U}(1) \hookrightarrow \mathbf{C}$ is the unit circle, then the tensor product $\mathbf{C} \otimes_{\mathbf{C}} \mathbf{C}$ is not in the universe. This choice of universe is compatible with what is done in Manolescu (2003), where desuspensions are only allowed with respect to $\mathbf{R}$ and $\mathbf{C}$.

Definition 3.53: When thinking of W as a universe, denote it by $\mathcal{W}$.

Definition 3.54: Given an interval $I \subset \mathbf{R}$, use W^I to denote the span of the eigenvectors of ℓ with eigenvalues in the interval I. Hence, e.g., $W^{\mu} = W^{(-\mu,\mu)}$.

Theorem 3.55: (Manolescu 2003, §7) Up to canonical isomorphism in $\bar{h}\mathrm{U}(1)\mathcal{SW}$, the spectrum

$$\Sigma^{-W^{(-\mu,0)}}\Sigma^{\infty}_{\mathcal{W}} I_{\mathrm{U}(1)}(S^{\mu}_{\lambda,r}, \varphi^{\mu}_{\lambda,r})$$

only depends on Y, the $\mathrm{Spin}^{\mathbf{C}}$ structure $\mathfrak{s}_\lambda$ and the metric g.

A disadvantage of working with the category $\bar{h}\mathrm{U}(1)\mathcal{SW}$ is that the universe $\mathcal{W}$ depends, at least superficially, on the metric g. This can be addressed by applying a *change of universe* to pass over to a standard choice of universe isomorphic to $\mathcal{W}$. In order to demonstrate that the choices involved are immaterial to the final result, the author shall invoke one more concept from stable homotopy theory.

Definition 3.56: Given two G-universes $\mathcal{U}_0$ and $\mathcal{U}_1$ and a linear isometry $f : \mathcal{U}_0 \to \mathcal{U}_1$, define the associated *(restrictive) change of universe functor,*

$$f^* : \bar{h}G\mathcal{SU}_1 \to \bar{h}G\mathcal{SU}_0,$$

by defining it on prespectra as the functor that sends a prespectrum E with structure maps σ to the prespectrum f^*E with structure maps $f^*\sigma$ such that

$$(f^*E)_V = E_{f(V)}, \quad (f^*\sigma)_{V,W} = \sigma_{f(V),f(W)}.$$

Likewise, define the associated *(inductive) change of universe functor,*

$$f_* : \bar{h}G\mathcal{SU}_0 \to \bar{h}G\mathcal{SU}_1,$$

by defining it on a prespectrum E with structure maps σ to be the prespectrum f_*E with structure maps $f_*\sigma$ satisfying the following. For finite dimensional subrepresentations $V \subset W \subset \mathcal{U}_1$, denote $V' := V \cap f(\mathcal{U}_0)$ and $W' := W \cap f(\mathcal{U}_0)$. Then, set

$$(f_*E)_V := E_{f^{-1}(V')} \wedge (V - V')^+,$$

and define the structure map $(f_*\sigma)_{V,W}$ to be the composite

$$\begin{aligned}
&E_{f^{-1}(V')} \wedge (V - V')^+ \wedge (W - V)^+ \\
&\xrightarrow{\sim} E_{f^{-1}(V')} \wedge (W' - V')^+ \wedge (W - W')^+ \\
&\xrightarrow{\mathrm{id}\wedge f^{-1}\wedge\mathrm{id}} E_{f^{-1}(V')} \wedge (f^{-1}(W') - f^{-1}(V'))^+ \wedge (W - W')^+ \\
&\xrightarrow{\sigma_{f^{-1}(V'),f^{-1}(W')}\wedge\mathrm{id}} E_{f^{-1}(W')} \wedge (W - W')^+.
\end{aligned}$$

Proposition 3.57: (Lewis Jr., May & Steinberger 1986, Proposition 1.2) f_* is left adjoint to f^*.

Theorem 3.58: (Lewis Jr., May & Steinberger 1986, Theorem 1.7 and Corollary 1.8) The functors $f^* : \bar{h}G\mathcal{S}\mathcal{U}_1 \to \bar{h}G\mathcal{S}\mathcal{U}_0$ defined by different choices of linear isometry f are *canonically and coherently isomorphic.* The same holds for the functors f_*. Moreover, if f be an isomorphism, f^* is an equivalence of categories with its inverse being f_*.

Remark 3.59: The author shall not expand on the precise meaning of canonical nor coherent in the present thesis; for that, the reader is directed to the proof of Lewis Jr., May & Steinberger (1986), Theorem 1.7. Suffice it to say that, by canonical, one means that the choices which appear in the proof are immaterial, and, by coherent, one means that certain diagrams that should commute do indeed commute.

Definition 3.60: Define $\mathcal{U}$ to be the U(1)-universe $\mathbf{R}^\infty \oplus \mathbf{C}^\infty$.

The essential point to be taken from Theorem 3.58 is that, since one may pass between the categories $\bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{U}$ and $\bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{W}$ in a natural fashion, one can think of the desuspension functor $\Sigma^{-W^{(-\mu,0)}}$ as being defined on $\bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{U}$ by intertwining with the changes of universe.

Definition 3.61: Given a finite dimensional subrepresentation $V \subset \mathcal{W}$, define the desuspension functor

$$\Sigma^{-V} : \bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{U} \to \bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{U}$$

as the composite

$$\bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{U} \xrightarrow{f_*} \bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{W} \xrightarrow{\Sigma^{-V}} \bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{W} \xrightarrow{f^*} \bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{U},$$

where $f : \mathcal{U} \to \mathcal{W}$ is any isometric isomorphism defined by a choosing a basis of eigenvectors for ℓ.

Proposition 3.62: The endofunctor Σ^{-V} on $\bar{h}\mathrm{U}(1)\mathcal{S}\mathcal{U}$ is well defined up to canonical isomorphism.

Proof: Immediate from Theorem 3.58. **QED**

With this understood, henceforth, the author shall often drop f^* and f_* from the notation.

Definition 3.63: Define the *metric dependent Seiberg-Witten Floer spectrum* as

$$\begin{aligned}\mathrm{SWF}(Y,\mathfrak{s}_\lambda,g) :=& \Sigma^{-W(-\mu,0)}\Sigma^\infty_{\mathcal{U}} I_{\mathrm{U}(1)}(S^\mu_{\lambda,r},\varphi^\mu_{\lambda,r})\\ \cong& f^*\Sigma^{-W(-\mu,0)}\Sigma^\infty_{\mathcal{W}} I_{\mathrm{U}(1)}(S^\mu_{\lambda,r},\varphi^\mu_{\lambda,r}) \in \bar{h}\mathrm{U}(1)\mathcal{SU},\end{aligned}$$

where $f : \mathcal{U} \to \mathcal{W}$ denotes any isometric isomorphism defined by a basis of eigenvectors for ℓ.

Corollary 3.64: The spectrum $\mathrm{SWF}(Y,\mathfrak{s}_\lambda,g)$ is well defined up to canonical isomorphism in $\bar{h}\mathrm{U}(1)\mathcal{SU}$.

Proof: Corollary of Theorem 3.55 and Theorem 3.58. **QED**

Next, the author proceeds to explain how to (de)suspend away the metric dependence.

Definition 3.65: Let X be an oriented 4-manifold-with-boundary such that $\partial X = Y$. Given a class $c \in \mathrm{H}^2(X,\mathbf{Z})$ define its square, $c^2 \in \mathbf{Q}$, as follows. Let $c' \in \mathrm{H}^2(X;\mathbf{Q})$ be the image of c under the change of coefficients $\mathrm{H}^2(X;\mathbf{Z}) \to \mathrm{H}^2(X;\mathbf{Q})$. Since $b_2(Y) = 0$, there is an exact sequence

$$\mathrm{H}^2(X,Y;\mathbf{Q}) \to \mathrm{H}^2(X;\mathbf{Q}) \to 0.$$

Pick any preimage $\tilde{c} \in \mathrm{H}^2(X,Y;\mathbf{Q})$ for c'. Define

$$c^2 := (\tilde{c} \smile \tilde{c})[X,Y] \in \mathbf{Q}.$$

Definition 3.66: Define a number $n(Y,\mathfrak{s}_\lambda,g) \in \mathbf{Q}$ as follows. Choose some simply-connected 4-manifold-with-boundary X with $\mathrm{Spin}^{\mathbf{C}}$ structure $\mathfrak{t}$ such that $\partial X = Y$ and such that $\mathfrak{t}$ agree with $\mathfrak{s}_\lambda$ on Y. Assume further that X have a neighbourhood of its boundary which be isometric to $[0,1] \times Y$. Fixing some connexion B on $\det\mathfrak{t}$ extending arbitrarily the connexion A_λ on $\det\mathfrak{s}_\lambda$, use $\mathfrak{D}^\pm_B$ to denote the Dirac operators of $(X,\mathfrak{t})$. Denote the signature of X by $\sigma(X)$. With this notation in place, let

$$n(Y,\mathfrak{s}_\lambda,g) := \mathrm{ind}_{\mathbf{C}}\,\mathfrak{D}^+_B - \frac{1}{8}(c_1(\det\mathfrak{t})^2 - \sigma(X)).$$

Proposition 3.67: (Manolescu 2003, §6) The number $n(Y,\mathfrak{s}_\lambda,g)$ does not depend on the choices involved in its definition. Indeed,

$$n(Y,\mathfrak{s}_\lambda,g) = \frac{1}{2}\left(\eta(\mathfrak{D}_{A_\lambda},0) - \dim_{\mathbf{R}}\mathrm{Ker}\,\mathfrak{D}_{A_\lambda} - \frac{1}{4}\eta(\mathrm{Sign},0)\right)$$

where $\eta(D,z)$ denotes the η function of an operator D (vid. Atiyah, Patodi & Singer 1975), and Sign is the operator on $\Omega^1(Y)\oplus\Omega^0(Y)$ given by

$$\begin{pmatrix} *\mathrm{d} & -\mathrm{d} \\ -\mathrm{d}^* & 0 \end{pmatrix}.$$

Proposition 3.68: (Manolescu 2003, §7) If N denote the cardinality of the finite set $\mathrm{H}_1(Y;\mathbf{Z})$, then $8Nn(Y,\mathfrak{s},g)$ is an integer and its residue modulo $8N$ does not depend on g.

Definition 3.69: Let $\mathfrak{n}(Y,\mathfrak{s}) \in \mathbf{Q}$ be such that $8N\mathfrak{n}(Y,\mathfrak{s}) \in \{0,\ldots,8N-1\}$ be the residue modulo $8N$ of $8Nn(Y,\mathfrak{s},g)$.

Theorem 3.70: (Manolescu 2003, §7) Up canonical to isomorphism in $\bar{h}\mathrm{U}(1)\mathcal{SU}$, the spectrum

$$\mathrm{SWF}(Y,\mathfrak{s}_\lambda) := \Sigma^{\mathbf{C}^{\mathfrak{n}(Y,\mathfrak{s})-n(Y,\mathfrak{s}_\lambda,g)}}\,\mathrm{SWF}(Y,\mathfrak{s}_\lambda,g)$$

only depends on Y and the $\mathrm{Spin}^{\mathbf{C}}$ structure $\mathfrak{s}_\lambda$.

Remark 3.71: The author is deviating from what is stated in Manolescu (2003) slightly in using the (de)suspension by $\mathbf{C}^{\mathfrak{n}(Y,\mathfrak{s})-n(Y,\mathfrak{s}_\lambda,g)}$ instead of by $\mathbf{C}^{-n(Y,\mathfrak{s}_\lambda,g)}$. The reason being his desire to operate in the standard category of spectra, $\mathrm{U}(1)\mathcal{SU}$, which does not allow for a desuspension by a "rational" dimensional representation in any obvious manner. In Manolescu (2003), a variant of the Spanier-Whitehead category is used, which can easily be made to formally admit such (de)suspensions.

The main result of Lidman & Manolescu (2018a) is that the U(1)-equivariant Borel cohomology of the SWF spectrum corresponds to the classical monopole Floer cohomology. In order to state this theorem, the author should firstly clarify what is meant by the cohomology of a spectrum in the present context. The rightful sort of G-equivariant cohomology theory to consider for G-spectra indexed on a universe $\mathcal{V}$ is that of $\mathrm{RO}(G;\mathcal{V})$-*graded cohomology* (vid. May & al. 1996, §XIII). In order to avoid having to deal with such complexities, the author decided to introduce the following language which shall simplify considerably the treatment.

Definition 3.72: Let h_G^* denote a $\mathbf{Z}$-graded G-equivariant (reduced) cohomology theory on G-spaces. The author shall say that h_G^* satisfies the *suspension axiom with respect to the universe* $\mathcal{V}$ when, for any G-space X and any finite dimensional subrepresentation $V \subset \mathcal{V}$, there be a natural isomorphism

$$h_G^n(X) \cong h_G^{n+\dim V}(\Sigma^V X).$$

Definition 3.73: Given a G-prespectrum E indexed on a universe $\mathcal{V}$ and G-equivariant cohomology theory h_G^* on G-spaces satisfying the suspension axiom with respect to the universe $\mathcal{V}$, define the *cohomology of* E as

$$h_G^n(E) := \operatorname*{Colim}_{V\subset\mathcal{V}} h_G^{n+\dim V}(E_V).$$

Remark 3.74: It is easy to see from this definition that, for a desuspension spectrum $\Sigma^{-V}\Sigma^{\infty}_{\mathcal{V}}X$, the cohomology is simply given by the cohomology of the space X but with a *grading shift.*

$$h^n_G(\Sigma^{-V}\Sigma^{\infty}_{\mathcal{V}}X) = h^{n+\dim V}_G(X).$$

Definition 3.75: The *Borel* G-equivariant cohomology theory for G-spaces with coefficients in an abelian ring R is defined as follows. Denote by EG a free contractible G-space and by BG the quotient EG/G. Then, for a G-space X,

$$\mathrm{c}\tilde{\mathrm{H}}^*_G(X;R) := \tilde{\mathrm{H}}^*((\mathrm{E}G \times X)/G;R).$$

Remark 3.76: Following May & al. (1996), the notation c indicates the "geometric completion" involved in obtaining the underlying spectrum from the equivariant Eilenberg-MacLane spectrum and helps one distinguish *Borel* from *Bredon* cohomology.

Proposition 3.77: For a G-universe $\mathcal{V}$ containing only finite dimensional subrepresentations V such that the vector bundle $(V \times \mathrm{E}G)/G$ over BG be *R-orientable,* the Borel cohomology theory with R-coefficients *satisfies the suspension axiom for $\mathcal{V}$.*

Proof: This follows directly from the Thom isomorphism theorem. **QED**

Remark 3.78: In the case at hand of $G = \mathrm{U}(1)$ and universe $\mathcal{U} = \mathbf{R}^\infty \oplus \mathbf{C}^\infty$, it is clear, therefore, that the Borel cohomology theory with $\mathbf{Z}$-coefficients satisfies the suspension axiom for $\mathcal{U}$ because a direct sum of a complex representation and a trivial real representation always define $\mathbf{Z}$-orientable vector bundles over BG. Hence, one can speak of the Borel cohomology of the SWF spectrum.

Theorem 3.79: (Lidman & Manolescu 2018a, Theorem 1.2.1) Letting $\widehat{\mathrm{HM}}^*(Y,\mathfrak{s})$ denote the $\mathbf{Q}$-graded "from" monopole Floer cohomology of Kronheimer & Mrowka (2007), there is an isomorphism

$$\widehat{\mathrm{HM}}^q(Y,\mathfrak{s}) \cong \mathrm{c}\tilde{\mathrm{H}}^{q-\mathfrak{n}(Y,\mathfrak{s})}_{\mathrm{U}(1)}(\mathrm{SWF}(Y,\mathfrak{s});\mathbf{Z}).$$

In particular, letting the "tilde" monopole Floer cohomology, $\widetilde{\mathrm{HM}}^*(Y,\mathfrak{s})$, be the mapping cone of the U-map on $\widehat{\mathrm{HM}}^*(Y,\mathfrak{s})$, there is an isomorphism

$$\widetilde{\mathrm{HM}}^q(Y,\mathfrak{s}) \cong \tilde{\mathrm{H}}^{q-\mathfrak{n}(Y,\mathfrak{s})}(\mathrm{SWF}(Y,\mathfrak{s});\mathbf{Z}).$$

Definition 3.80: For future reference, define the *metric dependent tilde monopole Floer cohomology* to be

$$\widetilde{\mathrm{HM}}^*(X, \mathfrak{s}, g) := \mathrm{H}^*(\mathrm{SWF}(X, \mathfrak{s}, g); \mathbf{Z}).$$

4. The Cohomotopical Contact Invariant

This section shall fulfil the first goal of the present thesis. The analytic results concerning the contact monopole C_λ shall allow the author to define a *cohomotopical* contact invariant in a manner that can be roughly outlined in the following way. Given a top dimensional cell in a CW-complex, if one were to quotient the complex by all other cells, one obtains a map to a sphere; this is an element of the cohomotopy of the complex. In the event that the cell not be top dimensional, but, instead, all the higher dimensional cells attach *null-homotopically* onto the given cell, one can still perform the same quotient and obtain an element in the cohomotopy set of the complex. In the case of a G-CW-complex, a similar story can be told about a G-cell; here, one needs to be more careful with what is meant by cohomotopy. In any event, one obtains a map from the G-CW-complex to the Thom space of a vector bundle over a G-orbit; analogously to how, in the non-equivariant setting, the sphere is the Thom space of a bundle over the orbit of the trivial group; that is, the point.

To achieve this in the present context, the author shall make use of a fundamental construction in the Conley theory; namely, the notion of *attractor-repeller pairs.* What has already been said about the contact configuration shall be summarised as saying that the orbit of the contact configuration defines a *repeller* in the isolated invariant set defining the Seiberg-Witten Floer spectrum. Well known results on Conley theory then provide a *cofibration* involving Conley indices. Due to the non-degeneracy of the contact configuration, the cofibre map of this cofibration can be interpreted, in the presence of a choice of U(1)-CW-structure on the Seiberg-Witten Floer spectrum, exactly as the quotient of all but one special U(1)-cell defined by the orbit of the contact monopole. This cofibre map shall be declared the contact invariant.

Lastly, one must take care to *stabilize* everything so as to make the cofibre map really an invariant with respect to the spectral cut-off parameter μ and the metric g. As it turns out, this does not provide any difficulty beyond what was already encountered in Manolescu (2003), and the proofs shall follow closely what is said there only with a few extra Conley theoretic inputs concerning the naturality of attractor-repeller pair cofibrations.

Let Y, λ and g be as in the preceding sections.

Remark 4.1: Notice that C_λ is, by definition, in the global Coulomb slice with respect to A_λ; that is, $C_\lambda \in W$. Moreover, since $\mathfrak{D}_{A_\lambda}\psi_\lambda = 0$, it is also true that $C_\lambda \in \operatorname{Ker}\ell$. Therefore, for $R > 0$ sufficiently large, $C_\lambda \in \mathrm{B}(W^\mu, R)$ for all $\mu > 0$. Henceforth, agree to set $R > 0$ large enough so that this last inclusion hold.

Remark 4.2: It is worth emphasising that the global Coulomb gauge does not fix a gauge with respect to the entire gauge group $\mathcal{G}(Y)$ but, rather, with respect to the normed gauge group $\mathcal{G}^\circ(Y)$. Hence, there is a circle's worth of fixed points of $\mathcal{X}^{\mathrm{GC}}_{\lambda,r}$ in W_k which are gauge equivalent to C_λ.

Definition 4.3: Denote by $U_\lambda \subset W_k$ the circle of configurations gauge equivalent to C_λ and call it the *contact circle.*

Definition 4.4: Use $\mathcal{J}^{\mathrm{GC}}_C$ to denote the tangent space to the $\mathrm{U}(1)$-orbit at $C \in W_k$.

Definition 4.5: Let $\tilde{g}$ denote the metric on W_k defined by assigning to tangent vectors $(a, \psi), (b, \phi) \in \mathrm{T}_C W_k \cong W_k$ the value $\Re\left\langle \Pi^{\mathrm{ELC}}_C(a, \psi), \Pi^{\mathrm{ELC}}_C(b, \phi)\right\rangle$, where Π^{ELC}_C is the projection to the enlarged local Coulomb gauge defined in Definition 3.12.

Remark 4.6: The metric $\tilde{g}$ is the one used in many of the technical results of Lidman & Manolescu (2018a). It is notable because it turns the Seiberg-Witten vector field $\mathcal{X}^{\mathrm{GC}}$ in global Coulomb gauge into the $\tilde{g}$-gradient of the $\mathrm{CSD}_{\lambda,r}$ functional restricted to W_k. In the present situation, it shall be necessary to invoke some of those results of Lidman & Manolescu (2018a) which make reference to this metric. The $\tilde{g}$ metric leads to the following definition.

Definition 4.7: Define the *local anticircular slice in the global Coulomb gauge* at $C \in W_k$, denoted $\mathcal{K}^{\mathrm{AGC}}_C$, as the $\tilde{g}$-orthogonal complement to $\mathcal{J}^{\mathrm{GC}}_C$ in W_k. Use $\mathcal{K}^{\mathrm{AGC}}_{j,C}$ for its Sobolev completion of regularity $j \in \mathbf{Z}$ and use $\Pi^{\mathrm{AGC}}_C : W_k \to \mathcal{K}^{\mathrm{AGC}}_C$ to denote the $\tilde{g}$-orthogonal projection.

Proposition 4.8: For sufficiently large $r > 0$, at any $C \in U_\lambda$, the derivative

$$\Pi^{\mathrm{AGC}}_C \circ \mathrm{D}_C\, \mathcal{X}_{\lambda,r} \circ \Pi^{\mathrm{ELC}}_C : \mathcal{K}^{\mathrm{AGC}}_{k,C} \to \mathcal{K}^{\mathrm{AGC}}_{k-1,C}$$

is surjective.

Proof: By gauge equivariance, it suffices to prove the result for $C = C_\lambda$. Theorem 2.23 has established that, for sufficiently large $r > 0$, the map

$$\Pi^{\mathrm{LC}}_{C_\lambda} \circ \mathrm{D}_{C_\lambda}\, \mathcal{X} : \mathcal{K}_{C_\lambda} \to \mathcal{K}_{\mathcal{X}_{\lambda,r}(C_\lambda)}$$

is surjective. Hence, the result follows directly from Lidman & Manolescu (2018a), Lemma 5.6.1, "(ii) $\Rightarrow$ (iv)"; cf. Lidman & Manolescu (2018a), Forumulae (97) and (94). **QED**

Remark 4.9: Henceforth, fix $r > 0$ so as to make Proposition 4.8 hold.

Remark 4.10: Bear in mind that the vector space W^μ is the direct sum of a real vector space and a complex vector space, whence comes its U(1)-action. Note that $U_\lambda \hookrightarrow W^\mu$ is the U(1)-equivariant embedding of a U(1)-manifold.

Definition 4.11: Let $E^\mu_\lambda \to U_\lambda$ denote the U(1)-equivariant normal bundle of U_λ as a submanifold of W^μ.

Proposition 4.12: For sufficiently large $\mu > 0$, U_λ is a hyperbolic fixed set of the flow $\varphi^\mu_{\lambda,r}$ in W^μ. In other words, for any $C \in U_\lambda$, the derivative

$$\mathrm{D}_C\, \mathcal{X}^\mu_{\lambda,r} : E^\mu_\lambda|_C \to E^\mu_\lambda|_C$$

has no eigenvalue with vanishing real part. In particular, U_λ is a non-degenerate fixed set.

Proof: By Proposition 4.8, U_λ consists of non-degenerate irreducible fixed points of the flow $\mathcal{X}^{\mathrm{GC}}_{\lambda,r}$ on W_k. Hence, apply the same argument used in the proof of Lidman & Manolescu (2018a), Proposition 7.3.1, to find that the same remains true when passing to a finite dimensional approximation provided one choose a sufficiently large μ. **QED**

Remark 4.13: Fix $\mu > 0$ large enough so as to make Proposition 4.12 hold.

Remark 4.14: Identify E^μ_λ with a sufficiently small tubular neighbourhood of U_λ so as to not contain any other fixed points of the flow $\varphi^\mu_{\lambda,r}$. This is possible due to the non-degeneracy ensured by Proposition 4.12. Now, as a vector bundle, one can split E^μ_λ into stable and unstable subbundles as $E^{\mathrm{s},\mu}_\lambda \oplus E^{\mathrm{u},\mu}_\lambda$, where

$$\begin{aligned} \forall v \in E^{\mathrm{s},\mu}_\lambda, \quad & \Big\langle v, \mathrm{D}_C\, \mathcal{X}^\mu_{\lambda,r}(v) \Big\rangle \geq m|v|^2, \\ \forall v \in E^{\mathrm{u},\mu}_\lambda, \quad & \Big\langle v, \mathrm{D}_C\, \mathcal{X}^\mu_{\lambda,r}(v) \Big\rangle \leq -m|v|^2 \end{aligned}$$

for some constant $m > 0$. Moreover, this splitting is preserved by $\mathrm{D}_C\, \mathcal{X}^\mu_{\lambda,r}$ and, hence, U_λ is an isolated invariant set with index pair $(\mathrm{D}(E^\mu_\lambda), \mathrm{S}(E^{\mathrm{u},\mu}_\lambda))$ where D and S denote the unit disk and unit sphere bundles.

Definition 4.15: For E a vector G-bundle over a compact Hausdorff space, denote its G-equivariant Thom space by $\Theta_G(E) := \mathrm{D}E/\mathrm{S}E$.

Corollary 4.16: The Conley index $I_{\mathrm{U}(1)}(U_\lambda, \varphi^\mu_{\lambda,r})$ is U(1)-equivariantly homotopy equivalent to the U(1)-equivariant Thom space $\Theta_{\mathrm{U}(1)}(E^{\mathrm{u},\mu}_\lambda)$ of the bundle $E^{\mathrm{u},\mu}_\lambda \to U_\lambda$.

As a direct consequence, by using a Morse decomposition, one finds that the contact circle U_λ defines a U(1)-cell in some U(1)-CW-complex decomposition of the space $I_{\mathrm{U}(1)}(S^\mu_{\lambda,r}, \varphi^\mu_{\lambda,r})$. If one were to add a *generic* perturbation on top of the canonical perturbation in use, as shall be done in §5, one would find that a possible choice of U(1)-CW-decomposition would have a one to one correspondence between its U(1)-cells and the set of fixed points and fixed circles of the Seiberg-Witten flow. It is preferable, of course, to avoid using such a generic perturbation. In order to derive a cohomotopical invariant from a given cell, it is necessary to establish something about the attaching maps of the higher dimensional cells. In particular, if the given cell be itself top dimensional or, more generally, if one find that all higher dimensional cells attach null-homotopically onto the given cell, then it follows that one can quotient all but the given cell in the U(1)-CW-complex and obtain a map to the quotient of the contact cell by its boundary. This is morally the strategy which shall be pursued next.

Theorem 4.17: For sufficiently large $r > 0$ and $\mu > 0$, there are no non-constant approximate Seiberg-Witten trajectories in the set $S^\mu_{\lambda,r}$ which have plus infinity limit in the contact circle U_λ.

Proof: The author starts with the argument in Step 1 of the proof of Proposition 3 in Manolescu (2003). Suppose the result not hold. Then, there exists an increasing sequence $\mu_n \to \infty$ and a sequence of non-constant trajectories $\gamma_n : \mathbf{R} \to \mathrm{D}(W^{\mu_n}, 2R)$ satisfying

$$\frac{\partial}{\partial t}\gamma_n(t) = -\,\mathcal{X}^{\mu_n}_{\lambda,r}(\gamma_n(t)), \quad \lim_{t\to\infty} \gamma_n(t) = C_\lambda.$$

Notice that there is a bound

$$\left\| \frac{\partial}{\partial t}\gamma_n(t) \right\|_{\mathrm{L}^2_{k-1}} \leq \|\ell\gamma_n(t)\|_{\mathrm{L}^2_{k-1}} + \|p^\mu c\gamma_n(t)\|_{\mathrm{L}^2_{k-1}} \leq KR.$$

where $K > 0$ is a constant independent of both n and t. This implies that the set of functions

$$\{\gamma_n : \mathbf{R} \to \mathrm{D}(W_{k-1}, 2R)\}$$

is equicontinuous. By use of the Arzelà-Ascoli theorem, one can replace this sequence by a subsequence which converge to some $\gamma : \mathbf{R} \to \mathrm{D}(W_{k-1}, 2R)$ in the compact-open topology. Now, due to compactness of c, the sequence of operators $(1 - p^{\mu_n})c$:

$W_k \to W_{k-1}$ converges to zero weakly. Given this, observe that

$$\frac{\partial}{\partial t}\gamma_n = -(\ell + c)\gamma_n + (1 - p^{\mu_n})c\gamma_n \xrightarrow[n\to\infty]{} -(\ell + c)\gamma.$$

uniformly as functions from compact subsets of $\mathbf{R}$ to W_{k-1}. Hence,

$$\gamma_n(t) - \gamma_n(0) = \int_0^t \frac{\partial}{\partial s}\gamma_n(s)\mathrm{d}s \xrightarrow[n\to\infty]{} -\int_0^t (\ell + c)\gamma(s)\mathrm{d}s.$$

Together, these two last assertions imply that

$$\frac{\partial}{\partial t}\gamma(t) = -(\ell + c)\gamma(t).$$

Moreover, observe that $\gamma(t) \xrightarrow[t\to\infty]{} U_\lambda$. Therefore, γ is the Coulomb projection of a Seiberg-Witten trajectory with positive infinity limit gauge equivalent to the contact configuration. By Theorem 2.25, such a trajectory must be constant. Let $C_n := \lim_{t\to-\infty}\gamma_n(t)$. By assumption, these cannot be equal to C_λ. Note also that $\lim_{n\to\infty} C_n = \lim_{t\to-\infty}\gamma(t) = C_\lambda$. However, Proposition 4.8 combined with Lidman & Manolescu (2018a), Proposition 7.2.2, guarantee that, for sufficiently large n and some small neighbourhood $N \supset W_k$ of C_λ, there cannot be any solution to $\mathcal{X}^{\mu}_{\lambda,r} = 0$ inside of N other than C_λ itself, thereby contradicting convergence of the sequence $\{C_n\}$ to C_λ. **QED**

A few more concepts from Conley theory shall be needed next. In what follows, suppose that G be a compact Lie group, Γ be a locally compact Hausdorff space with a continuous G-action and $\phi : \Gamma \times \mathbf{R} \to \Gamma$ a continuous and equivariant flow.

Definition 4.18: For a G-invariant subset $T \subset \Gamma$, define its *ω-limit,* $\omega(T)$, as the maximal invariant set of the closure of $\varphi(T \times [0, \infty))$. Likewise, define its *$\omega^*$-limit,* $\omega^*(T)$, as the maximal invariant set of the closure of $\varphi(T \times (-\infty, 0])$.

Definition 4.19: A G-invariant subset $A \subset S$ is an *attractor* when there is a neighbourhood $U \subset S$ of A as a subspace of S such that $A = \omega(U)$. Likewise, a G-invariant subset $R \subset S$ of an isolated invariant set $S \subset \Gamma$ is called an *repeller* when there is a neighbourhood $U \subset S$ of R as a subspace of S such that $R = \omega^*(U)$.

Definition 4.20: Given an attractor $A \subset S$ in the isolated invariant set $S \subset \Gamma$, one defines its *complementary repeller,* $A^* \subset S$, as the set $\{x \in S \mid \omega(\{x\}) \cap A = \emptyset\}$. Given a repeller $R \subset S$, one defines its *complementary attractor,* $R^* \subset S$, similarly.

Definition 4.21: An *attractor-repeller pair* (A, R) of an isolated invariant set $S \subset \Gamma$ consists of an attractor A and a repeller R in S such that $A = R^*$, or, equivalently, $R = A^*$.

Definition 4.22: Let (A, R) be an attractor-repeller pair for of an isolated invariant set $S \subset \Gamma$. A triple of G-invariant compact sets (L, M, N), $N \subset M \subset L \subset \Gamma$, is called an *index triple* for (A, R) when (L, M) is an index pair for R, (L, N) is an index pair for S and (M, N) is an index pair for A.

Theorem 4.23: (Conley 1978, §I.7, albeit non-equivariantly) For any attractor-repeller pair (A, R) of an isolated invariant set $S \subset \Gamma$, there exists an index triple (L, M, N) for it and the induced *cofibration,*

$$I_G(A, \phi) \to I_G(S, \phi) \to I_G(R, \phi),$$

is independent of the choice of (L, M, N) up to G-equivariant homotopy.

Now, one can reinterpret the analytic results concerning the contact circle in the Conley theoretic language.

Theorem 4.24: The contact circle $U_\lambda \subset S^{\mu}_{\lambda,r}$ is a *repeller* in $S^{\mu}_{\lambda,r}$. Hence, *there exists a cofibration*

$$I_{\mathrm{U}(1)}(U^*_\lambda, \varphi^{\mu}_{\lambda,r}) \to I_{\mathrm{U}(1)}(S^{\mu}_{\lambda,r}, \varphi^{\mu}_{\lambda,r}) \to I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu}_{\lambda,r}).$$

Proof: Follows from Theorem 4.17 and Theorem 4.23. **QED**

Definition 4.25: Define the *spectral cut-off and metric dependent cohomotopical contact invariant* as the cofibre map

$$\Psi(\lambda, g, \mu) : I_{\mathrm{U}(1)}(S^{\mu}_{\lambda,r}, \varphi^{\mu}_{\lambda,r}) \to I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu}_{\lambda,r})$$

of Theorem 4.24.

Proposition 4.26: $\Psi(\lambda, g, \mu)$ does not depend on the choice of $r > 0$.

Proof: Suppose one chose two sufficiently large values for r, call them $r_0 < r_1$. Without loss of generality, assume $|r_0 - r_1|$ to be small. Pick the value of $R > 0$ large enough so that it satisfy Theorem 3.28 for both $r = r_0$ and $r = r_1$. Then, $\mathrm{D}(W^\mu, 2R)$ serves as an isolating neighbourhood for all the isolated invariant sets $S^{\mu}_{\lambda,r}$ under the parametrised family of flows $\varphi^{\mu}_{\lambda,r}$ as r varies in $[r_0, r_1]$. The continuation properties of Conley theory then provide a diagram of the form

$$\begin{array}{ccc} I_{\mathrm{U}(1)}(S^{\mu}_{\lambda,r_0}, \varphi^{\mu}_{\lambda,r_0}) & \longrightarrow & I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu}_{\lambda,r_0}) \\ \Big\downarrow & & \Big\downarrow \\ I_{\mathrm{U}(1)}(S^{\mu}_{\lambda,r_1}, \varphi^{\mu}_{\lambda,r_1}) & \longrightarrow & I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu}_{\lambda,r_1}) \end{array}$$

which commutes up to homotopy and where the vertical arrows are homotopy equivalences (cf. Conley 1978, §III.3.1 and Kurland 1982 for the non-equivariant case).

QED

Definition 4.27: The *metric dependent contact Thom space* is the desuspension

$$\mathcal{T}(\lambda, g) := \Sigma^{-W^{(-\mu,0)}} \Sigma^{\infty}_{\mathcal{U}} I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu}_{\lambda,r}) \in \bar{h}\mathrm{U}(1)\mathcal{SU}.$$

Definition 4.28: The *metric dependent cohomotopical contact invariant,*

$$\Psi(\lambda, g) : \mathrm{SWF}(Y, \mathfrak{s}_\lambda, g) \to \mathcal{T}(\lambda, g).$$

is the desuspension $\Sigma^{-W^{(-\mu,0)}}\Psi(\lambda, g, \mu)$ as a morphism in $\bar{h}\mathrm{U}(1)\mathcal{SU}$.

Proposition 4.29: $\Psi(\lambda, g)$ does not depend on the choice of $\mu > 0$.

Proof: This proof shall start by recalling the setup of the first half of the proof of Theorem 1 of Manolescu (2003). Assume two values for μ be given; call them $0 < \mu_0 < \mu_1$ and assume both be large enough so as to satisfy Theorem 3.33. Consider $\mu \in [\mu_0, \mu_1]$. Denote $\tilde{\varphi}^{\mu}_{\lambda,r}$ the flow of the vector field $-u^{\mu_1} \cdot (\ell + p^{\mu} c p^{\mu})$ on W^{μ_1}. It is easy to check that, for any $\mu \in [\mu_0, \mu_1]$, all finite type trajectories of $-(\ell + p^{\mu} c p^{\mu})$ on W^{μ_1} are, in fact, contained in $W^{\mu} \subset W^{\mu_1}$. As a consequence, notice that the set $S^{\mu}_{\lambda,r} \subset W^{\mu}$ can be identified with the union of the finite type trajectories of $\tilde{\varphi}^{\mu}_{\lambda,r}$ contained in $\mathrm{B}(W^{\mu_1}, R)$. Hence, abuse notation and write $S^{\mu}_{\lambda,r} \subset W^{\mu_1}$ for all $\mu \in [\mu_0, \mu_1]$. Moreover, in this description, $\mathrm{D}(W^{\mu_1}, 2R)$ is an isolating neighbourhood for $S^{\mu}_{\lambda,r}$ for all $\mu \in [\mu_0, \mu_1]$. By the continuation properties of the Conley index,

$$I_{\mathrm{U}(1)}(S^{\mu_1}, \varphi^{\mu}_{\lambda,r}) \simeq I_{\mathrm{U}(1)}(S^{\mu_0}, \tilde{\varphi}^{\mu}_{\lambda,r}).$$

Write $W^{\mu_1} = W^{\mu_0} \oplus W'$, where W' is L^2-orthogonal to W^{μ_0}. Of course, W' is simply the span of the eigenvectors of ℓ with eigenvalues in $(-\mu_1, -\mu_0] \cup [\mu_0, \mu_1)$. Use $D \subset W'$ to denote a small disk around the origin. Then, if one care to check, one finds that

$$\mathrm{D}(W^{\mu_0}, 3R/2) \times D \subset W^{\mu_1}$$

is also an isolating neighbourhood for $S^{\mu_0}_{\lambda,r}$. Furthermore, with respect to this product, one can show that the flow $\tilde{\varphi}^{\mu}_{\lambda,r}$ is homotopic to a product flow $\varphi^{\mu_0} \times \mathfrak{f}$, where $\mathfrak{f}$ is the flow on D induced by $-\ell$. Notice that

$$I_{\mathrm{U}(1)}(\{0\}, \mathfrak{f}) = \left(W^{(-\mu_1, -\mu_0]}\right)^{+}.$$

By the behaviour of the Conley index under product flows, it follows that

$$I_{\mathrm{U}(1)}(S^{\mu_1}_{\lambda,r}, \varphi^{\mu_1}_{\lambda,r}) \simeq I_{\mathrm{U}(1)}(S^{\mu_0}_{\lambda,r}, \varphi^{\mu_0}_{\lambda,r}) \wedge \left(W^{(-\mu_1,-\mu_0]}\right)^+.$$

Now, focus is turned to the contact circle. Recall the assumption from Remark 4.1 that μ_0, μ_1 be large enough so that $U_\lambda \subset \mathrm{B}(W^{\mu_0}, R) \subset \mathrm{B}(W^{\mu_1}, R)$. As above, observe that

$$I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu_1}_{\lambda,r}) \simeq I_{\mathrm{U}(1)}(U_\lambda, \tilde{\varphi}^{\mu_0}_{\lambda,r}).$$

On the other hand, considering again the product flow $\varphi^{\mu_0}_{\lambda,r} \times \text{ᚠ}$, one finds

$$I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu_1}_{\lambda,r}) \simeq I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu_0}_{\lambda,r}) \wedge \left(W^{(-\mu_1,-\mu_0]}\right)^+.$$

This leads to a diagram of the form

$$\begin{array}{ccc}
I_{\mathrm{U}(1)}(S^{\mu_0}_{\lambda,r}, \varphi^{\mu_0}_{\lambda,r}) \wedge \left(W^{(-\mu_1,-\mu_0]}\right)^+ & \longrightarrow & I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu_0}_{\lambda,r}) \wedge \left(W^{(-\mu_1,-\mu_0]}\right)^+ \\
\downarrow & & \downarrow \\
I_{\mathrm{U}(1)}(S^{\mu_1}_{\lambda,r}, \varphi^{\mu_1}_{\lambda,r}) & \longrightarrow & I_{\mathrm{U}(1)}(U_\lambda, \varphi^{\mu_1}_{\lambda,r})
\end{array}$$

which, due to the naturality of the attractor-repeller cofibration under continuation, commutes up to homotopy. Desuspending everything as needed provides the required invariance. **QED**

Definition 4.30: Define the *contact Thom space* as the (de)suspension

$$\mathcal{T}(\lambda) := \Sigma^{\mathbf{C}^{\mathrm{n}(Y,\mathfrak{s}_\lambda) - n(Y,\mathfrak{s}_\lambda,g)}} \mathcal{T}(\lambda, g) \in \bar{h}\mathrm{U}(1)\mathcal{SU}.$$

Definition 4.31: Define the *cohomotopical contact invariant* of the contact rational homology sphere (Y, λ),

$$\Psi(\lambda) : \mathrm{SWF}(Y, \mathfrak{s}_\lambda) \to \mathcal{T}(\lambda),$$

to be the (de)suspension $\Sigma^{\mathbf{C}^{\mathrm{n}(Y,\mathfrak{s}_\lambda) - n(Y,\mathfrak{s}_\lambda,g)}} \Psi(\lambda, g)$ as a morphism of $\bar{h}\mathrm{U}(1)\mathcal{SU}$.

Proposition 4.32: $\Psi(\lambda)$ does not depend on the choice of metric g.

Proof: Again, the proof shall start by recalling what is said by Manolescu (2003) and then extending the argument to deal with the contact circle. Since the space of compatible metrics is connected, it suffices to prove the result for nearby metrics, so consider two such metrics g_0, g_1 and a smooth path $t \mapsto g_t$ interpolating them. One can choose $\mu > 0$ and $R > 0$ large enough so as to satisfy the usual requirements for all metrics along the path g_t. The author shall use a subscripted t to denote the versions of objects constructed with the metric g_t; for example, W^μ_t denotes the version of W^μ constructed with g_t. Notice that, perhaps after increasing μ slightly, one can assume that μ not be an eigenvalue of ℓ_t for any $t \in [0,1]$. As a consequence, the spaces W^μ_t have constant dimension as t varies. This means the spaces W^μ_t form a vector bundle over $[0,1]$ and, therefore, the vector spaces W^μ_t for different values of t may be identified via a trivialisation of this bundle; hence, use W^μ to denote any of the spaces W^μ_t. For different values of t, consider the balls $\mathrm{B}(W^\mu, R)_t$ all as subsets of this same W^μ. By assuming the metrics g_0, g_1 sufficiently close to one another, one also finds that, for any $t_1, t_2 \in [0,1]$, $\mathrm{B}(W^\mu, R)_{t_1} \subset \mathrm{B}(W^\mu, 2R)_{t_2}$. From this, it follows that

$$\bigcap_{t\in[0,1]} \mathrm{D}(W^\mu, 2R)_t$$

is an isolating neighbourhood for $(S^\mu_{\lambda,r})_t$ with respect to the flow $(\varphi^\mu_{\lambda,r})_t$ for all $t \in [0,1]$. The flow $(\varphi^\mu_{\lambda,r})_t$ varies continuously with $t \in [0,1]$; hence, by Conley theory,

$$I_{\mathrm{U}(1)}\left((S^\mu_{\lambda,r})_0, (\varphi^\mu_{\lambda,r})_0\right) \simeq I_{\mathrm{U}(1)}\left((S^\mu_{\lambda,r})_1, (\varphi^\mu_{\lambda,r})_1\right). \qquad (*)$$

Consider now the contact circle. One easily checks that, under the change of metric, the contact circle, $(U_\lambda)_t$, moves smoothly in W^μ (it is not fixed under changes of metric because it depends on the global Coulomb projection). By assuming the metrics to be sufficiently close, one also sees that

$$\bigcap_{t\in[0,1]} \mathrm{D}((E^\mu_\lambda)_t)$$

is an isolating neighbourhood for all $(U_\lambda)_t$, where $(E^\mu_\lambda)_t \to (U_\lambda)_t$ is the g_t version of the tubular neighbourhood introduced in Remark 4.14. Therefore,

$$I_{\mathrm{U}(1)}\left((U_\lambda)_0, (\varphi^\mu_{\lambda,r})_0\right) \simeq I_{\mathrm{U}(1)}\left((U_\lambda)_1, (\varphi^\mu_{\lambda,r})_1\right),$$

which is not much to say due to the characterisation of these as Thom spaces in Corollary 4.16; however, the fact that both this homotopy equivalence and the one of $(*)$ come from the same deformation of the flow allows one to use the naturality of the attractor-repeller cofibration sequence so as to have the following diagram commute up to homotopy.

$$\begin{array}{ccc}
I_{\mathrm{U}(1)}\left((S^\mu_{\lambda,r})_0, (\varphi^\mu_{\lambda,r})_0\right) & \longrightarrow & I_{\mathrm{U}(1)}\left((U_\lambda)_0, (\varphi^\mu_{\lambda,r_0})_0\right) \\
\downarrow & & \downarrow \\
I_{\mathrm{U}(1)}\left((S^\mu_{\lambda,r})_1, (\varphi^\mu_{\lambda,r})_1\right) & \longrightarrow & I_{\mathrm{U}(1)}\left((U_\lambda)_1, (\varphi^\mu_{\lambda,r_1})_1\right).
\end{array}$$

Now, the effect of the desuspensions shall be addressed. Consider the subspace $(W^{(-\mu,0)})_t \subset W^\mu$. Note that, despite all the $(W^\mu)_t \cong W^\mu$ being identified, the subspaces $(W^{(-\mu,0)})_t$ shall still vary with $t \in [0,1]$. Recall that W is defined as the direct sum of a real and a complex space; use $n_{\mu,t}$ to denote the complex dimension of this complex summand appearing in $(W^{(-\mu,0)})_t$. Notice that $n_{\mu,1} - n_{\mu,0}$ is the spectral flow of the family of Dirac operators $(\mathfrak{D}_{A_\lambda})_t$ defined as the metric g_t varies. One can then check with Proposition 3.67 that

$$n(Y, \mathfrak{s}_\lambda, g_0) - n(Y, \mathfrak{s}_\lambda, g_1) = n_{\lambda,1} - n_{\lambda,0}.$$

Without loss of generality, suppose $n(Y, \mathfrak{s}_\lambda, g_0) \leq n(Y, \mathfrak{s}_\lambda, g_1)$. Together with the fact that the operator family $*_t\mathrm{d} : \Omega^1(Y) \to \Omega^1(Y)$ has zero spectral flow due to $\mathrm{H}_1(Y; \mathbf{R}) = 0$, the above implies that

$$(W^{(-\mu,0)})_0 \cong (W^{(-\mu,0)})_1 \oplus \mathbf{C}^{n(Y,\mathfrak{s}_\lambda,g_1) - n(Y,\mathfrak{s}_\lambda,g_0)}.$$

The result follows by combining this with the commuting diagram above. **QED**

Remark 4.33: In the same vein as in Manolescu (2003), the metric invariance can be strengthened to invariance up to *canonical* isomorphism, which is to say, in this context, that the isomorphism does not depend on the path of metrics interpolating the given two metrics, but the details shall be left out.

5. Recovery of the Cohomological Invariant

In light of the equivalence, proved by Lidman & Manolescu (2018a), between the Borel U(1)-equivariant cohomology of the Seiberg-Witten Floer spectrum and the monopole Floer "from" cohomology, this section shall discuss the relation between the cohomotopical contact invariant defined in the previous section and the well known contact invariants in Floer cohomology.

In Kronheimer, Mrowka, Ozsváth & Szabó (2007), §6.3, a distinguished element of the monopole Floer cohomology group $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s}_\lambda)$, therein denoted $\check{\psi}$, is defined (up to sign) from the datum of a contact structure $\operatorname{Ker}\lambda$. Here, this class shall be denoted $\psi(\lambda)$. In fact, most of the groundwork for the definition of this invariant was done a decade earlier in Kronheimer & Mrowka (1997), except that the machinery of monopole Floer homology had not yet been developed. This same invariant was studied in Taubes (2009), §4, therein denoted $\mathfrak{t}_r$, and was shown (Taubes 2009, Proposition 4.3) to be generated by a single generator of the monopole Floer cochain complex. This generator is essentially the contact configuration, herein denoted C_λ, with the caveat that a generic perturbation must be used in that context, else the monopole Floer cohomology groups cannot be defined.

By work of Taubes (2010a, 2010b, 2010c, 2010d & 2010e), it was established that there is a natural equivalence between the monopole Floer cohomology $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s}_\lambda)$ and the embedded contact homology $\mathrm{ECH}_*(Y, \lambda; 0)$ of M. Hutchings (vid. Hutchings & Taubes 2007). Furthermore, in ECH, there is a very simply defined contact invariant, which is the class generated by the empty set of Reeb orbits. In Taubes (2010e), Taubes established that, under his isomorphism, the ECH contact invariant corresponds to $\psi(\lambda)$.

There is yet another guise under which $\psi(\lambda)$ appears. In Kutluhan, Lee & Taubes (2020a, 2020b, 2020c, 2021a & 2021b), the authors construct isomorphisms between a variant of the monopole Floer homologies, called the *balanced* monopole Floer homologies, and the *Heegaard* Floer homologies of Ozsváth & Szabó (2004); in case $b_1 = 0$, these balanced monopole Floer homologies agree with the usual monopole Floer homologies. In Ozsváth & Szabó (2005), an invariant of contact structures is defined (up to sign) which lives in the group $\mathrm{HF}^+(-Y, \mathfrak{s}_\lambda)$; this is usually denoted $c^+(Y, \lambda)$. This invariant was identified with the ECH version of the contact invariant in Colin, Ghiggini & Honda (2012a & 2012b).

The manner via which the cohomotopical contact invariant $\Psi(\lambda)$ recovers the cohomological invariant $\psi(\lambda)$ is fundamentally rather simple. Returning to the motif of the collapse of all but a single cell in a CW-complex, one can obtain a class in the cohomology of that complex by pulling back the generator of the cohomology of the sphere. In the case of a U(1)-CW-complex, as has been said in the preceding section, the same construction leads to a map from the complex to the Thom space of a vector bundle over an orbit space. One then proceeds by understanding the Borel cohomology of the Thom space as being generated by an equivariant Thom class and the pullback of this Thom class provides a cohomological invariant.

Since the classical cohomological contact invariant is the class in monopole Floer cohomology defined by the cochain consisting only of the generator associated to the contact monopole, it is not at all surprising that this approach shall work. However, some complexity emerges in that the author must make use of the appropriate sort of *generic* perturbation in line with what is done in Lidman & Manolescu (2018a). Therefore, this section shall inevitably assume a certain degree of familiarity on the reader's part with the technical apparatus of the canonical reference which is Kronheimer & Mrowka (2007).

The author starts this section by demonstrating that the construction of the invariant $\Psi(\lambda)$ is not affected by the addition of a *small* generic perturbation.

Definition 5.1: By an *(abstract) perturbation,* one means a gauge equivariant section $\mathfrak{q} : \mathcal{C}(Y) \to \mathrm{T}\mathcal{C}(Y)$. The *Seiberg-Witten vector field perturbed by* $\mathfrak{q}$ is defined as $\mathcal{X}_{\lambda,r;\mathfrak{q}} := \mathcal{X}_{\lambda,r} + \mathfrak{q}$.

Definition 5.2: Let $\mathcal{C}^\sigma(Y,\mathfrak{s}_\lambda)$ denote *the real blow-up of* $\mathcal{C}(Y,\mathfrak{s}_\lambda)$ *along the reducibles with respect to the* L^2 *norm*; that is,

$$\mathcal{C}^\sigma(Y,\mathfrak{s}_\lambda) := \{(A,s,\psi) \in \mathcal{A}(\det \mathfrak{s}_\lambda) \times \mathbf{R}_{\geq 0} \times \Gamma(\mathcal{S}_\lambda) \mid \|\psi\|_{\mathrm{L}^2} = 1\}.$$

Use $\mathcal{C}^\sigma(Y,\mathfrak{s}_\lambda)_k$ for the completion in the Sobolev norm L^2_k.

Definition 5.3: Noting that $\mathcal{G}(Y)$ acts freely on $\mathcal{C}^\sigma(Y,\mathfrak{s}_\lambda)$, denote its quotient by

$$\mathcal{B}^\sigma(Y,\mathfrak{s}_\lambda) := \mathcal{C}^\sigma(Y,\mathfrak{s}_\lambda)/\mathcal{G}(Y).$$

Use $\mathcal{B}^\sigma(Y,\mathfrak{s}_\lambda)_k$ for the quotient

$$\mathcal{B}^\sigma(Y,\mathfrak{s}_\lambda)_k := \mathcal{C}^\sigma(Y,\mathfrak{s}_\lambda)_k/\mathcal{G}(Y)_{k+1}.$$

Definition 5.4: Given a perturbation $\mathfrak{q}$, one can obtain the $\mathcal{G}(Y)$-equivariant vector field $\mathcal{X}^{\sigma}_{\lambda,r;\mathfrak{q}}$ on the blow-up $\mathcal{C}^{\sigma}(Y,\mathfrak{s}_\lambda)$. Also, this vector field descends to the quotient $\mathcal{B}^{\sigma}(Y,\mathfrak{s}_\lambda)$ to define a vector field $[\mathcal{X}_{\lambda,r;\mathfrak{q}}]^{\sigma}$.

Recall that one defines the monopole Floer cohomology $\widehat{\mathrm{HM}}^*(Y,\mathfrak{s}_\lambda)$ (Kronheimer & Mrowka 2007, Definition 22.3.4) as the cohomology of a cochain complex $\hat{C}^*$ (cf. Kronheimer & Mrowka 2007, Formula (22.2)) whose generators are the fixed points of the Seiberg-Witten vector field $[\mathcal{X}_{\lambda,r;\mathfrak{q}}]^{\sigma}$ on the space $\mathcal{B}^{\sigma}(Y,\mathfrak{s}_\lambda)_k$ and whose differentials are defined by counting Seiberg-Witten trajectories in a suitable way (cf. Kronheimer & Mrowka 2007, Definition 22.1.3). For this to be well defined, it is necessary to require non-degeneracy of the fixed points (Definition 2.20; cf. Kronheimer & Mrowka 2007, Definition 12.1.1) and to impose a certain *Morse-Smale* condition on the moduli spaces of trajectories between fixed points (Kronheimer & Mrowka 2007, Definition 14.5.6). This is achieved through a judicious choice of perturbation called an *admissible perturbation* (Kronheimer & Mrowka 2007, Definition 22.1.1), which will cause the perturbed Seiberg-Witten vector field $[\mathcal{X}_{\lambda,r;\mathfrak{q}}]^{\sigma}$ to satisfy these requirements.

To certify that there be enough perturbations to always achieve the required transversality, one produces a *large Banach space of tame perturbations* (Kronheimer & Mrowka 2007, Definition 11.6.3), which exists according to Kronheimer & Mrowka (2007), Theorem 11.6.1. Denote such a Banach space by $\mathcal{P}$. Then, according to Kronheimer & Mrowka (2007), Theorem 15.1.1, there always is some admissible perturbation in $\mathcal{P}$. However, the proof of this theorem achieves somewhat more than that. Indeed, it follows from Kronheimer & Mrowka (2007), Proposition 15.1.3, that there is a sequence $\mathfrak{q}_n$ of admissible perturbations converging to zero; hence, admissible perturbations may be assumed to have norm as small as desired.

Proposition 5.5: Given a large Banach space of tame perturbations $\mathcal{P}$, there exists a ball $\mathrm{B}(\mathcal{P},\varepsilon)$ of radius $\varepsilon > 0$ centred at zero and smooth map

$$\mathrm{B}(\mathcal{P},\varepsilon) \to \mathcal{C}(Y,\mathfrak{s}_\lambda)_k,$$

which shall be denoted $\mathfrak{q} \mapsto C_\lambda(\mathfrak{q})$, satisfying $C_\lambda(0) = C_\lambda$ and $\mathcal{X}_{\lambda,r}(C_\lambda(\mathfrak{q})) = 0$.

Proof: This follows from the implicit function theorem of Banach manifolds applied to the map

$$\mathcal{C}(Y,\mathfrak{s}_\lambda)_k \times \mathcal{P} \to \mathcal{K}, \quad (C,\mathfrak{q}) \mapsto \mathcal{X}_{\lambda,r;\mathfrak{q}}(C),$$

where transversality is ensured via Kronheimer & Mrowka (2007), Lemma 12.5.2.

QED

What this proposition is saying, morally, is that, if one make a small enough perturbation, it is possible to "track" the contact configuration to a nearby configuration, which shall still solve the perturbed Seiberg-Witten equations. Next, one desires to see that this perturbed contact configuration $C_\lambda(\mathfrak{q})$ still satisfies all the required theorems of §2 provided $\mathfrak{q}$ be made small enough.

Proposition 5.6: Given a large Banach space of tame perturbations $\mathcal{P}$ there exists some $\varepsilon > 0$ lower than that of Proposition 5.5 such that, for any $\mathfrak{q} \in \mathcal{P}$ with norm less than ε, the monopole $C_\lambda(\mathfrak{q})$ satisfies the same conclusions of Theorem 2.23, Theorem 2.24 and Theorem 2.25, where C_λ is traded for $C_\lambda(\mathfrak{q})$ and $\mathcal{X}_{\lambda,r}$ is traded for $\mathcal{X}_{\lambda,r;\mathfrak{q}}$.

Proof: Assume the contrary. Then, there is some sequence $\mathfrak{q}_n \subset \mathcal{P}$ converging to zero such that each $\mathfrak{q}_n$ violate one of Theorem 2.23, Theorem 2.24 or Theorem 2.25. Now, invoke Kronheimer & Mrowka (2007), Proposition 11.6.4, to produce a sequence of gauge transformations $u_i \in \mathcal{G}(Y)$ such that $u_i \cdot C_\lambda(\mathfrak{q}_i)$ converge to a solution C of

$$\mathcal{X}_{\lambda,r}(C) = 0.$$

But then it must be the case that C is gauge equivalent to C_λ, which implies that C_λ would violate one Theorem 2.23, Theorem 2.24 or Theorem 2.25. **QED**

The consequence of the last two propositions is that one can repeat the construction of the cohomotopical contact invariant from §4 using the perturbed setup provided by $\mathcal{X}_{\lambda,r;\mathfrak{q}}$ and $C_\lambda(\mathfrak{q})$, where $\mathfrak{q}$ be any sufficiently small admissible perturbation in the Banach space $\mathcal{P}$.

Definition 5.7: Given a perturbation $\mathfrak{q}$, denote by $\mathrm{SWF}(Y, \mathfrak{s}; \mathfrak{q})$ the version of the Seiberg-Witten Floer spectrum constructed just as $\mathrm{SWF}(Y, \mathfrak{s})$ was in §3 but using the finite dimensional approximations to the perturbed Seiberg-Witten vector field

$$\mathcal{X}^{\mathrm{GC}}_{\lambda,r;\mathfrak{q}} := \mathcal{X}^{\mathrm{GC}}_{\lambda,r} + \Pi^{\mathrm{GC}}_{*}\mathfrak{q}$$

instead of $\mathcal{X}^{\mathrm{GC}}_{\lambda,r}$.

Definition 5.8: Given a perturbation $\mathfrak{q}$ with norm less than the ε provided by Proposition 5.6, denote by

$$\Psi(\lambda; \mathfrak{q}) : \mathrm{SWF}(Y, \mathfrak{s}_\lambda; \mathfrak{q}) \to \mathcal{T}(\lambda; \mathfrak{q})$$

the version of the cohomotopical invariant constructed just as was done in §4 but by using $\mathcal{X}_{\lambda,r;\mathfrak{q}}$ and $C_\lambda(\mathfrak{q})$ instead of $\mathcal{X}_{\lambda,r}$ and C_λ.

Proposition 5.9: (Lidman & Manolescu 2018a, Proposition 6.1.6) There exists a large Banach space of tame perturbations, $\mathcal{P}$ called a *large Banach space of very tame perturbations* such that, for any $\mathfrak{q} \in \mathcal{P}$, $\mathrm{SWF}(Y, \mathfrak{s}_\lambda; \mathfrak{q})$ is U(1)-equivariantly stably homotopy equivalent to $\mathrm{SWF}(Y, \mathfrak{s}_\lambda)$.

Proposition 5.10: For $\mathcal{P}$ a large Banach space of very tame perturbations just as in Proposition 5.9, and $\mathfrak{q} \in \mathcal{P}$ with norm less than the ε provided by Proposition 5.6, there is a commuting square

$$\begin{array}{ccc} \mathrm{SWF}(Y, \mathfrak{s}_\lambda) & \longrightarrow & \mathcal{T}(\lambda) \\ \big\downarrow & & \big\downarrow \\ \mathrm{SWF}(Y, \mathfrak{s}_\lambda; \mathfrak{q}) & \longrightarrow & \mathcal{T}(\lambda; \mathfrak{q}), \end{array}$$

where the horizontal arrows are $\Psi(\lambda)$ and $\Psi(\lambda; \mathfrak{q})$, the left vertical arrow is the one provided by Proposition 5.9 and the right vertical arrow is a U(1)-equivariant stable homotopy equivalence.

Proof: The result follows by arguing similarly to Proposition 4.32, where $\Psi(\lambda)$ was shown not to depend on the metric, but, instead of interpolating between metrics, interpolating with $t\mathfrak{q}$, $t \in [0, 1]$, between the perturbation $\mathfrak{q}$ and zero and using Lidman & Manolescu (2018a), Proposition 6.1.2, to ensure the Conley indices remain unchanged. **QED**

Definition 5.11: Let $U_\lambda(\mathfrak{q})$ denote the U(1)-orbit of $C_\lambda(\mathfrak{q})$.

Definition 5.12: Given an admissible perturbation $\mathfrak{q}$ denote by $\hat{C}^*(\mathfrak{q})$ the cochain complex defining the monopole Floer cohomology $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s}_\lambda)$ (cf. Kronheimer & Mrowka 2007, Formula (22.2)).

At this point, it is also possible to identify a very simple form for the cohomological contact invariant. Firstly, notice that there is a class in the cochain complex $\hat{C}^*(\mathfrak{q})$ corresponding to the orbit $U_\lambda(\mathfrak{q})$ of the monopole $C_\lambda(\mathfrak{q})$. This class is in fact a cocycle due to Theorem 2.25.

Proposition 5.13: (Taubes 2009, Proposition 4.3; cf. Taubes 2010d) For sufficiently large parameter $r > 0$ and a perturbation $\mathfrak{q}$ with norm less than the ε provided by Proposition 5.6, the contact invariant $\psi(\lambda)$, as defined in Kronheimer, Mrowka, Ozsváth & Szabó 2007, is, up to sign, the cohomology class of the cocyle $U_\lambda(\mathfrak{q})$ in the cochain complex $\hat{C}^*(\mathfrak{q})$.

Remark 5.14: Henceforth, assume $r > 0$ to be large enough to make the preceding hold.

Now, the author begins to recall the setup used in Lidman & Manolescu (2018a) to equate $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s}_\lambda)$ and the Borel cohomology of $\mathrm{SWF}(Y, \mathfrak{s}_\lambda)$. The reader is cautioned that, therein, the letter λ is used for the spectral cut-off parameter, herein denoted μ. Moreover, because of the author's attempt to keep the notation as close to Lidman & Manolescu (2018a) as possible, the present need to emphasize the contact form λ and the parameter r shall, unfortunately, lead to a certain saturation of decorations on the symbols used.

Definition 5.15: Denote by W^σ the real blow-up of W along the reducibles; that is,

$$W^\sigma := \{(A_\lambda + a, s, \psi) \in \mathcal{A}(\det \mathfrak{s}_\lambda) \times \mathbf{R}_{\geq 0} \times \Gamma(\mathcal{S}_\lambda) \mid \mathrm{d}^* a = 0, \, \|\psi\|_{\mathrm{L}^2} = 1\}.$$

Use $(W_k)^\sigma$ to denote the L^2_k completion.

Definition 5.16: Define the *finite dimensional approximation to the perturbed Seiberg-Witten vector field* by

$$\mathcal{X}^\mu_{\lambda,r;\mathfrak{q}} := u^\mu \cdot \left(\ell + p^\mu(c + \Pi^{\mathrm{GC}}_* \mathfrak{q})\right).$$

Definition 5.17: For a finite dimensional approximation $W^\mu \subset W_k$, denote the blow-up along the reducibles by $(W^\mu)^\sigma$.

Definition 5.18: The *blown-up finite dimensional approximation to the perturbed Seiberg-Witten vector field,* $\mathcal{X}^{\mu,\sigma}_{\lambda,r;\mathfrak{q}}$, is the vector field on $(W^\mu)^\sigma$ uniquely determined by the $\mathrm{U}(1)$-equivariant vector field $\mathcal{X}^\mu_{\lambda,r;\mathfrak{q}}$.

Definition 5.19: Denote by $\mathcal{X}^{\mathrm{AGC},\mu,\sigma}_{\lambda,r;\mathfrak{q}}$ the vector field uniquely determined by $\mathcal{X}^{\mu,\sigma}_{\lambda,r;\mathfrak{q}}$ on the quotient $(W^\mu)^\sigma/\mathrm{U}(1)$.

Remark 5.20: The notation "AGC" here (cf. Definition 4.7) is in agreement with Lidman & Manolescu (2018a), §6.2.

Using the flow of $\mathcal{X}^{\mathrm{AGC},\mu,\sigma}_{\lambda,r;\mathfrak{q}}$ on $(W^\mu)^\sigma/\mathrm{U}(1)$, Lidman & Manolescu (2018a) proceed to define a Morse complex out of its fixed points and trajectories. For that, one needs to select a suitable perturbation $\mathfrak{q}$ to ensure that this vector field be a *Morse-Smale vector field* in a sense similar to that of Palis (1969); cf. Melo & Palis (1982).

The only difference is that, here, just as in Kronheimer & Mrowka (2007), special care must be taken with the fact that $(W^\mu)^\sigma$ is a *manifold-with-boundary* and trajectories can converge to or emanate from the boundary. The Morse-Smale condition in such a scenario is developed in detail in Kronheimer & Mrowka (2007), §2.4, for the case in which the flow be a gradient flow. In the case at hand, the flow is not a gradient flow, but, nonetheless, the Morse-Smale condition can be stated as follows.

Definition 5.21: A smooth flow $\phi : \mathbf{R} \times \Gamma \to \Gamma$ on a manifold-with-boundary Γ induced by a vector field v tangent to $\partial\Gamma$ is said to be a *Morse-Smale flow with no closed trajectories* whenever

(i) it has finitely many fixed points all of which are *hyperbolic* (that is, the derivative $\mathrm{D}\,v$ has no purely imaginary eigenvalues at that point);

(ii) given any two fixed points x, y for which one of x or y not be in $\partial\Gamma$, the stable manifold of x intersects the unstable manifold of y transversely in Γ;

(iii) given any two fixed points x, y in $\partial\Gamma$, the stable manifold of x intersects the unstable manifold of y transversely in $\partial\Gamma$;

(iv) it has no closed trajectories.

Given a Morse-Smale vector field with no closed trajectories, one can define its Morse complex, in just the same manner as is done in Kronheimer & Mrowka (2007), §2.4, by generating the abelian group from the fixed points and defining the differentials by counting trajectories. This is detailed in Lidman & Manolescu (2018a), §2.5. In fact, there is a stronger condition satisfied by the flow of $\mathcal{X}^{\mathrm{AGC},\mu,\sigma}_{\lambda,r;\mathfrak{q}}$, which Lidman & Manolescu (2018a) call a "quasi-gradient". However, for the purposes being pursued here, such details need not concern the reader.

With an eye towards establishing an isomorphism with $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s}_\lambda)$, one must also ensure this perturbation to be admissible in the sense of Kronheimer & Mrowka (2007). The existence of such a perturbation can only be guaranteed if the spectral cut-off parameter be chosen in appropriate fashion. For that, recall from Remark 3.21 that there was a condition imposed on the definition of the spectral cut-off projections $p^\mu : W_k \to W^\mu$ (vid. Definition 3.22), which consisted of requiring that, for an unbounded strictly increasing sequence $\{\mu_i\} \subset \mathbf{R}_{>0}$, one have $p^{\mu_i} = \tilde{p}^{\mu_i}$, where $\tilde{p}^\mu : W_k \to W^\mu$ are the L^2-orthogonal projections (vid. Definition 3.19).

Proposition 5.22: (Lidman & Manolescu 2018a, Propositions 7.4.1, 8.0.1 and 10.0.2) Let $\mathcal{P}$ be a large Banach space of very tame perturbations in sense of Proposition 5.9. There exists an integer $L > 0$ such that, for $i \geq L$, there exists an admissible perturbation $\mathfrak{q} \in \mathcal{P}$ for which $\mathcal{X}^{\mathrm{AGC},\mu_i,\sigma}_{\lambda,r;\mathfrak{q}}$ is a *Morse-Smale vector field with no closed trajectories* in the sense of Definition 5.21.

Remark 5.23: As was the case with Kronheimer & Mrowka (2007), Theorem 15.1.1, the proof of this proposition actually establishes something slightly stronger. If the reader care to check, in fact, the proof establishes that one can require that the perturbation have norm as small as desired.

Definition 5.24: Given L and $\mathfrak{q}$ as asserted to exist by Proposition 5.22 and $i \geq L$, use $\hat{C}^*(\mu_i, \mathfrak{q})$ to denote the Morse complex defined by the vector field $\mathcal{X}^{\mathrm{AGC},\mu_i,\sigma}_{\lambda,r;\mathfrak{q}}$. Note that this cochain complex receives a $\mathbf{Q}$-grading instead of a $\mathbf{Z}$-grading defined according to Lidman & Manolescu (2018a), Formula (231).

Remark 5.25: Likewise, whenever dealing with gradings of $\hat{C}^*(\mathfrak{q})$ and $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s}_\lambda)$, the author means the $\mathbf{Q}$-grading according to Kronheimer & Mrowka (2007), Definition 28.3.1.

Given this grading, by compactness of the moduli space of solutions to the Seiberg-Witten equations (Kronheimer & Mrowka 2007, Theorem 12.1.2), there is a finite range of gradings of $\hat{C}^*(\mathfrak{q})$ for which there be classes corresponding to irreducible monopoles.

Definition 5.26: Let $I_{\text{irred}} \in \mathbf{Q}$ denote the maximum value of $|i|$ for which $\hat{C}^i(\mathfrak{q})$ admit a generator defined by an irreducible monopole.

Proposition 5.27: (Lidman & Manolescu 2018a, Propositions 9.3.1, 13.1.1, 13.1.4 and 13.3.1) For L and $\mathfrak{q}$ as in Proposition 5.22, there exists a constant $N \in \mathbf{Q}_{>0}$ such that $N > I_{\text{irred}}$ and, for any $i > L$ there is a chain homomorphism $f : \hat{C}^*(\mathfrak{q}) \to \hat{C}^*(\mu_i, \mathfrak{q})$ satisfying the following properties.

(i) For any $j \in [-N, N]$, f induces an *isomorphism* in homology

$$\mathrm{H}^j(\hat{C}^*(\mathfrak{q})) \cong \mathrm{H}^j(\hat{C}^*(\mu_i, \mathfrak{q}));$$

(ii) for any $j \in [-N, N]$, f restricted to degree j is defined by a 1-to-1 correspondence between the generators of $\hat{C}^j(\mathfrak{q})$ (that is, monopoles) and the generators of $\hat{C}^j(\mu_i, \mathfrak{q})$ (that is, fixed points of $\mathcal{X}^{\mathrm{AGC},\mu,\sigma}_{\lambda,r;\mathfrak{q}}$).

Proposition 5.28: (Lidman & Manolescu 2018a, Formulae (273) and (274)) For $\mathfrak{q}$ and L as provided by Proposition 5.22 and $i \geq L$, there exists a constant $M_i \in \mathbf{Q}_{>0}$ such that, for any $j \in [-M_i, M_i]$,

$$\mathrm{c}\tilde{\mathrm{H}}^{j-\mathfrak{n}(Y,\mathfrak{s}_\lambda)}_{\mathrm{U}(1)}(\mathrm{SWF}(Y, \mathfrak{s}_\lambda; \mathfrak{q})) \cong \mathrm{H}^j(\hat{C}(\mu_i, \mathfrak{q})).$$

Moreover, as $i \to \infty$, so does $M_i \to \infty$.

Remark 5.29: The grading of $\mathrm{c}\tilde{\mathrm{H}}^{j-\mathfrak{n}(Y,\mathfrak{s}_\lambda)}_{\mathrm{U}(1)}(\mathrm{SWF}(Y, \mathfrak{s}_\lambda; \mathfrak{q}))$ is off by $\mathfrak{n}(Y, \mathfrak{s}_\lambda)$ because of the different conventions being used by the author (vid. Remark 3.71).

Before the author proceed to determine how the contact invariant shall recover the class $\psi(\lambda)$, the author should say something about how Proposition 5.28 is achieved. He also hopes to add some more detail to what is said in Lidman & Manolescu (2018a), §2.8. For that to be done, a review of some more concepts from Conley theory is needed. In what follows suppose Γ to be a manifold-with-boundary and $\phi : \mathbf{R} \times \Gamma \to \Gamma$ a smooth flow. Assume further that $\partial\Gamma$ be preserved by ϕ. In Hell (2009), §3.1.2, it is shown to be straightforward to extend the Conley theory to manifolds-with-boundary, such as Γ, provided that the flow be tangent to the boundary. The definitions of maximal invariant set and isolated invariant set (Definition 3.34 and Definition 3.35) are left unchanged. The only point that should be emphasized is that the notion of *neighbourhood* is, of course, in the *point-set-topological* sense. That is, for $S \subset \Gamma$ to be an isolated invariant subset, one asks for a compact set $U \subset \Gamma$ such that $S = \operatorname{Inv} U$ and that $S \subset \operatorname{int} U$, where $\operatorname{int} U$ is *the union of all opens sets contained in* U; no mention is made of $\partial\Gamma$. This means that, for example, if $U \subset \Gamma$ be a properly embedded manifold, it is perfectly permissible for S to intersect $\partial U \subset \partial\Gamma$. This understood, the definitions of index pair (Definition 3.36), attractor-repeller pair (Definition 4.21) and index triple (Definition 4.22) also remain unchanged; likewise for the existence theorems of index pairs (Theorem 3.37) and index triples (Theorem 4.23). Moreover, by arguing as in Floer (1987) or Floer & Zehnder (1988), all of this can be accomplished in the equivariant setting with little difficulty; hence, assume G to be a compact Lie group acting on Γ and let ϕ be G-equivariant. The notion of the Conley index $I_G(S, \phi)$ is still defined as above; however, one can also define a version of the Conley index *relative to the boundary of* Γ.

Definition 5.30: Let $S \subset \Gamma$ be an isolated invariant set with index pair (M, N). Define the *Conley index relative to the boundary,* $I^\partial_G(S, \phi)$, as the G-equivariant homotopy type of the space $M/(N \cup (M \cap \partial\Gamma))$.

Concepts of equivariant Conley theory generally remain true for $I^{\partial}_{G}(S, \phi)$ in analogous forms to how they hold in the case without boundary; therefore, the author shall refrain from listing all here.

Definition 5.31: Given an isolated invariant set $S \subset \Gamma$, by a *Morse decomposition of S*, one means a sequence $\{S_0, \ldots, S_n\}$ of pairwise disjoint subsets of S where each S_i be an isolated invariant set in Γ and such that, for each $x \in S \setminus (\bigcup_i S_i)$, there exist a pair of indices $i < j$ for which $\omega(x) \in S_i$ and $\omega^*(x) \in S_j$ (vid. Definition 4.18).

Definition 5.32: Given an isolated invariant set $S \subset \Gamma$ and a Morse decomposition $\{S_0, \ldots, S_n\}$ of S define the associated *Morse filtration* to be $\{S_{\leq 0}, \ldots, S_{\leq n}\}$ where $S_{\leq 0} := S_0$ and each $S_{\leq i}$ is defined successively by

$$S_{\leq i} := \{x \in S \mid \omega(x) \in S_{\leq i-1},\ \omega^*(x) \in S_i\}.$$

Remark 5.33: Notice that, for each i, $(S_{\leq i-1}, S_i)$ is an attractor-repeller pair decomposition of $S_{\leq i}$.

Now, the author specializes to the case in which the flow be *Morse-Smale with no closed trajectories* but he should clarify what that means in the equivariant case.

Definition 5.34: A ϕ-fixed G-orbit $\mathcal{O} \subset \Gamma$ is said to be *hyperbolic* whenever the derivative of the vector field $\frac{\partial}{\partial t}\phi(t, -)$ restrict to the normal bundle of $\mathcal{O}$ as a bundle endomorphism with no purely imaginary eigenvalues.

For hyperbolic fixed orbits, it makes sense to speak of their stable and unstable manifolds defined in the same manner and seen to be manifolds as in the non-equivariant case. Note that, in this case, they have the structure of a vector bundle over the orbit.

Definition 5.35: A G-equivariant smooth flow $\phi : \mathbf{R} \times \Gamma \to \Gamma$ over a manifold-with-boundary Γ is said to be a *Morse-Smale flow with no closed trajectories* when

(i) there are finitely many ϕ-fixed G-orbits, all of which are *hyperbolic*;

(ii) given any two ϕ-fixed G-orbits $\mathcal{O}_1$,$\mathcal{O}_2$, for which one of $\mathcal{O}_1$ or $\mathcal{O}_2$ not be in $\partial\Gamma$, the stable manifold of $\mathcal{O}_1$ intersects the unstable manifold of $\mathcal{O}_2$ transversely in Γ;

(iii) given any two ϕ-fixed G-orbits $\mathcal{O}_1$,$\mathcal{O}_2$, in $\partial\Gamma$, the stable manifold of $\mathcal{O}_1$ intersects the unstable manifold of $\mathcal{O}_2$ transversely in $\partial\Gamma$;

(iv) it has no closed trajectories.

Remark 5.36: Note that, for $\mathcal{O} \subset \Gamma$ a hyperbolic ϕ-fixed G-orbit, the Conley index $I_G(\mathcal{O})$ is of the form $(\mathrm{D}(V) \times \mathcal{O})/(\mathrm{S}(V) \times \mathcal{O})$ where V is some G-representation and $\mathrm{D}(V), \mathrm{S}(V)$ are the unit disk and sphere in a G-invariant inner product. The pair $(\mathrm{D}(V) \times \mathcal{O}, \mathrm{S}(V) \times \mathcal{O})$ is what is called a G-cell; cf. May & al. (1996), §I.3 and §X.

Henceforth, add the assumption that ϕ be a G-equivariant Morse-Smale flow with no closed trajectories. Then, for any isolated invariant set S, one can obtain a Morse decomposition $\{S_0, \ldots, S_n\}$ where each S_i is a ϕ-fixed G-orbit. In this case, the induced Morse filtration of $I_G(S, \phi)$ is simply a G-cell decomposition of S with a 1-to-1 correspondence between cells and elements of $\{S_0, \ldots, S_n\}$.

At this point, it becomes convenient to introduce the functoriality property of the Conley index. This theory has been developed by McCord (1986, 1991).

Definition 5.37: Given a pair of G-equivariant flow spaces (Γ_1, ϕ_1), (Γ_2, ϕ_2), a G-equivariant mapping $f : \Gamma_1 \to \Gamma_2$ satisfying $f(\partial\Gamma_1) \subset \partial\Gamma_2$ is called a *flow map* whenever it be equivariant with respect to the flows; that is, for all $t \in \mathbf{R}$ and $x \in \Gamma_1$, $\phi_2(t, f(x)) = f(\phi_1(t, x))$.

Proposition 5.38: (McCord 1986, Theorem 2.2) Given a flow map $f : (\Gamma_1, \phi_1) \to (\Gamma_2, \phi_2)$ and an isolated invariant set S_2 in Γ_2 with index pair (M_2, N_2), it follows that $S_1 := f^{-1}(S_2)$ is isolated invariant in Γ_1 with index pair $(M_1, N_1) := (f^{-1}(M_2), f^{-1}(N_2))$.

Remark 5.39: While this is proven in the non-equivariant case and without boundary, it is straightforward to extend it to these generalizations. In any event, for the purposes of the present thesis, in the only occasions in which this shall be used, it shall be evident that (M_1, N_1) shall define an index pair without recourse to this result.

Definition 5.40: Given a flow map $f : (\Gamma_1, \phi_1) \to (\Gamma_2, \phi_2)$ and index pairs (M_1, N_1), (M_2, N_2) for isolated invariant sets S_1, S_2 as in Proposition 5.38, define the induced maps on the Conley indices

$$I_G(f) : I_G(S_1, \phi_1) \to I_G(S_2, \phi_2), \quad I_G^\partial(f) : I_G^\partial(S_1, \phi_1) \to I_G^\partial(S_2, \phi_2),$$

respectively, by sending $[x] \in M_1/N_1$ to $[f(x)] \in M_2/N_2$ and $[x] \in M_1/(N_1 \cup (M_1 \cap \partial\Gamma_1)$ to $[f(x)] \in M_2/(N_2 \cup (M_2 \cap \partial\Gamma_2)$.

Specialize now to the case $G = \mathrm{U}(1)$.

Definition 5.41: A $\mathrm{U}(1)$-action on Γ is said to be *semifree* when all $\mathrm{U}(1)$-orbits be either fixed points or free.

Remark 5.42: The $\mathrm{U}(1)$-action on W^μ is semifree as it is of the form $\mathbf{R}^m \oplus \mathbf{C}^n$.

Now, suppose (Γ, ϕ) be a semifree $\mathrm{U}(1)$-equivariant Morse-Smale flow space with no closed trajectories and $\partial\Gamma = \emptyset$. Denote by Γ^σ the real blow-up of Γ along the fixed point set $\Gamma^{\mathrm{U}(1)}$. Then, it is easy to see that the induced $\mathrm{U}(1)$-action on Γ^σ is *free.* It is also easy to see that the flow ϕ then defines a $\mathrm{U}(1)$-equivariant flow ϕ^σ on Γ^σ. The isolated invariant set S also lifts to an isolated invariant set S^σ in the blow-up. Furthermore, It can be checked that the flow of ϕ^σ is Morse-Smale with no closed trajectories. The blow-down map $b : \Gamma^\sigma \to \Gamma$ defines a *flow map.* Now, given an index pair (M, N) for S, one can check directly that these define an index pair (M^σ, N^σ) for S^σ in the blow-up. Hence, without recourse to Proposition 5.38, it becomes clear that the induced map

$$I^\partial_{\mathrm{U}(1)}(b) : I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma) \to I^\partial_{\mathrm{U}(1)}(S, \phi) = I_{\mathrm{U}(1)}(S, \phi)$$

is well defined.

Next, note that, because the action of $\mathrm{U}(1)$ is free on Γ^σ, it follows that the action of $\mathrm{U}(1)$ is *free away from the base point* on $I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma)$; hence,

$$\mathrm{c}\tilde{\mathrm{H}}^*_{\mathrm{U}(1)}(I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma)) \cong \tilde{\mathrm{H}}^*(I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma)/\mathrm{U}(1)).$$

Because of this, it becomes desirable to compute the cohomology of the homotopy type $I_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma)/\mathrm{U}(1)$ via a Morse complex approach. To understand how that is done, introduce yet another flow space: $\tilde{\Gamma} := \Gamma^\sigma/\mathrm{U}(1)$ with the flow $\tilde{\phi}$ as induced by ϕ^σ. Denote the quotient map by $q : \Gamma^\sigma \to \tilde{\Gamma}$; note that q is a flow map and that the isolated invariant set S^σ descends via q to an isolated invariant set $\tilde{S}$ on $\tilde{\Gamma}$. Therefore, one has a map

$$I^\partial_{\mathrm{U}(1)}(q) : I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma) \to I^\partial_{\mathrm{U}(1)}(\tilde{S}, \tilde{\phi}) = I^\partial(\tilde{S}, \tilde{\phi})$$

where the $\mathrm{U}(1)$-action on $\tilde{\Gamma}$ is being taken to be trivial. It is not difficult to see that this map is nothing more than the quotient map

$$I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma) \to I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma)/\mathrm{U}(1).$$

Now, note that the maps induced by flow maps respect Morse decompositions in the following sense.

Proposition 5.43: (McCord 1991, Proposition 3.3) If $f : (\Gamma_1, \phi_1) \to (\Gamma_2, \phi_2)$ be a flow map, $S \subset \Gamma_2$ be an isolated invariant set and $\{S_0, \ldots, S_n\}$ be a Morse decomposition for S, then $\{f^{-1}(S_0), \ldots, f^{-1}(S_n)\}$ is a Morse decomposition for $f^{-1}(S)$.

Remark 5.44: As in the other result of McCord (1986) cited above, this was proven in the non-equivariant case and without boundary. Again, it is straightforward to apply McCord's argument with these generalizations in place; however, in the cases in which this result shall be used, it shall be clear that it holds in a direct way.

Remark 5.45: Given the assumption that the flow ϕ on Γ be Morse-Smale without closed trajectories, then it follows that there is a Morse decomposition $\{S_0, \ldots, S_n\}$ for S in which each S_i is simply a ϕ-fixed hyperbolic U(1)-orbit; by the semifree assumption, this means it is either a point or a free circle. Lifting this to the blow-up, one gets $\{S_0^\sigma, \ldots, S_n^\sigma\}$, which is easily seen, directly, to be a Morse decomposition for S^σ. It is no longer true, however, that each S_i^σ is a U(1)-orbit; when S_i be a U(1)-fixed point, S_i^σ may consist of multiple U(1)-orbits and trajectories joining them. Hence, one can perform further Morse decompositions for each S_i^σ. Let them be denoted $\{S_{i,0}^\sigma, \ldots, S_{i,m_i}^\sigma\}$. Note that, by concatenating them, these Morse decompositions define a combined Morse decomposition of S^σ in which each set is indeed a single ϕ-fixed hyperbolic U(1)-orbit. Now, one can descend each of these Morse decompositions to the quotient $\tilde{\Gamma}$ to obtain Morse decompositions $\{\tilde{S}_{i,0}, \ldots, \tilde{S}_{i,m_i}\}$. These Morse decompositions consist of hyperbolic fixed points of the flow $\tilde{\phi}$ and, it is standard to compute the cohomology of $I^\partial(\tilde{S}, \tilde{\phi})$ by building a Morse complex generated by $\{\tilde{S}_{i,j}\}$ with differentials defined by a careful count of trajectories as defined in Kronheimer & Mrowka (2007), §2.4. The fact that this complex computes the claimed cohomology is the content of Floer (1989), albeit not for the Conley indices with boundary as is the case here, but Floer's proof can be generalized to this case as is argued in Lidman & Manolescu (2018a), §2.8. Let this cochain complex be denoted by C^*. Now, if n denote the codimension of the submanifold $\Gamma^{\mathrm{U}(1)}$ in Γ, it follows that, for any $i \in \{0, \ldots, n-2\}$,

$$\mathrm{c}\tilde{\mathrm{H}}^i_{\mathrm{U}(1)}(I_{\mathrm{U}(1)}(S, \phi)) \cong \mathrm{c}\tilde{\mathrm{H}}^i_{\mathrm{U}(1)}(I^\partial_{\mathrm{U}(1)}(S^\sigma, \phi^\sigma)) \cong \mathrm{H}^i(C^*),$$

cf. Kronheimer & Mrowka (2007), §2.6.

Remark 5.46: If one care to check (vid. Lidman & Manolescu 2018a, Formula (272)), this is how the proof of Proposition 5.28 above is obtained; here, Γ is the flow space W^{μ_i} with the flow ϕ being that of the vector field $-\mathcal{X}^{\mu_i}_{\lambda,r;\mathfrak{q}}$. Note that, the codimension of the U(1)-fixed point set can be made as large as desired simply by increasing the spectral cut-off parameter μ_i. Therefore, the isomorphism above can be obtained between arbitrarily large degrees as is claimed.

Bearing this discussion in mind, it becomes possible to determine how the contact invariant comes into picture.

Definition 5.47: Suppose a G-equivariant (reduced) cohomology theory h^*_G be given. By a G-equivariant Thom class, one means, for a G-vector bundle $V \to X$, a class $\theta_G(V) \in h^*_G(\Theta_G(V))$, where $\Theta_G(V)$ denotes the Thom space, subject to the requirement that, given the inclusion of any G-orbit $i : \mathcal{O} \to X$, the restriction $i^*\theta_G(V)$ generate $h^*_G(\Theta_G(i^*V))$ as a free $h^*_G(\mathcal{O}_+)$-module, where $\mathcal{O}_+$ denotes the orbit space $\mathcal{O}$ with a disjoint base point added.

Remark 5.48: Specialising this definition to when G act trivially on the base space X, the only type of orbit is $i : G/G \to X$, so the requirement is that $i^*\theta_G(V)$ generate $h^*_G(\Theta_G(V|_p))$ as a free $h^*_G(S^0)$-module.

Remark 5.49: In the other extreme, specialising to when G act freely on X, the only type of orbit is $i : G/1 \to X$, so the requirement is that $\theta_G(V)$ generate $h^*_G(\Theta_G(V))$ as a free $h^*_G(X_+)$-module.

Remark 5.50: It is not at all certain if, given a bundle, such an equivariant Thom class exists, or even if the prerequisite that $h^*_G(\Theta_G(i^*V))$ be a free $h^*_G((G/H)_+)$-module with a single generator is satisfied. In the event that it exist, one shall say that V is h^*_G*-orientable.*

Remark 5.51: Note that, for $G = \mathrm{U}(1)$ acting on $X \cong \mathrm{U}(1)$ freely, there always exists a Thom class for any U(1)-bundle $E \to X$

Lemma 5.52: Suppose (Γ, ϕ) to be a semifree U(1)-equivariant Morse-Smale flow space with no closed trajectories such that $\partial\Gamma = \emptyset$. Suppose $S \subset \Gamma$ to be an isolated invariant set admitting a Morse decomposition $\{S_0, \ldots, S_n\}$ in which each S_i be a hyperbolic ϕ-fixed U(1)-orbit. Suppose that the last set, S_n, be a free U(1)-orbit whose unstable manifold be of dimension strictly less than the codimension of the fixed point set $\Gamma^{\mathrm{U}(1)} \subset \Gamma$. Consider the associated Morse filtration $\{S_{\leq 0}, \ldots, S_{\leq n}\}$. Denote by

$$p : I_{\mathrm{U}(1)}(S, \phi) \to I_{\mathrm{U}(1)}(S_n, \phi)$$

the cofibre map of the attractor-repeller pair cofibration

$$I_{\mathrm{U}(1)}(S_{\leq n-1},\phi) \to I_{\mathrm{U}(1)}(S,\phi) \to I_{\mathrm{U}(1)}(S_n,\phi)$$

associated to the last level of the Morse filtration. Denote by $E \to S_n$ the unstable normal bundle of S_n in Γ. Let

$$\Gamma^\sigma, \quad \tilde{\Gamma}, \quad \{m_i \mid i=0,\ldots,n\}, \quad \{\tilde{S}_{i,j} \mid i=0,\ldots,n;\ j=0,\ldots,m_i\}, \quad C^*$$

be as defined in Remark 5.45. Then, it follows that the following are true.

(i) $m_n = 0$;

(ii) the associated generator $S_{n,0}$ of the cochain complex C^* is a cocycle;

(iii) the cohomology class $[S_{n,0}]$ of this cocyle corresponds to $\pm p^*(\theta_{\mathrm{U}(1)}(E))$ under the isomorphism

$$\mathrm{c}\tilde{\mathrm{H}}^{\dim E}_{\mathrm{U}(1)}(I_{\mathrm{U}(1)}(S,\phi)) \cong \mathrm{H}^{\dim E}(C^*),$$

described in Remark 5.45.

Proof: For the first assertion, note that the assumption that S_n be U(1)-free means that it lifts to a single U(1)-free orbit, S^σ_n, in Γ^σ. Hence, the Morse decomposition of S^σ_n must consist of a single entry $\{S^\sigma_{n,0}\}$.

The second assertion is true because $\tilde{S}_{n,0}$ is the descent to the quotient of the free U(1)-orbit $S^\sigma_{n,0}$ which, in turn, comes, via the blow-up, from S_n. But, by assumption, S_n is the last entry of a Morse decomposition; therefore, there cannot be any trajectories coming into it. Hence, by Kronheimer & Mrowka (2007), Definition 2.4.4, this class is a cocycle, as the differential map of the cochain complex is defined by counting trajectories coming into the generator.

For the last assertion, consider firstly the combined Morse decomposition on $\tilde{\Gamma}$ for $\tilde{S}$ given by the sets $\{\tilde{S}_{i,j}\}$. Denote the resulting Morse filtration by $\{\tilde{S}_{\leq(i,j)}\}$. Associated to the last level of the filtration, is the attractor-repeller pair cofibration

$$I^\partial(\tilde{S}_{\leq(n-1,m_{n-1})},\tilde{\phi}) \to I^\partial(\tilde{S},\tilde{\phi}) \to I^\partial(\tilde{S}_{n,0},\tilde{\phi}).$$

Denote the cofibre map of this sequence by

$$\tilde{p} : I^\partial(\tilde{S},\tilde{\phi}) \to I^\partial(\tilde{S}_{n,0},\tilde{\phi}).$$

Also, as $\tilde{S}_{n,0}$ is a hyperbolic fixed point of $\tilde{\phi}$, denote by $\tilde{E} \to \tilde{S}_{n,0}$ its unstable normal bundle in $\tilde{\Gamma}$; this is simply a bundle over a *point*. It is therefore evident that $\tilde{p}^*\theta(\tilde{E})$

is, up to a sign, precisely the class of the cocycle $\tilde{S}_{n,0}$ under the isomorphism

$$\tilde{\mathrm{H}}^{\dim \tilde{E}}(I^{\partial}(\tilde{S}, \tilde{\phi})) \cong \mathrm{H}^{\dim \tilde{E}}(C^*).$$

Now, consider the corresponding Morse decomposition $\{S^{\sigma}_{i,j}\}$ of S^{σ} and denote by $\{S^{\sigma}_{\leq(i,j)}\}$ the associated Morse filtration. As above, for the cofibre map of the last level of this filtration, write

$$p^{\sigma} : I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}, \phi^{\sigma}) \to I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}_{n,0}, \phi^{\sigma}).$$

Let $E^{\sigma} \to S^{\sigma}_{n,0}$ be the the unstable normal bundle of $S^{\sigma}_{n,0}$ in Γ^{σ}; this bundle is $\mathrm{c}\tilde{\mathrm{H}}_{\mathrm{U}(1)}$-orientable by Remark 5.51. If $q : \Gamma^{\sigma} \to \tilde{\Gamma}$ denote the quotient map, then one obtains a map between Conley indices

$$I^{\partial}_{\mathrm{U}(1)}(q) : I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}, \phi^{\sigma}) \to I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}, \phi^{\sigma})/\mathrm{U}(1) \cong I^{\partial}(\tilde{S}, \tilde{\phi}).$$

Now, because this map preserves Morse decompositions (Proposition 5.43), there is a commuting diagram of the form

$$\begin{array}{ccc} I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}, \phi^{\sigma}) & \longrightarrow & I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}_{n,0}, \phi^{\sigma}) \\ \downarrow & & \downarrow \\ I^{\partial}(\tilde{S}, \tilde{\phi}) & \longrightarrow & I^{\partial}(\tilde{S}_{n,0}, \tilde{\phi}). \end{array}$$

where the horizontal maps are p^{σ} and $\tilde{p}$ respectively, and the vertical maps are induced by $I^{\partial}_{\mathrm{U}(1)}(q)$. Hence, it is not difficult to see that, under the isomorphism

$$\mathrm{c}\tilde{\mathrm{H}}^{\dim E}_{\mathrm{U}(1)}(I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}, \phi^{\sigma})) \cong \tilde{\mathrm{H}}^{\dim E}(I^{\partial}(\tilde{S}, \tilde{\phi})),$$

induced by $I^{\partial}_{\mathrm{U}(1)}(q)$, the class $\tilde{p}^*(\theta(\tilde{E}))$ corresponds to $\pm(p^{\sigma})^*(\theta_{\mathrm{U}(1)}(E^{\sigma}))$. Finally, reintroduce $b : \Gamma^{\sigma} \to \Gamma$ to denote the blow-down map. One then has a commuting diagram of the form

$$\begin{array}{ccc} I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}, \phi^{\sigma}) & \longrightarrow & I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}_{n,0}, \phi^{\sigma}) \\ \downarrow & & \downarrow \\ I_{\mathrm{U}(1)}(S, \phi) & \longrightarrow & I_{\mathrm{U}(1)}(S_n, \phi), \end{array}$$

where the horizontal maps are, respectively, p^{σ} and p and the vertical maps are induced by $I^{\partial}_{\mathrm{U}(1)}(b)$. Hence, it is not difficult to see that, under the isomorphism

$$\mathrm{c}\tilde{\mathrm{H}}^{\dim E}_{\mathrm{U}(1)}(I_{\mathrm{U}(1)}(S, \phi)) \cong \mathrm{c}\tilde{\mathrm{H}}^{\dim E}_{\mathrm{U}(1)}(I^{\partial}_{\mathrm{U}(1)}(S^{\sigma}, \phi^{\sigma}))$$

induced by $I^{\partial}_{\mathrm{U}(1)}(b)$, the class $(p^{\sigma})^*(\theta(E^{\sigma}))$ gets sent to $\pm p^*\theta(E)$. **QED**

Definition 5.53: Use $\theta_\lambda \in \mathrm{c}\tilde{\mathrm{H}}^*_{\mathrm{U}(1)}(\mathcal{T}(\lambda))$ to denote the desuspension of the Thom class $\theta_{\mathrm{U}(1)}(E^{\mathrm{u},\mu}_\lambda)$ suitably suspended or desuspended according to Definition 4.31, where $E^{\mathrm{u},\mu}_\lambda \to U_\lambda$ is the unstable bundle of U_λ in W^μ. Likewise, in the generically perturbed case, one defines a Thom class $\theta_\lambda(\mathfrak{q}) \in \mathrm{c}\tilde{\mathrm{H}}^*_{\mathrm{U}(1)}(\mathcal{T}(\lambda;\mathfrak{q}))$.

Remark 5.54: Of course, this is only defined *up to sign* which shall be consistent with the familiar sign ambiguity of the contact invariant in monopole Floer cohomology (cf. Lin, Ruberman & Saveliev 2018).

Theorem 5.55: The cohomological contact invariant is recovered by the cohomotopical invariant via

$$\pm\psi(\lambda) = \Psi(\lambda)^*(\theta_\lambda).$$

Proof: Let L and $\mathfrak{q}$ be as in Proposition 5.22. Let $i > L$ and assume also that the norm of $\mathfrak{q}$ be less than the ε of Proposition 5.10; this is permitted according to Remark 5.23. Then, according to Proposition 5.10 and Proposition 5.13 it suffices to prove that, under the isomorphisms of Proposition 5.28 and Proposition 5.27 one of the cohomology classes

$$\pm\Psi(\lambda;\mathfrak{q})^*(\theta_\lambda(\mathfrak{q})),$$

corresponds to the class in Morse cohomology expressed by the generator $U_\lambda(\mathfrak{q})$ of the complex $\hat{C}^*(\mu_i,\mathfrak{q})$.

Now, recalling from Remark 5.46 how Proposition 5.28 is proven in Lidman & Manolescu (2018a), it becomes clear that Lemma 5.52 can be applied in the case at hand with $\Gamma = W^{\mu_i}$, ϕ the flow of the vector field $-\mathcal{X}^{\mu_i}_{\lambda,r;\mathfrak{q}}$ and the isolated invariant set $S = S^{\mu_i}_{\lambda,r;\mathfrak{q}}$ defined just as in Definition 3.32 but using the generically perturbed Seiberg-Witten vector field. The Morse decomposition $\{S_0,\ldots,S_n\}$ is chosen to be any one whose entries consist of single hyperbolic ϕ-fixed U(1)-orbits while requiring that the last entry, S_n, be $U_\lambda(\mathfrak{q})$. This is permitted because, according to Proposition 5.6, $U_\lambda(\mathfrak{q})$ is a repeller. **QED**

Given Theorem 5.55, the author intends to use the newly constructed invariant to deduce results about $\psi(\lambda)$ in contexts in which it has proven difficult to do so while relying solely on the machinery of monopole Floer homology, Heegaard Floer homology and embedded contact homology. Note that it is *not* clear, at the moment, if there is any case in which $\Psi(\lambda)$ may hold any more information than $\psi(\lambda)$ does. The key advantage of $\Psi(\lambda)$ that the author wishes to emphasise is that it does not require a generic perturbation for its definition. This shall be exploited in the next section.

6. Finite Coverings

In Lidman & Manolescu (2018b), the authors studied the Seiberg-Witten Floer spectrum in the presence of a finite regular covering. The key to their results was the observation that the spectrum of the manifold upstairs in the covering acquires an action of the deck transformation group G and, upon taking appropriate fixed points of this action, one obtains the downstairs spectrum. A Smith-type inequality is then derived through actual application of Smith theory.

In this section, the author's goal is to formulate the contact invariant in this same scenario of a finite regular covering. This shall involve studying the attractor-repeller pair cofibration used to define the contact invariant in the G-equivariant setting. In doing so, one encounters no real difficulty and it is straightforward to obtain a G-equivariant cohomotopical contact invariant.

Consider a finite group G and a rational homology sphere Y equipped with a free G-action. Use $\pi : Y \to Y/G$ to denote the quotient map. Agree to fix a metric g on Y/G and use, on Y, the induced G-invariant metric π^*g. Suppose λ be a contact form on Y/G so that $\pi^*\lambda$ be a *G-equivariant contact form* on Y. Notice that the canonical $\mathrm{Spin}^{\mathbf{C}}$ structure defined by λ on Y/G naturally lifts to Y as the canonical $\mathrm{Spin}^{\mathbf{C}}$ structure defined by $\pi^*\lambda$; that is, $\pi^*\mathfrak{s}_\lambda = \mathfrak{s}_{\pi^*\lambda}$. Likewise, the connexion $A_{\pi^*\lambda}$ on $\det \mathfrak{s}_{\pi^*\lambda}$ is the lift π^*A_λ of the connexion A_λ on $\det \mathfrak{s}_\lambda$. Denote by W the global Coulomb slice with respect to $A_{\pi^*\lambda}$ on Y and by W' the global Coulomb slice with respect to A_λ on Y/G.

It is difficult to study this scenario in the classical setting of monopole Floer homology due to the need for generic perturbations in order to achieve the required Morse-Smale condition of Morse theory. Indeed, there is no guarantee that a sufficiently generic perturbation chosen for Y in order to satisfy the conditions for the construction of the group $\widehat{\mathrm{HM}}^*(Y, \pi^*\mathfrak{s})$ can be made G-equivariant so that it define a valid perturbation for the construction of $\widehat{\mathrm{HM}}^*(Y/G, \mathfrak{s})$. As a consequence, the behaviour of the monopole Floer homology groups under coverings has proven elusive to study via the classical Morse theoretic approach. As the construction of the SWF spectrum avoids the addition of a generic perturbation, one can say something significant using this machinery. The author starts by recalling the main observations of Lidman & Manolescu (2018b).

Remark 6.1: Note that G acts linearly on the Coulomb slice W. Moreover, the quotient map $\pi : Y \to Y/G$ induces an inclusion $W' \hookrightarrow W$ which identifies W' with the fixed point space W^G. This inclusion is *not,* however, an isometry in the L^2-norm. Nonetheless, the L^2 ball $\mathrm{B}(W', R')$ is identified with the L^2 ball $\mathrm{B}(W, |G|R')^G \subset W$, which allows one to construct the Sobolev norm on W so as to have it be G-invariant. As a consequence, in the Sobolev completions, one still has $W'_k = W^G_k$.

Remark 6.2: The Fredholm operator ℓ on Y is G-equivariant. Hence, its restriction to W^G agrees with the analogous operator defined on Y/G. Therefore, use ℓ to denote *both* these operators. Furthermore, note that $(W')^\mu = (W^\mu)^G$. The map c is also G-equivariant, so a finite-type Seiberg-Witten trajectory in W'_k is the same thing as a G-fixed finite-type Seiberg-Witten trajectory in W_k. The same identification can be made between the Seiberg-Witten trajectories in the *finite dimensional approximations.*

Use $R > 0$ to denote the constant provided by Theorem 3.28 for the case of the manifold Y and $R' > 0$ to denote this constant for the quotient manifold Y/G. By perhaps increasing R or R', one can ensure that $R' = R/|G|$. This means that R will also satisfy the conclusions of Theorem 3.28 for Y/G.

Next, recall from Remark 3.30 that the choice of bump functions u^μ for Y was made so as to have it constant on the spheres centred at zero in the Sobolev norm; therefore, the u^μ are automatically G-invariant. Therefore, their restrictions to $(W^\mu)^G$ can be used to define the bump functions required for Y/G. With these conditions, the finite dimensional Seiberg-Witten flow $\varphi^\mu_{\pi^*\lambda,r}$ of Y is G-equivariant and restricts to $(W^\mu)^G$ as the finite dimensional Seiberg-Witten flow of Y/G. Now, if the reader agree to fix $\mu > 0$ large enough so as to have Theorem 3.33 hold for both Y and Y/G, then, it follows that the isolated invariant set $S^\mu_{\pi^*\lambda,r}$ is G-invariant and its G-fixed subset, $(S^\mu_{\pi^*\lambda,r})^G$, is precisely the isolated invariant set used to define the SWF spectrum of Y/G. Moreover, since the G-action is linear on W^μ, the isolating neighbourhood $\mathrm{D}(W^\mu, 2R)$ of $S^\mu_{\pi^*\lambda,r}$ is G-invariant and $\mathrm{D}(W^\mu, R)^G = \mathrm{D}((W^\mu)^G, R)$ serves as an isolating neighbourhood for the construction of the SWF spectrum on Y/G.

A few more notions from equivariant stable homotopy theory shall be required. In what follows, use H to denote an arbitrary compact Lie group. As before, the reader is directed to May & al. (1996) for further details.

Theorem 6.3: Given a closed normal subgroup $K \subset H$, there exists a functor, called the *geometric fixed points functor,* from H-spectra indexed over a universe $\mathcal{V}$ to (H/K)-spectra indexed over the universe $\mathcal{V}^K$,

$$\Phi^K : H\mathcal{SV} \to (H/K)\mathcal{SV}^K,$$

satisfying the properties that

(i) $\Phi^K \Sigma^\infty_{\mathcal{V}} X \cong \Sigma^\infty_{\mathcal{V}^K} X^K$,

(ii) $\Phi^K E \wedge \Phi^K F \cong \Phi^K(E \wedge F)$.

Proof: Vid. May & al. (1996), §XVI.3. **QED**

Remark 6.4: Since the author is simply dealing with suspension spectra, these two properties suffice in understanding the geometric fixed points of the spectra at hand. In particular, note that, for an H-space X and an H-representation V, it follows that

$$\Phi^K \Sigma^{-V} \Sigma^\infty_{\mathcal{V}} X = \Sigma^{-V^K} \Sigma^\infty_{\mathcal{V}^K} X^K.$$

Definition 6.5: An H-universe is called *complete* if one can find, for any finite dimensional H-representation V, a sub-representation in $\mathcal{V}$ isomorphic to V.

Remark 6.6: For G the group of deck transformations of the covering $\pi : Y \to Y/G$, notice that the universe $\mathcal{W}$ defined by the Coulomb gauge of Y is naturally a $G \times \mathrm{U}(1)$-universe and its G-fixed point space, $\mathcal{W}^G$, is the $\mathrm{U}(1)$-universe defined by the Coulomb gauge of Y/G. Let $\mathcal{U}'$ denote a *complete* $\mathrm{U}(1) \times G$-universe. By intertwining with change of universe functors defined by an isometry $\mathcal{W} \to \mathcal{U}'$, as was done in Definition 3.61, one can consider the functor $\Sigma^{-W^{(-\mu,0)}}$ as an endofunctor of the category $\bar{h}(\mathrm{U}(1) \times G)\mathcal{SU}'$.

Definition 6.7: Define the *metric dependent G-equivariant Seiberg-Witten Floer spectrum* as

$$\mathrm{SWF}_G(Y, \pi^*\mathfrak{s}_\lambda, \pi^* g) := \Sigma^{-W^{(-\mu,0)}} \Sigma^\infty_{\mathcal{U}'} I_{\mathrm{U}(1)\times G}(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\pi^*\lambda,r}) \in \bar{h}(\mathrm{U}(1) \times G)\mathcal{SU}'.$$

Remark 6.8: The author believes it to be possible to (de)suspend away the metric dependence in an analogous fashion to the non-equivariant case; however, it seems the details are somewhat subtle, so he chose not to pursue that goal in this thesis. This would involve considering a localization of the representation ring $\mathrm{RO}(\mathrm{U}(1) \times G)$, equivariant spectral flow and equivariant eta invariants.

Theorem 6.9: (Lidman & Manolescu 2018b) The Seiberg-Witten Floer spectra for Y and Y/G are related by

$$\Phi^G \operatorname{SWF}_G(Y, \pi^*\mathfrak{s}, \pi^* g) = \operatorname{SWF}(Y/G, \mathfrak{s}, g).$$

Proof: Although the statement in Lidman & Manolescu (2018b) is made in terms of $\mathrm{U}(1) \times G$-spaces instead of spectra, the properties of the geometric fixed points functor mentioned above make clear that this is the correct statement for spectra.

QED

Now, the contact invariant construction shall be considered in this setting. If necessary, increase R, μ and the parameter r used in the Seiberg-Witten equations so that Remark 4.1, Proposition 4.8 and Theorem 4.17 be satisfied for *both* Y and Y/G.

Notice that the upstairs contact circle, $U_{\pi^*\lambda} \subset (W^\mu)^G \subset W^\mu$, is naturally identified with the downstairs contact circle, U_λ. Beware, however, that the dual attractors to $U_{\pi^*\lambda}$ in $S^\mu_{\pi^*\lambda, r}$ and in $(S^\mu_{\pi^*\lambda, r})^G$ are, of course, *not the same.*

Definition 6.10: Let

$$\mathcal{T}_G(\pi^*\lambda, g) := \Sigma^{W^{(-\mu,0)}} \Sigma^\infty_{\mathcal{U}'} I_{\mathrm{U}(1)\times G}(U_{\pi^*\lambda}, \varphi^\mu_{\pi^*\lambda, r}) \in \bar{h}(\mathrm{U}(1) \times G)\mathcal{SU}'$$

denote the Thom space appearing in the codomain of the cohomotopical contact invariant but now seen as a $(\mathrm{U}(1) \times G)$-spectrum (cf. Definition 4.27).

The attractor-repeller cofibration

$$I_{\mathrm{U}(1)\times G}\left((U_{\pi^*\lambda})^*, \varphi^\mu_{\pi^*\lambda,r}\right) \to I_{\mathrm{U}(1)\times G}\left(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\pi^*\lambda,r}\right) \to I_{\mathrm{U}(1)\times G}\left(U_{\pi^*\lambda}, \varphi^\mu_{\pi^*\lambda,r}\right)$$

can be formed G-equivariantly as well. This is done by constructing a $(\mathrm{U}(1) \times G)$-invariant index triple (L, M, N) according to Theorem 4.23. The consequence is that the contact invariant gains a G-equivariant version.

Definition 6.11: Define the *G-equivariant metric dependent contact invariant* as the map

$$\Psi_G(\pi^*\lambda, \pi^* g) : \operatorname{SWF}_G(Y, \pi^*\lambda, \pi^* g) \to \mathcal{T}_G(\pi^*\lambda, \pi^* g)$$

given by desuspending the above cofibre map by the $(\mathrm{U}(1) \times G)$-representation $W^{(-\mu,0)}$.

Theorem 6.12: The cohomotopical contact invariants of $(Y, \pi^*\lambda)$ and $(Y/G, \lambda)$ are related by

$$\Phi^G \Psi_G(\pi^*\lambda, \pi^* g) = \Psi(\lambda, g).$$

Proof: If (L, M, N) be a $\mathrm{U}(1) \times G$-invariant index triple for $S^{\mu}_{\pi^*\lambda,r}$, then the triple of fixed point sets (L^G, M^G, N^G) is a U(1)-invariant index triple for $(S^{\mu}_{\pi^*\lambda,r})^G = S^{\mu}_{\lambda,r}$. The result follows at once. **QED**

The extra data available due to the G-action allows one to define a G-equivariant (or $(\mathrm{U}(1) \times G)$-equivariant) version of the monopole Floer cohomologies via the G-equivariant cohomology of the SWF spectrum since the spectrum has naturally acquired a G-action. There are three choices of equivariant cohomology theory which spring to mind in this context: *Borel cohomology, equivariant K-theory* and *Bredon cohomology.* In the present thesis, the author shall mainly pursue the use of Borel cohomology as that turned out to be the most readily applicable. A few minor remarks shall be made about Bredon cohomology as well, but no deep results have followed from that.

7. Borel Cohomology

This section shall deal with the Borel G-equivariant cohomology of the spectrum SWF_G and consider the resulting equivariant contact class. Despite the spectrum SWF_G being a $\mathrm{U}(1) \times G$-spectrum, in the interests of simplicity, the author shall ignore the $\mathrm{U}(1)$-action and only consider cohomologies equivariant with respect to the G-action. Application of Borel cohomology to the cohomotopical contact invariant leads to a *G-equivariant cohomological contact invariant* ψ_G, which, as shall be seen, holds information about both the *upstairs* and *downstairs* contact structures of the covering. Verily, if one know the value of ψ_G, one can recover the value of both cohomological contact invariants of the covering.

Ideally, what one really wishes is to infer something about the *upstairs* contact invariant from knowledge about the *downstairs* contact invariant; the new *equivariant* contact invariant, therefore, may seem not to be leading in that direction. However, Borel cohomology, under appropriate circumstances, enjoys the very powerful property that the cohomology of the fixed points of a G-space conditions significantly the Borel cohomology of the whole space; this is the content of the *localization theorem.* As a consequence of localization, one *sometimes* can determine the equivariant contact invariant from knowledge of the downstairs contact invariant. And, as the equivariant contact invariant recovers the non-equivariant upstairs contact invariant, this allows one to infer the upstairs contact invariant starting only from knowledge of the downstairs contact invariant.

The applicability of localization, however, is limited to scenarios where one know precisely what the SWF_G spectrum is. This happens, for instance, in the event that there be a unique solution to the Seiberg-Witten equations, which implies that SWF_G is an equivariant sphere spectrum. Nonetheless, such manifolds are sufficiently abundant that the results derived say something quite non-trivial. A special case of interest shall be that of *elliptic manifolds,* where a very general theorem may be stated which aids greatly in determining whether the lift of a tight contact structure remains tight or becomes overtwisted.

For further simplicity, the author shall avoid the language of G-spectra and work instead at the level of G-spaces as much as possible in this section. This is similar to what is done in Lidman & Manolescu (2018b). The main reason being that the appropriate form of the localization theorem is more difficult to describe for spectra as questions about the various forms of fixed points functors come into play.

Let Y, G, π, g and λ be as in the previous section.

Definition 7.1: Define the *G-equivariant Borel metric dependent monopole Floer cohomology* as

$$\mathrm{c}\widetilde{\mathrm{HM}}^*_G(Y, \pi^*\mathfrak{s}, \pi^* g; R) := \mathrm{c}\tilde{\mathrm{H}}^*_G(\mathrm{SWF}_G(Y, \pi^*\mathfrak{s}, \pi^* g); R)$$

where R is some commutative ring, which will be left implicit in the notation henceforth.

Remark 7.2: In order to make the best out of Borel cohomology, it is best to focus on the case $G = \mathbf{Z}/p\mathbf{Z}$ for a prime p. In this case, the coefficient ring R shall be chosen to be $\mathbf{Z}/p\mathbf{Z}$. For the remainder of this section, these choices shall be implicit lest the notation become overloaded.

Remark 7.3: Since $\mathrm{SWF}_G(Y, \pi^*\mathfrak{s}, \pi^* g)$ depends on g only up to suspension by G-representations, it follows that $\mathrm{c}\widetilde{\mathrm{HM}}^*_G(Y, \pi^*\mathfrak{s}, \pi^* g)$ depends on g only up to shifts in grading, which, although cosmetically unpleasant, is not a major issue in practice.

Definition 7.4: Given a G-representation V and a ring R, one says that V is *G-equivariantly R-orientable* if the vector bundle $V_G := (V \times \mathrm{E}G)/G$ over $\mathrm{B}G$ be R-orientable. In which case, one writes $e_G(V) \in \mathrm{c}\tilde{\mathrm{H}}^*_G(S^0; R)$ for its *Euler class.*

Remark 7.5: Since the author is using coefficients $R = \mathbf{Z}/p\mathbf{Z}$ for the group $G = \mathbf{Z}/p\mathbf{Z}$ and p is a prime, it follows that all G-representations are G-equivariantly R-orientable. In the case $p = 2$, all vector bundles are R-orientable anyway. Otherwise, if p be an odd prime, then any non-trivial representation of G is complex and, therefore, the vector bundle defined over $\mathrm{B}G$ shall also be complex and therefore orientable over any field. So, in any event, an equivariant Euler class always exists for the purposes being pursued here.

Remark 7.6: Note that, due to the Thom isomorphism, for an orientable G-representation V, the ring $\mathrm{c}\tilde{\mathrm{H}}^*_G(V^+)$ is isomorphic to $\mathrm{c}\tilde{\mathrm{H}}^*_G(S^0)$ with its grading shifted by $\dim V$. Moreover, under this isomorphism, notice that the Thom class $\theta_G(V_G)$ of the bundle $V_G \to \mathrm{B}G$ gets sent to $1 \in \tilde{\mathrm{H}}^0_G(S^0)$. In other words, one can think of $\theta_G(V_G)$ as a G-equivariant fundamental class of the G-manifold V^+.

Theorem 7.7: (Localization Theorem) Let Γ be a finite G-CW-complex and $S \subset \tilde{\mathrm{H}}^*(\mathrm{B}G; \mathbf{Z}/p\mathbf{Z})$ consist of those elements which be Euler classes of G-representations having no trivial summand. The inclusion of fixed points $\Gamma^G \hookrightarrow \Gamma$ induces an *isomorphism between cohomologies localized with respect to S,*

$$S^{-1}\mathrm{c}\tilde{\mathrm{H}}^*_G(\Gamma; \mathbf{Z}/p\mathbf{Z}) \xrightarrow{\sim} S^{-1}\mathrm{c}\tilde{\mathrm{H}}^*_G(\Gamma^G; \mathbf{Z}/p\mathbf{Z}).$$

Proof: Vid. tom Dieck 1987, Theorem 3.13. **QED**

Example 7.8: Consider the inclusion of fixed points

$$(V^+)^G \hookrightarrow V^+$$

of a G-representation sphere coming from a G-representation V potentially having $V^G \neq 0$. In this example, assume $p > 2$. Firstly, recall that, in this case, the cohomology of $\mathrm{B}G$ is the ring

$$\mathrm{c}\tilde{\mathrm{H}}^*_G(S^0) = (\mathbf{Z}/p\mathbf{Z})[u, v]/(v^2)$$

where $\deg u = 2$, $\deg v = 1$. It is worth paying close attention to this ring. In the following diagram, the top row indicates the abelian subgroups at each degree, the middle row indicates the generator with the corresponding degree and the bottom row indicates the numerical value of the degree.

$$\begin{array}{ccccccccc} \cdots & 0 & 0 & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots \\ & & & \langle 1 \rangle & \langle v \rangle & \langle u \rangle & \langle uv \rangle & \langle u^2 \rangle & \\ & (-2) & (-1) & (0) & (1) & (2) & (3) & (4). & \end{array}$$

Now, because $p > 2$ is an odd prime, all representations of G are complex and therefore define orientable bundles over $\mathrm{B}G$. It follows from the Thom isomorphism that

$$\mathrm{c}\tilde{\mathrm{H}}^*_G(V^+) \cong \mathrm{c}\tilde{\mathrm{H}}^{*-\dim V}_G(S^0)$$

and likewise for $(V^+)^G$. Hence, *before localization,* the inclusion of fixed points $(V^+)^G \hookrightarrow V^+$ induces a map

$$\mathrm{c}\tilde{\mathrm{H}}^{*-\dim V}_G(S^0) \to \mathrm{c}\tilde{\mathrm{H}}^{*-\dim V^G}_G(S^0)$$

which, in the notation above, is depicted as

$$\begin{array}{ccccccc} \cdots & 0 & 0 & \cdots & 0 & \mathbf{Z}/p\mathbf{Z} & \cdots \\ & \downarrow & \downarrow & & \downarrow & \downarrow & \\ \cdots & 0 & \mathbf{Z}/p\mathbf{Z} & \cdots & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots, \end{array}$$

where all the $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ maps are isomorphisms. Now, consider the effect of localization. The set S with respect to which one must perform localization is the set $\{u^n \mid n > 0\}$; this can be seen from looking at the representation theory of G. The *localized* Borel cohomology therefore is

$$S^{-1}\mathrm{c}\tilde{\mathrm{H}}_G^*(S^0) = (\mathbf{Z}/p\mathbf{Z})[u, u^{-1}, v]/(v^2).$$

Schematically,

$$\begin{array}{ccccccccc} \cdots & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots \\ & \langle u^{-1}\rangle & \langle u^{-1}v\rangle & \langle 1\rangle & \langle v\rangle & \langle u\rangle & \langle uv\rangle & \langle u^2\rangle & \\ & (-2) & (-1) & (0) & (1) & (2) & (3) & (4). & \end{array}$$

After localization, the map induced by the inclusion of fixed points takes the form

$$\begin{array}{ccccccc} \cdots & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots \\ & \downarrow & \downarrow & & \downarrow & \downarrow & \\ \cdots & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots, \end{array}$$

where *all* maps are isomorphisms.

Example 7.9: Now, consider the same scenario but with $p = 2$. In this case, recall that the cohomology of $\mathrm{B}G$ is the ring

$$\mathrm{c}\tilde{\mathrm{H}}_G^*(S^0) = (\mathbf{Z}/2\mathbf{Z})[u]$$

where $\deg u = 1$. In particular, as an abelian group, this is the same as in the case $p > 2$; only the ring structures differ. Again, draw the same sort of diagram as before.

$$\begin{array}{ccccccccc} \cdots & 0 & 0 & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \cdots \\ & & & \langle 1\rangle & \langle u\rangle & \langle u^2\rangle & \langle u^3\rangle & \langle u^4\rangle & \\ & (-2) & (-1) & (0) & (1) & (2) & (3) & (4). & \end{array}$$

Unlike in the $p > 2$ case, here, there are non-trivial real representations of G which define non-orientable bundles over $\mathrm{B}G$. No matter; in this case, the coefficient ring being used for cohomology theories is $\mathbf{Z}/2\mathbf{Z}$, and all vector bundles are $\mathbf{Z}/2\mathbf{Z}$-orientable. Hence, again by the Thom isomorphism,

$$\mathrm{c}\tilde{\mathrm{H}}_G^*(V^+) \cong \mathrm{c}\tilde{\mathrm{H}}_G^{*-\dim V}(S^0)$$

and likewise for $(V^+)^G$. As before, the inclusion of fixed points $(V^+)^G \hookrightarrow V^+$ induces a map

$$\mathrm{c}\tilde{\mathrm{H}}_G^{*-\dim V}(S^0) \to \mathrm{c}\tilde{\mathrm{H}}_G^{*-\dim V^G}(S^0)$$

which one depicts as

$$\begin{array}{ccccccc} \cdots & 0 & 0 & \cdots & 0 & \mathbf{Z}/2\mathbf{Z} & \cdots \\ & \downarrow & \downarrow & & \downarrow & \downarrow & \\ \cdots & 0 & \mathbf{Z}/2\mathbf{Z} & \cdots & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \cdots, \end{array}$$

where all the $\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ maps are $1 \mapsto 1$. Localization, in this case, is with respect to the set $S = \{u^n \mid n > 0\}$. Therefore,

$$S^{-1}\mathrm{c}\tilde{\mathrm{H}}^*_G(S^0) = (\mathbf{Z}/2\mathbf{Z})[u, u^{-1}].$$

Schematically,

$$\begin{array}{ccccccccc} \cdots & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \cdots \\ & \langle u^{-2}\rangle & \langle u^{-1}\rangle & \langle 1\rangle & \langle u\rangle & \langle u^2\rangle & \langle u^3\rangle & \langle u^4\rangle & \\ & (-2) & (-1) & (0) & (1) & (2) & (3) & (4). & \end{array}$$

The end result is effectively the same as in the $p > 2$ case; the map induced by the inclusion of fixed points is

$$\begin{array}{ccccccc} \cdots & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \cdots & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \cdots \\ & \downarrow & \downarrow & & \downarrow & \downarrow & \\ \cdots & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \cdots & \mathbf{Z}/2\mathbf{Z} & \mathbf{Z}/2\mathbf{Z} & \cdots, \end{array}$$

where *all* maps are isomorphisms.

Remark 7.10: Recall the *unstable normal bundle* $E^{\mathrm{u},\mu}_{\pi^*\lambda} \to U_{\pi^*\lambda}$ from Remark 4.14. As was seen earlier, the G-equivariant Conley index of $U_{\pi^*\lambda}$ is the Thom space of this bundle,

$$I_G(U_{\pi^*\lambda}, \varphi^\mu_{\pi^*\lambda,r}) = \Theta_G(E^{\mathrm{u},\mu}_{\pi^*\lambda}).$$

Let $\text{ᛖ} \hookrightarrow E^\mu_\lambda$ denote a fibre of the bundle. Note that ᛖ is a G-representation and decompose it as $\text{ᛖ}^G \oplus \text{ᚠ}$ where ᚠ is a G-representation with trivial fixed points, $\text{ᚠ}^G = 0$. Next, consider the unstable normal bundle $E^{\mathrm{u},\mu}_\lambda \to U_\lambda$. The Conley index is again computed as

$$I(U_\lambda, \varphi^\mu_{\lambda,r}) = \Theta(E^{\mathrm{u},\mu}_\lambda).$$

One can ensure that the unstable normal bundles be related as

$$E^{\mathrm{u},\mu}_\lambda = (E^{\mathrm{u},\mu}_{\pi^*\lambda})^G.$$

Hence, the Conley indices of U_λ in $(W^\mu)^G$ and of $U_{\pi^*\lambda}$ in W^μ are related by

$$I_G(U_{\pi^*\lambda}, \varphi^\mu_{\pi^*\lambda,r}) = I(U_\lambda, \varphi^\mu_{\lambda,r}) \wedge \text{ᚠ}^+.$$

Since $I(U_\lambda, \varphi^\mu_{\lambda,r})$ is G-trivial, one has that

$$\mathrm{c}\tilde{\mathrm{H}}^*_G(I(U_\lambda, \varphi^\mu_{\lambda,r})) \cong \tilde{\mathrm{H}}^*(I(U_\lambda, \varphi^\mu_{\lambda,r})) \underset{\mathbf{Z}/p\mathbf{Z}}{\otimes} \mathrm{c}\tilde{\mathrm{H}}^*_G(S^0)$$

and, similarly,

$$\mathrm{c}\tilde{\mathrm{H}}^*_G(I(U_\lambda, \varphi^\mu_{\lambda,r}) \wedge \text{ᚠ}^+) \cong \tilde{\mathrm{H}}^*(I(U_\lambda, \varphi^\mu_{\lambda,r})) \underset{\mathbf{Z}/p\mathbf{Z}}{\otimes} \mathrm{c}\tilde{\mathrm{H}}^*_G(\text{ᚠ}^+).$$

Remark 7.11: Next, consider a *non-equivariant* Thom class

$$\theta(E^{\mu}_{\lambda}) \in \tilde{\mathrm{H}}^{\dim \text{ᛖ}^G}(I(U_\lambda, \varphi^{\mu}_{\lambda,r}))$$

for the bundle $E^{\mathrm{u},\mu}_{\lambda} \to U_\lambda$. Bearing in mind the isomorphisms outlined in the previous remark, define a class

$$\theta_G(E^{\mathrm{u},\mu}_{\pi^*\lambda}) := \theta(E^{\mathrm{u},\mu}_{\lambda}) \otimes \theta_G(\text{ᚠ}_G) \in \tilde{\mathrm{H}}^*(\Theta(E^{\mathrm{u},\mu}_{\lambda})) \otimes_{\mathbf{Z}/p\mathbf{Z}} \mathrm{c}\tilde{\mathrm{H}}^*_G(\text{ᚠ}^+) \cong \mathrm{c}\tilde{\mathrm{H}}^*_G(\Theta_G(E^{\mathrm{u},\mu}_{\pi^*\lambda})).$$

It is easy to verify that the class $\theta_G(E^{\mathrm{u},\mu}_{\pi^*\lambda})$ serves as a G-equivariant Thom class for the G-bundle $E^{\mathrm{u},\mu}_{\pi^*\lambda} \to U_{\pi^*\lambda}$.

Remark 7.12: Due to the difficulties of working with localization in the context of spectra, the author decided to proceed with certain arguments applied prior to desuspension. For that end, it becomes useful to define contact invariants dependent on the sufficiently large spectral cut-off parameter. In what follows, let $\mu > 0$ again be large enough to satisfy what is said in Theorem 3.33 and Proposition 4.12.

Definition 7.13: Let the *cohomological contact invariant in finite dimensional approximation* be the class

$$\psi(\lambda, g, \mu) \in \tilde{\mathrm{H}}^*(I(S^{\mu}_{\lambda,r}, \varphi^{\mu}_{\lambda,r}))$$

given as the pullback of $\theta(E^{\mathrm{u},\mu}_{\lambda})$ via the cofibre map

$$I(S^{\mu}_{\lambda,r}, \varphi^{\mu}_{\lambda,r}) \to \Theta(E^{\mathrm{u},\mu}_{\lambda}).$$

Definition 7.14: Let the *equivariant cohomological contact invariant in finite dimensional approximation* be the class

$$\psi_G(\pi^*\lambda, \pi^*g, \mu) \in \mathrm{c}\tilde{\mathrm{H}}^*_G(I_G(S^{\mu}_{\pi^*\lambda,r}, \varphi^{\mu}_{\pi^*\lambda,r}))$$

given as the pullback of $\theta_G(E^{\mathrm{u},\mu}_{\pi^*\lambda})$ via the cofibre map

$$I_G(S^{\mu}_{\pi^*\lambda,r}, \varphi^{\mu}_{\pi^*\lambda,r}) \to \Theta_G(E^{\mathrm{u},\mu}_{\pi^*\lambda}).$$

Proposition 7.15: Under the map induced by inclusion of fixed points

$$\mathrm{c}\tilde{\mathrm{H}}^*_G(I_G(S^{\mu}_{\pi^*\lambda,r}, \varphi^{\mu}_{\pi^*\lambda,r})) \to \mathrm{c}\tilde{\mathrm{H}}^*_G(I(S^{\mu}_{\lambda,r}, \varphi^{\mu}_{\lambda,r})) \cong \tilde{\mathrm{H}}^*(I(S^{\mu}_{\lambda,r}, \varphi^{\mu}_{\lambda,r})) \otimes \mathrm{c}\tilde{\mathrm{H}}^*_G(S^0)$$

the class $\psi_G(\pi^*\lambda, \pi^*g, \mu)$ gets sent to $\psi(\lambda, g, \mu) \otimes e_G(\text{ᚠ})$.

Proof: Consider the commuting diagram

$$\begin{array}{ccc} c\tilde{\mathrm{H}}^*_G(I_G(S^{\mu}_{\pi^*\lambda,r},\varphi^{\mu}_{\pi^*\lambda,r})) & \longleftarrow & c\tilde{\mathrm{H}}^*_G(\Theta_G(E^{\mathrm{u},\mu}_{\pi^*\lambda})) \\ \Big\downarrow & & \Big\downarrow \\ c\tilde{\mathrm{H}}^*_G(I(S^{\mu}_{\lambda,r},\varphi^{\mu}_{\lambda,r})) & \longleftarrow & c\tilde{\mathrm{H}}^*_G(\Theta(E^{\mathrm{u},\mu}_{\lambda})), \end{array}$$

where the horizontal arrows come from the cofibre maps and the vertical from the inclusion of fixed points. By the isomorphisms discussed in Remark 7.10, one can rewrite the right vertical map as

$$\tilde{\mathrm{H}}^*(\Theta(E^{\mathrm{u},\mu}_{\lambda}))\otimes c\tilde{\mathrm{H}}^*_G(\mathfrak{k}^+)\to \tilde{\mathrm{H}}^*(\Theta(E^{\mathrm{u},\mu}_{\lambda}))\otimes c\tilde{\mathrm{H}}^*_G(S^0).$$

Moreover, with respect to these tensor products, this map is simply $\mathrm{id}\otimes i^*$ where i is the inclusion of fixed points $i: S^0\hookrightarrow \mathfrak{k}^+$. But note that $i^*\theta_G(\mathfrak{k}_G)$ is, by definition, the Euler class $e_G(\mathfrak{k})$. The result then follows by commutativity of the diagram.

QED

Proposition 7.16: If $\psi_G(\pi^*\lambda,\pi^*g,\mu)=0$, then $\psi(\lambda,g,\mu)=0$.

Proof: Since the representation $\mathfrak{k}$ has no trivial summand, the Euler class $e_G(\mathfrak{k})$ is never zero. The result then follows immediately from Proposition 7.15. **QED**

One can now desuspend appropriately to obtain a result that is not dependent on the spectral cut-off μ.

Definition 7.17: Define the *metric dependent equivariant cohomological contact invariant*

$$\psi_G(\pi^*\lambda,\pi^*g)\in c\widetilde{\mathrm{HM}}^*_G(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g)$$

as the desuspension of the class $\psi_G(\pi^*\lambda,\pi^*g,\mu)$ by the G-representation $W^{(-\mu,0)}$.

Theorem 7.18: If $\psi_G(\pi^*\lambda,\pi^*g)=0$, then $\psi(\lambda,g)=0$.

Proof: Follows directly from Proposition 7.16 after desuspension. **QED**

Remark 7.19: Recall that there is a natural transformation $c\tilde{\mathrm{H}}^*_G\to\tilde{\mathrm{H}}^*$ induced by the inclusion of the fibre of the Borel construction. That is, if Γ be a G-space, $\{p\}\times\Gamma\hookrightarrow\Gamma_G:=(EG\times\Gamma)/G$. In the present context, this translates to a forgetful map

$$c\widetilde{\mathrm{HM}}^*_G(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g)\to\widetilde{\mathrm{HM}}^*(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g).$$

Theorem 7.20: Under the map

$$c\widetilde{\mathrm{HM}}^*_G(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g)\to\widetilde{\mathrm{HM}}^*(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g),$$

the class $\psi_G(\pi^*\lambda,\pi^*g)$ gets sent to $\psi(\pi^*\lambda,\pi^*g)$.

Proof: This follows simply from the fact that the equivariant Thom class

$$\theta_G(E^{\mathrm{u};\mu}_{\pi^*\lambda}) \in \mathrm{c}\tilde{\mathrm{H}}^*_G(I_G(U_{\pi^*\lambda}))$$

is sent to a *non-equivariant* Thom class for the bundle $E^{\mathrm{u};\mu}_{\pi^*\lambda} \to U_{\pi^*\lambda}$ via the map

$$\mathrm{c}\tilde{\mathrm{H}}^*_G(I_G(U_{\pi^*\lambda})) \to \tilde{\mathrm{H}}^*(I(U_{\pi^*\lambda})).$$

QED

Corollary 7.21: If the equivariant contact invariant $\psi_G(\pi^*\lambda, \pi^*g)$ vanish, then *the non-equivariant contact invariants of both Y and Y/G shall also vanish,*

$$\psi(\pi^*\lambda, \pi^*g) = 0, \quad \psi(\lambda, g) = 0.$$

Proof: Corollary of Theorem 7.18 and Theorem 7.20. **QED**

In order to seek computable examples, the author shall make use of the following definition first used in the work of Lin & Lipnowski (2022).

Definition 7.22: By saying that Y is a *minimal L-space,* one means that the perturbed Seiberg-Witten equations admit no irreducible solutions for some perturbation of arbitrarily small norm.

Example 7.23: Perhaps the best known examples of minimal L-spaces are the *elliptic manifolds* due to their metrics of positive scalar curvature (cf. Kronheimer & Mrowka (2007), §22.7).

Example 7.24: Another well known example of a minimal L-space is the *Hantzsche-Wendt* manifold, that is, the unique *flat* rational homology 3-sphere (cf. Kronheimer & Mrowka (2007), §37.4).

Example 7.25: By recent work of Lin (2020), it is known that all rational homology 3-spheres which have *sol geometry* are minimal L-spaces.

Example 7.26: According to the surprising work of Lin & Lipnowski (2022), it is known that certain well known *hyperbolic* rational homology 3-spheres are also minimal L-spaces.

Remark 7.27: Assume that X be a minimal L-space. In this case, it is easy to see that

$$\mathrm{SWF}(X, \mathfrak{s}, g) = \Sigma^{-W^{(-\mu,0)}} S$$

where $S \in \mathcal{S}\mathbf{R}^\infty$ is the sphere spectrum (cf. Manolescu 2003, §10, example about the Poincaré homology sphere). Likewise, supposing Y to be a minimal L-space upon which G freely act, it is just as easy to see that

$$\mathrm{SWF}_G(Y, \pi^*\mathfrak{s}, \pi^* g) = \Sigma^{-W^{(-\mu,0)}} S_G,$$

where $S_G \in G\mathcal{SU}'$ is the G-equivariant sphere spectrum for the universe $\mathcal{U}'$ seen as a complete G-universe; this is because, there being a unique solution to the Seiberg-Witten equations, G can only act trivially on it and, therefore, the action on the Conley index will come entirely from the linear action on W^μ.

Remark 7.28: In order to apply the localization theorem, it is easiest to work at the level of spaces instead of spectra and consider the inclusion of fixed points (cf. Theorem 6.9)

$$I(S^\mu_{\lambda,r}, \varphi^\mu_{\lambda,r}) \to I_G(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\pi^*\lambda,r}).$$

This is simply *the inclusion of fixed points of a G-representation sphere,* exactly as was studied in Example 7.8 and Example 7.9. The crux of the matter is to deduce what this says about the contact invariants at play. The first observation is the following.

Proposition 7.29: Let $S \subset \mathrm{H}^*(\mathrm{B}G; \mathbf{Z}/p\mathbf{Z})$ be the set of Euler classes of G-representations having no trivial summands. The map

$$S^{-1}\mathrm{c}\tilde{\mathrm{H}}^*_G(I_G(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\pi^*\lambda,r})) \to S^{-1}\mathrm{c}\tilde{\mathrm{H}}^*_G(I(S^\mu_{\lambda,r}, \varphi^\mu_{\lambda,r}))$$

on localized Borel cohomology induced by the inclusion of fixed points maps $\psi_G(\pi^*\lambda, \pi^* g, \mu)$ to $\psi(\lambda, g, \mu) \otimes e_G(\text{ᚠ})$, where these classes are being seen now as classes in localized Borel cohomology.

Proof: Follows from Proposition 7.15 after localizing. **QED**

Theorem 7.30: Suppose $\pi : Y \to Y/G$ be a regular p-fold covering of minimal L-spaces where p be prime and let λ be a contact form on Y/G. It follows that $\psi_G(\pi^*\lambda, \pi^* g) = 0$ if and only if $\psi(\lambda, g) = 0$.

Proof: It suffices to work with the invariants $\psi_G(\pi^*\lambda, \pi^* g, \mu)$ and $\psi(\lambda, g, \mu)$ dependent on the spectral cut-off parameter as $\psi_G(\pi^*\lambda, \pi^* g)$ and $\psi(\lambda, g)$ are just grading-shifted versions of those. The localization theorem asserts that

$$S^{-1}\mathrm{c}\tilde{\mathrm{H}}^*_G(I_G(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\pi^*\lambda,r})) \to S^{-1}\mathrm{c}\tilde{\mathrm{H}}^*_G(I(S^\mu_{\lambda,r}, \varphi^\mu_{\lambda,r}))$$

is an isomorphism. Since $I_G(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\pi^*\lambda,r})$ is a G-representation sphere, its Borel cohomology is a free one-dimensional $\mathrm{c}\tilde{\mathrm{H}}^*_G(S^0)$-module; hence, $\psi_G(\pi^*\lambda, \pi^* g, \mu)$ cannot be a torsion element (with respect to the $c\tilde{\mathrm{H}}^*_G(S^0)$-module structure) of

$$c\tilde{\mathrm{H}}^*_G(I_G(S^{\mu}_{\pi^*\lambda,r}, \varphi^{\mu}_{\pi^*\lambda,r})).$$

The result then follows from Proposition 7.29 and, again, the observation that $e_G(\text{ᚠ}) \neq 0$ since $\text{ᚠ}^G = 0$. **QED**

Theorem 7.31: Suppose $\pi : Y \to Y/G$ be a regular p-fold covering of minimal L-spaces where p be prime and let λ be a contact form on Y/G. If the contact invariant $\psi(\lambda)$ with $\mathbf{Z}/p\mathbf{Z}$ coefficients on Y/G vanish, the contact invariant $\psi(\pi^*\lambda)$ with $\mathbf{Z}/p\mathbf{Z}$ coefficients on Y shall also vanish.

Proof: Noting, again, that it suffices to work with the analogous invariants dependent on the metric and the spectral cut-off parameter, combine Theorem 7.30 and Theorem 7.20. **QED**

Definition 7.32: For a rational homology 3-sphere Y with $\mathrm{Spin}^{\mathbf{C}}$ structure $\mathfrak{s}$, the *Frøyshov invariant,* $h(Y, \mathfrak{s})$, is defined as follows. Recall firstly that, as a $\mathbf{Z}[U]$-module, $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s})$ splits as a rank one free summand and a torsion summand. Let $h(Y, \mathfrak{s})$ be minus one half of the minimal degree, with respect to the absolute $\mathbf{Q}$-grading, in which the free summand be non-zero.

Remark 7.33: There is another well known invariant extracted from the $\mathbf{Q}$-grading of a Floer theory; in this case, from the Heegaard Floer theory. This one is denoted $d(Y, \mathfrak{s})$ and was first defined by Ozsváth & Szabó (2003) in a similar fashion but referencing $\mathrm{HF}^+_*(Y, \mathfrak{s})$ instead of $\widehat{\mathrm{HM}}^*(Y, \mathfrak{s})$. According to the corpus that has equated Heegaard Floer homology, monopole Floer cohomology and embedded contact homology, the invariants h and d, in fact, hold the same information but, for historical reasons, they are related by $-2d(Y, \mathfrak{s}) = h(Y, \mathfrak{s})$. This is achieved via the isomorphisms that preserve the absolute grading of the Floer theories by *hyperplane fields* (vid. Cristofaro-Gardiner 2013, Ramos 2018) and the observation that the $\mathbf{Q}$-grading, in each case, is recovered through Gompf's d_3 invariant of the hyperplane fields, perhaps plus one half depending on one's conventions. The advantage of working with the Ozsváth-Szabó invariant is that it is readily computable for many important classes of manifolds.

Theorem 7.34: Suppose $\pi : Y \to Y/G$ be a regular p-fold covering of minimal L-spaces where p be prime and let λ be a contact form on Y/G. If $\psi(\lambda) \neq 0$ and $\deg \psi(\pi^*\lambda) = -2h(Y, \pi^*\mathfrak{s}_\lambda)$, it follows that $\psi(\pi^*\lambda) \neq 0$.

Proof: As before, work with the metric and spectral cut-off dependent versions. Then, $\psi(\lambda, g, \mu) \neq 0$ implies $\psi_G(\pi^*\lambda, \pi^*g, \mu) \neq 0$ by localization. Meanwhile, the map

$$c\tilde{\mathrm{H}}^*_G(I_G(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\lambda,r})) \to \tilde{\mathrm{H}}^*(I(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\lambda,r})),$$

which occurs in Theorem 7.20, can be written in terms of each degree as

$$\begin{array}{cccccc} \cdots & 0 & \mathbf{Z}/p\mathbf{Z} & \mathbf{Z}/p\mathbf{Z} & \cdots \\ & \downarrow & \downarrow & \downarrow & \cdots \\ \cdots & 0 & \mathbf{Z}/p\mathbf{Z} & 0 & \cdots, \end{array}$$

where the central map $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ is an isomorphism. Hence, the invariant $\psi(\pi^*\lambda, \pi^*g, \mu)$ is non-zero precisely when

$$\tilde{\mathrm{H}}^{\deg \psi(\pi^*\lambda, \pi^*g, \mu)}(I(S^\mu_{\pi^*\lambda,r}, \varphi^\mu_{\lambda,r})) = \mathbf{Z}/p\mathbf{Z}.$$

Or, equivalently, after desuspensions,

$$\widetilde{\mathrm{HM}}^{\deg \psi(\pi^*\lambda)}(Y, \pi^*\mathfrak{s}) = \mathbf{Z}/p\mathbf{Z}.$$

Now, to relate this to the Frøyshov invariant, recall that, for an L-space, the value of $-2h(Y, \pi^*\mathfrak{s}_\lambda)$ is the grading of the *only* degree in which $\widetilde{\mathrm{HM}}^*(Y, \pi^*\mathfrak{s}_\lambda)$ is non-trivial. **QED**

Remark 7.35: Note that $\deg \psi(\pi^*\lambda) = -2h(Y, \pi^*\mathfrak{s}_\lambda)$ is also an obvious *necessary* condition for $\psi(\pi^*\lambda) \neq 0$. Therefore, the above theorem can be seen as strengthening this to a necessary and *sufficient* condition.

Theorem 7.36: Suppose $\pi : Y \to Y/G$ be a regular p-fold covering of minimal L-spaces where p be prime and let λ be a contact form on Y/G. If $\deg \psi(\pi^*\lambda) < -2h(Y, \pi^*\mathfrak{s}_\lambda)$, then $\psi(\lambda) = 0$.

Proof: Note $\deg \psi_G(\pi^*\lambda, \pi^*g) = \deg \psi(\pi^*\lambda, \pi^*g)$, so, if $\deg \psi(\pi^*\lambda) < -2h(Y, \pi^*\mathfrak{s}\lambda)$, then $\psi_G(\pi^*\lambda, \pi^*g) = 0$ because $c\widetilde{\mathrm{HM}}_G^{\deg \psi_G(\pi^*\lambda, \pi^*g)}(Y, \pi^*\mathfrak{s}_\lambda, \pi^*g) = 0$. The result now follows from Theorem 7.18. **QED**

Remark 7.37: Recall that the grading of the contact invariant is given simply by the 3-dimensional obstruction theoretic invariant of hyperplane fields as

$$\deg \psi(\lambda) = d_3(\mathrm{Ker}\,\lambda) + \frac{1}{2}.$$

The reader can find a precise definition of d_3, originally due to Gompf (1998), in the following section, which shall de dedicated to understanding its behaviour under coverings.

Theorem 7.38: (Matkovič 2018; Ghiggini 2008; Ghiggini, Lisca & Stipsicz 2006; Wu 2006) The tight contact structures on a small Seifert fibred L-space all have non-vanishing contact invariant and no pair of non-isotopic tight contact structures share the same $\mathrm{Spin}^{\mathbf{C}}$ structure.

Remark 7.39: Elliptic manifolds are small Seifert fibred L-spaces, and this theorem allows one to state a stronger version of the above results.

Corollary 7.40: Suppose $\pi : Y \to Y/G$ be a regular p-fold covering of *elliptic manifolds* where p be prime and let λ be a *tight* contact form on Y/G. It follows that $\pi^*\lambda$ is tight if and only if $d_3(\pi^*\lambda) + \frac{1}{2} = -2h(Y, \pi^*\mathfrak{s}_\lambda)$.

Remark 7.41: One can avoid computing the Frøyshov invariant here if one know precisely the list of tight contact structures on Y together with their corresponding $\mathrm{Spin}^{\mathbf{C}}$ structures and d_3 invariants. In that event, it suffices for one to compute $\pi^*\mathfrak{s}_\lambda$ and $d_3(\pi^* \operatorname{Ker} \lambda)$ and compare with the entries of that list. If there be a match, $\pi^*\lambda$ *must* be tight, if not, $\pi^*\lambda$ must be overtwisted. This may be useful in some cases where computing Frøyshov invariants is difficult and the classification of tight contact structures is known by other methods, but it should be noted that, in many cases of interest, the calculations required to classify tight contact structures on small Seifert fibred L-spaces require comprehensive knowledge of the Frøyshov invariants (cf. Matkovič 2018). In summary, this can be phrased as follows.

Corollary 7.42: Suppose $\pi : Y \to Y/G$ be a regular p-fold covering of *elliptic manifolds* where p be prime and let λ be a *tight* contact form on Y/G. It follows that $\pi^*\lambda$ is *isotopic* to a given tight contact structure – and therefore is also tight – if and only if it is *homotopic* to said tight contact structure.

Use Y_{HW} to denote the *Hantzsche-Wendt* manifold, that is, the unique rational homology 3-sphere with Euclidean geometry. A curious fact about Y_{HW} is that there are cyclic *self-coverings* $Y_{\mathrm{HW}} \to Y_{\mathrm{HW}}$ of any odd degree (vid. Chelnokov & Mednykh 2020). As Y_{HW} is also known to be a minimal L-space, one could hope to apply the above methods here. Unfortunately, this fails because these self-coverings are never *regular* due to the first homology of Y_{HW} having even order whilst the coverings odd order.

The case of the rational homology spheres admitting *Sol geometry* may be one where the methods developed here can work. The problem, however, lies in the limited knowledge available about the contact topology of such manifolds; there is no known classification of the isotopy classes of tight contact structures. Hence, before being able to apply Theorem 7.34 in this case, one would have to produce tight contact structures and develop a way to compute the respective Frøyshov invariants. The author intends to pursue that route in later work.

A similar story can be told of the few hyperbolic manifolds known to be minimal L-spaces; although the methods above make non-trivial statements about lifting tight contact structures, one hardly has the ability to apply them due to lack of information about the contact topology of those manifolds. One case where the methods developed here *can be* applied is the double covering of the hyperbolic manifold $\mathtt{m007}(3,1)$ by $\mathtt{m036}(-3,2)$, where the notation being used is that of the Hodgeson-Weeks census of small volume closed hyperbolic 3-manifolds from the well known *SnapPy* software of Culler, Dunfield, Goerner & Weeks. That both these manifolds are minimal L-spaces is the consequence of the work of Lin & Lipnowski (2022), Theorem 1.

This leaves the elliptic manifolds currently as the best place in which to seek to perform concrete computations. In the next few sections, the author shall proceed to develop certain topological techniques which shall be required for these calculations. The goal shall be to apply Corollary 7.42 to find certain tight contact structures which lift to tight contact structures by showing simply that the lift have the same homotopy theoretic invariants as a known tight contact structure. In order to do that, one must address the problem of computing the obstruction theoretic invariants of the lifted contact structure. There are two obstruction theoretic invariants: the $\mathrm{Spin}^{\mathbf{C}}$ structure and the d_3 invariant of Gompf. Lifting either of them provides a challenge of its own that must be overcome. The next two sections shall deal with these two problems.

8. The d3 invariant and Finite Coverings

Consider the following problem. Say λ be a contact form on a rational homology 3-sphere Y/G regularly and finitely covered by $\pi : Y \to Y/G$. Given the value of $d_3(\operatorname{Ker}\lambda)$ is it possible to say something about the value of $d_3(\pi^* \operatorname{Ker}\lambda)$? This problem was studied by Khuzam (2012) via use of the *G-signature theorem.* In summary, one can solve the problem by seeking an almost complex 4-manifold-with-boundary having the contact manifold $(Y, \pi^*\lambda)$ as its almost complex boundary and extending the G-action into its interior. The action need not remain free inside the 4-manifold-with-boundary; it need only have a properly embedded closed surface as its G-fixed points. Having such an *equivariant almost complex filling,* one can infer the lifting behaviour of the d_3 invariant of the contact form λ.

Producing such a 4-manifold-with-boundary is typically a difficult task. The goal of this section is to describe a method that the author devised by use of Kirby calculus to construct such a filling. The method consists, in essence, of the following. One starts with a Kirby diagram of Y/G, which also defines a 4-dimensional 2-handlebody having Y/G as its boundary. Then, one removes a set of 2-handles judiciously so that the resulting handlebody evidently have a *branched* covering with branching locus a properly embedded surface with boundary at the curves along which the removed 2-handles were originally attached. Next, one computes this branched covering, and proceeds to *equivariantly* attach 2-handles in the hope that, in the quotient, these 2-handles be precisely the 2-handles that were removed initially. This procedure is not always easy to carry out, but, in simple cases such as when Y/G be a surgery on a knot, it can be readily done. One must also be careful with the almost complex structure on the filling. To achieve that, one considers *Legendrian* Kirby diagrams for the contact manifold $(Y/G, \lambda)$.

As an application of the method, the author shall calculate the d_3 invariant of the lift of a tight contact structure of the (-8)-surgery on the left-handed trefoil via a double covering. This is an example of a *prism* manifold, therefore elliptic, so one can apply the results of the preceding section to try and determine if the lifted contact structure remains tight. Recall that, for the lift to be tight, one needs the lift to have an appropriate pair of d_3 invariant *and* $\mathrm{Spin}^{\mathbf{C}}$ structure. It shall be shown that it does indeed have the appropriate d_3 invariant, whereas, understanding the behaviour of the $\mathrm{Spin}^{\mathbf{C}}$ structures under coverings is the content of the next section.

To begin, the author shall recall the relevant definitions and the standard results that shall be needed. Let Y, G, π, g and λ be as in the previous section.

Definition 8.1: A contact 3-manifold (Y, λ) is called the *almost complex boundary* of an almost complex 4-manifold-with-boundary (M, J) if $\partial M = Y$ and $\operatorname{Ker}\lambda = \mathrm{T}Y \cap J\mathrm{T}Y$.

Definition 8.2: (Gompf 1998) Given a contact structure $\operatorname{Ker}\lambda$ on a rational homology 3-sphere Y, the d_3 invariant is defined as follows. Choose some almost complex 4-manifold-with-boundary (M, J) such that Y be its almost complex boundary. Then, define

$$d_3(\operatorname{Ker}\lambda) := \frac{1}{4}\big(c_1(M,J)^2 - 2\chi(M) - 3\sigma(M)\big)$$

where the notation $c^2 \in \mathbf{Q}$, for a class $c \in \mathrm{H}^2(X;\mathbf{Z})$, is defined in Definition 3.65.

Remark 8.3: The author shall sometimes write $d_3(\lambda)$ instead of $d_3(\operatorname{Ker}\lambda)$ as that is more convenient in the present context.

Remark 8.4: Gompf originally called this invariant θ and the factor of $1/4$ was not present in his definition. There is another convention sometimes found in the literature, particularly in the context of Heegaard Floer theory, wherein the value of d_3 has $1/2$ added to what was defined above. The appeal of doing so is that the grading of the contact invariant becomes d_3 instead of $d_3 + 1/2$.

Now, consider again the case of finite coverings. The following result of Khuzam summarizes what can be said by means of the G-signature theorem about the lifting behaviour of d_3.

Theorem 8.5: (Khuzam 2012, Theorem 2) Suppose $\pi : Y \to Y/G$ be an m-fold regular cyclic covering. Let λ be a contact form on Y/G and $\pi^*\lambda$ its lift to Y. Suppose further that one have $\Pi : M \to M/G$ an m-fold cyclic *branched* covering of almost complex 4-manifolds-with-boundary where the branching locus be a closed surface $S \subset M$ satisfying $S \cap \partial M = \emptyset$ and where $(Y, \pi^*\lambda)$ be the almost complex boundary of M. Then, the d_3 invariants of λ and $\pi^*\lambda$ are related by

$$d_3(\pi^*\lambda) = md_3(\lambda) + \frac{3}{4}m\sigma(M/G) - \frac{3}{4}\sigma(M) - \frac{3}{4}\sum_{k=1}^{m-1}(S\cdot S)\csc^2(\gamma_k/2),$$

where γ_k denotes the angle of rotation given by the action of the element $k \in \mathbf{Z}/m\mathbf{Z}$ on the normal planes to the surface S when X be equipped with a $\mathbf{Z}/m\mathbf{Z}$-invariant metric. In fact, this metric can always be chosen so that $\gamma_k = 2\pi k/m$.

Remark 8.6: In the present thesis, the author follows the convention that the lens space $L(p,1)$ is the one given by $(-p)$-surgery on the unknot in S^3.

Example 8.7: In Khuzam (2012), the only example studied is that of the universal covering $S^3 \to L(p,1)$ of the lens space $L(p,1)$. As a first example here, the author shall extend that to the lens covering $L(p,1) \to L(mp,1)$. One can construct the branched covering $\Pi : M \to M/G$ as follows. Let M be the disk bundle over S^2 of Euler class $-p$ so that $\partial M = L(p,1)$. Note, $G := \mathbf{Z}/m\mathbf{Z}$ acts on M by rotating the disk fibres. The action is free away from the zero section, which is fixed; therefore, $S := M^G \cong S^2$. The manifold M/G is no other than the disk bundle of Euler class $-mp$ over S^2, so its boundary is $L(mp,1)$. Both M and M/G have almost complex structures coming from their disk bundle structures and Π is pseudoholomorphic with respect to these. Clearly, this is precisely the scenario of Theorem 8.5 and it becomes possible to compute $d_3(\pi^*\lambda)$ from the value of $d_3(\lambda)$. The lens space $L(mp,1)$ admits precisely $mp-1$ tight contact forms $\lambda_1, \ldots, \lambda_{mp-1}$ (vid. Honda 1999 or Giroux 2000) satisfying

$$d_3(\lambda_k) = \frac{1}{4}\left(-mp - 1 + 4k - \frac{4k^2}{mp}\right).$$

Noting that $S \cdot S = -p$, one computes

$$\begin{aligned}
&d_3(\pi^*\lambda_k)\\
=&\frac{m}{4}\left(-mp - 1 + 4k - \frac{4k^2}{mp}\right) + \frac{3}{4}m(-1) - \frac{3}{4}(-1) - \frac{3}{4}\sum_{j=1}^{m-1}(-p)\csc^2(j\pi/m)\\
=&-m + km - \frac{k^2}{p} + \frac{3}{4} - \frac{p}{4}\\
=&\frac{1}{4}\left(-p + 3 + 4m(k-1) - \frac{4k^2}{p}\right),
\end{aligned}$$

where the author used the well known identity $\sum_{j=1}^{m-1}\csc^2(j\pi/m) = (m^2-1)/3$, cf. Cauchy (1821), Note VIII. Comparing this with the possible values of the d_3 invariants for the tight contact structures on $L(p,1)$, one easily sees that the only values of k for which $d_3(\pi^*\lambda_k)$ matches the value of the d_3 invariant of some tight contact structure on $L(p,1)$ are $k=1$ and $k = mp-1$. These are the *universally* tight contact structures of $L(mp,1)$ and, in any event, one knows that they lift to the universally tight contact structures on $L(p,1)$, so there is nothing interesting happening in this family of examples; that is, all virtually overtwisted contact structures immediately lift to overtwisted contact structures.

Remark 8.8: As the reader may have foreseen, finding the equivariant almost complex filling of the contact G-manifold $(Y, \pi^*\lambda)$ in order to apply Theorem 8.5, is not typically an easy task. The remainder of this section, shall introduce a method for producing examples of such fillings by use of Kirby calculus. The rationale is the following. Firstly, one constructs a *branched covering* of 4-manifolds-with-boundary where the branching locus is allowed to intersect the boundary; secondly, a set of 2-handles is *equivariantly* glued in order to cap off the branching locus thereby making the boundary freely acted by the deck transformations.

Remark 8.9: Such matters shall require working with knot diagrams. The author prefers to draw what are called *grid diagrams.* These are knot diagrams where only vertical and horizontal lines occur, and apparent corners should be understood as being arcs with of a very small radius. This style of diagram has the advantage of bringing isotopies and Reidemeister moves more evidently into the realm of combinatorics. The following figure shows the example of the left-handed trefoil as a grid diagram.

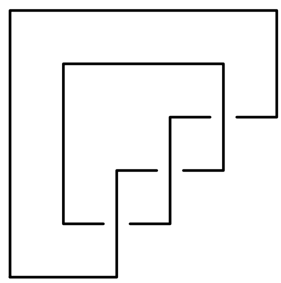


Figure 8.10

Remark 8.11: Now, follows a series of standard definitions and propositions which shall be needed in pursuing the goal of the remainder of this section.

Definition 8.12: By a *Legendrian knot or link* in a contact 3-manifold (Y, λ), one means a knot or link everywhere tangent to the contact structure $\operatorname{Ker}\lambda$.

Remark 8.13: It is standard to represent Legendrian knots in $S^3 = (\mathbf{R}^3)^+$, with its standard contact form $\lambda = \mathrm{d}z + x\mathrm{d}y$, via their *front projection diagrams.* Those are the knot diagrams one obtains when projecting a knot to the yz-plane. This leads to a diagram that possesses no tangencies parallel to the y-axis but may have *cusps* pointing in directions parallel to the z-axis and whose transverse crossings must always have the strand passing *underneath* be the one for which the slope $\mathrm{d}z/\mathrm{d}y$ be *highest.* Conversely, any such diagram where all crossings be transverse is the front projection of a Legendrian knot. The grid knot diagrams, which the author favours, can readily be used to represent front projection diagrams if the following convention be agreed: the y-axis is understood to point in the southeast direction and the z-axis in the northeast direction. Here, the corners of the curve pointing in the *southwest* and *northeast* directions are understood to be *smoothed* as before whereas those pointing in the *northwest* and *southeast* are understood to be *cusps.* Note that the diagram in Figure 8.10 can be reinterpreted as a front projection diagram as it satisfies the required conditions. The following figure shows another *inequivalent* example of a Legendrian left-handed trefoil. When the author desire for the reader to interpret a particular diagram as Legendrian, he shall make it clear from the context; otherwise, the diagrams are to be understood simply as representing smooth links.

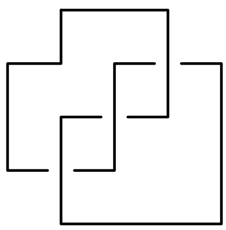


Figure 8.14

Definition 8.15: The *contact framing* of a Legendrian knot K in (Y, λ) is the framing induced by the contact structure $\operatorname{Ker} \lambda$.

Definition 8.16: The *Thurston-Bennequin invariant,* $\operatorname{tb}(K) \in \mathbf{Z}$, of a Legendrian knot $K \subset S^3$ is the value of the contact framing relative to the 0-framing; that is, relative to the framing induced by a Seifert surface.

Proposition 8.17: For a Legendrian knot $K \subset S^3$, if $w(K)$ denote its *writhe* and $c(K)$ denote the number of *cusps* in one of its front projection diagrams, then

$$\operatorname{tb}(K) = w(K) - \frac{1}{2}c(K).$$

Proof: Vid. e.g. Geiges (2008), Proposition 3.5.9. **QED**

Example 8.18: The Legendrian left-handed trefoil K in Figure 8.14 has $\operatorname{tb}(K) = -7$.

Definition 8.19: The *rotation number,* $r(K)$, of an *oriented* Legendrian knot in S^3 is the degree of the map $f : S^1 \to S^1$ defined as follows. Take some parametrization $\gamma : S^1 \to S^3$ of K. Let $\Sigma \subset S^3$ denote a Seifert surface for K. Trivialize the standard contact structure of S^3 over Σ yielding a bundle isomorphic to $\Sigma \times \mathbf{R}^2$. Since K is Legendrian, the derivative γ' together with this trivialization defines a map $S^1 \to \mathbf{R}^2 \setminus \{0\}$. Given a deformation retraction $\mathbf{R}^2 \setminus \{0\} \to S^1$, this defines the desired map $f : S^1 \to S^1$.

Definition 8.20: For an oriented Legendrian knot $K \subset S^3$, denote, respectively, by $c_+(K)$ and $c_-(K)$ the number upwardly and downwardly oriented cusps of a front projection diagram of K. In a Legendrian grid diagram of the sort being used by the author, $c_+(K)$ is the number of corners which *start by pointing east and finish by pointing north* as one traverses it along the orientation of K, while $c_-(K)$ is the number of corners which *start by pointing south and finish by pointing west.*

Proposition 8.21: For an oriented Legendrian knot $K \subset S^3$, the rotation number can be computed by

$$r(K) = \frac{1}{2}\left(c_-(K) - c_+(K)\right).$$

Proof: Vid. e.g. Geiges (2008), Proposition 3.5.19. **QED**

Example 8.22: The Legendrian left-handed trefoil K in Figure 8.14 has $r(K) = 0$.

Definition 8.23: A 3-manifold Y given as surgery on a Legendrian link in S^3 where the framing of a link component K is precisely *the contact framing minus 1* is called a *Legendrian surgery.*

A Legendrian surgery naturally inherits a *tight* contact structure from S^3. Moreover, the 4-manifold-with-boundary obtained from the corresponding Kirby diagram has a natural *complex structure;* indeed, it is a *Stein surface.* For a reference, vid. Gompf (1998). This means that d_3 invariants can be read off this Kirby diagram as follows.

Proposition 8.24: (Gompf 1998) Let (Y, λ) be a contact rational homology 3-sphere given as Legendrian surgery on a Legendrian link $\bigsqcup_{i=1}^{n} K_i$ in S^3. Use L to denote the linking matrix of the link and $\sigma(L)$ its signature as a symmetric bilinear form. Then,

$$d_3(\lambda) = \frac{1}{4}\left(\sum_{i,j=1}^{n} r(K_i)(L^{-1})_{ij} r(K_j) - 2(n+1) - 3\sigma(L)\right).$$

Remark 8.25: Let (X, λ) be a contact rational homology 3-sphere. Suppose one have a Kirby diagram for an almost complex 4-manifold-with-boundary (N, J) consisting entirely of 2-handles with its almost complex boundary being $\partial(N, J) =$

(X, λ). Label the 2-handles of N as $\{h_i \mid i \in I\}$ and their respective attaching knots in S^3 as $\{K_i \mid i \in I\}$. Now, consider a *subset* of these 2-handles, denote it $\{h_i \mid i \in I'\}$, and *remove* them from the Kirby diagram. This yields *another* almost complex 4-manifold-with-boundary, call it (N', J'), having as its 2-handles the set $\{h_i \mid i \in I \setminus I'\}$; denote its almost complex boundary by $(X', \lambda') := \partial(N', J')$. Consider the link $L := \sqcup_{i \in I'} K_i$ as a link in X'. Fix some *pseudoholomorphic curve* $S' \subset N'$ with $\partial S' = L$ and $\operatorname{int} S' \subset \operatorname{int} N'$. Now, the p-fold *branched covering* of N' with branching locus S' exists precisely when the homology class $[S'] \in \mathrm{H}_2(N', \partial N'; \mathbf{Z})$ be divisible by p. Assume that this be indeed the case. Then, in fact, the branched cover, call it M', can be made to come with an almost complex structure and the covering map $\Pi' : M' \to N'$ can be made to respect the almost complex structures. Write $Y' := \partial M'$; this is a contact manifold. The restriction to the boundaries gives a branched covering of 3-manifolds $\pi' : Y' \to X'$ having branching locus the link $L \subset X'$. Required now is the concept of *equivariant handle attachment.*

Definition 8.26: Consider a 2-handle h as a copy of $D^2 \times D^2$ to be attached along $S^1 \times D^2$. Regard $D^2 \times D^2$ as having the action of $G := \mathbf{Z}/p\mathbf{Z}$ defined by $e^{2\pi i/p} \cdot (z_1, z_2) := (z_1, e^{2\pi i/p} z_2)$. This action defines a g-fold branched covering $D^2 \times D^2 \to D^2 \times D^2$ with branching locus the core disk $D^2 \times \{0\}$. Think of this as the *G-2-handle.* When one be in possession of a G-4-manifold-with-boundary M with a G-fixed surface S having $\operatorname{int} S \subset \operatorname{int} M$ and $\partial S \subset \partial M$, define the *equivariant handle attachment* of h along a component K of ∂S to be the G-4-manifold-with-boundary $M \cup_\varphi h$ given by gluing h to M via an *equivariant framing* φ of K; that is, an equivariant diffeomorphism onto its image $\varphi : S^1 \times D^2 \to M$ such that its image be a tubular neighbourhood of K.

Remark 8.27: Notice that the resulting manifold $M \cup_\varphi h$ has one of the boundary components of the G-fixed surface S *capped* by the G-fixed core disk of h. Hence, $M \cup_\varphi h$ has one less G-fixed circle on its boundary compared to M.

Remark 8.28: Returning now to the context of Remark 8.25, one desires to equivariantly attach 2-handles to the manifold M' in order to convert the *branched* covering of 3-manifolds $\pi : Y' \to X'$ to a *genuine* covering $\pi : Y \to X$, where X be the 3-manifold with which one has started. Whether this is possible or not is a matter about the framings of the handles $\{h_i \mid i \in I'\}$ which were initially removed from N

resulting in the manifold N'. Consider a fixed $i \in I'$. Now, equivariantly attach a 2-handle $\tilde{h}_i$ along the knot $(\Pi')^{-1}(K_i)$ with framing determined by an integer $m_i \in \mathbf{Z}$. One hopes to be able to choose m_i so that the corresponding *quotient* handle $\tilde{h}_i/G$ have the same framing as the handle h_i of N; thereby allowing one to identify them. The main issue is that determining the behaviour of framings under the quotient turns out to be a subtle matter.

Instead of considering this general case, simplify the scenario as follows. Consider instead a manifold N as above but consisting of a *single* 2-handle h attached along a knot $K \subset S^3$ with framing n. Then, one forms N' by removing h to obtain, of course, the disk D^4. The manifold M', therefore, is a familiar sort of manifold; it is the branched covering of D^4 branched over the Seifert surface of K with its interior pushed into the interior of D^4. In this case, it is an easy matter to determine the behaviour of framings under the quotient. If one equivariantly attach a 2-handle $\tilde{h}$ to M' along $(\Pi')^{-1}(K)$ with framing m, the framing of the quotient handle $\tilde{h}/G$ shall be mp. Hence, for the procedure to work, one needed n to be a multiple of p.

The process of computing branched coverings in Kirby calculus is outlined in Gompf & Stipsicz (1999), §6.3. In general, it can be a difficult procedure where one obtains a Kirby diagram for the complement of a tubular neighbourhood of the branching surface, computes the genuine covering of the resulting manifold and then glues handles to fill in the hole left by the removal of the branching surface. There is a simpler approach, discussed in Rolfsen (2003), which has the disadvantage of requiring blow-ups to be performed. By a blow-up the author means the formation of the connected sum with a copy of $\overline{\mathbf{CP}}^2$ or $\mathbf{CP}^2$. In the case of $\overline{\mathbf{CP}}^2$, almost complex complex structures can be naturally carried to the blow-up; in the case of $\mathbf{CP}^2$, this is not the case. Hence, for the purposes being pursued in this section, one may only perform blow-ups of the type where one takes the connected sum with a copy $\overline{\mathbf{CP}}^2$. The procedure is as follows. Starting with the Kirby diagram of N consisting of a single 2-handle h attached along a knot K, perform a sequence of blow-ups in order to untie K without changing the framing n of the handle h. Now, the branched covering that one needs to compute is simply branched over a disk since K has become the unknot. This is a significantly easier task than the general case.

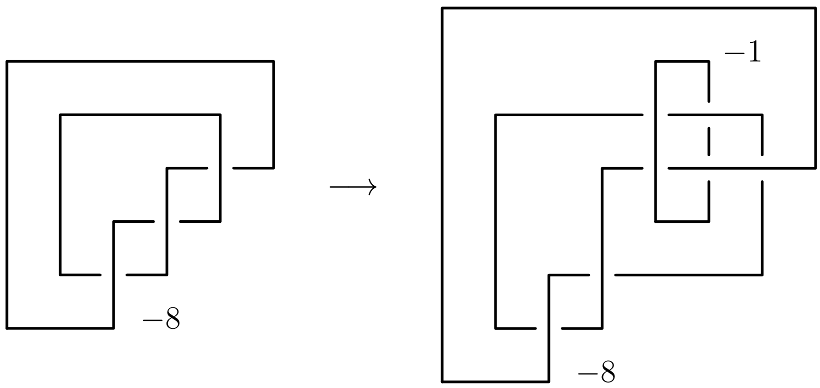


Figure 8.29

Consider now a concrete example in which to apply this procedure. Start with the manifold X given as (-8)-surgery on the left handed trefoil knot K. This manifold admits a double covering, which is the one that shall be studied here. One can perform a single blow-up to untie K as depicted in Figure 8.29. Hence, the manifold N shall be the 2-handlebody defined by the right hand side Kirby diagram of Figure 8.29. After a series of isotopies, one can achieve the form on the left hand side of the following figure.

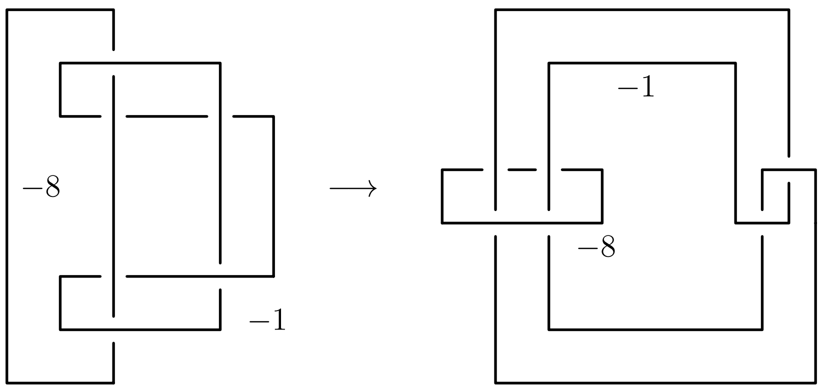


Figure 8.30

After another isotopy, one obtains the diagram in right hand side of Figure 8.30. What one desires to do now is to *remove* the (-8)-framed handle in order to form the manifold N' as depicted in the left hand side of the following figure. Once that be done, one can read off the double branched covering branched over the obvious Seifert disk for the knot K. The resulting manifold, M', is depicted on the right hand side of the following figure. The more lightly stroked curve indicates the handle which has been *removed,* which is, therefore, also the boundary of the branching locus.

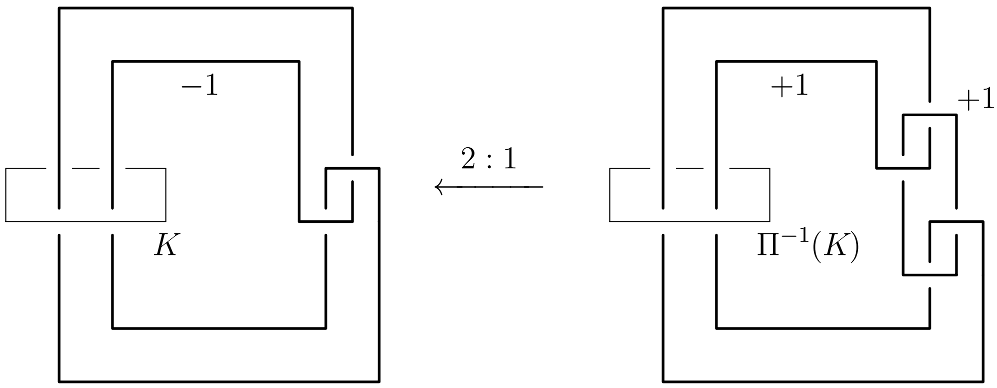


Figure 8.31

Remark 8.32: Notice that the framings behave under the branched covering in such a manner that the *blackboard framing* be preserved irrespective of the change in the writhes. This is why the (-1)-framed handle lifts to a pair of $(+1)$-framed handles.

Next, one equivariantly attaches a 2-handle along $(\Pi')^{-1}(K)$ in order to cap off the branching surface S leading to the branched covering of 4-manifolds-with-boundary seen in the following figure where the branching locus no longer intersects the boundary.

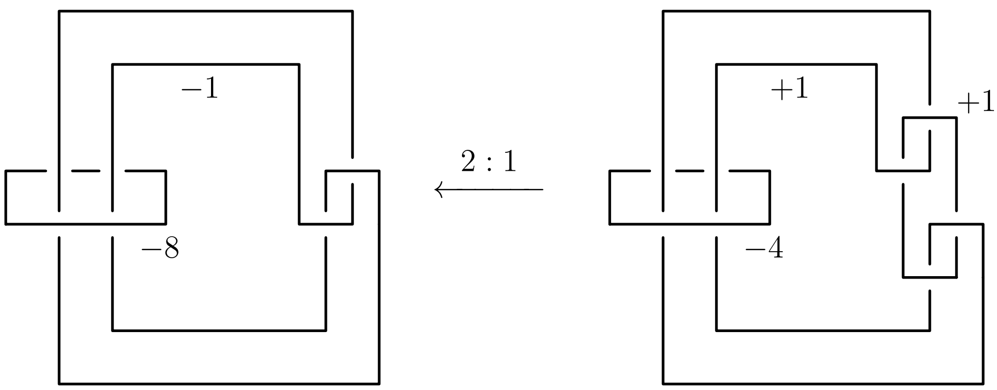


Figure 8.33

Proposition 8.34: The manifold X defined by (-8)-surgery on the left handed trefoil admits a contact form λ which lifts via a double covering $Y \to X$ to a contact form $\pi^*\lambda$ having $d_3(\pi^*\lambda) = 1/4$.

Proof: Define N and M to be the 4-manifolds-with-boundary on the left and right hand side of Figure 8.33 respectively. Here, the branching locus no longer intersects the boundaries $X = \partial N$ and $Y := \partial M$; therefore, one can apply Theorem 8.5 but only to the subset – and this subset may be proper – of the contact structures on X which occur as almost complex boundaries of N. The author shall focus on a particular contact structure. Let λ be the contact structure on X defined as the *Legendrian surgery* according to the Legendrian representative of K seen in Figure 8.14. Recall, from Example 8.18, that the Thurston-Bennequin invariant of this Legendrian knot is -7; hence, the Legendrian surgery is indeed the topological (-8)-surgery. Since, in defining the 4-manifold-with-boundary N, only a (-1)-blow-up was performed, this contact structure is still an almost complex boundary of N. From Example 8.22, one knows that the rotation number is 0. With these data, one applies Proposition 8.24 to find that

$$d_3(\pi^*\lambda) = \frac{1}{4}(0 - 4 + 3) = -\frac{1}{4}.$$

Now, using Theorem 8.5, one computes

$$\begin{aligned} &d_3(\pi^*\lambda) \\ &= -\frac{1}{2} + \frac{3}{4}\left(+2\sigma \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix} - \sigma \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{pmatrix} - \sum_{k=1}^{2} (-4)\csc^2(\pi k/m) \right) \\ &= \frac{1}{4}, \end{aligned}$$

where the matrices are read off from Figure 8.33. **QED**

Proposition 8.35: The manifold X defined by (-8)-surgery on the left handed trefoil is the *prism manifold* having Seifert fibration $(S^2; (1, -1), (3, 2), (2, 1), (2, 1))$.

Proof: The left-handed trefoil is a torus knot; hence, one can establish the Seifert invariants of X from a well known theorem of Moser (1971). **QED**

Corollary 8.36: The double cover Y of X is the lens space $L(12, 7)$.

Proof: One can compute coverings directly from the Seifert invariants. The Seifert manifold $(S^2; (1, -1), (3, 2), (2, 1), (2, 1))$ has a horizontal double covering (meaning it lifts Seifert fibres to Seifert fibres 1-to-1) with invariants

$$(S^2; (1, -1), (1, -1), (3, 2), (3, 2), (1, 1), (1, 1))$$

and this is one of the Seifert structures on the lens space $L(12, 7)$. **QED**

Remark 8.37: The lens space $L(12, 7)$ can be obtained as surgery on a chain of three unknots with framings respectively -2, -4 and -2. According to the classification of tight contact structures on lens spaces due, independently, to Honda

(1999) and Giroux (2000), the manifold Y has precisely 3 isotopy classes of tight contact structures. Each of these contact structures comes from Legendrian surgery on the three possible Legendrian stabilizations of this chain of unknots having the appropriate Thurston-Bennequin invariants. The three Legendrian surgery diagrams are depicted in the following figure.

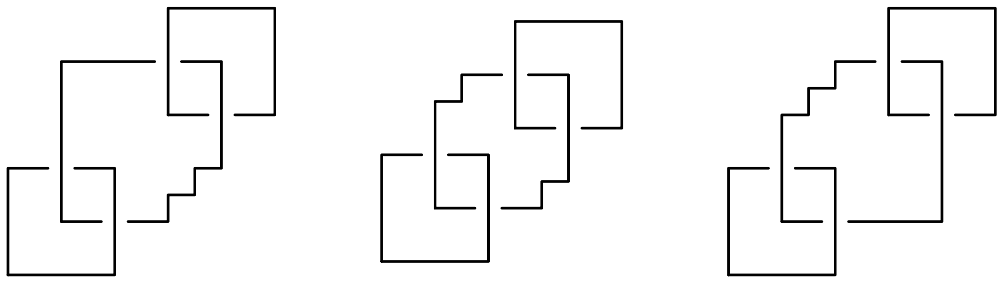


Figure 8.38

Proposition 8.39: The three tight contact structures on $Y = L(12, 7)$ depicted in Figure 8.38 have d_3 invariants, respectively, equal to $-1/12$, $1/4$ and $-1/12$.

Proof: Compute using Proposition 8.24. **QED**

Remark 8.40: In light of Proposition 8.34, the contact structure depicted in the middle of Figure 8.38 is the one that shall receive special attention as its d_3 invariant matches that of the lift of the tight contact structure on X via the double cover. In order to apply Corollary 7.42 and see that these two contact structure are one and the same, what remains to be shown is whether the $\mathrm{Spin}^{\mathbf{C}}$ structures also agree. This problem shall be studied in the next section.

9. The Spin-C Structure and Finite Coverings

This section shall deal with a fairly elementary problem but one that turns out to be quite difficult to solve in practice. The problem is that of lifting $\mathrm{Spin}^{\mathbf{C}}$ structures across finite coverings. There are various methods which one might try to use in approaching this problem. For instance, in the case of Seifert manifolds, there is a certain canonical $\mathrm{Spin}^{\mathbf{C}}$ structure defined by the Seifert fibration. This canonical $\mathrm{Spin}^{\mathbf{C}}$ structure is pulled back naturally via coverings and, therefore, it suffices to have a good description of it upstairs and downstairs in the covering in order to understand how every other $\mathrm{Spin}^{\mathbf{C}}$ structure lifts. This is so because the set of $\mathrm{Spin}^{\mathbf{C}}$ structures on a manifold Y is a $\mathrm{H}^2(Y;\mathbf{Z})$-torsor, and, under the map induced by a covering, this torsor structure is preserved; hence, it suffices to understand the induced map in the second cohomology together with how one single $\mathrm{Spin}^{\mathbf{C}}$ structure lifts in order to deduce the action on every other. Another approach is to use contact topology. Suppose one know of a certain *universally* tight contact structure λ on Y, then, its lift must always be universally tight via any covering. If there not be many of these upstairs, this reveals information about what the lift of the $\mathrm{Spin}^{\mathbf{C}}$ structure associated with λ is. This strategy is particularly useful in the case of lens spaces. Another method is to turn to *Spin structures.* To any Spin structure is associated a $\mathrm{Spin}^{\mathbf{C}}$ structure in a natural way. Hence, it is suffices to know how a Spin structure lifts via a covering to deduce how the $\mathrm{Spin}^{\mathbf{C}}$ structures lift as well.

This last approach shall be the one pursued in the present section. Spin structures can be studied with the aid of the Kirby calculus and this fits well into the picture of lifting d_3 invariants of the preceding section. Indeed, Gompf & Stipsicz (1999) describe not only how to express Spin structures on a 3-manifold given by a Kirby diagram by ascribing extra decorations to it, but also how this description changes under Kirby moves. The main feat of the current section is to describe the manner in which this description of Spin structures behaves under finite coverings.

The double covering of the (-8)-surgery on the left-handed trefoil considered in the previous section shall continue to furnish examples in this section. By the end, it shall be established that the tight contact structure whose lift was postulated to be tight indeed has its $\mathrm{Spin}^{\mathbf{C}}$ structure lift to the correct one, so that the theorems concerning the equivariant contact invariant assert that the lift be tight.

Definition 9.1: Let M be an n-manifold with $n \geq 3$ and use M_k to denote the k-skeleton of M in some cellular decomposition of M. By a *Spin structure* on M one means a trivialization of $\mathrm{T}M|_{M_1}$ which extend to $\mathrm{T}M|_{M_2}$.

According to Gompf & Stipsicz (1999), Spin structures can be understood in terms of Kirby calculus in a rather convenient fashion which shall be recalled next. Consider a 3-manifold Y given as the boundary of a 2-handlebody M. Denote by $\{K_i \subset S^3 \mid i \in I\}$ the set of knots onto which the 2-handles of M are attached and by $\{h_i \mid i \in I\}$ the corresponding 2-handles.

Definition 9.2: Given a Spin structure $\mathcal{S}$ on Y, define the class $w_2(M, \mathcal{S}) \in \mathrm{H}^2(M, Y; \mathbf{Z}/2\mathbf{Z})$ as the obstruction to extending $\mathcal{S}$ from Y to M.

Remark 9.3: By standard obstruction theory, this class can be characterized in terms of the its evaluations on each of the classes $[D_i, \partial D_i] \in \mathrm{H}_2(M, Y; \mathbf{Z}/2\mathbf{Z})$ associated to the cocores $\{D_i\}$ of the 2-handles $\{h_i\}$. One can ask whether $\mathcal{S}$ extends across h_i to a Spin structure on $h_i \cong D^2 \times D^2$. It follows that $w_2(M, \mathcal{S})$ evaluates on the class $[D_i, \partial D_i]$ as 0 when $\mathcal{S}$ extends across h_i and as 1 when it does not.

Definition 9.4: An element w of $\mathrm{H}^2(M, Y; \mathbf{Z}/2\mathbf{Z})$ is called *characteristic* when it gets mapped to the second Stiefel-Whitney class $w_2(M) \in \mathrm{H}^2(M; \mathbf{Z}/2\mathbf{Z})$ via the map $\mathrm{H}^2(M, Y; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^2(M; \mathbf{Z}/2\mathbf{Z})$.

For each i, suppose $F_i \subset M$ to be a closed surface constructed by taking a Seifert surface for the knot in S^3 along which the 2-handle h_i was attached, pushing the interior of this Seifert surface into the interior of D^4 and capping it by gluing it to the core disk of the handle h_i. Orient the closed surface F_i in a manner compatible to the orientation of the knot along which h_i was attached. Notice that the classes $[F_i]$ span $\mathrm{H}_2(M; \mathbf{Z})$ as a $\mathbf{Z}$-module.

Proposition 9.5: A class $w \in \mathrm{H}^2(M, Y; \mathbf{Z}/2\mathbf{Z})$ is characteristic if and only if, for all $i \in I$,

$$w([D_i, \partial D_i]) = [F_i] \cdot [F_i] \mod 2.$$

Proof: Follows from Wu's formula. Vid. Gompf & Stipsicz (1999), Exercise 5.7.3.

QED

Proposition 9.6: (Gompf & Stipsicz 1999) The class $w_2(M, \mathcal{S}) \in \mathrm{H}^2(M, Y; \mathbf{Z}/2\mathbf{Z})$ completely characterizes the Spin structure $\mathcal{S}$ on Y. Conversely, given any characteristic element $w \in \mathrm{H}^2(M, Y; \mathbf{Z}/2\mathbf{Z})$ there is some Spin structure $\mathcal{S}$ on Y such that $w_2(M, \mathcal{S}) = w$.

Remark 9.7: In a Kirby diagram, the author shall denote a Spin structure $\mathcal{S}$ by writing a 0 or a 1 following the framing coefficient separated from it by a comma next to each 2-handle. This signifies the value of the evaluation of $w_2(Y, \mathcal{S})$ on the class in $H_2(M, Y; \mathbf{Z}/2\mathbf{Z})$ corresponding to the respective 2-handle. The following figure shows this notation in the main example from the preceding section.

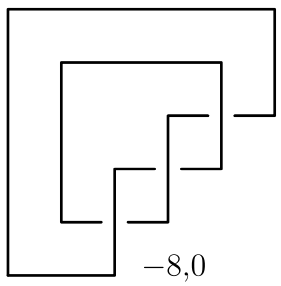


Figure 9.8

Remark 9.9: It is then possible to keep track of the Spin structure during the performance of Kirby moves. The rule when performing a handle slide of h_i over h_j, where the respective components of $w_2(M, \mathcal{S})$ be n_i and n_j, is that n_j changes precisely when $n_i = 1$, whereas n_i always stays unchanged. For a blow-up, the component of $w_2(M, \mathcal{S})$ associated to the newly attached (± 1)-framed 2-handle is always 1. The next figure illustrates the result of both these operations.

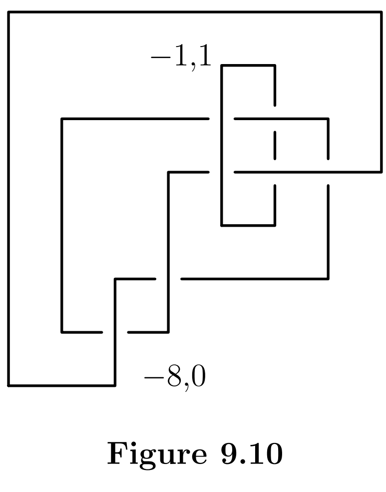


Figure 9.10

The matter is now to consider how a Spin structure lifts via a finite covering in the Kirby calculus setting. As in the previous section, the procedure shall be carried out in two parts. Firstly, one performs a *branched* covering over a 4-manifold-with-boundary obtained by judiciously removing 2-handles and then one equivariantly glues 2-handles in the branched covering to cap the branching locus. In the first part, one needs to take care of how a Spin structure behaves near the *free* 2-handles; in the second part, one needs to take care of how it behaves near the *non-free* 2-handles.

Consider a 3-manifold X, the boundary of a 4-manifold-with-boundary N having only a 0-handle and a set of 2-handles $\{h_i \mid i \in I\}$. Let $\mathcal{S}$ be a Spin structure on X. Denote by $\{K_i \subset S^3\}$ the set of attaching knots of the 2-handles and by $\{D_i\}$ the cocores. Given a subset of 2-handles $\{h_i \mid i \in I'\}$, denote by N' the 4-manifold-with-boundary given by removing this set of 2-handles from N; denote by X' its boundary. Now, *assume* that there exist the p-fold branched covering $\Pi' : M' \to N'$ branched at a Seifert surface S' for the link $L := \bigsqcup_{i \in I'} K_i$ with its interior pushed into the interior of D^4. Let $Y' := \partial M'$ and $\pi' : Y' \to X'$ be the restriction of Π'. For each 2-handle h_i of N', the lift $(\Pi')^{-1}(h_i)$ consists of p 2-handles of M' freely permuted by the deck transformations.

Proposition 9.11: For each handle h in $(\Pi')^{-1}(h_i)$ with cocore D,

$$w_2(Y', (\pi')^*\mathcal{S})([D, \partial D]) = w_2(X', \mathcal{S})([D_i, \partial D_i]).$$

Proof: Recall that $w_2(X', \mathcal{S})([D_i, \partial D_i])$ is characterized by whether the Spin structure $\mathcal{S}$ extends to the whole interior of the 2-handle h_i. That extension existing, the lifted Spin structure $(\pi')^*\mathcal{S}$ shall also extend over each of the preimage 2-handles h of h_i. Conversely, should the extension not exist downstairs, it cannot extend upstairs over any h either. **QED**

Example 9.12: The next figure exemplifies Proposition 9.11 in the already familiar setting of the branched double cover over the left-handed trefoil K from the preceding section (cf. Figure 8.31).

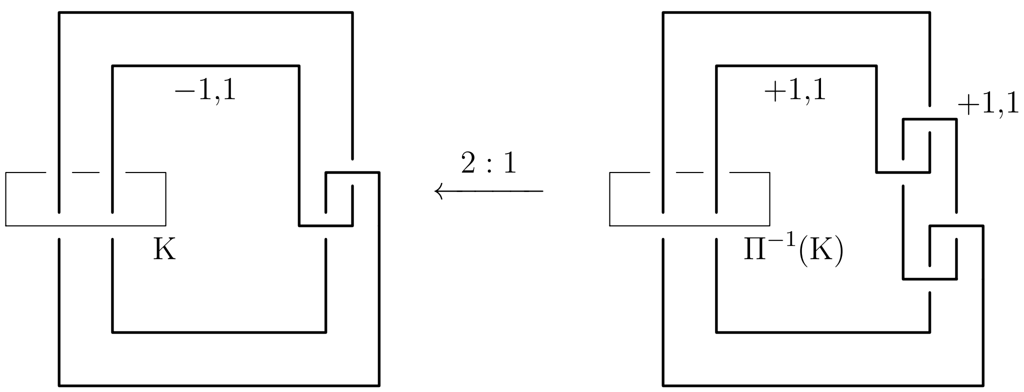


Figure 9.13

Now, one needs to understand what happens to the non-free 2-handles. Proceeding as in Remark 8.28, one equivariantly attaches a 2-handle $\tilde{h}_i$ along $(\Pi')^{-1}(K_i)$ for each $i \in I'$ in order to cap the branching locus thereby defining the manifolds $M := M' \cup \bigcup_{i \in I'} \tilde{h}_i$ and $N := M/G$. Let m_i denote the framing coefficient of the equivariant handle $\tilde{h}_i$. As before, one hopes to be able to choose m_i so that the original manifold Y be ∂N. Assume that this be the case. The question then becomes: given the value of $w_2(X, \mathcal{S})([D_i, \partial D_i])$ for an $i \in I'$, what is the value of $w_2(Y, \pi^*\mathcal{S})([\tilde{D}_i, \partial \tilde{D}_i])$ where $\tilde{D}_i$ denotes the cocore of $\tilde{h}_i$?

Proposition 9.14: For each $i \in I'$, if it be case that $w_2(X, \mathcal{S})([D_i, \partial D_i]) = 1$ then it follows $w_2(Y, \pi^*\mathcal{S})([\tilde{D}_i, \tilde{\partial} D_i]) = 1$. Conversely, if $w_2(X, \mathcal{S})([D_i, \partial D_i]) = 0$, then $w_2(Y, \pi^*\mathcal{S})([\tilde{D}_i, \partial \tilde{D}_i])$ is 1 if the covering multiplicity p be even and 0 otherwise.

Proof: Firstly, consider the 2-handle $D^2 \times D^2$ having the Spin structure $\mathcal{S}_0$ defined along $D^2 \times S^1$ which does not extend across $D^2 \times D^2$. Consider the p-fold branched cover $D^2 \times D^2 \to D^2 \times D^2$ given by rotating the second D^2 factor and keeping the first one fixed. One can trivialize the bundle $\mathrm{T}(D^2 \times D^2)|_{D^2 \times S^1}$ by splitting it as $\mathbf{R}^3 \oplus \mathrm{T}S^1$. It is clear now that this trivialization is pulled back to itself along the restriction of the covering to $D^2 \times S^1$. This means that the Spin structure $\mathcal{S}_0$ is pulled back to itself and the lift also does *not* extend across $D^2 \times D^2$. Secondly, consider instead the Spin structure $\mathcal{S}$ which extend across $D^2 \times D^2$. Then, to understand the lifting behaviour, make use of the other Spin structure already considered, that is $\mathcal{S}_0$, which does *not* extend across the covering. The obstruction class to a homotopy between $\mathcal{S}$ and $\mathcal{S}_0$ can be seen as a non-trivial class in the cohomology $\mathrm{H}^1(S^1; \pi_1(\mathrm{SO}(4))) \cong \pi_1(\mathrm{SO}(4))$. One now easily sees that, under the p-fold covering, this class in $\pi_1(\mathrm{SO}(4))$ gets traversed p times. Since $\pi_1(\mathrm{SO}(4))$ is the cyclic group in two elements, the result follows. **QED**

Example 9.15: The next figure continues the line of examples coming from the (-8)-surgery on the left-handed trefoil and shows the perhaps slightly unexpected behaviour in this case: the Spin structure does extend across the (-8)-framed handle downstairs but, upstairs, its lift does not extend across the lifted 2-handle because the cover is a *double* cover.

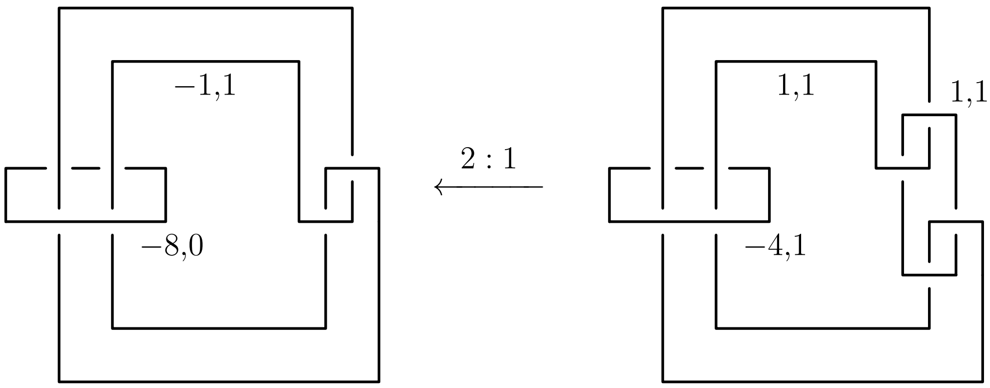
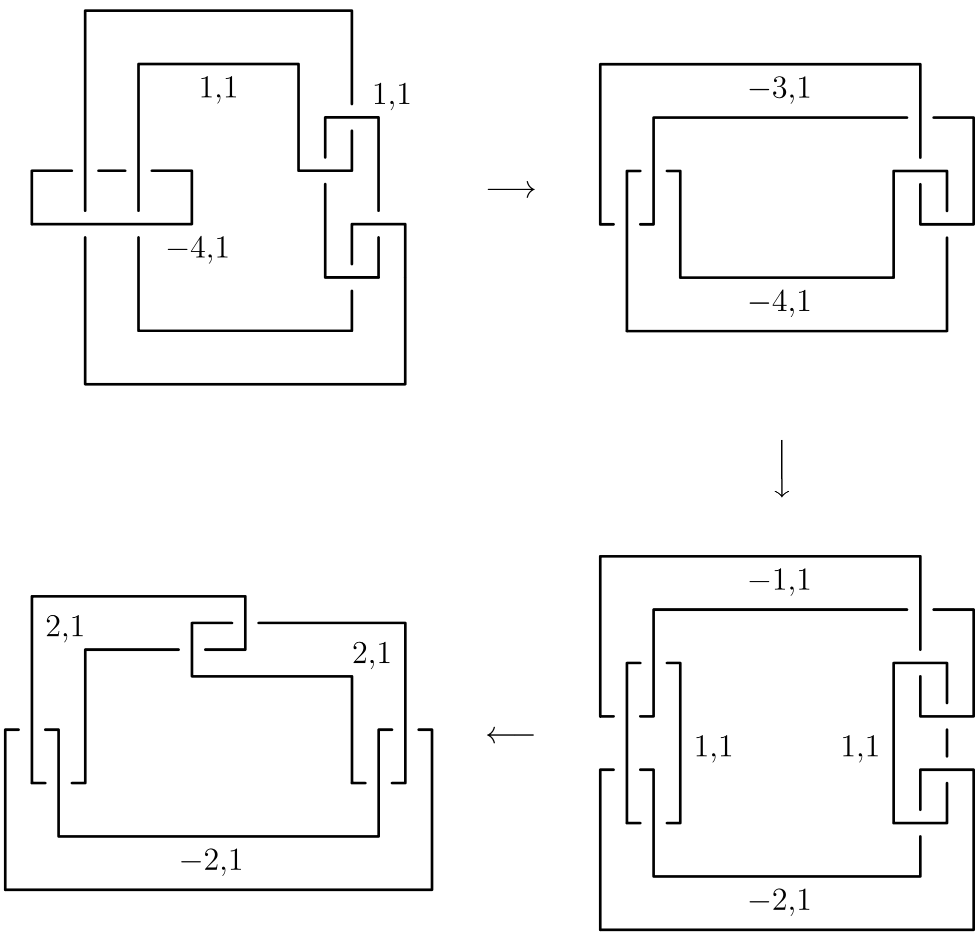


Figure 9.16

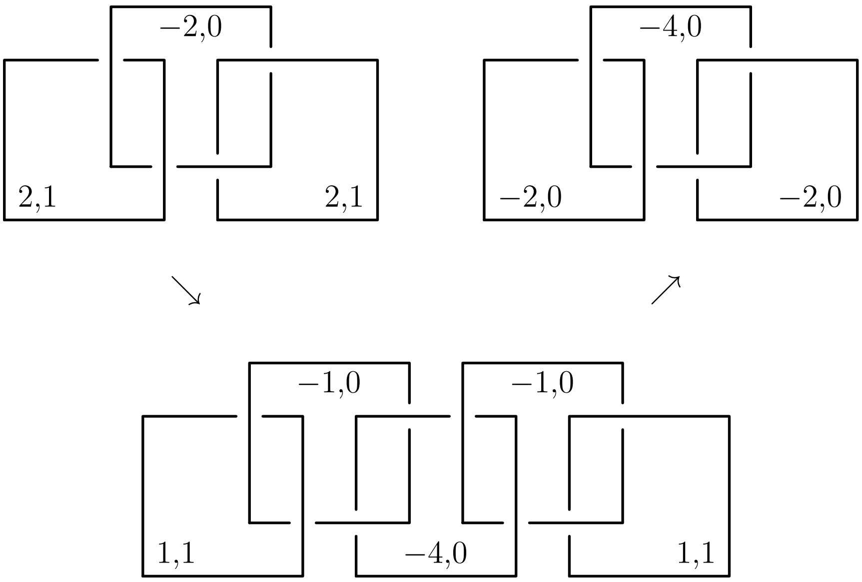
Proposition 9.17: Let X be the (-8)-surgery on the left-handed trefoil. Let $\mathcal{S}$ be the Spin structure on X as defined by the Kirby diagram in Figure 9.8. Let Y be the lens space $L(12,7)$. Denote by $\pi : Y \to X$ the double covering. Let M be the standard linearly plumbed 4-manifold-with-boundary whose boundary is $L(12,7)$. Then, the Spin structure $\pi^*\mathcal{S}$ on Y is the one for which $w_2(M, \pi^*\mathcal{S}) = 0$.

Proof: Start from the right hand side of Figure 9.16, and compute



Here, the first step blew down of one of the +1-framed unknots; the second step blew up twice by +1-framed unknots, and the third step blew down the −1-framed unknot.

Now, if one pay careful attention at the signs of the crossings on the last diagram, one sees that one can perform a handle slide of any one of the 2-framed handles over the (−2)-framed handle in order to obtain a linear chain of unknots. Thence, it is a straightforward matter to perform blow-ups and blow-downs to reach the standard plumbing diagram of the lens space $L(12, 7)$. The procedure is outlined in the following diagrams.



QED

Proposition 9.18: A $\mathrm{Spin}^{\mathbf{C}}$ structure on a 3-manifold Y consists of precisely the same data as a complex trivialization of $\mathrm{T}Y \oplus \mathbf{R}$ over the 2-skeleton of Y which extend across its 3-skeleton.

Proof: Follows from obstruction theory. Cf. Gompf & Stipsicz (1999), Remark 5.6.9(a), for the case of Spin structures. Also vid. Gompf (1998). **QED**

Remark 9.19: From this characterization of $\mathrm{Spin}^{\mathbf{C}}$ structures on Y, it is easy to see that a Spin structure $\mathcal{S}$ defines a $\mathrm{Spin}^{\mathbf{C}}$ structure by taking the trivialization of $\mathrm{T}Y|_{Y_2}$ defined by $\mathcal{S}$ and picking the trivial complex structure on $\mathrm{T}Y|_{Y_2} \oplus \mathbf{R}$. One can then check that this trivialization extends to $\mathrm{T}Y|_{Y_3}$.

Definition 9.20: Given a Spin structure $\mathcal{S}$, denote its induced $\mathrm{Spin}^{\mathbf{C}}$ structure by $\mathcal{S}^{\mathbf{C}}$.

Proposition 9.21: (Gompf 1998, Theorem 4.12) Let (Y, λ) be a contact 3-manifold given as Legendrian surgery on a Legendrian link $L := \bigsqcup_{i\in I} K_i$ in S^3. Let M be the 4-manifold-with-boundary defined by the same Kirby diagram. Suppose $\mathcal{S}$ be a Spin structure on Y and use $L' = \bigsqcup_{i\in I'} K_i \subset L$ to denote the sublink of L consisting of those components K_i for which $w_2(Y, \mathcal{S})([D_i, \partial D_i]) \neq 0$, where D_i denotes the cocore of the handle attached along K_i. Recall that the author uses $\mathfrak{s}_\lambda$ to denote the Spin$^{\mathbf{C}}$ structure defined by the contact form λ. Then, the difference class

$$\mathfrak{s}_\lambda - \mathcal{S}^{\mathbf{C}} \in \mathrm{H}^2(Y; \mathbf{Z})$$

of Spin$^{\mathbf{C}}$ structures is determined by the restriction to Y of a class $\rho \in \mathrm{H}^2(M; \mathbf{Z})$ defined by its evaluations on the classes $[F_i] \in \mathrm{H}_2(M; \mathbf{Z})$, according to the formula

$$(\mathfrak{s}_\lambda - \mathcal{S}^{\mathbf{C}})([F_i]) = \frac{1}{2}\left(r(K_i) + \sum_{j\in I'} [F_i]\cdot[F_j]\right).$$

Example 9.22: In the case of the Spin 3-manifold X defined by Figure 9.8 and studied in Proposition 9.17, consider the Spin$^{\mathbf{C}}$ structure $\mathfrak{s}_\lambda$ defined by the contact structure $\operatorname{Ker}\lambda$ produced by Legendrian surgery on the Legendrian left-trefoil depicted in Figure 8.14, that is, having Thurston-Bennequin invariant -7 and rotation number zero. Computing using Proposition 9.21, one readily finds that $\mathfrak{s} - \mathcal{S}^{\mathbf{C}} = 0$.

Remark 9.23: Using Proposition 9.21 and the Kirby calculus techniques developed above, one can compute the lift of a Spin$^{\mathbf{C}}$-structure $\mathfrak{s}$ via a finite covering by computing the lift of a Spin structure $\mathcal{S}$ and then computing the lift of the second cohomology class $\mathfrak{s} - \mathcal{S}^{\mathbf{C}}$.

Theorem 9.24: Consider the manifold X given by (-8)-surgery on the left-handed trefoil with tight contact structure $\operatorname{Ker}\lambda$ given by Legendrian surgery according to Figure 8.14. The lift of $\operatorname{Ker}\lambda$ via the double cover $\pi : Y \to X$ is the tight contact structure $\tilde{\lambda}$ on $Y \cong L(12, 7)$ given by Legendrian surgery on the middle diagram of Figure 8.38.

Proof: By Remark 8.40, $d_3(\pi^*\lambda) = d_3(\tilde{\lambda})$. Meanwhile, Proposition 9.17 and Example 9.22 combine to assert that the Spin$^{\mathbf{C}}$ structures also match; that is, $\pi^*\mathfrak{s}_\lambda \cong \mathfrak{s}_{\tilde{\lambda}}$. Since the d_3 invariant and the Spin$^{\mathbf{C}}$ structure completely characterize the homotopy class of a hyperplane field, $\pi^*\lambda$ and $\tilde{\lambda}$ are homotopic. Now, according to Corollary 7.42, this is a sufficient condition for $\pi^*\lambda$ and $\tilde{\lambda}$ to be isotopic. **QED**

Remark 9.25: To the best of the author's knowledge, there is no other way to obtain this result short of working in coordinates, which would probably prove itself to be untenable.

10. Bredon Cohomology

In this short section, the author shall apply the cohomotopical contact invariant to a different cohomology theory. The G-equivariant cohomology theory that shall be used is the $\mathrm{RO}(G)$*-graded Bredon cohomology with coefficients in the Burnside Mackey functor* $\underline{\mathrm{A}}(G)$ (vid. May & al. 1996, Chapter X). Bredon cohomology is the equivariant analogue of *ordinary cohomology* and is what is required for the formulation of *equivariant obstruction theory.*

This cohomology theory shall be denoted herein by $\mathrm{H}^*_G(-;\underline{\mathrm{A}}(G))$. The main reason to consider Bredon cohomology is its elegant behaviour with respect to passing to fixed points, which allows one to extend certain results obtained with Borel cohomology to more general coverings. The results derived herein are mostly of theoretical interest due to the difficulty in performing concrete computations with Bredon cohomology. Indeed, the Bredon cohomology of a point is known only for a select few finite groups and not even for all cyclic groups.

Definition 10.1: Define the *G-equivariant Bredon metric dependent Monopole Floer cohomology* as

$$\widetilde{\mathrm{HM}}^*_G(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g) := \mathrm{H}^*_G(\,\mathrm{SWF}_G(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g);\underline{\mathrm{A}}(G))).$$

Remark 10.2: Note that the grading of $\widetilde{\mathrm{HM}}^*_G$ is over the representation ring $\mathrm{RO}(G)$. As a consequence, the dependence of $\widetilde{\mathrm{HM}}^*_G$ on the metric g is only up to a shift of the grading; this is not entirely obvious, but shall not be proved in this thesis.

Let $\mathsf{M} \hookrightarrow E^{\mathrm{u},\mu}_\lambda$ denote a fibre of the unstable normal bundle $E^{\mathrm{u},\mu}_\lambda \to U_{\pi^*\lambda}$. Then, M is naturally a representation of G. Since $U_{\pi^*\lambda}$ is a connected trivial G-manifold, it follows that there exists a G-equivariant Thom class for the G-vector bundle $E^{\mathrm{u},\mu}_\lambda$ (vid. May & al. 1996, §XVI.9 or Lewis Jr., May & Steinberger 1986, §III.6). Denote this class by

$$\theta_{\pi^*\lambda,G} \in \mathrm{H}^{\mathsf{M}}_G(\mathcal{T}_G(\pi^*\lambda,\pi^*g)).$$

Notice that the grading in which this class lives may not be in $\mathbf{Z}$.

Definition 10.3: Define the *metric dependent G-equivariant Bredon cohomological contact invariant* as

$$\psi_G(\pi^*\lambda,\pi^*g) := \Psi_G(\pi^*\lambda,\pi^*g)^*(\theta_{\pi^*\lambda,G}) \in \widetilde{\mathrm{HM}}^{\mathsf{M}}_G(Y,\pi^*\mathfrak{s}_\lambda,\pi^*g)).$$

Next, the author shall consider how to recover the downstairs contact invariant.

Definition 10.4: For an element $\alpha \in \mathrm{RO}(G)$ and a G-space X, denote the natural map

$$\mathrm{H}_G^\alpha(X; \underline{\mathrm{A}}(G)) \to \mathrm{H}^{\dim \alpha^G}(X^G; \mathbf{Z})$$

by $x \mapsto x^G$ where the cohomology theory on the right is ordinary (i.e. Eilenberg-MacLane) cohomology, which is given by applying the functor Φ^G to the representing map $\Sigma^\infty X \to \Sigma^\alpha \mathrm{H}\underline{\mathrm{A}}(G)$ where $\mathrm{H}\underline{\mathrm{A}}(G)$ denotes the Eilenberg-MacLane G-spectrum which represents the cohomology theory $\mathrm{H}_G^*(-; \underline{\mathrm{A}}(G))$.

Theorem 10.5: $\psi(\lambda, g) = (\psi_G(\pi^*\lambda, \pi^*g))^G$.

Proof: Consider the commuting diagram

$$\begin{array}{ccc}
\widetilde{\mathrm{HM}}_G^{\mathtt{M}}(Y, \pi^*\mathfrak{s}_\lambda, \pi^*g) & \longleftarrow & \mathrm{H}_G^{\mathtt{M}}(\mathcal{T}(\pi^*\lambda, \pi^*g)) \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{HM}}^{\dim \mathtt{M}^G}(Y/G, \mathfrak{s}_\lambda, g) & \longleftarrow & \mathrm{H}^{\dim \mathtt{M}^G}(\mathcal{T}(\lambda, g))
\end{array}$$

where the vertical arrows are the natural restrictions to fixed points discussed above and the horizontal arrows are the pullbacks by $\Psi_G(\pi^*\lambda, \pi^*g)$ and $\Psi(\lambda, g)$ respectively. The result follows immediately after noting that, under the right vertical map, the Thom class $\theta_{\pi^*\lambda, G}$ gets sent to the Thom class θ_λ (cf. Costenoble & Waner 1992, Theorem C). **QED**

Definition 10.6: For an element $\alpha \in \mathrm{RO}(G)$, denote the natural map

$$\widetilde{\mathrm{HM}}_G^\alpha(Y, \pi^*\mathfrak{s}_\lambda, \pi^*g) \to \widetilde{\mathrm{HM}}^{\dim \alpha}(Y, \pi^*\mathfrak{s}_\lambda, \pi^*g).$$

given by passage to the subgroup $1 \subset G$ by $x \mapsto x|1$ (vid. Costenoble & Waner 2016, §1.10.1).

Theorem 10.7: $\psi_G(\pi^*\lambda, \pi^*g)|1 = \psi(\pi^*\lambda, \pi^*g)$.

Proof: This is immediate from the chain level description of passage to subgroups. **QED**

Corollary 10.8: If $\psi_G(\pi^*\mathfrak{s}_\lambda, \pi^*g) = 0$ then both $\psi(\pi^*\lambda, \pi^*g) = 0$ and $\psi(\lambda, g) = 0$.

11. Non-Degeneracy

This section shall present the author's original proof for Theorem 2.23; that is, the theorem which asserted that the contact monopole C_λ is non-degenerate for sufficiently large $r > 0$. As the author said in the beginning of the present thesis, unbeknownst to him, this result was originally proven in Taubes (2007). The proof presented here is in fact significantly different and uses a simpler sort of argument. The idea is to reformulate the linearized Seiberg-Witten equations as a pair of coupled Dirac equations for the connexion and the spinor and square them to obtain Laplace equations, which then, when the parameter $r > 0$ be made large, shall not admit any solutions. It shall prove necessary to rewrite the Seiberg-Witten equations in the language of strictly pseudoconvex CR-geometry — an analogue of sorts, in odd dimensions, to almost Kähler geometry.

As in §2, let Y be an oriented 3-manifold, $\lambda \in \Omega^1(Y)$ a contact form, $\xi := \operatorname{Ker}\lambda$ and g a metric such that $\lambda \wedge \mathrm{d}\lambda = \mathrm{Vol}_g$. Fix a complex structure $J : \xi \to \xi$ so as to have $g(-,-) = \mathrm{d}\lambda(-, J-)$. Use $R \in \Gamma(\mathrm{T}Y)$ to denote the Reeb field defined with respect to g. Extend J to a map $\mathrm{T}Y \to \mathrm{T}Y$ by setting $JR = 0$.

Remark 11.1: Notice that

$$\Lambda^k_{\mathbf{C}}(\xi^* \times \mathbf{C}) \cong \bigoplus_{p+q=k} \Lambda^{p,q}\xi^*.$$

Moreover, one has $\mathrm{T}^*Y \otimes \mathbf{C} = \langle\lambda\rangle_{\mathbf{C}} \oplus \Lambda^{1,0}\xi^* \oplus \Lambda^{0,1}\xi^*$, whence it follows that

$$\Lambda^k_{\mathbf{C}}(\mathrm{T}^*Y \otimes \mathbf{C}) \cong \left(\bigoplus_{p+q=k} \Lambda^{p,q}\xi^*\right) \oplus \left(\lambda \wedge \bigoplus_{p+q=k-1} \Lambda^{p,q}\xi^*\right).$$

Remark 11.2: Note that, since $\dim_{\mathbf{C}} \Lambda^{1,0}\xi = 1$, it is trivially true that

$$[\Gamma\Lambda^{1,0}\xi, \Gamma\Lambda^{1,0}\xi] \subset \Gamma\Lambda^{1,0}\xi.$$

This is a formal integrability condition that may not hold in higher dimensions; when it does, (Y, ξ, J) is called a *Cauchy-Riemann, or CR, manifold.* If, further, as is the case here, one has $\xi = \operatorname{Ker}\lambda$ for λ a contact form with $\mathrm{d}\lambda(-, J-)$ positive definite on ξ, the tuple (Y, ξ, J, λ) is called a *strictly pseudoconvex CR-manifold.*

Now, recall the definition of the spinor bundle $\mathcal{S}_\lambda$ and the Clifford multiplication map $c\ell : \mathrm{T}Y \to \mathrm{End}_{\mathbf{C}}(\mathcal{S}_\lambda)$ from Definition 2.1. Here, the author shall be considering the extension of the map $c\ell$ to the complexification $\mathrm{T}Y \otimes \mathbf{C}$ by imposing $\mathbf{C}$-*linearity.* Meanwhile, by precomposing with the duality map $\omega \mapsto \omega^*$ one obtains a map $\mathrm{T}^*Y \otimes \mathbf{C} \to \mathrm{End}_{\mathbf{C}}(\mathcal{S}_\lambda)$, which shall still be denoted $c\ell$. It is important to note that this last map is therefore $\mathbf{C}$-*anti*linear, because the duality $\omega \mapsto \omega^*$ is $\mathbf{C}$-antilinear. As a consequence of this convention, note that the difference of Dirac operators is given by

$$(\mathfrak{D}_{A_c+a} - \mathfrak{D}_{A_c})\,\psi = -a \cdot \psi.$$

The Clifford multiplication map can then be extended further to all complex k-forms according to the rule

$$c\ell(\omega \wedge \eta) := \frac{1}{2}[c\ell(\omega), c\ell(\eta)].$$

The resulting map $\Lambda^*_{\mathbf{C}}\mathrm{T}^*Y \otimes \mathbf{C} \to \mathrm{End}_{\mathbf{C}}(\mathcal{S}_\lambda)$ shall also be denoted $c\ell$.

Definition 11.3: Define the *Cauchy-Riemann* operators

$$\partial : \Gamma\Lambda^{p,q}\xi^* \to \Gamma\Lambda^{p+1,q}\xi^*, \quad \overline{\partial} : \Gamma\Lambda^{p,q}\xi^* \to \Gamma\Lambda^{p,q+1}\xi^*$$

by composing the exterior derivative $\mathrm{d} : \Gamma\Lambda^{p,q}\xi^* \to \Gamma\Lambda^{p+q+1}_{\mathbf{C}}(\mathrm{T}^*Y \otimes \mathbf{C})$ with the appropriate projection according to the direct sum decomposition in Remark 11.1.

Remark 11.4: The exterior derivative decomposes as

$$d = \partial + \overline{\partial} + \lambda \wedge \mathcal{L}_R\,.$$

Remark 11.5: The goal shall be now to express the Dirac operator in terms of the Cauchy-Riemann operators. For that end, it shall prove necessary to work in a connexion on Y other than the one of Levi-Civita.

Definition 11.6: The *Tanaka-Webster connexion* is the connexion on $\mathrm{T}Y$ defined so as to have the covariant derivative satisfy

$$\nabla_{\mathrm{TW}}\,\lambda = 0, \quad \nabla_{\mathrm{TW}}\,g = 0, \quad \nabla_{\mathrm{TW}}\,J = 0,$$

$$\mathrm{T}^{\mathrm{TW}}|_\xi = \mathrm{d}\lambda \otimes R, \quad \mathrm{T}^{\mathrm{TW}}(R) = -\frac{1}{2}J \circ (\mathcal{L}_R\,J),$$

where $\mathrm{T}^{\mathrm{TW}} \in \Omega^2(Y;\mathrm{T}Y) \cong \Omega^1(Y;\mathrm{End}(\mathrm{T}Y))$ signifies the torsion of the Tanaka-Webster connexion.

Remark 11.7: Recall that a connexion on $\mathrm{T}Y$ together with a connexion on $\det\mathfrak{s}_\lambda$ precisely define a connexion on $\mathcal{S}_\lambda$. Given a connexion A on $\det\mathfrak{s}_\lambda$, use $\mathrm{TW}\times A$ to denote the induced connexion on $\mathcal{S}_\lambda$.

Definition 11.8: By the *Tanaka-Webster-Dirac operator,* one means $\mathfrak{D}^{\mathrm{TW}}_A : \Gamma\mathcal{S}_\lambda \to \Gamma\mathcal{S}_\lambda$ defined as the composite

$$\Gamma\mathcal{S}_\lambda \xrightarrow{\nabla_{\mathrm{TW}\times A}} \Gamma(\mathrm{T}^*Y \otimes \mathcal{S}_\lambda) \xrightarrow{c\ell} \Gamma\mathcal{S}_\lambda.$$

Remark 11.9: On a CR-manifold, there is a notion of Chern connexion defined formally in the same manner as for a complex manifold using the operator $\overline{\partial}$ defined above. Let A_c denote the Chern connexion on $\det \mathfrak{s}_\lambda$.

Proposition 11.10: (Petit 2005) For $\alpha \in \Gamma\Lambda^{0,q}\xi^*$, one has

$$\mathfrak{D}^{\mathrm{TW}}_{A_c}\, \alpha = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)\alpha + i(-1)^{q+1}\, \nabla_{\mathrm{TW}}(\alpha)(R).$$

Proposition 11.11: (Petit 2005, Proposition 3.4) The Tanaka-Webster-Dirac and the Levi-Civita-Dirac operators are related by

$$\mathfrak{D}^{\mathrm{TW}}_{A} - \mathfrak{D}_A = \frac{1}{4}\, c\ell(\lambda \wedge \mathrm{d}\lambda) = -\frac{1}{4}.$$

Now, consider a Seiberg-Witten configuration $(A, \psi) \in \mathcal{C}(Y, \mathfrak{s}_\lambda)$. Write the spinor as $\psi = r^{-1/2}(\alpha + \beta)$ where $\alpha \in \Gamma\Lambda^{0,0}\xi^*$ and $\beta \in \Gamma\Lambda^{0,1}\xi^*$. Likewise, consider the connexion $A = A_c + 2a$ by writing $a = -f\lambda + \frac{i}{\sqrt{2}}(\eta + \overline{\eta})$ where $f \in \Lambda^{0,0}\xi^*$, $\Re f = 0$ and $\eta \in \Gamma\Lambda^{0,1}\xi^*$. The goal shall be to write the Seiberg-Witten equations in terms of α, β, f, η and the operators ∂, $\overline{\partial}$. The Dirac equation can easily be translated using the above facts.

Corollary 11.12: The Dirac equation $\mathfrak{D}_{A_c+a}\psi = 0$ is equivalent to the equations

$$\sqrt{2}\,\overline{\partial}^*\, \beta - i(\nabla_{\mathrm{TW}}\, \alpha)(R) + \left(\frac{1}{4} + if\right)\alpha - i\eta^*(\beta) = 0$$

$$\sqrt{2}\,\overline{\partial}\,\alpha + i(\nabla_{\mathrm{TW}}\, \beta)(R) + \left(\frac{1}{4} - if\right)\beta + i\eta\alpha = 0.$$

The curvature equation requires more careful consideration. It is in line to understand how the Hodge operator interacts with the present structure. Use $\overline{*}$: $\Lambda^*_{\mathbf{C}}\mathrm{T}^*Y \otimes \mathbf{C} \to \Lambda^*_{\mathbf{C}}\mathrm{T}^*Y$ to denote the $\mathbf{C}$-*anti*linear Hodge operator; that is, its defining property is that $\overline{*}\langle\alpha, \beta\rangle = \alpha \wedge \overline{*}\beta$.

Definition 11.13: Let $\overline{*}_\xi : \xi^*_{i,j} \to \xi^*_{n-i,n-j}$, (where, in the present case, $n = 1$) be the $\mathbf{C}$-*anti*linear Hodge operator on the bundle $\xi^* \otimes \mathbf{C}$ defined by the Hermitian inner product induced by $\mathrm{d}\lambda$ and J.

Remark 11.14: The formal adjoints of the Cauchy-Riemann operators are given by

$$\partial^* = -\overline{*}_\xi \partial \overline{*}_\xi, \quad \overline{\partial}^* = -\overline{*}_\xi\, \overline{\partial}\, \overline{*}_\xi.$$

Remark 11.15: The Hodge operator of Y restricts as

$$\overline{*} : \Lambda^{i,j}\xi^* \to \lambda \wedge \Lambda^{n-i,n-j}\xi^*,$$

where, in the present case, $n = 1$, and is given by $\delta \mapsto \lambda \wedge \overline{*}_\xi\delta$.

Remark 11.16: Since $\mathrm{d}\lambda \in \Gamma\Lambda^{1,1}\xi^*$ is the orientation form of ξ, it follows $\overline{*}\lambda = \frac{1}{2}\mathrm{d}\lambda$.

Remark 11.17: $\overline{*}_\xi$ acts on $\Lambda^{1,0}\xi^* \oplus \Lambda^{0,1}\xi^*$ as $\alpha \mapsto -J\overline{\alpha}$.

Proposition 11.18: The Seiberg-Witten curvature equation (vid. Definition 2.13) is equivalent to the equations

$$\sqrt{2}\,\overline{\partial}\, f + i\,\nabla_{\mathrm{TW}}\,\eta(R) - \frac{i}{\sqrt{2}}\overline{*}(F^{0,1}_{A_c} - F^{0,1}_{A_\lambda}) + m\overline{\eta} + i\overline{\alpha}\beta = 0$$
$$\frac{1}{\sqrt{2}}\left(\overline{\partial}^*\,\eta - \partial^*\overline{\eta}\right) - \langle \overline{*}(F_{A_c} - F_{A_\lambda}), \lambda\rangle + f + \frac{i}{2}\left(|\alpha|^2 - |\beta|^2\right) = 0$$

Proof: Consider the term

$$\overline{*}\mathrm{d}a = \overline{*}\mathrm{d}\left(-f\lambda + \frac{i}{\sqrt{2}}(\eta + \overline{\eta})\right).$$

The first term can be expanded as

$$\begin{aligned}\overline{*}\mathrm{d}(-f\lambda) &= -\overline{*}(-\lambda \wedge \mathrm{d}f + f\mathrm{d}\lambda)\\ &= i(\overline{\partial}\, f - \partial f) - f\lambda.\end{aligned}$$

Whilst the second term can be computed as

$$\begin{aligned}\overline{*}\mathrm{d}\left(\frac{i}{\sqrt{2}}(\eta + \overline{\eta})\right) &= \frac{-i}{\sqrt{2}}\overline{*}\mathrm{d}(\eta + \overline{\eta})\\ &= \frac{-i}{\sqrt{2}}\overline{*}\left((\partial + \overline{\partial})(\eta + \overline{\eta}) + \lambda \wedge \mathcal{L}_R(\eta + \overline{\eta})\right)\\ &= \frac{1}{\sqrt{2}}\left(-\lambda \wedge (\overline{\partial}^*\,\eta - \partial^*\overline{\eta}) - \nabla_{\mathrm{TW}}(\eta - \overline{\eta})(R) + im\overline{\eta} - i\overline{m}\eta)\right)\end{aligned}$$

where $m : Y \to \mathbf{C}$ is certain a function dependent on the torsion of the Tanaka-Webster connexion, which shall be described now. Firstly, note that $\mathrm{T}^{\mathrm{TW}}(R) = -\frac{1}{2}J\,\mathcal{L}_R\,J$ is self-adjoint and anticommutes with J. As a consequence, one checks that

$$\mathcal{L}_R(\eta + \overline{\eta}) = \nabla_{\mathrm{TW}}(\eta + \overline{\eta})(R) + \mathrm{T}^{\mathrm{TW}}(R)(\eta + \overline{\eta})$$

where $\mathrm{T}^{\mathrm{TW}}(R)$ is acting on T^*Y by precomposition. Moreover, another consequence of $\mathrm{T}^{\mathrm{TW}}(R)$ anticommuting with J is that it maps $\xi^*_{0,1}$ to $\xi^*_{1,0}$ and vice versa; so, due to these bundles being one complex dimensional, one sees that $\overline{*}_\xi\mathrm{T}^{\mathrm{TW}}(R)(\eta)$ is given by a complex valued function multiplying $\overline{\eta}$; hence, define m so as to have

$$m\overline{\eta} = i\mathrm{T}^{\mathrm{TW}}(R)(\overline{\eta}).$$

Now, consider the quadratic term

$$r\tau(\psi) = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\beta^* \\ \overline{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix} \in \mathrm{End}(\Lambda^{0,0}\xi^* \oplus \Lambda^{0,1}\xi^*).$$

By working directly with Definition 2.1, one shows that, under the isomorphism $i\mathrm{T}^*Y \cong i\mathfrak{su}(\mathcal{S}_\lambda)$, this becomes

$$r\tau(\psi) = -\frac{i}{2}\lambda(|\alpha|^2 - |\beta|^2) - \frac{1}{\sqrt{2}}(\overline{\alpha}\beta - \alpha\overline{\beta}).$$

QED

Proposition 11.19: The local Coulomb gauge condition (vid. Definition 2.17) is equivalent to the equation

$$-i\,\nabla_{\mathrm{TW}}(F)(R) + \frac{1}{\sqrt{2}}(\overline{\partial}^*\,H + \partial^*\overline{H}) - \frac{1}{2}(A^*(\alpha) + B^*(\beta) - \alpha^*(A) - \beta^*(B)) = 0,$$

where (F, H, A, B) is a tangent vector at (f, η, α, β).

Proof: Consider the coexterior derivative term

$$-\mathrm{d}^*\left(-F\lambda + \frac{i}{\sqrt{2}}(H + \overline{H})\right) = 0.$$

The first term reduces to

$$-\mathrm{d}^*(-F\lambda) = \overline{*}\mathrm{d}\overline{*}(-F\lambda) = -\,\mathcal{L}_R\,F = -\,\nabla_{\mathrm{TW}}(F)(R),$$

and the second to

$$\begin{aligned}
-\mathrm{d}^*\left(\frac{i}{\sqrt{2}}(H + \overline{H})\right) &= \frac{i}{\sqrt{2}}\overline{*}\mathrm{d}\left(\lambda \wedge \overline{*}_\xi(H + \overline{H})\right)\\
&= \frac{i}{\sqrt{2}}\overline{*}\left(-\lambda \wedge (\partial + \overline{\partial})\overline{*}_\xi(H + \overline{H})\right)\\
&= \frac{i}{\sqrt{2}}\left(\overline{*}_\xi(\partial + \overline{\partial})\overline{*}_\xi(H + \overline{H})\right)\\
&= \frac{-i}{\sqrt{2}}\left((\partial^* + \overline{\partial}^*)(H + \overline{H})\right)
\end{aligned}$$

.

QED

Corollary 11.20: The linearization of the Seiberg-Witten equations at (f, η, α, β) in the local Coulomb gauge is given by

$$\begin{aligned}
\sqrt{2}\,\overline{\partial}^*\,H - i\,\nabla_{\mathrm{TW}}(F)(R) + F - iB^*(\beta) + i\alpha^*A &= 0\\
\sqrt{2}\,\overline{\partial}\,F + i\,\nabla_{\mathrm{TW}}(H)(R) + m\overline{H} + i\overline{A}\beta + i\overline{\alpha}B &= 0\\
\sqrt{2}\,\overline{\partial}^*\,B - i\,\nabla_{\mathrm{TW}}(A)(R) + A\left(\frac{1}{4} + if\right) + i\alpha F - iH^*(\beta) - i\eta^*(B) &= 0\\
\sqrt{2}\,\overline{\partial}\,A + i\,\nabla_{\mathrm{TW}}(B)(R) + B\left(\frac{1}{4} - if\right) - i\beta F + iA\eta + i\alpha H &= 0,
\end{aligned}$$

where (F, H, A, B) is a tangent vector at (f, η, α, β).

Remark 11.21: The key observation to make is that these are a pair of Dirac equations.

Theorem 11.22: For sufficiently large r, the map

$$\Pi^{\mathrm{LC}}_{C_\lambda} \mathrm{D}_{C_\lambda} \mathcal{X}_{\lambda,r} : \mathcal{K}_{C_\lambda} \to \mathcal{K}_{C_\lambda}$$

is an isomorphism.

Proof: Note that this map is Fredholm and has index zero; therefore, it suffices to prove that its kernel vanishes. For that end, what one must do is to input C_λ as (f, η, α, β) into the equations of Corollary 11.20 and solve for (F, H, A, B) showing that the only solution is zero. Observe that, since the spinor part of the contact configuration is given by $\alpha = r^{1/2}$ and $\beta = 0$, Corollary 11.12 implies $ir^{1/2}\eta = 0$ and $(\frac{1}{4} + if)r = 0$; hence, the connexion part of the contact configuration is given by $\eta = 0$ and $f = \frac{i}{4}$. Plugging these into Corollary 11.20 yields

$$\begin{aligned}
\sqrt{2}\,\overline{\partial}\,F + i\,\nabla_{\mathrm{TW}}(H)(R) + m\overline{H} + ir^{1/2}B &= 0\\
\sqrt{2}\,\overline{\partial}^*\,H - i\,\nabla_{\mathrm{TW}}(F)(R) + F + ir^{1/2}A &= 0\\
\sqrt{2}\,\overline{\partial}\,A + i\,\nabla_{\mathrm{TW}}(B)(R) + \frac{1}{2}B + ir^{1/2}H &= 0\\
\sqrt{2}\,\overline{\partial}^*\,B - i\,\nabla_{\mathrm{TW}}(A)(R) + ir^{1/2}F &= 0.
\end{aligned}$$

With respect to the connexion $\mathfrak{k} := A_c + \frac{i}{4}\lambda$, one can conveniently pack most terms together into Dirac operators as

$$\begin{aligned}
\mathfrak{D}_{\mathfrak{k}}(F+H) + L(F+H) + ir^{1/2}(A+B) &= 0\\
\mathfrak{D}_{\mathfrak{k}}(A+B) + ir^{1/2}(H+F) &= 0,
\end{aligned}$$

where $L(F+H) := -\frac{1}{2}H + m\overline{H} + F$. Applying the Dirac operator to the first equation and substituting in the second gives

$$\mathfrak{D}^2_{\mathfrak{k}}(F+H) + \mathfrak{D}_{\mathfrak{k}} \circ L(F+H) + r(F+H) = 0.$$

Notice that the elliptic second-order differential operator $\mathfrak{D}^2_{\mathfrak{k}} + \mathfrak{D}_{\mathfrak{k}} \circ L + r$ has the same symbol as the Laplacian. As a consequence, choosing r sufficiently large ensures that it be an isomorphism between appropriate Sobolev spaces. To see why this is, think of this operator as an *elliptic operator with parameter* where the parameter is r taking values on the ray $(0, \infty) \subset \mathbf{C}$ (vid. Shubin (2000), §9); elliptic theory then implies that, for large r, the resulting operator is invertible. As a consequence, F and H must be zero, and, then, A and B must be zero as well. **QED**

Remark 11.23: Hereby the author concludes the proof of Theorem 11.22.

12. Uniqueness of Trajectories

This section shall deal with the proof of Theorem 2.25, where it was claimed that there are no non-trivial Seiberg-Witten trajectories with positive infinity limit the contact configuration. A very similar assertion is made by Taubes (2009), Proposition 5.15. The key difference is that Taubes deals with the case of a non-torsion $\det \mathfrak{s}_\lambda$, which is not the case here. The fact that $\det \mathfrak{s}_\lambda$ is non-torsion in Taubes (2009), Proposition 5.15, also causes its conclusion to be somewhat weaker than what is proved here. Another minor difference is that, in the present thesis, the author must not assume that the negative infinity limit of the trajectory be non-degenerate. These differences require only significant modifications to the final part of the proof; therefore, the proof presented here shall be very similar to what one can find in Taubes (2009) and other works of Taubes that have drawn significantly from it. However, Taubes' proof is difficult to follow for someone who has not read Taubes (2007) and Taubes (2009) in their entirety; therefore, the author felt that it was necessary to include all the details here for the sake of completeness.

The overall strategy shall consist of the following. On the one hand, it shall be shown that only the contact configuration and its gauge equivalent configurations have the property that the spinor component be bounded *away from zero* in a certain way. On the other, it shall be shown that the trajectory necessarily satisfies the same sort of bound on its spinor component for *all time.* As a consequence, both endpoints shall have to be gauge equivalent.

The author shall use the symbol ∇_A to denote the $\mathrm{Spin}^{\mathbf{C}}$ covariant derivative on $\mathcal{S}_\lambda$ induced by the connexion A on $\det \mathfrak{s}_\lambda$. This is to say that Clifford multiplication is covariantly constant for ∇_A. It shall also be necessary to work with covariant derivatives defined on each of the summands $\mathcal{S}_\lambda \cong \Lambda^{0,0}\xi^* \oplus \Lambda^{0,1}\xi^*$. Both these derivatives shall be denoted by the same symbol as

$$\nabla'_A : \Gamma(\Lambda^{0,k}\xi^*) \to \Gamma(\Lambda^{0,k}\xi^* \otimes \mathrm{T}^*Y)$$

and shall be defined by projecting ∇_A to the respective summand of $\mathcal{S}_\lambda$. This can be conveniently expressed in terms of Clifford multiplication by the formula

$$\nabla'_A = \frac{1}{2}\left(1 + i(-1)^k\, c\ell(\lambda)\right)\nabla_A.$$

Hence, the difference to ∇_A is given by

$$\nabla_A - \nabla'_A = i(-1)^k\, c\ell(\nabla_{\mathrm{LC}}\lambda),$$

where ∇_{LC} denotes the Levi-Civita connexion.

Henceforth, the author shall deal with Seiberg-Witten trajectories. Suppose $\gamma : \mathbf{R} \to \mathcal{C}$ to be a finite type trajectory satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = -\,\mathcal{X}_{\lambda,r}\,(\gamma(t)).$$

Assume that both limits $\lim_{t\to\pm\infty}\gamma(t)$ be well defined in the L^2_5 topology. Note that, by Sobolev embedding, the limits are also well defined in C^2. The author shall not *yet* be assuming that the positive time limit be the contact configuration C_λ; that assumption shall be added later. Write $\gamma(t) = (A(t), \psi(t)) \in \mathcal{C}(Y, \mathfrak{s}_\lambda)$ for its connexion and spinor components respectively. Write $A(t) := A_\lambda(t) + a(t)$ where $a(t)$ is a purely imaginary 1-form on Y. Decompose the spinor as $\psi(t) = (\alpha(t), \beta(t))$ according to the direct sum $\mathcal{S}_\lambda = \Lambda^{0,0}\xi^* \oplus \Lambda^{0,1}\xi^*$.

Remark 12.1: In what follows, a sequence of bounds on various quantities shall be established. These shall involve constants K_n and bundle endomorphisms E_n labelled by n the number of the lemma in which they appear. These quantities may depend on the metric of Y and the contact form λ, but they shall not depend on the particular solution (A, ψ) to the Seiberg-Witten equations *nor shall they shall not depend on the value of the parameter* $r > 0$.

Lemma 12.2: The spinor ψ satisfies the second order equation

$$-\frac{\partial^2}{\partial t^2}\psi + \nabla_A^*\nabla_A\psi - c\ell\left(\frac{1}{2}{*F_{A_\lambda}} + r\tau(\psi) - \frac{ir}{2}\lambda\right)\psi + \frac{s}{4}\psi = 0,$$

where $s : Y \to \mathbf{R}$ denotes the scalar curvature of Y.

Proof: The Dirac equation

$$\frac{\partial}{\partial t}\psi = -\mathfrak{D}_A\psi$$

implies the second order equation

$$\frac{\partial^2}{\partial t^2}\psi = -\frac{\partial}{\partial t}\mathfrak{D}_A\psi = -\mathfrak{D}_A\frac{\partial}{\partial t}\psi - c\ell\left(\frac{\partial}{\partial t}A\right)\psi = \mathfrak{D}_A^2\psi - c\ell\left(\frac{\partial}{\partial t}A\right)\psi.$$

Now, apply the the well known Weitzenböck formula (cf. Nicolaescu 2000, (1.3.11)),

$$\mathfrak{D}_A^*\mathfrak{D}_A\psi = \nabla_A^*\nabla_A\psi - \frac{1}{2}\,c\ell({*F_A})\psi + \frac{s}{4}\psi,$$

to find that

$$-\frac{\partial^2}{\partial t^2}\psi + \nabla_A^*\nabla_A\psi - c\ell\left(\frac{\partial}{\partial t}A + \frac{1}{2}{*F_A}\right)\psi + \frac{s}{4}\psi = 0.$$

The result then follows by applying the curvature component of the Seiberg-Witten trajectory equations. **QED**

Lemma 12.3: The norm squared of the spinor ψ satisfies the second order equation

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}|\psi|^2+\frac{1}{2}\mathrm{d}^*\mathrm{d}|\psi|^2+\left|\frac{\partial}{\partial t}\psi\right|^2+|\nabla_A\psi|^2+\frac{r}{2}\left(|\psi|^4-\Re\left\langle i\,c\ell(\lambda)\psi,\psi\right\rangle\right)$$
$$+\Re\left\langle c\ell(F_{A_\lambda})\psi,\psi\right\rangle+s|\psi|^2=0$$

where the inner product is the pointwise inner product of the Hermitian bundle $\mathcal{S}_\lambda$.

Proof: Follows by taking the pointwise inner product with ψ of both sides in the equation asserted by Lemma 12.2. **QED**

Lemma 12.4: There is a constant $K_{12.4}>0$ such that

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}|\psi|^2+\frac{1}{2}\mathrm{d}^*\mathrm{d}\,|\psi|^2+\frac{r}{2}|\psi|^2\left(|\psi|^2-1\right)\leq-\left|\frac{\partial}{\partial t}\psi\right|^2-|\nabla_A\psi|^2+K_{12.4}\,|\psi|^2\,.$$

Proof: Define the constant to be

$$K_{12.4}:=\max\left\{\sup_Y|F_{A_\lambda}|\,,\sup_Y|s|\right\}.$$

The equation of Lemma 12.3 implies that

$$-\frac{1}{2}|\psi|^2+\frac{1}{2}\mathrm{d}^*\mathrm{d}\,|\psi|^2+\frac{r}{2}|\psi|^2\left(|\psi|^2-1\right)+\left|\frac{\partial}{\partial t}\psi\right|^2+|\nabla_A\psi|^2\leq\left(|F_{A_\lambda}|+|s|\right)|\psi|^2$$
$$\leq K_{12.4}\,|\psi|^2\,.$$

QED

Lemma 12.5: There is a constant $K_{12.5}>0$ such that

$$|\alpha|^2+|\beta|^2\leq 1+K_{12.5}r^{-1}.$$

Proof: This is analogous to the first statement of Taubes (2009), Lemma 5.1. Start by setting the constant to be

$$K_{12.5}:=K_{12.4}$$

Consider the function $u:\mathbf{R}\times Y\to\mathbf{R}$ defined as $u:=|\psi|^2-1-K_{12.5}r^{-1}$. The following differential inequality is implied by Lemma 12.4.

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}u+\frac{1}{2}\mathrm{d}^*\mathrm{d}u+r|\psi|^2u\leq 0.$$

Consequently, the function u cannot admit a *positive local maximum* anywhere in $\mathbf{R}\times Y$. Consider now two cases. Firstly, suppose that the trajectory be *constant;* that is, that ψ be constant in the parameter t. Because Y is compact, not admitting a positive local maximum implies that $u\leq 0$ everywhere in Y. Secondly, suppose that the trajectory *not* be constant. Here, the limits $\lim_{t\to\pm\infty}\psi$ are constant trajectories as in the previous case, which implies that the limits $\lim_{t\to\pm\infty} u$ are everywhere non-positive on Y. Since u does not admit positive local maxima, it follows that $u \leq 0$ everywhere on $\mathbf{R} \times Y$. In other words,

$$|\alpha|^2 + |\beta|^2 = |\psi|^2 \leq 1 + K_{12.5} r^{-1}.$$

QED

Lemma 12.6: The norm squared of the component α of the spinor ψ satisfies the second order equation

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}|\alpha|^2 + \frac{1}{2}\mathrm{d}^*\mathrm{d}|\alpha|^2 + \left|\frac{\partial}{\partial t}\alpha\right|^2 + |\nabla_A \alpha|^2 - \frac{r}{2}(1 - |\alpha|^2 - |\beta|^2)|\alpha|^2$$
$$+ E_{12.6}(\alpha, \beta) + E'_{12.6}(\alpha, \nabla'_A \beta) + E''_{12.6}(\alpha, \alpha) = 0.$$

For bilinear forms $E_{12.6}$, $E'_{12.6}$ and $E''_{12.6}$ dependent only on Y, λ and the metric g.

Proof: This is analogous to the first equation in Taubes (2009), (5-9). The proof is similar to Lemma 12.3 but the author feels that certain points are worth clarifying here. Start with the equation of Lemma 12.2 and take the pointwise inner product of both sides with α.

$$-\Re\left\langle \frac{\partial^2}{\partial t^2}\psi, \alpha \right\rangle + \Re\left\langle \nabla_A^* \nabla_A \psi, \alpha \right\rangle - \Re\left\langle \frac{1}{2} c\ell(F_{A_\lambda})\psi, \alpha \right\rangle$$
$$+ \Re\left\langle r\, c\ell(\tau(\psi))\psi, \alpha \right\rangle - \Re\left\langle \frac{r}{2} c\ell(i\lambda)\psi, \alpha \right\rangle + \frac{s}{4}|\alpha|^2 = 0. \qquad (*)$$

Now, consider each term in turn. For the first term, one notes

$$\Re\left\langle \frac{\partial^2}{\partial t^2}\psi, \alpha \right\rangle = \Re\left\langle \frac{\partial^2}{\partial t^2}\alpha, \alpha \right\rangle = \frac{1}{2}\frac{\partial^2}{\partial t^2}|\alpha|^2 - \left|\frac{\partial}{\partial t}\alpha\right|^2.$$

For the second term of $(*)$, consider the following.

$$\mathrm{d}^*\mathrm{d}|\alpha|^2 = \mathrm{d}^*\mathrm{d}\Re\langle\psi, \alpha\rangle = \Re\langle \nabla_A^*\nabla_A\psi, \alpha\rangle - 2\Re\langle \nabla_A\psi, \nabla_A\alpha\rangle + \Re\langle\psi, \nabla_A^*\nabla_A\alpha\rangle$$
$$= 2\Re\langle \nabla_A^*\nabla_A\psi, \alpha\rangle - 2\Re\langle \nabla_A\psi, \nabla_A\alpha\rangle - \Re\langle\alpha, \nabla_A^*\nabla_A\beta\rangle + \Re\langle\beta, \nabla_A^*\nabla_A\alpha\rangle$$

Meanwhile, one also has

$$0 = \mathrm{d}^*\mathrm{d}\Re\langle\alpha, \beta\rangle = \Re\langle \nabla_A^*\nabla_A\alpha, \beta\rangle - 2\Re\langle \nabla_A\alpha, \nabla_A\beta\rangle + \Re\langle\alpha, \nabla_A^*\nabla_A\beta\rangle.$$

Combining the two yields

$$\mathrm{d}^*\mathrm{d}\,|\alpha|^2 = 2\Re\langle \nabla_A^*\nabla_A\psi, \alpha\rangle - 2\,|\nabla_A\alpha|^2 - 2\Re\langle\alpha, \nabla_A^*\nabla_A\beta\rangle.$$

To understand this last term in more detail, recall the relation between the covariant derivatives ∇_A and ∇'_A from above. One readily finds

$$\nabla_A^*\nabla_A\beta = (\nabla'_A - i\,c\ell(\nabla_{\mathrm{LC}}\lambda))^* \circ (\nabla'_A - i\,c\ell(\nabla_{\mathrm{LC}}\lambda))\,\beta$$
$$= (\nabla'_A)^*\nabla'_A\beta + f_1(\nabla'_A\beta) + f_2(\beta),$$

where f_1 and f_2 are two bundle endomorphisms of $\mathcal{S}_\lambda$ dependent on λ, ∇_{LC} and $\nabla^*_{\mathrm{LC}}\nabla_{\mathrm{LC}}\lambda$ but certainly not on r. When one take the pointwise inner product of this equation with α, one shall note that the term involving second derivatives of β vanishes.

$$\Re \langle \nabla_A^* \nabla_A \beta, \alpha \rangle = \Re \langle f_1(\nabla'_A \beta), \alpha \rangle + \Re \langle f_2(\beta), \alpha \rangle .$$

Applying this to the above formula for $\mathrm{d}^*\mathrm{d}\,|\alpha|^2$, one finds

$$\Re \langle \nabla_A^* \nabla_A \psi, \alpha \rangle = \frac{1}{2}\mathrm{d}^*\mathrm{d}\,|\alpha|^2 + |\nabla_A \alpha|^2 + f_3(\alpha, \beta) + f_4(\alpha, \nabla'_A \beta)$$

for two bilinear forms f_3 and f_4 on $\mathcal{S}_\lambda$ independent of r. Next, the third term of $(*)$ can be simply written as

$$-\left\langle \frac{1}{2}\, c\ell(F_{A_\lambda})\psi, \alpha \right\rangle = f_5(\alpha, \beta) + f_6(\alpha, \alpha),$$

where f_5 and f_6 are two bilinear forms on $\mathcal{S}_\lambda$ independent of r. For the fourth term of $(*)$, start by considering the endomorphism $\psi \otimes \psi^* - \frac{1}{2}|\psi|^2\mathrm{id}$ of $\mathcal{S}_\lambda$. According to the direct sum $\mathcal{S}_\lambda \cong \Lambda^{0,0}\xi^* \oplus \Lambda^{0,1}\xi^*$, one writes this endomorphism as a matrix

$$\begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\beta^* \\ \overline{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix},$$

where β^* here denotes the dual section to β of the dual bundle $\Lambda^{1,0}\xi$. Since $c\ell(\tau(\psi))$ is defined to be this endomorphism, it follows that

$$\Re \langle r\, c\ell(\tau(\psi))\psi, \alpha \rangle = \frac{r}{2}\left(|\alpha|^4 + |\alpha|^2 |\beta|^2 \right).$$

Meanwhile, the fifth term of $(*)$ can be resolved as

$$-\Re \left\langle \frac{r}{2}\, c\ell(i\lambda)\psi, \alpha \right\rangle = -\frac{r}{2}\,|\alpha|^2 .$$

At last, for the sixth term of $(*)$, one can write

$$\frac{s}{4}\,|\alpha|^2 = f_7(\alpha, \alpha)$$

for f_7 a bilinear form on $\mathcal{S}_\lambda$ independent of r. The result is then proven by declaring

$$E_{12.6} := f_3 + f_5, \quad E'_{12.6} := f_4, \quad E''_{12.6} := f_6 + f_7.$$

QED

Lemma 12.7: The squared norm of the component β of the spinor ψ satisfies the second order equation

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}|\beta|^2 + \frac{1}{2}\mathrm{d}^*\mathrm{d}|\beta|^2 + \left|\frac{\partial}{\partial t}\beta\right|^2 + |\nabla_A \beta|^2 + \frac{r}{2}(1 + |\alpha|^2 + |\beta|^2)|\beta|^2$$
$$+ E_{12.7}(\alpha, \beta) + E'_{12.7}(\nabla'_A \alpha, \beta) + E''_{12.7}(\beta, \beta) = 0.$$

For bilinear forms $E_{12.7}$, $E'_{12.7}$ and $E''_{12.7}$ on $\mathcal{S}_\lambda$ dependent only on Y, λ and the metric g.

Proof: The proof is analogous to that of Lemma 12.6. **QED**

Lemma 12.8: There exists a constant $K_{12.8} > 0$ such that, for any $K > K_{12.8}$, the function

$$v : \mathbf{R} \times Y \to \mathbf{R}, \quad v := 1 - |\alpha|^2 + r^{-1}K$$

satisfy the differential inequality

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}v + \mathrm{d}^*\mathrm{d}v + \frac{r}{2}|\alpha|^2 v \geq \left|\frac{\partial}{\partial t}\alpha\right|^2 + |\nabla_A \alpha|^2 + \frac{r}{2}|\alpha|^2|\beta|^2 + \left(\frac{K - K_{12.8}}{2}\right)|\alpha|^2 \\ - |\beta|^2 - |\nabla'_A \beta|^2 .$$

Proof: Since the bilinear forms $E_{12.6}$, $E'_{12.6}$ and $E''_{12.6}$, from Lemma 12.6 are all independent of t, one can take their supremum norms and define the desired constant to be

$$K_{12.8} := \max\left\{|E_{12.6}|^2, |E'_{12.6}|\right\} + 2|E''_{12.6}| .$$

To see that this constant satisfies the stated result, start with the following differential equation for the function v implied by Lemma 12.6.

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}v + \mathrm{d}^*\mathrm{d}v + \frac{r}{2}|\alpha|^2 v - \left|\frac{\partial}{\partial t}\alpha\right|^2 - |\nabla_A \alpha|^2 - \frac{r}{2}|\alpha|^2|\beta|^2 - \frac{K - K_{12.8}}{2}|\alpha|^2 \\ = \frac{K_{12.8}}{2}|\alpha|^2 + E_{12.6}(\alpha, \beta) + E'_{12.6}(\alpha, \nabla'_A \beta) + E''_{12.6}(\alpha, \alpha).$$

Now, using the supremum norms, bound the right hand side as follows.

$$\geq \left(\frac{K_{12.8}}{2} - |E''_{12.6}|\right)|\alpha|^2 - |E_{12.6}||\alpha||\beta| - |E'_{12.6}||\alpha||\nabla'_A \beta| .$$

By applying the arithmetic-geometric mean inequality twice, one then finds

$$\begin{aligned} &\geq - \frac{|E_{12.6}|^2}{K_{12.8} - 2|E''_{12.6}|}|\beta|^2 - \frac{|E'_{12.6}|^2}{K_{12.8} - 2|E''_{12.6}|}|\nabla'_A \beta|^2 \\ &\geq - \frac{\max\left\{|E_{12.6}|^2, |E'_{12.6}|^2\right\}}{K_{12.8} - 2|E''_{12.6}|}\left(|\beta|^2 + |\nabla'_A \beta|^2\right) \\ &= - |\beta|^2 - |\nabla'_A \beta|^2 . \end{aligned}$$

QED

Lemma 12.9: There exist constants $r_{12.9} > 0$ and $K_{12.9} > 0$ such that, for all $r > r_{12.9}$, the component β of the spinor satisfy

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}|\beta|^2 + \frac{1}{2}\mathrm{d}^*\mathrm{d}|\beta|^2 + \frac{r}{2}|\alpha|^2|\beta|^2 \leq - |\nabla_A \beta|^2 - \left|\frac{\partial}{\partial t}\beta\right|^2 - \frac{r}{2}|\beta|^4 - \frac{r}{4}|\beta|^2 \\ + \frac{K_{12.9}}{r}\left(|\alpha|^2 + |\nabla'_A \alpha|^2\right).$$

Proof: Start off by defining the desired constants as

$$r_{12.9} := 4\left|E_{12.7}\right|, \quad K_{12.9} := \max_{r>r_{12.9}}\left(\frac{r\max\left\{\left|E_{12.7}\right|^2, \left|E'_{12.7}\right|^2\right\}}{\frac{r}{2} - 2E''_{12.7}}\right).$$

Now, assuming $r > r_{12.9}$, consider the equation asserted by Lemma 12.7. By bounding with the supremum norms the terms involving the bilinear forms, one can state that

$$\begin{aligned}
&-\frac{1}{2}\frac{\partial^2}{\partial t^2}\left|\beta\right|^2 + \frac{1}{2}\mathrm{d}^*\mathrm{d}\left|\beta\right|^2 + \left|\frac{\partial}{\partial t}\beta\right|^2 + \left|\nabla_A\beta\right|^2 + \frac{r}{2}\left|\alpha\right|^2\left|\beta\right|^2 + \frac{r}{2}\left|\beta\right|^4 + \frac{r}{4}\left|\beta\right|^4 \\
\leq &-\frac{r}{4}\left|\beta\right|^2 + \left|E_{12.7}\right|\left|\alpha\right|\left|\beta\right| + \left|E'_{12.7}\right|\left|\nabla'_A\alpha\right|\left|\beta\right| + \left|E''_{12.7}\right|\left|\beta\right|^2.
\end{aligned}$$

Now, apply the arithmetic-geometric mean inequality twice to assert

$$\begin{aligned}
&\leq -\left(\frac{r}{4} - \left|E''_{12.7}\right|\right)\left|\beta\right|^2 + \left|E_{12.7}\right|\left|\alpha\right|\left|\beta\right| + \left|E'_{12.7}\right|\left|\nabla'_A\alpha\right|\left|\beta\right| \\
&\leq \left(\frac{\left|E_{12.7}\right|^2}{\frac{r}{2} - 2\left|E''_{12.7}\right|}\right)\left|\alpha\right|^2 + \left(\frac{\left|E'_{12.7}\right|^2}{\frac{r}{2} - 2\left|E''_{12.7}\right|}\right)\left|\nabla'_A\alpha\right|^2 \\
&\leq \frac{K_{12.9}}{r}\left(\left|\alpha\right|^2 + \left|\nabla'_A\alpha\right|^2\right).
\end{aligned}$$

QED

Lemma 12.10: There are constants $K_{12.10} > 0$ and $r_{12.10} > 0$ such that for any $r > r_{12.10}$ the following inequality hold.

$$\left|\beta\right|^2 \leq K_{12.10}\left(r^{-1}(1 - \left|\alpha\right|^2) + r^{-2}\right).$$

Proof: This is analogous to the second statement of Taubes (2009), Lemma 5.1. The details in this case are as follows. Start by declaring the constants to be

$$r_{12.10} := \max\left\{K_{12.9}, 2\sqrt{K_{12.9}}\right\} \quad K_{12.10} := K_{12.9}(K_{12.8} + 2).$$

Define the following functions.

$$\begin{aligned}
u : \mathbf{R}\times Y \to \mathbf{R}, &\quad u := 1 - \left|\alpha\right|^2 + \frac{K_{12.8} + 2}{r} \\
w : \mathbf{R}\times Y \to \mathbf{R}, &\quad w := \left|\beta\right|^2 - \left(\frac{K_{12.9}}{r}\right)u.
\end{aligned}$$

Combine Lemma 12.8 and Lemma 12.9 to assert a differential inequality for w of the form

$$\begin{aligned}
&-\frac{1}{2}\frac{\partial^2}{\partial t^2}w + \frac{1}{2}\mathrm{d}^*\mathrm{d}w + r\left|\alpha\right|^2 w \\
\leq &-\left|\nabla_A\beta\right|^2 - \left|\frac{\partial}{\partial\beta}\right|^2 - \frac{r}{2}\left|\beta\right|^4 - \frac{r}{4}\left|\beta\right|^2 + \frac{K_{12.9}}{r}\left(\left|\alpha\right|^2 + \left|\nabla'_A\alpha\right|^2\right) \\
&+ \frac{K_{12.9}}{r}\left(-\left|\frac{\partial}{\partial t}\alpha\right|^2 - \left|\nabla_A\alpha\right|^2 - \frac{r}{2}\left|\alpha\right|^2\left|\beta\right|^2 - \left|\alpha\right|^2 + \left|\beta\right|^2 + \left|\nabla'_A\beta\right|^2\right).
\end{aligned}$$

Neglecting some of the negative terms, one finds

$$\begin{aligned}&\leq -\left|\nabla_A\beta\right|^2 - \frac{r}{4}\left|\beta\right|^2 + \frac{K_{12.9}}{r}\left(\left|\alpha\right|^2 + \left|\nabla'_A\alpha\right|^2\right)\\&\quad + \frac{K_{12.9}}{r}\left(-\left|\nabla_A\alpha\right|^2 - \left|\alpha\right|^2 + \left|\beta\right|^2 + \left|\nabla'_A\beta\right|^2\right).\end{aligned}$$

Notice the key cancellation of the terms involving $|\alpha|^2$.

$$= -\left|\nabla_A\beta\right|^2 - \frac{r}{4}\left|\beta\right|^2 + \frac{K_{12.9}}{r}\left|\nabla'_A\alpha\right|^2 + \frac{K_{12.9}}{r}\left(-\left|\nabla_A\alpha\right|^2 + \left|\beta\right|^2 + \left|\nabla'_A\beta\right|^2\right).$$

Since $|\nabla'_A|\,\alpha \leq |\nabla_A\alpha|$ and $|\nabla'_A\beta| \leq |\nabla_A\beta|$, one finds

$$\leq -\left(1 - \frac{K_{12.9}}{r}\right)\left|\nabla_A\beta\right|^2 - \left(\frac{r}{4} - \frac{K_{12.9}}{r}\right)\left|\beta\right|^2.$$

From the definition of $r_{12.10}$, it follows that both of these terms are negative. Therefore, the following differential inequality has been established.

$$-\frac{1}{2}\frac{\partial^2}{\partial t^2}w + \frac{1}{2}\mathrm{d}^*\mathrm{d}w + \frac{r}{2}\left|\alpha\right|^2 w \leq 0. \tag{$*$}$$

This implies that the function w cannot admit a *positive local maximum* anywhere in $\mathbf{R}\times Y$. Consider two cases. Firstly, suppose that the trajectory be constant; that is, that α and β be constant in the parameter t. Because Y is compact, it follows that $w \leq 0$ everywhere in Y. Secondly, suppose that the trajectory not be constant; that is, α and β may depend on the parameter t. Here, the limits $\lim_{t\to\pm\infty}(\alpha,\beta)$ define constant trajectories as in the previous case; hence, the limits $\lim_{t\to\pm\infty} w$ are everywhere non-positive, and, combined with the fact that w does not admit positive local maxima, this implies that $w \leq 0$ everywhere in $\mathbf{R}\times Y$. Expanding the definition of w, one finds

$$\left|\beta\right|^2 \leq K_{12.9}r^{-1}(1+\left|\alpha\right|^2) + r^{-2}K_{12.9}(K_{12.8}+2) \leq K_{12.10}\left(r^{-1}(1+\left|\alpha\right|^2) + r^{-2}\right).$$

QED

The next goal is to derive a similar sort of bound for the curvature F_A and the derivative $\frac{\partial}{\partial t}A$. It shall prove advantageous to work with objects over $\mathbf{R}\times Y$ rather than time dependent objects over Y. For that end, introduce $\hat{\mathrm{d}}$ and $\hat{*}$ to denote, respectively, the exterior derivative and the Hodge operator over $\mathbf{R}\times Y$. The Spin$^{\mathbf{C}}$ structure $\mathfrak{s}_\lambda$ over Y also defines a Spin$^{\mathbf{C}}$ structure over $\mathbf{R}\times Y$, which shall be denoted $\hat{\mathfrak{s}}_\lambda$ and shall be described next. Let $\pi : \mathbf{R}\times Y \to Y$ be the projection. The spinor bundles of $\hat{\mathfrak{s}}_\lambda$ are both defined as $\hat{\mathcal{S}}^{\pm}_\lambda := \pi^*\mathcal{S}_\lambda$. Meanwhile, the Clifford multiplication map $\hat{c\ell} : \mathrm{T}(\mathbf{R}\times Y) \to \mathrm{End}_{\mathbf{C}}(\hat{\mathcal{S}}^+_\lambda \oplus \hat{\mathcal{S}}^-_\lambda)$ is defined by requiring

$$\hat{c\ell}\left(\frac{\partial}{\partial t}\right) = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and, for $v \in \mathrm{T}Y$, requiring

$$\hat{c\ell}(v) = \begin{pmatrix} 0 & -c\ell(v)^* \\ c\ell(v) & 0 \end{pmatrix}.$$

The time dependent connexion A over $\det \mathfrak{s}_\lambda$ also defines a connexion on the bundle $\det \hat{\mathfrak{s}}_\lambda$; denote this connexion by $\hat{A}$ and note that it is characterized by requiring its induced covariant derivative to be

$$\nabla_{\hat{A}} := \mathrm{d}t \otimes \frac{\partial}{\partial t} + \nabla_A.$$

Hence, one sees that its curvature is given by

$$F_{\hat{A}} = \mathrm{d}t \wedge \frac{\partial}{\partial t} A + F_A.$$

One can then define the pair of Dirac operators

$$\mathfrak{D}^\pm_{\hat{A}} : \Gamma \hat{\mathcal{S}}^\pm_\lambda \to \Gamma \hat{\mathcal{S}}^\mp_\lambda, \quad \mathfrak{D}^\pm_{\hat{A}} = \hat{c\ell} \circ \nabla_{\hat{A}}.$$

Next, let $\Lambda^\pm \mathrm{T}^*(\mathbf{R} \times Y) \subset \Lambda^2 \mathrm{T}^*(\mathbf{R} \times Y)$ denote the ± 1-eigenbundle of $\hat{*}$, and introduce the quadratic map

$$\hat{\tau} : \hat{\mathcal{S}}^+_\lambda \to i\Lambda^+ \mathrm{T}^*(\mathbf{R} \times Y),$$

defined by requiring

$$\hat{c\ell}(\hat{\tau}(\phi)) = \phi \otimes \phi^* - \frac{1}{2} \langle \phi, \phi \rangle \,\mathrm{id} \in \mathrm{End}_{\mathbf{C}}(\hat{\mathcal{S}}^+_\lambda).$$

That this uniquely defines $\hat{\tau}$ is a standard fact; vid. e.g. Kronheimer & Mrowka (2007), §1.1.

Proposition 12.11: The four dimensional connexion $\hat{A}$ and the spinor ψ seen as a spinor over $\mathbf{R} \times Y$ obey the four-dimensional Seiberg-Witten equations

$$\frac{1}{2}\left(F^+_{\hat{A}} - F^+_{\hat{A}_\lambda}\right) - r\hat{\tau}(\psi) + ir\mathrm{d}\lambda = 0, \quad \mathfrak{D}^+_{\hat{A}}\psi = 0.$$

Proof: This is standard; vid. e.g. Kronheimer & Mrowka (2007), §4.3. **QED**

The bounds on the curvature shall be obtained in two steps. Firstly, one can read off a bound for the self-dual part, $F^+_{\hat{A}}$, by simply using the Seiberg-Witten equations as noted in the next lemma. Secondly, one must work considerably harder to obtain a bound on the anti-self-dual part, $F^-_{\hat{A}}$.

Lemma 12.12: There exist constants $r_{12.12} > 0$ and $K_{12.12} > 0$ such that, for $r > r_{12.12}$,

$$|F^+_{\hat{A}}| \leq rK_{12.12}.$$

Proof: By the four-dimensional Seiberg-Witten equations (Proposition 12.11), one has

$$|F^+_{\hat{A}}| \leq 2r\,|\hat{\tau}(\psi)| + |F^+_{\hat{A}_\lambda}| + 2r\,|\mathrm{d}\lambda|\,.$$

Hence, the result follows by applying Lemma 12.5. **QED**

The strategy to obtain the bound on the anti-self-dual part of the curvature shall start by seeking a second order differential inequality in a manner similar to what was done for the spinor.

Lemma 12.13: The anti-self-dual part of the curvature satisfies the following second order equation.

$$\nabla^*_{\widehat{\mathrm{LC}}}\nabla_{\widehat{\mathrm{LC}}}F^-_{\hat{A}} + E_{12.13}(F^-_{\hat{A}}) = -\Pi^-\left(2r\hat{*}\hat{\mathrm{d}}\hat{\mathrm{d}}^*\hat{\tau}(\psi)\right),$$

where $E_{12.13}$ is some bundle endomorphism of $i\Lambda^-\mathrm{T}^*(\mathbf{R}\times Y)$ independent of r and the t coordinate function; $\Pi^- : \Lambda^2\mathrm{T}^*(\mathbf{R}\times Y) \to \Lambda^-\mathrm{T}^*(\mathbf{R}\times Y)$ denotes the projection with respect to the splitting $\Lambda^2\mathrm{T}^*(\mathbf{R}\times Y) \cong \Lambda^+\mathrm{T}^*(\mathbf{R}\times Y) \oplus \Lambda^-\mathrm{T}^*(\mathbf{R}\times Y)$, and $\nabla_{\widehat{\mathrm{LC}}}$ denotes the Levi-Civita connexion of $\mathbf{R}\times Y$.

Proof: By the second Bianchi identity and the fact that $\hat{\tau}(\psi)$ is self-dual, one has

$$\Pi^-\hat{\mathrm{d}}^*\hat{\mathrm{d}}F^-_{\hat{A}} = -\Pi^-\hat{\mathrm{d}}^*\hat{\mathrm{d}}F^+_{\hat{A}} = -\Pi^-\left(\hat{*}\hat{\mathrm{d}}\hat{\mathrm{d}}^*\hat{\tau}(\psi)\right), \tag{$*$}$$

Also note that

$$\Pi^-\hat{\mathrm{d}}^*\hat{\mathrm{d}}F^-_{\hat{A}} = (\hat{\mathrm{d}}^*\hat{\mathrm{d}} + \hat{\mathrm{d}}\hat{\mathrm{d}}^*)F^-_{\hat{A}}.$$

The well known Weitzenböck formula relating the connexion Laplacian and the Beltrami Laplacian provides a bundle homomorphism $E_{12.13}$ depending solely on the curvature of $\mathbf{R}\times Y$ such that

$$\Pi^-\hat{\mathrm{d}}^*\hat{\mathrm{d}}F^-_{\hat{A}} = \nabla^*_{\widehat{\mathrm{LC}}}\nabla_{\widehat{\mathrm{LC}}}F^-_{\hat{A}} + E_{12.13}(F^-_{\hat{A}}).$$

Combining this with ($*$) and the four-dimensional Seiberg-Witten curvature equation (Proposition 12.11) finishes the proof. **QED**

In order to make use of this equation, it is necessary to understand what the right-hand side says. For that end, the next two lemmata consider the quadratic map $\hat{\tau}$ in more detail.

Lemma 12.14: The differential 2-form $\hat{\tau}(\psi)$ may be alternatively described as

$$\hat{\tau}(\psi)(v,w) = \frac{1}{8}\left\langle \psi, [\hat{c\ell}(v), \hat{c\ell}(w)]\psi \right\rangle.$$

Proof: The proof shall be omitted as it is a tedious but straightforward calculation using the generators and relations of the Clifford algebra of $\mathbf{C}^4$. Cf. Nicolaescu (2000), Example 1.3.3 and Exercise 1.3.2. **QED**

Remark 12.15: Recall that $\mathfrak{D}^+_{\hat{A}}\psi = 0$.

Lemma 12.16: The following holds.

$$\hat{\mathrm{d}}^*\hat{\tau}(\psi) = \frac{i}{4}\Im\left\langle \nabla_{\hat{A}}\psi, \psi\right\rangle.$$

Proof: Let $I : \Lambda^k \mathrm{T}^*(\mathbf{R}\times Y)\otimes \mathrm{T}^*(\mathbf{R}\times Y) \to \Lambda^{k-1}\mathrm{T}^*(\mathbf{R}\times Y)$ be the trace map induced by the metric. Hence, one can express adjoint of the exterior the derivative as

$$\mathrm{d}^*\hat{\tau}(\psi) = I \circ \nabla_{\widehat{\mathrm{LC}}}\tau(\psi).$$

Next, use the expression of $\hat{\tau}(\psi)$ given by Lemma 12.14 to assert

$$\begin{aligned}
&8\left(\nabla_{\widehat{\mathrm{LC}}}\hat{\tau}(\psi)\right)(u,v)(w)\\
=&\left\langle (\nabla_{\hat{A}}\psi)(w), [\hat{c\ell}(u), \hat{c\ell}(v)]\psi\right\rangle + \left\langle \psi, [\hat{c\ell}(u), \hat{c\ell}(v)](\nabla_{\hat{A}}\psi)(w)\right\rangle.
\end{aligned}$$

Fix an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathrm{T}(\mathbf{R}\times Y)$ at a point p in order to compute the trace. Using the assumption that $\mathfrak{D}^+_{\hat{A}}\psi = 0$, one computes

$$\begin{aligned}
&8\left(I\circ\nabla_{\widehat{\mathrm{LC}}}\hat{\tau}(\psi)\right)(v)|_p\\
=&\sum_{i=1}^4\left(\left\langle (\nabla_{\hat{A}}\psi)(e_i), [\hat{c\ell}(e_i), \hat{c\ell}(v)]\psi\right\rangle + \left\langle \psi, [\hat{c\ell}(e_i), \hat{c\ell}(v)](\nabla_{\hat{A}}\psi)(e_i)\right\rangle\right)\\
=&\sum_{i=1}^4\left\langle (\nabla_{\hat{A}}\psi)(e_i), 2(\hat{c\ell}(e_i)\hat{c\ell}(v) + \langle v, e_i\rangle)\psi\right\rangle\\
&-\sum_{i=1}^4\left\langle \psi, 2(\hat{c\ell}(v)\hat{c\ell}(e_i) + \langle e_i, v\rangle)(\nabla_{\hat{A}}\psi)(e_i)\right\rangle\\
=&-\sum_{i=1}^4\left\langle \hat{c\ell}(e_i)(\nabla_{\hat{A}}\psi)(e_i), 2\hat{c\ell}(v)\psi\right\rangle + 2\left\langle \nabla_{\hat{A}}\psi, \psi\right\rangle\\
&-\left\langle \psi, 2\hat{c\ell}(v)(\mathfrak{D}^+_{\hat{A}}\psi)\right\rangle - 2\left\langle \psi, \nabla_{\hat{A}}\psi\right\rangle\\
=&-\left\langle \mathfrak{D}^+_{\hat{A}}\psi, 2\hat{c\ell}(v)\psi\right\rangle + 2\left\langle \nabla_{\hat{A}}\psi, \psi\right\rangle - 2\left\langle \psi, \nabla_{\hat{A}}\psi\right\rangle\\
=&2\left\langle \nabla_{\hat{A}}\psi, \psi\right\rangle - 2\left\langle \psi, \nabla_{\hat{A}}\psi\right\rangle
\end{aligned}$$

QED

Lemma 12.17: There exists a constant $K_{12.17} > 0$ such that

$$\frac{1}{2}\hat{\mathrm{d}}^*\hat{\mathrm{d}}|F^-_{\hat{A}}| + \frac{r}{4}|\psi|^2|F^-_{\hat{A}}| \le K_{12.17}\left(|F^-_{\hat{A}}| + r\left(\left|\nabla_{\hat{A}}\psi\right|^2 + |\psi|^2\right)\right).$$

Proof: Use $\hat{\mathrm{d}}_{\hat{A}}$ to denote the exterior covariant derivative on $\hat{\mathcal{S}}_\lambda$ induced by $\nabla_{\hat{A}}$. Then, one has that

$$\hat{\mathrm{d}}^2_{\hat{A}} = \frac{1}{2}F_{\hat{A}} + f_1,$$

where f_1 is some $\mathrm{End}(\hat{\mathcal{S}}_\lambda)$-valued 2-form dependent only on the curvature of Y. Hence, Lemma 12.16 implies that

$$\hat{*}\hat{\mathrm{d}}\hat{\mathrm{d}}^*\hat{\tau}(\psi) = \frac{1}{8}F_{\hat{A}}|\psi|^2 + f_2(\nabla_A\psi, \nabla_A\psi) + f_3(\psi,\psi),$$

where $f_2, f_3 : \hat{\mathcal{S}}_\lambda \otimes \hat{\mathcal{S}}_\lambda \to \Lambda^2\mathrm{T}^*(\mathbf{R}\times Y)$ are bundle homomorphisms dependent only on the metric of Y and the inner product of $\mathcal{S}_\lambda$. Now, incorporating this into the equation of Lemma 12.13, yields an equation of the form

$$\nabla^*_{\widehat{\mathrm{LC}}}\nabla_{\widehat{\mathrm{LC}}}F^-_{\hat{A}} + \frac{r}{4}F^-_{\hat{A}}|\psi|^2 = -E_{12.13}(F^-_{\hat{A}}) + f_4(\nabla_A\psi,\nabla_A\psi) + f_5(\psi,\psi),$$

where f_4 and f_5 are again bundle homomorphisms dependent only on the metric of Y, and the inner product of $\mathcal{S}_\lambda$. The result then follows by taking the inner product of both sides with $F^-_{\hat{A}}$, dividing by $|F^-_{\hat{A}}|$ and neglecting a variety of positive terms from the left hand side in order to obtain the claimed inequality. **QED**

Lemma 12.18: There exist constants $K_{12.18} > 0$, and $r_{12.18} > 0$ such that, for $r > r_{12.18}$, the function

$$q : \mathbf{R}\times Y \to \mathbf{R}, \quad q := |F^-_{\hat{A}}| + \frac{r}{4}K_{12.17}(|\psi|^2 - 1) - K_{12.18}$$

satisfy the second order inequality

$$\frac{1}{2}\hat{\mathrm{d}}^*\hat{\mathrm{d}}q + \frac{r}{4}|\psi|^2 q \le K_{12.17}|F^-_{\hat{A}}|$$

Proof: Define the constants to be

$$K_{12.18} := (1 + K_{12.17})^2, \quad r_{12.18} := 1.$$

Combining Lemma 12.17 with Lemma 12.4, one finds that

$$\begin{aligned}\frac{1}{2}\hat{\mathrm{d}}^*\hat{\mathrm{d}}q + \frac{r}{4}|\psi|^2 q \le & K_{12.17}\left(|F^-_{\hat{A}}| + r\left(\left|\nabla_{\hat{A}}\psi\right|^2 + |\psi|^2\right)\right) \\ & - rK_{12.17}\left|\nabla_{\hat{A}}\psi\right|^2 + rK^2_{12.4}|\psi|^2 - rK_{12.18} \\ \le & K_{12.17}\left(|F^-_{\hat{A}}| + r(1+K_{12.17})|\psi|^2\right) - rK_{12.18}.\end{aligned}$$

Invoke Lemma 12.5 to bound $|\psi|^2$ and assert that

$$\begin{aligned}&\le K_{12.17}\left(|F^-_{\hat{A}}| + r(1+K_{12.18})(1+K_{12.5}r^{-1})\right) - rK_{12.18} \\ &\le K_{12.17}|F^-_{\hat{A}}|.\end{aligned}$$

QED

Given just this differential inequality, it is not possible to directly obtain the pointwise bounds desired. It shall also be needed to obtain an L^2 bound for the curvature as shall be described next.

Lemma 12.19:

$$\int_{\mathbf{R}\times Y} \hat{*}\left(\left|\frac{\partial}{\partial t}A\right|^2 + \left|\frac{1}{2}*(F_A - F_{A_\lambda}) + r\tau(\psi) - \frac{ir}{2}\lambda\right|^2 + 2r\left|\frac{\partial}{\partial t}\psi\right|^2 + 2r\left|\mathfrak{D}_A\psi\right|^2\right)$$
$$= \mathrm{CSD}_{\lambda,r}\left(\lim_{t\to-\infty}(A,\psi)\right) - \mathrm{CSD}_{\lambda,r}\left(\lim_{t\to+\infty}(A,\psi)\right).$$

Proof: Follows from the well known fact that the Seiberg-Witten equations are the equations of the downward gradient flow of the Chern-Simons-Dirac functional (vid. Kronheimer & Mrowka 2007, §4.3). In the present context, with the canonical perturbations at use, it is easy to check that the flow induced by $\mathcal{X}_{\lambda,r}$ is the downward gradient flow of $\mathrm{CSD}_{\lambda,r}$. **QED**

In view of this last lemma, the goal shall be to bound the value of the Chern-Simons-Dirac functional for the end points of the trajectory. This shall be done by bounding all of its terms in turn; cf. Definition 3.24.

Lemma 12.20: There exist constants $K_{12.20} > 0$ and $r_{12.20} > 0$ such that, for any $r > r_{12.20}$,

$$\left|\int_Y \lambda \wedge \lim_{t\to\pm\infty}(F_A)\right| \leq rK_{12.20}.$$

Proof: Since the limit configurations $\lim_{t\to\pm\infty}\gamma$ satisfy the three dimensional Seiberg-Witten equations, one has

$$\left|\lim_{t\to\pm\infty} F_A\right| \leq |F_{A_\lambda}| + 2r\left|\tau\left(\lim_{t\to\pm\infty}\psi\right)\right| + r$$

The result then follows by applying Lemma 12.5 to bound the term involving the spinor and integrating. **QED**

Lemma 12.21: There exists a constant $K_{12.21} > 0$ such that

$$\left|\int_Y \lim_{t\to\pm\infty}(a \wedge \mathrm{d}a)\right| \leq K_{12.21}r^2.$$

Proof: Without loss of generality, suppose the limit to be $t \to +\infty$. Let $B = A_\lambda + b$ be a connexion on $\det \mathfrak{s}_\lambda$ which be gauge equivalent to $\lim_{t\to\infty} A$ and also satisfy $\mathrm{d}^*b = 0$. Likewise, let $\phi = \lim_{t\to\infty}\psi$ and write $\eta := -2*\tau(\phi) + i*\lambda$. By the three-dimensional Seiberg-Witten equations, one sees that b satisfies

$$\mathrm{d}b = r\eta, \quad \mathrm{d}^*b = 0.$$

Since the operator $\mathrm{d} + \mathrm{d}^*$ is elliptic, the following standard elliptic estimate holds (vid. Nicolaescu 2000, Theorem 1.2.18 (v)).

$$\|b - Pb\|_{\mathrm{L}^2_1} \leq K \|r\eta\|_{\mathrm{L}^2_0},$$

where P denotes the projection onto the kernel of $d : \Omega^1(Y) \cap \operatorname{Ker} \mathrm{d}^* \to \Omega^2(Y)$. Recall also that $b_1(Y) = 0$; hence, by the Hodge theorem, Pb is a harmonic 1-form, which must vanish. Meanwhile, by Lemma 12.5, one can bound the norm of η to assert

$$\|b\|_{\mathrm{L}^2_1} \leq K' r$$

for some constant $K' > 0$. This last inequality then implies that

$$\left| \int_Y b \wedge \mathrm{d}b \right| \leq K'' r^2.$$

Setting $K_{12.21} := K''$, the stated result follows from the fact that the Chern-Simons functional is gauge invariant when $b_1(Y) = 0$ (cf. Kronheimer & Mrowka 2007, Lemma 4.1.3). **QED**

Lemma 12.22: There exists a constant $K_{12.22} > 0$ such that

$$\mathrm{CSD}_{\lambda,r} \left(\lim_{t \to \pm\infty} (A, \psi) \right) \leq K_{12.22} r^2.$$

Proof: Vid. Definition 3.24; cf. Lemma 12.20 and Lemma 12.21. **QED**

Lemma 12.23: There exist constants $K_{12.23} > 0$ and $r_{12.23} > 0$ such that, for $r > r_{12.23}$ and any $t \in \mathbf{R}$,

$$\int_{[t,t+1] \times Y} \hat{*} \left(\left| \frac{\partial}{\partial t} A \right|^2 + |F_A|^2 \right) \leq K_{12.23} r^2.$$

Proof: Combining Lemma 12.19 with Lemma 12.22 and ignoring a few positive terms on the left hand side, one asserts that

$$\int_{\mathbf{R} \times Y} \hat{*} \left(\left| \frac{\partial}{\partial t} A \right|^2 + \left| \frac{1}{2} * (F_A - F_{A_\lambda}) + r\tau(\psi) - \frac{ir}{2} \lambda \right|^2 \right) \leq K_{12.22} r^2.$$

Meanwhile, using the triangle inequality, the arithmetic-geometric mean inequality and Lemma 12.5, one finds that there exists a constant $K > 0$ such that, for $r > r_{12.23} := 1$,

$$\left| \frac{1}{2} * (F_A - F_{A_\lambda}) + r\tau(\psi) - \frac{ir}{2} \lambda \right|^2 \geq \frac{1}{8} |F_A|^2 - K r^2.$$

The result then follows by combining these two inequalities and setting $K_{12.23} := 8 \max\{K, K_{12.22}\}$. **QED**

Lemma 12.24: There exist constants $K_{12.24} > 0$ and $r_{12.24} > 0$ such that, for $r > r_{12.24}$,

$$|F^-_{\hat{A}}| \leq K_{12.24} r.$$

Proof: Let p be any point of $\mathbf{R} \times Y$. Recall the function q from Lemma 12.18. The goal shall be to establish a bound on q and thence derive the bound on $|F^-_{\hat{A}}|$. Assume firstly that $q(p) \geq 0$. Use G_p to denote the Green's function with pole at p of the operator $\hat{\mathrm{d}}^* \hat{\mathrm{d}}$ acting on $\Omega^0(\mathbf{R} \times Y)$. Use $\mathrm{dist}_p : \mathbf{R} \times Y \to \mathbf{R}$ to denote the function whose value at x is the geodesic distance between p and x. Note that there exists some constant $C_0 > 0$, independent of p, such that,

$$0 \leq G_p \leq C_0 \mathrm{dist}_p^{-2}, \quad \left| \hat{\mathrm{d}} G_p \right| \leq C_0 \mathrm{dist}_p^{-3}.$$

Let $\rho > 0$ be some number smaller than the injectivity radius of Y and small enough so that q be positive on a ball of radius ρ centred at p. Let $\chi : [0, \infty) \to [0, 1]$ denote some smooth monotonic function satisfying $\chi|_{[0,\rho/4]} = 1$ and $\chi|_{[\rho/2,\infty)} = 0$; one must make sure to use the same function χ for all p. Then, set $\chi_p : \mathbf{R} \times Y \to \mathbf{R}$ to be $\chi_p := \chi \circ \mathrm{dist}_p$. Let $t_0 \in \mathbf{R}$ be such that the t coordinate of the point p be $t_0 + 1/2$. Use I to denote the interval $[t_0, t_0 + 1]$. Now, multiply both sides of the inequality asserted in Lemma 12.18 by $\chi_p G_p$ and integrate to find that

$$\frac{1}{2} \int_{I \times Y} \hat{*} \chi_p G_p \hat{\mathrm{d}}^* \hat{\mathrm{d}} q \leq K_{12.17} \int_{I \times Y} \hat{*} \chi_p G_p |F^-_{\hat{A}}| - \int_{I \times Y} \hat{*} \chi_p G_p \frac{r}{4} |\psi|^2 q. \qquad (*)$$

Consider firstly the left-hand side of this inequality. Integration by parts and two applications of the Cauchy-Schwarz inequality yield

$$\begin{aligned} &\frac{1}{2} q(p) + \frac{1}{2} \int_{I \times Y} \hat{*} \left\langle \hat{\mathrm{d}} \chi_p, \hat{\mathrm{d}} G_p \right\rangle q + \frac{1}{2} \int_{I \times Y} \hat{*} \hat{\mathrm{d}}^* \hat{\mathrm{d}} \chi_p G_p q. \\ \geq &\frac{1}{2} q(p) - C_1 \left(\int_{I \times Y} \hat{*} q^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where $C_1 > 0$ is a constant dependent only on C_0, ρ and the L^2_2 norm of χ_p. Since χ was chosen to be the same for all p, the constant C_1 may also be chosen independently of p. Next, consider the first term on the right-hand side of $(*)$. By applying the Cauchy-Schwarz inequality and Lemma 12.23, one finds

$$K_{12.17} \int_{I \times Y} \hat{*} \chi_p G_p |F^-_{\hat{A}}| \leq K_{12.17} C_2 \left(\int_{I \times Y} |F^-_{\hat{A}}|^2 \right)^{\frac{1}{2}} \leq r K_{12.17} C_2 \sqrt{K_{12.23}},$$

where $C_2 > 0$ is again a constant dependent only on C_0, ρ and $\|\chi_p\|_{\mathrm{L}^2_2}$. Now, consider the second term on the right-hand side of $(*)$. Note that, by the assumptions made above, one has $q > 0$ everywhere on the support of χ_p; hence,

$$-\int_{I\times Y} \hat{*}\chi_p G_p \frac{r}{4}\,|\psi|^2\, q \le 0.$$

At last, notice that by virtue of Lemma 12.23 and Lemma 12.5, one can bound the L^2 norm of q as

$$\left(\int_{I\times Y} \hat{*} q^2\right)^{\frac{1}{2}} \le rC_4,$$

for some constant C_4 depending only on $K_{12.23}$ and $K_{12.5}$. Incorporating all that has been said so far into $(*)$ reveals

$$q(p) \le rC_5.$$

where $C_5 := \max\{C_1, K_{12.17}C_2\sqrt{K_{12.23}}, C_4\}$. Recall that, initially, it was assumed that $q(p) \ge 0$; however, in the event that $q(p) \le 0$, one of course also has $q(p) \le rC_5$. Therefore, by expanding the definition of q (vid. Lemma 12.18), one finds

$$|F^-_{\hat{A}}| \le rC_5 + K_{12.18} + rK_{12.4}(1 - |\psi|^2).$$

The result is herefore proven if one apply Lemma 12.5 and set the constants to be

$$r_{12.24} := 1, \quad K_{12.24} := \max\{C_5, K_{12.4}, K_{12.18}\}.$$

QED

Lemma 12.25: There is a constant $K_{12.25} > 0$ such that

$$\left|\frac{\partial}{\partial t}A\right| + |F_A| \le K_{12.25}r.$$

Proof: Recall that

$$\left|\frac{\partial}{\partial t}A\right|^2 + |F_A|^2 = |F^+_{\hat{A}}|^2 + |F^-_{\hat{A}}|^2.$$

Hence, the result follows directly by combining Lemma 12.12 and Lemma 12.24.

QED

Henceforth, add the assumption that

$$\lim_{t\to+\infty} \gamma = C_\lambda.$$

The goal shall be bound $|\alpha|^2$ *away* from zero in the same manner as is the case for C_λ but for the entirety of the trajectory.

For that end, start by introducing the symplectic form $\omega := e^{2t}(\mathrm{d}t \wedge \lambda + *\lambda)$, on $\mathbf{R} \times Y$ and fixing some almost complex structure J compatible with ω. Also fix a non-decreasing C^∞ function $\sigma : [0,\infty) \to [0,1]$ satisfying $\sigma|_{[0,1/2]} = 0$ and $\sigma|_{[1,\infty)} = 1$. Use σ' to denote its derivative. Assume further that $\sigma' : \mathbf{R} \to [0,3]$.

Definition 12.26: Given $\delta > 0$, denote $\sigma_\delta : \mathbf{R} \times Y \to [0,1]$ the function

$$(t,y) \mapsto \sigma\left(\delta^{-1}\left(1 - |\alpha|^2\right)\right).$$

Definition 12.27: Given $\delta > 0$, denote $\sigma'_\delta : \mathbf{R} \times Y \to [0,1]$ the function

$$(t,y) \mapsto \sigma'\left(\delta^{-1}\left(1 - |\alpha|^2\right)\right).$$

In what follows, recall the covariant derivative, which was used earlier,

$$\nabla'_A := \frac{1}{2}(1 + i\, c\ell(\lambda))\nabla_A$$

defined on the summand $\Lambda^{0,0}\mathrm{T}^*Y$ of the spinor bundle $\mathcal{S}_\lambda$ This induces a covariant derivative on the trivial complex line bundle over $\mathbf{R} \times Y$ defined by

$$\nabla'_{\hat{A}} := \mathrm{d}t \otimes \frac{\partial}{\partial t} + \nabla'_A.$$

This covariant derivative can then be extended to a covariant exterior derivative on complex differential forms, which shall be denoted

$$\hat{\mathrm{d}}_{\hat{A}} : \Omega^k(\mathbf{R} \times Y) \otimes \mathbf{C} \to \Omega^{k+1}(\mathbf{R} \times Y) \otimes \mathbf{C}.$$

Definition 12.28: Given $\delta > 0$, define a 2-form $\wp_\delta \in \Omega^2(\mathbf{R} \times Y)$ by the formula

$$\wp_\delta := \frac{1}{\delta}\sigma'_\delta \cdot \hat{\mathrm{d}}_{\hat{A}}\alpha \wedge \hat{\mathrm{d}}_{\hat{A}}\overline{\alpha} + \sigma_\delta \cdot F_{\hat{A}}.$$

Remark 12.29: Notice that the form $\wp_\delta$ is closed.

Lemma 12.30: For all $\delta \in (0, \delta_{2.24})$, there exists $s_\delta \in \mathbf{R}$, such that, the 2-form $\wp_\delta$ vanishes identically on $[s_\delta, \infty) \times Y \subset \mathbf{R} \times Y$.

Proof: By Theorem 2.24 and the fact that $\lim_{t\to\infty} \gamma = C_\lambda$ in the C^0 topology, it follows that there exists $s_\delta \in \mathbf{R}$ such that, for $t > s_\delta$, $|\psi(t)| \geq 1 - \delta_{2.24}$. Since the function σ is supported on $[1/2, \infty)$, the claimed result follows. **QED**

Lemma 12.31: For $r > r_{2.24}$ and $\delta \in (0, \delta_{2.24})$, $\omega \wedge \wp_\delta$ is integrable over all of $\mathbf{R} \times Y$.

Proof: Notice that there is some $K > 0$, which may well depend on r, A or ψ here, such that, for sufficiently negative $s < 0$, $|\omega \wedge \wp_\delta|$ restricted to $(-\infty, s)$ is no greater than Ke^s. This is a consequence of the fact that the limit $\lim_{t\to-\infty} \gamma(t)$ is well defined in the Sobolev norm L^2_5 and, therefore, also in C^2 by Sobolev embedding. Together with Lemma 12.30, this implies integrability. **QED**

Lemma 12.32: Provided $r > r_{2.24}$ and $\delta \in (0, \delta_{2.24})$, it follows $\int_{\mathbf{R}\times Y} \omega \wedge \wp_\delta = 0$.

Proof: Note that ω is exact and $\wp_\delta$ is closed and integrate by parts. **QED**

Introduce the notation

$$\partial_{\hat{A}} : \Omega^{i,j}(\mathbf{R}\times Y) \to \Omega^{i+1,j}(\mathbf{R}\times Y), \quad \overline{\partial}_{\hat{A}} : \Omega^{i,j}(\mathbf{R}\times Y), \to \Omega^{i,j+1}(\mathbf{R}\times Y),$$

for the covariant Cauchy-Riemann operators associated to the connexion $\hat{A}$ and the almost complex structure J. It can be verified that, for $\eta \in \Omega^{0,0}(\mathbf{R}\times Y)$, they take the form

$$\partial_{\hat{A}}\eta := \frac{1}{2}\left(\hat{\mathrm{d}}_{\hat{A}}\eta + ie^{-2t}\hat{*}(\omega \wedge \hat{\mathrm{d}}_{\hat{A}}\eta)\right), \quad \overline{\partial}_{\hat{A}}\,\eta := \frac{1}{2}\left(\hat{\mathrm{d}}_{\hat{A}}\eta - ie^{-2t}\hat{*}(\omega \wedge \hat{\mathrm{d}}_{\hat{A}}\eta)\right).$$

Lemma 12.33: Vanishing of the integral in Lemma 12.32 amounts to saying

$$\int_{\mathbf{R}\times Y} \hat{*}e^{2t}\left(\delta^{-1}\sigma'_\delta\left(|\partial_{\hat{A}}\alpha|^2 - |\,\overline{\partial}_{\hat{A}}\,\alpha|^2\right) + r\sigma_\delta\left(1 - |\alpha|^2 + |\beta|^2\right)\right) = 0.$$

Proof: Can be checked directly by expanding with the expressions given above for $\partial_{\hat{A}}$ and $\overline{\partial}_{\hat{A}}$ acting on 0-forms and by using the four-dimensional Seiberg-Witten curvature equation (Proposition 12.11). **QED**

Lemma 12.34: (cf. Taubes 2009, Lemma 5.13) For a given $\delta \in (0, \delta_{2.24})$, there exist $r_{12.34} > 0$ and $K_{12.34} > 0$ such that, for all $r > r_{12.34}$,

$$|\,\overline{\partial}_{\hat{A}}\,\alpha| \leq K_{12.34}.$$

Proof: It is well known (cf. Nicolaescu 2000, §1.4.3), that, in terms of the notation above, the Dirac equation,

$$\mathfrak{D}^+_{\hat{A}}\psi = 0,$$

takes the more familiar form

$$\overline{\partial}_{\hat{A}}\,\alpha + \overline{\partial}^*_{\hat{A}}\,\beta = f_1(\psi),$$

where f_1 is some bundle homomorphism independent of r. Hence, a bound on $\overline{\partial}_{\hat{A}}\,\alpha$ of the variety required follows from a bound on $\frac{\partial}{\partial t}\beta$, $\nabla_A\beta$ and $|\psi|$. The bound on $|\psi|$ was already provided by Lemma 12.5. The rest of the proof shall concern itself with the other two bounds.

Let $r_{12.34}$ be large enough so that the injectivity radius of Y be strictly larger than $(r_{12.34})^{-1/2}$ and so that the bundle $\Lambda^{0,2}\mathrm{T}^*(\mathbf{R}\times Y)$ be trivial when restricted to any ball of radius $(r_{12.34})^{-1/2}$ in $\mathbf{R} \times Y$. Then, let $r > r_{12.34}$, and fix a point $p \in \mathbf{R} \times Y$. Introduce $\phi_{p,r}$ to denote the Gaussian chart centred at p rescaled so that the ball of radius 1 of $\mathbf{R}^4$ be mapped to the geodesic ball of radius $r^{-1/2}$. Now, let $\tilde{\beta} : \mathrm{B}(\mathbf{R}^4, 1) \to \mathbf{C}$ denote the pullback of β via this chart seen as a complex valued function by trivialising the bundle $\Lambda^{0,2}\mathrm{T}^*(\mathbf{R}\times Y)$ on this chart via the parallel transport map of the connexion $\hat{A}$. One can check that Lemma 12.2 implies a certain second order equation for $\tilde{\beta}$ of the form

$$\Delta\tilde{\beta} + \sum_{j=1}^{4} f_{j+1}\frac{\partial}{\partial x_j}\tilde{\beta} + f_6\tilde{\beta} + r^{-1}f_7 = 0,$$

where $f_2, \ldots, f_7$ are complex valued functions with norms bounded above by some constant $K_1 > 0$ independent of r and the point p. Recall Lemma 12.10; it implies that, for some constant $K_2 \geq K_1$, also independent of r and p, that $|\beta| \leq Kr^{-1/2}$. Standard elliptic theory then provides a bound of the form

$$\left|\frac{\partial}{\partial x_j}\tilde{\beta}(0)\right| \leq K_3 r^{-1/2}$$

where $K_3 \geq K_2$ is some potentially larger constant independent of r and p. Since the chart at hand was scaled so that the ball of radius 1 be mapped to the geodesic ball of radius $r^{-1/2}$, and $\Lambda^{0,2}\mathrm{T}^*(\mathbf{R} \times Y)$ was trivialized by the parallel transport of the connexion $\hat{A}$, whose curvature satisfies the bound of Lemma 12.25, if the reader care to check, it follows that, for some constant $K_4 \geq K_3$ independent of r and p,

$$\left|\frac{\partial}{\partial t}\beta\right| + |\nabla_A\beta| \leq K_4.$$

QED

The next few results needed shall require mention of a variant of the *vortex equations* on $\mathbf{R}^4 = \mathbf{C}^2$. These are defined next. In what follows, use ω_0 to denote the standard symplectic form on $\mathbf{R}^4$.

Definition 12.35: Consider a pair (a_0, α_0), where $a_0 \in i\Omega^1(\mathbf{R}^4)$ and $\alpha_0 \in \Omega^{0,0}(\mathbf{R}^4)$. Then, (a_0, α_0) satisfy the *vortex equations with bound* $K > 0$ when

$$\overline{\partial}\,\alpha_0 = 0, \quad |\alpha_0| \leq 1, \quad \mathrm{d}^+ a = \frac{1}{2}(1 - |\alpha_0|^2)\omega_0, \quad |\mathrm{d}^- a| \leq K.$$

Lemma 12.36: (Taubes 2009, Lemma 5.14) For any $K > 0$ and $\delta \in (0, 1)$, there exist $R_{12.36} > 2$ and $K_{12.36} > 1$ such that, for any vortex (a_0, α_0) with bound K, if one use V to denote the volume of the set

$$\left\{\, x \in \mathrm{B}(\mathbf{R}^4, R_{12.36}) \;\middle|\; (1 - |\alpha_0(x)|^2) > \delta \,\right\}$$

and V' denote the volume of the set

$$\left\{\, x \in \mathrm{B}(\mathbf{R}^4, \tfrac{1}{2}R_{12.36}) \;\middle|\; \delta > (1 - |\alpha_0|^2) \geq \tfrac{1}{2}\delta \,\right\},$$

then it follows that $V' \leq K_{12.36} V$.

Remark 12.37: The proof shall be omitted, because, unlike other results above, this one is exactly as stated in Taubes (2009) and its proof therein is self contained.

Remark 12.38: The ability to apply this lemma in the present context comes from the following.

Lemma 12.39: There exists $K_{12.39} > 0$ such that, for any $R \geq 1$ and $\varepsilon > 0$, there exists $r_{12.39} > 0$ such that, for all $p \in \mathbf{R} \times Y$ and $r > r_{12.39}$, it follows that there exists a vortex (a_0, α_0) bounded by $K_{12.39}$ and a gauge transformation u such that

$$\|(a_0, (\alpha_0, 0)) - (\phi_{p,r,R})^* u \cdot (a, (\alpha, \beta))\|_{\mathrm{C}^0(\mathrm{D}(\mathbf{R}^4, R))} < \varepsilon,$$

where $\phi_{p,r,R}$ denotes the rescaled Gaussian chart centred at p so that the geodesic ball of radius $Rr^{-1/2}$ be mapped to the ball of radius R of $\mathbf{R}^4$.

Proof: According to Taubes (2009), the proof is an adaptation of a similar statement made in Taubes (1996). The idea is as follows. Assume the contrary. Then, for any $K_{12.39} > 0$, there exists $R \geq 1$, $\varepsilon > 0$, an increasing unbounded sequence $\{r_n\}$, a sequence of points $p_n \in \mathbf{R} \times Y$ and a sequence of Seiberg-Witten trajectories $(A_\lambda + a_n, (\alpha_n, \beta_n))$ with parameter $r = r_n$ such that, if one define $(\tilde{a}_n, (\tilde{\alpha}_n, \tilde{\beta}_n))$ to mean the pullback of $(a_n, (\alpha_n, \beta_n))$ via the chart $\phi_{p_n, r_n, R}$ seen as functions of the ball of radius R of $\mathbf{R}^4$, then $(\tilde{a}_n, (\tilde{\alpha}_n, \tilde{\beta}_n))$ do not lie in a ball of radius ε around any vortex bounded by $K_{12.39}$ in the $\mathrm{C}^0(\mathrm{D}(\mathbf{R}^4, R))$ norm. Now, the Seiberg-Witten equations imply that, after redefining $(\tilde{a}_n, (\tilde{\alpha}_n, \tilde{\beta}_n))$ by applying some gauge transformation u_n, these functions obey *elliptic* equations of the form

$$\mathrm{d}^+ \tilde{a}_n = f_1(\tilde{\psi}_n, \tilde{\psi}_n) + f_2, \quad \mathrm{d}^* \tilde{a}_n = 0, \quad \overline{\partial}\, \tilde{\alpha}_n + \overline{\partial}^*\, \tilde{\beta}_n = f_3(\tilde{a}_n, \tilde{\psi}_n),$$

for certain polynomial functions f_1, f_2 and f_3. On the other hand, by using Lemma 12.5, Lemma 12.10, Lemma 12.25 and standard elliptic theory arguments, one can conclude that there must be a subsequence uniformly convergent on the disk of radius R. Let the limit of this convergent subsequence be denoted $(\tilde{a}_\infty, (\tilde{\alpha}_\infty, \tilde{\beta}_\infty))$. The point now is that, by Lemma 12.10, one can see further that $\beta_\infty = 0$. Hence, if one care to check, it follows, by examining carefully the terms f_1, f_2 and f_3, that the equations above reduce to the vortex equations in the limit $n \to \infty$. Consequently, $(a_\infty, \alpha_\infty)$ satisfies the vortex equations. But then note that, if $K_{12.39}$ be large enough relative to the constant $K_{12.25}$ provided by Lemma 12.25, $(a_\infty, \alpha_\infty)$ is a vortex bounded by $K_{12.39}$, which is a contradiction. **QED**

Set the following notation

$$\Omega_\delta := \int_{\mathbf{R}\times Y} \hat{*} e^{2t}\sigma_\delta, \quad \Omega'_\delta := \int_{\mathbf{R}\times Y} \hat{*} e^{2t}\sigma'_\delta.$$

Lemma 12.40: (cf. Taubes 2009, Lemma 5.17) Given $\delta \in (0, \delta_{2.24})$, there exist $r_{12.40} > 0$ and $K_{12.40} > 0$ such that, for all $r > r_{12.40}$,

$$\Omega'_\delta \le K_{12.40}\Omega_\delta.$$

Proof: The analogous statement in Taubes (2009) lacks a proof as it is claimed to be similar to a previous lemma; therefore, the author shall furnish the details here. Firstly, notice that the function σ_δ is non-zero only at points where $(1 - |\alpha|^2) \ge \delta$; meanwhile, σ'_δ is non-zero only at points where $\delta \ge (1 - |\alpha|^2) \ge \frac{1}{2}\delta$. Now, If g be the product metric of $\mathbf{R}\times Y$ consider instead the metric $e^{2t}g$. Since the set of points where $(1 - |\alpha|^2) \ge \delta$ has t coordinate bounded above, the volume of this set in the metric $e^{2t}g$ is, in fact, finite. Hence, note that, in order to prove the claimed result, it suffices to bound the volume of points where $\delta \ge (1-|\alpha|^2) \ge \frac{1}{2}\delta$ by some constant times the volume of the set of points where $(1 - |\alpha|^2) \ge \delta$; both volumes being with respect to the metric $e^{2t}g$. For the rest of this proof, the metric at use shall be $e^{2t}g$ whenever the author talk of volumes or geodesy. For brevity, use $R := R_{12.36}$ to denote the constant from Lemma 12.36. Consider a set Λ of disjoint geodesic balls in Y centred at points p where $\delta \ge (1-|\alpha(p)|^2) \ge \frac{1}{2}\delta$ and all having radius $\frac{1}{4}Rr^{-1/2}$. Due to compactness of Y, one can also assume Λ to be maximal with respect to inclusion. For each ball $B \in \Lambda$, let $B'' \supset B' \supset B$ denote the concentric geodesic balls having radii $Rr^{-1/2}$ and $\frac{1}{2}Rr^{-1/2}$ respectively. Now, suppose that some point $p \in \mathbf{R}\times Y$ at which $\delta \ge (1 - |\alpha(p)|^2) \ge \frac{1}{2}\delta$ not be in the set $\bigcup_{B\in\Lambda} B'$. Then, if r be sufficiently large compared to R, say larger than some $r_{12.40} > 0$, the ball of radius $\frac{1}{4}Rr^{-1/2}$ centred at p would not intersect any of the balls in the set Λ; so, if this were the case, Λ could not be maximal. Therefore, $\bigcup_{B\in\Lambda} B'$ covers the set of points where $\delta \ge (1 - |\alpha|^2) \ge \frac{1}{2}\delta$. Moreover, perhaps after an increase to $r_{12.40} > 0$, Riemannian geometry provides an upper bound for the maximal number n such that there be a set of balls $\{B_i\} \subset \Lambda$ satisfying $B''_1 \cap \cdots \cap B''_n \ne \emptyset$; this upper bound is independent of δ and r except that one must ensure $r > r_{12.40}$. The consequence of all of this is that it suffices to provide the desired sort of bound for each of the balls B. For that end, given a ball $B \in \Lambda$, denote by V_B the volume of the subset of B'' where $\delta \geq (1 - |\alpha(p)|^2) \geq \frac{1}{2}\delta$; likewise, denote by V'_B the volume of the subset of B' where $(1 - |\alpha|^2) \geq \delta$. Now, apply Lemma 12.36 in combination with Lemma 12.39 in order to obtain a constant $K_{12.40} > 0$ independent of B, satisfying $V'_B \leq K_{12.40} V_B$. **QED**

Lemma 12.41: Given $\delta \in (0, \delta_{2.24})$, there exists $r_{12.41}$ such that for all $r > r_{12.41}$, $\Omega_\delta = 0$.

Proof: To start, let $r_{12.41} = \max\{r_{12.40}, r_{12.34}\}$. Recall that Lemma 12.33 asserted that

$$\int_{\mathbf{R}\times Y} \hat{*} e^{2t} \left(\delta^{-1} \sigma'_\delta \left(|\partial_{\hat{A}} \alpha|^2 - |\,\overline{\partial}_{\hat{A}}\, \alpha|^2 \right) + r\sigma_\delta \left(1 - |\alpha|^2 + |\beta|^2 \right) \right) = 0.$$

Neglecting some positive terms yields the inequality

$$\int_{\mathbf{R}\times Y} \hat{*} e^{2t} \left(-\delta^{-1} \sigma'_\delta |\,\overline{\partial}_{\hat{A}}\, \alpha|^2 + r\sigma_\delta (1 - |\alpha|^2) \right) \leq 0.$$

Focusing on the second term of the integrand, note that, at a point where $\sigma_\delta \neq 0$, it is necessarily the case that $(1 - |\alpha|^2) \geq \delta$; hence,

$$\int_{\mathbf{R}\times Y} \hat{*} e^{2t} \left(-\delta^{-1} \sigma'_\delta |\,\overline{\partial}_{\hat{A}}\, \alpha|^2 + r\sigma_\delta \delta \right) \leq 0.$$

By applying Lemma 12.40 and Lemma 12.34, one finds

$$\left(- K_{12.40} K_{12.34}^2 \delta^{-1} + r\delta \right) \Omega_\delta \leq 0.$$

But $\Omega_\delta \geq 0$. Hence, perhaps after increasing $r_{12.41}$, it follows that, for $r > r_{12.41}$, $\Omega_\delta = 0$. **QED**

Theorem 12.42: Given $\delta \in (0, \delta_{2.24})$, for all $r > r_{12.41}$, it follows that, pointwise on all of $\mathbf{R} \times Y$,

$$1 - |\alpha|^2 \leq \delta.$$

Proof: Ω_δ is the integral of $e^{2t}\sigma_\delta$, which is non-negative and strictly positive wherever $(1 - |\alpha|^2) > \delta$. **QED**

Corollary 12.43: $C = \lim_{t\to-\infty} \gamma$ is gauge equivalent to C_λ.

Proof: C is a solution to the Seiberg-Witten equations on Y. Its $\Lambda^{0,0}\xi^*$ component is $\lim_{t\to-\infty} \alpha$. Therefore, it also satisfies the bound in Theorem 12.42. Theorem 2.24 then guarantees that C must be gauge equivalent to C_λ. **QED**

Corollary 12.44: γ is the constant trajectory at C_λ.

Proof: Assume the contrary. Because the Seiberg-Witten vector field $\mathcal{X}_{\lambda,r}$ is minus the gradient of the functional $\mathrm{CSD}_{\lambda,r}$, the value of $\mathrm{CSD}_{\lambda,r}$ never increases along the trajectory γ. Moreover, the contact configuration C_λ is a non-degenerate fixed point of the downward gradient flow of $\mathrm{CSD}_{\lambda,r}$ by Theorem 11.22, which means that the value of $\mathrm{CSD}_{\lambda,r}$ certainly decreases along γ for sufficiently large time as one approaches C_λ. But since both endpoints of γ are gauge equivalent and $b_1 = 0$, the values of $\mathrm{CSD}_{\lambda,r}$ are the same for gauge equivalent configurations. This cannot be. **QED**

Remark 12.45: Hereby, the author concludes the proof of Theorem 2.25.

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