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A linear logic framework for multimodal logics

Bruno Xavier[1], Carlos Olarte[2] and Elaine Pimentel[3*]

[1]DIMAp, Universidade Federal do Rio Grande do Norte, Natal, Brazil, [2]LIPN, CNRS UMR 7030, Université Sorbonne Paris Nord, Villetaneuse, France and [3]Computer Science Department, University College London, London WC1E 6BT, UK
*Corresponding author. Email: e.pimentel@ucl.ac.uk

Abstract

One of the most fundamental properties of a proof system is *analyticity*, expressing the fact that a proof of a given formula F only uses subformulas of F. In sequent calculus, this property is usually proved by showing that the cut rule is admissible, i.e., the introduction of the auxiliary lemma H in the reasoning "if H follows from G and F follows from H, then F follows from G" can be eliminated. The proof of cut admissibility is usually a tedious, error-prone process through several proof transformations, thus requiring the assistance of (semi-)automatic procedures. In a previous work by Miller and Pimentel, linear logic (LL) was used as a logical framework for establishing sufficient conditions for cut admissibility of object logical systems (OL). The OL's inference rules are specified as an LL theory and an easy-to-verify criterion sufficed to establish the cut-admissibility theorem for the OL at hand. However, there are many logical systems that cannot be adequately encoded in LL, the most symptomatic cases being sequent systems for modal logics. In this paper, we use a linear-nested sequent (LNS) presentation of MMLL (a variant of LL with subexponentials), and show that it is possible to establish a cut-admissibility criterion for LNS systems for (classical or substructural) multimodal logics. We show that the same approach is suitable for handling the LNS system for intuitionistic logic.

Keywords: Linear logic; cut elimination; multimodal logics; linear-nested systems

1. Introduction

Proof systems are frameworks for formalizing reasoning, where inference rules dictate how theorems can be derived from the given hypotheses. Since proofs themselves are mathematical objects of study, proof systems can also be used as (meta-level) tools for analyzing the structural properties of proofs. One of the most important of such properties is *analyticity*.

Analytic calculi consist solely of rules that decompose the formulas to be proved in a stepwise manner. As a result, proofs from an analytic calculus satisfy the subformula property: every formula that appears (anywhere) in the proof must be a subformula of the formulas to be proved. This is a powerful restriction on the shape of proofs, and it can be exploited to prove important meta-logical properties of logical systems such as consistency, decidability, and interpolation.

The best-known formalism for proposing analytic proof systems is Gentzen's *sequent calculus* (Gentzen 1969), which manipulates consequence relations between contexts of formulas, represented by $\Gamma \vdash \Delta$. In sequent calculus systems, analyticity is often guaranteed by proving a

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property called *cut admissibility*: if $\Gamma_1 \vdash \Delta_1, H$ is provable and $H, \Gamma_2 \vdash \Delta_2$ is provable, then so is $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$. In other words, one needs to prove that the cut rule below can be eliminated.

$$\frac{\Gamma_1 \vdash \Delta_1, H \quad H, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ cut}$$

Note the inherent *duality*: the cut formula H is both a conclusion of a statement and a hypothesis of another. In analytic systems, this duality is often an *invariant*, being preserved throughout the cut-elimination process. Developing general methods for detecting such invariants enables the use of meta-level frameworks to uniformly reasoning about object-level properties.

Unfortunately, sequent systems are not expressive enough for constructing analytic calculi for many logics of interest. The critical examples are modal logics: sequents appear to be too coarse for capturing the subtle behavior of modalities. As a result, many new formalisms extending sequent systems have been proposed over the last 30 years, including *hypersequent calculi* (Avron 1996), *nested calculi* (Brünnler 2009; Bull 1992; Kashima 1994; Poggiolesi 2009) and *labeled calculi* (Simpson 1994; Viganò 2000).

In this work, we study cut admissibility under the *linear nested system* formalism (LNS for short, Lellmann 2015), where a single sequent is replaced by a list of sequents, and the inference rules govern the transfer of formulas between the different sequents. LNS form a particular subclass of nested systems, where the tree structure is restricted to a line.

While such a more expressive formalism enables calculi for a broader class of logics, the greater bureaucracy makes it harder to prove logical properties, such as analyticity itself (Bruscoli and Guglielmi 2006). We hence lift to LNS the method developed by Miller and Pimentel (2013) and revisited by Felty et al. (2021). The idea is to use a meta-level framework in order to specify and reason about, in a uniform way, different logical (object-level) proof systems. Such a framework should be powerful enough so as to be able to handle a great amount of systems. In Miller and Pimentel (2013), the meta-logical framework was linear logic (LL Girard 1987). The advantage of having LL as a framework is that it is a relatively simple, resource aware system, where steps of computation can be naturally specified. The problem is that simplicity often comes with a price, and LL is not suitable for reasoning about modalities other than its own. In the present work, we adopt MMLL, a multimodal extension of LL, as the meta-level framework. Object-level logical systems (OL for short) are specified as MMLL theories, where OL rules are specified as MMLL clauses. Such clauses have a very special shape: they are *bipoles*, formulas that can be totally decomposed in one focused step (Andreoli 1992). In this way, the *focusing proof strategy* enforces a one-to-one correspondence between a OL-rule application and the decomposition of the respective MMLL formula in its focused system LNS$_{\text{FMLL}}$. This is called *adequacy*, and it completely ties OL-{formulas, rules, derivations} to MMLL-{formulas, bipoles, focused derivations}. We thus smoothly generalize the cut-admissibility criterion given in Miller and Pimentel (2013) to the linear nested setting, capturing at the meta-level the duality invariant mentioned above for the specified logical systems. In this work, we will apply this method for reasoning about classical/substructural normal multimodal logical systems, as well as the multiconclusion system for intuitionistic logic.

This paper extends our previous work (Olarte et al. 2020) in the following ways. (1) The meta-level frameworks adopted here and in Olarte et al. (2020) are based on LL with multimodalities (called *subexponentials*). Different from *op. cit.*, MMLL considers both unbounded (classical) and bounded (linear) subexponentials, depending if they assume or not the axioms of weakening and contraction. Hence, in this paper, we can also reason about (multimodal) substructural systems; (2) In Olarte et al. (2020), both the object- and meta-level logical systems had the sequent system presentation. In this work, all systems are represented as LNS, so we can reason about a broader class of OL (including multi-modal ones); (3) In the focused system proposed here (LNS$_{\text{FMLL}}$), the modal rules are completely deterministic, improving the proof search procedure; (4) The proof of cut admissibility of the meta-logic is done directly in LNS$_{\text{FMLL}}$, without the need of translating this system to the unfocused version and proving the completeness of focusing; (5) We provide

a companion repository available at Xavier and Olarte (2021) with the Coq formalization of the cut-admissibility theorem for $\mathsf{LNS_{FMLL}}$, as well as adequacy and the application of the cut-coherent criterion for some specified OL.

Organization and contributions. We start Section 2 by recalling linear nested systems, and presenting LNS systems for propositional classical and intuitionistic logics, as well as for modal logics that are extensions of the basic modal logic K with axioms from the set {T, 4, D}. In Section 3, we show how to lift the LNS formalism to LL with subexponentials, considering not only the structural axioms for contraction and weakening, but also the subexponential version of modal axioms (system MMLL). Section 3.1 presents $\mathsf{LNS_{FMLL}}$, a focused linear nested system for MMLL, which will be the adopted logical framework for specifying and characterizing OL cut-admissibility. In Section 3.2, we prove that $\mathsf{LNS_{FMLL}}$ is itself analytic, with the proof formalized in Coq. As far as we know, this is the first formalization of cut admissibility of a LNS-focused system. Section 4 shows how object logics can be specified as MMLL theories. The encoding is natural and direct, and the proof of adequacy is immediate due to the focusing discipline. Different from Olarte et al. (2020), such encoding is general enough to consider multimodal substructural logics. Finally, the criteria for establishing OL cut admissibility are presented in Section 5. We show that such criteria can be easily checked. Section 6 concludes the paper.

2. Linear Nested Systems

In this section, we present a brief introduction to linear nested systems. For further details, please refer to Lellmann (2015), Lellmann and Pimentel (2019).

One of the main problems of using sequent systems as a framework is that sequents are often not adequate for expressing modal behaviors. Indeed, the usual introduction rule for the box modality in the modal logic K

$$\frac{G_1, \ldots, G_n \vdash F}{\Box G_1, \ldots, \Box G_n \vdash \Box F} \; \mathsf{k}$$

is somehow unsatisfactory: it modifies the context (by adding boxes to the hypothesis), and the distinction between left and right rules for the modal connective box is lost. In order to propose a better formulation, we need a tighter control of formulas in the context, something that sequents do not provide. Hence, the need for extending the notion of sequent systems.

Definition 1. *Let $\mathscr{L}$ be a formal language consisting of well-formed formulas. A* sequent *is an expression of the form $\Gamma \vdash \Delta$ where Γ (the* antecedent*) and Δ (the* succedent*) are finite multisets of formulas in $\mathscr{L}$, and $\vdash$ is the* consequence *symbol.*

The set LNS *of* linear nested sequents *is given recursively by*

   (i) if $\Gamma \vdash \Delta$ is a sequent then $\Gamma \vdash \Delta \in$ LNS
   (ii) if $\Gamma \vdash \Delta$ is a sequent and $\mathscr{G} \in$ LNS then $\mathscr{G} /\!/ \Gamma \vdash \Delta \in$ LNS.

We call each sequent in a linear nested sequent a component *and slightly abuse notation, abbreviating "linear nested sequent" to* LNS. *We shall denote by* $\mathsf{LNS}_{\mathscr{L}}$ *a linear nested sequent system for the logic $\mathscr{L}$.*

In words, a linear nested sequent is simply a finite list of sequents. This data structure matches exactly that of a *history* in a backwards proof search in an ordinary sequent calculus (Lellmann 2015). The local behavior of modalities in the rule k is obtained as follows:

$$\frac{\mathscr{G} /\!/ \Gamma \vdash \Delta /\!/ \cdot \vdash F}{\mathscr{G} /\!/ \Gamma \vdash \Delta, \Box F} \; \Box_R \qquad \frac{\mathscr{G} /\!/ \Gamma \vdash \Delta /\!/ \Gamma', F \vdash \Delta'}{\mathscr{G} /\!/ \Gamma, \Box F \vdash \Delta /\!/ \Gamma' \vdash \Delta'} \; \Box_L$$

$$\frac{}{\mathscr{G}\,/\!/\,\Gamma, A \vdash A, \Delta}\ \mathsf{init} \qquad \frac{}{\mathscr{G}\,/\!/\,\Gamma, \mathtt{f} \vdash \Delta}\ \mathtt{f}_L \qquad \frac{}{\mathscr{G}\,/\!/\,\Gamma \vdash \mathtt{t}, \Delta}\ \mathtt{t}_R$$

$$\frac{\mathscr{G}\,/\!/\,\Gamma, F \vdash \Delta \quad \mathscr{G}\,/\!/\,\Gamma, G \vdash \Delta}{\mathscr{G}\,/\!/\,\Gamma, F \vee G \vdash \Delta}\ \vee_L \qquad \frac{\mathscr{G}\,/\!/\,\Gamma \vdash F, \Delta}{\mathscr{G}\,/\!/\,\Gamma \vdash F \vee G, \Delta}\ \vee_{R1} \qquad \frac{\mathscr{G}\,/\!/\,\Gamma \vdash G, \Delta}{\mathscr{G}\,/\!/\,\Gamma \vdash F \vee G, \Delta}\ \vee_{R2} \qquad \frac{\mathscr{G}\,/\!/\,\Gamma \vdash F, \Delta \quad \mathscr{G}\,/\!/\,\Gamma \vdash G, \Delta}{\mathscr{G}\,/\!/\,\Gamma \vdash F \wedge G, \Delta}\ \wedge_R$$

$$\frac{\mathscr{G}\,/\!/\,\Gamma, F \vdash \Delta}{\mathscr{G}\,/\!/\,\Gamma, F \wedge G \vdash \Delta}\ \wedge_{L1} \qquad \frac{\mathscr{G}\,/\!/\,\Gamma, G \vdash \Delta}{\mathscr{G}\,/\!/\,\Gamma, F \wedge G \vdash \Delta}\ \wedge_{L2} \qquad \frac{\mathscr{G}\,/\!/\,\Gamma_1 \vdash F, \Delta_1 \quad \mathscr{G}\,/\!/\,\Gamma_2, G \vdash \Delta_2}{\mathscr{G}\,/\!/\,\Gamma_1, \Gamma_2, F \to G \vdash \Delta_1, \Delta_2}\ \to_L \qquad \frac{\mathscr{G}\,/\!/\,\Gamma, F \vdash G, \Delta}{\mathscr{G}\,/\!/\,\Gamma \vdash F \to G, \Delta}\ \to_R$$

Figure 1. System LNS$_G$ for classical logic.

$$\frac{\mathscr{G}\,/\!/\,\Gamma, F, F \vdash \Delta}{\mathscr{G}\,/\!/\,\Gamma, F \vdash \Delta}\ \mathsf{C}_L \qquad \frac{\mathscr{G}\,/\!/\,\Gamma \vdash F, F, \Delta}{\mathscr{G}\,/\!/\,\Gamma \vdash F, \Delta}\ \mathsf{C}_R \qquad \frac{\mathscr{G}\,/\!/\,\Gamma \vdash \Delta}{\mathscr{G}\,/\!/\,\Gamma, F \vdash \Delta}\ \mathsf{W}_L \qquad \frac{\mathscr{G}\,/\!/\,\Gamma \vdash \Delta}{\mathscr{G}\,/\!/\,\Gamma \vdash F, \Delta}\ \mathsf{W}_R$$

Figure 2. The structural rules of contraction and weakening.

$$\frac{\Gamma \vdash \Delta \,/\!\!/\, \Sigma, F \vdash \Pi}{\Gamma, F \vdash \Delta \,/\!\!/\, \Sigma \vdash \Pi}\ \mathtt{lift} \qquad \frac{\Gamma \vdash \Delta \,/\!\!/\, F \vdash G}{\mathscr{G}\,/\!/\,\Gamma \vdash F \supset G, \Delta}\ \supset_R \qquad \frac{\mathscr{G}\,/\!/\,\Gamma_1 \vdash F, \Delta_1 \quad \mathscr{G}\,/\!/\,\Gamma_2, G \vdash \Delta_2}{\mathscr{G}\,/\!/\,\Gamma_1, \Gamma_2, F \supset G \vdash \Delta_1, \Delta_2}\ \supset_L$$

Figure 3. Rules for intuitionistic implication in the system LNS$_I$.

Reading bottom up, while in $\Box_R$ a new nesting/component is created and F is moved in it, in $\Box_L$ *exactly one* boxed formula is moved into an existing nesting, losing its modality.

We will explore the local/linear structure of LNS in two ways. First, components have a tight connection to *worlds* in Kripke-like semantics, so that LNS is an adequate framework for describing alethic modalities in logical systems driven by this kind of semantics. Second, since information is fragmented into components, rules act locally on formulas and are usually context independent. Hence, the movement of formulas on derivations can be better predicted and controlled. This implies that both: we will be able to adequately specify a representative class of logical systems (see Section 4); and the techniques developed in Miller and Pimentel (2013) will remain valid in the proposed framework.

A further advantage of this framework is that it is often possible to restrict the list of sequents in a LNS to the last two components, that we call *active*.

Definition 2. (End-active). *An application of a linear nested sequent rule is* end-active *if the rightmost components of the premises are active and the only active components (in premises and conclusion) are the two rightmost ones. The* end-active variant *of a* LNS *calculus is the calculus with the rules restricted to end-active applications.*

All the logical systems studied in Lellmann (2015), Lellmann et al. (2017), Lellmann and Pimentel (2019) can be restricted to the end-active version. This means that, for example, when restricted to classical logic, new components are never created (this reflects the fact that the Kripke structure for classical logic is flat). Hence, the LNS collapses to the usual sequent system LK (Gentzen 1969). Figure 1 presents the (end-active) LNS rules for classical propositional logic where the history "$\mathscr{G}\,/\!/$" is always empty. Figure 2 presents the structural rules of weakening and contraction.

A more interesting case is the linear-nested system for propositional intuitionistic logic. LNS$_I$ (Lellmann 2015) is the system sharing with LNS$_G$ the axioms, structural rules , and rules for conjunction and disjunction, but adding the rules for intuitionistic implication $\supset$ shown in Figure 3. Observe that, bottom-up, the rule for implication right creates a new component, adds the sequent $F \vdash G$ there, and erases the back history. The lift rule, on the other hand, moves left formulas into the next component. The consecutive application of these rules mimics, possibly in many steps, the behavior of the sequent right rule for implication in the multiconclusion

Axioms: $\mathsf{K}: \Box(F \to G) \to (\Box F \to \Box G)$ $\mathsf{D}: \neg(\Box F \wedge \Box\neg F)$ $\mathsf{T}: \Box F \to F$ $\mathsf{4}: \Box F \to \Box\Box F$

$$\dfrac{\Gamma \vdash \Delta /\!\!/ \Sigma, F \vdash \Pi}{\Gamma, \Box F \vdash \Delta /\!\!/ \Sigma \vdash \Pi} \;\Box_L \quad \dfrac{\Gamma \vdash \Delta /\!\!/ \cdot \vdash F}{\mathscr{G} /\!\!/ \Gamma \vdash \Delta, \Box F} \;\Box_R \quad \dfrac{\Gamma \vdash \Delta /\!\!/ \cdot \vdash \cdot}{\mathscr{G} /\!\!/ \Gamma \vdash \Delta} \;\mathsf{d} \quad \dfrac{\mathscr{G} /\!\!/ \Gamma, F \vdash \Delta}{\mathscr{G} /\!\!/ \Gamma, \Box F \vdash \Delta} \;\mathsf{t} \quad \dfrac{\Gamma \vdash \Delta /\!\!/ \Sigma, \Box F \vdash \Pi}{\Gamma, \Box F \vdash \Delta /\!\!/ \Sigma \vdash \Pi} \;\mathsf{4}$$

Figure 4. Some modal axioms and their linear-nested sequent rules.

intuitionistic sequent system mLJ (Maehara 1954)

$$\dfrac{\dfrac{\Gamma, F \vdash G}{\Gamma \vdash \Delta, F \supset G} \;\supset_R}{\vdots \; \mathscr{G}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\cdot \vdash \Delta /\!\!/ \Gamma, F \vdash G}{\Gamma \vdash \Delta /\!\!/ F \vdash G} \;\mathtt{lift}}{\mathscr{G} /\!\!/ \Gamma \vdash \Delta, F \supset G} \;\supset_R}{}$$

where the double line indicates multiple applications of the `lift` rule.

This also interprets, proof theoretically, the definition of satisfaction for intuitionistic logic (see Pimentel 2018). Observe that, once all formulas in the left context are lifted, the only possible action is the application of rules in the last (right-most) component. Hence, the right context Δ is *forgotten*. This shows an interesting dynamic in end-active systems: apply first rules that do not involve *moving-between* or *creating-new* components; then apply a rule that creates a new component, followed by the lift rules as much as possible; finally, come back to the beginning of the cycle, with local rules applied to the (active) last component.

The possibility of having such a notion of "proof normalization" was studied in Pimentel et al. (2019) in the nested systems framework. In that work, it was shown that end-active nested systems with very specific rules' shape can be *sequentialized*. This implies that such nested systems correspond to well-known sequent systems. In this work, we will use this result in a very pragmatic way. Namely, since some sequent systems are not adequate for specification and reasoning, we will consider the corresponding (end-active) LNS that: have the same logical properties as the original systems; can be easily specified in our framework; and entails easy-to-check meta-level conditions for cut admissibility.

In the present work, besides reasoning about intuitionistic and classical logics, we shall also reason about linear nested systems for some notable extensions of the normal modal logic K. Figure 4 presents some modal axioms and the respective linear nested rules.[1] The calculus $\mathsf{LNS_K}$ contains the rules of $\mathsf{LNS_G}$ together with the rules $\Box_R$ and $\Box_L$. Let $\mathscr{A} = \{\mathsf{T}, \mathsf{4}, \mathsf{D}\}$. Extensions of the logic K are represented by $\mathsf{K}\ell$, where $\ell \subseteq \mathscr{A}$. As usual, we write $\mathsf{S4} = \mathsf{KT4}$.

Modalities can be combined, giving rise to multimodal logics.

Definition 3. *Simply dependent multimodal logics (Demri 2000) are characterized by a triple* $(N, \preccurlyeq, f)$, *where N is a denumerable set, $(N, \preccurlyeq)$ is a partial order, and f is a mapping from N to the set $\mathfrak{L}$ of extensions of modal logic K with axioms from the set $\mathscr{A}$. The order $\preccurlyeq$ is upwardly closed with respect to f, that is, if $i \preccurlyeq j$ then $f(i) \subseteq f(j)$. The logic described by $(N, \preccurlyeq, f)$ has modalities $\Box^i$ for every $i \in N$, with axioms for the modality i given by the logic $f(i)$ and interaction axioms $\Box^j F \to \Box^i F$ for every $i, j \in N$ with $i \preccurlyeq j$.*

The linear nested system $\mathsf{LNS}_{(N, \preccurlyeq, f)}$ for the simply dependent multimodal logic given by the description $(N, \preccurlyeq, f)$ is given in Lellmann and Pimentel (2019, Figure 6).

3. LL and its Variants

Classical LL (Girard 1987) is a resource conscious logic, in the sense that formulas are consumed when used during proofs, unless marked with the exponential ? (whose dual is !). Formulas

marked with ? behave *classically*, i.e., they can be contracted and weakened during proofs. LL connectives include the additive conjunction & and disjunction ⊕ and their multiplicative versions ⊗ and ⅋, together with their units and the first-order quantifiers:

$$F, G, \ldots \ ::= \quad A \ \mid \ F \otimes G \ \mid \ 1 \ \mid \ F \oplus G \ \mid \ 0 \ \mid \ \exists x.F \ \mid \ !F$$
$$\mid \ A^{\perp} \ \mid \ F \⅋ G \ \mid \ \perp \ \mid \ F \& G \ \mid \ \top \ \mid \ \forall x.F \ \mid \ ?F$$

LITERALS   MULTIPLICATIVES   ADDITIVES   QUANTIF.   EXP.

Note that $(\cdot)^{\perp}$ (negation) has atomic scope. For an arbitrary formula F, $F^{\perp}$ denotes the result of moving negation inward until it has atomic scope. We shall refer to atomic (A) and negated atomic $(A^{\perp})$ formulas as *literals*. The connectives in the first line denote the de Morgan dual of the connectives in the second line. Hence, for atoms A, B, the expression $(\perp \& (A \otimes (!B)))^{\perp}$ denotes $1 \oplus (A^{\perp} \⅋ (?B^{\perp}))$. The linear implication $F \multimap G$ is a short hand for $F^{\perp} \⅋ G$. The equivalence $F \equiv G$ is defined as $(F \multimap G) \& (G \multimap F)$.

The rules for the exponentials in LL are promotion, dereliction, weakening, and contraction, presented below in the one-sided sequent presentation

$$\frac{\vdash ?G_1, \cdots, ?G_n, F}{\vdash ?G_1, \cdots, ?G_n, !F} \ ! \qquad \frac{\vdash \Gamma, F}{\vdash \Gamma, ?F} \ \mathsf{der} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?F} \ ?_W \qquad \frac{\vdash \Gamma, ?F, ?F}{\vdash \Gamma, ?F} \ ?_C$$

Note that, in the promotion rule, the banged formula $!F$ can be introduced only if all the formulas in the context are marked with the exponential ?. That is, promotion is not context-independent, just like the rule k in the previous section.

In the quest for locality, Guerrini et al. (1998) proposed 2-sequents systems for LL variants, with separate rules for the exponentials. In Lellmann et al. (2017), this work was revisited, establishing a lighter notation and extending the discussion to multimodalities.

$\mathsf{LNS_{LL}}$ (Lellmann et al. 2017) is an end-active, linear nested system for linear logic. In this system, the promotion rule is split into the following local rules:

$$\frac{\vdash \Gamma /\!/ \vdash F}{\mathscr{E} /\!/ \vdash \Gamma, !F} \ ! \qquad \frac{\vdash \Gamma /\!/ \vdash \Delta, ?F}{\vdash \Gamma, ?F /\!/ \vdash \Delta} \ ?$$

Observe that no checking must be done in the context in order to apply the ? rule: the only checking is in the ! rule, where $\mathscr{E}$ should be the empty sequent or an empty list of components. Note the similarities between the LNS rules ! and $\square_R$, as well as ? and 4 are shown in Figure 4. Indeed, in Lellmann et al. (2017) such similarities were exploited in order to propose extensions of $\mathsf{LNS_{LL}}$ with multimodalities, called *subexponentials*, allowing for different modal behaviors.

Since the proof of adequacy of the proposed encodings in Section 4 is greatly alleviated if a *focusing discipline* is used, we introduce next $\mathsf{LNS_{FMLL}}$, a novel focused version of the LNS for linear logic with subexponentials. As we shall see, the modal/subexponential phase of the system $\mathsf{LNS_{FMLL}}$ has a better control of the flow of formulas than the system in Olarte et al. (2020).

3.1 Multimodalities in linear logic and the focused system $\mathsf{LNS_{FMLL}}$

Similar to modal connectives, exponentials in LL are not *canonical* (Danos et al. 1993), in the sense that, even having the same scheme for introduction rules, marking the exponentials with different labels does not preserve equivalence: if $i \neq j$ then $!^i F \not\equiv !^j F$ and $?^i F \not\equiv ?^j F$.

Let $\mathscr{I}$ be a set of *labels*, organized in a preorder $\preceq$. We will call $!^i, ?^i$, for $i \in \mathscr{I}$, *subexponentials*, which can be seen as substructural multimodalities.

There are, however, two main differences between multimodalities in normal modal logics and subexponentials in linear logic:

(i). Substructural behavior. Subexponentials carry the possibility of being *unbounded* (or classical) if weakening and contraction are allowed or *bounded* otherwise (thus having a linear behavior);

(ii). *Nature of modalities.* Normal modal logics start from the weakest version, assuming only the axiom K. Then, extensions are considered, by adding other axioms. Exponentials in linear logic, on the other side "take for granted" the behaviors expressed by axioms K, T, and 4 (Martini and Masini 1994) – and this is inherited by subexponentials.

We observe that (i) opened a venue for proposing different multimodal substructural logical systems, that encountered a number of different applications *e.g.* in the specification and verification of concurrent systems (Nigam et al. 2017), biological systems (Olarte et al. 2016), bigraphs (Chaudhuri and Reis 2015), applications in linguistics (Kanovich et al. 2019), and the specification of systems with multiple contexts, which may be represented by sets or multisets of formulas (Nigam et al. 2016). Regarding (ii), Guerrini et al. (1998) unified the modal and LL approaches, with the exponentials assuming only the linear version of K, with the possibility of adding modal extensions to it.

In Lellmann et al. (2017), we brought this discussion to the multimodal case, extending the concept of simply dependent multimodal logics to the substructural setting, where subexponentials consider not only the structural axioms for contraction and weakening:

$$C : \ !^i(F) \multimap !^iF \otimes !^iF \qquad W : \ !^iF \multimap 1 \qquad U = \{C, W\}$$

but also the subexponential version of axioms $\{K, 4, D, T\}$:

$$K : \ !^i(F \multimap G) \multimap !^iF \multimap !^iG \qquad 4 : \ !^iF \multimap !^i!^iF \qquad D : \ !^iF \multimap ?^iF \qquad T : \ !^iF \multimap F$$

This means that $?^i$ can behave classically (U) or not, but also with exponential behaviors different from those in LL. Hence, by assigning different modal axioms one obtains, in a modular way, a class of different substructural modal logics. For instance, subexponentials assuming T allow for dereliction, those assuming 4 are persistent, and those assuming only K are *functorial* (Girard 1998). In particular, substructural KD can be seen as a fragment of elementary linear logic ELL (Guerrini et al. 1998).

The main goal of the present work is to show how this new class of subexponentials can be applied to the problem of characterizing cut admissibility of object-level logical systems. This line of work of using LNS systems at the meta-level started in Olarte et al. (2020), where we have considered a finite class of unbounded subexponentials for specifying classical modal logics. In this work, we extend the meta-logic to support an infinite set of subexponentials, which can be bounded or unbounded, for specifying classical and linear LNS systems for multimodal logics.

Before presenting the formal definition, let us start with some intuitions. First of all, observe that systems at meta- *and* object levels are presented in the linear nested formulation. This allows not only broadening the range of specified systems, but also a more granular control of modalities. However, this comes with a price: the meta-level role of exponentials becomes highly overloaded. Indeed, they should not only handle the internal structural properties of formulas, but also the external structural properties of modalities. For example, how a classical, nonpersistent OL box modality □ should be encoded at the meta-level? The natural answer would be: via an unbounded, non persistent subexponential $?^i$. Note that the object rule $□_L$ suppresses the modality of the boxed formula when moving between components (reading the rule in Figure 4 bottom-up). If the subexponential modality $?^i$ is also removed at the meta-level (when moving to the next component), this would impose a linear nature on formulas. On the other hand, keeping the subexponential while moving between components implies that the box will be persistent (a mismatch with respect to the OL behavior).

We solve this problem by introducing the unbounded, *confined* subexponentials $!^c, ?^c$, matching the exact same behavior of the LL exponentials in sequent systems. That is, promotion is restricted to a component, and it can be applied only when the linear context is empty (see rule $!^c$ in Figure 5). Although it seems that we are sacrificing locality for the sake of modularity and uniformity, this is not the case: banged formulas with label c will never appear in the encodings – the confined classical behavior is encoded using the question-mark exponential. Hence, the nonlocal
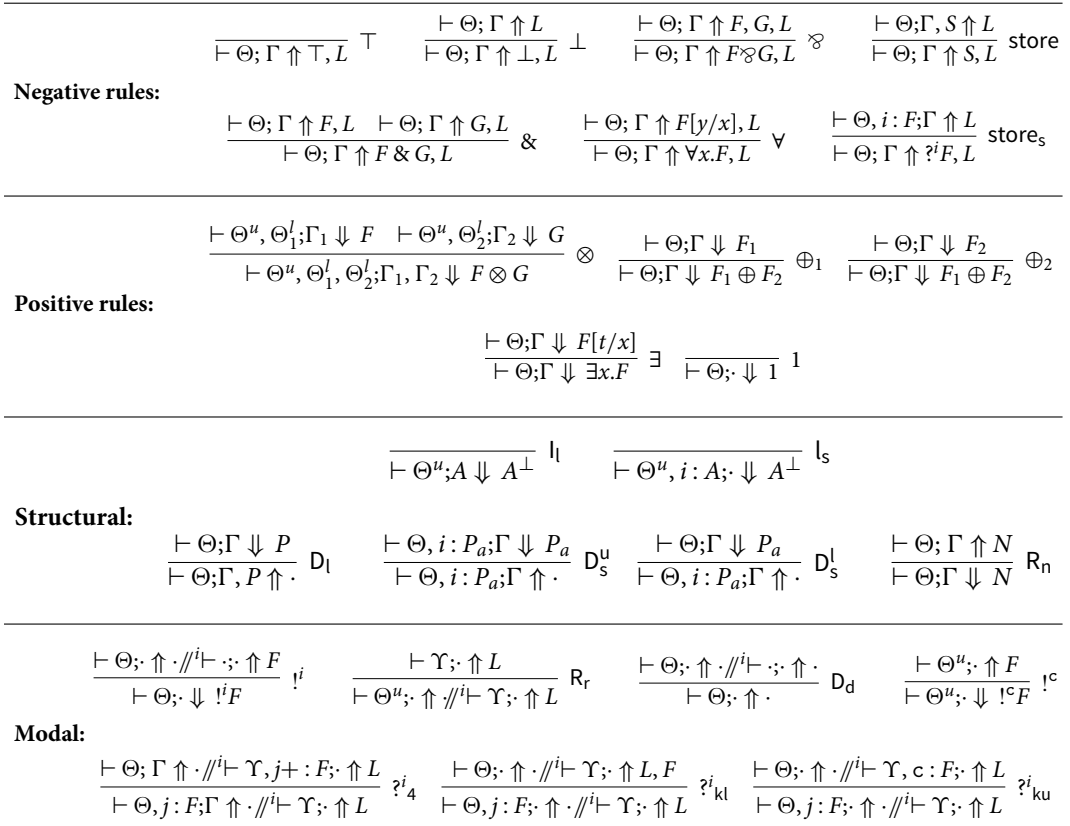
Negative rules:

$$\frac{}{\vdash \Theta; \Gamma \Uparrow \top, L} \top \qquad \frac{\vdash \Theta; \Gamma \Uparrow L}{\vdash \Theta; \Gamma \Uparrow \bot, L} \bot \qquad \frac{\vdash \Theta; \Gamma \Uparrow F, G, L}{\vdash \Theta; \Gamma \Uparrow F \parr G, L} \parr \qquad \frac{\vdash \Theta; \Gamma, S \Uparrow L}{\vdash \Theta; \Gamma \Uparrow S, L} \text{ store}$$

$$\frac{\vdash \Theta; \Gamma \Uparrow F, L \quad \vdash \Theta; \Gamma \Uparrow G, L}{\vdash \Theta; \Gamma \Uparrow F \,\&\, G, L} \,\& \qquad \frac{\vdash \Theta; \Gamma \Uparrow F[y/x], L}{\vdash \Theta; \Gamma \Uparrow \forall x.F, L} \,\forall \qquad \frac{\vdash \Theta, i : F; \Gamma \Uparrow L}{\vdash \Theta; \Gamma \Uparrow ?^i F, L} \text{ store}_s$$

Positive rules:

$$\frac{\vdash \Theta^u, \Theta_1^l; \Gamma_1 \Downarrow F \quad \vdash \Theta^u, \Theta_2^l; \Gamma_2 \Downarrow G}{\vdash \Theta^u, \Theta_1^l, \Theta_2^l; \Gamma_1, \Gamma_2 \Downarrow F \otimes G} \otimes \qquad \frac{\vdash \Theta; \Gamma \Downarrow F_1}{\vdash \Theta; \Gamma \Downarrow F_1 \oplus F_2} \oplus_1 \qquad \frac{\vdash \Theta; \Gamma \Downarrow F_2}{\vdash \Theta; \Gamma \Downarrow F_1 \oplus F_2} \oplus_2$$

$$\frac{\vdash \Theta; \Gamma \Downarrow F[t/x]}{\vdash \Theta; \Gamma \Downarrow \exists x.F} \exists \qquad \frac{}{\vdash \Theta; \cdot \Downarrow 1} 1$$

Structural:

$$\frac{}{\vdash \Theta^u; A \Downarrow A^\perp} I_l \qquad \frac{}{\vdash \Theta^u, i : A; \cdot \Downarrow A^\perp} I_s$$

$$\frac{\vdash \Theta; \Gamma \Downarrow P}{\vdash \Theta; \Gamma, P \Uparrow \cdot} D_l \qquad \frac{\vdash \Theta, i : P_a; \Gamma \Downarrow P_a}{\vdash \Theta, i : P_a; \Gamma \Uparrow \cdot} D_s^u \qquad \frac{\vdash \Theta; \Gamma \Downarrow P_a}{\vdash \Theta, i : P_a; \Gamma \Uparrow \cdot} D_s^l \qquad \frac{\vdash \Theta; \Gamma \Uparrow N}{\vdash \Theta; \Gamma \Downarrow N} R_n$$

Modal:

$$\frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow F}{\vdash \Theta; \cdot \Downarrow !^i F} !^i \qquad \frac{\vdash \Upsilon; \cdot \Uparrow L}{\vdash \Theta^u; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} R_r \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow \cdot}{\vdash \Theta; \cdot \Uparrow \cdot} D_d \qquad \frac{\vdash \Theta^u; \cdot \Uparrow F}{\vdash \Theta^u; \cdot \Downarrow !^c F} !^c$$

$$\frac{\vdash \Theta; \Gamma \Uparrow \cdot /\!/^i \vdash \Upsilon, j{+} : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \Gamma \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} ?^i_4 \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L, F}{\vdash \Theta, j : F; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} ?^i_{kl} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon, c : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} ?^i_{ku}$$

Figure 5. End-active focused system LNS_{FMLL}. Θ^u (resp. Θ^l) contains only unbounded (resp. linear) subexponentials. In I_s and I_l, A is atomic. In $\forall$, y is fresh. In store, S is a literal or a positive formula. In R_n, N is a negative formula. In D_l, P is positive, and in D_s^u, D_s^l, P_a is not atomic. In D_s^u, D_s^l and I_s, $T \in \mathscr{U}(i)$. In all question-marked rules $i \preceq j$. Moreover, $i \neq c$ in $!^i$; $D \in \mathscr{U}(i)$ in D_d; $4 \in \mathscr{U}(j)$ in $?^i_4$; $\{4, C, W\} \cap \mathscr{U}(j) = \emptyset$ in $?^i_{kl}$; $4 \notin \mathscr{U}(i)$ and $U \subseteq \mathscr{U}(i)$ in $?^i_{ku}$ and in D_s^u.

promotion rule $!^c$ will never be applied. It is part of the systems, however, for completeness of the cut-elimination procedure (see Section 3.2).

Definition 4. (MMLL). *Multimodal linear logic (MMLL) shares with linear logic all connectives except the exponentials:* MMLL *contains labeled subexponentials* $!^i, ?^i$, *specified by the subexponential signature given by* $\Sigma = \langle \mathscr{I}, \preceq, \mathscr{U}(i) \rangle$, *where*

- $\mathscr{I}$ *is a set of labels;*
- $\mathscr{U}(i)$ *represents the set of axioms within* $\{C, W, K, 4, D, T\}$ *that the subexponential* $i \in \mathscr{I}$ *assumes, with* $K \in \mathscr{U}(i)$ *for all* $i \in \mathscr{I}$;
- $\preceq$ *is a preorder among the elements of* $\mathscr{I}$ *that is upwardly closed with respect to* $\mathscr{U}(i)$, *i.e., if an axiom* $A \in \mathscr{U}(i)$ *and* $i \preceq j$, *then* $A \in \mathscr{U}(j)$.

We assume $c \in \mathscr{I}$ *to be a distinguished element s.t.* $\mathscr{U}(c) = \{C, W, K, T\}$ *and* c *is only* $\preceq$-*related with itself. Finally, for every subexponential* $i \in \mathscr{I}$, *we assume a subexponential* $i{+} \in \mathscr{I}$ *s.t.* $\mathscr{U}(i{+}) = \mathscr{U}(i) \cup \{T\}$, $i \preceq i{+}$, *and if* $i \preceq j$ *then* $i{+} \preceq j{+}$.

As usual, the upwardly condition is needed for cut elimination (Danos et al. 1993). Since we are considering extensions of normal modal logics, all the subexponentials are assumed to have

the axiom K. Finally, the subexponential $i+$, that adds $\top$ to $\mathscr{U}(i)$, will be used to give a more deterministic rule for exponentials containing the axiom 4 (more on this below).

The focused proof system for MMLL is presented in Figure 5 and explained in the following. *Focusing* (Andreoli 1992) is a discipline on proofs aiming at reducing nondeterminism during proof search. Focusing in LL-based systems is grounded on two kinds of separations: (a) classical/linear behavior of formulas and (b) invertible/non invertible introduction rules.

For (a), observe that it is possible to incorporate the structural rules of contraction and weakening for formulas of the shape $?F$ into the LL introduction rules. This is reflected into the syntax in the so called *dyadic sequents* where the context is split into two, a classical (set of formulas Θ) and a linear (multiset of formulas Γ). The dyadic sequent $\vdash \Theta; \Gamma$ is then interpreted as the linear logic sequent $\vdash ?\Theta, \Gamma$ where $?\Theta = \{?F \mid F \in \Theta\}$. This idea can be generalized to the case of the subexponentials. The *subexponential context* is a multiset of the form $\Theta = \{i : F \mid i \in \mathscr{I}\}$, and the dyadic sequent $\vdash \Theta; \Gamma$ is interpreted as the (subexponential) linear logic sequent $\vdash \{?^i F \mid i : F \in \Theta\}, \Gamma$. That is, $i : F \in \Theta$ represents that F is *stored* in the (subexponential) context in Θ labeled by i. We extend the indexed notation to contexts as follows: given a linear context $\Gamma = \{F_1, \ldots, F_n\}$, we will write $i : \Gamma$ for the subexponential context $\{i : F_1, \ldots, i : F_n\}$. Finally, we shall use the superindex Θ^u (resp. Θ^l) to denote a context where all the subexponentials are unbounded (resp. linear). When we simply write Θ, the context may contain both, linear and unbounded subexponentials.

For (b), it turns out that proofs can be organized in two alternating phases: *negative* containing only invertible rules, and *positive* containing only noninvertible rules. The connectives $\bindnasrepma, \bot, \&, \top, ?^i, \forall$ have invertible introduction rules and are thus classified as negative. The remaining connectives $\otimes, 1, \oplus, 0, !^i, \exists$ are positive. Formulas inherit their polarity from their main connective, *e.g.*, $F \otimes G$ is positive and $F \bindnasrepma G$ is negative. Although the bias assigned to atoms does not interfere with provability (Andreoli 1992; Miller and Saurin 2007), here we follow Andreoli's convention of classifying atomic formulas as *negative*, thus negated atoms as positive.

The inference rules of the focused system $\mathsf{LNS_{FMLL}}$ in Figure 5 involve two kinds of sequents

$$\Theta; \Gamma \Uparrow L \quad \text{and} \quad \Theta; \Gamma \Downarrow F$$

where L is a (possibly empty) list of formulas. The formula occurrence F in a $\Downarrow$-sequent is called the *focus* of that sequent. Moreover, linear nested sequents in $\mathsf{LNS_{FMLL}}$ have the shape

$$\vdash \mathscr{G} /\!/^i Seq$$

where $\mathscr{G}$ is either an unfocused sequent or it is empty (in this case $/\!/^i$ is not present). The superscript in "$/\!/^i$" indicates that the component was created by a subexponential labeled with i. Seq, above, is a focused or unfocused $\mathsf{LNS_{FMLL}}$ sequent.

The different LNS above reflect not only the negative/positive proof phases but also the behavior of the promotion rule:

- $\vdash \Theta; \Gamma \Uparrow L$ belongs to the *negative phase*. During this phase, all negative formulas in the list L are introduced and all positive formulas and literals are moved to the linear context Γ;
- $\vdash \Theta; \Gamma \Downarrow F$ belongs to the *positive phase*, where all positive connectives at the root of F are introduced; and
- $\vdash \mathscr{G} /\!/^i \vdash \Theta; \Gamma \Uparrow F$ belongs to the *exponential/modal phase*. During this phase, only applications of the rules for $?^i$ are allowed, ending with an application of $\mathsf{R_r}$ (release rule).
  The dynamics of the different phases is detailed below.

Negative Phase. Negative rules can be applied eagerly, in any order. This process includes storing formulas. Note that the rule store moves, to the linear zone Γ, the literal or positive formula S since it cannot be decomposed during the negative phase; $\mathsf{store_s}$ stores formulas marked with $?^i$ into the correspondent subexponential context.

The negative phase ends when the list L in unfocused sequents is empty. A decision rule is then applied. There are four possibilities.

- D_l, D_s^u, D_s^l: a positive phase starts by *focusing* on a non-atomic formula F such that
  - D_l: F is taken from the linear context (and thus erased from it);
  - D_s^u: a copy of F is taken from the subexponential context i, thus making an implicit contraction ($U \subseteq \mathcal{U}(i)$) and a dereliction ($T \in \mathcal{U}(i)$);
  - D_s^l: analogous to D_s^u, with the difference that $U \cap \mathcal{U}(i) = \emptyset$ and the formula is not contracted.
- D_d: starts an exponential phase.

Positive Phase. Once we focus on a formula, the proof follows by applying positive rules, where the focus persists on the decomposed subformulas until either: the proof ends with an instance of an axiom; a negative formula is reached, and the positive phase ends with the application of R_n; or a banged formula is derived, which either moves the proof to the negative case (if the subexponential is $!^c$) or triggers an exponential phase execution (otherwise). In the latter, only the rules for $?^i$, moving formulas between components, are allowed. When this moving is over, the exponential phase ends with an application of the rule R_r, starting again a negative phase.

There are two initial rules. In I_l, the atom A is in the linear zone and the subexponential context must be unbounded (being implicitly weakened); in I_s, the atom is placed in a subexponential context i with $T \in \mathcal{U}(i)$. The need of the axiom T comes from the fact that it embodies the capacity of using the formula in the same component, corresponding to dereliction in linear logic.

Exponential Phase. The exponential phase starts with the application of a $!^i$ rule, with $i \neq c$, or the rule D_d. The distinguished subexponential $!^c$ behaves as the LL bang exponential. Hence, its introduction rule simply ends the positive phase and assumes that the subexponential context is unbounded. For any other subexponential in MMLL, the rule $!^i$ creates a new component, moving the active formula to it. Once the new component is created, the $?^i$ rules dictate the movement of formulas between components, as described in the following.

Usually, nested-based systems for modal/linear logics present one core box left/question mark rule, and other modal rules with different behaviors are added on the top of it, depending on the modal axioms of the underlying modality/subexponential. In Olarte et al. ([2020](#)), we have adopted such approach, lifting to the meta-level the OL modal rules. Although resulting into *adequate* specifications, in the sense of successfully capturing the object-level behavior, this approach is not satisfactory for the meta-level focused system, since it adds a great deal of nondeterminism during proof search.

In this work, we propose more efficient modal rules from the proof theoretical point of view using all the subexponential features present in MMLL. First of all, when $4 \in \mathcal{U}(j)$, the rules $?^i_{kl}$ and $?^i_{ku}$ cannot be applied. Dually, if $4 \notin \mathcal{U}(j)$, then the rule $?^i_4$ is not enabled and the use of $?^i_{kl}$ (resp. $?^i_{ku}$) is only possible if $U \not\subseteq \mathcal{U}(j)$ (resp. $U \subseteq \mathcal{U}(j)$). Second, the proposed rules have a better control of contraction, thus avoiding the need of guessing the number of times a formula must be copied to the next component. Note that the rule $?^i_4$ moves the formula F stored in the context j to the context $j+$. This has two immediate effects: the formula F can be copied to yet another component (once it is created) reflecting the behavior of the modal rule 4 (persistence); moreover, since $T \in \mathcal{U}(j+)$, the formula F can be also used in the last component by applying the decision rule. Finally, when $U \subseteq \mathcal{U}(j)$, $j+$ is also unbounded and decision/dereliction on the same formula can be performed several times. In other words, the rule $?^i_4$ embeds both the behavior of K (moving formulas between components) and also 4 (by keeping the modality of the formula). On the other hand, the behavior of K, without 4, is specified by the rules $?^i_{kl}$ and $?^i_{ku}$. In the first case, j is linear and then F is not contracted. In the second case, F is placed in the context c. Since $T \in \mathcal{U}(c)$, F can be used as many times as needed in the last component (by using the decision

rule during the positive phase). Also, since c is not related to any subexponential, the formula F cannot be moved to other components and thus confined to the last component.

All this is illustrated in the next example.

Example 3.1. Let $4, 4+, k, k+$ be subexponentials s.t. $\mathscr{U}(4) = \{K, 4\}$, $\mathscr{U}(4+) = \{K, 4, T\}$, $\mathscr{U}(k) = \{K, C, W\}$ and $\mathscr{U}(k+) = \{K, C, W, T\}$. The following derivations show, respectively, that the axioms K and 4 are provable:

$$
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\frac{
\frac{\overline{\vdash \cdot; A \Downarrow A^\perp}\ I_l}{\vdash 4{+}: A^\perp; A \Uparrow \cdot}\ D_s^l}{\vdash \cdot; \cdot \Uparrow /\!\!/^4 \vdash 4{+}: A^\perp; \cdot \Uparrow A}\ R_r, \text{store}
}{\vdash 4{+}: A^\perp; \cdot \Uparrow\ !^4 A}\ \text{store}, D_l, !^4, ?^4_4
}{\vdash \cdot; \cdot \Uparrow /\!\!/^4 \vdash 4{+}: A^\perp; \cdot \Uparrow\ !^4 A}\ R_r
}{\vdash 4: A^\perp; \cdot \Uparrow /\!\!/^4 \vdash \cdot; \cdot \Uparrow\ !^4 A}\ ?^4_4
}{\vdash 4: A^\perp; \cdot \Downarrow\ !^4 !^4 A \Uparrow \cdot}\ !^4
}{\vdash 4: A^\perp; !^4 !^4 A \Uparrow \cdot}\ D_l
}{\vdash \cdot; \cdot \Uparrow\ ?^4 A^\perp \otimes !^4 !^4 A}\ \otimes, \text{store}_c, \text{store}
$$

$$
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\frac{\overline{\vdash c:(A \otimes B^\perp), c:A^\perp; A \Downarrow A^\perp}\ I_l}{\vdash c:(A \otimes B^\perp), c:A^\perp; \cdot \Downarrow A}\ R_n, \text{store}, D_s^u \quad \overline{\vdash c:(A \otimes B^\perp), c:A^\perp; B \Downarrow B^\perp}\ I_l}{\vdash c:(A \otimes B^\perp), c:A^\perp; B \Downarrow A \otimes B^\perp}\ \otimes
}{\vdash c:(A \otimes B^\perp), c:A^\perp; \cdot \Uparrow B}\ \text{store}, D_s^u
}{\vdash \cdot; \cdot \Uparrow /\!\!/^k \vdash c:(A \otimes B^\perp), c:A^\perp; \cdot \Uparrow B}\ R_r
}{\vdash k:(A \otimes B^\perp), k:A^\perp; \cdot \Uparrow /\!\!/^k \vdash \cdot; \cdot \Uparrow B}\ 2 \times ?^k_{ku}
}{\vdash k:(A \otimes B^\perp), k:A^\perp; \cdot \Downarrow\ !^k B}\ !^k
}{\vdash k:(A \otimes B^\perp), k:A^\perp; !^k B \Uparrow \cdot}\ D_l
}{\vdash \cdot; \cdot \Uparrow\ !^k(A \multimap B) \multimap (!^k A \multimap !^k B)}\ \otimes, \text{store}_c, \otimes, \text{store}_c, \text{store}
$$

The first derivation involves three components (two of them created using the $!^4$ rule). The information $A^\perp$ is passed from the 1st to the 2nd and 3rd components through the subexponential $4+$ via rule $?^4_4$. Since this subexponential is linear, the context $4+$ disappear after the application of D_s^l. The second derivation illustrates the application of the rule $?^i_{ku}$ in the unbounded subexp. k. Note that the formula $A \otimes B^\perp$ is copied to the context c. All the other axioms in Figure 4 can be proved similarly, using a subexponential featuring the needed modal behaviors.

Finally, it should be noted that the proposed focused system is complete w.r.t. the unfocused version. For the plain linear logic rules, completeness is guaranteed by Andreoli's result in Andreoli (1992). For the modal rules, it is straightforward to simulate the LNS rules from the focused ones, following the same lines as the process of obtaining sequent rules from the LNS ones (see the example for implication right rule for mLJ in Section 2).

3.2 *Cut Admissibility for* LNS$_{\text{FMLL}}$

In this section, we present a cut-admissibility result for the system LNS$_{\text{FMLL}}$. It should be noted that this is, for the best of our knowledge, the first formally verified proof of cut admissibility for *focused* LNS.[2] This *is not* a simple enhancement w.r.t. other formalizations: usually cut-admissibility of focused systems is done in two steps, first translating it into the unfocused version, then proving the completeness of focusing. Here, we prove the cut admissibility of LNS$_{\text{FMLL}}$ *directly*, observing that the multimodal flavor of the system makes the task even less trivial.

We proceed by double induction on the complexity of the cut formula and the sum of the premises' heights of the cut rule. Since we are proving the theorem directly on a focused system, we require to eliminate, simultaneously, different rules dealing with the different phases of the proof (see e.g., Liang and Miller 2009). The cut rules are in Figure 6.

In the following, we show the most interesting proof reductions needed to establish the main result (Theorem 3.5). In the repository (Xavier and Olarte 2021), the reader can find the formalization in Coq of all the results of this section, together with a companion PDF with extra details about the proof.

Auxiliary results. We start with a series of auxiliary results for the system LNS$_{\text{FMLL}}$. They are not only needed for the proof of the main theorem, but they will also help on understanding better the dynamics of the system.

Notation 5. *We use $\vdash \Theta; \Gamma \Updownarrow X$ to represent a sequent that can belong to the negative phase – and X is a list of formulas, or to the positive phase – and X is a formula. We use $\vdash_n \mathscr{S}$ to indicate that the linear nested sequent $\mathscr{S}$ has a derivation of height at most n. If $\Theta = \{i_1 : F_1, ..., i_n : F_n\}$ is a subexponential context, then $\Theta+$ denotes the subexponential context $\{i_1+ : F_1, ..., i_n+ : F_n\}$ and $\mathscr{F}(\Theta)$ denotes the list of formulas $F_1, \ldots, F_n$. For easing the notation, we will write $i : \Theta$ instead of $i : \mathscr{F}(\Theta) = \{i : F_1, \ldots, i : F_n\}$. Finally, $\Theta = (\Theta_1, \ldots, \Theta_n)$ denotes that $\Theta_1, \ldots, \Theta_n$ is a partition of Θ. In this section, Ω will be used to denote a context assuming axiom 4 (called 4-context), and Ψ a context without 4.*

Recall that the axiom T allows for dereliction. The following result shows some particular behaviors of T-contexts.

Theorem 3.1. (The axiom T). *The following propositions hold:*

(T1) *If $\{\mathsf{T}, \mathsf{C}, \mathsf{W}\} \subseteq \mathscr{U}(i)$ and $\vdash_n \Theta, i : F, \mathsf{c} : F; \Gamma \Updownarrow X$, then $\vdash_n \Theta, i : F; \Gamma \Updownarrow X$.*
(T2) *If $\{\mathsf{T}, \mathsf{C}, \mathsf{W}\} \subseteq \mathscr{U}(i)$ and $\vdash_n \Theta, i : F; \Gamma, F \Updownarrow X$, then $\vdash_n \Theta, i : F; \Gamma \Updownarrow X$.*
(T3) *If $\mathsf{T} \in \mathscr{U}(i)$ and $\vdash_n \Theta; \Gamma, F \Updownarrow X$, then $\vdash_n \Theta, i : F; \Gamma \Updownarrow X$.*
($\mathscr{L}A_1$) *If $\{\mathsf{T}, \mathsf{C}, \mathsf{W}\} \subseteq \mathscr{U}(i)$ and $\vdash \Theta, i : F; \Gamma \Uparrow L, F$ then $\vdash \Theta, i : F; \Gamma \Uparrow L$.*
($\mathscr{L}A_2$) *If $\mathsf{T} \in \mathscr{U}(i)$ and $\vdash \Theta; \Gamma \Uparrow L, F$ then $\vdash \Theta, i : F; \Gamma \Uparrow L$.*

All these results can be easily generalized to contexts featuring the corresponding axioms. For example, in the case of the absorption lemma ($\mathscr{L}A_2$), if $\vdash \Theta; \Delta \Uparrow L, \Gamma$, then $\vdash \Theta, i : \Gamma; \Delta \Uparrow L$. Moreover, (T1) states that if a formula is stored in a context i where T holds, then it is "redundant" to place that formula also in the local context c. The other results allow us for moving formulas in the different zones of a sequent where T holds.

The next theorem highlights the difference between multisets and lists in contexts. Namely, while permuting a list preserves provability, it may not preserve the height of a derivation. On the other hand, weakening and contraction are height-preserving admissible for unbounded contexts.

Theorem 3.2. (Structural Rules). *The following propositions hold.*

($\equiv_P$) *If L_1 is a permutation of L_2 and $\vdash \Theta; \Gamma \Uparrow L_1$ then $\vdash \Theta; \Gamma \Uparrow L_2$*
(W) *If $\mathsf{U} \subseteq \mathscr{U}(i)$ and $\vdash_n \Theta; \Gamma \Updownarrow X$ then $\vdash_n \Theta, i : F; \Gamma \Updownarrow X$.*
(C) *If $\mathsf{U} \subseteq \mathscr{U}(i)$ and $\vdash_n \Theta, i : F, i : F; \Gamma \Updownarrow X$ then $\vdash_n \Theta, i : F; \Gamma \Updownarrow X$.*

As usual in focused systems, the shape of phases can be totally characterized. In the subexponential phase, since the only action allowed is moving formulas between components until the release rule (R_r) is applied, every derivation has the shape

$$\cfrac{\cfrac{\vdash \Omega+, \mathsf{c} : \Psi_1; \cdot \Uparrow L, \mathscr{F}(\Psi_2)}{\vdash \Omega, \Psi, \Xi; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow L}}{}$$

$$\cfrac{\vdash \Omega, \Psi, \Xi; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow P}{\vdash \Omega, \Psi, \Xi; \cdot \Downarrow \ !^i P} \ !^i \qquad\qquad \cfrac{\vdash \Omega, \Psi, \Xi; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow \cdot}{\vdash \Omega, \Psi, \Xi; \cdot \Uparrow \cdot} \ \mathsf{D_d}$$

where Ξ contains all the unbounded subexponentials not related to i, $\Psi = (\Psi_1^u, \Psi_2^l)$ and $i \preceq j$ for all $j : F$ in $\Omega \cup \Psi_1$. Observe that Ξ is weakened when the release rule is applied. The formulas in Ω gain the axiom T, the unbounded formulas in Ψ are stored in c and the linear ones are placed in the linear context ($\mathscr{F}(\Psi_2)$). On the left, the subexponential phase is triggered by the promotion rule and $L = P$. On the right, the rule $\mathsf{D_d}$ is applied and L is empty.

$$\dfrac{\vdash \Lambda, \Theta_1, i:H;\Gamma \Downarrow F \quad \vdash \Lambda, \Theta_2;\cdot \Downarrow\ !^i H^\perp}{\vdash \Lambda, \Theta;\Gamma \Uparrow F} \ \Downarrow CC \qquad \dfrac{\vdash \Lambda, \Theta_1, i:H;\Gamma \Uparrow L \quad \vdash \Lambda, \Theta_2;\cdot \Downarrow\ !^i H^\perp}{\vdash \Lambda, \Theta;\Gamma \Uparrow L} \ \Uparrow CC$$

$$\dfrac{\vdash \Lambda, \Theta_1;\Gamma \Uparrow H, L \quad \vdash \Lambda, \Theta_2;\Delta \Downarrow H^\perp}{\vdash \Lambda, \Theta;\Gamma, \Delta \Uparrow L} \ \Uparrow C$$

$$\dfrac{\vdash \Lambda, \Theta_1;\Gamma \Uparrow L_1, H, L_2 \quad \vdash \Lambda, \Theta_2;\Delta \Uparrow H^\perp, L_3}{\vdash \Lambda, \Theta;\Gamma, \Delta \Uparrow L_1, L_2, L_3} \ \Uparrow C* \qquad \dfrac{\vdash \Lambda, \Theta_1;\Gamma, H \Uparrow L_1 \quad \vdash \Lambda, \Theta_2;\Delta \Uparrow H^\perp, L_2}{\vdash \Lambda, \Theta;\Gamma, \Delta \Uparrow L_1, L_2} \ \Uparrow LC*$$

Figure 6. Cut-rules for the system $\mathsf{LNS_{FMLL}}$. Λ is unbounded and $\Theta = (\Theta_1, \Theta_2)$ is linear.

Theorem 3.3. (Subexp. phase). *Assume that Ξ is unbounded, $\Psi = (\Psi_1^u, \Psi_2^l)$, $\Omega = (\Omega_1^u, \Omega_2^l)$, and if $j : F \in \Omega \cup \Psi_1$ then $i \preceq j$. Then*

(S) $\vdash_m \Omega+, c : \Psi_1; \cdot \Uparrow L, \mathscr{F}(\Psi_2)$ *iff* $\vdash_{m+n+1} \Omega, \Psi, \Xi; \cdot \Uparrow \cdot /\!/^i \cdot; \cdot \Uparrow L$
   where $n = |\Omega, \Psi|$.

This result can be further specialized depending whether the index i in the subexponential phase $(\cdot /\!/^i \cdot)$ is unbounded or $4 \in \mathscr{U}(i)$. Remember the upwardly closeness condition: if $i \preceq j$ and some axiom $\mathsf{A} \in \mathscr{U}(i)$ then $\mathsf{A} \in \mathscr{U}(j)$. Hence, if i is unbounded, there is no linear subexponentials in Ω, Ψ, Ξ and, thus, Ψ_2 and Ω_2 are necessarily empty in **(S)**. Moreover, if $4 \in \mathscr{U}(i)$, Ψ (containing the indices without 4) must be necessarily unbounded and weakened in the application of $\mathsf{R_r}$. More precisely:

(SU) $\vdash_m \Omega_{1+}, c : \Psi_1; \cdot \Uparrow L$ iff $\vdash_{m+k+1} \Omega_1, \Psi_1, \Xi; \cdot \Uparrow \cdot /\!/^i \cdot; \cdot \Uparrow L$
(S4) $\vdash_m \Omega +; \cdot \Uparrow L$ iff $\vdash_{m+r+1} \Omega, \Xi; \cdot \Uparrow \cdot /\!/^i \cdot; \cdot \Uparrow L$

where $k = |\Omega_1, \Psi_1|$ and $r = |\Omega|$.

It is also interesting to revisit the two different promotion rules: $!^c$ and $!^i$. The following theorem shows that they coincide when $\{\mathsf{C}, \mathsf{W}, \mathsf{T}\} \subseteq \mathscr{U}(i)$.

Theorem 3.4. *Let i be unbounded and $\mathsf{T} \in \mathscr{U}(i)$. If $\vdash_n \Theta; \cdot \Downarrow\ !^i P$ then $\vdash_{n-1} \Theta; \cdot \Uparrow P$.*

Cut-rules and reductions. The five cut-rules admissible in $\mathsf{LNS_{FMLL}}$ are presented in Figure 6. In all cases, Λ contains only unbounded subexponentials whereas $\Theta = (\Theta_1, \Theta_2)$ is a linear context. Below we explain the interdependencies between these rules during the cut-elimination procedure. For the sake of readability, in most of the derivations, we omit the unbounded context Λ.

Elimination of $\Uparrow C$. Since the cut formula is the first element in the list of formulas in the left premise, it must be necessarily principal. Take for instance the following case:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1 \\ \vdash \Theta_1;\Gamma \Uparrow P, Q, L \end{array}}{\vdash \Theta_1;\Gamma \Uparrow P \otimes Q, L} \otimes \quad \dfrac{\begin{array}{c}\pi_2 \\ \vdash \Theta_2;\Delta_1 \Downarrow P^\perp\end{array} \ \begin{array}{c}\pi_3 \\ \vdash \Theta_3;\Delta_2 \Downarrow Q^\perp\end{array}}{\vdash \Theta_2, \Theta_3;\Delta \Downarrow P^\perp \otimes Q^\perp} \otimes}{\vdash \Theta_1, \Theta_2, \Theta_3;\Gamma, \Delta \Uparrow L} \ \Uparrow C$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\pi_1 \\ \vdash \Theta_1;\Gamma \Uparrow P, Q, L\end{array} \ \begin{array}{c}\pi_2 \\ \vdash \Theta_2;\Delta_1 \Downarrow P^\perp\end{array}}{\vdash \Theta_1, \Theta_2;\Gamma, \Delta_1 \Uparrow Q, L} \Uparrow C \quad \begin{array}{c}\pi_3 \\ \vdash \Theta_3;\Delta_2 \Downarrow Q^\perp\end{array}}{\vdash \Theta_1, \Theta_2, \Theta_3;\Gamma, \Delta \Uparrow L} \ \Uparrow C$$

When the cut formula has the shape $?^i F$ (resp. a positive formula or a literal), a shorter cut using $\Uparrow CC$ (resp. $\Uparrow LC*$) is used. For example:

$$\dfrac{\dfrac{\vdash \Theta_1, i:H;\Gamma \Uparrow L}{\vdash \Theta_1;\Gamma \Uparrow ?^i H, L} \ \mathsf{stores} \quad \vdash \Theta_2;\cdot \Downarrow\ !^i H^\perp}{\vdash \Theta_1, \Theta_2;\Gamma \Uparrow L} \ \Uparrow C \qquad \rightsquigarrow \qquad \dfrac{\vdash \Theta_1, i:H;\Gamma \Uparrow L \quad \vdash \Theta_2;\cdot \Downarrow\ !^i H^\perp}{\vdash \Theta_1, \Theta_2;\Gamma \Uparrow L} \ \Uparrow CC$$

When the cut formula is an atom, the right premise of the cut must necessarily finish with the initial rule. We show below the case where A is in the (linear) subexponential context i (where, necessarily, $\mathsf{T} \in \mathscr{U}(i)$) and we use the *admissible rule* (T3), which is is justified by the respective statement in Theorem 3.1.

$$\dfrac{\dfrac{\vdash \Theta_1; A, \Gamma \Uparrow L}{\vdash \Theta_1; \Gamma \Uparrow A, L} \text{ store}}{\vdash \Theta_1, i : A; \Gamma \Uparrow L} \quad \dfrac{}{\vdash i : A; \cdot \Downarrow A^\perp} \mathsf{I}_s \quad \Uparrow C \qquad \rightsquigarrow \qquad \dfrac{\vdash \Theta_1; A, \Gamma \Uparrow L}{\vdash \Theta_1, i : A; \Gamma \Uparrow L} \text{ T3}$$

In the following, we will adopt the same procedure of reading the results in Theorem 3.1 as rules. **Elimination of $\Uparrow$CC.** The elimination of this rule is easy when the list L is not empty: it suffices to permute down the application of the introduction rule.

When L is empty, the left premise necessarily starts with a decision rule or an application of $\mathsf{D_d}$. We show below the case where Q is taken from the linear subexponential context a and, necessarily, $\mathsf{T} \in \mathscr{U}(a)$. We proceed with a (smaller) application of $\Downarrow$CC:

$$\dfrac{\dfrac{\vdash \Theta_1, i : H; \Gamma \Downarrow Q}{\vdash a : Q, \Theta_1, i : H; \Gamma \Uparrow \cdot} \mathsf{D_s^l}}{\vdash a : Q, \Theta_1, \Theta_2; \Gamma \Uparrow \cdot} \quad \dfrac{}{\vdash \Theta_2; \cdot \Downarrow \; !^i H^\perp} \; \Uparrow\text{CC} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Theta_1, i : H; \Gamma \Downarrow Q \qquad \vdash \Theta_2; \cdot \Downarrow \; !^i H^\perp}{\vdash \Theta_1, \Theta_2; \Gamma \Uparrow Q} \; \Downarrow\text{CC}}{\vdash a : Q, \Theta_1, \Theta_2; \Gamma \Uparrow \cdot} \; \mathscr{L}A_2$$

Consider now the case where the left premise starts with an application of $\mathsf{D_d}$. Assume that $\mathsf{D} \in \mathscr{U}(a)$, $a \preceq i$ and $4 \in \mathscr{U}(i)$. Moreover, let $\Psi = (\Psi_1^u, \Psi_2^l)$ and $\Omega = (\Omega_1^u, \Omega_2^l, \Omega_3^l)$. Hence

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash \Omega_1^{\succeq a}+, \Omega_2+, \mathsf{c} : \Psi_1^{\succeq a}, i+ : H; \cdot \Uparrow \mathscr{F}(\Psi_2)}}{\vdash \Omega_1, \Omega_2, \Psi, i : H; \cdot \Uparrow /\!\!/^a \vdash \cdot; \cdot \Uparrow \cdot} \; ?^a, \mathsf{R_r}}{\vdash \Omega_1, \Omega_2, \Psi, i : H; \cdot \Uparrow \cdot} \; \mathsf{D_d} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdash \Omega_1^{\succeq i}+, \Omega_3 +; \cdot \Uparrow H^\perp}}{\vdash \Omega_1, \Omega_3, \Psi_1; \cdot \Downarrow \; !^i H^\perp} \; !^i, ?^i, \mathsf{R_r}}{}}{\vdash \Omega, \Psi; \cdot \Uparrow \cdot} \; \Uparrow CC$$

The leaves in the above derivation are obtained from Theorem 3.3. The notation $\Omega_1^{\succeq a}$ represents the subset of formulas in Ω_1 whose indices are above a and that were not weakened in the end of the subexponential phase. Note that, due to the upwardly closeness condition, the linear context Ψ_2 must necessarily be moved to the left premise (since 4 is not present in Ψ). Also, Ψ_1 is necessarily weakened in the right premise after applying the release rule. The reduction consists in cutting with $!^{i+} H^\perp$ instead of $!^i H^\perp$ as follows:

$$\dfrac{\dfrac{\overset{\pi_1^w}{\vdash \Omega'+, \Omega_2+, \mathsf{c} : \Psi_1^{\succeq a}, i+ : H; \cdot \Uparrow \mathscr{F}(\Psi_2)} \qquad \overset{\pi_2^w}{\vdash \Omega'+, \Omega_3 +; \cdot \Downarrow \; !^{i+} H^\perp}}{\vdash \Omega'+, \Omega_2+, \Omega_3+, \mathsf{c} : \Psi_1^{\succeq a}; \cdot \Uparrow \mathscr{F}(\Psi_2)} \; \Uparrow CC}{\vdash \Omega, \Psi; \cdot \Uparrow \cdot} \; \mathsf{D_d}, ?^i, \mathsf{R_r}$$

In the above derivation, Ω' contains all the formulas in both $\Omega_1^{\succeq a}$ and $\Omega_1^{\succeq i}$, *i.e.*, the formulas from Ω_1 that were not weakened in neither of the derivations. Note that $a \preceq i$ and then, the application of $\mathsf{D_d}$ can preserve all the formulas in $\Omega_1^{\succeq i}$ (similarly for Ω_3). Derivations π_i^w for $i \in \{1, 2\}$ are obtained from π_i by weakening the formulas in Ω_1' not needed in each derivation.

Elimination of $\Downarrow$CC. When the focused formula on the left premise is principal and focus cannot be lost, an invertibility lemma is used. Below a representative case using the invertibility lemma $(\mathscr{L}\oplus)$: If $\vdash \Theta; \Delta \Uparrow L, F$ then $\vdash \Theta; \Delta, F \oplus G \Uparrow L$.

$$\dfrac{\dfrac{\vdash \Theta_1, i : H; \Gamma \Downarrow P}{\vdash \Theta_1, i : H; \Gamma \Downarrow P \oplus Q} \mathsf{D_s^l}}{\vdash \Theta_1, \Theta_2; \Gamma \Uparrow P \oplus Q} \quad \dfrac{}{\vdash \Theta_2; \cdot \Downarrow \; !^i H^\perp} \; \Downarrow\text{CC} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\vdash \Theta_1, i : H; \Gamma \Downarrow P \qquad \vdash \Theta_2; \cdot \Downarrow \; !^i H^\perp}{\vdash \Theta_1, \Theta_2; \Gamma \Uparrow P} \; \Downarrow\text{CC}}{\vdash \Theta_1, \Theta_2; \Gamma, P \oplus Q \Uparrow \cdot} \; \mathscr{L}\oplus}{\vdash \Theta_1, \Theta_2; \Gamma \Uparrow P \oplus Q} \text{ store}$$

When the left premise starts with the promotion rule, we also have several cases depending on the axioms of the introduced subexponential. Consider the case where i is unbounded and $4 \notin \mathscr{U}(i)$. Let $\Psi = (\Psi_1^u, \Psi_2^l)$ and $\Omega = (\Omega_1^u, \Omega_2^l)$:

$$\cfrac{\cfrac{\vdash \Omega +^{\succeq a}, \mathsf{c} : H, \mathsf{c} : \Psi_1^{\succeq a}; \cdot \Uparrow P, \mathscr{F}(\Psi_2)}{\vdash \Omega, \Psi, i : H; \cdot \Downarrow \; !^a P} \; !^a, ?^a, \mathsf{R_r} \qquad \cfrac{\cfrac{\vdash \Omega_1^{\succeq i} +, \mathsf{c} : \Psi_1^{\succeq i}; \cdot \Uparrow H^\perp}{\vdash \Omega_1, \Psi_1; \cdot \Downarrow \; !^i H^\perp} \; !^i, ?^i, \mathsf{R_r}}{\vdash \Omega, \Psi; \cdot \Uparrow \; !^a P} \; \Downarrow CC$$

Note that H is stored in the context c. The reduction considers a cut with $?^c H$ (instead of $?^i H$):

$$\cfrac{\cfrac{\vdash \Omega' +, \mathsf{c} : H, \mathsf{c} : \Psi_1'; \cdot \Uparrow P, \mathscr{F}(\Psi_2) \qquad \cfrac{\cfrac{\vdash \Omega' +, \mathsf{c} : \Psi_1'; \cdot \Uparrow H^\perp}{\vdash \Omega' +, \mathsf{c} : \Psi_1'; \cdot \Downarrow \; !^c H^\perp} \; !^c}{}}{\cfrac{\cfrac{\vdash \Omega' +, \mathsf{c} : \Psi_1'; \cdot \Uparrow P, \mathscr{F}(\Psi_2)}{\vdash \Omega, \Psi; \cdot \Downarrow \; !^a P} \; !^a, ?^a, \mathsf{R_r}}{\vdash \Omega, \Psi; \cdot \Uparrow \; !^a P} \; \text{store, } \mathsf{D_l}}}{} \; \Uparrow CC$$

Ψ_1' contains all the formulas in both $\Psi_1^{\succeq a}$ and $\Psi_1^{\succeq i}$ (the same with Ω'). Since i is unbounded, note that the application of $!^c$ above is valid.

Another interesting case, similar to the previous one, is when $\{\mathsf{C}, \mathsf{W}, 4\} \subseteq \mathscr{U}(i)$. Hence, H is moved to the context $i+$ and the reduction considers a (smaller) cut with $i+$ (instead of i).

Elimination of $\Uparrow\mathsf{C^*}$. When the list L_1 is not empty, the first formula in L_1 is necessarily principal in the left premise. Hence, we simply permute down the application of that rule.

When L_1 is empty, we consider two cases: either the cut formula is stored in the left premise, or it is necessarily stored in the right premise. The reductions for these cases are as follows:

$$\cfrac{\cfrac{\vdash \Theta_1; \Gamma, H \Uparrow L_1}{\vdash \Theta_1; \Gamma \Uparrow H, L_1} \; \text{store} \qquad \vdash \Theta_2; \Delta \Uparrow H^\perp, L_2}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L_1, L_2} \; \Uparrow\mathsf{C^*} \qquad \rightsquigarrow \qquad \cfrac{\vdash \Theta_1; H, \Gamma \Uparrow L_1 \qquad \vdash \Theta_2; \Delta \Uparrow H^\perp, L_2}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L_1, L_2} \; \Uparrow\mathsf{LC^*}$$

$$\cfrac{\vdash \Theta_1; \Gamma \Uparrow H, L_1 \qquad \cfrac{\vdash \Theta_2; \Delta, H^\perp \Uparrow L_2}{\vdash \Theta_2; \Delta \Uparrow H^\perp, L_2} \; \text{store}}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L_1, L_2} \; \Uparrow\mathsf{C^*} \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\vdash \Theta_2; H^\perp, \Delta \Uparrow L_2 \qquad \vdash \Theta_1; \Gamma \Uparrow H, L_1}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L_2, L_1} \; \Uparrow\mathsf{LC^*}}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L_1, L_2} \; \equiv_P$$

Elimination of $\Uparrow\mathsf{LC^*}$ Similarly to the previous case, when the list L_1 is not empty the reduction is easy. When this is not the case, the left premise must start with a decision rule. When the focus is on the cut formula, the reduction is as follows:

$$\cfrac{\cfrac{\vdash \Theta_1; \Gamma \Downarrow H}{\vdash \Theta_1; \Gamma, H \Uparrow \cdot} \; \mathsf{D_l} \qquad \vdash \Theta_2; \Delta \Uparrow H^\perp, L}{\vdash \Theta_1, \Theta_2 : \Gamma, \Delta \Uparrow L} \; \Uparrow\mathsf{LC^*} \qquad \rightsquigarrow \qquad \cfrac{\vdash \Theta_2; \Delta \Uparrow H^\perp, L \qquad \vdash \Theta_1; \Gamma \Downarrow H}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L} \; \Uparrow\mathsf{C}$$

When the proof continues by focusing on a formula different from the cut formula, an invertibility result is needed. Here, the case of $\oplus$ when F is a positive formula and $F \oplus G$ was taken from the unbounded context Λ (omitted in the derivation):

$$\cfrac{\cfrac{\cfrac{\vdash \Theta_1; \Gamma, H \Downarrow F}{\vdash \Theta_1; \Gamma, H \Downarrow F \oplus G} \; \oplus}{\vdash \Theta_1; \Gamma, H \Uparrow \cdot} \; \mathsf{D_s^u} \qquad \vdash \Theta_2; \Delta \Uparrow H^\perp, L}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L} \; \Uparrow\mathsf{LC^*} \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\cfrac{\cfrac{\vdash \Theta_1; \Gamma, H \Downarrow F}{\vdash \Theta_1; \Gamma, F, H \Uparrow \cdot} \; \mathsf{D_l} \qquad \vdash \Theta_2; \Delta \Uparrow H^\perp, L}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta, F \Uparrow L} \; \Uparrow\mathsf{LC^*}}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L, F} \; \mathscr{L}U}{\vdash \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow L} \; \mathscr{L}'\oplus}{}$$

where $\mathscr{L}U$ is the following lemma: if F is a positive formula and $\vdash \Theta; \Delta, F \Uparrow L$ then $\vdash \Theta; \Delta \Uparrow L, F$.

$\mathscr{L}'\oplus$ is a tailored version of the invertibility result we have already seen ($\mathscr{L}\oplus$) when $F \oplus G$ is stored in a unbounded context.

Theorem 3.5. (Admissibility of Cut). *The cut rules in Figure 6 are admissible.*

3.3 Discussion

Before concluding this section, let us explain some further motivations on the design of the five rules in Figure 6, as well as the proof of the above theorem in Coq.

Cut-rules. The Rule $\Uparrow C$ is the starting point of the procedure since the cut-formula H is principal in both premises. The other rules appear during the elimination process. For instance, as we already know, the elimination of $\Uparrow C$ depends on $\Uparrow LC^*$ when H is stored. Notice that the same case can be also solved with an application of $\Uparrow C^*$. In fact, one may inline the elimination procedure of $\Uparrow LC^*$ directly in the elimination of $\Uparrow C$ and $\Uparrow C^*$ and hence, only four rules will be needed. However, this implies replicating the same procedure several times during the proof. In particular, during the elimination of $\Uparrow C^*$, we need to consider whether H or $H^\perp$ is stored when L_1 is empty and, in each case, we have to apply a similar procedure from $\Uparrow LC^*$. The inclusion of $\Uparrow LC^*$ thus makes easier to understand the elimination process (and more amenable for mechanization).

The set of rules we have chosen is not the only possible alternative. For instance, instead of $\Uparrow LC^*$, one may think in reducing the same case in $\Uparrow C$ (when H is stored) with the following rule:

$$\frac{\vdash \Lambda, \Theta_1; \Gamma, H \Uparrow L \qquad \vdash \Lambda, \Theta_2; \Delta \Downarrow H^\perp}{\vdash \Lambda, \Theta; \Gamma, \Delta \Uparrow L} \;\Uparrow LC$$

In this case, $H^\perp$ is a negative atom or it must lose focus. In the first case, the cut can be eliminated without any additional cut. In the second case, we obtain an instance of $\Uparrow LC^*$. Therefore, $\Uparrow LC$ can be seen as an instance of $\Uparrow LC^*$.

Consider now the elimination of $\Uparrow CC$ when the list L is empty and the left premise focuses on some formula F. One may eliminate this case by applying the following rule:

$$\frac{\vdash \Lambda, \Theta_1, i : H; \Gamma \Downarrow F \qquad \vdash \Lambda, \Theta_2; \cdot \Downarrow \,!^i H^\perp}{\vdash \Lambda, \Theta; \Gamma \Downarrow F} \;\Downarrow CC^\star$$

However, this rule requires a side condition: H cannot be a positive atom. Without that condition, the (nonprovable) sequent $\vdash i : \top; \cdot \Downarrow A^\perp$ can be proved by using $\Downarrow CC^\star$ with $H = A$. Hence, the atomic case in the elimination of $\Uparrow CC$ must be treated separately and some additional lemmas need to be proved. In particular, it is required to show that if $\vdash \Lambda, \Theta_1, (i : A); \Gamma \Downarrow F$ and $\vdash \Lambda, \Theta_2; \cdot \Downarrow \,!^i A^\perp$ then $\vdash \Lambda, \Theta; \Gamma \Uparrow F$. Note that this lemma is an instance of the rule $\Downarrow CC$.

Mechanization in Coq. Besides the inherent complexity of dealing with the several cases appearing during the cut-elimination procedure, in this formalization we have to cope also with the problem of encoding the higher-order quantifiers. There are several alternatives for representing binders in proof assistants (see e.g., Chlipala 2008; Heberlin et al. 2017; Laurent 2021; Pitts 2003). The current formalization builds on our previous work (Felty et al. 2021), where the Hybrid system (Felty and Momigliano 2009) is used to represent the syntax. From the positive side, this choice has the immediate effect of allowing us to define binders at two levels: the meta-logic (*specification logic* in the terminology of the two-level approach in Hybrid) and the *object logic*, the logical system to be encoded as a theory in the specification logic. This separation reflects the different logics defined here: the specification logic is MMLL (and its system LNS$_{\mathsf{FMLL}}$) used to reason about different object logics (more on this in the next section). From the negative side, sequents cannot be defined as an inductive Type but as an inductive proposition (Prop). Hence, we cannot manipulate derivations as objects (due to the impossibility of doing pattern matching on a Prop

value). Therefore, we cannot extract a program (the cut-elimination procedure) from the proof of the theorem above and what we have is a proof of cut admissibility. The work of Laurent (2021) allows for defining first-order binders and sequents in Type. However, such binders cannot do pattern matching as in $\exists F, G.P(F \star G)$, where $P(\cdot)$ is a predicate symbol and $\star$ is a connective of the OL, nor to encode first-order object logics. As far as we know, it is not possible to represent in Coq quantifiers at the specification level and also at the object level and keep the definition of sequents in Type. The reader can find in Felty et al. (2021) details about the representation of syntax in Hybrid. The repository (Xavier and Olarte 2021) also includes further documentation on how to define the subexponential signature in MMLL as well as some tactics to improve automatization when using the library in Coq.

It is worth noticing that there are some recent advances towards providing native support in Coq for reasoning about binders. Frameworks, mainly using de Bruijn terms, have been developed to automatize some of the tasks in such reasoning. The work of Stark (2020) covers the mechanization of binders and it makes use of the *Autosubst 2* framework, that translates second-order HOAS specifications into potentially mutual inductive term sorts (Stark et al. 2019). We also refer the reader to Forster (2021), that uses de Bruijn terms to formalize Computability Theory in the Calculus of Inductive Constructions.

4. LNS **Systems as** MMLL **Theories**

This section dwells on the specification of a class of LNS systems as MMLL theories. More precisely, we encode the inference rules of a given OL as an MMLL theory. The encoding is modular and it allows for the specification of multimodal logics in a uniform way. We prove that the resulting specifications are *adequate* in the sense that an OL-sequent $\mathscr{S}$ is provable iff the encoding of $\mathscr{S}$ together with the resulting theory given by OL's rules is also provable in $\mathsf{LNS_{FMLL}}$. This level of adequacy (Nigam and Miller 2010) is usually known as **FCP** (*full completeness of proofs*).

In Miller and Pimentel (2013), LL was used as a logical framework for specifying a number of logical systems. Here we shall proceed similarly, but with MMLL as the meta-level system. In what follows, we assume that the subexponential signature of MMLL includes a distinguished label ω for storing theories, such that $i \preceq \omega$ for all $i \in \mathscr{I}$, and $\mathscr{U}(\omega) = \{\mathsf{C}, \mathsf{W}, \mathsf{K}, \mathsf{T}, \mathsf{D}, 4\}$. Other subexponentials will be added to the base signature depending on the logical system encoded.

OL formulas are specified using the meta-level predicates $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, that identify the occurrence of such formulas on the left and on the right side of the sequent respectively. Hence, OL sequents of the form $F_1, \ldots, F_n \vdash G_1, \ldots, G_m$, $n, m \geq 0$, are specified as the multiset of atomic MMLL formulas $\lfloor F_1 \rfloor, \ldots, \lfloor F_n \rfloor, \lceil G_1 \rceil, \ldots, \lceil G_m \rceil$. As a mnemonic, formulas on the (**L**)eft side of OL sequents are specified with the predicate starting with $\lfloor$. Given a set of OL-formulas Γ, we shall use $\lfloor \Gamma \rfloor$ to denote the set of MMLL-formulas $\{\lfloor F \rfloor \mid F \in \Gamma\}$, similarly for $\lceil \Gamma \rceil$.

OL-*Inference rules* are specified as rewriting clauses that replace the principal formula in the conclusion of the rule by the active formulas in the premises. The LL connectives indicate how these OL-formulas are connected: contexts are copied (&) or split ($\otimes$), in different inference rules ($\oplus$) or in the same sequent ($\mathscr{B}$). Such clauses are members of the MMLL theory $\mathscr{T}_{\mathscr{L}}$, specifying the behavior of the logical system $\mathscr{L}$. Theories will be stored in the subexponential ω, since these clauses can be used as many times as needed and also copied to any component. Figure 7 presents the specification of the LNS rules in Figure 1.

Consider the rules $\wedge_{Li}$ and $\wedge_R$ in Figure 1, and the corresponding clauses in Figure 7. The derivations in Figure 8 illustrate the behavior of these clauses once we focus on them. The first derivation in Figure 8 corresponds to focusing on the clause $\wedge_L$ from the theory $\mathscr{T}_{\mathsf{G}}$ where $\mathsf{I} = \mathsf{I_l}$ or $\mathsf{I} = \mathsf{I_s}$ and, accordingly, $\Gamma_1 = \lfloor F \wedge G \rfloor$, or $\Gamma_1 = \emptyset$ and $\lfloor F \wedge G \rfloor \in \Theta$. Bottom-up, the active formula $F \wedge G$ is taken from the linear or the classical context, and the whole positive phase (after the resulting negative phase) ends by storing the atom $\lfloor F \rfloor$ into the linear context. This derivation mimics *exactly* an application of the rule $\wedge_{L1}$ at the object level. Similarly, if instead of $\oplus_1$ we

$$\wedge_L : \lfloor F \wedge G \rfloor^{\perp} \otimes (\lfloor F \rfloor \oplus \lfloor G \rfloor) \qquad \wedge_R : \lceil F \wedge G \rceil^{\perp} \otimes (\lceil F \rceil \,\&\, \lceil G \rceil) \qquad \mathsf{f}_L : \lfloor \mathsf{f} \rfloor^{\perp} \otimes \top$$

$$\vee_L : \lfloor F \vee G \rfloor^{\perp} \otimes (\lceil F \rceil \,\&\, \lfloor G \rfloor) \qquad \vee_R : \lceil F \vee G \rceil^{\perp} \otimes (\lceil F \rceil \oplus \lceil G \rceil) \qquad \mathsf{t}_R : \lceil \mathsf{t} \rceil^{\perp} \otimes \top$$

$$\rightarrow_L : \lfloor F \rightarrow G \rfloor^{\perp} \otimes (\lceil F \rceil \otimes \lfloor G \rfloor) \qquad \rightarrow_R: \lceil F \rightarrow G \rceil^{\perp} \otimes (\lfloor F \rfloor \,\invamp\, \lceil G \rceil) \qquad \mathsf{init} : \lfloor F \rfloor^{\perp} \otimes \lceil F \rceil^{\perp}$$

Figure 7. Encoding of propositional rules of the system $\mathsf{LNS_G}$ for classical logic. In all the specification clauses, there is an implicit existential quantification on F and G.

Figure 8. Examples of derivations focusing on the clauses $\wedge_L$ and $\wedge_R$.

apply $\oplus_2$, the atom $\lfloor G \rfloor$ is stored, thus reflecting the behavior of $\wedge_{L2}$. The second derivation in Figure 8 starts by focusing on the clause $\wedge_R$, ending up with two premises corresponding exactly to the two premises of the OL-rule. The clauses for the other connectives and units work similarly. Moreover, when the clause corresponding to the initial rule is focused on, the proof must finish by showing that F is on the left and on the right of the OL-sequent. Given the identifier of a clause r (*i.e.*, an OL-rule), we shall use $[r]$ in $\mathsf{LNS_{FMLL}}$ derivations to collapse all the steps involved in the decomposition of the clause r. For instance:

$$\frac{\vdash \Theta, \omega : \mathscr{T}_{\mathsf{G}}; \lfloor F \rfloor \Uparrow \cdot}{\vdash \Theta, \omega : \mathscr{T}_{\mathsf{G}}; \lfloor F \wedge G \rfloor \Uparrow \cdot} \; [\,\wedge_{L1}\,]$$

Regarding the *structural rules* of weakening and contraction, it may be the case that an OL admits some of them on the left, right, or both sides of sequents. We mimic those behaviors by adding the structural rules in Figure 9 according to each case. For instance, if pos_i (weakening/-contraction for the left context) is in $\mathscr{T}_{\mathscr{L}}$, we can prove the equivalence $\lfloor F \rfloor \equiv ?^i \lfloor F \rfloor$. Hence, under the presence of pos_i, formulas of the shape $\lfloor F \rfloor$ can be stored in a unbounded subexponential context and they are free to be weakened and contracted (if i is unbounded). Similarly for neg_i and right formulas. Observe that pos_i and neg_i are parametric w.r.t. the subexponential label. For instance, if $i = \mathsf{c}$, then the use of such clauses is *restricted* to a component (*cf.* Definition 4).

Theorem 4.1. (Adequacy of $\mathscr{T}_{\mathsf{G}}$). *Let $\mathscr{T}_{\mathsf{G}}$ consist of the set of clauses in Figure 7 together with the structural rules pos_c and neg_c. Then $\mathscr{T}_{\mathsf{G}}$ is adequate **FCP** w.r.t. $\mathsf{LNS_G}$:*

   (1) **Soundness:** *If the sequent $\Gamma \vdash \Delta$ is provable in $\mathsf{LNS_G}$ then the sequent $\vdash \omega : \mathscr{T}_{\mathsf{G}}, \mathsf{c} : (\lfloor \Gamma \rfloor, \lceil \Delta \rceil); \cdot \Uparrow \cdot$ is provable in $\mathsf{LNS_{FMLL}}$.*
   (2) **Completeness:** *If the sequent $\vdash \omega : \mathscr{T}_{\mathsf{G}}, \mathsf{c} : (\lfloor \Gamma \rfloor, \lceil \Delta \rceil); \lfloor \Gamma' \rfloor, \lceil \Delta' \rceil \Uparrow \cdot$ is provable in $\mathsf{LNS_{FMLL}}$ then the sequent $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is provable in $\mathsf{LNS_G}$.*

Proof. As illustrated in Figure 8, we can prove that focusing on a clause is in one-to-one correspondence to the application of the corresponding rule in the OL. This is the case for all the rules introducing connectives. However, we cannot establish such a tight correspondence for the structural rules. In the proof of soundness, we are forced to use pos_c and neg_c to store all the atoms in

Structural rules:	$\mathsf{pos}_i : \lfloor F \rfloor^\perp \otimes (?^i \lfloor F \rfloor)$	$\mathsf{neg}_i : \lceil F \rceil^\perp \otimes (?^i \lceil F \rceil)$
Intuitionistic implication:	$\supset_L : \lfloor F \supset G \rfloor^\perp \otimes (\lceil F \rceil \otimes \lfloor G \rfloor)$	$\supset_R : \lceil F \supset G \rceil^\perp \otimes {!}^{t4}(\lfloor F \rfloor \,\vartheta\, \lceil G \rceil)$
Modal rules:	$\square_{Li} : \lfloor \square F \rfloor^\perp \otimes ?^i \lfloor F \rfloor$	$\square_{Ri} : \lceil \square F \rceil^\perp \otimes {!}^i \lceil F \rceil$

Figure 9. Encoding of structural, intuitionistic implication, and modal rules.

the unbounded subexponential context c. Moreover, contraction and weakening at the object level are mimicked by a meta-level result in $\mathsf{LNS_{FMLL}}$ (Theorem 3.2), but no clause in $\mathscr{T}_\mathsf{G}$ is applied. For completeness, the applications of pos_c and neg_c do not have an object-level counterpart (since there is no linear/classical distinction at the object-level). Note that the images of the $\mathsf{LNS_{FMLL}}$ sequents before and after the application of these rules are the same OL-sequent. $\qquad\square$

In the intuitionistic system $\mathsf{LNS_I}$ (see Figures 1, 2 and 3), the rule $\supset_R$ creates a new component while $\mathtt{lift}$ moves formulas across components. Such behavior will be specified with the subexponential $t4$ where $\mathscr{U}(t4) = \{\mathsf{K, C, W, T, 4}\}$. The encoding is thus obtained by considering the clauses in Figure 7 but replacing the rules for implication for those in Figure 9.

The proof of adequacy for this encoding requires an intermediary result. Consider the $\mathsf{LNS_I}$ derivation on the left:

$$
\cfrac{\cfrac{\cfrac{\cdot \vdash G /\!\!/ F, C \vdash A}{F \vdash G /\!\!/ C \vdash A}\ \mathtt{lift}}{F \vdash G /\!\!/ C \vdash (A \vee B)}\ \vee_{R1}}{F \vdash G, C \supset (A \vee B)}\ \supset_R
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{F, C \vdash A}{F, C \vdash (A \vee B)}\ \vee_{R1}}{\cdot \vdash G /\!\!/ F, C \vdash (A \vee B)}\ \mathsf{R}}{F \vdash G /\!\!/ C \vdash (A \vee B)}\ \mathtt{lift}}{F \vdash G, C \supset (A \vee B)}\ \supset_R
\tag{1}
$$

This OL-derivation does not have a corresponding $\mathsf{LNS_{FMLL}}$-derivation: once we focus on the clause $\supset_R$, necessarily the formula F is moved to the next component ($\mathtt{lift}$) and, after finishing the subexponential phase, we are free to focus on the clause $\vee_{R1}$. Hence, we shall show adequacy w.r.t. an alternative equivalent system, following the sequentialization procedure for LNS proposed in Pimentel et al. (2019).

More precisely, we consider the system $\mathsf{LNS_I^S}$ which is as $\mathsf{LNS_I}$ where the list $\mathscr{G}$ is empty in all the rules in Figures 1, 2 and 3 and the following *release* rule is added:

$$
\cfrac{\Gamma' \vdash \Delta'}{\Gamma \vdash \Delta /\!\!/^i \Gamma' \vdash \Delta'}\ \mathsf{R}
\tag{2}
$$

The restriction on $\mathscr{G}$ guarantees that the introduction rules of connectives are applied only after the modal phase as in $\mathsf{LNS_{FMLL}}$. This strategy is complete (Pimentel et al. 2019) since it is always possible to organize LNS proofs by first applying all the lift operations to later apply the introduction rules on the last component. For that, it is shown that the application of rules (*e.g.*, $\vee_{R_1}$ above) permutes down w.r.t the lift rule. See for instance the right derivation in (1).

Theorem 4.2. (Adequacy of $\mathscr{T}_I$). *Let $\mathscr{T}_I$ contain pos_{t4}, neg_c plus the introduction clauses of $\mathscr{T}_\mathsf{G}$ with the clauses for implication substituted by the clauses in Figure 9. Then $\mathscr{T}_I$ is adequate **FCP** w.r.t. $\mathsf{LNS_I^S}$:*

(1) **Soundness:** *If the sequent $\Gamma \vdash \Delta$ is provable in $\mathsf{LNS_I^S}$ then the sequent $\vdash \omega : \mathscr{T}_I, t4 : \lfloor \Gamma \rfloor, \mathsf{c} : \lceil \Delta \rceil; \Uparrow \cdot$ is provable in $\mathsf{LNS_{FMLL}}$.*
(2) **Completeness:** *If the sequent $\vdash \omega : \mathscr{T}_I, t4 : \lfloor \Gamma \rfloor, \mathsf{c} : \lceil \Delta \rceil; \lfloor \Gamma' \rfloor, \lceil \Delta' \rceil \Uparrow \cdot$ is provable in $\mathsf{LNS_{FMLL}}$ then the sequent $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is provable in $\mathsf{LNS_I^S}$.*

Proof. For all the rules but implication right, the discussion is similar to the classical case. Consider the following derivation:

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{\vdash t4:\Theta, t4:\lfloor F\rfloor, \mathsf{c}:\lceil G\rceil\omega:\mathscr{T}_{\mathsf{l}};\cdot\Uparrow\cdot}
{\vdash t4:\Theta, \omega:\mathscr{T}_{\mathsf{l}};\lfloor F\rfloor, \lceil G\rceil\Uparrow\cdot}\ [\mathsf{pos}_{t4}],[\mathsf{neg}_{\mathsf{c}}]}
{\vdash\cdot;\cdot\Uparrow/\!\!/^{t4}\vdash t4:\Theta, \omega:\mathscr{T}_{\mathsf{l}};\cdot\Uparrow\lfloor F\rfloor\otimes\lceil G\rceil}\ R_r,\otimes, 2\times\mathsf{store}}
{\vdash\Theta, \omega:\mathscr{T}_{\mathsf{l}};\cdot\Uparrow/\!\!/^{t4}\vdash\cdot;\cdot\Uparrow\lfloor F\rfloor\otimes\lceil G\rceil}\ ?^{t4}_4}
{
\cfrac{
\cfrac{\vdash\Theta,\omega:\mathscr{T}_{\mathsf{l}};\cdot\Downarrow\lceil F\supset G\rceil^{\perp}\ \mathsf{I_s}\qquad \vdash\Theta,\omega:\mathscr{T}_{\mathsf{l}};\cdot\Downarrow !^{t4}(\lfloor F\rfloor\otimes\lceil G\rceil)}
{\vdash\Theta,\omega:\mathscr{T}_{\mathsf{l}};\cdot\Downarrow\exists F, G.(\lceil F\supset G\rceil^{\perp}\otimes !^{t4}(\lfloor F\rfloor\otimes\lceil G\rceil))}}
{\vdash\Theta,\omega:\mathscr{T}_{\mathsf{l}};\cdot\Uparrow\cdot}\ \mathsf{D_s}
}
$$

Here, the context Θ is split into $\mathsf{c}:\Theta$ and $t4:\Theta$. The former contains all the right formulas that will be forgotten in the subexponential phase. The latter contains all the left formulas that will be carried over the components. Note also that $\mathsf{T}\in\mathscr{U}(t4)$ and then, $t4+=t4$. Moreover, since $t4\preceq\omega$, the theory $\mathscr{T}_{\mathsf{l}}$ always moves between components. This derivation corresponds, one-to-one, to an application of $\supset_R$ at the object level followed by zero or more application of `lift` and ending with the release rule R in $\mathsf{LNS}^{S}_{\mathsf{I}}$. □

We now analyze the specification of modal logics. In order to propose a parametric and modular encoding, we consider different subexponentials for each modal behavior. For that, the following notation will be handy:

- the subexponential structure will include the set of labels $M=\{k, t, d, 4, t4, d4\}$, where $\{\mathsf{K},\mathsf{C},\mathsf{W}\}\subseteq\mathscr{U}(l)$ for all $l\in M$;
- the name of each $l\in M$ matches the respective assumed axioms. For example, $\mathscr{U}(d4)=\{\mathsf{K},\mathsf{C},\mathsf{W},\mathsf{D},4\}$.

The (parameterized) clauses specifying the rules for box are given in Figure 9. Remember that ℓ represents a (possibly empty) subset of $\mathscr{A}=\{\mathsf{T},\mathsf{D},4\}$ and $\mathsf{K}\ell$ is the modal logic resulting from extending K with the axioms in ℓ. The theory $\mathscr{T}_{\mathsf{K}\ell}$ for $\mathsf{K}\ell$ is given by the clauses of $\mathscr{T}_{\mathsf{G}}$ (Figure 7) plus the clauses $\mathsf{neg}_{\mathsf{c}}$ and $\mathsf{pos}_{\mathsf{c}}$ (Figure 9) and the clauses $\Box_{Ll}$ and $\Box_{Rl}$ (Figure 9), where $l\in M$ is the subexponential corresponding to ℓ. For example, $\mathscr{T}_{\mathsf{KT4}}=\mathscr{T}_{\mathsf{G}}\cup\{\mathsf{neg}_{\mathsf{c}},\mathsf{pos}_{\mathsf{c}}\}\cup\{\Box_{Lt4},\Box_{Rt4}\}$.

For adequacy, we also consider a sequentialized LNS system. As we did for the intuitionistic case, all the rules introducing the logical connectives rules in $\mathsf{LNS}^{S}_{\mathsf{K}\ell}$ assume that the history $\mathscr{G}$ is empty. Moreover, the release rule (2) is added to the system.

Theorem 4.3. (Adequacy of $\mathscr{T}_{\mathsf{K}\ell}$). $\mathscr{T}_{\mathsf{K}\ell}$ *is adequate **FCP** w.r.t.* $\mathsf{LNS}^{S}_{\mathsf{K}\ell}$.

(1) **Soundness:** *If the sequent* $\Box\Phi,\Gamma\vdash\Delta$ *is provable in* $\mathsf{LNS}^{S}_{\mathsf{K}\ell}$ *then the sequent* $\vdash\omega:\mathscr{T}_{\mathsf{K}\ell}, l:\lfloor\Phi\rfloor, \mathsf{c}:(\lfloor\Gamma\rfloor,\lceil\Delta\rceil);\cdot\Uparrow\cdot$ *is provable in* $\mathsf{LNS}_{\mathsf{FMLL}}$.
(2) **Completeness:** *If the sequent* $\vdash\omega:\mathscr{T}_{\mathsf{K}\ell}, l:\lfloor\Phi\rfloor, \mathsf{c}:(\lfloor\Gamma\rfloor,\lceil\Delta\rceil);\lfloor\Gamma'\rfloor,\lceil\Delta'\rceil\Uparrow\cdot$ *is provable in* $\mathsf{LNS}_{\mathsf{FMLL}}$ *then the sequent* $\Box\Phi,\Gamma,\Gamma'\vdash\Delta,\Delta'$ *is provable in* $\mathsf{LNS}^{S}_{\mathsf{K}\ell}$.

Proof. Consider the following derivation.

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overset{\pi}{\vdash\Theta(\mathsf{c}/k), \Theta(4+), t4:\Theta, \omega:\mathscr{T}_{\mathsf{K}\ell};\lceil A\rceil\Uparrow\cdot}}
{\vdash\Theta(j);\cdot\Uparrow/\!\!/^{l}\vdash\Theta(\mathsf{c}/k),\Theta(4+), t4:\Theta, \omega:\mathscr{T}_{\mathsf{K}\ell};\cdot\Uparrow\lceil A\rceil}\ R_r,\mathsf{store}}
{\vdash\Theta,\omega:\mathscr{T}_{\mathsf{K}\ell};\cdot\Uparrow/\!\!/^{l}\vdash\cdot;\cdot\Uparrow\lceil A\rceil}\ ?^{l}_{ku}, ?^{l}_4}
{
\cfrac{
\cfrac{\vdash\Theta,\omega:\mathscr{T}_{\mathsf{K}\ell};\cdot\Downarrow\lceil\Box A\rceil^{\perp}\ \mathsf{I_s}\qquad \vdash\Theta,\omega:\mathscr{T}_{\mathsf{K}\ell};\cdot\Downarrow !^{l}\lceil A\rceil}
{\vdash\Theta,\omega:\mathscr{T}_{\mathsf{K}\ell};\cdot\Downarrow\exists F.(\lceil\Box F\rceil^{\perp}\otimes !^{l}\lceil F\rceil)}\ \exists,\otimes}}
{\vdash\Theta,\omega:\mathscr{T}_{\mathsf{K}\ell};\cdot\Uparrow\cdot}\ \mathsf{D_s}
}
$$

The context Θ can be split into: $\Theta(j)$ with subexponentials not related to l; $\Theta(k)$ containing subexponentials without 4; $\Theta(4)$ containing subexponentials with 4 and without T; and $\Theta(t4)$ is

a context with subexponentials featuring both 4 and T. The context $\Theta(j)$ contains all the right formulas, together with all the left formulas in contexts not related to i that will be "forgotten" after the application of the release rule; using the rule $?!_4$, the formulas in context $\Theta(4)$ are moved to the next component and the axiom T is added. Using the same rule, the context $\Theta(t4)$ remains the same when passing to the next component. Moreover, the formulas in the context $\Theta(k)$ are moved without the box and then, stored in the subexponential c (notation $\Theta(c/k)$). Derivation π continues by using $\mathsf{neg_c}$ and storing $\lceil A \rceil$ in the context c. This derivation adequately captures the behavior of $\square$ in the logic LNS_ℓ. $\qquad\square$

It is worth noticing the *modularity* of the encodings: all the modal systems have *exactly* the same encoding, only differing on the meta-level modality. This is a direct consequence of locality, granted by LNS. Therefore, we are able to spot the core characteristics of the logical systems, allowing punctual actions to be taken at the meta-level. This opens the possibility of being able to adequately encode a larger class of modal systems inside $\mathsf{LNS_{FMLL}}$. For instance, if we are considering a (modal) substructural logic where formulas not necessarily behave classically, it suffices to remove pos and/or neg accordingly and decree that $\mathsf{U} \not\subset \mathscr{U}(l)$ for the context l storing left boxed formulas.

The method proposed here is also adequate for specifying simply dependent multimodal OL logics (see Definition 3). For each $i \in N$, we consider a subexponential l_i and decree that $l_i \preceq l_j$ iff $i \preccurlyeq j$ (and hence, $!^{l_j}F \multimap !^{l_i}F$). Moreover, $\mathscr{U}(l_i) = f(i)$. That is, each modal behavior specified by the indices in N is mapped to a subexponential with the same modal behavior. The system $\mathsf{LNS}_{(N,\preccurlyeq,f)}$ can be then specified in $\mathsf{LNS_{FMLL}}$ as the theory $\mathscr{T}_{\mathsf{K}\ell_N}$ as we did before for the one-modality case but considering a clause $\square_{L_i}$ (Figure 9) for each $i \in N$. As expected, this clause stores F in the context l_i when the atom $\lfloor \square_i F \rfloor$ is present in the linear or subexponential context. Similarly, we have an instance of $\square_{R_i}$ in Figure 9 for each $i \in N$ that starts a subexponential phase indexed with l_i.

Theorem 4.4. (Adequacy of multimodal systems). $\mathscr{T}_{\mathsf{K}\ell_N}$ is adequate FCP w.r.t. $\mathsf{LNS}^S_{(N,\preccurlyeq,f)}$.

(1) **Soundness:** *If the sequent* $\{\square_i \Phi_i\}, \Gamma \vdash \Delta$ *is provable in* $\mathsf{LNS}^S_{(N,\preccurlyeq,f)}$ *then the sequent*
$\vdash \omega : \mathscr{T}_{\mathsf{K}\ell_N}, \{l_i : \lfloor \Phi_i \rfloor\}, \mathsf{c} : (\lfloor \Gamma \rfloor, \lceil \Delta \rceil); \Uparrow \cdot$ *is provable in* $\mathsf{LNS_{FMLL}}$.
(2) **Completeness:** *If the sequent* $\vdash \omega : \mathscr{T}_{\mathsf{K}\ell_N}, \{l_i : \lfloor \Phi_i \rfloor\}, \mathsf{c} : (\lfloor \Gamma \rfloor, \lceil \Delta \rceil); \lfloor \Gamma' \rfloor, \lceil \Delta' \rceil \Uparrow \cdot$ *is provable in* $\mathsf{LNS_{FMLL}}$ *then the sequent* $\{\square_i \Phi_i\}, \Gamma, \Gamma' \vdash \Delta, \Delta'$ *is provable in* $\mathsf{LNS}^S_{\mathsf{K}\ell}$.

Let us give a further example comparing derivations in the object- and meta-systems. In particular, we illustrate the reason of the apparently missing clause encoding the modal rule T, and the advantage, from the proof search perspective, of the stricter control of the modal rules in Figure 5, where applications of the rule K on a subexponential with 4 are not allowed.

Example 4.1. Consider the following OL-derivations:

$$
\dfrac{\dfrac{\overline{A \vdash A}\ \text{init}\quad \overline{B \vdash B}\ \text{init}}{\dfrac{A \supset B, A \vdash B}{\dfrac{\square(A \supset B), A \vdash B}{\square(A \supset B) \vdash A \supset B}\ \supset_R}\ \mathsf{T}}\ {\supset_L}}{}
$$

$$
\dfrac{\overline{A, \square A \vdash A}\ \text{init}\qquad \dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ \text{init}}{A, \square A \vdash \cdot /\!/ \cdot \vdash A}\ \mathsf{K, R}}{A, \square A \vdash \square A}\ \square_R}{}}{\dfrac{A, \square A \vdash A \wedge \square A}{\dfrac{\dfrac{\square A \vdash \cdot /\!/ \square A \vdash A \wedge \square A}{\dfrac{\square A, \square A \vdash \cdot /\!/ \cdot \vdash A \wedge \square A}{\dfrac{\square A, \square A \vdash \square(A \wedge \square A)}{\square A \vdash \square(A \wedge \square A)}\ \mathsf{C}_L}\ \square_R}\ 4}\ \mathsf{K, R}}\ \wedge_R}
$$

The underlying system in the first (resp. second) derivation is $\mathsf{LNS_{KT}}$ (resp. $\mathsf{LNS_{K4}}$). Below we show the corresponding meta-level derivations in the system $\mathsf{LNS_{FMLL}}$. For simplicity, we omit

the context ω containing the theory with the clauses of each system, and we do not use pos/neg, keeping the $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ atoms in the linear context:

$$
\dfrac{\dfrac{}{\vdash t : \lfloor A \supset B \rfloor; \cdot \Downarrow \lfloor A \supset B \rfloor^{\perp}} \; \mathsf{I_s} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \cdot; \lfloor A \rfloor, \lceil A \rceil \Uparrow \cdot}\; [\mathsf{init}] \quad \dfrac{}{\vdash \cdot; \lfloor B \rfloor, \lceil B \rceil \Uparrow \cdot}\; [\mathsf{init}]}{\vdash \cdot; \lfloor A \rfloor, \lceil B \rceil \Downarrow \lceil A \rceil \otimes \lfloor B \rfloor}\; \otimes, \mathsf{R_n}, \mathsf{store}}{\vdash t : \lfloor A \supset B \rfloor; \lfloor A \rfloor, \lceil B \rceil \Uparrow \cdot}\; [\supset_L]}{\dfrac{\vdash \cdot; \lfloor \Box(A \supset B) \rfloor, \lfloor A \rfloor, \lceil B \rceil \Uparrow \cdot}{\vdash \cdot; \lfloor \Box(A \supset B) \rfloor, \lceil A \supset B \rceil \Uparrow \cdot}\; [\supset_R]}\; [\Box_L]
$$

$$
\dfrac{\dfrac{\dfrac{}{\vdash 4+ : \lfloor A \rfloor; \lceil A \rceil \Uparrow \cdot}\; [\mathsf{init}] \qquad \dfrac{\dfrac{}{\vdash 4+ : \lfloor A \rfloor; \lceil A \rceil \Uparrow \cdot}\; [\mathsf{init}]}{\vdash 4+ : \lfloor A \rfloor; \lceil \Box A \rceil \Uparrow \cdot}\; [\Box_R]}{\dfrac{\vdash 4+ : \lfloor A \rfloor; \lceil A \wedge \Box A \rceil \Uparrow \cdot}{\vdash 4 : \lfloor A \rfloor; \lceil \Box(A \wedge \Box A) \rceil \Uparrow \cdot}\; [\Box_R]}\; [\wedge_R]}{\vdash \cdot; \lfloor \Box A \rfloor, \lceil \Box(A \wedge \Box A) \rceil \Uparrow \cdot}\; [\Box_L]
$$

It is worth noticing that we do not have an explicit clause for capturing the modal rule T. Instead, by storing the formula $\lfloor A \supset B \rfloor$ into the context t, such atom can be used in the current context by applying the rule $\mathsf{I_s}$, as in the leftmost leaf in the first derivation.

In the second meta-level derivation there is no explicit contraction, unlike in the OL derivation. The reason is the neater control of the rules for 4 and K in $\mathsf{LNS_{FMLL}}$. Note that the formula $\lfloor A \rfloor$, after the application of $[\Box_R]$, is located at 4+. Due to the presence of T in 4+, the left most application of [init] is valid. This branch of the derivation simulates the application of the rule K at the object level. The topmost application of $[\Box_R]$ (with the corresponding applications of $?_4^4$) simulates the application of 4 at the object level.

The current version of the companion Coq library of this paper contains the mechanization of some of the adequacy results presented in this section. For each OL, there is a separated directory defining the derivability relation of the corresponding logic as an inductive proposition. Then, the clauses of the MMLL theory are introduced and the adequacy theorem is established. Defined tactics allow for some automation when dealing with the bipoles in the theory. In some cases, such tactics complete the decomposition of the bipole once the terms substituting the existential quantified variables have been chosen. For the time being of this publication, the adequacy results for classical (Theorem 4.1) and intuitionistic (Theorem 4.2) logics have been mechanized, as well as the instance $\ell = \{\mathsf{T}, 4\}$ (i.e., the modal logic S4) for Theorem 4.3.

5. Cut Admissibility for Object Logics

In the previous section, we showed how to adequately encode a logical systems $\mathscr{L}$ as an MMLL theory $\mathscr{T}_{\mathscr{L}}$. In this section, we introduce clauses that adequately capture the cut rule at the object level, leading to the theory $\mathscr{T}_{\mathscr{L}+\mathsf{cut}}$. More interestingly, if the encoding satisfies a condition called *cut-coherence*, then both $\mathscr{T}_{\mathscr{L}}$ and $\mathscr{T}_{\mathscr{L}+\mathsf{cut}}$ prove the same (encoded) sequents, thus showing that the cut rule is admissible at the object level. This result is proved by relying on the meta-theory of MMLL. As we shall see, testing cut-coherence is straightforward and hence, $\mathsf{LNS_{FMLL}}$ becomes a suitable logical framework for proving analyticity for a large class of systems, including several well known modal logical systems and also multi- and substructural-modal logics.

We start by setting some requirements that OLs should comply in order to be amenable for the MMLL specification.

Definition 6. (Canonical-bipoles). *A MMLL-formula is a* bipole *(Andreoli 1992) if no positive connective is in the scope of a negative one, bangs have negative scope while question marks have atomic scope. A MMLL-formula F is a* canonical-bipole *if F is a bipole built from MMLL connectives and atomic formulas of the shape $\lceil G \rceil$, $\lfloor G \rfloor$ where G is an OL-formula.*

Observe that all the clauses introducing connectives in Figures 7 and 9 have the shape $\exists \overline{F}.(H^{\perp} \otimes G)$, where H is atomic and G a canonical-bipole, *e.g.* $\exists F, G.(\lceil F \supset G \rceil^{\perp} \otimes \,!^{t4}(\lfloor F \rfloor \,\mathbin{⅋}\, \lceil G \rceil))$ and $\exists F.(\lfloor \Box F \rfloor^{\perp} \otimes \,?^{i}\lfloor F \rfloor)$. As seen in Section 4, focusing on this kind of formulas produces specific and controlled shapes of derivations in $\mathsf{LNS_{FMLL}}$.

Requirement 5.1. (Canonical theories and encodings). Let $\mathscr{C}$ be the set of connectives of the object logic $\mathscr{L}$. The *encoding* of $\mathscr{L}$ as an MMLL theory is a pair of functions $\mathsf{B}\lfloor \cdot \rfloor$ and $\mathsf{B}\lceil \cdot \rceil$ from $\mathscr{C}$ to MMLL canonical-bipoles. The *encoding of left and right introduction rules* for a given n-ary connective $\star \in \mathscr{C}$ is defined as, respectively

$$\mathsf{E}\lfloor \star \rfloor = \exists F_1, ..., F_n.(\lfloor \star(F_1, ..., F_n) \rfloor^{\perp} \otimes \mathsf{B}\lfloor \star \rfloor) \qquad \mathsf{E}\lceil \star \rceil = \exists F_1, ..., F_n.(\lceil \star(F_1, ..., F_n) \rceil^{\perp} \otimes \mathsf{B}\lceil \star \rceil)$$

The *canonical theory* for $\mathscr{L}$ is the least set $\mathscr{T}_{\mathscr{L}}$ s.t. (1) for each $\star \in \mathscr{C}$, $\mathsf{E}\lfloor \star \rfloor, \mathsf{E}\lceil \star \rceil \in \mathscr{T}_{\mathscr{L}}$; (2) $\mathsf{pos}_i, \mathsf{neg}_j$ may belong to $\mathscr{T}_{\mathscr{L}}$; and (3) $\mathsf{init} \in \mathscr{T}_{\mathscr{L}}$ (see Figure 7). Moreover, the theory $\mathscr{T}_{\mathscr{L}}$ must be adequate for the OL $\mathscr{L}$: a sequent is provable in $\mathscr{L}$ iff its encoding is provable in $\mathsf{LNS}_{\mathsf{FMLL}}$ with the clauses $\mathscr{T}_{\mathscr{L}}$.

In words, $\mathscr{T}_{\mathscr{L}}$ includes the encoding of left and right introduction rules as well as the initial rule. Depending on the structural behavior of the OL, $\mathscr{T}_{\mathscr{L}}$ may include also the encoding of the rules for weakening and contraction. As already shown in the previous section, the encoded inference rules determine completely the shape of derivations in $\mathsf{LNS}_{\mathsf{FMLL}}$. Also, we have showed that the proposed encodings are adequate.

Regarding the specification of structural rules via pos_i and neg_j clauses, recall that, in intuitionistic and modal systems, the right formulas never move between components. This explains why neg should be instantiated with c. Hence, as an invariant, the use of the rules in the theory $\mathscr{T}_{\mathscr{L}}$ can only store atoms of the form $\lceil F \rceil$ in the context c (if neg_c is part of the theory).

In LNS_I, left formulas can always be moved to the next component. Moreover, besides c and ω, the encoding requires only an extra subexponential: $t4$ (Theorem 4.2). Hence, pos_i is instantiated with $t4$. On the other hand, in modal systems, only boxed formulas on the left can be moved between components. Hence, in these logical systems, pos_i must be instantiated with $i = \mathsf{c}$ (and unboxed formulas cannot be moved between components).

In any case, the following technical requirement, needed in the proof of Theorem 5.2, is valid.

Requirement 5.2. (pos and neg). If $\mathsf{neg}_j \in \mathscr{T}_{\mathscr{L}}$ then $j = \mathsf{c}$. Moreover, if $\mathsf{pos}_i \in \mathscr{T}_{\mathscr{L}}$ then $\{\mathsf{C}, \mathsf{W}, \mathsf{T}\} \subseteq \mathscr{U}(i)$ and either: (1) $i = \mathsf{c}$ or (2) $i \neq \mathsf{c}$ and $i \preceq m$ for all $m \in \mathscr{I}, m \neq \mathsf{c}$.

5.1 Cut rules, cut coherence, and admissibility of cut

The (multiplicative) OL-cut rule can be specified as the bipole $\mathsf{cut} = \exists F.(\lfloor F \rfloor \otimes \lceil F \rceil)$. In fact, focusing on that formula mimics exactly the behavior of the cut rule at the object level:

$$\frac{\Gamma_1 \vdash \Delta_1, F \quad \Gamma_2, F \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ \mathsf{cut} \quad \Longleftrightarrow \quad \frac{\dfrac{\vdash \omega : \mathsf{cut}; \lfloor \Gamma_1 \rfloor, \lceil \Delta_1 \rceil, \lceil F \rceil \Uparrow \cdot \quad \vdash \omega : \mathsf{cut}; \lfloor \Gamma_2 \rfloor, \lfloor F \rfloor, \lceil \Delta_2 \rceil \Uparrow \cdot}{\dfrac{\vdash \omega : \mathsf{cut}; \lfloor \Gamma_1 \rfloor, \lfloor \Gamma_2 \rfloor, \lceil \Delta_1 \rceil, \lceil \Delta_2 \rceil \Downarrow \mathsf{cut}}{\vdash \omega : \mathsf{cut}; \lfloor \Gamma_1 \rfloor, \lfloor \Gamma_2 \rfloor, \lceil \Delta_1 \rceil, \lceil \Delta_2 \rceil \Uparrow \cdot} \ \mathsf{D_s}} \ \exists, \otimes, \mathsf{R_n}, \mathsf{store}}$$

We shall use cut_n to denote the rule cut applied to (OL-) formulas of size strictly smaller than n. For instance, if $G = G_1 \star G_2$, a valid application of $\mathsf{cut}_{|G|}$ can instantiate the existentially quantified variable F in cut with either G_1 or G_2 (but not with G).

Using the formulas cut and init, we can prove that $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are *duals* in the sense that the $\mathsf{LNS}_{\mathsf{FMLL}}$ sequent $\vdash \omega : \{\mathsf{cut}, \mathsf{init}\}; \Uparrow \lfloor F \rfloor \equiv \lceil F \rceil^{\perp}$ is provable. More interestingly, this duality can be generalized to the right and left bodies of OL rules as well. More precisely,

Definition 7. (Cut coherence). *Let $\mathscr{T}_{\mathscr{L}}$ be the canonical theory of the OL $\mathscr{L}$. We say that $\mathscr{T}_{\mathscr{L}}$ is cut-coherent if, for each connective $\star \in \mathscr{C}$, and $F = \star(F_1, ..., F_n)$, the sequent below is provable*

$$\vdash \omega : \mathsf{cut}_{|F|}; \Uparrow \forall F_1, ..., F_n.((\mathsf{B}\lfloor \star \rfloor)^{\perp} \otimes (\mathsf{B}\lceil \star \rceil)^{\perp})$$

Theorem 5.1. *All the encodings in Section 4 are cut coherent.*

Proof. Proving this kind of sequents in $\mathsf{LNS_{FMLL}}$ is almost immediate. Here the example for the encoding of the rules introducing $\square$ in the logic K:

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\cfrac{\vdash \omega : \mathsf{cut}, \mathsf{c} : \lceil F \rceil^{\perp}; \lfloor F \rfloor^{\perp}, \lfloor F \rfloor \Uparrow \cdot \quad \mathsf{D_l}, \mathsf{I_l} \qquad \vdash \omega : \mathsf{cut}, \mathsf{c} : \lceil F \rceil^{\perp}; \lceil F \rceil \Uparrow \cdot \quad \mathsf{D_s}, \mathsf{I_l}}{\vdash \omega : \mathsf{cut}, \mathsf{c} : \lceil F \rceil^{\perp}; \lfloor F \rfloor^{\perp} \Uparrow \cdot} [\mathsf{cut}(F)]
}{\vdash \omega : \mathsf{cut}, k : \lceil F \rceil^{\perp}; \cdot \Uparrow /\!\!/^k \vdash \cdot; \cdot \Uparrow \lfloor F \rfloor^{\perp}} \; ?^k_{\mathsf{ku}}, ?^{\omega}, \mathsf{R_r}, \mathsf{store}
}{\vdash \omega : \mathsf{cut}, k : \lceil F \rceil^{\perp}; \cdot \Downarrow \; !^k \lfloor F \rfloor^{\perp}} \; !^k
}{\vdash \omega : \mathsf{cut}, k : \lceil F \rceil^{\perp}; !^k \lfloor F \rfloor^{\perp} \Uparrow \cdot} \; \mathsf{D_l}
}{\vdash \omega : \mathsf{cut}; \cdot \Uparrow \forall F.((?^k \lfloor F \rfloor)^{\perp} \otimes (!^k \lceil F \rceil)^{\perp})}
}
$$

Note that cut is instantiated with F (a subformula of $\square F$). $\qquad\square$

We now state the main result of the section. Consider two cut-free proof derivations from the object-level point of view (i.e., using only $\mathscr{T}_{\mathscr{L}}$) introducing the cut formula F. It is possible to prove the same sequent using the rule cut (at the object-level) with strict subformulas of F.

Theorem 5.2. (OL cut admissibility). *Let $\mathscr{T}_{\mathscr{L}}$ be the canonical theory of a given OL $\mathscr{L}$, Γ, Δ be multisets, Θ_1, Θ_2 bounded and Λ unbounded subexponential contexts containing only atoms of the form $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$. If the sequents $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_1; \Gamma, \lfloor F \rfloor \Uparrow \cdot$ and $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_2; \Delta, \lceil F \rceil \Uparrow \cdot$ are both provable, then the sequent $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}_{|F|}\}, \Lambda, \Theta_1, \Theta_2; \Gamma, \Delta \Uparrow \cdot$ is also provable.*

Proof. When the theory includes the structural rules pos_i and/or $\mathsf{neg_c}$, the proof of this theorem requires to count the number of times a formula stored in a unbounded subexponential context is used in a derivation. Since this is difficult to formalize, we introduce two additional clauses (depending on the structural behavior of the OL at hand):

(a) $\mathsf{cutC} = \exists F.(?^i \lfloor F \rfloor \otimes ?^c \lceil F \rceil)$ (when both pos_i and $\mathsf{neg_c}$ are present)
(b) $\mathsf{cutN} = \exists F.(?^i \lfloor F \rfloor \otimes !^c \lceil F \rceil)$ (when only pos_i is present)

We will write cutC_n, cutN_n to denote the rules cutC, cutN applied to (OL) formulas of size strictly smaller than n. Observe that, in the presence of the corresponding structural rules, cutC and cutN are equivalent to cut (remember that with pos_i, it is possible to prove the equivalences $\lfloor F \rfloor \equiv ?^i \lfloor F \rfloor$ and $\lceil F \rceil \equiv !^i \lceil F \rceil$).

Assume that the OL features both, pos_i and $\mathsf{neg_c}$. Hence, we consider the rules cut and cutC and we show, simultaneously, both:

(1) if $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_1; \Omega, \lfloor F \rfloor \Uparrow \cdot$ and $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_2; \Psi, \lceil F \rceil \Uparrow \cdot$ are both provable then the sequent
   $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}_{|F|}, \mathsf{cutC}_{|F|}\}, \Lambda, \Theta_1, \Theta_2; \Omega, \Psi \Uparrow \cdot$ is also provable; and
(2) if $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_1, i : \lfloor F \rfloor; \Omega \Uparrow \cdot$ and $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_2, \mathsf{c} : \lceil F \rceil; \Psi \Uparrow \cdot$ are both provable then the sequent $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}_{|F|}, \mathsf{cutC}_{|F|}\}, \Lambda, \Theta_1, \Theta_2; \Omega, \Psi \Uparrow \cdot$ is also provable.

In the proof of (1), if the structural rules are used on $\lfloor F \rfloor$ or $\lceil F \rceil$, we produce a shorter derivation with (2), *i.e.*, with cutC.

For the proof of (2), consider the case where $\lfloor F \rfloor$ and $\lceil F \rceil$ are principal in both premises thus using, respectively, the left and right introduction rules for the same connective:

$$\dfrac{\dfrac{\overset{\pi}{\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_1, i : \lfloor F \rfloor ; \Omega \Downarrow \mathtt{B}\lfloor \star \rfloor(F)}}{\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_1, i : \lfloor F \rfloor ; \Omega \Downarrow \mathtt{E}\lfloor \star \rfloor(F)}}{\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_1, i : \lfloor F \rfloor ; \Omega \Uparrow \cdot} \quad \text{and} \quad \dfrac{\dfrac{\overset{\pi'}{\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_2, \mathtt{c} : \lceil F \rceil ; \Psi \Downarrow \mathtt{B}\lceil \star \rceil(F)}}{\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_2, \mathtt{c} : \lceil F \rceil ; \Psi \Downarrow \mathtt{E}\lceil \star \rceil(F)}}{\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda, \Theta_2, \mathtt{c} : \lceil F \rceil ; \Psi \Uparrow \cdot}$$

Here, $\mathtt{E}\lfloor \star \rfloor(F)$ means that the existential quantifiers of the clause were instantiated with the formula F. As we already know, the bodies $\mathtt{B}\lfloor \star \rfloor(F)$ and $\mathtt{B}\lceil \star \rceil(F)$ produce well controlled forms of derivations. They can either end the proof (e.g., with the clause $\mathtt{f}_L$) or produce derivations with one or two premises.

The resulting generated premises, after focusing on the body of the clause, can only add new atoms to the linear or classical context before ending the negative phase. For simplicity, consider that both π and π' produce one premise. We then have

- Π produces a sequent of the form $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda', \Theta_1', i : \lfloor F \rfloor ; \Omega' \Uparrow \cdot$.
- Π' produces a sequent of the form $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Lambda'', \Theta_2', \mathtt{c} : \lceil F \rceil ; \Psi' \Uparrow \cdot$.

The primed contexts are extensions of the original ones with atoms of the form $\lfloor \cdot \rfloor, \lceil \cdot \rceil$. By induction, the sequents:

- $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}_{|F|}, \mathsf{cutC}_{|F|}\}, \Lambda', \Theta_1'; \Omega' \Uparrow \cdot$; and
- $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}_{|F|}, \mathsf{cutC}_{|F|}\}, \Lambda'', \Theta_2'; \Psi' \Uparrow \cdot$

are both provable. Hence, the sequents below are also provable:

- $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}_{|F|}, \mathsf{cutC}_{|F|}\}, \Lambda, \Theta_1; \Omega \Downarrow \mathtt{B}\lfloor \star \rfloor(F)$; and
- $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}_{|F|}, \mathsf{cutC}_{|F|}\}, \Lambda, \Theta_2; \Psi \Downarrow \mathtt{B}\lceil \star \rceil(F)$.

At this point, we rely on cut coherence (see Definition 7) and cut admissibility in $\mathsf{LNS}_{\mathsf{FMLL}}$:

$$\dfrac{\vdash \omega : \mathscr{T}, \Lambda, \Theta_2; \Psi \Downarrow \mathtt{B}\lceil \star \rceil \quad \dfrac{\vdash \omega : \mathscr{T}, \Lambda, \Theta_1; \Omega \Downarrow \mathtt{B}\lfloor \star \rfloor \quad \overline{\vdash \omega : \mathscr{T}, \Lambda, ; \cdot \Uparrow (\mathtt{B}\lceil \star \rceil)^{\perp}, (\mathtt{B}\lfloor \star \rfloor)^{\perp}} \; \text{cut-coherence}}{\vdash \omega : \mathscr{T}, \Lambda, \Theta_1; \Omega \Uparrow (\mathtt{B}\lceil \star \rceil)^{\perp}} \; \mathsf{FMLL} - \mathsf{cut}}{\vdash \omega : \mathscr{T}, \Lambda, \Theta_1, \Theta_2; \Omega, \Psi \Uparrow \cdot} \; \mathsf{FMLL} - \mathsf{cut}$$

where $\mathscr{T} = \mathscr{T}_{\mathscr{L}} \cup \{\mathsf{cut}_{|F|}, \mathsf{cutC}_{|F|}\}$.

Now let us illustrate a nonprincipal case involving modal behaviors. Consider a derivation where the left and right premises introduce, respectively, the modal formulas G_n and F_m

$$\dfrac{\dfrac{\dfrac{\vdash \{\Lambda, \Theta_1\}^{\succeq n}, \Gamma_G, i : \lfloor F_m \rfloor ; \Delta_G \Uparrow \cdot}{\vdash \Lambda, \Theta_1, \mathtt{c} : \lceil G_n \rceil, i : \lfloor F_m \rfloor ; \cdot \Uparrow \cdot} \; [G_n] \quad \dfrac{\vdash \{\Lambda, \Theta_2\}^{\succeq m}, \Gamma_F; \Delta_F \Uparrow \cdot}{\vdash \Lambda, \Theta_2, \mathtt{c} : \lceil G_n \rceil, \mathtt{c} : \lceil F_m \rceil ; \cdot \Uparrow \cdot} \; [F_m]}{\vdash \Lambda, \Theta_1, \Theta_2, \mathtt{c} : \lceil G_n \rceil ; \cdot \Uparrow \cdot}}{} \; [\mathsf{cutC}_i]$$

where Γ_F, Δ_f (resp. Γ_G, Δ_G) represent the atomic subformulas appearing in the decomposition of $\lceil F_m \rceil$ (resp. $\lceil G_n \rceil$). In each case, the subindices n and m denote the underlying modalities of these formulas. Note that n and m cannot be $\mathtt{c}$ since modalities of the OL are not mapped to this index. Also, $i \neq \mathtt{c}$, otherwise, $\lfloor F_m \rfloor$ must be necessarily weakened in the left derivation (and the end sequent is provable by using $[G_n]$). We then have $n \preceq i$ and, by Requirement 5.2, $i \preceq n$, $i \preceq m$ and hence, $n \preceq m$. Since $\mathsf{T} \in \mathscr{U}(i)$, such axiom is also present in n and m and Theorem 3.1 applies on the context Λ, Θ_i. We conclude that $\Lambda, \Theta_i \supseteq \{\Lambda, \Theta_i\}^{\succeq n} \supseteq \{\Lambda, \Theta_i\}^{\succeq m}$ and the resulting reduction is:

$$\dfrac{\dfrac{\overset{\Pi}{\vdash \{\Lambda, \Theta_1\}^{\succeq n}, \Gamma_G, i/\mathsf{c} : \{\Delta_G\}, i : \lfloor F_m \rfloor; \cdot \Uparrow \cdot} \quad \overset{\Psi}{\vdash \{\Lambda, \Theta_2\}^{\succeq n}, \Gamma_G, i/\mathsf{c} : \{\Delta_G\}, \mathsf{c} : \lceil F_m \rceil; \cdot \Uparrow \cdot}}{\dfrac{\dfrac{\vdash \{\Lambda, \Theta_1, \Theta_2\}^{\succeq n}, \Gamma_G, i/\mathsf{c} : \Delta_G; \cdot \Uparrow \cdot}{\vdash \{\Lambda, \Theta_1, \Theta_2\}^{\succeq n}, \Gamma_G; \Delta_G \Uparrow \cdot} \; \text{store, R}_r, [\mathsf{pos}_i, \mathsf{neg}_\mathsf{c}]}{\vdash \Lambda, \Theta_1, \Theta_2, \mathsf{c} : \lceil G_n \rceil; \cdot \Uparrow \cdot} \; [G_n]}}{} \; [\mathsf{cutC}_i]$$

Π is obtained by Theorem 3.1 and Ψ is the same as the derivation introducing F_m but weakening the formulas in Γ_G and Δ_G in the release rule.   □

Since for every OL formula F, $|F| > 0$, by induction we conclude the following.

Corollary 8. *Let $\mathscr{T}_{\mathscr{L}}$ be the theory of a given OL $\mathscr{L}$, and let Ψ be a multiset and Θ a subexponential context containing only atoms of the form $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$. The sequent $\vdash \omega : \{\mathscr{T}_{\mathscr{L}}, \mathsf{cut}\}, \Theta; \Psi \Uparrow \cdot$ is provable iff $\vdash \omega : \mathscr{T}_{\mathscr{L}}, \Theta; \Psi \Uparrow \cdot$ is provable.*

The mechanization of Theorem 5.1 is unfortunately not feasible due to the impossibility of reasoning about derivations (see discussion in the end of Section 3). For instance, it is not possible to specify, in a general way, the number of premises resulting after the decomposition of a bipole. The current version of the library offers definitions, in the form of class types, for cut coherence of constants, connectives, and OL binders. Hence, the clauses encoding OL rules must be instances of such classes (thus satisfying cut coherence). The proof of OL cut admissibility, for each specific case, follows the proof shown above: the principal cases rely on cut coherence and cut-admissibility of MMLL. The current version of the repository exemplifies this procedure for classical and intuitionistic logics, as well as for the logic S4.

6. Discussion and Conclusion

In this paper, we have extended the sufficient criterion for cut admissibility of object logical systems given in Miller and Pimentel (2013). For that, we moved from LL to a variant of linear logic with subexponentials called MMLL, where different modal behaviors were embodied into the subexponential connectives. This allowed to establish a *simple* yet powerful criterion –*cut-coherence* – for proving analyticity for a large class of systems. This criterion neatly captures the *duality* of inference rules. In fact, checking cut coherence is equivalent to checking $\mathsf{B} \lfloor \star \rfloor^{\perp} \equiv \mathsf{B} \lceil \star \rceil$, and vice versa. And *this* is the spirit of cut admissibility.

We start the discussion by exploring the similarities and differences between MMLL and other LL based frameworks. First of all, since LL is a resource conscious logic, it is natural representing OL rules as meta-level rewriting clauses. The proposal in Miller and Pimentel (2013) explores this in a very elegant way, with natural and direct encodings, together with an extremely simple sufficient condition for cut-admissibility of the specified OLs. For broadening the spectrum of systems amenable for specification, Nigam et al. (2016) considered an extension of linear logic with subexponentials (SELL, Nigam and Miller 2010) as the logical framework. However, encoding modalities different from those in LL remained difficult (or even impossible) using only the linear logic exponentials. In Olarte et al. (2020), we approached this problem by adopting a framework (SLL) allowing a finite set of unbounded subexponentials with different modal characteristics, giving a partial solution to the specification of modal systems. MMLL intends to close this series of works, enabling not only the specification of an infinite set of bounded or unbounded OL systems (*e.g.*, multimodal normal logics) but also keeping the simplicity of the cut admissibility characterization given in Miller and Pimentel (2013). As a side effect, we obtained a system (LNS$_{\mathsf{FMLL}}$), with a more efficient proof search strategy when compared with SLL, and more general than SELL.

Cut elimination and focusing are central for the development of logical frameworks. For this reason, the above series of works have been followed by different system implementations and formalizations in proof assistants. For instance, cut elimination and invertibility lemmas were proved for several linear logic calculi in Abella (Chaudhuri et al. 2019). The library YALLA[3] also formalizes several meta-properties of different variants of LL in Coq. Linear nested systems have been formalized in D'Abrera et al. (2021), together with a cut elimination procedure for the tense logic Kt. The works by Nigam et al. (2014) and Olarte et al. (2018) allow for proving, semi-automatically, invertibility and cut admissibility theorems for different logical systems, including LL-based systems. The aforementioned work by Nigam et al. (2016) is also supported by a tool that automatically checks the cut-coherence conditions proposed there. The work by Xavier et al. (2017) was the first formalization of completeness of focusing and cut admissibility for first-order linear logic in Coq. The representation of the binders, needed for the quantifiers, followed the technique PHOAS (parametric higher order abstract syntax, Chlipala 2008). That work also formalized the adequacy of the encoding of different logical systems as theories in LL. This formalization effort was extended in Felty et al. (2021), by introducing first-order object logics and using the cut coherence criteria in Miller and Pimentel (2013) to show cut admissibility for some OLs. Defining quantifiers at the LL-level and also at the OL level was only possible by replacing the PHOAS representation of syntax with a definitional representation using a de Bruijn style supported by the Hybrid library (Felty and Momigliano 2012).

Logical frameworks, based on type systems, have also been used for characterizing and proving cut-elimination theorems of object logics (see e.g., Licata et al. 2017; Pfenning 2000). Usually, the embedding of the OL into the logical framework is not simple/direct. The approach followed here is rather different: we provide easy-to-check conditions that guarantee that the property holds.

The key feature for achieving all this is *modularity*. Everywhere. Starting from the choice of LNS, a generalization of sequent systems, as the base framework. This allows for the *locality* of rules, enabling the central behavior of connectives to be shared among several different logics and leaving to the subexponentials the work of separating modal behaviors. That is, modalities reflect modalities, while (vanilla) LL captures rules as rewriting clauses (as it should be). Second, structural rules are parametric w.r.t. subexponentials, allowing for a clear separation between modals and local structural behaviors. And last, but not least, since subexponentials in MMLL also reflect Kripke models, logics having the same semantic behavior share the same modal characteristics.

As future work, it would be interesting to analyze the case of non-normal modal logics (Lellmann and Pimentel 2019), as well as to explore the failure cases: is it possible, at the meta-level, to identify the reasons for the lack of analyticity? This would push the line of investigation towards finding *necessary* conditions for cut admissibility.

Notes

1 Axiom D usually appears in the literature as $\Box F \to \Diamond F$, where $\Diamond$ is the *possibility* modality. Here we prefer not to introduce the diamond connective in the syntax. Instead, we adopt the system with box and negation, where $\neg A$ is a shorthand for $A \to$ f.

2 We note that, in D'Abrera et al. (2021), a cut-elimination *procedure* was mechanized in Coq for the LNS system for the tense logic Kt.

3 https://perso.ens-lyon.fr/olivier.laurent/yalla/.

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