Gauge Theory in Co-homogeneity One

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Wen dieser Engel überwand,
welcher so oft auf Kampf verzichtet,
der geht gerecht und aufgerichtet
und groß aus jener harten Hand,
die sich, wie formend, an ihn schmiegte.
Die Siege laden ihn nicht ein.
Sein Wachstum ist: der Tiefbesiegte
von immer Größerem zu sein.

-from *Der Schauende*
by Rainer Maria Rilke

Abstract

We use co-homogeneity one symmetries to construct new families of instantons over Riemannian manifolds with special holonomy groups and asymptotically conical geometry. In doing so, we give a complete description of the behaviour of Calabi-Yau instantons and monopoles with an $SU(2)^2$-symmetry, by considering gauge theory on the smoothing and small resolution of the conifold, and on the canonical bundle of $\mathbb{CP}^1 \times \mathbb{CP}^1$, with their known asymptotically conical co-homogeneity one Calabi-Yau metrics.

Furthermore, we classify $SU(2)^3$-invariant G_2-instantons on the spinor bundle of the 3-sphere, equipped with the asymptotically conical co-homogeneity one G_2-metrics of Bryant-Salamon, and show that if any non-invariant instanton shares the same asymptotic behaviour, its deformation theory must be obstructed.

Declaration

I, Jakob R. Stein confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Impact Statement

The application of gauge theory to geometry touches upon aspects of algebraic geometry, geometric analysis, and has led to deep results in lower-dimensional topology [DK90]. More recently, proposals by Donaldson–Thomas–Segal (DTS) for a programme to extend these ideas to higher dimensions [DT98], [DS11] has garnered interest from both the mathematical and theoretical physics communities.

A natural setting for this higher-dimensional gauge theory is on Riemannian manifolds with special holonomy groups. In particular, two special-holonomy manifolds known as G_2-manifolds and Calabi-Yau 3-folds play an important role in string-theoretic models of fundamental physics, e.g. via M-theory [AW02].

One of the aims of this programme is to construct invariants of these manifolds by using solutions to certain gauge-theoretic equations called instantons. However, obtaining a precise description of the moduli-space of these instantons remains a significant challenge: this thesis aims to redress some of these difficulties by constructing new families of examples of instantons on G_2-manifolds and Calabi-Yau 3-folds with symmetries, and studying the moduli-space in this symmetric setting.

Author Contribution Statement

The results of §4 originate in unpublished joint work with Matthew Turner.

Self-Plagiarism Statement

Most of the contents of §2, §3 and the appendices can be found in the single-author arXiv pre-print [Ste21].

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A final thanks goes to my family, who may never read this document, but who helped shape it nonetheless.

Contents

Introduction

Inspired by the conjectural picture outlined by Donaldson-Thomas in [DT98], later expanded upon in [DS11], also [Wal17], significant progress has been made in generalising the classical study of gauge theory in dimensions three and four to higher dimensions.

One goal of this programme is to mimic the construction found in [DK90] of invariants for smooth 4-manifolds using moduli-spaces of *anti-self-dual instantons*: solutions to a first-order system of partial differential equations implying the Yang-Mills instanton equations. The Yang-Mills equations, which are discussed in detail in §1.1, can be defined on a bundle over any oriented Riemannian manifold: in higher dimensions, finding solutions to these equations via a first-order system requires the presence of additional geometric structure.

In particular, instanton moduli-spaces over Riemannian manifolds with special holonomy groups possess many of the desirable features found in the four-dimensional case, see e.g. [DS11, §2.2]. In this thesis, we will focus on two kinds of manifolds admitting special-holonomy metrics: Calabi-Yau 3-folds in (real) dimension six, and G_2-manifolds in dimension seven. These geometries are discussed in §1.2, §1.3 respectively.

Obtaining an analytic description of the moduli-space of these instantons in higher dimensions presents a significant challenge to the Donaldson-Thomas programme, stemming from issues of compactness: in general, bubbling phenomena are expected to occur along calibrated currents with Hausdorff co-dimension four [Tia00]. We wish to find examples clarifying the relationship between compactifications of the moduli-space, and the calibrated geometry of special-holonomy metrics. This is particularly difficult in the G_2-case, where we cannot rely on the input from algebraic geometry.

In this thesis, we will exploit symmetries, both of the bundle data and of the underlying Riemannian manifolds, to construct new examples of instantons on metrics with holonomy G_2 and $SU(3)$. Restricting to this setting, and structure groups of rank one, we will be able to give explicit descriptions of the moduli-space of these symmetric solutions in §2, §3, and §4.

Since metrics with holonomy contained in $SU(3)$ and G_2 are Ricci-flat, outside of flat metrics, the maximal symmetries we could hope to exploit are *co-homogeneity one*, i.e. we have a Lie group of isometries acting on the Riemannian manifold with generic orbits of co-dimension one. For this reason, co-homogeneity one symmetries have played a historically significant role in the study of special holonomy, and we will discuss these symmetries in detail in §1.1, as well as recalling explicit examples in §1.1, §1.2, and §1.3.

Moreover, if an irreducible Riemannian manifold has holonomy equal to G_2 or $SU(3)$ as the case may be, it must be non-compact to admit any continuous symmetry at all. Fortunately, there are non-compact G_2-metrics admitting S^1-symmetries recently constructed by Foscolo-Haskins-Nordström [FHN21a] occurring in families of arbitrarily large dimension [FHN21a, Corollary 9.5], and infinitely many one-parameter families of examples with a co-homogeneity one symmetry of $SU(2)^2 \times U(1)$ by the same authors [FHN21b]. Thus, studying their gauge theory has the potential to provide examples of G_2-instantons on a large class of geometries: carrying out this project motivated many of the results contained in this thesis.

To explain in more detail, we note that the co-homogeneity one manifolds of [FHN21b] come in one-parameter families with asymptotically locally conical (ALC) geometry at infinity, i.e. outside of a compact subset, these G_2-metrics converge to a circle fibration over a Calabi-Yau cone, with fibres of some length $\ell > 0$. In the limit as $\ell \to 0$, these G_2-manifolds collapse to a Calabi-Yau 3-fold with asymptotically conical (AC) geometry at infinity, i.e. the Calabi-Yau is diffeomorphic to a cone outside of a compact subset, with a Riemannian metric converging to the corresponding metric cone.

The AC Calabi-Yau metrics arising in this limit all admit a co-homogeneity one action of $SU(2)^2$, and share the same asymptotic cone, up to double-cover: the *conifold*, the $SU(2)^2$-invariant Calabi-Yau cone metric on the ordinary double-point singularity $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid \sum_i z_i^2 = 0\}$. The first of these $SU(2)^2$-invariant Calabi-Yau metrics, defined on the *small resolution* $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ of the ordinary double point, is unique up to overall scale, as is the second on the *smoothing* T^*S^3. These examples were first described in [CdlO90], with the second discovered independently in [Ste93]. Finally, the canonical bundle $K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$ has a one-parameter family of $SU(2)^2$-invariant AC Calabi-Yau metrics, up to scale, which arise by varying the relative volume of each copy of $\mathbb{CP}^1$. These are described in [PZT01], generalising earlier work of [Cal79] when these two volumes are equal.

We investigate $SU(2)^2$-invariant gauge theory on these AC Calabi-Yau 3-folds in §2, §3: as well as instantons, we also consider *Calabi-Yau monopoles*, which are a slight generalisation of instantons on non-compact Calabi-Yau 3-folds cf. (2.1). These Calabi-Yau monopoles model the asymptotic behaviour of instantons on ALC G_2-manifolds, and are analogous to *Bogomol'yni monopoles* found in dimension three: see also §1.1, (1.3). We expect that the instantons constructed in §2 can be used to construct G_2-instantons on the co-homogeneity one ALC G_2-metrics of [FHN21b] near the collapsed limit.

Furthermore, in §3.3, we prove that the relevant bubbling and compactness theorems hold for the instantons constructed in §2, §3 in terms of anti-self-dual instantons along complex curves in the AC Calabi-Yau, in line with the general picture laid out in [DT98], [DS11] for the compact case.

Far from the collapsed limit, the families of co-homogeneity one G_2-metrics in [FHN21b] have AC geometry when $\ell \to \infty$, see [LO18] for partial results comparing invariant instantons on G_2-metrics with ALC and AC asymptotics. Two of these families, referred to as the $\mathbb{B}_7$ and $\mathbb{D}_7$ families in the physics literature [CGLP02], share the same limiting AC G_2 metric: the metric constructed by Bryant-Salamon [BS89] on the spinor bundle $\mathbf{S}(S^3)$ of the 3-sphere S^3, which admits a co-homogeneity one action of $SU(2)^3$. In §4.2, we settle a question posed in [LO18], by classifying $SU(2)^3$-invariant instantons over $\mathbf{S}(S^3)$ and constructing a new family of examples. Moreover, in §4.3, using the deformation theory of instantons on AC G_2 metrics carried out in [Dri20], we show that the symmetric solutions from §4.2 are the only solutions with unobstructed deformations in the moduli-space of instantons sharing the same asymptotic behaviour.

A slightly more speculative suggestion is that, via a gluing procedure, we may be able to use the instantons on $\mathbf{S}(S^3)$ to construct instantons on the ALC metrics of the $\mathbb{B}_7$, $\mathbb{D}_7$ families for ℓ sufficiently large, by considering instantons on some of the incomplete *conically-singular* (CS) co-homogeneity one G_2-metrics constructed in [FHN21b]. These metrics have a complete ALC end at infinity modelled on a circle-fibration over the conifold, and an incomplete end with a conical singularity modelled on the asymptotic cone of $\mathbf{S}(S^3)$. It is suggested in [FHN21b], adapting earlier arguments of [Kar09], that the $\mathbb{B}_7$, $\mathbb{D}_7$ families may be viewed as de-singularisations of these CS metrics by gluing in a rescaled copy of $\mathbf{S}(S^3)$, and one may be able to produce instantons on these families by following a similar procedure.

Plan of Thesis

Chapter 1: We introduce the study of gauge theory using co-homogeneity one symmetries in §1.1, drawing on the more familiar setting of instantons in dimension four for examples. This is partly as a warm-up for the slightly more complicated analysis in higher dimensions, but also because these 4-dimensional instantons will appear as bubbling limits for these higher-dimensional instantons in §3.3.

Throughout the following, if a manifold M has a co-homogeneity one action by Lie group K, with exactly one exceptional isotropy subgroup H', and generic isotropy subgroup H, we will denote the sequence $H \subset H' \subseteq K$ as the *group diagram* of M. We will refer to the generic K-orbit K/H as the *principal* orbit, the exceptional orbit K/H' as the *singular* orbit, and the union of generic K-orbits as the *space of principal orbits*.

In order to fix conventions, we give a preliminary introduction to the geometry of co-homogeneity one Calabi Yau metrics in §1.2, in the case $K = SU(2)^2$, and H is either the diagonal subgroup $\triangle U(1)$ or $\triangle U(1) \times \mathbb{Z}_2$, i.e. we describe the Calabi-Yau metrics on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, T^*S^3, and $\mathcal{O}(-2,-2)$. We will also give a preliminary introduction to the geometry of co-homogeneity one G_2-metrics in §1.3, in the case $K = SU(2)^3$, and H is the diagonal subgroup $\triangle SU(2)$ in all three factors, i.e. we describe the G_2 metrics on $\mathbf{S}(S^3)$.

Chapter 2: We proceed with the main aims of this thesis in §2. We will begin this chapter with an overview of the results proved in §3: namely, classifying $SU(2)^2$-invariant solutions to the Calabi-Yau monopole equations. We consider the space of principal $SU(2)$-bundles and connections over $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, T^*S^3, and $\mathcal{O}(-2,-2)$ that are invariant under the $SU(2)^2$-action, and we describe the gauge theory on the complement of the singular orbit by pulling back bundles over the principal orbit in §2.3. Invariant bundles over the principal orbit are classified by an integer, but only one of these bundles, denoted P_1, admits irreducible connections. We write down the Calabi-Yau monopole equations for this bundle explicitly in Proposition 2.3.3, as an non-autonomous, non-linear ODE system in four variables. We also briefly mention reducible solutions to these equations in §2.4, which are explicit.

Chapter 3: We cannot generically expect to find explicit solutions in the irreducible case, but by imposing that the bundle data extends to the singular orbit, we can describe the space of solutions to the ODEs near the singular orbit using a power-series. In §3.1, we will find that these local solutions to the monopole equations are always in a two-parameter family for each extension of the bundle P_1 to the singular orbit, and we can obtain a local one-parameter family of instantons by setting one of these two parameters to zero.

The remaining sections of this chapter are dedicated to finding a qualitative description of the behaviour of the local solutions in §3.1 as we move away from the singular orbit. In §3.2, using the existence of invariant sets for these ODE systems, we determine the asymptotic behaviour of the local instanton solutions to obtain new one-parameter families of instantons with curvature decaying exactly quadratically at infinity: two families on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, one on T^*S^3, and countably many on $\mathcal{O}(-2,-2)$.

We also prove that the families of instantons on $\mathcal{O}(-2,-2)$ admit a natural compactifications by instantons with faster than quadratic curvature decay: these fast-decaying instantons can also be found via a result of [Ban93]. To prove their existence in the invariant setting, we employ a rescaling argument along the fibres of $\mathcal{O}(-2,-2)$, and we prove that these fast-decaying instantons are unique on a fixed bundle via some comparison results allowing us to compare different members of each one-parameter family of solutions as they move away from the singular orbit.

We continue discussing rescaling arguments in §3.3. To show the relevant bubbling result, we can consider an adiabatic limit in which we shrink the metric on $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ along the fibre. We prove that, if we rescale one of the families of instantons on this metric, in the limit the size of fibre shrinks to zero, this family converges to the standard anti-self-dual instanton on $\mathbb{C}^2$. We use this to prove the compactification result: that if we do not rescale, this family converges on compact subsets of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \setminus \mathbb{CP}^1$ to an (abelian) instanton in the other family.

Although ultimately unnecessary for proving the results contained in chapters §2, §3, we also show a bubbling phenomena for the families of instantons on $\mathcal{O}(-2,-2)$. We consider a limit in which the metric is close to the simplest (non-trivial) example of an asymptotically locally Euclidean (ALE) fibration: a copy of the Eguchi-Hansion metric on the total space of the co-tangent bundle of $\mathbb{CP}^1$, fibred over the standard metric on $\mathbb{CP}^1$. As one might expect, in this limit, we find that the families of instantons on $\mathcal{O}(-2,-2)$, suitably rescaled, are close to corresponding families of anti-self-dual connections for the Eguchi-Hanson metric constructed by Nakajima in [Nak90], which are described explicitly in §1.1.

Finally, in §3.4, we analyse the behaviour of the full system of the monopole equations away from the singular orbit to prove the non-existence of invariant monopoles on $\mathcal{O}(-1)\oplus \mathcal{O}(-1)$ and $\mathcal{O}(-2,-2)$. We show that, aside from the one-parameter family of invariant monopoles on T^*S^3 found previously in [Oli16] and the instantons described in the previous sections, any other member of the local two-parameter families of solutions to the monopole equation from §3.1 cannot have quadratically decaying curvature. This corrects a gap in the proof of [Oli16] for the uniqueness of the family of monopoles on T^*S^3.

Chapter 4: The last chapter of this thesis is based on a joint work with Matthew

Turner. In this chapter, we discuss $SU(2)^3$-invariant instantons on the metrics of Bryant-Salamon on $\mathbf{S}(S^3)$, mostly following [LO18]: there is only one bundle admitting irreducible invariant connections over the principal orbits, and we re-write the ODE system in [LO18] corresponding to the $SU(2)^3$-invariant instanton equations in Proposition 4.2.1.

As shown in [LO18], there are two ways of extending this bundle to the singular orbit, and each gives a local one-parameter family of solutions to the instanton ODEs, which are given in Proposition 4.2.3. One family has an closed-form expression (4.11), which describes the one-parameter family of instantons on $\mathbf{S}(S^3)$ found previously in [Cla14]. As shown in [LO18], this one-parameter family converges on compact subsets of $\mathbf{S}(S^3) \setminus S^3$ to an explicit member (4.12) in the other family of local solutions, which is defined on all of $\mathbf{S}(S^3)$.

However, these authors were unable to determine the behaviour of the other members of this one-parameter family of local solutions moving away from the singular orbit. We do so in §4.2, using the existence of invariant sets for this ODE system, and the work of [Mar56] on non-autonomous ODE systems with autonomous large-time limits. In doing so, we obtain a new one-parameter family of (non-explicit) solutions, containing the explicit solution (4.12) of [LO18].

All these invariant examples on $\mathbf{S}(S^3)$ share the same asymptotic behaviour: they converge to the $SU(2)^3$-invariant *nearly-Kähler instanton*, cf. (4.4), studied in [CH16] on the homogeneous space $SU(2)^3 \to SU(2)^3/\triangle SU(2)$. In §4.3, we use a computation in [Dri20] to show that if an instanton on $\mathbf{S}(S^3)$ has these asymptotics, and its deformation theory is unobstructed, then the instanton is necessarily $SU(2)^3$-invariant. Thus, by the results of the previous section, it must lie in one of the two families given in §4.2, up to gauge.

Appendix A: We must impose boundary conditions for the invariant Calabi-Yau structures on the space of principal orbits found in §1.2 to extend smoothly over the singular orbit. Likewise, we must impose boundary conditions for the invariant bundle data on the space of principal orbits in §2, §3 to extend to the singular orbit. These discussions are relegated to Appendices A.1 and A.2 respectively: using the analysis of Eschenburg-Wang [EW00] on invariant tensors, which can be adapted to (adjoint-valued) forms, these are representation-theoretic computations.

Appendix B: To aid the flow of the exposition in §3, most of the explicit computations for parametrizing local solutions to the ODEs in §3.1 are consigned to this appendix.

Chapter 1

Background

In this chapter, we will give an introductory exposition of the necessary background material used in the later chapters, mostly following standard references.

In §1.1, we will use the familiar setting of instantons in dimension four to introduce the study of gauge theory using co-homogeneity one symmetries. Although much of the material in this section is well-known, Example 1.1.3 recovers the anti-self-dual instantons on $T^*\mathbb{CP}^1$ constructed in [BJCC80a] in the co-homogeneity one set-up.

We give an introduction to the geometry of co-homogeneity one Calabi-Yau metrics in §1.2. In order to fix conventions for the Calabi-Yau gauge theory in §2, §3, we will briefly describe the general theory of Calabi-Yau metrics foliated by parallel hyper-surfaces, and explicitly construct the $SU(2)^2$-invariant co-homogeneity one Calabi-Yau metrics on $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, T^*S^3, and $\mathcal{O}(-2,-2)$ following [FH17]. The only new material in this section is the co-homogeneity one description of the family of metrics found by [PZT01] on $\mathcal{O}(-2,-2)$.

The final section §1.3 has a similar structure to §1.2: to fix conventions for the G_2-gauge theory in §4, we will briefly describe the general theory of G_2-metrics foliated by parallel hyper-surfaces, and explicitly construct the $SU(2)^3$-invariant co-homogeneity one G_2-metrics of [BS89] on $\mathbf{S}(S^3)$, following [LO18].

1.1 Gauge Theory in Four Dimensions

Let $\left(M^4, g\right)$ be an oriented Riemannian manifold, and fix a principal G-bundle $P \to M$ with a compact, semi-simple Lie group G. We will denote the adjoint bundle associated to P by $\mathrm{ad}P$. There is a natural gauge-invariant energy functional on the space of connections on P, referred to as the *Yang-Mills energy* $\mathcal{YM}(A) := \int_M |F_A|^2$, where $F_A \in \Omega^2\left(\mathrm{ad}P\right)$ is the curvature of a connection A on P, and we take point-wise norms with respect to some ad-invariant metric on the Lie algebra of G.

A connection on P is referred to as being *Yang-Mills* if it satisfies the Euler-Lagrange equations for $\mathcal{YM}(A)$:

$$d_A^* F_A = 0 \tag{1.1}$$

where d_A^* is the formal adjoint of the induced exterior covariant derivative $d_A : \Omega^1\left(\mathrm{ad}P\right) \to \Omega^2\left(\mathrm{ad}P\right)$.

In dimension four, connections minimising $\mathcal{YM}(A)$ can be found by solving the *(anti-)self-dual instanton equations*:

$$*F_A = \pm F_A \tag{1.2}$$

where $*$ denotes the Hodge star with respect to the Riemannian metric g and the chosen orientation. In this section, we shall be concerned with *anti-self-dual instantons*, i.e. solutions to (1.2) with $*F_A = -F_A$[1], admitting continuous symmetries.

Example 1.1.1. Take a non-compact oriented Riemannian 3-manifold (N, g) equipped with a principal bundle $P \to N$ and a *Bogomol'nyi monopole*: a pair (A, Φ), for some connection A on P and $\Phi \in \Omega^0(\mathrm{ad}P)$ satisfying:

$$F_A = \pm * d_A \Phi \tag{1.3}$$

Pulling back (A, Φ) to the product $g + d\theta^2$ of g with the flat metric $d\theta^2$ on the circle S^1, gives an S^1-invariant solution $\mathbf{A} = A + \Phi d\theta$ to (1.2) on the pull-back of P to $N \times S^1$. We will see in §2, §4.3 that the Bogomol'nyi monopole equations (1.3) have higher-dimensional analogues.

As it will be useful to us later, let us write down (1.2) in co-ordinates near a hyper-surface: if we suppose that N is an oriented hyper-surface in M, then a tubular neighbourhood of $N \subset M$ can be identified with $N \times I$ for some interval $I \subseteq \mathbb{R}$ via the exponential map. In these coordinates, the metric on M appears as $g = dt^2 + g_t$ for some t-dependent metric g_t on N.

In this neighbourhood, we may always write P as the pull-back of some bundle on N. Similarly, if P is equipped with a connection form A, then we may write $A = A_t + \gamma_t dt$, where A_t is a one-parameter family of connections over N, and $\gamma_t \in \Omega^0(\mathrm{ad}P)$ is a one-parameter family of sections of the adjoint bundle.

Via a gauge transformation, we can always choose to set $\gamma_t = 0$: for each $t \in I$, take $g_t \in G$ such that $\gamma_t + g_t^{-1}(\partial_t g_t) = 0$. We will refer to this choice of gauge as the *temporal gauge*, and the curvature of $A = A_t$ in this gauge is given by $F_A = F_{A_t} - \partial_t A_t \wedge dt$.

Moreover, given some t-dependent orthonormal co-frame $(\sigma_1, \sigma_2, \sigma_3)_{t \in I}$ for g_t[2], we can write an orthonormal basis $\omega_i^{\pm} = dt \wedge \sigma_i \pm \sigma_j \wedge \sigma_k$ of (anti-)self-dual two-forms on M, where (ijk) are cyclic permutations of (123), so that (1.2) appears in these coordinates as:

$$F_{A_t} \wedge \sigma_i \mp \partial_t A_t \wedge \sigma_j \wedge \sigma_k = 0 \tag{1.4}$$

using the temporal gauge, for (ijk) cyclic permutations of (123).

A natural context where such a family of parallel hyper-surfaces occur is when N is the orbit of a Lie group of isometries acting on M:

Definition 1.1.1. A compact Lie group K of isometries acts with *co-homogeneity one* on a connected Riemannian manifold (M, g) if there is a co-dimension one K-orbit in M.

[1] note, however, we can identify anti-self-dual instantons with *self-dual instantons*, i.e. with solutions of $*F_A = F_A$, by changing the orientation on M.

[2] note that this always exists, since every oriented 3-manifold is parallelisable.

It is not difficult to show that the K-orbits of a co-homogeneity one action foliate a dense open subset of M into parallel hyper-surfaces, and that these parallel hyper-surfaces can be written as a homogeneous space K/H, where H denotes the generic isotropy subgroup of the K-action [AA93]. Moreover, the complete co-homogeneity one manifolds we will encounter in this thesis will have a unique singular isotropy subgroup[3] $H' \subset K$, and in this case, M can be equivariantly identified with the total space of a vector-bundle $K \times_{H'} V \to K/H'$ for some orthogonal H'-representation V, with $H \subset H'$ realised as the stabiliser subgroup of some non-zero $v \in V$.

Returning now to the gauge theory: suppose the connection A_t in (1.4) is invariant under some lift of a co-homogeneity one action to the total space of the bundle, then we can reduce (1.4) to a system of ordinary differential equations in a single variable. In order to make sense of this, we introduce the following definition:

Definition 1.1.2. Let K be a compact Lie group acting on a smooth manifold M, and $P \to M$ be a principal G-bundle for some Lie group G. P is said to be K-*invariant* if there is a lift of the K-action to the total space of P. Furthermore, if K acts transitively on M, then P is called K-*homogeneous.*

Homogeneous bundles have been studied in detail in [Wan58]: K-homogeneous bundles over the homogeneous space K/H can be equivariantly identified with $K \times_H G$, where H acts on G via some group homomorphism $\lambda : H \to G$. Moreover, this homomorphism classifies homogeneous bundles up to equivariant isomorphism.

If K/H is the principal orbit of a co-homogeneity one manifold M with diagram $H \subset H' \subseteq K$, a homogeneous bundle over the principal orbit can be extended over to an invariant bundle over M by extending the group homomorphism $H \to G$ to a group homomorphism $H \subset H' \to G$. The bundle defined by this extension is just the pull-back via the projection $M \to K/H'$ of a homogeneous bundle over K/H'.

The advantage of using homogenous bundles for gauge theory is that K-invariant connection forms can be identified with linear maps on the tangent space at a single point: if we write $\mathfrak{h} \subset \mathfrak{k}$, $\mathfrak{g}$ for the Lie algebras of $H \subset K$, G respectively, then by [Wan58, Thm.A], any K-invariant connection one-form can be written as a linear map $A : \mathfrak{k} \to \mathfrak{g}$, such that A intertwines the H-representations $\mathfrak{k}$ and $\mathfrak{g}$, and $A|_{\mathfrak{h}} = d\lambda$. In other words, on $\mathfrak{h}$, A is the image of the canonical connection on $K \to K/H$ under λ, and since K is compact, we have the H-invariant splitting $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ for some $\mathfrak{m} \subset \mathfrak{k}$. Thus the connection is uniquely determined by the H-equivariant map $A|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{g}$.

If we work outside of the singular orbit, using the temporal gauge, we can use this description to view any invariant connection on a co-homogeneity one manifold as a one-parameter family of linear maps $A_t : \mathfrak{k} \to \mathfrak{g}$, so that (1.4) becomes a finite-dimensional ODE system.

Before seeing some explicit examples, which all have a co-homogeneity one action of $SU(2)$, we will fix the following piece of notation- throughout this thesis, we will denote

[3] in particular, this will always be the case when the metric is complete, irreducible and Ricci-flat.

E_1, E_2, E_3 a basis for the Lie algebra $\mathfrak{su}(2)$ given by the matrices:

$$E_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad E_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad E_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

so that $[E_i, E_j] = 2E_k$ for cyclic permutations of (123), and the action of $U(1)$ on $SU(2)$ is generated by E_1. Clearly, we can identify the span of E_2, E_3 under the adjoint action of $U(1)$ with $\mathbb{C}_2$, where $\mathbb{C}_n$ denotes n^{th} tensor power of the standard representation of $U(1)$ on $\mathbb{C}$. We will denote the one-forms dual to E_1, E_2, E_3 as e_1, e_2, e_3.

Example 1.1.2. ($\mathbb{C}^2$, BPST) $\mathbb{C}^2$ is equipped with a co-homogeneity one action of $SU(2)$ via the standard representation, with diagram $\{1\} \subset SU(2) \subseteq SU(2)$. We can define an $SU(2)$-invariant $SU(2)$-bundle over $\mathbb{C}^2$ by the homomorphism $\mathrm{Id} : SU(2) \to SU(2)$, i.e. the singular isotropy group $SU(2)$ acts via the identity homomorphism on the fibre $SU(2)$.

Outside of the singular $SU(2)$-orbit at the origin in $\mathbb{C}^2$, this bundle is equivariantly trivial, and can be pulled back from the trivial bundle over the principal orbit $S^3 \subset \mathbb{C}^2$. So, up to the adjoint action of $SU(2)$ on $\mathfrak{su}(2)$ by equivariant gauge transformations, we can always put any $SU(2)$-invariant connection A on this bundle into the diagonal form $A = \sum_i \alpha_i E_i \otimes e_i$, for some $\alpha_i(t)$ depending on the radial distance $t \in \mathbb{R}_{\geq 0}$ from the origin. Furthermore, we require that $\alpha_i(0) = 1$, so that this connection extends smoothly over the origin as the Maurer-Cartan connection, the unique $SU(2)$-invariant connection on the restriction of this bundle to the origin.

Now, since $\mathbb{C}^2$ has an additional (right) co-homogeneity one action of $SU(2)$, we will focus on the case that the connection shares this symmetry, i.e. $\alpha := \alpha_1 = \alpha_2 = \alpha_3$. Using the standard basis of self-dual forms for the flat metric, we can then write the $SU(2)^2$-invariant anti-self-dual equations as the ODE $t\dot{\alpha} = 2\alpha(\alpha - 1)$. This has the explicit solution $\alpha = \left(1 + \kappa t^2\right)^{-1}$ for any $\kappa \in \mathbb{R}$, extending smoothly over the singular orbit at $t = 0$.

Clearly, this exists for all $t \geq 0$ if and only if $\kappa \geq 0$, and $\kappa = 0$ defines a flat connection. If $\kappa > 0$, we can always fix the solution to have $\kappa = 1$ by an overall rescaling of the metric, giving the classical Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton: the absolute minimum of the Yang-Mills energy over $\mathbb{C}^2$ with charge one.

Example 1.1.3. ($T^*\mathbb{CP}^1$, BCC): $T^*\mathbb{CP}^1$ admits a co-homogeneity one action of $SU(2)$ with diagram $\mathbb{Z}_2 \subset U(1) \subset SU(2)$, by viewing $T^*\mathbb{CP}^1$ as the co-tangent bundle of the homogeneous space $\mathbb{CP}^1 = SU(2)/U(1)$. Here, $T^*\mathbb{CP}^1$ is equipped with the $SU(2) \times U(1)$-invariant metric of Eguchi-Hanson [EGH80], which can be written, up to scale, as:

$$g = dt^2 + \varphi^2 \left((1 - \varphi^{-4}) e_1^2 + e_2^2 + e_3^2 \right) \tag{1.5}$$

on the space of principal orbits $T^*\mathbb{CP}^1 \setminus \mathbb{CP}^1 = \mathbb{R}_{>0} \times SU(2)/\mathbb{Z}_2$, where $t \geq 0$ is the radial geodesic arc-length extending over $\mathbb{CP}^1$ at $t = 0$, and $\varphi(t)$ is the unique solution to $\dot{\varphi}^2 = 1 - \varphi^{-4}$ on $[0, \infty)$ with $\varphi(0) = 1$, $\ddot{\varphi}(0) = 2$.

This metric is complete, with a unique asymptotically locally Euclidean (ALE) end modelled on the flat orbifold $\mathbb{C}^2/\mathbb{Z}_2$. Outside of the origin, we can identify $\mathbb{C}^2/\mathbb{Z}_2$ with the space of principal orbits as smooth manifolds, and $|g - g_{\mathbb{C}^2/\mathbb{Z}_2}| = O(t^{-4})$ as $t \to \infty$, where

$g_{\mathbb{C}^2/\mathbb{Z}_2}$ denotes the flat metric, and we take norms with respect to the Eguchi-Hanson metric g.

We can define a family of $SU(2)$-invariant $SU(2)$-bundles on $T^*\mathbb{CP}^1$ by the homomorphism $\lambda^l : U(1) \to SU(2)$ given by taking l'th power of the diagonal embedding $\lambda : U(1) \hookrightarrow SU(2)$ for some $l \in \mathbb{Z}_{>0}$. Outside of the singular orbit, these bundles can be pulled back from bundles over the principal orbit $SU(2)/\mathbb{Z}_2 \cong SO(3)$: either the trivial bundle $SU(2) \times SO(3)$ when l is even, or the non-trivial bundle $SO(4) \to SO(3)$ when l is odd.

In either case, using the action of $SU(2)$ on $\mathfrak{su}(2)$ by equivariant gauge-transformations, any $SU(2)$-invariant connection on the space of principal orbits can be written in the diagonal form $A = \sum_i \alpha_i E_i \otimes e_i$ for some $\alpha_i(t)$, smooth functions of $t \in \mathbb{R}_{>0}$. This connection extends smoothly over the singular orbit at $t = 0$ if and only if α_1, $t^{1-l}(\alpha_2 + \alpha_3)$, $t^{-l-1}(\alpha_2 - \alpha_3)$ are smooth, even functions near $t = 0$, and $\alpha_1(0) = l$, cf. Proposition A.3.1 in appendix A.

As in the previous example, since the Eguchi-Hanson metric admits an additional $U(1)$-symmetry, we will assume the connection shares this symmetry, i.e. $\alpha_2 = \alpha_3$. The corresponding $SU(2) \times U(1)$-invariant anti-self-dual equations, up to gauge, are:

$$\dot{\alpha}_1 = 2\tfrac{\dot{\varphi}}{\varphi}\left(\alpha_2^2 - \alpha_1\right) \qquad \dot{\alpha}_2 = 2\tfrac{1}{\dot{\varphi}\varphi}\alpha_2\left(\alpha_1 - 1\right) \tag{1.6}$$

The anti-self-dual equations (1.6) have a one-parameter family of solutions for each $l \in \mathbb{Z}_{>0}$. This family can be written down explicitly as:

$$\alpha_1 = \frac{l}{\varphi^2}\frac{1+\zeta_\kappa^l(\varphi)}{1-\zeta_\kappa^l(\varphi)} \qquad \alpha_2 = \frac{2l}{1-\zeta_\kappa^l(\varphi)}\sqrt{\frac{\zeta_\kappa^l(\varphi)}{\varphi^4-1}} \tag{1.7}$$

where $\zeta_\kappa^l(\varphi) := \kappa\left(\frac{\varphi^2-1}{\varphi^2+1}\right)^l$, and parameter κ lies in the interval $0 \leq \kappa \leq 1$.

These invariant one-parameter families were found previously by Boutaleb-Joutei, Chakrabarti, and Comtet (BCC) in [BJCC80a], see also [BJCC80b]. For each fixed l, the solution (1.7) with $\kappa = 0$ is a unique abelian solution to (1.6) extending over the singular orbit, and the solutions with $0 \leq \kappa < 1$ are asymptotic to the flat connection $(\alpha_1, \alpha_2) = (0, 0)$. At $\kappa = 1$, there is a transition in asymptotic behaviour: the solution (1.7) with $\kappa = 1$ is asymptotic to the flat connection $(\alpha_1, \alpha_2) = (1, 1)$. When $l = 1$, the solution with $\kappa = 1$ is just the flat connection $(\alpha_1, \alpha_2) = (1, 1)$ for all time, but these solutions are irreducible otherwise.

For each of these families, we can explain the limit $\kappa \to 1$ as a bubbling phenomenon at infinity, see [Nak90, Theorem 5.2]. In terms of solutions (1.7), in the limit $\kappa \to 1$, the smooth trajectories $\{(\alpha_1, \alpha_2)(t) \mid t \geq 0\} \subset \mathbb{R}^2$ of the solutions for $0 \leq \kappa < 1$ converge to a piece-wise smooth trajectory consisting of two components. The first component, traversed in some sufficiently large time $T(\kappa) \to \infty$ as $\kappa \to 1$, is the trajectory of the solution (1.7) with $\kappa = 1$. The second component, which is only traversed after time $T(\kappa)$, is the trajectory of the anti-self-dual instanton $\alpha_1 = \alpha_2 = \frac{1}{1+t^2}$ on $\mathbb{C}^2/\mathbb{Z}_2$ descending from the BPST instanton on $\mathbb{C}^2$ from Example 1.1.2.

As is computed in [BFRM96, §6.1], in the special case $l = 1$, the invariant one-parameter family (1.7) with $l = 1$ gives all the solutions of the anti-self-duality equations

(1.2) on this bundle, up to gauge-equivalence. This observation can also be understood in terms of the *framed* moduli-space of anti-self-dual connections for the Eguchi-Hanson metric constructed in [Nak90], which considers the moduli-space of anti-self-dual connections up to gauge transformations asymptotic to the identity. This framed moduli-space, equipped with a natural metric, is isometric to the underlying Eguchi-Hanson space [Nak90, Theorem 0.3], see also [BFRM96, §7]. We can recover the description of this moduli-space in our invariant set-up, at least as a co-homogeneity one manifold, by considering the orbit of the family (1.7) under constant (t-independent) $SU(2)$-invariant gauge-transformations. Here, the parameter κ corresponds to a parametrisation of the radial parameter t in (1.5) such that $\kappa(0) = 0$, $\kappa(\infty) = 1$, and the gauge orbits correspond to the orbits of the co-homogeneity one action of $SU(2)$ on $T^*\mathbb{CP}^1$.

With these two examples in place, we will now move on to the higher-dimensional setting. For this purpose, and in order to fix conventions for the gauge theory, we will introduce two kinds of manifolds admitting Riemannian metrics with special holonomy groups in the following sections: Calabi-Yau 3-folds in (real) dimension six, and G_2-manifolds in dimension seven.

1.2 Calabi-Yau structures

We first recall the following definition:

Definition 1.2.1. An $SU(3)$-*structure* on a 6-manifold M is a pair (ω, Ω), for some non-degenerate 2-form ω, and complex volume form $\Omega = \mathrm{Re}\Omega + i\mathrm{Im}\Omega$, satisfying:

$$\omega \wedge \mathrm{Re}\Omega = 0 \qquad\qquad \tfrac{1}{6}\omega^3 = \tfrac{1}{4}\mathrm{Re}\Omega \wedge \mathrm{Im}\Omega \tag{1.8}$$

Moreover, it is enough to specify the pair of real forms $(\omega, \mathrm{Re}\Omega)$ to determine the $SU(3)$-structure [Hit00, §2].

Definition 1.2.2. A *Calabi-Yau structure* on a 6-manifold M is an $SU(3)$-structure (ω, Ω) that is torsion-free, i.e. $d\omega = d\Omega = 0$. A *Calabi-Yau 3-fold* (M, ω, Ω) is a (real) 6-manifold M, together with a Calabi-Yau structure (ω, Ω).

We will see some concrete examples of Calabi-Yau structures in the co-homogeneity one setting later in this section. In order to understand their construction, it will be useful to recall some generalities on Calabi-Yau geometry in co-dimension one, following [CS07]:

We start by noting that if $\iota : N \hookrightarrow M$ is an oriented immersion of a real 5-dimensional manifold N into a 6-manifold M, then an $SU(3)$-structure on M naturally equips N with an $SU(2)$-structure i.e. an $SU(2)$-reduction of the frame bundle. To be more explicit, by [CS07, Proposition 1], an $SU(2)$-reduction is equivalent to the following:

Definition 1.2.3. An $SU(2)$-*structure* on a 5-manifold N is a triple of 2-forms $(\omega_1, \omega_2, \omega_3)$ and a nowhere-vanishing 1-form η, satisfying:

1. $\omega_i \wedge \omega_j = \delta_{ij} v$, with v a fixed 4-form such that $v \wedge \eta$ is nowhere-vanishing: i.e. v is a volume form on the distribution $\mathcal{H} := \ker \eta$.

2. $X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X,Y) \geq 0$, i.e. $(\omega_1, \omega_2, \omega_3)$ is an oriented orthonormal basis of self-dual two-forms on $\mathcal{H}$, with respect to the volume form v, and a Riemannian metric g on $\mathcal{H}$ defined via $g(X,Y)v = \omega_1 \wedge (X \lrcorner \omega_2) \wedge (Y \lrcorner \omega_3)$ for $X, Y \in \mathcal{H}$.

So, given some 6-manifold M with an $SU(3)$-structure (ω, Ω), and an oriented immersion $\iota : N \hookrightarrow M$, an $SU(2)$-structure on N arises as follows:

$$\eta = \iota^* (\hat{n} \lrcorner \omega) \qquad \omega_1 = \iota^* \omega \qquad \omega_2 = \iota^* (\hat{n} \lrcorner \mathrm{Re}\Omega) \qquad \omega_3 = \iota^* (\hat{n} \lrcorner \mathrm{Im}\Omega) \tag{1.9}$$

where $\hat{n}$ the canonical unit normal to $\iota(N) \subset M$, defined by the chosen orientation and the Riemannian metric g induced by (ω, Ω).

If ι is an embedding, then we can view $N \subset M$ as an oriented hyper-surface, and a tubular neighbourhood $N \subset M$ can be identified with $N \times I$ for some interval $I \subseteq \mathbb{R}$ using the exponential map. In these coordinates, the metric on M appears as $g = dt^2 + g_t$ for some t-dependent metric g_t on N, and (1.9) gives rise to a family of $SU(2)$-structures $(\eta, \omega_i)_{t \in I}$ on N inducing g_t. In this neighbourhood, the $SU(3)$-structure (ω, Ω) on M takes the form:

$$\omega = dt \wedge \eta + \omega_1 \qquad \Omega = (dt + i\eta) \wedge (\omega_2 + i\omega_3) \tag{1.10}$$

and if (ω, Ω) is Calabi-Yau, i.e. ω, Ω are closed, then $(\eta, \omega_i)_{t \in I}$ satisfy the following structure equations on N:

$$d\omega_1 = 0 \qquad d(\omega_3 \wedge \eta) = 0 \qquad d(\omega_2 \wedge \eta) = 0 \tag{1.11}$$

along with the evolution equations for $t \in I$:

$$d\eta = \partial_t \omega_1 \qquad d\omega_2 = -\partial_t(\omega_3 \wedge \eta) \qquad d\omega_3 = \partial_t(\omega_2 \wedge \eta) \tag{1.12}$$

Observe that (1.11) is preserved under (1.12). This allows us to interpret a Calabi-Yau structure (at least locally) as a flow by (1.12) in the space of $SU(2)$-structures satisfying (1.11) on some fixed 5-manifold. This motivates the following definition:

Definition 1.2.4. A *hypo-structure* on a 5-manifold N is an $SU(2)$-structure satisfying (1.11). We refer to (1.12) as the *hypo-evolution equations.*

Putting aside completeness of the resulting metrics for a moment, the Riemannian cone is an important class of examples for this construction of Calabi-Yau metrics:

Example 1.2.1. (Calabi-Yau cone) Let N be a 5-manifold equipped with a fixed hypo-structure $(\eta^{se}, \omega_i^{se})$ satisfying the following structure equations:

$$d\eta^{se} = 2\omega_1^{se} \qquad d\omega_2^{se} = -3\omega_3^{se} \wedge \eta^{se} \qquad d\omega_3^{se} = 3\omega_2^{se} \wedge \eta^{se} \tag{1.13}$$

Then the following 1-parameter family $(\eta, \omega_i)_{t \in \mathbb{R}_{>0}}$ of $SU(2)$-structures:

$$\eta = t\eta^{se} \qquad \omega_i = t^2 \omega_i^{se} \tag{1.14}$$

satisfy (1.11), (1.12) iff $(\eta^{se}, \omega_i^{se})$ satisfies (1.13). As in (1.10), this family defines the *conical* $SU(3)$*-structure* (ω_C, Ω_C) on $\mathbb{R}_{>0} \times N$:

$$\omega_C = tdt \wedge \eta^{se} + t^2\omega_1^{se} \qquad\qquad \Omega_C = t^2\left(\omega_2^{se} + i\omega_3^{se}\right) \wedge \left(dt + it\eta^{se}\right) \tag{1.15}$$

which is Calabi-Yau iff $(\eta^{se}, \omega_i^{se})$ satisfies the structure equations (1.13). We refer to an $SU(2)$-structure $(\eta^{se}, \omega_i^{se})$ satisfying (1.13) as being *Sasaki-Einstein*: one can show that such an $SU(2)$-structure induces a Sasaki-Einstein metric g^{se} on N, or in other words, the Calabi-Yau metric g_C induced by (ω_C, Ω_C) on $\mathbb{R}_{>0} \times N$ is a metric cone $g_C = dt^2 + t^2 g^{se}$ over g^{se}. Sasaki-Einstein metrics exist in abundance, see e.g. [Spa11], so this construction yields many (incomplete) examples of Calabi-Yau metrics.

Another class of examples of Calabi-Yau metrics that can be constructed using solutions to (1.11), (1.12) are in the co-homogeneity one setting: concretely, there are four known examples of complete co-homogeneity one Calabi-Yau 3-folds in the literature. Three of these share a symmetry group[4] $SU(2)^2$, and can be written the form $M = SU(2)^2 \times_{H'} V$ for some singular isotropy subgroup $H' \subset SU(2)^2$, and H'-representation V:

Example 1.2.2. $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$, with a metric obtained by Candelas and de la Ossa in [CdlO90], also known as the *small resolution of the conifold.* The metric is unique up to rescaling by a constant factor, and as a co-homogeneity one manifold we have the diagram $\triangle U(1) \subset U(1) \times SU(2) \subset SU(2)^2$, where $\triangle U(1)$ is the diagonal $U(1)$ subgroup. The $U(1) \times SU(2)$ representation is given by the following: viewing $v \in V \cong \mathbb{C}^2$ as a quaternion, and $q \in SU(2)$ as a unit quaternion, then $(e^{i\theta}, q).v = qve^{-i\theta}$. By applying the outer automorphism exchanging the factors of $SU(2) \subset SU(2)^2$, we can get another co-homogeneity one metric from the small resolution, with singular isotropy group $U(1) \times SU(2) \subset SU(2)^2$, but this metric is distinct only up to equivariant isometries.

Example 1.2.3. T^*S^3 over S^3, with a metric also considered in [CdlO90] and found independently by Stenzel in [Ste93]. This is also referred to as the *smoothing of the conifold* and again, this metric is unique up to overall scale. The group diagram is $\triangle U(1) \subset \triangle SU(2) \subset SU(2)^2$, and we have as a $\triangle SU(2)$ representation $V \cong \mathfrak{su}(2)$, i.e. $SU(2)$ acts via the adjoint representation. As a smooth manifold, it is diffeomorphic to $\mathbb{R}^3 \times S^3$, the only rank 3 vector-bundle over S^3 up to diffeomorphism.

Example 1.2.4. $\mathcal{O}(-2,-2)$, the total space of the canonical bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$, with a metric found by Calabi in [Cal79] (unique up to overall scaling), which was later generalised to a one-parameter family of metrics by Pando-Zayas and Tseytlin in [PZT01]. This parameter represents the relative volume of each $\mathbb{CP}^1$ as the zero-section of $\mathcal{O}(-2,-2)$, and Calabi's construction considers the case when these two volumes are equal. The group diagram is $K_{2,-2} \subset U(1)^2 \subset SU(2)^2$, where $K_{2,-2}$ is the kernel of the map $U(1)^2 \to U(1)$ given by $(e^{i\theta_1}, e^{i\theta_2}) \mapsto e^{2i\theta_1 - 2i\theta_2}$, and as a $U(1)^2$-representation we have $V \cong \mathbb{C}_{2,-2}$, i.e. for complex number $V \ni v$, $(e^{i\theta_1}, e^{i\theta_2}).v = e^{2i(\theta_1-\theta_2)}v$. Note that there is a (non-unique)

[4]the construction of Calabi [Cal79] on the canonical bundle $\mathcal{O}(-3) \to \mathbb{CP}^2$ has co-homogeneity one symmetry group $SU(3)$, but the asymptotic geometry is less relevant to our considerations. We also thank Udhav Fowdar for pointing out that $\mathcal{O}(-3)$ cannot admit irreducible invariant instantons for the rank-one gauge groups considered in this thesis.

isomorphism $K_{2,-2} \cong \triangle U(1) \times \mathbb{Z}_2 \subset U(1)^2$, where we define $\triangle U(1) \times \mathbb{Z}_2 \subset U(1)^2$ as the (internal) direct product of the diagonal subgroup $\triangle U(1)$ and the $\mathbb{Z}_2$-subgroup generated by $(e^{2i\pi}, e^{i\pi})$, by sending $K_{2,-2} \ni (e^{i\theta_1}, e^{i\theta_2}) \mapsto (e^{i\theta_1}, e^{i\theta_1}).(e^{2i\pi}, e^{i(\theta_2-\theta_1)}) \in \triangle U(1) \times \mathbb{Z}_2$.

The asymptotic model for the geometry of these spaces (up to $\mathbb{Z}_2$-cover) is the unique co-homogeneity one Calabi-Yau metric cone over $SU(2)^2/\triangle U(1) \cong S^2 \times S^3$, referred to as the *conifold* in [CdlO90]. In the co-homogeneity one setting, there is an obvious diffeomorphism identifying the space of principal orbits with the smooth manifold underlying the conifold, and pulling back any of these asymptotically conical metrics to a metric on the conifold via this diffeomorphism, by [CdlO90], [PZT01], we have $|i^*g - g_C| \to 0$ as $t \to \infty$, where t denotes the radial parameter on the cone, i^*g denotes the pulled-back metric, and we take norms with respect to the conical metric g_C.

To have a uniform set-up for the gauge theory in later sections, we will describe the construction of these examples in detail. This will involve finding solutions to the evolution equations (1.12) in the space of invariant hypo-structures on the principal orbit $S^2 \times S^3 = SU(2)^2/\triangle U(1)$, and extending to the singular orbits in the complete cases.

We will use the basis E_1, E_2, E_3 for $\mathfrak{su}(2)$ such that $[E_i, E_j] = 2E_k$ for cyclic permutations of (123), and fix a basis for left-invariant vector fields on $SU(2)^2$:

$$\begin{aligned} U^1 &:= (E_1, 0) \quad V^1 := (E_2, 0) \quad W^1 := (E_3, 0) \\ U^2 &:= (0, E_1) \quad V^2 := (0, E_2) \quad W^2 := (0, E_3) \end{aligned} \tag{1.16}$$

with respective dual one-forms $u^1, v^1, w^1, u^2, v^2, w^2$. Denote $U^\pm := U^1 \pm U^2$, $V^\pm := V^1 \pm V^2$, $W^\pm := W^1 \pm W^2$ with respective dual one-forms $u^\pm, v^\pm, w^\pm$. Here, the vector field U^+ generates the diagonal subgroup $\triangle U(1)$.

We let $\mathfrak{m} \subset \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ be the $\triangle U(1)$-invariant complement of the span of U^+. Observe that the adjoint action of the isotropy subgroup $\triangle U(1)$ on $\mathfrak{m}$ is given by $\mathfrak{m} = \langle U^-, V^1, W^1, V^2, W^2 \rangle \cong \mathbb{R} \oplus \mathbb{C}_2 \oplus \mathbb{C}_2$, where U^- spans a trivial representation of $\triangle U(1)$, and $\{V^i, W^i\}_{i=1,2}$ each span a complex one-dimensional representation with weight two.

Note that if $\mathbb{Z}_2 \subset SU(2)^2$ is a subgroup of the flow generated by the vector field U^-, then the adjoint action of $\mathbb{Z}_2$ on $\mathfrak{m}$ is trivial. Hence, without loss of generality, we will always assume the principal isotropy subgroup of the $SU(2)^2$-action is $\triangle U(1)$ in this section.

With this in place, let us define the *standard* invariant Sasaki-Einstein structure $(\eta^{se}, \omega_i^{se})$ on $SU(2)^2/\triangle U(1)$ as:

$$\begin{aligned} \eta^{se} &:= \tfrac{4}{3} u^- & \omega_1^{se} &:= -\tfrac{2}{3}(v^1 \wedge w^1 - v^2 \wedge w^2) \\ \omega_2^{se} &:= \tfrac{2}{3}(v^1 \wedge v^2 + w^1 \wedge w^2) & \omega_3^{se} &:= \tfrac{2}{3}(v^1 \wedge w^2 - w^1 \wedge v^2) \end{aligned} \tag{1.17}$$

It is easy to check that $(\eta^{se}, \omega_i^{se})$ satisfies the Sasaki-Einstein structure equations (1.13). The corresponding Calabi-Yau cone over $SU(2)^2/\triangle U(1)$ has the conical $SU(3)$-structure (ω_C, Ω_C), as in (1.15), and we refer to this cone as the *conifold*[5].

[5] note that any invariant Sasaki-Einstein structure on $SU(2)^2/\triangle U(1)$ can be obtained from (1.17) by rotating the plane spanned by $(\omega_2^{se}, \omega_3^{se})$. However, since any of two of these structures induce the same Sasaki-Einstein metric g^{se}, we will make this particular choice without loss of generality.

Furthermore, it is not hard to show that the space of invariant two-forms on $SU(2)^2/\triangle U(1)$ is four-dimensional, and spanned by $\omega_0^{se}, \omega_1^{se}, \omega_2^{se}, \omega_3^{se}$, where we define:

$$\omega_0^{se} := \tfrac{2}{3}(v^1 \wedge w^1 + v^2 \wedge w^2) \tag{1.18}$$

By using this basis of invariant two-forms and the invariant one-form η^{se}, we have the following description of the space of hypo-structures:

Proposition 1.2.1 ([FH17]). *Up to transformations by isometries with respect to the induced metric, any invariant non-degenerate family of hypo-structures* $(\eta, \omega_1, \omega_2, \omega_3)_{t \in I}$ *on* $SU(2)^2/\triangle U(1)$ *can be written:*

$$\eta = \lambda \eta^{se} \quad \omega_1 = u_0 \omega_0^{se} + u_1 \omega_1^{se} \quad \omega_2 = \mu \omega_2^{se} \quad \omega_3 = v_0 \omega_0^{se} + v_3 \omega_3^{se} \tag{1.19}$$

for some $\lambda(t), u_0(t), u_1(t), v_0(t), v_3(t)$ *such that* $\mu^2 := -u_0^2 + u_1^2 = -v_0^2 + v_3^2 > 0$*,* $\lambda > 0$*, and* $v_0 u_0 = 0$*.*

At least one of v_0 or u_0 must vanish: if v_0 vanishes, we will refer to this family as a *hypo-structure of type* $\mathcal{I}$, while if u_0 vanishes, we will refer to this family as a *hypo-structure of type* $\mathcal{II}$. We will write these two situations explicitly below, along with corresponding hypo-evolution equations (1.12):

1. Type $\mathcal{I}$:

$$\eta = \lambda \eta^{se} \qquad \omega_1 = u_0 \omega_0^{se} + u_1 \omega_1^{se} \qquad \omega_2 = \mu \omega_2^{se} \qquad \omega_3 = \mu \omega_3^{se} \tag{1.20}$$

   The corresponding hypo-evolution equations are:

$$\partial_t u_0 = 0 \qquad \partial_t u_1 = 2\lambda \qquad \partial_t(\lambda \mu) = 3\mu \tag{1.21}$$

2. Type $\mathcal{II}$:

$$\eta = \lambda \eta^{se} \qquad \omega_1 = \mu \omega_1^{se} \qquad \omega_2 = \mu \omega_2^{se} \qquad \omega_3 = v_0 \omega_0^{se} + v_3 \omega_3^{se} \tag{1.22}$$

   The corresponding hypo-evolution equations are:

$$\partial_t \mu = 2\lambda \qquad \partial_t(\mu\lambda) = 3v_3 \qquad \partial_t(\lambda v_3) = 3\mu \qquad \partial_t(\lambda v_0) = 0 \tag{1.23}$$

The constants u_0, $-\lambda v_0$ appearing in (1.21), (1.23) are fixed by topological data: they are the coefficients of co-homology classes $[\omega], [\mathrm{Re}\Omega]$ on $\mathbb{R}_{>0} \times S^2 \times S^3$ with respect to the generators $[\omega_0^{se}]$, $[\omega_0^{se} \wedge \eta^{se}]$. If both these constants vanish, then $u_1 = v_3 = \mu$, and clearly $\lambda = t, \mu = t^2$ is a solution to the resulting evolution equations:

$$\partial_t \mu = 2\lambda \qquad \partial_t(\mu\lambda) = 3\mu$$

giving rise to the conical Calabi-Yau structure (ω_C, Ω_C) of the conifold.

For each of the families, one can write down the corresponding invariant Calabi-Yau metric $g = dt^2 + g_t$ explicitly on the space of principal orbits, cf. [FH17, Prop.2.16]:

1. Type $\mathcal{I}$:

$$g = dt^2 + \lambda^2(\eta^{se})^2 + \tfrac{2}{3}(u_1 - u_0)\left((v^1)^2 + (w^1)^2\right) + \tfrac{2}{3}(u_1 + u_0)\left((v^2)^2 + (w^2)^2\right) \tag{1.24}$$

2. Type $\mathcal{II}$:

$$g = dt^2 + \lambda^2(\eta^{se})^2 + \tfrac{4}{3}(v_3 - v_0)\left((v^-)^2 + (w^-)^2\right) + \tfrac{4}{3}(v_3 + v_0)\left((v^+)^2 + (w^+)^2\right) \tag{1.25}$$

With this description in hand, the problem of finding invariant Calabi-Yau metrics on the space of principal orbits is reduced to finding solutions to the evolution equations (1.21) or (1.23). We can write the complete Calabi-Yau metrics as solutions extending to the singular orbits at $t = 0$:

Lemma 1.2.2 ([PZT01],[CdlO90]). *Up to transformations by isometries with respect to the induced metric, the space of $SU(2)^2$-invariant Calabi-Yau structures (ω, Ω) on M can be identified with:*

(i) *For $M = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, the ray $\{(U_0, U_1) \in \mathbb{R}^2 \mid U_1 = -U_0 < 0\}$.*

(ii) *For $M = \mathcal{O}(-2, -2)$, the open convex cone $\{(U_0, U_1) \in \mathbb{R}^2 \mid U_1 > |U_0| \geq 0\}$.*

These invariant Calabi-Yau structures induce a hypo-structure of type $\mathcal{I}$ on the principal orbits, with $(U_0, U_1) := (u_0(0), u_1(0))$, and:

$$\mu^2 = u_1^2 - U_0^2 \qquad \lambda^2 = \frac{u_1^3 - 3U_0^2 u_1 + U_1(3U_0^2 - U_1^2)}{u_1^2 - U_0^2} \tag{1.26}$$

The proof of part (i) in the co-homogeneity one set-up can be found in [FH17, Thm.2.27], part (ii) is similar, and verifying that these Calabi-Yau structures extend to the singular orbits is left to the appendix A.1.

We will comment on the parameters (U_0, U_1) appearing in Lemma 1.2.2: clearly $\{(U_0, U_1) \in \mathbb{R}^2 \mid U_1 > |U_0| \geq 0\}$ can be identified with the Kähler cone of $\mathcal{O}(-2, -2)$ generated by the Kähler classes of the two copies of $\mathbb{CP}^1 \subset \mathcal{O}(-2, -2)$. It is not hard to see that multiplicative rescalings of the cone are equivalent to constant rescalings of the metric, and the point $U_0 = U_1 = 0$ is identified with the conifold $u_1 = \mu = t^2$, $\lambda = t$. Furthermore, the diffeomorphism arising from exchanging the two copies of $\mathbb{CP}^1$ acts on this cone via reflection $U_0 \to -U_0$, and the Calabi construction in [Cal79] produces exactly the metrics in the subset fixed by this action.

Calabi-Yau structures on the cone boundary $U_1 = \pm U_0$ (excluding the origin) are not quite the same as those found on the boundary of the Kähler cone of $\mathcal{O}(-2, -2)$ however, which generically have $\mathbb{Z}_2$-quotient singularities. Rather, they are a (smooth) branched double-covering[6]: up to exchanging the factors of $\mathbb{CP}^1$, this boundary gives the Calabi-Yau structure on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1 = SU(2)^2/U(1) \times SU(2)$. In the rest

[6]these quotient singularities do not appear in our set-up, as we only define λ, μ, u_0, u_1 at the identity coset on the principal orbit.

of this thesis, for ease of notation, we will fix the scaling convention for this metric to be $(U_0, U_1) = (-1, 1)$.

Finally, for the Calabi-Yau structure on T^*S^3, we give the explicit solutions to (1.23) extending to the singular orbit S^3 cf. [FH17, Thm.2.27]:

Lemma 1.2.3 ([Ste93])**.** *Up to scale, and transformations by isometries with respect to the induced metric, there is a unique $SU(2)^2$-invariant Calabi-Yau structure on T^*S^3. It induces a hypo-structure of type $\mathcal{II}$ on the principal orbits, with:*

$$\begin{aligned} \lambda &= \left(\frac{2}{3}\right)^{\frac{1}{3}} \frac{\sinh 3s}{(\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}}} & \mu &= \left(\frac{2}{3}\right)^{\frac{2}{3}} (\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}} \\ v_0 &= -\left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{(\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}}}{\sinh 3s} & v_3 &= \left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{(\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}}}{\tanh 3s} \end{aligned} \tag{1.27}$$

for $s \in [0, \infty)$, where $s(t) := \int_0^t \lambda^{-1}(\hat{t}) d\hat{t}$

1.3 G_2 structures

Recall that a G_2-structure on a 7-manifold M is a reduction of the frame bundle to the exceptional Lie group G_2. This is equivalent to existence of a non-degenerate 3-form $\varphi \in \Omega^3(M)$ which is fixed by the point-wise action of $G_2 \subset SO(7)$ in some framing of the tangent space at every point. Furthermore, this data determines a Riemannian metric, volume-form, and orientation on M.

Existence of such a structure on some oriented 7-manifold is purely topological: it is equivalent to the existence of a spin structure [Joy07, Ch.11]. On the other hand, constructing torsion-free G_2-structures is much more difficult:

Definition 1.3.1. A G_2-manifold (M, φ) is a 7-manifold equipped with a torsion-free G_2-structure, i.e. $d\varphi = d^*\varphi = 0$.

Although their existence was first suggested by the work of Berger [Ber55], the first examples of complete, irreducible G_2-manifolds were only constructed much later in [BS89], by exploiting co-homogeneity one symmetries. Subsequently, another co-homogeneity one example was found in [BGGG01], which was later generalised to a one-parameter family in [Bog13], and a partial proof of existence for a second one-parameter family was given in [BB13]. More recently, infinitely many families of complete, co-homogeneity one G_2-metrics have been found in [FHN21b], confirming earlier predictions in the physics literature [CGLP02]. In order to understand these constructions, it will be useful to recall some definitions regarding G_2 geometry in co-dimension one:

If $\iota : N \hookrightarrow M$ is an oriented immersion of a 6-manifold N into M, then a G_2-structure φ on M naturally equips N with an $SU(3)$-structure:

$$\omega = \iota^* (\hat{n} \lrcorner \varphi) \qquad \mathrm{Re}\Omega = \iota^* \varphi \qquad \mathrm{Im}\Omega = \iota^* (-\hat{n} \lrcorner * \varphi) \tag{1.28}$$

where $\hat{n}$ the canonical unit normal to $\iota(N) \subset M$, defined by the chosen orientation and the Riemannian metric induced by φ.

If ι is an embedding, then we can view $N \subset M$ as an oriented hyper-surface, and a tubular neighbourhood of $N \subset M$ can be identified with $N \times I$ for some interval $I \subseteq \mathbb{R}$ using the exponential map. In these coordinates, the metric on M appears as $g = dt^2 + g_t$ for some t-dependent metric g_t on N, and (1.28) gives rise to a family of $SU(3)$-structures $(\omega, \Omega)_{t\in I}$ inducing g_t. Meanwhile, the G_2-structure on M appears as:

$$\varphi = dt \wedge \omega + \mathrm{Re}\Omega \qquad *\varphi = -dt \wedge \mathrm{Im}\Omega + \tfrac{1}{2}\omega^2 \tag{1.29}$$

and if this G_2-structure is torsion-free, then $(\omega, \Omega)_{t\in I}$ satisfy the following structure equations on N:

$$d\omega \wedge \omega = 0 \qquad d\mathrm{Re}\Omega = 0 \tag{1.30}$$

subject to the evolution equations:

$$d\omega = \partial_t \mathrm{Re}\Omega \qquad d\mathrm{Im}\Omega = -\tfrac{1}{2}\partial_t \left(\omega^2\right) \tag{1.31}$$

Observe that (1.30) is preserved under (1.31). This allows us to interpret a torsion-free G_2-structure (at least locally) as a flow by (1.30) in the space $SU(3)$-structures satisfying (1.30) on some fixed 6-manifold, cf. [Hit01]. This motivates the following definition:

Definition 1.3.2. A *half-flat structure* on a 6-manifold N is an $SU(3)$-structure (ω, Ω) on N satisfying (1.30). We refer to (1.31) as the *half-flat evolution equations.*

Furthermore, by [Bry10, Thm.17.4], every real-analytic half-flat structure arises this way, i.e. is induced by an embedding into a G_2-manifold.

Example 1.3.1. (G_2 cone) Let N be a 6-manifold equipped with a fixed $SU(3)$-structure $\left(\omega^{nK}, \Omega^{nK}\right)$ satisfying the following structure equations:

$$d\omega^{nK} = 3\mathrm{Re}\Omega^{nK} \qquad d\mathrm{Im}\Omega^{nK} = -2\left(\omega^{nK}\right)^2 \tag{1.32}$$

Then the following 1-parameter family $(\omega, \Omega)_{t\in\mathbb{R}_{>0}}$ of $SU(3)$-structures:

$$\omega = t^2\omega^{nK} \qquad \Omega = t^3\Omega^{nK} \tag{1.33}$$

satisfy (1.30), (1.31) iff $\left(\omega^{nK}, \Omega^{nK}\right)$ satisfies (1.32). As in (1.29), this family defines the *conical G_2-structure* φ_C on $\mathbb{R}_{>0} \times N$:

$$\varphi_C = t^2 dt \wedge \omega^{nK} + t^3\mathrm{Re}\Omega^{nK} \qquad *\varphi_C = -t^3 dt \wedge \mathrm{Im}\Omega^{nK} + \tfrac{1}{2}t^4\left(\omega^{nK}\right)^2 \tag{1.34}$$

which is torsion-free iff $\left(\omega^{nK}, \Omega^{nK}\right)$ satisfies the structure equations (1.32). We refer to an $SU(3)$-structure $\left(\omega^{nK}, \Omega^{nK}\right)$ satisfying (1.32) as being *nearly-Kähler*: one can show that such an $SU(3)$-structure induces a nearly-Kähler metric g^{nK} on N, or in other words, the G_2-metric g_C induced by φ_C on $\mathbb{R}_{>0} \times N$ is a metric cone $g_C = dt^2 + t^2 g^{nK}$. The handful of known nearly-Kähler metrics, either those with homogeneous [WG68] or co-homogeneity one [FH17] symmetries, provide (incomplete) examples of G_2-metrics using this construction.

Example 1.3.2. (Locally-conical G_2) Take a 5-manifold N' equipped with a Sasaki-Einstein structure $(\eta^{se}, \omega_i^{se})$, and a principal S^1-bundle $S^1 \hookrightarrow N \to N'$ equipped with a connection 1-form θ such that the curvature $d\theta$ satisfies $d\theta \wedge \omega_i^{se} = 0$, i.e. $d\theta$ is an anti-self-dual 2-form for the induced metric on $\mathcal{H} = \ker \eta^{se}$, cf. [FHN21a, Lemma 3.5]. Then, for some constant $\ell > 0$, the family of $SU(3)$-structures $(\omega, \Omega)_{t \in \mathbb{R}_{>0}}$ on N:

$$\omega = \ell t \eta^{se} \wedge \theta + t^2 \omega_2^{se} \qquad \Omega = t^2 \left(\ell\theta - it\eta^{se}\right) \wedge \left(\omega_1^{se} - i\left(\omega_3^{se} - \tfrac{\ell}{2t} d\theta\right)\right) \tag{1.35}$$

satisfies (1.30), (1.31). If we define the *locally conical* G_2-structure φ_{LC} using this family as in (1.29), then metric g_{LC} induced by φ_{LC} satisfies $g_{LC} = dt^2 + t^2 g^{se} + \ell^2\theta^2 + O(t^{-1})$ as $t \to \infty$, where g^{se} is the Sasaki-Einstein metric on N' induced by $(\eta^{se}, \omega_i^{se})$.

Moreover, if we denote ξ as the unit vector-field with respect to g_{LC} associated to the infinitesimal S^1-action on N, then we can recover an asymptotically conical $SU(3)$-structure (ω', Ω') on $N' \times \mathbb{R}_{>0}$ by defining:

$$\omega' := \xi \lrcorner \varphi_{LC} \qquad \mathrm{Re}\Omega' := \varphi_{LC}|_{\ker\theta} \qquad \mathrm{Im}\Omega' := -\xi \lrcorner * \varphi_{LC} \tag{1.36}$$

The torsion of this $SU(3)$-structure (1.36) vanishes as $\ell \to 0$ and we recover the conical Calabi-Yau-structure (1.15) from (1.36) in this limit.

Another class of examples of G_2-metrics that can be constructed from solutions to (1.30), (1.31) are the co-homogeneity one setting:

Example 1.3.3. The spinor bundle $\mathbf{S}(S^3)$ of the 3-sphere admits a one-parameter family of co-homogeneity one G_2-metrics described by Bryant-Salamon in [BS89]. This parameter represents the volume of the zero-section $S^3 \subset \mathbf{S}(S^3)$, or alternatively, the coefficient of the co-homology class $[\varphi]$ of $\mathbf{S}(S^3)$.

The total space of the spinor bundle can be written as $\mathbf{S}(S^3) = SU(2)^2 \times_{\Delta SU(2)} \mathbb{H}$ where $\Delta SU(2)$ acts on the right diagonally, and admits co-homogeneity one action of $SU(2)^3$, viewed here as acting on the left. The corresponding group diagram is $\triangle_{1,2,3}SU(2) \subset \triangle_{1,2}SU(2) \times SU(2) \subset SU(2)^3$. Here, $\triangle_{1,2}SU(2), \triangle_{1,2,3}SU(2) \subset SU(2)^3$ denotes the diagonal $SU(2)$-subgroup in the first two factors of $SU(2)^3$, and the diagonal subgroup in all three factors respectively.

As a smooth manifold, $\mathbf{S}(S^3)$ is diffeomorphic to $S^3 \times \mathbb{R}^4$. This diffeomorphism can be written $SU(2)^3$-equivariantly as a map $\mathbf{S}(S^3) \to SU(2) \times \mathbb{H}$ descending from the map $SU(2)^2 \times \mathbb{H} \to SU(2) \times \mathbb{H}$ given by $(p_1, p_2, v) \mapsto (p_1\bar{p}_2, v\bar{p}_1)$. Here, we identify $S^3 \times \mathbb{R}^4$ with $SU(2) \times \mathbb{H}$, equipped with the $SU(2)^3$-action

$$(q_1, q_2, q_3) \cdot (p, v) \mapsto (q_1 p \bar{q}_2, q_3 v \bar{q}_1) \tag{1.37}$$

The asymptotic model for the geometry of these metrics, up to overall scale, is the co-homogeneity one G_2-cone over the nearly-Kähler $S^3 \times S^3 = SU(2)^3/\triangle_{1,2,3}SU(2)$. Note that the group of outer-automorphisms permuting the factors of $SU(2)^3$ yield non-equivalent isometric actions of $SU(2)^3$ on $\mathbf{S}(S^3)$ in general, although these induce equivalent actions on the cone. In particular, the subgroup of cyclic permutations gives three non-equivariantly isometric realisations of $\mathbf{S}(S^3)$.

Example 1.3.4. $S^3 \times \mathbb{R}^4$ admits another co-homogeneity one G_2-metric, first described in [BGGG01], and later generalised to a one-parameter family by [Bog13], referred to as $\mathbb{B}_7$ family. Here, the group acting is $SU(2)^2 \times U(1) \subset SU(2)^3$ via (1.37). These metrics have asymptotically locally conical geometry, with end modelled on a circle bundle fibred over the conifold, pulled back from $SU(2)^2$ fibred over the link $SU(2)^2/\triangle U(1)$. This bundle is equipped with a connection one-form coming from the canonical connection on $SU(2)^2 \to SU(2)^2/\triangle U(1)$, and the freedom to rescale the length of the circle fibre by some constant, with fixed overall scale, gives rise to the one-parameter family of $\mathbb{B}_7$ metrics.

It can be shown that, as the length ℓ of this asymptotic circle fibre collapses to zero in this one-parameter family, the G_2-metric degenerates to an $SU(2)^2$-invariant Calabi-Yau metric on the $U(1)$-quotient of $S^3 \times \mathbb{R}^4$: this collapsed limit is precisely the co-homogeneity one Calabi-Yau metric on the smoothing T^*S^3.

Example 1.3.5. $S^3 \times \mathbb{R}^4$, admits yet another distinct family of co-homogeneity one G_2-metrics, referred to as the $\mathbb{D}_7$ family in [CGLP02], with a rigorous proof of existence found later in [FHN21b]. Again, the group acting is $SU(2)^2 \times U(1)$, but with $SU(2)^2 \times U(1) \subset SU(2)^3$ acting as (1.37) with a cyclic permutation[7] $(123) \in S_3$. These have the same asymptotically locally conical geometry as the $\mathbb{B}_7$-family, with ends modelled on a circle bundle fibred over the conifold, pulled back from $SU(2)^2$ fibred over the link.

It can be shown that, as the length ℓ of this asymptotic circle fibre collapses to zero in this one-parameter family, the G_2-metric degenerates to an $SU(2)^2$-invariant Calabi-Yau metric on the $U(1)$-quotient of $S^3 \times \mathbb{R}^4$: this collapsed limit is precisely the co-homogeneity one Calabi-Yau metric on the small resolution $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

For the remainder of this section, we will describe Example 1.3.3 on the spinor bundle $\mathbf{S}(S^3)$ in further detail, following [LO18]. To align with the notation in §1.2, we will $SU(2)^3$-equivariantly identify the principal orbit $SU(2)^3/\triangle_{1,2,3}SU(2)$ with $SU(2)^2$ via the inclusion map $SU(2)^2 \times \{1\} \hookrightarrow SU(2)^3$, where we view $SU(2)^2 \times \{1\}$ acting on the left of $SU(2)^2$ in the obvious way, and $\{1\} \times \{1\} \times SU(2)$ acting diagonally on the right. Since $\{1\} \times \{1\} \times SU(2)$ acts trivially on the singular orbit, we can identify the singular orbit $SU(2)^3/\triangle_{1,2}SU(2) \times SU(2)$ with $SU(2)^2/\triangle_{1,2}SU(2)$ in the same way.

Using basis U^+, V^+, W^+, U^-, V^-, W^- of left-invariant vector-fields on $SU(2)^2$ as in (1.16), we can then write the tangent space of $SU(2)^2$ as:

$$\langle U^+, V^+, W^+\rangle \oplus \langle U^-, V^-, W^-\rangle \cong \mathfrak{su}_+(2) \oplus \mathfrak{su}_-(2) \tag{1.38}$$

i.e. $\triangle SU(2)$ acts as two copies of the adjoint representation.

As is shown in [MS13], up to isometries with respect to the induced metric, the space of $SU(2)^3$-invariant half-flat structures on $S^3 \times S^3$ is at most two-dimensional. Up to these isometries, any non-degenerate family of $SU(2)^3$-invariant half-flat structures $(\omega, \Omega)_{t \in \mathbb{R}_{>0}}$

[7] taking the inverse permutation yields another non-equivariantly isometric copy of the $\mathbb{D}_7$ family.

appears as:

$$\begin{aligned}\mathrm{Re}\Omega &= +8\beta^3 u^- \wedge v^- \wedge w^- - 8\alpha^2\beta \sum_{(uvw)} u^+ \wedge v^+ \wedge w^- \\ \mathrm{Im}\Omega &= -8\alpha^3 u^+ \wedge v^+ \wedge w^+ + 8\alpha\beta^2 \sum_{(uvw)} u^- \wedge v^- \wedge w^+ \\ \omega &= 4\alpha\beta\left(u^- \wedge u^+ + v^- \wedge v^+ + w^- \wedge w^+\right)\end{aligned} \tag{1.39}$$

for some $\alpha > 0, \beta > 0$ real-valued functions of t, where $\sum_{(uvw)}$ denotes the sum over cyclic permutations of (uvw). The induced metric $dt^2 + g_t$ on $\mathbb{R}_{>0} \times S^3 \times S^3$ is given by:

$$g = dt^2 + 4\alpha^2\left(\left(u^+\right)^2 + (v^+)^2 + (w^+)^2\right) + 4\beta^2\left((u^-)^2 + (v^-)^2 + (w^-)^2\right) \tag{1.40}$$

This metric has holonomy contained in G_2 iff $(\omega, \Omega)_{t\in\mathbb{R}_{>0}}$ solve the evolution equations (1.31), i.e. α, β are solutions to:

$$\dot{\alpha} = \frac{1}{2}\left(1 - \frac{\alpha^2}{\beta^2}\right) \qquad \dot{\beta} = \frac{\alpha}{\beta} \tag{1.41}$$

Clearly, taking $\alpha = \frac{1}{3}t$, $\beta = \frac{\sqrt{3}}{3}t$ is a solution to (1.41): this corresponds to taking the conical G_2-structure (1.34) over the unique (up to isometries) $SU(2)^3$-invariant nearly-Kähler $SU(3)$-structure $\left(\omega^{nK}, \Omega^{nK}\right)$ on $S^3 \times S^3$, given by setting $\alpha = \frac{1}{3}$, $\beta = \frac{\sqrt{3}}{3}$ in (1.39). It is straightforward to verify that the $SU(3)$-structure defined this way satisfies the nearly-Kähler structure equations (1.39).

There is also a one-parameter family of complete solutions to (1.41) extending to the singular orbit $S^3 = SU(2)^2/\triangle_{1,2}SU(2)$, representing the volume of this orbit with respect to the induced metric. However, without loss of generality, we will fix this volume by the following transformation:

Lemma 1.3.1. *$(\alpha, \beta) \mapsto (\alpha_\delta, \beta_\delta)$ is a symmetry of (1.41), where $\alpha_\delta(t) := \frac{\alpha(\delta t)}{\delta}$, $\beta_\delta(t) := \frac{\beta(\delta t)}{\delta}$ for some $\delta > 0$.*

Up to overall rescaling of the metric on $\mathbf{S}(S^3)$, this transformation describes pulling back the metric by the diffeomorphism rescaling the fibres of $\mathbf{S}(S^3)$ by a constant δ. Up to this rescaling, we can uniquely solve (1.41) extending to the singular orbit by change of variable $r(t) = \sqrt{3}\beta(t)$, $r \in [1, \infty)$, so that the solution to (1.41) appears as:

$$\alpha = \tfrac{r}{3}\sqrt{1 - r^{-3}} \qquad \beta = \tfrac{r}{\sqrt{3}} \tag{1.42}$$

As previously mentioned, the asymptotic model for this geometry is the cone over the $SU(2)^3$-invariant nearly-Kähler $SU(3)$-structure $\left(\omega^{nK}, \Omega^{nK}\right)$ on $S^3 \times S^3$. Outside of the singular orbit, we can identify $\mathbf{S}(S^3)$ with the smooth manifold $\mathbb{R}_{>0} \times S^3 \times S^3$ underlying the cone, so that the complete G_2-metric g as in (1.40) with (α, β) defined by (1.42) satisfies $|g - g_C| = O(t^{-3})$ as $t \to \infty$, where t denotes the radial parameter on the cone, and we take norms with respect to the conical metric g_C, see [BS89].

Chapter 2

Calabi-Yau Gauge Theory: Set-up and Results

In the next two chapters, we will give a complete description of the behaviour of Calabi-Yau instantons and monopoles with an $SU(2)^2$-symmetry, on Calabi-Yau 3-folds with asymptotically conical geometry and $SU(2)^2$ acting with co-homogeneity one.

We consider gauge theory on the smoothing and small resolution of the conifold, and on the canonical bundle of $\mathbb{CP}^1 \times \mathbb{CP}^1$, with their known asymptotically conical co-homogeneity one Calabi-Yau metrics, and find new one-parameter families of invariant instantons. We also entirely classify the relevant moduli-spaces of instantons and monopoles satisfying a natural curvature decay condition, and show that the expected bubbling phenomena occur in these families.

Before carrying this out, in §2.1 we will give an overview of the Calabi-Yau monopole and instanton equations, and of the results of the analysis contained in Chapter §3.

Once this is done, in §2.3 Proposition 2.8, we will write the $SU(2)^2$-invariant monopole equation on the smoothing, small resolution, and the canonical bundle of $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a system of ODEs. Before analysing the full system in the next chapter §3, in §2.4, we briefly mention explicit solutions to these ODEs corresponding to reducible connections.

2.1 Overview

Let $\left(M^6, \omega, \Omega\right)$ be a Calabi-Yau 3-fold as in Definition 1.2.2, where ω denotes the Kähler form, and $\Omega = \mathrm{Re}\Omega + i\mathrm{Im}\Omega$ denotes the holomorphic volume form on M such that $\frac{1}{6}\omega^3 = \frac{1}{4}\mathrm{Re}\Omega \wedge \mathrm{Im}\Omega$, and fix a principal G-bundle $P \to M$ with a compact semi-simple Lie group G. The pair (A, Φ), for some connection A on P and non-trivial $\Phi \in \Omega^0\left(\mathrm{ad}P\right)$, is called a (Calabi-Yau) *monopole* if it satisfies the *Calabi-Yau monopole equations*:

$$F_A \wedge \omega^2 = 0 \qquad\qquad F_A \wedge \mathrm{Re}\Omega = *d_A\Phi \tag{2.1}$$

where $*$ is the Hodge star of the Riemannian metric defined by (ω, Ω), $F_A \in \Omega^2\left(\mathrm{ad}P\right)$ is the curvature of A, and $d_A : \Omega^0\left(\mathrm{ad}P\right) \to \Omega^1\left(\mathrm{ad}P\right)$ is the induced covariant derivative. We refer to the section Φ as the *Higgs field* for this monopole.

We obtain the *Calabi-Yau instanton equations* for a connection A on P by setting

$\Phi = 0$ in (2.1):

$$F_A \wedge \omega^2 = 0 \qquad\qquad F_A \wedge \mathrm{Re}\Omega = 0 \tag{2.2}$$

Note that if a monopole (A, Φ) has $d_A\Phi = 0$, then A is also a (Calabi-Yau) *instanton*, i.e. a solution of (2.2), but the existence of a non-trivial parallel section Φ implies that the connection A must be reducible in this case.

In terms of the complex geometry, the first condition of (2.2) says that F_A is a primitive Lie algebra-valued two-form, while the second condition says it is of type $(1,1)$. Furthermore, it is not hard to prove that instantons minimize the Yang-Mills energy functional $\mathcal{YM}(A) := \int_M |F_A|^2$ on the space of connections on P, where we take point-wise norms with respect to some ad-invariant metric on the Lie algebra of G. Hence, on the special unitary frame bundle $SU(E)$ of some hermitian vector bundle E over M with trivial determinant bundle, a Calabi-Yau instanton is also referred to as a Hermitian Yang-Mills (HYM) connection in the literature.

When G is abelian, (2.1) and (2.2) are linear equations, and the moduli-space of their solutions are well-understood: if $G = U(1)$ for example, any two-form on M which is an instanton in the sense of (2.2) is harmonic, with the converse holding when (M, ω, Ω) is compact with full holonomy $SU(3)$. Even when M is non-compact, every $U(1)$-bundle carries a unique Calabi-Yau instanton with decaying curvature when (ω, Ω) is asymptotically conical with full holonomy $SU(3)$ by [FHN21a, Theorem 5.12]. For non-abelian gauge groups, one usually seeks a description of the gauge theory starting with the next simplest case of rank one groups: in particular, without loss of generality[1], we will always take the gauge group to be $SU(2)$ in this chapter.

We will study the Calabi-Yau monopole equations (2.1) with the assumption of $SU(2)^2$-symmetry for both the Calabi-Yau structure and the bundle data: the Calabi-Yau monopole equations were first studied in this setting by [Oli16], for the asymptotically conical metric of Stenzel [Ste93] on the cotangent bundle of S^3. There is a one-parameter family of invariant monopoles for this metric, with a single explicit instanton [Oli16, Theorem 2] appearing at the boundary of this family when the Higgs field vanishes. In this thesis, we will independently verify this claim using new proofs, as well as proving that the explicit instanton actually lies in a one-parameter family of invariant instantons for this metric. We also describe the invariant gauge theory for all the other known examples of $SU(2)^2$-invariant AC Calabi-Yau metrics, namely the metric of Candelas and de la Ossa [CdlO90] on the small resolution of the conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$, and the metric of Calabi [Cal79], later generalised to a one-parameter family by Pando-Zayas and Tseytlin [PZT01], on the canonical bundle $\mathcal{O}(-2,-2)$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

To understand the various components of these gauge-theoretic moduli-spaces, we must first discuss fixing the asymptotic behaviour of solutions. A natural condition on a solution of (2.1) on an asymptotically conical metric is that it converges at the conical end to some model solution (A_∞, Φ_∞) on the cone, pulled back from the link. Concretely, up to double-cover, the metrics on T^*S^3, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and $\mathcal{O}(-2,-2)$ all share the same asymptotic cone with link $S^2 \times S^3$, and we have the following potential invariant

[1]gauge group $SO(3)$ always lifts to $SU(2)$ in our invariant setting, see Proposition A.2.2 of the appendix.

model solutions: either we have a flat connection with a trivial Higgs field, or we have the unique non-flat invariant instanton pulled back from $S^2 \times S^3$, which we denote A^{can}, with a possibly non-trivial parallel Higgs field[2].

On these asymptotically conical metrics, we find four distinct possibilities for any invariant irreducible solution (A, Φ) to the monopole equations: (i) the curvature does not decay quadratically, i.e. $t^2|F_A|$ is unbounded as $t \to \infty$, where t is the radial parameter of the cone, and we take norms with respect to the cone metric, (ii) (A, Φ) is an invariant monopole which is asymptotic to A^{can} with a non-trivial Higgs field as $t \to \infty$, (iii) $\Phi = 0$, A is an invariant instanton which is asymptotic to A^{can} as $t \to \infty$, (iv) $\Phi = 0$, A is an invariant instanton which is asymptotic to a flat connection as $t \to \infty$[3].

We shall restrict to cases (ii)-(iv) by only considering invariant solutions with quadratic curvature decay. In general, this is a natural assumption to make for solutions on asymptotically conical metrics, since solutions on the cone converging to some model solution have curvature decaying (at least) as a two-form on the link of the cone. As far as the author is aware, this thesis is the first situation for which we have a complete description of this moduli-space for the invariant co-homogeneity one gauge theory. Also, although we were unable to prove this in full generality, cf. Remarks 3.3.3, 3.3.6, 3.4.13, we conjecture that situation (i) does not actually arise, i.e. any invariant solution to the monopole equations on T^*S^3, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and $\mathcal{O}(-2,-2)$ without quadratic curvature decay must blow up in finite time.

We now summarise the main results of chapters §2, §3. For the metric of Stenzel, there is a single $SU(2)$-bundle admitting irreducible invariant connections, which we denote P_{Id}, and we find a one-parameter family of instantons, and a one-parameter family of monopoles:

Theorem A. *In a neighbourhood of $S^3 \subset T^*S^3$, up to gauge, invariant solutions to the monopole equations are in a two-parameter family $(S, \Phi)_{\xi,\chi}$, $\xi, \chi \in (-\infty, \infty)$, containing a one-parameter family of invariant instantons with $\chi = 0$. Moreover $(S, \Phi)_{\xi,\chi}$ extends over all of T^*S^3 when:*

(i) $\xi \in (-1, 1)\,, \chi = 0$, *as an irreducible instanton asymptotic to A^{can} at infinity,*

(ii) $\xi = \pm 1, \chi = 0$, *as a flat connection,*

(iii) $\xi = 0, \chi \in (0, \infty)$, *as an irreducible monopole asymptotic to A^{can} with a non-trivial parallel Higgs field at infinity.*

Otherwise, $(S, \Phi)_{\xi,\chi}$ cannot extend over T^*S^3 with quadratically decaying curvature.

See Proposition 3.1.3 for a proof of the local statement, Theorem 3.2.1 for parts (i), (ii), and Proposition 3.4.6 for (iii). The existence of the one-parameter family of monopoles $(S, \Phi)_\chi := (S, \Phi)_{0,\chi}$, $\chi \in (0, \infty)$, and the instanton $S_0 := (S, \Phi)_{0,0}$ was already established in [Oli16], which considered only local solutions $(S, \Phi)_{\xi,\chi}$ with $\xi = 0$: we fix a gap in the

[2] these monopoles (A^{can}, Φ_m) pulled back from the link actually come in a one-parameter family, parametrised by the *mass* $m = |\Phi_m| > 0$. This is explained in more detail in [Oli16].

[3] one can also show that for (ii), (iii), solutions have *exactly* quadratic curvature decay, while for (iv), solutions have curvature decaying *faster* than quadratically, and moreover, have curvature bounded in L^2-norm.

proof of [Oli16, Theorem 1] by showing these are all the invariant monopoles with quadratic curvature decay. We also note here that there is a (non-equivariant) isometric involution of T^*S^3, arising from the map exchanging the factors of $SU(2)$ in each $SU(2)^2$-orbit, which sends $(S,\Phi)_{\xi,\chi} \mapsto (S,\Phi)_{-\xi,\chi}$.

For the small resolution $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, there are two $SU(2)$-bundles admitting invariant irreducible connections, denoted $P_{0,\mathrm{Id}}$ and $P_{1,\mathbf{0}}$, and these are equivariantly isomorphic over the complement $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\setminus\mathbb{CP}^1$. We find that each bundle carries a one-parameter family of instantons R_ϵ, $R'_{\epsilon'}$, respectively, and the family $R'_{\epsilon'}$ contains an invariant abelian instanton R'_0:

Theorem B. *In a neighbourhood of $\mathbb{CP}^1 \subset \mathcal{O}(-1)\oplus\mathcal{O}(-1)$, invariant instantons are in two one-parameter families R_ϵ, $\epsilon\in(-\infty,\infty)$ and $R'_{\epsilon'}$, $\epsilon'\in[0,\infty)$, up to gauge. Moreover R_ϵ, $R'_{\epsilon'}$ extends over all of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ when:*

(i) *$\epsilon\in(0,\infty)$, as an irreducible instanton asymptotic to A^{can} at infinity,*

(ii) *$\epsilon'\in[0,1)$, as an instanton asymptotic to A^{can} at infinity, which is abelian if $\epsilon'=0$ and irreducible otherwise,*

(iii) *$\epsilon=0$ or $\epsilon'=1$, as a flat connection.*

Otherwise, R_ϵ, $R'_{\epsilon'}$ cannot extend over $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ with quadratically decaying curvature.

See Propositions 3.1.5, 3.1.6 for a proof of the local statement, Theorems 3.2.6, 3.2.7 for parts (i)-(iii).

We can also show that, as the curvature of the invariant family R_ϵ blows up on the calibrated co-dimension four $\mathbb{CP}^1$ in the limit $\epsilon\to\infty$, we get the expected bubbling and removable-singularity phenomena:

Theorem C. *Let R_ϵ be the one-parameter family of invariant instantons and R'_0 the invariant abelian instanton extending over $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Then, in the limit $\epsilon\to\infty$:*

(i) *Up to an appropriate rescaling, R_ϵ bubbles off a family of anti-self-dual connections along $\mathbb{CP}^1\subset\mathcal{O}(-1)\oplus\mathcal{O}(-1)$.*

(ii) *Without this rescaling, R_ϵ converges uniformly to R'_0 on compact subsets of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\setminus\mathbb{CP}^1$.*

See Theorem 3.3.2 for proofs and a more precise statement of these results. The proof of Theorem C is more involved than for a similar co-homogeneity one bubbling theorem for instantons found in [LO18, Theorem 2]: everything was explicit in that case, whereas we must genuinely prove (i) here to obtain the relevant compactification result (ii).

There are countably many bundles over $\mathcal{O}(-2,-2)$ admitting irreducible invariant connections, which we denote $P_{1-l,l}$ for $l\in\mathbb{Z}$. The number $l\in\mathbb{Z}$ can be understood topologically by associating a rank two complex vector bundle to $P_{1-l,l}$ via the standard representation: this associated bundle splits into a direct sum of line bundles pulled back from $\mathcal{O}(\pm(1-l),\pm l)\to\mathbb{CP}^1\times\mathbb{CP}^1$. Each bundle $P_{1-l,l}$ carries a one-parameter family of instantons $Q^l_{\alpha_l}$ similar to the family $R'_{\epsilon'}$ of Theorem B: Q^l_0 is abelian, $Q^l_{\alpha_l}$ is asymptotic to A^{can} at infinity when the parameter $\alpha_l \geq 0$ is less than some finite critical value α_l^{crit}, and the asymptotic behaviour of this family jumps to a flat connection at the critical value. However, there is a new phenomenon on $\mathcal{O}(-2,-2)$, as the instantons $Q^l_{\alpha_l^{\text{crit}}}$ are not themselves flat when $l \neq 0,1$, and they are rigid in the moduli-space of invariant, irreducible instantons with this asymptotic behaviour.

Theorem D. *In a neighbourhood of $\mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathcal{O}(-2,-2)$, invariant instantons are in countably many one-parameter families $Q^l_{\alpha_l}$, $l \in \mathbb{Z}$, $\alpha_l \in [0,\infty)$, up to gauge. Moreover, $Q^l_{\alpha_l}$ extends over all of $\mathcal{O}(-2,-2)$ when:*

(i) *$\alpha_l \in \left[0, \alpha_l^{\text{crit}}\right)$ for some $\alpha_l^{\text{crit}} \in (0,\infty)$, as an instanton asymptotic to A^{can} at infinity, which is abelian if $\alpha_l = 0$ and irreducible otherwise,*

(ii) *$l = 0,1$, $\alpha_l = \alpha_l^{\text{crit}}$ as a flat connection,*

(iii) *$l \neq 0,1$, $\alpha_l = \alpha_l^{\text{crit}}$ as an irreducible instanton asymptotic to a flat connection at infinity.*

Otherwise, $Q^l_{\alpha_l}$ cannot extend over $\mathcal{O}(-2,-2)$ with quadratically decaying curvature.

See Proposition 3.1.1 for a proof of the local statement, and Theorems 3.2.8, 3.2.9 for parts (i)-(iii). See also the end of §3.3 for a further discussion of the behaviour of instantons on $\mathcal{O}(-2,-2)$.

The existence and uniqueness of these fast-decaying instantons on $\mathcal{O}(-2,-2)$ verifies the results of [Ban93] in the $SU(2)^2$-invariant Calabi-Yau 3-fold setting: if we fix the holomorphic structure on the rank two complex vector-bundle pulled back from the direct sum of $\mathcal{O}(\pm(1-l), \pm l) \to \mathbb{CP}^1 \times \mathbb{CP}^1$, then there is a unique Hermitian-Yang-Mills connection that is asymptotic to the flat connection at infinity, which is compatible with this holomorphic structure.

In the final result, the proof of which can be found in Proposition 3.4.1, we show that Theorems A - D fully describe the moduli-space of the $SU(2)^2$-invariant Calabi-Yau gauge theory:

Theorem E. *There are no irreducible, invariant monopoles on $\mathcal{O}(-2,-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ with quadratically decaying curvature.*

Remark 2.1.1. *One may be able to replace the assumption of invariance in Theorem E with the assumption of finite intermediate energy $E_I(A,\Phi) := \int_M |F_A \wedge \text{Re}\Omega|^2 + |d_A\Phi|^2$, by adapting the techniques found in [FNO20] for G_2-monopoles.*

2.2 Monopole and Instanton Equations

Consider again the monopole equations (2.1), for a connection A and $\Phi \in \Omega^0(\text{Ad}P)$ on some principal bundle P over a Calabi-Yau 3-fold (M,ω,Ω). As it is more convenient for our purposes, we can rewrite (2.1) as:

$$F_A \wedge \omega^2 = 0 \tag{2.3a}$$

$$F_A \wedge \text{Im}\Omega = -\frac{1}{2} d_A\Phi \wedge \omega^2 \tag{2.3b}$$

Let us assume we are in the general set-up of §1.2: we let $N \subset M$ be a (real) hyper-surface, and we suppose that N foliates M into parallel hyper-surfaces, up to working on a tubular neighbourhood $N \times I \subseteq M$ for some $I \subseteq \mathbb{R}$. As in (1.10), we can write the Calabi-Yau structure (ω, Ω) on M in terms of a one-parameter family of hypo-structures $(\eta, \omega_i)_t$ on N.

Moreover, in this neighbourhood, we may always write $P \to M$ as the pull-back of some bundle on N, and we can view any $\Phi \in \Omega^0(\mathrm{Ad}P)$ as a one-parameter family of sections Φ_t over N. So, using the Calabi-Yau structure (1.10), and the temporal gauge on a tubular neighbourhood of N, (2.3) takes the form:

$$F_{A_t} \wedge \omega_2 \wedge \eta + \frac{1}{2} d_{A_t}\Phi \wedge \omega_1^2 = 0 \tag{2.4a}$$

$$F_{A_t} \wedge \omega_1 \wedge \eta + \frac{1}{2} \partial_t A_t \wedge \omega_1^2 = 0 \tag{2.4b}$$

$$F_{A_t} \wedge \omega_3 + \partial_t A_t \wedge \omega_2 \wedge \eta = d_{A_t}\Phi \wedge \omega_1 \wedge \eta - \frac{1}{2}\partial_t \Phi \omega_1^2 \tag{2.4c}$$

We refer to the equation (2.4a) as the *static* monopole equation, and (2.4b), (2.4c), as the monopole *evolution* equations, where (2.4b) is just the condition (2.3a), and the other two arise from (2.3b). Furthermore, it is not difficult to compute that the static equation (2.4a) is preserved by the evolution equations. Similarly, in the case $\Phi = 0$, we will refer to the respective equations as the static and dynamic instanton equations.

Remark 2.2.1. *The instanton equations are equivalent to the gradient flow*

$$*(F_{A_t} \wedge \omega_1) = -\partial_t A_t$$

of a Chern-Simons functional $CS_{\omega_1} : \mathcal{A} \times I \to \mathbb{R}$ on the space of connections $\mathcal{A}$ on P pulled back to N:

$$CS_{\omega_1}(A_0 + a, t) := \frac{1}{2}\int_N \mathrm{tr}\left(a \wedge \left(2F_{A_0} + d_{A_0}a + \frac{2}{3}a \wedge a\right)\right) \wedge \omega_1(t)$$

with initial conditions $A_{t=t_0}$ satisfying (2.4a).

2.3 Invariant Monopole and Instanton ODEs

Away from the singular orbit, the general set-up of (2.4) clearly applies to the co-homogeneity one metrics on $\mathcal{O}(-2,-2)$, T^*S^3, and $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We will also suppose that the bundle, connection, and Higgs field are invariant under the $SU(2)^2$-action, so that (2.4) is a system of ODEs for the invariant connection and Higgs field on $SU(2)^2/H$, where the relevant principal isotropy subgroup H is given by $H = K_{2,-2}$ or $H = \triangle U(1)$.

Recall §1.1: by [Wan58], we can write such invariant bundles as $SU(2)^2 \times_H G \to SU(2)^2/H$ for some compact gauge group G and homomorphism $\lambda : H \to G$. These bundles are referred to as $SU(2)^2$-*homogeneous*. Recall also that an invariant connection on this bundle can be written as an H-equivariant linear map $A : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{g}$, such that $A|_{\mathfrak{h}} = d\lambda$. Here, $\mathfrak{g}$, $\mathfrak{h} \cong \mathfrak{u}(1)$ denotes the Lie algebra of G, $H \subset SU(2)^2$, and $d\lambda$ is the image of the canonical connection on $SU(2)^2 \to SU(2)^2/H$ under λ.

If $H = K_{2,-2}$, $G = SU(2)$ then the defining homomorphism $K_{2,-2} \to SU(2)$ of the homogeneous bundle is classified by a pair $(n, j) \in \mathbb{Z}\times\mathbb{Z}_2$. Using the isomorphism $K_{2,-2} \cong \triangle U(1) \times \mathbb{Z}_2 \subset U(1)^2$, we can write these as:

$$(e^{i\theta}, e^{i\theta}).(e^{2i\pi}, e^{i\pi}) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^j \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} \tag{2.5}$$

for some $(n, j) \in \mathbb{Z} \times \mathbb{Z}_2$, and similarly (with $j = 0$) for every homomorphism $\triangle U(1) \to SU(2)$. We will denote the corresponding homogeneous $SU(2)$-bundles over $SU(2)^2/H$ as $P_{n,j}$, P_n respectively, although since the action of $\mathbb{Z}_2$ in (2.5) is trivial on the Lie algebra of the gauge group $SU(2)$, for the following section, it will suffice just to consider P_n.

The canonical connection on P_n appears as $nE_1 \otimes u^+$, and the space of invariant connections can be identified as an affine space for intertwiners of $\triangle U(1)$-representations given by left-invariant one-forms on $SU(2)^2/\triangle U(1)$ and the composition of (2.5) with the adjoint action on $\mathfrak{su}(2)$. We summarise the results in the following proposition, and compute curvatures:

Proposition 2.3.1. *$SU(2)^2$-invariant connections A on P_n, and corresponding curvatures F_A, are of the following form:*

(i) *If $n = 0$, then for some $a_1, a_2, a_3 \in \mathbb{R}$,*

$$A = a_1E_1 \otimes u^- + a_2E_2 \otimes u^- + a_3E_3 \otimes u^- \quad F_A = \tfrac{3}{2}(a_1E_1 + a_2E_2 + a_3E_3) \otimes \omega_1^{se} \tag{2.6}$$

(ii) *Otherwise, for some $a_0, a_1, a_2, b_1, b_2 \in \mathbb{R}$, where $a_1 = a_2 = b_1 = b_2 = 0$ if $n \neq 1$:*

$$\begin{aligned} A = {} & a_1(E_2 \otimes v^1 + E_3 \otimes w^1) + b_1(E_3 \otimes v^1 - E_2 \otimes w^1) \\ & + a_2(E_2 \otimes v^2 + E_3 \otimes w^2) + b_2(E_3 \otimes v^2 - E_2 \otimes w^2) + a_0E_1 \otimes u^- + nE_1 \otimes u^+ \end{aligned}$$

$$\begin{aligned} F_A = {} & 3(a_1a_2 + b_1b_2)E_1 \otimes \omega_3^{se} + 3(a_1b_2 - b_1a_2)E_1 \otimes \omega_2^{se} \\ & + \tfrac{3}{2}\left(a_1^2 + b_1^2 + a_2^2 + b_2^2 - n\right) E_1 \otimes \omega_0^{se} + \tfrac{3}{2}\left(-a_1^2 - b_1^2 + a_2^2 + b_2^2 + a_0\right) E_1 \otimes \omega_1^{se} \\ & + \tfrac{3}{2}(a_0 - 1)\left(a_1\left(E_2 \otimes w^1 - E_3 \otimes v^1\right) + b_1\left(E_2 \otimes v^1 + E_3 \otimes w^1\right)\right) \wedge \eta^{se} \\ & + \tfrac{3}{2}(a_0 + 1)\left(a_2\left(E_2 \otimes w^2 - E_3 \otimes v^2\right) + b_2\left(E_2 \otimes v^2 + E_3 \otimes w^2\right)\right) \wedge \eta^{se} \end{aligned} \tag{2.7}$$

In a similar way, we can classify $SU(2)^2$-invariant sections of the adjoint bundle: an $SU(2)^2$-invariant section of $\mathrm{Ad}P_n$ appears as an element of the Lie algebra $\mathfrak{su}(2)$ invariant under the $\triangle U(1)$ action. This understood, the following proposition is immediate:

Proposition 2.3.2. *$SU(2)^2$-invariant sections of $\mathrm{Ad}P_n$ are of the form $\Phi = \phi_1E_1 + \phi_2E_2 + \phi_3E_3$ for some $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, where $\phi_2 = \phi_3 = 0$ if $n \neq 0$.*

There are some useful facts about the bundle data of Propositions 2.3.1 and 2.3.2 that should be noted before continuing: firstly, it is clear that the bundles P_n admit only reducible invariant connections when $n \neq 1$, and when $n = 1$, from the explicit expressions for curvature, we see that an invariant connection given by (2.7) is reducible iff either $a_0 = 1, a_2 = b_2 = 0$, or $a_0 = -1, a_1 = b_1 = 0$, or $a_1 = b_1 = a_2 = b_2 = 0$.

Secondly, on $P_n \to SU(2)^2/\triangle U(1)$, there is an invariant gauge transformation generated by the vector field E_1 on the fibre, which acts by rotation on the plane spanned by E_2, E_3, and leaves E_1 fixed. In the notation of Propositions 2.3.1, 2.3.2 for $n \neq 0$, this acts as a rotation $(a_1 + ib_1, a_2 + ib_2) \mapsto \left(e^{i\theta}(a_1 + ib_1), e^{i\theta}(a_2 + ib_2)\right)$ by some common angle θ, and acts trivially on (a_0, ϕ_1).

Using Propositions 2.3.1, 2.3.2, we can now write down (2.4) on $\mathcal{O}(-2,-2)$, T^*S^3, and $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ as ODE systems for the coefficients appearing in these two propositions. Since we are primarily interested in finding non-abelian solutions to (2.4), it will suffice to consider the case for $n = 1$[4]:

Proposition 2.3.3. *On $P_1 \to \mathbb{R}_{>0} \times SU(2)^2/\triangle U(1)$ with Calabi-Yau structure (1.19), invariant monopoles (A, Φ) can be written, up to gauge, as:*

$$A = a_1(E_2 \otimes v^1 + E_3 \otimes w^1) + a_2(E_2 \otimes v^2 + E_3 \otimes w^2) + a_0 E_1 \otimes u^- + E_1 \otimes u^+ \quad \Phi = \phi E_1$$

with (a_0, a_1, a_2, ϕ) real-valued functions satisfying the following ODE system:

$$\begin{aligned}
\dot{a_0} &= \frac{4\lambda}{\mu^2}\left((a_1^2 + a_2^2 - 1)u_0 - (a_0 - a_1^2 + a_2^2)u_1\right) \\
\dot{a_1} &= \frac{3}{2\lambda\mu}\left((a_0 - 1)a_1 v_3 - (a_0 + 1)a_2 v_0\right) - 2\frac{u_1 - u_0}{\mu}a_2\phi \\
\dot{a_2} &= \frac{3}{2\lambda\mu}\left((a_0 - 1)a_1 v_0 - (a_0 + 1)a_2 v_3\right) - 2\frac{u_1 + u_0}{\mu}a_1\phi \\
\dot{\phi} &= \frac{3}{\mu^2}\left(\left(a_1^2 + a_2^2 - 1\right)v_0 - 2a_1 a_2 v_3\right)
\end{aligned} \tag{2.8}$$

Proof. We use Propositions 2.3.1 and 2.3.2 in the monopole equations (2.4): we use the temporal gauge to put the connection into the form $A_t = a_1 I_1 + b_1 J_1 + a_2 I_2 + b_2 J_2 + a_0 E_1 \otimes u^- + n E_1 \otimes u^+$, where I_1, I_2, J_1, J_2 are defined by:

$$\begin{aligned}
I_1 &:= E_2 \otimes v^1 + E_3 \otimes w^1 & \qquad J_1 &:= E_3 \otimes v^1 - E_2 \otimes w^1 \\
I_2 &:= E_2 \otimes v^2 + E_3 \otimes w^2 & \qquad J_2 &:= E_3 \otimes v^2 - E_2 \otimes w^2
\end{aligned}$$

Then $d_{A_t}\Phi = [A_t, \Phi] = \phi\,[A_t, E_1] = 2\phi\,(-a_1 J_1 + b_1 I_1 - a_2 J_2 + b_2 I_2)$. This implies $d_{A_t}\Phi \wedge \omega_1^2$ vanishes, so the static equation (2.4a) is just the single condition $a_1 b_2 - b_1 a_2 = 0$. Equation (2.4b) also only has a single component, giving:

$$\dot{a_0} = \frac{4\lambda}{\mu^2}\left((a_1^2 + b_1^2 + a_2^2 + b_2^2 - 1)u_0 - (a_0 - a_1^2 - b_1^2 + a_2^2 + b_2^2)u_1\right)$$

Splitting (2.4c) into E_1, E_2, E_3 components, the E_1 component gives:

$$\dot{\phi} = \frac{3}{\mu^2}\left(\left(a_1^2 + b_1^2 + a_2^2 + b_2^2 - 1\right)v_0 - 2\left(a_1 a_2 + b_1 b_2\right)v_3\right)$$

[4]See §2.4 for explicit abelian solutions in the case $n = 1$: the solutions for $n \neq 1$ are similar.

Meanwhile the E_2, E_3 components together give:

$$
\begin{aligned}
\dot{a_1} &= \frac{3}{2\lambda\mu}\left((a_0-1)a_1v_3-(a_0+1)a_2v_0\right)-2\frac{u_1-u_0}{\mu}a_2\phi \\
\dot{b_1} &= \frac{3}{2\lambda\mu}\left((a_0-1)b_1v_3-(a_0+1)b_2v_0\right)-2\frac{u_1-u_0}{\mu}b_2\phi \\
\dot{a_2} &= \frac{3}{2\lambda\mu}\left((a_0-1)a_1v_0-(a_0+1)a_2v_3\right)-2\frac{u_1+u_0}{\mu}a_1\phi \\
\dot{b_2} &= \frac{3}{2\lambda\mu}\left((a_0-1)b_1v_0-(a_0+1)b_2v_3\right)-2\frac{u_1+u_0}{\mu}b_1\phi
\end{aligned}
$$

We can now use the invariant gauge-transformation generated by E_1 to simplify this ODE system, which appears as the symmetry of the equations. Using the static condition $a_1b_2 - a_2b_1 = 0$, we will use this symmetry to set $b_1 = b_2 = 0$, thus giving the ODEs in the form stated. □

We note here that (2.8) displays some further discrete symmetries:

Proposition 2.3.4. *The following involution is a discrete symmetry of* (2.8)*:*

$$(a_0, a_1, a_2, \phi) \mapsto (a_0, -a_1, -a_2, \phi) \tag{2.9}$$

Specialising to the case of (2.8) *with* $u_0 = 0$*, we have an additional symmetry:*

$$(a_0, a_1, a_2, \phi) \mapsto (-a_0, a_2, a_1, \phi) \tag{2.10}$$

Remark 2.3.5. *If one is also free to vary the Calabi-Yau structure,* (2.10) *becomes a symmetry of the full system* (2.8) *with* $u_0 \mapsto -u_0$*.*

Proof. One can easily check that the symmetries of this proposition are indeed symmetries of the ODE systems in question. We comment instead on the origin of such symmetries: (2.9) is a residual symmetry from the invariant gauge transformation that we used to set $b_1 = b_2 = 0$: it is simply the rotation by angle π of the plane spanned by E_2, E_3. Meanwhile, (2.10) is the symmetry arising from interchanging the two factors of $SU(2)$ on the principal orbits: this explains why one must alter the Calabi-Yau structure to see it as a symmetry of (2.8). □

We also recall that a natural condition on solutions to the monopole equations on asymptotically conical CY 3-folds is to require quadratically decaying curvature. In terms of our ODE system (2.8), this requirement takes the following form:

Lemma 2.3.6. *An invariant solution* (A, Φ) *to the monopole equations determined by a solution* (a_0, a_1, a_2, ϕ) *to* (2.8) *has quadratically decaying curvature if and only if* a_i*,* $ta_1\phi$*,* $ta_2\phi$ *are bounded.*

Proof. Using the expression for curvature $F_A = F_{A_t} - \partial_t A_t \wedge dt$ in the temporal gauge, the explicit expressions for F_{A_t}, A_t, given in (2.6), and the scaling of k-forms on the cone, it is clear that $t^2|F_A|$ is bounded if $a_0, a_1, a_2, t\dot{a}_0, t\dot{a}_1, t\dot{a}_2$ are bounded. The converse is clear for $t\dot{a}_0, t\dot{a}_1, t\dot{a}_2$, and note that $t^2|F_{A_t}|$ is bounded only if $a_1^2 + a_2^2 - 1$, and $-a_1^2 + a_2^2 + a_0$ are: the first of these implies a_1, a_2 must be bounded, and since $|-a_1^2 + a_2^2 + a_0| \geq |a_0| - |a_1^2 - a_2^2|$ this implies a_0 must be bounded also.

Up to terms decaying faster than $O(t^{-1})$, as $t \to \infty$, the ODE system (2.8) is asymptotic to (2.8) on the conifold:

$$\begin{aligned} \dot{a}_0 &= -\frac{4}{t}\left(a_0 - a_1^2 + a_2^2\right) \\ \dot{a}_1 &= \frac{3}{2t}\left(a_0 - 1\right) a_1 - 2a_2\phi \\ \dot{a}_2 &= -\frac{3}{2t}\left(a_0 + 1\right) a_2 - 2a_1\phi \\ \dot{\phi} &= -\frac{6}{t^2} a_1 a_2 \end{aligned} \tag{2.11}$$

and comparing the expressions for $t\dot{a}_0, t\dot{a}_1, t\dot{a}_2$ gives the statement of the lemma. □

2.4 Reducible Solutions

Before conducting an analysis of the full system (2.8), we will briefly say something about the reducible case, i.e. if we consider abelian or flat connections. Firstly, note that the trivial flat connection $A^\flat$ on $P_1 \to SU(2)^2/\triangle U(1) \cong S^2 \times S^3$ appears in two distinct $SU(2)^2$-invariant gauge-equivalence classes[5], which can be represented by:

$$A_1^\flat := E_1 \otimes u^1 + E_2 \otimes v^1 + E_3 \otimes w^1 \qquad A_2^\flat := E_1 \otimes u^2 + E_2 \otimes v^2 + E_3 \otimes w^2 \tag{2.12}$$

i.e. in terms of Proposition 2.3.1, we have $a_0 = 1, a_1 = 1, b_1 = a_2 = b_2 = 0$, or $a_0 = -1, a_2 = 1, b_1 = a_1 = b_2 = 0$ respectively. These are clearly just lifts of the standard Maurer-Cartan form on $SU(2)$ to P_1, and $A_1^\flat, A_2^\flat$ are exchanged via non-equivariant diffeomorphism obtained via exchanging the factors of $SU(2)$ in $SU(2)^2/\triangle U(1)$.

Secondly, note that if both $a_1 = a_2 = 0$ then the connection is abelian, and we can solve (2.8) explicitly on the space of principal orbits:

$$a_0(t) = \frac{C - 2u_0u_1}{\mu^2} \qquad\qquad \phi = -3I(t) \tag{2.13}$$

where $\dot{I}(t) = \frac{v_0}{\mu^2}$, and C is a constant of integration.

Using the results of Appendix A.2, we see that a generic solution (2.13) can extend over the singular orbits S^2, or $S^2 \times S^2$ only if[6] $C = 2u_0u_1(0)$, $\dot{I} = 0$, and can never extend over the singular orbit S^3.

Remark 2.4.1. *For later reference, we note that the generic abelian solution* (2.13) *on* $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \setminus \mathbb{CP}^1$ *is also unbounded near* $\mathbb{CP}^1$ *unless* $C = 2u_0u_1(0)$.

Remark 2.4.2. *Note that the curvature of the abelian instanton* (2.13)*, viewed as a harmonic two-form, has norm decaying to* $O(t^{-2})$ *as* $t \to \infty$ *with respect to the cone metric. The existence of abelian instantons on* $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O}(-2, -2)$ *with this curvature decay, and their uniqueness on a fixed bundle, follows from [HHM04, Thm. 1A] and its refinement [FHN21a, Thm. 5.12] for AC Calabi-Yau 3-folds.*

[5] although these represent the same connection up to non-equivariant gauge, at least on $S^2 \times S^3$.
[6] the converse will also hold for a suitable choice of bundle extension.

Chapter 3

Calabi-Yau Gauge Theory: Analysis

In the following chapter, we will classify quadratically decaying solutions to the $SU(2)^2$-invariant Calabi-Yau monopole equations (2.8), beginning by parametrizing solutions near the singular orbit at $t = 0$ in §3.1. This uses the theory of singular-initial value-problems, and the computations in Appendix A.2 to extend the solutions over the singular orbit at $t = 0$.

Once we have parametrized solutions to (2.8) near $t = 0$, we will consider the long-time behaviour of these local solutions as we move away from the singular orbit. We carry this out in two parts: firstly for the subsystem of (2.8) when the Higgs field ϕ vanishes in §3.2, i.e. for the Calabi-Yau instanton equations. With this done, in §3.4 we investigate solutions for which the Higgs field is non-vanishing.

We also prove the relevant bubbling and compactness theorems in §3.3 for the families of instantons constructed in §3.2.

3.1 Local Solutions

We consider the full system (2.8). Unlike with the reducible case, in general this will not have explicit solutions, and instead, we will analyse the qualitative behaviour of solutions as they move away from the singular orbit.

To determine their behaviour near the singular orbit, we will apply the theory of singular initial value problems of the form [FH17, Thm.4.7]: $t\dot{y} = M_{-1}(y) + M(t, y)$, where $M(t, y)t^{-1}$, $M_{-1}(y)$ are smooth functions of their arguments. To have unique solution near $t = 0$, we require that $M_{-1}(y_0) = 0$ at the initial value $y(0) = y_0$, and that the linearisation $d_{y_0}M_{-1}$ has no positive integer eigenvalues. This theory, combined with the boundary conditions found in Appendix A.2, will allow us to construct local solutions to (2.8) extending over the singular orbits at $t = 0$.

In all cases, we will find that solutions to the monopole equations are in a local two-parameter family for each bundle extending P_1 over the singular orbit, with the vanishing of the second parameter corresponding to the vanishing of the Higgs field ϕ, and thus a local one-parameter family of instantons.

First of all, there is a countable family of bundles $P_{1-l,l}$, $l \in \mathbb{Z}$ extending P_1 over the singular orbit $S^2 \times S^2$ by Proposition A.2.1. However, we can reduce our computations to the case $l > 0$ by the diffeomorphism exchanging the factors of $SU(2)$ in the $SU(2)^2$-orbits on the total space of the bundle, since this map sends $P_{1-l,l} \mapsto P_{l,1-l}$. As this map acts on the underlying Calabi-Yau structure (1.20) by sending the constant $u_0 \mapsto -u_0$, the monopole ODEs (2.8) also transform, but solutions of the transformed system are equivalent to solutions of (2.8) under the symmetry (2.10):

Proposition 3.1.1. *In a neighbourhood of the singular orbit of $P_{1-l,l} \to \mathcal{O}(-2,-2)$ local solutions to (2.8) are in a two-parameter family $\left(Q^l, \Theta^l\right)_{\alpha_l,\beta_l} := (a_0, a_1, a_2, \phi)_{\alpha_l,\beta_l}$ for each $l \in \mathbb{Z}$. For $l > 0$, these solutions satisfy:*

$$a_0 = 1 - 2l + O(t^2) \qquad \phi = \beta_l + O(t^2)$$
$$a_1 = -\frac{1}{l}\beta_l\alpha_l\sqrt{\frac{U_1 - U_0}{U_1 + U_0}}t^l + O(t^{l+2}) \qquad a_2 = \alpha_l t^{l-1} + O(t^{l+1})$$

Proof. We write (2.8) for a Calabi-Yau structure of type $\mathcal{I}$:

$$\begin{aligned}
\dot{a_0} &= \frac{4\lambda}{\mu^2}\left((a_1^2 + a_2^2 - 1)u_0 - (a_0 - a_1^2 + a_2^2)u_1\right)\\
\dot{\phi} &= -\frac{6}{\mu}a_1a_2\\
\dot{a}_1 &= \frac{3}{2\lambda}(a_0 - 1)a_1 - 2\frac{u_1 - u_0}{\mu}a_2\phi\\
\dot{a}_2 &= -\frac{3}{2\lambda}(a_0 + 1)a_2 - 2\frac{u_1 + u_0}{\mu}a_1\phi
\end{aligned} \tag{3.1}$$

We consider solutions to (3.1) with this Calabi-Yau structure given by (1.26), for any U_1, U_0 with $U_1 > |U_0| \geq 0$. Recall the power-series of λ, u_1, μ near $t = 0$ is given by:

$$\lambda(t) = 3t + O(t^3) \qquad u_1 = U_1 + O(t^2) \qquad \mu = \sqrt{U_1^2 - U_0^2} + O(t^2)$$

Although we cannot apply [FH17, Thm.4.7] directly, we can use the boundary conditions for extending smoothly to the singular orbit in Appendix A.2 to re-write this system in the correct form.

First, we assume $l > 0$. Then, using Proposition A.2.3, we can define smooth functions X_1, X_2 such that $a_1 = t^l X_1$, $a_2 = t^{l-1}X_2$, and (3.1) becomes:

$$\begin{aligned}
\dot{a_0} &= O(t)\\
\dot{\phi} &= O(t^{2l-1})\\
\dot{X}_1 &= \frac{1}{t}\left(\frac{1}{2}\left(a_0 - 1 - 2l\right)X_1 - 2X_2\phi\sqrt{\frac{U_1 - U_0}{U_1 + U_0}}\right) + O(t)\\
\dot{X}_2 &= -\frac{1}{2t}\left(a_0 - 1 + 2l\right)X_2 + O(t)
\end{aligned}$$

Since the extension conditions also require $a_0(0) = 1-2l$, once we fix constants $\alpha_l := X_2(0)$, $\delta_l := \phi(0)$ such that $lX_1(0) + X_2(0)\phi(0)\sqrt{\frac{U_1-U_0}{U_1+U_0}} = 0$, then $y(t) = (a_0, X_1, X_2, \phi)$ satisfies a singular initial-value problem with linearisation:

$$d_{y_0}M_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}X_1(0) & -2\sqrt{\frac{U_1-U_0}{U_1+U_0}}\alpha_l & -2l & -2\sqrt{\frac{U_1-U_0}{U_1+U_0}}\delta_l \\ -\frac{1}{2}\alpha_l & 0 & 0 & 0 \end{pmatrix}$$

at initial value $y_0 = \left(1-2l, -\frac{1}{l}\alpha_l\beta_l\sqrt{\frac{U_1-U_0}{U_1+U_0}}, \alpha_l, \beta_l\right)$. This has a unique solution once we fix y_0, since $\det\left(k\mathrm{Id} - d_{y_0}M_{-1}\right) = (k+2l)\,k^3 > 0$ for $k > 0$.

Recovering the local solutions extending smoothly over the singular orbit when $l \leq 0$, for a fixed Calabi-Yau structure, is a straightforward procedure: we consider the solutions for $l > 0$ as previously, but with $U_0 \mapsto -U_0$. Applying the transformation (2.10) to these solutions gives solutions to (2.8), and it is easy to verify from Proposition A.2.3 that these solutions extend to $P_{l,1-l}$. □

Remark 3.1.2. *By setting $\beta_l = 0$ in $\left(Q^l, \Theta^l\right)_{\alpha_l,\beta_l}$, we obtain a local one-parameter family of instantons i.e. solutions to (2.8) with $\phi = 0$, and for $l > 0$ these solutions have:*

$$a_0 - a_0^{\mathrm{ab}} = -\frac{6\alpha_l^2}{l(U_0+U_1)}t^{2l} + O(t^{2l+2}) \tag{3.2}$$

near $t = 0$, where a_0^{ab} denotes the abelian solution to (2.8) extending over the singular orbit of $P_{1-l,l}$.

Moreover, when $l = 1$, these solutions have:

$$a_0 = -1 - \frac{6}{U_1+U_0}(\alpha_1^2-1)t^2 + O(t^4) \quad a_2 = \alpha_1 + \frac{3}{2(U_1+U_0)}\alpha_1(\alpha_1^2-1)t^2 + O(t^4) \tag{3.3}$$

As their proofs are similar, we will state the results for solutions extending over singular orbits S^2 and S^3 without proof. The correct re-parametrisations, corresponding initial values y_0, and linearisations $d_{y_0}M_{-1}$ as in [FH17, Thm.4.7] can be found in Appendix B.

Starting with S^3, we note that by Proposition A.2.1, the bundle P_1 extends uniquely over this orbit, and we denote this extension P_{Id}:

Proposition 3.1.3. *In the neighbourhood of the singular orbit, solutions to (2.8) on $P_{\mathrm{Id}} \to T^*S^3$ are in a two-parameter family $(S,\Phi)_{\xi,\chi} := (a_0, a_+, a_-, \phi)_{\xi,\chi}$, where $a_+ = a_1 + a_2$, $a_- = a_1 - a_2$. These solutions satisfy:*

$$a_0 = \xi + O(t^2) \quad a_+ = 1 + \left(\frac{9}{8}(\xi^2-1) - \chi\right)t^2 + O(t^4) \quad a_- = \xi + O(t^2) \quad \phi = \chi t + O(t^3)$$

Remark 3.1.4. *By setting $\chi = 0$ in $(S,\Phi)_{\xi,\chi}$, we obtain a local one-parameter family of instantons i.e. solutions to (2.8) with $\phi = 0$. For later reference, we give some additional* *terms in the resulting Taylor series:*

$$\begin{aligned} a_0 &= \xi + \frac{9}{10}\xi(-1+\xi^2)t^2 + O(t^4) \\ a_+ &= 1 + \frac{9}{8}(-1+\xi^2)t^2 + O(t^4) \\ a_- &= \xi + \frac{27}{40}\xi(-1+\xi^2)t^2 + O(t^4) \end{aligned} \tag{3.4}$$

Finally, the bundle P_1 extends in exactly two ways over $S^2 = SU(2)^2/U(1) \times SU(2)$, and we denote these possible extensions $P_{0,\mathrm{Id}}$ and $P_{1,\mathbf{0}}$ cf. Proposition A.2.1:

Proposition 3.1.5. *In the neighbourhood of the singular orbit of $P_{0,\mathrm{Id}} \to \mathcal{O}(-1)\oplus\mathcal{O}(-1)$, solutions to (2.8) are in a two-parameter family $(R,\Psi)_{\epsilon,\delta} := (a_0,a_1,a_2,\phi)_{\epsilon,\delta}$, with:*

$$a_0 = -1 + \epsilon t^2 + O(t^4) \quad a_1 = -\frac{\delta}{\sqrt{3}}t^2 + O(t^4) \quad a_2 = 1 - \frac{1}{2}\epsilon t^2 + O(t^4) \quad \phi = \delta t^2 + O(t^4)$$

Proposition 3.1.6. *In a neighbourhood of the singular orbit of $P_{1,\mathbf{0}} \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, solutions to (2.8) are in a two-parameter family $(R',\Psi')_{\epsilon',\delta'} := (a_0,a_1,a_2,\phi)_{\epsilon',\delta'}$, with:*

$$a_0 = 1 + O(t^2) \qquad a_1 = \epsilon' + O(t^2) \qquad a_2 = O(t^2) \qquad \phi = \delta' + O(t^2)$$

Remark 3.1.7. *For later reference, we note that $a_2 = -\frac{\sqrt{3}}{4}\epsilon'\delta' t^2 + O(t^4)$ for the resulting power-series of $(R',\Psi')_{\epsilon',\delta'}$.*

Having computed these two-parameter families of local solutions to the monopole equations (2.8), by uniqueness, we see that the following one-parameter families are the local solutions to the instanton equations, i.e. (2.8) with $\phi = 0$:

$$S_\xi := (S,\Phi)_{\xi,0} \qquad R_\epsilon := (R,\Psi)_{\epsilon,0} \qquad R'_{\epsilon'} := (R',\Psi')_{\epsilon',0} \qquad Q^l_{\alpha_l} := \left(Q^l,\Theta^l\right)_{\alpha_l,0}$$

For later reference, we have already computed some additional terms in the power-series of S_ξ, $Q^l_{\alpha_l}$, in (3.4), (3.2), (3.3). For the analysis of the family $R'_{\epsilon'}$, it will be more useful to first apply the transformation (2.10), and then compute higher-order terms with respect to (2.8) with $u_0 \mapsto -u_0$. To explain why, observe that the instanton equations for a hypo-structure of type $\mathcal{I}$, i.e. (2.8) with ϕ and v_0 vanishing, has at least one of a_1 or a_2 vanishing identically, and if both vanish we have the abelian solution. From the boundary conditions of Propositions A.2.3, A.2.7, which of a_1 or a_2 must necessarily vanish will depend on how we extend the bundle P_1 to the singular orbit: we have a_1 vanishing for $P_{0,\mathrm{Id}}$ and $P_{1-l,l}$ for $l > 0$, while a_2 vanishes for $P_{1,\mathbf{0}}$ and $P_{1-l,l}$ for $l \leq 0$.

However, we can always reduce our analysis to a single ODE system with, say, a_1 vanishing identically by applying (2.10) to (2.8) and mapping $u_0 \mapsto -u_0$: this is the same as pulling back these equations by the diffeomorphism exchanging the factors of $SU(2)$ in the $SU(2)^2$-orbits on the total space of the bundle. This has been previously explained for the solutions $Q^l_{\alpha_l}$, and we can apply the same reasoning to the family $R'_{\epsilon'}$: the caveat here is that if we exchange the factors on the singular orbit S^2, then the bundle P_1 and the Calabi-Yau structure on the principal orbits now extends over $S^2 = SU(2)^2/SU(2) \times U(1)$ rather than our convention $S^2 = SU(2)^2/U(1) \times SU(2)$.

With this explained, let us denote $P_{\mathbf{0},1}$ as the bundle obtained from $P_{1,\mathbf{0}}$ by exchanging the factors of $SU(2)^2$, and pull back the local one-parameter family of invariant instantons $R'_{\epsilon'}$ on $P_{1,\mathbf{0}}$ to a local one-parameter family of invariant instantons on $P_{\mathbf{0},1}$. Corollary A.2.8 ensures that these solutions actually extend to $SU(2)^2/SU(2)\times U(1)$, and for later reference, we compute some higher order terms in the power-series:

Lemma 3.1.8. *In a neighbourhood of the singular orbit, solutions to (2.8) on $P_{\mathbf{0},1}$ with $\phi = 0$ are in a one-parameter-family, pulled back via (2.10) from the one-parameter family $R'_{\epsilon'}$, and these satisfy:*

$$a_0 = -1 - \frac{3}{4}\left(\epsilon'^2 - 1\right)t^2 + O(t^4) \qquad a_2 = \epsilon' + \frac{3}{8}\epsilon'\left(\epsilon'^2 - 1\right)t^2 + O(t^4) \tag{3.5}$$

3.2 Solutions to the Instanton Equations

Using the description of solutions to (2.8) near the singular orbit, we will now describe the qualitative behaviour of the solutions as we move away from this orbit. We will focus first on the case ϕ vanishes i.e. instantons: in this case, by Lemma 2.3.6, the requirement of quadratic curvature decay is equivalent to considering bounded solutions.

Our general strategy of proof will be to find *forward-invariant* sets for this system, where we define a subset $\mathcal{S} \subset \mathbb{R}^n$ to be *forward-invariant* for an ODE system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ if a solution $\mathbf{x}(t)$ contained in $\mathcal{S}$ at some non-singular initial time t^*, must remain in $\mathcal{S}$ for all forward time $t \geq t^*$ for which the solution exists. After determining the long-time behaviour of any solutions lying in these sets, we will then take the power-series solutions of §3.1, valid for small times $t \geq 0$, and establish which of these local solutions enter our forward-invariant sets, and hence remain there for all time.

This strategy will be sufficient in most cases, i.e. for Theorem 3.2.1 on T^*S^3, Theorems 3.2.6, 3.2.7 on $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ and Theorem 3.2.8 for special cases of bundles on $\mathcal{O}(-2,-2)$, as the power-series solutions will lie in our invariant sets at arbitrarily small non-zero times. However the general case of $\mathcal{O}(-2,-2)$ in Theorem 3.2.9 requires more care, and we will employ a rescaling argument to determine which of the local solutions enter our invariant sets.

We will start with the smoothing T^*S^3. Recall from §2.4 (2.12) that the equivariant equivelance classes of the flat connection A_1^b, A_2^b are trivial solutions to (2.8). Recall also the canonical connection $A^{\text{can}} := E_1 \otimes u^+$ is the unique non-trivial invariant instanton on the conifold pulled back from the link $SU(2)^2/\triangle U(1)$.

A single explicit non-trivial solution to (2.8) on the smoothing was found in [Oli16, Theorem 2]:

$$a_0 = \phi = 0 \qquad a_1 = a_2 = \frac{1}{2}\sqrt{\frac{4}{3\lambda(v_3 - v_0)}} \tag{3.6}$$

given locally by the power-series S_ξ in (3.4) with $\xi = 0$. We now show that this instanton actually lies in a one-parameter family:

Theorem 3.2.1. *Invariant instantons with quadratic curvature decay on $P_{\mathrm{Id}} \to T^*S^3$ are in a one-parameter family S_ξ, $-1 \leq \xi \leq 1$, up to gauge. Moreover:*

(i) *The isometry exchanging the factors of $SU(2)$ on the principal orbits of T^*S^3 sends $S_\xi \mapsto S_{-\xi}$, with explicit fixed point S_0 given by (3.6).*

(ii) *$S_1 = A_1^b$, $S_{-1} = A_2^b$, and S_ξ, $-1 < \xi < 1$ are irreducible with $\lim_{t\to\infty} S_\xi(t) = A^{\mathrm{can}}$.*

Proof of Theorem 3.2.1. We will prove that the local solutions S_ξ given by the power-series (3.4) near the singular orbit exist for all time if $|\xi| \leq 1$ and are otherwise unbounded.

First, we formulate (2.8) with $\phi = 0$ in terms of $a_+ = a_1 + a_2$, $a_- = a_1 - a_2$:

$$\dot{a}_0 = f_0(a_+a_- - a_0) \qquad \dot{a}_+ = f_+(a_0a_- - a_+) \qquad \dot{a}_- = f_-(a_0a_+ - a_-) \tag{3.7}$$

where we define:

$$f_0 := \frac{4\lambda}{\mu} \qquad f_+ := \frac{3(v_3 + v_0)}{2\lambda\mu} \qquad f_- := \frac{3(v_3 - v_0)}{2\lambda\mu}$$

By non-degeneracy of the hypo-structure (1.19), the functions f_0, f_+, f_- are all strictly positive on $(0, \infty)$, so the following lemma is immediate:

Lemma 3.2.2. *Critical points of (3.7) for $t \in (0, \infty)$ are given by the following triples (a_0, a_+, a_-):*

$$(1, 1, 1) \qquad (1, -1, -1) \qquad (-1, 1, -1) \qquad (-1, -1, 1) \qquad (0, 0, 0)$$

Proof. This follows by a simple computation: note that these critical points are just the canonical connection A^{can} and the flat connections $A_1^\flat$, $A_2^\flat$ under the symmetries (2.9) and (2.10). □

Lemma 3.2.3. *The following sets in $\mathbb{R}^3$ are forward-invariant for (3.7):*

$$(0, \infty)^3 \qquad (0, 1)^3 \qquad (1, \infty)^3$$

Proof. (i) We bound a solutions (a_0, a_+, a_-) lying in the quadrant $(0, \infty)^3$ with boundary $a_0 = 0$, $a_+ = 0$, and $a_- = 0$. We can exclude the axes at intersections of these planes by local uniqueness to ODEs, since (3.7) has three families of solutions given by setting any two of (a_0, a_+, a_-) to be identically zero.

At $a_0 = 0, a_+ \geq 0, a_- \geq 0$, $\dot{a}_0 = f_0a_+a_- \geq 0$, with equality iff $a_+ = 0$ or $a_- = 0$. Since a solution cannot hit any of the axes, this implies both are zero if $\dot{a}_0 = 0$, but since $(0, 0, 0)$ is a critical point, by uniqueness one cannot have this situation either, and hence the inequality is strict. This implies a solution with $a_0 > 0, a_+ \geq 0, a_- \geq 0$ for some non-zero time cannot leave this region at $a_0 = 0, a_+ \geq 0, a_- \geq 0$.

One obtains the same result for a_+ and a_- by repeating the proof with permuted subscripts $0, +, -$.

(ii) We show that the boundary of the unit cube also bounds solutions lying inside it. By the symmetry of permuting $0, +, -$, and the previous result, it will be enough to show this for the top face of the cube: i.e. prove that a solution with $1 > a_0 > 0, 1 \geq a_+ > 0, 1 \geq a_- > 0$ cannot leave this region via $a_0 = 1, 1 \geq a_+ > 0, 1 \geq a_- > 0$. We have, at $a_0 = 1, 1 \geq a_+ > 0, 1 \geq a_- > 0$, $\dot{a}_0 = f_0\left(a_+a_- - 1\right) \leq 0$, with equality iff both $a_+ = a_- = 1$. However since $(1,1,1)$ is a critical point for (3.7), this cannot be the case, hence the inequality is strict, and we cannot have a solution with $1 > a_0 > 0, 1 \geq a_+ > 0, 1 \geq a_- > 0$ leaving this region at $a_0 = 1, 1 \geq a_+ > 0, 1 \geq a_- > 0$, arguing as before.

(iii) The proof that the quadrant $(1,\infty)^3$ bounded by the planes $a_0 = 1$, $a_+ = 1$, and $a_- = 1$ goes almost exactly as for the previous part of the lemma: at $a_0 = 1, a_+ \geq 1, a_- \geq 1$, $\dot{a}_0 = f_0 (a_+ a_- - 1) \geq 0$ with equality iff both $a_+ = a_- = 1$, hence the inequality is strict, and we cannot have a solution leaving this region via $a_0 = 1, a_+ \geq 1, a_- \geq 1$.

□

Having established these results, we can immediately see from the local solutions S_ξ in (3.4) for some sufficiently small non-zero time $(a_0, a_+, a_-)_\xi \in (0,1)^3$ for $0 < \xi < 1$, and $(a_0, a_+, a_-)_\xi \in (1,\infty)^3$ for $1 < \xi$, so we have a rough bound on the behaviour of our solutions as $t \to \infty$. However, we can use Lemma 3.2.3 to show an improved statement:

Lemma 3.2.4. *The following sets are forward-invariant for* (3.7)*:*

(i) $\mathcal{S}_\infty := \{(a_0, a_+, a_-) \in (1,\infty)^3 \mid a_+ a_- > a_0, a_0 a_- > a_+, a_0 a_+ > a_-\}$

(ii) $\mathcal{S}_0 := \{(a_0, a_+, a_-) \in (0,1)^3 \mid a_+ a_- < a_0, a_0 a_- < a_+, a_0 a_+ < a_-\}$.

Proof. Given an ODE system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ in $\mathbb{R}^n$, if one has a hypersurface $h(\mathbf{x}) = 0$ such that $\nabla h \cdot \mathbf{F}(\mathbf{x}, t) > 0$, where ∇ is the gradient of h, and "$\cdot$" denotes the standard dot product on $\mathbb{R}^n$, then for all time for which a smooth solution $\mathbf{x}(t)$ exists, it can only cross hypersurface $h(\mathbf{x}) = 0$ in the same direction as ∇h.

In the case of (3.7), we use the hypersurfaces $\{(a_0, a_+, a_-) \in \mathbb{R}^3 \mid a_0 = a_+ a_-\}$, $\{(a_0, a_+, a_-) \in \mathbb{R}^3 \mid a_+ = a_0 a_-\}$, and $\{(a_0, a_+, a_-) \in \mathbb{R}^3 \mid a_- = a_+ a_0\}$:

(i) $\mathcal{S}_\infty$ is the region in $(1,\infty)^3$ bounded by these three hyperbolic paraboloids, with triple intersection at $(1,1,1)$, and intersecting pairwise along three line segments in $\mathbb{R}^3$. We can exclude the intersections by noting that

$$\{(a_0, a_+, a_-) \in [1,\infty)^3 \mid a_+ = a_0 a_-, a_- = a_0 a_+\} = \{(a_0, a_+, a_-) \in [1,\infty)^3 \mid a_- = a_+, a_0 = 1\}$$

which lies in the boundary of $(1,\infty)^3$. So using the previous lemma, and the symmetry of permuting $0, +, -$, it will be enough to prove that a solution contained in $\mathcal{S}_\infty$, at some initial time, cannot leave via $\{(a_0, a_+, a_-) \in [1,\infty)^3 \mid a_0 = a_+ a_-\}$. We calculate for $h = a_+ a_- - a_0$, with $a_0 > 1, a_+ > 1, a_- > 1$:

$$\nabla h \cdot (\dot{a}_0, \dot{a}_+, \dot{a}_-)|_{h=0} = a_+ a_- \left(f_+ ({a_-}^2 - 1) + f_- ({a_+}^2 - 1)\right) > 0$$

Repeating the proof with indices $0, +, -$ permuted gives the result for surfaces defined by $a_0 a_- - a_+ = 0$ and $a_0 a_+ - a_- = 0$ respectively.

(ii) $\mathcal{S}_0$ is also bounded by these three hyperbolic paraboloids, but in $(0,1)^3$, ∇h (as we have defined it) points outward. As for the intersections, we can again exclude them, as before for $(a_0, a_+, a_-) \in (0, \infty)^3$, but also for

$$\{(a_0, a_+, a_-) \in [0,1]^3 \mid a_+ = a_0 a_-, a_- = a_0 a_+\} = \\ \{(a_0, a_+, a_-) \in [0,1]^3 \mid a_+ = a_-, a_0 = 1\} \cup \{(a_0, a_+, a_-) \in [0,1]^3 \mid a_+ = a_- = 0\}$$

which lies in the boundary of the unit cube. Now the calculation is exactly the same as the previous part of the lemma, with $0 < a_0 < 1, 0 < a_+ < 1, 0 < a_- < 1$ and $h = a_+ a_- - a_0$, giving $\nabla h \cdot (\dot{a}_0, \dot{a}_+, \dot{a}_-)|_{h=0} < 0$.

□

Note that solutions (a_0, a_+, a_-) to (3.7) lying inside $\mathcal{S}_0$, $\mathcal{S}_\infty$ have a_0, a_+, a_- monotonic in t. We can then use this fact to determine their asymptotic behaviour:

Lemma 3.2.5. *A solution (a_0, a_+, a_-) to (3.7) lying inside $\mathcal{S}_0$ at some time $t^* > 0$, exists for all forward time $t \geq t^*$, and is asymptotic as $t \to \infty$ to $(0,0,0)$. A solution (a_0, a_+, a_-) lying inside $\mathcal{S}_\infty$ at some time t^* cannot be bounded for all $t \geq t^*$.*

Proof. We begin by looking at solutions lying in $\mathcal{S}_0$. Forward-time existence and boundedness of these solutions follows from the boundedness of $\mathcal{S}_0$, and since a_0, a_+, a_- are all (strictly) monotonically decreasing in $\mathcal{S}_0$, the solution (a_0, a_+, a_-) must have a limit lying in the closure. To determine that limit, we reparameterize (3.7) in terms of the variable s, as in the explicit solutions given by (1.27):

$$\dot{a}_0 = \frac{4\lambda^2}{\mu}(a_+ a_- - a_0) \quad \dot{a}_+ = \frac{3(v_3 + v_0)}{2\mu}(a_0 a_- - a_+) \quad \dot{a}_- = \frac{3(v_3 - v_0)}{2\mu}(a_0 a_+ - a_-) \tag{3.8}$$

In particular, by using (1.27), one can check that $\lambda f_0 \to C_0 > 0$ as $s \to \infty$ for some strictly positive constant C_0, and similarly $\lambda f_\pm \to C_\pm > 0$. If a solution (a_0, a_+, a_-) to (3.8) lying in $\mathcal{S}_0$ does not have $a_+ a_- - a_0 \to 0$ as $s \to \infty$, then we get a contradiction: otherwise for s sufficiently large we can bound $\dot{a}_0$ above, away from 0. Said more explicitly, if we do not have $a_+ a_- - a_0 \to 0$, then we do not have $\dot{a}_0 \to 0$, so for some constant $C_0^* < 0$, there exists $s^* \gg 0$ such that $\dot{a}_0(s) < C_0^*$ for all $s \geq s^*$. Integrating this inequality would give the contradiction $a_0 \to -\infty$ as $s \to \infty$, thus we must have $a_+ a_- - a_0 \to 0$ as $s \to \infty$.

One then repeats this argument for $a_\pm(s)$, to obtain that a solution in $\mathcal{S}_0$ must tend to a critical point of this system in the closure of $\mathcal{S}_0$ as $s \to \infty$: either $(0,0,0)$, or $(1,1,1)$ by Lemma 3.2.2. Since a_0, a_+, a_- are all strictly decreasing, we must have $(a_0, a_+, a_-) \to (0,0,0)$.

Now we deal with solutions (a_0, a_+, a_-) to (3.7) lying in $\mathcal{S}_\infty$. These have a_0, a_+, a_- strictly increasing as long as the solution exists, so again, if a solution is bounded and exists for all time, it must have limit lying in the closure of $\mathcal{S}_\infty$. Let us assume this is the case and derive a contradiction: since the right-hand side of (3.8) has a limit as $s \to \infty$, this implies that $(\dot{a}_0, \dot{a}_+, \dot{a}_-)$ must also have a limit. Since $\lambda f_0 \to C_0 > 0$ we have, for a fixed constant $C_0^* > 0$, some $S > 0$ such that for all $s > S$:

$$\dot{a}_0 > C_0^*(a_+ a_- - a_0)$$

and likewise for $\dot{a}_{\pm}$. As such, a bounded solution existing for all time cannot have simultaneously $\dot{a}_0, \dot{a}_+, \dot{a}_- \to 0$ as $s \to \infty$, since this would require $(a_0, a_+, a_-) \to (1,1,1)$, which is impossible by the monotonicity of a_0, a_+, a_-. Therefore, at least one of $\dot{a}_0, \dot{a}_+, \dot{a}_-$ must be bounded below away from 0 for s sufficiently large, and hence the corresponding a_0, a_+, a_- must be unbounded above as $s \to \infty$. □

We can now conclude the proof of Theorem 3.2.1: the first point is clear by applying the symmetry outlined in (2.10) to the local power-series of $(a_0, a_+, a_-)_\xi$, i.e. (3.4), and noting that the fixed point $\xi = 0$ is the explicit solution (3.6). For the rest, by using (3.4), one finds the flat connection $(a_0, a_+, a_-)_1 = (1,1,1)$ is a critical point, and:

$$
\begin{aligned}
a_0 - a_- a_+ &= -\frac{9}{10}\left(\xi^2 - 1\right)\xi t^2 + O(t^4) \\
a_+ - a_0 a_- &= 1 - \xi^2 - \frac{45 - 63\xi^2}{40}\left(\xi^2 - 1\right) t^2 + O(t^4) \\
a_- - a_+ a_0 &= -\frac{27}{20}\left(\xi^2 - 1\right)\xi t^2 + O(t^4)
\end{aligned}
$$

In particular, for non-zero t sufficiently small, and $0 < \xi < 1$, we have $(a_0, a_+, a_-)_\xi(t) \in \mathcal{S}_0$, while for $1 < \xi$ we have $(a_0, a_+, a_-)_\xi(t) \in \mathcal{S}_\infty$. Using the symmetry (2.10) for $\xi < 0$, Theorem 3.2.1 follows. □

On $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $\mathcal{O}(-2,-2)$ there are multiple ways of extending the invariant bundle P_1 to the singular orbit. Each extension carries a distinct one-parameter family of irreducible instantons:

Theorem 3.2.6. *Invariant instantons with quadratic curvature decay on* $P_{0,\mathrm{Id}} \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ *are in a one-parameter family* R_ϵ, $\epsilon \geq 0$, *up to gauge. Moreover:*

(i) $R_0 = A_2^\flat$, *and* R_ϵ *are irreducible for* $\epsilon > 0$.

(ii) $\lim_{t\to\infty} R_\epsilon(t) = A^{\mathrm{can}}$ *for* $\epsilon > 0$.

Theorem 3.2.7. *Invariant instantons with quadratic curvature decay on* $P_{1,\mathbf{0}} \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ *are in a one-parameter family* $R'_{\epsilon'}$, $0 \leq \epsilon' \leq 1$, *up to gauge. Moreover:*

(i) R'_0 *is abelian,* $R'_1 = A_1^\flat$, *and* $R'_{\epsilon'}$ *are irreducible for* $0 < \epsilon' < 1$.

(ii) $\lim_{t\to\infty} R'_{\epsilon'}(t) = A^{\mathrm{can}}$ *for* $0 \leq \epsilon' < 1$.

Here, the canonical connection $A^{\mathrm{can}} := E_1 \otimes u^+$ is the unique non-trivial invariant instanton on the conifold pulled back from the link $SU(2)^2/\triangle U(1)$, and A_1^b, A_2^b are the equivariant equivalence classes of the flat connection given by (2.12).

For instantons over $\mathcal{O}(-2,-2)$, we also split the statement of the theorem into two cases. The first case is similar to the situation of Theorem 3.2.7:

Theorem 3.2.8. *Invariant instantons with quadratic curvature decay on* $P_{1-l,l} \to \mathcal{O}(-2,-2)$ *with* $l = 0, 1$, *are in a one-parameter family* $Q^l_{\alpha_l}$, $0 \leq \alpha_l \leq 1$, *up to gauge. Moreover:*

(i) Q^l_0 *is abelian,* $Q^0_1 = A_1^\flat$, $Q^1_1 = A_2^\flat$, *and* $Q^l_{\alpha_l}$ *are irreducible for* $0 < \alpha_l < 1$.

(ii) $\lim_{t\to\infty} Q^l_{\alpha_l}(t) = A^{\mathrm{can}}$ *for* $0 \leq \alpha_l < 1$.

The second case exhibits a new phenomenon: now, the instantons appearing at the boundary of the moduli-space of instantons with asymptotics A^{can} are not themselves flat, but are asymptotic to the flat connection:

Theorem 3.2.9. *Invariant instantons with quadratic curvature decay on $P_{1-l,l} \to \mathcal{O}(-2,-2)$ with $l \neq 0, 1$, are in a one-parameter family $Q^l_{\alpha_l}$, $0 \leq \alpha_l \leq \alpha_l^{\mathrm{crit}}$ for some $\alpha_l^{\mathrm{crit}} > 0$, up to gauge. Moreover:*

(i) *Q^l_0 are abelian, and $Q^l_{\alpha_l}$ are irreducible for $0 < \alpha_l \leq \alpha_l^{\mathrm{crit}}$.*

(ii) *$\lim_{t\to\infty} Q^l_{\alpha_l}(t) = A^{\mathrm{can}}$ for $0 \leq \alpha_l < \alpha_l^{\mathrm{crit}}$, $\lim_{t\to\infty} Q^l_{\alpha_l^{\mathrm{crit}}}(t) = A_1^\flat$ for $l < 0$, and $\lim_{t\to\infty} Q^l_{\alpha_l^{\mathrm{crit}}}(t) = A_2^\flat$ for $l > 1$.*

Proof of Theorems 3.2.6, 3.2.7, 3.2.8. Most of what is required to prove these theorems boils down to studying the qualitative behaviour of a single ODE system. We study solutions to (2.8) with $\phi = 0$, $a_1 = 0$:

$$\dot{a}_0 = -\frac{4\lambda}{\mu^2}\left(a_2^2(u_1 - u_0) + a_0 u_1 + u_0\right) \qquad \dot{a}_2 = -\frac{3}{2\lambda} a_2 (a_0 + 1) \tag{3.9}$$

where we have a family of Calabi-Yau structures defined by hypo-structures of type $\mathcal{I}$, so that u_0, u_1, μ, λ are non-degenerate solutions to hypo-evolution equations (1.21).

We consider forward-invariant sets for this system, see also Fig. 3.1 below:

Lemma 3.2.10. *The following sets are forward-invariant for (3.9):*

(i) $H_\pm := \{(a_0, a_2) \in \mathbb{R}^2 \mid \pm a_2 > 0\}$

(ii) $\mathcal{R}_\infty := \{(a_0, a_2) \in \mathbb{R}^2 \mid a_0 < -1, 1 < a_2\}$

(iii) $\mathcal{R}_0 := \{(a_0, a_2) \in \mathbb{R}^2 \mid -1 < a_0 < 1, 0 < a_2 < 1\}$

Proof. (i) Since there is always a non-trivial abelian solution $(a_0, 0)$ to (3.9), by uniqueness a solution hitting $a_2 = 0$ at some time $t^* > 0$ must be there for all time $t > 0$. Furthermore, since the symmetry (2.9) exchanges the upper/lower-half planes, we can reduce to the case of $a_2 > 0$ in what follows.

(ii) In the following, we will split the upper-half plane into four quadrants centred around the critical point $(-1, 1)$, and look at the sign of $\dot{a}_0$ along $a_0 = -1$ and $\dot{a}_2$ along $a_2 = 1$.

Since $\lambda > 0$ for all $t > 0$, and $a_2 > 0$ by assumption, the sign of $\dot{a}_2$ is the same as that of $-(a_0+1)$, and the sign of $\dot{a}_0$ is the same as that of $-\big((a_2^2 - 1)(u_1 - u_0) + (a_0 + 1)u_1\big)$. Then $\dot{a}_2 > 0$ for all $a_0 < -1$. Since $\lambda > 0$, solutions to the hypo equations (1.21) must have $u_1 \pm u_0$ strictly increasing. In addition we must have $\mu = \sqrt{u_1^2 - u_0^2} > 0$ for all time $t > 0$, so $u_1 \pm u_0$ must be strictly positive for all time $t > 0$, and hence also u_1. Thus at $a_0 = -1$, we have that $\dot{a}_0 < 0$ iff $a_2 > 1$. Thus a solution in $\mathcal{R}_\infty$ at some initial time $t^* > 0$, cannot leave via either of its boundaries $a_0 = -1$ or $a_2 = 1$, and since the intersection $(-1, 1)$ is a critical point, the solution must remain in $\mathcal{R}_\infty$ for all time $t > t^*$.

(iii) As shown in the first part of the lemma, no solution can hit $a_2 = 0$, the bottom of $\mathcal{R}_0$, unless it is contained in $a_2 = 0$ for all time. From the proof of the second part of the lemma, we see that a solution in $\mathcal{R}_0$ cannot exit $\mathcal{R}_0$ via. the top $a_2 = 1, a_0 > -1$, or the side $a_0 = -1, a_2 \leq 1$. All that remains to show is that the side $a_0 = 1$, $1 > a_2 > 0$ is bounding. This follows from the fact that $u_1 \pm u_0$ must be strictly positive, since at $a_0 = 1$, $\dot{a}_0 = -\frac{4\lambda}{\mu^2}((a_2^2(u_1 - u_0) + u_1 + u_0) < 0$.

□

These sets determine the behaviour of solutions lying inside them:

Lemma 3.2.11. *A solution (a_0, a_2) to (3.9) lying inside $\mathcal{R}_0$ at initial time $t^* > 0$ exists for all forward time $t \geq t^*$, and converges to $(0,0)$ as $t \to \infty$. A solution lying inside $\mathcal{R}_\infty$ at initial time $t^* > 0$ cannot be uniformly bounded for all $t \geq t^*$.*

Proof. For the bounded set $\mathcal{R}_0$, it is clear that solutions exist for all time, but it remains to prove their asymptotic behaviour. Since $\dot{a}_2 < 0$ in $\mathcal{R}_0$, a_2 is strictly decreasing, and as it is bounded below, a_2 must have a limit $\hat{a}_2 \in [0, 1)$ as $t \to \infty$. To get a limit for a_0, notice that the first equation of (3.9), together with the hypo-evolution equations (1.21), gives:

$$\frac{d}{dt}\left(a_0\mu^2\right) = -4\lambda\left(a_2^2(u_1 - u_0) + u_0\right) \tag{3.10}$$

Written in integral form on the interval $t \geq t^*$, this is the equation:

$$a_0(t) = -\frac{1}{\mu^2}\left(\left(\int_{t^*}^{t} 4\lambda\left(a_2^2(u_1 - u_0) + u_0\right)\right) + a_0(t^*)\mu^2(t^*)\right) \tag{3.11}$$

Since the hypo-structure λ, u_1, u_0, μ is asymptotically conical as a function of t and a_0 bounded, as $t \to \infty$ (3.11) gives:

$$a_0(t) \sim -\frac{1}{t^4}\int_T^t 4t\left(\hat{a}_2^2t^2 + (\hat{a}_2^2 - 1)u_0\right) \sim -\hat{a}_2^2 - \frac{2u_0\left(\hat{a}_2^2 - 1\right)}{t^2} + O(t^{-4}) \sim -\hat{a}_2^2$$

for some $T \geq t^*$ sufficiently large. Hence we also have a limit $a_0 \to -\hat{a}_2^2$ as $t \to \infty$. Since $a_2 > 0$, integrating the second equation of (3.9) gives us, as $t \to \infty$:

$$a_2(t) = a_2(T)\exp\left(-\int_T^t \frac{3}{2\lambda}(a_0 + 1)\right) \sim a_2(T)\exp\left((\hat{a}_2^2 - 1)\int_T^t \frac{3}{2t}\right) = Ct^{\frac{3}{2}(\hat{a}_2^2 - 1)}$$

where C is some constant of integration. As $\hat{a}_2 < 1$, this implies $a_2 \to 0$, and thus also $a_0 \to 0$.

Now we come to solutions lying in $\mathcal{R}_\infty$. Since $\mathcal{R}_\infty$ is forward-invariant, and a solution lying in $\mathcal{R}_\infty$ has $\dot{a}_2 > 0$ for all finite t, the statement for finite t follows directly from the previous lemmas. All that is left is to prove that if a solution exists for all time in $\mathcal{R}_\infty$, then it cannot be bounded. We will assume that it is, and derive a contradiction:

If a solution is bounded, then since a_2 is strictly increasing in $\mathcal{R}_\infty$, a_2 must have a limit as $t \to \infty$, and as before, the integral formula (3.11), and the boundedness of a_0 gives that (a_0, a_2) must have a limit lying on the curve $a_0 = -a_2^2$. Since a_2 is strictly increasing, we can bound a_2 away from 1, thus for some t large enough, we can also bound a_0 away from

-1. Call this bound C, i.e. there exists T, such that for $t > T$ we have $a_0 < C < -1$. Then we also have that:

$$\dot{a}_2 > -\frac{3}{2\lambda} a_2 (C+1)$$

So by integrating this inequality, we get:

$$a_2(t) \geq a_2(T) \exp\left(-\frac{3}{2\lambda}(C+1)\int_T^t \frac{1}{\lambda}\right)$$

but the right-hand side grows to $O(t^{-\frac{3(C+1)}{2}})$ as $t \to \infty$, hence we have a contradiction. $\square$

We now conclude the proof of Theorem 3.2.6, by applying our analysis above to the local power-series R_ϵ of Proposition 3.1.5 with $\delta = 0$, so that $(a_0, a_2)_\epsilon \in \mathcal{R}_0$ for $\epsilon > 0$ while $(a_0, a_2)_\epsilon \in \mathcal{R}_\infty$ for $\epsilon < 0$ at sufficiently small non-zero time. Taking $\epsilon = 0$ gives the flat connection $(a_0, a_2)_0 = (-1, 1)$, which is a critical point of (3.9).

Theorems 3.2.7, and 3.2.8, also follow from what has been said. In the first case, in order to apply the results of the previous lemmas, one must first pull-back the Calabi-Yau structure via the involution $u_0 \mapsto -u_0$ by exchanging the factors of $SU(2)$ on the principal orbits, which pulls back the local solutions to solutions of the form (3.5). These invariant instantons extend on the singular orbit $SU(2)^2/SU(2) \times U(1)$ rather than $SU(2)^2/U(1) \times SU(2)$ as is our convention, but one can fix this by again applying the involution lifted to the total space of the principal bundle i.e. (2.10). Similarly for the latter case, to consider $l = 0$, one considers the local solutions on $P_{0,1}$ for the original Calabi-Yau structure pulled-back via the involution, and then applies the involution again on the total space of $P_{0,1}$ to get the result on $P_{1,0}$.

With this in mind, we can apply our analysis to the local power-series (3.5) and (3.3). We see that these situations exhibit the same behaviour: up to invariant gauge transformation (2.9), for some sufficiently small $t^* > 0$, for $1 > \epsilon' > 0$ (respectively $1 > \alpha_1 > 0$) we have $(a_0, a_2)_{\epsilon'}(t^*) \in \mathcal{R}_0$ (respectively $(a_0, a_2)_{\alpha_1}$), while for $\epsilon' > 1$, we have $(a_0, a_2)_{\epsilon'}(t^*) \in \mathcal{R}_\infty$ (respectively $(a_0, a_2)_{\alpha_1}(t^*)$). We also see that $(a_0, a_2)_0(0) = (-1, 0)$, hence by uniqueness $(a_0, a_2)_0$ must correspond to the abelian solution to (3.9), and $(a_0, a_2)_1(0) = (-1, 1)$. $\square$

The proof for the remaining case of Theorem 3.2.9 requires slightly more care:

Proof of Theorem 3.2.9. We are again studying solutions to the ODE (3.9). Looking at the local power-series solutions in Proposition 3.1.1, we see that they do not initially lie in the sets $\mathcal{R}_0$ or $\mathcal{R}_\infty$ covered in our previous analysis. However, we will show that the only possibilities are that such solutions either enter $\mathcal{R}_0$ or $\mathcal{R}_\infty$ in finite time, or are otherwise asymptotic to the flat connection $A_2^\flat$:

Lemma 3.2.12. *Let $\mathcal{R}_1 := \{(a_0, a_2) \in \mathbb{R}^2 \mid 1 > a_2 > 0, a_0 < -1\}$. A solution (a_0, a_2) to (3.9) lying in $\mathcal{R}_1$ at initial time $t^* > 0$ can remain in $\mathcal{R}_1$ for all forward time $t \geq t^*$ only if it is asymptotic to $(-1, 1)$ as $t \to \infty$, and must otherwise enter one of $\mathcal{R}_0$, $\mathcal{R}_\infty$ in finite time.*

Proof. Since the hypo-structure (1.20) is non-degenerate, $\mu > 0$, $\lambda > 0$, $u_1 \pm u_0 > 0$ for $t > 0$. So, in $\mathcal{R}_1$:

$$\dot{a}_0 = -\frac{4\lambda}{\mu^2}\left((a_2^2 - 1)(u_1 - u_0) + (a_0 + 1)u_1\right) > 0 \qquad \dot{a}_2 = -\frac{3}{2\lambda}a_2\left(a_0 + 1\right) > 0$$

Hence, a solution lying in $\mathcal{R}_1$ can only leave in finite time via the boundaries $\{a_2 = 1, a_0 < -1\}$ or $\{a_0 = -1, 0 < a_2 < -1\}$, since $(-1, 1)$ is a critical point for (3.9). Since $\dot{a}_2$ is strictly positive on the first boundary, and $\dot{a}_0$ is strictly positive on the second, this proves that if a solution leaves $\mathcal{R}_1$ in finite time, it must actually cross the boundary and end up in the regions $\mathcal{R}_\infty$, $\mathcal{R}_0$ respectively.

If a solution remains in $\mathcal{R}_1$ for all forward-time, then by monotonicity (a_0, a_2) has a limit lying in the closure. The existence of a limit, combined with the integral formula (3.11), gives that (a_0, a_2) must also converge to a point lying on the curve $a_0 = -a_2^2$, which only intersects the closure of $\mathcal{R}_1$ at $(-1, 1)$. □

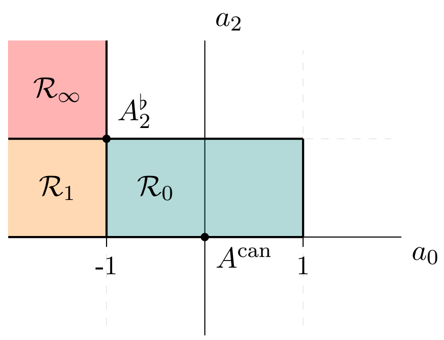

Figure 3.1: Distinguished sets for (3.9), and possible asymptotics: the flat connection $A_2^\flat$ at $(a_0, a_2) = (-1, 1)$ and A^{can} at $(a_0, a_2) = (0, 0)$.

We must also prove a comparison lemma for two solutions to (3.9), which will allow us to compare our power-series solutions away from the singular orbit at $t = 0$:

Lemma 3.2.13 (Forward-Comparison)**.** *Let (a_0, a_2), $(\hat{a}_0, \hat{a}_2)$ be two solutions to (3.9). If $a_0(t^*) < \hat{a}_0(t^*)$, $a_2(t^*) > \hat{a}_2(t^*) \geq 0$, at initial time $t^* > 0$, then $a_0(t) < \hat{a}_0(t)$, $a_2(t) > \hat{a}_2(t) \geq 0$, for all forward time $t \geq t^*$ for which these solutions exist.*

Proof. Let $t > t^* > 0$ be the first time for which the condition $a_0 < \hat{a}_0$, $a_2 > \hat{a}_2$ fails. By uniqueness of solutions to ODEs, we cannot have both $a_0(t) = \hat{a}_0(t)$ and $a_2(t) = \hat{a}_2(t)$, hence we must have exactly one of these. In the first case, at t:

$$\dot{a}_0 - \dot{\hat{a}}_0 = -\frac{4\lambda}{\mu^2}((a_2^2 - \hat{a}_2^2)(u_1 - u_0)) < 0$$

but this implies $a_0(t^{**}) - \hat{a}_0(t^{**}) > 0$ for some $t^* < t^{**} < t$, which contradicts t being the first time the condition fails. In the second case, at t:

$$\dot{a}_2 - \dot{\hat{a}}_2 = -\frac{3}{2\lambda}((a_0 - \hat{a}_0)a_2 > 0$$

but this implies $a_2(t^{**}) - \hat{a}_2(t^{**}) < 0$ for some $t^* < t^{**} < t$, which is again a contradiction. □

Another ingredient we will need is a slight improvement on the comparison lemma, restricted to solutions lying in $\mathcal{R}_1$:

Lemma 3.2.14 (Improved Comparison). *Let (a_0, a_2), $(\hat{a}_0, \hat{a}_2)$ be two solutions to (3.9), with $a_0(t^*) < \hat{a}_0(t^*)$, $a_2(t^*) > \hat{a}_2(t^*) \geq 0$, at some initial time $t^* > 0$. Then $a_2 - \hat{a}_2$ is strictly increasing $\forall t \geq t^*$ for which $(a_0, a_2)(t) \in \mathcal{R}_1$.*

Proof. By the forward-comparison lemma, $a_0 < \hat{a}_0$, $a_2 > \hat{a}_2 \geq 0$ for all time $t \geq t^*$, and by definition $(a_0 + 1) < 0$ for all time $t \geq t^*$ such that $(a_0, a_2)(t) \in \mathcal{R}_1$. Rewriting $\dot{a}_2 - \dot{\hat{a}}_2$ using (3.9):

$$\dot{a}_2 - \dot{\hat{a}}_2 = \frac{3}{2\lambda}\left(\hat{a}_2\left(\hat{a}_0 - a_0\right) + \left(\hat{a}_2 - a_2\right)\left(a_0 + 1\right)\right) > 0$$

for all such t, and hence $a_2 - \hat{a}_2$ is strictly increasing in t as claimed. □

With these out of the way, we are almost ready to prove the theorem. First of all, it is clear that the one-parameter family $Q^l_{\alpha_l}$ of local solutions to the ODEs (3.9) given by Proposition 3.1.1 with $\beta_l = 0, \alpha_l > 0$, are all contained in $\mathcal{R}_1$ for some $t^* > 0$ sufficiently small, and up to gauge transformation (2.9) we can assume this one-parameter family has $\alpha_l \geq 0$. The local solution with $\alpha_l = 0$ is clearly the abelian solution by uniqueness.

If $(a_0, a_2)_\alpha = (a_0^\alpha, a_2^\alpha)$ and $(a_0, a_2)_{\alpha'} = (a_0^{\alpha'}, a_2^{\alpha'})$ are any two of these solutions, then near the singular orbit:

$$\begin{aligned} a_0^\alpha - a_0^{\alpha'} &= -\frac{6}{l(U_1 + U_0)}\left(\alpha^2 - \alpha'^2\right)t^{2l} + O(t^{2l+2}) \\ a_2^\alpha - a_2^{\alpha'} &= \left(\alpha - \alpha'\right)t^{l-1} + O(t^{l+1}) \end{aligned} \tag{3.12}$$

So, by the forward-comparison lemma, if $(a_0, a_2)_{\alpha_l}$ hits the boundary of $\mathcal{R}_0$ in finite time (and thus enters it if $\alpha_l > 0$) then so does $(a_0, a_2)_{\alpha_l'}$ for all $0 \leq \alpha_l' \leq \alpha_l$. Similarly, if $(a_0, a_2)_{\alpha_l}$ hits the boundary of $\mathcal{R}_\infty$ in finite time (and thus enters it), then so does $(a_0, a_2)_{\alpha_l'}$, for all $\alpha_l' \geq \alpha_l$. By continuous dependence on initial conditions for singular initial-value problems, these sets are disjoint open intervals in $\mathbb{R}_{\geq 0}$. Clearly, the set $\alpha_l \in \mathbb{R}_{\geq 0}$ for which $(a_0, a_2)_{\alpha_l}$ hits the boundary of $\mathcal{R}_0$ in finite time is non-empty since it contains 0, so to complete the theorem, we must prove:

1. There exists $\alpha_l > 0$ such that $(a_0, a_2)_{\alpha_l}$ enters $\mathcal{R}_\infty$ in finite time.

2. There is at most one α_l such that $(a_0, a_2)_{\alpha_l}$ remains in $\mathcal{R}_1$ for all time.

The latter statement follows directly from our improved forward-comparison lemma: if $\alpha > \alpha'$ then $a_0^\alpha(t) < a_0^{\alpha'}(t), a_2^\alpha(t) > a_2^{\alpha'}(t)$ for all $t > 0$, and $\dot{a_2}^\alpha(t) - \dot{a}_2^{\alpha'}(t) > 0$ when the solutions are in $\mathcal{R}_1$. However, if two distinct solutions lie in $\mathcal{R}_1$ for all time $t > 0$, they must both be asymptotic as $t \to \infty$ to $(-1, 1)$ which would be a contradiction as $a_2^\alpha - a_2^{\alpha'}$ can be bounded below away from 0.

The former statement can be proved via a rescaling argument, which we state below as a proposition:

Proposition 3.2.15. *Fix $l > 1$. Then $\exists\, \alpha_l > 0, t^* > 0$ such that $(a_0, a_2)_{\alpha_l}(t^*) \in \mathcal{R}_\infty$, where $(a_0, a_2)_{\alpha_l}$ is the one-parameter family of solutions $Q^l_{\alpha_l}$ to (3.9) near $t = 0$ given in Proposition 3.1.1.*

Proof. As has been said previously, every local solution $(a_0, a_2)_{\alpha_l}$ is contained in the region $\mathcal{R}_1$ for small $t > 0$. Since these solutions can only fail to exist for all forward time if they leave in finite time via $\mathcal{R}_\infty$, it suffices to consider the case that, for all α_l, these solutions exist for all time.

We start by rescaling the Calabi-Yau structure along the fibre $\mathbb{C}_{2,-2}$ of $\mathcal{O}(-2,-2)$, by defining, for some $\delta > 0$:

$$\lambda_\delta(t) := \frac{\lambda(\delta t)}{\delta} \qquad\qquad (u_1)_\delta(t) := u_1(\delta t) \tag{3.13}$$

Near the singular orbit $S^2 \times S^2$, we have the power-series expansions $\lambda = 3t + O(t^3)$, $u_1 = U_1 + O(t^2)$ for some fixed Calabi-Yau structure, and for fixed t, $\lambda_\delta(t) \to 3t$, $(u_1)_\delta(t) \to U_1$ as $\delta \to 0$[1].

In terms of the rescaled Calabi-Yau structure, the instanton equations (3.9) for $\left(a_0^\delta, a_2^\delta\right)(t) = (a_0, a_2)(\delta t)$ become the family of ODEs, parametrized by δ:

$$\dot{a}_0^\delta = -\frac{4\delta^2\lambda_\delta}{(u_1)_\delta^2 - u_0^2}\left(a_2^2((u_1)_\delta - u_0) + a_0\left(u_1\right)_\delta + u_0\right) \qquad \dot{a}_2^\delta = -\frac{3}{2\lambda_\delta}a_2\left(a_0 + 1\right) \tag{3.14}$$

One can always rescale the one-parameter family of (local) solutions $(a_0, a_2)_{\alpha_l}$ to (3.9) to obtain solutions to (3.14) for fixed $\delta > 0$, but one can show that there is a one-parameter family of (local) solutions extending to the singular orbit for any $\delta \geq 0$.

To verify this claim, we apply the boundary conditions in Proposition A.2.3 for extending an invariant connection to the singular orbit, which allows us to write $a_2^\delta = t^{l-1}X_2^\delta$ for some smooth X_2^δ. The ODEs (3.14) can now be written as the singular initial-value problem:

$$\dot{a}_0^\delta = O(t) \qquad\qquad \dot{X}_2^\delta = -\frac{a_0 + 2l - 1}{2t}X_2^\delta + O(t) \tag{3.15}$$

which, for every $\delta \geq 0$, has a one-parameter family of solutions by fixing $X_2^\delta(0)$ as some constant κ_l.

These solutions are determined by the local power-series:

$$a_0^\delta = 1 - 2l + O(t^2) \qquad\qquad a_2^\delta = \kappa_l t^{l-1} + O(t^{l+1}) \tag{3.16}$$

and by comparing the two power-series, it is clear that the rescaled solutions $(a_0, a_2)_{\alpha_l}(\delta t)$ to (3.14) for any $\delta > 0$ have $\kappa_l = \alpha_l \delta^{l-1}$.

Meanwhile, for $\delta = 0$, (3.14) can be solved explicitly:

$$a_0^0 = 1 - 2l \qquad\qquad a_2^0 = \kappa_l t^{l-1} \tag{3.17}$$

We can always fix $\kappa_l = 1$ for this solution by a further rescaling of t, so as $\delta \to 0$, a solution $\left(a_0^\delta, a_2^\delta\right)$ to (3.14) has $\left(a_0^\delta, a_2^\delta\right)(t) \to \left(1 - 2l, t^{l-1}\right)$. By assumption, for all δ these solutions exist for all time, and therefore we can always find $T > 1$, $\delta \ll 1$, such that $\left(a_0^\delta, a_2^\delta\right)(T) \in \mathcal{R}_\infty$. If we set $\delta(\alpha_l)$ such that $\delta^{1-l} = \alpha_l$, and take $t^* = T\delta$, then the

[1] if we consider the metric on $\mathcal{O}(-2,-2)$, rescaled by the diffeomorphism $t \mapsto \delta t$, this rescaling is the adiabatic limit as $\delta \to 0$ of the product of the rescaled metric on the fibres and the two copies of $\mathbb{CP}^1$ of fixed volume. See Prop. 3.3.4 in §3.3 for a similar discussion.

solution $(a_0, a_2)_{\alpha_l}$ to the instanton equations (3.9) can be rescaled to a solution of (3.14), so it must satisfy $(a_0, a_2)_{\alpha_l}(t^*) \in \mathcal{R}_\infty$ for some α_l sufficiently large. □

This concludes the proof of Theorem 3.2.9, in the case $l > 1$. As before, one can consider the case $l < 0$ in the same way, by first considering solutions for the pulled-back Calabi-Yau structure by exchanging the factors of $SU(2)$ on the underlying manifold, and then applying this diffeomorphism again on the total space of the principal bundle. □

3.3 Bubbling

Having described the one-parameter family R_ϵ of solutions to (2.8) on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in Theorem 3.2.6, a natural question would be to ask about the behaviour of solutions as $\epsilon \to \infty$. We will show that there is a familiar bubbling phenomenon [Tia00, Thm.4.3.3] in this setting: after a suitable rescaling of the metric, the one-parameter family of Calabi-Yau instantons R_ϵ converges as $\epsilon \to \infty$ to an anti-self-dual connection along the co-dimension four calibrated singular orbit $S^2 = \mathbb{CP}^1 \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We use this result to obtain the expected removable-singularity statement [Tia00, §5.2], which says that as R_ϵ bubbles off this anti-self-dual connection, if we do not perform this rescaling, it will uniformly converge on compact subsets of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \setminus \mathbb{CP}^1$ to the instanton R'_0 of Theorem 3.2.7, which extends smoothly over $\mathbb{CP}^1$. Recall here that the abelian instanton R'_0 is determined by the unique solution to the ODE (3.9) on $[0, \infty)$ with $a_2 = 0$, which has explicit form (2.13) with $C = -2$, $u_0 = -1, u_1(0) = 1$.

In terms of the solutions to (2.8), the heuristic picture is that the smooth trajectory $\{(a_0, a_2)(t) \mid t \geq 0\}$ of the solution R_ϵ, $\epsilon > 0$ degenerates in the limit $\epsilon \to \infty$ to a piecewise-smooth trajectory consisting of two components: the first corresponding to the trajectory of an anti-self-dual connection, which is only traversed in non-zero time t if we rescale, and the second being the trajectory of the abelian instanton R'_0.

Let us first discuss this rescaling in detail: as $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ has the structure of a vector bundle, fibre-wise multiplication equips it with an natural $SU(2)^2$-equivariant action of $\mathbb{R}_{>0}$. Let s_δ denote the corresponding $\mathbb{R}$-action for some $\delta > 0$, i.e. the map fixing the singular orbit and sending $t \mapsto \delta t$ on the space of principal orbits. Pulling back the Riemannian metric g on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ as in (1.24) gives:

$$s_\delta^* g = \delta^2 \left(dt^2 + \lambda_\delta^2 (\eta^{se})^2 + \tfrac{2}{3} u_\delta^+ \left((v^2)^2 + (w^2)^2 \right) \right) + \tfrac{2}{3} u_\delta^- \left((v^1)^2 + (w^1)^2 \right) \tag{3.18}$$

where λ_δ, u_δ^+, and u_δ^- are defined as:

$$\lambda_\delta(t) := \frac{\lambda(\delta t)}{\delta} \qquad u_\delta^+(t) := \frac{(u_1 + u_0)(\delta t)}{\delta^2} \qquad u_\delta^-(t) := (u_1 - u_0)(\delta t) \tag{3.19}$$

We will refer to the limit $\delta \to 0$ as the *adiabatic limit*, and recall that here, $u_0 = -1$, and that $\lambda(t) = \frac{3}{2}t + O(t^3)$, $u_1 = 1 + \frac{3}{2}t^2 + O(t^4)$ near $t = 0$. We claim that λ_δ, $u_\delta^\pm$ have well-defined point-wise limits as $\delta \to 0$[2]:

[2] c.f. Prop. 3.3.4., the proof of Prop. 3.3.1 is similar.

Proposition 3.3.1. *Let u_1, u_0, λ, μ be the solution to the hypo-evolution equations (1.21) extending to S^2 with $u_1(0) = -u_0(0) = -1$ fixed. Define λ_δ, $u_\delta^\pm$ as in (3.19) for some $\delta > 0$. As $\delta \to 0$,*

$$\tfrac{2}{3}u_\delta^+ \to t^2 \qquad\qquad u_\delta^- \to 2 \qquad\qquad \tfrac{2}{3}\lambda_\delta \to t^2$$

Near the adiabatic limit, restricted to any finite distance from the singular orbit, $s_\delta^* g$ is approximated by the metric $\delta^2 g_F + g_B$ for some δ sufficiently small, where g_F denotes a lift of the Euclidian metric on the fibre $\mathbb{C}^2$ and g_B denotes a round metric on the base S^2. Here, the lift of the Euclidian metric on $\mathbb{C}^2$ to the fibres identifies $\frac{3}{2}\eta^{se} = u^1 - u^2$, v^2, w^2 with the standard orthonormal basis of one-forms on $S^3 \subset \mathbb{C}^2$, and v^1, w^1 as an orthonormal basis of one-forms for the singular orbit S^2, viewed upstairs on $SU(2)^2 \to SU(2)^2/U(1) \times SU(2)$.

One can always obtain a solution to the Calabi-Yau instanton equation (2.2) on the flat Calabi-Yau 3-fold $\mathbb{C}^3$ by pulling-back any anti-self-dual connection on $\mathbb{C}^2$ to $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}$, so at least at the level of tangent spaces, if we pull back some Calabi-Yau instanton by s_δ on some sufficiently large neighbourhood of $\mathbb{CP}^1 \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in the adiabatic limit, there should appear an anti-self-dual connection pulled back from the fibre. However, the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is non-trivial, so to make this a global statement, one must first choose a connection on this bundle: as we shall see, this connection will be fixed by the assumption of symmetry.

We first explain how to view the $SU(2)^2$-invariant bundle $P_{0,\mathrm{Id}}$ over $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ as a bundle over the $\mathbb{C}^2$-fibres: there is an obvious $SU(2)^2$-equivariant $U(1)$-action on $S^3 \times \mathbb{C}^2$, viewed as $SU(2) \times \mathbb{H}$, where $SU(2)^2$ acts on the left and $U(1)$ on the right, and this $U(1)$-action induces a quotient map $q : S^3 \times \mathbb{C}^2 \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. By the definition of $P_{0,\mathrm{Id}}$, its pull-back via q is also the pull-back of an $SU(2)$-invariant bundle over $\mathbb{C}^2$, via the projection $\pi : S^3 \times \mathbb{C}^2 \to \mathbb{C}^2$ onto the second factor. Here, we view $\mathbb{C}^2$ as a co-homogeneity one manifold with group diagram $\{1\} \subset SU(2) \subseteq SU(2)$, and define the $SU(2)$-invariant $SU(2)$-bundle over $\mathbb{C}^2$ by the homomorphism $\mathrm{Id} : SU(2) \to SU(2)$, i.e. the singular isotropy group $SU(2)$ acts via the identity homomorphism on the fibre $SU(2)$ cf. Example 1.1.2.

The canonical connection on $P_{0,\mathrm{Id}}$ over the singular orbit $S^2 = SU(2)^2/U(1) \times SU(2)$ is just the flat Maurer-Cartan form $A_2^\flat$, and this pulls back via q over $S^3 \times \mathbb{C}^2$ as the canonical connection (pulled-back via π) on the singular orbit $\{0\} = SU(2)/SU(2)$ of the $SU(2)$-invariant bundle over $\mathbb{C}^2$. Using this choice of reference connection, a connection defined on $q^* P_{0,\mathrm{Id}}$ over $S^3 \times \mathbb{C}^2$ descends to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ if and only if the corresponding adjoint-valued one-form is basic with respect to the $U(1)$-action.

Furthermore, the one-form u^1 is the unique $SU(2)$-invariant connection on the principal $U(1)$-bundle $S^3 \to S^2$, and this induces an $SU(2)$-invariant connection on the associated vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to S^2$. We can use this connection to project any (adjoint-valued) one-form on $S^3 \times \mathbb{C}^2$ to its semi-basic component, and thus uniquely lift any $U(1) \times SU(2)$-invariant $SU(2)$-connection over $\mathbb{C}^2$ to a connection over $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

The upshot of this discussion is that, if we recall the $SU(2)^2$-invariant anti-self-dual instanton over $\mathbb{C}^2$ given in Example 1.1.2:

$$A^{\text{asd}} := \frac{1}{1+t^2}\left(E_1 \otimes u^2 + E_2 \otimes v^2 + E_3 \otimes w^2\right) \tag{3.20}$$

then since this connection is implicitly $U(1) \times SU(2)$-invariant, it can be uniquely lifted to the connection $\bar{A}^{\text{asd}} := A^{\text{asd}} - \left(\frac{1}{1+t^2} - 1\right) E_1 \otimes u^1$.

With these preliminaries out of the way, we can now state the main theorem of this section:

Theorem 3.3.2. *Let $\delta(\epsilon) = \sqrt{2\epsilon^{-1}}$. Then, as $\epsilon \to \infty$:*

(i) $s_\delta^* R_\epsilon(t) \to \bar{A}^{\text{asd}}(t)$.

(ii) $R_\epsilon(t) \to R_0'(t)$ *uniformly on compact subsets of* $(0, \infty)$.

Proof of Theorem 3.3.2 (i). We start by rewriting the instanton equations in terms of the rescaling (3.18), and the lift from an invariant connection on the fibre. If we define $\left(a_0^\delta, a_2^\delta\right)(t) := \left(\frac{1-a_0}{2}, a_2\right)(\delta t)$, and consider some invariant connection A defined by (a_0, a_2), as in Proposition 2.3.1, then:

$$s_\delta^* A(t) = a_2^\delta(t)\left(E_2 \otimes v^2 + E_3 \otimes w^2\right) + a_0^\delta(t) E_1 \otimes \left(u^2 - u^1\right) + E_1 \otimes u^1$$

Written in this way, the instanton equations (3.9) for (a_0, a_2) becomes the following one-parameter family of ODEs for $\left(a_0^\delta, a_2^\delta\right)$:

$$\dot{a}_0^\delta = \frac{2\lambda_\delta}{u_\delta^+}\left(\left(a_2^\delta\right)^2 - a_0^\delta\right) + \frac{2\delta^2\lambda_\delta}{u_\delta^-}\left(\left(1 - a_0{}^\delta\right)\right) \qquad \dot{a}_2^\delta = -\frac{3}{\lambda_\delta}\left(1 - a_0{}^\delta\right) a_2^\delta \tag{3.21}$$

By rescaling the family of solutions R_ϵ to (3.9), we obtain a one-parameter family of solutions of (3.21) for each $\delta > 0$. We now show that one still obtains a one-parameter family as $\delta \to 0$.

Considering the boundary conditions in Proposition A.2.7 for extending $\left(a_0^\delta, a_2^\delta\right)$ to $t = 0$, we can write $a_0^\delta = 1 - t^2 X_0$, $a_2^\delta = 1 - t^2 X_2$ for some smooth functions X_0, X_2, so that, in a neighbourhood of $t = 0$, (3.21) becomes the well-defined initial-value problem:

$$t\dot{X}_0 = 2\left(X_0 - X_2\right) + O(t^2) \qquad t\dot{X}_2 = 2\left(X_0 - X_2\right) + O(t^2) \tag{3.22}$$

hence, once we fix the parameter $\kappa := X_0(0) = X_2(0)$, the continuous dependence of (3.21) on δ gives existence of a sufficiently small open neighbourhood of $t = 0$ such that, for each $\delta \geq 0$, solutions to (3.21) are in a one-parameter family. These are determined by the power-series:

$$a_0^\delta = 1 - \kappa t^2 + O(t^4) \qquad a_2^\delta = 1 - \kappa t^2 + O(t^4) \tag{3.23}$$

Comparing with the power-series in Proposition 3.1.5 for R_ϵ, we see that the rescaled solutions $\left(\frac{1-a_0}{2}, a_2\right)_\epsilon(\delta t)$ for $\delta > 0$ have $\kappa = \delta^2\epsilon/2$, and so the family of solutions $\left(a_0^\delta, a_2^\delta\right)$ exist for all time if $\kappa \geq 0$, $\delta > 0$ by Theorem 3.2.6.

By continuity, the solutions must also exist for all time for $\kappa \geq 0$ as $\delta \to 0$, but as the resulting equations:

$$t\dot{a}_0^0 = 2\left(\left(a_2^0\right)^2 - a_0^0\right) \qquad\qquad t\dot{a}_2^0 = 2\left({a_0}^0 - 1\right)a_2^0 \tag{3.24}$$

have the explicit solutions $a_0^0 = a_2^0 = \left(1+\kappa t^2\right)^{-1}$ cf. Example 1.1.2, this is already guaranteed.

We now set $\delta(\epsilon) = \sqrt{2\kappa\epsilon^{-1}}$ for some given $\kappa > 0$, which we can always fix to be 1 by a further rescaling. By rescaling the family $R_\epsilon = (a_0, a_2)_\epsilon$ to the instanton equations (3.9), we get a solution $\left(a_0^\delta, a_2^\delta\right)(t) = \left(\frac{1-a_0}{2}, a_2\right)_\epsilon(\delta t)$ to (3.21). As we have just shown, the solution has $a_0^\delta(t)$, $a_2^\delta(t) \to \left(1+t^2\right)^{-1}$ as $\delta \to 0$, hence $s_\delta^* R_\epsilon(t) \to \bar{A}^{\text{asd}}(t)$. □

Remark 3.3.3. *Since $\left(1+\kappa t^2\right)^{-1}$ blows up in finite time if $\kappa < 0$, by keeping the freedom to vary κ, this rescaling argument can be used to show that local solutions of Theorem 3.2.6 with $\epsilon < 0$ are not only unbounded but blow up in finite time.*

Proof of Theorem 3.3.2 (ii). Let $(a_0, a_2)_\epsilon(t) = \left(a_0^\epsilon, a_2^\epsilon\right)(t)$ denote the one-parameter family of solutions to (3.9) corresponding to R_ϵ. Using the local power-series in Proposition 3.1.5, and the forward-comparison lemma, it follows that $a_0^\epsilon(t^*)$, $a_2^\epsilon(t^*)$ are both strictly monotonic (increasing and decreasing, respectively) in $\epsilon > 0$ for fixed $t^* > 0$, and so $(a_0, a_2)_\epsilon(t)$ converges point-wise on $(0, \infty)$ as $\epsilon \to \infty$.

Since $a_2^\epsilon(t)$ is strictly decreasing in t for all $t > 0$, $\epsilon > 0$, if we assume that the pointwise-limit $\inf_\epsilon a_2^\epsilon(t^*)$ is non-zero, then we can use the inequality $a_2^\epsilon(t) \geq a_2^\epsilon(t^*) \geq \inf_\epsilon a_2^\epsilon(t^*) \geq 0$ for all $\epsilon > 0$, $t \leq t^*$ to uniformly bound a_2^ϵ away from zero on $(0, t^*)$ and derive a contradiction. Explicitly, by part (i), for any $\varepsilon > 0$, $T > 0$, $\exists\epsilon(\varepsilon, T) > 0$ such that $\forall\epsilon \geq \epsilon(\varepsilon, T)$, the rescaled solution $(a_0, a_2)_\epsilon(\sqrt{2\epsilon^{-1}}T)$ satisfies $|a_2^\epsilon(\sqrt{2\epsilon^{-1}}T) - (1+T^2)^{-1}| < \varepsilon$. By the assumption $L := \inf_\epsilon a_2^\epsilon(t^*) > 0$, we can pick ε, T such that $0 < \varepsilon < L - (1+T^2)^{-1}$, and then apply our inequality to any $\epsilon \geq \max\{\epsilon(\varepsilon, T), 2\left(\frac{T}{t^*}\right)^2\}$:

$$\varepsilon < L - (1+T^2)^{-1} \leq |a_2^\epsilon(\sqrt{2\epsilon^{-1}}T)| - |(1+T^2)^{-1}| \leq |a_2^\epsilon(\sqrt{2\epsilon^{-1}}T) - (1+T^2)^{-1}|$$

since $\sqrt{2\epsilon^{-1}}T \leq t^*$. However, this demonstrates the existence of an $\epsilon \geq \epsilon(\varepsilon, T)$ such that the inequality $|a_2^\epsilon(\sqrt{2\epsilon^{-1}}T) - (1+T^2)^{-1}| < \varepsilon$ fails.

This previous discussion implies that a_2^ϵ converges uniformly to zero on any compact interval contained in $(0, \infty)$, and by using (3.10) to express the derivative $\dot{a}_0^\epsilon$ purely in terms of a_2^ϵ (up to some fixed functions of t), we also get the uniform convergence of a_0^ϵ. Now consider the initial value problem defined by the ODE (3.9) with $(a_0, a_2)(t^*) = (a_0, a_2)_\epsilon(t^*)$ at fixed initial time $t^* > 0$: this has unique solution $(a_0, a_2)_\epsilon(t)$ on $(0, \infty)$, and continuous dependence on initial conditions guarantees that the limit as $\epsilon \to \infty$ is the unique solution to (3.9) with $a_2 = 0$, and $a_0(t^*) = \sup_\epsilon a_0^\epsilon(t^*)$. Since this solution must be contained in the closure of $\mathcal{R}_0$, by Remark 2.4.1, this must be identified with R_0' on $(0, \infty)$, the unique solution to (3.9) bounded on $(0, \infty)$ with $a_2 = 0$. □

We will now move on to discussing a somewhat similar bubbling phenomena for instantons on $\mathcal{O}(-2, -2)$. Recall Example 1.1.3, there is a one-parameter family of anti-self-dual instantons for the Eguchi-Hanson metric on $T^*\mathbb{CP}^1$: we will show that we can recover these instantons fibred along a co-dimension four calibrated sub-manifold $\mathbb{CP}^1 \subset \mathcal{O}(-2, -2)$, by considering an appropriate adiabatic limit of the one-parameter families of Calabi-Yau instantons $Q^l_{\alpha_l}$ on $\mathcal{O}(-2,-2)$.

Firstly, without loss of generality, we can fix an overall scale for the family of metrics on $\mathcal{O}(-2,-2)$ such that one of the copies of $\mathbb{CP}^1 \subset \mathcal{O}(-2,-2)$ has fixed volume. If we let the volume of the other $\mathbb{CP}^1 \subset \mathcal{O}(-2,-2)$ go to zero, while rescaling the metric by s_δ as in (3.18), (3.19) with $\delta \to 0$, then the metric $s_\delta^* g$ is approximated by a fibration $\delta^2 g_{\mathrm{EH}} + g_B$ of the lift of a rescaled Eguchi-Hanson metric g_{EH} described in (1.5) over a round metric on the base $\mathbb{CP}^1$, for some δ sufficiently small. More precisely:

Proposition 3.3.4. *Let u_1, u_0, λ, μ be the one-parameter family of solutions to the hypo-evolution equations (1.21) extending to $S^2 \times S^2$ with $U_1 - U_0 = u_1(0) - u_0(0) = 1$ fixed. Define λ_δ, $u_\delta^\pm$ as in (3.19), with $\delta = \sqrt{\frac{2}{3}\left(U_1 + U_0\right)}$. As $\delta \to 0$,*

$$\tfrac{2}{3} u_\delta^+ \to \varphi^2 \qquad\qquad u_\delta^- \to 1 \qquad\qquad \tfrac{2}{3}\lambda_\delta \to \varphi\sqrt{1-\varphi^{-4}}$$

where $\varphi(t)$ is the unique solution to $\dot{\varphi}^2 = 1 - \varphi^{-4}$ on $[0,\infty)$ with $\varphi(0) = 1$, $\ddot{\varphi}(0) = 2$.

Proof. Let u_1, u_0, λ, μ be solutions to (1.21) extending over $S^2 \times S^2$ at $t = 0$. We define:

$$u_\delta^+(t) := \tfrac{1}{\delta^2}\left(u_1 + u_0\right)(\delta t) \qquad u_\delta^-(t) := \left(u_1 - u_0\right)(\delta t) \qquad \left(\mu\lambda\right)_\delta(t) := \tfrac{1}{\delta^2}\mu\lambda(\delta t)$$

with $\delta = \sqrt{\frac{2}{3}\left(U_1 + U_0\right)}$. Then $u_\delta^\pm$, $\left(\mu\lambda\right)_\delta$ solve the rescaled system:

$$\dot{u}_\delta^+ = \frac{2(\mu\lambda)_\delta}{\sqrt{u_\delta^+ u_\delta^-}} \qquad \dot{u}_\delta^- = \frac{2\delta^2(\mu\lambda)_\delta}{\sqrt{u_\delta^+ u_\delta^-}} \qquad \dot{(\mu\lambda)}_\delta = 3\sqrt{u_\delta^+ u_\delta^-} \tag{3.25}$$

with initial conditions $u_\delta^+(0) = \frac{3}{2}$, $\left(\mu\lambda\right)_\delta(0) = 0$, and $u_\delta^-(0) > 0$ a fixed constant. By varying initial condition $u_\delta^-(0)$ for this family of smooth initial-value problems, we obtain a continuous one-parameter family of solutions to (3.25) for each fixed $\delta \geq 0$: for $\delta > 0$, these arise by rescaling solutions to (1.21), with $u_\delta^-(0) = U_1 - U_0$.

The parameter $u_\delta^-(0)$ corresponds to varying the overall scale of the resulting Eguchi-Hanson metric in the limit $\delta \to 0$; we will fix this by setting $u_\delta^-(0) = 1$. We can then can explicitly solve the system (3.25) with $\delta = 0$, by taking:

$$u_0^+ = \tfrac{3}{2}\varphi^2 \qquad\qquad u_0^- = 1 \qquad\qquad \left(\mu\lambda\right)_0 = \sqrt{\left(\tfrac{3}{2}\right)^3\left(\varphi^4 - 1\right)}$$

Since $\left(\mu\lambda\right)_\delta = \lambda_\delta\sqrt{u_\delta^+ u_\delta^-}$ by definition, the claim is proved. □

With the geometry in the adiabatic limit made explicit, we now discuss the procedure for lifting invariant connections on $T^*\mathbb{CP}^1$ to invariant connections on $\mathcal{O}(-2,-2)$. This follows a very similar strategy as previously discussed for lifting invariant connections on $\mathbb{C}^2$ to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and we will only sketch the details here:

First, note that there is an $SU(2)^2$-equivariant $U(1)$-action on $S^3 \times T^*\mathbb{CP}^1$ given by the product $e^{i\theta} \mapsto \left(e^{i\theta}, e^{-2i\theta}\right)$ of the standard Hopf action on S^3 and fibre-wise multiplication on $T^*\mathbb{CP}^1$. Taking the quotient by this action, by definition, gives a map $S^3 \times T^*\mathbb{CP}^1 \to \mathcal{O}(-2,-2)$.

Pulling back the $SU(2)^2$-invariant bundle $P_{1-l,l}$ by this quotient map identifies it with the pull-back of an $SU(2)$-invariant bundle over $T^*\mathbb{CP}^1$ via projection $S^3 \times T^*\mathbb{CP}^1 \to T^*\mathbb{CP}^1$ onto the second factor. Here, we view $T^*\mathbb{CP}^1$ as a co-homogeneity one manifold with group diagram $\mathbb{Z}_2 \subset U(1) \subset SU(2)$, with the $SU(2)$-invariant $SU(2)$-bundle defined by the homomorphism $\lambda^l : U(1) \to SU(2)$, i.e. the singular isotropy group $U(1)$ acts as the l^{th}-power of the standard diagonal embedding $\lambda : U(1) \hookrightarrow SU(2)$ on the fibre $SU(2)$ cf. Example 1.1.3.

Then, using the canonical connection on the singular orbit S^2 as a reference connection, and the $SU(2)$-invariant one-form u^1 on $S^3 \to S^2$, we can (uniquely) lift $U(1) \times SU(2)$-invariant connections on $T^*\mathbb{CP}^1$ to $SU(2)^2$-invariant connections on $\mathcal{O}(-2,-2)$. In particular, the one-parameter family of $U(1) \times SU(2)$-invariant anti-self-dual instantons A^l_κ over $T^*\mathbb{CP}^1$ described in Example 1.1.3:

$$A^l_\kappa := \alpha_1 E_1 \otimes u^2 + \alpha_2 \left(E_2 \otimes v^2 + E_3 \otimes w^2\right) \tag{3.26}$$

with α_1, α_2 given explicitly by (1.7), can be lifted to a one-parameter family of $SU(2)^2$-invariant connections $\bar{A}^l_\kappa := A^l_\kappa - (\alpha_1 - 1)\, E_1 \otimes u^1$.

Theorem 3.3.5. *Fix $U_1 - U_0 = 1$, $l > 0$, $0 \leq \kappa \leq 1$, and let $\delta = \sqrt{\frac{2}{3}\,(U_1 + U_0)}$, $\alpha_l = l\sqrt{\kappa}\delta^{1-l}$. Then $s^*_\delta Q^l_{\alpha_l}(t) \to \bar{A}^l_\kappa(t)$ as $\delta \to 0$.*

Proof. We consider solutions to the rescaled system (3.21) with $\delta = \sqrt{\frac{2}{3}\,(U_1 + U_0)}$. Using extension conditions to $t = 0$ given by Proposition A.2.3, we can write $a^\delta_2 = t^{l-1} X_2$ for some smooth function $X_2(t)$, so that a^δ_0, X_2 solve a family of singular initial problems of the form:

$$t\dot{a}^\delta_0 = O(t^2) \qquad\qquad t\dot{X}_2 = \left(a^\delta_0 - l\right) X_2 + O(t^2)$$

This system has a one-parameter family of solutions all the way to $\delta = 0$, parametrised by $\gamma := X_2(0)$. Comparing with the power-series for $Q^l_{\alpha_l}$ in Proposition 3.1.1, we see that $\gamma = \alpha_l \delta^{l-1}$ for $\delta > 0$.

In the limit $\delta \to 0$, the rescaled system (3.21) converges to the anti-self-dual equations (1.6) for $\left(a^\delta_0, a^\delta_2\right)$:

$$\dot{a}^\delta_0 = 2\tfrac{\dot{\varphi}}{\varphi}\left(\left(a^\delta_2\right)^2 - a^\delta_0\right) \qquad\qquad \dot{a}^\delta_2 = 2\tfrac{1}{\dot{\varphi}\varphi} a^\delta_2 \left(a^\delta_0 - 1\right)$$

which can be explicitly solved by (1.7) with $\kappa = \left(\frac{\gamma}{l}\right)^2$. □

Remark 3.3.6. *Since the explicit solution (1.7) blows up in finite time if $\kappa > 1$, up to exchanging the copies of $\mathbb{CP}^1 \subset \mathcal{O}(-2,-2)$, one can use this rescaling argument to show that the local solutions $Q^l_{\alpha_l}$ for metrics on $\mathcal{O}(-2,-2)$ with $U_1 \pm U_0$ sufficiently close to 0 must also blow up in finite time for some α_l sufficiently large.*

3.4 Solutions to the Monopole Equations

In this section, we analyse the qualitative behaviour of solutions to the monopole equations (2.8) with non-zero Higgs field Φ away from the singular orbit. Assuming that the connection is not an instanton, we first show that there are no solutions for $\mathcal{O}(-2,-2)$, or $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ with quadratic curvature decay:

Proposition 3.4.1. *There are no irreducible invariant monopoles on $\mathcal{O}(-2,-2)$ or $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ with quadratic curvature decay.*

Proof. We look at solutions to the monopole equations (2.8) with $v_0 = 0, v_3 = \mu$, i.e. for a hypo-structure of type $\mathcal{I}$ given by (1.20):

$$\begin{aligned}
\dot{a}_0 &= \frac{4\lambda}{\mu^2}\left((a_1^2 + a_2^2 - 1)u_0 - (a_0 - a_1^2 + a_2^2)u_1\right)\\
\dot{\phi} &= -\frac{6}{\mu}a_1 a_2\\
\dot{a}_1 &= \frac{3}{2\lambda}(a_0 - 1)a_1 - 2\frac{u_1 - u_0}{\mu}a_2\phi\\
\dot{a}_2 &= -\frac{3}{2\lambda}(a_0 + 1)a_2 - 2\frac{u_1 + u_0}{\mu}a_1\phi
\end{aligned} \tag{3.27}$$

Recall from Lemma 2.3.6 that a weak condition for solutions (a_0, a_1, a_2, ϕ) was to assume at least boundedness of a_0, a_1, a_2. We will show that there are no such solutions to (3.27) existing for all time: in particular, we show that if a solution exists for all time with a_0 bounded and ϕ non-zero, then $a_1 - a_2$ must have at least exponential growth at infinity, provided certain initial conditions are satisfied. These initial conditions will be satisfied by the local solutions extending to $t = 0$ obtained in the previous propositions, up to certain easily-verified symmetries:

Lemma 3.4.2. *The following involutions are symmetries of* (3.27)*:*

$$(a_0, a_1, a_2, \phi) \mapsto (a_0, -a_1, -a_2, \phi) \tag{3.28}$$

$$(a_0, a_1, a_2, \phi) \mapsto (a_0, -a_1, a_2, -\phi) \tag{3.29}$$

We now find the set in which our solutions remain for all time:

Lemma 3.4.3. *The set $\mathcal{R}^+_\infty := \{(a_0, a_1, a_2, \phi) \in \mathbb{R}^4 \mid a_1 > 0 > a_2, \phi > 0\}$ is forward-invariant under* (3.27).

Proof. We let t be the first time a solution (a_0, a_1, a_2, ϕ) leaves $\mathcal{R}^+_\infty$. However, none of the possibilities for a solution to leave $\mathcal{R}^+_\infty$ can hold at t:

(i) $a_1 = 0, a_2 < 0, \phi > 0$, since it implies $\dot{a}_1 > 0$.

(ii) $a_2 = 0, a_1 > 0, \phi > 0$, since it implies $\dot{a}_2 < 0$.

(iii) $a_1 = a_2 = 0$, $\phi \geq 0$ coincides with the reducible solution $a_1 \equiv a_2 \equiv 0$. By uniqueness, this would imply that the solution must be reducible for all time.

(iv) Since $\phi(t_0) > 0$ for some $t_0 < t$, if we assume $\phi(t) = 0$, then by the mean value theorem $\dot{\phi}(t_1) < 0$ for some $t_0 < t_1 < t$ which implies $a_1(t_1)a_2(t_1) > 0$.

∎

Remark 3.4.4. *Although we have shown explicitly from the ODEs that solutions preserve $|\phi| > 0$, it also follows more generally, since the function $|\Phi|^2 : M \to \mathbb{R}$ is sub-harmonic for any monopole (A, Φ) over an arbitrary Calabi-Yau 3-fold M. In the invariant co-homogeneity one setting, this implies that if $|\Phi(t)|^2 = 0$ for some $t > 0$, then it vanishes for all t by the maximum principle.*

We also prove that solutions lying in this set, if they exist for all time, are (exponentially) unbounded as $t \to \infty$:

Lemma 3.4.5. *If (a_0, a_1, a_2, ϕ) is a solution to (3.27) existing for all time $t \geq t^*$ with a_0 bounded, lying in $\mathcal{R}^+_\infty$ at initial time t^*, then $a_1(t) - a_2(t)$ cannot be uniformly bounded for all $t \geq t^*$.*

Proof. Since the CY-structure is AC, $\lambda \sim t$ as $t \to \infty$, hence $\frac{3}{2\lambda}(a_0 \pm 1) \to 0$ as $t \to \infty$ by assumption. We also have that $\frac{u_1 \pm u_0}{\mu} \to 1$ in the same limit. Since $a_1 > 0 > a_2$, then for every $\epsilon > 0$, $\exists T^* \gg 0$ such that $\forall t > T^*$ the following inequalities hold:

$$\begin{aligned} \frac{3}{2\lambda}(a_0 - 1)a_1 &> -\epsilon a_1 & \quad -\frac{3}{2\lambda}(a_0 + 1)a_2 &< -\epsilon a_2 \\ -\frac{u_1 - u_0}{\mu}a_2 &> -(1-\epsilon)a_2 & \quad -\frac{u_1 - u_0}{\mu}a_1 &< -(1-\epsilon)a_1 \end{aligned} \tag{3.30}$$

Let $T := \max\{t^*, T^*\}$ for our fixed initial time t^*, and $\bar{\phi} := \phi(t^*) > 0$. Since ϕ is strictly increasing in $\mathcal{R}^+_\infty$, we have $\phi(t) > \bar{\phi}$ for $t > t^*$. Putting all our inequalities together on $t > T$, we obtain the following:

$$\dot{a}_1 - \dot{a}_2 > (2\bar{\phi} - \epsilon(2\bar{\phi} + 1))(a_1 - a_2)$$

and if we choose $\epsilon < \dfrac{\bar{\phi}}{(2\bar{\phi} + 1)}$, then by integrating:

$$a_1(t) - a_2(t) \geq (a_1(T) - a_2(T)) \exp((t - T)\bar{\phi})$$

□

This completes the proof the proposition, since in all cases, using the symmetries (3.28), (3.29), for the power series solutions near the singular orbit, one can reduce to the case of the monopole lying in $\mathcal{R}^+_\infty$ for some small initial time:

1. For local solutions $(R', \Psi')_{\epsilon', \delta'}$ defined by Proposition 3.1.6, since $\epsilon', \delta' \neq 0$ by assumption i.e. we do not have an instanton, then up to symmetry one can assume $\epsilon', \delta' > 0$. Hence $(a_0, a_1, a_2, \phi)_{\epsilon', \delta'}$ lies in $\mathcal{R}^+_\infty$.

2. For local solutions $\left(Q^l, \Theta^l\right)_{\alpha_l, \beta_l}$ defined by Proposition 3.1.1, since $\alpha_l, \delta_l \neq 0$ by assumption, up to symmetry one can assume $\alpha_l < 0, \beta_l > 0$. Hence $(a_0, a_1, a_2, \phi)_{\alpha_l, \beta_l}$ lies in $\mathcal{R}^+_\infty$. This also covers the case $l \leq 0$, by exchanging the factors of $SU(2)$ on the principal orbits, and considering the Calabi-Yau structure on $\mathcal{O}(-2, -2)$ pulled-back via this diffeomorphism.

3. For local solutions $(R, \Psi)_{\epsilon, \delta}$ defined by Proposition 3.1.5, since $\delta \neq 0$ by assumption, then up to (3.29), one can also assume $\delta > 0$. While the image of a solution under

(3.28) may not extend to the singular orbit, existence of a bounded solution extending to the singular orbit would imply existence of a bounded solution in $\mathcal{R}_\infty^+$ under the symmetry.

□

The existence of invariant monopoles on T^*S^3 was shown in [Oli16]: by restricting to monopoles that are independent of the base S^3-directions, i.e. solving (2.8) with $a_0 = 0$, $a_1 = a_2$, Oliveira constructed a one-parameter family of invariant monopoles for T^*S^3, first by considering the local solutions $(S, \Phi)_{\xi,\chi}$ with $\xi = 0$ of this system, and then applying PDE methods for monopoles on a family of metrics on the fibre. Due to a computational error in [Oli16, Lemma 6, Appendix A], Oliveira did not consider local solutions $(S, \Phi)_{\xi,\chi}$ with ξ non-zero, but we fix the resulting gap in the proof of the main theorem [Oli16, Theorem 1] by imposing quadratic curvature decay:

Proposition 3.4.6. *Invariant monopoles with quadratic curvature decay on $P_{\mathrm{Id}} \to T^*S^3$ are in a one-parameter family $(S, \Phi)_\chi := (S, \Phi)_{0,\chi}$, $\chi > 0$, up to gauge. Moreover:*

(i) $\lim_{t\to\infty} (S, \Phi)_\chi (t) = (A^{\mathrm{can}}, \Phi_\chi)$*, where Φ_χ is a constant non-trivial Higgs field.*

(ii) $(S, \Phi)_{0,0} = S_0$ *where S_0 is the instanton of Theorem 3.2.1, with a trivial Higgs field.*

Proof. We rewrite the monopole equations (2.8) with $a_\pm := a_1 \pm a_2$:

$$\begin{aligned}
\dot{a_0} &= \frac{4\lambda}{\mu}\left(a_+a_- - a_0\right) \\
\dot{a}_+ &= \frac{3(v_3+v_0)}{2\lambda\mu}(a_0a_- - a_+) - 2a_+\phi \\
\dot{a}_- &= \frac{3(v_3-v_0)}{2\lambda\mu}(a_0a_+ - a_-) + 2a_-\phi \\
\dot{\phi} &= \frac{3}{\mu^2}\left(\left(\frac{1}{2}(a_+^2+a_-^2) - 1\right)v_0 - \frac{1}{2}(a_+^2 - a_-^2)v_3\right)
\end{aligned} \tag{3.31}$$

for μ, λ, v_3, v_0 explicit solutions to the hypo-equations given in (1.27), and recall from Lemma 2.3.6, that we are interested in solutions with $a_0, a_+, a_-, t\phi a_+, t\phi a_-$ bounded. There are three parts to the proof:

1. Solutions to (3.31) extending over the singular orbit with $a_0, a_- \not\equiv 0$ are unbounded.
2. Solutions to (3.31) with $a_0, a_- \equiv 0$, which have local power-series $(S, \Phi)_{\xi,\chi}$ given in Proposition 3.1.3 for $\xi = 0$, are bounded iff $\chi \geq 0$.
3. In this case, solutions with $\chi > 0$ have $a_+ \to 0$, $\phi \to \phi_\chi$ as $t \to \infty$ for some constant $\phi_\chi > 0$, and $ta_+\phi$ bounded. The solution with $\xi, \chi = 0$ is the explicit instanton (3.6) found in [Oli16].

To prove the first part, we will recall the symmetries (2.9) and (2.10) of the problem:

Lemma 3.4.7. *The following involutions are symmetries of (3.31):*

$$(a_0, a_+, a_-, \phi) \mapsto (a_0, -a_+, -a_-, \phi) \tag{3.32}$$

$$(a_0, a_+, a_-, \phi) \mapsto (-a_0, a_+, -a_-, \phi) \tag{3.33}$$

We can also prove a strict monotonicity condition for ϕ:

Lemma 3.4.8 (Monotonicity)**.** *A solution (a_0, a_+, a_-, ϕ) to (3.31) with $\pm\phi(t^*) > 0, \pm\dot{\phi}(t^*) > 0$ at some initial time $t^* > 0$, has $\pm\phi(t) > 0, \pm\dot{\phi}(t) > 0$ for all $t \geq t^*$.*

Proof. We calculate:

$$\ddot{\phi}\Big|_{\dot{\phi}=0} = \frac{6}{\mu^2}\left(a_+^2\left(v_3 - v_0\right) + a_-^2\left(v_3 + v_0\right)\right)\phi$$

In particular, since $\dot{\phi} \neq 0$ for $a_+ = a_- = 0$, and $(v_3 \pm v_0) > 0$ for $t \neq 0$, we have $\ddot{\phi} > 0$ at $\dot{\phi} = 0$ iff $\phi > 0$. Hence any critical point for $\phi > 0$ must be minimum, and since $\dot{\phi}(t^*) > 0$, we must have $\dot{\phi}(t) > 0, \phi(t) > 0$ for all $t > t^*$. The proof for $\phi(t) < 0, \dot{\phi}(t) < 0$ is similar. □

Using this, we find a set that contains our solutions for all time:

Lemma 3.4.9. *A solution (a_0, a_+, a_-, ϕ) to (3.31) lying in the set*

$$\mathcal{S}_\infty^\pm := \{(a_0, a_+, a_-, \phi) \in \mathbb{R}^4 \mid a_0 > 0, a_+ > 0, a_- > 0, \pm\phi > 0\}$$

at some initial time t^* with $\pm\dot{\phi}(t^*) > 0$, remains there for all forward time $t \geq t^*$.

Proof. Let t be the first time a solution (a_0, a_+, a_-, ϕ) leaves $\mathcal{S}_\infty^\pm$. However, none of the possibilities for a solution to leave $\mathcal{S}_\infty^\pm$ can hold at t:

(i) $a_0 = 0, a_- > 0, a_+ > 0$, since it implies $\dot{a}_0 > 0$. The same is true if we permute indices $0, +, -$.

(ii) $a_+ = 0, a_- = 0, a_0 > 0$, since then $\dot{a}_+ = \dot{a}_- = 0$. By local uniqueness and existence for ODEs, from (3.31), one sees that the solution must have $a_+ \equiv 0, a_- \equiv 0$, at least for some small interval $(t - \epsilon, t + \epsilon)$. Again one obtains similar results by permuting indices.

(iii) $a_0 = a_+ = a_- = 0$ coincides with solution $(0, 0, 0, -3I)$, where $\dot{I} = \frac{v_0}{\mu^2}$, which is a solution to (3.31) for any choice of initial condition $\phi(t)$ for $t > 0$.

(iv) $\phi = 0$ is impossible by monotonicity.

□

We can now use monotonicity to bound ϕ away from zero, which will show that solutions in $\mathcal{S}_\infty^\pm$ must be unbounded as $t \to \infty$:

Lemma 3.4.10. *A solution (a_0, a_+, a_-, ϕ) to (3.31) lying in $\mathcal{S}_\infty^\pm$ with $\pm\dot{\phi} > 0$ at some initial time $t^* > 0$ cannot have $a_\mp$ uniformly bounded for all forward-time $t \geq t^*$.*

Proof. We start with the case $\mathcal{S}_\infty^+$. Since $a_+ a_0 > 0$, we have the following inequality:

$$\dot{a}_- > \left(2\phi - \frac{3(v_3 - v_0)}{2\lambda\mu}\right) a_-$$

As the Calabi-Yau structure (1.27) is asymptotically conical, we have $\frac{3(v_3-v_0)}{2\lambda\mu} \to 0$ as $t \to \infty$. So, as ϕ strictly increasing, then for fixed t^*, $\exists T > t^*$ such that $\forall t > T$:

$$\bar{\phi} := \phi(t^*) > \frac{3(v_3 - v_0)}{2\lambda\mu}(t)$$

Then, since $a_- > 0$, integrating the inequality for $\dot{a}_-$ gives:

$$a_-(t) \geq a_-(T) \exp((t - T)\bar{\phi})$$

The proof for $\mathcal{S}_\infty^-$ is almost identical, since now:

$$\dot{a}_+ > \left(-2\phi - \frac{3(v_3 + v_0)}{2\lambda\mu}\right) a_+ \tag{3.34}$$

with $\phi < 0$ monotonically decreasing and $\frac{3(v_3+v_0)}{2\lambda\mu} \to 0$. $\square$

To complete the proof of the first part of the theorem, one only need apply this lemma to the power-series solution $(S, \Phi)_{\xi,\chi}$ of Proposition 3.1.3. Up to symmetry, we can take $\chi, \xi > 0$, so for some $0 < t^*$ sufficiently small, the solution $(a_0, a_+, a_-, \phi)_{\xi,\chi}(t^*)$ lies in $\mathcal{S}_\infty^+$ with $\dot{\phi}(t^*) > 0$, and hence we obtain that these solutions are unbounded.

To prove the second and third parts of the theorem (cf. [Oli16, Theorem 1]), we need to prove local solutions $(S, \Phi)_{0,\chi}$ with $\xi = 0$, i.e. solutions to the ODE:

$$\dot{a}_+ = -a_+ \left(\frac{3(v_3 + v_0)}{2\lambda\mu} + 2\phi\right) \qquad \dot{\phi} = -\frac{3}{\mu^2}\left(\frac{1}{2}a_+^2(v_3 - v_0) + v_0\right) \tag{3.35}$$

have fixed asymptotics $a_+ \to 0$, $\phi \to \phi_\chi > 0$ only in the case $\chi > 0$, and if $\chi < 0$ are these solutions are unbounded as $t \to \infty$. By uniqueness, the local solution with $\chi = 0$, $\xi = 0$ is the instanton (3.6) with $\phi \equiv 0$.

We first note that the sign of a_+ is preserved by (3.35), hence by using the gauge symmetry (3.32) we can always reduce to the case $a_+ > 0$ in the following. Assuming this, we can prove the existence of a set in which solutions become unbounded:

Lemma 3.4.11. *Solutions to (3.35) with $a_+ > 0, \phi < 0, \dot{\phi} < 0$ at some initial time $t^* > 0$, cannot have a_+ uniformly bounded for all forward-time $t \geq t^*$.*

Proof. This proceeds almost identically to the proof of Lemma 3.4.10, only now we have the inequality (3.34) is an equality. Again we have $\phi < 0$ monotonically decreasing by Lemma 3.4.8, and integrating the inequality for $\dot{a}_+$ in terms of $\phi(t^*)$, we have that there exists $T > t^*$, such that for all $t \geq T$:

$$a_+(t) \geq a_+(T) \exp(-(t - T)\phi(t^*))$$

$\square$

We also prove the existence of a set in which solutions are bounded for all time, and have the desired asymptotics:

Lemma 3.4.12. *Solutions to (3.35) with $a_+ > 0, \phi > 0, \dot{\phi} > 0$ at some initial time t^*, are bounded for all $t \geq t^*$, and have $(a_+, \phi) \to (0, \phi_\chi)$ as $t \to \infty$, for some constant $\phi_\chi > 0$. Moreover, $ta_+\phi$ is bounded for all $t \geq t^*$.*

Proof. We already have lower bounds for a_+ and ϕ. We now prove an upper bound for ϕ: we have the inequality $\dot{\phi} < -\frac{3v_0}{\mu^2}$, and hence by integrating ϕ must be bounded above. Since ϕ is also strictly increasing, this implies the existence of a limit $\phi \to \phi_\chi > 0$ as $t \to \infty$.

For a_+, since $\frac{3(v_3+v_0)}{2\lambda\mu} > 0$, and $\phi > 0$ strictly increasing, we have the inequality $\dot{a}_+ \leq -2a_+\phi(t^*)$. Integrating this, we get:

$$0 < a_+(t) \leq a_+(t^*) \exp(-2(t - t^*)\phi(t^*))$$

giving us the required asymptotics for a_+, $t\phi a_+$. □

The final two parts of the proof of the Theorem 3.4.6 are now immediate, since the local solutions $(S, \Phi)_{0,\chi}$ to (3.35) given by Proposition 3.1.3 with $\xi = 0$, $\chi < 0$ satisfies the conditions of Lemma 3.4.11, while for $\chi > 0$ they satisfy the conditions of Lemma 3.4.12. □

Remark 3.4.13. *Since the local solutions (a_0, a_1, a_2, ϕ) to (2.8) satisfying the conditions of Lemmas 3.4.11, 3.4.5, or 3.4.10 grow at least exponentially in t as $t \to \infty$ one could also consider weakening the assumption of quadratic curvature decay to e.g. bounded curvature, in the statements of Propositions 3.4.1 and 3.4.6 .*

Chapter 4

G_2 Gauge Theory

In this final chapter, which is based on a joint work with Matthew Turner, we will give a complete description of the behaviour of G_2-instantons with an $SU(2)^3$-symmetry, on G_2-manifolds with asymptotically conical geometry and $SU(2)^3$ acting with co-homogeneity one. Note that a result of [FHN21b, Thm.E] implies that the only complete, simply-connected G_2-manifold with co-homogeneity one symmetry of $SU(2)^3$ is the spinor bundle $\mathbf{S}(S^3)$ of S^3, equipped with the family of metrics of Bryant-Salamon [BS89].

We consider G_2-instantons on $\mathbf{S}(S^3)$ with its one-parameter family of G_2-metrics, completing the analysis of [LO18] by constructing a new one-parameter family of G_2-instantons, and classifying the relevant moduli-spaces satisfying a natural curvature decay condition.

To carry this out, we will first give an overview of the G_2-instanton equations in §4.1, before focusing on the invariant setting in §4.2. We will recall from [LO18] the system of ODEs corresponding to the $SU(2)^3$-invariant equations in Proposition 4.2.1, and the parametrization of its local solutions in Proposition 4.2.3. The new results contained in this section are Theorem 4.2.5, classifying global solutions to the G_2-instanton ODEs using the theory of asymptotically autonomous ODE systems from [Mar56].

In the final section §4.3, we use a computation of [Dri20] to show Proposition 4.3.5, which describes the moduli-space of G_2-instantons on $\mathbf{S}(S^3)$ away from the invariant regime of §4.2.

4.1 G_2 Gauge Theory: Set-Up

Let (M^7, φ) be a G_2-manifold, equipped with a principal G-bundle $P \to M$ for a compact, semi-simple Lie group G, where $\varphi \in \Omega^3(M)$ is a non-degenerate torsion-free G_2-structure on M. A connection A on P is called a *G_2-instanton* if it satisfies the *G_2-instanton equations*:

$$*F_A = -F_A \wedge \varphi \tag{4.1}$$

where $*$ is the Hodge star of the Riemannian metric defined by φ, and $F_A \in \Omega^2(\mathrm{ad}P)$ is the curvature of A.

From (4.1), it is not hard to see that G_2-instantons minimize the Yang-Mills energy functional $\mathcal{YM}(A) := \int_M |F_A|^2$ on the space of finite-energy connections on P.

We will write (4.1) in the case (M,φ) is foliated by parallel hyper-surfaces. Recall (1.29) from §1.3: in this case, we can write the G_2-structure on M in terms of a one-parameter family of half-flat structures $(\omega,\Omega)_{t\in I}$:

$$\varphi = dt\wedge\omega + \mathrm{Re}\Omega \qquad\qquad *\varphi = -dt\wedge\mathrm{Im}\Omega + \tfrac{1}{2}\omega^2$$

As it will be more convenient for the following computations, we note that the G_2-instanton equations (4.1) are equivalent to:

$$F_A\wedge *\varphi = 0 \tag{4.2}$$

So, written in the temporal gauge, the G_2-instanton equations for $A=A_t$ appear as:

$$F_{A_t}\wedge\omega^2 = 0 \tag{4.3a}$$

$$F_{A_t}\wedge\mathrm{Im}\Omega - \frac{1}{2}\partial_t A_t\wedge\omega^2 = 0 \tag{4.3b}$$

If the G_2-structure φ is torsion-free, in particular if φ is co-closed, then $(\omega,\Omega)_{t\in I}$ is subject to the evolution equation $d\mathrm{Im}\Omega = -\frac{1}{2}\partial_t\left(\omega^2\right)$. Thus, the G_2-instanton equation (4.3a) is preserved under evolution by (4.3b).

If (M,φ) is a G_2 cone, i.e. $(\omega,\Omega)_{t\in\mathbb{R}_{>0}}$ is constructed from a fixed nearly-Kähler half-flat structure $\left(\omega^{nK},\Omega^{nK}\right)$ on the link via $\omega=t^2\omega^{nK}$, $\Omega=t^3\Omega^{nK}$ as in (1.34), t-invariant solutions to (4.3) are pulled back from solutions to the Hermitian Yang-Mills equations on the link:

$$F_A\wedge\mathrm{Re}\Omega^{nK}=0 \qquad\qquad F_A\wedge\left(\omega^{nK}\right)^2=0 \tag{4.4}$$

Solutions of (4.4) are referred to as *nearly-Kähler instantons*, and appear naturally as asymptotic limits of G_2-instantons on asymptotically conical G_2-manifolds, cf. [CH16] .

4.2 Invariant Instanton ODEs

We recall [LO18] for (4.3) in the invariant setting: we consider the $SU(2)^3$-invariant co-homogeneity one G_2-metrics on the spinor bundle $\mathbf{S}(S^3)$ from Example 1.3.3 in §1.3, and assume the bundle and connection form are also invariant under some lift of the $SU(2)^3$-action to the total space of the bundle.

$SU(2)^3$-homogeneous bundles over the principal orbit $SU(2)^3/\triangle SU(2)$ of $\mathbf{S}(S^3)$ with gauge group $SU(2)$ are classified by homomorphisms $\triangle SU(2)\to SU(2)$. This gives exactly two non-equivariantly equivalent bundles over the principal orbit, defined by the trivial homomorphism $\triangle SU(2)\to SU(2)$, and the identity homomorphism $\triangle SU(2)\to SU(2)$ respectively.

By [Wan58, Thm. A], the space of invariant connections on these homogeneous bundles is an affine space of intertwiners of the $\triangle SU(2)$-action on left-invariant one-forms on $SU(2)^3/\triangle SU(2)$ and on the Lie-algebra of the gauge group. Recall also from §1.3 that the action of $\triangle SU(2)$ acts on the tangent space of $SU(2)^3/\triangle SU(2) = SU(2)^2$ as two copies of the adjoint representation $\mathfrak{su}_+(2)\oplus\mathfrak{su}_-(2)$.

Since $\triangle SU(2)$ acts trivially on the gauge group for the trivial homogeneous bundle, the only $SU(2)^3$-invariant connection on this bundle is the trivial flat connection. Meanwhile, for the non-trivial homogeneous bundle, $\triangle SU(2)$ acts on $\mathfrak{su}(2)$ via the adjoint representation, thus the space of invariant connections is two-dimensional.

Using this description of invariant connections, and the description of $SU(2)^3$-invariant torsion-free G_2-structures (1.39) in §1.3, [LO18, Prop. 5] write the G_2-instanton equations (4.3) as follows:

Proposition 4.2.1 ([LO18])**.** *On $\mathbb{R}_{>0} \times SU(2)^2$ with a torsion-free G_2-structure given by (1.39), $SU(2)^3$-invariant instantons can be written, up to gauge, as:*

$$A = x_+ \left(E_1 \otimes u^+ + E_2 \otimes v^+ + E_3 \otimes w^+\right) + x_- \left(E_1 \otimes u^- + E_2 \otimes v^- + E_3 \otimes w^-\right) \quad (4.5)$$

with (x_+, x_-) real-valued functions satisfying the following ODE system:

$$\dot{x}_+ = \frac{x_+}{\alpha}\left(1 - \frac{\alpha^2}{\beta^2} - x_+\right) + x_-^2 \frac{\alpha}{\beta^2} \qquad \dot{x}_- = \frac{2x_-}{\alpha}(x_+ - 1) \quad (4.6)$$

Here, we can identify both $SU(2)^3$-homogeneous bundles with the trivial $SU(2)^2$-homogeneous bundle over $SU(2)^2$, up to $SU(2)^2$-equivariant isomorphism. We can retrieve the case of the flat connection on the trivial $SU(2)^3$-homogeneous bundle by taking $x_+ = x_- = 0$.

We note here an additional discrete symmetry of (4.6), arising from the non-equivariant isometry exchanging the factors of $SU(2)^2$ on the principal orbits:

Lemma 4.2.2. *$(x_+, x_-) \to (x_+, -x_-)$ is a symmetry of (4.6).*

Moreover, to find solutions of (4.6) for the one-parameter family of torsion-free G_2-structures (α, β) extending over the singular orbit S^3, in what follows, it will suffice to consider the fixed member (1.42). Namely, if we recall Lemma 1.3.1: this one-parameter family of G_2-structures are related by $(\alpha, \beta) \mapsto (\alpha_\delta, \beta_\delta)$, where $\alpha_\delta(t) := \frac{\alpha(\delta t)}{\delta}$, $\beta_\delta(t) := \frac{\beta(\delta t)}{\delta}$ for some $\delta > 0$. So, we get a solution of (4.3) by pulling back the connection via the rescaling $t \mapsto \delta t$, i.e. if $(x_+(t), x_-(t))$ is a solution to (4.6), $(x_+(\delta t), x_-(\delta t))$ is a solution to (4.6) with $(\alpha, \beta) \mapsto (\alpha_\delta, \beta_\delta)$.

Now, up to $SU(2)^2$-equivariant isomorphism, we can extend the trivial bundle over the principal orbit $SU(2)^2$ to the singular orbit $S^3 = SU(2)^2/\triangle SU(2)$ in one of two ways: using either the identity homomorphism $\triangle SU(2) \to SU(2)$ or the trivial homomorphism. As is shown in [LO18], each extension gives a one-parameter family of solutions to (4.6) near $t = 0$:

Proposition 4.2.3 ([LO18])**.** *In a neighbourhood of the singular orbit at $t = 0$, solutions to (4.6) are in two one-parameter families T_γ, $T'_{\gamma'}$ for parameters $\gamma, \gamma' \in \mathbb{R}$:*

1. *The family T_γ extends over the trivial $SU(2)^2$-homogeneous bundle over the singular orbit, and these solutions satisfy near $t = 0$:*

$$x_+ = \gamma t^2 + O(t^4) \qquad x_- = 0 \quad (4.7)$$

2. *The family $T'_{\gamma'}$ extends over the non-trivial $SU(2)^2$-homogeneous bundle over the singular orbit, and these solutions satisfy near $t = 0$:*

$$x_+ = 1 + O(t^2) \qquad\qquad x_- = \gamma' + O(t^2) \tag{4.8}$$

Remark 4.2.4. *For later use, we compute some additional terms in the Taylor series of $T'_{\gamma'}$ near $t = 0$:*

$$x_+ = 1 + \frac{3}{8}\left(\gamma'^2 - 1\right)t^2 + O(t^4) \qquad\qquad x_- = \gamma' + \frac{3}{4}\left(\gamma'^2 - 1\right)\gamma' t^2 + O(t^4) \tag{4.9}$$

Far away from $t = 0$, the G_2-structure on $\mathbf{S}(S^3)$ is asymptotic to the conical $SU(2)^3$-invariant G_2-structure over $S^3 \times S^3$: more precisely $\alpha = \frac{t}{3} + O(t^{-2})$, $\beta = \frac{t}{\sqrt{3}} + O(t^{-2})$ for t sufficiently large. So if (x_+, x_-) are bounded a-priori, the system (4.6) differs from the corresponding instanton equations on the cone

$$\dot{x}_+ = \frac{1}{t}\left(2x_+ - 3x_+^2 + x_-^2\right) \qquad\qquad \dot{x}_- = \frac{6}{t}x_-\left(x_+ - 1\right) \tag{4.10}$$

by $O(t^{-4})$ terms.

We will see later that bounded solutions of (4.6) will converge to critical points of (4.10): $(1, 1)$, $(0, 0)$, $(1, -1)$, $(\frac{2}{3}, 0)$. These critical points correspond to $SU(2)^3$-invariant nearly-Kähler instantons: the critical points $(1, 1)$, $(0, 0)$, $(1, -1)$ are all the flat connection in different non-equivariant gauges, while the only non-trivial instanton is $A^{nK} := (\frac{2}{3}, 0)$. This is identified with the canonical connection on $SU(2)^3 \to SU(2)^3/\triangle SU(2)$, and has been studied previously as a nearly-Kähler instanton in [CH16].

With the two regimes $t \to 0$, $t \to \infty$ understood, we will now discuss complete solutions to (4.6). The family of local solutions T_γ in Proposition 4.2.3 can be obtained explicitly by solving (4.6) with $x_- = 0$, and was found previously in [Cla14]. In terms of variable $r(t) = \sqrt{3}\beta(t)$, these solutions are given by:

$$x_+ = \frac{2}{3}\left(1 + \frac{2\gamma(r-1) - 3r}{2\gamma r(r^2 - 1) + 3r}\right) \qquad\qquad x_- = 0 \tag{4.11}$$

Clearly, these solutions exist for all time if and only if $\gamma \geq 0$, and $\gamma = 0$ is just the flat connection $(0, 0)$. Furthermore, in the limit $\gamma \to \infty$, the solution (4.11) converges outside the singular orbit at $r = 1$ to another explicit solution of (4.6):

$$x_+ = \tfrac{2}{3}\left(1 + \tfrac{1}{r(r+1)}\right) \qquad\qquad x_- = 0 \tag{4.12}$$

The limiting solution (4.12) still extends over the singular orbit, but on a different invariant bundle: it extends to $r = 1$ as the member of the family $T'_{\gamma'}$ with $\gamma' = 0$. This solution was found previously in [LO18], but we now show it lies in a one-parameter family of solutions with γ' non-zero:

Theorem 4.2.5. *$SU(2)^3$-invariant instantons with quadratic curvature decay on $\mathbf{S}(S^3)$ are in two one-parameter families T_γ, $\gamma \geq 0$, and $T'_{\gamma'}$, $-1 \leq \gamma' \leq 1$. Moreover:*

1. *The isometry exchanging the factors of $SU(2)^2$ on the principal orbits sends $T'_{\gamma'} \mapsto T'_{-\gamma'}$.*

2. T_0, T_1', T_{-1}' *are flat, otherwise* T_γ, $T_{\gamma'}'$ *are irreducible and asymptotic to* A^{nK}.

Proof. The analysis for the family T_γ follows from its explicit form (4.11), and the transformation $T_{\gamma'}' \mapsto T_{-\gamma'}'$ is not hard to see from applying Lemma 1.3.1 to the local expression (4.8) for $T_{\gamma'}'$. For the rest of this section, we will prove Theorem 4.2.5 by showing that the local solutions $T_{\gamma'}'$ exist for all time if $-1 \leq \gamma' \leq 1$, and otherwise cannot be bounded.

Lemma 4.2.6. *The following sets are forward-invariant for* (4.6)*:*

(i) $H_\pm := \{(x_+, x_-) \in \mathbb{R}^2 \mid \pm x_- > 0\}$

(ii) $\mathcal{T}_\infty := \{(x_+, x_-) \in \mathbb{R}^2 \mid x_+ > 1, x_- > 1\}$

(iii) $\mathcal{T}_0 := \{(x_+, x_-) \in \mathbb{R}^2 \mid \frac{2}{3} < x_+ < 1, 0 < x_- < 1\}$.

Proof. (i) As previously mentioned, setting $x_- = 0$ gives a family of solutions to (4.6). Hence, by symmetry of Lemma 4.2.2, we will reduce to the case $x_- > 0$ in what follows.

(ii) For $x_- > 0$, the sign of $\dot{x}_-$ is given by the sign of $x_+ - 1$, hence a solution cannot leave $\mathcal{T}_\infty$ via the line $x_+ > 1$, $x_- = 1$. Secondly, $\dot{x}_+|_{x_+=1} = \frac{\alpha}{\beta^2}\left(x_-^2 - 1\right)$, hence a solution cannot leave via the line $x_+ = 1$, $x_- > 1$ either. Finally, the intersection $x_+ = x_- = 1$ is a critical point of (4.6), corresponding to the flat connection.

(iii) By part (i), we can always assume $x_- > 0$. Using the same argument as part (ii), we see that $\dot{x}_- < 0$ when $1 > x_+ > 0$, $\dot{x}_+|_{x_+=1} < 0$ when $1 > x_- > 0$, and $x_+ = x_- = 1$ is a critical point. Thus, it only remains to show a solution cannot leave $\mathcal{T}_0$ via the line segment $x_+ = \frac{2}{3}$, $x_- > 0$. This follows from the inequality, $3\alpha^2 < \beta^2$ on $t > 0$, which can easily be seen from (1.42). With this inequality, it is clear that:

$$\dot{x}_+\Big|_{x_+=\frac{2}{3}} = \frac{2}{3\alpha}\left(\frac{1}{3} - \frac{\alpha^2}{\beta^2}\right) + x_-^2\frac{\alpha}{\beta^2} > 0$$

□

Lemma 4.2.7. *A solution* (x_+, x_-) *to* (4.6) *lying in* $\mathcal{T}_\infty$ *at some initial time* $t_0 > 0$, *cannot be uniformly bounded for all* $t \geq t_0$.

Proof. Since x_- is strictly increasing in $\mathcal{T}_\infty$, if the solution (x_+, x_-) blows up at finite time T, then necessarily the solution cannot be bounded for all $t < T$. On the other hand, recalling the asymptotic behaviour (4.10) of the system, if we re-parametrise (4.6) by $t \mapsto e^t$, then for t sufficiently large and (x_+, x_-) lying in a compact subset of $\mathcal{T}_\infty$, this re-parametrised system is asymptotic to the autonomous system:

$$\dot{x}_+ = 2x_+ - 3x_+^2 + x_-^2 \qquad \dot{x}_- = 6x_-(x_+ - 1) \tag{4.13}$$

up to terms decaying exponentially in t. The theory of non-autonomous systems asymptotic to autonomous systems can be found in [Mar56]: here, we apply [Mar56, Thm.3], which says that if solutions to (4.13) in $\mathcal{T}_\infty$ cannot be uniformly bounded for sufficiently large times, then neither can solutions to (4.6).

So let us assume for contradiction a solution to (4.13) exists for all time in $\mathcal{T}_\infty$, and is uniformly bounded. Since x_- is monotonically increasing in $\mathcal{T}_\infty$, there exists an $\epsilon > 0$ such that $x_- > 1+\epsilon$ for $t > t_0$. If we let $x_+(\epsilon) > 1$ be the unique solution to $2x_+ - 3x_+^2 + (1+\epsilon)^2 = 0$ in $\mathcal{T}_\infty$, then x_+ is strictly increasing in $1 < x_+ < x_+(\epsilon)$ for time $t > t_0$, and hence x_+ is uniformly bounded below away from 1. But this is a contradiction, since it implies $\dot{x}_-$ is bounded below away from zero, and hence x_- cannot be bounded. □

Lemma 4.2.8. *A solution (x_+, x_-) to (4.6) lying in $\mathcal{T}_0$ at some initial time $t_0 > 0$ converges to $A^{nK} = \left(\frac{2}{3}, 0\right)$ as $t \to \infty$*

Proof. The key to proving this statement will be to show that a solution in $\mathcal{T}_0$ must get arbitrarily close to the critical point $\left(\frac{2}{3}, 0\right)$ of (4.13) at *some* forward time. Once we have proved this, we can apply [Mar56, Thm.2]: since the linearisation of (4.13) near $\left(\frac{2}{3}, 0\right)$ has only (real) negative eigenvalues, it is asymptotically stable for (4.6).

So, let (x_+, x_-) be a solution to the re-parametrisation $t \mapsto e^t$ of (4.6) which lies in $\mathcal{T}_0$. Since x_- is strictly decreasing, there must be an $\epsilon \in (0,1)$ such that $x_- < 1 - \epsilon$ for all forward time. Then we can take $T(\epsilon) > t_0$ sufficiently large such that $\dot{x}_+ < 2x_+ - 3x_+^2 + x_-^2 + \epsilon$ for all $t > T$, and an $x_+(\epsilon)$ sufficiently close to 1 such that $\dot{x}_+ < 2x_+ - 3x_+^2 + (1-\epsilon)^2 + \epsilon < 0$ on $x_+(\epsilon) < x_+ < 1$, $t > T$. Hence, we can bound x_+ away from 1 for $t > T$.

On the other hand $x_-(x_+ - 1)$ cannot be bounded above away from zero, since this would imply that $\dot{x}_-$ would be bounded above away from zero after some sufficiently large time, and hence x_- would be unbounded. Combined with the previous observation, this implies x_- cannot be bounded away from 0, and hence $x_- \to 0$ as $t \to \infty$ since x_- is decreasing. Similarly, $|2x_+ - 3x_+^2 + x_-^2|$ cannot be bounded below away from 0, and hence x_+ cannot be bounded away from $\frac{2}{3}$, and we are done. □

The proof of the main theorem now follows from the local power-series solutions $T'_{\gamma'} = (x_+, x_-)_{\gamma'}$ of (4.8): The solution with $\gamma' = 0$ is the explicit solution (4.12), and one can take $\gamma' > 0$ otherwise, up to the symmetry of Lemma 4.2.2. Then $(x_+, x_-)_{\gamma'} \in \mathcal{T}_0$ when $0 < \gamma' < 1$, $(x_+, x_-)_{\gamma'} \in \mathcal{T}_\infty$ when $1 < \gamma'$, and $T'_{\gamma'}$ with $\gamma' = 1$ is the critical point $(1,1)$ of (4.6) corresponding to the flat connection. □

Remark 4.2.9. *There is a solution to the conical equations (4.10) on $t \in (0, \infty)$:*

$$x_+ = \frac{\sqrt{1+2t^2} - 1 + 2t^2}{3t^2} \qquad x_- = \frac{\sqrt{1+2t^2} - 1}{t^2} \tag{4.14}$$

interpolating between the flat connection $A^\flat = (1,1)$ as $t \to 0$, and the nearly-Kähler instanton $A^{nK} = (\frac{2}{3}, 0)$ as $t \to \infty$. One may also be able to apply a construction similar to [MNT22] to recover the long-time existence of the family of instantons $T'_{\gamma'}$ near the limit $\gamma' \to 1$ using the flat connection near $t = 0$ and (4.14) as an asymptotic model.

4.3 Deformation Theory

In the previous section, we classified $SU(2)^3$-invariant solutions to the G_2-instanton equations, giving two families asymptotic to the non-trivial invariant nearly-Kähler instanton

A^{nK} on $S^3 \times S^3$. One might then hope to produce more examples of G_2-instantons on the Bryant-Salamon metric by considering deformations of these symmetric solutions away from the symmetric regime.

However, using the deformation theory of G_2-instantons on asymptotically conical G_2-manifolds worked out in [Dri20], we will find that these invariant families actually classify all G_2-instantons on $\mathbf{S}(S^3)$ asymptotic to A^{nK}, at least if their deformations are unobstructed. For completeness, we will first briefly recount this deformation theory, following [Dri20], also [Nak90].

Let (M^7, φ) be an AC G_2-manifold, with asymptotic cone $C(\Sigma)$, and $P \to M$ be a principal G-bundle with G compact, semi-simple. Extending the radial parameter on $C(\Sigma) \cong \mathbb{R}_{>0} \times \Sigma$ to a smooth positive function t on M, we define the weighted norms for smooth compactly-supported adjoint-valued p-forms $\Phi \in \Omega^p_c(\mathrm{ad}P)$:

$$||\Phi||_{W^{k,2}_\mu} := \left(\sum_{j=0}^{k} \int_M \left| t^{j-\mu} \nabla^j_A \Phi \right|^2 t^{-7} \right)^{\frac{1}{2}} \qquad ||\Phi||_{C^k_\mu} := \sum_{j=0}^{k} \sup_M \left| t^{j-\mu} \nabla^j_A \Phi \right|$$

for some $\mu < 0$, and a fixed connection A on P.

We will use $\Omega^p_{k,\mu}(\mathrm{ad}P)$ to denote the completion of $\Omega^p_c(\mathrm{ad}P)$ with respect to the weighted Sobolev norm $W^{k,2}_\mu$, and define $\Omega^p_\mu(\mathrm{ad}P) := \cap_{k \geq 0} \Omega^p_{k,\mu}(\mathrm{ad}P)$. A weighted version of the standard Sobolev embedding in dimension seven [Dri20, Thm.2.5.5] can be used to show that $\Phi \in \Omega^p_\mu(\mathrm{ad}P)$ implies that $||\Phi||_{C^k_\mu} < \infty$ for all $k \geq 0$, i.e. $|\nabla^j_A \Phi| = O\left(t^{\mu - j}\right)$.

To consider the space of connections on P with fixed asymptotic behaviour, we fix a *framing at infinity*: a pair (P_∞, A_∞) consisting of a bundle $P_\infty \to \Sigma$ equipped with a connection A_∞, such that $P \to M$ is identified with P_∞ pulled back over the conical end of M. We will define a connection A on P as *asymptotic to A_∞ at polynomial rate $\mu < 0$* if $||A - A_\infty||_{W^{k,2}_\mu} < \infty$ for all $k \geq 0$, where we pull back A_∞ to the end of M and use the $W^{k,2}_\mu$-norm defined using the covariant derivative associated to A_∞.

The relevant space of connections we will consider is the set $\mathcal{A}_{\mu-1}$, $\mu < 0$ of all connections asymptotic to A_∞ with polynomial rate strictly less than -1. While $\mathcal{A}_{\mu-1}$ is not preserved under arbitrary gauge transformations of P, we consider the subgroup $\mathcal{G}_\mu$ of *framed gauge transformations with weight* μ: gauge transformations of P which are asymptotic to the identity on P_∞ at rate μ, see [Nak90], [Dri20] for precise details of how to set-up these weights. The gauge group $\mathcal{G}_\mu$ has the property that the point-wise exponential map $\exp : \Omega^0_\mu(\mathrm{ad}P) \to \mathcal{G}_\mu$ is surjective on a open neighbourhood of the identity in $\mathcal{G}_\mu$, and the tangent space to the $\mathcal{G}_\mu$-orbit through some $A \in \mathcal{A}_{\mu-1}$ is spanned by elements of the form $d_A\Phi$ for some $\Phi \in \Omega^0_\mu$.

Using this framework, we can now begin describing solutions to the G_2-instantons equations with fixed asymptotics. However, instead of defining their moduli-space directly, it turns out to be more convenient to describe G_2-instantons in $\mathcal{A}_{\mu-1}$ as solutions to the G_2-monopole equations:

$$F_A \wedge *\varphi = *d_A\Phi \tag{4.15}$$

with $(\Phi, A) \in \Omega^0_{\mu-1}(\mathrm{ad}P) \oplus \mathcal{A}_{\mu-1}$. Since any solution (Φ, A) to (4.15) has $|\Phi|^2$ sub-harmonic, if $|\Phi|^2$ decays to 0 as $t \to \infty$, we must have $\Phi = 0$ by the maximum principle: thus we do not actually enlarge the solution space by switching to this set-up.

We are now in a position to define the *framed moduli-space* of G_2-instantons:

$$\mathcal{M}\left(A_\infty, \mu\right) := \{(\Phi, A) \in \Omega^0_{\mu-1}\left(\mathrm{ad}P\right) \oplus \mathcal{A}_{\mu-1} \mid F_A \wedge *\varphi = *d_A\Phi\}/\mathcal{G}_\mu$$

for some given weight $\mu < 0$. In a neighbourhood of a solution $(0, A)$ to (4.15), we can fix a gauge such that a neighbourhood of $[(0, A)] \in \mathcal{M}\left(A_\infty, \mu\right)$ is described by the zeroes of the non-linear operator

$$d_A\phi + *\left(*\varphi \wedge d_A a + a \wedge a\right) = 0 \qquad\qquad d_A^* a = 0 \tag{4.16}$$

on $(\phi, a) \in \Omega^0_{\mu-1}\left(\mathrm{ad}P\right) \oplus \Omega^1_{\mu-1}\left(\mathrm{ad}P\right)$, where $d_A^* a = 0$ is the standard gauge-fixing condition.

The advantage of passing to the monopole equations (4.15) is that linearising the gauge-fixed equations (4.16) at some fixed solution $(0, A)$ yields the elliptic PDE $D_A\left(\phi, a\right) = 0$, where D_A is the Dirac operator:

$$D_A := \begin{pmatrix} 0 & d_A^* \\ d_A & *\left(*\varphi \wedge d_A \cdot\right) \end{pmatrix} : \Omega^0_{\mu-1}\left(\mathrm{ad}P\right) \oplus \Omega^1_{\mu-1}\left(\mathrm{ad}P\right) \to \Omega^0_{\mu-2}\left(\mathrm{ad}P\right) \oplus \Omega^1_{\mu-2}\left(\mathrm{ad}P\right)$$

We also note by [Dri20, Theorem 4.2.12], for weights $-5 < \mu < 0$, the kernel of D_A can be identified with the *deformation space* $H^1_{\mu-1}(A)$ of the G_2-instanton equations at A:

$$H^1_{\mu-1}\left(A\right) := \frac{\ker\left(*\left(*\varphi \wedge d_A \cdot\right) : \Omega^1_{\mu-1}\left(\mathrm{ad}P\right) \to \Omega^1_{\mu-2}\left(\mathrm{ad}P\right)\right)}{\mathrm{im}\left(d_A : \Omega^0_{\mu}\left(\mathrm{ad}P\right) \to \Omega^1_{\mu-1}\left(\mathrm{ad}P\right)\right)}$$

defined as the space of solutions to the linearised instanton equations $*\varphi \wedge d_A a = 0$, modulo linearised gauge transformations $d_A\Phi$ for some $\Phi \in \Omega^0_\mu$.

By [Dri20, Prop.4.4.2], outside of some discrete set of critical weights depending only on (P_∞, A_∞) and the geometry of asymptotic cone, the elliptic operator $D_A : \Omega^0_{k,\mu-1} \oplus \Omega^1_{k,\mu-1} \to \Omega^0_{k-1,\mu-2} \oplus \Omega^1_{k-1,\mu-2}$ is Fredholm, so the standard results apply see e.g. [DK90, Prop.4.2.19], [Dri20, Theorem 4.5.3]:

Proposition 4.3.1 ([Dri20]). *Let $-5 < \mu < 0$ be a non-critical weight. Then $D_A : \Omega^0_{\mu-1}\left(\mathrm{ad}P\right) \oplus \Omega^1_{\mu-1}\left(\mathrm{ad}P\right) \to \Omega^0_{\mu-2}\left(\mathrm{ad}P\right) \oplus \Omega^1_{\mu-2}\left(\mathrm{ad}P\right)$ has finite-dimensional kernel and co-kernel. Moreover, there exists an open neighbourhood $U \subset \ker D_A$ of 0, and smooth map $\pi : U \to \mathrm{coker} D_A$, $\pi(0) = 0$, $d\pi(0) = 0$, such that a neighbourhood of $[A] \in \mathcal{M}\left(A_\infty, \mu\right)$ is homeomorphic to a neighbourhood of 0 in $\pi^{-1}(0)$.*

The point is that if D_A is surjective, then we can apply the implicit function theorem to obtain smooth solutions of the non-linear equation (4.16) in a sufficiently small neighbourhood of A, and a neighbourhood of $[A] \in \mathcal{M}(A_\infty, \mu)$ has the structure of a smooth manifold of dimension $\mathrm{ind} D_A = \dim\ker D_A$. We define A to be *obstructed* if the co-kernel is non-empty: in general, this neighbourhood is only smooth of dimension $\mathrm{ind} D_A$ if $\pi^{-1}(0)$ contains only regular values.

In the case of interest, on the spinor-bundle $\mathbf{S}(S^3)$, we wish to consider the moduli-space $\mathcal{M}\left(A^{nK}, \mu\right)$ of G_2-instantons with gauge group $SU(2)$, asymptotic to the unique non-trivial $SU(2)^3$-invariant nearly-Kähler instanton A^{nK} on $S^3 \times S^3$ with rate $\mu - 1$. To understand this moduli-space in more detail, we now consider the role of symmetries in the weighted set-up of [Dri20], adapting arguments of [Bra89, Thm. 1.3].

As before, let (M, φ) be an AC G_2-manifold with asymptotic cone $C(\Sigma)$, and let $P \to M$ be a principal G-bundle with compact Lie group G. Denote by $\mathrm{Aut}(M, \varphi)$ the subgroup of diffeomorphisms of M fixing the G_2-structure φ, and $\mathrm{Aut}(P)$ the subgroup of diffeomorphisms of P commuting with the G-action. Finally, fix an asymptotic framing (P_∞, A_∞) over Σ for P, and denote by $\mathrm{Aut}(P_\infty, A_\infty)$ the subgroup of diffeomorphisms of P_∞ commuting with the G-action and fixing A_∞.

Suppose that (M, φ) has a connected Lie sub-group of automorphisms $K \subset \mathrm{Aut}(M, \varphi)$ which restrict to automorphisms of $(\Sigma, \varphi|_\Sigma)$ along the conical end. Moreover, assume we are given a lift of the K-action on Σ to P_∞ for which A_∞ is invariant, i.e. a Lie group homomorphism $K \to \mathrm{Aut}(P_\infty, A_\infty)$, $k \mapsto k_\infty$, and for simplicity we will also assume this lift has an extension $K \to \mathrm{Aut}(P)$ to the interior.

Then there is a short exact sequence:

$$1 \longrightarrow \mathcal{G}_\mu \hookrightarrow \mathcal{H}_\mu \longrightarrow K \longrightarrow 1$$

where $\mathcal{H}_\mu \subset \mathrm{Aut}(P)$ denotes the subgroup of automorphisms covering some element of K, such that $\hat{k}.k_\infty^{-1} \in \mathcal{G}_\mu$ for all $\hat{k} \in \mathcal{H}_\mu$ covering $k \in K$, i.e. the automorphism $\hat{k}$ is asymptotic to k_∞ with rate μ. The map $\mathcal{G}_\mu \to \mathcal{H}_\mu$ is just the inclusion as automorphisms of P covering the identity map on M, and the map $\mathcal{H}_\mu \to K$ is induced by the projection $P \to M$.

Given some equivalence class $[A] \in \mathcal{A}_{\mu-1}/\mathcal{G}_\mu$, denote $K^{[A]} \subset K$ the subgroup such that $k \in K^{[A]}$ if $[k_\infty^* A] = [A]$. We claim that, given $A \in [A]$, we can uniquely lift $K^{[A]}$ to a group of automorphisms of P fixing A, asymptotic to the lift $k \to k_\infty$ with rate μ:

Lemma 4.3.2. *For all $A \in [A]$ there exists a unique homomorphism $k \mapsto k_A$ lifting $K^{[A]} \to \mathcal{H}_\mu$, such that $k_A^* A = A$.*

Proof. To see this, let $\mathcal{G}_\mu^A \subset \mathcal{G}_\mu$, $\mathcal{H}_\mu^A \subset \mathcal{H}_\mu$ denote the subgroups fixing $A \in [A]$. Then $K^{[A]} \subset K$ fits into the exact sequence:

$$1 \longrightarrow \mathcal{G}_\mu^A \hookrightarrow \mathcal{H}_\mu^A \longrightarrow K^{[A]} \longrightarrow 1$$

In other words, $k_\infty^*.A = g^* A$ for some $g \in \mathcal{G}_\mu$ if and only if there exists $k_A \in \mathcal{H}_\mu$ covering k such that $k_A^* A = A$, with k_A unique up to an element in $\mathcal{G}_\mu^A$. However, in contrast to the un-weighted case, $\mathcal{G}_\mu$ acts freely on $\mathcal{A}_{\mu-1}$ [Dri20, p.47], and thus k_A is unique. □

We can repeat this discussion at the infinitesimal level: denote by $\mathfrak{aut}(M, \varphi)$ the Lie-algebra of vector-fields on M fixing the G_2-structure, and $\mathfrak{aut}(P_\infty, A_\infty)$ the Lie-algebra of vector fields on P_∞ fixing the connection A_∞.

Suppose we have a Lie sub-algebra $\mathfrak{k} \subset \mathfrak{aut}(M, \varphi)$ of vector-fields which restrict to vector-fields pulled back from Σ along the end, and we are given a lift $X \mapsto X_\infty$ of $\mathfrak{k}$ to P_∞ for which A_∞ is invariant, i.e. a Lie-algebra homomorphism $\mathfrak{k} \to \mathfrak{aut}(P_\infty, A_\infty)$.

Moreover, assume there exists an extension of this lift to the interior i.e. a lift $X \mapsto \tilde{X}_\infty$ to a vector-field on the total space of P, such that the vertical vector-field $\tilde{X}_\infty - X_\infty$, viewed here as a section of the adjoint bundle, lies in $\Omega^0_\mu(\mathrm{ad}P)$[1].

In this set-up, we prove the following lemma:

Lemma 4.3.3. *If $A \in \mathcal{A}_{\mu-1}$ is a G_2-instanton on P asymptotic to A_∞, then there is well-defined linear map:*

$$L : \mathfrak{k} \to H^1_{\mu-1}(A) \qquad\qquad L : X \mapsto \left[\mathcal{L}_{\tilde{X}_\infty} A\right] \tag{4.17}$$

Moreover, $\ker L \subset \mathfrak{k}$ *is a Lie-sub-algebra.*

Proof. To verify that (4.17) is well-defined, we will use the identity

$$\mathcal{L}_{\tilde{X}} A := d(\tilde{X} \lrcorner A) + \tilde{X} \lrcorner dA = X \lrcorner F_A + d_A\left(\tilde{X} \lrcorner A\right)$$

for any lift $\tilde{X}$ to P of a vector field X on M, where we view the G-equivariant map $\tilde{X} \lrcorner A$ from P to the Lie algebra of G as a section of the adjoint bundle. We can show $\mathcal{L}_{X_\infty} A \in \Omega^1_{\mu-1}(\mathrm{ad}P)$ by restricting to the end of M and setting $a = A - A_\infty$. Then for any $\Phi \in \Omega^0(\mathrm{ad}P)$:

$$F_{A_\infty} = F_A - d_A a - [a \wedge a] \qquad\qquad d_{A_\infty}\Phi = d_A \Phi - [a, \Phi]$$

Since by assumption, $\mathcal{L}_{X_\infty} A_\infty = X \lrcorner F_{A_\infty} + d_{A_\infty}(X_\infty \lrcorner A_\infty) = 0$, we have:

$$\begin{aligned} \mathcal{L}_{\tilde{X}_\infty} A &= X \lrcorner F_{A_\infty} + X \lrcorner (d_A a + [a \wedge a]) + d_A\left(\tilde{X}_\infty \lrcorner A\right) \\ &= d_A(X \lrcorner a) + [a, (X_\infty \lrcorner A_\infty)] + X \lrcorner (d_A a + [a \wedge a]) + d_A\left(\tilde{X}_\infty - X_\infty\right) \end{aligned}$$

To show that this lies in $\Omega^1_{\mu-1}(\mathrm{ad}P)$, we note that $X_\infty \lrcorner A_\infty \in \Omega^0(\mathrm{ad}P_\infty)$ has constant norm along the end, and X restricts to a vector-field pulled back from Σ, so $|X|$ grows linearly. Moreover, $a \in \Omega^1_{\mu-1}(\mathrm{ad}P)$, $\tilde{X}_\infty - X_\infty \in \Omega^0_\mu(\mathrm{ad}P)$ by assumption, thus $\mathcal{L}_{\tilde{X}_\infty} A \in \Omega^1_{\mu-1}(\mathrm{ad}P)$.

As in the previous discussion for $K^{[A]}$, $L(X) = 0$ if and only if there is a unique lift $X \mapsto X_A$ to a vector-field on P such that $\mathcal{L}_{X_A} A = 0$, and the vertical vector-field $X_A - \tilde{X}_\infty$ on P lies in $\Omega^0_\mu(\mathrm{ad}P)$, viewed here as a section of the adjoint bundle. This section $X_A - \tilde{X}_\infty$ is precisely the one for which $\mathcal{L}_{\tilde{X}_\infty} A = d_A\left(X_A - \tilde{X}_\infty\right)$, so uniqueness follows from the injectivity of $d_A : \Omega^0_\mu(\mathrm{ad}P) \to \Omega^1_{\mu-1}(\mathrm{ad}P)$ [Dri20, Cor.4.2.6].

Now, since the lift of $[X, Y]_\infty$ can be identified with the commutator $[X_\infty, Y_\infty]$ on P_∞ for all $X, Y \in \mathfrak{k}$, then it is not hard to see that $[X, Y]_A := [X_A, Y_A]$ also satisfies the two conditions for lifting $[X, Y]$ to P if $L(X) = L(Y) = 0$, and so $\ker L \subset \mathfrak{k}$ is a Lie sub-algebra. □

With this general picture understood, let us return to the Bryant-Salamon metric on $\mathbf{S}(S^3)$. Any principal bundle $P \to \mathbf{S}(S^3)$ must be trivial for gauge group $SU(2)$, and we fix an asymptotic framing by the homogeneous bundle $P_\infty = SU(2)^3 \times_{\triangle SU(2)} SU(2) \to$

[1] note that such an extension always exists if P admits a K-invariant connection asymptotic to A_∞ with rate $\mu - 1$.

$SU(2)^3/\triangle SU(2)$, where $\triangle SU(2)$ acts on the gauge group via the identity map. Recall that the $SU(2)^3$-invariant canonical connection associated to this homogeneous bundle is the nearly-Kähler instanton A^{nK} considered in §4.2.

Consider the Dirac operator $D_A : \Omega^0_{\mu-1}(\mathrm{ad}P)\oplus\Omega^1_{\mu-1}(\mathrm{ad}P) \to \Omega^0_{\mu-2}(\mathrm{ad}P)\oplus\Omega^1_{\mu-2}(\mathrm{ad}P)$, associated to a G_2-instanton $A \in \mathcal{A}_{\mu-1}$ on $\mathbf{S}(S^3)$ asymptotic to A^{nK}. By [Dri20, Thm.6.5.5], this operator is Fredholm with index $\mathrm{ind}D_A = 1$ between $-2 < \mu < 0$, and below the critical rate $\mu = -2$ the index jumps to $\mathrm{ind}D_A = -1$.

Remark 4.3.4. *The $SU(2)^3$-invariant family T_γ, $\gamma > 0$ given by (4.11) decays to A^{nK} at the critical rate with $\mu = -2$: i.e. $|T_\gamma - A^{nK}| = O(t^{-3})$ as $t \to \infty$. We note an error in [LO18, Prop. 5], as the invariant instanton T'_0 given by (4.12) shares this asymptotic decay rate. Moreover, we expect this holds for the whole invariant family $T'_{\gamma'}$, $-1 < \gamma' < 1$.*

We will use this computation to prove the following proposition:

Proposition 4.3.5. *Any G_2-instanton on $\mathbf{S}(S^3)$ asymptotic to A^{nK} with rate $-2 < \mu - 1 < 0$ is either obstructed or gauge-equivalent to an instanton in the one of the families T_γ, $T'_{\gamma'}$.*

Proof. We will use the computation of the index in [Dri20] to show that, if an instanton $A \in \mathcal{A}_{\mu-1}$ is not obstructed, then it must be $SU(2)^3$-invariant, for some lift of the action of $SU(2)^3$ to P asymptotic to the action of $SU(2)^3$ on the framing bundle $P_\infty = SU(2)^3 \times_{\triangle SU(2)} SU(2) \to SU(2)^3/\triangle SU(2)$. Once this is proven, the result follows from the existence and uniqueness results of Theorem 4.2.5 in the previous section, since any connection asymptotic to the non-trivial connection A^{nK} on the asymptotic link must have quadratic curvature decay.

So to prove invariance, we note that if A is not obstructed, the deformation space $H^1_{\mu-1}(A)$ is one-dimensional for weights $-2 < \mu < 0$ by [Dri20, Thm.6.5.5]. Since the map $L : \mathfrak{su}(2)^3 \to H^1_{\mu-1}(A)$ defined in Lemma 4.3.3 is linear, then the kernel has co-dimension at most one in $\mathfrak{su}(2)^3$. However, since this kernel is a Lie sub-algebra of $\mathfrak{su}(2)^3$, it cannot have co-dimension one, and so L must vanish on all of $\mathfrak{su}(2)^3$.

As previously discussed, this implies that we can uniquely lift $\mathfrak{su}(2)^3$ to a Lie-algebra of vector-fields on P fixing A, such that these vector-fields are asymptotic to the infinitesimal action of $SU(2)^3$ on the homogeneous bundle $P_\infty = SU(2)^3 \times_{\triangle SU(2)} SU(2) \to SU(2)^3/\triangle SU(2)$. Since these vector-fields are complete, and $SU(2)^3$ is simply-connected, it follows by [Pal57, Ch.3 Thm.7, Ch.4 Thm.3] that these vector-fields integrate to give a unique lift of the $SU(2)^3$-action to P fixing A. □

Appendix A

Extending invariant data to the singular orbit

A.1 Calabi-Yau structures

By considering $SU(2)^2$-invariant $SU(3)$-structures on the space of principal orbits of $\mathcal{O}(-2,-2)$, $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, and T^*S^3 in §1.2, and imposing that these $SU(3)$-structures be closed, we obtained ordinary differential equations (1.21), (1.23) depending on geodesic parameter $t\in\mathbb{R}_{>0}$. In this appendix, we compute the boundary conditions that these $SU(3)$-structures must satisfy in order to extend smoothly to the singular orbits at $t=0$. For $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, this analysis has been carried out in [FH17, Lemma 4.1] and partially carried out for T^*S^3 in [FH17, Lemma 4.2]. However, for completeness, we will include proofs of these results to match the conventions in this thesis.

We will use the method of computing these boundary conditions developed by Eschenburg-Wang in [EW00]. We will briefly recall this technique:

Suppose K is a compact Lie group acting with co-homogeneity one on a smooth manifold M, and for simplicity, we will assume there is exactly one singular K-orbit, with corresponding group diagram of M given by $H\subset H'\subseteq K$. Recall that, in this case, M can be equivariantly identified with $K\times_{H'}V$ for some orthogonal H'-representation V, such that H is identified with the stabiliser subgroup of some non-zero $v_0\in V$.

Let $\mathfrak{p}$ denote the space of left-invariant vector fields on the singular orbit K/H'. H' acts on the tangent space of M as the direct sum $\mathfrak{p}\oplus V$, so that any K-invariant form on M can be viewed as an H'-equivariant map from V into the exterior algebra of forms on $\mathfrak{p}\oplus V$. The evaluation of this map at $v_0\in V$ is an H-invariant form on $\mathfrak{p}\oplus V$, or in other words, a K-invariant form on the principal orbit.

To extend K-invariant k-forms to the singular orbit, one writes down a basis of the finite-dimensional space of H-invariant elements of $\bigwedge^k(\mathfrak{p}^*\oplus V^*)$, obtained by evaluating H'-equivariant maps $V\to\bigwedge^k(\mathfrak{p}^*\oplus V^*)$ that are specified by homogeneous polynomials in the coordinates of $v\in V$.

If $\alpha\in\bigwedge^k(\mathfrak{p}^*\oplus V^*)$ denotes such a basis element, then it has a well-defined degree $d\in\mathbb{Z}_{\geq 0}$, defined as the degree of the equivariant homogeneous polynomial map $V\to\bigwedge^k(\mathfrak{p}^*\oplus V^*)$ evaluating to α at v_0. As is shown in [EW00, Lemma 1.1] any K-invariant form $a(t)\alpha$ on the space of principal orbits, for some smooth function $a(t)$ of the geodesic parameter t, extends smoothly over the singular orbit at $t = 0$ if and only if $a(t)t^{-d}$ is a smooth, even function of t near $t = 0$. We will refer to such a function $a(t)$ as having *degree* d.

We will use this procedure to write down the boundary conditions for $SU(2)^2$-invariant two-forms and three-forms on the co-homogeneity one manifolds $\mathcal{O}(-2,-2)$, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and T^*S^3. We summarise below:

Proposition A.1.1. *Let* $\alpha = \lambda dt \wedge \eta^{se} + u_0\omega_0^{se} + u_1\omega_1^{se} + u_2\omega_2^{se} + u_3\omega_3^{se}$ *be an* $SU(2)^2$*-invariant two-form on* $\mathbb{R}_{>0} \times SU(2)^2/K_{2,-2}$*. Then* α *extends smoothly over the singular orbit* $S^2 \times S^2 = SU(2)^2/U(1)^2$ *at* $t = 0$ *if and only if* u_2, u_3, λ *are odd, and* u_0, u_1 *are even.*

By a result of Hitchin in [Hit00, §2], it is enough to determine ω and $\mathrm{Re}\Omega$ to determine the whole $SU(3)$-structure, so we only need to determine the additional boundary extension conditions for the three-form $\mathrm{Re}\Omega$ to extend (ω, Ω) to the singular orbit:

Proposition A.1.2. *Let* $\mathrm{Re}\Omega = \mu\omega_2^{se} \wedge dt - \lambda\left(v_0\omega_0^{se} + v_3\omega_3^{se}\right) \wedge \eta^{se}$ *be an* $SU(2)^2$*-invariant three-form on* $\mathbb{R}_{>0} \times SU(2)^2/K_{2,-2}$*. Then* $\mathrm{Re}\Omega$ *extends smoothly over the singular orbit* $S^2 \times S^2 = SU(2)^2/U(1)^2$ *at* $t = 0$ *iff* μ*,* λv_0 *are even,* λv_3 *is odd,* $(\lambda v_3)'(0) = 3\mu(0)$ *and* $\lambda v_0(0) = 0$*.*

For $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, it will suffice to check the boundary conditions for 2-forms:

Proposition A.1.3. *(c.f. [FH17, Lemma 4.1]) Let* $\alpha = \lambda dt \wedge \eta^{se} + u_0\omega_0^{se} + u_1\omega_1^{se} + u_2\omega_2^{se} + u_3\omega_3^{se}$ *be an* $SU(2)^2$*-invariant two-form on* $\mathbb{R}_{>0} \times SU(2)^2/\triangle U(1)$*. Then* α *extends smoothly over the singular orbit* $S^2 = SU(2)^2/U(1) \times SU(2)$ *at* $t = 0$ *if and only if* u_0, u_1, u_2, u_3 *are even,* λ *is odd,* $u_0(0) + u_1(0) = 0$*,* $u_2(0) = u_3(0) = 0$*, and* $u_0''(0) + u_1''(0) = 2\lambda'(0)$*.*

Since $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ arises as the small resolution of the conifold, realised as the affine variety $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \mid (z_1, z_2, z_3, z_4) \in \mathbb{C}^4\}$, the complex structure on this singular variety extends automatically to $\mathbb{CP}^1 \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We can use this fact to obtain the following corollary to Proposition A.1.3:

Corollary A.1.4. *Let* (ω, Ω) *be an* $SU(2)^2$*-invariant Calabi-Yau structure on* $\mathbb{R}_{>0} \times SU(2)^2/\triangle U(1)$*. Then* Ω *extends smoothly over the singular orbit* $S^2 = SU(2)^2/U(1) \times SU(2)$ *at* $t = 0$ *if* ω *extends.*

For the smoothing T^*S^3, we compute the following boundary conditions for two- and three-forms:

Proposition A.1.5. *(c.f. [FH17, Lemma 4.2]) Let* $\alpha = \lambda dt \wedge \eta^{se} + u_0\omega_0^{se} + u_1\omega_1^{se} + u_2\omega_2^{se} + u_3\omega_3^{se}$ *be an* $SU(2)^2$*-invariant two-form on* $\mathbb{R}_{>0} \times SU(2)^2/\triangle U(1)$*. Then* α *extends smoothly over the singular orbit* $S^3 = SU(2)^2/\triangle SU(2)$ *at* $t = 0$ *if and only if* u_0, u_1, u_3 *are odd,* u_2, λ *are even,* $u_2(0) = 0$*,* $u_0'(0) + u_3'(0) = 0$*, and* $2\lambda(0) = u_1'(0)$*.*

Proposition A.1.6. *Let* $\beta = \sum_i u_i\omega_i^{se} \wedge dt + \sum_i v_i\omega_i^{se} \wedge \eta^{se}$ *be an* $SU(2)^2$*-invariant three-form on* $\mathbb{R}_{>0} \times SU(2)^2/\triangle U(1)$*. Then* β *extends smoothly over the singular orbit* $S^3 = SU(2)^2/\triangle SU(2)$ *at* $t = 0$ *if and only if* $u_0, v_0, u_1, v_1, u_3, v_3$ *are even,* u_2, v_2 *are odd,* $u_1(0) = v_1(0) = 0$*,* $u_0(0) + u_3(0) = v_0(0) + v_3(0) = 0$*,* $3(u_0(0) - u_3(0)) + 2v_2'(0) = 0$*, and* $v_0''(0) + v_3''(0) + 3u_2'(0) = 0$*.*

The rest of this section will be devoted to the proofs of these claims:

Proof of Proposition A.1.1. First of all, we describe how to rewrite the Cartesian coordinates on the fibre $\mathbb{C}_{2,-2}$ in terms of vector fields on $SU(2)^2/\triangle U(1)\times\mathbb{Z}_2$. Here, let dx^0, dx^1 denote the Cartesian coordinate one-forms on the fibre $\mathbb{C}_{2,-2}$ at the identity coset in $SU(2)^2/U(1)^2$. The flow (e^{is}, e^{-is}, t) of the vector field U^- on $SU(2)^2\times\mathbb{C}$ along the curve $\gamma(t) = (1, 1, t)$ is identified with $(1, 1, e^{4is}t)$ by the $U(1)^2$ action, so along γ, the polar coordinate vector field on $\mathbb{C}_{2,-2}$ is identified with $\frac{1}{4}U^-$ and we may write $dx^0 = 4tu^- = 3t\eta^{se}$ and $dx^1 = dt$.

Here we will denote the tangent space to the fibre $\mathbb{C}_{2,-2}$ as V, and $\mathfrak{p}$ as the span of left-invariant vector fields $\langle V^1, W^1, V^2, W^2\rangle$: clearly, the tangent space at a point of $\mathcal{O}(-2,-2)$ can be identified with $\mathfrak{p}\oplus V$. As $\triangle U(1)\times\mathbb{Z}_2$-representations, V is trivial and $\mathfrak{p}\cong\mathbb{C}_2\oplus\mathbb{C}_2$, so the space of $\triangle U(1)\times\mathbb{Z}_2$ invariant two-forms on $\mathfrak{p}\oplus V$ is (real) 5-dimensional: explicitly, if $p^{1,2} := v^{1,2} + iw^{1,2}$, then a basis for this space can be written as the real and imaginary parts of $dx^0\wedge dx^1$, $p^1\wedge\bar{p}^1$, $p^2\wedge\bar{p}^2$, $p^1\wedge\bar{p}^2$.

As a $U(1)^2$-representation, the space of $\triangle U(1)\times\mathbb{Z}_2$ invariant two-forms decomposes into the following irreducible representations:

$$\textstyle\bigwedge^2(\mathfrak{p}^*\oplus V^*)\supset\langle dx^0\wedge dx^1, p^1\wedge\bar{p}^1, p^2\wedge\bar{p}^2\rangle\oplus\langle p^1\wedge\bar{p}^2\rangle\cong\mathbb{R}^3\oplus\mathbb{C}_{2,-2} \tag{A.1}$$

To apply the general machinery of [EW00, Lemma 1.1], we find $U(1)^2$-equivariant maps $V\to\bigwedge^2(\mathfrak{p}^*\oplus V^*)$ into the space of invariant two-forms defined by a homogeneous polynomial of fixed degree. The two-forms arising from degree zero homogeneous polynomials are the $U(1)^2$-invariant two-forms $dx^0\wedge dx^1$, $p^1\wedge\bar{p}^1$, $p^2\wedge\bar{p}^2$, while the degree one homogeneous polynomials have the single generator $z\mapsto zp^1\wedge\bar{p}^2$, where $z\in\mathbb{C}_{2,-2}$. This evaluates at $z = 1$ to a multiple of the two-forms ω_2^{se} and ω_3^{se} by taking real and imaginary parts.

We now rewrite the generic smooth $SU(2)^2$-invariant two-form α on the space of principal orbits:

$$\begin{aligned}\alpha &= \lambda dt\wedge\eta^{se} + u_0\omega_0^{se} + u_1\omega_1^{se} + u_2\omega_2^{se} + u_3\omega_3^{se}\\ &= \frac{\lambda}{3t}\left(dx^0\wedge dx^1\right) + \frac{2}{3}(u_0-u_1)(v^1\wedge w^1) + \frac{2}{3}(u_0+u_1)(v^2\wedge w^2) + u_2\omega_2^{se} + u_3\omega_3^{se}\end{aligned}$$

thus, by the previous calculation, α extends to the singular orbit at $t = 0$ if and only if u_2, u_3, λ are odd, and u_0, u_1 are even. □

Proof of Proposition A.1.2. We denote V, $\mathfrak{p}$, p^1, p^2, dx^0, dx^1 as in the previous proof. As irreducible $U(1)^2$-representations, the (real) 8-dimensional space of $\triangle U(1)\times\mathbb{Z}_2$-invariant 3-forms on $\mathfrak{p}\oplus V$ can be written $\mathbb{C}_{0,0}\oplus\mathbb{C}_{4,-4}\oplus 2\mathbb{C}_{2,-2}$, corresponding to a splitting:

$$\langle(dx^0-idx^1)\wedge p^1\wedge\bar{p}^2\rangle\oplus\langle(dx^0+idx^1)\wedge p^1\wedge\bar{p}^2\rangle\oplus\langle(dx^0+idx^1)\wedge p^j\wedge\bar{p}^j\rangle_{j=1,2} \tag{A.2}$$

and as before, we must find generators for $U(1)^2$-equivariant polynomial maps from V into the space of invariant 3-forms.

The $U(1)^2$-invariant 3-forms corresponding to a degree zero polynomial are spanned by $(dx^0-idx^1)\wedge p^1\wedge\bar{p}^2$, with real and imaginary parts $dt\wedge\omega_3^{se} + 3t\eta^{se}\wedge\omega_2^{se}$ and $dt\wedge$

$\omega_2^{se} - 3t\eta^{se} \wedge \omega_3^{se}$. Homogeneous degree one polynomials have the two generators $z \mapsto z(dx^0 + idx^1) \wedge p^j \wedge \bar{p}^j$ for $z \in \mathbb{C}_{2,-2}$, which evaluate at $z = 1$ to $dt \wedge v^j \wedge w^j$ and $3t\eta^{se} \wedge v^j \wedge w^j$ by taking real and imaginary parts, and finally homogeneous polynomials of degree two have the single generator $z \mapsto z^2(dx^0 + idx^1) \wedge p^1 \wedge \bar{p}^2$, which evaluates at $z = 1$ to $dt \wedge \omega_3^{se} - 3t\eta^{se} \wedge \omega_2^{se}$ and $dt \wedge \omega_2^{se} + 3t\eta^{se} \wedge \omega_3^{se}$.

This splitting of the space of invariant 3-forms provides the boundary extension conditions for a generic invariant 3-form. Applying this to the specific 3-form:

$$\mathrm{Re}\Omega = \mu dt \wedge \omega_2^{se} - \frac{\lambda}{3t}\left(v_0\omega_0^{se} + v_3\omega_3^{se}\right) \wedge 3t\eta^{se}$$

defined by a generic hypo-structure (1.19), shows that this 3-form extends if and only if $\left(\mu + \frac{\lambda v_3}{3t}\right)$ is even, $\left(\mu - \frac{\lambda v_3}{3t}\right)$ has degree two and $\frac{\lambda v_0}{3t}$ is odd. □

Proof of Proposition A.1.3. We first describe a change of coordinates from left-invariant vector fields on the principal orbits of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ to a coordinate system adapted to the boundary extension problem. We identify $S^2 \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ with $S^2 \subset \mathrm{im}\mathbb{H}$, and the fibre over $i \in S^2$ with $\mathbb{H}$, so that $SU(2)^2$ acts on $S^2 \times \mathbb{H}$ as:

$$(q_1, q_2) \cdot (x, y) \mapsto (q_1 x \bar{q}_1, q_2 y \bar{q}_1)$$

With this description of the $SU(2)^2$-action, we will write vector fields on $SU(2)^2$ as vector fields on $S^2 \times \mathbb{R}^4$:

Here, let (x^0, x^1, x^2, x^3) denote Cartesian coordinates on the fibre. Along the curve $\gamma(t) = (i, t) \in S^2 \times \mathbb{H}$, the vector fields $U^1, U^2, V^1, V^2, W^1, W^2$ are given by:

$$U^1 = \frac{d}{ds}\bigg|_{s=0} \left(i, te^{-is}\right) = (0, -it) \qquad U^2 = \frac{d}{ds}\bigg|_{s=0} \left(1, e^{is}t\right) = (0, it)$$
$$V^1 = \frac{d}{ds}\bigg|_{s=0} \left(e^{js} i e^{-js}, te^{-js}\right) = (-2k, -jt) \qquad V^2 = \frac{d}{ds}\bigg|_{s=0} \left(i, e^{js}t\right) = (0, jt)$$
$$W^1 = \frac{d}{ds}\bigg|_{s=0} \left(e^{ks} i e^{-ks}, te^{-ks}\right) = (2j, -kt) \qquad W^2 = \frac{d}{ds}\bigg|_{s=0} \left(i, e^{ks}t\right) = (0, kt)$$

Clearly $\frac{\partial}{\partial t} = \frac{\partial}{\partial x^0}$, and:

$$V^2 = t\frac{\partial}{\partial x^2} \qquad W^2 = t\frac{\partial}{\partial x^3} \qquad U^- = -2t\frac{\partial}{\partial x^1}$$

so that:

$$dx^0 = dt \qquad dx^1 = -2tu^- \qquad dx^2 = tv^2 \qquad dx^3 = tw^2 \tag{A.3}$$

We will now describe the boundary conditions for extending 2-forms over the singular orbit. Here, V will denote the tangent space to the fibre over $i \in S^2$, and $\mathfrak{p}$ denote the complement of the Lie algebra of $U(1) \times SU(2)$ in $SU(2)^2$, i.e. $\mathfrak{p} = \langle V^1, W^1 \rangle$, and V is the span of the coordinate vector fields $\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$.

Recall that the $\triangle U(1)$-invariant subspace of $\bigwedge^2(\mathfrak{p}^* \oplus V^*)$ is 5-dimensional: it contains the real and imaginary parts of $\left(dx^0 + idx^1\right) \wedge \left(dx^2 + idx^3\right)$ in $\bigwedge^2(V^*)$, $v^1 \wedge w^1$ in $\bigwedge^2(\mathfrak{p}^*)$, and the rest is spanned by the real and imaginary parts of $\left(dx^2 + idx^3\right) \wedge \left(v^1 - iw^1\right)$. We find a basis for these in terms of $U(1) \times SU(2)$-equivariant homogeneous polynomials $V \to \bigwedge^2(\mathfrak{p}^* \oplus V^*)$, evaluated at some $v_0 \in V$.

Firstly, since $\{1\} \times SU(2) \subset U(1) \times SU(2)$ acts trivially on $\bigwedge^2(\mathfrak{p}^*)$ and $\mathfrak{p} \cong \mathbb{C}_2$ as a $U(1)$-representation, it is clear that $v^1 \wedge w^1 = \frac{3}{4}(\omega_0^{se} + \omega_1^{se})$ is $U(1) \times SU(2)$-invariant, thus identified with a degree zero homogeneous polynomial map.

Second, observe that the action of $U(1) \times SU(2)$ on V arises from the restriction of the natural action of $SU(2)^2$ on the left and right of $\mathbb{C}^2$ respectively: viewing the $U(1) \times SU(2)$ action this way, it not hard to see that $\bigwedge^2(V^*) \cong (\mathbb{R} \oplus \mathbb{C}_2) \oplus \mathfrak{su}(2)$ as a $U(1) \times SU(2)$-representation, where $U(1)$ is understood as acting trivially on the $SU(2)$-representation and vice-versa.

The trivial sub-representation in $\bigwedge^2(V^*)$ is spanned by the standard symplectic form $dx^0 \wedge dx^1 + dx^2 \wedge dx^3$ on V, and the $\mathfrak{su}(2)$ is spanned by the standard anti-self-dual two-forms on V, i.e. $dx^0 \wedge dx^1 - dx^2 \wedge dx^3$ under cyclic permutations of (123). Identifying V with $\mathbb{H}$, and $\mathfrak{su}(2)$ with im$\mathbb{H}$, a degree two equivariant homogeneous polynomial map $L : \mathbb{H} \to \mathrm{im}\mathbb{H}$ is given by $L(v) = vi\bar{v}$ for $v \in \mathbb{H}$, which evaluates at $v = 1$ to $dx^0 \wedge dx^1 - dx^2 \wedge dx^3$. In terms of coordinates (A.3), the images of these degree zero and two homogeneous polynomials $V \to \bigwedge^2(V^*)$ respectively, are:

$$-\frac{3t}{2} dt \wedge \eta^{se} + \frac{3t^2}{4}(\omega_0^{se} + \omega_1^{se}) \qquad\qquad -\frac{3t}{2} dt \wedge \eta^{se} - \frac{3t^2}{4}(\omega_0^{se} + \omega_1^{se})$$

Third, a map $L : V \to V \otimes \mathfrak{p}^*$ is $U(1) \times SU(2)$-equivariant if $L(qve^{-i\theta}).z = q\left(L(v).ze^{-2i\theta}\right)e^{-i\theta}$, where $z \in \mathbb{C}$, $v \in V$ (identified as elements of $\mathfrak{p}$, $\mathbb{H}$ respectively), and $(e^{i\theta}, q) \in U(1) \times SU(2)$. There are two degree one homogeneous polynomial maps $L_l(v).z = vl\bar{z}$ for $l \in \{j, k\}$. Identifying $L_l(v)$ with an anti-symmetric endomorphism in $\mathrm{End}(V \oplus \mathfrak{p})$, so naturally an element of $\bigwedge^2(\mathfrak{p}^* \oplus V^*)$, and evaluating at $v = 1$, gives $dx^2 \wedge v^1 + dx^3 \wedge w^1$ and $dx^3 \wedge v^1 - dx^2 \wedge w^1$, or $-\frac{3t}{2}\omega_2^{se}$ and $-\frac{3t}{2}\omega_3^{se}$ in terms of coordinates (A.3).

This splitting provides the boundary extension conditions for a generic invariant two-form $\alpha = \lambda dt \wedge \eta^{se} + \sum_i u_i \omega_i^{se}$ on the space of principal orbits. Rewriting α with respect to our basis:

$$\begin{aligned}
\alpha = &\frac{1}{4}\left(\frac{u_1 + u_0}{t^2} - \frac{\lambda}{t}\right)\left(-2t dt \wedge \eta^{se} + t^2(\omega_0^{se} + \omega_1^{se})\right) \\
&+ \frac{1}{4}\left(\frac{u_1 + u_0}{t^2} + \frac{\lambda}{t}\right)\left(2t dt \wedge \eta^{se} + t^2(\omega_0^{se} + \omega_1^{se})\right) \\
&+ \frac{u_1 - u_0}{2}(\omega_0^{se} - \omega_1^{se}) + \frac{u_2}{t}\omega_2^{se} + \frac{u_3}{t}\omega_3^{se}
\end{aligned}$$

we see that, to extend α to $t = 0$, u_0, u_1, u_2, u_3 must be even, λ must be odd, $u_0(0) + u_1(0) = 0$, $u_2(0) = u_3(0) = 0$, and $u_0''(0) + u_1''(0) = 2\lambda'(0)$ as claimed. □

Proof of Corollary A.1.4. By Proposition A.1.3, if the Kähler form ω of an invariant Calabi-Yau structure (ω, Ω) extends over the singular orbit $S^2 = SU(2)^2/U(1) \times SU(2)$ at $t = 0$, then (ω, Ω) must it induce a hypo-structure of type $\mathcal{I}$ on the principal orbits: if the evolution equations (1.23) are satisfied with $\lambda(t)$ odd, $\mu(t)$ even, and $\mu(0)=0$, then $v_0(t)$ must vanish for all t. For a hypo-structure of type $\mathcal{I}$ solving the evolution equations (1.21), re-parametrising $t \mapsto r$, $r(t) := \sqrt[3]{\lambda\mu}$ pulls back the corresponding holomorphic volume form from the conifold. Hence if ω extends, then $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\setminus\mathbb{CP}^1$ has a holomorphic volume form pulled back from the conifold, and hence it extends automatically to the singular orbit $SU(2)^2/U(1)\times SU(2)=\mathbb{CP}^1\subset\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ at $r(0)=0$ of the small resolution $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. □

Proof of Proposition A.1.5. We first describe a change of coordinates from left-invariant vector fields on the principal orbits of T^*S^3 to a coordinate system adapted to the boundary extension problem. The coordinate system we will need is achieved via the isomorphism:

$$T^*S^3 = SU(2)^2\times_{\triangle SU(2)}\mathfrak{su}(2)\cong S^3\times \mathrm{im}\mathbb{H}\subset\mathbb{H}\times\mathrm{im}\mathbb{H}$$

Identifying $\mathfrak{su}(2)$ with the imaginary quaternions $\mathrm{im}\mathbb{H}$ and $SU(2)$ with the unit quaternions $S^3\subset\mathbb{H}$, this map can be explicitly given along with its inverse:

$$(p,q,y)\mapsto(q\bar{p},py\bar{p}) \qquad\qquad (x,y)\mapsto(\bar{x},1,xy\bar{x})$$

By requiring this map is equivariant, we get a description of an $SU(2)^2$-action in this new coordinate system via the embedding in $\mathbb{H}\times\mathrm{Im}\mathbb{H}$:

$$(q_1,q_2)\,.\,(x,y)\mapsto(q_2x\bar{q}_1,q_1y\bar{q}_1)$$

With this description in hand, we will write vector fields on $SU(2)^2$ as vector fields on T^*S^3: here, let (x^1,x^2,x^3) denote the Euclidean coordinates on the fibre of $S^3\times\mathrm{im}\mathbb{H}$. Along the curve $\gamma(t)=(1,it)$, the vector fields U^1,U^2,V^1,V^2,W^1,W^2 are given by:

$$U^1=\frac{d}{ds}\bigg|_{s=0}\left(e^{-is},it\right)=(-i,0) \qquad U^2=\frac{d}{ds}\bigg|_{s=0}\left(e^{is},it\right)=(i,0)$$
$$V^1=\frac{d}{ds}\bigg|_{s=0}\left(e^{-js},e^{js}ite^{-js}\right)=(-j,-2tk) \qquad V^2=\frac{d}{ds}\bigg|_{s=0}\left(e^{js},it\right)=(j,0)$$
$$W^1=\frac{d}{ds}\bigg|_{s=0}\left(e^{ks},e^{ks}ite^{-ks}\right)=(-k,2tj) \qquad W^2=\frac{d}{ds}\bigg|_{s=0}\left(e^{ks},it\right)=(k,0)$$

Clearly $\frac{\partial}{\partial t}=\frac{\partial}{\partial x^1}$, and:

$$V^+ := V^1+V^2=(0,-2tk)=-2t\frac{\partial}{\partial x^3} \qquad W^+ := V^1+V^2=(0,2tj)=2t\frac{\partial}{\partial x^2}$$

so that:

$$dx^1=dt \qquad dx^2=2tw^+ \qquad dx^3=-2tv^+ \tag{A.4}$$

We proceed with the calculation describing the boundary conditions for extending 2-forms over the singular orbit: here, V will denote the tangent space to the fibre of T^*S^3, and $\mathfrak{su}(2)^-$ will denote the tangent space to the singular orbit i.e. $\mathfrak{su}(2)^-=\langle U^-,V^-,W^-\rangle$, and V is the span of the coordinate vector fields $\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}$. As $\triangle SU(2)$-representations, both $\mathfrak{su}(2)^-$ and V are isomorphic to the adjoint representation, and isomorphic to $\mathbb{R}\oplus\mathbb{C}_2$ as $\triangle U(1)$-representations, where the trivial component of this representation is spanned by U^- and $\frac{\partial}{\partial x^1}$ respectively.

The 5-dimensional space of $\triangle U(1)$-invariant 2-forms on $\mathfrak{su}(2)^-\oplus V$ can be written as the span of $u^-\wedge dx^1$, $dx^2\wedge dx^3$, $v^-\wedge w^-$, and the real and imaginary parts of $\left(dx^2+idx^3\right)\wedge\left(v^- - iw^-\right)$, and as before, we associate an integer degree to each component in a splitting of this space, by taking the degree of some $\triangle SU(2)$-equivariant homogeneous polynomials in $v\in V$, and evaluating at some non-zero $v_0\in V$.

To achieve this, we first identify $\mathfrak{su}(2)^-$ and V with the imaginary quaternions $\mathrm{im}\mathbb{H}$. Then as $\triangle SU(2)$- representations:

$$\bigwedge^2(\mathrm{im}\mathbb{H}\oplus\mathrm{im}\mathbb{H})\cong 2\bigwedge^2(\mathrm{im}\mathbb{H})\oplus(\mathrm{im}\mathbb{H}\otimes\mathrm{im}\mathbb{H}^*)\cong 2\mathrm{im}\mathbb{H}\oplus(\mathrm{im}\mathbb{H}\otimes\mathrm{im}\mathbb{H}^*)$$

where the second of these isomorphisms arises from the equivariant isomorphism $*:\bigwedge^2(\mathrm{im}\mathbb{H})\to\mathrm{im}\mathbb{H}$ given by the standard Hodge star on $\mathbb{R}^3$:

$$*dx^1=dx^2\wedge dx^3 \qquad *dx^2=dx^3\wedge dx^1 \qquad *dx^3=dx^1\wedge dx^2$$

First of all, is not hard to see equivariant polynomial maps $L:\mathrm{im}\mathbb{H}\to\mathrm{im}\mathbb{H}$ have a single degree one generator $L(v)=v$, which evaluates at $v=i$ to $*u^-=v^-\wedge w^-$, $*dx^1=dx^2\wedge dx^3$ respectively. The latter can be rewritten as $-4t^2v^+\wedge w^+$ in the coordinate system (A.4).

To determine the space of equivariant maps $\mathrm{im}\mathbb{H}\to\mathrm{im}\mathbb{H}\otimes\mathrm{im}\mathbb{H}^*$, we identify the adjoint action of $SU(2)$ on $\mathrm{im}\mathbb{H}$ with the standard action of $SO(3)$ on $\mathbb{R}^3$: equivariant polynomial maps from $\mathbb{R}^3$ into $\mathbb{R}^3\otimes\left(\mathbb{R}^3\right)^*$ with this action are spanned by the constant map $\mathrm{id}:\mathbb{R}^3\to\mathbb{R}^3$, the degree-one polynomial $v\mapsto .\times v$, and the degree-two polynomial $v\mapsto\langle\,.\,,v\rangle v$, where $\times$ denotes the cross product on $\mathbb{R}^3$ and $\langle\,.\,,v\rangle$ denotes the Euclidian dot product with v. These evaluate at $v=(1,0,0)$ to $\mathrm{id}_{\mathbb{R}}+\mathrm{id}_{\mathbb{C}_2}$, $j_{\mathbb{C}_2}$, and $\mathrm{id}_{\mathbb{R}}$ respectively, where we denote the map between the trivial $\triangle U(1)$-representations in $\mathrm{im}\mathbb{H}$ as $\mathrm{id}_{\mathbb{R}}$, and the maps corresponding to the identity and multiplication by complex number i between the $\triangle U(1)$-representations $\mathbb{C}_2$ as $\mathrm{id}_{\mathbb{C}_2}$ and $j_{\mathbb{C}_2}$ respectively.

Considering these as 2-forms on $\mathfrak{su}(2)^-\oplus V$, from the linear map $\mathrm{id}_{\mathbb{R}}+\mathrm{id}_{\mathbb{C}_2}$ we get $dx^1\wedge u^-+dx^2\wedge v^-+dx^3\wedge w^-$ which can be rewritten as $dt\wedge u^-+2t\left(w^+\wedge v^- - v^+\wedge w^-\right)$ using our coordinate system (A.4). Meanwhile $j_{\mathbb{C}_2}$ gives $dx^3\wedge v^- - dx^2\wedge w^-$, which can be rewritten as $-2t\left(v^+\wedge v^- + w^+\wedge w^-\right)$, and $\mathrm{id}_{\mathbb{R}}$ gives $dx^1\wedge u^-$ which is rewritten as $dt\wedge u^-$.

This splitting provides the boundary extension conditions for a generic invariant two-form $\alpha=\lambda dt\wedge\eta^{se}+\sum_i u_i\omega_i^{se}$ on the space of principal orbits. Rewriting α with respect to our basis:

$$\begin{aligned}\alpha=&\frac{2}{3}\left(2\lambda-\frac{u_1}{t}\right)dt\wedge u^-+\frac{4\left(u_0+u_3\right)}{3t^2}v^+\wedge w^++\frac{4}{3}\left(u_0-u_3\right)v^-\wedge w^-\\&+\frac{4u_2}{3t}\left(v^+\wedge v^-+w^-\wedge w^+\right)+\frac{2u_1}{3t}\left(dt\wedge u^-+2t\left(w^+\wedge v^- - v^+\wedge w^-\right)\right)\end{aligned}$$

we see that for α to extend we must have u_0, u_1, u_3 are odd, u_2, λ even, $u_2(0) = 0$, $u_0'(0) + u_3'(0) = 0$, and $2\lambda(0) = u_1'(0)$ as claimed. $\square$

Proof of Proposition A.1.6. We denote V, $\mathfrak{su}^-(2)$, dx^1, dx^2, dx^3 as in the previous proof. Identifying V, $\mathfrak{su}^-(2)$ with the imaginary quaternions $\mathrm{im}\mathbb{H}$, then the space of 3-forms on $\mathfrak{su}^-(2) \oplus V$ can be written as a $\triangle SU(2)$-representation:

$$\bigwedge^3(\mathrm{im}\mathbb{H} \oplus \mathrm{im}\mathbb{H}) \cong 2\bigwedge^3(\mathrm{im}\mathbb{H}) \oplus 2\left(\mathrm{im}\mathbb{H} \otimes \bigwedge^2(\mathrm{im}\mathbb{H}^*)\right) \cong \mathbb{R}^2 \oplus 2\left(\mathrm{im}\mathbb{H} \otimes \mathrm{im}\mathbb{H}^*\right)$$

where again, the second of these equivalences arises from the equivariant isomorphism $* : \bigwedge^2(\mathrm{im}\mathbb{H}) \to \mathrm{im}\mathbb{H}$ given by the standard Hodge star on $\mathbb{R}^3$.

Now the space of equivariant maps from $\mathrm{im}\mathbb{H} \to \mathrm{im}\mathbb{H} \otimes \mathrm{im}\mathbb{H}^*$ was given in the previous proof: its image, when evaluated at $i \in \mathrm{im}\mathbb{H}$, is spanned by $\mathrm{id}_{\mathbb{R}} + \mathrm{id}_{\mathbb{C}_2}$, $j_{\mathbb{C}_2}$, and $\mathrm{id}_{\mathbb{R}}$ with degrees 0, 1, and 2 respectively. So, denoting u^-, v^-, w^- as e^1, e^2, e^3, the degree zero maps on $\mathrm{im}\mathbb{H} \otimes \mathrm{im}\mathbb{H}^*$ corresponding to $\mathrm{id}_{\mathbb{R}} + \mathrm{id}_{\mathbb{C}_2}$ are $\sum_i e^i \wedge *dx^i$ and $\sum_i dx^i \wedge *e^i$, which can be rewritten as:

$$\frac{3}{8}\left(dt \wedge (\omega_0^{se} - \omega_3^{se}) - 3t\omega_2^{se} \wedge \eta^{se}\right) \qquad -\frac{3}{8}\left(4dt \wedge \omega_2^{se} - 3t^2(\omega_0^{se} + \omega_3^{se}) \wedge \eta^{se}\right)$$

The degree one maps on $\mathrm{im}\mathbb{H} \otimes \mathrm{im}\mathbb{H}^*$ corresponding to $j_{\mathbb{C}_2}$ are $e^3 \wedge *dx^2 - e^2 \wedge *dx^3$ and $dx^3 \wedge *e^2 - dx^2 \wedge *e^3$, which can be rewritten as:

$$\frac{9}{8}t\omega_1^{se} \wedge \eta^{se} \qquad \frac{9}{8}t\omega_1^{se} \wedge dt$$

and the degree two maps on $\mathrm{im}\mathbb{H} \otimes \mathrm{im}\mathbb{H}^*$ corresponding to $\mathrm{id}_{\mathbb{R}}$ are $e^1 \wedge *dx^1$ and $dx^1 \wedge *e^1$, which can be rewritten as:

$$-\frac{9}{8}t^2(\omega_0^{se} + \omega_3^{se}) \wedge \eta^{se} \qquad \frac{3}{8}dt \wedge (\omega_0^{se} - \omega_3^{se})$$

Furthermore, the invariant (degree zero) maps spanning $\mathrm{im}\mathbb{H} \to \bigwedge^3(\mathrm{im}\mathbb{H})$ are the constant map $dx^1 \wedge dx^2 \wedge dx^3$ and $e^1 \wedge e^2 \wedge e^3$ respectively, which can be rewritten as:

$$\frac{3}{8}t^2 dt \wedge (\omega_0^{se} - \omega_3^{se}) \qquad \frac{3}{8}t^2 dt \wedge (\omega_0^{se} - \omega_3^{se}) \wedge \eta^{se}$$

We now rewrite an invariant 3-form $\beta = \sum_i u_i \omega_i^{se} \wedge dt + \sum_i v_i \omega_i^{se} \wedge \eta^{se}$ in terms of this basis:

$$\begin{aligned}\beta = &\frac{v_2}{3t}\left(dt \wedge (\omega_0^{se} - \omega_3^{se}) - 3t\omega_2^{se} \wedge \eta^{se}\right) - \frac{u_2}{t}\left(dt \wedge \omega_2^{se} - \frac{3}{4}t^2(\omega_0^{se} + \omega_3^{se}) \wedge \eta^{se}\right) \\ &+ \frac{1}{2}(u_0 + u_3)\, dt \wedge (\omega_0^{se} + \omega_3^{se}) + \left(\frac{1}{2}(u_0 + u_3) + \frac{v_2}{3t}\right) dt \wedge (\omega_0^{se} - \omega_3^{se}) \\ &+ \frac{1}{2}(v_0 - v_3)(\omega_0^{se} - \omega_3^{se}) \wedge \eta^{se} + \left(\frac{1}{2}(v_0 + v_3) + \frac{3tu_2}{4}\right)(\omega_0^{se} + \omega_3^{se}) \wedge \eta^{se} \\ &+ u_1\omega_1^{se} \wedge dt + v_1\omega_1^{se} \wedge \eta^{se}\end{aligned}$$

So for β to extend, applying [EW00], $\frac{v_2}{t}$, $\frac{u_2}{t}$, $\frac{u_0+u_3}{t}$, and $v_0 - v_3$ must be even, $\frac{u_1}{t}$ and $\frac{v_1}{t}$ must be odd, and $\frac{1}{t^2}\left(\frac{1}{2}(v_0 + v_3) + \frac{3tu_2}{4}\right)$, $\frac{1}{2}(u_0 + u_3) - \frac{v_2}{3t}$ must be degree two. Unravelling these conditions, we get $u_0, v_0, u_1, v_1, u_3, v_3$ must be even, u_2, v_2 must be odd, $u_1(0) = v_1(0) = 0$, $u_0(0) + u_3(0) = v_0(0) + v_3(0) = 0$, $3(u_0(0) - u_3(0)) + 2v_2'(0) = 0$, and $v_0''(0) + v_3''(0) + 3u_2'(0) = 0$ as claimed. □

A.2 Bundle Data

By considering $SU(2)^2$-invariant Calabi-Yau instantons and monopoles on the space of principal orbits $\mathbb{R}_{>0} \times SU(2)^2/\triangle U(1)$ of T^*S^3, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and $\mathcal{O}(-2,-2)$ in §2, we obtained ordinary differential equations depending on the geodesic parameter $t \in \mathbb{R}_{>0}$. In this appendix, we check the boundary conditions used in §3.1 for these data to extend smoothly to the singular orbits at $t = 0$.

Firstly, we must extend the invariant bundle: let H denote the principal isotropy subgroup of the $SU(2)^2$ action, and H' the singular isotropy subgroup. Recall from §1.1 that the homogeneous $SU(2)$-bundles over the principal orbit, defined by the homomorphism $H \to SU(2)$, can be extended to an $SU(2)^2$-invariant bundle over the total space by extending the homomorphism $H \to SU(2)$ to a homomorphism $H \subset H' \to SU(2)$.

Recall also §2.3: homogeneous bundles $P_{n,j}$ over the principal orbits of $\mathcal{O}(-2,-2)$ are classified by a pair $(n, j) \in \mathbb{Z} \times \mathbb{Z}_2$. Using the isomorphism of the principal isotropy subgroup $K_{2,-2} \cong \triangle U(1) \times \mathbb{Z}_2 \subset U(1)^2$, we can write the associated homomorphism $K_{2,-2} \to SU(2)$ as:

$$(e^{i\theta}, e^{i\theta}).(e^{2i\pi}, e^{i\pi}) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^j \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} \tag{A.5}$$

and similarly for the bundles P_n over the principal orbits of T^*S^3, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ associated to homomorphisms $\triangle U(1) \to SU(2)$ of the principal isotropy subgroup $\triangle U(1)$.

Proposition A.2.1. *Up to equivariant isomorphism, the $SU(2)^2$-invariant $SU(2)$-bundles extending P_n, $P_{n,j}$ to the singular orbit are given by:*

(i) *Extending over $S^3 = SU(2)^2/\triangle SU(2)$: P_1, P_0 extend to P_{Id}, $P_{\mathbf{0}}$ defined by homomorphisms $\mathrm{Id}, \mathbf{0}$ respectively.*

(ii) *Extending over $S^2 = SU(2)^2/U(1) \times SU(2)$: P_n extends to $P_{n,\mathbf{0}}$ defined by homomorphism $i^n \times \mathbf{0}$ for all n, and P_1 also extends to $P_{0,\mathrm{Id}}$ defined by homomorphism $i^0 \times \mathrm{Id}$.*

(iii) *Extending over $S^2 = SU(2)^2/SU(2) \times U(1)$: P_n extends to $P_{\mathbf{0},n}$ defined by homomorphism $\mathbf{0} \times i^n$ for all n, and P_1 also extends to $P_{\mathrm{Id},0}$ defined by homomorphism $\mathrm{Id} \times i^0$.*

(iv) *Extending over $S^2 \times S^2 = SU(2)^2/U(1)^2$: $P_{n,j}$ extends to $P_{l,m}$ defined by homomorphism $i^l \times i^m$, where $l + m = n$, and either $j = m \bmod 2$ or $j = l \bmod 2$.*

where Id, $\mathbf{0} : SU(2) \to SU(2)$ denote the identity and the trivial homomorphism respectively, and i^n denotes the n^{th}-power of the diagonal embedding $i : U(1) \hookrightarrow SU(2)$.

Proof. The first two parts of the proposition follow directly from the previous discussion, and the group diagrams $\triangle U(1) \subset \triangle SU(2) \subset SU(2)^2$, and $\triangle U(1) \subset U(1) \times SU(2) \subset$

$SU(2)^2$ for T^*S^3 and $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, respectively. The third part follows via exchanging the factors of $SU(2)^2$ for $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, i.e. writing the group diagram as $\triangle U(1)\subset SU(2)\times U(1)\subset SU(2)^2$.

The group diagram for $\mathcal{O}(-2,-2)$ is given by $K_{2,-2}\subset U(1)^2\subset SU(2)^2$, so the singular homomorphisms are classified by a pair of integers (l,m):

$$(e^{i\theta_1},e^{i\theta_2})\mapsto\begin{pmatrix}e^{il\theta_1+im\theta_2} & 0\\ 0 & e^{-il\theta_1-im\theta_2}\end{pmatrix}\tag{A.6}$$

where the principal isotropy group $K_{2,-2}$ is uniquely defined as the kernel of (A.6) with $(l,m)=(2,-2)$. One can realise the isomorphism $K_{2,-2}\cong\triangle U(1)\times\mathbb{Z}_2\subset U(1)^2$ in exactly two ways, either with $\mathbb{Z}_2\subset U(1)^2$ defined as the subgroup generated by $(e^{2i\pi},e^{i\pi})$ or $\mathbb{Z}_2\subset U(1)^2$ defined as the subgroup generated by $(e^{i\pi},e^{2i\pi})$, equivalent up to the automorphism exchanging the factors of $U(1)\subset U(1)^2$.

The first of these isomorphisms $K_{2,-2}\to\triangle U(1)\times\mathbb{Z}_2\subset U(1)^2$ is given by the map $(e^{i\theta_1},e^{i\theta_2})\mapsto(e^{i\theta_1},e^{i\theta_1}).(e^{2i\pi},e^{i(\theta_2-\theta_1)})$, and if we re-write (A.6) as:

$$(e^{i\theta_1},e^{i\theta_2})\mapsto\begin{pmatrix}e^{i\theta_2-i\theta_1} & 0\\ 0 & e^{-i\theta_2+i\theta_1}\end{pmatrix}^m\begin{pmatrix}e^{i(l+m)\theta_1} & 0\\ 0 & e^{-i(l+m)\theta_1}\end{pmatrix}$$

and fix the $\mathbb{Z}_2$-generator $(e^{2i\pi},e^{i\pi})$, then (A.6) restricts to $\triangle U(1)\times\mathbb{Z}_2\subset U(1)^2$ as the homomorphism (A.5) with $j=m\,\mathrm{mod}2$ and $l+m=n$. By exchanging the factors of $U(1)\subset U(1)^2$, which also exchanges (l,m) in (A.6), we get the homomorphism (A.5) with $j=l\,\mathrm{mod}2$ and $l+m=n$. □

With a little extra work, the following proposition can also be seen from the previous discussion:

Proposition A.2.2. *Any $SU(2)^2$-invariant $SO(3)$-bundle over $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, T^*S^3 or $\mathcal{O}(-2,-2)$ admitting irreducible invariant connections has a lift to an $SU(2)^2$-invariant $SU(2)$-bundle.*

Proof. By definition, an invariant $SO(3)$-bundle lifts if the homomorphism $H'\to SO(3)$ from the singular isotropy subgroup H' to the gauge group lifts to $SU(2)$. On the other hand, to admit irreducible invariant connections, the invariant $SO(3)$-bundle restricted to the space of principal orbits must lift to the invariant $SU(2)$-bundle P_1 i.e. if we denote the principal isotropy group by H, the homomorphism $H\to SO(3)$ lifts to the homomorphism $H\to SU(2)$ given by (2.5) with $n=1$.

The statement for T^*S^3 and $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ is then immediate from Proposition A.2.1. As for $\mathcal{O}(-2,-2)$, the $SO(3)$-bundles are classified by the singular homomorphisms $U(1)^2\to SO(2)\subset SO(3)$:

$$(e^{i\theta_1},e^{i\theta_2})\mapsto\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(l\theta_1+m\theta_2) & \sin(l\theta_1+m\theta_2)\\ 0 & -\sin(l\theta_1+m\theta_2) & \cos(l\theta_1+m\theta_2)\end{pmatrix}\tag{A.7}$$

which lift to the $SU(2)$-homomorphism (A.6) when l,m are both even.

By the assumption of irreducibility, we require $l+m=2$, so it suffices to consider the case where l,m are also both odd. Restricted to $K_{2,-2}\subset U(1)^2$, this gives:

$$(e^{i\theta_1},e^{i\theta_2})\mapsto\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\cos(2\theta_1)&\sin(2\theta_1)\\0&-\sin(2\theta_1)&\cos(2\theta_1)\end{pmatrix}\tag{A.8}$$

up to the automorphism exchanging the factors of $U(1)\subset U(1)^2$. Recall from §1.10 that both $(e^{i\pi},e^{2i\pi}),(e^{2i\pi},e^{i\pi})\in K_{2,-2}$ act trivially on the tangent space of the principal orbits of $\mathcal{O}(-2,-2)$ given in terms of basis (1.16), but one of $(e^{i\pi},e^{2i\pi}),(e^{2i\pi},e^{i\pi})$ acts non-trivially on $\mathfrak{so}(3)$ by (A.8), and so every invariant $\mathfrak{so}(3)$-valued connection one-form on the principal orbit can only take values in the set of fixed-points $\mathfrak{u}(1)\subset\mathfrak{so}(3)$ and must therefore be reducible. □

In any case, now we have classified possible extensions of invariant bundles over the singular orbit in Proposition A.2.1, for the rest of this section, we will describe the conditions for smoothly extending invariant connections and sections of the adjoint bundle.

For any $SU(2)^2$-invariant connection A, it will suffice to describe the conditions for extending $SU(2)^2$-invariant adjoint-valued one-forms: if we denote $\lambda: H'\to SU(2)$ the homomorphism extending the bundle over the singular isotropy subgroup H', then we can use the canonical connection $d\lambda$ over the singular orbit as an $SU(2)^2$-invariant reference connection to get an $SU(2)^2$-invariant adjoint-valued one form $A-d\lambda$. In order to calculate these extension conditions, we apply a similar analysis as in [EW00, Lemma 1.1] applied to adjoint-valued forms, cf. [LO18].

For brevity, we will restrict to the case that the bundle extends the homogeneous bundle P_1 over the singular orbit, i.e. the bundle admits irreducible invariant connections: recall Propositions 2.3.1, 2.3.2, on P_1 pulled back to the space of principal orbits, an $SU(2)^2$-invariant connection A can be written:

$$\begin{aligned}A=\,&a_1(E_2\otimes v^1+E_3\otimes w^1)+b_1(E_3\otimes v^1-E_2\otimes w^1)\\&+a_2(E_2\otimes v^2+E_3\otimes w^2)+b_2(E_3\otimes v^2-E_2\otimes w^2)+a_0E_1\otimes u^-+E_1\otimes u^+\end{aligned}\tag{A.9}$$

for some (a_0,a_1,a_2,b_1,b_2) functions of geodesic distance $t\in\mathbb{R}_{>0}$ from the singular orbit at $t=0$. Meanwhile, an $SU(2)^2$-invariant section Φ of the adjoint bundle can be written:

$$\Phi=\phi E_1\tag{A.10}$$

for some function ϕ of $t\in\mathbb{R}_{>0}$.

Proposition A.2.3. (A.9) *extends to the singular orbit $S^2\times S^2=SU(2)^2/U(1)^2$ on $P_{l,m}$ with $l+m=1$, if and only if $a_0(0)=l-m$, a_0 is even, and:*

(i) *If $l\geq 1$, then a_1,b_1 must be of degree $l-1$ and a_2,b_2 of degree l*

(ii) *If $m\geq 1$, then a_1,b_1 must be of degree m and a_2,b_2 of degree $m-1$.*

Proposition A.2.4. (A.10) *extends to the singular orbit $S^2\times S^2=SU(2)^2/U(1)^2$ on $P_{l,m}$ with $l+m=1$, if and only if ϕ is even.*

Proposition A.2.5. (A.9) *extends to the singular orbit* $S^3 = SU(2)^2/\triangle SU(2)$ *on* P_{Id} *if and only if* a_1, a_2, a_0 *even,* b_1, b_2 *odd,* $b_1'(0) = -b_2'(0)$, $a_1(0) - a_2(0) = a_0(0)$, *and* $a_1(0) + a_2(0) = 1$.

Proposition A.2.6. (A.10) *extends to the singular orbit* $S^3 = SU(2)^2/\triangle SU(2)$ *on* P_{Id} *if and only if* ϕ *is odd.*

Proposition A.2.7. (A.9) *extends to the singular orbit* $S^2 = SU(2)^2/U(1) \times SU(2)$ *on* $P_{0,\mathrm{Id}}$, $P_{1,\mathbf{0}}$ *iff:*

(1) *On* $P_{0,\mathrm{Id}}$*:* a_1, a_2, a_0 b_1, b_2 *even,* $a_1(0) = b_1(0) = b_2(0) = 0$, $a_2(0) = -a_0(0) = 1$, *and* $a_0''(0) + 2a_2''(0) = b_2''(0) = 0$.

(2) *On* $P_{1,\mathbf{0}}$*:* a_1, b_1, a_2, b_2, a_0 *even,* $a_2(0) = b_2(0) = 0$, $a_0(0) = 1$.

By exchanging the factors of $SU(2) \subset SU(2)^2$ in Proposition A.2.7, we also obtain the following corollary:

Corollary A.2.8. (A.9) *extends to the singular orbit* $S^2 = SU(2)^2/SU(2) \times U(1)$ *on* $P_{\mathrm{Id},0}$, $P_{\mathbf{0},1}$ *iff:*

(1) *On* $P_{\mathrm{Id},0}$*:* a_1, a_2, a_0 b_1, b_2 *even,* $a_2(0) = b_2(0) = b_1(0) = 0$, $a_1(0) = a_0(0) = 1$, *and* $a_0''(0) - 2a_1''(0) = b_1''(0) = 0$.

(2) *On* $P_{\mathbf{0},1}$*:* a_1, b_1, a_2, b_2, a_0 *even,* $a_1(0) = b_1(0) = 0$, *and* $a_0(0) = -1$.

Proposition A.2.9. (A.9) *extends to the singular orbit* $S^2 = SU(2)^2/U(1) \times SU(2)$ *on* $\mathrm{Ad}P_{0,\mathrm{Id}}$, $\mathrm{Ad}P_{1,\mathbf{0}}$ *if and only if:*

(1) *On* $P_{0,\mathrm{Id}}$*:* ϕ *even, and* $\phi(0) = 0$.

(2) *On* $P_{1,\mathbf{0}}$*:* ϕ *even.*

In the remainder of this section, we will explicitly prove these claims:

Proof of A.2.3. We first calculate the boundary extension conditions for invariant sections of $\Omega^1(\mathrm{Ad}P_{l,m})$. Here, denote $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{p} = \langle V^1, W^1, V^2, W^2\rangle$, and V the tangent space of the fibre $\mathbb{C}_{2,-2}$, spanned by the Cartesian coordinate vector fields $\frac{\partial}{\partial x^0}$, $\frac{\partial}{\partial x^1}$. Clearly $\mathfrak{p} \oplus V$ is a $U(1)^2$-invariant splitting of the tangent space of $\mathcal{O}(-2,-2)$, and as $U(1)^2$-representations:

$$\mathfrak{g} = \langle E_1\rangle \oplus \langle E_2, E_3\rangle \cong \mathbb{R} \oplus \mathbb{C}_{2l,2m} \quad \mathfrak{p} = \langle V^1, W^1\rangle \oplus \langle V^2, W^2\rangle \cong \mathbb{C}_{2,0} \oplus \mathbb{C}_{0,2} \quad V \cong \mathbb{C}_{2,-2} \tag{A.11}$$

while as $\triangle U(1) \times \mathbb{Z}_2$-representations[1]:

$$\mathfrak{g} = \langle E_1\rangle \oplus \langle E_2, E_3\rangle \cong \mathbb{R} \oplus \mathbb{C}_{2(l+m)} \quad \mathfrak{p} = \langle V^1, W^1\rangle \oplus \langle V^2, W^2\rangle \cong \mathbb{C}_2 \oplus \mathbb{C}_2 \quad V \cong \mathbb{R}^2 \tag{A.12}$$

[1] note the factor of $\mathbb{Z}_2$ in $K_{2-2} \cong \triangle U(1) \times \mathbb{Z}_2$ does not appear in the representation theory, as it always acts trivially on the tangent space and the Lie algebra of the gauge group.

Recall that, since $l+m=1$, the space of $\triangle U(1)\times\mathbb{Z}_2$-invariant adjoint-valued one-forms in $\mathfrak{g}\otimes(V^*\oplus\mathfrak{p}^*)$ is spanned by the real and imaginary parts of $E_1\otimes\left(dx^0+idx^1\right)$, $(E_2+iE_3)\otimes\left(v^1-iw^1\right),(E_2+iE_3)\otimes\left(v^2-iw^2\right)$.

To apply the analysis of [EW00, Lemma 1.1], we use (A.11) to look for a basis in terms of $U(1)^2$-equivariant homogeneous polynomials $p: V\to\mathfrak{g}\otimes(V^*\oplus\mathfrak{p}^*)$, evaluated at $1\in V=\mathbb{C}$.

First assume $l>0$. By making the identification of the fibre Cartesian coordinate one-forms $dx^0=dt$ and $dx^1=3t\eta^{se}=4tu^-$ along $\gamma(t)=(1,1,t)\in SU(2)^2\times\mathbb{C}$, and by taking real and imaginary parts, we obtain the following splitting:

degree	polynomial $p(z)$	evaluation at $z=1$
1	$zE_1\otimes\left(dx^0+idx^1\right)$	$E_1\otimes dt, E_1\otimes 4tu^-$
$l-1$	$z^{l-1}(E_2+iE_3)\otimes\left(v^1-iw^1\right)$	$E_2\otimes v^1+E_3\otimes w^1, -E_2\otimes w^1+E_3\otimes v^1$
l	$z^l(E_2+iE_3)\otimes\left(v^2-iw^2\right)$	$E_2\otimes v^2+E_3\otimes w^2, -E_2\otimes w^2+E_3\otimes v^2$

We can recover the case $l\le 0$ by exchanging $P_{l,m}\mapsto P_{m,l}$, since clearly, the polynomials of degree $l-1$ and l are exchanged by this map, and we are working under the assumption $l+m=1$.

We now apply this calculation to an invariant connection A of the proposition: the canonical connection $d\lambda$ on $P_{l,m}$ is given by $d\lambda=lE_1\otimes u^1+mE_1\otimes u^2$, so writing the $SU(2)^2$-invariant connection A in (A.9) as an invariant section $A-d\lambda\in\Omega^1(\mathrm{Ad}P_{l,m})$, we get:

$$\begin{aligned}A-d\lambda&=a_1(E_2\otimes v^1+E_3\otimes w^1)+b_1(E_3\otimes v^1-E_2\otimes w^1)\\&\quad+a_2(E_2\otimes v^2+E_3\otimes w^2)+b_2(E_3\otimes v^2-E_2\otimes w^2)+(a_0-2l+1)E_1\otimes u^-\end{aligned}$$

So if $l>0$, we require $a_0(0)=2l-1$, a_0 be even, a_1,b_1 to have degree $l-1$ and a_2,b_2 to have degree l to extend A. Again, one gets the corresponding claim for $l\le 0$ by exchanging the factors of $SU(2)$. □

Proof of A.2.4. The degree of a function appearing as the coefficient of an $SU(2)^2$-invariant element in $\Omega^0(\mathrm{Ad}P_{l,m})$ on the principal orbits is determined by a $U(1)^2$-equivariant homogeneous polynomial from $V=\mathbb{C}_{2,-2}$ to $\mathfrak{g}=\langle E_1\rangle\oplus\langle E_2,E_3\rangle\cong\mathbb{R}\oplus\mathbb{C}_{2l,2m}$.

When $l+m=1$, there is a single degree zero polynomial given by the constant map E_1, so the invariant section $\Phi=\phi_1E_1$ must have ϕ_1 even. □

Proof of A.2.5. Let $\mathfrak{g}$ denote the Lie algebra of the gauge group $SU(2)$, V denote the tangent space to the fibre of T^*S^3, and $\mathfrak{su}(2)^-$ denote the complement of the Lie algebra of $\triangle SU(2)$ in $SU(2)^2$, i.e. $\mathfrak{g}=\langle E_1,E_2,E_3\rangle$, $\mathfrak{su}(2)^-=\langle U^-,V^-,W^-\rangle$, and V is the span of the coordinate vector fields $\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}$ as in (A.4). Recall that the tangent space of T^*S^3 can be written as the $\triangle SU(2)$-invariant splitting $\mathfrak{su}(2)^-\oplus V$, with $\triangle SU(2)$ acting as the adjoint on $\mathrm{im}\mathbb{H}\oplus\mathrm{im}\mathbb{H}$. An invariant element of $\Omega^1(\mathrm{Ad}P)$ is determined by an $\triangle SU(2)$-equivariant map $V\to\mathfrak{g}\otimes(\mathfrak{su}(2)^-\oplus V)^*$, where $\triangle SU(2)$ acts as a $\triangle SU(2)$-representations:

$$\mathfrak{g}\otimes\left(\mathfrak{su}(2)^-\oplus V\right)\cong\mathrm{im}\mathbb{H}\otimes(\mathrm{im}\mathbb{H}\oplus\mathrm{im}\mathbb{H})^*$$

So that the space of $SU(2)^2$-invariant one-forms on $\mathrm{ad}P_{\mathrm{Id}}$ is identified with the space of $\triangle SU(2)$-equivariant maps:

$$L : \mathrm{im}\mathbb{H} \to \mathrm{im}\mathbb{H} \otimes (\mathrm{im}\mathbb{H} \oplus \mathrm{im}\mathbb{H})^*$$

Recall from the proof of A.1.5, the space of equivariant polynomial maps $\mathrm{im}\mathbb{H} \to \mathrm{im}\mathbb{H} \otimes \mathrm{im}\mathbb{H}^*$ evaluated at $i \in \mathrm{im}\mathbb{H}$ is spanned by $\mathrm{id}_{\mathbb{R}} + \mathrm{id}_{\mathbb{C}_2}$, $j_{\mathbb{C}_2}$, and $\mathrm{id}_{\mathbb{R}}$ with respective degrees 0,1 and 2. Here, we are denoting the map between the trivial $\triangle U(1) \subset \triangle SU(2)$-representations in $\mathrm{im}\mathbb{H}$ as $\mathrm{id}_{\mathbb{R}}$, and the maps corresponding to the identity and multiplication by complex number i between the $\triangle U(1)$-representations $\mathbb{C}_2$ as $\mathrm{id}_{\mathbb{C}_2}$ and $j_{\mathbb{C}_2}$ respectively.

The maps $\mathrm{id}_{\mathbb{R}} + \mathrm{id}_{\mathbb{C}_2}$, in this setting, are $E_1 \otimes u^- + E_2 \otimes v^- + E_3 \otimes w^-$ and $E_1 \otimes dx^1 + E_2 \otimes dx^2 + E_3 \otimes dx^3$, and in the coordinates on the principal orbits (A.4), these are written:

$$E_1 \otimes u^- + E_2 \otimes v^- + E_3 \otimes w^- \qquad\qquad E_1 \otimes dt + 2t(E_2 \otimes w^+ - E_3 \otimes v^+)$$

Similarly, the maps $j_{\mathbb{C}_2}$ are $E_2 \otimes w^- - E_3 \otimes v^-$ and $E_2 \otimes dx^3 - E_3 \otimes dx^2$ here, and in coordinates (A.4):

$$E_2 \otimes w^- - E_3 \otimes v^- \qquad\qquad -2t\left(E_2 \otimes v^+ + E_3 \otimes w^+\right)$$

and finally, the maps $\mathrm{id}_{\mathbb{R}}$ correspond to $E_1 \otimes dx^1$ and $E_1 \otimes u^-$, where the first of these is rewritten $E_1 \otimes dt$ in coordinates on the principal orbits.

With this grading of the space of $\triangle U(1)$-invariant subspace of adjoint-valued one-forms, we can now check the extension of any $SU(2)^2$-invariant adjoint-valued one-form defined on the principal orbits to $t = 0$, by writing any such adjoint-valued one-form as a t-dependent linear combination of these Lie algebra-valued one-forms. To apply this result to the connection A in (A.9), we need to use the canonical connection to get a section of $\Omega^1(\mathrm{ad}P_{\mathrm{Id}})$. In this case, the canonical connection is:

$$d\lambda = E_1 \otimes u^+ + E_2 \otimes v^+ + E_3 \otimes w^+$$

Writing $A - d\lambda$ in terms of the basis above, this is:

$$\begin{aligned}
A - d\lambda = &\left(\frac{a_1 + a_2 - 1}{2t}\right) 2t\left(E_2 \otimes v^+ + E_3 \otimes w^+\right) \\
&- \left(\frac{b_1 + b_2}{2t}\right)\left(E_1 \otimes dt - 2t\left(E_3 \otimes v^+ - E_2 \otimes w^+\right)\right) \\
&+ (a_1 - a_2)\left(E_1 \otimes u^- + E_2 \otimes v^- + E_3 \otimes w^-\right) + (b_1 - b_2)\left(E_3 \otimes v^- - E_2 \otimes w^-\right) \\
&+ (a_0 - (a_1 - a_2))\, E_1 \otimes u^- + \left(\frac{b_1 + b_2}{2t}\right) E_1 \otimes dt
\end{aligned}$$

So we require $a_1 - a_2$ to be even, $\frac{a_1+a_2-1}{t}$, $b_1 - b_2$ be odd, and $\frac{b_1+b_2}{t}$, $a_0 - a_1 + a_2$ be degree 2 for connection A in (A.9) to extend to the singular orbit at $t = 0$. This is equivalent to the requirement a_1, a_2, a_0 even, b_1, b_2 odd, $b_1'(0) = -b_2'(0)$, $a_1(0) - a_2(0) = a_0(0)$, and

$a_1(0) + a_2(0) = 1$, as claimed. □

Proof of A.2.6. Write, V, $\mathfrak{g}$ as in the previous proof. The space of $SU(2)^2$-invariant sections of ad$P$ is identified with the space of $\triangle SU(2)$-equivariant maps $L : V \to \mathfrak{g}$, and homogeneous generators for the polynomial elements gives extension conditions for $SU(2)^2$-invariant sections restricted to the principal orbits. L is a $\triangle SU(2)$-equivariant map:

$$L : \mathrm{im}\mathbb{H} \to \mathrm{im}\mathbb{H}$$

This has a single polynomial generator $L(v) = v$ for $v \in \mathrm{im}\mathbb{H}$, so that for $q \in \triangle SU(2)$, $qL(v)q^{-1} = L(qvq^{-1})$, and the corresponding invariant section evaluated at $i \in \mathrm{im}\mathbb{H}$ is E_1. This generator is degree one, so $\triangle U(1)$-invariant section $\Phi = \phi E_1$ must have ϕ odd. □

Proof of A.2.7. We first calculate the boundary extension conditions for invariant adjoint-valued one-forms. Here, denote Lie algebra of the gauge group $\mathfrak{g} = \mathfrak{su}(2) = \langle E_1, E_2, E_3\rangle$, the space of left-invariant tangent-vectors on the singular orbit $\mathfrak{p} = \langle V^1, W^1\rangle$, and V the tangent space of the fibre of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over a fixed point in S^2, spanned by the co-ordinate vector fields $\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$ as in (A.4). Recall that $U(1) \times SU(2)$ acts on $\mathfrak{p}$ trivially for the $SU(2)$-factor and as $\mathbb{C}_2$ for the $U(1)$-factor, and on $v \in V$ (identified as an element of $\mathbb{H}$) as $(e^{i\theta}, q).v = qve^{-i\theta}$.

An $SU(2)^2$-invariant adjoint-valued one-forms is determined by a $U(1)\times SU(2)$-equivariant map $V \to \mathfrak{g} \otimes (\mathfrak{p} \oplus V)^*$, and we find the subset of homogeneous polynomial maps in each of the cases of the proposition:

1. $P = P_{0,\mathrm{Id}}$: In this case, $U(1) \times SU(2)$ acts on $\mathfrak{g}$ as $\mathrm{im}\mathbb{H}$ for $SU(2)$, and trivially for $U(1)$.

   We first find homogeneous polynomial generators for $SU(2)^2$-invariant adjoint-valued one-forms on the singular orbit S^2, i.e. equivariant maps $L : V \to \mathfrak{g} \otimes \mathfrak{p}^*$, where the domain is identified with $\mathbb{H}$, and the image is understood as a space of real linear maps $\mathbb{H} \supset \mathbb{C} \to \mathrm{im}\mathbb{H}$. In this setting, $SU(2) \times U(1)$-equivariance is the requirement that the map evaluated at $v \in \mathbb{H}$ given by $L(v) : \mathbb{C} \to \mathrm{im}\mathbb{H}$ satisfies $L(qve^{-i\theta}).p = q\left(L(v).(e^{2i\theta}p)\right)\bar{q}$ where $p \in \mathbb{C}$, $(e^{i\theta}, q) \in U(1) \times SU(2)$.

   There are two such polynomial generators, both of degree two: $L_l(v).p = vlp\bar{v}$, for $l \in \{j, k\}$. Evaluating at $1 \in \mathbb{H}$ to the maps $L_l(1).p = lp$, and writing in terms of the explicit basis for $\mathfrak{g}$, $\mathfrak{p}$, these are $E_2 \otimes v^1 + E_3 \otimes w^1$ and $E_3 \otimes v^1 - E_2 \otimes w^1$ respectively.

   We also find homogeneous polynomial generators for $SU(2)^2$-invariant adjoint-valued one-forms on the fibre V, i.e. equivariant maps $L : V \to \mathfrak{g} \otimes V^*$, where the domain is identified with $\mathbb{H}$, and the image is understood as the space of real linear maps $\mathbb{H} \to \mathrm{im}\mathbb{H}$. Here, $SU(2) \times U(1)$-equivariance is the requirement that the map evaluated at $v \in \mathbb{H}$ given by $L(v) : \mathbb{H} \to \mathrm{im}\mathbb{H}$ satisfies $L(qve^{-i\theta}).p = q\left(L(v).(\bar{q}pe^{i\theta})\right)\bar{q}$ where $p \in \mathbb{H}$, $(e^{i\theta}, q) \in U(1) \times SU(2)$.

There are two polynomial generators of degree one, given by $L_l(v).p = pi\bar{l}\bar{v} + \langle p, vli\rangle$, where $l \in \{1, i\}$ and $\langle .,.\rangle$ denotes the Euclidian dot-product on $\mathbb{R}^4$. Evaluating at $1 \in \mathbb{H}$ to the maps $p \mapsto pi + \langle p, i\rangle$ and $p \mapsto \mathrm{Im}\,(p)$, and writing in terms of the explicit basis for $\mathfrak{g}$, V in co-ordinates (A.3), these are $E_1 \otimes dt + t\left(E_2 \otimes w^2 - E_3 \otimes v^2\right)$ and $t\left(-2E_1 \otimes u^- + E_2 \otimes v^2 + E_3 \otimes w^2\right)$ respectively.

There are also two polynomial generators of degree three, given by $L_l(v).p = \langle vl, p\rangle vi\bar{v}$ for $l \in \{1, i\}$. Evaluating at $1 \in \mathbb{H}$ to the map $p \mapsto \langle l, p\rangle i$, and writing in terms of the explicit basis for $\mathfrak{g}$, V in co-ordinates (A.3), these are $E_1 \otimes dt$ and $-2tE_1 \otimes u^-$ respectively.

Now we have obtained this splitting of the $\triangle U(1)$-invariant subspace of adjoint-valued one-forms, we can check the extension to the singular orbit of any $SU(2)^2$-invariant element of $\Omega^1\left(\mathrm{ad}P_{\mathrm{Id}}\right)$ defined on the principal orbits. To apply this result to the connection A in (A.9), we need to use the canonical connection to get a section of $\Omega^1\left(\mathrm{ad}P_{\mathrm{Id}}\right)$. In this case, the canonical connection is:

$$d\lambda = E_1 \otimes u^2 + E_2 \otimes v^2 + E_3 \otimes w^2$$

So writing $A - d\lambda$ in terms of our basis above:

$$\begin{aligned}
A - d\lambda = {} & a_1(E_2 \otimes v^1 + E_3 \otimes w^1) + b_1(E_3 \otimes v^1 - E_2 \otimes w^1) \\
& + \frac{a_2 - 1}{t}\left(t(E_2 \otimes v^2 + E_3 \otimes w^2) - 2tE_1 \otimes u^-\right) \\
& + \frac{b_2}{t}\left(t(E_3 \otimes v^2 - E_2 \otimes w^2) - E_1 \otimes dt\right) \\
& + \frac{(a_0 + 2a_2 - 1)}{t}\left(tE_1 \otimes u^-\right) + \frac{b_2}{t}E_1 \otimes dt
\end{aligned}$$

it follows we require a_1, a_2, a_0 b_1, b_2 even, $a_1(0) = b_1(0) = b_2(0) = 0$, $a_2(0) = -a_0(0) = 1$, and $a_0''(0) + 2a_2''(0) = b_2''(0) = 0$ to extend A to $t = 0$.

2. $P = P_{1,\mathbf{0}}$: In this case, $U(1) \times SU(2)$ acts on $\mathfrak{g} = \langle E_1\rangle \oplus \langle E_2, E_3\rangle$ as $\mathbb{R} \oplus \mathbb{C}_2$ for the $U(1)$-factor, and trivially for the $SU(2)$-factor.

We first find homogeneous polynomial generators for $SU(2)^2$-invariant adjoint-valued one-forms on the fibre V, i.e. $U(1) \times SU(2)$-equivariant maps $L : V \to \mathfrak{g} \otimes V^*$, where the domain is identified with $\mathbb{H}$, and the image is understood as the space of real linear maps $\mathbb{H} \to \mathrm{im}\mathbb{H}$. Here, $SU(2) \times U(1)$-equivariance is the requirement that the map evaluated at $v \in \mathbb{H}$ given by $L(v) : \mathbb{H} \to \mathrm{im}\mathbb{H}$ satisfies $L(qve^{-i\theta}).p = e^{in\theta}\left(L(v).(\bar{q}pe^{i\theta})\right)e^{-in\theta}$ where $p \in \mathbb{H}$, $(e^{i\theta}, q) \in U(1) \times SU(2)$.

There are two polynomial generators of degree one given by $L_l(v).p = \langle p, vl\rangle i$, where $l \in \{1, i\}$, and $\langle .,.\rangle$ denotes the Euclidian dot-product on $\mathbb{R}^4$. Evaluating at $1 \in \mathbb{H}$ to the maps $p \mapsto \langle p, l\rangle i$ and writing in terms of the explicit basis for $\mathfrak{g}$, V in co-ordinates (A.3), these are $E_1 \otimes dt$ and $-2tE_1 \otimes u^-$ respectively.

Finally, there are an additional two polynomial generators of degree one, given by $L_l(v).p = \langle p, vli\rangle + i\bar{l}\bar{v}p$ for $l \in \{1, i\}$. Evaluating at $1 \in \mathbb{H}$ to the maps $p \mapsto \langle p, i\rangle + ip$, $p \mapsto \mathrm{Im}\,(p)$, and writing in terms of the explicit basis for $\mathfrak{g}$, V in co-ordinates (A.3), these are $E_1 \otimes dt + t\left(E_3 \otimes v^2 - E_2 \otimes w^2\right)$ and $t\left(E_2 \otimes v^2 + E_3 \otimes w^2 - 2E_1 \otimes u^-\right)$ respectively.

Next, we find homogeneous polynomial generators for $SU(2)^2$-invariant adjoint-valued one-forms on the singular orbit, i.e. equivariant maps $L : V \to \mathfrak{g} \otimes \mathfrak{p}^*$, where the domain is identified with $\mathbb{H}$, and the image is understood as the space of real linear maps $\mathbb{H} \supset \mathbb{C} \to \mathrm{im}\mathbb{H}$. Here, $SU(2) \times U(1)$-equivariance is the requirement that the map evaluated at $v \in \mathbb{H}$ given by $L(v) : \mathbb{C} \to \mathrm{im}\mathbb{H}$ satisfies $L(qve^{-i\theta}).p = e^{i\theta}\left(L(v).(pe^{-2i\theta})\right)e^{-i\theta}$ where $p \in \mathbb{C}$, $(e^{i\theta}, q) \in U(1) \times SU(2)$.

There are two polynomial generators of degree zero, the $U(1)\times SU(2)$ invariant maps $L_l(v).p = pl$, where $l \in \{j, k\}$. Writing these in terms of the explicit basis for $\mathfrak{g}$, $\mathfrak{p}$, these are $E_2 \otimes v^1 + E_3 \otimes w^1$ and $E_3 \otimes v^1 - E_2 \otimes w^1$ respectively.

Given our description of the $\triangle U(1)$-invariant subspace of adjoint-valued one-forms, we can now check the extension of any $SU(2)^2$-invariant section of $\Omega^1\left(\mathrm{ad}P_{1,\mathbf{0}}\right)$ defined on the principal orbits to the zero section, since any such section must be a linear combination of these Lie algebra-valued one-forms with associated degree.

To apply this result to the connection A in (A.9), we need to use the canonical connection to get adjoint-valued one-form. In this case, the canonical connection is:

$$d\lambda = nE_1 \otimes u^1$$

so writing $A - d\lambda$ in terms of our basis above:

$$\begin{aligned}
A - d\lambda &= a_1(E_2 \otimes v^1 + E_3 \otimes w^1) + b_1(E_3 \otimes v^1 - E_2 \otimes w^1) \\
&\quad + \frac{a_2}{t}\left((tE_2 \otimes v^2 + E_3 \otimes w^2) - 2tE_1 \otimes u^-\right) \\
&\quad + \frac{b_2}{t}\left(t(E_3 \otimes v^2 - E_2 \otimes w^2) + E_1 \otimes dt\right) \\
&\quad + \frac{a_0 + 2a_2 - 1}{t}\left(tE_1 \otimes u^-\right) - \frac{b_2}{t}E_1 \otimes dt
\end{aligned}$$

and so we require a_1, b_1, a_2, b_2, a_0 even, $a_2(0) = b_2(0) = 0$, and $a_0(0) = 1$ as claimed.

□

Proof of A.2.9. Write, V, $\mathfrak{g}$ as in the previous proof. The space of $SU(2)^2$-invariant sections of $\mathrm{ad}P$ is identified with the space of $U(1) \times SU(2)$-equivariant maps $L : V \to \mathfrak{g}$, and homogeneous generators for the polynomial elements gives extension conditions for $SU(2)^2$-invariant sections restricted to the principal orbits.

1. $P = P_{0,\mathrm{Id}}$: in this case, a map $L : \mathbb{H} \to \mathrm{im}\mathbb{H}$ is $U(1) \times SU(2)$-equivariant if $L(qve^{-i\theta}) = qL(v)\bar{q}$, where $v \in \mathbb{H}$, $(e^{i\theta}, q) \in U(1) \times SU(2)$. This has a single degree two homogeneous polynomial generator $L(v) = vi\bar{v}$, which evaluates at $1 \in \mathbb{H}$ to $i \in \mathrm{im}\mathbb{H}$ corresponding to the element $E_1 \in \mathfrak{g}$. Hence, the invariant section $\Phi = \phi E_1$ must have $\phi(0) = 0$, and ϕ even.

2. $P = P_{1,\mathbf{0}}$: in this case, a map $L : \mathbb{H} \to \mathrm{im}\mathbb{H}$ is $U(1) \times SU(2)$-equivariant if $L(qve^{-i\theta}) = e^{i\theta}L(v)e^{-i\theta}$, where $v \in \mathbb{H}$, $(e^{i\theta}, q) \in U(1) \times SU(2)$. This has a single degree zero homogeneous polynomial generator given by the constant map i, corresponding to the element $E_1 \in \mathfrak{g}$. Hence, the invariant section $\Phi = \phi E_1$ must have ϕ even.

□

A.3 Bundle Data: Eguchi-Hanson

In this section of the appendix, we consider extending $SU(2)$-invariant connections on $T^*\mathbb{CP}^1$, equipped with the Eguchi-Hanson metric (1.5), smoothly over the singular orbit $\mathbb{CP}^1$ via the method of [EW00].

Recall Example 1.1.3, if we identify the underlying space of $T^*\mathbb{CP}^1$ with $T^*\mathbb{CP}^1 = SU(2) \times_{U(1)} \mathbb{C}_2$, where $\mathbb{C}_2$ is the irreducible $U(1)$-representation with weight two, then the obvious $SU(2)$-action on $T^*\mathbb{CP}^1$ has the group diagram $\mathbb{Z}_2 \subset U(1) \subset SU(2)$.

There is a family of $SU(2)$-invariant $SU(2)$-bundles on $T^*\mathbb{CP}^1$ defined by the homomorphism $\lambda^l : U(1) \to SU(2)$ given by taking l'th power of the diagonal embedding $\lambda : U(1) \hookrightarrow SU(2)$ for some $l \in \mathbb{Z}_{>0}$. Recall that on these bundles, any smooth $SU(2)$-invariant connection over the space of principal orbits can be put into the diagonal form:

$$A = \alpha_1 E_1 \otimes e_1 + \alpha_2 E_2 \otimes e_2 + \alpha_3 E_3 \otimes e_3 \tag{A.13}$$

for some $\alpha_1, \alpha_2, \alpha_3$ smooth functions of the geodesic distance $t \in \mathbb{R}_{>0}$ from the singular orbit at $t = 0$.

To ensure this connection extends smoothly to the singular orbit at $t = 0$, we compute:

Proposition A.3.1. (A.13) *extends smoothly over the singular orbit* $\mathbb{CP}^1 = SU(2)/U(1)$ *on the invariant bundle defined by* λ^l *if and only if* α_1 *is even,* $\alpha_1(0) = l$*,* $\alpha_2 + \alpha_3$ *has degree* $l - 1$*, and* $\alpha_2 - \alpha_3$ *has degree* $l + 1$*.*

Here, we refer to a function $\alpha(t)$ defined on some open neighbourhood of $0 \in [0, \infty)$ as having *degree* $d \geq 0$ if $\alpha(t)t^{-d}$ is a smooth, even function of t in this neighbourhood.

Proof. First, we note that the singular isotropy subgroup $U(1)$ acts on the tangent space of $T^*\mathbb{CP}^1$ over the identity coset in the base $\mathbb{CP}^1 = SU(2)/U(1)$ as two copies of $\mathbb{C}_2$: one spanned by left-invariant vector-fields E_2, E_3 on $\mathbb{CP}^1$, and one spanned by the vector-fields in the fibre $\mathbb{C}_2$. Moreover for each $l \in \mathbb{Z}$, $U(1)$ acts on the Lie algebra of the gauge group as $\mathbb{R} \oplus \mathbb{C}_{2l}$, where the trivial representation is spanned by E_1, and the weight $2l$-representation is spanned by E_2, E_3.

By [EW00], an $SU(2)$-invariant Lie-algebra valued one-form extends smoothly over the singular orbit if it is in the image of the evaluation map at some non-zero $z \in \mathbb{C}_2$, on the space of $U(1)$-equivariant homogoneous polynomial maps $\mathbb{C}_2 \mapsto (\mathbb{R} \oplus \mathbb{C}_2) \otimes_{\mathbb{R}} (\mathbb{C}_2 \oplus \mathbb{C}_2)^*$ of degree d, with coefficients given by smooth functions of degree d.

To understand this in more detail, we shall see an example: we can construct two $U(1)$-equivariant degree-one homogeneous polynomials Re_z, Im_z taking $z \in \mathbb{C}_2$ to $\mathbb{R}$-linear maps $\mathrm{Re}_z, \mathrm{Im}_z : \mathbb{C}_2 \to \mathbb{R}$ defined by $\mathrm{Re}_z(w) := \mathrm{Re}(\bar{z}w)$, $\mathrm{Im}_z(w) := \mathrm{Im}(\bar{z}w)$ for $w \in \mathbb{C}_2$.

Evaluating Re_z, Im_z at $z = 1$ gives the corresponding $SU(2)$-invariant adjoint-valued one-forms $E_1 \otimes e_2$, $E_1 \otimes e_3$, $E_1 \otimes dx_0$, and $E_1 \otimes dx_1$, where dx_0, dx_1 denote the Cartesian coordinate one-forms on the fibre $\mathbb{C}_2$.

We can repeat a similar construction for equivariant homogeneous polynomial maps $p_z : \mathbb{C}_2 \mapsto \mathbb{C}_2 \otimes_{\mathbb{R}} (\mathbb{C}_2 \oplus \mathbb{C}_2)^*$. By evaluating at $z = 1$ and taking real and imaginary parts, we get the following table:

degree	polynomial p_z	evaluation at $z = 1$
1	$\bar{z} E_1 \otimes (e_2 + ie_3)$	$E_1 \otimes e_2, E_1 \otimes e_3$
$l - 1$	$z^{l-1} (E_2 + iE_3) \otimes \left(e^2 - ie^3\right)$	$E_2 \otimes e^2 + E_3 \otimes e^3, -E_2 \otimes e^3 + E_3 \otimes e^2$
$l + 1$	$z^{l+1} (E_2 + iE_3) \otimes \left(e^2 + ie^3\right)$	$E_2 \otimes e^2 - E_3 \otimes e^3, E_2 \otimes e^3 - E_3 \otimes e^2$

with a similar table replacing e_2, e_3 with the Cartesian coordinate one-forms dx_0, dx_1 on the fibre. Note that by switching to polar coordinates in $\mathbb{C}_2$, we can identify $dx^0 = dt$ and $dx^1 = 2te_1$ along the ray $\gamma(t) = t \in \mathbb{C}_2$.

Now, to apply [EW00] to the connection (A.13), we can use the canonical connection $d\lambda^l = lE_1 \otimes e^1$ on this bundle as a reference connection, to get the adjoint-valued one form $A - d\lambda^l$, which extends over $t = 0$ if and only if A extends. Along $\gamma(t)$, we can write $A - d\lambda^l$ in terms of our basis:

$$\begin{aligned} A - d\lambda^l = \tfrac{\alpha_1 - l}{2t} (2tE_1 \otimes e_1) + \tfrac{1}{2}(\alpha_2 + \alpha_3) (E_2 \otimes e_2 + E_3 \otimes e_3) \\ + \tfrac{1}{2}(\alpha_2 - \alpha_3) (E_2 \otimes e_2 - E_3 \otimes e_3) \end{aligned}$$

So, for A to extend smoothly over the singular orbit at $t = 0$, we require $\alpha_1(0) = l$, α_1 to be even, $\alpha_2 + \alpha_3$ to be degree $l - 1$, and $\alpha_2 - \alpha_3$ to be degree $l + 1$. $\square$

Appendix B

Singular Initial Value Problems

For use in the following proof, we note that the Calabi-Yau structure on T^*S^3 is given by (1.27), and we compute the power-series near $t = 0$ of the following expressions:

$$\frac{4\lambda}{\mu} = \frac{2}{t} + O(t) \qquad \frac{3(v_3 - v_0)}{2\lambda\mu} = \frac{1}{t} + O(t) \qquad \frac{3(v_3 + v_0)}{2\lambda\mu} = \frac{9}{4}t + O(t^3)$$
$$\frac{3v_0}{\mu^2} = -\frac{1}{2t^2} + \frac{3}{2} + O(t^2) \qquad \frac{3v_3}{\mu^2} = \frac{1}{2t^2} + \frac{3}{4} + O(t^2)$$

Proof of Proposition 3.1.3. Let (a_0, a_1, a_2, ϕ) be a solution to (2.8) on T^*S^3. Using Prop. A.2.5, define smooth functions a_-, A_+, ψ such that $a_1 - a_2 = a_-$, $a_1 + a_2 = 1 + t^2 A_+$, $\phi = t\psi$. Then $y(t) = (a_0, \psi, A_+, a_-)$ must satisfy a singular initial value problem with linearisation:

$$d_{y_0}M_{-1} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & \frac{9}{4}\xi \\ \frac{9}{4}\xi & -2 & -2 & \frac{9}{4}\xi \\ 2 & 0 & 0 & -2 \end{pmatrix}$$

at initial value $y_0 = \left(\xi, \frac{9}{8}(\xi^2 - 1) - \chi, \xi, \chi\right)$ for some $\xi, \chi \in \mathbb{R}$. This initial value problem has a unique solution once we fix y_0, since $\det(k\mathrm{Id} - d_{y_0}M_{-1}) = (k+3)^2 k^2$. $\square$

For use in the following proofs, we note that the Calabi-Yau structure on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is given by (1.26) with $U_1 = -U_0 = -u_0 = 1$, and the power-series of λ, u_1, μ near $t = 0$ satisfy:

$$\lambda(t) = \frac{3}{2}t + O(t^3) \qquad u_1 = 1 + \frac{3}{2}t^2 + O(t^4) \qquad \mu = \sqrt{3}t + O(t^3)$$

Proof of Propositions 3.1.6. Let (a_0, a_1, a_2, ϕ) be a solution to (2.8) on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Using Prop. A.2.7 for extending on $P_{1,\mathbf{0}}$, $y(t) = (a_0, a_1, a_2, \phi)$ satisfies a singular initial value problem with linearisation:

$$d_{y_0}M_{-1} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{3}\epsilon' \\ \epsilon' & 0 & 0 & -\frac{4}{\sqrt{3}}\delta' \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

at initial value $y_0 = (1, \epsilon', 0, \delta')$ for some $\epsilon', \delta' \in \mathbb{R}$. This initial value problem has a unique solution once we fix y_0, since $\det(k\mathrm{Id} - d_{y_0}M_{-1}) = (k+2)^2\, k^2$. □

Proof of Proposition 3.1.5. Let (a_0, a_1, a_2, ϕ) be a solution to (2.8) on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Using Prop. A.2.5 for extending on $P_{0,\mathrm{Id}}$, we define smooth functions X_0, X_1, X_2, ψ such that $a_0 = -1 + t^2 X_0$, $a_1 = t^2 X_1$, $a_2 = 1 + t^2 X_2$, and $\phi = t^2 \psi$. Then $y(t) = (X_0, X_1, X_2, \psi)$ satisfies a singular initial value problem with linearisation:

$$d_{y_0}M_{-1} = \begin{pmatrix} -4 & 0 & 0 & -8 \\ 0 & -2 & -2\sqrt{3} & 0 \\ 0 & -\frac{4}{\sqrt{3}} & -4 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix}$$

at initial value $y_0 = \left(\epsilon, -\frac{1}{\sqrt{3}}\delta, -\frac{1}{2}\epsilon, \delta\right)$. This initial value problem has a unique solution once we fix y_0, since $\det(k\mathrm{Id} - d_{y_0}M_{-1}) = (k+6)^2\, k^2$. □

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