Discontinuous Galerkin Methods for Friedrichs Systems with Irregular Solutions

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Abstract

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This work is concerned with the numerical solution of Friedrichs systems by discontinuous Galerkin finite element methods (DGFEMs). Friedrichs systems are boundary value problems with symmetric, positive, linear first-order partial differential operators and allow the unified treatment of a wide range of elliptic, parabolic, hyperbolic and mixed-type equations. We do not assume that the exact solution of a Friedrichs system belongs to a Sobolev space, but only require that it is contained in the associated graph space, which amounts to differentiability in the characteristic direction.

We show that the numerical approximations to the solution of a Friedrichs system by the DGFEM converge in the energy norm under hierarchical h- and p- refinement. We introduce a new compatibility condition for the boundary data, from which we can deduce, for instance, the validity of the integration-by-parts formula. Consequently, we can admit domains with corners and allow changes of the inertial type of the boundary, which corresponds in special cases to the componentwise transition from in- to outflow boundaries.

To establish the convergence result we consider in equal parts the theory of graph spaces, Friedrichs systems and DGFEMs. Based on the density of smooth functions in graph spaces over Lipschitz domains, we study trace and extension operators and also investigate the eigensystem associated with the differential operator. We pay particular attention to regularity properties of the traces, that limit the applicability of energy integral methods, which are the theoretical underpinning of Friedrichs systems. We provide a general framework for Friedrichs systems which incorporates a wide range of singular boundary conditions. Assuming the aforementioned compatibility condition we deduce well-posedness of admissible Friedrichs systems and the stability of the DGFEM. In a separate study we prove hp-optimality of least-squares stabilised DGFEMs.

To my parents

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Contents

Introduction

In 1958 Friedrichs introduced a class of boundary value problems which admits the study of a wide range of differential equations in a unified framework. Although these boundary value problems, today known as Friedrichs systems, consist of systems of linear differential equations of first-order, transformations of second-order elliptic, parabolic and hyperbolic equations into the setting of Friedrichs are known. Also, many other equations, such as first-order symmetric hyperbolic systems and second-order equations with anisotropic diffusion, can be analysed with Friedrichs' methodology. Yet, while the exploration of such diverse boundary value problems in a unified framework is of great theoretical value, Friedrichs' main motivation was of more practical nature. A number of important physical phenomena are attributed in the mathematical model to a change of type of the governing equations. We highlight, in particular, the study of transonic flow, which is of great practical significance. In this setting, regions of subsonic flow correspond to a local model of elliptic type while regions of supersonic flow are represented by a hyperbolic equation.

With Friedrichs' aim in mind to provide a unified treatment for these diverse classes of equations, it is inevitable that solutions to Friedrichs systems incorporate the mathematical characteristics of solutions of the individual differential equations. Thus a solution to a Friedrichs system may be discontinuous as this is not an unusual feature when considering hyperbolic equations. A solution may exhibit poles at corners of the domain as is well-documented in the case of elliptic boundary value problems. The list could be extended further, giving reference to holomorphy, boundary and interior layers and many other familiar features. Clearly, trying to capture all of these characteristics simultaneously is a major challenge which remains an unresolved problem at present. The main difficulty is the correct implementation of the boundary conditions and the treatment of corners of the domain. Indeed, in order to prove existence and uniqueness of solutions Friedrichs had to impose constraints on the boundary conditions which severely restricted the range of problems he was able to analyse. In the language of fluid dynamics he demanded that the number of in- and outflow components of the differential operator is constant on each simply connected component of the domain boundary. In this case one says that the boundary is of constant multiplicity. Also, he only covered corners by means of case studies rather than within the general theory. Many authors have since then attempted to overcome these limitations; we name Morawetz, Lax, Rauch, Phillips and Sarason in particular. Significant progress has been achieved on individual problems such as the Frankl equation, which serves as a prototype for transonic flow, cf. (Morawetz 1958) and (Lax and Phillips 1960). Also the general theory has been advanced steadily. Under certain conditions one can now ensure the well-posedness of Friedrichs systems with a boundary of non-constant multiplicity, cf., for instance, (Rauch 1994). Similarly, a number of criteria for boundary value problems on non-smooth domains have been introduced to guarantee the existence and uniqueness of solutions. We give a more detailed account on these results in Section 2.6 of this dissertation.

Already in his original paper Friedrichs considered the numerical solution of Friedrichs systems by means of a finite difference scheme. Several alternative methods have been proposed to find approximate solutions of Friedrichs systems, including finite volume and finite element methods. In this dissertation we focus on the application of discontinuous Galerkin finite element methods (DGFEMs) to Friedrichs systems. These finite element methods include jump terms originating from a weak formulation of the boundary value problem due to a lack of continuity within the finite-dimensional approximation space.

DGFEMs exhibit a number of advantages over competing schemes, which are crucial for the numerical solution of Friedrichs systems. We emphasise their main features:

- DGFEMs for Friedrichs systems are stable in the associated energy norm, which is stronger than the L^2-norm, without requiring the presence of additional stabilising terms. The constant in the stability bound does not depend on the exact solution u and is therefore insensitive to discontinuities of u. However, we remark that this constant does depend on the coefficients of the differential operator; hence it is advisable to stabilise the method, when it is employed at an intermediate stage, in the course of the solution of a nonlinear problem.

- Typically, the regularity of the exact solution of a Friedrichs system varies significantly throughout the domain. Results from approximation theory show that in regions of high-regularity the exact solution is well approximated by polynomials of high degree, while in regions of low regularity it is preferable to locally decrease the mesh size. DGFEMs allow the coupling of elements with high and low polynomial degrees without loss of efficiency. It is far more intricate to deal with a non-uniform distribution of polynomial degrees when continuous finite elements are used. In this case care must be taken that the numerical solution in elements with a high polynomial degree is continuously linked to the solution in neighbouring elements where a lower degree polynomial solution has been used.

- The weak continuity requirements between finite elements in the formulation of the discontinuous Galerkin finite element method allows one to easily combine elements of different shape with each other. Indeed, curved boundaries and other geometrical features can be naturally implemented in the discontinuous Galerkin framework. This makes the method suitable for computations on complicated domains.

- The implementation of inhomogeneous boundary conditions follows naturally from the underlying weak formulation of the boundary value problem. In contrast, continuous finite element schemes such as the one proposed in (LeSaint 1973/74) enforce the boundary conditions by restriction of the trial space. Yet, if the inhomogeneous boundary conditions cannot be satisfied exactly by functions in the trial space, e.g. because they are not of polynomial type, then a technique has to be devised to single out functions which satisfy the boundary conditions in an approximate sense. Ensuring the stability of such a technique, in particular in view of type-changes, is a non-trivial task that can be avoided in the framework of DGFEMs.

- For equations of hyperbolic type with a uniform direction of hyperbolicity, cf. Section 2.5, the discontinuous Galerkin solution can be calculated element by element. This is closely related to the fact that discontinuous Galerkin methods can also be utilised for the time-discretisation of partial differential equations.

These benefits highlighted the fact that the discontinuous Galerkin finite element method is capable of solving a wide range of Friedrichs systems in a natural and efficient manner. Clearly, the advantages of the DGFEMs over other schemes are most apparent for problems of variable regularity and for complicated domains. In fact, we point out that if the exact solution is globally smooth and if the computational domain is sufficiently simple, then it should not be expected that the discontinuous Galerkin method outperforms carefully selected finite difference schemes and simpler continuous finite element methods. Nevertheless, in early publications, *a priori* estimates for the discontinuous Galerkin method were limited to Friedrichs systems whose solution u is globally smooth, i.e. where u is contained in a Sobolev space $W^{k,p}(\Omega)$, Ω being the domain of the boundary value problem and $k \in \mathbb{N}$, see (LeSaint and Raviart 1974) or (Bey and Oden 1996). This shortfall was addressed in (Houston, Schwab and Süli 2000b) where scalar differential equations were considered. In this paper the exact solution is only required to be elementwise contained in a Sobolev space so that local differences in the regularity could be accounted for by the error bound. Nevertheless, this paper excludes a very important situation when the exact solution exhibits a discontinuity along a characteristic curve. There is only one exception, namely when the computational mesh is exactly aligned with the discontinuity of u, in which case the error bound in (Houston et al. 2000b) can be applied.

The aim of this dissertation is to extend the analysis of the discontinuous Galerkin method to Friedrichs systems with discontinuous solutions. More specifically we want to give a meaningful description of the method in this more general setting and prove its convergence. It is easily seen that for this endeavour the scale of Sobolev spaces is quite unsuitable. On the one hand, for the spaces $W^{s,2}(\Omega) = H^s(\Omega)$ with index $s \in [0, {}^1\!/_2)$ the construction of a trace operator by means of density of smooth functions is not possible and thereby the notion of the boundary value problem and also of the DGFEM becomes unclear. On the other hand if $s \geq {}^1\!/_2$ then discontinuous solutions cannot be treated in sufficient generality, as is demonstrated by the following example.

Example 1 *Let χ be the characteristic function of the interval $(-{}^1\!/_2, {}^1\!/_2)$ in $\mathbb{R}$ and let $x = (x_1, x_2, \dots, x_n) = (x_1, x')$ where $x' = (x_2, \dots, x_n)$. We consider the function*

$$u(x) = \chi(x_1)\, \mathrm{e}^{-|x'|^2/2}, \qquad x \in \mathbb{R}^n.$$

The Fourier transformation of this function is

$$(\mathscr{F}u)(\xi) = (2\pi)^{n-1} \frac{\sin(\xi_1/2)}{\xi_1/2}\, \mathrm{e}^{-|\xi'|^2/2}, \qquad \xi \in \mathbb{R}^n.$$

Therefore the Sobolev norm $\|u\|^2_{W^{s,2}(\Omega)}$ is equal to

$$\begin{aligned}\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathscr{F}u|^2 \,\mathrm{d}\xi &= (2\pi)^{2n-2} \int_{\mathbb{R}_{\xi_1}} (1+|\xi_1|^2)^s \Big|\frac{\sin(\xi_1/2)}{\xi_1/2}\Big|^2 \int_{\mathbb{R}_{\xi'}} \frac{(1+|\xi|^2)^s}{(1+|\xi_1|^2)^s}\, \mathrm{e}^{-|\xi'|^2/2} \,\mathrm{d}\xi' \,\mathrm{d}\xi_1 \\ &\geq C \int_{\mathbb{R}_{\xi_1}} (1+|\xi_1|^2)^s \Big|\frac{\sin(\xi_1/2)}{\xi_1/2}\Big|^2 \,\mathrm{d}\xi_1,\end{aligned}$$

where C is a positive real constant. The last integral is divergent if $s \geq {}^1\!/_2$.

In view of the example we carry out the analysis of Friedrichs systems in a different class of spaces, namely in the so-called graph spaces. Given a differential operator $\mathfrak{L}$ a function v is contained in the associated graph space if v and the image $\mathfrak{L}v$ are L^q-integrable. Conceptually, the graph space contains all functions which are weakly differentiable along the characteristics of $\mathfrak{L}$.

In order to carry out the analysis of Friedrichs systems and of discontinuous Galerkin methods within the framework of graph spaces we develop, in Chapter 1, the relevant mathematical properties of these spaces since we require an array of results which are not available from the literature.

To introduce the notion of the trace of functions in the graph space we start our investigations with the density of smooth functions. We only need to require the segment property of Ω to show that bounded C^{∞}-functions are dense in the graph space. The density results in (Friedrichs 1958) and (Rauch 1994), concerned with graph spaces on smooth domains, do not allow us to incorporate geometric singularities such as corners when Ω is, say, a polyhedron. The handling of corners makes up an essential part in the analysis of finite element methods for Friedrichs systems. Even if the computational domains, consisting of the polygonal finite elements, approximate the original smooth domain of a well-posed Friedrichs system arbitrarily well in the refinement process, it is an open problem whether from a certain point onwards in the course of the refinement the associated Friedrichs systems on the computational domains are well-posed, too. Sarason, who addressed a related problem in (Sarason 1962), had to impose strong assumptions on polygonal domains to prove results in this direction. For instance, he required that the polygonal domain under consideration can be uniformly approximated by smooth domains which are of uniform constant multiplicity.

We can now define the trace operator on the basis of the integration by parts formula for smooth functions. The vector space of all traces is called the trace space. It naturally inherits a norm from the graph space by infimising the graph space norm over all functions with the same trace. This norm is distinguished by a number of important properties. In particular, it is, up to homeomorphy, the only norm under which boundary operators on the graph space can be continuously factorised in terms of the trace operator; we clarify the notion of boundary operators in the text, however we remark here that they are of great significance in the statement of boundary conditions for Friedrichs systems. The proof of this factorisation relies on a description of the kernel of the trace operator in terms of zero extensions, which we study in Section 1.4. Conceptually, the factorisation confirms that the trace operator collects all information available, near the boundary, about elements of the graph space. The existence of a bounded extension operator $\mathcal{E}_{\mathcal{L}}$ from the trace space into the graph space follows directly from the uniform convexity and reflexivity of the graph space.

We point out that the formal extension of the integration-by-parts formula to the entire graph space does not anymore represent an identity between integrals over the domain of the boundary value problem and its boundary. The failure of the integration-by-parts formula in its classical sense has profound implications on our investigations into Friedrichs systems as will be seen in Chapters 2 and 3.

Another difficulty in understanding the well-posedness of Friedrichs systems is that it is not possible at present to give an intrinsic definition of the norm of the trace space. An intrinsic definition of the trace norm does not make reference to functions whose support is not contained in the boundary $\partial\Omega$. A different example of an intrinsic definition is, for instance, formula (1.10) on page 25 which describes the norm of the Besov space $B^{q',q',q-1/q'}(\partial\Omega)$, which is the trace space of $W^{1,q'}(\Omega)$. An intrinsic definition of the trace space leads to an intrinsic description of which boundary conditions can be satisfied by functions in the graph space and provides insight into the dependence of the solution of the Friedrichs system on the boundary data.

Although we do not give a complete intrinsic description of the trace norm, we elucidate important aspects. In the Hilbert space setting, that is when $q = 2$, we begin the analysis of the trace space with the characterisation of the trace norm by identification with the formal boundary integral

$$\Big(\int_{\partial\Omega} \mathcal{L}\mathcal{E}_{\mathcal{L}}v \cdot \mathcal{E}_{\mathcal{L}}v \,\mathrm{d}S\Big)^{1/2}. \tag{1}$$

In order to turn (1) into an intrinsic formula one needs to eliminate the reference to $\mathcal{L}$ in the integrant. This is a delicate task, which depends in part on understanding the eigenspaces of $\mathcal{L}$, an issue we address later in Chapter 1. We remark here that (1) may also serve as a starting point for an investigation of the trace spaces in the spirit of Dirichlet-to-Neumann maps arising in the theory of inverse problems, cf. (Uhlmann 2003), where operators similar to $\mathcal{L}$ are studied by means of pseudo-differential operators.

For the analysis of Friedrichs systems we can concentrate on differential operators with a Hermitian principal part. In this setting we show that functions from $\partial\Omega$ to $\mathbb{R}^m$ whose support is either contained in the in- or outflow boundary of the domain, are integrable in an L^2-sense; yet, that coupling in the tangential direction of in- and outflow components may result in traces of $B^{2,2,-1/2}(\partial\Omega)$-type. This difference decides if energy integral methods are applicable since these methods are restricted to functions which have a square product, which is not defined for functions in $B^{2,2,-1/2}(\partial\Omega)$. We also exemplify the presence of strong poles in the vicinity of characteristic points and corners of the boundary. Still assuming that the principal part of $\mathcal{L}$ is Hermitian, we return to the spectral properties of $\mathcal{L}$. In particular, we give an explicit description of the eigenvalues and eigenprojections of $\mathcal{L}$ for functions in the image of $\mathcal{E}_{\mathcal{L}}$, making use of a modification of the reaction term of $\mathcal{L}$.

After a detailed analysis of trace spaces, we turn to general boundary value problems on graph spaces. Bearing in mind the low degree of regularity of traces of the graph space, we introduce a general framework for boundary value problems which underlies the mild assumption that edges of the domain and points of non-constant multiplicity are a null set in the Hausdorff measure of the boundary. In this setting we also consider weak and adjoint formulations of the original boundary value problem.

We conclude the first chapter with an outlook on operators with skew-Hermitian coefficients and remarks on the duality properties of graph spaces for $q \neq 2$. In particular, we represent the dual space of the graph space as cone in a vector-valued L^q-space and extend the construction of $W^{q,-1/q}(\Omega)$ to the graph space setting.

In Chapter 2 we pursue the analysis of Friedrichs systems in two stages. In the first stage we focus on incorporating a large class of boundary value problems into the analysis. We extend the definition of Friedrichs systems to the setting of boundary value problems considered in Chapter 1. We then generalise the proof by Friedrichs on the existence of solutions to our framework. The boundary conditions we impose at that point are formulated abstractly. How boundary conditions can be implemented by means of matrix functions is the topic of Section 2.2. Here we also make a connection to the pointwise descriptions of the boundary conditions introduced by Friedrichs. Our main interest in this section concerns the smoothness requirements on the matrix functions to render the boundary value problem in our framework meaningful. We underline that these requirements can be significantly weakened if the trace space is of L^2-type.

When we turn to the question of well-posedness of Friedrichs systems, it becomes evident that in our general framework, which we designed to incorporate a wide variety of domains and boundary conditions, solutions may lack important properties. We exemplify the failure of the integration-by-parts formula and discuss the continuous dependence of the solution on the boundary data. Other authors have explored the loss of the well-posedness of Friedrichs systems in the presence of corners and type-changes as well; we refer for instance to (Rauch 1994). However, these investigations are typically targeted at understanding certain analytical phenomena in the vicinity of singularities and do not provide a suitable foundation for the error analysis of the discontinuous Galerkin method we have in mind. For more details we refer to Section 2.6.

In the second stage of our analysis of Friedrichs systems we identify that the loss of well-posedness is connected to an imbalance between the rank of the boundary conditions and the rank of the graph space trace operator. Taking this observation into consideration we present a setting for Friedrichs systems in which we can verify the well-posedness of the boundary value problem while still admitting type-changes and corners. Certain ill-posed problems, such as the famous example by Moyer, are discarded automatically. In consequence, this formulation is a suitable basis for the error analysis of the discontinuous Galerkin method. We end the second chapter with an outline of a number of important differential equations which can be transformed into Friedrichs systems and present a review of the relevant literature on the subject.

Based on the work in Chapters 1 and 2, we are in the position to construct the discontinuous Galerkin method for Friedrichs systems with discontinuous solutions. We demonstrate that the DGFEM is stable and has a unique solution for approximation spaces which are contained in the so-called broken graph space. The broken graph space is the product of the graph spaces over the finite elements. Furthermore, we estimate the error of the discontinuous Galerkin method with the distance between the exact solution of the Friedrichs system and the continuous functions within the finite element space. An immediate consequence of this result is the convergence of the discontinuous Galerkin solution to the exact solution for polynomial approximation spaces. We explicitly include Friedrichs systems with discontinuous solutions in our analysis.

After having established the main result of the dissertation, namely the convergence of the discontinuous Galerkin method in broken graph spaces, we focus our attention on the more classical problem of studying the performance of the DGFEM in Sobolev spaces. At present, error bounds which are simultaneously optimal in h and p are only known under additional assumptions. Indeed, optimal error bounds are only known for differential operators, which have elementwise constant coefficients, and for the streamline diffusion stabilised discontinuous Galerkin method, cf. (Houston et al. 2000b) and (Houston, Schwab and Süli 2002b). The error bound for general scalar linear first-order differential operators in (Houston et al. 2002b) is suboptimal in p by $p^{3/2}$. The error bound due to Georgoulis is suboptimal in h by $h^{1/2}$ and in p by one order, cf. (Georgoulis 2003). We improve the bound, for certain problems, by half an order in p. In the remaining part of Chapter 3 we investigate the influence of least-squares terms in the discontinuous Galerkin framework. While our main interest is the stabilisation of the original discontinuous Galerkin method, the parameterised family of finite element methods we introduce to facilitate a systematic error analysis also contains a least-squares method of discontinuous type. We prove for all members of the family an *a priori* error bound that is optimal in both h and p. We conclude the dissertation with numerical examples which clarify that for a range of parameters the additional least-squares terms not only lead to a stronger stability bound but also improve the approximation properties of the method.

Notation

Given normed spaces X and Y, we denote the space of all bounded linear operators from X to Y by $\mathscr{B}(X,Y)$. For $\Lambda \in \mathscr{B}(X,Y)$, we call Y the codomain of Λ. The codomain is in general not equal to the image or range of Λ, which is the set

$$\operatorname{Im}\Lambda := \{y \in Y : \exists\, x \in X : \Lambda x = y\}.$$

We put $\mathscr{B}(X) := \mathscr{B}(X, X)$. We write the operator norms in $\mathscr{B}(X, Y)$ and $\mathscr{B}(X)$ as $\|\cdot\|_{\mathscr{B}(X,Y)}$ and $\|\cdot\|_{\mathscr{B}(X)}$, respectively. Consider the product space

$$X^m := \prod_{i=1}^{m} X = \underbrace{X \times \cdots \times X}_{m\text{-times}}.$$

Let $x = (x_1, \dots, x_m) \in X^m$ and $q \in [1, \infty]$. Then

$$\|x\|_{X,q} := \begin{cases} \sqrt[q]{\sum_i \|x_i\|_X^q} & : \text{ for } q \in [1, \infty), \\ \max_i \|x_i\|_X & : \text{ for } q = \infty \end{cases}$$

defines a norm on X^m. Given a submanifold M of $\mathbb{R}^n$, we generally assume that the function spaces $L^q(M)^m$ and $W^{1,q}(M)^m$ are normed by

$$\|\cdot\|_{L^q(M)^m} := \|\cdot\|_{L^q(M),q} \quad \text{and} \quad \|\cdot\|_{W^{1,q}(M)^m} := \|\cdot\|_{W^{1,q}(M),q},$$

respectively. The most important cases are when M is equal to an open subset $\Omega \subset \mathbb{R}^n$ or to its boundary $\partial\Omega$.

Most theorems in this text which concern function spaces, hold for the codomains $\mathbb{C}^l$ and $\mathbb{R}^l$, $l \in \mathbb{N}$. However occasionally we have to distinguish between the real and complex case. In such situations we extend the notation of the function space by an additional argument, which identifies the codomain. For instance, we write $L^q(M, \mathbb{R}^m)$ for the space of all L^q-integrable functions which map M into $\mathbb{R}^m$. The space of complex-valued L^q-functions is denoted by $L^q(M, \mathbb{C}^m)$. In accordance with our notation we also write $L^q(M, \mathbb{R})^m$ and $L^q(M, \mathbb{C})^m$.

We shall often employ the Einstein summation convention. This means that when an index occurs more than once in the same expression then the expression is implicitly summed over this index. For instance, the term

$$\sum_i a_i\, b_i$$

is abbreviated by $a_i\, b_i$. If the summation convention is employed the range of the index will always be clear from the context.

The complex conjugate of $z \in \mathbb{C}$ is $\overline{z}$. The Hermitian conjugate of a matrix $B = (B_{ij})_{ij}$ is the matrix $(\overline{B_{ji}})_{ij}$, which we denote by B^{H}.

By $\langle f, g\rangle_M$ we mean the sesquilinear product $\int_M f \cdot \overline{g}\,\mathrm{d}x$ over M, where the functions f and g are chosen so that their dot product is integrable. We do not require that f and g are elements of the same function space.

Finally, we denote the ball with radius δ centred at x by $B_\delta(x)$ and the identity matrix by I.

We make use of the space of distributions $\mathscr{D}'(\Omega)$ over open subsets $\Omega \subset \mathbb{R}^n$. The space of test functions of $\mathscr{D}'(\Omega)$ is $\mathscr{D}(\Omega)$.

Chapter 1

Graph Spaces

1.1 Definition of Graph Spaces

Consider a non-empty open set $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$. To simplify the notation we assume that n is greater than or equal to 2 and remark that the theorems in the subsequent text can be extended to the one-dimensional case with only minor modifications of the proofs. Choose a conjugate pair q, q', i.e., $q, q' \in \mathbb{R}$ such that

$$1 < q < \infty, \qquad q' = \frac{q}{q-1}.$$

Let $l, m \in \mathbb{N}$. Given a tensor $B \in W^{1,\infty}(\Omega)^{l\times m\times n}$ and a matrix $C \in L^\infty(\Omega)^{l\times m}$, we are interested in the graph space of the linear differential operator

$$\mathcal{L} : L^q(\Omega)^m \to \mathscr{D}'(\Omega)^l, v \mapsto \partial_k(B_{ijk}\, v_j) + C_{ij}\, v_j.$$

The formal adjoint of $\mathcal{L}$ is defined as

$$\mathcal{L}' : L^{q'}(\Omega)^l \to \mathscr{D}'(\Omega)^m, w \mapsto -\partial_k(\overline{B_{jik}}\, w_j) + (\overline{C_{ji}} + \partial_k\overline{B_{jik}})w_j.$$

The graph space of $\mathcal{L}$ is the set

$$W^q_{\mathcal{L}}(\Omega) := \{v \in L^q(\Omega)^m : \mathcal{L}v \in L^q(\Omega)^l\},$$

which is equipped with the graph norm

$$\|v\|_{\mathcal{L},q} := \|v\|_{\mathcal{L},q,\Omega} := \sqrt[q]{\|v\|^q_{L^q(\Omega)^m} + \|\mathcal{L}v\|^q_{L^q(\Omega)^l}}.$$

Generally, we assume that functions in $W^q_{\mathcal{L}}(\Omega)$ map into $\mathbb{C}^l$, however we also allow the case that $W^q_{\mathcal{L}}(\Omega)$ contains only the functions which map into $\mathbb{R}^l$. Where it is necessary for clarity, we denote the graph space of complex-valued functions by $W^q_{\mathfrak{L}}(\Omega, \mathbb{C}^l)$ and the graph space of real-valued functions by $W^q_{\mathfrak{L}}(\Omega, \mathbb{R}^l)$. We shall require that the coefficients B and C of $\mathfrak{L}$ are real whenever we consider the space $W^q_{\mathfrak{L}}(\Omega, \mathbb{R}^l)$.

In addition to the graph norm, we use the graph semi-norm

$$|v|_{\mathfrak{L},q} := |v|_{\mathfrak{L},q,\Omega} := \|\mathfrak{L}v\|_{L^q(\Omega)^l}$$

and, for $v, w \in W^2_{\mathfrak{L}}(\Omega)$, the graph scalar product

$$\langle v, w\rangle_{\mathfrak{L}} := \langle v, w\rangle_{\mathfrak{L},\Omega} := \langle v, w\rangle_\Omega + \langle \mathfrak{L}v, \mathfrak{L}w\rangle_\Omega.$$

By the graph of $\mathfrak{L}$ we understand the subspace

$$\Gamma(\mathfrak{L}) := \{(v, w) \in L^q(\Omega)^m \times L^q(\Omega)^l : \mathfrak{L}v = w\},$$

which is endowed with the norm inherited from $L^q(\Omega)^m \times L^q(\Omega)^l$. The embedding

$$\mathfrak{I} : W^q_{\mathfrak{L}}(\Omega) \to L^q(\Omega)^m \times L^q(\Omega)^l, v \mapsto (v, \mathfrak{L}v) \tag{1.1}$$

is an isometry between $W^q_{\mathfrak{L}}(\Omega)$ and $\Gamma(\mathfrak{L})$. Thus, depending on our preference, we can understand $W^q_{\mathfrak{L}}(\Omega)$ as a subspace of $L^q(\Omega)^m$ or of $L^q(\Omega)^m \times L^q(\Omega)^l$. The latter perception is the motivation for the name 'graph space'. We also consider the adjoint graph which is

$$\Gamma'(\mathfrak{L}') := \{(v, w) \in L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l : v = -\mathfrak{L}'w\}.$$

Example 2 *Assume that $l = n$ and $m = 1$. The scalar-valued Sobolev space $W^{1,q}(\Omega)$ coincides with the graph space $W^q_{\mathfrak{L}}(\Omega)$ if $\mathfrak{L}$ is chosen to be gradient, that is if $B_{ijk} = \delta_{ik}$ where δ_{ik} is the Kronecker delta. Then*

$$\|v\|^q_{W^{1,q}(\Omega)} = \|v\|^q_{L^q(\Omega)} + \|\operatorname{grad} v\|^q_{L^q(\Omega)^n} = \|v\|^q_{\mathfrak{L},q}.$$

Example 3 *Assume that $l = 1$ and $m = n$. If $\mathfrak{L} = \operatorname{div}$ then $W^q_{\mathfrak{L}}(\Omega)$ is equal to the space*

$$W^q(\operatorname{div}, \Omega) = \{v \in L^q(\Omega)^m : \operatorname{div} v \in L^q(\Omega)\}$$

since the graph norm is in this case

$$\|v\|^q_{W^q(\operatorname{div},\Omega)} := \|v\|^q_{L^q(\Omega)^m} + \|\operatorname{div} v\|^q_{L^q(\Omega)} = \|v\|^q_{\mathfrak{L},q}.$$

Example 4 *Assume that $l = m = n = 3$ and let*

$$\mathfrak{L}v = \operatorname{rot} v = \Big(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big).$$

Then $W^q_{\mathfrak{L}}(\Omega)$ is equal to $W^q(\operatorname{rot}, \Omega) := \{v \in L^q(\Omega)^3 : \operatorname{rot} v \in L^q(\Omega)^3\}$ since

$$\|v\|^q_{W^q(\operatorname{rot},\Omega)} := \|v\|^q_{L^q(\Omega)^3} + \|\operatorname{rot} v\|^q_{L^q(\Omega)^3} = \|v\|^q_{\mathfrak{L},q}.$$

Remark 1 *Consider the notation* $\dot{\mathfrak{L}}v = B_{ijk}(\partial_k v_j) + C_{ij}v_j$*. This does not have an immediate meaning if* $v \in W^q_{\mathfrak{L}}(\Omega)$ *because there is no standard definition for the product between distributions* $\partial_k v_j$ *and* $W^{1,\infty}$*-functions* B_{ijk}*. For our analysis, we shall suppose that the term* $B_{ijk}(\partial_k v_j) + C_{ij}v_j$ *represents the distribution* $\partial_k(B_{ijk}v_j) + (C_{ij} - \partial_k B_{ijk})v_j$*, to which we can apply the above definition for graph spaces. Adopting this convention for the formal adjoint, we obtain the more common definition*

$$\mathfrak{L}' : w \mapsto -\overline{B_{jik}}\,\partial_k w_j + \overline{C_{ji}}\,w_j.$$

1.2 Density

Our first investigations of the graph space $W^q_{\mathfrak{L}}(\Omega)$ concern the density of the set of smooth functions in this space. Consider the approximate identity $\delta \mapsto \psi_\delta$, $\delta > 0$, with non-negative $C^\infty(\Omega)$-functions $\psi_\delta(x) = \delta^{-n}\,\psi_1(\delta^{-1}x)$, satisfying the conditions that the support of ψ_1 lies within the unit ball centred at the origin and that $\|\psi_1\|_{L^1(B_1(0))} = 1$. We write, for $v \in W^q_{\mathfrak{L}}(\Omega)$,

$$v^\delta = (v_1^\delta, \dots, v_m^\delta) = (v_1 * \psi_\delta, \dots, v_m * \psi_\delta).$$

We shall write $\Omega' \Subset \Omega$ if Ω' is relatively compact in Ω, i.e. $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ compact.

Theorem 1 *Let* $\Omega' \Subset \Omega$ *and suppose that for* $v \in W^q_{\mathfrak{L}}(\Omega)$*,* $1 < q < \infty$*, the support of* v *is a subset of* Ω'*. Then, for every* $\varepsilon > 0$ *there exists a function* $v_\varepsilon \in C_0^\infty(\Omega')^m$ *with*

$$\|v - v_\varepsilon\|_{\mathfrak{L},q} < \varepsilon.$$

Proof. Step I: Transformation of $\partial(Bv) * \psi$ *and of* $\partial(Bv^\delta)$
Choose v as in the statement of the theorem. Let $u \in L^{q'}(\Omega')^l$ and assume that $\delta < \operatorname{dist}(\operatorname{supp}(v), \partial\Omega')$ and $\delta' < \operatorname{dist}(\partial\Omega', \partial\Omega)$. We denote differentiation with respect to the variable $\dot{x}$ by $\dot{\partial}_k$; this implies that

$$\dot{\partial}_k \psi_\delta(x - \dot{x}) = -\partial_k \psi_\delta(x - \dot{x}).$$

Hence, by extending u to Ω by setting $u(x) = 0$ outside Ω', we have that

$$\begin{aligned}\int_\Omega (\partial_k(B_{ijk}\,v_j) * \psi_\delta)(x)\,\overline{u_i(x)}\,\mathrm{d}x &= \int_\Omega\int_\Omega \dot{\partial}_k(B_{ijk}(\dot{x})\,v_j(\dot{x}))\,\psi_\delta(x - \dot{x})\,\mathrm{d}\dot{x}\,\overline{\lim_{\delta'\to 0} u_i^{\delta'}(x)}\,\mathrm{d}x \\ &= \lim_{\delta'\to 0}\int_\Omega \dot{\partial}_k B_{ijk}(\dot{x})\,v_j(\dot{x})\int_\Omega \psi_\delta(x - \dot{x})\,\overline{u_i^{\delta'}(x)}\,\mathrm{d}x\,\mathrm{d}\dot{x}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\delta'\to 0} \int_\Omega -B_{ijk}(\dot{x})\, v_j(\dot{x})\, \dot{\partial}_k \int_\Omega \psi_\delta(x-\dot{x})\, \overline{u_i^{\delta'}(x)}\, \mathrm{d}x\, \mathrm{d}\dot{x} \\
&= \lim_{\delta'\to 0} \int_\Omega \int_\Omega -B_{ijk}(\dot{x})\, v_j(\dot{x})\, \dot{\partial}_k \psi_\delta(x-\dot{x})\, \mathrm{d}\dot{x}\, \overline{u_i^{\delta'}(x)}\, \mathrm{d}x \\
&= \int_\Omega \int_\Omega \partial_k(B_{ijk}(\dot{x})\, \psi_\delta(x-\dot{x}))\, v_j(\dot{x})\, \mathrm{d}\dot{x}\, \overline{u_i(x)}\, \mathrm{d}x.
\end{aligned}
$$

In the course of integration by parts we used the fact that

$$
\Big(\dot{x} \mapsto \int_\Omega \psi_\delta(x-\dot{x})\, \overline{u_i^{\delta'}(x)}\, \mathrm{d}x\Big) \in \mathscr{D}(\Omega)^l.
$$

As $L^{q'}(\Omega')^l$ is the dual space of $L^q(\Omega')^l$ and $\operatorname{supp}(\psi_\delta * \partial_k(B_{ijk}\, v_j)) \subset \Omega'$, we obtain the identity of the $L^q(\Omega')^l$ functions

$$
x \mapsto (\partial_k(B_{ijk}\, v_j) * \psi_\delta)(x) = x \mapsto \int_\Omega \partial_k(B_{ijk}(\dot{x})\, \psi_\delta(x-\dot{x}))\, v_j(\dot{x})\, \mathrm{d}\dot{x}.
$$

Similarly, we have that

$$
\begin{aligned}
x \mapsto \partial_k(B_{ijk}\, (\psi_\delta * v_j))(x) &= x \mapsto \partial_k\Big(B_{ijk}(x) \int_\Omega v_j(\dot{x})\, \psi_\delta(x-\dot{x})\, \mathrm{d}\dot{x}\Big) \\
&= x \mapsto \int_\Omega \partial_k(B_{ijk}(x)\, \psi_\delta(x-\dot{x}))\, v_j(\dot{x})\, \mathrm{d}\dot{x}.
\end{aligned}
$$

Step II: Boundedness of $(B(x)-B(\dot{x}))(\partial\psi_\delta)v$

Suppose that $\delta < \operatorname{dist}(\operatorname{supp}(v), \partial\Omega')$. We consider the operator

$$
T_\delta : L^1_{\mathrm{loc}}(\Omega)^m \to L^1_{\mathrm{loc}}(\Omega)^l,\ (T_\delta v)_i(x) = \int_\Omega (B_{ijk}(\dot{x}) - B_{ijk}(x))\, \partial_k \psi_\delta(x-\dot{x})\, v_j(\dot{x})\, \mathrm{d}\dot{x}.
$$

The support of $\dot{x} \mapsto \partial_k\psi_\delta(x-\dot{x})$ is contained in $B_\delta(x)$. Since the Sobolev embedding theorem holds in $B_\delta(x)$, the restrictions of components B_{ijk} to $B_\delta(x)$ are Lipschitz continuous functions for which

$$
\|B\| := \|B\|_{W^{1,\infty}(\Omega)^{l\times m\times n}}
$$

is a Lipschitz constant. Therefore we obtain continuity of T_δ on the subspace $L^1(\Omega)^m$:

$$
\begin{aligned}
\|T_\delta v\|_{L^1(\Omega)^l} &= \sum_{i,j,k} \int_\Omega \Big| \int_\Omega (B_{ijk}(\dot{x}) - B_{ijk}(x))\, \partial_k\psi_\delta(x-\dot{x})\, v_j(\dot{x})\, \mathrm{d}\dot{x} \Big|\, \mathrm{d}x \\
&\le \sum_{i,j,k} \int_\Omega \int_\Omega \Big| \frac{B_{ijk}(\dot{x}) - B_{ijk}(x)}{\delta} \Big|\, |\delta\, \partial_k\psi_\delta(x-\dot{x})|\, |v_j(\dot{x})|\, \mathrm{d}\dot{x}\, \mathrm{d}x \\
&\le \sum_{j,k} l\, \|B\| \int_\Omega \int_\Omega |\delta\, \partial_k\psi_\delta(x-\dot{x})|\, \mathrm{d}x\, |v_j(\dot{x})|\, \mathrm{d}\dot{x} \\
&= l\, \|B\|\, \|\nabla\psi_1\|_{L^1(\Omega)^n}\, \|v\|_{L^1(\Omega)^m},
\end{aligned}
$$

where we used that $\delta\, \partial_k \psi_\delta(x) = \delta^{-n}\, (\partial_k \psi_1)(\delta^{-1}x)$ and the transformation of variables $x \mapsto \delta x$. In the case of $v \in L^\infty(\Omega)^m$, a similar bound holds with respect to the L^∞-norm, since

$$\begin{aligned}
\|T_\delta v\|_{L^\infty(\Omega)^l} &= \max_i \ \operatorname*{ess-sup}_x \Big| \sum_{j,k} \int_\Omega (B_{ijk}(\dot x) - B_{ijk}(x))\, \partial_k \psi_\delta(x - \dot x)\, v_j(\dot x)\, \mathrm{d}\dot x \Big| \\
&\le \max_i \ \operatorname*{ess-sup}_x \sum_{j,k} \int_\Omega \Big| \frac{B_{ijk}(\dot x) - B_{ijk}(x)}{\delta} \Big| |\delta\, \partial_k \psi_\delta(x - \dot x)|\, |v_j(\dot x)|\, \mathrm{d}\dot x \\
&\le m\, \|B\| \Big(\sum_k \int_\Omega |\delta\, \partial_k \psi_\delta(x - \dot x)|\, \mathrm{d}\dot x \Big)\, \|v\|_{L^\infty(\Omega)^m} \\
&= m\, \|B\|\, \|\nabla \psi_1\|_{L^1(\Omega)^n}\, \|v\|_{L^\infty(\Omega)^m}.
\end{aligned}$$

Next we apply the Riesz-Thorin interpolation theorem to derive a bound for $v \in L^q(\Omega)^m$, cf. Theorem 58 in the Appendix. We select $\theta = 1 - 1/q = 1/q'$ and $p = q$, so that, for $v \in L^q(\Omega)^m$,

$$\begin{aligned}
&\Big\| \int_\Omega (B_{ijk}(\dot x) - B_{ijk}(\,\cdot\,))\, \partial_k \psi_\delta((\,\cdot\,) - \dot x)\, v_j(\dot x)\, \mathrm{d}\dot x \Big\|_{L^q(\Omega)^l} \\
&\qquad = \|T_\delta v\|_{L^q(\Omega)^l} \le 2\, l^q\, m^{1-q}\, \|B\|\, \|\nabla \psi_1\|_{L^1(\Omega)^n}\, \|v\|_{L^q(\Omega)^m}.
\end{aligned}$$

The constant 2 is included to cover both the real and the complex case, cf. Theorem 58.

Step III: Boundedness of $(\mathcal{L}v)^\delta - \mathcal{L}v^\delta$

Since $\psi_\delta(x) \ge 0$ for all $x \in \mathbb{R}^n$, we deduce that

$$\begin{aligned}
&\Big\| \sum_{j,k} \int_\Omega \partial_k (B_{ijk}(\dot x) - B_{ijk}(\,\cdot\,))\, \psi_\delta((\,\cdot\,) - \dot x)\, v_j(\dot x)\, \mathrm{d}\dot x \Big\|_{L^q(\Omega)^l} \qquad (1.2) \\
&\le \Big(\sum_{i=1}^{l} \int_\Omega \Big(\sum_{j,k} \int_\Omega |\partial_k (B_{ijk}(\dot x) - B_{ijk}(x))|\, |\psi_\delta(x - \dot x)\, v_j(\dot x)|\, \mathrm{d}\dot x \Big)^q \mathrm{d}x \Big)^{1/q} \\
&\le 2\, l\, m\, \|B\|\, \|\, \psi_\delta * |v|\, \|_{L^q(\Omega)^m} \le 2\, l\, m\, \|B\|\, \|\psi_1\|_{L^1(\Omega)}\, \|v\|_{L^q(\Omega)^m}.
\end{aligned}$$

After applying the identities from Step I, we use the product rule:

$$\begin{aligned}
&\|\partial_k (B_{ijk}\, v_j) * \psi_\delta - \partial_k (B_{ijk}\, v_j^\delta)\|_{L^q(\Omega)^l} \\
&\qquad = \Big\| \int_\Omega \partial_k \big((B_{ijk}(\dot x) - B_{ijk}(\,\cdot\,))\, \psi_\delta((\,\cdot\,) - \dot x) \big) v_j(\dot x)\, \mathrm{d}\dot x \Big\|_{L^q(\Omega)^l} \qquad (1.3) \\
&\qquad \le 2\, l\, m\, \|B\|\, \|\psi_1\|_{W^{1,1}(\Omega)}\, \|v\|_{L^q(\Omega)^m}.
\end{aligned}$$

Notice that the parentheses in (1.2) and (1.3) are set differently. Finally, as in (1.2),

$$\begin{aligned}
\|(C_{ij} v_j) * \psi_\delta - C_{ij}\, v_j^\delta\|_{L^q(\Omega)^l} &= \Big\| \int_\Omega ((C_{ij}(\dot x) - C_{ij}(\,\cdot\,))\, \psi_\delta((\,\cdot\,) - \dot x))\, v_j(\dot x)\, \mathrm{d}\dot x \Big\|_{L^q(\Omega)^l} \\
&\le 2\, l\, \|C\|_{L^\infty(\Omega)^{l \times m}}\, \|\psi_1\|_{L^1(\Omega)}\, \|v\|_{L^q(\Omega)^m}.
\end{aligned}$$

Step IV: Weakly converging sequence
The last step shows that the sequence

$$s_1(\ell) = \mathcal{L}(v) * \psi_{1/\ell} - \mathcal{L}(v * \psi_{1/\ell})$$

is bounded in $L^q(\Omega)^l$. Hence by the Banach-Alaoglu theorem there exists a sequence $t_1 : \mathbb{N} \to \mathbb{N}$ such that $s_1 \circ t_1$ is weakly converging to an element $\dot{v} \in L^q(\Omega)^l$. We want to show that $\dot{v} = 0$. Since $\operatorname{supp}(v) \subset \Omega'$ it is enough to test $\dot{v}$ with functions $w \in L^{q'}(\Omega')^l$. We let $\dot{\ell} = 1/t_1(\ell)$ and $\delta' < \operatorname{dist}(\partial\Omega', \partial\Omega)$:

$$\begin{aligned}\int_\Omega \dot{v}_i \overline{w_i}\, \mathrm{d}x &= \lim_{\delta' \to 0} \int_\Omega \dot{v}_i\, \overline{w_i^{\delta'}}\, \mathrm{d}x \\ &= \lim_{\delta' \to 0} \lim_{\ell \to \infty} \int_\Omega \big(\mathcal{L}(v) * \psi_{\dot{\ell}} - \mathcal{L}\big(v * \psi_{\dot{\ell}}\big)\big)\, \overline{w^{\delta'}}\, \mathrm{d}x \\ &= \lim_{\delta' \to 0} \lim_{\ell \to \infty} \int_\Omega \big(v_j - v_j * \psi_{\dot{\ell}}\big)\, \overline{\mathcal{L}' w^{\delta'}}\, \mathrm{d}x = 0,\end{aligned}$$

and so $\dot{v} = 0$. We used that $w^{\delta'} \in \mathscr{D}(\Omega)^l$.
Step V: Strongly converging sequence
Given $\varepsilon > 0$, we can select a sequence $t_2 : \mathbb{N} \to \mathbb{N}$, such that $t_3 = t_1 \circ t_2$ has the property that for all $\ell \in \mathbb{N}$ we have $\ddot{\ell} := 1/t_3(\ell) < \delta$ and

$$\|v - v * \psi_{\ddot{\ell}}\|_{L^q(\Omega)^m} < \frac{\varepsilon}{3 \cdot 2^\ell}, \qquad \|\mathcal{L}(v) - \mathcal{L}(v) * \psi_{\ddot{\ell}}\|_{L^q(\Omega)^l} < \frac{\varepsilon}{3 \cdot 2^\ell}.$$

Using Mazur's theorem, cf. (Rudin 1991, p. 67), there exists a finite convex combination

$$v_\varepsilon = \sum_{\ell=0}^{s} \lambda_\ell\, v * \psi_{\ddot{\ell}}, \qquad \sum_{\ell=0}^{s} \lambda_\ell = 1, \;\; \lambda_\ell \in [0,1], \;\; s \in \mathbb{N},$$

such that

$$\Big\| \Big(\sum_{\ell=0}^{s} \lambda_\ell\, \mathcal{L}(v) * \psi_{\ddot{\ell}} \Big) - \mathcal{L}(v_\varepsilon) \Big\|_{L^q(\Omega)^l} = \Big\| \sum_{\ell=0}^{s} \lambda_\ell\, s_1(\ell) \Big\|_{L^q(\Omega)^l} < \frac{\varepsilon}{3}. \tag{1.4}$$

Hence

$$\|v - v_\varepsilon\|_{L^q(\Omega)^m} \le \sum_{\ell=0}^{s} \lambda_\ell \|v - v * \psi_{\ddot{\ell}}\|_{L^q(\Omega)^m} < \sum_{\ell=0}^{\infty} \frac{\varepsilon}{3 \cdot 2^\ell} < \frac{\varepsilon}{3}.$$

Similarly, but by using (1.4) and the triangle inequality, we have that

$$\|\mathcal{L}(v) - \mathcal{L}(v_\varepsilon)\|_{L^q(\Omega)^l} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Consequently, $\|v - v_\varepsilon\|_{\mathcal{L},q} < \varepsilon$ and $v_\varepsilon \in C_0^\infty(\Omega')^m$. ////

Corollary 1 *For $\delta, \varepsilon > 0$ there exists a non-negative function $\phi \in C_0^\infty(B_\delta(0))$ such that, for v as in Theorem 1,*

$$\|v - v * \phi\|_{\mathcal{L},q} < \varepsilon.$$

Proof. Choose $\phi = \sum_\ell \lambda_\ell \psi_{\ddot{\ell}}$ for λ_ℓ and $\psi_{\ddot{\ell}}$ defined in Step V. ////

The transformations in Step I and the first bound in Step II are based on ideas in (Friedrichs 1954). We can extend Theorem 1 to all functions in $W^q_{\mathcal{L}}(\Omega)$ using the techniques introduced by Meyers and Serrin in (Meyers and Serrin 1964), see also (Adams and Fournier 2003, p. 67). In this case it becomes necessary to admit approximations by functions with noncompact support.

Theorem 2 *The space $C^\infty(\Omega)^m \cap W^q_{\mathcal{L}}(\Omega)$ is dense in $W^q_{\mathcal{L}}(\Omega)$, $1 < q < \infty$.*

Proof. Let Ω_i be open subsets in Ω such that $\Omega_i \Subset \Omega_{i+1}$ and

$$\bigcup_{i=1}^{\infty} \Omega_i = \Omega.$$

Let $\mathcal{F}$ be a partition of unity of Ω subordinate to the covering $(\Omega_{i+1} \setminus \overline{\Omega_{i-1}})_{i\in\mathbb{N}}$, where Ω_{-1} is taken as the empty set. Let $\dot{f}_i$ be the sum of all $f_j \in \mathcal{F}$ for which i is the smallest index such that $\operatorname{supp}(f_j) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$. Then the $\dot{f}_i$ sum to one, too. Choose $\varepsilon > 0$. For $i \in \mathbb{N}$ and $v \in W^q_{\mathcal{L}}(\Omega)$ there exists, according to the last theorem, a function $v_{\varepsilon,i}$ in $C_0^\infty(\Omega_{i+1} \setminus \overline{\Omega_{i-1}})^m$ such that

$$\|\dot{f}_i\, v - v_{\varepsilon,i}\|_{\mathcal{L},q} < \frac{\varepsilon}{2^i}.$$

By construction $\Omega_{j+1} \setminus \overline{\Omega_{j-1}}$ is only intersected by the supports of $v_{\varepsilon,j-2}$, $v_{\varepsilon,j-1}$ and $v_{\varepsilon,j}$. Hence the sum

$$v_\varepsilon = \lim_{j\to\infty} \sum_{i=0}^{j} v_{\varepsilon,i}$$

is defined and is a member of $C^\infty(\Omega)^m$. Notice that because of the layout of the supports of the $v_{\varepsilon,i}$, the sequence

$$j \mapsto \Big(\sum_{i=0}^{j+1} v_{\varepsilon,i}\Big)_+\Big|_{\Omega_j}$$

of $L^q(\Omega)^m$-functions exhibits monotonic and pointwise convergence to $(v_\varepsilon)_+$ as $j \to \infty$. Its members are bounded in the $L^q(\Omega)^m$-norm by $\|v\|_{L^q(\Omega)^m} + \varepsilon$ because

$$\Big\|\Big(\sum_{i=0}^{j+1} v_{\varepsilon,i}\Big)_+\Big\|_{L^q(\Omega_j)^m} \leq \Big\|\sum_{i=0}^{j+1} \dot{f}_i\, v\Big\|_{L^q(\Omega_j)^m} + \sum_{i=0}^{j+1} \|\dot{f}_i\, v - v_{\varepsilon,i}\|_{L^q(\Omega_j)^m}.$$

Hence by the monotone convergence theorem $(v_\varepsilon)_+$ is $L^q(\Omega)^m$-integrable. The same argument applied to $(v_\varepsilon)_-$ and to $(\mathfrak{L}v_\varepsilon)_+$ and $(\mathfrak{L}v_\varepsilon)_-$ asserts that $v_\varepsilon \in W^q_{\mathfrak{L}}(\Omega)$. We conclude that

$$\|v - v_\varepsilon\|_{\mathfrak{L},q} \leq \sum_{i=0}^{\infty} \left\| f_i\, v - v_{\varepsilon,i} \right\|_{\mathfrak{L},q} < \varepsilon.$$

This proves the density of smooth functions in $W^q_{\mathfrak{L}}(\Omega)$. ////

Theorem 3 *The space $W^q_{\mathfrak{L}}(\Omega)$ is a Banach space.*

Proof. Theorem 2 states that the graph space is a subset of the completion of $C^\infty(\Omega)^m \cap W^q_{\mathfrak{L}}(\Omega)$ in the graph norm. Let $(v_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $C^\infty(\Omega)^m \cap W^q_{\mathfrak{L}}(\Omega)$. Then $(v_i)_i$ converges to an element $\dot{v} \in L^q(\Omega)^m$ in the $L^q(\Omega)^m$-norm and $(\mathfrak{L}v_i)_i$ converges to an element $\dot{v}_{\mathfrak{L}} \in L^q(\Omega)^l$ in the $L^q(\Omega)^l$-norm. But $\dot{v}_{\mathfrak{L}}$ is the image of $\dot{v}$ under $\mathfrak{L}$ in the distributional sense since, for all $\phi \in \mathscr{D}(\Omega)^l$,

$$\int_\Omega v\, \overline{\mathfrak{L}'\phi}\, \mathrm{d}x = \lim_{i\to\infty} \int_\Omega v_i\, \overline{\mathfrak{L}'\phi}\, \mathrm{d}x = \lim_{i\to\infty} \int_\Omega \mathfrak{L}v_i\, \overline{\phi}\, \mathrm{d}x = \int_\Omega \mathfrak{L}v\, \overline{\phi}\, \mathrm{d}x.$$

Therefore, $\overline{C^\infty(\Omega)^m \cap W^q_{\mathfrak{L}}(\Omega)}$ is a subset of $W^q_{\mathfrak{L}}(\Omega)$. ////

As for $W^{1,q}(\Omega)$, one needs to consider density of $C^\infty(\Omega)^m$ and of $C_0^\infty(\mathbb{R}^n)^m$ separately. In our context $C_0^\infty(\mathbb{R}^n)^m$ is the set of all functions on Ω which are the restriction of a smooth function with compact support in $\mathbb{R}^n$.

Example 5 *Given $\Omega = (-1,0) \cup (0,1)$, $B = 1$ and $C = 0$, consider the function*

$$v : \Omega \to \{-1, 1\}, x \mapsto \operatorname{sign}(x).$$

The function v lies in $C^\infty(\Omega)$ and therefore also in the $W^q_{\mathfrak{L}}(\Omega)$-closure of $C^\infty(\Omega)$. However it is not a member of $\overline{C_0^\infty(\mathbb{R})}$. A good approximation by a $C_0^\infty(\mathbb{R})$-function to v in the L^2-sense has a large gradient around the origin and hence it is a bad approximation in the $W^q_{\mathfrak{L}}(\Omega)$ semi-norm. For details we refer to (Adams and Fournier 2003, p. 68).

A sufficient condition for the density of $C_0^\infty(\mathbb{R}^n)^m$ in $W^q_{\mathfrak{L}}(\Omega)$ is that Ω has the segment property. This means that there exists a family $\mathfrak{N}$ which associates to every $x \in \partial\Omega$ a neighbourhood N_x of x and a nonzero vector y_x such that for $z \in \overline{\Omega} \cap N_x$ and $0 < \tau < 1$ the vector $z + \tau\, y_x$ is an element of Ω. Visually the definition means that Ω is not allowed to lie on both sides of its boundary. The definition of the segment property admits $\partial\Omega = \emptyset$, i.e. $\Omega = \mathbb{R}^n$.

Theorem 4 *If Ω fulfills the segment property, then $C_0^\infty(\mathbb{R}^n)^m$ is dense in $W_{\mathcal{L}}^q(\Omega)$.*

Proof: Let f be a fixed function in $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$\|x\| < 1 \Rightarrow f(x) = 1 \quad \text{and} \quad \|x\| > 2 \Rightarrow f(x) = 0.$$

Let $f_\varepsilon(x) := f(\varepsilon x)$ for $\varepsilon \in (0,1)$. Then $f_\varepsilon(x) = 1$ if $\|x\| \leq 1/\varepsilon$. If $v \in W_{\mathcal{L}}^q(\Omega)$, then

$$v_\varepsilon := (f_\varepsilon\, v_1, \ldots, f_\varepsilon\, v_n)$$

belongs to $W_{\mathcal{L}}^q(\Omega)$ and has bounded support. We abbreviate $\|B\| := \|B\|_{L^\infty(\Omega)^{l\times m\times n}}$ and $\Omega_\varepsilon := \{x \in \Omega : \|x\| > 1/\varepsilon\}$. It follows from the chain of inequalities

$$|v_\varepsilon|_{\mathcal{L},q,\Omega_\varepsilon} \leq \|f_\varepsilon\|_{L^\infty(\Omega)}\, |v|_{\mathcal{L},q,\Omega_\varepsilon} + \|B\|\, |f_\varepsilon|_{W^{1,\infty}(\Omega)}\, \|v\|_{L^q(\Omega_\varepsilon)^m} \leq (1 + \|B\|)\, \|f\|_{W^{1,\infty}(\Omega)}\, \|v\|_{\mathcal{L},q,\Omega_\varepsilon}$$

that

$$\|v - v_\varepsilon\|_{\mathcal{L},q,\Omega} = \|v - v_\varepsilon\|_{\mathcal{L},q,\Omega_\varepsilon} \leq \|v\|_{\mathcal{L},q,\Omega_\varepsilon} + \|v_\varepsilon\|_{\mathcal{L},q,\Omega_\varepsilon} \leq C\, \|v\|_{\mathcal{L},q,\Omega_\varepsilon}.$$

The right-hand side tends to zero as ε tends to 0. In other words, all $v \in W_{\mathcal{L}}^q(\Omega)$ can be approximated by $W_{\mathcal{L}}^q(\Omega)$-functions with bounded support. In order to prove density we may therefore assume, in combination with Theorem 2, that v is an element of $C^\infty(\Omega)^m \cap W_{\mathcal{L}}^q(\Omega)$ and that $\text{supp}(v)$ is bounded.

We define $\mathcal{N}$ to be the family of neighbourhoods N_x referred to in the definition of the segment property. Thus the set

$$F = \text{supp}(v) \setminus \bigcup_{N_x \in \mathcal{N}} N_x$$

is compact and is contained in Ω. There exists an open set N_0 such that $F \Subset N_0 \Subset \Omega$. Since $\text{supp}(v)$ is compact, we can select a finite number of neighbourhoods $N_1, \ldots, N_\kappa \in \mathcal{N}$, such that $\text{supp}(v) \subset N_0 \cup N_1 \cup \cdots \cup N_\kappa$. Moreover, we can select subsets $\dot{N}_i \Subset N_i$, $i \in \{0, \ldots, \kappa\}$, such that $\text{supp}(v) \subset \dot{N}_0 \cup \dot{N}_1 \cup \cdots \cup \dot{N}_\kappa$, too.

Let $\mathcal{F}$ be a finite partition of unity of the union of the $\dot{N}_i$ subordinate to the $\dot{N}_i$. Let $\dot{f}_i$ be the locally finite sum of all $f_j \in \mathcal{F}$ for which i is the smallest index such that $\text{supp}(f_j) \subset N_i$. Let $v_i = \dot{f}_i v$ on Ω. Suppose that for each i we can find a $v_{\varepsilon,i} \in C_0^\infty(\mathbb{R}^n)^m$ such that

$$\|v_i - v_{\varepsilon,i}\|_{\mathcal{L},q} < \frac{\varepsilon}{\kappa + 1}. \tag{1.5}$$

Then, putting $v_\varepsilon = \sum_{i=0}^\kappa v_{\varepsilon,i}$, we obtain

$$\|v - v_\varepsilon\|_{\mathcal{L},q} \leq \sum_{i=0}^{\kappa} \|v_i - v_{\varepsilon,i}\|_{\mathcal{L},q} < \varepsilon.$$

Since $\operatorname{supp}(v_0) \Subset \dot{N}_0 \Subset \Omega$, clearly $v_0 \in C_0^\infty(\mathbb{R}^n)^m$ and therefore $v_{\varepsilon,0} := v_0$.

Fix $i \in \{1, \ldots, \kappa\}$. We extend v_i to be identically zero outside Ω. Then $v_i \in C^\infty(\Omega \setminus \Gamma)^m$ where $\Gamma := \partial\Omega \cap \dot{N}_i$. Let y be the nonzero vector associated with N_i in the definition of the segment property. We define, for $\tau \in \mathbb{R}$,

$$v_\tau(x) := v_i(x + \tau\, y). \tag{1.6}$$

A positive τ corresponds to a translation out of Ω. We first show that finite convex combinations of functions v_{τ_i} for suitably chosen values τ_i approximate v_i arbitrarily well. In a second step we replace the convex combination by functions in $C_0^\infty(\mathbb{R}^n)^m$.

Select $\varphi \in \mathscr{D}(\Omega)^m$. The support of φ is bounded away from $\partial\Omega$. Therefore, there exists a $v_\varphi \in C_0^\infty(\mathbb{R}^n)^m \subset L^q(\Omega)^m$ such that v_φ equals v_i on the restriction to $\operatorname{supp}(\varphi) + B_\delta(0)$ where $0 < \delta < \operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega)$. Similarly to (1.6), we set, for $\tau \in \mathbb{R}$,

$$v_{\varphi,\tau}(x) := v_\varphi(x + \tau\, y).$$

Because translation and pointwise multiplication are continuous in $L^q(\Omega)^m$, we observe weak convergence of $\mathcal{L}v_\tau$ as $\tau \to 0$: if $\tau \in (0, \delta)$

$$\int_\Omega \mathcal{L}v_\tau\, \overline{\varphi}\, \mathrm{d}x = \int_\Omega \mathcal{L}v_{\varphi,\tau}\, \overline{\varphi}\, \mathrm{d}x \to \int_\Omega \mathcal{L}v_\varphi\, \overline{\varphi}\, \mathrm{d}x = \int_\Omega \mathcal{L}v_i\, \overline{\varphi}\, \mathrm{d}x \quad \text{as} \quad \tau \to 0.$$

According to (1.6), $v_{1/(j+\ell)}$ denotes v_τ with $\tau = 1/(j+\ell)$. By Mazur's theorem we can select a sequence $(\dot{v}_j)_{j\in\mathbb{N}}$ of finite convex combinations

$$\dot{v}_j = \sum_{\ell=0}^{s} \lambda_\ell\, v_{1/(j+\ell)}, \qquad \sum_{\ell=0}^{s} \lambda_\ell = 1, \;\; \lambda_\ell \in [0,1], \;\; s \in \mathbb{N}, \tag{1.7}$$

such that $(\dot{v}_j)_{j\in\mathbb{N}}$ converges strongly in the first component of the operator:

$$\|(\mathcal{L}\dot{v}_j)_1 - (\mathcal{L}v_i)_1\|_{L^q(\Omega)} < 1/j.$$

In view of

$$\Big|\int \mathcal{L}(\dot{v}_j - v_i)\, \overline{\varphi}\, \mathrm{d}x\Big| \le \sum_{\ell=0}^{s} \lambda_\ell \Big|\int \mathcal{L}(v_{(j+\ell)} - v_i)\, \overline{\varphi}\, \mathrm{d}x\Big|,$$

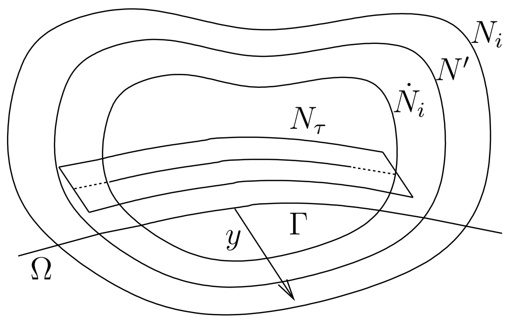
the sequence $(\dot{v}_j)_{j\in\mathbb{N}}$ is weakly convergent in the remaining components of the operator. Repeating the application of Mazur's theorem inductively for the second, third up to the n-th component, we construct a sequence $(\ddot{v}_j)_{j\in\mathbb{N}}$ which converges to v_i in the graph space semi-norm. Continuity of translation in $L^q(\Omega)^m$ implies strong convergence of $(\ddot{v}_j)_{j\in\mathbb{N}}$ to v_i in $W^q_{\mathcal{L}}(\Omega)$.

We now replace $\ddot{v}_j$ by a function in $C_0^\infty(\mathbb{R}^n)^m$. Let $\alpha := \min(1, \operatorname{dist}(\dot{N}_i, \mathbb{R}^n \setminus N_i)/\|y\|)$. We then have $\dot{N}_i \cap \Omega \Subset \mathbb{R}^n \setminus (\Gamma - \tau\, y)$ for $\tau \in (0, \alpha)$. To prove this claim, choose an N' with $\dot{N}_i \Subset N' \Subset N_i$ such that $\operatorname{dist}(N', \mathbb{R}^n \setminus N_i)/\|y\| > \tau$.

Hence $\overline{\Gamma' - \tau\, y}$ with $\Gamma' := \partial\Omega \cap N'$ lies compactly in N_i. Choose $\beta > 0$ such that $\tau + \beta < 1$ and $\beta < \min(\tau, \operatorname{dist}(\Gamma' - \tau\, y, \mathbb{R}^n \setminus N_i))$. Then

$$N_\tau := \{x \in \mathbb{R}^n : \exists\, \tau' \in (-\beta, \beta) : x + \tau'\, y \in (\Gamma' - \tau\, y)\}$$

is a neighbourhood of $\Gamma - \tau\, y$ which does not intersect $\dot{N}_i \cap \Omega$.



We select $j \in \mathbb{N}$ such that

$$\|v_i - \ddot{v}_j\|_{\mathcal{L},q} < \frac{\varepsilon}{\kappa + 1} \quad \text{and} \quad 1/j < \alpha.$$

By construction $\ddot{v}_j$ is a finite convex combination of translations v_τ with $\tau \in (0, \alpha)$, so that we can expand $\ddot{v}_j$ like in (1.7). For each $v_{1/(j+\ell)}$ there exists a function $\dddot{v}_{1/(j+\ell)} \in C_0^\infty(\mathbb{R}^n)^m$ which coincides with $v_{1/(j+\ell)}$ on $\mathbb{R}^n \setminus N_{1/(j+\ell)}$. Then

$$v_{i,\varepsilon} = \sum_{\ell=0}^{s} \lambda_\ell\, \dddot{v}_{1/(j+\ell)}$$

is a $C_0^\infty(\mathbb{R}^n)^m$-function which fulfills inequality (1.5) as required. ////

Corollary 2 *The set of $W^q_{\mathcal{L}}(\Omega)$-functions with compact support is dense in $W^q_{\mathcal{L}}(\Omega)$.*

The proof uses ideas from the proof of the corresponding result for $W^{1,q}(\Omega)$; compare with (Adams and Fournier 2003, p. 68). However, because the proof in (Adams and Fournier 2003) relies on the fact that $W^{1,q}(\Omega)$ is closed under translation, we reversed the order in which smooth approximation and translation are used and included weak convergence and Mazur's theorem in the argument.

Remark 2 *It is an interesting question, whether the above density results carry over to higher-order graph spaces; these are spaces* $\{v \in L^q(\Omega)^m : \mathcal{L}v \in L^q(\Omega)^l\}$ *where* $\mathcal{L}$ *contains higher-order derivatives. We would like to draw the reader's attention to Step II in the proof of Theorem 1. Observe how* T_δ *is bounded by cancelling the* $1/\delta$*-term, which is introduced by differentiating* ψ_δ*, and by using the Lipschitz continuity of* B*. In the case of an* ℓ*th-order operator, differentiation would introduce a factor* $1/\delta^\ell$ *instead. Suppose one continues the argument of Theorem 1 by requiring that*

$$\left|\frac{B_{ij\alpha}(\dot{x}) - B_{ij\alpha}(x)}{\delta^\ell}\right| \tag{1.8}$$

is bounded independently of x *and* $\dot{x}$*. We assume here that* $B_{ij\alpha}$ *is the coefficient tensor of the symbol of* $\mathcal{L}$*, where* α *is the multi-index of the partial derivatives. With this requirement the proof of the higher-order analogue of Theorem 1 could be analogously completed like most of the remainder of this section. However (1.8) implies also that* B *is constant for* $\ell > 1$*. But if* $\mathcal{L}$ *has constant coefficients one can in any case bypass Theorem 1 and directly use the sequence* $i \mapsto (v * \psi_{1/i})$ *in the other proofs of this section. It is beyond the scope of this dissertation to resolve in which circumstances smooth functions are dense in higher-order graph spaces if* $\mathcal{L}$ *has non-constant coefficients.*

1.3 Lipschitz Domains and Besov Spaces

The properties of $W^q_{\mathcal{L}}(\Omega)$ strongly depend on the regularity of Ω. In addition to the segment property, we consider Lipschitz domains, domains with a strong local Lipschitz condition, polyhedra as well as C^k-regular and analytic domains, $k \in \mathbb{N}$.

We say that Ω is a Lipschitz domain or, equivalently, has a Lipschitz boundary if for each point $x \in \partial\Omega$ there exists a neighbourhood N_x of x, an orthogonal coordinate transformation and a Lipschitz continuous function $f_x : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap N_x = \{y = (y_1, \dots, y_n) \in N_x : y_n > f(y_1, \dots, y_{n-1})\},$$

where $(y_1, \dots, y_n)$ are the transformed coordinates of y.

Let Ω be a Lipschitz domain. We say that Ω is a domain with strong local Lipschitz property if there exist positive numbers $\delta, L \in \mathbb{R}$, $R \in \mathbb{N}$, and a locally finite subcover $\mathcal{N}$ out of the N_x above such that:

1. no point in $\mathbb{R}^n$ is contained in more than R of the $N_x \in \mathcal{N}$;
2. the functions $f_x|_{N_x}$ have the Lipschitz constant L;

3. for all $y, \dot{y} \in \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) < \delta\}$ with $\operatorname{dist}(y, \dot{y}) < \delta$ there exists an $N_x \in \mathcal{N}$ such that $y, \dot{y} \in \{y \in N_x : \operatorname{dist}(y, \partial N_x) > \delta\}$.

The definition of the strong local Lipschitz property is due to (Adams and Fournier 2003). For a comparison with similar notions of Lipschitz regularity we refer to the classification of domains by Fraenkel, cf. (Fraenkel 1979). Lipschitz domains with compact boundary, for example bounded domains, have the strong local Lipschitz property. Lipschitz domains satisfy the segment property.

Let x be in $\partial\Omega$ and choose N_x and f_x as above. According to Rademacher's theorem, the classical derivative of a Lipschitz function exists almost everywhere, cf. (Federer 1969, p. 216). Consider

$$x = (y_1, \ldots, y_{n-1}, f_x(y_1, \ldots, y_{n-1}))$$

in the local coordinate system. If the classical derivative of f exists at x we assign to x the outward normal

$$\nu(x) := \frac{(\partial_1 f_x(y_1, \ldots, y_{n-1}), \ldots, \partial_{n-1} f_x(y_1, \ldots, y_{n-1}), -1)}{\|(\partial_1 f_x(y_1, \ldots, y_{n-1}), \ldots, \partial_{n-1} f_x(y_1, \ldots, y_{n-1}), -1)\|_2}. \tag{1.9}$$

We remark that if there is a second neighbourhood $\dot{N}_x \in \mathcal{N}$ of x to which the functions $\dot{f}_x$ is associated, then $\dot{f}_x$ applied to formula (1.9) defines almost everywhere in $\dot{N}_x \cap N_x \cap \partial\Omega$ the same outward normal as f_x.

We call a domain C^k-regular, $k \geq 1$, or analytic if its boundary is a C^k-differentiable or analytic manifold, respectively.

Let us consider a domain Ω satisfying the strong local Lipschitz condition which is the intersection of finitely many C^k-regular domains Ω_i. Then $\partial\Omega$ is a subset of the union of the boundaries $\partial\Omega_i$. Let E be the set of all $x \in \partial\Omega$ which are contained in more than one $\partial\Omega_i$. If the closure of E is a null set in the $(n-1)$-dimensional Lebesgue measure, we call Ω a C^k-regular polyhedron. Let F_i be the set of all points in $\partial\Omega$ which are only contained in the boundary $\partial\Omega_i$. Then we name the simply connected components of F_i faces of Ω. Finally, a non-empty intersection between the closures of two distinct faces is called an facet.

We introduce Besov spaces on the boundary of Ω in order to define a trace operator of the graph space. To avoid unnecessary technicalities we only consider the Besov space $B^{q',q',1-1/q'}(\partial\Omega)$ and its dual $B^{q,q,-1/q}(\partial\Omega)$, $q' \in (1, \infty)$. A function f is contained in $B^{q',q',1-1/q'}(\partial\Omega)$ if, and only if, f is an $L^{q'}(\partial\Omega)$-function, for which the Besov space norm

$$\|f\|_{B^{q',q',1-1/q'}(\partial\Omega)} := \Big(\|f\|^{q'}_{L^{q'}(\partial\Omega)} + \int_{\substack{x,y\in\partial\Omega \\ \operatorname{dist}(x,y)<1}} \frac{|f(x) - f(y)|^{q'}}{\operatorname{dist}(x, y)^{n+q'}} \,\mathrm{d}S(x, y)\Big)^{1/q'} \tag{1.10}$$

is finite with respect to the Hausdorff measure on $\partial\Omega$.

Some authors call the space $B^{q',q',1-1/q'}(\partial\Omega)$ a fractional-order Sobolev space and use the notation $W^{1-1/q',q'}(\partial\Omega)$. However this terminology is not used consistently in the literature. For instance, in (Adams and Fournier 2003) fractional-order Sobolev space are constructed by the complex method of interpolation and differ from the class of Besov spaces, which is defined via the real method of interpolation, cf. (Adams and Fournier 2003, pp. 230 & 250).

Provided Ω is a Lipschitz domain, we have the characterisation

$$B^{q',q',1-1/q'}(\partial\Omega) = \{v \in L^{q'}(\partial\Omega) : \exists\, w \in W^{1,q'}(\mathbb{R}^n) : v = w|_{\partial\Omega}\}, \tag{1.11}$$

where the restriction operation is defined, for instance, by virtue of a density argument. Equation (1.11) describes a special case of a theorem in (Jonsson and Wallin 1984). Jonsson and Wallin replace $\partial\Omega$ by a closed set for which it is merely required that it has a Hausdorff dimension which lies strictly between 0 and n. In the proof of (1.11), the authors demonstrate the existence of an extension operator

$$\mathcal{E}_{\partial\Omega,\mathbb{R}^n} : B^{q',q',1-1/q'}(\partial\Omega) \to W^{1,q'}(\mathbb{R}^n),\ (\mathcal{E}_{\partial\Omega,\mathbb{R}^n} v)|_{\partial\Omega} = v.$$

If Ω fulfills the strong local Lipschitz property, there also exists an extension operator

$$\mathcal{E}_{\Omega,\mathbb{R}^n} : W^{1,q'}(\Omega) \to W^{1,q'}(\mathbb{R}^n),\ (\mathcal{E}_{\Omega,\mathbb{R}^n} v)|_{\Omega} = v.$$

This result is due to Stein, cf. (Stein 1970). Hence we can define the surjective $W^{1,q'}(\Omega)$-trace operator

$$\mathcal{T}_{W^{1,q'}(\Omega)} : W^{1,q'}(\Omega) \to B^{q',q',1-1/q'}(\partial\Omega), v \mapsto (\mathcal{E}_{\Omega,\mathbb{R}^n} v)|_{\partial\Omega}$$

and its right inverse

$$\mathcal{E}_{\partial\Omega,\Omega} : B^{q',q',1-1/q'}(\partial\Omega) \to W^{1,q'}(\Omega), v \mapsto (\mathcal{E}_{\partial\Omega,\mathbb{R}^n} v)|_{\Omega}.$$

We remark that all the operators considered in this section are continuous and linear. As usual, we shall represent linear functionals Λ contained in $B^{q,q,-1/q}(\partial\Omega)$ with the bracket notation:

$$\Lambda v =: \langle \Lambda, v\rangle_{\partial\Omega}, \qquad v \in B^{q',q',1-1/q'}(\partial\Omega). \tag{1.12}$$

Definition (1.12) is motivated by the observation that $L^q(\partial\Omega)$ is dense in $B^{q,q,-1/q}(\partial\Omega)$. Thus we can find a sequence $(\Lambda_i)_{i\in\mathbb{N}}$ with $\Lambda_i \in L^q(\partial\Omega)$ which converges to Λ. Then,

$$\langle \Lambda, v\rangle_{\partial\Omega} = \lim_{i\to\infty} \int_{\partial\Omega} \Lambda_i\, \overline{v}\, \mathrm{d}S.$$

For the reader's convenience we prove the density of $L^q(\partial\Omega)$ in $B^{q,q,-1/q}(\partial\Omega)$ next.

Theorem 5 *The Besov space $B^{q',q',1-1/q'}(\partial\Omega)$ is reflexive.*

Proof: Let

$$M := \{(x,y) \in \partial\Omega \times \partial\Omega : \operatorname{dist}(x,y) < 1\}.$$

Observe that $\partial\Omega$ and M are disjoint subsets of $\partial\Omega \cup M$ since the elements of M are pairs. We define on $\partial\Omega \cup M$ the measure

$$\mu(\kappa) := \int_{\kappa\cap\partial\Omega} \mathrm{d}S + \int_{\kappa\cap M} \frac{\mathrm{d}S(x,y)}{\operatorname{dist}(x,y)^{n+q'-2}},$$

where κ is a subset in $\partial\Omega \cup M$, element in the σ-algebra induced by the Hausdorff measure on $\partial\Omega$. Consider the operator

$$\Phi : B^{q',q',1-1/q'}(\partial\Omega) \to L^{q'}(\partial\Omega \cup M, \mu), v \mapsto w,$$

where

$$w(z) = \begin{cases} v(z) & : \text{ if } z \in \partial\Omega \\ v(x) - v(y) : & \text{ if } z = (x,y) \in M. \end{cases}$$

Then Φ maps $B^{q',q',1-1/q'}(\partial\Omega)$ isometrically onto a closed subspace of $L^{q'}(\partial\Omega \cup M, \mu)$. We now use that closed subspaces of reflexive Banach spaces are reflexive themselves. ////

Given two reflexive Banach spaces X and Y and a continuous linear operator $A : X \to Y$, we have that A is injective if, and only if, the image of the dual operator $A' : Y' \to X'$ is dense in X', cf. (Wloka 1987, pp. 261-262). Clearly, the embedding of $B^{q',q',1-1/q'}(\partial\Omega)$ into $L^{q'}(\partial\Omega)$ is injective and therefore the dual $L^q(\partial\Omega)$ is dense in $B^{q,q,-1/q}(\partial\Omega)$.

1.4 The Closure of $\mathscr{D}(\Omega)^m$

We denote the closure of $\mathscr{D}(\Omega)^m$ in $W^q_{\mathfrak{L}}(\Omega)$ by

$$W^q_{\mathfrak{L},0}(\Omega) := \overline{\mathscr{D}(\Omega)^m}. \tag{1.13}$$

Just as for Sobolev spaces, the closure of $\mathscr{D}(\Omega)^m$ constitutes an important subspace of $W^q_{\mathfrak{L}}(\Omega)$. It is particularly helpful that there are a number of different but equivalent descriptions of $W^q_{\mathfrak{L},0}(\Omega)$. In this section we investigate how $W^q_{\mathfrak{L},0}(\Omega)$ can be defined in terms of zero extensions, which we shall define next.

Given a function $v \in W^q_{\mathfrak{L}}(\Omega)$ we call the function

$$\dot{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

the zero extension of v. Let $\dot{B}$ be a tensor function in $W^{1,\infty}(\mathbb{R}^n)^{l\times m\times n}$ and let $\dot{C}$ be a matrix function in $L^\infty(\mathbb{R}^n)^{l\times m}$ so that $\dot{B}|_\Omega = B$ and $\dot{C}|_\Omega = C$. Given

$$\dot{\mathfrak{L}} : L^q(\mathbb{R}^n)^m \to \mathscr{D}'(\mathbb{R}^n)^m, v \mapsto \partial_k(\dot{B}_{ijk}v_j) + \dot{C}_{ij}\, v_j,$$

we call the graph space $W^q_{\dot{\mathfrak{L}}}(\mathbb{R}^n)$ an extension of $W^q_{\mathfrak{L}}(\Omega)$.

Suppose that Ω is a domain with strong local Lipschitz condition; then by the Sobolev embedding theorem the coefficient tensor B is Lipschitz continuous, cf. (Adams and Fournier 2003, p. 85). Kirszbraun's theorem states that every Lipschitz function defined on a subset of $\mathbb{R}^n$ can be extended to all of $\mathbb{R}^n$ without increasing the Lipschitz constant, cf. (Federer 1969, p. 201). Therefore the strong local Lipschitz condition ensures the existence of an extension $W^q_{\dot{\mathfrak{L}}}(\mathbb{R}^n)$ of $W^q_{\mathfrak{L}}(\Omega)$. However in many cases $W^q_{\mathfrak{L}}(\Omega)$ can also be extended from more general domains, provided the coefficients B and C possess additional regularity. For instance if B and C are constant then $W^q_{\mathfrak{L}}(\Omega)$ can be extended from all open sets Ω to $\mathbb{R}^n$.

Theorem 6 *Let Ω be a Lipschitz domain. Suppose that $\dot{B}$ and $\dot{C}$ are chosen as above so that $W^q_{\dot{\mathfrak{L}}}(\mathbb{R}^n)$ is an extension of $W^q_{\mathfrak{L}}(\Omega)$. Then $v \in W^q_{\mathfrak{L}}(\Omega)$ is a member of $W^q_{\mathfrak{L},0}(\Omega)$ if, and only if, the zero extension $\dot{v}$ of v belongs to $W^q_{\dot{\mathfrak{L}}}(\mathbb{R}^n)$.*

Proof. Suppose that v is a member of $W^q_{\mathfrak{L},0}(\Omega)$. Let $(v^i)_{i\in\mathbb{N}}$ be a sequence in $\mathscr{D}(\Omega)^m$ converging to v. The image $\dot{\mathfrak{L}}\dot{v}$ is the distribution which is defined for $w \in \mathscr{D}(\mathbb{R}^n)^l$ by

$$\int_{\mathbb{R}^n} \dot{v}\, \overline{\dot{\mathfrak{L}}'w}\, \mathrm{d}x = \int_\Omega v\, \overline{\mathfrak{L}'w}\, \mathrm{d}x = \lim_{i\to\infty} \int_\Omega v^i\, \overline{\mathfrak{L}'w}\, \mathrm{d}x = \lim_{i\to\infty} \int_\Omega (\mathfrak{L}v^i)\, \overline{w}\, \mathrm{d}x = \int_{\mathbb{R}^n} (\mathfrak{L}v)^{\dot{}}\, \overline{w}\, \mathrm{d}x,$$

where $(\mathfrak{L}v)^{\dot{}}$ denotes the zero extension of $\mathfrak{L}v$ and $\dot{\mathfrak{L}}'$ the formal adjoint of $\dot{\mathfrak{L}}$. Therefore $\dot{\mathfrak{L}}\dot{v} \in L^q(\mathbb{R}^n)^l$ and $\dot{v} \in W^q_{\dot{\mathfrak{L}}}(\mathbb{R}^n)$. Notice that this part of the proof does not require the Lipschitz continuity of $\partial\Omega$.

Suppose that the zero extension $\dot{v}$ of v is an element of $W^q_{\dot{\mathfrak{L}}}(\mathbb{R}^n)$. For f_ε as in the proof of Theorem 4, the product $f_\varepsilon\dot{v}$ is a member of $W^q_{\dot{\mathfrak{L}}}(\mathbb{R}^n)$ and approximates $\dot{v}$ as $\varepsilon \to 0$. Hence on replacing $\dot{v}$ with $f_\varepsilon\dot{v}$, we may assume that $\dot{v}$ has bounded support.

Since Ω is a Lipschitz domain, we can assign to every $x \in \partial\Omega$ a bounded neighbourhood N_x and an orthonormal coordinate system $\{y_1, \dots, y_n\}$ of $\mathbb{R}^n$ such that $\partial\Omega \cap N_x$ is the graph of a

Lipschitz function $M \subset \operatorname{span}(y_1, \ldots, y_{n-1}) \to \mathbb{R}\, y_n$ and such that $\Omega \cap N_x$ lies above $\partial\Omega \cap N_x$. Select, for every N_x, a neighbourhood $\dot{N}_x \Subset N_x$ and define the set

$$F = \operatorname{supp}(\dot{v}) \setminus \bigcup_{x \in \partial\Omega} \dot{N}_x,$$

which is compact and contained in Ω. There exists an open set $\dot{N}_0$ such that $F \Subset \dot{N}_0 \Subset \Omega$. Since $\operatorname{supp}(\dot{v})$ is compact, we can select a finite number of neighbourhoods $\dot{N}_i := \dot{N}_{x_i}$, $i \in \{1, \ldots, \kappa\}$, such that $\operatorname{supp}(\dot{v}) \subset \dot{N}_0 \cup \dot{N}_1 \cup \cdots \cup \dot{N}_\kappa$.

As in the proof of Theorem 4, define the functions $\dot{f}_i$ by a partition of unity subordinate to the $\dot{N}_i$ and set $\dot{v}_i := \dot{f}_i \dot{v}$. By Theorem 1, $\dot{v}_0$ is an element of $W^q_{\mathcal{L},0}(\Omega)$. Fix $i \in \{1, \ldots, \kappa\}$ and let L be a Lipschitz constant of $\partial\Omega \cap N_x$ with respect to the local coordinate system. We denote by C the cone

$$\Big\{ \sum_{j=1}^{n} \mu_j y_j \in \mathbb{R}^n : \mu_n > L \Big(\sum_{j=1}^{n-1} \mu_j^2 \Big)^{1/2} \Big\}.$$

Select an approximate identity $\delta \mapsto \psi_\delta$ such that $\operatorname{supp}(\psi_1) \Subset C$. By applying Corollary 1 with an $\varepsilon > 0$, $\delta := \operatorname{dist}(\partial N_x, \partial \dot{N}_x)/2$ and with the above approximate identity, we obtain a mollifier ϕ which satisfies

$$\|\dot{v}_i - \dot{v}_i * \phi\|_{\mathcal{L},q} < \varepsilon.$$

The construction of ϕ implies that $\operatorname{supp}(\phi) \Subset B_\delta \cap C$. Hence the support of $\dot{v}_i * \phi$ is subset of $\operatorname{supp}(\dot{v}_i) + (B_\delta \cap C) \subset \Omega \cap N_x$. We define, for $\tau \in \mathbb{R}$,

$$\dot{v}_\tau(x) := (\dot{v}_i * \phi)(x - \tau\, y_n).$$

Notice that, in contrast with equation (1.6), the translation is directed inwards for positive τ. For $\tau \in (0, \delta)$ the support of $\dot{v}_\tau$ remains in $\Omega \cap N_x$ and therefore $\dot{v}_\tau$ is an element of $C_0^\infty(\mathbb{R}^n)^m$. Like in the proof of Theorem 4, we use the weak continuity of the translation operator to construct a sequence of finite convex combinations of $\dot{v}_\tau$ which converges strongly to the element $\dot{v}_i * \phi$. ////

The corresponding theorem for $W^{1,q}(\Omega)$ is given, for instance, in (Adams and Fournier 2003, p. 71, p. 159). However, for $W^{1,q}(\Omega)$, it is not necessary to construct $\dot{v}_i * \phi$, which is the reason for requiring Lipschitz continuity of $\partial\Omega$. Indeed if B and C are translation invariant, i.e. constant, then it is possible to prove the theorem by demanding only the segment property of Ω.

1.5 Trace and Integration by Parts

The construction of the trace operator of $W^{1,p}(\Omega)$ via density of smooth functions is well known. Similar definitions for $W^q(\mathrm{div},\Omega)$ and $W^q(\mathrm{rot},\Omega)$ exist, too. In this section we extend the concept of the trace operator to general first-order graph spaces, subject to the assumption that Ω fulfills the strong local Lipschitz condition.

We set $B(\nu) := B(\nu,x) := (B_{ijk}(x)\,\nu_k(x))_{ij}$. For $v \in W^{1,q}(\Omega)^m$ and $w \in W^{1,q'}(\Omega)^l$,

$$\langle B(\nu)\,v, w\rangle_{\partial\Omega} = \langle \mathcal{L}v, w\rangle_\Omega - \langle v, \mathcal{L}'w\rangle_\Omega. \tag{1.14}$$

By the Hölder inequality

$$|\langle B(\nu)\,v, w\rangle_{\partial\Omega}| \le \|v\|_{\mathcal{L},q}\,\|w\|_{\mathcal{L}',q'}. \tag{1.15}$$

This, together with the continuity of $\mathcal{E}_{\partial\Omega,\Omega}$, shows that for all $v \in W^{1,q}(\Omega)^m$ the functional

$$\langle B(\nu)\,v,\cdot\rangle_{\partial\Omega} : B^{q',q',1-1/q'}(\partial\Omega)^l \to \mathbb{R},\ g \mapsto \langle B(\nu)\,v, g\rangle_{\partial\Omega}$$

is a continuous mapping. Therefore, $\langle B(\nu)\,v,\cdot\rangle$ belongs to the dual $B^{q,q,-1/q}(\partial\Omega)^l$. From (1.15) we also obtain the continuity of the linear operator

$$\dot{\mathcal{T}}_\mathcal{L} : W^{1,q}(\Omega)^m \to B^{q,q,-1/q}(\partial\Omega)^l,\ v \mapsto \langle B(\nu)\,v,\cdot\rangle_{\partial\Omega}$$

in $\|\cdot\|_{\mathcal{L},q}$. As $W^{1,q}(\Omega)^m$ is dense in $W^q_\mathcal{L}(\Omega)$, $\dot{\mathcal{T}}_\mathcal{L}$ extends to a continuous linear operator, which is defined on the entire graph space. We have therefore proven the following theorem.

Theorem 7 *Let Ω be a Lipschitz domain. Suppose that, for a given v, the sequence $(v_i)_{i\in\mathbb{N}}$ of $W^{1,q}(\Omega)^m$-functions converges to v in $W^q_\mathcal{L}(\Omega)$ as $i \to \infty$. Then, the operator*

$$\dot{\mathcal{T}}_\mathcal{L} : W^q_\mathcal{L}(\Omega) \to B^{q,q,-1/q}(\partial\Omega)^l, v \mapsto \lim_{i\to\infty}\langle B(\nu)\,v_i,\cdot\rangle_{\partial\Omega} =: \langle B(\nu)\,v,\cdot\rangle_{\partial\Omega}$$

exists and is continuous. The operator norm of $\dot{\mathcal{T}}_\mathcal{L}$ depends on the norms $\|B\|_{W^{1,\infty}(\Omega)^{l\times m\times n}}$ and $\|\mathcal{E}_{\partial\Omega,\Omega}\|_{\mathscr{B}(B^{q',q',1-1/q'}(\partial\Omega)^l, W^{1,q'}(\Omega))}$ only. Here $\mathcal{E}_{\partial\Omega,\Omega}$ refers to the extension operator defined on page 26.

As a byproduct of the above construction, we can formally extend the integration by parts formula to all functions v in $W^q_\mathcal{L}(\Omega)$ and w in $W^{q'}_{\mathcal{L}'}(\Omega)$. We define the continuous bilinear form

$$W^q_\mathcal{L}(\Omega) \times W^{q'}_{\mathcal{L}'}(\Omega) \to \mathbb{R}, (v,w) \mapsto \langle v,w\rangle_{B(\nu)} := \langle \mathcal{L}v, w\rangle_\Omega - \langle v, \mathcal{L}'w\rangle_\Omega. \tag{1.16}$$

Assume that $v \in W^q_\mathcal{L}(\Omega)$ and $w \in W^{q'}_{\mathcal{L}'}(\Omega)$ and let $(v_i)_{i\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ be sequences of smooth functions which converge to v and w, respectively. Then, due to equation (1.14),

$$\lim_{i\to\infty}\lim_{j\to\infty}\langle B(\nu)\,v_i, w_j\rangle_{\partial\Omega} = \langle v,w\rangle_{B(\nu)}.$$

Remark 3 *We emphasise here we are dealing with three different bilinear forms which are conceptually identical, but have different domains. In equation (1.14) we find on the left-hand side the mapping*

$$B^{q,q,1-1/q}(\partial\Omega)^m \times B^{q',q',1-1/q'}(\partial\Omega)^l, (v,w) \mapsto \langle B(\nu)v, w\rangle_{\partial\Omega}.$$

In Theorem 7, we exchange the domain of the first argument but keep the same notation:

$$W^q_{\mathfrak{L}}(\Omega) \times B^{q',q',1-1/q'}(\partial\Omega)^l, (v,w) \mapsto \langle B(\nu)v, w\rangle_{\partial\Omega}.$$

Finally, in equation (1.16) we consider the bilinear form $\langle v, w\rangle_{B(\nu)}$. We use a different notation here to emphasise that $\langle\cdot,\cdot\rangle_{B(\nu)}$ acts in both arguments on functions with domain Ω rather than on functions with domain $\partial\Omega$, cf. also Example 25.

We do not want to conceal that there is a certain degree of arbitrariness in our choice of the codomain $B^{q,q,-1/q}(\partial\Omega)^l$ of $\dot{\mathfrak{T}}_{\mathfrak{L}}$. On the one hand, the image of $\dot{\mathfrak{T}}_{\mathfrak{L}}$ can be a proper subspace of $B^{q,q,-1/q}(\partial\Omega)^l$. Consider for example a hyperbolic convection equation which has characteristics tangential to the boundary. In this case $B(\nu)$ vanishes on $\partial\Omega$ and therefore the image of $\dot{\mathfrak{T}}_{\mathfrak{L}}$ is $\{0\}$. On the other hand, we could have defined $\dot{\mathfrak{T}}_{\mathfrak{L}}$ with an even larger codomain, for instance by choosing a smaller space of test functions in equation (1.14) and the subsequent construction of $\dot{\mathfrak{T}}_{\mathfrak{L}}$.

Of all possible choices there is certainly one distinguished codomain, namely the image of $\dot{\mathfrak{T}}_{\mathfrak{L}}$. We therefore call $\operatorname{Im}\dot{\mathfrak{T}}_{\mathfrak{L}}$ the trace space of $W^q_{\mathfrak{L}}(\Omega)$ and denote it by $W^q_{\mathfrak{T}_{\mathfrak{L}}}(\partial\Omega)$. We equip $W^q_{\mathfrak{T}_{\mathfrak{L}}}(\partial\Omega)$ with the norm

$$\|v\|_{\mathfrak{T}_{\mathfrak{L}},q} := \inf\{\|w\|_{\mathfrak{L},q} \in \mathbb{R} : \dot{\mathfrak{T}}_{\mathfrak{L}}(w) = v\}, \tag{1.17}$$

which we call the trace norm of $W^q_{\mathfrak{T}_{\mathfrak{L}}}(\partial\Omega)$. We then introduce, as a substitution for $\dot{\mathfrak{T}}_{\mathfrak{L}}$, the operator

$$\mathfrak{T}_{\mathfrak{L}} : W^q_{\mathfrak{L}}(\Omega) \to W^q_{\mathfrak{T}_{\mathfrak{L}}}(\partial\Omega), v \mapsto \dot{\mathfrak{T}}_{\mathfrak{L}} v.$$

We call $\mathfrak{T}_{\mathfrak{L}}$ the trace operator of $W^q_{\mathfrak{L}}(\Omega)$. Where unambiguous we abbreviate $\mathfrak{T}_{\mathfrak{L}}$ by $\mathfrak{T}$. By construction $\|\mathfrak{T}\|_{\mathscr{B}(W^q_{\mathfrak{L}}(\Omega), W^q_{\mathfrak{T}}(\partial\Omega))} = 1$.

Theorem 8 *Let $v \in W^q_{\mathfrak{T}}(\partial\Omega)$. There exists a unique element $\dot{v} \in W^q_{\mathfrak{L}}(\Omega)$ with $\mathfrak{T}\dot{v} = v$ such that $\|\dot{v}\|_{\mathfrak{L},q} = \|v\|_{\mathfrak{T},q}$.*

Proof. Recall the embedding $\mathfrak{I}$ from page 14. Since $W^q_{\mathfrak{L}}(\Omega)$ is complete, it is isometric to a closed subset of $L^q(\Omega)^m \times L^q(\Omega)^l$. A closed subspace W of a uniformly convex Banach space

X is uniformly convex itself. Equally, reflexivity of X implies reflexivity of W. The inverse image $\mathcal{T}^{-1}v$ is a closed convex set in $W^q_{\mathcal{L}}(\Omega)$. The result now follows since in uniformly convex and reflexive Banach spaces, closed convex sets contain a unique point which is closest to the origin, cf. (Benyamini and Lindenstrauss 2000, p. 40). ////

Corollary 3 *The graph space is reflexive and uniformly convex.*

Theorem 8 asserts that the infimum in the definition of the trace norm, i.e. in equation (1.17), is attained. Choosing $\dot{v}$ as above, we define the extension map

$$\mathcal{E}_{\mathcal{L},q} : W^q_{\mathcal{T}}(\partial\Omega) \to W^q_{\mathcal{L}}(\Omega), v \mapsto \dot{v}. \tag{1.18}$$

If $q = 2$ then $\mathcal{E}_{\mathcal{L},q}\, v$ is the orthogonal projection of $0 \in W^q_{\mathcal{L}}(\Omega)$ to $\mathcal{T}^{-1}v$. In particular $\mathcal{E}_{\mathcal{L},2}$ is linear.

One consequence of the next theorem is that $W^q_{\mathcal{T}}(\partial\Omega)$ is complete, cf. (Rudin 1991, Theorem 1.41).

Theorem 9 *The kernel of $\mathcal{T}$ is equal to $W^q_{\mathcal{L},0}(\Omega)$.*

Proof. Every function in $\mathscr{D}(\Omega)^m$ has a trivial boundary trace and therefore, by continuity, $W^q_{\mathcal{L},0}(\Omega) \subset \ker \mathcal{T}$. It remains to show the implication in the other direction.

Suppose that v lies in the kernel of $\mathcal{T}_{\mathcal{L}}$. Let $\dot{v}$ be the zero extension of v and let $\dot{\mathcal{L}}$ be an extension of $\mathcal{L}$. By $\dot{\mathcal{L}}'$ we understand the formal adjoint of $\dot{\mathcal{L}}$, which is an extension of $\mathcal{L}'$. According to Theorem 6 we only need to prove that $\dot{v}$ is an element of $W^q_{\mathcal{L}}(\mathbb{R}^n)$. It is apparent that $\dot{v} \in L^q(\Omega)^m$. We apply $\dot{\mathcal{L}}$ to $\dot{v}$ in the sense of distributions. Let $\phi \in \mathscr{D}(\mathbb{R}^n)^l$, then

$$\int_{\mathbb{R}^n} \dot{v}\, \overline{\dot{\mathcal{L}}'\phi}\, \mathrm{d}V = \int_{\Omega} v\, \overline{\mathcal{L}'\phi}\, \mathrm{d}V = \int_{\Omega} \mathcal{L}v\, \overline{\phi}\, \mathrm{d}V - \langle B(\nu)v, \phi\rangle_{\partial\Omega} = \int_{\Omega} \mathcal{L}v\, \overline{\phi}\, \mathrm{d}V.$$

Hence the zero extension of $\mathcal{L}v$ is equal to $\dot{\mathcal{L}}\dot{v}$ and therefore $\dot{\mathcal{L}}\dot{v} \in L^{q'}(\Omega)^l$. ////

The operator $\mathcal{T}$ earns its designation 'trace operator' by virtue of the next theorem. Let V be a metrisable vector space. An operator $\mathcal{J} : W^q_{\mathcal{L}}(\Omega) \to V$ is called boundary operator if $\mathcal{J}$ is continuous and if

$$\forall\, v, \dot{v} \in C^\infty_0(\mathbb{R}^n) : v|_{\partial\Omega} = \dot{v}|_{\partial\Omega} \Rightarrow \mathcal{J}v = \mathcal{J}\dot{v}. \tag{1.19}$$

Clearly, $\mathcal{T}$ and $\dot{\mathcal{T}}_{\mathcal{L}}$ are boundary operators.

Theorem 10 *The continuous operator $\mathcal{J} : W^q_{\mathcal{L}}(\Omega) \to V$ is a boundary operator if, and only if, $\mathcal{T}v = \mathcal{T}\dot{v}$ implies that $\mathcal{J}v = \mathcal{J}\dot{v}$ for $v, \dot{v} \in W^q_{\mathcal{L}}(\Omega)$.*

Proof. Suppose that $\mathcal{J}$ is a boundary operator and that $\mathcal{T}v = \mathcal{T}\dot{v}$. Hence $v - \dot{v} \in \ker \mathcal{T}$. Approximation by $\mathscr{D}(\Omega)^m$-functions shows that $\mathcal{J}v = \mathcal{J}\dot{v}$. The implication in the other direction follows since $v|_{\partial\Omega} = \dot{v}|_{\partial\Omega}$ forces $\mathcal{T}v = \mathcal{T}\dot{v}$ for $v, \dot{v} \in \mathscr{D}(\Omega)^m$. ////

Theorem 11 *For every boundary operator $\mathcal{J} : W^q_{\mathcal{L}}(\Omega) \to V$ there exists a unique linear operator $\dot{\mathcal{J}} : W^q_{\mathcal{J}}(\partial\Omega) \to V$ such that $\mathcal{J} = \dot{\mathcal{J}} \circ \mathcal{T}$, i.e. such that the following diagram is commutative:*

$$\begin{array}{ccc} W^q_{\mathcal{L}}(\Omega) & \xrightarrow{\ \mathcal{T}\ } & W^q_{\mathcal{J}}(\partial\Omega) \\ & \searrow^{\mathcal{J}} & \Big\downarrow \dot{\mathcal{J}} \\ & & V \end{array} \tag{1.20}$$

Moreover if V is a normed vector space then the operator norms of $\mathcal{J}$ and $\dot{\mathcal{J}}$ are equal, that is $\|\dot{\mathcal{J}}\|_{\mathscr{B}(W^q_{\mathcal{J}}(\partial\Omega),V)} = \|\mathcal{J}\|_{\mathscr{B}(W^q_{\mathcal{L}}(\Omega),V)}$.

Proof. The existence of $\dot{\mathcal{J}}$ is a direct consequence of the previous theorem. The chain of equalities

$$\|\dot{\mathcal{J}}\|_{\mathscr{B}(W^q_{\mathcal{J}}(\partial\Omega),V)} = \sup_{\substack{w \in W^q_{\mathcal{L}}(\Omega) \\ w \neq 0}} \frac{\|(\dot{\mathcal{J}} \circ \mathcal{T})w\|_V}{\|\mathcal{T}w\|_{\mathcal{T},q}} = \sup_{\substack{w \in W^q_{\mathcal{L}}(\Omega) \\ w \neq 0}} \frac{\|(\dot{\mathcal{J}} \circ \mathcal{T})w\|_V}{\|w\|_{\mathcal{L},q}} = \|\mathcal{J}\|_{\mathscr{B}(W^q_{\mathcal{L}}(\Omega),V)}$$

implies that $\mathcal{J}$ and $\dot{\mathcal{J}}$ have the same operator norm. ////

By Theorem 11 we can identify the set of boundary operators with the set of continuous operators whose domain is $W^q_{\mathcal{J}}(\partial\Omega)$. The trace space is, up to equivalence of norms, the only space for which this identification is valid.

Theorem 12 *Consider a normed space W and a continuous operator $\mathcal{T}_W : W^q_{\mathcal{L}}(\Omega) \to W$. Suppose that for all boundary operators $\mathcal{J} : W^q_{\mathcal{L}}(\Omega) \to V$ there exist continuous operators $\dot{\mathcal{J}} : W \to V$ such that $\mathcal{J} = \dot{\mathcal{J}} \circ \mathcal{T}_W$. Then $W^q_{\mathcal{J}}(\partial\Omega)$ is homeomorphic to the image of $\mathcal{T}_W$.*

Proof. Consider the associated operators $\ddot{\mathcal{T}}_{\mathcal{L}} : W \to W^q_{\mathcal{J}}(\partial\Omega)$ so that $\mathcal{T}_{\mathcal{L}} = \ddot{\mathcal{T}}_{\mathcal{L}} \circ \mathcal{T}_W$ and $\ddot{\mathcal{T}}_W : W \to W^q_{\mathcal{J}}(\partial\Omega)$ so that $\mathcal{T}_W = \ddot{\mathcal{T}}_W \circ \mathcal{T}_{\mathcal{L}}$. Hence

$$\mathcal{T}_{\mathcal{L}} = \ddot{\mathcal{T}}_{\mathcal{L}} \circ \mathcal{T}_W = \ddot{\mathcal{T}}_{\mathcal{L}} \circ \ddot{\mathcal{T}}_W \circ \mathcal{T}_{\mathcal{L}}, \qquad \mathcal{T}_W = \ddot{\mathcal{T}}_W \circ \mathcal{T}_{\mathcal{L}} = \ddot{\mathcal{T}}_W \circ \ddot{\mathcal{T}}_{\mathcal{L}} \circ \mathcal{T}_W.$$

Thus $\ddot{\mathcal{T}}_{\mathcal{L}} \circ \ddot{\mathcal{T}}_W$ is the identity operator of $W^q_{\mathcal{J}}(\partial\Omega)$ and $\ddot{\mathcal{T}}_W \circ \ddot{\mathcal{T}}_{\mathcal{L}}$ is the identity operator of $\operatorname{Im} \mathcal{T}_W$. Therefore $\ddot{\mathcal{T}}_{\mathcal{L}}|_{\operatorname{Im} W} = \ddot{\mathcal{T}}_W^{-1}$. By the hypotheses $\ddot{\mathcal{T}}_{\mathcal{L}}|_{\operatorname{Im} W}$ and $\ddot{\mathcal{T}}_W$ are continuous. ////

We have learned that boundary operators can be viewed as operators which map $W^q_{\mathcal{J}}(\partial\Omega)$-functions instead of $W^q_{\mathcal{L}}(\Omega)$-functions. However, in order to assess the continuity of a mapping $W^q_{\mathcal{J}}(\partial\Omega) \to V$, we need to return to the norm of $W^q_{\mathcal{L}}(\Omega)$ as before; this is due to the definition of the trace norm. In order to remedy this shortcoming one needs to construct a norm on $W^q_{\mathcal{J}}(\partial\Omega)$ which is equivalent to the trace norm but is defined with only intrinsic properties of $W^q_{\mathcal{J}}(\partial\Omega)$. For a number of well-known spaces an intrinsic definition of the trace norm is known; we consider three classical examples.

Example 6 *As illustrated in Example 2, $W^{1,q}(\Omega)$ is isomorphic to $W^q_{\mathcal{L}}(\Omega)$ if $\mathcal{L} = \mathrm{grad}$. On page 26 we already introduced the surjective trace operator*

$$\mathcal{T}_{W^{1,q}(\Omega)} : W^{1,q}(\Omega) \to B^{q,q,1-1/q}(\partial\Omega).$$

The operator $\mathcal{T}_{\mathrm{grad}}$, defined via Theorem 7, is

$$\mathcal{T}_{\mathrm{grad}} : W^q_{\mathrm{grad}}(\Omega) \to B^{q,q,-1/q}(\partial\Omega)^l, v \mapsto (w \mapsto \langle v, \nu \cdot w\rangle_{\partial\Omega}).$$

Since

$$B^{q,q,-1/q}(\partial\Omega)^l \to B^{q,q,-1/q}(\partial\Omega), w \mapsto \nu \cdot w.$$

maps surjectively, we obtain the natural identification of $\mathcal{T}_{\mathrm{grad}}v$ with the functional $w \mapsto \langle v, w\rangle_{\partial\Omega}$, which is an element of $B^{q,q,-1/q}(\partial\Omega)$. Similarly $\mathcal{T}_{W^{1,q}(\Omega)}v$ can be understood as a functional in $B^{q,q,-1/q}(\partial\Omega)$ by using the canonical embedding of $B^{q,q,1-1/q}(\partial\Omega)$ into the space $B^{q,q,-1/q}(\partial\Omega)$:

$$v \mapsto (w \in B^{q,q,1-1/q}(\partial\Omega) \mapsto \langle v, w\rangle_{\partial\Omega}).$$

Both identifications are consistent with each other because both are based on density of smooth functions. Hence $\mathrm{Im}\,\mathcal{T}_{\mathrm{grad}}$ is equal to $B^{q,q,1-1/q}(\partial\Omega)$.

Example 7 *For domains with smooth boundary the trace space of $W^2(\mathrm{rot}, \Omega)$ is isomorphic to $H^{-1/2}(\mathrm{div}, \partial\Omega)$, cf. (Cessenat 1996). The result has recently been generalised to bounded Lipschitz domains. It remains true if $H^{-1/2}(\mathrm{div}, \partial\Omega)$ is defined as*

$$\{v \in H^{-1/2}(\partial\Omega)^3 : \exists\, \eta \in H^{-1/2}(\partial\Omega)\, \forall\, \phi \in H^2(\Omega) : \langle v, \mathrm{grad}\phi\rangle = \langle \eta, \phi\rangle\};$$

for details we refer to (Buffa, Costabel and Sheen 2002) and (Tartar 1997).

Example 8 *So far all examples of trace spaces we have considered were proper subsets of the codomain $B^{q,q,-1/q}(\partial\Omega)$. In contrast, the trace space of $W^2(\mathrm{div}, \Omega)$ equals $B^{2,2,-1/2}(\partial\Omega)$. The proof for bounded Lipschitz domains can be found, for instance, in volume 3, p. 204 in (Dautray and Lions 1988-93).*

We conclude the section with a theorem which is similar to Theorem 9. In both cases we examine density of smooth functions v_ε in the pre-image $\mathcal{T}^{-1}w$: for Theorem 13 assuming that $w \in W^q_{\mathcal{T}}(\partial\Omega)$, while for Theorem 9 we only considered $w = 0$. In contrast to Theorem 13, in Theorem 9 we required that the functions for which density is assumed have compact support.

Theorem 13 *For each $v \in W^q_{\mathcal{L}}(\Omega)$ and $\varepsilon > 0$ there exists a $v_\varepsilon \in C^\infty(\Omega)^m \cap W^q_{\mathcal{L}}(\Omega)$ such that $\|v - v_\varepsilon\|_{\mathcal{L},q} < \varepsilon$ and such that $\mathcal{J}v = \mathcal{J}v_\varepsilon$ for every boundary operator $\mathcal{J}$.*

Proof. Let $\varepsilon > \delta > 0$. Select v_ε, $v_{\varepsilon,i}$ as well as v_δ and $v_{\delta,i}$ like in the proof of Theorem 2. We define

$$v_j := \sum_{i=1}^{j-1} v_{\delta,i} + \sum_{i=j}^{\infty} v_{\varepsilon,i}.$$

By the monotone convergence theorem, v_j belongs to $C^\infty(\Omega)^m \cap W^q_{\mathcal{L}}(\Omega)$. Because $v_j - v_\varepsilon$ is an element of $C_0^\infty(\Omega)^m$, we deduce from (1.19) that $\mathcal{J}v_j = \mathcal{J}v_\varepsilon$. However, by (1.5),

$$\|v_\delta - v_j\|_{\mathcal{L},q} \leq \sum_{i=j}^{\infty} \|v_\delta - v_\varepsilon\|_{\mathcal{L},q} \leq \sum_{i=j}^{\infty} \frac{2\,\varepsilon}{2^i} = 2^{2-j}\,\varepsilon.$$

Therefore,

$$\|\mathcal{J}v_\delta - \mathcal{J}v_\varepsilon\|_V = \|\mathcal{J}v_\delta - \mathcal{J}v_j\|_V \leq \|\mathcal{J}\|\, 2^{2-j}\,\varepsilon.$$

Letting $j \to \infty$ proves that $\mathcal{J}v_\delta$ and $\mathcal{J}v_\varepsilon$ have the same trace, which coincides, by continuity, with the trace of v. ////

1.6 Hilbert Space Setting

We assume in this section that $W^q_{\mathcal{L}}(\Omega)$ is a Hilbert space; that is, we require that $q = 2$. We continue our study of the trace space of $W^2_{\mathcal{L}}(\Omega)$. Obviously, $W^2_{\mathcal{T}}(\partial\Omega)$ is a Hilbert space equipped with the scalar product

$$W^2_{\mathcal{T}}(\partial\Omega) \times W^2_{\mathcal{T}}(\partial\Omega) \to \mathbb{R}, (v, w) \mapsto \langle \mathcal{E}_{\mathcal{L},2}v, \mathcal{E}_{\mathcal{L},2}w\rangle_{\mathcal{L}}.$$

The starting point of our analysis is the observation that the extension operator $\mathcal{E}_{\mathcal{L}} := \mathcal{E}_{\mathcal{L},2}$ maps into a subset of $W^2_{\mathcal{L}}(\Omega)$ which contains smoother functions than the graph space in general.

We introduce the second-order linear differential operator

$$\mathcal{O} : L^2(\Omega)^m \to \mathscr{D}(\Omega)^m, v \mapsto \mathcal{L}'\mathcal{L}v + v$$

and define the space

$$W^2_{\mathcal{L},\mathcal{O}}(\Omega) := \{v \in L^2(\Omega)^m : \mathcal{L}v \in L^2(\Omega)^l \text{ and } \mathcal{O}v \in L^2(\Omega)^m\}.$$

We equip the space $W^2_{\mathcal{L},\mathcal{O}}(\Omega)$ with the norm

$$\|v\|_{\mathcal{L},\mathcal{O}} := \|v\|_{L^2(\Omega)^m} + \|\mathcal{L}v\|_{L^2(\Omega)^l} + \|\mathcal{O}v\|_{L^2(\Omega)^m}.$$

Since $W^2_{\mathcal{L},\mathcal{O}}(\Omega)$ is a subspace of $W^2_{\mathcal{L}}(\Omega)$, the trace operator $\mathcal{T}_{\mathcal{L}}$ is defined for all functions in $W^2_{\mathcal{L},\mathcal{O}}(\Omega)$. However since the image of $\mathcal{L}|_{W^2_{\mathcal{L},\mathcal{O}}(\Omega)}$ is contained in $W^2_{\mathcal{L}'}(\Omega)$, we can define the second trace operator

$$\mathcal{T}_{\mathcal{L},\mathcal{O}} : W^2_{\mathcal{L},\mathcal{O}}(\Omega) \to B^{2,2,-1/2}(\partial\Omega)^m, v \mapsto \langle B(\nu)^{\mathsf{H}}\, \mathcal{L}v, \cdot\rangle_{\partial\Omega} := (-\mathcal{T}_{\mathcal{L}'} \circ \mathcal{L})v.$$

The integration by parts formula (1.14) applied to $\mathcal{L}'$ takes for smooth function v, w the form

$$\langle -B(\nu)^{\mathsf{H}}\, v, w\rangle_{\partial\Omega} = \langle \mathcal{L}'v, w\rangle_\Omega - \langle v, \mathcal{L}w\rangle_\Omega.$$

Substituting v by $-\mathcal{L}v$, we obtain for $v \in W^2_{\mathcal{L},\mathcal{O}}(\Omega)$ and $w \in W^{1,2}(\Omega)^m$

$$\langle B(\nu)^{\mathsf{H}}\, \mathcal{L}v, w\rangle_{\partial\Omega} = -\langle \mathcal{L}'\mathcal{L}v, w\rangle_\Omega + \langle \mathcal{L}v, \mathcal{L}w\rangle_\Omega = \langle v, w\rangle_{\mathcal{L}} - \langle \mathcal{O}v, w\rangle_\Omega. \tag{1.21}$$

We formally extend the integration-by-parts formula (1.21) and introduce the following notation, motivated by Remark 3:

$$W^2_{\mathcal{L},\mathcal{O}}(\Omega) \times W^2_{\mathcal{L}}(\Omega) \to \mathbb{R}, (v, w) \mapsto \langle \mathcal{L}v, w\rangle_{B(\nu)^{\mathsf{H}}} := \langle v, w\rangle_{\mathcal{L}} - \langle \mathcal{O}v, w\rangle_\Omega.$$

Theorem 14 *The image of $\mathcal{E}_{\mathcal{L}}$, that is the set of minimisers in the graph norm for given fixed traces, is equal to the kernel of $\mathcal{O}$:*

$$\ker \mathcal{O} := \{v \in W^2_{\mathcal{L},\mathcal{O}}(\Omega) : \mathcal{O}v = 0\}.$$

Proof. Let $v \in \operatorname{Im} \mathcal{E}_{\mathcal{L}}$. It follows that v is the smallest element in $v + W^2_{\mathcal{L},0}(\Omega)$ with respect to the graph norm and therefore $\langle v, w\rangle_{\mathcal{L}} = 0$ for all $w \in W^2_{\mathcal{L},0}(\Omega)$. Yet by (1.21), $\langle v, w\rangle_{\mathcal{L}} = \langle \mathcal{O}v, w\rangle_\Omega$ and therefore $\mathcal{O}v = 0$ in the sense of distributions. Now let $v \in \{v \in W^2_{\mathcal{L},\mathcal{O}}(\Omega) : \mathcal{O}v = 0\}$ and $\dot{v} = \mathcal{E}_{\mathcal{L}}\mathcal{T}_{\mathcal{L}}v$. Clearly $\mathcal{T}_{\mathcal{L}}\dot{v} = \mathcal{T}_{\mathcal{L}}v$ and $\mathcal{O}(\dot{v} - v) = \mathcal{O}\dot{v} - \mathcal{O}v = 0$. Hence $\langle \dot{v} - v, w\rangle_{\mathcal{L}} = 0$ for $w \in W^2_{\mathcal{L}}(\Omega)$ and therefore $\dot{v} = v$. ////

Corollary 4 *The image of $\mathcal{E}_{\mathcal{L}}$ is the orthogonal complement of $W^2_{\mathcal{L},0}(\Omega)$ in $W^2_{\mathcal{L}}(\Omega)$.*

We carry over the result of Theorem 14 to the adjoint operator $\mathcal{L}'$ by reversing the role of $\mathcal{L}'$ and $\mathcal{L}$. Let $\mathcal{O}_{\mathcal{L}'} := \mathcal{L}\mathcal{L}' + I$. Then the image of $\mathcal{E}_{\mathcal{L}'}$ is the space

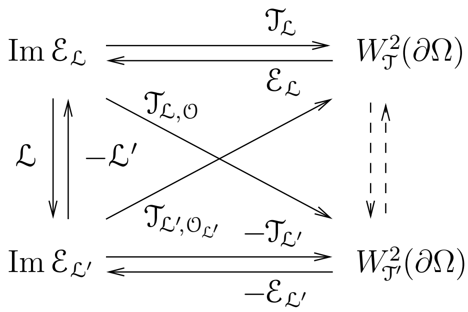
$$\{v \in W^2_{\mathcal{L}',\mathcal{O}_{\mathcal{L}'}}(\Omega) : \mathcal{O}_{\mathcal{L}'} v = 0\}.$$

According to Theorem 14 the trace norm of $v \in W^2_{\mathcal{T}}(\partial\Omega)$ fulfills the identities

$$\|v\|^2_{\mathcal{T}} = \langle \mathcal{E}_{\mathcal{L}} v, \mathcal{E}_{\mathcal{L}} v \rangle_{\mathcal{L}} = \langle \mathcal{L}\mathcal{E}_{\mathcal{L}} v, \mathcal{E}_{\mathcal{L}} v \rangle_{B(\nu)^{\mathsf{H}}}. \tag{1.22}$$

If v is smooth, then the term on the right-hand side is an integral over $\partial\Omega$ instead of over Ω; this property is advantageous for the construction of an intrinsic representation of the trace norm. Thus, in order to make $\langle \mathcal{L}\mathcal{E}_{\mathcal{L}} v, \mathcal{E}_{\mathcal{L}} v \rangle_{B(\nu)^{\mathsf{H}}}$ fully intrinsic, one would want to have a better understanding of the mapping indicated in the diagram below by the dashed lines.

Theorem 15 *The following diagram, not considering the dashed arrows, is commutative:*



Proof. We first consider the vertical arrows on the left. Let $v \in \operatorname{Im}\mathcal{E}_{\mathcal{L}}$. Then

$$\mathcal{O}_{\mathcal{L}'}\mathcal{L}v = (\mathcal{L}\mathcal{L}')\mathcal{L}v + \mathcal{L}v = \mathcal{L}\mathcal{O}v = \mathcal{L}0 = 0.$$

From $\mathcal{O}v = 0$ it follows that $\mathcal{L}v \in W^2_{\mathcal{L}'}(\Omega)$:

$$\mathcal{L}'\mathcal{L}v = -v. \tag{1.23}$$

Therefore $\mathcal{L}v \in \operatorname{Im}\mathcal{E}_{\mathcal{L}'}$. We also have that $\mathcal{O}_{\mathcal{L}'} w = 0$ for all $w \in \operatorname{Im}\mathcal{E}_{\mathcal{L}'}$; therefore

$$\mathcal{L}(-\mathcal{L}')w = w. \tag{1.24}$$

The combination of (1.23) and (1.24) implies that $-\mathcal{L}'|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}'}}$ is the inverse of $\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}$. The commutativity along the horizontal arrows on the top and also along the horizontal arrows at the bottom follows directly from the definitions of $\mathcal{T}_{\mathcal{L}}$ and $\mathcal{E}_{\mathcal{L}}$. Commutativity along the descending diagonal arrow is a consequence of the definition of $\mathcal{T}_{\mathcal{L},\mathcal{O}}$. Finally, the ascending arrow follows by reversing the roles of $\mathcal{L}$ and of $\mathcal{L}'$. ////

Corollary 5 *The spaces* $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$ *and* $\operatorname{Im} \mathcal{E}_{\mathcal{L}'}$ *are isometric with the isomorphism* $\mathcal{L}$.

Proof. We denote $\breve{v} := \mathcal{L}v$. Then

$$\langle v, v\rangle_{\mathcal{L}} = \langle v, v\rangle_{\Omega} + \langle \mathcal{L}v, \mathcal{L}v\rangle_{\Omega} = \langle \mathcal{L}'\breve{v}, \mathcal{L}'\breve{v}\rangle_{\Omega} + \langle \breve{v}, \breve{v}\rangle_{\Omega} = \langle \breve{v}, \breve{v}\rangle_{\mathcal{L}'}$$

proves the result. ////

The isometry between $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$ and $W^2_{\mathcal{J}}(\partial\Omega)$ as well as between $\operatorname{Im} \mathcal{E}_{\mathcal{L}'}$ and $W^2_{\mathcal{J}'}(\partial\Omega)$ is a direct consequence of Theorem 8 and (1.18). Therefore, according to the proof of Corollary 4, the graph space is isometric to the orthogonal sum of the image and the kernel of the trace operator.

1.7 Square Systems

If $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$ and $\operatorname{Im} \mathcal{E}_{\mathcal{L}'}$ coincide, we can examine the eigenvalues and eigenfunctions of $\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}$. We can use the information about the eigensystem of $\mathcal{L}$ to characterise the trace space in more detail.

Let $m = l$. We decompose the coefficients of $\mathcal{L}$ into their Hermitian and skew-Hermitian parts

$$\begin{aligned} B^h_{ijk} &:= {}^1\!/_2\, B_{ijk} + {}^1\!/_2\, \overline{B_{jik}}, \qquad & C^h_{ij} &:= {}^1\!/_2\, C_{ij} + {}^1\!/_2\, \overline{C_{ji}}, \\ B^s_{ijk} &:= {}^1\!/_2\, B_{ijk} - {}^1\!/_2\, \overline{B_{jik}}, \qquad & C^s_{ij} &:= {}^1\!/_2\, C_{ij} - {}^1\!/_2\, \overline{C_{ji}} \end{aligned}$$

and define

$$\mathcal{L}^h : v \mapsto \partial_k(B^h_{ijk} v_j) + C^h_{ij} v_j, \qquad \mathcal{L}^s : v \mapsto \partial_k(B^s_{ijk} v_j) + C^s_{ij} v_j.$$

Analogously to Remark 1, $\mathcal{L}^h$ and $(\mathcal{L}^h)'$ may be rewritten as

$$\begin{aligned} \mathcal{L}^h : v &\mapsto {}^1\!/_2\, B^h_{ijk}(\partial_k v_j) + {}^1\!/_2\, \partial_k(B^h_{ijk} v_j) + D^h_{ij} v_j, \\ (\mathcal{L}^h)' : v &\mapsto -{}^1\!/_2\, B^h_{ijk}(\partial_k v_j) - {}^1\!/_2\, \partial_k(B^h_{ijk} v_j) + D^h_{ij} v_j, \end{aligned} \tag{1.25}$$

where $D^h_{ij} := C^h_{ij} + {}^1\!/_2\, \partial_k B^h_{ijk}$. Similarly, $\mathcal{L}^s$ and $(\mathcal{L}^s)'$ can be transformed to

$$\begin{aligned} \mathcal{L}^s : v &\mapsto {}^1\!/_2\, B^s_{ijk}(\partial_k v_j) + {}^1\!/_2\, \partial_k(B^s_{ijk} v_j) + D^s_{ij} v_j, \\ (\mathcal{L}^s)' : v &\mapsto {}^1\!/_2\, B^s_{ijk}(\partial_k v_j) + {}^1\!/_2\, \partial_k(B^s_{ijk} v_j) - D^s_{ij} v_j, \end{aligned} \tag{1.26}$$

where $D^s_{ij} := C^s_{ij} + {}^1\!/_2\, \partial_k B^s_{ijk}$. One can read off directly from (1.25) and (1.26) that

$$\mathcal{L}^h v + (\mathcal{L}^h)' v = 2D^h v, \qquad \mathcal{L}^s v - (\mathcal{L}^s)' v = 2D^s v. \tag{1.27}$$

Hence, for $v \in \operatorname{Im} \mathcal{E}_{\mathcal{L}}$,

$$\mathcal{L}'v = (\mathcal{L}^h)'v + (\mathcal{L}^s)'v = -\mathcal{L}^h v + \mathcal{L}^s v + 2(D^h - D^s)v.$$

The two situations in which $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$ and $\operatorname{Im} \mathcal{E}_{\mathcal{L}'}$ are guaranteed to coincide are the purely Hermitian case, that is $\mathcal{L} = \mathcal{L}^h$, and the purely skew-Hermitian case, that is $\mathcal{L} = \mathcal{L}^s$. In the next two sections we concentrate on these two cases.

Replacing the coefficient matrix C in $\mathcal{L}$ by another matrix in $L^\infty(\Omega)^{m\times m}$ leaves the graph space unchanged as set. However, the graph norm changes up to equivalence, at least if the graph space is understood as a subset of $L^2(\Omega)^m$ and not as a subset of $L^q(\Omega)^m \times L^q(\Omega)^l$, cf. definition (1.1). Therefore we may select C freely as far as the definition of the trace space is concerned. Later, in order to simplify the search for eigenvalues, we choose $C_{ij} = -{}^1\!/_2\, \partial_k B_{ijk}$, which results in $D^h = D^s = 0$.

1.8 The Hermitian Case

In this section we make the additional assumption that $\mathcal{L} = \mathcal{L}^h$. Our aim is to characterise the trace space of $W^2_{\mathcal{L}}(\Omega)$. In particular, we compare the trace space with the space $L^2_B(\partial\Omega)$, which is similar to a weighted L^2-space on the boundary. While an intrinsic definition of the trace space for all differential operators with Hermitian coefficients remains out of reach, we identify conditions under which an intrinsic description of $W^2_{\mathcal{T}}(\partial\Omega)$ can be given.

The Vector Space $L^2_B(\partial\Omega)$

Given that $B(\nu, x)$ is Hermitian at $x \in \partial\Omega$, we can find an unitary matrix X and a real diagonal matrix Λ such that $B(\nu, x) = X^{\mathsf{H}}\Lambda X$. We define the $(m \times m)$-matrices

$$(E_+)_{ij} := \begin{cases} 1 : \Lambda_{ij} > 0, \\ 0 : \text{otherwise} \end{cases} \qquad \text{and} \qquad (E_-)_{ij} := \begin{cases} 1 : \Lambda_{ij} < 0, \\ 0 : \text{otherwise} \end{cases}$$

and $E_0 := I - E_+ - E_-$. We set

$$B_+(\nu, x) := X^{\mathsf{H}}\Lambda E_+ X, \quad B_-(\nu, x) := X^{\mathsf{H}}\Lambda E_- X, \quad |B|(\nu, x) := B_+(\nu, x) - B_-(\nu, x).$$

The matrices $B_+(\nu, x)$ and $B_-(\nu, x)$ split $B(\nu, x)$ into its positive and negative semi-definite part, i.e. $B(\nu, x) = B_+(\nu, x) + B_-(\nu, x)$ and for all $v \in \mathbb{R}^m$ we observe that $v^{\mathsf{H}}B_+(\nu, x)v \geq 0$ and that $v^{\mathsf{H}}B_-(\nu, x)v \leq 0$. Clearly, $B_+(\nu)$, $B_-(\nu)$ and $|B|(\nu)$ are Hermitian matrices. A splitting with these properties is unique and in particular it does not depend on the choice of X.

Next, let us consider the vector space

$$L^2_B(\partial\Omega) := \{v : \partial\Omega \mapsto \mathbb{R}^m : v \text{ is measurable and } \|v\|_B < \infty\}$$

where $\|\cdot\|_B$ is the norm which is induced by the scalar product

$$(v, w) \mapsto \int_{\partial\Omega} v^{\mathsf{H}}\, |B|(\nu)\, w \, \mathrm{d}S.$$

The space $L^2_B(\partial\Omega)$ is isometric to a weighted L^2-space. We consider the union of m disjoint copies of $\partial\Omega$. Formally that is the set

$$\partial\Omega_m := \bigcup_{i=1}^{m} \{\{x, i\} : x \in \partial\Omega\}.$$

We define on $\partial\Omega_m$ the measure

$$\mu(\kappa) = \sum_{i=1}^{m} \int_{\kappa\cap\partial\Omega_i} |\Lambda_i| \, \mathrm{d}S, \qquad \kappa \subset \partial\Omega_m.$$

Then the operator

$$\Phi : L^2_B(\partial\Omega) \to L^2(\partial\Omega_m, \mu), v \mapsto w \quad \text{such that} \quad w|_{\partial\Omega_i} = (Xv)_i$$

is an isometry between $L^2_B(\partial\Omega)$ and the Hilbert space $L^2(\partial\Omega_m, \mu)$, provided we agree on the assumption that X is measurable.

We decompose $L^2_B(\partial\Omega)$ by means of projections

$$\begin{aligned} \mathcal{P}_+ &: L^2_B(\partial\Omega) \to L^2_B(\partial\Omega)\, , v \mapsto X^{\mathsf{H}} E_+ X v, \\ \mathcal{P}_- &: L^2_B(\partial\Omega) \to L^2_B(\partial\Omega)\, , v \mapsto X^{\mathsf{H}} E_- X v. \end{aligned}$$

Conceptually, $\operatorname{Im}\mathcal{P}_+$ contains the functions which vanish outside the outflow boundary and $\operatorname{Im}\mathcal{P}_-$ contains the functions which vanish outside the inflow boundary. Clearly, $\mathcal{P}_+$ and $\mathcal{P}_-$ are projections and $\operatorname{Im}\mathcal{P}_+$ is the orthogonal complement of $\operatorname{Im}\mathcal{P}_-$ in $L^2_B(\partial\Omega)$.

Sometimes we prefer to use the matrix functions $X^{\mathsf{H}}E_+X$ and $X^{\mathsf{H}}E_-X$ instead of to $\mathcal{P}_+$ and $\mathcal{P}_-$. For that purpose we define the abbreviations

$$\begin{aligned} P_+ &: \partial\Omega \to \mathbb{R}^{n\times n}\, , x \mapsto X^{\mathsf{H}} E_+ X, \\ P_- &: \partial\Omega \to \mathbb{R}^{n\times n}\, , x \mapsto X^{\mathsf{H}} E_- X. \end{aligned}$$

Like $B_+(\nu)$ and $B_-(\nu)$ also the matrix functions P_+ and P_- are independent of the choice of X because they are the projections onto the sum of the eigenspaces which are associated to the positive and to the negative eigenvalues, respectively.

Suppose that $w \in W^2_{\mathcal{L}}(\Omega)$ is the $W^2_{\mathcal{L}}(\Omega)$-limit of the sequence $(v_i)_{i\in\mathbb{N}}$ where $v_i \in C(\mathbb{R}^n)^m \cap W^2_{\mathcal{L}}(\Omega)$ and $v_i|_{\partial\Omega} \in \operatorname{Im}\mathcal{P}_+$. Then (v_i) is a Cauchy sequence in $L^2_B(\partial\Omega)$ because, according to (1.15), for $i, j \in \mathbb{N}$

$$\|v_i - v_j\|^2_B = \langle B(\nu)\,(v_i - v_j), (v_i - v_j)\rangle_{\partial\Omega} \le \|v_i - v_j\|^2_{\mathcal{L}}.$$

Let v be the limit of (v_i) in $L^2_B(\partial\Omega)$. Then $B(\nu)\,v$ and $\mathcal{T}_{\mathcal{L}}w$ are equal in the sense of $B^{2,2,-1/2}(\partial\Omega)^m$ since for all $\phi \in B^{2,2,1/2}(\partial\Omega)^m$ we have

$$\langle B(\nu)\,v, \phi\rangle_{\partial\Omega} = \lim_{i\to\infty} \langle B(\nu)\,v, \phi\rangle_{\partial\Omega} = \langle B(\nu)\,w, \phi\rangle_{\partial\Omega}.$$

The first equality holds because $v \to B(\nu)\,v$ is a continuous mapping in $L^2_B(\partial\Omega)$; the second follows from the continuity of $\mathcal{T}_{\mathcal{L}}$. To summarise the observation in a theorem, we introduce the notation

$$W^2_{\mathcal{L},+}(\Omega) := \overline{\{v \in C(\mathbb{R}^n)^m \cap W^2_{\mathcal{L}}(\Omega) : v|_{\partial\Omega} \in \operatorname{Im}\mathcal{P}_+\}}^{W^2_{\mathcal{L}}(\Omega)},$$
$$W^2_{\mathcal{L},-}(\Omega) := \overline{\{v \in C(\mathbb{R}^n)^m \cap W^2_{\mathcal{L}}(\Omega) : v|_{\partial\Omega} \in \operatorname{Im}\mathcal{P}_+\}}^{W^2_{\mathcal{L}}(\Omega)}.$$

Theorem 16 *The operators $v \in W^2_{\mathcal{L},+}(\Omega) \to v|_{\partial\Omega}$ and $v \in W^2_{\mathcal{L},-}(\Omega) \to v|_{\partial\Omega}$ map continuously into $\operatorname{Im}\mathcal{P}_+ \subset L^2_B(\partial\Omega)$ and $\operatorname{Im}\mathcal{P}_- \subset L^2_B(\partial\Omega)$, respectively.*

Proof. The result $W^2_{\mathcal{L},+}(\Omega)$ is proven above; the result for $W^2_{\mathcal{L},-}(\Omega)$ follows analogously. ////

In general, $W^2_{\mathcal{L},+}(\Omega)$ and $W^2_{\mathcal{L},-}(\Omega)$ are not dense in $W^2_{\mathcal{T}}(\partial\Omega)$. We consider an example with constant coefficients.

Example 9 *Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and*

$$\mathcal{L}v = \partial_x \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} v + \frac{\partial_y}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} v. \tag{1.28}$$

The space $W^2_{\mathcal{L},+}(\Omega)$ contains at least all functions in $W^{1,2}(\Omega) \times \{0\} \times W^{1,2}(\Omega)$ and $W^2_{\mathcal{L},-}(\Omega)$ contains at least all functions in $\{0\} \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. Consider the change of coordinates

$$v \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} v,$$

under which the operator $\mathcal{L}$ transforms to

$$\dot{\mathcal{L}}v = \partial_x \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v + \partial_y \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} v.$$

A function (v_1, v_2, v_3) is an element of $W^2_{\mathfrak{L}}(\Omega)$ if, and only if, $v_1 \in W^{1,2}(\Omega)$ and $(v_2, v_3) \in W^2(\mathrm{div}, \Omega)$. Consequently, $W^2_{\mathfrak{L}}(\Omega)$ is spanned by functions of the type $(-v_1/\sqrt{2}, v_1/\sqrt{2}, 0)$ and $(v_2/\sqrt{2}, v_2/\sqrt{2}, v_3)$. The trace space of $W^2_{\mathfrak{L}}(\Omega)$ is

$$\{(u_1 - u_2, u_1 + u_2, 0) : u_1 \in B^{2,2,1/2}(\partial\Omega)^m \text{ and } u_2 \in B^{2,2,-1/2}(\partial\Omega)^m\}$$

with the intrinsic norm $(\|u_1\|^2_{B^{2,2,1/2}(\partial\Omega)^m} + \|u_2\|^2_{B^{2,2,-1/2}(\partial\Omega)^m})^{1/2}$. Since u_2 is in general not an element of $L^2_B(\partial\Omega)$, we deduce that $W^2_{\mathfrak{L},+}(\Omega) + W^2_{\mathfrak{L},-}(\Omega)$ is a proper subspace of $W^2_{\mathfrak{L}}(\Omega)$.

In the above example the spaces $W^2_{\mathfrak{L},+}(\Omega)$ and $W^2_{\mathfrak{L},-}(\Omega)$ are coupled by the second coefficient matrix in (1.28). The situation is different for graph spaces for which $W^2_{\mathfrak{L},+}(\Omega)$ and $W^2_{\mathfrak{L},-}(\Omega)$ are independent of each other. Let x be an element of $\partial\Omega$ and $N \subset \mathbb{R}^n$ be a neighbourhood of x. For a fixed $k \in \{1, \ldots, n\}$ the entries B_{ijk} form an $(m \times m)$-matrix which we denote by B^k. Suppose that in $N \cap \overline{\Omega}$ there is a $W^{1,\infty}$-coordinate transformation such that all B^k have the same block diagonal structure, that is there are numbers $r_1, \ldots, r_s$ such that $1 = r_1 < r_2 < \ldots < r_s = m + 1$ and for $t \in \{1, \ldots, s\}$

$$r_t \le i < r_{t+1} \text{ and } (j < r_t \text{ or } r_{t+1} \le j) \Rightarrow B^k_{ij} = B_{ijk} = 0. \tag{1.29}$$

It then follows that also $B(\nu)$ fulfills (1.29). Let us suppose that the s submatrices

$$(B(\nu)_{ij})_{r_t \le i,j < r_{t+1}}$$

are either positive or negative semi-definite on $N \cap \partial\Omega$. The block structure of the coefficient matrices implies that we can separate $W^2_{\mathfrak{L}}(\Omega)$ into s individual graph spaces, that are the subspaces with the coefficient matrices $(B^k_{ij})_{r_t \le i,j < r_{t+1}}$. By means of a partition of unity we can then show that $v|_{\partial\Omega \cap N}$ is contained in $L^2_B(\partial\Omega \cap N)$ for any $v \in W^2_{\mathfrak{L}}(\Omega)$. Indeed, locally the spaces which are associated to a positive semi-definite block in $B(\nu)$ constitute $W^2_{\mathfrak{L},+}$ while the subspaces associated to a negative semi-definite block in $B(\nu)$ constitute $W^2_{\mathfrak{L},-}$. Hence, if $\partial\Omega$ can be covered by neighbourhoods such as N then $W^2_{\mathfrak{L}}(\Omega)$ is equal to $W^2_{\mathfrak{L},+}(\Omega) + W^2_{\mathfrak{L},-}(\Omega)$.

Theorem 17 *Suppose that there is a neighbourhood $N \subset \mathbb{R}^n$ of a point $x \in \partial\Omega$ and a $W^{1,\infty}(N \cap \Omega)^{m\times m}$-coordinate transformation such that the transformed coefficient matrices of the principal part of $\mathfrak{L}$ satisfy (1.29) on $\Omega \cap N$. Moreover, assume that the blocks on the diagonal of $B(\nu)$ are either positive or negative semi-definite on $\partial\Omega \cap N$. Then, the operator*

$$\mathfrak{T}_N : C^\infty_0(\mathbb{R}^n)^m \to L^2_B(\partial\Omega \cap N), v \mapsto v|_{\partial\Omega\cap N}$$

has a continuous extension to $W^2_{\mathfrak{L}}(\Omega)$. In addition, $\langle B(\nu)\, \mathfrak{T}_N\, v, \phi\rangle_{\partial\Omega} = (\mathfrak{T}_{\mathfrak{L}}\, v)(\phi)$ for all test functions $\phi \in B^{2,2,1/2}(\partial\Omega \cap N)^m$.

It is evident that a general principle for the intrinsic formulation of the trace norm cannot depend on the coefficient $B(\nu)$ only.

Example 10 *Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and $\mathcal{L}v = (-\partial_x v_1, \partial_x v_2, 0)^{\mathsf{H}}$. According to Theorem 17 the trace space is a subset of $L^2_B(\partial\Omega)$. The boundary matrix $B(\nu)$ coincides with the corresponding term of the graph space considered in Example 9. However, the trace space considered in this example is not contained in $L^2_B(\partial\Omega)$.*

With Theorem 17 we do not cover the case where individual blocks of $B(\nu)$ change from positive to negative semi-definite type. Theorem 17 cannot be extended to such settings.

Example 11 *Let $\Omega := \{(x,y) \in \mathbb{R}^2 : y > |x| \text{ and } y < 1\}$ and $\mathcal{L} : v \mapsto y^\alpha\, \partial_x v$, $\alpha \in (0,\infty)$. We denote the union of the in- and outflow boundary by $\partial\Omega'$, that is $\partial\Omega' = \partial\Omega \setminus \{(x,y) \in \partial\Omega : y = 1\}$. Given $x \in \partial\Omega'$, $B(\nu)$ is negative definite if $x < 0$ and positive definite if $x > 0$. The extension $\mathcal{E}_{\mathcal{L}} g$ of a function $g : \partial\Omega' \to \mathbb{R}$ is*

$$(x,y) \mapsto \operatorname{cosech}\Big(\frac{2\,y}{y^\alpha}\Big)\,\Big(g(-y,y)\sinh\Big(\frac{y-x}{y^\alpha}\Big) + g(y,y)\sinh\Big(\frac{y+x}{y^\alpha}\Big)\Big).$$

It follows that the trace norm of g is

$$\Big(\int_0^1 y^\alpha \operatorname{cosech}(2\,y^{1-\alpha})\,\big(\cosh(2\,y^{1-\alpha})(g(y,y)^2 + g(-y,y)^2) - 2\,g(y,y)\,g(-y,y)\big)\,\mathrm{d}y\Big)^{1/2}. \tag{1.30}$$

Suppose that

$$g(y,y) = g(-y,y). \tag{1.31}$$

Then the integrand of (1.30) simplifies to $2\,y^\alpha \tanh(y^{1-\alpha})\,g(y,y)^2$. Since $\tanh(y)$ is asymptotically equal to y as $y \to 0$, the trace norm is, for functions which satisfy (1.31), equivalent to

$$\Big(\int_0^1 y\,g(y,y)^2\,\mathrm{d}y\Big)^{1/2}.$$

Therefore

$$\mathcal{T}_{\partial\Omega'} : C_0^\infty(\mathbb{R}^n)^m \to L^2_B(\partial\Omega), v \mapsto v|_{\partial\Omega} \tag{1.32}$$

can only have a continuous extension to $W^2_{\mathcal{L}}(\Omega)$ if $\alpha \geq 1$. We now examine boundary functions for which $g(-y,y) = 0$. Then, the integrand of the trace norm is $y^\alpha \coth(2\,y^{1-\alpha})\,g(y,y)^2$. For these functions the trace norm is equivalent to

$$\Big(\int_0^1 y^{2\alpha-1}\,g(y,y)^2\,\mathrm{d}y\Big)^{1/2}.$$

Again we find that $\alpha \geq 1$ is a necessary condition for the existence of a continuous extension of (1.32) to $W^2_{\mathcal{L}}(\Omega)$. A corresponding result also holds if $g(y,y) = 0$.

Taking inspiration from this example, we illustrate a technique which can be used to clarify when the restriction $v|_{\partial\Omega}$ is meaningful and contained in $L^2_B(\partial\Omega)$ for $v \in W^2_{\mathcal{L}}(\Omega)$. Suppose we can find a positive function $\psi : \Omega \to (0,1)$ such that the trace $\psi|_{\partial\Omega}$ equals 1 on the inflow and 0 on the outflow boundary. Furthermore, let us assume that $B_{ijk}\,\psi$ is contained in $W^{1,\infty}(\Omega)$ for all admissible indices i, j, k. Then $W^2_{\mathcal{L}}(\Omega)$ is the intersection of the graph spaces associated to the operators $v \mapsto \partial_k(B_{ijk}\,\psi\,v_j)$ and $v \mapsto \partial_k(B_{ijk}\,(1-\psi)\,v_j)$. Since these operators have as boundary matrices $B_-(\nu)$ and $B_+(\nu)$, respectively, their trace operators map, by Theorem 17, into $L^2_{B\psi}(\partial\Omega)$ and $L^2_{B(1-\psi)}(\partial\Omega)$, respectively. This in turn proves that the trace operator of $W^2_{\mathcal{L}}(\Omega)$ maps into $L^2_B(\partial\Omega)$. In the example above, choosing the function $\psi(x,y) = (y-x)^\alpha/(2\,y)^\alpha$, we get that $B_{1,1,x}\,\psi = -{}^1\!/_2\,(y-x)^\alpha$ is contained in $W^{1,\infty}(\Omega)$ if, and only if, $\alpha \geq 1$.

Example 11 makes it evident that $L^2_B(\partial\Omega)$ is in general not large enough to contain all traces of $W^2_{\mathcal{L}}(\Omega)$-functions. However, if singularities only occur at certain locations we can embed $W^2_{\mathcal{T}}(\partial\Omega)$ into a local version of the space $L^2_B(\partial\Omega)$. Let M be the union of all neighbourhoods N which satisfy the hypotheses of Theorem 17, that is in N the operator $\mathcal{L}$ is of block structure after a coordinate transformation and each block in $B(\nu)$ is either positive or negative semi-definite throughout $N \cap \partial\Omega$. We then introduce the space

$$L^2_{B,\mathrm{loc}}(\partial\Omega) := \Big\{ v : \partial\Omega \to \mathbb{R}^m : v \text{ measurable and } \forall\, K \Subset M \cap \partial\Omega : \int_K v\,|B|(\nu)\,v\,\mathrm{d}S < \infty \Big\}.$$

We would like to know if the restrictions $v|_{\partial\Omega}$ of functions $v \in W^2_{\mathcal{L}}(\Omega)$ are contained in $L^2_{B,\mathrm{loc}}(\partial\Omega)$. Suppose that $K \Subset M$, that $v \in W^2_{\mathcal{L}}(\Omega)$ and that $(v_k)_{k\in\mathbb{N}}$ is a sequence of smooth functions which converges to v in $W^2_{\mathcal{L}}(\Omega)$. Since $\overline{K}$ is compact, we can find a finite covering of K with bounded neighbourhoods N_i of the kind described in Theorem 17. Let $\mathcal{F}$ be a partition of unity subordinate to the N_i. Moreover, let $\dot{f}_i$ be the finite sum of all $f_j \in \mathcal{F}$ with $\mathrm{supp}(f_j) \cap K \neq \emptyset$ for which i is the smallest index such that $\mathrm{supp}(f_j) \subset N_i$. We set $v_i := \dot{f}_i\,v$ and $v_{k,i} := \dot{f}_i\,v_k$ on Ω. Then $v_{k,i} \to v_i$ as $k \to \infty$ in $W^2_{\mathcal{L}}(\Omega)$. Applying Theorem 17 we verify that

$$\int_{K\cap N_i} v_i\,|B|(\nu)\,v_i\,\mathrm{d}S < \infty.$$

Thus $v|_{\partial\Omega}$ is a member of $L^2_{B,\mathrm{loc}}(\partial\Omega)$.

We typically require in addition that $S = \partial\Omega \setminus M$ is a null set in the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ because only then is the embedding of $W^2_{\mathcal{T}}(\partial\Omega)$ into $L^2_{B,\mathrm{loc}}(\partial\Omega)$ injective.

We conclude the section with remarks on the regularity of $|B|(\nu)$ and the transformation X. If $\partial\Omega$ is C^2-regular in a neighbourhood of $x \in \partial\Omega$, then $B(\nu)$ is locally an element of $W^{1,\infty}$. A partial answer when $B_+(\nu)$ and $B_-(\nu)$ share this degree of regularity is given in the following theorem. The proof of the theorem is based on perturbation theory of linear operators. For an overview of the relevant prerequisites see the Appendix.

Theorem 18 *Let $x \in \partial\Omega$ and assume that $\partial\Omega$ can be represented in a neighbourhood of x by the graph of a twice continuously differentiable function. If there is an $\varepsilon > 0$ such that $B_\varepsilon(0)$ contains at most one simple eigenvalue of $B(\nu, x)$, then $B_+(\nu)$ and $B_-(\nu)$ are of the class $B^{2,2,1/2}$ in a neighbourhood of x.*

Proof. Because the eigenvalues of $B(\nu, x)$ change continuously with x, there is a neighbourhood N_x of x such that there is at most one eigenvalue contained in $B_{\varepsilon/3}(0)$ and no additional eigenvalue contained in $B_{2\varepsilon/3}(0)$. As pointed out in the Appendix, by possibly reducing the size of N_x, we can assume that all λ-groups and therefore all total projections are well-defined in N_x. Clearly, the projection onto the sum of eigenspaces which are associated with eigenvalues larger than $2\varepsilon/3$ is continuously differentiable. If $B_{\varepsilon/3}(0)$ contains an eigenvalue λ then the eigenprojection is continuously differentiable as it is a total projection. Moreover, in such a case λ itself is continuously differentiable so that $|\lambda|$ is a Lipschitz function. Therefore $B_+(\nu)$ is Lipschitz continuous in N_x which implies that $B_+(\nu)$ is contained in $B^{2,2,1/2}$. An analogous argument applies to $B_-(\nu)$. ////

Corollary 6 *If Ω is a C^2-domain and the differential operator is scalar, i.e. $m = 1$, then $B_+(\nu)$ and $B_-(\nu)$ are members of $B^{2,2,1/2}(\partial\Omega)^{m\times m}$.*

Proof. If $m = 1$, then $B(\nu)$ has only one eigenvalue. ////

The following example, which goes back to (Rellich 1937), demonstrates that even for very smooth functions $B(\nu)$ the transformation X can be singular if a multiple eigenvalue branches at the origin.

Example 12 *Consider*

$$T : \mathbb{R} \to \mathbb{R}^{2\times 2}, x \mapsto \mathrm{e}^{-\frac{1}{x^2}} \begin{pmatrix} \cos\frac{2}{x} & \sin\frac{2}{x} \\ \sin\frac{2}{x} & -\cos\frac{2}{x} \end{pmatrix}, \quad \textit{with } T(0) := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The function is infinitely differentiable for all $x \in \mathbb{R}$ and the same is true for the eigenvalues, which are $\pm\exp(-1/x^2)$ for $x \neq 0$ and zero for $x = 0$. But the associated projections onto *the eigenspaces are given, for $x \neq 0$, by*

$$\begin{pmatrix} \cos^2 \frac{1}{x} & \cos \frac{1}{x} \sin \frac{1}{x} \\ \cos \frac{1}{x} \sin \frac{1}{x} & \sin^2 \frac{1}{x} \end{pmatrix}, \begin{pmatrix} \sin^2 \frac{1}{x} & -\cos \frac{1}{x} \sin \frac{1}{x} \\ -\cos \frac{1}{x} \sin \frac{1}{x} & \cos^2 \frac{1}{x} \end{pmatrix}.$$

These matrix functions are infinitely differentiable on $\mathbb{R} \setminus \{0\}$, but they cannot be extended continuously to $x = 0$ and do not have a uniform Lipschitz constant on $\mathbb{R} \setminus \{0\}$. Furthermore, there is no function of eigenvectors of T that is continuous in a neighbourhood of 0 and that does not vanish at 0.

As noted in the Appendix, singularities of the eigenprojections cannot occur under analytic perturbations if the matrix function is real and symmetric. Therefore on a part $\partial\Omega'$ of the boundary which is analytic and on which $B(\nu)$ is real, symmetric and analytic, we find that the transformation X is analytic, too.

Eigenvalues and Eigenvectors

Under the condition that not only $\mathcal{L} = \mathcal{L}^h$ but also $D^h = 0$ holds, we can fully characterise the eigensystem of $\mathcal{L}$ on the restriction to $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$. We can use this information on the eigenspaces to improve our understanding of the trace space.

From $D^h = 0$ we deduce that

$$\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}} = -\mathcal{L}'|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}} = \mathcal{L}^{-1}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}. \tag{1.33}$$

It follows directly that $\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}$ is normal. Moreover the graph norms with respect to $\mathcal{L}$ and $\mathcal{L}'$ coincide in $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$:

$$\langle v, w \rangle_{\mathcal{L}} = \langle v, w \rangle_{\Omega} + \langle \mathcal{L}v, \mathcal{L}w \rangle_{\Omega} = \langle v, w \rangle_{\Omega} + \langle \mathcal{L}'v, \mathcal{L}'w \rangle_{\Omega} = \langle v, w \rangle_{\mathcal{L}'}.$$

We obtain, for $v \in \operatorname{Im} \mathcal{E}_{\mathcal{L}}$ and $\breve{v} := \mathcal{L}v$, the two identities

$$\begin{aligned} v + \breve{v} &= v + \mathcal{L}v = \mathcal{L}\mathcal{L}v + \mathcal{L}v = \mathcal{L}(\mathcal{L}v + v) = \quad (v + \breve{v})\breve{}, \\ v - \breve{v} &= v - \mathcal{L}v = \mathcal{L}\mathcal{L}v - \mathcal{L}v = \mathcal{L}(\mathcal{L}v - v) = -\,(v - \breve{v})\breve{}. \end{aligned} \tag{1.34}$$

Theorem 19 *The space $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$ is the orthogonal sum of the eigenspaces $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1)$ and $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)$. In addition, the operators*

$$\begin{aligned} \mathcal{P}_1 &: \operatorname{Im} \mathcal{E}_{\mathcal{L}} \to \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1) \quad, v \mapsto {}^1\!/_2\,(v + \breve{v}), \\ \mathcal{P}_{-1} &: \operatorname{Im} \mathcal{E}_{\mathcal{L}} \to \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)\,, v \mapsto {}^1\!/_2\,(v - \breve{v}), \end{aligned}$$

are projections onto $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1)$ and $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)$, respectively.

Proof. Equations (1.34) show that 1 and -1 are eigenvalues of $\mathcal{L}$, i.e. the elements of the point spectrum of the operator. Given $v \in \operatorname{Im} \mathcal{E}_{\mathcal{L}}$, we write $v = {}^1\!/_2\,(v + \breve{v}) + {}^1\!/_2\,(v - \breve{v})$ where ${}^1\!/_2\,(v + \breve{v}) \in \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1)$ and ${}^1\!/_2\,(v - \breve{v}) \in \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)$. The functions $v_1 \in \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1)$ and $v_{-1} \in \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)$ are orthogonal since

$$\langle v_1, v_{-1} \rangle_{\mathcal{L}} = \langle v_1, v_{-1} \rangle_\Omega + \langle v_1, -v_{-1} \rangle_\Omega = 0.$$

The operators $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ are surjective because $\mathcal{P}_1 v_1 = v_1$ and $\mathcal{P}_{-1} v_{-1} = v_{-1}$, which also shows that $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ are projections. ////

Since the complement $\mathcal{L}|_{W^2_{\mathcal{L},0}(\Omega)}$ may have the eigenvalues 1 or -1 too, the eigenspaces $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1)$ and $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)$ do not have to coincide with $\operatorname{Eig}(\mathcal{L}, 1)$ and $\operatorname{Eig}(\mathcal{L}, -1)$.

Notice that the mappings $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ are nothing else but the orthogonal projections onto $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1)$ and $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)$. We extend their meaning to all of $W^2_{\mathcal{T}}(\partial\Omega)$: For $v \in W^2_{\mathcal{T}}(\partial\Omega)$ we mean by $\mathcal{P}_1 v$ and $\mathcal{P}_{-1} v$ the elements of, respectively, $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, 1)$ and $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im} \mathcal{E}_{\mathcal{L}}}, -1)$ which are closest to v.

We remark that one could also prove Theorem 19 by means of the factorisation

$$\mathcal{O} = (\mathcal{L} + I)(-\mathcal{L} + I).$$

Example 13 *Let $\Omega = (0, 1)$ and $\mathcal{L}v = \partial_x v$. Then the space $\operatorname{Im} \mathcal{E}_{\mathcal{L}}$ is equal to the space of solutions of*

$$-\partial_x^2 v + v = 0,$$

which is the span of e^x and e^{-x}. In agreement with Theorem 19, we observe that $\mathcal{L}\mathrm{e}^x = \mathrm{e}^x$, that $\mathcal{L}\mathrm{e}^{-x} = -\mathrm{e}^{-x}$ and that $\langle \mathrm{e}^x, \mathrm{e}^{-x} \rangle_{\mathcal{L}} = 0$.

In the previous section we have used the projections $\mathcal{P}_+$ and $\mathcal{P}_-$ to illustrate that under certain conditions the restrictions $v|_{\partial\Omega}$ of functions in the graph space are contained in $L^2_B(\partial\Omega)$. Similarly, we can utilise the projections $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ to identify classes of functions v in $L^2_B(\partial\Omega)$ for which we can guarantee that $B(\nu)\, v$ is a member of the trace space.

We record that given a boundary function $v \in W^2_{\mathcal{T}}(\partial\Omega)$, all members $u \in W^2_{\mathcal{L}}(\Omega)$ with trace $\mathcal{T}u = v$ differ only by $W^2_{\mathcal{L},0}(\Omega)$-functions. Because $\operatorname{Im} \mathcal{P}_1$ and $W^2_{\mathcal{L},0}(\Omega)$ are orthogonal to each other, all extensions of v have the same $\mathcal{P}_1$-component. In this sense we extend the meaning of $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ by defining

$$\mathcal{P}_1 : W^2_{\mathcal{T}}(\partial\Omega) \to W^2_{\mathcal{T}}(\partial\Omega), v \mapsto \mathcal{T}_{\mathcal{L}}\, \mathcal{P}_1\, \mathcal{E}_{\mathcal{L}}\, v,$$
$$\mathcal{P}_{-1} : W^2_{\mathcal{T}}(\partial\Omega) \to W^2_{\mathcal{T}}(\partial\Omega), v \mapsto \mathcal{T}_{\mathcal{L}}\, \mathcal{P}_{-1}\, \mathcal{E}_{\mathcal{L}}\, v.$$

Consider two functions $v, w \in B^{2,2,1/2}(\partial\Omega)^m$ such that $v \in \operatorname{Im} \mathcal{P}_1$ and $w \in \operatorname{Im} \mathcal{P}_{\text{-}1}$. Then

$$\begin{aligned} \|v\|_{\mathcal{J}}^2 &= \langle \mathcal{L}v, v\rangle_{B(\nu)^{\mathsf{H}}} = \quad \langle B(\nu)\, v, v\rangle_{\partial\Omega} \le \|v\|_B^2, \\ \|w\|_{\mathcal{J}}^2 &= \langle \mathcal{L}w, w\rangle_{B(\nu)^{\mathsf{H}}} = -\, \langle B(\nu)\, w, w\rangle_{\partial\Omega} \le \|w\|_B^2. \end{aligned}$$

Consequently, the sets

$$\begin{aligned} &\{B(\nu)\, v : v \in \overline{B^{2,2,1/2}(\partial\Omega)^m \cap \operatorname{Im} \mathcal{P}_1}^{\|\cdot\|_B}\}, \\ &\{B(\nu)\, w : w \in \overline{B^{2,2,1/2}(\partial\Omega)^m \cap \operatorname{Im} \mathcal{P}_{\text{-}1}}^{\|\cdot\|_B}\} \end{aligned} \tag{1.35}$$

are contained in $W^2_{\mathcal{J}}(\partial\Omega)$. Equation (1.35) gives a lower bound on the size of the trace space. In general, passing v through $\mathcal{P}_1$ and $\mathcal{P}_{\text{-}1}$ leads to a loss in regularity in the sense of Besov spaces. This is even true if the coefficients of $\mathcal{L}$ are of class C^∞.

Example 14 *Let $\Omega = (-1,1) \times (0,1)$ and let*

$$\mathcal{L}v(x,y) = B(x)\, \partial_y v(x,y), \quad \textit{where } B(x) := \begin{cases} \mathrm{e}^{-1/x} & \textit{for } x > 0, \\ 0 & \textit{for } x \le 0. \end{cases}$$

The coefficient B belongs to $C^\infty(\mathbb{R}^2)$. We consider the constant function $w : \partial\Omega \to \mathbb{R}, x \mapsto 1$ on the boundary. For $x \le 0$ it follows immediately that $(\mathcal{E}_{\mathcal{L}} w)(x,y) = 0$. Thus also $\mathcal{P}_1 w$ and $\mathcal{P}_{\text{-}1} w$ vanish in this region. For $x > 0$ we can construct the projections by the method of characteristics:

$$\mathcal{P}_1 w(x,y) = \frac{\exp(y \exp 1/x)}{1 + \exp\exp 1/x}, \quad \mathcal{P}_{\text{-}1} w(x,y) = \frac{\exp((1-y)\exp 1/x)}{1 + \exp\exp 1/x}.$$

The functions $\mathcal{P}_1 w$ and $\mathcal{P}_{\text{-}1} w$ are elements of $C^\infty(\Omega)$. In contrast the traces $(\mathcal{P}_1 w)|_{\partial\Omega}$ and $(\mathcal{P}_{\text{-}1} w)|_{\partial\Omega}$ are discontinuous at the points $(0,0)$ and $(0,1)$.

In order to more clearly understand the previous example we consider the operator $\mathcal{O}$ in more detail; when fully expanded it reads

$$\dot{A}_{ik\iota\kappa}\, \partial_{k\kappa} v_\iota + \dot{B}_{i\iota\kappa}\, \partial_\kappa v_\iota + \dot{C}_{i\iota}\, v_\iota,$$

where

$$\begin{aligned} \dot{A}_{ik\iota\kappa} &:= 4\, B_{ijk}\, B_{\iota j\kappa}, \\ \dot{B}_{i\iota\kappa} &:= 2\partial_k(B_{ijk} B_{\iota j\kappa}) + 2B_{ijk}(\partial_k B_{\iota j\kappa}) + 2B_{ij\kappa}(\partial_k B_{\iota jk}), \\ \dot{C}_{i\iota} &:= 2B_{ijk}(\partial_{k\kappa} B_{\iota j\kappa}) + (\partial_k B_{ijk})(\partial_\kappa B_{\iota j\kappa}) + \delta_{i\iota}. \end{aligned} \tag{1.36}$$

If $\mathcal{O}$ is a scalar operator then its Fichera function is

$$\partial\Omega \to \mathbb{R}, x \mapsto (\dot{B}_{i\iota\kappa} - \partial_k \dot{A}_{ik\iota\kappa})\, \nu_\kappa =: (\dot{B}_\kappa - \partial_k \dot{A}_{k\kappa})\, \nu_\kappa.$$

Because of the particular choice of the coefficients in (1.36), the Fichera function of $\mathcal{O}$ vanishes. For a general class of second-order degenerate elliptic-parabolic equations Fichera (Fichera 1956) decomposed the boundary of Ω into three regions: Σ_3 is the set of all points of $\partial\Omega$ for which $\dot{A}_{k\kappa}\,\nu_k\,\nu_\kappa$ is positive; Σ_2 is the set of all points where $\dot{A}_{k\kappa}\,\nu_k\,\nu_\kappa = 0$ and $(\dot{B}_\kappa - \partial_k \dot{A}_{k\kappa}) > 0$; finally Σ_1 is the complement $\partial\Omega \setminus (\Sigma_3 \cup \Sigma_2)$. The 'Generalised Dirichlet Problem' for second-order degenerate elliptic-parabolic equations is to determine a solution of $\mathcal{O}u = f$ which vanishes on $\Sigma_2 + \Sigma_3$. This problem has been analysed by a number of authors. In particular, we refer to the work by Oleĭnik and her student Radkevič. For details see (Oleĭnik and Radkevič 1973) and (Magenes 1996). Important contributions are also due to Kohn and Nirenberg. We highlight (Kohn and Nirenberg 1965), (Kohn and Nirenberg 1967) and (Kohn 1978).

The next theorem, which we cite from (Kohn and Nirenberg 1967, p. 801) in abbreviated form, shows that, by possibly rescaling $\mathcal{L}$, one can ensure basic smoothness properties of the projections $\mathcal{P}_1$ and $\mathcal{P}_{-1}$. In view of the special structure of our problem we simplify the statement of the theorem by making the assumption that $\Sigma_2 = \emptyset$. For the general case we refer to the original publication.

Theorem 20 *Let Ω be a bounded domain with C^∞-boundary. Suppose that the coefficients of the linear differential operator*

$$\mathcal{O}u = \dot{A}_{k\kappa}\,\partial_{k\kappa}v + \dot{B}_\kappa\,\partial_\kappa v + \dot{C}\,v$$

are real and of class C^∞ in $\overline{\Omega}$. The leading part of the operator is degenerate elliptic-parabolic, i.e. $\dot{A}_{k\kappa}(x)\,\xi_k\,\xi_\kappa \geq 0$ for all $\xi \in \mathbb{R}^m$ and $x \in \overline{\Omega}$. The Fichera function $(\dot{B}_\kappa - \partial_k \dot{A}_{k\kappa})\,\nu_\kappa$ vanishes on $\partial\Omega$. If Σ_3 is closed and if $-\dot{C}$ is sufficiently large in comparison to $\|\dot{A}_{k\kappa}\|_{W^{3,\infty}(\Omega)}$ then for every $r \geq 1$ there exists a constant $C(r)$ such that

$$\|u\|_{W^{r,2}(\Omega)} \leq C(r)\,\|f\|_{W^{r,2}(\Omega)} \tag{1.37}$$

for all $f \in W^{r,2}(\Omega)$ and all solutions u of $\mathcal{O}u = f$, $u|_{\Sigma_3} = 0$.

The requirement that $-\dot{C}$ needs to be of adequate size is not a severe restriction. For $\alpha \in (0,\infty)$ we introduce the operator $\mathcal{L}_\alpha := \alpha\,\mathcal{L}$. Clearly, the norms $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}_\alpha}$ are equivalent and consequently the same holds for $\|\cdot\|_{\mathcal{T}}$ and $\|\cdot\|_{\mathcal{T}_\alpha}$, where $\mathcal{T}_\alpha$ is the trace operator associated to $\mathcal{L}_\alpha$. However, inspection of (1.36) shows that by choosing α sufficiently small we can increase the ratio between $\dot{A}$ and $\dot{C}$ as necessary because the component $\delta_{\iota\iota}$ in the definition of $\dot{C}_{\iota\iota}$ is not affected by the rescaling.

In contrast, the condition that Σ_3 is closed limits the applicability of Theorem 20. It implies that the boundary $\partial\Omega$ consists of a finite number of components each of which either belongs completely to Σ_3 or to its complement. We have seen in Example 14 that if Σ_3 is not closed then the solution of $\mathcal{O}u = f$ is in general not smooth on $\overline{\Omega}$.

We now apply Theorem 20 to obtain a regularity estimate. Suppose that (1.37) holds. Let $u \in W^{r+2,2}(\Omega)$ and $\dot{u} = \mathcal{E}_{\mathcal{L}}\mathcal{T}_{\mathcal{L}}u \in W^2_{\mathcal{L}}(\Omega)$ where $r \geq 2$. We set $v := \dot{u} - u$. Since $B(\nu, x) \neq 0$ if, and only if, $x \in \Sigma_3$, we know that $v|_{\Sigma_3} = 0$. Taking (1.37) into account, we deduce from $\mathcal{O}v = -\mathcal{O}u \in W^{r,2}(\Omega)$ that $\dot{u} \in W^{r,2}(\Omega)$. Moreover $\mathcal{L}\,\dot{u} \in W^{r-1,2}(\Omega)$. According to the definition of $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ and recalling Theorem 19, the restrictions $\mathcal{P}_1|_{B^{2,2,r+3/2}(\partial\Omega)}$ and $\mathcal{P}_{-1}|_{B^{2,2,r+3/2}(\partial\Omega)}$ map into $B^{2,2,r-3/2}(\partial\Omega)$.

Certainly, it would be desirable to obtain similar regularity estimates for systems of equations. However, we are not aware of any results corresponding to Theorem 20 for degenerate elliptic-parabolic systems. For a more general investigation of degenerate elliptic-parabolic systems we refer, for instance, to (Bertiger and Cosner 1979) and the subsequent publications by the authors.

The Vector Space $W^2_{\mathcal{L},B}(\Omega)$

In the next chapter we refer to the closure

$$W^2_{\mathcal{L},B}(\Omega) := \overline{\{v|_\Omega : v \in C_0^\infty(\mathbb{R}^n)^m\}}^{\|\cdot\|_{\mathcal{L},B}} \tag{1.38}$$

of $C_0^\infty(\mathbb{R}^n)|_\Omega$ in the norm

$$\|\cdot\|_{\mathcal{L},B} : C_0^\infty(\mathbb{R}^n)^m \to \mathbb{R}, v \mapsto \left(\|v\|^2_{\mathcal{L}} + \|v\|^2_B\right)^{1/2}.$$

This space is equal to $W^2_{\mathcal{L}}(\Omega)$ if $W^2_{\mathcal{T}}(\partial\Omega)$ is a subset of $L^2_B(\partial\Omega)$. For $W^2_{\mathcal{L},B}(\Omega)$ we can introduce the trace operator $\mathcal{T}_B$. We define

$$\mathcal{T}_B : W^2_{\mathcal{L},B}(\Omega) \to L^2_B(\partial\Omega), v \mapsto v|_{\partial\Omega}$$

by continuous extension from the space of smooth functions.

Example 11 from the previous section shows that $\mathcal{T}_B$ is in general not surjective. However, $\mathcal{T}_B$ is an injective operator on the restriction to $\operatorname{Im}\mathcal{E}_{\mathcal{L}} \cap W^2_{\mathcal{L},B}(\Omega)$. Namely, choose $v \in \operatorname{Im}\mathcal{E}_{\mathcal{L}} \cap W^2_{\mathcal{L},B}(\Omega) \setminus \{0\}$. Then, there is a $w \in B^{2,2,1/2}(\partial\Omega)^m$ so that $\langle B(\nu)\, v, w\rangle_{\partial\Omega} \neq 0$. Since $w - 2\,\mathcal{P}_-\, w$ is contained in $L^2_B(\partial\Omega)$ and

$$\langle B(\nu)\, v, w\rangle_{\partial\Omega} = \langle |B|(\nu)\, v, w - 2\,\mathcal{P}_-\, w\rangle_{\partial\Omega},$$

it follows that $\mathcal{T}_B\, v \neq 0$. We have the following alternative characterisation of $W^2_{\mathcal{L},B}(\Omega)$.

Theorem 21 *A function* $u \in W^2_{\mathfrak{L}}(\Omega)$ *belongs to* $W^2_{\mathfrak{L},B}(\Omega)$ *if, and only if, there is a function* $v \in L^2_B(\partial\Omega)$ *such that* $\langle B(\nu)\, u, w\rangle_{\partial\Omega} = \langle B(\nu)\, v, w\rangle_{\partial\Omega}$ *for all* $w \in B^{2,2,1/2}(\partial\Omega)^m$.

Proof. Suppose $u \in W^2_{\mathfrak{L}}(\Omega)$ and $v \in L^2_B(\partial\Omega)$. We have that $\langle B(\nu)\, u, w\rangle_{\partial\Omega} = \langle B(\nu)\, v, w\rangle_{\partial\Omega}$ for all $w \in B^{2,2,1/2}(\partial\Omega)^m$. Then $\langle B(\nu)\, u, w\rangle_{\partial\Omega} \leq \|v\|_B\, \|w\|_B$ for $w \in B^{2,2,1/2}(\partial\Omega)^m$. Thus $w \mapsto \langle B(\nu)\, u, \cdot\rangle_{\partial\Omega}$ has a continuous extension to $L^2_B(\partial\Omega)$. Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^n)^m$-functions converging to u. Then we have

$$\lim_{i\to\infty} \langle B(\nu)\,(v - u_i), w\rangle_{\partial\Omega} = \lim_{i\to\infty} \langle B(\nu)\,(u - u_i), w\rangle_{\partial\Omega} = 0$$

for all $w \in L^2_B(\partial\Omega)$. Thus the sequence $(u_i|_{\partial\Omega})_{i\in\mathbb{N}}$ converges weakly in $L^2_B(\partial\Omega)$ to v. By applying Mazur's theorem we can pass to a sequence $(\dot{u}_i)_{i\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R}^n)^m$-functions which converge to u in $W^2_{\mathfrak{L}}(\Omega)$ and strongly to v in $L^2_B(\partial\Omega)$. Thus $(\dot{u}_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in the $W^2_{\mathfrak{L},B}(\Omega)$-norm and $u \in W^2_{\mathfrak{L},B}(\Omega)$. The implication in the other direction follows by setting $v = \mathfrak{T}_B u$ for $u \in W^2_{\mathfrak{L},B}(\Omega)$. ////

Unlike for $L^2(\Omega)^m$, matrix functions in $L^\infty(\partial\Omega)^{m\times m}$ do not define endomorphisms on $L^2_B(\partial\Omega)$. Consider the following example.

Example 15 *Suppose that* $\partial\Omega = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ *and that*

$$B(\nu, x) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 x \end{pmatrix}, \qquad x \in \partial\Omega.$$

The matrix function

$$J(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad x \in \partial\Omega$$

does not define an endomorphism on $L^2_B(\partial\Omega)$. *For instance the image of* $(0, 1/x)^{\mathsf{H}} \in L^2_B(\partial\Omega)$ *is the function* $(1/x, 0))^{\mathsf{H}}$ *which does not belong to* $L^2_B(\partial\Omega)$.

For this reason we introduce the set

$$L^\infty_B(\partial\Omega)^{m\times m} := \{J \in L^\infty(\partial\Omega)^{m\times m} : \exists\, C > 0\, \forall\, x \in \partial\Omega : J^{\mathsf{H}}\, |B|(\nu)\, J|_x \leq C^2\, |B|(\nu)|_x\}.$$

Here we denoted with '$\leq$' the partial ordering of positive semi-definite matrices, i.e. at each $x \in \partial\Omega$ the matrix $C\,|B|(\nu) - J^{\mathsf{H}}\,|B|(\nu)\,J|_x$ is positive semi-definite. We now have as desired

$$\|J\, v\|_B \leq C\|v\|_B$$

for $v \in L^2_B(\partial\Omega)$ and C as in the definition of $L^\infty_B(\partial\Omega)^{m\times m}$. We remark that P_+ and P_- are members of $L^\infty_B(\partial\Omega)^{m\times m}$ with $C = 1$.

1.9 Boundary Value Problems

In the context of first-order graph spaces we understand under the term boundary value problem the following task:

BVP 1 *Let $\mathfrak{J} : W^2_{\mathfrak{L}}(\Omega) \to V$ be a boundary operator with respect to the graph space of $\mathfrak{L}$. Given $f \in L^2(\Omega)^m$ and $h \in V$, find $u \in W^2_{\mathfrak{L}}(\Omega)$ such that $\mathfrak{L}u = f$ and $\mathfrak{J}u = h$.*

Clearly, for general $\mathfrak{L}$, $\mathfrak{J}$, f and h neither existence nor uniqueness of u is guaranteed. We call BVP 1 the classical formulation of the boundary value problem.

In order to analyse Friedrichs systems, it is necessary to consider the weak formulation of the boundary value problem. In contrast to the classical formulation, here the boundary conditions are imposed via bilinear forms.

Before we can state the weak formulation, we have to agree on an appropriate space of test functions. For that purpose we introduce the set of smooth functions which vanish on a closed subset S of $\partial\Omega$ with Hausdorff measure zero. The test space consists of the closure of this set in the $W^{1,2}(\Omega)$-norm:

$$W_0^{1,2}(\mathbb{R}^n \backslash S)^m = \overline{\{v \in W^{1,2}(\mathbb{R}^n)^m : \operatorname{supp}(v) \Subset \mathbb{R}^n \setminus S\}}.$$

Consequently $\mathscr{D}(\Omega)^m \subset W_0^{1,2}(\mathbb{R}^n \backslash S)^m$. The restriction of functions in $W_0^{1,2}(\mathbb{R}^n \backslash S)^m$ to $\partial\Omega$ constitute the set

$$B_0^{2,2,1/2}(\partial\Omega \backslash S)^m := \overline{\{v \in B^{2,2,1/2}(\partial\Omega)^m : \operatorname{supp}(v) \Subset \partial\Omega \setminus S\}}.$$

The spaces $B_0^{2,2,1/2}(\partial\Omega \backslash S)^m$ and $W_0^{1,2}(\mathbb{R}^n \backslash S)^m$ are equipped with the $B^{2,2,1/2}(\partial\Omega)^m$- and $W^{1,2}(\mathbb{R}^n)^m$-norms, respectively.

We restrict our attention to boundary operators $\mathfrak{J}$ which map $W^2_{\mathfrak{L}}(\Omega)$-functions into the dual of $B_0^{2,2,1/2}(\partial\Omega \backslash S)^m$, which we denote by $B_0^{2,2,-1/2}(\partial\Omega \backslash S)^m$. By construction $v_1, v_2 \in B_0^{2,2,-1/2}(\partial\Omega \backslash S)^m$ coincide if, and only if, $\langle v_1, w\rangle_{\partial\Omega} = \langle v_2, w\rangle_{\partial\Omega}$ for all $w \in B_0^{2,2,1/2}(\partial\Omega \backslash S)^m$. The introduction of the pair of spaces $(B_0^{2,2,1/2}(\partial\Omega \backslash S)^m, B_0^{2,2,-1/2}(\partial\Omega \backslash S)^m)$ allows us to consider boundary operators which are smooth with the exception of the set S, where we allow the operator to be singular.

BVP 2 *Let $\mathfrak{J} : W^2_{\mathfrak{L}}(\Omega) \to B_0^{2,2,-1/2}(\partial\Omega \backslash S)^m$ be a boundary operator with respect to the graph space of $\mathfrak{L}$. Given $f \in L^2(\Omega)^m$, $h \in B_0^{2,2,-1/2}(\partial\Omega \backslash S)^m$, find $u \in W^2_{\mathfrak{L}}(\Omega)$ such that $\forall\, w \in W_0^{1,2}(\mathbb{R}^n \backslash S)^m$:*

$$\langle \mathfrak{L}u, w\rangle_\Omega + \langle \mathfrak{J}u, w\rangle_{\partial\Omega} = \langle f, w\rangle_\Omega + \langle h, w\rangle_{\partial\Omega}.$$

Let $V = B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$. Then every solution of BVP 1 solves BVP 2. Now consider a solution u of BVP 2. Testing u with functions in $\mathscr{D}(\Omega)^m$ implies that $\mathfrak{L}u = f$. Consequently $\langle \mathfrak{J}u, w\rangle_{\partial\Omega}$ equals $\langle h, w\rangle_{\partial\Omega}$ for all $w \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$, which implies equality of $\mathfrak{J}u$ and h in $B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$. Therefore every solution of BVP 2 solves BVP 1.

Example 16 *Recall the definition of Ω and $\mathfrak{L}$ from Example 11. We select $\mathfrak{J}$ so that it imposes the inflow boundary conditions on this domain:*

$$(\mathfrak{J}v)|_{\partial\Omega_i} = v|_{\partial\Omega_i} \quad \text{and} \quad (\mathfrak{J}v)|_{\partial\Omega\backslash\partial\Omega_i} = 0$$

where $\partial\Omega_i := \{(x, y) \in \partial\Omega : x \leq 0 \text{ and } y \in (0, 1)\}$. Let $S = \{(0, 0), (-1, 1), (1, 1)\}$. In Example 11 we noted that, independently of α, the function $v(x, y) = 1/\sqrt{y}$ is a member of the graph space. We are interested in the continuity of the linear form

$$\mathfrak{J} : B_0^{2,2,1/2}(\partial\Omega\backslash S)^m \to \mathbb{R}, w \mapsto \langle \mathfrak{J}\, v, w\rangle_{\partial\Omega} = \int_{\partial\Omega_i} \frac{1}{\sqrt{x}}\, w \,\mathrm{d}(x, y).$$

Because of density it is sufficient to consider elements $w \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ whose support is bounded away from S. Without loss of generality we can assume that $\operatorname{supp}(w) \subset \partial\Omega_i$. However then $\mathfrak{J}w = \langle B(\nu)\, v, w\rangle_{\partial\Omega}$. Thus $\|\mathfrak{J}\|_{\mathscr{B}(B_0^{2,2,1/2}(\partial\Omega\backslash S)^m,\mathbb{R})}$ is bounded by the operator norm of $\mathfrak{T}$.

Consider a family of functions $(w_t)_{t\in(\varepsilon,1-\varepsilon)}$ which are translations of each other and whose support is contained in the set

$$\{(-y, y) \in \partial\Omega_i : y \in (t - \varepsilon, t + \varepsilon)\}.$$

A decrease in t results in an increase of the term $\mathfrak{J}w_t$. It is evident that $\mathfrak{J}$ is not continuous with respect to the $L^2(\partial\Omega)^m$-norm since the $L^2(\partial\Omega)^m$-norm of w_t is invariant under translation and since the support of w can be concentrated arbitrarily close to the origin by passing to families $(w_t)_{t\in(\varepsilon,1-\varepsilon)}$ with sufficiently small ε. In contrast, the $B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$-norm of w_t increases as $t \searrow \varepsilon$ as an inspection of (1.10) reveals. This difference makes the continuity $\mathfrak{J}$ with respect to the $B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$-norm possible.

Finally, we observe that if we replace the domain of $\mathfrak{J}$ by $W^2_{\mathfrak{L}}(\Omega)$ we do not obtain a continuous linear map anymore. This is easily seen be choosing $w(x, y) = 1/\sqrt{y}$.

Example 16 brings us to the delicate and important issue as to whether we can assume that the bilinear form $\langle \mathfrak{J}\cdot, \cdot\rangle_{\partial\Omega}$ has a continuous extension to $W^2_{\mathfrak{L}}(\Omega) \times W^2_{\mathfrak{L}'}(\Omega)$. We are interested in the extension of $\langle \mathfrak{J}\cdot, \cdot\rangle_{\partial\Omega}$ because the analysis of Friedrichs systems is based on energy integral methods which rely on the positivity of

$$v \mapsto \langle \mathfrak{L}v, v\rangle_\Omega + \langle \mathfrak{J}v, v\rangle_{\partial\Omega}.$$

Certainly for the boundary value problem considered in Example 16 such an extension does not exist.

If the image of $\mathcal{J}$ is contained in $W^2_{\mathcal{T}}(\partial\Omega)$ and $\mathcal{J} : W^2_{\mathcal{L}}(\Omega) \to W^2_{\mathcal{T}}(\partial\Omega)$ is continuous then $\mathcal{J}$ naturally induces a continuous bilinear form on the graph space via density of smooth functions. Let $v \in C_0^\infty(\mathbb{R}^n)^m$ and $w \in C_0^\infty(\mathbb{R}^n)^m$; then

$$\begin{aligned}\langle \mathcal{J}v, w\rangle_{\partial\Omega} &= \langle \mathcal{L}\mathcal{E}_{\mathcal{L}}\,\mathcal{J}\,v, w\rangle_\Omega - \langle \mathcal{E}_{\mathcal{L}}\,\mathcal{J}\,v, \mathcal{L}'w\rangle_\Omega \le \|\mathcal{E}_{\mathcal{L}}\,\mathcal{J}\,v\|_{\mathcal{L}}\|w\|_{\mathcal{L}'} \\ &= \|\mathcal{J}\,v\|_{\mathcal{T}}\|w\|_{\mathcal{L}'} \le \|\mathcal{J}\|_{\mathcal{B}(W^2_{\mathcal{L}}(\Omega), W^2_{\mathcal{T}}(\partial\Omega))}\,\|v\|_{\mathcal{L}}\|w\|_{\mathcal{L}'}.\end{aligned} \tag{1.39}$$

For the first equality we used that $\mathcal{J}v$ lies in the domain of $\mathcal{E}_{\mathcal{L}}$ and integration by parts. The meaning of $\langle \mathcal{J}v, w\rangle_{\partial\Omega}$ in (1.39) is given by the embedding of $W^2_{\mathcal{T}}(\partial\Omega)$ into $B^{2,2,-1/2}(\partial\Omega)^m$ in the sense of Theorem 7. Because of the bound (1.39) we can extend $(v, w) \mapsto \langle \mathcal{J}v, w\rangle_{\partial\Omega}$ to a bilinear form which acts in both arguments on functions which are contained in a graph space. This extension is comparable with the extension of $\langle B(\nu)\cdot, \cdot\rangle_{\partial\Omega}$ in Section 1.5; confer in particular with Remark 3.

Conversely, we can question whether a boundary operator $\mathcal{J} : W^2_{\mathcal{L}}(\Omega) \to V$ for which $\langle \mathcal{J}\cdot, \cdot\rangle_{\partial\Omega}$ has a continuous extension must have the codomain $W^2_{\mathcal{T}}(\partial\Omega)$. For our needs it is sufficient to consider the Hermitian case $\mathcal{L} = \mathcal{L}^h$. We make the assumption that the scalar product

$$V \times C_0^\infty(\mathbb{R}^n)^m, (v, w) \mapsto \langle v, w\rangle_{\partial\Omega} = \langle v, (w|_{\partial\Omega})\rangle_{\partial\Omega} \tag{1.40}$$

is meaningful and that

$$\forall\, u \in W^2_{\mathcal{L}}(\Omega)\;\; \forall\, v \in C_0^\infty(\mathbb{R}^n)^m : |\langle \mathcal{J}\,u, w\rangle_{\partial\Omega}| < C\,\|u\|_{\mathcal{L}}\,\|w\|_{\mathcal{T}'} \tag{1.41}$$

holds for some constant $C > 0$. Then for fixed u the linear mapping $w \mapsto \langle \mathcal{J}\,u, w\rangle_{\partial\Omega}$ has a unique continuous extension to $W^2_{\mathcal{T}}(\partial\Omega)$. Thus by the Riesz representation theorem there is a $T(u) \in W^2_{\mathcal{T}}(\partial\Omega)$ such that

$$\forall\, w \in W^2_{\mathcal{T}}(\partial\Omega) : \langle \mathcal{J}\,u, w\rangle_{\partial\Omega} = \langle \mathcal{E}_{\mathcal{L}}T(u), \mathcal{E}_{\mathcal{L}}w\rangle_{\mathcal{L}} = \langle B(\nu)\,\mathcal{L}\,\mathcal{E}_{\mathcal{L}}T(u), w\rangle_{\partial\Omega}.$$

Since $B(\nu)\,\mathcal{L}\,\mathcal{E}_{\mathcal{L}}T(u)$ is a member of the trace space, we deduce that there is a boundary operator $\dot{\mathcal{J}} : W^2_{\mathcal{L}}(\Omega) \to W^2_{\mathcal{T}}(\partial\Omega)$ such that $\langle \mathcal{J}u, v\rangle_{\partial\Omega} = \langle \dot{\mathcal{J}}u, v\rangle_{\partial\Omega}$ for all $u \in W^2_{\mathcal{L}}(\Omega)$ and $v \in C_0^\infty(\mathbb{R}^n)^m$. In this sense a continuous extension of $\langle \mathcal{J}\cdot, \cdot\rangle_{\partial\Omega}$ exists if, and only if, the codomain of $\mathcal{J}$ is $W^2_{\mathcal{T}}(\partial\Omega)$. Having said this it is easily seen that we can also allow codomains which can be continuously embedded into $W^2_{\mathcal{T}}(\partial\Omega)$.

Theorem 22 *Let $\mathcal{L} = \mathcal{L}^h$. Consider a boundary operator $\mathcal{J} : W^2_{\mathcal{L}}(\Omega) \to V$ such that (1.40) is meaningful and (1.41) is satisfied. Then $\langle \mathcal{J}\cdot, \cdot\rangle_{\partial\Omega}$ has a continuous extension to $W^2_{\mathcal{L}}(\Omega) \times W^2_{\mathcal{L}'}(\Omega)$ if, and only if, V is continuously embedded into $W^2_{\mathcal{T}}(\partial\Omega)$.*

Remark 4 *Theorem 22 and Example 16 now make a fundamental difficulty in the analysis of Friedrichs systems quite apparent. On the one hand we desire to allow boundary operators with a large codomain in order to implement boundary conditions which exhibit singularities. On the other hand we also want to limit the size of the codomain to* $W^2_{\mathfrak{J}}(\partial\Omega)$ *so that the applicability of energy integral methods is not restricted. We shall return to this issue in the next chapter.*

To familiarise ourselves with the concept of weak boundary value problems we consider an example. We use the opportunity to introduce the $^\circ$-adjoint boundary operator of $\mathfrak{J} : W^2_{\mathfrak{L}}(\Omega) \to B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$ with respect to the pivot $\langle\cdot,\cdot\rangle_{\partial\Omega}$. We denote the dual space of $W^2_{\mathfrak{L}}(\Omega)$ by $W^2_{\mathfrak{L}}(\Omega)'$. For all $v \in W^2_{\mathfrak{L}}(\Omega)$ and $w \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ we have the bound

$$\langle \mathfrak{J}\, v, w\rangle_{\partial\Omega} \leq \|\mathfrak{J}\|_{\mathscr{B}(W^2_{\mathfrak{L}}(\Omega), B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m)}\, \|v\|_{\mathfrak{L}}\, \|w\|_{B_0^{2,2,1/2}(\partial\Omega\backslash S)^m}.$$

Therefore the operator

$$\mathfrak{J}^\circ : B_0^{2,2,1/2}(\partial\Omega\backslash S)^m \to W^2_{\mathfrak{L}}(\Omega)', w \mapsto (v \mapsto \langle \mathfrak{J} v, w\rangle_{\partial\Omega})$$

is continuous. Moreover for all $v, w \in W_0^{1,2}(\mathbb{R}^n\backslash S)^m$ the identity

$$\langle v, \mathfrak{J}^\circ w\rangle_{\partial\Omega} = \langle \mathfrak{J}\, v, w\rangle_{\partial\Omega} \tag{1.42}$$

holds.

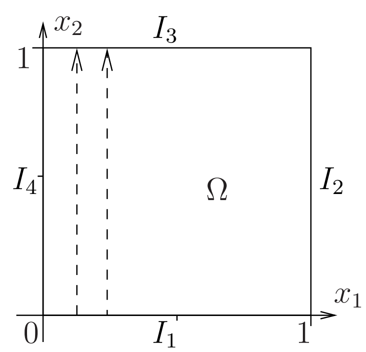
Example 17 *Let* $\Omega = (0,1)^2$ *and let* $\mathfrak{L}$ *be*

$$\mathfrak{L} : L^2(\Omega)^2 \to \mathscr{D}'(\Omega)^2, (v_1(x,y), v_2(x,y)) \mapsto (\partial_y v_1, x\,\partial_y v_2)^{\mathsf{H}}.$$

Accordingly,

$$B(\nu) = \begin{pmatrix} \nu_2 & 0 \\ 0 & x\,\nu_2 \end{pmatrix}.$$

We denote the four edges of the domain by I_1, I_2, I_3 *and* I_4 *as indicated in the figure.*



Clearly, $\mathcal{T}$ vanishes on I_2 and I_4. On I_1 and I_3 the matrix $B(\nu)$ is invertible and therefore the function values of v are well-defined on I_1 and I_3:

$$v(x,y) = B(\nu)^{-1}(\mathcal{T}v)(x,y), \qquad (x,y) \in I_1 \cup I_3. \tag{1.43}$$

On each characteristic the graph space is isomorphic to $W^{1,2}(0,1)^2$ and the trace defined via (1.43) agrees, up to a null set, with the $W^{1,2}(0,1)^2$-trace taken point-by-point on I_1 and I_3. Thus we can define the boundary operator

$$\mathcal{J} : W^2_{\mathcal{L}}(\Omega) \to W^2_{\mathcal{T}}(\partial\Omega), (v_1, v_2) \mapsto ((x,0) \mapsto (0, \sqrt{x}\, v_1)^{\mathsf{H}})$$

We want to show that $\mathcal{J}$ has a continuous extension to $W^2_{\mathcal{L}}(\Omega) \times W^2_{\mathcal{L}'}(\Omega)$.

In Example 11 we used a function $\psi : \Omega \to (0,1)$ to write $W^2_{\mathcal{L}}(\Omega)$ as the intersection of the graph spaces associated to the operators $v \mapsto \partial_k(B_{ijk}\,\psi\, v_j)$ and $v \mapsto \partial_k(B_{ijk}\,(1-\psi)\, v_j)$. By choosing $\psi(x,y) = (1-x)$ for the current example we demonstrate along the same lines that the restriction of $(v_1, v_2) \in W^2_{\mathcal{L}}(\Omega)$ to $I_1 \cup I_3$ is contained in $L^2_B(I_1 \cup I_3)$. Since the coefficient in the first component of $\mathcal{L}$ is constant, the set of restrictions $v_1|_{I_1 \cup I_3}$ of functions $(v_1, v_2) \in W^2_{\mathcal{L}}(\Omega)$ is actually equal to $L^2(I_1 \cup I_3)$. This can also be seen by combining the eigenspace decomposition of Example 13 with (1.35).

Let $S = \{(0,0),(0,1),(1,0),(1,1)\}$. Requiring that $\mathcal{J}^\circ$ satisfies equation (1.42) for all $v, w \in W^{1,2}_0(\mathbb{R}^n \backslash S)^m$ implies that

$$\mathcal{J}^\circ(v_1, v_2) = (\sqrt{x}\, v_2, 0)^{\mathsf{H}}.$$

Then $\sqrt{x}\, v_2$ is an element of $L^2(I_1 \cup I_3)$ if v_2 is contained in the set

$$L^2(I_1 \cup I_3, x) = \Big\{ x\, v|_{I_1 \cup I_3} : \int_{I_1 \cup I_3} v^2\, x \,\mathrm{d}x < \infty \Big\}.$$

Using the continuity of $\mathcal{T}_B$ and that $\operatorname{Im}\mathcal{J} \subset W^2_{\mathcal{T}'}(\partial\Omega)$ we establish that $\langle \mathcal{J}^\circ\cdot, \cdot\rangle_{\partial\Omega}$ has a continuous extension to $W^2_{\mathcal{L}}(\Omega) \times W^2_{\mathcal{L}'}(\Omega)$. Because of (1.42) also the bilinear form $\langle \mathcal{J}\cdot, \cdot\rangle_{\partial\Omega}$ can be continuously extended. Thus $\mathcal{J}$ maps into $W^2_{\mathcal{T}}(\partial\Omega)$. Taking into account that

$$L^2(I_1 \cup I_3, x) \to L^2(I_1 \cup I_3, x), v_1 \mapsto \sqrt{x}\, v_1$$

is a surjective mapping, we deduce that the set $L^2_B(I_1 \cup I_3)$ is in fact equal to the trace space.

We now come to the statement of the weak adjoint formulation of a boundary value problem. The $'$-adjoint boundary operator to $\mathcal{J} : W^2_{\mathcal{L}}(\Omega) \to B^{2,2,-1/2}_0(\partial\Omega\backslash S)^m$ is

$$\mathcal{J}' : W^{1,2}_0(\mathbb{R}^n \backslash S)^m \to W^2_{\mathcal{L}}(\Omega)', v \mapsto \mathcal{J}^\circ v + \mathcal{T}^\circ v. \tag{1.44}$$

For the definition of $\mathcal{T}^\circ$ we substitute the codomain $B^{2,2,-1/2}(\partial\Omega)^m$ of $\mathcal{T}$ by the larger space $B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$. Then (1.42) is applicable. The motivation for the construction of $\mathcal{J}'$ is that for all $v \in W^2_{\mathcal{L}}(\Omega)$, $w \in W_0^{1,2}(\mathbb{R}^n\backslash S)^m$

$$\langle \mathcal{L}v, w\rangle_\Omega + \langle \mathcal{J}v, w\rangle_{\partial\Omega} = \langle v, \mathcal{L}'w\rangle_\Omega + \langle v, \mathcal{J}'w\rangle_{\partial\Omega}. \tag{1.45}$$

The weak adjoint formulation of the boundary value problem then has the following form:

BVP 3 *Let $\mathcal{J} : W^2_{\mathcal{L}}(\Omega) \to B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$ be a boundary operator with respect to the graph space of $\mathcal{L}$. Given $f \in L^2(\Omega)^m$, $h \in B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$, find $u \in W^2_{\mathcal{L}}(\Omega)$ such that $\forall\, w \in W_0^{1,2}(\mathbb{R}^n\backslash S)^m$:*

$$\langle u, \mathcal{L}'w\rangle_\Omega + \langle u, \mathcal{J}'w\rangle_{\partial\Omega} = \langle f, w\rangle_\Omega + \langle h, w\rangle_{\partial\Omega}.$$

Obviously BVP 2 and BVP 3 are equivalent. Finally, we consider the strong formulation of the boundary value problem.

Observe that we can extend the domain of $\mathcal{J}'$ to the whole graph space $W^2_{\mathcal{L}'}(\Omega)$ if, and only if, $\langle \mathcal{J}\cdot, \cdot\rangle_{\partial\Omega}$ has a continuous extension to $W^2_{\mathcal{L}}(\Omega) \times W^2_{\mathcal{L}'}(\Omega)$. Similarly by identifying $\mathcal{J}$ and $\dot{\mathcal{J}}$ in (1.20), we can state that $\mathcal{J}'$ is contained in $\mathscr{B}(W^2_{\mathcal{T}'}(\partial\Omega), W^2_{\mathcal{T}}(\partial\Omega)')$ if, and only if, there is a continuous extension of $\langle \mathcal{J}\cdot, \cdot\rangle_{\partial\Omega}$. In general, $\mathcal{J}'$ is only contained in the larger space $\mathscr{B}(B_0^{2,2,1/2}(\partial\Omega\backslash S)^m, W^2_{\mathcal{T}}(\partial\Omega)')$.

BVP 4 *Let $\mathcal{J} : W^2_{\mathcal{L}}(\Omega) \to B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$ be a boundary operator with respect to the graph space of $\mathcal{L}$. Given $f \in L^2(\Omega)^m$, $h \in B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$, find a function $u \in W^2_{\mathcal{L}}(\Omega)$ and a sequence $(u_i)_{i\in\mathbb{N}}$ such that $u_i \in W^{1,2}(\Omega)^m$ and $\mathcal{J}u_i = h$ for $i \in \mathbb{N}$ and*

$$\lim_{i\to\infty} \|u - u_i\|_{\mathcal{L}} = 0.$$

Strong solutions, i.e. solutions of BVP 4, are weak solutions, i.e. solutions of BVP 2. Suppose that $u \in W^2_{\mathcal{L}}(\Omega)$ is a strong solution which is approximated by a sequence $(u_i)_{i\in\mathbb{N}}$ whose elements satisfy the boundary conditions exactly. Then for all $w \in W_0^{1,2}(\mathbb{R}^n\backslash S)^m$ we have

$$\langle f, w\rangle_\Omega + \langle h, w\rangle_{\partial\Omega} = \langle \lim_{i\to\infty} \mathcal{L}u_i, w\rangle_\Omega + \langle \lim_{i\to\infty} \mathcal{J}u_i, w\rangle_{\partial\Omega} = \langle \mathcal{L}u, w\rangle_\Omega + \langle \mathcal{J}u, w\rangle_{\partial\Omega}.$$

In contrast, not every weak solution is a strong solution. Evidently, if a weak solution u satisfies boundary conditions which are not contained in $\mathcal{J}(W^{1,2}(\Omega)^m)$ then u cannot be strong. However the difference between weak and strong solutions is more subtle than this. For instance, in Example 25 consider a weak solution of a homogeneous boundary value problem which is not strong.

In the framework of Friedrichs systems to understand when weak solutions are also strong is closely related to the question of well-posedness. The reason is that for a large class of problems it is possible to prove that weak solutions of Friedrichs systems exist and that strong solutions are unique.

Remark 5 *We introduced the adjoint operator of $\mathcal{L}$ formally in (1.1). We shall now clarify how the formal adjoint is connected to the adjoint operator in the sense of unbounded operators. We denote the adjoint operator in the sense of unbounded operators by $\mathcal{L}^\diamond$ in order to distinguish it from the formal adjoint. We assume that $l = m$. The domain $\mathcal{D}(\mathcal{L}^\diamond)$ is the space of all $w \in L^2(\Omega)^m$ for which*

$$\exists\, C \in \mathbb{R}\ \forall\, v \in W^2_{\mathcal{L}}(\Omega) : C\,\|v\|_{L^2(\Omega)^m} \geq \langle \mathcal{L}v, w\rangle_{\partial\Omega}. \tag{1.46}$$

If $w \in \mathcal{D}(\mathcal{L}^\diamond)$, then by the Hahn-Banach theorem the functional $v \mapsto \langle \mathcal{L}v, w\rangle_{\partial\Omega}$ can be extended to a continuous functional on $L^2(\Omega)^m$. The extension is unique since $W^2_{\mathcal{L}}(\Omega)$ is dense in $L^2(\Omega)^m$. By the Riesz representation theorem we can assign to every $w \in \mathcal{D}(\mathcal{L}^\diamond)$ a unique element $\mathcal{L}^\diamond w \in L^2(\Omega)^m$ such that

$$\forall\, v \in W^2_{\mathcal{L}}(\Omega) : \langle \mathcal{L}v, w\rangle_{\partial\Omega} = \langle v, \mathcal{L}^\diamond w\rangle_{\partial\Omega}.$$

Adapting Theorem 13.8 in (Rudin 1991) to our notation, we obtain that

$$\begin{aligned}\Gamma(\mathcal{L}^\diamond) &= \Gamma'(\mathcal{L})^\perp \\ &= \{(w_1, w_2) \in L^2(\Omega)^m \times L^2(\Omega)^m : \langle -\mathcal{L}v, w_1\rangle_\Omega + \langle v, w_2\rangle_\Omega = 0\},\end{aligned} \tag{1.47}$$

where $^\perp$ marks the orthogonal complement in $L^2(\Omega)^m \times L^2(\Omega)^m$. Choosing $v \in \mathscr{D}(\Omega)^m$ in (1.47) forces $w_2 = \mathcal{L}'w_1$. Then, for $v \in W^2_{\mathcal{L}}(\Omega)$, the integration by parts formula (1.14) implies that $w_1 \in \ker \mathcal{T}_{\mathcal{L}'}$. Since pairs $(w, \mathcal{L}'w)$ with $w \in \ker \mathcal{T}_{\mathcal{L}'}$ are members of the complement of $\Gamma'(\mathcal{L})$, the adjoint in the sense of unbounded operators is the restriction of the formal adjoint, defined by (1.1), to $\ker \mathcal{T}_{\mathcal{L}'}$:

$$\mathcal{L}^\diamond = \mathcal{L}'|_{\ker \mathcal{T}_{\mathcal{L}'}}. \tag{1.48}$$

Applying Theorem 13.13 from (Rudin 1991), we learn that the restriction of $\mathcal{O}$ to

$$\mathcal{D}(\mathcal{O}) = \mathcal{D}(\mathcal{L}^\diamond\mathcal{L}) = \{x \in W^2_{\mathcal{L}}(\Omega) : \mathcal{L} \in \ker \mathcal{T}_{\mathcal{L}'}\}$$

is a one-to-one mapping onto $L^2(\Omega)^m$, where $\mathcal{D}(\mathcal{O})$ and $\mathcal{D}(\mathcal{L}^\diamond\mathcal{L})$ are defined by substituting in (1.46) the operator $\mathcal{L}$ with $\mathcal{O}$ and $\mathcal{L}^\diamond\mathcal{L}$, respectively. Moreover it follows from Theorem 13.13 in (Rudin 1991) that $\mathcal{O}|_{\mathcal{D}(\mathcal{O})}$ is a self-adjoint operator. We therefore remark that $\mathcal{O}_{\mathcal{L}'}$ is not to be mistaken for the adjoint of $\mathcal{O}$, despite the fact that $\mathcal{O}_{\mathcal{L}'}$ describes the set of minimisers of the adjoint of $\mathcal{L}$.

1.10 The Skew-Hermitian Case

Graph spaces of operators with skew-Hermitian coefficients in the first-order component of the operator share many properties with their Hermitian counterpart discussed in Section 1.8. We exemplify this close relationship by exhibiting a decomposition of $\operatorname{Im}\mathcal{E}_{\mathcal{L}}$ into eigenspaces which resembles the decomposition we gave in Section 1.8. We shall not consider the analysis of the skew-Hermitian case further than that since investigations in this direction are not the objective of this dissertation.

We assume that $\mathcal{L} = \mathcal{L}^s$ and that $D^s = 0$. Then, according to Theorem 15,

$$\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}} = \mathcal{L}'|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}} = -\mathcal{L}^{-1}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}. \tag{1.49}$$

Clearly, $\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}$ is normal and the graph norms with respect to $\mathcal{L}$ and $\mathcal{L}'$ coincide in $\operatorname{Im}\mathcal{E}_{\mathcal{L}}$. We derive, for $v \in \operatorname{Im}\mathcal{E}_{\mathcal{L}}$, the two identities:

$$\begin{aligned} v + i\,\breve{v} &= -\mathcal{L}\mathcal{L}v + i\,\mathcal{L}v = \mathcal{L}(-\breve{v} + i\,v) = i\,\breve{(v + i\,\breve{v})}, \\ v - i\,\breve{v} &= -\mathcal{L}\mathcal{L}v - i\,\mathcal{L}v = -\mathcal{L}(\breve{v} + i\,v) = -i\,\breve{(v + i\,\breve{v})}. \end{aligned} \tag{1.50}$$

Theorem 23 *The space* $\operatorname{Im}\mathcal{E}_{\mathcal{L}}$ *is the orthogonal sum of the eigenspaces* $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}, i)$ *and* $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}, -i)$. *In addition the operators*

$$\begin{aligned} &\mathcal{P}_i : \operatorname{Im}\mathcal{E}_{\mathcal{L}} \to \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}, i), v \mapsto {}^1\!/_2\,(v + i\,\breve{v}), \\ &\mathcal{P}_{-i} : \operatorname{Im}\mathcal{E}_{\mathcal{L}} \to \operatorname{Eig}(\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}, -i), v \mapsto {}^1\!/_2\,(v - i\,\breve{v}), \end{aligned}$$

are projections onto $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}, i)$ *and* $\operatorname{Eig}(\mathcal{L}|_{\operatorname{Im}\mathcal{E}_{\mathcal{L}}}, -i)$, *respectively.*

Proof. Almost identical to the proof of Theorem 19. ////

1.11 Remarks on Duality

We end the chapter with general remarks on the dual space of $W^q_{\mathcal{L}}(\Omega)$, $q \in (1, \infty)$. We allow $m \neq l$ and do not impose symmetry conditions on the coefficients of the differential operator. As before, we assume that Ω satisfies a strong local Lipschitz condition. In Corollary 3 we already stated that $W^q_{\mathcal{L}}(\Omega)$ is reflexive and uniformly convex. Since the graph space is isometric to a closed subset of $L^q(\Omega)^m \times L^q(\Omega)^l$ it also inherits separability from this space. We turn our attention to the question of how the dual $W^q_{\mathcal{L}}(\Omega)'$ of $W^q_{\mathcal{L}}(\Omega)$ can be represented explicitly.

Theorem 24 *For every* $\Lambda \in W^q_{\mathfrak{L}}(\Omega)'$ *there exist elements* $(w_1, w_2) \in L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l$ *such that, for all* $v \in W^q_{\mathfrak{L}}(\Omega)$,

$$\Lambda(v) = \langle w_1, v\rangle_\Omega + \langle w_2, \mathfrak{L}v\rangle_\Omega. \tag{1.51}$$

Let V *be the set of* $(w_1, w_2) \in L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l$ *which fulfill (1.51). Then, there exists a unique element* $\bar{w} \in V$ *such that*

$$\|\Lambda\|_{W^q_{\mathfrak{L}}(\Omega)'} = \|\bar{w}\|_{L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l} = \inf\{\|w\|_{L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l} : w \in V\}. \tag{1.52}$$

Proof. The dual of $L^q(\Omega)^m \times L^q(\Omega)^l$ is $L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l$. We identify $W^q_{\mathfrak{L}}(\Omega)$ with $\Gamma(\mathfrak{L})$ through $\mathfrak{I}$, cf. (1.1). Then the elements of the dual of $W^q_{\mathfrak{L}}(\Omega)$ are functionals defined on the subspace $\Gamma(\mathfrak{L})$ of $L^q(\Omega)^m \times L^q(\Omega)^l$. By the Hahn-Banach theorem in uniformly convex Banach spaces the norm-preserving extension of a linear functional is uniquely defined, cf. (Werner 2000, Exercises I.4.13, I.4.18 and III.6.7). Let $\Lambda \in W^q_{\mathfrak{L}}(\Omega)'$ and let $(\bar{w}_1, \bar{w}_2) \in L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l$ represent the norm-preserving extension of Λ. Clearly, $(\bar{w}_1, \bar{w}_2)$ satisfies (1.51) and (1.52). All elements in V are extensions of Λ from $\Gamma(\mathfrak{L})$ to $L^q(\Omega)^m \times L^q(\Omega)^l$, thus their norms are larger than or equal to $\|\Lambda\|_{W^q_{\mathfrak{L}}(\Omega)'}$. ////

Theorem 25 *The set* V *is equal to the complete affine space* $\bar{w} + \Gamma'(\mathfrak{L}'|_{\ker \mathfrak{T}_{\mathfrak{L}'}})$.

Proof. Let $w \in V$. Since for all $v \in \mathscr{D}(\Omega)$ we have

$$\langle w_1 - \bar{w}_1, v\rangle_\Omega + \langle w_2 - \bar{w}_2, \mathfrak{L}v\rangle_\Omega = 0,$$

we know that $w_2 - \bar{w}_2 \in W^{q'}_{\mathfrak{L}'}(\Omega)$ with $\mathfrak{L}'(w_2 - \bar{w}_2) = \bar{w}_1 - w_1$. Hence for $v \in W^q_{\mathfrak{L}}(\Omega)$ it follows that

$$0 = \langle w_1 - \bar{w}_1, v\rangle_\Omega + \langle w_2 - \bar{w}_2, \mathfrak{L}v\rangle_\Omega = \langle w_2 - \bar{w}_2, v\rangle_{B(\nu)^{\mathsf{H}}}$$

and therefore $w \in \bar{w} + \Gamma'(\mathfrak{L}'|_{\ker \mathfrak{T}_{\mathfrak{L}'}})$. To prove the implication in the other direction, suppose that $w \in \bar{w} + \Gamma'(\mathfrak{L}'|_{\ker \mathfrak{T}_{\mathfrak{L}'}})$. Then,

$$\langle w_1 - \bar{w}_1, v\rangle_\Omega + \langle w_2 - \bar{w}_2, \mathfrak{L}v\rangle_\Omega = \langle w_1 - \bar{w}_1, v\rangle_\Omega + \langle \mathfrak{L}'(w_2 - \bar{w}_2), v\rangle_\Omega = 0$$

and thus $w \in V$. The set is closed due to the continuity of the trace operator. ////

The proof of Theorem 24 is similar to the corresponding proof for $W^{1,q}(\Omega)$-spaces, cf. (Adams and Fournier 2003, p. 62). Recall that we already encountered $\Gamma'(\mathfrak{L}'|_{\ker \mathfrak{T}_{\mathfrak{L}'}})$ in equation (1.48) in the Hilbert space setting. The exponents in the next theorem are applied to each component. We characterise $\bar{w}$ in more detail.

Theorem 26 *Let q' be a positive even integer. Then $\tilde{w} \in \bar{w} + \Gamma'(\mathcal{L}'|_{\ker \mathcal{T}_{\mathcal{L}'}})$ fulfills*

$$\mathcal{L}(\tilde{w}_1^{q'-1}) = \tilde{w}_2^{q'-1} \tag{1.53}$$

in the sense of distributions if, and only if, $\tilde{w} = \bar{w}$.

Proof: We define, for $(-\mathcal{L}'w, w) \in \Gamma'(\mathcal{L}'|_{\ker \mathcal{T}_{\mathcal{L}'}})$ and $(\tilde{w}_1, \tilde{w}_2) \in \bar{w} + \Gamma'(\mathcal{L}'|_{\ker \mathcal{T}_{\mathcal{L}'}})$, the functions

$$\phi_{w,\tilde{w}} : \mathbb{R} \to \mathbb{R}, t \mapsto \|\tilde{w}_1 - t\,\mathcal{L}'w\|^{q'}_{L^{q'}(\Omega)} + \|\tilde{w}_2 + t\,w\|^{q'}_{L^{q'}(\Omega)}.$$

The $L^{q'}(\Omega)$-norm is Fréchet differentiable. Indeed, if q' is an even number, then the derivative $(D\|\cdot\|^{q'}_{L^{q'}(\Omega)})(w)$ is $q'\,w^{q'-1}$, cf. (Werner 2000, p. 114). Therefore $\phi'_{w,\tilde{w}}(t)$ equals, by the chain rule,

$$\int_\Omega q'(\tilde{w}_1 - t\mathcal{L}'w)^{q'-1} \cdot (\mathcal{L}'w) + q'(\tilde{w}_2 + tw)^{q'-1} \cdot w \,\mathrm{d}x.$$

For $t = 0$ the derivative simplifies to

$$\phi'_{w,\tilde{w}}(0) = \int_\Omega -q'\tilde{w}_1^{q'-1} \cdot (\mathcal{L}'w) + q'\tilde{w}_2^{q'-1} \cdot w \,\mathrm{d}x. \tag{1.54}$$

Since $\bar{w}$ is the minimiser of the $L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l$-norm in $\bar{w} + \Gamma'(\mathcal{L}'|_{\ker \mathcal{T}_{\mathcal{L}'}})$, necessarily $\phi'_{w,\bar{w}}(0) = 0$ for all $w \in \ker \mathcal{T}_{\mathcal{L}'}$. Hence, choosing $w \in \mathscr{D}(\Omega)$, it follows from (1.54) that the distribution

$$q'\,(-\mathcal{L}(\bar{w}_1^{q'-1}) + \bar{w}_2^{q'-1})$$

equals zero and therefore that $\bar{w}$ fulfills (1.53). Now suppose that $\tilde{w} \in \bar{w} + \Gamma'(\mathcal{L}'|_{\ker \mathcal{T}_{\mathcal{L}'}})$ satisfies equation (1.53). This implies by (1.54) that $\phi'_{w,\tilde{w}}(0) = 0$ for $w \in \mathscr{D}(\Omega)$. Also, by (1.54), there is a constant C, depending on $\tilde{w}$, such that for all $w \in \ker \mathcal{T}_{\mathcal{L}'}$ the inequality $\phi'_{w,\tilde{w}}(0) \le C\|w\|_{\mathcal{L}',q'}$ holds. Hence, by Theorem 9, $\phi'_{w,\tilde{w}}(0) = 0$ holds for all w which are elements of the closure $\ker \mathcal{T}_{\mathcal{L}'}$. Since the $L^{q'}(\Omega)$-norm is convex, $\phi_{w,\tilde{w}}$ attains a minimum at the origin and therefore $\tilde{w}$ is the minimiser of the $L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l$-norm in the space $\bar{w} + \Gamma'(\mathcal{L}'|_{\ker \mathcal{T}_{\mathcal{L}'}})$. ////

Corollary 7 *The function $\bar{w}_1^{q'-1}$ is an element of $W^{q'}_{\mathcal{L}}(\Omega)$.*

For general $q \in (1, \infty)$ one needs to include the sign function into the Fréchet derivative of $v \mapsto \|v\|^q_{L^q(\Omega)}$, namely

$$(D\|\cdot\|^q_{L^q(\Omega)})(w) = q\,\mathrm{sign}(w)|w|^{q-1}.$$

Observe that $\bar{w}^{q'-1} \in L^q(\Omega)^m \times L^q(\Omega)^l$ since $(q'-1)q = q'$. Obviously, each element in $L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l$ which fulfills (1.53) is a norm-preserving extension of a functional over $W^q_{\mathcal{L}}(\Omega)$. If $q = 2$, Theorem 26 recovers the Riesz Representation Theorem for Hilbert spaces.

Example 18 *Let q' be a positive even integer. Suppose that $\mathfrak{L}$ is the mapping $W^2_{\mathfrak{L}}(0,1) \to L^2(0,1), v \mapsto \partial_x v$. According to Theorem 26, the functions*

$$f : (0,1) \to \mathbb{R}^2, x \mapsto (\mathrm{e}^x, (q'-1)^{1/(q'-1)} e^x),$$
$$g : (0,1) \to \mathbb{R}^2, x \mapsto (1,0)$$

represent norm-preserving extensions of linear functionals over $W^q_{\mathfrak{L}}(\Omega)$. The sum $f+g$ does not satisfy (1.53) unless $q' = 2$. Hence the set of minimising elements is, in general, not a vector space and the operator

$$W^q_{\mathfrak{L}}(\Omega)' \to L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l, \Lambda \to \bar{w} \tag{1.55}$$

is nonlinear.

Nevertheless, inserting $(\alpha\,\bar{w}_1, \alpha\,\bar{w}_2)$, $\alpha \in \mathbb{R}$, into equation (1.53) proves that the set of minimising elements $\bar{w}$ is closed under scalar multiplication. Therefore it is a cone. Example 18 discourages us from searching for a generalisation of the Riesz Representation Theorem for $q \neq 2$, considering the distinguished role of $\bar{w}$ as norm-defining element in equation (1.52).

Instead, we take Theorem 24 as our starting point for investigations in a different direction. First we illustrate how, by utilising the L^2-scalar product instead of the graph space scalar product as pivot, the dual space can be embedded into the set of distributions.

Select $w \in \bar{w} + \Gamma'(\mathfrak{L}'|_{\ker \mathfrak{T}_{\mathfrak{L}'}})$. The representation of the functional Λ by formula (1.51) shows that the restriction of Λ to $\mathscr{D}(\Omega)^m$ is equal to the distribution

$$T : \mathscr{D}(\Omega)^m \to \mathbb{R}, v \mapsto \langle w_1 + \mathfrak{L}' w_2, v\rangle_\Omega. \tag{1.56}$$

If two functionals $\Lambda_1, \Lambda_2 \in W^q_{\mathfrak{L}}(\Omega)'$ have the same restriction to $\mathscr{D}(\Omega)^m$, continuity of Λ_1 and Λ_2 implies that they coincide at least on $W^q_{\mathfrak{L},0}(\Omega)$. Since, by the Hahn-Banach theorem, every continuous functional on $W^q_{\mathfrak{L},0}(\Omega)$ can be extended to a continuous functional over $W^q_{\mathfrak{L}}(\Omega)$, we know that every element in $W^q_{\mathfrak{L},0}(\Omega)'$ can be identified with a distribution which has a representation like (1.56). Given, instead, a distribution Λ of type (1.56), integration by parts makes apparent that $\Lambda \in W^q_{\mathfrak{L},0}(\Omega)'$. We summarise our findings in the next theorem.

Theorem 27 *Let $W^{q,\prime}_{\mathfrak{L},0}(\Omega)$ be the image of the mapping*

$$\Psi : L^q(\Omega)^m \times L^q(\Omega)^l \to \mathscr{D}(\Omega)^m, (w_1, w_2) \mapsto \langle w_1 + \mathfrak{L}' w_2, \cdot\rangle_\Omega$$

and equip $W^{q,\prime}_{\mathfrak{L},0}(\Omega)$ by the norm

$$\|\langle w_1 + \mathfrak{L}' w_2, \cdot\rangle_\Omega\|_{W^{q,\prime}_{\mathfrak{L},0}(\Omega)} := \min\{\|(\tilde{w}_1, \tilde{w}_2)\|_{L^{q'}(\Omega)^m \times L^{q'}(\Omega)^l} : \tilde{w} \in (w_1, w_2) + \Gamma'(\mathfrak{L}')\}.$$

Then $W^{q,\prime}_{\mathfrak{L},0}(\Omega)$ is isometric to the dual of $W^q_{\mathfrak{L},0}(\Omega)$.

Proof: It remains to show that the minimum in the definition of $\|\cdot\|_{W^{q,\prime}_{\mathcal{L},0}(\Omega)}$-norm is attained and that the norm is isometric to the operator norm of the dual. The reasoning that the set of elements in $L^q(\Omega)^m \times L^q(\Omega)^l$, which are mapped by Ψ to the same functional as (w_1, w_2), coincides with $(w_1, w_2) + \Gamma'(\mathcal{L}')$ is similar to the proof of Corollary 25. Now the theorem follows from the existence of a norm-preserving extension of functionals in $W^q_{\mathcal{L},0}(\Omega)'$ to the graph space. ////

Chapter 2

Friedrichs Systems

Friedrichs systems are boundary value problems which fulfill three requirements: these are a symmetry and a positivity condition on the differential operator, and an adjointness condition on the boundary conditions.

We discuss the boundary conditions in two stages. First we introduce semi-admissible conditions, which ensure only existence of solutions in a weakened sense; then we examine under which circumstances the boundary conditions become admissible, which guarantees that the just mentioned solutions solve the boundary value problem as described by BVP 1. In practice boundary conditions are usually imposed by means of matrix functions $J : \partial\Omega \to \mathbb{R}^{m\times m}$. We therefore derive criteria for the abstract definitions of admissibility and semi-admissibility which are easily applicable to matrix functions J. We also address the delicate question of well-posedness of Friedrichs systems. Using Moyer's example we highlight difficulties involved in the analysis of general admissible boundary conditions. These findings prompt us to focus on Friedrichs systems which satisfy an additional new constraint. These systems satisfy a stability estimate which will also play an important role in the next chapter on discontinuous Galerkin finite element methods for Friedrichs systems. After we have discussed these more theoretical issues we turn to a number of examples of Friedrichs systems. We illustrate how first-order hyperbolic systems and second-order hyperbolic, elliptic and parabolic equations can be transferred to the framework of Friedrichs. We then turn to the Frankl problem which is a representative for the important class of mixed-type equations. We end the chapter with a literature review.

For Friedrichs systems we consider the case of $q = 2$. Only then we are able to exploit symmetry fully. We make the underlying assumption that all functions in the graph space are real-valued.

2.1 Definition of Friedrichs Systems

Symmetric Coefficients: Let $l = m$. We call the operator

$$\mathcal{L} : L^2(\Omega)^m \to \mathscr{D}'(\Omega)^m, v \mapsto \partial_k(B_{ijk}\, v_j) + C_{ij}\, v_j$$

Friedrichs symmetric if the coefficients B and C are real and if B is symmetric in the first two indices i and j, i.e. if $B_{ijk} = B_{jik}$ for all $i, j \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\}$. In contrast with the operators we considered in Section 1.8 we do not require that $C_{ij} = -{}^1\!/_2\, \partial_k B_{ijk}$ but allow any $C \in L^\infty(\Omega, \mathbb{R})^{m\times m}$. We sometimes rewrite $\mathcal{L}$ as

$$\mathcal{L} : v \mapsto {}^1\!/_2\, B_{ijk}(\partial_k v_j) + {}^1\!/_2\, \partial_k(B_{ijk} v_j) + D_{ij} v_j,$$

where $D_{ij} := C_{ij} + {}^1\!/_2\, \partial_k B_{ijk}$. The symmetric component of the assignment

$$W^2_{\mathcal{L}}(\Omega) \times W^2_{\mathcal{L}}(\Omega) \to \mathbb{R}, (v, w) \mapsto \langle \mathcal{L}v, w\rangle_\Omega$$

is the bilinear form

$$ {}^1\!/_2\, \langle \mathcal{L}v, w\rangle_\Omega + {}^1\!/_2\, \langle v, \mathcal{L}w\rangle_\Omega = \langle D^h v, w\rangle_\Omega + {}^1\!/_2\, \langle v, w\rangle_{B(\nu)}, \tag{2.1}$$

where $D^h = {}^1\!/_2\, D + {}^1\!/_2\, D^{\mathsf{H}}$ is defined as on page 38. Notice that the symmetric part of $\langle \mathcal{L}v, w\rangle_\Omega$ does not contain any derivatives. If $\mathcal{L}$ is Friedrichs symmetric, then the adjoint $\mathcal{L}'$ is Friedrichs symmetric, too. Obviously, the fact that $\mathcal{L}$ is Friedrichs symmetric does not mean that $\mathcal{L}$ is a symmetric operator. In fact, if $D = 0$ then the symmetric first-order coefficients imply that $\mathcal{L}$ is a skew-symmetric operator on the restriction to $W^2_{\mathcal{L},0}(\Omega)$:

$$\forall\, v, w \in W^2_{\mathcal{L},0}(\Omega) : \langle \mathcal{L}v, w\rangle_\Omega = -\langle v, \mathcal{L}w\rangle_\Omega. \tag{2.2}$$

Positivity: We call the operator $\mathcal{L}$ positive if D^h is uniformly positive definite on Ω, i.e. if there exists a constant $\gamma > 0$ such that for all $x \in \Omega, v \in \mathbb{R}^n : v^{\mathsf{H}}\, D^h\, v|_x \geq \gamma\, v^{\mathsf{H}}\, v|_x$. Since

$$\langle \mathcal{L}v, v\rangle_\Omega = \langle D^h v, v\rangle_\Omega + {}^1\!/_2\, \langle v, v\rangle_{B(\nu)}, \tag{2.3}$$

it is apparent that for $v \in W^2_{\mathcal{L},0}(\Omega)$ the product $\langle \mathcal{L}v, v\rangle_\Omega + \langle \mathcal{J}\, v, v\rangle_{\partial\Omega} = \langle \mathcal{L}v, v\rangle_\Omega$ is positive if $v \neq 0$. Equation (1.25) shows that if a Friedrichs symmetric operator is positive then its adjoint is positive, too. Positive Friedrichs symmetric operators are called accretive operators in short.

Semi-Admissible Boundary Conditions: Let S be a closed subset of $\partial\Omega$ with Hausdorff measure zero. Semi-admissible boundary conditions are chosen so that $\langle \mathcal{L}v, v\rangle_\Omega + \langle \mathcal{J}\, v, v\rangle_{\partial\Omega}$ is positive definite on the function space $W^{1,2}_0(\mathbb{R}^n \backslash S)^m$. Generally, we say that a boundary value operator

$$\mathcal{R} : W^2_{\mathcal{L}}(\Omega) \to B^{2,2,-1/2}_0(\partial\Omega \backslash S)^m, v \mapsto \mathcal{R}\, v$$

is positive semi-definite if for all $v \in W_0^{1,2}(\mathbb{R}^n \backslash S)^m$ the term $\langle \mathcal{R}\, v, v\rangle_{\partial\Omega}$ is non-negative. We call a boundary operator $\mathcal{J}$ semi-admissible whenever

$$\mathcal{J} + {}^1\!/_2\, \mathcal{T} =: \mathcal{R} \tag{2.4}$$

is positive semi-definite. The definition of semi-admissability is motivated by the bound

$$\langle \mathcal{L}v, v\rangle_\Omega + \langle \mathcal{J}\, v, v\rangle_{\partial\Omega} = \langle D^h v, v\rangle_\Omega + \langle \mathcal{R}\, v, v\rangle_{\partial\Omega} \geq \gamma\, \langle v, v\rangle_\Omega, \quad v \in W_0^{1,2}(\mathbb{R}^n \backslash S)^m, \tag{2.5}$$

which is derived from (2.3), recalling γ from the last paragraph. If $\mathcal{J}$ is semi-admissible with respect to $\mathcal{T} = \mathcal{T}_\mathcal{L}$ then the $'$-adjoint boundary operator $\mathcal{J}' : W_0^{1,2}(\mathbb{R}^n \backslash S)^m \to W_\mathcal{L}^2(\Omega)'$ is semi-admissible with respect to the adjoint trace operator $\mathcal{T}_{\mathcal{L}'}$: For all $v \in W_\mathcal{L}^2(\Omega)$, $w \in W_0^{1,2}(\mathbb{R}^n \backslash S)^m$

$$\langle v, \mathcal{J}'\, w\rangle_{\partial\Omega} - {}^1\!/_2\, \langle B(\nu)\, v, w\rangle_{\partial\Omega} = \langle \mathcal{J}\, v, w\rangle_{\partial\Omega} + {}^1\!/_2\, \langle B(\nu)\, v, w\rangle_{\partial\Omega} = \langle \mathcal{R}\, v, w\rangle_{\partial\Omega}.$$

This implies that, for all $v \in W_0^{1,2}(\mathbb{R}^n \backslash S)^m$,

$$\langle v, \mathcal{L}'v\rangle_\Omega + \langle v, \mathcal{J}'\, v\rangle_{\partial\Omega} \geq \gamma\, \langle v, v\rangle_\Omega. \tag{2.6}$$

We label the spaces that contain the functions which satisfy the boundary condition $\mathcal{J}v = 0$ by the subscript $\mathcal{J}$:

$$W_{\mathcal{L},\mathcal{J}}^2(\Omega) := \{v \in W_\mathcal{L}^2(\Omega) : \mathcal{J}v = 0\}, \quad C_{0,\mathcal{J}}^\infty(\mathbb{R}^n)^m := \{v \in C_0^\infty(\mathbb{R}^n)^m : \mathcal{J}v = 0\}, \quad \ldots$$

In particular, we have

$$W_{0,\mathcal{J}'}^{1,2}(\mathbb{R}^n \backslash S)^m = \{v \in W_0^{1,2}(\mathbb{R}^n \backslash S)^m : \forall\, u \in W_\mathcal{L}^2(\Omega) : \langle (\mathcal{J} + \mathcal{T})\, u, v\rangle_{\partial\Omega} = 0\}.$$

From (2.5), (2.6) and the Cauchy-Schwarz inequality we deduce the stability of the boundary value problem on these spaces: for $v \in W_{0,\mathcal{J}}^{1,2}(\mathbb{R}^n \backslash S)^m$, $w \in W_{0,\mathcal{J}'}^{1,2}(\mathbb{R}^n \backslash S)^m$:

$$\|\mathcal{L}v\|_{L^2(\Omega)^m} \geq \gamma\, \|v\|_{L^2(\Omega)^m}, \quad \|\mathcal{L}'w\|_{L^2(\Omega)^m} \geq \gamma\, \|w\|_{L^2(\Omega)^m}. \tag{2.7}$$

Before we turn our attention to boundary value problems as specified in Section 1.9, we shall first examine a class of more general solutions. Employing in BVP 3 only elements of $W_{0,\mathcal{J}'}^{1,2}(\mathbb{R}^n \backslash S)^m$ as test functions, we can use stability on $W_{0,\mathcal{J}'}^{1,2}(\mathbb{R}^n \backslash S)^m$ to derive the following result:

Theorem 28 *Let $\mathcal{L} : W_\mathcal{L}^2(\Omega) \to L^2(\Omega)^m$ be an accretive operator and $\mathcal{J} : W_\mathcal{L}^2(\Omega) \to B_0^{2,2,-1/2}(\partial\Omega \backslash S)^m$ be a semi-admissible boundary operator. Suppose that $g \in W_\mathcal{L}^2(\Omega)$, $f \in L^2(\Omega)^m$. Then there exists a function $u \in W_\mathcal{L}^2(\Omega)$ such that for all $v \in W_{0,\mathcal{J}'}^{1,2}(\mathbb{R}^n \backslash S)^m$*

$$\langle u, \mathcal{L}'v\rangle_\Omega = \langle f, v\rangle_\Omega + \langle \mathcal{J}\, g, v\rangle_{\partial\Omega}. \tag{2.8}$$

Proof. We define $\dot f := f + \mathcal{L}g$. Consider the problem

$$\text{find } \dot u \in W^2_{\mathcal{L}}(\Omega) \text{ s.t.} \forall\, v \in W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m : \langle \dot u, \mathcal{L}'v\rangle_\Omega = \langle \dot f, v\rangle_\Omega. \tag{2.9}$$

Inequality (2.7) implies that $\mathcal{L}'$ is injective on $W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m$. Hence we have a unique correspondence between functions $v \in W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m$ and functions w in the image $W := \mathcal{L}'(W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m)$. Accordingly, we can assign to every w the scalar product $\langle \dot f, v\rangle_\Omega$:

$$\Lambda : W \to \mathbb{R}, w \mapsto \langle \dot f, (\mathcal{L}')^{-1}w\rangle_\Omega = \langle \dot f, v\rangle_\Omega.$$

This mapping is linear and bounded; for the latter consider

$$|\langle \dot f, v\rangle_\Omega| \le \|\dot f\|_{L^2(\Omega)^m}\,\|v\|_{L^2(\Omega)^m} \le \frac{1}{\gamma}\,\|\dot f\|_{L^2(\Omega)^m}\,\|w\|_{L^2(\Omega)^m}.$$

By the Hahn-Banach theorem there exists a continuous extension of Λ to $L^2(\Omega)^m$. By the Riesz representation theorem we can select a function $\dot u \in L^2(\Omega)^m$ such that $\langle \dot u, \mathcal{L}'v\rangle_\Omega = \langle \dot u, w\rangle_\Omega = \langle \dot f, v\rangle_\Omega$ for all $v \in W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m$. From $\mathscr{D}(\Omega)^m \subset W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m$ it follows that $\mathcal{L}\dot u = \dot f$ in the sense of distributions and therefore that $\dot u \in W^2_{\mathcal{L}}(\Omega)$ and that $\dot u$ solves equation (2.9). Since

$$\forall\, v \in W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m : \langle \dot u - g, \mathcal{L}'v\rangle_\Omega = \langle \dot f, v\rangle_\Omega - \langle \mathcal{L}g, v\rangle_\Omega + \langle g, v\rangle_{\mathcal{J}} = \langle f, v\rangle_\Omega + \langle g, v\rangle_{\mathcal{J}}$$

the function $u = \dot u - g$ satisfies equation (2.8). ////

The proof of Theorem 28 is an adaptation of the corresponding proof in (Friedrichs 1958). A closely related argument has been utilised by Morawetz (Morawetz 1958) to demonstrate the existence of a weak solution of the Tricomi equation. The result has been published in the same issue of 'Communications on Pure and Applied Mathematics' in which also (Friedrichs 1958) appeared.

Clearly the solutions of BVP 3 solve (2.8). But do the solutions of (2.8) satisfy BVP 3? As far as the differential operator is concerned, the answer is affirmative. By assuming that v in equation (2.8) is a member of $\mathscr{D}(\Omega)^m$ we deduce that $\mathcal{L}u = f$. However, regarding the validity of the boundary conditions, more thought is needed. We deduce from $\mathcal{L}u = f$ and (2.8), that

$$\langle \mathcal{J}\, u, v\rangle_{\partial\Omega} = \langle \mathcal{J}\, g, v\rangle_{\partial\Omega} \quad \text{with} \quad v \in B^{2,2,1/2}_{0,\mathcal{J}'}(\mathbb{R}^n\backslash S)^m. \tag{2.10}$$

This in itself does not imply that $\mathcal{J}u = \mathcal{J}g$. Consider the following example in which the solution of (2.8) does not satisfy the boundary conditions.

Example 19 *Let $\Omega = (-1,1)$ and $\mathfrak{L}v = x\,v' + v$. The solution of the ordinary differential equation $\mathfrak{L}u = 0$ can be written as*

$$u(x) = \begin{cases} c_- \, \exp\left(\int_{x_0}^{x} -\frac{1}{\tau}\,\mathrm{d}\tau\right) = \frac{c_-\,x_0}{x} & \text{for } x, x_0 < 0, \\ \frac{c_+\,x_0}{x} & \text{for } x, x_0 > 0, \end{cases} \tag{2.11}$$

allowing a discontinuity at 0. Clearly $\mathfrak{L}$ is Friedrichs symmetric and positive with γ equal to $^1/_2$. Since $B(\nu) = 1$ on the boundary, we can choose the semi-admissible boundary operator $\mathfrak{J} = 0$. Although it might be surprising at first, (2.8) has only one solution, namely $u = 0$. All other solutions in (2.11) are inadmissible, because they do not lie in $W^2_{\mathfrak{L}}(\Omega)$ due to a pole at the origin.

Suppose we had chosen $\mathfrak{J}v = v$ instead. Still the only solution of (2.8) is $u = 0$, independent of g. Evidently, for $\mathfrak{J}\,g \neq 0$ this u is not a solution to BVP 3.

Theorem 28 induces an equivalence relation in $W^2_{\mathfrak{L}}(\Omega)$. We say that two functions $g, \dot{g}$ share the same equivalence class $[\![g]\!]$ if for all $f \in L^2(\Omega)^m$ both functions result in the same set of solutions $\{u_f\}$ of (2.8), keeping in mind that we have not proven uniqueness of solutions of (2.8). Equivalently $\dot{g} \in [\![g]\!]$ if, and only if, $\langle \mathfrak{J}(g - \dot{g}), v\rangle_{\partial\Omega} = 0$ for all $v \in B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m$. Therefore the definition of the equivalence relation is independent of the choice of f.

We use the annihilator of $B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m$ with respect to $\langle \mathfrak{J}\,\cdot, \cdot\rangle_{\partial\Omega}$ to rephrase the above statement:

$$\begin{aligned}(B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m)^{\perp_{\mathfrak{J}}} := &\{w \in \operatorname{Im}\mathfrak{J} :\ \forall\, v \in B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m : \langle w, v\rangle_{\partial\Omega} = 0\} \\ &\subset B^{2,2,-1/2}_{0}(\partial\Omega \backslash S)^m.\end{aligned}$$

Then

$$[\![g]\!] = g + \mathfrak{J}^{-1}(B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m)^{\perp_{\mathfrak{J}}}.$$

Clearly $[\![0]\!] = \mathfrak{J}^{-1}(B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m)^{\perp_{\mathfrak{J}}}$.

Admissible Boundary Conditions: If

$$(B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m)^{\perp_{\mathfrak{J}}} = \{0\} \tag{2.12}$$

we call $\mathfrak{J}$ and $\mathfrak{J}'$ strictly adjoint. Notice that $\mathfrak{J}$ and $\mathfrak{J}'$ are strictly adjoint if, and only if, whenever $u \in W^2_{\mathfrak{L}}(\Omega)$ we have that

$$(\forall\, v \in B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n \backslash S)^m : \langle \mathfrak{J}u, v\rangle_{\mathfrak{J}} = 0) \Rightarrow \mathfrak{J}u = 0. \tag{2.13}$$

If the boundary conditions are strictly adjoint and semi-admissible, we say that they are admissible.

We subsume how strict adjointness and semi-admissibility are interconnected: The boundary operator $\mathfrak{J}$ has the codomain $B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$. In Theorem 28 we test the equality of $\mathfrak{J}u$ and $\mathfrak{J}g$ with functions in $B_{0,\mathfrak{J}'}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m$. However, since $B_{0,\mathfrak{J}'}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m$ is in the general case not dense in $B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$ this method can only distinguish functions on a subspace V of $B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$. Strict adjointness demands that $\mathfrak{J}$ maps into V to ensure that the boundary conditions are in fact attained. We formulated this condition by saying that the complement $(B_{0,\mathfrak{J}'}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m)^{\perp_{\mathfrak{J}}}$ of V in $\operatorname{Im}\mathfrak{J}$ is trivial.

Friedrichs Systems: A boundary value problem consisting of an accretive first-order linear differential operator on $\Omega \subset \mathbb{R}^n$ and an admissible boundary condition on $\partial\Omega$ is called a Friedrichs system.

Theorem 29 *A boundary value problem of type BVP 1 which consists of an accretive operator $\mathfrak{L}$, an admissible boundary operator $\mathfrak{J} : W^2_{\mathfrak{L}}(\Omega) \to B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m$, boundary conditions $h = \mathfrak{J}g \in \operatorname{Im}\mathfrak{J}$ and a right-hand side $f \in L^2(\Omega)^m$ has a solution u in $W^2_{\mathfrak{L}}(\Omega)$.*

Proof. The existence of u follows from Theorem 28 and (2.13). ////

We attend to the question of uniqueness in a later section about well-posedness. First we transfer the definition of Friedrichs systems to a less abstract setting.

2.2 Matrix-Valued Boundary Conditions

The boundary operators considered in practice can almost exclusively be represented by matrix functions which are defined on the boundary. By that we mean that there is a mapping

$$J : \partial\Omega \to \mathbb{R}^{m\times m}$$

such that for all $v \in C_0^\infty(\mathbb{R}^n)^m$

$$(\mathfrak{J}\, v)(x) = J(x)\, v(x). \tag{2.14}$$

We say that J is a boundary operator, is semi-admissible, is strictly adjoint or is admissible if $\mathfrak{J}$ can be extended to a continuous operator

$$\mathfrak{J} : W^2_{\mathfrak{L}}(\Omega) \to B_0^{2,2,-1/2}(\partial\Omega\backslash S)^m, v \mapsto J\, v|_{\partial\Omega} \tag{2.15}$$

which has the respective property. To analyse J we introduce

$$J' := B(\nu) + J^{\mathsf{H}} \qquad \text{and} \qquad J^* := B(\nu) + J. \tag{2.16}$$

Notice the similarity between (1.44) and (2.16).

Theorem 30 *Consider a mapping $J \in L^\infty(\partial\Omega)^{m\times m}$ such that, almost everywhere on $\partial\Omega$, the matrices*

$$R(x) := J(x) + {}^1\!/_2\, B(\nu)(x) \tag{2.17}$$

are positive semi-definite. Suppose that J has a continuous extension in the sense of (2.15). Then J is a semi-admissible boundary operator.

Proof. For all functions $v \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ the term

$$\langle R\, v, v\rangle_{\partial\Omega} \tag{2.18}$$

is non-negative. ////

If one had to pin down the meaning of strict adjointness for individual matrices $J(x)$ and $B(\nu, x)$ for a fixed $x \in \partial\Omega$, one would probably think of

$$\{0\} = \{v \in \operatorname{Im} J(x) : \forall\, w \in \ker J'(x) : v \cdot w = 0\} \tag{2.19}$$

in the view of (2.12). Analysing (2.19) is a first step towards understanding the more complex concept of strict adjointness of boundary operators. In the next theorem we present conditions which are equivalent to (2.19) and which are in practice often easier to validate. We remark that the results beginning with Theorem 31 and ending with Corollary 8 are, up to minor modifications, due to (Friedrichs 1958).

Theorem 31 *Consider the matrices $J(x)$ and $B(\nu, x)$ for an $x \in \partial\Omega$. Then (2.19) holds if, and only if,*

$$\operatorname{Im} J^*(x) \cap \operatorname{Im} J(x) = 0 \tag{2.20}$$

or, equivalently,

$$\ker J'(x) + \ker J^{\mathsf{H}}(x) = \mathbb{R}^m. \tag{2.21}$$

Proof: We consider (2.20) first. We use that the kernel of $J'(x)$ is equal to the orthogonal complement of the image of the adjoint operator. We choose the orthogonal complement with respect to the canonical scalar product in $\mathbb{R}^m$:

$$\ker J'(x) = (\operatorname{Im} J'(x)^{\mathsf{H}})^{\perp_{\mathbb{R}^m}} = (\operatorname{Im} J^*(x))^{\perp_{\mathbb{R}^m}}.$$

Let $v \in \mathbb{R}^m$. Then $J(x)\, v \cdot w = 0$ for all $w \in (\operatorname{Im} J^*)^{\perp_{\mathbb{R}^m}} \subset \mathbb{R}^m$ if, and only if, $J(x)\, v \in \operatorname{Im} J^*(x)$. We now turn to (2.21). It holds if, and only if, (2.20) holds because

$$\begin{aligned} \operatorname{Im} J^* \cap \operatorname{Im} J = 0 &\Leftrightarrow (\ker J')^{\perp_{\mathbb{R}^m}} \cap (\ker J^{\mathsf{H}})^{\perp_{\mathbb{R}^m}} = 0 \\ &\Leftrightarrow (\ker J' + \ker J^{\mathsf{H}})^{\perp_{\mathbb{R}^m}} = 0 \\ &\Leftrightarrow \ker J' + \ker J^{\mathsf{H}} = \mathbb{R}^m. \end{aligned} \tag{2.22}$$

For completeness we show that

$$(\ker J')^{\perp_{\mathbb{R}^m}} \cap (\ker J^{\mathsf{H}})^{\perp_{\mathbb{R}^m}} = (\ker J' + \ker J^{\mathsf{H}})^{\perp_{\mathbb{R}^m}}.$$

Let $v \in (\ker J')^{\perp_{\mathbb{R}^m}} \cap (\ker J^{\mathsf{H}})^{\perp_{\mathbb{R}^m}}$ and split $w = w_1 + w_2$ with $w_1 \in \ker J'$ and $w_2 \in \ker J^{\mathsf{H}}$. Then $\langle v, w\rangle_{\partial\Omega} = \langle v, w_1\rangle_{\partial\Omega} + \langle v, w_2\rangle_{\partial\Omega} = 0$. Conversely, suppose that $v \in (\ker J' + \ker J^{\mathsf{H}})^{\perp_{\mathbb{R}^m}}$. Let $w_1 \in \ker J'$ and $w_2 \in \ker J^{\mathsf{H}}$. Then $\langle v, w_1\rangle_{\partial\Omega} = 0$ and $\langle v, w_2\rangle_{\partial\Omega} = 0$. ////

It proves to be helpful to express condition (2.19) in terms of projections. We do not have to require that these projections are orthogonal; that is, we consider all pairs of matrices $P_a, P_b \in \mathbb{R}^{m\times m}$ for which

$$P_a + P_b = I, \qquad P_a\, P_b = P_b\, P_a = 0.$$

It follows that $P_a\, P_a = P_a$ and that $P_b\, P_b = P_b$.

Theorem 32 *Consider a matrix function $J \in L^\infty(\partial\Omega)^{m\times m}$. At a point $x \in \partial\Omega$ condition (2.19) is fulfilled if, and only if, there is a pair of projections $P_{J^{\mathsf{H}}}, P_{J'} \in \mathbb{R}^{m\times m}$ such that*

$$J^{\mathsf{H}}(x) = -B(\nu, x)\, P_{J^{\mathsf{H}}}, \qquad J'(x) = B(\nu, x)\, P_{J'} \tag{2.23}$$

or equivalently such that

$$J(x) = -(P_{J^{\mathsf{H}}})^{\mathsf{H}}\, B(\nu, x), \qquad J^*(x) = (P_{J'})^{\mathsf{H}}\, B(\nu, x). \tag{2.24}$$

Proof. We first show that (2.19) implies (2.23). The identity $\ker J^{\mathsf{H}}(x) + \ker J'(x) = \mathbb{R}^m$ holds if, and only if, there is a pair of projections $P_{J^{\mathsf{H}}}(x)$ and $P_{J'}(x)$ into $\ker J'(x)$ and $\ker J^{\mathsf{H}}(x)$, respectively. By definition of J^{H} and J' there is a matrix function $R \in \mathbb{R}^{m\times m}$ such that $J^{\mathsf{H}}(x) = R^{\mathsf{H}} - {}^1\!/_2\, B(\nu, x)$ and $J'(x) = R^{\mathsf{H}} + {}^1\!/_2\, B(\nu, x)$. We then have the relation

$$\begin{aligned} R^{\mathsf{H}} &= (J^{\mathsf{H}}(x) + {}^1\!/_2\, B(\nu, x))\, P_{J'} + (J'(x) - {}^1\!/_2\, B(\nu, x))\, P_{J^{\mathsf{H}}} \\ &= {}^1\!/_2\, B(\nu, x)\, P_{J'} - {}^1\!/_2\, B(\nu, x)\, P_{J^{\mathsf{H}}}. \end{aligned}$$

Consequently, $J^{\mathsf{H}}(x) = -B(\nu)(x)\, P_{J^{\mathsf{H}}}$ and $J'(x) = B(\nu)\, P_{J'}$. Condition (2.24) is equivalent to (2.23) by transposition.

Now we assume that (2.24) is satisfied. Let $w \in \operatorname{Im} J^*(x) \cap \operatorname{Im} J(x)$. Thus $w \in \operatorname{Im} P_{J^{\mathsf{H}}} \cap \operatorname{Im} P_{J'}$ which in turn implies that $w = 0$ and that (2.20) holds. ////

The following theorem is a tool we may use to interchange the role of J with the adjoint J'.

Theorem 33 *If $R \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix then* $\ker R = \ker R^{\mathsf{H}}$.

Proof. If $v \in \ker R$ then $2\, v^{\mathsf{H}} R^{\mathsf{H}} v = v^{\mathsf{H}} R^h\, v$. Noting that R^h is symmetric, we conclude

$$R^h\, v \cdot R^h\, v \le \lambda_{\max}\, (v \cdot R^h\, v) = 2\, \lambda_{\max}\, (v \cdot R\, v) = 0,$$

where $\lambda_{\max}$ is the largest eigenvalue of R^h. Thus $R^{\mathsf{H}} v = Rv = 0$. ////

Theorem 34 *Consider the matrices $J(x)$ and $B(\nu, x)$ for an $x \in \partial\Omega$. Suppose that (2.17) is satisfied. Then (2.19) holds if, and only if,*

$$\operatorname{Im} J'(x) \cap \operatorname{Im} J^{\mathsf{H}}(x) = 0 \tag{2.25}$$

or, equivalently,

$$\ker J^*(x) + \ker J(x) = \mathbb{R}^m. \tag{2.26}$$

Proof. Suppose that (2.19) holds. We choose $P_{J'}$ and $P_{J^{\mathsf{H}}}$ as in Theorem 32. Let $w = J'(x)\, v_1 = J^{\mathsf{H}}(x)\, v_2$. We may assume that $v_1 = P_{J'}\, v_1$ and $v_2 = P_{J^{\mathsf{H}}}\, v_2$. Then

$$R^{\mathsf{H}}(v_1 - v_2) = {}^1\!/_2\, J'(x)(v_1 - v_2) + {}^1\!/_2\, J(x)^{\mathsf{H}}(v_1 - v_2) = J'(x)\, v_1 - J^{\mathsf{H}}(x)\, v_2 = 0$$

where $R = J(x) + {}^1\!/_2\, B(\nu, x)$. According to Theorem 33 it follows that $Rv_1 = Rv_2$. Furthermore,

$$B(\nu, x)(v_1 + v_2) = B(\nu, x) P_{J'}\, v_1 + B(\nu, x) P_{J^{\mathsf{H}}}\, v_2 = J'(x)\, v_1 - J^{\mathsf{H}}(x)\, v_2 = 0.$$

Hence

$$R(v_1 + v_2) = (^1/_2\, J + {}^1/_2\, J^*)\,(v_1 + v_2) = (-(P_{J^{\mathsf{H}}})^{\mathsf{H}} + (P_{J'})^{\mathsf{H}})\, B(\nu, x)\,(v_1 + v_2) = 0$$

so that $Rv_1 = -Rv_2 = 0$. We also have $R^{\mathsf{H}} v_1 = -R^{\mathsf{H}} v_2 = 0$. Thus

$$w = {}^1/_2\, J' v_1 + {}^1/_2\, J^{\mathsf{H}} v_2 = {}^1/_2\, J'(v_1 + v_2) + {}^1/_2\, J^{\mathsf{H}}(v_1 + v_2) = R^{\mathsf{H}}(v_1 + v_2) = 0,$$

which proves (2.25). Condition (2.26) follows by an argument analogous to (2.22).

In (2.20) and (2.25) the roles of J and J' are interchanged. Since J is the $'$-adjoint of J' we conclude that (2.25) implies (2.20) if (2.17) is fulfilled. ////

Corollary 8 *Consider a matrix function $J \in L^\infty(\partial\Omega)^{m\times m}$. At a point $x \in \partial\Omega$ condition (2.19) is fulfilled if, and only if, there is a pair of projections $P_J, P_{J^*} \in \mathbb{R}^{m\times m}$ such that*

$$J(x) = -B(\nu, x)\, P_J, \qquad J^*(x) = B(\nu, x)\, P_{J^*} \tag{2.27}$$

or, equivalently, such that

$$J^{\mathsf{H}}(x) = -(P_J)^{\mathsf{H}}\, B(\nu, x), \qquad J'(x) = (P_{J^*})^{\mathsf{H}}\, B(\nu, x). \tag{2.28}$$

Proof. After an interchange of J and J' the corollary takes the form of Theorem 32. ////

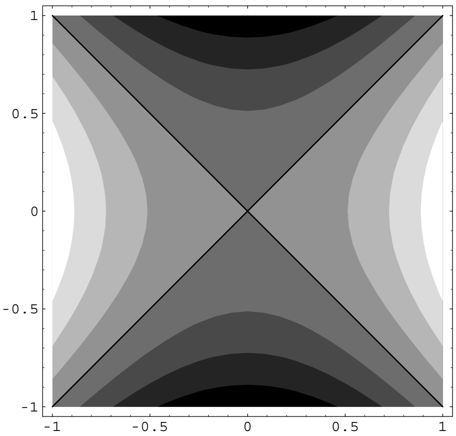
Example 20 *Consider a boundary matrix $B(\nu)$ which has at $x \in \partial\Omega$ the value*

$$B(\nu, x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For elements $v = (v_1, v_2) \in \mathbb{R}^2$ the bilinear form $v^{\mathsf{H}}\, B(\nu, x)\, v$ takes the following signs:

$$\begin{aligned} v^{\mathsf{H}}\, B(\nu)\, v > 0 &\Leftrightarrow v \in V_+ := \{(w_1, w_2) \in \mathbb{R}^2 : |w_1| > |w_2|\},\\ v^{\mathsf{H}}\, B(\nu)\, v = 0 &\Leftrightarrow v \in V_0 := \{(w_1, w_2) \in \mathbb{R}^2 : |w_1| = |w_2|\},\\ v^{\mathsf{H}}\, B(\nu)\, v < 0 &\Leftrightarrow v \in V_- := \{(w_1, w_2) \in \mathbb{R}^2 : |w_1| < |w_2|\}.\end{aligned}$$

For the analysis of Friedrichs systems it is often helpful to visualise these regions by means of a contour plot of $v^{\mathsf{H}}\, B(\nu, x)\, v$:



Clearly, V_+ consists of the left and right quadrant, V_- of the upper and lower quadrant and V_0 of the remaining two diagonals. Using $R = B(\nu)\,(-P_J + P_{J^*})$ and $R^{\mathsf{H}} = B(\nu)\,(-P_{J^{\mathsf{H}}} + P_{J'})$, we deduce that

$$\operatorname{Im}(P_J) \cup \operatorname{Im}(P_{J^{\mathsf{H}}}) \subset V_- \cup V_0 \quad \textit{and} \quad \operatorname{Im}(P_{J^*}) \cup \operatorname{Im}(P_{J'}) \subset V_+ \cup V_0$$

are necessary conditions to satisfy (2.17).

Over the last pages we have worked out a number of criteria to establish condition (2.19). We will now combine these conditions with regularity assumptions to ensure that the J can be continuously extended to a strictly adjoint boundary operator.

Our first investigations concern the smoothness of the projections utilised in Theorem 32. Given a projection $P \in L^\infty(\partial\Omega)^{m\times m}$, we need to ensure that Pv is contained in the space $B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ whenever $v \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$. Like in Section 1.9 we let S be a closed subset of $\partial\Omega$ which has Hausdorff measure zero. We assume that for every simply connected component F of $\partial\Omega \setminus S$ there exists a C^1-diffeomorphism

$$\Phi : N \to M$$

such that $N \cap F = F$ and $\Phi F = \{(x_1, \ldots, x_n) \in M : x_1 = 0\}$. Alternatively we could say that each component F is a C^1-manifold with an atlas which only contains one chart. We require that Φ and Φ^{-1} are bounded and have bounded derivatives, that is they are of class $W^{1,\infty}$. Then $\Phi W^{1,2}(N) = W^{1,2}(M)$. It follows that $\Phi B^{2,2,1/2}(F) = B^{2,2,1/2}(\Phi F)$ by restriction to F; we refer to Section 1.3 and to the respective theorems in (Jonsson and Wallin 1984).

The projection P is componentwise continuous with Hölder exponent $1/2$ if for all entries P_{ij} of P the term

$$\|P_{ij}\|_{L^\infty(\partial\Omega)} + \sup_F \sup_{(x,y)\in F^2} \frac{|P_{ij}(x) - P_{ij}(y)|}{\|x-y\|^{1/2}} \tag{2.29}$$

is finite. Here the first supremum ranges over all components of $\partial\Omega \setminus S$. If a component F is unbounded we consider under the second supremum only pairs (x, y) with $\|x - y\| < 1$. If P satisfies (2.29) we write $P \in C^{1/2}(\partial\Omega \setminus S)^{m\times m}$.

Suppose that $v \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ and therefore that $v\,\Phi^{-1}|_{\Phi F} \in B^{2,2,1/2}(\Phi F)^m$. We can now apply a corollary from (Triebel 1992, p. 205) which states that if P is contained $C^{1/2}(\partial\Omega \setminus S)^{m\times m}$ then the product $(P\,v)\,\Phi^{-1}$ is an element of $B^{2,2,1/2}(\Phi F)^m$. Moreover for fixed P the product depends continuously on v. It follows that P defines an endomorphism on $B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$.

We have now the tools to prove a sufficient condition for the admissibility of a matrix function in terms of projections. Among other things, we obtain an explicit factorisation of $\mathcal{J}$ in the sense of (1.20).

Theorem 35 *Consider a matrix function $J \in L^\infty(\partial\Omega)^{m\times m}$ and a pair of projections $P_{J^{\mathsf{H}}}, P_{J'}$ in $C^{1/2}(\partial\Omega \setminus S)^{m\times m}$. Then J is a strictly adjoint boundary operator if, for $x \in \partial\Omega \setminus S$,*

$$J^{\mathsf{H}}(x) = -B(\nu, x)\, P_{J^{\mathsf{H}}}(x), \qquad J'(x) = B(\nu, x)\, P_{J'}(x) \tag{2.30}$$

or, equivalently,

$$J(x) = -(P_{J^{\mathsf{H}}})^{\mathsf{H}}(x)\, B(\nu, x), \qquad J^*(x) = (P_{J'})^{\mathsf{H}}(x)\, B(\nu, x). \tag{2.31}$$

Proof. Clearly, (2.31) follows from transposition of (2.30) and therefore (2.30) and (2.31) are equivalent. We need to check that J can be continuously extended in the sense of (2.15). Let u be an element of $W^2_{\mathcal{L}}(\Omega)$ which is the limit of a sequence $(u_i)_{i\in\mathbb{N}}$ consisting of $C_0^\infty(\mathbb{R}^n)^m$-functions. Then for every test function $v \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ we have the bound

$$\begin{aligned}\langle Ju, v\rangle_{\partial\Omega} &= \lim_{i\to\infty} \langle -(P_{J^{\mathsf{H}}})^{\mathsf{H}}\, B(\nu)\, u_i, v\rangle_{\partial\Omega} = \lim_{i\to\infty} \langle B(\nu)\, u_i, -(P_{J^{\mathsf{H}}})^{\mathsf{H}}\, v\rangle_{\partial\Omega} \\ &\le \|\mathcal{T}\|_{\mathscr{B}(W^2_{\mathcal{L}}(\Omega), B^{2,2,-1/2}(\partial\Omega)^m)}\, \|(P_{J^{\mathsf{H}}})^{\mathsf{H}}\|_{\mathscr{B}(B_0^{2,2,1/2}(\partial\Omega\backslash S)^m)}\, \|u\|_{\mathcal{L}}\, \|v\|_{B_0^{2,2,1/2}(\partial\Omega\backslash S)^m}.\end{aligned}$$

Therefore J is a boundary operator. It follows from the respective definitions of J^*, J' and J^{H} that these mappings are also boundary operators.

We now investigate the strict adjointness of J. We split the test function v into two components: $v = v_1 + v_2$, where $v_1 = (P_{J^{\mathsf{H}}})v$ and $v_2 = (P_{J'})v$. We record that $v_1 \in \ker J'$. Therefore

$$\langle Ju, v\rangle_{\partial\Omega} = \langle Ju, v_1\rangle_{\partial\Omega} - \lim_{i\to\infty} \langle (P_{J^{\mathsf{H}}})^{\mathsf{H}}\, B(\nu)\, u_i, (P_{J'})v\rangle_{\partial\Omega} = \langle Ju, v_1\rangle_{\partial\Omega}.$$

Thus if $\langle Ju, v\rangle_{\partial\Omega} = 0$ for all $v \in B_{0,\mathcal{J}'}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m$ then $\langle Ju, v\rangle_{\partial\Omega} = 0$ for all functions $v \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$. This implies that J is strictly adjoint. ////

The results we have obtained so far in this section provide us with a recipe to check if a matrix function is an admissible boundary operator. The first step is to ensure that (2.17) holds almost everywhere on $\partial\Omega$. After that one verifies that (2.21) or (2.25) are satisfied for all $x \in \partial\Omega \setminus S$. Once the kernels or images of J' and J^{H} are known one constructs the associated projections $P_{J'}$ and $P_{J^{\mathsf{H}}}$. Finally one checks whether $P_{J'}$ and $P_{J^{\mathsf{H}}}$ are members of $C^{1/2}(\partial\Omega \setminus S)^{m\times m}$.

Applying the argument of Theorem 35 to the projections P_J and P_{J^*} proves that if P_J and P_{J^*} are of class $C^{1/2}$ and if (2.17) is satisfied then J' is a strictly adjoint boundary operator with respect to the adjoint boundary value problem. It also follows that J is a boundary operator.

In the statement of the next theorem (2.32) is motivated by (2.26).

Theorem 36 *Suppose the matrix function $J \in L^\infty(\partial\Omega)^{m\times m}$ is extendable to a boundary operator. Then J is admissible if for each $v \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ there is a splitting*

$$v = v_1 + v_2 \tag{2.32}$$

where $v_1 \in B_{0,J^{\mathsf{H}}}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m$ and $v_2 \in B_{0,J'}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m$.

Proof. We split $v \in B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ according to (2.32). Let u be the $W_{\mathcal{L}}^2(\Omega)$-limit of the sequence $(u_i)_{i\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R}^n)^m$-functions. Then

$$\langle J\,u, v\rangle_{\partial\Omega} = \lim_{i\to\infty} \langle J\,u_i, v\rangle_{\partial\Omega} = \lim_{i\to\infty} \langle u_i, J^{\mathsf{H}}\,v\rangle_{\partial\Omega} = \lim_{i\to\infty} \langle u_i, J^{\mathsf{H}}\,v_2\rangle_{\partial\Omega} = \langle J\,u, v_2\rangle_{\partial\Omega}.$$

Thus J is strictly adjoint. ////

We learned in Section 1.8 that in general the kernel of a boundary operator has lower regularity than the boundary operator itself. If J^{H} and J' are analytic then, according to Theorem 60 in the Appendix, the projection onto the kernel is analytic with the exception of a set of measure zero. We can assume that this set of exceptional points is a subset of S. However, it remains unclear to us if poles in the vicinity of S can prevent $B_0^{2,2,1/2}(\partial\Omega\backslash S)^m$ to have a splitting into $B_{0,J^{\mathsf{H}}}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m$ and $B_{0,J'}^{2,2,1/2}(\mathbb{R}^n\backslash S)^m$. We illustrate the problem with the following example.

Example 21 *Let $I = (-1, 1)$. We might think of I as being a section of the boundary of a smooth domain. On I we define the matrix function*

$$B(\nu, x) := \begin{pmatrix} x^2 & 0 \\ 0 & -x^2 \end{pmatrix}$$

and the pair of projections

$$P_J(x) := \frac{1}{x^2-1}\begin{pmatrix} -1 & x \\ -x & x^2 \end{pmatrix}, \quad P_{J^*}(x) := \frac{1}{x^2-1}\begin{pmatrix} x^2 & -x \\ x & -1 \end{pmatrix}.$$

The matrix function $J = -B(\nu)\,P_J$ fulfills (2.17) and (2.19); we omit the details of the calculation. The kernel of J' is spanned by the function $x \mapsto (1/x, 1)^{\mathsf{H}}$. Thus $B^{2,2,1/2}_{0,J'}(\mathbb{R}^n\backslash S)^m$ is equal to $\{0\}$ and Theorem 36 is not applicable. However, we find that $P_{J^{\mathsf{H}}} = P_J$ and $P_{J^*} = P_{J'}$. Consequently, the matrix function J defines an admissible boundary operator.

After we have discussed criteria to establish admissibility of boundary operators we should also assure that admissible boundary operators are actually available for a large class of equations. The next example ensures that admissible boundary operators generally exist, at least if $B(\nu)$ is sufficiently smooth.

Example 22 *Let $J = -B_-(\nu)$. Since $-B_-(\nu) + {}^1\!/_2\,B(\nu) = {}^1\!/_2\,|B|(\nu)$ the boundary operator is semi-admissible. Since $J = -P_-B(\nu)$ and $J' = B_+(\nu) = P_+B(\nu) = (I - P_-)B(\nu)$ the boundary conditions also satisfy (2.19). Therefore J and J' are admissible if P_- is contained in $C^{1/2}(\partial\Omega \setminus S)^{m\times m}$.*

In particular for scalar problems only P_+ and P_- define admissible boundary conditions. Theorem 37 is an adaptation of Lemma 4 in (LeSaint 1995).

Theorem 37 *Consider a matrix function J for which (2.17) and (2.19) hold. If at $x \in \partial\Omega$ the matrix $B(\nu, x)$ is either positive or negative semi-definite then $J(x)$ is equal to*

$$-P_-(x)\,B(\nu,x) = 0 \quad \text{or} \quad -P_-(x)\,B(\nu,x) = -B(\nu,x),$$

respectively.

Proof. We consider the positive semi-definite case first. Let $v \in \ker J'(x)$. Then

$$0 \le {}^1\!/_2\,B(\nu,x)\,v\cdot v = -(J(x) - {}^1\!/_2\,B(\nu,x))\,v\cdot v \le 0.$$

Thus $B(\nu,x)\,v = 0$ and $J(x)\,v - {}^1\!/_2\,B(\nu,x)\,v = 0$. We conclude that $v \in \ker J(x)$. Consequently,

$$\ker J(x) = \ker J(x) + \ker J'(x) = \mathbb{R}^m,$$

and thus $J(x) = 0$. We deduce the negative semi-definite case by a duality argument. Because $J(x) = 0$ we have $P_J|_{(\ker J)^\perp} = 0$. Therefore $P_{J^*}(x)|_{(\ker J)^\perp} = I|_{(\ker J)^\perp}$ and $J'(x) = B(\nu,x)$.

The result follows because $-B(\nu,x)$ is the boundary matrix associated to the graph space of $\mathfrak{L}'$.
////

Finally, we point to the definition of maximal boundary conditions by Lax, which are closely related to admissible boundary conditions. A boundary matrix J is called maximal at x with respect to the relation $B(\nu,x)\,v\cdot v\geq 0$ if there exists no subspace U of $\mathbb{R}^m$ which contains $\ker J(x)$ properly and such that $B(\nu,x)\,v\cdot v\geq 0$ for all $v\in U$. From the definition of maximality one can infer that if (2.17) holds then

$$\dim\ker J = m - \operatorname{rank} B_-(\nu). \tag{2.33}$$

Under the assumption that (2.17) holds, all maximal boundary conditions satisfy (2.19) and all boundary conditions which satisfy (2.19) can be transformed into maximal boundary conditions by adjusting $J|_{\ker B(\nu)}$ so that J fulfills (2.33). For the rather technical proof we refer to the original sources (Lax and Phillips 1960) and (Friedrichs 1958, pp. 355-357).

2.3 The Codomain $L^2_{B,\mathrm{loc}}(\partial\Omega)$

In the last chapter we identified settings in which the trace space is contained in $L^2_{B,\mathrm{loc}}(\partial\Omega)$. In this situation the regularity requirements on the boundary conditions of Friedrichs systems can be relaxed. To allow a more succinct presentation we consider in this section only boundary conditions which are defined by matrix functions.

Suppose that $W^2_{\mathfrak{T}}(\partial\Omega)\subset L^2_{B,\mathrm{loc}}(\partial\Omega)$ and that the set $S:=\overline{\partial\Omega\setminus M}$ is a null set in the Hausdorff measure on $\partial\Omega$. Here M is defined as on page 44. We are interested in matrix functions $J:\partial\Omega\to\mathbb{R}^{m\times m}$ which have a continuous extension

$$\mathfrak{J}: W^2_{\mathfrak{L}}(\Omega)\to L^2_{B,\mathrm{loc}}(\partial\Omega),\, v\mapsto J\,v|_{\partial\Omega}. \tag{2.34}$$

If $P_{J^{\mathsf{H}}}\in L^\infty(\partial\Omega)^{m\times m}$ then the matrix function $J=-(P_{J^{\mathsf{H}}})^{\mathsf{H}}B(\nu)\in L^\infty(\partial\Omega)^{m\times m}$ is extendable in the sense of (2.34).

Theorem 38 *Let $\mathfrak{L}: W^2_{\mathfrak{L}}(\Omega)\to L^2(\Omega)^m$ be accretive and let $J\in L^\infty(\partial\Omega)^{m\times m}$ be a matrix function which has a continuous extension $\mathfrak{J}$ in the form of (2.34) and which satisfies (2.17) and (2.19). Suppose that $B^{2,2,1/2}_{0,\mathfrak{J}'}(\mathbb{R}^n\backslash S)^m$ is dense in $L^2_{B,\mathrm{loc},\mathfrak{J}'}(\partial\Omega)$. Then for all $g\in W^2_{\mathfrak{L}}(\Omega)$ and $f\in L^2(\Omega)^m$ there exists a function $u\in W^2_{\mathfrak{L}}(\Omega)$ such that $\mathfrak{L}u=f$ and $\mathfrak{J}u=\mathfrak{J}g$.*

Proof. Analogously to the proof of Theorem 28 we show that there is a function $u\in W^2_{\mathfrak{L}}(\Omega)$

such that for all $v \in W^{1,2}_{0,\mathcal{J}'}(\mathbb{R}^n \backslash S)^m$ with $\operatorname{supp}(v|_{\partial\Omega}) \Subset M$

$$\langle u, \mathcal{L}'v\rangle_\Omega = \langle f, v\rangle_\Omega + \langle \mathcal{J}\, g, v\rangle_{\partial\Omega}.$$

By density, $\langle \mathcal{J}u - \mathcal{J}g, v\rangle_{\partial\Omega} = 0$ for all $v \in L^2_{B,\mathrm{loc},\mathcal{J}'}(\partial\Omega)$ with $\operatorname{supp}(v|_{\partial\Omega}) \Subset M$. Hence $\mathcal{J}u = \mathcal{J}g$ by (2.19). ////

The relevance of Theorem 38 is that it covers Friedrichs systems with rough boundary conditions.

Example 23 *We denote the Heaviside function with H and use $y_+ = \max(y, 0)$ and $y_- = \min(y, 0)$. We let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and*

$$\mathcal{L}v = \partial_x \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -1 \end{pmatrix} v + \partial_y \begin{pmatrix} 0 & y_+ & 0 \\ y_+ & 0 & y_- \\ 0 & y_- & 0 \end{pmatrix} v + v.$$

Then, according to Theorems 32 and 38, the boundary operator $J := -(P_{J^\mathsf{H}})^\mathsf{H} B(\nu)$ with

$$P_{J^\mathsf{H}}(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & H(y) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is admissible.

Example 24 *Consider an operator $\dot{\mathcal{L}}$ on $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ which has the boundary matrix*

$$B(\nu) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathrm{e}^{-\frac{1}{y^2}} \sin\frac{1}{y} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

assuming that $W^2_{\mathcal{T}}(\partial\Omega)$ is a subset of $L^2_{B,\mathrm{loc}}(\partial\Omega)$. We impose the boundary conditions

$$J(x, y) = \begin{pmatrix} 1 & 0 & 0 \\ (-\mathrm{e}^{-\frac{1}{y^2}} \sin\frac{1}{y})_- & (-\mathrm{e}^{-\frac{1}{y^2}} \sin\frac{1}{y})_+ & (-\mathrm{e}^{-\frac{1}{y^2}} \sin\frac{1}{y})_+ \\ 0 & 0 & 0 \end{pmatrix}.$$

To show the existence of solutions of the associated boundary value problem, we merely need to verify (2.17) and (2.21) or (2.25) and argue that the associated projections are bounded. The fact that J rapidly changes the components on which it imposes the boundary conditions does not lead to any difficulties in this setting.

Certainly one can cover far more complicated examples in the framework of Theorem 38. For instance, one could include discontinuities like the ones seen in the boundary conditions of Example 23 with the rapid type changes present in the boundary operator J of Example 24. But provided that $W^2_{\mathfrak{J}}(\partial\Omega) \subset L^2_{B,\mathrm{loc}}(\partial\Omega)$, we find that however complex the boundary conditions we consider are, the hardest step of their analysis in the framework of Friedrichs is determining the algebraic properties of J and $B(\nu)$.

2.4 Well-Posedness

Having established the existence of solutions to Friedrichs systems, we turn our attention to the other ingredients of the concept of well-posedness of a boundary value problem, namely to uniqueness and to the continuous dependence of the solution on the data f and g. The simplest case is when $W^2_{\mathfrak{J}}(\partial\Omega)$ is homeomorphic to $L^2_B(\partial\Omega)$.

Theorem 39 *Consider an accretive operator $\mathfrak{L}$ and a matrix function $J = -(P_{J^{\mathsf{H}}})^{\mathsf{H}} B(\nu)$, $P_{J^{\mathsf{H}}} \in L^\infty_B(\partial\Omega)^{m\times m}$, which satisfies (2.17). Suppose that $W^2_{\mathfrak{J}}(\partial\Omega)$ is homeomorphic to $L^2_B(\partial\Omega)$. Then, the boundary value problem $\mathfrak{L}u = f$, $Ju = Jg$ is well-posed for $f \in L^2(\Omega)^m$ and $g \in L^2_B(\partial\Omega)$.*

Proof. A solution u which exists according to Theorem 28. Uniqueness and continuous dependence on the data follows from

$$\begin{aligned}\langle Ju, u\rangle_{\partial\Omega} &= \langle |B|(\nu)(P_+ - P_-)\, P_J\, g, u\rangle_{\partial\Omega}\\ &\le 1/(2\beta)\|(P_+ - P_-)\, P_J\, g\|^2_B + C\,\beta/2\|u\|^2_{\mathfrak{L}}\\ &\le 1/(2\beta)\|(P_+ - P_-)\, P_J\|_{\mathscr{B}(L^2_B(\partial\Omega), L^2_B(\partial\Omega))}\, \|g\|^2_B + C\,\beta/2(\|u\|^2_{L^2(\Omega)^m} + \|f\|^2_{L^2(\Omega)^m})\end{aligned}$$

and (2.5), where $\beta > 0$, $C > 0$. ////

If $W^2_{\mathfrak{J}}(\partial\Omega)$ is not homeomorphic to $L^2_B(\partial\Omega)$ we can still prove the uniqueness of strong solutions.

Theorem 40 *Consider an accretive operator $\mathfrak{L} : W^2_{\mathfrak{L}}(\Omega) \to L^2(\Omega)^m$ and an admissible boundary operator $\mathfrak{J} : W^2_{\mathfrak{L}}(\Omega) \to B^{2,2,-1/2}_0(\partial\Omega\backslash S)^m$. Then there is at most one strong solution u of the Friedrichs system $\mathfrak{L}u = f$, $\mathfrak{J}u = \mathfrak{J}g$ where $f \in L^2(\Omega)^m$ and $g \in W^2_{\mathfrak{L}}(\Omega)$.*

Proof. Suppose that u and $\dot{u}$ in $W^2_{\mathfrak{L}}(\Omega)$ are strong solutions of the boundary value problem. Then $u - \dot{u}$ is a strong solution of the homogeneous boundary value problem. Let $(u_i)_{i\in\mathbb{N}}$

be a sequence of $W^{1,2}(\Omega)^m$-functions converging to $u - \dot{u}$. Then all u_i are contained in $W^{1,2}_{0,\mathfrak{J}}(\mathbb{R}^n \backslash S)^m$. We deduce now from (2.7) and continuity that $u = \dot{u}$. ////

However even under relatively strong assumptions we cannot assume that the solutions of a given boundary value problem are strong. Consider, for instance, the following example which is an adaptation to Friedrichs systems of a problem discussed in (Moyer 1968). It is constructed from a differential operator with constant coefficients and boundary matrices $B(\nu)$ of full rank.

Example 25 *We abbreviate the Cauchy-Riemann operator by*

$$\mathfrak{L}_{\mathrm{CR}}\, v = \begin{pmatrix} -\partial_x & \partial_y \\ \partial_y & \partial_x \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

We choose $\mathfrak{L} = \mathfrak{L}_{\mathrm{CR}} + I$ and $\Omega = (0,1)^2$. The boundary matrix $B(\nu)$ is then

$$\pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on the boundary segments $x = 1$, $x = 0$, $y = 1$ and $y = 0$, respectively. Correspondingly, we define

$$J(x,y) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad J(x,y) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad J(x,y) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad J(x,y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

on $x = 1$, $x = 0$, $y = 1$ and $y = 0$. Comparison with (2.17), (2.20) and (2.31) shows that J is extendable to an admissible boundary operator $\mathfrak{J} : W^2_{\mathfrak{L}}(\Omega) \to B^{2,2,-1/2}_0(\partial\Omega \backslash S)^m$ where S contains the four corners of the domain. Using the polar coordinates (ϕ, r), we set

$$v(\phi, r) := r^{-1/2}(\cos\phi/2, -\sin\phi/2).$$

Since the components of v are the real and imaginary parts of the holomorphic function $z^{-1/2}$, it follows that $\mathfrak{L}_{\mathrm{CR}} v = 0$ and that $\mathfrak{L} v = v$. We note that v belongs to $W^2_{\mathfrak{L}}(\Omega)$ and that v satisfies the homogeneous boundary conditions on the bottom and on the left face of Ω. We select a radially symmetric function $\psi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(v) \Subset B_1(0)$ such that ψ is equal to 1 in a neighbourhood of the origin. Then $u = \psi\, v$ satisfies the homogeneous boundary conditions in $B^{2,2,-1/2}_0(\partial\Omega \backslash S)^m$. Moreover, $u \in W^2_{\mathfrak{L}}(\Omega)$ since

$$\mathfrak{L} u = (\psi'(r) + \psi(r))\, v.$$

Yet, due to the pole at the origin, u is not contained in $W^2_{\mathfrak{L},B}(\Omega)$ and thus the above proof of the uniqueness of solutions does not apply.

Moyer highlights the closely related fact that the integration by parts formula (1.14) is not valid for u in the classical sense. Indeed for bounded smooth functions w which satisfy the homogeneous boundary conditions, the Cauchy-Riemann operator $\mathfrak{L}_{\mathrm{CR}}$ is formally self-adjoint. Thus for them we have

$$\int_\Omega \mathfrak{L}_{\mathrm{CR}} w \cdot w \,\mathrm{d}V = 0.$$

In contrast, for u we find that

$$\int_\Omega \mathfrak{L}_{\mathrm{CR}} u \cdot u \,\mathrm{d}V = -\int_0^1 \int_0^{\pi/2} r^{-1} \psi(r) \psi'(r) \,(r \,\mathrm{d}\phi)\,\mathrm{d}r = \pi/4.$$

By continuity, this implies that there cannot be a sequence of $C^1(\mathbb{R}^2)^2$-functions which fulfill the boundary conditions exactly and converge to u. In other words, while u is a weak solution of the boundary value problem it is not strong in the sense of Section 1.9.

The failure of the integration by parts formula in this example sheds new light on Remark 3. There we introduced the bilinear form $\langle\cdot,\cdot\rangle_{B(\nu)}$ to formally extend integration by parts to the entire graph space. We now see a setting in which the boundary integral

$$\int_{\partial\Omega} v_i \, B_{ijk} \, \nu_k \, w_j \,\mathrm{d}S$$

is meaningful and differs from $\langle\cdot,\cdot\rangle_{B(\nu)}$.

With Moyer's example we acquainted ourselves with a solution which on the one hand satisfies homogeneous boundary conditions and on the other hand possesses a pole which prevents it from being contained in $W^2_{\mathfrak{L},B}(\Omega)$. For completeness, we mention the subsequent publication (Sarason 1984) in which the example of Moyer has been expanded and embedded into a class of boundary value problems for which the solutions are or are not contained in $W^2_{\mathfrak{L},B}(\Omega)$ depending on the choice of certain parameters.

Moyer's construction is different from Example 16 we have seen in the last chapter. Also there the solutions of the boundary value problem could exhibit a singularities. However these singularities are always accompanied by singularities in the boundary conditions. Owing to this interrelation, in Example 16 singularities in the solution can be avoided by appropriately selecting the space from which $h = \mathfrak{J} g$ is chosen. Because of (2.7), controlling singularities is an important step towards establishing the uniqueness of solutions to the boundary value problem.

Suppose that J imposes inflow boundary conditions, i.e. $J = -B_-(\nu)$. Then we have the relation

$$B(\nu) = |B|(\nu)\,(P_+ - P_-) = 2\,R\,(P_+ - P_-) = (R + R^{\mathsf{H}})\,(P_+ - P_-)$$

where R is the matrix $J + {}^1\!/_2\, B(\nu)$ we defined in (2.17). In the next theorem we consider a generalisation of this setting, namely Friedrichs systems for which there is a factorisation

$$B(\nu) = R^h\, T \tag{2.35}$$

where T is a matrix function in $L^\infty_B(\partial\Omega)^{m\times m}$ and $R^h = R + R^{\mathsf{H}}$. Taking the transpose of (2.35) shows that $\ker R^h$ is a subset of $\ker B(\nu)$. From

$$R^h = \left(-(P_{J^{\mathsf{H}}})^{\mathsf{H}} + (P_{J'})^{\mathsf{H}}\right) B(\nu) + \left(B(\nu)\,(-P_J + P_{J^*})\right)^{\mathsf{H}} \tag{2.36}$$

$$= \left(-P_{J^{\mathsf{H}}} + P_{J'} - P_J + P_{J^*}\right)^{\mathsf{H}} B(\nu) \tag{2.37}$$

and Theorem 33 it follows that $\ker R = \ker R^h = \ker B(\nu)$. We abbreviate $\dot{T} := -P_{J^{\mathsf{H}}} + P_{J'} - P_J + P_{J^*}$. Since B and R^h are hermitian, we have $R^h = B\,\dot{T}$. Hence $R^h = R^h\, T\, \dot{T}$ and $B(\nu) = B(\nu)\,\dot{T}\,T$. Therefore by possibly modifying T on the restriction to $\ker R$ we can assume that T is invertible and that T^{-1} belongs to $L^\infty_B(\partial\Omega)^{m\times m}$ as well.

Now let us consider two boundary operators $J, \dot{J}$ for which there are matrix functions $T, \dot{T}$ such that (2.35) is satisfied. Denoting $\dot{R} = \dot{J} + {}^1\!/_2\, B(\nu)$ we have $R^h = \dot{T}^{\mathsf{H}}\,(\dot{R} + \dot{R}^{\mathsf{H}})\,T^{-1}$ and

$$\int_{\partial_{\rm int}\kappa} R^h\, v\cdot v\,\mathrm{d}S = \int_{\partial_{\rm int}\kappa} \dot{R}^h\, T^{-1} v\cdot \dot{T}\, v\,\mathrm{d}S \tag{2.38}$$

$$\leq \frac{\|T^{-1}\|^2_{L^\infty_B(\partial\Omega)^{m\times m}} + \|\dot{T}\|^2_{L^\infty_B(\partial\Omega)^{m\times m}}}{2} \int_{\partial_{\rm int}\kappa} \dot{R}^h\, v\cdot v\,\mathrm{d}S.$$

We consider the space W of all measurable functions v on $\partial\Omega$ for which the norm

$$\|v\|_R := \sqrt{\langle v, v\rangle_R}, \quad \langle v, w\rangle_R := \int_{\partial\Omega} R^h\, v\cdot w\,\mathrm{d}S,$$

is finite. If $J = -B_-(\nu)$ then $L^2_B(\partial\Omega)$ is equal to this space. The bound (2.38) shows that for all J which satisfy (2.35) the spaces W and $L^2_B(\partial\Omega)$ coincide and that the norms $\|\cdot\|_R$ and $\|\cdot\|_B$ are equivalent on $L^2_B(\partial\Omega)$.

Theorem 41 *Let Ω be a domain which satisfies a strong local Lipschitz condition and let $\mathfrak{L}$ be an accretive operator on this domain. Given the pair of projections $P_J, P_{J^*} \in L^\infty_B(\partial\Omega)^{m\times m}$, we define the matrix function $J = -B(\nu)\,P_J$. We assume that there is a second pair of projections $P_{J^{\mathsf{H}}}, P_{J'} \in L^\infty_B(\partial\Omega)^{m\times m}$ such that $J' = B(\nu)\,P_{J'}$. We also adopt the hypothesis that the matrices*

$$R(x) := J(x) + {}^1\!/_2\, B(\nu, x), \qquad x \in \partial\Omega,$$

are positive semi-definite. Suppose that there is a matrix function $T \in L^\infty_B(\partial\Omega)^{m\times m}$ such that $B(\nu) = R^h\, T$ on $\partial\Omega$. Then for each $f \in L^2(\Omega)^m$ and $g \in L^2_B(\partial\Omega)$ there exists a unique *function $u \in W^2_{\mathfrak{L},B}(\Omega)$ which solves $\mathfrak{L}u = f$ and $Ju = Jg$. Moreover, we have the stability estimate*

$$\|u\|_\Omega \le \gamma^{-1} \|f\|_\Omega + \|T\,P_J\,g\|_R \le \gamma^{-1} \|f\|_\Omega + C\,\|g\|_B, \qquad C > 0.$$

Proof. We note that

$$J = -B(\nu)\,P_J = R^h\,(-T\,P_J) \quad \text{and} \quad J' = B(\nu)\,P_{J'} = R^h\,(T\,P_{J'}).$$

Hence, for all $v \in C_0^\infty(\mathbb{R}^n)^m$,

$$\begin{aligned}\langle D^h v, v\rangle_\Omega + \langle v, v\rangle_R &= \langle \mathfrak{L}' v, v\rangle_\Omega + \langle J'\,v, v\rangle_{\partial\Omega} \\ &\le 1/(2\gamma)\,\|\mathfrak{L}'v\|^2_\Omega + \gamma/2\,\|v\|^2_\Omega + {}^1\!/_2\;\langle T\,P_{J'}\,v, T\,P_{J'}\,v\rangle_R + {}^1\!/_2\;\langle v, v\rangle_R.\end{aligned}$$

We conclude that

$$\langle D^h v, v\rangle_\Omega + \langle v, v\rangle_R \le \gamma^{-1}\,\|\mathfrak{L}'v\|^2_\Omega + \|T\,P_{J'}\,v\|^2_R. \tag{2.39}$$

Consequently, the mapping

$$\Phi : C_0^\infty(\mathbb{R}^n)^m \to L^2(\Omega)^m \times L^2_B(\partial\Omega), v \mapsto (\mathfrak{L}'v, T\,P_{J'}\,v)$$

is injective. Clearly, there is a positive constant $C = C(f, g)$ such that

$$\langle f, v\rangle_\Omega + \langle Jg, v\rangle_{\partial\Omega} = \langle f, v\rangle_\Omega + \langle T\,P_{J'}\,g, v\rangle_R \le C\,(\langle D^h v, v\rangle_\Omega + \langle v, v\rangle_R)^{1/2}. \tag{2.40}$$

We equip $W := \operatorname{Im}\Phi$ with the norm

$$(\mathfrak{L}'v, T\,P_{J'}\,v) \mapsto (\|\mathfrak{L}'v\|^2_\Omega + \|T\,P_{J'}\,v\|^2_R)^{1/2}.$$

Combining (2.39) and (2.40) shows that the assignment

$$\Psi : W \to \mathbb{R},\; w = (\mathfrak{L}'v, T\,P_{J'}\,v) \mapsto \langle f, v\rangle_\Omega + \langle Jg, v\rangle_{\partial\Omega}$$

is continuous. By the Hahn-Banach theorem and the Riesz representation theorem there is a pair (u_1, u_2) in $L^2(\Omega)^m \times L^2_B(\partial\Omega)$ such that, for all $(\mathfrak{L}'v, T\,P_{J'}\,v) \in W$,

$$\langle u_1, \mathfrak{L}'v\rangle_\Omega + \langle u_2, T\,P_{J'}\,v\rangle_R = \langle f, v\rangle_\Omega + \langle Jg, v\rangle_{\partial\Omega}.$$

Testing with $v \in \mathscr{D}(\Omega)^m$ shows that $\mathfrak{L}u_1 = f$. Thus $u_1 \in W^2_{\mathfrak{L}}(\Omega)$. Moreover,

$$\langle u_2, T\,P_{J'}\,v\rangle_R - \langle B(\nu)\,u_1, v\rangle_{\partial\Omega} = \langle u_2, J'\,v\rangle_{\partial\Omega} - \langle B(\nu)\,u_1, v\rangle_{\partial\Omega} = \langle Jg, v\rangle_{\partial\Omega}. \tag{2.41}$$

Thus $B(\nu)\,u_1 = J^*u_2 - Jg$ in $B^{2,2,-1/2}(\partial\Omega)^m$. However, since $J^*u_2 - Jg$ is an element of $L^2_B(\partial\Omega)$, also $B(\nu)\,u_1 \in L^2_B(\partial\Omega)$. Therefore u_1 is contained in $W^2_{\mathcal{L},B}(\Omega)$ by Theorem 21. Hence the linear functional

$$L^2_B(\partial\Omega) \to \mathbb{R}, v \mapsto \langle -(P_{J^{\mathsf{H}}})^{\mathsf{H}}\, B(\nu)\, u_1, v\rangle_{\partial\Omega} = \langle J\, u_1, v\rangle_{\partial\Omega}$$

is meaningful and continuous. In consequence, we can rearrange (2.41) to $J(u_1 - g) = J^*(u_1 - u_2)$. Recalling (2.20), we deduce that $Ju_1 = Jg$.

Now suppose there is a second function $\dot{u} \in W^2_{\mathcal{L}}(\Omega)$ for which Ju is contained in $L^2_B(\partial\Omega)$ and which satisfies $\mathcal{L}\dot{u} = f$ and $J\dot{u} = Jg$. Then $\mathcal{L}(u_1 - \dot{u}) = 0$ and $J(u_1 - \dot{u}) = 0$. It follows from (2.7) that $\dot{u} = u_1$.

With Ju_1 also $Ru_1 = Ju_1 + J^*u_1$ is meaningful. Thus we find

$$\langle D^h u_1, u_1\rangle_\Omega + \langle u_1, u_1\rangle_R \le \gamma^{-1}\,\|\mathcal{L}u_1\|^2_\Omega + \|T\,P_J\,u_1\|^2_R = \gamma^{-1}\,\|f\|^2_\Omega + \|T\,P_J\,g\|^2_R \tag{2.42}$$

similarly to (2.39). ////

Observe that the hypotheses of Theorem 41 do not require that the boundary matrix $B(\nu)$ and boundary conditions are continuous as matrix functions. Nevertheless, the fact that J has a continuous extension to a boundary operator follows automatically.

Example 26 *Recall the setting of Example 25. Let us concentrate on the boundary segment* $x = 0$. *Here we have*

$$B(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1/2 & 0 \\ -1 & 1/2 \end{pmatrix}, \quad R^h = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues of R^h *are* 0 *and* 2. *Consequently* $\ker R^h \neq \ker B(\nu)$ *and the last theorem is not applicable. In Example 20 we learned that the images of the projections*

$$P_J = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad P_{J^*} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad P_{J^{\mathsf{H}}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad P_{J'} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

have to be contained in $V_- \cup V_0$ *and* $V_+ \cup V_0$, *respectively. In this example if the image of one of the projections is contained in* V_0 *then condition (2.35) cannot be satisfied. Let* P *be one of the projections and suppose that* $\operatorname{Im} P \subset V_0$ *and* $v \in \operatorname{Im} P$. *From (2.36) it follows that* $R^h\,v \cdot v = \pm B(\nu)\,v \cdot v = 0$. *However, then* $\operatorname{Im} P$ *is a subset of the kernel of* R. *Therefore the rank of* R *is at most* $m - \dim \operatorname{Im} P \neq \operatorname{rank} B(\nu)$. *Clearly, the projections* P_{J^*} *and* $P_{J^{\mathsf{H}}}$ *are contained in* V_0.

We bring the section to a close with an example of a Friedrichs system which is not covered by Theorem 41 but for which the uniqueness of solutions follows easily from the method of characteristics. Yet, for the system the solutions do not depend continuously on the boundary data with respect to the $L^2_B(\partial\Omega)$- and $L^2(\Omega)^m$-norms.

Example 27 *We choose the same domain Ω as in Example 16. We consider the differential operator $\mathfrak{L}v = (-\partial_x v_1 + v_1, \partial_x v_2 + v_2)^{\mathsf{H}}$. Then the assignment*

$$J : \partial\Omega \to \mathbb{R}^{2\times 2}, (x, y) \mapsto \operatorname{sign}(x) \begin{pmatrix} {}^1/_2 & {}^1/_2 \\ -{}^1/_2 & -{}^1/_2 \end{pmatrix}$$

defines an admissible boundary operator. We assume that $f = 0$. The solutions of the boundary value problem are of the form

$$(c_1(y)\,\mathrm{e}^{x}, c_2(y)\,\mathrm{e}^{-x}).$$

The graph norm of these solutions is

$$\left(\int_0^1 (c_1^2(y) + c_2^2(y)) \sinh(y)\,\mathrm{d}y\right)^{1/2}. \tag{2.43}$$

For the boundary data $g = (g_1, g_2)^{\mathsf{H}}$ the coefficients c_1 and c_2 are given by

$$\begin{pmatrix} c_1(y) \\ c_2(y) \end{pmatrix} = \frac{1}{\mathrm{e}^{-2y} - \mathrm{e}^{2y}} \begin{pmatrix} \mathrm{e}^{-y} & \mathrm{e}^{y} \\ \mathrm{e}^{y} & \mathrm{e}^{-y} \end{pmatrix} \begin{pmatrix} g_1(-y) + g_2(-y) \\ g_1(y) \ + \ g_2(y) \end{pmatrix}.$$

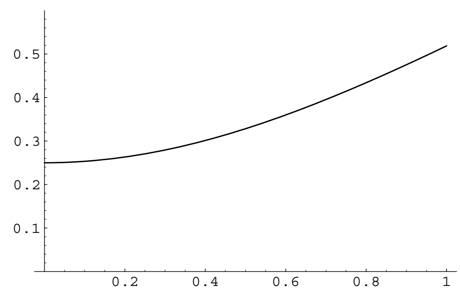
Suppose that

$$g_1(-y) + g_2(-y) = 0 \quad \text{and that} \quad g_1(y) + g_2(y) = 1. \tag{2.44}$$

We plot the function

$$y \mapsto y\,(c_1^2(y) + c_2^2(y)) \sinh(y)$$

for this boundary data, i.e. we plot the product of the integrand of (2.43) with y.



The plot illustrates that with the boundary data (2.44) the integrand is asymptotic to the function $y \mapsto 1/y$. Hence for this data there is no solution of the Friedrichs system. We also consider the family of boundary functions

$$g_i(x,y) = \begin{cases} (0,1)^{\mathsf{H}} : x > 1/i, \\ (0,0)^{\mathsf{H}} : x \le 1/i, \end{cases} \qquad i \in \mathbb{N}.$$

While for these functions g_i solutions u_i of the Friedrichs system exist, the sequence of ratios between $\|g_i\|_B$ and the graph norm of the respective solution u_i diverges as $i \to \infty$. Also the boundary value problem is ill-posed with respect to the $L^2(\partial\Omega)^2$-norm because this norm is equivalent to $\|\cdot\|_B$ in this example.

2.5 Examples of Friedrichs Systems

In this section we give an overview of a wide range of boundary value problems which can be analysed in the framework of Friedrichs systems. In this sense the section can also serve us as a reference. Of particular interest is Example 32 concerning the Frankl equation.

Example 28 (First-Order Hyperbolic Systems) Of the different kinds of equations which we examine in this section, symmetric first-order hyperbolic systems can be transformed into Friedrichs systems most directly. Consider the Friedrichs symmetric operator

$$\mathfrak{L} : v \mapsto \partial_k(B_{ijk} v_j) + C_{ij}\, v_j.$$

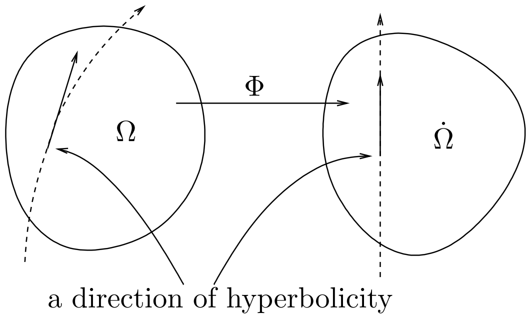
We call $\mathfrak{L}$ uniformly hyperbolic on the domain Ω if there is a vector $\alpha \in \mathbb{R}^n$ such that $B_{ijk}(x)\,\alpha_k$ is positive definite for all $x \in \overline{\Omega}$. Given $\beta \in \mathbb{R}$, the differential operator satisfies the identity

$$\mathfrak{L}(\mathrm{e}^{\beta\alpha\cdot x} u) = \partial_k(B_{ijk}\, u_j) + B_{ijk}\, u_j\, \partial_k \mathrm{e}^{\beta\alpha\cdot x} + C_{ij}\, u\, \mathrm{e}^{\beta\alpha\cdot x} = \mathrm{e}^{\beta\alpha\cdot x}(\mathfrak{L} + \beta\, B(\alpha))\, u. \tag{2.45}$$

We used $B(\alpha)_{ij} := B_{ijk}\,\alpha_k$. The parameter β can be chosen sufficiently large to ensure that $\dot{\mathfrak{L}} := \mathfrak{L} + \beta\,\alpha_k B_{ijk}$ is accretive. Thus, under the transformation $u = \mathrm{e}^{-\beta\alpha\cdot x}\, v$, the boundary value problem $\mathfrak{L}v = f$, $\mathfrak{J}v = \mathfrak{J}g$ is equivalent to the Friedrichs system $\dot{\mathfrak{L}}u = \mathrm{e}^{-\beta\alpha\cdot x} f$, $\mathfrak{J}u = \mathfrak{J}\mathrm{e}^{-\beta\alpha\cdot x} g$, provided $\mathfrak{J}$ is admissible with respect to $\dot{\mathfrak{L}}$. Notice that transformation (2.45) does not modify the principal part of the differential operator and therefore $W^2_{\mathfrak{L}}(\Omega)$ and $W^2_{\dot{\mathfrak{L}}}(\Omega)$ share the same set of associated admissible boundary operators.

Suppose that the differential operator under consideration is not uniformly hyperbolic but that there exists a vector field $\alpha : \Omega \to \mathbb{R}^n$ such that $B_{ijk}(x)\,\alpha_k(x)$ is positive definite for all

$x \in \overline{\Omega}$. In the language of fluid dynamics, if α does not have circular streamlines, vortices or similar features then there might be a coordinate transformation which turns $\mathfrak{L}$ into a uniformly hyperbolic operator.



We need to make the statement mathematically more precise. Suppose there a diffeomorphism Φ of class $W^{1,\infty}$ from Ω to a domain $\dot{\Omega}$ such that the vector field $\partial\Phi/\partial(x_1, \dots, x_n)\,\alpha$ consists of vectors which are parallel to each other; here $\partial\Phi/\partial(x_1, \dots, x_n)$ is the Jacobian of Φ. Then the change of coordinates Φ turns $\mathfrak{L}$ into a uniformly hyperbolic operator $\dot{\mathfrak{L}}$ which has, up to the change of coordinates, the same solutions as the original system. We use here that if $B_{ijk}(x)\,\alpha_k(x)$ is positive definite then also $\beta(x)\,B_{ijk}(x)\,\alpha_k(x)$ is positive definite for all $\beta(x) > 0$. Thus, by rescaling, we can always alter $\partial\Phi/\partial(x_1, \dots, x_n)\,\alpha$ into a constant vector field which points into a direction of uniform hyperbolicity of $\dot{\mathfrak{L}}$.

Example 29 (Second-Order Hyperbolic Equations) An attractive feature of Friedrichs systems is their close connection to second-order hyperbolic equations. Again, a uniform direction of hyperbolicity plays an important role in the reduction of the boundary value problem to a Friedrichs system. Consider the second-order differential operator

$$\mathfrak{L} : v \mapsto \partial_k\left(A_{jk}\,\partial_j v\right) + \partial_j(B_j v) + C\,v \tag{2.46}$$

for which we assume that $A \in W^{2,\infty}(\Omega)^{n\times n}$, $B \in W^{1,\infty}(\Omega)^n$ and $C \in L^\infty(\Omega)$. We also require that A is symmetric. The operator $\mathfrak{L}$ is uniformly hyperbolic if there is an $\alpha \in \mathbb{R}^n$ such that for all $x \in \overline{\Omega}$ the scalar $\alpha_k\,A_{jk}(x)\,\alpha_j$ is negative and $A(x)$ is positive definite in the orthogonal complement of α in $\mathbb{R}^n$. Often the direction of uniform hyperbolicity has the physical interpretation of time. Analogously to the previous example, some operators are only uniformly hyperbolic after a coordinate transformation.

By an orthogonal change of coordinates we may assume that $\alpha = (1, 0, \dots, 0)^{\mathsf{H}}$, that $A_{1,1}$ is negative and that $A_{1k} = 0$ for $k \in \{2, \dots, n\}$. Then we rename $u_0 := v$ and $u_1 := \partial_i v$ for

$i = 1, \ldots, n$. Hence we have the relationship

$$\sum_{i=2}^{n} A_{jk}\,(\partial_1 u_i - \partial_i u_1) = 0, \qquad j \in \{2, \ldots, n\}.$$

We pass to a symmetric hyperbolic operator of first order $v \mapsto \partial_k(\dot{B}^k\, v) + \dot{C}\, v$ with the coefficients

$$\dot{B}^1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -A_{11} & 0 & \cdots & 0 \\ 0 & 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & A_{n2} & \cdots & A_{nn} \end{pmatrix}, \qquad \dot{B}^k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -A_{2k} & \cdots & -A_{nk} \\ 0 & -A_{2k} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -A_{nk} & 0 & \cdots & 0 \end{pmatrix}, \quad k \in \{2, \ldots, n\},$$

and

$$\dot{C} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ -C - \partial_i B_i & -B_1 & -B_2 & \cdots & -B_n \\ 0 & \partial_j A_{2j} & -\partial_1 A_{22} & \cdots & -\partial_1 A_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \partial_j A_{nj} & -\partial_1 A_{n2} & \cdots & -\partial_1 A_{nn} \end{pmatrix}, \tag{2.47}$$

where the summation index i ranges from 1 to n and the index j from 2 to n. The right-hand side is defined as

$$\dot{f} = (0, -f, 0, \ldots, 0)^{\mathsf{H}}.$$

Observe that $\dot{B}_1$ is positive definite. Thus the system is of the type considered in the last example and it can be therefore transformed into a Friedrichs system by means of (2.45).

We now turn to the boundary conditions. Let us assume that $\Omega = [0, T] \times \Omega_0$, where $T > 0$ and $\Omega_0 \subset \mathbb{R}^{n-1}$, that the direction of uniform hyperbolicity is $(1, 0, \ldots, 0)^{\mathsf{H}}$ and that $A_{11} = -1$. Suppose that we wish to impose initial conditions on $\{0\} \times \Omega_0$. By this we mean that we prescribe the value of $v = g_0$ and $\partial_1 v = g_1$ on this surface. Because $\dot{B}(\nu, x) = -\dot{B}_1(x)$ is negative definite we have to choose $J(x) = -\dot{B}(\nu, x)$ in order to satisfy admissibility at $x \in \{0\} \times \Omega_0$. We implement the initial conditions by setting

$$J(u_0, u_1, \ldots, u_n)^{\mathsf{H}} = J(g_0, g_1, \partial_2 g_0, \ldots, \partial_n g_0)^{\mathsf{H}}.$$

On $[0, T] \times \{0\}$ one can choose from a greater variety of boundary conditions. We shall investigate them by means of an example. Let $\Omega_0 = (0, 1)$ corresponding to $n = 2$ and let $\mathcal{L}v = -u_{tt} + u_{xx} = 0$. Then, for $x = 0$ we find that

$$\dot{B}(\nu) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{2.48}$$

We remark that due to the structure of $B(\nu)$ we cannot directly impose the Dirichlet condition $u_0 = g_0$. Let us parameterise the set of admissible matrices by $a, b, c, d \in \mathbb{R}$:

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b + {}^1/_2 \\ 0 & c + {}^1/_2 & d \end{pmatrix}.$$

We consider three types of matrices which satisfy (2.17) and (2.19):

	a	b	c	d
Type 1	≥ 0	$-{}^1/_2$	${}^1/_2$	0
Type 2	0	${}^1/_2$	$-{}^1/_2$	≥ 0
Type 3	> 0	$\lvert b\rvert < {}^1/_2$	$-b$	$({}^1/_4 - b^2)/a$

Type 1 conditions fix the value of $u_1 = \partial_1 v$ on the left side of the boundary. Thus with Type 1 conditions, in combination with the initial condition $v(0,0) = g_0(0,0)$, we can determine the values of v on the segment $x = 0$ and impose in this manner Dirichlet conditions. For instance, with $J_{\mathrm{Type\,I}}(x_1, 0) = 0$ together with $v(0,0) = 0$ we can model a reflecting boundary. Type 2 boundary operators correspond to Neumann conditions since they fix the value of $\partial_x v$. Finally one uses matrices of Type 3 to implement Robin conditions.

Example 30 (Elliptic Equations) We study the elliptic equation

$$\mathcal{L}v = \partial_k \left(A_{jk}\, \partial_j v\right) + \partial_j (B_j v) + C\, v = f \tag{2.49}$$

for which we assume that $A \in W^{2,\infty}(\Omega)^{n\times n}$, $B \in W^{1,\infty}(\Omega)^n$ and $C \in L^\infty(\Omega)$ as well as that A is symmetric and positive definite on $\overline{\Omega}$. Suppose there exists a function $(p_1, \ldots, p_n) \in C^1(\mathbb{R}^n)^n$ such that $\partial_k(A_{jk}\, p_j)$ is positive on $\overline{\Omega}$. As for second-order hyperbolic problems, we let $u_0 = v$ and $u_j = \partial_j v$. Thus the new variables satisfy the relationship

$$A\,(u_1, \ldots, u_n)^{\mathsf{H}} = A\,(\partial_1 u_0, \ldots, \partial_n u_0)^{\mathsf{H}}.$$

We wish to transform (2.49) into a first-order system of the form

$$\sum_{j=0}^{n}\sum_{k=1}^{n} \partial_k(\dot{B}_{ijk}\, u_j) + \sum_{j=0}^{n} \dot{C}_{ij}\, u_j = \dot{f}_i, \qquad i \in \{0, \ldots, n\}.$$

We choose

$$\dot{B}^k = \begin{pmatrix} -\delta\, A_{jk}\, p_j & -A_{k1} & \ldots & -A_{kn} \\ -A_{1k} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ -A_{nk} & 0 & \ldots & 0 \end{pmatrix}, \quad k \in \{1, \ldots, n\},$$

and

$$\dot{C} = \begin{pmatrix} -(C + \sum_j B_j) + \partial_k(\delta\, A_{jk}\, p_j) & -B_1 + \delta\, A_{j1}\, p_j & \dots & -B_n + \delta\, A_{jn}\, p_j \\ \partial_k A_{1k} & A_{11} & \dots & A_{1n} \\ \vdots & \vdots & & \vdots \\ \partial_k A_{nk} & A_{n1} & \dots & A_{nn} \end{pmatrix},$$

where δ is a positive real number, where the summation indices range from 1 to n and where $\dot{B}^k$ are submatrices of B which consist of the entries B_{ijk} with a fixed index k. The right-hand side is defined as

$$\dot{f} = (-f, 0, \dots, 0)^{\mathsf{H}}.$$

The matrix D^h, introduced on page 65, equals

$$\begin{pmatrix} -(C + \sum_j B_j) + {}^1\!/_2\, \partial_k(\delta A_{jk} p_j) & -{}^1\!/_2\, B_1 + {}^1\!/_2\, \delta A_{j1} p_j & \dots & -{}^1\!/_2\, B_n + {}^1\!/_2\, \delta A_{jn} p_j \\ -{}^1\!/_2\, B_1 + {}^1\!/_2\, \delta\, A_{j1}\, p_j & A_{11} & \dots & A_{1n} \\ \vdots & \vdots & & \vdots \\ -{}^1\!/_2\, B_n + {}^1\!/_2\, \delta\, A_{jn}\, p_j & A_{n1} & \dots & A_{nn} \end{pmatrix}.$$

A symmetric matrix is positive definite if, and only if, all its principal minors are positive. Because of the positive definiteness of A it is sufficient to show that the determinant of D^h is positive. Let us first consider the case $B = 0$ and $C = 0$. Then, positive definiteness can be guaranteed by choosing δ sufficiently small. If B and C do not vanish, we use the identity

$$\mathfrak{L}(\mathrm{e}^{\beta\alpha\cdot x}\, u) = \mathrm{e}^{\beta\alpha\cdot x}\, (\mathfrak{L}u + \beta\, \alpha_j\, \partial_k(A_{jk}\, u) + (\beta^2\, \alpha_j\, A_{jk}\, \alpha_k + \beta\, B_j\, \alpha_j)\, u) \tag{2.50}$$

for $\beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$. By selecting β large and δ small enough we ensure the positive definiteness of D^h. For some operators such as $u \mapsto -\Delta u + u$ one obtains already for $\delta = 0$ and $\beta = 0$ the positive definiteness of D^h. For them neither the functions $(p_1, \dots, p_n)$ nor the change of variables (2.50) are needed.

Having transformed the differential operator we turn our attention to the boundary conditions. Clearly,

$$\dot{B}(\nu) = \begin{pmatrix} -\delta\, A_{jk}\, \nu_k\, p_j & -A_{k1}\nu_k & \dots & -A_{kn}\nu_k \\ -A_{1k}\nu_k & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -A_{nk}\nu_k & 0 & \dots & 0 \end{pmatrix}.$$

We assign to each x where a Dirichlet boundary condition is to be implemented two positive numbers q_+, q_- such that

$$-\delta\, A_{jk}\, \nu_k\, p_j = q_+ - q_-\,.$$

Then we choose

$$J = \begin{pmatrix} q_- & 0 & \dots & 0 \\ A_{1k}\nu_k & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{nk}\nu_k & 0 & \dots & 0 \end{pmatrix}, \quad R = \begin{pmatrix} {}^1/_2\, q_+ + {}^1/_2\, q_- & -{}^1/_2 A_{k1}\nu_k & \dots & -{}^1/_2 A_{kn}\nu_k \\ {}^1/_2\, A_{1k}\nu_k & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ {}^1/_2\, A_{nk}\nu_k & 0 & \dots & 0 \end{pmatrix}.$$

Clearly, $R = J + {}^1/_2\, \dot{B}(\nu)$ is positive semi-definite and thus J is semi-admissible. The image of J^* is the span of $(1, 0, \dots, 0)^{\mathsf{H}}$. Since the columns of A are linearly independent, we know that for no ν the image J contains $(1, 0, \dots, 0)^{\mathsf{H}}$. Therefore J is pointwise admissible in the sense of (2.17) and (2.19).

In order to impose oblique Neumann boundary conditions at $x \in \partial\Omega$ one needs to select $(p_1, \dots, p_n)$ so that $-\delta\, A_{jk}\, \nu_k\, p_j$ is positive at x, e.g. by choosing $p_j = -\nu_j$. Then, the boundary operator

$$J = \begin{pmatrix} 0 & A_{k1}\nu_k & \dots & A_{kn}\nu_k \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

is admissible. Since, for instance, solutions of the Laplace equation subject to a Neumann boundary condition are not unique, we can, in general, not assume that the positivity of $-\delta\, A_{jk}\, \nu_k\, p_j$ can be fulfilled everywhere on $\partial\Omega$. For the details of the implementation of Robin boundary conditions we refer to (Friedrichs 1958, p. 359).

Example 31 (Parabolic Equations) We turn to parabolic operators of the form

$$\mathfrak{L} : v \mapsto \partial_k \left(A_{jk}\, \partial_j v \right) + \partial_j (B_j v) + C\, v \tag{2.51}$$

where $A \in W^{2,\infty}(\Omega)^{n \times n}$, $B \in W^{1,\infty}(\Omega)^n$, $C \in L^\infty(\Omega)$. In addition, we assume that A is symmetric, that $A_{jk} = 0$ if $j = 1$ and that the submatrix $(A_{jk})_{2 \le j,k \le n}$ is positive definite. We also demand that B_1 is negative on $\overline{\Omega}$. Again x_1 usually has the meaning of time.

We reduce $\mathfrak{L}$ to the first-order operator with the coefficients

$$\dot{B}^1 = \begin{pmatrix} -B_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \dot{B}^k = \begin{pmatrix} 0 & -A_{2k} & \cdots & -A_{nk} \\ -A_{2k} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -A_{nk} & 0 & \cdots & 0 \end{pmatrix}$$

and

$$\dot{C} = \begin{pmatrix} -(C + \sum_j B_j) & -B_2 & \dots & -B_n \\ \partial_k A_{2k} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ \partial_k A_{nk} & A_{n2} & \dots & A_{nn} \end{pmatrix},$$

where j ranges from 1 to n and k from 2 to n. The matrix D_h is equal to

$$\begin{pmatrix} -(C + \sum_j B_j) & -{}^1\!/_2\, B_2 & \dots & -{}^1\!/_2\, B_n \\ -{}^1\!/_2\, B_2 & A_{11} & \dots & A_{1n} \\ \vdots & \vdots & & \vdots \\ -{}^1\!/_2\, B_n & A_{n1} & \dots & A_{nn} \end{pmatrix}.$$

Following (2.45), we can pass to an accretive operator by setting

$$(\dot{u}_1, \dot{u}_2, \dots, \dot{u}_n) := (\mathrm{e}^{-\beta x_1} u_1, u_2, \dots, u_n),$$

where β is a sufficiently large real number. The right-hand side is defined as

$$\dot{f} = (-\mathrm{e}^{-\beta x_1} f, 0, \dots, 0)^{\mathsf{H}}.$$

We consider again the domain $[0, T] \times \Omega_0$ from Example 29. On $\{0\} \times \Omega_0$, the matrix $\dot{B}(\nu)$ equals $-\dot{B}_1$ so that the boundary condition $v = g_0$ has to be imposed in order to satisfy admissibility. Just as for second-order hyperbolic equations, we investigate boundary conditions on $[0, T] \times \{0\}$ by virtue of an example. Let $\Omega_0 = (0, 1)$ and $\mathfrak{L}v = -u_t + u_{xx} = 0$. At $x = 0$ we have the boundary matrix

$$\dot{B}(\nu) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Thus $\dot{B}(\nu)$ is a submatrix of (2.48) and so we can reuse the parametrisation with a, b, c and d. Again Type 1 corresponds to a Dirichlet, Type 2 to a Neumann and Type 3 to a Robin boundary condition.

Example 32 (Frankl equation I: Friedrichs' Method) Friedrichs' main motivation behind the study of accretive operators and admissible boundary conditions was not so much the desire to handle elliptic, parabolic and hyperbolic equations in a unified framework but rather the need to treat differential equations which are in some parts of the domain elliptic and in other parts hyperbolic. A prototype of these equations is the Frankl equation

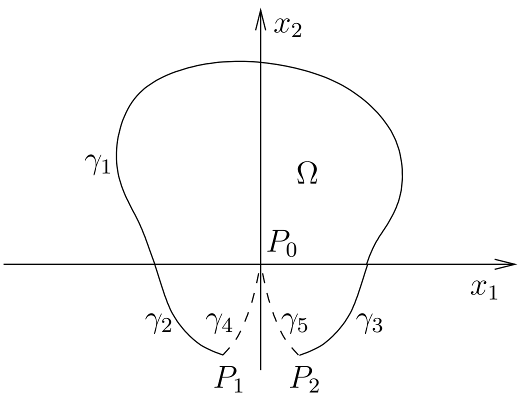
$$\Big(A(x_2)\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)v = 0 \tag{2.52}$$

for which one assumes that

$$\frac{\partial A}{\partial x_2} > 0 \text{ for } x_2 > c \text{ for some } c < 0,$$
$$x_2\, A(x_2) \geq 0 \text{ for all } x_2 \in \mathbb{R}.$$

If $A(y) = y$ then one calls (2.52) the Tricomi equation. Differential operators of this kind play an important role in compressible gas dynamics. Typically, the areas where the operator is hyperbolic correspond to regions of supersonic flow while areas of elliptic type correspond to regions of subsonic flow. We outline this relationship in more detail in the next example. However, let us first investigate how the Frankl equation is related to Friedrichs systems.

The authors of (Morawetz 1958) and (Lax and Phillips 1960) studied the Frankl equation on domains Ω of the type depicted in the figure below.



The subdomain $\{(x_1, x_2) \in \Omega : x_2 > 0\}$ is star-shaped with respect to the origin in the (x_1, y)-coordinate system, where

$$y = \int_0^{x_2} \sqrt{A(\xi)}\, \mathrm{d}\xi.$$

The curves γ_2 and γ_3 are subcharacteristic, that is we assume that

$$A\Big(\frac{\partial x_2}{\partial x_1}\Big)^2 + 1 > 0$$

along γ_2 and γ_3. Finally, γ_4 and γ_5 characteristic curves, i.e.

$$A\Big(\frac{\partial x_2}{\partial x_1}\Big)^2 + 1 = 0$$

along γ_4 and γ_5. Following (Morawetz 1958), we introduce the new unknowns $u_1 = \partial_1 v$, $u_2 = \partial_2 v$ to transform equation (2.52) into a system of equations. Thus (2.52) becomes

$$\begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial u_1}{\partial x_1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial u_2}{\partial x_2} = 0.$$

The equation can be made positive by multiplying from the left with the weight

$$\begin{pmatrix} a & -A\,b \\ b & a \end{pmatrix}.$$

Then the equation is transformed into

$$\begin{pmatrix} A\,a & A\,b \\ A\,b & -a \end{pmatrix} \frac{\partial u_1}{\partial x_1} + \begin{pmatrix} -A\,b & a \\ a & 0 \end{pmatrix} \frac{\partial u_2}{\partial x_2} = 0.$$

The matrix D^h equals

$$1/2 \begin{pmatrix} -A\,\partial_1 a + \partial_2(Ab) & -A\,\partial_1 a - \partial_2 a \\ -A\,\partial_1 a - \partial_2 a & \partial_1 a \end{pmatrix}.$$

Morawetz proposes the weights

$$\begin{aligned} a &= x_1, \quad b = c\,|x_1| && \text{for } x_2 \le 0, \\ a &= x_1, \quad b = \big(c\sqrt{A}\,|x_1| + \textstyle\int_0^{x_2} \sqrt{A(\xi)}\,\mathrm{d}\xi\big)/\sqrt{A} && \text{for } x_2 \ge 0, \end{aligned}$$

where c is a real constant which is to be determined with regard to the boundary conditions. Lax and Phillips verified that on $\gamma_1 \cup \gamma_2 \cup \gamma_3$ homogeneous Dirichlet boundary conditions are semi-admissible and maximal, cf. p. 78. Requiring appropriate regularity conditions on A ensures that admissible conditions can be imposed.

Morawetz treats the singularity at the origin P_0 by introducing weighted L^2-spaces in order to demonstrate the existence of a weak solution of the Frankl equation. Therefore her methodology does not quite fall into the framework of Friedrichs. However it coincides with it in its essential points. Based on the findings in (Morawetz 1958), Lax and Phillips proved that the weak solution is unique and that it can be approximated by smooth functions which satisfy the homogeneous boundary conditions exactly. They have accounted for the corners P_1 and P_2 using the semi-definiteness of the matrix $B(\nu)$ on γ_4 and γ_5. Using smooth scalar multiplier functions they were able to control the effect of the singularity at P_0.

Example 33 (Frankl equation II: The Physical Motivation) As already pointed out in the previous example, the Frankl equation plays an important role in compressible gas dynamics. We highlight the important steps in the derivation of the physical model; however, we leave certain details to the references (Morawetz 1981) and (Dautray and Lions 1988-93, Chap. X).

We consider a two-dimensional steady flow which is described by the density ρ, the pressure p and the velocity (u, v). We use the physical coordinates (x, y) and denote partial differentiation by subscripts. Conservation of mass is expressed by the relation

$$(\rho u)_x + (\rho v)_y = 0. \tag{2.53}$$

Postulating that shock waves are either weak or of constant intensity, we deduce that the flow is isentropic and thus irrotational, that is $v_x = u_y$. It follows that p is a function of the density ρ alone: $p = p(\rho)$. One generally assumes that $\partial p/\partial\rho > 0$. Because the flow is irrotational we can reduce the momentum equation to Bernoulli's law:

$$\frac{u^2+v^2}{2} + \int \frac{\mathrm{d}p}{\rho} = \text{constant}.$$

Consequently, ρ is a function of V where $V = u^2 + v^2$. For irrotational flow there exists a velocity potential φ such that $(u, v) = \nabla\varphi$. By virtue of (2.53), there is also a stream function ψ which satisfies $\rho\,(u, -v) = \nabla\psi$.

We now change the coordinate system and regard u and v instead of x and y as independent variables. The space spanned by u and v is called the velocity plane or the hodograph plane. We use the velocity potential φ and the stream function ψ to set up the system

$$\begin{aligned} \mathrm{d}x &= \frac{1}{\rho V^2}\,(-v\,\mathrm{d}\psi + \rho\,u\,\mathrm{d}\varphi), \\ \mathrm{d}y &= \frac{1}{\rho V^2}\,(\quad u\,\mathrm{d}\psi + \rho\,v\,\mathrm{d}\varphi), \end{aligned} \tag{2.54}$$

where $\rho = \rho(V)$. By setting $u + iv = V\,e^{i\theta}$ we introduce the polar coordinates (V, θ) on the hodograph plane. In these coordinates (2.54) takes the form of the Chaplyguine-Molenbroek equations:

$$\varphi_\theta = \frac{V}{\rho}\,\psi_V, \qquad \psi_\theta = \frac{1}{V(1/\rho V)_V}\,\varphi_V. \tag{2.55}$$

This system is linear since the coefficients only depend on the coordinate V. We define

$$A(\sigma) := V\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(\frac{1}{\rho V}\Big), \qquad \sigma := -\int_c^V \frac{\rho}{V}\,\mathrm{d}V,$$

where $c = (\mathrm{d}p/\mathrm{d}\rho)^{1/2}(\rho)$ is the speed of sound. By eliminating φ and substituting V by σ in (2.55) we finally arrive at the Frankl equation:

$$A(\sigma)\,\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\sigma^2} = 0.$$

In conclusion then, we are able to reduce the original nonlinear system to a linear differential equation. However we perform this simplification by means of a nonlinear coordinate transformation.

To gain a better understanding of the representation of the physical solution on the hodograph plane we discuss the transformation to the (θ, σ)-coordinates considering the important example of an exterior flow surrounding an airfoil. Our presentation is based on the description of this setting in (Dautray and Lions 1988-93).

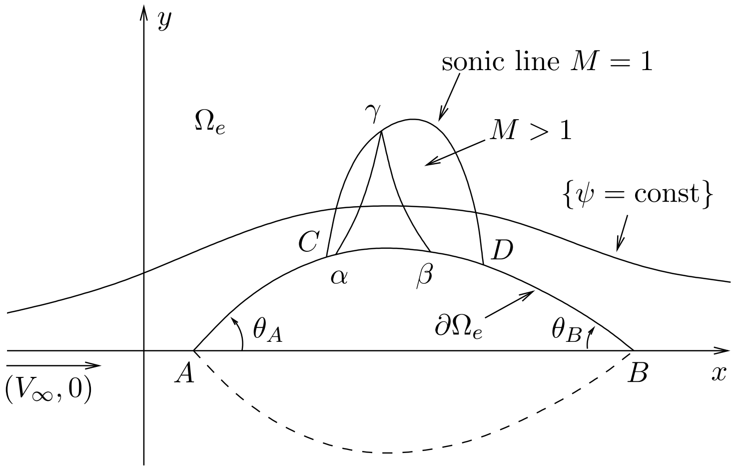


Figure 1. Steady compressible flow around an airfoil.

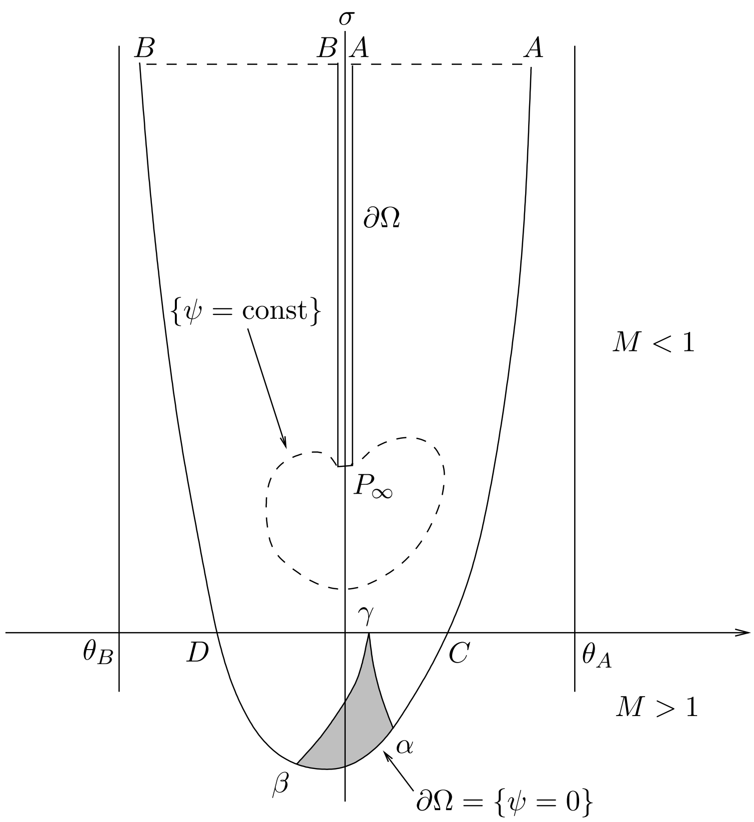


Figure 2. Representation of the flow on the hodograph plane.

Figures 1 and 2 on page 97 depict the physically important setting of a flow around an airfoil which is subsonic and uniform at infinity and has a bounded supersonic pocket attached to the wing profile. We denote the exterior domain by Ω_e, the corresponding domain on the hodograph plane is Ω. By construction, the velocity field (u, v) is tangential to the streamlines $\psi(x, y) =$ constant. Dautray and Lions impose a no-slip boundary condition from which they deduce that ψ is constant on the airfoil boundary. Therefore on the hodograph plane the contour line $\psi = 0$ constitutes the image of $\partial\Omega_e$; Ω is its interior. The domain Ω is unbounded in the direction of the σ-axis. Furthermore the boundary has two vertical asymptotes which correspond to the angles θ_A and θ_B at the leading and trailing edge of the airfoil. The flow in the physical plane is uniform at infinity. This point corresponds in (θ, σ)-coordinates to the position $P_\infty : \theta = 0, \sigma = \sigma_\infty > 0$.

Since the boundary $\psi(\theta, \sigma) = 0$ is a priori unknown the problem of finding the flow field remains nonlinear. Yet, in order to solve the problem numerically, one does not need to handle a nonlinear differential equation anymore but instead one iterates the computational boundary. Based on the analysis we outlined in Example 32, one typically solves the Frankl problem on Ω in two stages. One first considers the boundary value problem on the domain Ω with the shaded triangle $\Delta(\alpha, \beta, \gamma)$ removed. Here γ is arbitrarily chosen on the open segment DC and $\overline{\gamma\beta}$, $\overline{\gamma\alpha}$ are characteristic arcs. Dirichlet boundary conditions are imposed consistently with the findings of the previous example. Then one solves a purely hyperbolic equation on $\Delta(\alpha, \beta, \gamma)$ with Dirichlet data on $\overline{\gamma\beta}$, $\overline{\gamma\alpha}$. We refer for further details to (Dautray and Lions 1988-93, vol. 4, p. 14), (Morawetz 1981, pp. 117) and to the references therein.

Example 34 We bring the section to a close with a less common example to illustrate that besides the above well-known types of boundary value problems there are many other classes of equations which can be converted to the framework by Friedrichs. The problem considered is based on Example 8 in (LeSaint 1995). We analyse on the domain $\Omega := [0, Z] \times \Omega_0 \subset \mathbb{R}^3$ the equation

$$A\frac{\partial^2}{\partial x_1^2} v + B_1 \frac{\partial}{\partial x_1} v + B_2 \frac{\partial}{\partial x_2} v + B_3 \frac{\partial}{\partial x_3} v = f, \tag{2.56}$$

where $\operatorname{div} B = 0$ and A is a negative constant. We impose the homogeneous Dirichlet boundary condition $v(0, x_2, x_3) = v(Z, x_2, x_3) = 0$ for $(x_2, x_3) \in \partial\Omega_0$. We also fix v on the set $\partial_-\Omega = \{(x_1, x_2, x_3) \in [0, Z] \times \partial\Omega_0 : B_2\nu_2 + B_3\nu_3 < 0\}$. Solutions of this problem exhibit, in general, boundary layers.

We derive from (2.56) the first-order differential system

$$\partial_1 \begin{pmatrix} B_1 & A \\ A & 0 \end{pmatrix} u + \partial_2 \begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix} u + \partial_3 \begin{pmatrix} B_3 & 0 \\ 0 & 0 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 0 & -A \end{pmatrix} u = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{2.57}$$

Because $\operatorname{div} B = 0$, the system is not accretive. A multiplication of (2.57) with

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \qquad a, b \in C^1(\Omega),$$

leads to a new system of equations for which

$$D^h = \begin{pmatrix} -{}^1\!/_2\, B.\nabla a - A/2\,\partial_1 b & -A/2\,(b + \partial_1 a) \\ -A/2\,(b + \partial_1 a) & -A\,a \end{pmatrix}.$$

Hence we can guarantee the positive definiteness of D^h by choosing $a = 1$ and $b = -(1+x_1)^{-1}$. To impose the boundary conditions, we define the operator

$$J = \pm{}^1\!/_2 \begin{pmatrix} -(1+x_1)^{-1} A + B_1 & 0 \\ 2\,A & 0 \end{pmatrix}, \qquad J = -B_-(\nu)$$

on $x_1 = 0$, $x_1 = Z$ and $(x_2, x_3) \in \partial\Omega_0$, respectively. J satisfies (2.17) and (2.19) and is admissible if A and B_1 are sufficiently smooth.

2.6 Literature Review

To put our results into a larger context we outline selected results from the literature on Friedrichs systems. We focus in particular on endeavours concerning Friedrichs systems on non-smooth domains.

The analysis of Friedrichs systems was initiated by the publication (Friedrichs 1958). As we pointed out already, Friedrichs' primary motivation was to provide a unified treatment for equations of mixed type. In the main part of the publication, Friedrichs considers boundary value problems on smooth domains with the condition that the boundary matrix $B(\nu, x)$ does not change its inertial type, by that we mean that the number of positive and negative eigenvalues of $B(\nu, x)$ is a constant function near the boundary. Here the vector field ν is locally extended into the domain. Moreover, Friedrichs restricts his investigations to homogeneous boundary conditions. In this setting he shows the existence and uniqueness of solutions. For his proofs he adapts ideas from (Friedrichs 1954) and (Lax 1955) to his setting. In the last section of the paper, Friedrichs investigates a number of specific boundary value problems on polygonal domains. He begins with differential operators for which on each face of Ω the matrix $B(\nu)$ is either positive or negative semi-definite. He concludes with case studies of the Tricomi and the Cauchy-Riemann equations.

In the list of contributors to the theory of Friedrichs systems the names of three of Friedrichs' former Ph.D. students appear repeatedly, namely Cathleen Morawetz, Peter Lax and Leonard

Sarason. In Example 32 we already referred to Morawetz' studies (Morawetz 1958) on related energy integral methods for the Frankl equation. Although similar in character, the results in (Morawetz 1958) differ from Friedrichs' work in certain aspects, for example in the domains considered and in the function spaces employed. While Morawetz is able to prove the existence of a weak solution, the paper leaves the question of uniqueness open.

The publication (Lax and Phillips 1960) is concerned with a class of dissipative symmetric operators with maximal boundary conditions. As remarked on page 78, if sufficiently regular, maximal boundary conditions can be imposed in terms of admissible boundary operators. Neglecting a possible rescaling, the definitions of dissipative symmetric and accretive operators are equivalent. Retaining the condition of constant inertial type on each face of a polygonal domain Ω, Lax and Phillips demonstrate existence of a unique solution if at each edge on one of the faces the function $B(\nu, x)$ is either positive or negative definite. The authors also single out a class of so-called unessential points by introducing certain multiplier functions. Although they to do not provide a general theory for unessential points, they gather sufficient results to show that the solution of the Tricomi problem constructed by Morawetz is strong and unique.

A different approach to Friedrichs systems is presented in (Sarason 1962). Here the question of existence and uniqueness is addressed on subdomains of $\mathbb{R}^2$ by locally separating the differential operator into components of hyperbolic, parabolic and elliptic type. By posing additional uniformity constraints on the boundary conditions the author shows that weak solutions with square integrable traces are strong and in consequence unique. Sarason extends his results to a class of cylindrical domains of higher dimension. He also demonstrates the existence of strong solutions on Lipschitz domains which can be approximated in a uniform sense by sets with non-characteristic smooth boundary.

The authors of (Phillips and Sarason 1966) consider a category of Friedrichs systems for which $B(\nu)$ changes its rank at a countable number of points. Since type-changes in $B(\nu)$ typically take place on subsets of $\partial\Omega$ of dimension $n-2$, their results are most valuable for boundary value problems on two-dimensional domains. They call a point x singular if in a neighbourhood of x an inequality fails which, on a global level, characterises accretive operators. They show that if the set of singular points is discrete then a locally maximal accretive extension of $\mathcal{L}$ exists. In a setting based on these local operators the authors are able to show existence and uniqueness of solutions.

Using the method of elliptic regularisation Sarason proved in (Sarason 1967) that provided certain *a priori* bounds are satisfied the solutions of Friedrichs systems are contained in $W^{1,2}(\Omega)^m$. The author carries over his result to boundary value problems on manifolds.

The publication (Peyser 1975) is concerned with Friedrichs systems on the domain $\Omega := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0, x_2 > 0\}$. The author demonstrates that if a bilinear form, dependent on the coefficients of the differential operator $\mathfrak{L}$, is uniformly positive definite on $\partial\Omega$ then the sets of weak and strong solutions coincide. In consequence such a Friedrichs system has a unique solution. Peyser terms the constraints on $\mathfrak{L}$ torsion conditions.

With the exception of the settings covered by the publications named above, the requirement by Friedrichs that $B(\nu)$ is of constant inertial type near the domain boundary remained a necessary part for the proof of uniqueness for Friedrichs systems. In (Rauch 1985) the author weakens the requirement to the condition that $B(\nu)$ does not change its rank on boundary $\partial\Omega$. Jeffrey Rauch, incidently a student of Peter Lax, starts his analysis at a similar point as we in the sense that he begins with the observation that the image of the trace operator maps into $H^{-1/2}(\partial\Omega)$. However he only considers boundary value problems on smooth domains. The main result of the publication is that, under suitable regularity constraints on the boundary value problem, the solution is contained in a weighted Sobolev space $H^s_{\tan}(\Omega)$, in which in comparison to the unweighted space $H^s(\Omega)$ only a weakened regularity condition is imposed in the normal direction near the boundary of the domain.

Applying the method of elliptic regularisation, in (Smoczyński 1987) it is proved that, provided certain dominance criteria hold, weak solutions of Friedrichs systems are contained in $W^{1,2}(\Omega)^m$ and therefore are strong. In contrast to the analysis of (Sarason 1967), which takes a similar approach, the results of Smoczyński also cover rank-changes of $B(\nu)$ which do not affect the dominance criteria. While the proofs are only given for C^∞-domains, Smoczyński indicates that his findings can be transferred to less regular domains which satisfy a curvature condition.

Another technique to analyse type changes of $B(\nu)$ is investigated in (Rauch 1994). Suppose that there is a $(n-2)$-dimensional C^1-submanifold Γ of $\partial\Omega$ on which $\operatorname{rank} B(\nu)$ changes. Rauch's main assumption to prove that weak solutions are strong is that locally there are two C^1-vector subbundles N_{big} and N_{small} such that $N_{\text{big}} \supset \ker B(\nu)$, $N_{\text{big}} \supset \ker N_{\text{small}}$ and that $\ker B(\nu)$ equals N_{big} on one side of Γ and N_{small} on the other side of Γ.

In (Secchi 1998) the author complements the results in (Rauch 1994) with regularity estimates for solutions of Friedrichs systems which have type changes of the kind analysed by Rauch. However Secchi requires in addition that $B(\nu)$ is definite on the one side of Γ and semi-definite on the other side. This condition has been relaxed in (Secchi 2000) where only one-sided definiteness is demanded.

This brief survey of available publications on the theory of Friedrichs systems is by no means complete. For instance we have completely neglected the analysis of Friedrichs systems by means of pseudo-differential operators, cf. (Friedrichs and Lax 1965) and (Sarason 1969/70), or semi-groups in the case of evolution equations, cf. (Bardos 1970). For the sake of brevity we also have not given further details about (Tartakoff 1971/72), (Osher 1973), (Friedrichs 1974), (Bardos and Rauch 1982), (Secchi 1996), which give insight into other aspects of Friedrichs systems, and we did not survey in detail lecture notes or textbooks on the subject such as (Friedrichs 1961) or (LeSaint 1995).

Chapter 3

The Discontinuous Galerkin Finite Element Method

In the two previous chapters we have built up the framework of Friedrichs systems, investigated a number of their properties and illustrated their relationship to other well-known types of boundary value problems. Now we turn to the question of how to find solutions of Friedrichs systems.

While it is in some cases possible and preferable to compute the analytic solution of the system, the alternative approach of finding a numerical approximation instead is often more appropriate. This might be because of the complexity of the problem, or because the numerical computation provides all the information one requires in a suitable form, or indeed because Friedrichs systems appear in intermediate stages of more elaborate numerical computations such as in the numerical solution of a nonlinear systems of PDEs.

A wide range of techniques exist for solving Friedrichs systems numerically. Indeed, already in Friedrichs' original paper we find a section which is devoted to the analysis of a finite difference schemes on rectangular domains, cf. (Friedrichs 1958, p. 346-347). In this dissertation, however, we shall concentrate on Galerkin methods. These are techniques where the approximate solution is obtained by restricting the weak formulation of the boundary value problem to finite-dimensional test and trial spaces. The weak formulation, though based on the description in Section 1.9, usually incorporates a decomposition of the computational domain into subdomains called finite elements. It is assumed that the test and trial space coincide, otherwise one speaks of Petrov-Galerkin methods.

Two important families of Galerkin methods are the discontinuous and the continuous Galerkin finite element methods. Our primary interest are the discontinuous Galerkin finite element methods, or DGFEMs in short. We also call these schemes discontinuous Galerkin methods. Occasionally we make comparisons with continuous Galerkin finite element methods (CGFEMs). For an introduction to CGFEMs for Friedrichs systems we refer to (LeSaint 1973/74), (Johnson 1987), (LeSaint 1995) and to the references therein. We shall not consider the numerical solution of Friedrichs systems by other methods. The reader interested in the construction of a finite volume scheme should consult (Vila and Villedieu 1997); for finite difference schemes (Lees 1961) serves as a starting point.

3.1 The Broken Graph Space

We restrict our attention to boundary value problems of the following type.

BVP 5 *Let Ω satisfy a strong local Lipschitz condition and let $\mathcal{L}$ be accretive. Given the pair of projections $P_J, P_{J^*} \in L^\infty_B(\partial\Omega)^{m\times m}$ we set $J = -B(\nu)\,P_J$. We assume there is a pair of projections $P_{J^\mathsf{H}}, P_{J'} \in L^\infty_B(\partial\Omega)^{m\times m}$ so that $J' = B(\nu)P_{J'}$. In addition we require that the matrices*

$$R(x) := J(x) + {}^1\!/_2\, B(\nu, x), \qquad x \in \partial\Omega,$$

are positive semi-definite and that there is a matrix function $T \in L^\infty_B(\partial\Omega)^{m\times m}$ such that

$$B(\nu) = {}^1\!/_2\,(R + R^\mathsf{H})\,T \tag{3.1}$$

on $\partial\Omega$. Then, according to Theorem 41, the boundary value problem

$$\mathcal{L}u = f, \qquad \mathcal{J}u = g \tag{3.2}$$

has a unique solution u in $W^2_{\mathcal{L},B}(\Omega)$ for $f \in L^2(\Omega)^m$ and $g \in L^2_B(\partial\Omega)$.

As before we sometimes write $\mathcal{J}u$ instead of Ju to emphasise that J has a continuous extension to a boundary operator $\mathcal{J}$.

Let $\mathcal{T} = \{\kappa_1, \kappa_2, \ldots, \kappa_N\}$ be a finite decomposition of Ω into open elements κ_i which also satisfy a strong local Lipschitz condition:

$$\Omega = \bigcup_{i=1}^{N} \overline{\kappa_i}, \qquad \text{and} \qquad i \neq j \Rightarrow \kappa_i \cap \kappa_j = \emptyset.$$

Consider an element $\kappa \in \mathcal{T}$. We abbreviate the interior boundary $\partial\kappa \setminus \partial\Omega$ by $\partial_{\text{int}}\kappa$. As for Ω we denote the outward normal of κ by ν. In order to formulate the discontinuous Galerkin method, we need to equip $\partial_{\text{int}}\kappa$ with a boundary condition. Analogously to the original boundary value problem on Ω, we choose the pair of projections $P_J, P_{J^*} \in L_B^\infty(\partial\Omega)^{m\times m}$ and set $J = -B(\nu)\, P_J$. We require that there is a pair of projections $P_{J^{\mathsf{H}}}, P_{J'} \in L_B^\infty(\partial\Omega)^{m\times m}$ so that $J' = B(\nu)P_{J'}$. In addition we have to ensure that the matrices

$$R(x) := J(x) + {}^1\!/_2\, B(\nu, x), \qquad x \in \partial_{\text{int}}\kappa,$$

are positive semi-definite and that there is a matrix function $T_\kappa \in L_B^\infty(\partial\Omega)^{m\times m}$ such that $B(\nu) = {}^1\!/_2\,(R + R^{\mathsf{H}})\, T_\kappa$ on $\partial_{\text{int}}\kappa$. Then we obtain, for every $\kappa \in \mathcal{T}$, a restricted boundary value problem: Find u such that

$$\mathcal{L}u|_\kappa = f|_\kappa, \quad Ju|_{\partial\kappa\cap\partial\Omega} = Jg|_{\partial\kappa\cap\partial\Omega}, \quad J\,u|_{\partial_{\text{int}}\kappa} = J\,\dot{g}|_{\partial_{\text{int}}\kappa}, \tag{3.3}$$

were, for the moment, $\dot{g}$ is an unspecified function in $L_B^2(\partial_{\text{int}}\kappa)$. The solution of the restricted boundary value problem (3.3) on κ is contained in $W^2_{\mathcal{L},B}(\kappa)$. This motivates us to define the broken graph space

$$W^2_{\mathcal{L},B}(\Omega, \mathcal{T}) := \bigoplus_{\kappa\in\mathcal{T}}^{n} W^2_{\mathcal{L},B}(\kappa).$$

At the boundary $\partial\kappa_i \cap \partial\kappa_j$ between the element κ_i and a neighbour κ_j, a member v of $W^2_{\mathcal{L},B}(\Omega, \mathcal{T})$ has, in general, two distinct traces: one from the restriction $v|_{\kappa_i}$ and one from $v|_{\kappa_j}$. We denote the internal trace $(v|_{\kappa_i})|_{\partial\kappa_i}$ of κ_i by v^+ and the external trace $(v|_{\kappa_j})|_{\partial\kappa_i\cap\partial\kappa_j}$ of κ_i by v^-. Altogether the external trace v^- is composed from the traces of all elements neighbouring κ_i. The difference $v^+ - v^-$ is denoted by $[v]$.

We equip $W^2_{\mathcal{L},B}(\Omega, \mathcal{T})$ with the broken graph norm

$$\|v\|^2_{\mathcal{L},\mathcal{T}} := \sum_{\kappa\in\mathcal{T}} \|\, v|_\kappa \,\|^2_{\mathcal{L},\kappa}\,.$$

In this norm $W^2_{\mathcal{L},B}(\Omega, \mathcal{T})$ is not a complete space. In the next section we will consider another norm on $W^2_{\mathcal{L},B}(\Omega, \mathcal{T})$.

3.2 Definition of the DGFEM

To define the discontinuous Galerkin finite element method we introduce the bilinear form

$$B_{\text{DG}} : W^2_{\mathcal{L},B}(\Omega, \mathcal{T}) \times W^2_{\mathcal{L},B}(\Omega, \mathcal{T}) \to \mathbb{R}, (v, w) \mapsto \langle \mathcal{L}v, w\rangle_\Omega + \langle \mathcal{J}v, w\rangle_{\partial\Omega} + \sum_{\kappa\in\mathcal{T}} \langle \mathcal{J}[v], w^+\rangle_{\partial_{\text{int}}\kappa}$$

and the linear form

$$\ell_{\text{DG}} : W^2_{\mathcal{L},B}(\Omega, \mathcal{T}) \to \mathbb{R}, w \mapsto \langle f, w\rangle_\Omega + \langle \mathcal{J}g, w\rangle_{\partial\Omega}.$$

The trial and test spaces are equal to a finite-dimensional subspace V of $W^2_{\mathcal{L},B}(\Omega,\mathcal{T})$. A discontinuous Galerkin approximation of the solution u of BVP 5 is a function $u_{\mathrm{DG}} \in V$ such that

$$\forall\, w \in V : B_{\mathrm{DG}}(u_{\mathrm{DG}}, w) = \ell_{\mathrm{DG}}(w). \tag{3.4}$$

We need to introduce a compatibility constraint for the boundary conditions of neighbouring elements to ensure that (3.4) has a unique solution. Let κ_j be an element neighbouring κ_i and let J_i and J_j be the boundary conditions of κ_i and κ_j. We then demand that

$$J_i = J_j' \quad \text{and} \quad J_i' = J_j \tag{3.5}$$

on the restriction to $\partial\kappa_i \cap \partial\kappa_j$. If (3.5) holds then

$$\begin{aligned}
&\sum_{\kappa\in\mathcal{T}} \langle \mathcal{J}[v], v^+\rangle_{\partial_{\mathrm{int}}\kappa} + {}^1\!/_2\, \langle B(\nu)\, v^+, v^+\rangle_{\partial_{\mathrm{int}}\kappa} \\
&= {}^1\!/_2 \sum_{\kappa\in\mathcal{T}} \langle \mathcal{J}[v], v^+\rangle_{\partial_{\mathrm{int}}\kappa} + {}^1\!/_2\, \langle B(\nu)\, v^+, v^+\rangle_{\partial_{\mathrm{int}}\kappa} - \langle \mathcal{J}'[v], v^-\rangle_{\partial_{\mathrm{int}}\kappa} - {}^1\!/_2\, \langle B(\nu)\, v^-, v^-\rangle_{\partial_{\mathrm{int}}\kappa} \\
&= {}^1\!/_2 \sum_{\kappa\in\mathcal{T}} \langle \mathcal{J}[v], [v]\rangle_{\partial_{\mathrm{int}}\kappa} + {}^1\!/_2\, \langle B(\nu)\, [v], [v]\rangle_{\partial_{\mathrm{int}}\kappa} = {}^1\!/_2 \sum_{\kappa\in\mathcal{T}} \langle \mathcal{R}[v], [v]\rangle_{\partial_{\mathrm{int}}\kappa}.
\end{aligned}$$

Thus, the bilinear form B_{DG} is positive definite:

$$\forall\, v \in W^2_{\mathcal{L},B}(\Omega,\mathcal{T}) : B_{\mathrm{DG}}(v,v) = \langle D^h\, v, v\rangle_\Omega + \langle \mathcal{R}\, v, v\rangle_{\partial\Omega} + {}^1\!/_2 \sum_{\kappa\in\mathcal{T}} \langle \mathcal{R}\,[v], [v]\rangle_{\partial_{\mathrm{int}}\kappa}.$$

Therefore we can equip $W^2_{\mathcal{L},B}(\Omega,\mathcal{T})$ with the energy norm

$$\|v\|_{\mathrm{DG}} := \|v\|_{\mathrm{DG},\mathcal{T}} := \sqrt{B_{\mathrm{DG}}(v,v)}, \qquad v \in W^2_{\mathcal{L},B}(\Omega,\mathcal{T}).$$

We note that $W^2_{\mathcal{L},B}(\Omega,\mathcal{T})$ is not complete in the energy norm. Occasionally we use the energy scalar product which is the bilinear form

$$\langle v, w\rangle_{\mathrm{DG}} := \langle D^h\, v, w\rangle_\Omega + \langle \mathcal{R}\, v, w\rangle_{\partial\Omega} + {}^1\!/_2 \sum_{\kappa\in\mathcal{T}} \langle \mathcal{R}\,[v], [w]\rangle_{\partial_{\mathrm{int}}\kappa}.$$

Theorem 42 *For each finite-dimensional approximation space V in $W^2_{\mathcal{L},B}(\Omega,\mathcal{T})$ there exists a unique solution u_{DG} of the discontinuous Galerkin finite element method. The solution satisfies the stability estimate*

$$\|u_{\mathrm{DG}}\|_{\mathrm{DG}} \le \gamma^{-1}\, \|f\|_{L^2(\Omega)^m} + \|T\, P_J\, g\|_R \le \gamma^{-1}\, \|f\|_{L^2(\Omega)^m} + C\, \|\, g\|_B, \tag{3.6}$$

where C is a suitable constant.

Proof. Because V is finite-dimensional, the existence and uniqueness of u_{DG} follows from the positive definiteness of B_{DG}. Using

$$B_{\mathrm{DG}}(u_{\mathrm{DG}}, u_{\mathrm{DG}}) = \ell_{\mathrm{DG}}(u_{\mathrm{DG}}) \leq 1/(2\gamma)\|f\|_{\Omega}^2 + \gamma/2\|u_{\mathrm{DG}}\|_{\Omega}^2 + {}^1\!/_2\, \|T\, P_J\, g\|_R^2 + {}^1\!/_2\, \|u_{\mathrm{DG}}\|_R^2,$$

we deduce (3.6). ////

If $J = -B_-(\nu)$ on $\partial\Omega$ then (3.6) simplifies to

$$\|u_{\mathrm{DG}}\|_{\mathrm{DG}} \leq \gamma^{-1}\, \|f\|_{L^2(\Omega)^m} + \sqrt{\langle -B_-(\nu)\, g, g\rangle_{\partial\Omega}}.$$

Remark 6 *Let us consider a Friedrichs system which does not satisfy all of the conditions required in the statement of Theorem 41, like, for instance, Moyer's example. In this setting the definitions of B_{DG} and ℓ_{DG} are still meaningful. Moreover, if the compatibility condition (3.5) is satisfied then B_{DG} is positive definite and therefore induces a norm on $W^2_{\mathcal{L},B}(\Omega, \mathcal{T})$. However, as is seen from Moyer's example, the term $\langle v, v\rangle_R$ might vanish. Therefore for such systems the above proof of the stability estimate (3.6) is not applicable.*

The formulation of the discontinuous Galerkin method with interior boundary conditions satisfying (3.5) is stated in (Johnson, Nävert and Pitkäranta 1984) and (LeSaint 1995). Two questions arise instantaneously: Is it always possible to equip the interior elemental boundaries with boundary conditions satisfying (3.5)? And if there is more than one set of interior boundary conditions which fulfills this requirement as well as the conditions we stated on the basis of Theorem 41, which one should be selected?

Suppose that we set $J = -B_-(\nu)$ on $\partial_{\mathrm{int}}\kappa$ for $\kappa \in \mathcal{T}$. If κ_j is a neighbour of κ_i and ν_i and ν_j are the respective outward normals, then

$$J = -B_-(\nu_i) = B_+(\nu_j) = J'$$

on $\partial\kappa_i \cap \partial\kappa_j$. Therefore, choosing inflow boundary conditions on the interior boundaries is consistent with (3.5). Putting

$$P_J = P_{J^{\mathsf{H}}} = P_-, \quad P_{J'} = P_{J^*} = P_+ \quad \text{and} \quad T_\kappa = P_+ - P_- \tag{3.7}$$

on $\partial_{\mathrm{int}}\kappa$, the remaining conditions on the interior boundary conditions are satisfied. This settles the first question.

Suppose $\dot{J}$ is another permissible set of interior boundary conditions. It then follows from (2.38) that the energy norms defined with J and $\dot{J}$ are equivalent. From this point of view it appears appropriate to always choose inflow boundary conditions on the interior boundaries.

We will therefore focus our attention on this choice. However, where no additional effort is involved we adopt the general setting.

At the end of this section we state, for future reference, the adjoint form of B_{DG}:

$$B_{\mathrm{DG}}(v,w) = \langle v, \mathcal{L}'w\rangle_\Omega + \langle v, \mathcal{J}'w\rangle_{\partial\Omega} + \sum_{\kappa\in\mathcal{T}} \langle v^+, \mathcal{J}'w^+ - \mathcal{J}^*w^-\rangle_{\partial_{\mathrm{int}}\kappa}. \tag{3.8}$$

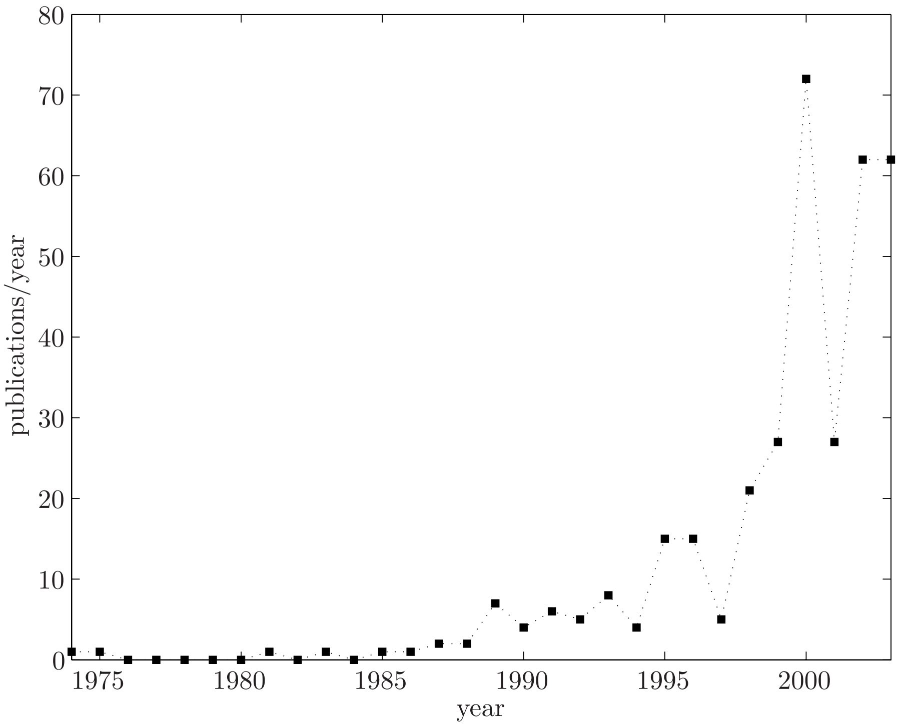
If the interior boundary conditions of B_{DG} are defined via (3.7) then

$$B_{\mathrm{DG}}(v,w) = \langle v, \mathcal{L}'w\rangle_\Omega + \langle v, \mathcal{J}'w\rangle_{\partial\Omega} + \sum_{\kappa\in\mathcal{T}} \langle v^+, B_+(\nu)\,[w]\rangle_{\partial_{\mathrm{int}}\kappa}. \tag{3.9}$$

Notice that the interior boundary integrals in (3.9) vanish if w is continuous.

3.3 Literature Review

Since 1973 the discontinuous Galerkin finite element method has been regarded by the numerical analysis and scientific computing communities with increasing interest. In particular, in the last five years the efforts to gain a better understanding of the method and to widen its range of application have intensified. We substantiate this observation with the figure below in which the numbers of entries per year in MathSciNet are recorded which refer explicitly to the designation "discontinuous Galerkin". In total the database lists 350 entries of this type referring to publications before 2004.



Due to the number of contributions on the discontinuous Galerkin finite element methods we can only give short descriptions of selected publications. For more comprehensive reviews on the subject we refer to (Cockburn 1999), (Cockburn, Karniadakis and Shu 2000) and (Cockburn 2003).

The discontinuous Galerkin finite element method was introduced in 1973 by Reed and Hill in (Reed and Hill 1973a) and (Reed and Hill 1973b). The authors consider the one-velocity neutron transport equation

$$\mu\,\partial_x\psi + \eta\,\partial_y\psi + \sigma\,\psi(x, y, \mu, \eta) = S(x, y, \mu, \eta).$$

According to (LeSaint and Raviart 1974), the function ψ represents the flux of neutrons in the direction (μ, η) at the location (x, y) of the physical domain, σ is the nuclear cross section and S represents scattering, fission and inhomogeneous source terms. While for the one-velocity neutron transport equation the vector (μ, η) is fixed, for the full neutron transport equation (μ, η) is an element of the unit circle $\mathbb{S}^1$. After defining a discontinuous Galerkin finite element method, Reed and Hill compare the DGFEM with a CGFEM by means of numerical experiments. In their examples they highlight the good stability properties the DGFEM and demonstrate that oscillations in the numerical solution, caused by the presence of discontinuities in the exact solutions, are damped rapidly.

We remark that already a few years earlier Nitsche proposed for the numerical solution of the Poisson equation a Galerkin method for which the *exterior* boundary conditions were implemented weakly, cf. (Nitsche 1971). However, in contrast to Reed and Hill, he assumed that the finite-dimensional approximation space is contained in $W^{1,2}(\Omega)$.

Motivated by the findings in (Reed and Hill 1973a), LeSaint and Raviart analysed the discontinuous Galerkin finite element method in (LeSaint and Raviart 1974). The authors consider the discontinuous Galerkin scheme for the one-velocity neutron transport equation. They demonstrate that the scheme has a unique solution and that there is always an ordering of the elements such that the solution can be calculated element by element. Assuming shape-regularity, the authors prove, for meshes with triangular and quadrilateral elements, the suboptimal error bound

$$\|u - u_{\mathrm{DG}}\|_{L^2(\Omega)^m} \leq C\,h^p\,\|u\|_{W^{p+1,2}(\Omega)^m}, \qquad C > 0,$$

for solutions u in $W^{p+1,2}(\Omega)^m$. Here h is the diameter of the finite elements. Based on superconvergence properties on quadrilateral elements, LeSaint and Raviart improve the error bound by one order for solutions in u in $W^{p+2,2}(\Omega)^m \cap W^{p+1,\infty}(\Omega)^m$. In a separate section they also show that the discontinuous Galerkin method is equivalent to a Runge-Kutta scheme when applied to ordinary differential equations.

The authors of (Johnson et al. 1984) apply the discontinuous Galerkin method to a scalar hyperbolic equation, which in contrast to the neutron equation, does not have a constant principal part:

$$\mathcal{L}v = B_k \partial_k v + Cv = f.$$

Moreover, they equip the method with an additional term which originates from the streamline diffusion finite element method; that is their scheme consists of the bilinear form

$$B_{\mathrm{DG}}(v, w) + \sum_{\kappa \in \mathcal{T}} \langle \mathcal{L}\, v, \delta B_k \partial_k w \rangle_\Omega$$

and of the linear form

$$\ell_{\mathrm{DG}}(w) + \sum_{\kappa \in \mathcal{T}} \langle f, \delta B_k \partial_k w \rangle_\Omega.$$

Besides $\delta = 0$ the authors consider the case $\delta = h$. They present a number of results and refer for their proofs to (Johnson and Pitkäranta 1983). We shall return to this publications in a moment. Johnson and his coworkers also consider a formulation of the discontinuous Galerkin method for Friedrichs systems. They introduce the condition that there is a constant $C > 0$ such for all $v, w \in L^2(\partial\kappa)^m$

$$\langle \mathcal{J}\, v, w \rangle_{\partial\Omega} \leq \langle \mathcal{R}\, v, v \rangle_{\partial\Omega} + C \, \|w\|^2_{L^2(\partial\Omega)^m}. \tag{3.10}$$

The authors state that under condition (3.10) the error bound

$$\|u - u_{\mathrm{DG}}\|_{L^2(\Omega)^m} \leq C\, h^{p+1/2} \, \|u\|_{W^{p+1,2}(\Omega)^m}, \qquad C > 0, \tag{3.11}$$

holds for $u \in W^{p+1,2}(\Omega)^m$.

For the proof the reader is referred to (Johnson and Huang Mingyou n.d.), a publication which still had to appear according to the references in (Johnson et al. 1984). However since this paper is not listed in MathSciNet and since the authors do not refer to it in later publications such as (Johnson 1987) it is unclear to us whether (Johnson and Huang Mingyou n.d.) has been published or not. Nevertheless (3.11) can be verified by extending the analysis of the scalar problem in (Johnson et al. 1984) and in (Johnson and Pitkäranta 1983).

While it is sensible to introduce (3.10) for an error analysis of the discontinuous Galerkin method under the assumption that the exact solution of the Friedrichs system is smooth, condition (3.10) appears less suitable if one is interested in investigating problems of low regularity. In view of the next example our assumption that there is a function T such that

$B(\nu) = R^h T$ seems more appropriate. We remark that our assumption is sufficient for (3.10) because

$$\begin{aligned}\langle \mathcal{J} v, w\rangle_{\partial\Omega} &= -\langle v, T P_{J^{\mathsf{H}}} w\rangle_R \le \langle \mathcal{R} v, v\rangle_{\partial\Omega} + \|T P_{J^{\mathsf{H}}}\|_{\mathscr{B}(L^2_B(\partial\Omega),L^2_B(\partial\Omega))} \|w\|_R^2 \\ &\le \langle \mathcal{R} v, v\rangle_{\partial\Omega} + \|T P_{J^{\mathsf{H}}}\|_{\mathscr{B}(L^2_B(\partial\Omega),L^2_B(\partial\Omega))} \|R^h\|_{L^\infty(\partial\Omega)^{m\times m}} \|w\|^2_{L^2(\partial\Omega)^m}.\end{aligned}$$

Example 35 *Representative of a boundary segment of a domain Ω we consider the interval $(-{}^1\!/_2\,, {}^1\!/_2\,)$ on which we define the matrix functions*

$$B(\nu) := \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \quad P_1 := \begin{pmatrix} 0 & 0 \\ x-1 & 1 \end{pmatrix}, \quad P_2 := \begin{pmatrix} 1 & 0 \\ 1-x & 0 \end{pmatrix}.$$

We introduce on $(-{}^1\!/_2\,, {}^1\!/_2\,)$ the matrix function

$$J(x) := \begin{cases} -B(\nu,x)\,P_1 & \text{if } x \ge 0, \\ -B(\nu,x)\,P_2 & \text{if } x < 0. \end{cases}$$

Then J satisfies (2.17) because

$$R(x) := J(x) + {}^1\!/_2\, B(\nu,x) = \operatorname{sign}(x) \begin{pmatrix} x/2 & 0 \\ x(x-1) & x/2 \end{pmatrix} \tag{3.12}$$

is positive definite at $x \neq 0$. Consequently, there is a factorisation $B(\nu) = (R + R^{\mathsf{H}})\,T$. A calculation shows that T is equal to

$$\frac{\operatorname{sign}(x)}{x^2 - 2\,x} \begin{pmatrix} -1 & 1-x \\ x-1 & 1 \end{pmatrix}.$$

Observe that T is not contained in $L^\infty_B(\partial\Omega)^{m\times m}$ due to the pole at the origin. Nevertheless J satisfies (3.10):

$$\begin{aligned}\langle J\,v, w\rangle_I &= \int_{-1/2}^{1/2} (-(1-x)\,v_1 + v_2)\,w_2\,x\,\mathrm{d}x \\ &\le \underbrace{\frac{2}{13}\int_{-1/2}^{1/2} (-(1-x)\,v_1 + v_2)^2\,x\,\mathrm{d}x}_{=:A}\ \underbrace{\frac{13}{2}\int_{-1/2}^{1/2} w_2^2\,x\,\mathrm{d}x}_{=:B}.\end{aligned}$$

Term A is bounded by $\langle R\,v, v\rangle_I$ because

$$\begin{aligned}\frac{(-(1-x)\,v_1+v_2)^2\,x}{R\,v\cdot v} &= 2 + 2\frac{((x-1)^2-1)\,v_1^2}{v_1^2 + 2(x-1)\,v_1\,v_2 + v_2^2} + 2\frac{(x-1)\,v_1\,v_2}{v_1^2 + 2(x-1)\,v_1\,v_2 + v_2^2} \\ &\le 2 + 2\frac{((x-1)^2-1)}{(1-x)}\frac{v_1^2}{v_1^2+v_2^2} + 2\frac{((x-1)^2-1)}{(1-x)}\frac{v_1\,v_2}{v_1^2+v_2^2} \\ &\le 2 + 3 + 3/2.\end{aligned}$$

Moreover, term B is bounded by $13/2\,\|w\|^2_{L^2(\partial\Omega)^m}$*. Thus the boundary conditions satisfy condition (3.10). However the matrices* P_1 *and* P_2 *are a pair of projections which converge in each entry to the projections* P_J *and* P_{J^*} *on the boundary segment* $x = 0$ *in Moyer's example, cf. Examples 25 and 26. Similarly, the boundary operator*

$$\frac{J(x)}{x}$$

converges in each entry to the boundary operator in Moyer's example on this segment as $x \to 0$*. We remark that pointwise the boundary condition* $(J(x)/x)\,u(x) = (J(x)/x)\,g(x)$ *is equivalent to* $J(x)\,u(x) = J(x)\,g(x)$*.*

The above example raises the question whether there are solutions u of Friedrichs systems which satisfy (3.10) and which have poles of the type seen in Moyer's example. In such a situation a number of problems appear. For instance, it is not clear that such a solution u has a finite energy norm $\|u\|_{\mathrm{DG}}$. Moreover, the integration-by-parts formula might not be valid and therefore should not be employed without further verification. Observe that in the construction of Example 35 we used that the boundary segment I has a characteristic point.

We now turn to (Johnson and Pitkäranta 1983). The authors consider the scalar transport equation on a convex domain in $\mathbb{R}^2$. The first part of the publication addresses the proof of (3.11) for quasi-uniform triangulations. The authors then extend the error bound to $L^p(\Omega)$, $2 \le p \le \infty$, for piecewise uniform meshes and constant or linear elements.

Assuming that the angle between the element edges and the direction of transport is uniformly bounded away from zero, Richter deduces in (Richter 1988) for the transport equation an optimal *a priori* error bound for the $L^2(\Omega)^m$-norm on semi-uniform meshes

$$\|u - u_{\mathrm{DG}}\|_{L^2(\Omega)^m} \le C\, h^{p+1}\, \|u\|_{W^{p+1,2}(\Omega)^m}, \qquad C > 0. \tag{3.13}$$

For the definition of semi-uniform meshes we refer to (Richter 1988) and (Reed and Hill 1973a).

Three years later Peterson verified numerically that (3.13) does not hold on general triangulations, cf. (Peterson 1991). Peterson uses an example in which certain edges of the triangulation are aligned with the characteristic direction. He points out that by displacing the mesh the optimal rate of convergence can be recovered.

Bey and Oden extend the analysis of (Johnson and Pitkäranta 1983) for the transport equation to approximation spaces with non-uniform polynomial degree, cf. (Bey and Oden 1996). Assuming that $u \in W^{k+1,2}(\Omega)$ they prove for families of quasi-uniform refinements that there is a constant $C > 0$ such that

$$\|u - u_{\mathrm{DG}}\|^2_{\mathrm{DG}} \le C \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa+1}}{p_\kappa^{2s_\kappa}}\, \max\{1, h_\kappa/p_\kappa^2\} \|u\|^2_{W^{k+1,2}(\kappa)}, \tag{3.14}$$

where h_κ is the diameter of κ and p_κ the polynomial degree of the element. Here the parameter s_κ is equal to $\min\{p_\kappa, k\}$ and the streamline-diffusion parameter δ introduced in (Johnson et al. 1984) is set to h_κ/p_κ^2. Bey and Oden complement their *a priori* analysis with residual-based *a posteriori* error bounds and comprehensive numerical experiments.

Lin and Zhou study in (Lin and Zhou 1993) the convergence of the numerical solution of the discontinuous Galerkin method to the exact solution u of a scalar hyperbolic problem under the assumption that u is only contained in $W^{1/2,2}(\Omega)$, $\Omega \subset \mathbb{R}^2$. To our understanding, the arguments of Lin and Zhou need to be refined in subtle points. For instance, we believe their analysis requires, in fact, the stronger condition that $u \in W^{1/2+\varepsilon,2}(\Omega)$, $\varepsilon > 0$; for details we refer to (Triebel 1992, p. 220).

Falk and Richter analyse in (Falk and Richter 2000) linear symmetric hyperbolic equations with a uniform direction of hyperbolicity. As remarked in the last chapter such equations can be transformed to the framework of Friedrichs. Falk and Richter illustrate how the computational domain can by triangulated so that the discontinuous Galerkin solution can be computed elementwise. Their construction relies on the observation that the boundary matrix $B(\nu)$ of a finite element κ is definite if the angle between the direction of uniform hyperbolicity and of the outward normal ν of κ is sufficiently small.

Houston, Schwab and Süli improved in (Houston et al. 2000b) the *a priori* error bound by Bey and Oden in several respects. They demonstrated that the suboptimality in p in Bey and Oden's bound can be circumvented by choosing the stabilisation parameter δ equal to h_κ/p_κ:

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}}^2 \leq C \sum_{\kappa \in \mathcal{T}} \Big(\frac{h_\kappa}{p_\kappa}\Big)^{2s_\kappa+1} \|u\|_{W^{s_\kappa+1,2}(\kappa)}^2, \tag{3.15}$$

Moreover the error bounds in (Houston et al. 2000b) depend on the regularity of the exact solution u on individual elements and do not require that u is contained globally in $W^{k,2}(\Omega)$. The authors prove the convergence of the discontinuous Galerkin method under p-refinement and show that the convergence is of exponential rate if the solution is patchwise analytic.

In (Houston, Schwab and Süli 2000a) the error bound (3.15) is extended to the discontinuous Galerkin method without streamline diffusion stabilization for scalar operators with elementwise constant coefficients.

Houston, Jensen and Süli analyse in (Houston, Jensen and Süli 2002a) methods of discontinuous Galerkin type for Friedrichs systems with inflow boundary conditions. They construct a parameterised space of Galerkin methods, the so-called t-DG family. The t-DG family is based on the classical discontinuous Galerkin method and incorporates a number of least-squares stabilised schemes. The authors demonstrate that the error bounds in (Houston et al. 2000b) can be transferred to this setting. This thesis contains the results of (Houston et al. 2002a) in generalised form. In particular, we consider the t-DG family with a wider range boundary conditions. Moreover, (Houston et al. 2002a), like the other publications mentioned so far, do assume that the analytical solution of the boundary value problem is contained in a Sobolev space.

Restricting their attention to h-convergence, Brezzi, Marini and Süli give a streamlined and extended analysis of (Houston et al. 2000b) and (Houston et al. 2002a) in (Brezzi, Marini and Süli 2004). In particular, the authors clarify that the internal boundary conditions of the discontinuous Galerkin finite element method can be understood as the composition of an averaging term and a jump term. Based on this observation they identify a range internal boundary conditions for which the discontinuous Galerkin method satisfies an *a priori* error bound optimal in h. We remark that despite the modifications of the internal boundary conditions in (Houston et al. 2002a) and (Brezzi et al. 2004), the internal boundary conditions are conceptually still inflow boundary conditions, i.e. the associations (3.7) hold. Therefore they generalise (3.9) in a different aspect than (3.8) does.

3.4 Convergence of the DGFEM in Broken Graph Spaces

The aim of this section is to show that as the approximation space V is enlarged the discontinuous Galerkin solution converges to the exact solution of the Friedrichs system. Hereby we do not assume that the decomposition $\mathcal{T}$ of Ω is fixed but allow it to be refined. However, since for different domain decompositions the respective discontinuous Galerkin solutions are contained in different broken graph spaces we have to introduce a new function space to study the convergence process.

Let us consider a family of decompositions $\mathscr{T} := (\mathcal{T}_i)_{i\in\mathbb{N}}$ of Ω such that $W^2_{\mathcal{L},B}(\Omega, \mathcal{T}_i) \subset W^2_{\mathcal{L},B}(\Omega, \mathcal{T}_j)$ if $i < j$. We equip the vector space

$$W^2_{\mathcal{L},B}(\Omega, \mathscr{T}) := \bigcup_{i\in\mathbb{N}} W^2_{\mathcal{L},B}(\Omega, \mathcal{T}_i)$$

with the scalar products

$$\langle v, w\rangle_{\mathcal{L},\mathscr{T}} := \langle v, w\rangle_{\mathcal{L},\mathcal{T}_i}, \qquad \langle v, w\rangle_{\mathrm{DG},\mathscr{T}} := \langle v, w\rangle_{\mathrm{DG},\mathcal{T}_i}, \qquad v, w \in W^2_{\mathcal{L},B}(\Omega, \mathcal{T}_i). \tag{3.16}$$

The scalar products are well-defined since if v and w belong to $W^2_{\mathcal{L},B}(\Omega, \mathcal{T}_i)$ and $W^2_{\mathcal{L},B}(\Omega, \mathcal{T}_j)$ then $\langle v, w\rangle_{\mathcal{L},\mathcal{T}_i} = \langle v, w\rangle_{\mathcal{L},\mathcal{T}_j}$ and $\langle v, w\rangle_{\mathrm{DG},\mathcal{T}_i} = \langle v, w\rangle_{\mathrm{DG},\mathcal{T}_j}$. We also introduce the energy norm $\|v\|^2_{\mathrm{DG},\mathscr{T}} := \langle v, v\rangle_{\mathrm{DG},\mathscr{T}}$ and the broken graph norm $\|v\|^2_{\mathcal{L},\mathscr{T}} := \langle v, v\rangle_{\mathcal{L},\mathscr{T}}$.

The next theorem is the central result about the convergence of the discontinuous Galerkin method in graph spaces. In particular, we do not need to require that the interior boundary conditions are defined by (3.7).

Theorem 43 *Let $(V_i)_{i\in\mathbb{N}}$ be a family of subspaces of $W^2_{\mathfrak{L},B}(\Omega,\mathscr{T})$ such that $C_0^\infty(\mathbb{R}^n)^m$ is contained in the closure of the set*

$$W^2_{\mathfrak{L}}(\Omega) \cap \bigcup_{i\in\mathbb{N}} V_i$$

in the broken graph norm and in the energy norm. We assume that the family $(V_i)_{i\in\mathbb{N}}$ is hierarchical, that means that $V_i \subset V_j$ if $i < j$. Let u_i be the discontinuous Galerkin approximation in V_i of the exact solution u of BVP 5. Then the sequence $(u_i)_{i\in\mathbb{N}}$ converges to u in the energy norm. Moreover, we have the bound

$$\|u - u_i\|_{\mathrm{DG}} \leq C \inf\{\|u - v_i\|_{\mathfrak{L},B} : v_i \in W^2_{\mathfrak{L}}(\Omega) \cap V_i\}, \tag{3.17}$$

where C is a constant independent of u and i.

Proof. Since u is contained in $W^2_{\mathfrak{L},B}(\Omega)$ we can find a $v \in C_0^\infty(\mathbb{R}^n)^m$ such that

$$\|v - u\|_{\mathfrak{L},\mathscr{T}} + \|v - u\|_{\mathrm{DG},\mathscr{T}} \leq C_1 \|v - u\|_{\mathfrak{L},B} \leq \varepsilon,$$

where C_1 is a suitable constant independent of u and v. According to the hypotheses there is an $i \in \mathbb{N}$ and a $v_i \in W^2_{\mathfrak{L}}(\Omega) \cap V_i$ such that

$$\|v - v_i\|_{\mathfrak{L},\mathscr{T}} + \|v - v_i\|_{\mathrm{DG},\mathscr{T}} \leq \varepsilon.$$

By Galerkin orthogonality we have that

$$B_{\mathrm{DG}}(v_i - u_i, v_i - u_i) = -B_{\mathrm{DG}}(u - v_i, v_i - u_i).$$

Using that $\mathfrak{J}[u - v_i]$ vanishes on interior boundaries, we deduce

$$\|v_i - u_i\|^2_{\mathrm{DG}} = -\langle \mathfrak{L}(u - v_i), v_i - u_i\rangle_\Omega - \langle \mathfrak{J}(u - v_i), v_i - u_i\rangle_{\partial\Omega}.$$

Thus there is a constant C_2 such that

$$C_2 \|v_i - u_i\|_{\mathrm{DG},\mathscr{T}} \leq \|\mathfrak{L}(u - v_i)\|_\Omega + \|T(P_J)(u - v_i)\|_{R,\partial\Omega}. \tag{3.18}$$

With

$$\forall\, w \in L^2_B(\partial\Omega) : \|w\|_{R,\partial\Omega} \leq C_3 \|w\|_{B,\partial\Omega},$$

we conclude that

$$\|u - u_i\|_{\mathrm{DG},\mathscr{T}} \le (2 + C_2^{-1} + C_2^{-1} C_3 \|TP_J\|_{L_B^\infty(\partial\Omega)^{m\times m}})\,\varepsilon$$

and that $(u_i)_{i\in\mathbb{N}}$ converges to u. The bound (3.17) follows from the triangle inequality and formula (3.18). ////

Corollary 9 *Under the above hypotheses the sequence $(u_i)_{i\in\mathbb{N}}$ converges to u in $L^2(\Omega)^m$.*

LeSaint proves in (LeSaint 1995) a corresponding result for *continuous* Galerkin finite element methods for Friedrichs systems. However to show strong convergence in the $L^2(\Omega)^m$-norm he requires that the solution of the Friedrichs system is strong, i.e. a solution of BVP 4. Under less restrictive conditions he demonstrates that a subsequence of $(u_i)_{i\in\mathbb{N}}$ converges weakly to u in $L^2(\Omega)^m$.

We now restrict our attention to the important case that κ is an affine image of a fixed master element $\hat{\kappa}$, i.e. $\kappa = F_\kappa(\hat{\kappa})$ for all $\kappa \in \mathcal{T} \in \mathscr{T}$, where F_κ is an injective affine mapping and where $\hat{\kappa}$ is either the open unit simplex or the open unit hypercube in $\mathbb{R}^m$. We denote by $\mathcal{P}_{p_\kappa}(\hat{\kappa})$ the space of polynomials on $\hat{\kappa}$ with total degree p_κ. If $\hat{\kappa}$ is the hypercube then we also consider the space $\mathcal{Q}_{p_\kappa}(\hat{\kappa})$ of tensor-polynomials on $\hat{\kappa}$ with degree p_κ in each coordinate direction.

Let p be the mapping which associates to each κ in $\mathcal{T}$ an index $p_\kappa \in \mathbb{N}$. Then we introduce the finite element space

$$S^p(\Omega, \mathcal{T}) := \{v \in L^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{R}_{p_\kappa}(\hat{\kappa})\},$$

where $\mathcal{R}_{p_\kappa}(\hat{\kappa})$ is either $\mathcal{P}_{p_\kappa}(\hat{\kappa})$ or $\mathcal{Q}_{p_\kappa}(\hat{\kappa})$.

Under h- and p-refinement, elements of $C_0^\infty(\mathbb{R}^n)^m$ can be arbitrarily well approximated by functions in $S^p(\Omega, \mathcal{T})$ with respect to the energy and broken graph norm. Thus Theorem 43 is applicable and the solutions of the discontinuous Galerkin method converge to the possibly discontinuous solution of the Friedrichs system under consideration with at least the rate specified in (3.17).

3.5 Convergence of the DGFEM in Broken Sobolev Spaces

We now turn to the question how the rate of convergence of $(u_i)_{i\in\mathbb{N}}$ can be classified if the solution is elementwise contained in Sobolev spaces. We restrict our attention hereby to bounds in the energy norm of the discontinuous Galerkin method.

We concentrate in the derivation of *a priori* error bounds on shape-regular decompositions of Ω, that is we assume that there is a parameter σ such that for all $\kappa \in \mathcal{T}$ and all $\mathcal{T}$ in $\mathscr{T}$ the ratio between the radius of the circumcircle R and the inscribed circle r of κ is bounded from above:

$$\frac{R}{r} \leq \sigma.$$

For the extension of the analysis to anisotropic meshes we refer to (Georgoulis 2003).

Let k be a mapping which assigns to every κ in $\mathcal{T}$ an index $k_\kappa \in \mathbb{N}$. Then the broken Sobolev space of regularity k is the function space

$$W^{k,q}(\Omega, \mathcal{T}) := \{v \in L^q(\Omega) : v|_\kappa \in W^{k_\kappa,q}(\kappa)\},$$

which is normed by

$$\|v\|_{W^{k,q}(\Omega,\mathcal{T})} := \Big(\sum_{\kappa\in\mathcal{T}} \|v\|^q_{W^{k_\kappa,q}(\kappa)}\Big)^{1/q}.$$

The rate of convergence typically observed in numerical experiments is

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \leq C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{s_\kappa - 1/2}}{(p_\kappa+1)^{s_\kappa - 1/2}} |u|_{W^{s_\kappa,2}(\kappa)^m}\,, \tag{3.19}$$

cf. for instance (Houston et al. 2000b). The question arises if this bound can be substantiated theoretically. According to (Jonsson and Wallin 1984) there is a continuous restriction operator from $W^{1/2+\varepsilon,2}(\kappa)^m$ to $W^{\varepsilon,2}(\partial\kappa)^m$ for $\varepsilon > 0$. Thus the $W^{1/2+\varepsilon,2}(\kappa)^m$-norm is stronger than the energy norm. The optimal order of convergence with respect to the $W^{1/2+\varepsilon,2}(\kappa)^m$-norm is

$$\|u - \dot{u}\|_{W^{\varepsilon,2}(\partial\kappa)^m} \leq C(\varepsilon) \frac{h_\kappa^{s_\kappa - 1/2 - \varepsilon}}{(p_\kappa+1)^{s_\kappa - 1/2 - \varepsilon}} |u|_{W^{s_\kappa,2}(\kappa)}, \tag{3.20}$$

where $\dot{u}$ is a suitable element $S^p(\Omega, \mathcal{T})$. Clearly, the optimal rate of the elementwise contribution to the energy norm is at least that of (3.20), $\varepsilon > 0$. However, within the class of Friedrichs systems we consider, there are certain examples for which the optimal rate is higher. For instance, if $\mathcal{L}$ is the identity operator $\mathcal{L}v = v$ then the energy norm coincides with the $L^2(\Omega)^m$-norm which has the optimal convergence rate

$$\|u - u_\pi\|_{L^2(\Omega)^m} \leq C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{s_\kappa}}{(p_\kappa+1)^{s_\kappa}} |u|_{W^{s_\kappa,2}(\kappa)^m}\,. \tag{3.21}$$

Since the energy norm is always stronger than or equivalent to the $L^2(\Omega)^m$-norm the optimal rate of the energy can never be higher than the one given in (3.21). Yet, we remark that it is rather untypical to observe the rate (3.21) in numerical examples. For instance, in the case $\mathcal{L}v = v$ we observe the higher rate because the first-order part of the operator vanishes. Thus it is a problem which in practice one would not solve with the discontinuous Galerkin method. We shall therefore call an error bound optimal if it recovers the exponents in (3.19).

The *a priori* error bounds in the literature are either based on the L^2-projection or on the so-called H^1-projection which we define later in Theorem 51. While the analysis applying the L^2-projection is suboptimal in p by $p^{3/2}$, the error analysis utilising the H^1-projection is suboptimal in h by $h^{1/2}$ and in p by one order, cf. (Houston et al. 2002b) and (Georgoulis 2003). Only under the requirement that the first-order coefficient B of $\mathcal{L}$ is elementwise constant a sharp estimate in h and p can be found, cf. for example (Houston et al. 2002b, Remark 3.13); we note that an optimal h estimate can be proved without the hypothesis that the first-order coefficient B of $\mathcal{L}$ is elementwise constant.

Here, we shall derive an error bound which is suboptimal in h and p by half an order under the assumption that the underlying meshes have at most one hanging node per element edge and are quasi-uniform. Thereby we reduce the total degree of suboptimality in h and p from $1^1/_2$ to 1 order. For completeness we show at the end of this section that the error analysis based on the L^2- and H^1-projections can be extended from the scalar setting with inflow boundary conditions to that of BVP 5.

We cite Theorem 4.72 from (Schwab 1998) in a modified form.

Theorem 44 *Let $\Omega \subset \mathbb{R}^2$ be a polygon and let $\mathscr{T}$ be a quasi-uniform family of quadrilateral meshes on Ω, with at most one hanging node per edge. Then, for any $u \in W^{2,2}(\Omega)$, there exists a $Pu \in S^p(\Omega, \mathcal{T}) \cap C^1(\Omega)$ such that*

$$\|u - Pu\|^2_{W^{1,2}(\Omega)} \le C \sum_{\kappa \in \mathcal{T}} \max\left\{ \frac{(p_\kappa - s_\kappa + 1)!}{(p_\kappa + s_\kappa - 1)!}, \frac{(p_\kappa - s_\kappa + 2)!}{p_\kappa(p_\kappa + 1)(p_\kappa + s_\kappa - 2)!} \right\} h_\kappa^{2s_\kappa - 2} |u|^2_{W^{s_\kappa,2}(\kappa)}$$

for $2 \le s_\kappa \le p_\kappa + 1$ such that the right-hand side is finite. Here C is independent of p_κ, s_κ, h_κ and $\mathcal{T} \in \mathscr{T}$. If $u \in W^{s_\kappa,2}(\kappa)$ for some $s_\kappa \ge 1$ and $\kappa \in \mathcal{T}$, then as $p \to \infty$

$$\|u - Pu\|^2_{W^{1,2}(\Omega)} \le C(s_\kappa) \sum_{\kappa \in \mathcal{T}} \left(\frac{h_\kappa}{p_\kappa} \right)^{2s_\kappa - 2} |u|^2_{W^{s_\kappa,2}(\kappa)}.$$

Although at the first sight the projection Pu seems to be more suitable for the analysis of continuous finite element methods, it allows us to eliminate the jump terms on interior boundaries of the discontinuous Galerkin method without weakening the error bound.

Theorem 45 *Let $\Omega \subset \mathbb{R}^2$ be a polygon and let $\mathscr{T}$ be a quasi-uniform family of quadrilateral meshes, with at most one hanging node per edge. If the solution u of BVP 5 is contained in*

$W^{2,2}(\Omega)$ *then*

$$\|u - u_{\rm DG}\|_{\rm DG}^2 \le C \sum_{\kappa\in\mathcal{T}} \max\Big\{ \frac{(p_\kappa - s_\kappa + 1)!}{(p_\kappa + s_\kappa - 1)!}, \frac{(p_\kappa - s_\kappa + 2)!}{p_\kappa(p_\kappa+1)(p_\kappa + s_\kappa - 2)!} \Big\}\, h_\kappa^{2s_\kappa - 2}\, |u|^2_{W^{s_\kappa,2}(\kappa)}$$

for $2 \le s_\kappa \le p_\kappa + 1$ *such that the right-hand side is finite. Here* C *is independent of* p_κ, s_κ, h_κ *and* $\mathcal{T} \in \mathscr{T}$. *If* $u \in W^{s_\kappa,2}(\kappa)$ *for some* $s_\kappa \ge 1$ *and* $\kappa \in \mathcal{T}$, *then as* $p \to \infty$

$$\|u - u_{\rm DG}\|_{\rm DG}^2 \le C(s_\kappa) \sum_{\kappa\in\mathcal{T}} \Big(\frac{h_\kappa}{p_\kappa}\Big)^{2s_\kappa-2} |u|^2_{W^{s_\kappa,2}(\kappa)}.$$

Proof. We abbreviate $\eta := u - Pu$ and $\xi := Pu - u_{\rm DG}$. Then

$$\|\xi\|_{\rm DG}^2 = -B_{\rm DG}(\eta,\xi) = \langle \mathcal{L}\eta, \xi\rangle_\Omega + \langle \mathcal{J}\eta, \xi\rangle_{\partial\Omega} \le C\left(\|\xi\|_\Omega + \|\xi\|_R\right)\|\eta\|_{W^{1,2}(\Omega)}$$

demonstrates the result. ////

A similar error bound can be demonstrated for quasi-uniform triangulations with polynomial spaces $\mathcal{P}_{p_\kappa}(\hat\kappa)$. However, on the right-hand side of the error bound the semi-norm $|u|_{W^{s_\kappa,2}(\kappa)}$ needs to be replaced by the norm $\|u\|_{W^{s_\kappa,2}(\kappa)}$, cf. (Schwab 1998, p. 206). We remark that while our error bound is sharper than the H^1-error bound where applicable, it does not cover dimensions higher than two and meshes which are not quasi-uniform.

We now show, for completeness, that the analysis in (Houston et al. 2000a) for scalar problems can be extended to boundary value problems of type BVP 5. The next theorem relates the approximation error of the discontinuous Galerkin method to the distance between the solution u and a suitably chosen projection u_π of u onto the approximation space. We abbreviate

$$\eta := u - u_\pi, \qquad \xi := u_\pi - u_{\rm DG}.$$

Theorem 46 *Let* u_π *be an element of the approximation space* V. *We assume that the internal boundary conditions satisfy (3.7). Then*

$$\|\xi\|_{\rm DG}^2 \le 2\,|\langle \eta, \mathcal{L}'\xi\rangle_\Omega| + \|T(P_{J^*})\eta\|^2_{R,\partial\Omega} + \sum_{\kappa\in\mathcal{T}} \|\eta^+\|^2_{B,\partial_{\rm int}\kappa}. \tag{3.22}$$

Proof. From Galerkin orthogonality we have

$$B(\xi,\xi) = -B(u - u_{\rm DG}, \xi) + B(\xi,\xi) = -B(\eta,\xi).$$

Thus, according to (3.9),

$$\begin{aligned}
&\langle D^h \xi, \xi\rangle_\Omega + \langle \mathcal{R}\,\xi, \xi\rangle_{\partial\Omega} + {}^1\!/_4 \sum_{\kappa\in\mathcal{T}} \langle |B|(\nu)\,[\xi], [\xi]\rangle_{\partial_{\rm int}\kappa} \\
&\quad = -\langle \eta, \mathcal{L}'\xi\rangle_\Omega - \langle \mathcal{J}^*\eta, \xi\rangle_{\partial\Omega} - \sum_{\kappa\in\mathcal{T}} \langle \eta^+, B_+(\nu)\,[\xi]\rangle_{\partial_{\rm int}\kappa} \\
&\quad \le |\langle \eta, \mathcal{L}'\xi\rangle_\Omega| + {}^1\!/_2\,\|T(P_{J^*})\eta\|^2_{R,\partial\Omega} + {}^1\!/_2\,\|\xi\|^2_{R,\partial\Omega} + \sum_{\kappa\in\mathcal{T}} \Big({}^1\!/_2\,\|\eta^+\|_{B,\partial_{\rm int}\kappa} + {}^1\!/_8\,\|[\xi]\|_{B,\partial_{\rm int}\kappa} \Big).
\end{aligned}$$

The result follows now by subtracting ${}^1\!/_2\,\|\xi\|^2_{R,\partial\Omega} + \sum_\kappa {}^1\!/_8\,\|[\xi]\|_{B,\partial_{\rm int}\kappa}$. ////

We use on the reference element the $L^2(\hat\kappa)$-projection $\hat P_{2,p_\kappa}$ of $L^2(\hat\kappa)^m$ onto $\mathcal{Q}_{p_\kappa}(\hat\kappa)^m$. On Ω we utilise the projection

$$P_{2,p} : L^2(\Omega)^m \to S^p(\Omega, \mathcal{T}), v|_\kappa \mapsto (\hat P_{2,p_\kappa}(v \circ F_\kappa)) \circ F_\kappa^{-1}.$$

Where unambiguous we write P_2 and $\hat P_2$ instead of $P_{2,p}$ and $\hat P_{2,p_\kappa}$.

Theorem 47 *Let $\hat\kappa = (-1,1)^n$ and $u \in W^{k,2}(\hat\kappa)$ for an integer $k_\kappa \ge 1$. Let s_κ be a non-negative integer which smaller than or equal to $\min\{p_\kappa+1, k\}$, $p_\kappa \ge 0$. Then*

$$\|u - \hat P_2\, u\|_{L^2(\hat\kappa)} \le C(n) \left(\frac{(p_\kappa + 1 - s_\kappa)!}{(p_\kappa + 1 + s_\kappa)!} \right)^{1/2} |u|_{W^{s_\kappa,2}(\hat\kappa)},$$

where $C(n)$ is a positive real number which only depends on the dimension n.

Proof. We refer to (Houston et al. 2002b) and (Schwab 1998, Theorem 3.11). ////

Corollary 10 *If $u \in W^{k,2}(\kappa)^m$ and $0 \le s_\kappa \le \min\{p_\kappa+1, k\}$, $k_\kappa \ge 1$, $p_\kappa \ge 0$ then*

$$\|u - P_2\, u\|_{L^2(\kappa)^m} \le C(n, s_\kappa, \sigma)\, \frac{h_\kappa^{s_\kappa}}{(p_\kappa+1)^{s_\kappa}}\, |u|_{W^{s_\kappa,2}(\kappa)^m},$$

where σ is the parameter introduced in the definition of shape-regularity.

Proof. The corollary is a consequence of the chain rule and Stirling's formula. ////

In the subsequent analysis we want to select $u_\pi := P_2\, u$. In view of the structure of (3.22) we need to ensure that we have control over the difference $u - P_2\, u$ on the boundaries of the finite elements. The next theorem, which addresses this issue, is cited in modified form from (Houston et al. 2002b).

Theorem 48 *Suppose that $u \in W^{k,2}(\hat{\kappa})$ for some integer $k \geq 1$, and let s_κ be an integer such that $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $k_\kappa \geq 1$, $p_\kappa \geq 0$; then, we have that*

$$\|u - \hat{P}_2\, u\|_{L^2(\partial\hat{\kappa})} \leq C(n)\, \Phi(s_\kappa, p_\kappa)\, |u|_{W^{s_\kappa,2}(\hat{\kappa})}$$

where $\Phi(s_\kappa, p_\kappa)$ is defined by

$$\Phi(s_\kappa, p_\kappa) = \frac{1}{\sqrt{2p+1}} \left[\left(\frac{(p_\kappa + 1 - s_\kappa)!}{(p_\kappa - 1 + s_\kappa)!} \right)^{1/2} + \left(\frac{(p_\kappa + 2 - s_\kappa)!}{(p_\kappa + s_\kappa)!} \right)^{1/2} \right] + \left(\frac{(p_\kappa + 1 - s_\kappa)!}{(p_\kappa + 1 + s_\kappa)!} \right)^{1/4} \left(\frac{(p_\kappa + 2 - s_\kappa)!}{(p_\kappa + s_\kappa)!} \right)^{1/4} + \left(\frac{(p_\kappa + 1 - s_\kappa)!}{(p_\kappa + 1 + s_\kappa)!} \right)^{1/2}.$$

Corollary 11 *If $u \in W^{k,2}(\kappa)$ and $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k\}$, $k_\kappa \geq 1$, $p_\kappa \geq 0$ then*

$$\|u - \hat{P}_2\, u\|_{L^2(\partial\kappa)} \leq C(n, s_\kappa, \sigma)\, \frac{h_\kappa^{s_\kappa - 1/2}}{(p_\kappa + 1)^{s_\kappa - 1/2}}\, |u|_{W^{s_\kappa,2}(\kappa)}.$$

Proof. The corollary is again a consequence of Stirling's formula. ////

To bound the term $\mathcal{L}'\xi$ in (3.22) we need to introduce an inverse inequality.

Theorem 49 *Let $\hat{\kappa}$ be the unit hypercube in $\mathbb{R}^n$. Then polynomials v in $\mathcal{Q}_{p_\kappa}(\hat{\kappa})$ satisfy the estimate*

$$|v|_{W^{1,2}(\hat{\kappa})} \leq C p_\kappa^2 \|v\|_{L^2(\hat{\kappa})}, \qquad p_\kappa \geq 1,$$

where C is a constant independent of p_κ.

Proof. The inequality is proved in (Schwab 1998, p. 208). ////

We have now collected all supporting results to prove the *a priori* bound for the discontinuous Galerkin method under the assumption that the approximation space consists of piecewise tensor product polynomials.

Theorem 50 *Let $\mathscr{T}$ be a family of shape-regular decompositions of a polyhedron Ω such that each element of the decompositions is an affine image of the hypercube in $\mathbb{R}^n$. Then there is a constant $C > 0$ which only depends on the dimension n, the parameter of shape-regularity σ and the differential operator $\mathcal{L}$ such that for all $u \in W^{k,2}(\Omega, \mathcal{T})$ and discontinuous Galerkin solutions $u_{\mathrm{DG}} \in S^p(\Omega, \mathcal{T})$, $\mathcal{T} \in \mathscr{T}$, the following bound is satisfied:*

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \leq C \sum_{\kappa \in \mathcal{T}} h_\kappa^{s_\kappa - 1/2} \left[\Phi(s_\kappa, p_\kappa) + (1 + p_\kappa^2) \left(\frac{(p_\kappa + 1 - s_\kappa)!}{(p_\kappa + 1 + s_\kappa)!} \right)^{1/2} \right] |u|_{W^{s_\kappa,2}(\kappa)}.$$

Here Φ is defined as in Theorem 48 and $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $k_\kappa \geq 1$, $p_\kappa \geq 1$

Proof. It follows directly from Theorem 48 that

$$\|T(P_{J^*})\,\eta\|^2_{R,\partial\Omega} + 2\sum_{\kappa\in\mathfrak{T}} \|\eta^+\|^2_{B,\partial_{\mathrm{int}}\kappa} \le C \sum_{\kappa\in\mathfrak{T}} h_\kappa^{s_\kappa - 1/2}\ \Phi(s_\kappa, p_\kappa)\, |u|_{W^{s_\kappa,2}(\kappa)^m}\,. \tag{3.23}$$

We turn to integral over Ω in (3.22). Recall that we denote the projection onto the space of elementwise constant functions by $P_{2,0}$. Then, by adapting the constant C in the course of the calculation,

$$\begin{aligned}
\langle \eta, \mathfrak{L}'\xi\rangle_\Omega &= \sum_{\kappa\in\mathfrak{T}} \int_\kappa \eta_i\,(-B_{ijk}\,\partial_k\xi_j + C_{ji}\,\xi_j)\,\mathrm{d}x \\
&= \sum_{\kappa\in\mathfrak{T}} \int_\kappa \eta_i\,(-(B_{ijk} - P_{2,0}\,B_{ijk})\,\partial_k\xi_j + C_{ji}\,\xi_j)\,\mathrm{d}x \\
&\le C \sum_{\kappa\in\mathfrak{T}} \Big(\|B_{ijk} - P_{2,0}\,B_{ijk}\|_{L^\infty(\kappa)^{m\times m\times n}}\, |\xi|_{W^{1,2}(\kappa)^m} + \|\xi\|_{L^2(\kappa)^m} \Big) \|\eta\|_{L^2(\kappa)^m} \\
&\le C \sum_{\kappa\in\mathfrak{T}} \Big(h_\kappa\, |B_{ijk} - P_{2,0}\,B_{ijk}|_{W^{1,\infty}(\kappa)^{m\times m\times n}}\, \frac{p_\kappa^2}{h_\kappa}\, \|\xi\|_{L^2(\kappa)^m} + \|\xi\|_{L^2(\kappa)^m} \Big) \|\eta\|_{L^2(\kappa)^m} \\
&\le C \sum_{\kappa\in\mathfrak{T}} (1 + p_\kappa^2)\, \|\xi\|_{L^2(\kappa)^m} \|\eta\|_{L^2(\kappa)^m} \\
&\le \gamma/2\, \|\xi\|^2_{L^2(\Omega)^m} + C \sum_{\kappa\in\mathfrak{T}} (1 + p_\kappa^2)^2\, h_\kappa^{2s_\kappa} \left(\frac{(p_\kappa + 1 - s_\kappa)!}{(p_\kappa + 1 + s_\kappa)!} \right) |u|^2_{W^{2_\kappa,2}(\kappa)^m}\,.
\end{aligned} \tag{3.24}$$

We used that the pull-backs of $\partial_k\xi_j$ and η_i to the reference element are $L^2(\hat\kappa)$-orthogonal. Combining (3.23) and (3.24) proves the theorem. ////

Corollary 12 *Under the hypotheses of the above theorem the approximation error of the discontinuous Galerkin method satisfies the error bound*

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \le C \sum_{\kappa\in\mathfrak{T}} \frac{h_\kappa^{s_\kappa - 1/2}}{(p_\kappa + 1)^{s_\kappa - 2}}\, |u|_{W^{s_\kappa,2}(\kappa)^m}\,, \tag{3.25}$$

where $C = C(n, \sigma, \mathfrak{L}, s)$ depends on the dimension n, the parameter of shape-regularity σ, the differential operator $\mathfrak{L}$ and the function of regularity indices s.

We already indicated on page 118 that the proof of Theorem 50 can be improved if the coefficient B of $\mathfrak{L}$ is elementwise constant.

Corollary 13 *Besides the hypotheses of Theorem 50 we assume that the coefficient B of the differential operator $\mathfrak{L}$ is constant on each element $\kappa \in \mathfrak{T} \in \mathscr{T}$. Then the approximation error of the discontinuous Galerkin method satisfies the error bound*

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \le C \sum_{\kappa\in\mathfrak{T}} \frac{h_\kappa^{s_\kappa - 1/2}}{(p_\kappa + 1)^{s_\kappa - 1/2}}\, |u|_{W^{s_\kappa,2}(\kappa)},$$

where $C = C(n, \sigma, \mathfrak{L}, s)$ depends on the dimension n, the parameter of shape-regularity σ, the differential operator $\mathfrak{L}$ and the function of regularity indices s.

Proof. Because B is elementwise constant the scalar product $\langle \eta, \mathfrak{L}'\xi \rangle_\Omega$ vanishes due to L^2-orthogonality. ////

Remark 7 *The error analysis above can be applied without significant modifications to Friedrichs systems which satisfy condition (3.10) by Johnson and his coworkers.*

The *a priori* error bound due to Georgoulis is also valid for boundary value problems of type BVP 5, cf. (Georgoulis 2003). We recall from (Houston et al. 2002b) the following approximation result.

Theorem 51 *Let the reference element $\hat{\kappa}$ be a hypercube in $\mathbb{R}^n$. Suppose that u belongs to $W^{k_\kappa,2}(\kappa)^m$ for $\kappa \in \mathfrak{T}$. Then, there exists a $P_{\mathfrak{Q}} u \in S^p(\Omega, \mathfrak{T})$ such that, for $p_\kappa \geq 1$, $k_\kappa \geq 1$ and $1 \leq s_\kappa \leq \min(p_\kappa + 1, k_\kappa)$, we have*

$$\|u - P_{\mathfrak{Q}} u\|^2_{L^2(\Omega)^m} \leq C \Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa} \frac{1}{p_\kappa(p_\kappa+1)} \Psi(p_\kappa, s_\kappa) |u|^2_{H^{s_\kappa}(\kappa)^m}$$

and

$$|u - P_{\mathfrak{Q}} u|^2_{W^{1,2}(\Omega)^m} \leq C \Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa - 2} \Psi(p_\kappa, s_\kappa) |u|^2_{H^{s_\kappa}(\kappa)^m}$$

and

$$\|u - P_{\mathfrak{Q}} u\|^2_{L^2(\partial\Omega)^m} \leq C \Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa - 1} \frac{\Psi(p_\kappa, s_\kappa)}{\sqrt{p_\kappa(p_\kappa+1)}} |u|^2_{H^{s_\kappa}(\kappa)^m}$$

where

$$\Psi(p_\kappa, s_\kappa) = \frac{(p_\kappa - s_\kappa + 1)!}{(p_\kappa + s_\kappa - 1)!} + \frac{1}{p_\kappa(p_\kappa+1)} \frac{(p_\kappa - s_\kappa + 2)!}{(p_\kappa + s_\kappa - 2)!}, \qquad 0 \leq s_\kappa \leq p_\kappa,$$

and C is a constant independent of h_κ, p_κ and u.

For the proof we choose the above quasi-interpolation operator to derive the *a priori* bound.

Theorem 52 *Let $\mathscr{T}$ be a family of shape-regular decompositions of a polyhedron Ω such that each element of the decompositions is an affine image of the hypercube in $\mathbb{R}^n$. Then there are constants $C_1 = C_1(n, \sigma, \mathfrak{L}) > 0$ and $C_2 = C_2(n, \sigma, \mathfrak{L}, s) > 0$ such that for all $u \in W^{k,2}(\Omega, \mathfrak{T})$ and discontinuous Galerkin solutions $u_{\mathrm{DG}} \in S^p(\Omega, \mathfrak{T})$, $\mathfrak{T} \in \mathscr{T}$, the following bound is satisfied:*

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \leq C_1 \sum_{\kappa \in \mathfrak{T}} h_\kappa^{s_\kappa - 1} p^{1/2} \, \Psi(p_\kappa, s_\kappa)^{1/2} \, |u|_{W^{s_\kappa,2}(\kappa)} \leq C_2 \sum_{\kappa \in \mathfrak{T}} \frac{h_\kappa^{s_\kappa - 1}}{p_\kappa^{s_\kappa - 3/2}} \, |u|_{W^{s_\kappa,2}(\kappa)}$$

Here Ψ is defined as in Theorem 51 and $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $k_\kappa \geq 1$, $p_\kappa \geq 1$.

Proof. We abbreviate

$$u_\pi := P_{\mathcal{Q}} u, \qquad \eta := u - u_\pi, \qquad \xi := u_\pi - u_{\mathrm{DG}}.$$

Then

$$\langle \mathcal{L}\eta, \xi\rangle_\kappa \le \|\xi\|_\kappa \left(\frac{h_\kappa}{2}\right)^{s_\kappa - 1} \Psi(p_\kappa, s_\kappa)^{1/2} |u|_{H^{s_\kappa}(\kappa)^m}$$

and

$$\begin{aligned}\langle \eta, \xi\rangle_{B,\partial\kappa} &\le \|B\|_{L^\infty(\partial\kappa)^{m\times m\times n}} \|\xi\|_{\partial\kappa} \|\eta\|_{\partial\kappa} \le C\, \|B\|_{L^\infty(\partial\kappa)^{m\times m\times n}} \frac{p_\kappa}{h_\kappa^{1/2}} \|\xi\|_\kappa \, \|\eta\|_{\partial\kappa} \\ &\le C\, \|B\|_{L^\infty(\partial\kappa)^{m\times m\times n}} \|\xi\|_\kappa \left(\frac{h_\kappa}{2}\right)^{s_\kappa - 1} p_\kappa^{1/2}\, \Psi(p_\kappa, s_\kappa)^{1/2} |u|_{H^{s_\kappa}(\kappa)^m}\end{aligned}$$

using the inequality $\|\xi\|_{\partial\kappa} < C\, p_\kappa / h_\kappa^{1/2}\, \|\xi\|_\kappa$, cf. (Georgoulis 2003, p. 75). The second inequality follows from Stirling's formula. ////

Remark 8 *In the proof of Theorem 45 and of Theorem 50 the suboptimality of the error analysis originates from the term $\langle \mathcal{L}\eta, \xi\rangle_\Omega$. Understanding this term better is crucial for the construction of an error bound which sharply reflects the rates of convergence observed in numerical experiments.*

The projections $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ defined in Section 1.8 exhibit interesting structural properties of this term. Let u_π be an element of the approximation space V_h and set $\eta := u - u_\pi$ and $\xi := u_\pi - u_{\mathrm{DG}}$. We decompose η and ξ into the components in $\mathrm{Eig}(\mathcal{L}|_{\mathrm{Im}\,\mathcal{E}_\mathcal{L}}, 1)$, $\mathrm{Eig}(\mathcal{L}|_{\mathrm{Im}\,\mathcal{E}_\mathcal{L}}, -1)$ and $W^2_{\mathcal{L},0}(\Omega)$:

$$\eta = \eta_1 + \eta_{-1} + \eta_0, \qquad \xi = \xi_1 + \xi_{-1} + \xi_0,$$

where $\eta_1, \xi_1 \in \mathrm{Eig}(\mathcal{L}|_{\mathrm{Im}\,\mathcal{E}_\mathcal{L}}, 1)$, $\eta_{-1}, \xi_{-1} \in \mathrm{Eig}(\mathcal{L}|_{\mathrm{Im}\,\mathcal{E}_\mathcal{L}}, -1)$ and $\eta_0, \xi_0 \in W^2_{\mathcal{L},0}(\Omega)$. Then

$$\langle \mathcal{L}\eta, \xi\rangle_\Omega = \int_\Omega \left(\begin{pmatrix} D+I & D-I & D-I \\ D+I & D-I & D+I \\ D+I & D-I & \mathcal{L} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_{-1} \\ \eta_0 \end{pmatrix} \right) \cdot \begin{pmatrix} \xi_1 \\ \xi_{-1} \\ \xi_0 \end{pmatrix} \mathrm{d}x. \tag{3.26}$$

In the course of calculating (3.26) we used that

$$\begin{aligned}\textstyle\int_\Omega \mathcal{L}\eta_0 \cdot \xi_1 \,\mathrm{d}x &= \textstyle\int_\Omega \eta_0 \cdot \mathcal{L}'\xi_1 \,\mathrm{d}x &&= \textstyle\int_\Omega \eta_0 \cdot (D+I)\,\xi_1 \,\mathrm{d}x, \\ \textstyle\int_\Omega \mathcal{L}\eta_0 \cdot \xi_{-1} \,\mathrm{d}x &= \textstyle\int_\Omega \eta_0 \cdot \mathcal{L}'\xi_{-1} \,\mathrm{d}x &&= \textstyle\int_\Omega \eta_0 \cdot (D-I)\,\xi_{-1} \,\mathrm{d}x.\end{aligned}$$

Thus the only part in which $\langle \mathcal{L}\eta, \xi\rangle_\Omega$ explicitly contains first-order derivatives is the component $\langle \mathcal{L}\eta_0, \xi_0\rangle_\Omega$. If $\xi_0 \in V_h \cap W^2_{\mathcal{L},0}(\Omega)$ then it is possible to eliminate this term by selecting u_π so that η_0 satisfies the relationship

$$\forall\, v \in V_h \cap W^2_{\mathcal{L},0}(\Omega) : \langle \mathcal{L}\eta_0, v\rangle_\Omega = 0. \tag{3.27}$$

Notice that the bilinear form $\langle \mathcal{L} w, v\rangle_\Omega$ *underlying (3.27) is positive definite on* $V_h \cap W^2_{\mathcal{L},0}(\Omega)$*. In that sense there is mathematical structure available which can be used to incorporate* $\langle \mathcal{L}\eta_0, \xi_0\rangle_\Omega$ *into the error analysis.*

A more significant difficulty when deriving an error bound by virtue of $\mathcal{P}_1$ *and* $\mathcal{P}_{-1}$ *arises from the angles between the components of* ξ*. While* ξ_1*,* ξ_{-1} *and* ξ_0 *are orthogonal to each other in the graph scalar product, we do not have control over the angles between these functions in the* $L^2(\Omega)^m$*-scalar product. So it is in this indirect way that the derivatives in* $\mathcal{L}$ *come into equation (3.26). At the moment we are not able to overcome this problem.*

Finally, we point out that the matrix in (3.26) is skew-symmetric in the off-diagonal entries. This implies, for instance, that $\langle \mathcal{L}\,\xi, \xi\rangle_\Omega$ *is equal to* $D\,\xi\cdot\xi + \xi_1\cdot\xi_1 - \xi_{-1}\cdot\xi_{-1}$ *and does not contain mixed terms.*

We remark that it is possible to expand the term $\langle \eta, \mathcal{L}'\xi\rangle_\Omega$ *analogously, which is part of the adjoint definition (3.8) of* B_{DG}*.*

3.6 Least-Squares DGFEM

We turn our attention to a closely related but distinct topic concerning the numerical solution of Friedrichs systems. In a generalised framework of discontinuous Galerkin methods we investigate the influence of terms of least-squares type on the numerical solution.

We are motivated to study this problem for a number of reasons. For instance, if one considers the error analysis of the DGFEM, one sees that the underlying difficulty in proving optimal error bounds is related to an imbalance in the norms which naturally appear in the analysis of the method, owing to the fact that the symmetric part of B_{DG} does not contain derivatives. In the context of least-squares methods this imbalance can be avoided as we demonstrate below.

Furthermore, by ensuring that the symmetric part of the bilinear form of the Galerkin method includes terms of first-order we can also strengthen the stability bound of the method. In Theorem 42 we obtained control over u_{DG} in terms of the energy norm of the method. However, one would like to have an explicit bound on the size of $\mathcal{L}u_{\mathrm{DG}}$ with respect to a suitable norm as well. Such an enhanced stability bound can be guaranteed in the setting of least-squares stabilised methods.

Finally we point out that while within the *continuous* finite element community least-squares schemes have established a significant position, the application of least-squares methods in the framework of discontinuous Galerkin methods did not attract much attention. This motivates us to assess the performance of least-squares schemes of discontinuous type.

We remark that the current and the next section are an extension of the investigations in (Houston et al. 2002a) and consequently majority of the findings in this part of the thesis result from cooperative work with Professor Endre Süli and Professor Paul Houston.

To introduce least-squares stabilised discontinuous Galerkin finite element methods, we initially restrict our attention to a prototype of these schemes, namely to the LS-DGFEM. For $v, w \in W^2_{\mathcal{L}}(\Omega)$ we consider the bilinear form

$$B_{\text{LS}}(v,w) = \langle Jv, Jw\rangle_{\partial\Omega} + \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa}{p_\kappa}\langle \mathcal{L}v, \mathcal{L}w\rangle_\kappa + \langle B_-(\nu)[v], B_-(\nu)[w]\rangle_{\partial_{\text{int}}\kappa} \tag{3.28}$$

and the linear form

$$\ell_{\text{LS}}(v,w) = \langle Jg, Jw\rangle_{\partial\Omega} + \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa}{p_\kappa}\langle f, \mathcal{L}w\rangle_\kappa.$$

The LS-DGFEM approximation of the exact solution u on the approximation space V_h is the function $u_{\text{LS}} \in V_h$ which satisfies the condition that

$$\forall\, w \in V_h : B_{\text{LS}}(u_{\text{LS}}, w) = \ell_{\text{LS}}(w).$$

The bilinear form B_{LS} is symmetric and positive definite and therefore a scalar product on $W^2_{\mathcal{L}}(\Omega)$. It in particular induces the LS-norm

$$\|v\|_{\text{LS}} = \sqrt{B_{\text{LS}}(v,v)}$$

on the broken graph space. Since $\ell_{\text{LS}}(w) = B_{\text{LS}}(u,w)$, the function u_{LS} is the orthogonal projection of u to V_h in the LS-DGFEM scalar product. Consequently, the method is stable and optimal in the LS-norm. This is even the case if u is not contained in a Sobolev space.

Again we can explore the rate of convergence if the solution u of the boundary value problem is contained in a broken Sobolev space. For the LS-DG method we use the quasi-interpolation operator defined in Theorem 51.

Theorem 53 *Let the reference element $\hat\kappa$ be a hypercube in $\mathbb{R}^n$. Suppose that u belongs to $W^{k_\kappa,2}(\kappa)^m$ for $\kappa \in \mathcal{T}$. Then, there exists a $P_{\mathcal{Q}}u \in S^p(\Omega,\mathcal{T})$ such that, for $p_\kappa \geq 1$, $k_\kappa \geq 1$ and $1 \leq s_\kappa \leq \min(p_\kappa + 1, k_\kappa)$, we have*

$$\|u - P_{\mathcal{Q}}u\|^2_{L^2(\Omega)^m} \leq C\Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa} \frac{1}{p_\kappa(p_\kappa+1)}\Psi(p_\kappa, s_\kappa)|u|^2_{H^{s_\kappa}(\kappa)^m}$$

and

$$|u - P_{\mathcal{Q}}u|^2_{W^{1,2}(\Omega)^m} \leq C\Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa-2} \Psi(p_\kappa, s_\kappa)|u|^2_{H^{s_\kappa}(\kappa)^m}$$

and

$$\|u - P_{\mathrm{Q}}u\|^2_{L^2(\partial\Omega)^m} \le C\Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa - 1} \frac{\Psi(p_\kappa, s_\kappa)}{\sqrt{p_\kappa(p_\kappa+1)}} |u|^2_{H^{s_\kappa}(\kappa)^m}$$

where

$$\Psi(p_\kappa, s_\kappa) = \frac{(p_\kappa - s_\kappa + 1)!}{(p_\kappa + s_\kappa - 1)!} + \frac{1}{p_\kappa(p_\kappa+1)} \frac{(p_\kappa - s_\kappa + 2)!}{(p_\kappa + s_\kappa - 2)!}, \qquad 0 \le s_\kappa \le p_\kappa,$$

and C is a constant independent of h_κ, p_κ and u.

We employ the projection P_{Q} in the proof of the next theorem.

Theorem 54 *Let $\mathscr{T}$ be a family of shape-regular decompositions of a polyhedral domain Ω such that each element of the decompositions is an affine image of the hypercube in $\mathbb{R}^n$. Then, there is a constant $C > 0$ which only depends on the dimension n, the parameter of shape-regularity σ and the differential operator $\mathfrak{L}$ such that for all $u \in W^{k,2}(\Omega, \mathfrak{T})$ and LS-DGFEM solutions u_{LS} in $S^p(\Omega, \mathfrak{T})$, $\mathfrak{T} \in \mathscr{T}$, the following bound is satisfied:*

$$\|u - u_{\mathrm{LS}}\|^2_{\mathrm{LS}} \le C \sum_{\kappa \in \mathfrak{T}} \Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa - 1} L_\kappa \frac{\Psi(p_\kappa, s_\kappa)}{\sqrt{p_\kappa(p_\kappa+1)}} |u|^2_{H^{s_\kappa}(\kappa)}. \tag{3.29}$$

Here Ψ is defined as in the previous theorem and $1 \le s_\kappa \le \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \ge 1$, $k_\kappa \ge 1$. The designation L_κ stands for

$$L_\kappa := \|B\|^2_{L^\infty(\kappa)^{m\times m\times n}} + \|D\|^2_{L^\infty(\kappa)^{m\times m}} + \|J\|^2_{L^\infty(\partial\kappa\cap\partial\Omega)^{m\times m}} + \|B_-(\nu)\|^2_{L^\infty(\partial_{\mathrm{int}}\kappa)^{m\times m}}.$$

Proof. From Theorem 51 and Stirling's formula it follows that (3.29) below holds if u_{LS} is substituted by $P_{\mathrm{Q}}u$. We also have $\|u - u_{\mathrm{LS}}\|_{\mathrm{LS}} \le \|u - P_{\mathrm{Q}}u\|_{\mathrm{LS}}$. ////

Corollary 14 *Under the assumptions of Theorem 54 have the bound*

$$\|u - u_{\mathrm{LS}}\|^2_{\mathrm{LS}} \le C \sum_{\kappa \in \mathfrak{T}} \Big(\frac{h_\kappa}{p_\kappa}\Big)^{2s_\kappa - 1} L_\kappa\, |u|^2_{H^{s_\kappa}(\kappa)} \tag{3.30}$$

where C in addition depends on the choice of s_κ and $p_\kappa \ge 1$, $k_\kappa \ge 1$, $1 \le s_\kappa \le \min(p_\kappa + 1, k_\kappa)$.

As before, a higher rate of convergence can occur if the first-order part of $\mathfrak{L}$ is degenerate. The rate in (3.29) coincides with the rate we obtained in Corollary 13. We remark that for fixed parameters h_κ and p_κ the LS-norm is in general stronger than the energy norm of the original discontinuous Galerkin finite element method. That both norms have the same optimal convergence rate results from coefficient h_κ/p_κ in (3.28). We comment that the coefficient makes the LS-norm mesh dependent. By that we mean that it is not possible to define an extension of the LS-norm to $W^2_{\mathcal{L},B}(\Omega, \mathscr{T})$ in an analogous way to (3.16).

Dropping the coefficient h_κ/p_κ in (3.28) leads to a modified method and a modified rate of convergence in (3.29) which would still be optimal but of lower order. In consequence, we would not be able to guarantee the optimal order of convergence in the boundary terms $\langle Jv, Jw\rangle_{\partial\Omega}$ and $\langle B_-(\nu)[v], B_-(\nu)[w]\rangle_{\partial_{\text{int}}\kappa}$ this way.

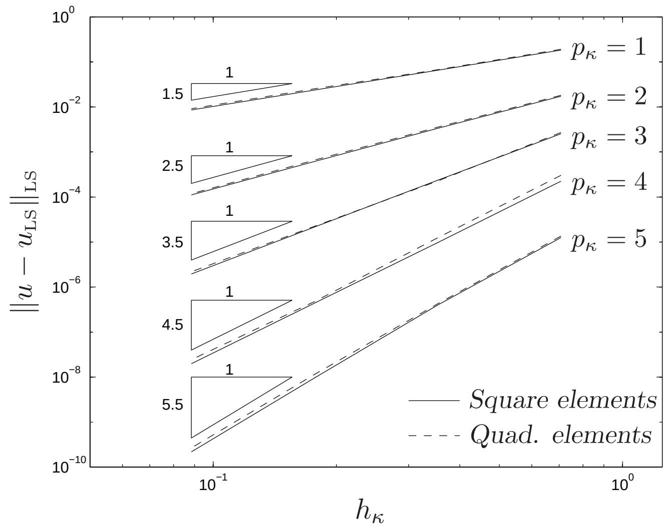
Before we continue the mathematical analysis of least-squares stabilised discontinuous Galerkin methods we assess the practical performance of the LS-DGFEM in a series of numerical experiments. We remark that the computations for Examples 36, 39, 40 and 41 were conducted by Professor Paul Houston. We use the opportunity to also verify the *a priori* error estimates derived in this section.

Example 36 *We consider a scalar advection-reaction problem with smooth analytical solution. To this end, we let* $\Omega = (-1,1)^2$, $B = (8/10, 6/10)$, $C = 1$ *and* $g = 1$. *The function* f *is chosen so that the analytical solution of the associated boundary value problem is*

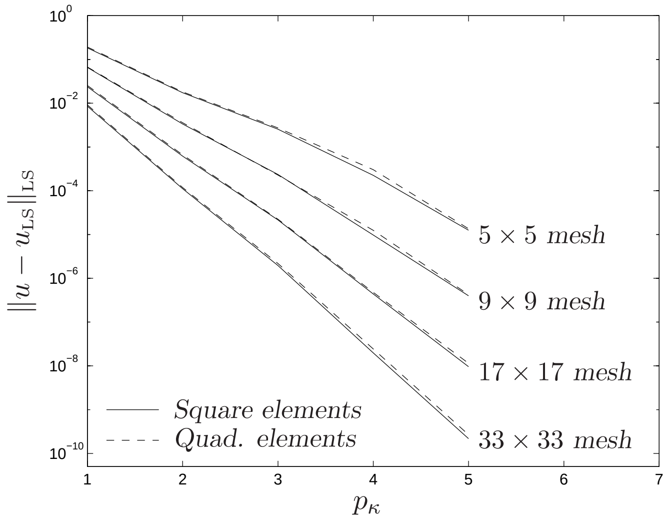
$$u(x,y) = 1 + \sin(\pi(1+x)(1+y)^2/8). \tag{3.31}$$

This is the same test problem as the one that was used in (Bey and Oden 1996) and (Houston et al. 2000b).

We investigate the asymptotic behaviour of the least-squares discontinuous Galerkin method on a sequence of successively finer square and quadrilateral meshes for different p_κ. *In each case the quadrilateral mesh is constructed from a uniform* $N \times N$ *square mesh by randomly perturbing each of the* interior *nodes by up to 10% of the local mesh-size, cf. (Houston et al. 2000b).*



In the above figure we compare the LS-norm of the error with the mesh function h_κ for $1 \le p_\kappa \le 5$. We observe that $\|u - u_{\mathrm{LS}}\|_{\mathrm{LS}}$ converges for fixed p_κ at the rate $\mathcal{O}(h_\kappa^{p_\kappa+1/2})$ to zero as the mesh is refined. This is in agreement with (3.29).



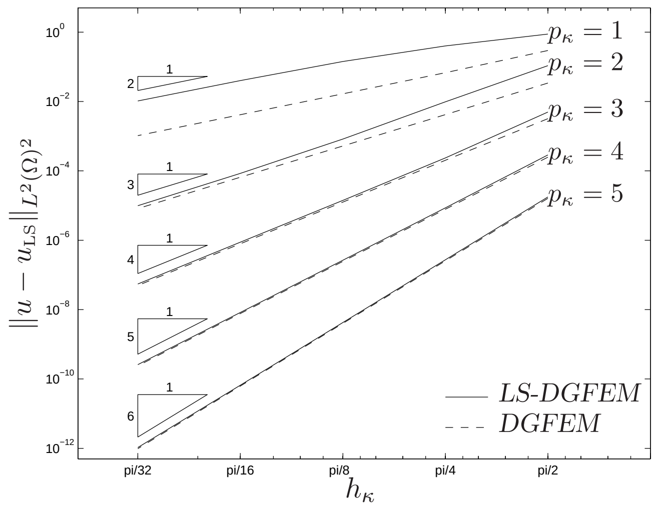
Finally, we investigate the convergence of the LS-DGFEM with p-enrichment on a fixed mesh. In the above figure we plot the least-squares norm of the error against p_κ on four different square and quadrilateral meshes. In each case, we observe that on a linear-log scale, the convergence plots become straight lines as the spectral order p_κ is increased, thereby indicating exponential convergence in p_κ. This finding is consistent with (3.29). Later in this chapter we prove, in a more general setting, that if the exact solution (3.31) is an analytic function then the rate of convergence is always exponential as p_κ increases.

The above figures indicate that the h- and p- convergence of the LS-DGFEM is robust with respect to mesh distortion.

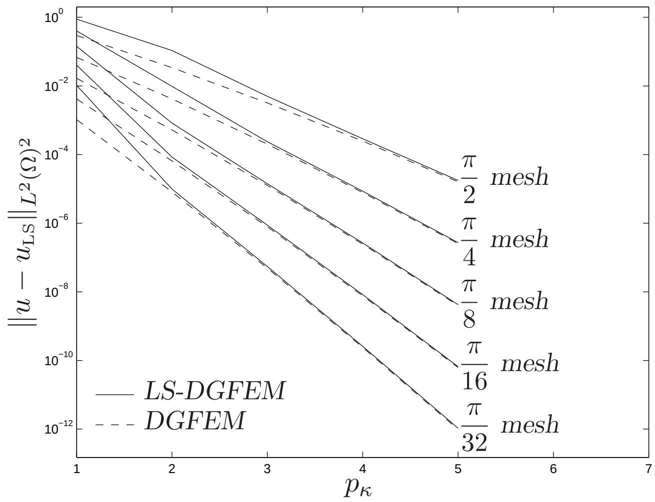
Example 37 *We compare the accuracy of the LS-DGFEM with that of the original DGFEM for a one-dimensional boundary value problem which has a smooth solution. We let $\Omega = (0, \pi)$ and select*

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad g(0) = -g(\pi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The solution to this system of equations is $u = (\cos(x), \sin(x))^{\mathsf{H}}$. The figure below indicates that both LS-DGFEM and DGFEM achieve optimal convergence rates in the L^2-norm for this system. However, while we observe that for $p_\kappa \ge 3$ the errors of the two schemes are virtually identical; this is not so for $p_\kappa \le 2$; indeed, in the latter case DGFEM delivers more *accurate results than LS-DGFEM on each of the uniform subdivisions of the interval* $(0, \pi)$ *considered.*



This can also be seen in the next figure which shows the L^2*-error against the polynomial degree.*



In order to understand the observed behaviour, let us consider a somewhat simpler scalar problem on $\Omega = (-1, 1)$, *with* $\mathcal{T} = \{\Omega\}$, $B = 1$ *and* $C = 0$. *Here the LS-DGFEM approximation can be expressed in closed form in terms of Legendre polynomials as*

$$u_{\mathrm{LS}} = \sum_{i=0}^{p_\kappa - 2} \frac{2i+1}{2} \langle L_i, u \rangle_\Omega \, L_i + \sum_{i=p_\kappa - 1}^{p_\kappa} \frac{1}{2} \langle L_{i-1}, u' \rangle_\Omega \, L_i.$$

Similarly, for DGFEM we obtain

$$u_{\mathrm{DG}} = \sum_{i=0}^{p_\kappa - 1} \frac{2i+1}{2} \langle L_i, u \rangle_\Omega \, L_i + \Big(\sum_{j=p_\kappa}^{\infty} \frac{2j+1}{2} \langle L_j, u \rangle_\Omega \Big) \, L_{p_\kappa}.$$

To derive these identities we used integration by parts, the formula

$$L_n = (L'_{n+1} - L'_{n-1})/(2n+1), \quad n \geq 1,$$

and the orthogonality of Legendre polynomials L_n *in the scalar product of* $L^2(-1,1)$*. Comparing* u_{LS} *with* u_{DG} *we see that for* $p_\kappa \geq 3$ *the first* $p_\kappa - 2$ *terms in the expansions of* u_{LS} *and* u_{DG} *coincide. Due to the fact that* u *is an entire analytic function on* $\mathbb{R}$*, the higher-order Legendre modes decay very quickly and hence the difference between* u_{LS} *and* u_{DG} *is small for large* p_κ*.*

Example 38 *In this example we address the superconvergence properties of the LS-DGFEM. We define superconvergence with respect to* h *as follows: let* $(u_h)_h$ *be a family of numerical solutions generated by a finite element method* $(B_h, \ell_h, \mathcal{T}_h)_h$*. Here* B_h *and* ℓ_h *are respectively the bilinear and linear form of the Galerkin method and* $\mathcal{T}_h$ *is an underlying quasi-uniform mesh. Finally,* h *is a positive scalar representative for the size of elements of* $\mathcal{T}_h$*. Suppose that* $r \in \mathbb{R}$ *is the largest positive number such that there is a constant* C*, depending on* u *but not on* h*, with*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C(u)\, h^r.$$

A family of points $(\xi_h)_h$ *is called superconvergent of order* $\sigma > 0$ *if*

$$|(u - u_h)(\xi_h)| \leq C(u)\, h^{r+\sigma}$$

for all $h > 0$ *and the solutions* u *considered.*

In order to test the LS-DGFEM for superconvergence we introduce the model problem

$$\Omega = (-1,1), \quad u(x) = \frac{1}{x^2+x+3}, \quad B(x) = x^2 + x + 3, \quad f = 0, \quad g(-1) = \frac{1}{3}.$$

We choose this admittedly simple equation because we wish to track the error decay at superconvergence points over several refinement stages, using high precision arithmetic with an 80 digit mantissa. This is necessary as the approximation error at superconvergence points quickly drops below the usual machine epsilon of 16 digits mantissa.

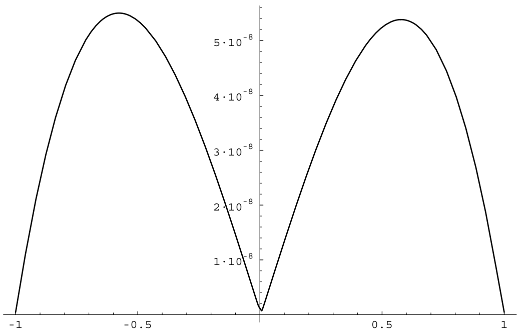
Overall the rate of h*-convergence in the* L^∞*-norm observed for the parameters of* p_κ *and* h *is* $h^{p_\kappa+1}$*. Indeed this is the optimal rate which can be achieved by any polynomial* L^∞*-approximation of our test problem. In order to obtain a clear understanding of the behaviour of the error* $e_\kappa = u|_\kappa - u_{\mathrm{LS}}|_\kappa$ *on the elements* κ*, we pull back the error to the reference element and monitor the average over all* κ *thereof:*

$$\overline{e} : \hat{\kappa} \to \mathbb{R}, x \mapsto \Big(\sum_{\kappa \in \mathcal{T}_h} \frac{(e_\kappa \circ F_\kappa)^2}{|\mathcal{T}|} \Big)^{1/2}.$$

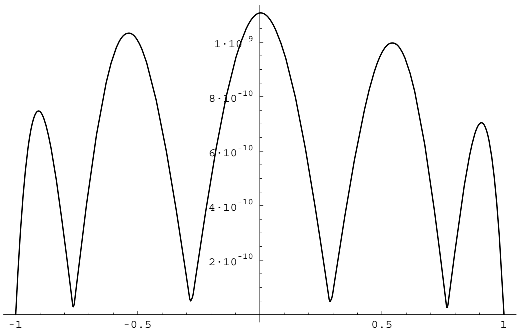
		$h = 2^{-k} \to h = 2^{-(k+1)}$						guess
p	sc. pt.	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	
1	1	2.24325	2.14096	2.07629	2.03976	2.02030	2.01026	2
	2	2.24059	2.14539	2.08051	2.04247	2.02183	2.01107	2
2	1	4.12836	4.06922	4.03588	4.01826	4.00921	4.00462	4
	2	4.05784	4.03082	4.01588	4.00805	4.00406	4.00203	4
	3	4.14355	4.08086	4.04296	4.02215	4.01125	4.00567	4
3	1	6.15645	6.09337	6.05094	6.02662	6.01361	6.00688	6
	2	5.01286	4.99940	4.99699	4.99728	4.99817	4.99890	5
	3	5.12498	5.05608	5.02643	5.01205	5.00565	5.00265	5
	4	6.15309	6.09655	6.05392	6.02850	6.01465	6.00743	6
4	1	8.10292	8.05079	8.02531	8.01263	8.00631	8.00315	8
	2	6.01554	5.99772	6.00092	5.99980	6.00006	6.00000	6
	3	5.99354	5.99097	5.99871	5.99898	5.99970	5.99984	6
	4	6.00543	5.99688	6.00106	6.00011	6.00029	6.00014	6
	5	8.12865	8.06714	8.03455	8.01754	8.00884	8.00444	8
5	1	10.0700	10.0514	10.0298	10.0160	10.0082	10.0042	10
	2	7.00043	6.99520	6.99961	6.99957	6.99980	6.99990	7
	3	7.00728	6.99842	7.00115	7.00033	7.00018	7.00009	7
	4	6.99310	6.99187	6.99814	6.99889	6.99946	6.99974	7
	5	7.00935	7.00025	7.00224	7.00092	7.00049	7.00025	7
	6	10.0888	10.0656	10.0382	10.0205	10.0106	10.0054	10
6	1	12.0710	12.0406	12.0214	12.0110	12.0056	12.0028	12
	2	8.02229	8.00539	7.99902	8.00033	7.99982	8.00002	8
	3	8.01104	8.00035	7.99646	7.99931	7.99933	7.99977	8
	4	8.02713	8.00886	8.00120	8.00123	8.00030	8.00027	8
	5	7.99814	7.99540	7.99422	7.99845	7.99893	7.99957	8
	6	8.02661	8.01046	8.00261	8.00183	8.00068	8.00045	8
	7	12.1012	12.0596	12.0320	12.0166	12.0084	12.0043	12

Table 1: Pointwise superconvergence rates. The first column lists the polynomial degree, the second column enumerates the superconvergence points. The subsequent columns collect the convergence rates as the elemental diameter is reduced. The last column shows the hypothetical convergence rate indicated by the numerical experiment

This may not reflect the calculation for the individual element but can indicate some general qualitative features of the approximation. Two typical averaged errors are depicted on the graphs in figure below.



(a)



(b)

While the left plot shows $\overline{e}$ *computed in a calculation with quadratics, we see on the right a corresponding result for* $p_\kappa = 5$. *Both graphs exhibit minima near the Gauß-Lobatto points of the respective polynomial degree. We therefore make the hypothesis that the LS-DG method exhibits superconvergence at these locations.*

In Table 1 on p. 132 we confirm this conjecture by listing the convergence rates at the Gauß-Lobatto points for $p_\kappa = 1$ *to* $p_\kappa = 6$ *and* h *between* $1/2$ *and* $1/256$. *In this table strong evidence is given, that at the boundary points* -1 *and* 1 *a convergence rate of* h^{2p_κ} *is attained by the finite element method, while at the interior points a decay of order* $h^{p_\kappa+2}$ *is achieved.*

Example 39 *We assess the practical performance of the LS-DGFEM for a scalar linear advection problem with discontinuous boundary data. We let* $\Omega = (0,2) \times (0,1)$, $B(x,y) = (1 + \sin(\pi y/2), 2)$, $C = 0$, $f = 0$ *and*

$$g(x,y) = \begin{cases} 1, & x = 0,\ 0 \le y \le 1\,, \\ \sin^6(\pi x), & 0 < x \le 1,\ y = 0\,, \\ 0, & 1 \le x \le 2,\ y = 0\,. \end{cases}$$

In Figure 3 we compare the performance of the LS-DGFEM with the standard discontinuous Galerkin method and the streamline-diffusion stabilised discontinuous Galerkin method (SD-DGFEM), cf. (Houston et al. 2000b). In each case, we show the outflow profile along the horizontal edge $0 \le x \le 2$, $y = 1$ *on a* 65×33 *uniform square mesh with discontinuous piecewise bilinear elements* ($p_\kappa = 1$). *We observe that the performance of the DGFEM and the SD-DGFEM are very similar, cf. (Houston et al. 2000b); in each case the smooth hill is very well approximated, with some under-shoots and over-shoots present in the vicinity of* *the discontinuity in the analytical solution. In contrast, the LS-DGFEM is overly-diffusive leading to the excessive smearing of both the discontinuity and the smooth hill present in the analytical solution, cf. Figure 3 (c).*

We remark that the numerical dissipation inherent in the LS-DGFEM is due to the inclusion of the least-squares stabilisation into the interelement jump terms rather than the presence of the least-squares stabilisation in the element-integral terms. Indeed, in Figure 3 (d) we show the solution generated by employing an unsymmetric least-squares discontinuous Galerkin finite element method where the element integral terms are identical to those in the LS-DGFEM while the interelement jump terms and the boundary terms are identical to those arising in the standard DGFEM. Here, we observe that even though this unsymmetric least-squares method is more dissipative than both DGFEM and SD-DGFEM, the excessive smearing inherent in the LS-DGFEM has now been eradicated.

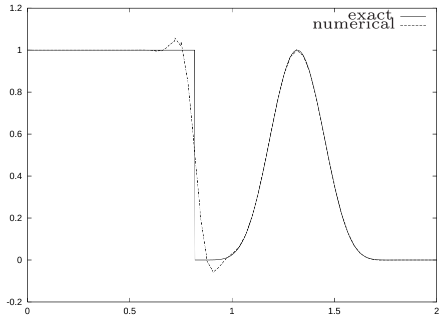
We conclude the example with a comparison of the performance of the LS-DGFEM with the standard, streamline-diffusion stabilised and the Galerkin least-squares finite element methods based on continuous *piecewise polynomials. In Figure 4 we show the outflow profiles of each of the aforementioned schemes; here, we again observe that the LS-CGFEM excessively smears out the solution, though the level of dissipation is very slightly less than when the LS-DGFEM is employed.*

3.7 A general family of discontinuous Galerkin methods

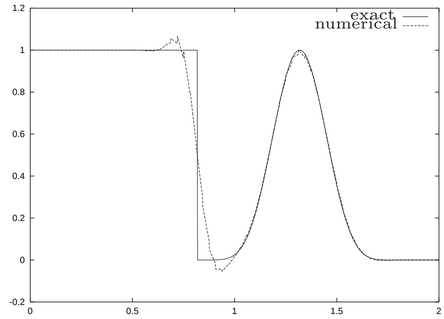
The stability estimate of the LS-DGFEM gives control over the residual $\mathcal{L}u_{\mathrm{LS}}$. In addition we have for this scheme an optimal *a priori* error bound. However Example 39 highlighted that the LS-DGFEM suffers from an excessive amount of numerical dissipation. Consequently, it would be desirable to find a method which combines the advantages of the LS-DGFEM with those of the original discontinuous Galerkin method. In order to conduct the construction of such a method in a systematic framework, we introduce a family of Galerkin methods which is parameterised by suitably chosen weight functions.

We introduce the parameterisation in two stages. First, we generalise the LS-DG scheme, then we combine it with the original discontinuous Galerkin method. We associate to each element $\kappa \in \mathcal{T}$ the matrix functions $M_\kappa : \kappa \to [0,1]^{m\times m}$ and $N_\kappa : \partial\kappa \to [0,1]^{m\times m}$. The generalised least-squares method consists of the bilinear form

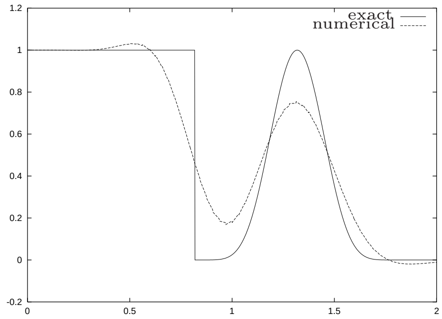
$$B_{\mathrm{LS}}(v,w) = \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa}{p_\kappa} \langle M_\kappa\, \mathcal{L}v, \mathcal{L}w\rangle_\kappa + \langle N_\kappa\, Jv, Jw\rangle_{\partial\Omega\cap\partial\kappa} + \langle N_\kappa\, B_-(\nu)[v], B_-(\nu)[w]\rangle_{\partial_{\mathrm{int}}\kappa}$$



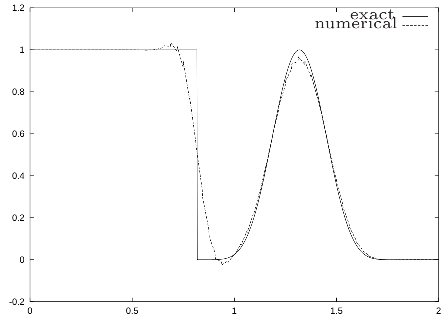
(a) original DGFEM



(b) SD-DGFEM

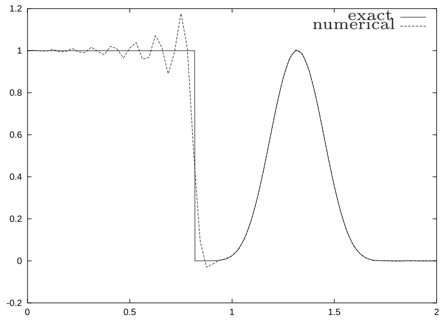


(c) LS-DGFEM

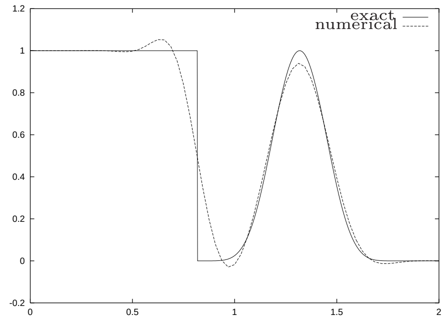


(d) unsymmetric LS-DGFEM

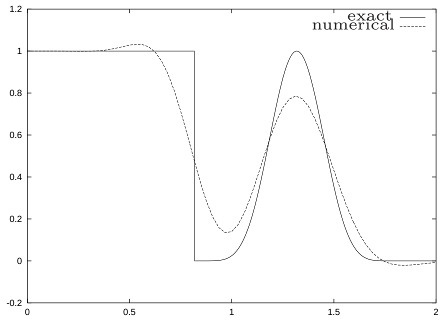
Figure 3: discontinuous Galerkin methods.



(a) standard CGFEM



(b) SD-CGFEM



(c) least-squares CGFEM

Figure 4: continuous Galerkin methods.

and the linear form

$$\ell_{\mathrm{LS}}(w) = \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa}{p_\kappa} \langle M_\kappa\, f, \mathcal{L} w\rangle_\kappa + \langle N_\kappa\, Jg, Jw\rangle_{\partial\Omega}.$$

The fully parameterised family of methods we consider incorporates terms of the original DGFEM by means of a convex combination. Let t be contained in the interval $[0, 1]$. Then the bilinear form B_t is defined by

$$B_t = t\, B_{\mathrm{DG}} + (1 - t)\, B_{\mathrm{LS}}.$$

Similarly, the linear form ℓ_t is defined by the identity

$$\ell_t = t\, \ell_{\mathrm{DG}} + (1 - t)\, \ell_{\mathrm{LS}}.$$

A t-DGFEM approximation of BVP 5 is a function $u_t \in V_h$ such that

$$\forall\, w \in V_h : B_t(u_t, w) = \ell_t(w).$$

If $t = 1$ then the t-DGFEM coincides with the original discontinuous Galerkin method. If $M_\kappa = I$, $N_\kappa = I$ and $t = 0$ for all $\kappa \in \mathcal{T}$ then the t-DGFEM is equal to the original LS-DGFEM.

However, if $M_\kappa = 0$, $N_\kappa = 0$ and $t = 0$ for all $\kappa \in \mathcal{T}$ then B_t and ℓ_t are identical to zero. Therefore we have to further restrict the set of admissible weights in order to guarantee a working Galerkin method. In view of the subsequent analysis the following conditions appear natural: we assume that there are constants $\mu \in (0, t)$ and $\dot{\mu} \geq 0$ such that

$$\begin{aligned} &\text{for } x \in \kappa &&: \quad M_\kappa(x) \geq \mu\, I, \\ &\text{for } x \in \partial_{\mathrm{int}}\kappa &&: \quad B_-(\nu)\, N_\kappa(x)\, B_-(\nu) \geq \frac{t - \mu}{2(1 - t)}\, B_-(\nu), \\ &\text{for } x \in \partial\Omega \cap \partial\kappa &&: \quad \dot{\mu} R \geq J^{\mathsf{H}}(x)\, N_\kappa(x)\, J(x) \geq \frac{\mu - t}{1 - t}\, R. \end{aligned} \tag{3.32}$$

There are two observations to be made. Firstly, condition (3.32) implicitly demands that $t > 0$ and that M_κ is positive definite, i.e. we exclude here the original DGFEM and LS-DGFEM. Secondly, from (3.32) it does not follow that the weight N_κ is positive semi-definite. Indeed, we may select $N_\kappa = 0$ for $\kappa \in \mathcal{T}$.

Let $v \in W^2_{\mathcal{L},B}(\Omega, \mathcal{T})$. Then

$$\begin{aligned} B_t(v,v) = &\sum_{\kappa\in\mathcal{T}} t\,\langle D^h\, v, v\rangle_\Omega + (1-t)\,\frac{h_\kappa}{p_\kappa}\,\langle M_\kappa\,\mathcal{L}v, \mathcal{L}v\rangle_\Omega \\ &+ \sum_{\kappa\in\mathcal{T}} {}^1\!/_2\,\langle(-B_-(\nu))[v], (t\,I + 2\,(1-t)\,N_\kappa(-B_-(\nu)))[v]\rangle_{\partial_{\mathrm{int}}\kappa} \\ &+ \sum_{\kappa\in\mathcal{T}} t\,\langle R\,v, v\rangle_{\partial\Omega\cap\partial\kappa} + (1-t)\,\langle N_\kappa\,J\,v, J\,v\rangle_{\partial\Omega\cap\partial\kappa} \\ \geq &\sum_{\kappa\in\mathcal{T}} t\,\gamma\,\|v\|^2_\kappa + (1-t)\,\mu\frac{h_\kappa}{p_\kappa}\,\|\mathcal{L}v\|^2_\kappa + \frac{\mu}{4}\,\|[v]\|^2_{B,\partial_{\mathrm{int}}\kappa} + \mu\|v\|^2_{R,\partial\kappa\cap\partial\Omega}. \end{aligned}$$

Thus if (3.32) holds then B_t is positive definite and defines the norm

$$\|v\|_t := \sqrt{B_t(v,v)}, \qquad v \in W^2_{\mathcal{L},B}(\Omega, \mathcal{T}),$$

which we call the t-norm. Notice that the t-norm is stronger than all other norms on $W^2_{\mathcal{L},B}(\Omega, \mathcal{T})$ we considered so far. Due to the parameter h_κ/p_κ we do not extend the t-norm to $W^2_{\mathcal{L},B}(\Omega, \mathscr{T})$.

Theorem 55 *For each finite-dimensional approximation space V_h there exists a unique t-DGFEM solution u_t. This solution satisfies the stability estimate*

$$\|u_t\|_t \leq C\,(t + (1-t)\,\max_{\kappa\in\mathcal{T}} h_\kappa/p_\kappa)\,\|f\|_\Omega + C\,\|g\|_R,$$

where C is a constant depending on γ, μ and $\dot\mu$.

Proof. Existence and uniqueness of u_t follow from the coercivity of B_t. Using

$$\begin{aligned} B_t(u_t, u_t) = \ell_t(u_t) \leq\; &t/(2\gamma)\,\|f\|^2_\Omega + t\,\gamma/2\,\|u_t\|^2_\Omega + t/(2\,\mu)\,\|T\,(P_J)\,g\|^2_R + t\,\mu/2\,\|u_t\|^2_R \\ &+ (1-t)\,\dot\mu^2/(2\,\mu)\,\|g\|^2_R + (1-t)\,\mu/2\,\|u_t\|^2_R, \\ &+ \sum_{\kappa\in\mathcal{T}} \Big((1-t)\,h_\kappa/(2\,\mu\,p_\kappa)\,\|f\|^2_\kappa + (1-t)\,h_\kappa\,\mu/(2\,p_\kappa)\,\|\mathcal{L}u_t\|^2_\kappa\Big), \end{aligned}$$

we deduce the stability bound. ////

We now turn to the convergence properties of the t-DGFEM. In the next theorem we make use of the abbreviations

$$\|B\| := \|B\|_{L^\infty(\Omega)^{m\times m\times n}}, \quad \|N\| := \max_{\kappa\in\mathcal{T}} \|N_\kappa\|_{L^\infty(\partial\kappa)^{m\times m}}, \quad \|M\| := \max_{\kappa\in\mathcal{T}} \|M_\kappa\|_{L^\infty(\kappa)^{m\times m}}.$$

Theorem 56 *Let $\mathscr{T}$ be a shape-regular family of decompositions into quadrilateral elements of a bounded polyhedral domain Ω. Suppose that the solution u of BVP 5 is contained in the broken Sobolev space $W^{k,2}(\Omega,\mathcal{T})^m$, $\mathcal{T}\in\mathscr{T}$. Then*

$$\|u-u_t\|_t^2 \le C\sum_{\kappa\in\mathcal{T}}\Big(\frac{h_\kappa}{2}\Big)^{2s_\kappa-1}\frac{\Psi(p_\kappa,s_\kappa)}{\sqrt{p_\kappa(p_\kappa+1)}}|u|^2_{H^{s_\kappa}(\kappa)^m},$$

where the constant C depends on t, $\|M\|$, $\|N\|$, μ, $\dot\mu$, $\mathcal{L}$ and where $1\le s_\kappa\le\min\{p_\kappa+1,k_\kappa\}$ and $p_\kappa\ge 1$, $k_\kappa\ge 1$.

Proof. Similarly to the error analysis of the original discontinuous Galerkin method we set

$$u_\pi := P_{\mathrm{Q}}u,\qquad \eta := u-u_\pi,\qquad \xi := u_\pi-u_t.$$

Then, by Galerkin orthogonality,

$$B_t(\xi,\xi) = -t\,B_{\mathrm{DG}}(\eta,\xi)-(1-t)\,B_{\mathrm{LS}}(\eta,\xi).$$

We first focus on the term $B_{\mathrm{DG}}(\eta,\xi)$. Recalling (3.7) and $\mathcal{L}'=\mathcal{L}+2D^h$, we deduce that

$$\begin{aligned}B_{\mathrm{DG}}(\eta,\xi)\le{}&\Big(\sum_{\kappa\in\mathcal{T}}\frac{p_\kappa}{h_\kappa}\|\eta\|_\kappa^2\Big)^{1/2}\Big(\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa}\|\mathcal{L}\,\xi\|_\kappa^2\Big)^{1/2}+2\,\|C\|_{L^\infty(\Omega)^{m\times m}}\|\eta\|_\Omega\|\xi\|_\Omega\\&+\|\eta\|^2_{R,\partial\Omega}\,\|T\,(P_{J'})\,\xi\|^2_{R,\partial\Omega}+\Big(\sum_{\kappa\in\mathcal{T}}\|\eta\|^2_{B,\partial_{\mathrm{int}}\kappa}\Big)^{1/2}\Big(\sum_{\kappa\in\mathcal{T}}\|[\xi]\|^2_{B,\partial_{\mathrm{int}}\kappa}\Big)^{1/2}.\end{aligned}$$

The least-squares term satisfies the following bound:

$$\begin{aligned}B_{\mathrm{LS}}(\eta,\xi)\le{}&\|M\|\,\Big(\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa}\|\mathcal{L}\,\eta\|_\kappa^2\Big)^{1/2}\Big(\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa}\|\mathcal{L}\,\xi\|_\kappa^2\Big)^{1/2}\\&+\|N\|\,\|B\|\,\Big(\sum_{\kappa\in\mathcal{T}}\dot\mu\,\|\eta\|^2_{R,\partial\Omega\cap\partial\kappa}\Big)^{1/2}\Big(\sum_{\kappa\in\mathcal{T}}\dot\mu\,\|\xi\|^2_{R,\partial\Omega\cap\partial\kappa}\Big)^{1/2}\\&+\|N\|\,\|B\|\,\Big(\sum_{\kappa\in\mathcal{T}}\|[\eta]\|^2_{B,\partial_{\mathrm{int}}\kappa}\Big)^{1/2}\Big(\sum_{\kappa\in\mathcal{T}}\|[\xi]\|^2_{B,\partial_{\mathrm{int}}\kappa}\Big)^{1/2}.\end{aligned}$$

Combining the last two inequalities we find that there is a constant C dependent on t, $\|M\|$, $\|N\|$, μ, $\dot\mu$ and $\mathcal{L}$ such that

$$\|\xi\|_t^2\le C\sum_{\kappa\in\mathcal{T}}\frac{p_\kappa}{h_\kappa}\|\eta\|_\kappa^2+\frac{h_\kappa}{p_\kappa}\|\mathcal{L}\,\eta\|_\kappa^2+\|[\eta]\|^2_{B,\partial_{\mathrm{int}}\kappa}+\|\eta\|^2_{B,\partial_{\mathrm{int}}\kappa}+\|\eta\|^2_{R,\partial\Omega\cap\partial\kappa}.$$

Now the result follows from Theorem 51. ////

Corollary 15 *Suppose that the hypotheses of the above theorem hold. Then, there is a constant C which depends on t, $\|M\|$, $\|N\|$, μ, $\dot\mu$, $\mathcal{L}$ and s such that*

$$\|u-u_t\|_t^2\le C\sum_{\kappa\in\mathcal{T}}\Big(\frac{h_\kappa}{p_\kappa+1}\Big)^{2s_\kappa-1}|u|^2_{H^{s_\kappa}(\kappa)^m}$$

for $1\le s_\kappa\le\min\{p_\kappa+1,k_\kappa\}$, $p_\kappa\ge1$, $k_\kappa\ge1$.

If u is elementwise analytic we can strengthen the theorem significantly. The argument behind the proof of the next theorem is due to (Houston et al. 2000b).

Theorem 57 *Let $\mathscr{T}$ be a shape-regular family of decompositions into quadrilateral elements of a bounded polyhedral domain Ω and let u be a solution of BVP 5. Suppose that there is a finite decomposition $\mathcal{T}'$ of Ω such that $u|_{\kappa'}$, $\kappa' \in \mathcal{T}'$, is extendible to an analytic vector function in a neighbourhood of κ. Assume in addition that all $\mathcal{T} \in \mathscr{T}$ are a refinement of $\mathcal{T}'$, i.e. for all $\kappa \in \mathcal{T}$ exists a $\kappa' \in \mathcal{T}'$ such that $\kappa \subset \kappa'$. Then the finite element error is bounded by*

$$\|u - u_t\|_t^2 \leq C(u) \sum_{\kappa \in \mathcal{T}_h} \left(\frac{h_\kappa}{2}\right)^{2s_\kappa - 1} p_\kappa^2 e^{-2bp_\kappa} \mathrm{meas}(\kappa),$$

where $1 \leq s_\kappa \leq p_\kappa$, b is a fixed element of $(0,1)$ and $C(u)$ is a constant depending on u.

Proof. Fix a $\kappa \in \mathcal{T}'$ and define for any $y \in \overline{\kappa}$ the closed line segment

$$S_{j,y} := \{x \in \overline{\kappa} : \text{ for } j \neq i : x_i = y_i\}.$$

Since $u|_{S_{j,y}}$ is analytically extendible, it can be transformed into a holomorphic function on a disk $D(x')$ with arbitrary origin $x' \in S_{j,y}$. We use the Cauchy estimate

$$\|\partial_j^{(k)} u_i\|_{L^\infty(D(x'))} \leq \frac{k!}{R^k} \|u_i\|_{L^\infty(D(x'))}, \qquad k \in \mathbb{N};\ i = 1, \ldots, m,$$

where R is the radius of $D(x')$, cf. (Rudin 1987, Theorem 10.26). Since $S_{j,y}$ is compact, we can find $R_{\min} > 0$ and $M > 0$ such that

$$\|\partial_j^{(k)} u_i\|_{L^\infty(S_{j,y})} \leq \frac{k!}{R_{\min}^k} M.$$

Due to the analyticity of u in κ, the values of $R_{\min}$ and M can be chosen so that they change continuously with y. Since κ is compact and $\mathcal{T}'$ is finite we can adjust $R_{\min}$ and M so that one can employ these constants for all κ, y and j simultaneously:

$$\exists\, R_{\min} > 0 \ \exists\, M > 0 \ \forall\, \kappa \in \mathcal{T}' \ \forall\, s \in \mathbb{N} : |u|_{W^{s,\infty}(\kappa)^m} \leq \frac{s!}{R_{\min}^s} M,$$

where we denote by $|\cdot|_{W^{s,\infty}(\kappa)^m}$ the semi-norm over $W^{s,\infty}(\kappa)^m$. Consider $s = \alpha\, p_\kappa + 1 \in \mathbb{N}$ with $\alpha \in (0,1)$. Using a generic constant C, applying Stirling's formula gives

$$\begin{aligned}
\Psi(p_\kappa, s)\, |u|^2_{W^{s,\infty}(\kappa)^m} &\leq 2\, \frac{(p_\kappa - s + 1)!}{(p_\kappa + s - 1)!} \left(\frac{s!\, M}{R_{\min}^s}\right)^2 \\
&\leq C\, \frac{((1-\alpha)p_\kappa)^{(1-\alpha)p_\kappa} e^{-(1-\alpha)p_\kappa}}{((1+\alpha)p_\kappa)^{(1+\alpha)p_\kappa} e^{-(1+\alpha)p_\kappa}}\ \frac{(\alpha p_\kappa + 1)^{2\alpha p_\kappa + 3} e^{-2\alpha p_\kappa - 2}}{R_{\min}^{2\alpha p_\kappa + 2}} \\
&\leq C\, (F(\alpha, 1/R_{\min}))^{p_\kappa}\, p_\kappa^3,
\end{aligned}$$

where

$$F(\alpha,\rho) := \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}} (\alpha\rho)^{2\alpha}.$$

For $\rho > 1$,

$$\min_{0<\alpha<1} F(\alpha,\rho) = \frac{\sqrt{1+\rho^2}-1}{\sqrt{1+\rho^2}+1} = F(\alpha_{\min},\rho) < 1, \qquad \alpha_{\min} = \frac{1}{\sqrt{1+\rho}}.$$

Hence there are $\varepsilon > 0$, $\delta > 0$ such that for all $\alpha' \in (\alpha_{\min}-\varepsilon, \alpha_{\min}+\varepsilon)$ we have $F(\alpha', 1/R_{\min}) < 1-\delta$. Thus for $p_\kappa > 1/\varepsilon$ there is $s' \in \mathbb{N}$ such that $F(s'/p_\kappa, 1/R_{\min}) < 1-\delta$. On the other hand for $p_\kappa \le 1/\varepsilon$ the terms $F(s'/p_\kappa, 1/R_{\min})^p_\kappa$, $s' := 1$, are bounded. Therefore by possibly enlarging C we obtain with $b = |\log(1-\delta)|$, that

$$\Psi(p_\kappa, s')\, |u|^2_{W^{s',\infty}(\kappa)^m} \le C\, e^{-bp}\, p_\kappa^3.$$

If $\kappa_j \in \mathcal{T}$ is subset of $\kappa' \in \mathcal{T}'$

$$|u|^2_{W^{s',2}(\kappa)^m} \le \frac{\text{meas}(\kappa_j)}{\text{meas}(\kappa)}\, |u|^2_{W^{s'\infty}(\kappa)^m}.$$

Summation over κ_j proves the theorem. ////

The key result of the proof is the bound on the term $\Psi(p_\kappa, s_\kappa)\, |u|_{W^{s_\kappa,2}(\kappa)}$. Consequently, the proof can be easily transferred to the respective error bounds of original DGFEM and LS-DGFEM. We omit the details here and refer to (Houston et al. 2000b) and (Houston et al. 2002a).

In a numerical example we compare the performance of a t-method with the original DGFEM and LS-DGFEM.

Example 40 *We consider the one-dimensional wave equation*

$$u_{tt} - c^2\, u_{xx} = 0. \tag{3.33}$$

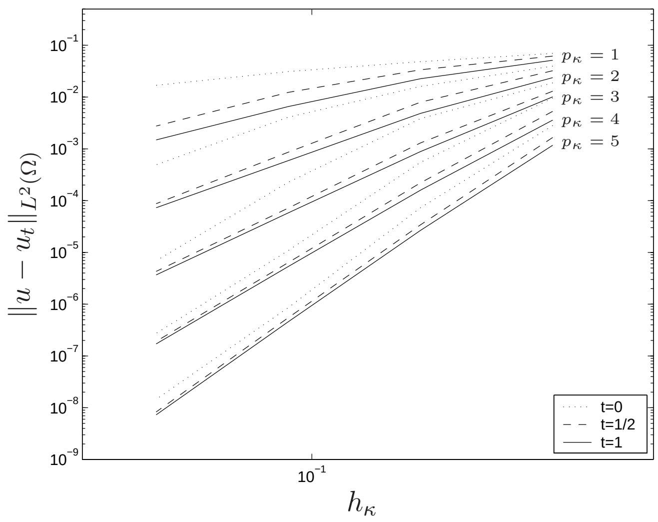
The equation can be rewritten as the symmetric first-order system

$$\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \partial_x \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.34}$$

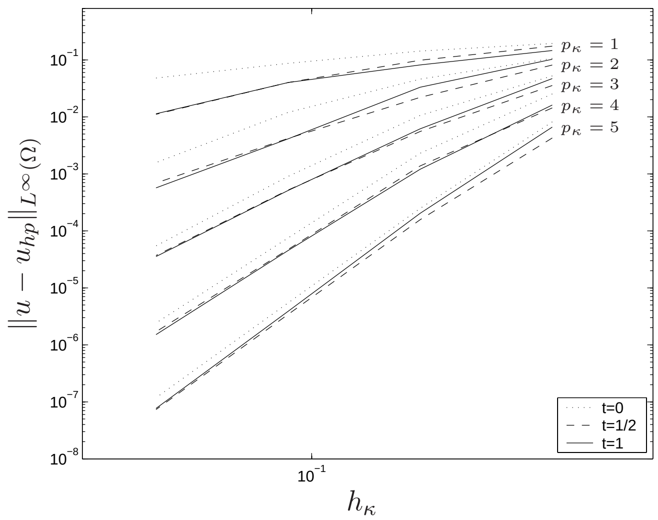
where $u_1 = u_t$ and $u_2 = c\, u_x$. Here we let $\Omega = (0,1) \times (0, {}^1\!/_2)$ and $c = {}^1\!/_2$, with

$$u_1(x_1, 0) = 0 \qquad \text{and} \qquad u_2(x_1,0) = c\, e^{-100(x_1 - {}^1\!/_2)^2},$$

for $0 \le x_1 \le 1$. We remark that according to Example 28 the equation can be transformed into a positive system by applying $u \mapsto \mathrm{e}^{-t}u$. However, since this transformation is a non-singular operation on the discrete approximation space $S^p(\Omega, \mathcal{T})$ as well, we do not need to carry it out explicitly for the computation of a numerical solution.



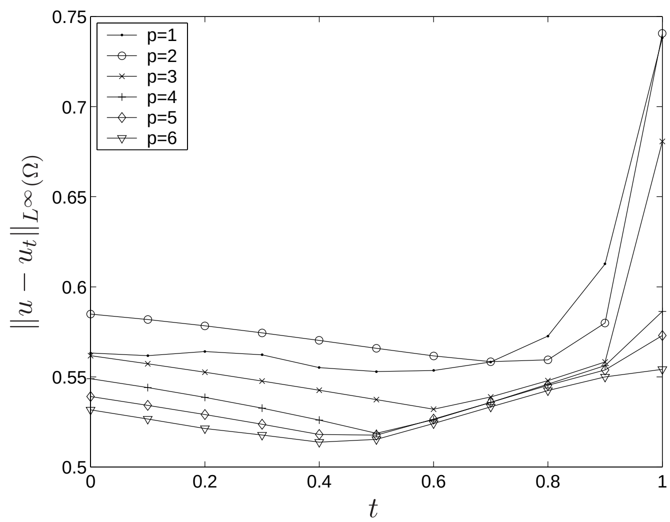
In the above figure we compare the performance of the original DGFEM ($t = 1$) *and the LS-DGFEM* ($t = 0$) *with that of the t-DGFEM with* $M_\kappa = I$, $N_\kappa = I$ *and* $t = {}^1\!/_2$, *using h-refinement on uniform square meshes for* $1 \leq p_\kappa \leq 5$. *For consistency, in each case the error is measured in terms of the* $L^2(\Omega)$*-norm. Here, we observe that the error of the DGFEM is always smaller than for the t-DGFEM with* $t = {}^1\!/_2$; *though the error for this latter scheme is always smaller than for the LS-DGFEM.*



However, if we measure the error with respect to the $L^\infty(\Omega)$*-norm, then we see that the t-DGFEM with* $t = {}^1\!/_2$ *now has clear advantages over both the DGFEM and the LS-DGFEM. Indeed, in the figure above we see that for* $p_\kappa > 1$, *the t-DGFEM with* $t = 1/2$ *now outperforms both DGFEM and LS-DGFEM on some of the meshes employed; this is particularly noticeable on coarser grids.*

The differences between the three schemes in the last example are quite small, owing to the smoothness of the exact solution. Motivated by our findings there, we now consider a boundary value problem with a non-smooth analytical solution.

Example 41 *We investigate the performance of the t-DGFEM on the non-smooth linear advection problem considered in Example 39. To this end, in the below figure we plot the* $L^\infty(\Omega)$*-norm of the error in the t-DGFEM for* $0 \leq t \leq 1$ *on a* 33×17 *uniform square mesh for* $1 \leq p_\kappa \leq 6$ *and* $M_\kappa = I$, $N_\kappa = I$. *Here, we observe that as t increases from* $t = 0$, *corresponding to the LS-DGFEM, the infinity norm of the error first decreases before increasing sharply as t approaches one corresponding to the DGFEM. This implies that for each* p_κ *there is an optimal value of the parameter* $t = t_{p_\kappa}$, *for which the infinity norm of the error in the t-DGFEM is minimised. We remark that the error curves for each* p_κ *are fairly flat around this hypothetical value* t_{p_κ}, *which is good from the practical point of view, since there is a fairly large range of t around* t_{p_κ} *which gives roughly the same error as* t_{p_κ} *itself.*



We postulate that the dissipation present in the t-DGFEM with t close to t_{p_κ} *is large enough to provide additional stabilisation for higher* p_κ, *cf. (Houston et al. 2000b), yet small enough to ensure that the error is not adversely affected by the excessive dissipation present in the LS-DGFEM. Of course, the level of dissipation added into the standard DGFEM by combining it with the LS-DGFEM can be controlled not only by varying t, but also by allowing* M_κ *and* N_κ *to change from element to element in the mesh* $\mathcal{T}$.

The last two examples substantiate that for suitable parameters t, M_κ and N_κ the numerical solution of the t-DGFEM improves or at least recovers the result which would be achieved be the original discontinuous Galerkin method. However, more importantly, we obtain additional control over the stability of the method. This is reflected in our analysis by the improved stability estimate for members of the t-DG family which includes a bound of the term $\mathcal{L}u_t$, cf. Theorem 55. In contrast to other stabilisation techniques such as slope limitation the introduction of least-squares terms does not lead to a violation of Galerkin orthogonality. This characteristic of the least-squares stabilisation makes the t-DG family particularly suitable in connection with duality based *a posteriori* error estimators. We point out that the application of *a posteriori* techniques plays a central role in the numerical solution of Friedrichs systems since the regularity of the exact solutions of these systems can vary significantly throughout the computational domain. Investigations in this direction including the analysis of least-squares stabilised methods have been pursued in (Süli and Houston 2003).

Conclusions

In this dissertation we extended the definition of the discontinuous Galerkin method to a mathematical framework which is suitable for the application to Friedrichs systems with discontinuous solutions. On the basis of this construction we demonstrated that the discontinuous Galerkin solution converges under h- and p-refinement to the exact solution of the boundary value problem. This finding relies on an error bound which relates the discontinuous Galerkin error to the distance between the exact solution and the set of continuous functions within the approximation space.

It was a particular concern to us to include into the analysis type-changes of boundary value problem and corners of the underlying domain. In order to incorporate these features we had to identify a mechanism that rules out solutions to Friedrichs systems which do not satisfy basic requirements such as the validity of the integration-by-parts formula in the classical sense of domain and boundary integrals. The technique we built our analysis on introduces a compatibility condition between the trace operator and the boundary conditions by virtue of a transformation matrix T on the boundary.

In the current literature the analysis of the discontinuous Galerkin method is limited to the study within the scale of Sobolev spaces. We already highlighted in the Introduction that this setting is not suitable for the study of Friedrichs systems with discontinuous solutions. However, numerical computations show that for such problems the discontinuous Galerkin method is a very competitive numerical technique. Indeed, this observation was already emphasised in the original publication by Reed and Hill, cf. (Reed and Hill 1973a). Therefore, we believe that the analysis of the DGFEM in the framework of graph spaces is a valuable addition to the existing literature.

In the course of constructing the mathematical underpinning for the application of DGFEMs to boundary value problems with discontinuous solutions we established a range of results regarding graph spaces, Friedrichs systems and numerical techniques.

We proved the density of smooth functions in graph spaces over domains which satisfy the segment property. This result allowed us to define a trace operator $\mathcal{T}_{\mathcal{L}}$ for graph spaces on Lipschitz domains. We emphasised the distinguished role of $\mathcal{T}_{\mathcal{L}}$ through the factorisation of boundary operators with $\mathcal{T}_{\mathcal{L}}$, a property which determines $\mathcal{T}_{\mathcal{L}}$ and the trace space $W^q_{\mathcal{T}}(\partial\Omega)$ up to homeomorphy. We thereby ensured that conceptually the trace defined by $\mathcal{T}_{\mathcal{L}}$ comprises all the information about graph space functions near the boundary. Since graph spaces are in general not translation invariant, our proofs differ, in part significantly, from the analysis for other function spaces such as $W^{1,q}(\Omega)$ and $W^2(\mathrm{div}, \Omega)$, for which the corresponding results are well-established.

We also introduced an extension operator $\mathcal{E}_{\mathcal{L}}$ from the trace space into the graph space and we highlighted its various features. For instance, we gave a characterisation of the image of $\mathcal{E}_{\mathcal{L}}$ in terms of the second-order differential operator $\mathcal{O}$ and elucidated the isometric correspondence between the image of $\mathcal{E}_{\mathcal{L}}$ and the image of the adjoint extension operator by virtue of $\mathcal{L}$.

In view of the examination of Friedrichs systems we illuminated the characteristics of trace spaces associated to differential operators with Hermitian coefficients. We distinguished between traces which are concentrated in the in- or outflow components of the boundary and which are of $L^2_B(\partial\Omega)$-type and traces which are not contained in $L^2_B(\partial\Omega)$ arising from effects related to corners, tangential coupling and changes in the inertial type of $B(\nu)$.

We also investigated the eigensystem of $\mathcal{L}$ in the image of $\mathcal{E}_{\mathcal{L}}$ and gave an explicit description of the eigenprojections and considered their regularity properties. As indicated in Remark 8, the understanding of the eigensystem of $\mathcal{L}$ might also prove valuable for the derivation of an optimal error bound in h and p for the discontinuous Galerkin method.

We transferred Friedrichs systems into a setting which accommodates a large class of phenomena originating from corners and type changes in the boundary conditions. In this framework singularities at the boundary can be incorporated if they are confined to a null set in the Hausdorff measure of the boundary. We carried over the existence proof by Friedrichs for boundary value problems with admissible boundary conditions. We gave here the characterisation of admissibility in abstract form but detailed in addition the connection to the definition by Friedrichs in terms of matrix functions on the boundary. We paid particular attention to the regularity constraints on the matrix functions which arise from our formulation of Friedrichs systems. This included conditions on the projection P_J, $P_{J'}$, P_{J^*} and $P_{J^{\mathsf{H}}}$ and a separate description of the special case when the range of the boundary operator is contained in $L^2_{B,\mathrm{loc}}(\partial\Omega)$.

Embedded into this general framework, we selected a class of boundary conditions under which Friedrichs systems are well-posed und which provided the basis for the above mentioned convergence proof of the discontinuous Galerkin method mentioned above. As has been already highlighted, our arguments ensure the compatibility of the trace operator with the boundary conditions by virtue of a factorisation of $B(\nu)$.

Having established the necessary foundation, in Chapter 3 we investigated into the performance of the discontinuous Galerkin method for Friedrichs systems with discontinuous solutions. To this end, we introduced the class of broken graph spaces, in which the convergence process of the DG method can be studied. In this setting we documented, prior to the convergence proof, the stability of the method which leads directly to the existence of an unique numerical solution. We also compared our compatibility condition based on T to the conditions by Johnson, Nävert and Pitkäranta, which have been proposed for the analysis of the discontinuous Galerkin method assuming that the exact solution is contained in a Sobolev space.

In a somewhat separate study we addressed the convergence properties of the DGFEM in broken Sobolev spaces. We improved, for certain problems, the error bound by Georgoulis by half an order in p. We also derived an *a priori* error bound for least-squares stabilised discontinuous Galerkin methods which is optimal in h and p. Furthermore, we presented numerical evidence that least-squares stabilisation leads not only to an improved stability bound of the method but also influences the approximation properties of the method advantageously. We then substantiated numerically superconvergence properties of the LS-DGFEM. For completeness we extended the analysis by Houston, Schwab and Süli on the one hand and by Georgoulis on the other hand from the class of scalar problems to Friedrichs systems satisfying our compatibility condition.

Outlook

We wish to highlight two questions for future research which arise from this dissertation. The first concerns the intrinsic definition of the trace space. There remain a number of techniques we have not utilised to derive an intrinsic definition. We already pointed in the Introduction at the theory of pseudo-differential operators which may be capable of shedding new light on already established results. In a classical problem in electrical impedance tomography one has to find for a section of the boundary the Neumann condition which leads to the same solution of a boundary value problem as a given Dirichlet condition. Solving this problem intrinsically, that is without solving the differential equation, seems to bear similarities to turning equation (1.22) into an intrinsic formulation.

The second question we stress concerns optimality of the discontinuous Galerkin method in h

and p in the framework of Sobolev spaces. It would be interesting to clarify whether the ideas in Remark 8 can be extended to prove an hp-optimal error bound in absence of streamline diffusion or least-squares stabilisation. Understanding orthogonality properties of the term $\langle \mathcal{L}\eta, \xi \rangle_\Omega$ appears to play a crucial role in developing an error analysis which reflects the rates of convergence observed in numerical examples.

Appendix

Riesz-Thorin Theorem

We cite the Riesz-Thorin theorem from (Werner 2000, p. 74) in abbreviated form.

Theorem 58 *Assume that* $p_0, p_1, q_0, q_1 \in (1, \infty)$ *and that* $p_0 \neq p_1$ *and* $q_0 \neq q_1$*. Let* $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ *and set* $c := 1$ *if* $\mathbb{K} = \mathbb{C}$ *and* $c := 2$ *if* $\mathbb{K} = \mathbb{R}$*. Let* $0 < \theta < 1$*, and define* p *and* q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Suppose μ *and* ν *are* σ*-finite measures. If* T *is a linear mapping with*

$$\begin{aligned} T : L^{p_0}(\mu, \mathbb{K}) &\to L^{q_0}(\nu, \mathbb{K}) \quad \textit{continuous with norm } M_0, \\ T : L^{p_1}(\mu, \mathbb{K}) &\to L^{q_1}(\nu, \mathbb{K}) \quad \textit{continuous with norm } M_1, \end{aligned}$$

then

$$\|Tf\|_{L^q} \leq c\, M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \qquad \forall\, f \in L^{p_0}(\mu, \mathbb{K}) \cap L^{p_1}(\mu, \mathbb{K}).$$

Hence the operator is extendable to a continuous linear mapping

$$T : L^p(\mu, \mathbb{K}) \to L^q(\nu, \mathbb{K})$$

with norm $c\, M_0^{1-\theta} M_1^{\theta}$*.*

Perturbation of Linear Operators

We give a short account on the necessary perturbation theory of linear operators. We follow the exposition by (Kato 1995) and refer the reader to this book for details. Consider the interval $I \subset \mathbb{R}$ and let $T : I \to \mathbb{C}^{m \times m}$ be a continuous function which assigns to every point in I a complex-valued matrix. Then, according to (Kato 1995, pp. 106-110), there are m

continuous functions $\mu_i : I \to \mathbb{C}$, $i \in \{1, \dots, m\}$, such that if at $x \in I$ the multiplicity of the eigenvalue λ of the matrix $T(x)$ is k then there are exactly k functions μ_i which attain the value λ at this point.

Consider another choice of functions $\dot{\mu}_i$ which represents the eigenvalues of T. Then, in a sufficiently small neighbourhood N_x of x, the graphs of the functions μ_i which pass through λ are equal to the graphs of the functions $\dot{\mu}_i$ which pass through λ. Therefore in N_x we can define the λ-group as union of the graphs of μ_i which pass through λ independently of the choice of the representing functions, cf. (Kato 1995, pp. 66,107).

Under the total projection of the λ-group at $x \in N_x$ we understand the projection to the sum of all eigenspaces which are associated to an eigenvalue in the λ-group. We restate Theorem 5.4 from (Kato 1995, p. 111) in abbreviated form.

Theorem 59 *Let $T(x)$ be differentiable at $x = 0$. Then the total projection of the λ-group is differentiable at $x = 0$. If T is diagonalisable, then the functions μ_i are differentiable at $x = 0$.*

According to the remark on page 115 in Kato's book, for diagonalisable operators continuous differentiability of $T(x)$ indeed implies continuous differentiability of the total projections and of the eigenvalues.

However we also need to consider the mapping which associates to every x the eigenprojection associated to the eigenvalue $\mu_i(x)$. In Example 12 from page 45 we illustrate that this mapping is generally less smooth than the total projections. The situation changes if we assume that T, is (in each entry T_{ij}) analytic, cf. (Kato 1995, p. 73).

Theorem 60 *Let the function $x \mapsto T(x)$ be analytic. Then, with the exception of a discrete set, the eigenprojections are also analytic.*

We also cite, again in abbreviated form, Theorem 6.1 on page 120 in Kato's book.

Theorem 61 *If the analytic family $T(x)$ is real and symmetric, then the eigenvalues $\mu_i(x)$ and the eigenprojections are analytic on the real axis.*

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