

Geometric construction of metaplectic covers of GL_n in characteristic zero

Richard Hill

October 17, 2010

Abstract

This paper presents a new construction of the m -fold metaplectic cover of GL_n over an algebraic number field k , where k contains a primitive m -th root of unity. A 2-cocycle on $\mathrm{GL}_n(\mathbb{A})$ representing this extension is given and the splitting of the cocycle on $\mathrm{GL}_n(k)$ is found explicitly. The cocycle is smooth at almost all places of k . As a consequence, a formula for the Kubota symbol on SL_n is obtained. The construction of the paper requires neither class field theory nor algebraic K-theory, but relies instead on naive techniques from the geometry of numbers introduced by W. Habicht and T. Kubota. The power reciprocity law for a number field is obtained as a corollary.

Preface

Recall that according to the Gauß-Schering Lemma, one can express the quadratic residue symbol in terms of the number of lattice points in a triangle. This paper is an attempt to generalize this formula, together with the associated proof of quadratic reciprocity. To make things more precise, let α and β be integers with β a prime not dividing 2α , and let S be the subset $\{1, 2, \dots, (\beta - 1)/2\}$ of \mathbb{Z}/β . Define a function $f : \mathbb{Z}/\beta \rightarrow \mathbb{Z}$ by $f(x) = 1$ for $x \in S$, and $f(x) = 0$ otherwise. The Gauß Lemma states that

$$\left(\frac{\alpha}{\beta}\right)_2 = (-1)^\Sigma, \quad \Sigma = \sum_{x \in (\mathbb{Z}/\beta) - \{0\}} f(\alpha x) f(-x).$$

In fact we may replace f by any function $\mathbb{Z}/\beta \rightarrow \mathbb{Z}$ satisfying for all non-zero x the relation

$$f(x) + f(-x) = 1.$$

It was noted by Schering [27] that the formula remains true when β is any integer coprime to 2α . This was done by showing that the right hand side of the formula is multiplicative in β . Going in a slightly different direction, Zolotareff [32] showed that the lemma may also be expressed in the form

$$\left(\frac{\alpha}{\beta}\right)_2 = \text{sign}(\sigma),$$

where σ is the permutation of $(\mathbb{Z}/\beta) - \{0\}$ which is defined by $\sigma(x) = \alpha x$.

It was already known to Gauß [7] that the lemma can be generalized to number fields. For this, suppose that k is a number field containing a primitive m -th root of unity, and let μ_m be the group of all m -th roots of unity in k . We shall write \mathfrak{o} for the ring of algebraic integers in k . Choose elements $\alpha, \beta \in \mathfrak{o}$ such that $m\alpha$ and β are coprime. Then the m -th power residue symbol $(\alpha/\beta)_m$ is defined, and we have the following formula:

$$\left(\frac{\alpha}{\beta}\right)_m = \prod_{\zeta \in \mu_m} \zeta^{\Sigma(\zeta)}, \quad \Sigma(\zeta) = \sum_{x \in (\mathfrak{o}/\beta) - \{0\}} f(\alpha x) f(\zeta x). \quad (1)$$

In this formula, $f : \mathfrak{o}/\beta \rightarrow \mathbb{Z}$ is a function satisfying for all non-zero x in \mathfrak{o}/β :

$$\sum_{\zeta \in \mu_m} f(\zeta x) = 1. \quad (2)$$

This more general version of the Gauß-Schering lemma has been used to give proofs of the m -th power reciprocity law in the field k (see for example [7, 9, 13, 19, 20]).

In this paper, we study the right hand side of (1) when the following generalizations are made: α and β become invertible $n \times n$ -matrices over \mathfrak{o} rather than scalars, and x runs through vectors in $\mathfrak{o}^n/\beta\mathfrak{o}^n - \{0\}$. We also require that β is invertible modulo m .

The first problem one encounters with this generalization is that the right hand side of (1) now depends on the function f chosen. In order to deal with

this, we would like to fix a function $f : k^n - \{0\} \rightarrow \mathbb{Z}$ satisfying (2). However this is also problematic, since the function f is required to be periodic modulo $\beta\mathfrak{o}^n$ for all possible values of β , and there is no such function. To get around this problem, we use the following trick. We shall call a sublattice L of k^n *admissible* if L is invariant under multiplication by μ_m and there is a number $\delta \in \mathfrak{o}$ coprime to m , such that

$$\delta\mathfrak{o}^n \subset L \subset \delta^{-1}\mathfrak{o}^n.$$

We may choose a function $f : k^n - \{0\} \rightarrow \mathbb{Z}$ satisfying (2), satisfying the following additional regularity condition:

For all admissible lattices $L_1 \subset L_2$ there is an admissible lattice L_3 , such that for every $v \in L_2 - L_1$, the function f is constant on the coset $v + L_3$.

Using such an f , we define our main object of study:

$$\text{Dec}(\alpha, \beta) = \prod_{\zeta \in \mu_m} \zeta^{\Sigma(\zeta)}, \quad \Sigma(\zeta) = \sum_{v \in \mathfrak{o}^n/L - \beta\mathfrak{o}^n/L} f(\alpha v) f(\zeta v).$$

In this formula, L is an admissible lattice, and is small enough so that the functions f and $f \circ \alpha$ are both well-defined on the set $\mathfrak{o}^n/L - \beta\mathfrak{o}^n/L$. The choice of admissible lattice will not alter the sum $\Sigma(\zeta)$ modulo m .

When one attempts to generalize Schering's proof that the right hand side of (1) is multiplicative in β , one instead obtains (as an easy exercise) the following 2-cocycle relation:

$$\text{Dec}(\alpha, \beta\gamma)\text{Dec}(\beta, \gamma) = \text{Dec}(\alpha, \beta)\text{Dec}(\alpha\beta, \gamma).$$

The dependence of $\text{Dec}(\alpha, \beta)$ on the function f can now be better understood: if one changes f , then the 2-cocycle Dec is only multiplied by a coboundary, and so its cohomology class is independent of f .

The language of admissible lattices is quite cumbersome, and there is a more natural way of defining the function $\text{Dec}(\alpha, \beta)$ as follows. Let S be the set of prime ideals of k which divide m , together with the infinite places of k . Let $\mathbb{A}(S)$ be the ring of S -adeles, i.e. the restricted topological product of the local fields k_v for primes v not dividing m . The field k is dense in the topological ring $\mathbb{A}(S)$, and our previous regularity condition is equivalent to saying that f is the restriction of a continuous function $f : \mathbb{A}(S)^n - \{0\} \rightarrow \mathbb{Z}$.

Contained in $\mathbb{A}(S)$ we have a compact open subring $\bar{\mathfrak{o}} = \prod_{v \notin S} \mathfrak{o}_v$ (this is simply the closure of \mathfrak{o}). We shall normalize the additive Haar measure on $\mathbb{A}(S)^n$ so that $\bar{\mathfrak{o}}^n$ has measure 1. With this new notation, our definition becomes

$$\text{Dec}(\alpha, \beta) = \prod_{\zeta \in \mu_m} \zeta^{\Sigma(\zeta)}, \quad \Sigma(\zeta) = \left\{ \int_{\bar{\mathfrak{o}}^n} - \int_{\beta \bar{\mathfrak{o}}^n} \right\} f(\alpha v) f(\zeta v) dv$$

Although the numbers $\Sigma(\zeta)$ are now rational, their denominators are coprime to m , and so the powers of ζ are well-defined. Using this new definition, one sees that the matrices α and β may be taken from the larger group $\text{GL}_n(\mathbb{A}(S))$, and that Dec is a continuous 2-cocycle on this group.

The body of the paper is concerned with proving the reciprocity law for $\text{Dec}(\alpha, \beta)$, which we now describe. We shall write k_∞ for the product of the archimedean completions of k ; k_m for the product of the non-archimedean completions at primes dividing m , and \mathbb{A} for the adèle ring of k , so we have $\mathbb{A} = \mathbb{A}(S) \times k_\infty \times k_m$.

We recall that a 2-cocycle on $\text{GL}_n(\mathbb{A})$, which restricts to a coboundary on the subgroup $\text{GL}_n(k)$ is called a *metaplectic* cocycle. Our main result is that Dec is the restriction to $\text{GL}_n(\mathbb{A}(S))$ of a metaplectic cocycle. The resulting metaplectic 2-cocycle corresponds to the m -fold metaplectic extension of GL_r . The paper gives a new and independent proof of the existence of this extension, without the need for class field theory or algebraic K-theory.

In more elementary terms, there is an analogous construction of a measurable cocycle Dec_∞ on the group $\text{GL}_n(k_\infty)$. There is also a continuous cocycle Dec_m on $\text{GL}_n(k_m)$, and we let $\text{Dec}_\mathbb{A}$ be the product of the three cocycles, which is therefore a cocycle on $\text{GL}_n(\mathbb{A})$. We write down an explicit function $\tau : \text{GL}_n(k) \rightarrow \mu_m$, such that for $\alpha, \beta \in \text{GL}_n(k)$ we have

$$\text{Dec}_\mathbb{A}(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)}.$$

To see how this is related to more familiar reciprocity laws, note that if α and β commute, then we immediately have

$$\text{Dec}_\mathbb{A}(\alpha, \beta) = \text{Dec}_\mathbb{A}(\beta, \alpha).$$

In particular, in the case of GL_1 where all elements commute, we recover the reciprocity law for the field k .

The function τ is defined in terms of numbers of lattice points in certain polyhedra, and the proof of the main result is a generalization of Gauß' proof of quadratic reciprocity. The arguments throughout are naive but rather long, and at times complicated.

I'm very grateful to the anonymous referee for taking the time to go through this long and technical paper, and also to the editor Sinai Robins for his interest in my work.

Contents

1	Introduction.	5
1.1	Metaplectic Groups	5
1.2	The Kubota symbol	10
1.3	Organization of the paper	12
2	Notation	13
2.1	Acyclic $(\mathbb{Z}/m)[\mu_m]$ -modules.	13
2.2	Central extensions.	14
2.3	Commutators and symmetric cocycles	14
2.4	Singular homology groups.	15
3	The arithmetic and geometric cocycles.	17
3.1	The arithmetic cocycle	17
3.2	Reduction to a discrete space.	21
3.3	Examples of arithmetic cocycles	22
3.4	The Gauss–Schering Lemma	23
3.5	The geometric cocycle	25
3.6	Example: $GL_2(\mathbb{R})$	30
3.7	Stability of the cocycles	31
3.8	Example: $GL_1(\mathbb{C})$	34
3.9	The totally complex case.	35
3.10	The real case.	39

4	Construction of fundamental functions.	39
4.1	The space X .	40
4.2	The complex \mathfrak{X} .	41
4.3	The complex \mathfrak{S} .	44
4.4	Modified Parallelotopes.	46
4.5	Statement of the results in the generic case.	48
4.6	A generic formula for the pairing.	49
4.7	The order of $\mathcal{H} \rtimes \mathcal{G}$ at 0.	50
4.8	Dependence of $\text{ord}_0(\mathcal{H} \rtimes \mathcal{G})$ on \mathcal{H} .	51
5	Deformation of fundamental functions.	54
5.1	Existence of certain limits.	54
5.2	Deformations of cells.	55
5.3	Deforming paths.	63
5.4	Deformation of homotopies.	67
5.5	The limit defining Dec_∞ .	68
6	The relation between the arithmetic and geometric cocycles.	69
6.1	Main Results.	69
6.2	A formula for the Kubota symbol.	71
6.3	Proof of Theorem ??.	72
6.4	Proof of Lemma ??.	73
6.5	Proof of Lemma ??.	75
6.6	Proof of Lemma ??.	77
7	Extending the cocycle to $\text{GL}_n(\mathbb{A})$.	82
7.1	The cocycle on $\text{SL}_n(k_m)$.	82
7.2	Application : The power reciprocity law.	85
7.3	Extending the cocycle to GL_n .	87

1 Introduction.

1.1 Metaplectic Groups

Let k be a global field with adèle ring \mathbb{A} and let G be a linear algebraic group over k . We shall regard $G(\mathbb{A})$ as a locally compact topological group with the

topology induced by that of \mathbb{A} . For a finite abelian group A , a *metaplectic extension* of G by A is a topological central extension:

$$1 \rightarrow A \rightarrow \tilde{G}(\mathbb{A}) \rightarrow G(\mathbb{A}) \rightarrow 1,$$

which splits over the discrete subgroup $G(k)$ of $G(\mathbb{A})$. Such extensions are of use in the theory of automorphic forms since certain automorphic forms (for example the classical theta-functions, see [31]) may be regarded as functions on $\tilde{G}(\mathbb{A})$, which are invariant under translation by the lift of $G(k)$ to $\tilde{G}(\mathbb{A})$. The groups $\tilde{G}(\mathbb{A})$ are topological groups but they are not in general groups of adèle valued points of an algebraic group.

If k contains a primitive m -th root of unity, then the group SL_n has a canonical metaplectic extension with kernel the group μ_m of all m -th roots of unity in k . This extension is always non-trivial; in fact if m is the total number of roots of unity in k and if $n \geq 3$, then the canonical extension is universal amongst metaplectic extensions.

As the groups $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_n(k)$ are not perfect, GL_n has no universal metaplectic extension. However the canonical extension of SL_n may be continued in various ways to give a metaplectic extension of GL_n by μ_m . This has been done by embedding GL_n in SL_r for $r > n$ (see [14]); we shall call the metaplectic extensions on GL_n obtained in this way the *standard twists*. In this paper we shall give an elementary construction of a metaplectic extension $\widetilde{\mathrm{GL}}_n(\mathbb{A})$ of GL_n , which in fact is not one of the standard twists, but which nevertheless restricts to the canonical metaplectic extension of SL_n . The method used gives an independent construction of the canonical metaplectic extension of SL_n .

There are other ways of constructing the group $\widetilde{\mathrm{GL}}_n(\mathbb{A})$. The advantages of the method of construction employed in this paper are as follows:

- Other methods of construction require class field theory and algebraic K-theory. In contrast the method here is very elementary. In fact one can deduce certain theorems of class field theory as corollaries of the results here.
- The method described here is very explicit in the sense that a 2-cocycle $\mathrm{Dec}_{\mathbb{A}}$ is given which represents the group extension. This means $\widetilde{\mathrm{GL}}_n(\mathbb{A})$ may be realized as a set of pairs $(\alpha, \chi) \in \mathrm{GL}_n(\mathbb{A}) \times \mu_m$ with multiplication given by

$$(\alpha, \chi)(\beta, \xi) = (\alpha\beta, \chi\xi\mathrm{Dec}_{\mathbb{A}}(\alpha, \beta)).$$

Thus the cohomology class of $\text{Dec}_{\mathbb{A}}$ is an element of $H^2(\text{GL}_n(\mathbb{A}), \mu_m)$ which splits on the subgroup $\text{GL}_n(k)$. An expression for $\text{Dec}_{\mathbb{A}}$ as a coboundary on $\text{GL}_n(k)$ is also obtained. In contrast the usual method of construction gives only a cocycle on the standard Borel subgroup. An expression for the whole cocycle has been obtained (after various incorrect formulae obtained by other authors) in [3], but the cocycle obtained there is more complicated than ours. In particular the formula of [3] involves first decomposing α , β and $\alpha\beta$ in the Bruhat decomposition, and then decomposing each of the three Weyl group elements as a minimal product of simple reflections.

- The cocycle $\text{Dec}_{\mathbb{A}}$ is smooth on the non-archimedean part of $\text{GL}_n(\mathbb{A})$; in fact if k has no real places then we obtain a cocycle which is smooth everywhere. The cocycle may therefore be used to study the smooth representations of the metaplectic group. More precisely suppose π is an irreducible representation of $\widetilde{\text{GL}}_n(\mathbb{A})$ on a space V of smooth functions on $\widetilde{\text{GL}}_n(\mathbb{A})$, on which $\widetilde{\text{GL}}_n(\mathbb{A})$ acts by right translation:

$$(\pi(g)\phi)(h) := \phi(hg), \quad g, h \in \widetilde{\text{GL}}_n(\mathbb{A}). \quad (3)$$

Let $\epsilon : \mu_m \rightarrow \mathbb{C}^\times$ be the restriction of the central character to the subgroup μ_m . Then V is isomorphic to a space V' of smooth functions on $\text{GL}_n(\mathbb{A})$ with the twisted action:

$$(\pi(\alpha, \chi)\phi')(\beta) = \epsilon(\chi \text{Dec}_{\mathbb{A}}(\beta, \alpha))\phi'(\beta\alpha). \quad (4)$$

The isomorphism $V \rightarrow V'$ is given by $\phi \mapsto \phi'$, where ϕ' is defined by

$$\phi'(\alpha) = \phi(\alpha, 1).$$

Although the action (4) looks more complicated than (3), it is perhaps easier to use in calculations as one is dealing with elements of GL_n . One cannot do this with the cocycle of [3] as it is not smooth (in fact on $\text{GL}_n(\mathbb{A})$ with $n \geq 2$ it is nowhere continuous).

There are two disadvantages to the method of construction described in this paper. First, the construction is long and quite difficult. Second, the part of the cocycle on the subgroup $\text{GL}_n(k_v)$ for $v|m$ is not very explicit. In a sense one has the same problem with the cocycle of [3], since it is expressed in terms of ramified Hilbert symbols.

The method of construction. The case that k is a function field is described in [10], [11]; in this paper we shall deal with the more difficult case that k is an algebraic number field. Let S be the set of places v of k for which $|m|_v \neq 1$, and let $\mathbb{A}(S)$ denote the restricted topological product of the fields k_v for $v \notin S$. Let k_∞ be the sum of the archimedean completions of k and let k_m be the sum of the fields k_v for non-archimedean places $v \in S$. We then have

$$\mathbb{A} = \mathbb{A}(S) \oplus k_\infty \oplus k_m,$$

and hence:

$$\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}(S)) \oplus \mathrm{GL}_n(k_\infty) \oplus \mathrm{GL}_n(k_m).$$

We shall write down explicit 2-cocycles $\mathrm{Dec}_{\mathbb{A}(S)}$ on $\mathrm{GL}_n(\mathbb{A}(S))$ and Dec_∞ on $\mathrm{GL}_n(k_\infty)$. Then, for a certain compact, open subgroup U_m of $\mathrm{GL}_n(k_m)$, we find a function $\tau : \mathrm{GL}_n(k) \cap U_m \rightarrow \mu_m$ such that

$$\mathrm{Dec}_{\mathbb{A}(S)}(\alpha, \beta) \mathrm{Dec}_\infty(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)}, \quad \alpha, \beta \in \mathrm{GL}_n(k) \cap U_m. \quad (5)$$

It follows fairly easily from this that we can extend τ to $\mathrm{SL}_n(k)$ in such a way that there is a unique continuous cocycle Dec_m on $\mathrm{SL}_n(k_m)$ defined by the formula

$$\mathrm{Dec}_{\mathbb{A}(S)}(\alpha, \beta) \mathrm{Dec}_\infty(\alpha, \beta) \mathrm{Dec}_m(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)}, \quad \alpha, \beta \in \mathrm{SL}_n(k).$$

Finally we show that there is a cocycle $\mathrm{Dec}_\mathbb{A}$ on $\mathrm{GL}_n(\mathbb{A})$ which is metaplectic and which extends all our cocycles.

Note that this definition of Dec_m is global; it is defined on the dense subgroup $\mathrm{SL}_n(k)$ and then extended by continuity to $\mathrm{SL}_n(k_m)$. It would be of some interest to find a local construction of the cocycle Dec_m , as the ramified Hilbert symbols may be expressed in terms of this cocycle via the isomorphism (6) below. However I do not know how to make such a construction.

If m is even then the cocycle $\mathrm{Dec}_\mathbb{A}$ has the surprising property that it is not, even up to a coboundary, a product of cocycles Dec_v on the groups $\mathrm{GL}_n(k_v)$ (in contrast the standard twists are products of local cocycles). In fact if we write $\widetilde{\mathrm{GL}}_n(k_v)$ for the preimage of $\mathrm{GL}_n(k_v)$ in $\widetilde{\mathrm{GL}}_n(\mathbb{A})$, then the various subgroups $\widetilde{\mathrm{GL}}_n(k_v)$ do not even commute with each other. This means that irreducible representations π of $\widetilde{\mathrm{GL}}_n(\mathbb{A})$ cannot be expressed

as restricted tensor products of irreducible representations of the groups $\widetilde{\mathrm{GL}}_n(k_v)$. Thus the usual local-to-global approach to studying automorphic representations must be modified to deal with $\widetilde{\mathrm{GL}}_n(\mathbb{A})$.

Matsumoto's Construction. We now review the usual construction of the canonical metaplectic extension of SL_n . Let F be any field. Recall (or see [23] or [12]) that for $n \geq 3$ there is a universal central extension

$$1 \rightarrow K_2(F) \rightarrow \mathrm{St}_n(F) \rightarrow \mathrm{SL}_n(F) \rightarrow 1,$$

where St_n denotes the Steinberg group. Hence, for any abelian group A we have

$$H^2(\mathrm{SL}_n(F), A) \cong \mathrm{Hom}(K_2(F), A). \quad (6)$$

Matsumoto (see [22] or [23]) proved that for any field F the group $K_2(F)$ is the quotient of $F^\times \otimes_{\mathbb{Z}} F^\times$ by the subgroup generated by $\{\alpha \otimes (1 - \alpha) : \alpha \in F \setminus \{0, 1\}\}$. The isomorphism (6) may be described as follows. If σ is a 2-cocycle representing a cohomology class in $H^2(\mathrm{SL}_n(F), A)$ then for diagonal matrices $\alpha, \beta \in \mathrm{SL}_n(F)$ we have

$$\frac{\sigma(\alpha, \beta)}{\sigma(\beta, \alpha)} = \prod_{i=1}^n \phi(\alpha_i, \beta_i), \quad \alpha = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_n \end{pmatrix},$$

where $\phi : K_2(F) \rightarrow A$ is the image of σ . Here we are writing the group law in A multiplicatively.

Suppose that k is a global field containing a primitive m -th root of unity. For any place v of k the m -th power Hilbert symbol gives a map $K_2(k_v) \rightarrow \mu_m$. Corresponding to this map there is a cocycle $\sigma_v \in H^2(\mathrm{SL}_n(k_v), \mu_m)$. For any place v of k we shall write \mathfrak{o}_v for the ring of integers in k_v . For almost all places v the cocycle σ_v splits on $\mathrm{SL}_n(\mathfrak{o}_v)$. One may therefore define a cocycle $\sigma_{\mathbb{A}}$ on $\mathrm{SL}_n(\mathbb{A})$ by $\sigma_{\mathbb{A}} = \prod_v \sigma_v$. This corresponds to a topological central extension:

$$1 \rightarrow \mu_m \rightarrow \widetilde{\mathrm{SL}}_n(\mathbb{A}) \rightarrow \mathrm{SL}_n(\mathbb{A}) \rightarrow 1. \quad (7)$$

Now recall the following.

Theorem 1 (Power Reciprocity Law) *For any place v of k let $(-, -)_{v,m}$ denote the m -th power Hilbert symbol on k_v . For $\alpha, \beta \in k^\times$ we have*

$$\prod_v (\alpha, \beta)_{v,m} = 1,$$

where the product is taken over all places of k .

(For a proof, see Chapter 12, Verse 4, Theorem 12 of [1].)

If one restricts $\sigma_{\mathbb{A}}$ to $\mathrm{SL}_n(k)$, this restriction corresponds under (6) to the map $K_2(k) \rightarrow \mu_m$ induced by the m -th power Hilbert symbol on k_v . Hence the restriction of $\sigma_{\mathbb{A}}$ to $\mathrm{SL}_n(k)$ corresponds to the map $K_2(k) \rightarrow \mu_m$ induced by the product of all the m -th power Hilbert symbols. By the reciprocity law this map is trivial. Therefore $\sigma_{\mathbb{A}}$ splits on $\mathrm{SL}_n(k)$, so the extension (7) is metaplectic.

1.2 The Kubota symbol

One of the results of this paper is a formula (see §6.2) for the Kubota symbol on SL_n . We recall here the definition of the Kubota symbol.

Let k be an algebraic number field and let \mathfrak{o} denote the ring of algebraic integers in k . Given an ideal \mathfrak{a} of \mathfrak{o} , we define $\mathrm{SL}_n(\mathfrak{o}, \mathfrak{a})$ to be the subgroup of matrices in $\mathrm{SL}_n(\mathfrak{o})$ which are congruent to the identity matrix modulo \mathfrak{a} . A subgroup of $\mathrm{SL}_n(k)$ is said to be an *arithmetic* subgroup if it is commensurable with $\mathrm{SL}_n(\mathfrak{o})$. An arithmetic subgroup is said to be a *congruence subgroup* if it contains $\mathrm{SL}_n(\mathfrak{o}, \mathfrak{a})$ for some non-zero ideal \mathfrak{a} . The *congruence subgroup problem* is the question: “is every arithmetic subgroup a congruence subgroup?” This question has been answered by Bass–Milnor–Serre [4] for $n \geq 3$ and by Serre [28] in the more difficult case $n = 2$ (see for example [26] for generalizations). To rule out some pathological cases, assume either that k has at least two archimedean places or that $n \geq 3$. If k has a real place then the answer to the congruence subgroup problem is “yes” whereas if k is totally complex the answer is “no”. We shall describe what happens when k is totally complex. To make statements clearer suppose μ_m is the group of all roots of unity in k . In this case there is an ideal \mathfrak{f} and a surjective homomorphism $\kappa_m : \Gamma(\mathfrak{f}) \rightarrow \mu_m$ with the following properties:

- $\ker(\kappa_m)$ is a non-congruence subgroup.
- For any arithmetic subgroup Γ of $\mathrm{SL}_n(k)$ there is an ideal \mathfrak{a} such that $\Gamma \supset \mathrm{SL}_n(\mathfrak{o}, \mathfrak{a}) \cap \ker(\kappa_m)$.

The map κ_m is called the Kubota symbol. In the case of SL_2 it is given by

the following formula (see [18]):

$$\kappa_m \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{c}{d}\right)_m & \text{if } c \neq 0, \\ 1 & \text{if } c = 0. \end{cases}$$

Here $\left(\frac{c}{d}\right)_m$ denotes the m -th power residue symbol, which we recall is defined by

$$\left(\frac{c}{d}\right)_m = \prod_{v|d} (c, d)_{v,m} = \left[\frac{k(\sqrt[m]{c})/k}{d} \right].$$

The symbol on the right is the global Artin symbol; the Galois group $\text{Gal}(k(\sqrt[m]{c})/k)$ may be identified with μ_m .

Connection between the congruence subgroup problem and metaplectic groups. As before suppose the number field k is totally complex. We shall introduce two topologies on $\text{SL}_n(k)$. For the first topology we take the congruence subgroups as a basis of neighbourhoods of the identity. The completion of $\text{SL}_n(k)$ with respect to this topology is the group $\text{SL}_n(\mathbb{A}_f)$, where \mathbb{A}_f denotes the ring of finite adèles of k . For our second topology we take the arithmetic subgroups of $\text{SL}_n(k)$ as a basis of neighbourhoods of the identity. This is a finer topology, and the completion of $\text{SL}_n(k)$ with respect to this topology is a group extension $\widetilde{\text{SL}}_n(\mathbb{A}_f)$ of $\text{SL}_n(\mathbb{A}_f)$. The homomorphism κ_m identifies the kernel of the extension with μ_m . We therefore have a short exact sequence:

$$1 \rightarrow \mu_m \rightarrow \widetilde{\text{SL}}_n(\mathbb{A}_f) \rightarrow \text{SL}_n(\mathbb{A}_f) \rightarrow 1. \quad (8)$$

This must be a central extension as $\text{SL}_n(\mathbb{A}_f)$ is perfect. On the other hand $\text{SL}_n(k)$ is contained in both completions, so the extension splits on $\text{SL}_n(k)$. Adding the group $\text{SL}_n(k_\infty)$ to both $\text{SL}_n(\mathbb{A}_f)$ and $\widetilde{\text{SL}}_n(\mathbb{A}_f)$ we obtain the canonical metaplectic extension of SL_n .

Conversely we may reconstruct the Kubota symbol from the metaplectic group as follows. Let $\widetilde{\text{SL}}_n(\mathbb{A}_f)$ be the preimage of $\text{SL}_n(\mathbb{A}_f)$ in the canonical metaplectic extension of SL_n . By restriction we have an exact sequence (8). Since k is totally complex, the group $\text{SL}_n(k_\infty)$ is both connected and simply connected. This implies that the restriction map gives an isomorphism

$$H^2(\text{SL}_n(\mathbb{A}), \mu_m) \rightarrow H^2(\text{SL}_n(\mathbb{A}_f), \mu_m).$$

Hence, if we regard $\mathrm{SL}_n(k)$ as a subgroup of $\mathrm{SL}_n(\mathbb{A}_f)$, then this subgroup lifts to a subgroup of $\widetilde{\mathrm{SL}}_n(\mathbb{A}_f)$.

There are two subgroups of $\mathrm{SL}_n(\mathbb{A}_f)$ on which the extension (8) splits. First, since (8) is a topological central extension, there are neighbourhoods \tilde{U} of the identity in $\widetilde{\mathrm{SL}}_n(\mathbb{A}_f)$ and U of the identity in $\mathrm{SL}_n(\mathbb{A}_f)$ such that the projection map restricts to a homeomorphism $\tilde{U} \rightarrow U$. As $\mathrm{SL}_n(\mathbb{A}_f)$ is totally disconnected we may take \tilde{U} , and hence U , to be a compact open subgroup. The inverse map $U \rightarrow \tilde{U}$ is a continuous splitting of the extension. Secondly we have a splitting of the extension on $\mathrm{SL}_n(k)$. By the Strong Approximation Theorem (Theorem 3.3.1 of [5]), $\mathrm{SL}_n(k)$ is dense in $\mathrm{SL}_n(\mathbb{A}_f)$. Hence the splitting on $\mathrm{SL}_n(k)$ cannot be continuous, since otherwise it would extend by continuity to a splitting of the whole extension.

Let $\Gamma = U \cap \mathrm{SL}_n(k)$. The group Γ is a congruence subgroup of $\mathrm{SL}_n(k)$, and we have two different splittings of the extension on Γ . Dividing one splitting by the other we obtain a homomorphism $\kappa_m : \Gamma \rightarrow \mu_m$. This homomorphism is not continuous with respect to the induced topology from $\mathrm{SL}_n(\mathbb{A}_f)$, so its kernel is a non-congruence subgroup. The map κ_m is the Kubota symbol.

In the construction of this paper the splittings of the extension are described explicitly. As a consequence we obtain a formula for the Kubota symbol on SL_n .

1.3 Organization of the paper

The paper is organized into the following sections:

- §2 We fix some standard notation from group cohomology and singular homology. To avoid confusions of signs later, the normalizations of various maps are fixed. Some known results are stated for later reference.
- §3 The cocycles $\mathrm{Dec}_{\mathbb{A}(S)}$ and Dec_∞ on the groups $\mathrm{GL}_n(\mathbb{A}(S))$ and $\mathrm{GL}_n(k_\infty)$ are defined. The cocycle $\mathrm{Dec}_{\mathbb{A}(S)}$ has been studied in [10]; some of the results from there are recalled. Analogous results are obtained for the cocycle Dec_∞ . On their own the results of this section concerning Dec_∞ are of little interest since they describe group extensions which are already well understood. It is the relation (5) between Dec_∞ and $\mathrm{Dec}_{\mathbb{A}(S)}$ that is interesting. This relation is stated in §4 and proved in §6.

- §4 The function τ is defined, and the formalism used in the proof of the relation (5) between $\text{Dec}_{\mathbb{A}(S)}$, Dec_{∞} and τ is introduced.
- §5 This is a technical section on the existence of certain limits.
- §6 The relation (5) between $\text{Dec}_{\mathbb{A}(S)}$, Dec_{∞} and τ is proved.
- §7 The cocycles are extended to $\text{GL}_n(\mathbb{A})$.

Sections 3 and 4 are easy, although section 4 has a lot of notation. Section 7 is quite formal. In contrast, sections 5 and 6 are difficult.

2 Notation

2.1 Acyclic $(\mathbb{Z}/m)[\mu_m]$ -modules.

Throughout the paper, μ_m will denote a cyclic group of order m . As μ_m will often be taken to be a group of roots of unity, we shall write the group law in μ_m multiplicatively. We shall deal with various modules over the group-ring $(\mathbb{Z}/m)[\mu_m]$. We shall write $[\mu_m]$ for the sum of the elements of μ_m . We also fix once and for all a generator ζ of μ_m . There is an exact sequence (see §7 of [2]):

$$(\mathbb{Z}/m)[\mu_m] \xleftarrow{[\mu_m]} (\mathbb{Z}/m)[\mu_m] \xleftarrow{1-[\zeta]} (\mathbb{Z}/m)[\mu_m] \xleftarrow{[\mu_m]} (\mathbb{Z}/m)[\mu_m].$$

Applying the function $\text{Hom}_{(\mathbb{Z}/m)[\mu_m]}(-, M)$ we obtain the chain complex:

$$M \xrightarrow{[\mu_m]} M \xrightarrow{1-[\zeta]} M \xrightarrow{[\mu_m]} M. \tag{9}$$

The cohomology of this complex is the Tate cohomology $\hat{H}^\bullet(\mu_m, M)$ (see [30]). We shall call M an *acyclic* module if (9) is exact, i.e. if its Tate cohomology is trivial.

Free modules are clearly acyclic. Injective modules are acyclic, since for these the functor $\text{Hom}(-, M)$ is exact. More generally, by Shapiro's Lemma (see [30]), any $(\mathbb{Z}/m)[\mu_m]$ -module which is induced from a \mathbb{Z}/m -module is acyclic.

Lemma 1 *Let M be an acyclic $(\mathbb{Z}/m)[\mu_m]$ -module and suppose we have a $(\mathbb{Z}/m)[\mu_m]$ -module homomorphism*

$$\Phi : M \rightarrow \mathbb{Z}/m.$$

Then there is a $(\mathbb{Z}/m)[\mu_m]$ -homomorphism $\hat{\Phi} : (1 - [\zeta])M \rightarrow \mu_m$ defined by

$$\hat{\Phi}((1 - [\zeta])a) = \zeta^{\Phi(a)}.$$

The lifted map $\hat{\Phi}$ is independent of the choice of generator ζ . Here \mathbb{Z}/m and μ_m are regarded as $(\mathbb{Z}/m)[\mu_m]$ -modules with the trivial action of μ_m .

The definition of $\hat{\Phi}$ looks more natural if we identify μ_m with $\mathfrak{a}/\mathfrak{a}^2$, where \mathfrak{a} denotes the augmentation ideal of $(\mathbb{Z}/m)[\mu_m]$.

Proof. We need only show that $\hat{\Phi}$ is well-defined. Suppose $(1 - [\zeta])a = (1 - [\zeta])b$. By the exact sequence (9), there is a $c \in M$ such that $b - a = [\mu_m]c$. Therefore $\Phi(b) - \Phi(a) = [\mu_m]\Phi(c)$. Since the action of $[\mu_m]$ on \mathbb{Z}/m is zero, we have $\Phi(b) = \Phi(a)$. \square

2.2 Central extensions.

Let G be an abstract group. We shall regard μ_m as a G -module with the trivial action. Given an inhomogeneous 2-cocycle σ on G with values in μ_m , one defines a central extension of G by μ_m normalized as follows:

$$\tilde{G} = G \times \mu_m, \quad (g, \xi)(h, \psi) := (gh, \xi\psi\sigma(g, h)).$$

Conversely, given a central extension

$$1 \rightarrow \mu_m \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

we may recover a 2-cocycle σ by choosing, for every $g \in G$, a preimage $\hat{g} \in \tilde{G}$ and defining

$$\sigma(g, h) = \widehat{\hat{g}h\hat{g}^{-1}}.$$

If G is a locally compact topological group then by a *topological central extension* of G by μ_m we shall mean a short exact sequence of topological groups and continuous homomorphisms:

$$1 \rightarrow \mu_m \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

such that the map $\tilde{G} \rightarrow G$ is locally a homeomorphism. The isomorphism classes of topological central extensions of G by μ_m correspond to elements of $H_{\text{meas}}^2(G, \mu_m)$, where H_{meas}^\bullet denotes the group cohomology theory based on Borel-measurable cochains (see [24]). As all our cochains will be Borel-measurable we shall write H^\bullet instead of H_{meas}^\bullet .

2.3 Commutators and symmetric cocycles

Proofs of the following facts may be found in [16], where they were used to determine the metaplectic extensions of D^\times , where D is a quaternion algebra over k .

Let G be a group and suppose G_1 and G_2 are two subgroups of G , such that every element of G_1 commutes with every element of G_2 . Given a 2-cocycle $\sigma \in Z^2(G, \mu_m)$, we define for $a \in G_1$ and $b \in G_2$ the *commutator*:

$$[a, b]_\sigma := \frac{\sigma(a, b)}{\sigma(b, a)}.$$

The commutator map is bimultiplicative and skew symmetric, and depends only on the cohomology class of σ . If G is a locally compact topological group and σ is Borel-measurable, then the commutator map is a continuous function on $G_1 \times G_2$.

If G is an abelian group, then we shall call a 2-cocycle σ on G *symmetric* if $[\cdot, \cdot]_\sigma$ is trivial on $G \times G$. This amounts to saying that the corresponding central extension \tilde{G} is abelian. We shall write $H_{sym}^2(G, \mu_m)$ for the subgroup of symmetric classes. If G and H are two abelian groups, then the restriction maps give an isomorphism:

$$H_{sym}^2(G \oplus H, \mu_m) \cong H_{sym}^2(G, \mu_m) \oplus H_{sym}^2(H, \mu_m).$$

The restriction map gives an isomorphism:

$$H_{sym}^2(G, \mu_m) \cong H_{sym}^2(G[m], \mu_m),$$

where $G[m]$ denotes the subgroup of m -torsion elements of G . Furthermore there is a canonical isomorphism (independent of ζ):

$$\begin{aligned} H_{sym}^2(\mu_m, \mu_m) &\cong \mathbb{Z}/m \\ \sigma &\mapsto b, \quad \text{where} \quad \prod_{i=1}^m \sigma(\zeta, \zeta^i) = \zeta^b. \end{aligned}$$

2.4 Singular homology groups.

We will need some notation from singular homology, which we now introduce. In order to avoid sign errors we must be clear about the precise definition of our chain complex, which is rather non-standard.

For $r \geq 0$ we define the r -simplex Δ^r by

$$\Delta^r = \left\{ \underline{x} \in \mathbb{R}^{r+1} : x_0, \dots, x_r \geq 0 \text{ and } \sum x_i = 1 \right\}.$$

By an r -prism we shall mean a product of finitely many simplices, i.e. an expression of the form $\prod_{i=1}^s \Delta^{a(i)}$, where $\sum a(i) = r$. Let Y be a topological space. By a *singular r -cell* in Y we shall mean a continuous map from an r -prism to Y . The cell $\mathcal{T} : \prod_{i=1}^s \Delta^{a(i)} \rightarrow Y$ will be said to be *degenerate* if $\mathcal{T}(\underline{x}_1, \dots, \underline{x}_s)$ is independent of one of the variables $\underline{x}_i \in \Delta^{a(i)}$ ($a(i) > 0$). We shall write $C_r(Y)$ for the $\mathbb{Z}/m\mathbb{Z}$ -module generated by the set of all singular r -cells in Y , with the following relations:

- Suppose A is an a -prism and B is a b -prism with $a + b = r$. If $\mathcal{T} : A \times B \rightarrow Y$ is a singular r -cell then we define another singular r -cell $\mathcal{T}' : B \times A \rightarrow Y$ by $\mathcal{T}'(b, a) = \mathcal{T}(a, b)$. For every such $\mathcal{T}, \mathcal{T}'$ we impose a relation:

$$\mathcal{T} = (-1)^{ab} \mathcal{T}';$$

- $\mathcal{T} = 0$ for every degenerate r -prism \mathcal{T} .

For any simplex Δ^r we define the j -th face map ($j = 0, \dots, r$) to be the map

$$\mathfrak{F}_j : \Delta^{r-1} \rightarrow \Delta^r, \quad (x_0, \dots, x_{r-1}) \mapsto (x_0, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_r).$$

The boundary of a singular cell $\mathcal{T} : \prod_{i=1}^s \Delta^{a(i)} \rightarrow Y$ is defined to be the sum

$$\partial \mathcal{T} = \sum_{i=1}^s \sum_{j=0}^{a(i)} (-1)^{a(1)+\dots+a(i-1)+j} \mathcal{T} \circ \mathfrak{F}_{i,j},$$

$$\text{where } \mathfrak{F}_{i,j}(\underline{x}_1, \dots, \underline{x}_s) = (\underline{x}_1, \dots, \underline{x}_{i-1}, \mathfrak{F}_j(\underline{x}_i), \underline{x}_{i+1}, \dots, \underline{x}_s).$$

This boundary map extends by linearity to a map $\partial : C_r(Y) \rightarrow C_{r-1}(Y)$.

For any subspace $Z \subseteq Y$ there is an inclusion $C_r(Z) \subseteq C_r(Y)$, and we define $C_\bullet(Y, Z) = C_\bullet(Y)/C_\bullet(Z)$. The homology groups of the complexes $C_\bullet(Y)$ and $C_\bullet(Y, Z)$ are the usual singular homology groups with coefficients in $\mathbb{Z}/m\mathbb{Z}$ (see for example [21]). We have taken a non-standard definition of the chain complex because we will write down singular cells explicitly and these will be as described above.

The base set $|\mathcal{T}|$ of a singular r -cell \mathcal{T} is defined to be the image of \mathcal{T} if \mathcal{T} is non-degenerate, and the empty set if \mathcal{T} is degenerate. The base set of an

element of $C_r(Y)$ is defined to be the union of all base-sets of singular r -cells in its support.

If Y is a vector space over \mathbb{R} then for vectors $v_0, \dots, v_r \in Y$ we shall denote by $[v_0, \dots, v_r]$ the r -cell $\Delta^r \rightarrow Y$ given by

$$[v_0, \dots, v_r](\underline{x}) := \sum_{i=0}^r x_i v_i.$$

The image of this map is the convex hull of the set $\{v_0, \dots, v_r\}$. To simplify our formulae we shall sometimes substitute the closed interval $I = [0, 1]$ for Δ^1 , by identifying 0 with $(1, 0)$ and 1 with $(0, 1)$.

Orientations. Let Y be a d -dimensional manifold. If $y \in Y$ then $H_d(Y, Y \setminus \{y\})$ is non-canonically isomorphic to $\mathbb{Z}/m\mathbb{Z}$. The manifold Y is said to be $\mathbb{Z}/m\mathbb{Z}$ -orientable if one can associate to each point $y \in Y$ an isomorphism

$$\text{ord}_y : H_d(Y, Y \setminus \{y\}) \longrightarrow \mathbb{Z}/m\mathbb{Z},$$

with the property that for every $y \in Y$ there is a neighbourhood U of y , such that for every $z \in U$ the following diagram commutes.

$$\begin{array}{ccc} & H_d(Y, Y \setminus U) & \\ & \swarrow \quad \searrow & \\ H_d(Y, Y \setminus \{y\}) & & H_d(Y, Y \setminus \{z\}) \\ \swarrow \text{ord}_y & & \swarrow \text{ord}_z \\ & \mathbb{Z}/m\mathbb{Z} & \end{array}$$

Such a collection of isomorphisms will be called an *orientation*.

Suppose Y is $\mathbb{Z}/m\mathbb{Z}$ -orientable and fix an orientation $\{\text{ord}_y\}$ on Y . Let $\mathcal{J} \in C_d(Y)$. Suppose that $y \in Y$ does not lie in the base set of $\partial\mathcal{J}$. Then \mathcal{J} represents a homology class in $H_d(Y, Y \setminus \{y\})$. From our condition on ord , the map $y \mapsto \text{ord}_y(\mathcal{J})$ is a locally constant function $Y \setminus |\partial\mathcal{J}| \rightarrow \mathbb{Z}/m\mathbb{Z}$. For a discrete subset $M \subseteq Y$ we shall use the notation

$$\{\mathcal{J}|M\} = \sum_{y \in M} \text{ord}_y \mathcal{J}.$$

3 The arithmetic and geometric cocycles.

3.1 The arithmetic cocycle

Let μ_m be a cyclic group of order m . Let V_{arith} be a totally disconnected locally compact abelian group and suppose that multiplication by m induces a measure-preserving automorphism of V_{arith} . Suppose μ_m acts on V_{arith} in such a way that every non-zero element of V_{arith} has trivial stabilizer in μ_m . Given this data we shall construct a 2-cocycle $\text{Dec}_{\text{arith}}$ on the group $G_{\text{arith}} = \text{Aut}_{\mu_m}(V_{\text{arith}})$ with values in μ_m . This cocycle $\text{Dec}_{\text{arith}}$ was first studied in [10] and we shall keep to the notation of that paper. Those results, which are stated here without proof, are proved in [10].

The Cocycle. We choose a compact, open, μ_m -invariant neighbourhood L of 0 in V_{arith} and we normalize the Haar measure dv on V_{arith} so that L has measure 1. By our condition on V_{arith} , it follows that for any compact open subset U of V_{arith} , the measure of U is a rational number and is integral at every prime dividing m . Thus for any locally constant function $\varphi : V_{\text{arith}} \rightarrow \mathbb{Z}/m$ of compact support, we may define its integral to be an element of \mathbb{Z}/m . This “modulo m integration” is independent of the neighbourhood L used to normalize the measure.

Finally, we choose an open and closed fundamental domain F for the action of μ_m on $V_{\text{arith}} \setminus \{0\}$; we write f for the characteristic function of F . The cocycle is given by the formula:

$$\text{Dec}_{\text{arith}}^{f,L}(\alpha, \beta) = \prod_{\xi \in \mu_m} \xi \left\{ \int_L - \int_{\beta L} \right\} f(\alpha v) f(\xi v) dv, \quad \alpha, \beta \in G_{\text{arith}}. \quad (10)$$

Up to a coboundary, this is independent of the choices of f and L . If we regard G_{arith} as a topological group with the compact-open topology then the cocycle $\text{Dec}_{\text{arith}}$ is a locally constant function. This may be deduced using the cocycle relation from the following fact.

Lemma 2 *If $\alpha, \beta \in G_{\text{arith}}$ and $\beta L = L$ then we have $\text{Dec}_{\text{arith}}^{(f,L)}(\alpha, \beta) = 1$.*

Proof. This follows immediately from the definition of $\text{Dec}_{\text{arith}}$. □

A Pairing. Before proceeding, we shall reformulate the definition of $\text{Dec}_{\text{arith}}$ in a more useful notation. We shall call a function $\varphi : V_{\text{arith}} \rightarrow \mathbb{Z}/m$ *symmetric* if $\varphi(\xi v) = \varphi(v)$ for all $\xi \in \mu_m$. We define \mathcal{C} to be the space of locally constant symmetric functions of compact support on V_{arith} . On the other hand a function $g : V_{\text{arith}} \setminus \{0\} \rightarrow \mathbb{Z}/m$ will be called *cosymmetric* if the sum

$$\sum_{\xi \in \mu_m} g(\xi v)$$

is a constant, independent of v . The value of the constant will be called the *degree* of g . We define \mathcal{F} to be the space of locally constant, cosymmetric functions $V_{\text{arith}} \setminus \{0\} \rightarrow \mathbb{Z}/m$. A cosymmetric function $f \in \mathcal{F}$ of degree 1 will be called a *fundamental function*. For example the function f described above is a fundamental function. Thus fundamental functions generalize the notion of a fundamental domain. The group G_{arith} acts on \mathcal{C} and \mathcal{F} on the right by composition. Let \mathcal{C}^o be the submodule of functions $M \in \mathcal{C}$ satisfying $M(0) = 0$ and let \mathcal{F}^o be the submodule of functions in \mathcal{F} of degree 0. There is a pairing $\mathcal{F}^o \times \mathcal{C}^o \rightarrow \mu_m$ given by

$$\langle g|M \rangle = \prod_{\xi \in \mu_m} \xi \int g(v) f(\xi v) M(v) dv,$$

where f is a fundamental function. The pairing is independent of f , and is G_{arith} -invariant in the sense that for $\alpha \in G_{\text{arith}}$ we always have:

$$\langle h\alpha|M\alpha \rangle = \langle h|M \rangle. \quad (11)$$

With this notation we can express $\text{Dec}_{\text{arith}}$ as follows:

$$\text{Dec}_{\text{arith}}^{(f,L)}(\alpha, \beta) = \langle f - f\alpha|\beta L - L \rangle. \quad (12)$$

Here we are writing L and βL for the characteristic functions of these sets. It is now clear that $\text{Dec}_{\text{arith}}$ is a 2-cocycle, since it is the cup-product of the 1-cocycles $f - f\alpha$ and $\beta L - L$.

Another expression for the pairing. To aid calculation we shall describe the pairing in a different way. Let \mathcal{M}^c be the space of locally constant functions of compact support $\varphi : V_{\text{arith}} \setminus \{0\} \rightarrow \mathbb{Z}/m$. Let \mathcal{M} be the space

of locally constant functions $\varphi : V_{\text{arith}} \setminus \{0\} \rightarrow \mathbb{Z}/m$. There is a right action of G_{arith} by composition on the spaces \mathcal{M}^c and \mathcal{M} :

$$(\phi\alpha)(v) := \phi(\alpha v), \quad \alpha \in G_{\text{arith}}.$$

We shall also regard \mathcal{M} and \mathcal{M}^c as left $(\mathbb{Z}/m)[\mu_m]$ -modules with the action given by

$$(\xi\phi)(v) = \phi(\xi^{-1}v), \quad \xi \in \mu_m.$$

The two actions commute.

As $(\mathbb{Z}/m)[\mu_m]$ -modules, both \mathcal{M} and \mathcal{M}^c are acyclic, since they are induced from spaces of functions on the fundamental domain F . Hence, by Lemma 1, the integration map $\int : \mathcal{M}^c \rightarrow \mathbb{Z}/m$ lifts to a map

$$\widehat{\int} : (1 - [\zeta])\mathcal{M}^c \rightarrow \mu_m,$$

defined by

$$\widehat{\int} (\varphi - \varphi\zeta^{-1})(v)dv = \widehat{\int} (1 - [\zeta])\varphi dv := \zeta^{-1} \int \varphi(v)dv.$$

The modules \mathcal{F}^o and \mathcal{C}^o introduced above may be expressed as follows:

$$\mathcal{C}^o = \ker \left((1 - [\zeta]) : \mathcal{M}^c \rightarrow \mathcal{M}^c \right) = [\mu_m]\mathcal{M}^c,$$

$$\mathcal{F}^o = \ker \left([\mu_m] : \mathcal{M} \rightarrow \mathcal{M} \right) = (1 - [\zeta])\mathcal{M}.$$

Proposition 1 *Given $g \in \mathcal{F}^o$ and $M \in \mathcal{C}^o$, the product $g \cdot M$ is in $(1 - [\zeta])\mathcal{M}^c$. The pairing $\mathcal{F}^o \times \mathcal{C}^o \rightarrow \mu_m$ is given by*

$$\langle g|M \rangle = \widehat{\int} (g \cdot M)(v)dv. \tag{13}$$

Proof. Since $[\mu_m]g = 0$, we have $g = (1 - [\zeta])h$ for some $h \in \mathcal{M}$. As M is symmetric we have $g \cdot M = (1 - [\zeta])(h \cdot M)$. Since M has compact support, so does $h \cdot M$. Therefore $g \cdot M \in (1 - [\zeta])\mathcal{M}^c$ and we have

$$\widehat{\int} g \cdot M = \zeta \int h(v)M(v)dv.$$

We now calculate the pairing:

$$\langle g|M \rangle = \prod_{\xi \in \mu_m} \xi \int f(\xi v)(h(v) - h(\zeta^{-1}v))M(v)dv.$$

Replacing ξ by $\zeta^{-1}\xi$ in the second term we obtain:

$$\langle g|M \rangle = \prod_{\xi \in \mu_m} \xi \int f(\xi v)h(v)M(v)dv - \int f(\zeta^{-1}\xi v)h(\zeta^{-1}v)M(v)dv.$$

Replacing v by ζv in the second term and using the symmetry of M we obtain:

$$\langle g|M \rangle = \prod_{\xi \in \mu_m} \xi \int f(\xi v)h(v)M(v)dv - \int f(\xi v)h(v)M(v)dv.$$

This cancels to give:

$$\langle g|M \rangle = \prod_{\xi \in \mu_m} \zeta \int f(\xi v)h(v)M(v)dv.$$

Since f is fundamental we have:

$$\langle g|M \rangle = \zeta \int h(v)M(v)dv.$$

□

3.2 Reduction to a discrete space.

The arithmetic cocycle $\text{Dec}_{\text{arith}}$ depends on a choice of an open and closed fundamental domain F for μ_m in $V_{\text{arith}} \setminus \{0\}$. In practice it is unnecessary to describe such an F completely. We shall show that a large part of the cocycle depends only on a fundamental domain in a discrete quotient of V_{arith} .

We shall fix once and for all a μ_m -invariant, compact, open subgroup $L \subset V_{\text{arith}}$. We shall write X for the (discrete) quotient group V_{arith}/L . The idea is that it is easier to find a fundamental domain in $X \setminus \{0\}$ than in $V_{\text{arith}} \setminus \{0\}$.

We introduce modules of functions on X analogous to \mathcal{F}^o and \mathcal{C}^o above. We let \mathcal{M}_X denote the space of all functions $X \setminus \{0\} \rightarrow \mathbb{Z}/m$ and we let \mathcal{M}_X^c

denote the space of all such functions with finite support. As the action of μ_m on $X \setminus \{0\}$ has no fixed points, these are acyclic $(\mathbb{Z}/m)[\mu_m]$ modules. We define

$$\mathcal{F}_X^o = \ker([\mu_m] : \mathcal{M}_X \rightarrow \mathcal{M}_X), \quad \mathcal{C}_X^o = \ker(1 - [\zeta] : \mathcal{M}_X^c \rightarrow \mathcal{M}_X).$$

There is a pairing $\mathcal{F}_X^o \times \mathcal{C}_X^o \rightarrow \mu_m$ given by

$$\langle (1 - [\zeta])g|M \rangle_X = \zeta \sum_{x \in X \setminus \{0\}} g(x)M(x). \quad (14)$$

We have canonical inclusions $\iota : \mathcal{M}_X \rightarrow \mathcal{M}$ and $\iota : \mathcal{M}_X^c \rightarrow \mathcal{M}^c$ and we have

$$\langle g|M \rangle_X = \langle \iota(g)|\iota(M) \rangle. \quad (15)$$

We shall write G_{arith}^+ for the semi-group of elements $\alpha \in G_{\text{arith}}$ such that $\alpha L \supseteq L$. Let F_X be a fundamental domain for the action of μ_m on $X \setminus \{0\}$. We shall suppose our fundamental domain F is chosen so that for $v \notin L$ we have $v \in F$ if and only if $v + L \in F_X$. As before we shall write f for the characteristic function of F .

Lemma 3 *For $\alpha, \beta \in G_{\text{arith}}^+$ we have:*

- *The restriction of $f\alpha^{-1}$ to $V_{\text{arith}} \setminus \alpha L$ is L -periodic, and therefore induces a function on $X \setminus \alpha L$.*
- *The set $\alpha\beta L - \alpha L$ is L -periodic. Its characteristic function therefore induces a function on X which is zero on αL .*
- *We have*

$$\text{Dec}_{\text{arith}}^{(f,L)}(\alpha, \beta) = \langle f\alpha^{-1} - f|\alpha\beta L - \alpha L \rangle_X.$$

Proof. The first two statements are easy to check; the third follows from (11), (12) and (15). \square

3.3 Examples of arithmetic cocycles

Let k be a global field containing a primitive m -th root of unity, and let μ_m be the group of m -th roots of unity in k . Consider the vector space $V = k^n$.

Local examples. Let v be a place of k such that $|m|_v = 1$, where $|\cdot|_v$ is the absolute value on k_v , normalized so that $d(mx) = |m|_v dx$ for any Haar measure dx on k_v . We shall write V_v for the vector space $V \otimes_k k_v = k_v^n$. The action of μ_m by scalar multiplication on V_v satisfies the conditions of V_{arith} of §3.1. We therefore obtain a 2-cocycle on $\text{GL}_n(k_v)$ with values in μ_m . We shall refer to this cocycle as Dec_v .

The commutator of Dec_v on the maximal torus has been calculated in Theorems 3 and 5 of [10]. It is given by

$$[\alpha, \beta]_{\text{Dec}_v} = (-1)^{\frac{(|\det \alpha|_v - 1)(|\det \beta|_v - 1)}{m^2}} \prod_{i=1}^n (-1)^{\frac{(|\alpha_i|_v - 1)(|\beta_i|_v - 1)}{m^2}} (\alpha_i, \beta_i)_{m,v}, \quad (16)$$

where

$$\alpha = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_n \end{pmatrix}$$

and $(\cdot, \cdot)_{m,v}$ denotes the m -th power Hilbert symbol on k_v . Our restriction on v amounts to requiring that $(\cdot, \cdot)_{m,v}$ is the tame symbol.

Semi-global examples. Let S be the set of all places v of k , such that $|m|_v \neq 1$. We shall write $\mathbb{A}(S)$ for the ring of S -adèles of k , i.e. the restricted topological product of the fields k_v for $v \notin S$. Let $V_{\mathbb{A}(S)} = V \otimes_k \mathbb{A}(S) = \mathbb{A}(S)^n$. The action of μ_m on $V_{\mathbb{A}(S)}$ by scalar multiplication satisfies the conditions of V_{arith} of §3.1. We therefore obtain a 2-cocycle on $\text{GL}_n(\mathbb{A}(S))$ with values in μ_m . We shall refer to this cocycle as $\text{Dec}_{\mathbb{A}(S)}$. The cocycle $\text{Dec}_{\mathbb{A}(S)}$ is not quite the product of the local cocycles Dec_v for $v \notin S$. In fact we have (Theorem 3 of [10]) up to a coboundary:

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \prod_{v \notin S} \text{Dec}_v(\alpha, \beta) \prod_{v < w} (-1)^{\frac{(|\det \alpha|_v - 1)(|\det \beta|_w - 1)}{m^2}}.$$

Here we have chosen an ordering on the set of places $v \notin S$; up to a coboundary the right hand side of the above formula is independent of the choice of ordering. As a consequence of this and (16), we have on the maximal torus in $\text{GL}_n(\mathbb{A}(S))$:

$$[\alpha, \beta]_{\text{Dec}_{\mathbb{A}(S)}} = (-1)^{\frac{(|\det \alpha|_{\mathbb{A}(S)} - 1)(|\det \beta|_{\mathbb{A}(S)} - 1)}{m^2}} \prod_{v \notin S} \prod_{i=1}^n (-1)^{\frac{(|\alpha_i|_v - 1)(|\beta_i|_v - 1)}{m^2}} (\alpha_i, \beta_i)_{v,m}.$$

The positive characteristic case. Suppose for a moment that k has positive characteristic. In this case the set S is empty, so in fact we have a cocycle $\text{Dec}_{\mathbb{A}}$ on the whole group $\text{GL}_n(\mathbb{A})$, where \mathbb{A} is the full adèle ring of k . The proof that $\text{Dec}_{\mathbb{A}}$ splits on $\text{GL}_n(k)$ is easy to understand.

We may also take $V_{\text{arith}} = \mathbb{A}^n/k^n$. This is acted on by $\text{GL}_n(k)$, so we obtain a corresponding cocycle Dec_k on $\text{GL}_n(k)$ with values in μ_m . One easily shows that, up to a coboundary, Dec_k is the restriction of $\text{Dec}_{\mathbb{A}}$. In this case however, V_{arith} is compact, so we may take $L = V_{\text{arith}}$ in the definition (10) of the cocycle. With this choice of L one sees immediately that the cocycle is trivial. This shows that the restriction of $\text{Dec}_{\mathbb{A}}$ to $\text{GL}_n(k)$ is a coboundary.

3.4 The Gauss–Schering Lemma

We next indicate the connection between the cocycle $\text{Dec}_{\mathbb{A}(S)}$ and the Gauss–Schering Lemma. Let k , μ_m and S be as in §3.3. We shall write \mathfrak{o}^S for the ring of S -integers in k . Recall that for non-zero, coprime $\alpha, \beta \in \mathfrak{o}^S$, the m -th power Legendre symbol is defined by

$$\left(\frac{\alpha}{\beta}\right)_{S,m} = \prod_{v \notin S, v|\beta} (\alpha, \beta)_{m,v}.$$

The Gauss–Schering Lemma is a formula for the Legendre symbol, commonly used to prove the quadratic reciprocity law in undergraduate courses. Choose a set R_β of representatives of the non-zero μ_m -orbits in $\mathfrak{o}^S/(\beta)$. Such sets are called m -th sets modulo β . For any $\xi \in \mu_m$ define

$$r(\xi) = |\{x \in R_\beta : \alpha x \in \xi R_\beta\}|.$$

The Gauss–Schering Lemma is the statement

$$\left(\frac{\alpha}{\beta}\right)_{S,m} = \prod_{\xi \in \mu_m} \xi^{r(\xi)}.$$

Consider the cocycle $\text{Dec}_{\mathbb{A}(S)}$ on $\text{GL}_1(\mathbb{A}(S))$ of §3.3. Let $L = \prod_{v \notin S} \mathfrak{o}_v$; we shall also write L for the characteristic function of this set. The subset $L \subset \mathbb{A}(S)$ satisfies the conditions of §3.2. The semi-group G_{arith}^+ of §3.2 contains β^{-1} for all non-zero $\beta \in \mathfrak{o}^S$. Choose a fundamental domain F for the action of μ_m on $(\mathbb{A}(S)/L) \setminus \{0\}$ and let f be its characteristic function. We shall write $f_{\mathbb{A}(S)}$ for an extension of f to $\mathbb{A}(S) \setminus \{0\}$. The Gauss–Schering Lemma may be reformulated as follows.

Proposition 2 *For non-zero, coprime $\alpha, \beta \in \mathfrak{o}_S$ we have:*

$$\left(\frac{\alpha}{\beta}\right)_{S,m} = \text{Dec}_{\mathbb{A}(S)}^{f_{\mathbb{A}(S)}, L}(\alpha, \beta^{-1})^{-1}.$$

Proof. As in §3.2 we let $X = \mathbb{A}(S)/L$ and we let F be a set of representatives for μ_m -orbits in X . Let $X[\beta] = \{x \in X : \beta x = 0\}$. We shall identify $X[\beta]$ with $\mathfrak{o}^S/(\beta)$. Let $F_\beta = F \cap X[\beta]$. The set F_β is an m -th set modulo β in the sense described above. Therefore the Gauss-Schering Lemma states that $(\alpha/\beta)_{S,m} = \prod \xi^{r(\xi)}$, where $r(\xi)$ may be rewritten as:

$$r(\xi) = \sum_{x \in F_\beta} f(\xi^{-1}\alpha x).$$

This is equivalent to

$$r(\xi) = \sum_{x \in X[\beta] \setminus \{0\}} f(x)f(\xi^{-1}\alpha x).$$

As we have normalized the Haar measure to give L measure 1, the sum can be rewritten as an integral:

$$r(\xi) = \left\{ \int_{\beta^{-1}L} - \int_L \right\} f(v)f(\xi^{-1}\alpha v)dv.$$

The proposition follows from the definition (10) of $\text{Dec}_{\text{arith}}$. □

It is common in proofs of reciprocity laws using the Gauss-Schering Lemma to change from a summation over β -division points to a summation over $\alpha\beta$ -division points when calculating $(\alpha/\beta)_m$ (see [9, 8]). The reason why this technique is useful, is because whereas the sum over β -division points is $\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta^{-1})$, the sum over $\alpha\beta$ -division points is, as we can see from Lemma 3, related to $\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta)$.

3.5 The geometric cocycle

In this section we let V_∞ be a vector space over \mathbb{R} with an action of the cyclic group μ_m such that every non-zero vector in V_∞ has trivial stabilizer in μ_m (such representations are called linear space forms). We shall write d for the dimension of V_∞ as a vector space over \mathbb{R} . Let G_∞ be the group $\text{Aut}_{\mu_m}(V_\infty)$

and let \mathfrak{g} be its Lie algebra $\text{End}_{\mu_m}(V_\infty)$. We shall construct a 2-cocycle Dec_∞ on G_∞ with values in μ_m . The construction is analogous to the construction of $\text{Dec}_{\text{arith}}$ of §3.1.

If $m = 2$ then the group μ_2 acts on V_∞ with the non-trivial element acting by multiplication by -1 . In this case $G_\infty = \text{GL}_d(\mathbb{R})$. In contrast if $m \geq 3$ then V_∞ is a direct sum of irreducible two-dimensional representations of μ_m . The group G_∞ is then a direct sum of groups isomorphic to $\text{GL}_{r(i)}(\mathbb{C})$, with $d = 2 \sum_i r(i)$.

For any singular r -cell $\mathcal{J} : A \rightarrow \mathfrak{g}$ in \mathfrak{g} and any singular s -cell $\mathcal{U} : B \rightarrow V_\infty$ we define a singular $r + s$ -cell $\mathcal{J} \cdot \mathcal{U} : A \times B \rightarrow V_\infty$ by

$$(\mathcal{J} \cdot \mathcal{U})(\underline{x}, \underline{y}) = \mathcal{J}(\underline{x}) \cdot \mathcal{U}(\underline{y}).$$

This operation extends to a bilinear map $C_r(\mathfrak{g}) \times C_s(V_\infty) \rightarrow C_{r+s}(V_\infty)$.

We fix an orientation ord on V_∞ . The following lemma will often be used in what follows.

Lemma 4 (i) For any $\alpha \in G_\infty$, $v \in V_\infty$ and any $\mathcal{J} \in Z_d(V_\infty, V_\infty \setminus \{v\})$ we have in \mathbb{Z}/m :

$$\text{ord}_{\alpha v}(\alpha \mathcal{J}) = \text{ord}_v(\mathcal{J}).$$

(ii) For every $\mathcal{J} \in C_d(V_\infty, V_\infty \setminus \{0\})$ we have in \mathbb{Z}/m :

$$\text{ord}_0((1 - [\zeta])\mathcal{J}) = \text{ord}_0([\mu_m]\mathcal{J}) = 0.$$

Proof. (i) Since $\mathbb{Z}/2\mathbb{Z}$ has no non-trivial automorphisms, there is nothing to prove in the case $m = 2$. Assume now $m > 2$. The group G_∞ is a direct sum of groups isomorphic to $\text{GL}_r(\mathbb{C})$, and is therefore connected. Hence the action of G_∞ on the set of orientations of V_∞ is trivial.

(ii) By (i) we have $\text{ord}_0((1 - [\zeta])\mathcal{J}) = 0$. On the other hand, the following holds in $(\mathbb{Z}/m)[\mu_m]$:

$$[\mu_m] = \left([\zeta] + 2[\zeta^2] + \dots + (m-1)[\zeta^{m-1}] \right) (1 - [\zeta]).$$

Hence $\text{ord}_0([\mu_m]\mathcal{J}) = 0$. □

Let d be the dimension of V_∞ as a vector space over \mathbb{R} . We may choose a finite $d - 1$ -dimensional cell complex \mathfrak{S} in $V_\infty \setminus \{0\}$ with the following properties:

- The inclusion $\mathfrak{S} \hookrightarrow V_\infty \setminus \{0\}$ is a homotopy equivalence;

- The group μ_m permutes the cells in \mathfrak{S} and each cell has trivial stabilizer in μ_m .

One could for example take \mathfrak{S} to be a triangulation of the unit sphere in V_∞ . We shall write \mathfrak{S}_\bullet for the corresponding chain complex with coefficients in \mathbb{Z}/m . It follows from the second condition above that each \mathfrak{S}_r is a free $(\mathbb{Z}/m)[\mu_m]$ -module. A basis consists of a set of representatives of μ_m -orbits of r -cells. Thus each \mathfrak{S}_r satisfies the exact sequence (9).

Choose a cycle $\omega \in \mathfrak{S}_{d-1}$ whose homology class is mapped to 1 by the isomorphisms:

$$H_{d-1}(\mathfrak{S}) \rightarrow H_{d-1}(V_\infty \setminus \{0\}) \xrightarrow{\partial^{-1}} H_d(V_\infty, V_\infty \setminus \{0\}) \xrightarrow{\text{ord}_0} \mathbb{Z}/m.$$

By Lemma 4 we know that $(1 - [\zeta])[\omega] = 0$ in $H_{d-1}(\mathfrak{S})$. As $\mathfrak{S}_d = 0$, this implies that $(1 - [\zeta])\omega = 0$ in \mathfrak{S}_{d-1} . Thus by the exact sequence (9) there is a $\mathcal{D} \in \mathfrak{S}_{d-1}$ such that $\omega = [\mu_m]\mathcal{D}$. We may think of \mathcal{D} as a fundamental domain for μ_m in \mathfrak{S} . As ω is a cycle we have:

$$[\mu_m]\partial\mathcal{D} = \partial[\mu_m]\mathcal{D} = \partial\omega = 0.$$

Thus, by the exact sequence (9), there is an $\mathcal{E} \in \mathfrak{S}_{d-2}$ such that

$$\partial\mathcal{D} = (1 - [\zeta])\mathcal{E}.$$

Definition 1 We may define the geometric cocycle $\text{Dec}_\infty^{(\mathcal{D})}$ for generic $\alpha, \beta \in G_\infty$ as follows:

$$\text{Dec}_\infty^{(\mathcal{D})}(\alpha, \beta) = \zeta^{-\text{ord}_0}([1, \alpha, \alpha\beta] \cdot \mathcal{E}).$$

Remark 1 In fact we have $\text{Dec}_\infty^{(\mathcal{D})}(\alpha, \beta) = \widehat{\text{ord}}_0(-[1, \alpha, \alpha\beta] \cdot \partial\mathcal{D})$, where $\widehat{\text{ord}}_0$ is a lifted map in the sense of Lemma 1. The $(\mathbb{Z}/m)[\mu_m]$ -module $Z_d(V_\infty, V_\infty \setminus \{0\})$ is free. However we shall not use this interpretation.

If the choice of \mathcal{D} is clear or irrelevant, then we shall omit it from the notation. In order to define $\text{Dec}_\infty(\alpha, \beta)$ on all pairs (α, β) rather than just generically, we must define $\text{ord}_0([1, \alpha, \alpha\beta] \cdot \mathcal{E})$ when 0 is on the boundary. To do this we slightly modify our definition. We define $[1, \alpha, \alpha\beta]$ to be the map $\mathfrak{g}^3 \rightarrow C_2(\mathfrak{g})$ defined by

$$(\epsilon_1, \epsilon_2, \epsilon_3) \mapsto [1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)].$$

Then $[1, \alpha, \alpha\beta] \cdot \mathcal{E}$ becomes a map $\mathfrak{g}^3 \rightarrow C_d(V_\infty)$. We fix a basis $\{b_1, \dots, b_r\}$ for \mathfrak{g} over \mathbb{R} . For $\epsilon = \sum x_i b_i$ we shall use the notation:

$$\text{“}\mathbf{lim}\text{”} = \lim_{\epsilon \rightarrow 0^+} \cdots \lim_{x_r \rightarrow 0^+}.$$

With this notation we define

$$\text{ord}_0([1, \alpha, \alpha\beta] \cdot \mathcal{E}) = \mathbf{lim}_{\epsilon_1 \rightarrow 0^+} \mathbf{lim}_{\epsilon_2 \rightarrow 0^+} \mathbf{lim}_{\epsilon_3 \rightarrow 0^+} \text{ord}_0\left([1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)] \cdot \mathcal{E}\right).$$

We will prove in §5 that, for a suitable choice of \mathfrak{S} , the above limit exists for all $\alpha, \beta \in G_\infty$. It is worth mentioning that the above limits do not necessarily commute.

Theorem 2 *Dec $_\infty$ is a 2-cocycle.*

Proof. Let $\alpha, \beta, \gamma \in G_\infty$ and consider the 3-cell in \mathfrak{g} :

$$\mathcal{A} = [1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3), \alpha\beta\gamma(1 + \epsilon_4)].$$

We have straight from the definition:

$$\partial \text{Dec}_\infty(\alpha, \beta, \gamma) = \mathbf{lim}_{\epsilon_1 \rightarrow 0^+} \mathbf{lim}_{\epsilon_2 \rightarrow 0^+} \mathbf{lim}_{\epsilon_3 \rightarrow 0^+} \mathbf{lim}_{\epsilon_4 \rightarrow 0^+} \zeta^{-\text{ord}_0((\partial \mathcal{A}) \cdot \mathcal{E})}.$$

The boundary of $\mathcal{A} \cdot \mathcal{E}$ is given by

$$\partial(\mathcal{A} \cdot \mathcal{E}) = \partial(\mathcal{A}) \cdot \mathcal{E} + \mathcal{A} \cdot \partial \mathcal{E}.$$

(Since coefficients are in \mathbb{Z}/m , the sign is only important for $m \geq 3$, and in these cases d is even). Taking the order of this at 0 and then taking limits we obtain:

$$\partial \text{Dec}_\infty(\alpha, \beta, \gamma) = \mathbf{lim}_{\epsilon_1 \rightarrow 0^+} \mathbf{lim}_{\epsilon_2 \rightarrow 0^+} \mathbf{lim}_{\epsilon_3 \rightarrow 0^+} \mathbf{lim}_{\epsilon_4 \rightarrow 0^+} \zeta^{\text{ord}_0(\mathcal{A} \cdot \partial \mathcal{E})}.$$

We must show that the right hand side here is 1. We know that $(1 - [\zeta])\mathcal{E} = \partial \mathcal{D}$. Therefore $(1 - [\zeta])\partial \mathcal{E} = 0$, so by the exact sequence (9), there is a $\mathcal{B} \in \mathfrak{S}_{d-3}$ such that

$$\partial \mathcal{E} = [\mu_m] \mathcal{B},$$

This implies

$$\mathcal{A} \cdot \partial \mathcal{E} = [\mu_m](\mathcal{A} \cdot \mathcal{B}).$$

The result now follows from Lemma 4. □

We next verify that our definition is independent of the various choices made.

Proposition 3 (i) The cocycle $\text{Dec}_\infty^{(\mathcal{D})}$ is independent of the choice of \mathcal{E} , the orientation ord and the generator ζ of μ_m .
(ii) The cohomology class of $\text{Dec}_\infty^{(\mathcal{D})}$ is independent of the choice of \mathcal{D} .

Proof. (i) We first fix \mathcal{D} and choose another \mathcal{E}' such that

$$(1 - [\zeta])\mathcal{E} = (1 - [\zeta])\mathcal{E}' = \partial\mathcal{D}.$$

By the exact sequence (9) there is an $\mathcal{A} \in \mathfrak{S}_{d-2}$ such that

$$\mathcal{E}' - \mathcal{E} = [\mu_m]\mathcal{A}.$$

This implies by Lemma 4:

$$\begin{aligned} \text{Dec}_\infty^{\mathcal{E}'}(\alpha, \beta) &= \text{Dec}_\infty^{\mathcal{E}}(\alpha, \beta)\zeta^{-\text{ord}_0([\mu_m][1, \alpha, \alpha\beta] \cdot \mathcal{A})} \\ &= \text{Dec}_\infty^{\mathcal{E}}(\alpha, \beta). \end{aligned}$$

Now suppose we choose a different orientation ord' . We have $\text{ord}' = u \cdot \text{ord}$ for some $u \in (\mathbb{Z}/m)^\times$. We may therefore choose $\omega' = u^{-1}\omega$, $\mathcal{D}' = u^{-1}\mathcal{D}$ and $\mathcal{E}' = u^{-1}\mathcal{E}$. With these choices we have

$$\text{ord}'_0([1, \alpha, \alpha\beta] \cdot \mathcal{E}') = \text{ord}_0([1, \alpha, \alpha\beta] \cdot \mathcal{E}).$$

Finally suppose $\zeta'^u = \zeta$ for some $u \in (\mathbb{Z}/m)^\times$. We then have in $(\mathbb{Z}/m)[\mu_m]$:

$$(1 - \zeta) = (1 - \zeta')(1 + \zeta' + \dots + \zeta'^{u-1}).$$

We may therefore take

$$\mathcal{E}' = (1 + \zeta' + \dots + \zeta'^{u-1})\mathcal{E}.$$

This implies by Lemma 4

$$\text{ord}_0([1, \alpha, \alpha\beta] \cdot \mathcal{E}') = u \cdot \text{ord}_0([1, \alpha, \alpha\beta] \cdot \mathcal{E}).$$

Therefore

$$\zeta'^{-\text{ord}_0([1, \alpha, \alpha\beta] \cdot \mathcal{E}')} = \zeta^{-\text{ord}_0([1, \alpha, \alpha\beta] \cdot \mathcal{E})}.$$

(ii) We now allow \mathcal{D} to vary. We choose \mathcal{D}' to satisfy

$$[\mu_m]\mathcal{D}' = [\mu_m]\mathcal{D}.$$

By the exact sequence (9) there is a $\mathcal{B} \in \mathfrak{S}_{d-1}$ such that

$$\mathcal{D}' = \mathcal{D} + (1 - [\zeta])\mathcal{B}.$$

Thus

$$\partial\mathcal{D}' = \partial\mathcal{D} + (1 - [\zeta])\partial\mathcal{B}.$$

We may therefore choose $\mathcal{E}' = \mathcal{E} + \partial\mathcal{B}$. Note that we have

$$\begin{aligned} \partial([1, \alpha, \alpha\beta] \cdot \mathcal{B}) &= \partial([1, \alpha, \alpha\beta]) \cdot \mathcal{B} + [1, \alpha, \alpha\beta] \cdot \partial\mathcal{B} \\ &= [1, \alpha] \cdot \mathcal{B} - [1, \alpha\beta] \cdot \mathcal{B} + [\alpha, \alpha\beta] \cdot \mathcal{B} + [1, \alpha, \alpha\beta] \cdot \partial\mathcal{B}. \end{aligned}$$

This implies using Lemma 4:

$$\begin{aligned} \text{ord}_0([1, \alpha, \alpha\beta] \cdot \partial\mathcal{B}) &= \text{ord}_0([1, \alpha\beta] \cdot \mathcal{B} - [1, \alpha] \cdot \mathcal{B} - [\alpha, \alpha\beta] \cdot \mathcal{B}) \\ &= \text{ord}_0([1, \alpha\beta] \cdot \mathcal{B}) - \text{ord}_0([1, \alpha] \cdot \mathcal{B}) - \text{ord}_0([1, \beta] \cdot \mathcal{B}). \end{aligned}$$

Putting things back together we obtain:

$$\text{Dec}_\infty^{(\mathcal{D}')}(\alpha, \beta) = \text{Dec}_\infty^{(\mathcal{D})}(\alpha, \beta) \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)},$$

where

$$\tau(\alpha) = \zeta^{\text{ord}_0([1, \alpha] \cdot \mathcal{B})}.$$

□

3.6 Example: $\text{GL}_2(\mathbb{R})$

If V_∞ is 1-dimensional then the fundamental domain \mathcal{D} is zero-dimensional, and so we have $\mathcal{E} = 0$ and Dec_∞ is always 1. The smallest non-trivial example is the case $m = 2$ and $V_\infty = \mathbb{R}^2$ with the group μ_2 acting on \mathbb{R}^2 by scalar multiplication. We shall consider this example now. We have $G_\infty = \text{GL}_2(\mathbb{R})$.

As $m = 2$ there is no need to worry about a choice of orientation. We may take our fundamental domain \mathcal{D} to be the half-circle:

$$\mathcal{D} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 = 1, y \geq 0 \right\}.$$

The boundary of this consists of the two points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. We may therefore take $\mathcal{E} = [v]$, where $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The cocycle is then given (generically) by:

$$\text{Dec}_\infty^{(\mathcal{D})}(\alpha, \beta) = \begin{cases} 1 & \text{if } 0 \notin [v, \alpha v, \alpha\beta v], \\ -1 & \text{if } 0 \in [v, \alpha v, \alpha\beta v]. \end{cases}$$

By choosing a different \mathcal{D} we may replace v by any other non-zero vector to obtain a cohomologous cocycle.

For later use we calculate the commutator of Dec_∞ on the standard torus in $\text{GL}_2(\mathbb{R})$.

Proposition 4 *The commutator of Dec_∞ on the subgroup of diagonal matrices in $\text{GL}_2(\mathbb{R})$ is given by:*

$$\left[\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix} \right]_{\text{Dec}_\infty} = (\alpha_1, \beta_1)_{\mathbb{R},2} (\alpha_2, \beta_2)_{\mathbb{R},2} (\det \alpha, \det \beta)_{\mathbb{R},2}.$$

The right hand side here consists of quadratic Hilbert symbols on \mathbb{R} .

Proof. Commutators are is continuous, bimultiplicative and alternating. The right hand side of the above formula is also continuous, bimultiplicative and alternating in α, β . It is therefore sufficient to check the formula in the case $\alpha = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \beta = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$. To calculate the commutator there we choose $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. With this choice we have

$$\begin{aligned} [(1 + \epsilon_1)v, \alpha(1 + \epsilon_2)v, \alpha\beta(1 + \epsilon_3)v] &= \left[(1 + \epsilon_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, (1 + \epsilon_2)^\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}, (1 + \epsilon_3) \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right], \\ [(1 + \epsilon_1)v, \beta(1 + \epsilon_2)v, \beta\alpha(1 + \epsilon_3)v] &= \left[(1 + \epsilon_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, (1 + \epsilon_2)^\beta \begin{pmatrix} -1 \\ 1 \end{pmatrix}, (1 + \epsilon_3) \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right]. \end{aligned}$$

We therefore have in $\mathbb{Z}/2$:

$$\lim_{\epsilon_1 \rightarrow 0^+, \epsilon_2 \rightarrow 0^+, \epsilon_3 \rightarrow 0^+} \begin{pmatrix} \text{ord}_0([(1 + \epsilon_1)v, \alpha(1 + \epsilon_2)v, \alpha\beta(1 + \epsilon_3)v]) \\ -\text{ord}_0([(1 + \epsilon_1)v, \beta(1 + \epsilon_2)v, \beta\alpha(1 + \epsilon_3)v]) \end{pmatrix} = 1.$$

This implies $\frac{\text{Dec}_\infty(\alpha, \beta)}{\text{Dec}_\infty(\beta, \alpha)} = -1$, which verifies the result. \square

3.7 Stability of the cocycles

Suppose V_∞ is the direct sum of the representations V_1 and V_2 of μ_m . The group G_∞ contains the direct sum of $G_1 = \text{Aut}_{\mu_m}(V_1)$ and $G_2 = \text{Aut}_{\mu_m}(V_2)$. We have defined cocycles Dec_∞ on G_∞ , $\text{Dec}_\infty^{(1)}$ on G_1 and $\text{Dec}_\infty^{(2)}$ on G_2 . The next result describes how these are related.

Proposition 5 *Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ denote elements of $G_1 \oplus G_2$. We have up to a coboundary:*

$$\text{Dec}_\infty(\alpha, \beta) = \text{Dec}_\infty^{(1)}(\alpha_1, \beta_1) \text{Dec}_\infty^{(2)}(\beta_1, \beta_2) (\det(\alpha_1), \det(\beta_2))_{\mathbb{R}, 2}.$$

Here $(\cdot, \cdot)_{\mathbb{R}, 2}$ denotes the quadratic Hilbert symbol on \mathbb{R} and \det is the determinant over the base field \mathbb{R} .

Note that for $m \geq 3$, α_i and β_i have positive determinant, so in this case the final term above vanishes.

Proof. We shall first consider the case $m = 2$. Thus we have $G_1 = \text{GL}_a(\mathbb{R})$, $G_2 = \text{GL}_b(\mathbb{R})$ and $G_\infty = \text{GL}_{a+b}(\mathbb{R})$. There is an isomorphism:

$$H^2(G_1 \oplus G_2, \mu_2) \cong H^2(G_1, \mu_2) \oplus H^1(G_1, H^1(G_2, \mu_2)) \oplus H^2(G_2, \mu_2)$$

The middle component of the isomorphism is given by the commutator $[\alpha_1, \beta_2]$ ($\alpha_1 \in G_1$, $\beta_2 \in G_2$); the other two components are given by restriction. We must show that the image of Dec_∞ is that described in the proposition.

We first examine the restriction of Dec_∞ to G_1 . We may assume without loss of generality that $V_2 = \mathbb{R}$. We shall assume for the moment that $V_1 = \mathbb{R}^n$ with $n \geq 2$; The case $n = 1$ will be dealt with separately.

For any r -cell $\mathcal{A} : A \rightarrow \mathbb{R}^n$ we define we define two $r + 1$ -cells $\mathcal{A}^+, \mathcal{A}^- : A \times I \rightarrow \mathbb{R}^{n+1}$ by:

$$(\mathcal{A}^+)(\underline{x}, t) = (1 - t)\mathcal{A}(\underline{x}) + te_{n+1},$$

$$(\mathcal{A}^-)(\underline{x}, t) = (1 - t)\mathcal{A}(\underline{x}) - te_{n+1},$$

and we shall write $\mathcal{A}^\pm = \mathcal{A}^+ - \mathcal{A}^-$. Here e_{n+1} is the $n + 1$ -st standard basis element in \mathbb{R}^{n+1} . The above construction has the following properties which are easily checked:

1. For $r \geq 1$, we have modulo degenerate cells: $\partial(\mathcal{A}^\pm) = (\partial\mathcal{A})^\pm$.

2. For $\alpha \in G_1 = \mathrm{GL}_n(\mathbb{R})$ we have $\alpha(\mathcal{A}^\pm) = (\alpha\mathcal{A})^\pm$. Furthermore $([-1] \cdot \mathcal{A})^\pm = [-1] \cdot (\mathcal{A}^\pm)$ (here $[-1]$ is acting by scalar multiplication on V_∞ , rather than on the coefficients of chains).
3. If \mathcal{A} is an n -cell in \mathbb{R}^n then $\mathrm{ord}_{0, \mathbb{R}^n}(\mathcal{A}) = \mathrm{ord}_{0, \mathbb{R}^{n+1}}(\mathcal{A}^\pm)$. (The choice of orientation here is unnecessary as $m = 2$). One should understand this formula as meaning that if one side is defined then so is the other and they are equal.

Let ω_1 be the generator of $H_{n-1}(V_1 \setminus \{0\})$ as in §3.5. By the first and third properties above, we may take $\omega = \omega_1^\pm$ as our generator for $H_n(V_\infty \setminus \{0\})$. By the second property, we may choose $\mathcal{D} = \mathcal{D}_1^\pm$. Since $n \geq 2$, the first property implies that we may take $\mathcal{E} = \mathcal{E}_1^\pm$.

Let $\alpha_1, \beta_1 \in G_1$. By the second property we have:

$$[1, \alpha_1, \alpha_1\beta_1] \cdot \mathcal{E} = ([1, \alpha_1, \alpha_1\beta_1] \cdot \mathcal{E}_1)^\pm.$$

Hence by the third property, it follows that:

$$\mathrm{Dec}_\infty^{(\mathcal{D}_1)}(\alpha_1, \beta_1) = \mathrm{Dec}_\infty^{(\mathcal{D})}(\alpha_1, \beta_1),$$

so the restriction of Dec_∞ to G_1 is $\mathrm{Dec}_\infty^{(1)}$.

We may check by hand that this still holds in the case $n = 1$, where $\mathrm{Dec}_\infty^{(1)}$ is trivial (simply take $\mathcal{E} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ and draw a picture). By the same reasoning we also know that the restriction of Dec_∞ to G_2 is cohomologous to $\mathrm{Dec}_\infty^{(2)}$.

It remains only to prove the formula for the commutator $[\alpha_1, \beta_2]_{\mathrm{Dec}_\infty}$, where $\alpha_1 \in G_1$, $\beta_2 \in G_2$. As commutators are bimultiplicative, this only depends on $\det(\alpha_1)$ and $\det(\beta_2)$ (as $\mathrm{SL}_n(\mathbb{R})$ is the commutator subgroup of $\mathrm{GL}_n(\mathbb{R})$). Furthermore, by what we have already proved, we may assume without loss of generality that $V_1 = V_2 = \mathbb{R}$. We are reduced to calculating the commutator:

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]_{\mathrm{Dec}_\infty}.$$

The result now follows from Proposition 4.

Finally suppose $m \geq 3$. In this case G_1 and G_2 are connected, so the middle commutator term is trivial. We need only verify that the restriction of Dec_∞ to G_1 is $\mathrm{Dec}_\infty^{(1)}$. By induction it is sufficient to prove this in the case

that V_2 is a simple μ_m -module. Since V_2 is a linear space form of μ_m , this implies that V_2 is 2-dimensional. The result follows as in the case $m = 2$ but replacing the construction \mathcal{A}^\pm by a construction which increases the dimensions of cells by 2. This is left to the reader. \square

Corollary 1 *Let $m = 2$ and let Dec_∞ be the cocycle on $\text{GL}_n(\mathbb{R})$ constructed from the action of μ_2 on \mathbb{R}^n . Then, on the standard torus in $\text{GL}_n(\mathbb{R})$, the commutator of Dec_∞ is given by*

$$[\alpha, \beta]_{\text{Dec}_\infty} = (\det \alpha, \det \beta)_{\mathbb{R}, 2} \prod_{i=1}^n (\alpha_i, \beta_i)_{\mathbb{R}, 2},$$

where

$$\alpha = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_n \end{pmatrix}.$$

Proof. This follows by induction from Proposition 5. \square

3.8 Example: $\text{GL}_1(\mathbb{C})$

We now calculate a specific example which we shall need in the next section. We choose an embedding of $\iota : \mu_m \hookrightarrow \mathbb{C}^\times$ and we let $V_\infty = \mathbb{C}$ with the action of μ_m given by ι . Thus Dec_∞ defines an element of $H^2(\mathbb{C}^\times, \mu_m)$ and we shall now calculate this element. As \mathbb{C}^\times is abelian, 2-cocycles on this group may be studied by studying their commutators. However since \mathbb{C}^\times is connected, the commutator of every 2-cocycle is trivial. We therefore have

$$H^2(\mathbb{C}^\times, \mu_m) = H_{\text{sym}}^2(\mathbb{C}^\times, \mu_m).$$

By Klose's isomorphism (see §2.3), it follows that

$$H^2(\mathbb{C}^\times, \mu_m) \cong \mathbb{Z}/m,$$

$$\sigma \mapsto a, \quad \text{where } \zeta^a = \prod_{i=1}^{m-1} \sigma(\iota(\zeta)^i, \iota(\zeta)).$$

Proposition 6 *The image under the above isomorphism of Dec_∞ is 1.*

Proof. We choose the generator ζ so that $\iota(\zeta) = e^{2\pi i/m}$. The group $H_2(\mathbb{C}, \mathbb{C}^\times)$ is generated by the following element:

$$\mathcal{A} = [1, \iota(\zeta), \iota(\zeta)^2] + [1, \iota(\zeta)^2, \iota(\zeta)^3] + \dots + [1, \iota(\zeta)^{m-1}, \iota(\zeta)^m].$$

We shall fix our orientation on \mathbb{C} such that $\text{ord}_0(\mathcal{A}) = 1$. We may therefore take

$$\omega = \partial\mathcal{A} = [1, \iota(\zeta)] + [\iota(\zeta), \iota(\zeta)^2] + \dots + [\iota(\zeta)^{m-1}, 1].$$

We may then choose \mathcal{D} to be the line segment $[1, e^{2\pi i/m}]$. Thus

$$\partial\mathcal{D} = [e^{2\pi i/m}] - [1] = (1 - \iota(\zeta))(-[1]).$$

We may therefore choose $\mathcal{E} = -[1]$. With this choice of \mathcal{E} we have as required:

$$\prod_{i=1}^{m-1} \text{Dec}_\infty(\iota(\zeta)^i, \iota(\zeta)) = \zeta^{\text{ord}_0(\mathcal{A})} = \zeta.$$

□

The corresponding central extension (normalized as described in §2.2) is as follows:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mu_m & \xrightarrow{\iota} & \mathbb{C}^\times & \rightarrow & \mathbb{C}^\times & \rightarrow & 1, \\ & & & & \alpha & \mapsto & \alpha^m. & & \end{array}$$

3.9 The totally complex case.

In this section we suppose k is a totally complex number field containing a primitive m -th root of unity and let μ_m be the group of all m -th roots of unity in k . We let $V_\infty = k_\infty^n$, where $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$. The action of μ_m by scalar multiplication on V_∞ satisfies the conditions of §3.5. The group G_∞ contains $\text{GL}_n(k_\infty)$. We therefore obtain by restriction a cocycle Dec_∞ on $\text{GL}_n(k_\infty)$. In this section we shall study this cocycle.

3.9.1 The cocycle up to a coboundary.

The determinant map gives an isomorphism:

$$\text{GL}_n(k_\infty)/\text{SL}_n(k_\infty) \stackrel{\det}{\cong} k_\infty^\times.$$

This gives us an inflation map:

$$H^2(k_\infty^\times, \mu_m) \rightarrow H^2(\mathrm{GL}_n(k_\infty), \mu_m).$$

As $\mathrm{SL}_n(k_\infty)$ is both connected and simply connected, the groups $H^1(\mathrm{SL}_n(k_\infty), \mu_m)$ and $H^2(\mathrm{SL}_n(k_\infty), \mu_m)$ are both trivial. Hence by from the Hochschild–Serre spectral sequence, it follows that the above map is an isomorphism.

As k_∞^\times is abelian we may speak of the commutators of cocycles; however as k_∞^\times is connected, these commutators are all trivial. Thus we have

$$H^2(\mathrm{GL}_n(k_\infty), \mu_m) \cong H_{sym}^2(k_\infty^\times, \mu_m),$$

Let S_∞ be the set of archimedean places of k . We have a decomposition:

$$k_\infty^\times = \bigoplus_{v \in S_\infty} k_v^\times.$$

By the results described in §2.3, we have

$$H^2(\mathrm{GL}_n(k_\infty), \mu_m) \cong \bigoplus_{v \in S_\infty} H_{sym}^2(k_v^\times, \mu_m).$$

Klose’s isomorphism (§2.3) now gives:

$$H^2(\mathrm{GL}_n(k_\infty), \mu_m) \cong \bigoplus_{v \in S_\infty} \mathbb{Z}/m.$$

By the results of the previous two sections 3.7 and 3.8, we know that the image of Dec_∞ under the above isomorphism is $(1, \dots, 1)$.

3.9.2 The group extension.

We now describe the central extension of $\mathrm{GL}_n(k_\infty)$, corresponding to Dec_∞ . Fix $v \in S_\infty$, so k_v is non-canonically isomorphic to \mathbb{C} . Define a subgroup $\widetilde{\mathrm{GL}}_n(k_v)$ of $\mathrm{GL}_{n+1}(k_v)$ as follows:

$$\widetilde{\mathrm{GL}}_n(k_v) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha \in \mathrm{GL}_n(k_v), \beta \in k_v^\times, \det \alpha = \beta^m \right\}.$$

The group extension of $\mathrm{GL}_n(k_v)$ defined by the restriction of Dec_∞ , is concretely realized as follows:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mu_m & \rightarrow & \widetilde{\mathrm{GL}}_n(k_v) & \rightarrow & \mathrm{GL}_n(k_v) \rightarrow 1, \\ & & \zeta & \mapsto & \begin{pmatrix} I_n & 0 \\ 0 & \iota_v(\zeta) \end{pmatrix}, & & \\ & & & & \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} & \mapsto & \alpha. \end{array}$$

Here $\iota_v : k \hookrightarrow k_v$ denotes the embedding corresponding to the place v . Let $\mu_m(k_\infty)$ be the m -torsion subgroup of k_∞^\times . Define a subgroup K of $\mu_m(k_\infty)$ to be the kernel of the homomorphism $h : \mu_m(k_\infty) \rightarrow \mu_m$ defined by

$$h(\xi_v) = \prod_{v \in S_\infty} \iota_v^{-1}(\xi_v).$$

The full group extension $\widetilde{\mathrm{GL}}_n(k_\infty)$ is the quotient

$$\left(\bigoplus_{v \in S_\infty} \widetilde{\mathrm{GL}}_n(k_v) \right) / K.$$

3.9.3 How the cocycle splits.

We will need to calculate precisely how the cocycle Dec_∞ splits. This is essential in order to find a formula for the Kubota symbol. Consider the set

$$U = \{\alpha \in \mathrm{GL}_n(k_\infty) : \alpha \text{ has no negative real eigenvalue}\}.$$

The set U is contractible, as it is a star body from 1. It is also a dense open subset of $\mathrm{GL}_n(k_\infty)$, as may be seen from the Jordan canonical form.

Lemma 5 *If $\alpha \in U$ then the function Dec_∞ is locally constant at the point $(1, \alpha)$.*

Proof. By definition we have:

$$\mathrm{Dec}_\infty(1, \alpha) = \zeta^{\mathrm{ord}_0([1, 1, \alpha] \cdot \mathcal{E})}.$$

If we suppose that Dec_∞ is discontinuous at $(1, \alpha)$ then this implies that 0 is on the base set of $[1, 1, \alpha] \cdot \mathcal{E}$. Thus there is a v in the base set of \mathcal{E} such that

$$(t + (1 - t)\alpha) \cdot v = 0, \quad t \in (0, 1).$$

This implies that v is a (non-zero) eigenvector of α with eigenvalue $\frac{-t}{1-t} < 0$. \square

Let $\bar{\mu} = k_\infty^\times/K$. Using the map h , we may identify μ_m with the subgroup $\mu_m(k_\infty)/K$ of $\bar{\mu}$, and hence we may regard Dec_∞ as taking values in $\bar{\mu}$; as such it is a coboundary. We define a function $w : \text{GL}_n(k_\infty) \rightarrow \bar{\mu}$ which splits Dec_∞ . First suppose $\alpha \in U$ and consider the path in \mathfrak{g} :

$$\wp_\alpha = [1, \alpha].$$

As α has no negative real eigenvalues, this path is contained in $\text{GL}_n(k_\infty)$. There is a unique continuous path $q_\alpha : I \rightarrow k_\infty^\times$ defined by:

$$q_\alpha(0) = 1, \quad q_\alpha(t)^m = \det \wp_\alpha(t).$$

We define $w(\alpha) = q_\alpha(1)$. More generally for $\alpha \in \text{GL}_n(k_\infty)$ we define

$$w(\alpha) = \mathbf{lim}_{\epsilon \rightarrow 0^+} w(\alpha(1 + \epsilon)).$$

Clearly $w(\alpha)^m = \det \alpha$. Hence, for $\alpha \in \text{SL}_n(k_\infty)$ we have $w(\alpha) \in \mu_m(k_\infty)$.

Theorem 3 *For $\alpha, \beta \in \text{GL}_n(k_\infty)$ we have $\frac{w(\alpha)w(\beta)}{w(\alpha\beta)} \in \mu_m(k_\infty)$. Furthermore in μ_m we have:*

$$\text{Dec}_\infty(\alpha, \beta) = h \left(\frac{w(\alpha)w(\beta)}{w(\alpha\beta)} \right).$$

For $\alpha, \beta \in \text{SL}_n(k_\infty)$ we have

$$\text{Dec}_\infty(\alpha, \beta) = \frac{hw(\alpha)hw(\beta)}{hw(\alpha\beta)}.$$

The theorem gives an explicit splitting of the image of Dec_∞ in $Z^2(\text{GL}_n(k_\infty), \bar{\mu})$ and also a splitting of the restriction of Dec_∞ to $\text{SL}_n(k_\infty)$.

Proof. The first statement is trivial and the third statement follows immediately from the second. We shall prove the second statement. By the limiting definition of both sides of the formula, it is sufficient to prove this in the case $\alpha, \beta, \alpha\beta \in U$. As U is simply connected there is a unique continuous section $\tau : U \rightarrow \widehat{\text{GL}}_n(k_\infty)$ such that $\tau(1) = 1$. This section is given by

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & w(\alpha) \end{pmatrix}.$$

Now consider the realization of $\widetilde{\mathrm{GL}}_n(k_\infty)$ as $\mathrm{GL}_n(k_\infty) \times \mu_m$ with multiplication given by $(\alpha, \xi)(\beta, \chi) = (\alpha\beta, \xi\chi\mathrm{Dec}_\infty(\alpha, \beta))$. To prove the theorem it is sufficient to show that the map $U \rightarrow \widetilde{\mathrm{GL}}_n(k_\infty)$ given by $\alpha \mapsto (\alpha, 1)$ is continuous, and hence coincides with τ .

There is a neighbourhood U_0 of 1 in $\mathrm{GL}_n(k_\infty)$ such that for $\alpha, \beta \in U_0$ we have $\mathrm{Dec}(\alpha, \beta) = 1$. Therefore the map $\beta \rightarrow (\beta, 1)$ is a local homomorphism on U_0 , and is therefore continuous. On the other hand by the previous lemma, for $\alpha \in U$ there is a neighbourhood U_α of 1 in U_0 , such that for $\beta \in U_\alpha$ we have:

$$(\alpha\beta, 1) = (\alpha, 1)(\beta, 1).$$

Thus the left hand side is a continuous function of $\beta \in U_\alpha$, so the theorem is proved. \square

Remark 2 *The above theorem shows that in the complex case the cocycle $\mathrm{Dec}_\infty^{(\mathcal{D})}$ does not depend on the fundamental domain \mathcal{D} . The cocycle does depend on the basis of \mathfrak{g} used to define the limits, but this dependence is only for α, β or $\alpha\beta$ in $\mathrm{GL}_n(k_\infty) \setminus U$.*

3.10 The real case.

If k is a number field, which is not totally complex, then k contains only two roots of unity. We describe the result analogous to Theorem 3 in this case.

Suppose that $\mu_m = \{1, -1\}$. We shall identify V_∞ with \mathbb{R}^d for purposes of notation. With this identification, G_∞ is the group $\mathrm{GL}_d(\mathbb{R})$. We may choose \mathfrak{S} to be a triangulation of the unit sphere S^{d-1} in \mathbb{R}^d . Our fundamental domain \mathcal{D} may be taken to be the cell:

$$\mathcal{D} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in S^{d-1} : x_1 \geq 0 \right\}.$$

Thus \mathcal{E} can be taken to be the cell

$$\mathcal{E} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in S^{d-1} : x_1 = 0, x_2 \geq 0 \right\}.$$

The cell \mathcal{E} is contained in the following subspace $W = \left\{ \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \right\}$. Define

$$U' = \left\{ \alpha \in \mathrm{GL}_n(\mathbb{R}) : \begin{array}{l} \alpha \text{ has no eigenvector in } W \\ \text{with a negative real eigenvalue} \end{array} \right\}.$$

The set U' is a dense, open, contractible subset of $\mathrm{GL}_n(\mathbb{R})$. One may prove analogously to the totally complex case:

Theorem 4 *If α , β and $\alpha\beta$ are all in U' then $\mathrm{Dec}_\infty^{(\mathcal{D})}$ is locally constant at (α, β) .*

This shows that $\mathrm{Dec}_\infty^{(\mathcal{D})}$ is the cocycle obtained from a section $G_\infty \rightarrow \tilde{G}_\infty$, which takes the identity to the identity and is continuous on U' .

4 Construction of fundamental functions.

Let k be an algebraic number field containing a primitive m -th root of unity and consider the vector space $V = k^n$. As before, we let S be the set of places v of k such that $|m|_v \neq 1$. We define $V_{\mathbb{A}(S)} = \mathbb{A}(S)^n$ and $V_\infty = k_\infty^n$, where $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$. From the previous section we have an arithmetic cocycle $\mathrm{Dec}_{\mathbb{A}(S)}^{(f,L)}$ on $\mathrm{GL}_n(\mathbb{A}(S))$ and a geometric cocycle $\mathrm{Dec}_\infty^{(\mathcal{D})}$ on $\mathrm{GL}_n(k_\infty)$. We shall relate the two. However the arithmetic cocycle is dependent on a choice of fundamental function f on $\mathbb{A}(S)^n \setminus \{0\}$ and the geometric cocycle is dependent (in the real case at least) on a fundamental domain \mathcal{D} . In order to describe the relation between $\mathrm{Dec}_{\mathbb{A}(S)}$ and Dec_∞ , we must first fix our choices of f and \mathcal{D} . In this section we choose a fundamental function f (at least generically) and a related fundamental domain \mathcal{D} and describe the mechanism by which the two cocycles will be related. In section 5 we deal with the problem of defining f everywhere, In section 6 we prove the relation between the cocycles, based on the methods of this section.

From now on we shall assume that m is a power of a prime p . There is no loss of generality here, but we will save on notation by doing this. We shall write ρ for a primitive p -th root of unity in k .

4.1 The space X .

We fix once and for all a lattice $L \subset V$ which is free as a $\mathbb{Z}[\mu_m]$ -module. For any finite place v we shall write L_v for the closure of L in V_v . We shall also write $L_{\mathbb{A}(S)}$ for the closure of L in $V_{\mathbb{A}(S)}$. Hence $L_{\mathbb{A}(S)} = \prod_{v \notin S} L_v$.

Let

$$V_m = V \cap \bigcap_{v|m} L_v,$$

and consider the group $X = V_m/L$. There are two ways of thinking about X . First, the diagonal embedding of k in $\mathbb{A}(S)$ induces an isomorphism

$$X \cong V_{\mathbb{A}(S)}/L_{\mathbb{A}(S)}. \quad (17)$$

Secondly, we can regard X as a dense subgroup of the group $X_\infty = V_\infty/L$. The two ways of thinking about X give the connection between the arithmetic and geometric cocycles defined in §3.

The semi-group Υ . Consider the following semigroup in $\mathrm{GL}_n(k)$:

$$\Upsilon = \{\alpha \in \mathrm{GL}_n(k) : \alpha L \supseteq L\}.$$

Let f be a fundamental function on $X \setminus \{0\}$, and define a fundamental function $f_{\mathbb{A}(S)}$ on $V_{\mathbb{A}(S)} \setminus \{0\}$ by

$$f_{\mathbb{A}(S)}(v) = \begin{cases} f(v + L_{\mathbb{A}(S)}) & v \notin L_{\mathbb{A}(S)}, \\ f_o(v) & v \in L_{\mathbb{A}(S)}, \end{cases}$$

where f_o is any fixed fundamental function. By abuse of notation we shall write L and $L_{\mathbb{A}(S)}$ for the characteristic functions of these sets.

Lemma 6 *With the above notation we have for $\alpha, \beta \in \Upsilon$:*

$$\mathrm{Dec}_{\mathbb{A}(S)}^{(f_{\mathbb{A}(S)}, L_{\mathbb{A}(S)})}(\alpha, \beta) = \langle f\alpha^{-1} - f|\alpha\beta L - \alpha L \rangle_X.$$

Proof. This follows immediately from Lemma 3. □

4.2 The complex \mathfrak{X} .

Our method of calculating $\mathrm{Dec}_{\mathbb{A}(S)}(\alpha, \beta)$ on Υ involves constructing fundamental functions on X quite explicitly by embedding X in $X_\infty := V_\infty/L$ and finding a fundamental domain \mathcal{F} for the action of μ_m on X_∞ . In this section we will find the fundamental domain \mathcal{F} .

4.2.1 Parallelotopes and \diamond -products.

Let \mathcal{T} be a singular r -cube and \mathcal{U} a singular s -cube in V_∞ . We can define an $(r + s)$ -cube $\mathcal{T} \diamond \mathcal{U}$ by:

$$(\mathcal{T} \diamond \mathcal{U})(x_1, \dots, x_r, y_1, \dots, y_s) = \mathcal{T}(x_1, \dots, x_r) + \mathcal{U}(y_1, \dots, y_s).$$

Note that for any $v \in V_\infty$, the cell $[v] \diamond \mathcal{T}$ is simply a translation of \mathcal{T} by v . Let $v, a_1, \dots, a_r \in V_\infty$. By the *parallelootope* $\tilde{\mathcal{P}}\text{ar}(v, a_1, \dots, a_r)$ in V_∞ we shall mean the following cell $I^r \rightarrow X_\infty$:

$$\left(\tilde{\mathcal{P}}\text{ar}(v, a_1, \dots, a_r) \right) (\underline{x}) = v + \sum_{i=1}^r x_i a_i.$$

Hence this is just a \diamond -product of line-segments. We shall more often deal with the projections $\mathcal{P}\text{ar}$ of $\tilde{\mathcal{P}}\text{ar}$ in X_∞ . We do not assume that the vectors a_i are linearly independent or even non-zero.

4.2.2 Construction of \mathfrak{X} .

We shall require a cell decomposition \mathfrak{X} of X_∞ in which the cells are parallelotopes. To describe this cell decomposition it is sufficient to describe the highest dimensional cells.

We begin with a cell decomposition of $\mathbb{Q}(\rho)_\infty / \mathbb{Z}[\rho]$. The highest dimensional cells are of the form

$$\mathcal{P} = \mathcal{P}\text{ar} \left(0, \frac{\rho^i}{1-\rho}, \frac{\rho^{i+1}}{1-\rho}, \dots, \frac{\rho^{i+p-2}}{1-\rho} \right), \quad i = 1, \dots, p.$$

Lemma 7 *The cells \mathcal{P} above form the highest dimensional cells of a cell decomposition of $\mathbb{Q}(\rho)_\infty / \mathbb{Z}[\rho]$.*

Proof. This is Theorem 1.1 of [19]. □

We shall refer to the corresponding cell decomposition of $\mathbb{Q}(\rho)_\infty / \mathbb{Z}[\rho]$ as $\mathfrak{X}(p)$.

We next introduce a cell decomposition of $\mathbb{Q}(\zeta)_\infty / \mathbb{Z}[\zeta]$. We have a decomposition

$$\mathbb{Q}(\zeta)_\infty / \mathbb{Z}[\zeta] = \bigoplus_{i=1}^{m/p} \zeta^i \cdot \mathbb{Q}(\rho)_\infty / \mathbb{Z}[\rho].$$

We may therefore define a cell decomposition $\mathfrak{X}(p^a)$ of $\mathbb{Q}(\zeta)_\infty/\mathbb{Z}[\zeta]$ by

$$\mathfrak{X}(p^a) = \prod_{i=1}^{m/p} \zeta^i \cdot \mathfrak{X}(p).$$

Thus the cells of $\mathfrak{X}(p^a)$ are of the form $\diamond_{i=1}^{m/p} \zeta^i \mathcal{P}_i$ with \mathcal{P}_i a cell of $\mathfrak{X}(p)$.

As we are assuming that L is free over $\mathbb{Z}[\zeta]$, there is a basis $\{b_1, \dots, b_n\}$ for V over $\mathbb{Q}(\zeta)$ such that $L = \sum_{i=1}^n \mathbb{Z}[\zeta]b_i$. Again we have a decomposition

$$X_\infty = \bigoplus_{i=1}^n (\mathbb{Q}(\zeta)_\infty/\mathbb{Z}[\zeta]) \cdot b_i.$$

We may therefore define

$$\mathfrak{X} = \prod_{i=1}^n \mathfrak{X}(p^a) \cdot b_i.$$

Lemma 8 *The cell decomposition \mathfrak{X} of X_∞ has the following properties:*

- (i) *The group μ_m permutes the cells of \mathfrak{X} .*
- (ii) *Every positive dimensional cell has trivial stabilizer in μ_m .*
- (iii) *Every r -cell \mathcal{P} in \mathfrak{X} is of the form*

$$\mathcal{P} = \text{pr}\tilde{\mathcal{P}}, \quad \tilde{\mathcal{P}} = \tilde{\mathcal{P}}\text{ar}(v_{\mathcal{P}}, a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}),$$

with $v_{\mathcal{P}} \in \frac{1}{1-\rho}L$ and $a_{\mathcal{P},i} \in \frac{1}{1-\rho}L \setminus L$. For any \mathcal{P} the set $\{a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}\}$ is linearly independent over \mathbb{R} .

- (iv) *We have $|\tilde{\mathcal{P}}| \cap L \subseteq \{0\}$ and $v_{\mathcal{P}} = 0$ if and only if $0 \in |\tilde{\mathcal{P}}|$.*

Much of this result is contained in Theorem 1.3 of [19], although it is stated there in a rather different notation. A sketch of the proof is included for completeness.

Proof. (i) and (iii) are clear from the construction. It is sufficient to verify (iv) for $\mathfrak{X}(p)$ and this is not difficult. It remains to show that no positive dimensional cell is fixed by a non-trivial subgroup of μ_m . Let \mathcal{P} be a positive

dimensional cell and suppose $\xi\mathcal{P} = \mathcal{P}$ for some $\xi \in \mu_m \setminus \{1\}$. We have an expression for \mathcal{P} as a parallelotope:

$$\mathcal{P} = \text{Par}(v_{\mathcal{P}}, a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}).$$

The set $\{a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}\}$ is permuted by ξ . This implies that the elements $\xi^i a_{\mathcal{P},1}$ are all in this set. However, the sum of these elements is zero. This contradicts the fact (iii), that the $a_{\mathcal{P},i}$ are linearly independent. \square

We shall write \mathfrak{X}_\bullet for the corresponding chain complex with coefficients in \mathbb{Z}/m . Thus \mathfrak{X}_r is the free \mathbb{Z}/m -module on the r -cells of the decomposition. By part (i) of the lemma, we have an action of μ_m on \mathfrak{X}_\bullet .

Lemma 9 *For $r = 1, 2, \dots, d$ the $(\mathbb{Z}/m)[\mu_m]$ -module \mathfrak{X}_r is free.*

Proof. A basis consists of a set of representatives for the μ_m -orbits of r -cells in \mathfrak{X} . To show that this is a basis we use the fact that such cells have trivial stabilizer. \square

4.2.3 The fundamental function f .

We shall fix an orientation ord_V on V_∞ . Using this, we define a corresponding orientation ord_X on X_∞ by the formula

$$\text{ord}_{X,x}(\text{pr}(\mathcal{F})) = \sum_{v \in V_\infty: \text{pr}(v)=x} \text{ord}_{V,v}(\mathcal{F}).$$

Let $\omega_X \in \mathfrak{X}_d$ be the generator of $H_d(\mathfrak{X})$, for which $\text{ord}_{X,x}(\omega_X) = 1$ holds for every $x \in X_\infty$. By Lemma 4, $(1 - [\zeta])\omega_X = 0$ holds in $H_d(\mathfrak{X})$. Since $\mathfrak{X}_{d+1} = 0$, we have $(1 - [\zeta])\omega_X = 0$ in \mathfrak{X}_d . Hence by the exact sequence (9) there is an element $\mathcal{F} \in \mathfrak{X}_d$ such that $[\mu_m]\mathcal{F} = \omega_X$. We fix such an \mathcal{F} once and for all. We may regard \mathcal{F} as a fundamental domain for the action of μ_m on X_∞ .

Define a function $f : X_\infty \setminus |\partial\mathcal{F}| \rightarrow \mathbb{Z}/m$ by

$$f(x) = \text{ord}_x(\mathcal{F}).$$

Lemma 10 *The function f is fundamental on the set of $x \in X_\infty$ whose μ_m -orbit does not meet the boundary of \mathcal{P} .*

Proof. For such an x , we have by Lemma 4:

$$\sum_{\xi \in \mu_m} f(\xi x) = \sum_{\xi \in \mu_m} \text{ord}_{\xi x} \mathcal{F} = \sum_{\xi \in \mu_m} \text{ord}_x (\xi^{-1} \cdot \mathcal{F}).$$

By linearity of ord_x we have:

$$\sum_{\xi \in \mu_m} f(\xi x) = \text{ord}_x ([\mu_m] \mathcal{F}) = \text{ord}_x (\omega_X) = 1.$$

□

We shall worry about how to extend the definition of f to $|\partial \mathcal{F}|$ in §5.3. The solution will be to take a limit over fundamental domains tending to \mathcal{F} .

4.3 The complex \mathfrak{S} .

Now that we have a fundamental function f at least generically, we shall describe the fundamental domain \mathcal{D} and the cell complex $\mathfrak{S} \subset V_\infty \setminus \{0\}$ used in the definition of $\text{Dec}_\infty^{(\mathcal{D})}$ in §3.5.

We have a cell decomposition \mathfrak{X} of X_∞ . Each r -cell in this decomposition is of the form

$$\mathcal{P} = \text{Par}(v_{\mathcal{P}}, a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}).$$

Corresponding to each such cell, we define an $r - 1$ -chain $\mathfrak{s}(\mathcal{P}) \in C_{r-1}(V_\infty \setminus \{0\})$ as follows. If $v_{\mathcal{P}} \neq 0$ then we simply define $\mathfrak{s}\mathcal{P} = 0$. If $v_{\mathcal{P}} = 0$ then we define $\mathfrak{s}\mathcal{P} : \Delta^{r-1} \rightarrow V_\infty \setminus \{0\}$ by $\mathfrak{s}\mathcal{P} = [a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}]$. Roughly speaking $\mathfrak{s}\mathcal{P}$ is the set of unit tangent vectors to \mathcal{P} at 0.

The cells $\mathfrak{s}\mathcal{P}$ form a cell complex, which we shall denote \mathfrak{S} . It follows easily that \mathfrak{S} satisfies the conditions of §3.5. We extend \mathfrak{s} by linearity to a map $\mathfrak{s} : \mathfrak{X}_\bullet \rightarrow \mathfrak{S}_{\bullet-1}$.

Lemma 11 *The map $\mathfrak{s} : \mathfrak{X}_\bullet \rightarrow \mathfrak{S}_{\bullet-1}$ commutes with the action of μ_m and anticommutes with ∂ . That is, for any cell \mathcal{P} of \mathfrak{X} , we have*

$$\partial \mathfrak{s}\mathcal{P} = -\mathfrak{s}\partial \mathcal{P}.$$

(Here the minus sign is acting on the coefficients, rather than on V_∞).

Proof. The first statement is clear. For the second we need to examine some separate cases. Let \mathcal{P} be an r -cell in \mathfrak{X} . First suppose $v_{\mathcal{P}} \neq 0$. In this

case, we know by part (iv) of Lemma 8, that 0 is not in the base set of \mathcal{P} . It follows that 0 is not in the base set of any face \mathcal{Q} of \mathcal{P} . Hence $v_{\mathcal{Q}} \neq 0$ for every face \mathcal{Q} of \mathcal{P} . We therefore have $\mathfrak{s}\mathcal{P} = 0$ and $\mathfrak{s}\mathcal{Q} = 0$ for every face \mathcal{Q} . As $\partial\mathcal{P}$ is a sum of faces, the result follows in this case.

Now suppose $v_{\mathcal{P}} = 0$. Let \mathcal{Q} be a face of \mathcal{P} . If \mathcal{Q} is a front face then we have $v_{\mathcal{Q}} = 0$ and if \mathcal{Q} is a back face we have $v_{\mathcal{Q}} \neq 0$. Thus, when calculating $\mathfrak{s}\partial\mathcal{P}$, we need only take into account the front faces of \mathcal{P} . It follows that we have

$$\mathfrak{s}\partial\mathcal{P} = \sum_{i=1}^r (-1)^i \mathfrak{s} \left(\diamond_{j \neq i} [0, a_{\mathcal{P},j}] \right).$$

By the definition of \mathfrak{s} , this gives

$$\begin{aligned} \mathfrak{s}\partial\mathcal{P} &= \sum_{i=1}^r (-1)^i [a_{\mathcal{P},1}, \dots, a_{\mathcal{P},i-1}, a_{\mathcal{P},i+1}, \dots, a_{\mathcal{P},r}] \\ &= -\partial[a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}] = -\partial\mathfrak{s}\mathcal{P}. \end{aligned}$$

□

We define $\mathcal{D} = \mathfrak{s}\mathcal{F}$, where \mathcal{F} is the fundamental domain in \mathfrak{X}_d chosen in §4.2.

Lemma 12 *The element \mathcal{D} satisfies the conditions of §3.5.*

Proof. By Lemma 11 we have:

$$\partial[\mu_m]\mathcal{D} = \partial\mathfrak{s}[\mu_m]\mathcal{F} = \partial\mathfrak{s}\omega_X = -\mathfrak{s}\partial\omega_X = 0.$$

Hence $[\mu_m]\mathcal{D}$ is a cycle. It remains to check that the image of $[\mu_m]\mathcal{D}$ under the maps

$$H_{d-1}(V_{\infty} \setminus \{0\}) \xrightarrow{\partial^{-1}} H_d(V_{\infty}, V_{\infty} \setminus \{0\}) \xrightarrow{\text{ord}_0} \mathbb{Z}/m,$$

is 1. For any cell \mathcal{P} of \mathfrak{X} centred at 0 we shall use the notation

$$\mathfrak{t}\mathcal{P} = [0, a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}].$$

We extend \mathfrak{t} by linearity to a map $\mathfrak{X}_{\bullet} \rightarrow C_{\bullet}(V_{\infty})$. With this notation we have:

$$\partial(\mathfrak{t}\mathcal{P}) = \mathfrak{s}\mathcal{P} - \mathfrak{t}(\partial\mathcal{P}).$$

Applying this relation to $[\mu_m]\mathcal{F}$, we obtain by Lemma 11:

$$\partial([\mu_m]\mathfrak{t}\mathcal{F}) = [\mu_m]\mathcal{D}.$$

We are therefore reduced to showing that $\text{ord}_{V,0}([\mu_m]\mathfrak{t}\mathcal{F}) = \text{ord}_{X,0}([\mu_m]\mathcal{F})$. This is a little messy, but it can be proved by induction on the dimension d of V_∞ . \square

4.4 Modified Parallelotopes.

In this section we shall discuss a method for constructing more general fundamental functions on X_∞ .

We have a cell decomposition \mathfrak{X} of X_∞ in which the r -cells are of the form

$$\mathcal{P} = \text{Par}(v_{\mathcal{P}}, a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}).$$

Recall that $\mathfrak{g} = \text{End}_{\mu_m}(V_\infty)$. We shall write 1 for the identity matrix in \mathfrak{g} . Suppose \wp is a path from 0 to 1 in \mathfrak{g} and \mathcal{P} is an r -cell in the complex \mathfrak{X} . We define an r -cell $\wp \bowtie \mathcal{P} : I^r \rightarrow X_\infty$ as follows:

$$(\wp \bowtie \mathcal{P})(x_1, \dots, x_r) = \text{pr} \left(v_{\mathcal{P}} + \sum_{i=1}^r \wp(x_i) \cdot a_{\mathcal{P},i} \right).$$

We extend the operator “ $\wp \bowtie$ ” by linearity to a map $(\wp \bowtie) : \mathfrak{X}_r \rightarrow C_r(X_\infty)$. Following Kubota, [19] we shall refer to $\wp \bowtie \mathcal{P}$ as a *modified parallelotope*.

Lemma 13 *The maps $(\wp \bowtie) : \mathfrak{X}_r \rightarrow C_r(X_\infty)$ commute with the boundary maps and with the action of μ_m .*

Proof. This is a routine verification. \square

Suppose \wp_1 and \wp_2 are two paths from 0 to 1 in \mathfrak{g} and \mathcal{H} is a homotopy from \wp_1 to \wp_2 . By this, we shall mean $\mathcal{H} : I^2 \rightarrow \mathfrak{g}$ is a continuous map, satisfying for all $t, x \in I$:

$$\mathcal{H}(0, x) = \wp_1(x), \quad \mathcal{H}(1, x) = \wp_2(x), \quad \mathcal{H}(t, 0) = 0, \quad \mathcal{H}(t, 1) = 1.$$

Let \mathcal{P} be an r -cell in the complex \mathfrak{X} :

$$\mathcal{P} = \text{Par}(v_{\mathcal{P}}, a_{\mathcal{P},1}, \dots, a_{\mathcal{P},r}).$$

We define an $r + 1$ -cell $(\mathcal{H} \bowtie \mathcal{P}) : I^{r+1} \rightarrow V_\infty$ by

$$(\mathcal{H} \bowtie \mathcal{P})(t, x_1, \dots, x_r) = \text{pr} \left(v_{\mathcal{P}} + \sum_{i=1}^r \mathcal{H}(t, x_i) \cdot a_{\mathcal{P}, r} \right).$$

We extend the operators “ $\mathcal{H} \bowtie$ ” to linear maps $(\mathcal{H} \bowtie) : \mathfrak{X}_r \rightarrow C_{r+1}(X_\infty)$.

Lemma 14 *The maps $\mathcal{H} \bowtie$ commute with the action of μ_m . Let \mathcal{H} be a homotopy from \wp_1 to \wp_2 . Then $\mathcal{H} \bowtie$ is a chain homotopy from $\wp_1 \bowtie$ to $\wp_2 \bowtie$. In other words, for any $\mathcal{P} \in \mathfrak{X}_r$ we have in $C_r(X_\infty)$:*

$$\partial(\mathcal{H} \bowtie \mathcal{P}) + \mathcal{H} \bowtie \partial \mathcal{P} = \wp_2 \bowtie \mathcal{P} - \wp_1 \bowtie \mathcal{P}.$$

Proof. This is proved by calculating $\partial(\mathcal{H} \bowtie \mathcal{P})$ directly for a cell \mathcal{P} of \mathfrak{X} .
□

Lemma 15 *For any path \wp , the following holds in $H_d(X_\infty)$:*

$$[\mu_m]_{\wp} \bowtie \mathcal{F} = [\mu_m] \mathcal{F}.$$

(This lemma is implicit in [19]).

Proof. This follows immediately from the previous two lemmata. □

We define a function $f^\wp : X_\infty \setminus |\wp \bowtie \partial \mathcal{F}| \rightarrow \mathbb{Z}/m$ by

$$f^\wp(x) = \text{ord}_x(\wp \bowtie \mathcal{F}).$$

It follows from the previous lemma, that f^\wp is fundamental at all x , whose μ_m -orbit avoids the base set of $\wp \bowtie \partial \mathcal{F}$. However f^\wp need not be the characteristic function of a fundamental domain, since it may take other values apart from 0 and 1.

Equivalence of paths. Suppose \wp and \wp' are two paths from 0 to 1 in \mathfrak{g} . We shall say that \wp and \wp' are *equivalent* if there is an increasing continuous bijection $\phi : I \rightarrow I$, such that for all $x \in I$ we have:

$$\wp(\phi(x)) = \wp'(x).$$

Lemma 16 *If \wp and \wp' are equivalent paths then the corresponding functions f^\wp and $f^{\wp'}$ have the same domains of definition and are equal.*

Proof. To prove that f^φ and $f^{\varphi'}$ have the same domains of definition, we need only show that $|\partial(\varphi \rtimes \mathcal{F})| = |\partial(\varphi' \rtimes \mathcal{F})|$. This follows by breaking the boundary into faces and using the fact that ϕ is bijective.

The map ϕ is homotopic to the identity map. In other words there is a map $\psi : I \times I \rightarrow I$ such that $\psi(1, x) = \phi(x)$, $\psi(0, x) = x$, $\psi(t, 0) = 0$ and $\psi(t, 1) = 1$ for all $x, t \in I$. Using the function ψ we define a homotopy \mathcal{H} from φ to φ' by $\mathcal{H}(t, x) = \varphi(\psi(t, x))$. As $\mathcal{H} \rtimes$ is a chain homotopy we have

$$\varphi' \rtimes \mathcal{F} - \varphi \rtimes \mathcal{F} = \partial(\mathcal{H} \rtimes \mathcal{F}) + \mathcal{H} \rtimes \partial\mathcal{F}.$$

Note that for any cell \mathcal{P} of \mathfrak{X} we have $|\mathcal{H} \rtimes \mathcal{P}| = |\varphi \rtimes \mathcal{P}| = |\varphi' \rtimes \mathcal{P}|$. Therefore for $x \notin |\varphi \rtimes \partial\mathcal{F}|$ we have:

$$\text{ord}_x(\varphi' \rtimes \mathcal{F}) - \text{ord}_x(\varphi \rtimes \mathcal{F}) = 0.$$

In other words $f^\varphi(x) = f^{\varphi'}(x)$. □

In view of the above lemma, we may specify piecewise linear paths simply as sequences of line segments:

$$\varphi = [0, a_1] + [a_1, a_2] + \cdots + [a_s, 1],$$

without worrying about the precise parametrization.

4.5 Statement of the results in the generic case.

The $d-1$ -chain \mathcal{G} . For the moment we shall assume that $d \geq 2$. Thus we are ruling out the case $k = \mathbb{Q}$, $n = 1$. By the definition of \mathcal{F} , we have

$$\partial([\mu_m]\mathcal{F}) = 0.$$

Thus $[\mu_m](\partial\mathcal{F}) = 0$. It follows from the exact sequence (9), that there is an element $\mathcal{G} \in \mathfrak{X}_{d-1}$ satisfying

$$\partial\mathcal{F} = (1 - [\zeta])\mathcal{G}.$$

We shall fix such a \mathcal{G} . Note that by Lemma 11, the $d-2$ -chain \mathcal{E} used in the definition of Dec_∞ may be taken to be $-\mathfrak{s}\mathcal{G}$.

The semigroup $\Upsilon_{\mathfrak{f}}$. For an ideal \mathfrak{f} of the ring of integers in k , let $G_{\mathfrak{f}}$ be the subgroup of $\text{GL}_n(k)$ consisting of matrices which are integral at every prime dividing \mathfrak{f} and are congruent to the identity matrix modulo \mathfrak{f} . We shall fix $\mathfrak{f} = (1 - \rho)m^2$ if m is odd and $\mathfrak{f} = 4m^2$ if m is even. Next let $\Upsilon_{\mathfrak{f}} = \Upsilon \cap G_{\mathfrak{f}}$.

The results. For $\alpha \in \Upsilon_{\mathfrak{f}}$, we define a path

$$\wp(\alpha) = [0, \alpha] + [\alpha, 1],$$

and a homotopy \mathcal{H}_α^1 from $\wp(1)$ to $\wp(\alpha)$:

$$\mathcal{H}_\alpha^1(t, x) = (1 - t)x + t\wp(\alpha)(x).$$

Finally we define

$$\tau(\alpha) = \zeta\{\mathcal{H}_\alpha^1 \bowtie \mathcal{G}|\alpha L\}.$$

(One may interpret this definition as a lifted map in the sense of Lemma 1)

We shall prove that $\text{Dec}_\infty^{(\mathcal{D})}$ and $\text{Dec}_{\mathbb{A}(S)}^{(f, L)}$ are related on $\Upsilon_{\mathfrak{f}}$ by the following formula:

$$\text{Dec}_{\mathbb{A}(S)}^{(f, L)}(\alpha, \beta) \text{Dec}_\infty^{(\mathcal{D})}(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)}, \quad \alpha, \beta \in \Upsilon_{\mathfrak{f}}.$$

As a consequence, we will deduce that for totally complex k , the Kubota symbol on $\text{SL}_n(\mathfrak{o}, \mathfrak{f})$ is given by

$$\kappa(\alpha) = \frac{\tau(\alpha)}{hw(\alpha)}$$

Here hw is the splitting of Dec_∞ of §3.9, Theorem 3.

4.6 A generic formula for the pairing.

Given paths \wp^1 and \wp^2 from 0 to 1 in \mathfrak{g} , we have constructed fundamental functions f^1 and f^2 in \mathcal{F}_X . We now describe a geometric method for calculating the pairing $\langle f^1 - f^2 | M - L \rangle_X$, where $M \supset L$ is a lattice contained in V_m .

Proposition 7 *Suppose $M \setminus L$ does not intersect the base set of $\partial(\mathcal{H} \bowtie \mathcal{G})$. Then we have:*

$$\langle f^{\wp^2} - f^{\wp^1} | M - L \rangle_X = \zeta\{\mathcal{H} \bowtie \mathcal{G} | M - L\}.$$

The right hand side is a lifted map in the sense of Lemma 1.

Proof. As $\mathcal{H} \bowtie$ is a chain homotopy from $\wp_1 \bowtie$ to $\wp_2 \bowtie$, the following holds in $C_d(X_\infty)$:

$$\wp_2 \bowtie \mathcal{F} - \wp_1 \bowtie \mathcal{F} = \partial(\mathcal{H} \bowtie \mathcal{F}) + \mathcal{H} \bowtie \partial \mathcal{F}.$$

This implies for $x \in M \setminus L$:

$$f^{\wp_2}(x) - f^{\wp_1}(x) = \text{ord}_x(\wp_2 \bowtie \mathcal{F} - \wp_1 \bowtie \mathcal{F}) = \text{ord}_x(\mathcal{H} \bowtie \partial \mathcal{F}).$$

By the definition of \mathcal{G} and Lemma 4, we have:

$$f^{\wp_2}(x) - f^{\wp_1}(x) = \text{ord}_x(\mathcal{H} \bowtie (1 - [\zeta])\mathcal{G}) = \text{ord}_x(\mathcal{H} \bowtie \mathcal{G}) - \text{ord}_{\zeta^{-1}x}(\mathcal{H} \bowtie \mathcal{G}).$$

The proposition now follows from the definition (14) in §3.2 of the pairing $\langle - | - \rangle_x$. \square

4.7 The order of $\mathcal{H} \bowtie \mathcal{G}$ at 0.

The statement of the results in §4.5 involves numbers of the form $\text{ord}_0(\mathcal{H} \bowtie \mathcal{G})$. However, this is not as yet defined, since 0 is always in the base set of $\partial(\mathcal{H} \bowtie \mathcal{G})$. To avoid this problem, for any r -cell \mathcal{P} in \mathfrak{X} containing 0 we cut $\mathcal{H} \bowtie \mathcal{P}$ into a singular part $\mathcal{H} \bowtie \mathcal{P}^0$ and a non-singular part $\mathcal{H} \bowtie \mathcal{P}^+$.

Let U be a small neighbourhood of 0 in I^{d-1} . We define $\mathcal{H} \bowtie \mathcal{P}^0$ to be the restriction of $\mathcal{H} \bowtie \mathcal{P}$ to $I \times U$ and we define $\mathcal{H} \bowtie \mathcal{P}^+$ to be the restriction of $\mathcal{H} \bowtie \mathcal{P}$ to $I \times (I^{d-1} \setminus U)$. We define the order of $\mathcal{H} \bowtie \mathcal{P}$ at 0 to be the limit as U gets smaller of the order of $\mathcal{H} \bowtie \mathcal{P}^+$ at 0.

To make things a little more precise we define for $\epsilon > 0$ an r -cell \mathcal{P}^ϵ in X_∞ . If $v_{\mathcal{P}} = 0$ then we define \mathcal{P}^ϵ to be the restriction of $\mathcal{P} : I^r \rightarrow X_\infty$ to the set

$$\left\{ (x_1, \dots, x_r) \in I^r : \sum x_i \leq \epsilon \right\}.$$

If $v_{\mathcal{P}} \neq 0$ then we define $\mathcal{P}^\epsilon = 0$.

Lemma 17 *For $\epsilon > 0$ sufficiently small we have*

$$\partial(\mathcal{P}^\epsilon) - (\partial \mathcal{P})^\epsilon = \text{pr}(\epsilon \cdot \mathfrak{s}\mathcal{P}).$$

Proof. We shall suppose that \mathcal{P} has 0 as its origin, since otherwise both sides of the formula are zero. Under this assumption, we have:

$$\mathcal{P} = \diamond_{i=1}^r [0, a_{\mathcal{P},i}], \quad \mathcal{P}^\epsilon = [0, \epsilon a_{\mathcal{P},1}, \epsilon a_{\mathcal{P},2}, \dots, \epsilon a_{\mathcal{P},r}].$$

Therefore

$$\begin{aligned}\partial(\mathcal{P}^\epsilon) &= [\epsilon a_{\mathcal{P},1}, \dots, \epsilon a_{\mathcal{P},r}] + \sum_{i=1}^r (-1)^i [0, \epsilon a_{\mathcal{P},1}, \dots, \epsilon a_{\mathcal{P},i-1}, \epsilon a_{\mathcal{P},i+1}, \dots, \epsilon a_{\mathcal{P},r}] \\ &= [\epsilon a_{\mathcal{P},1}, \dots, \epsilon a_{\mathcal{P},r}] + (\partial\mathcal{P})^\epsilon.\end{aligned}$$

The result follows. \square

4.8 Dependence of $\text{ord}_0(\mathcal{H} \bowtie \mathcal{G})$ on \mathcal{H} .

Let \wp_1 and \wp_2 be two paths from 0 to 1 in \mathfrak{g} . Suppose we have two homotopies \mathcal{H} and \mathcal{J} from \wp_1 to \wp_2 . In this section, we investigate the relation between $\text{ord}_{0,x}(\mathcal{H} \bowtie \mathcal{G})$ and $\text{ord}_{0,x}(\mathcal{J} \bowtie \mathcal{G})$.

We begin by choosing a homotopy \mathcal{U} from \mathcal{H} to \mathcal{J} . Thus, $\mathcal{U} : I^3 \rightarrow \mathfrak{g}$ satisfies the following conditions:

$$\begin{aligned}\mathcal{U}(0, t, x) &= \mathcal{H}(t, x), & \mathcal{U}(1, t, x) &= \mathcal{J}(t, x), \\ \mathcal{U}(u, 0, x) &= \wp_1(x), & \mathcal{U}(u, 1, x) &= \wp_2(x), \\ \mathcal{U}(u, t, 0) &= 0, & \mathcal{U}(u, t, 1) &= 1.\end{aligned}$$

We shall also suppose that there is some $\epsilon > 0$ such that for $x < 1$ we have

$$\mathcal{U}(u, t, \epsilon x) = x\mathcal{U}(u, t, \epsilon).$$

We shall define, under this assumption, maps $(\mathfrak{s}\mathcal{U}) : I^2 \rightarrow \mathfrak{g}$, $(\mathfrak{s}\mathcal{H}) : I \rightarrow \mathfrak{g}$ and $(\mathfrak{s}\mathcal{J}) : I \rightarrow \mathfrak{g}$ by

$$\begin{aligned}(\mathfrak{s}\mathcal{U})(u, t) &= \epsilon^{-1}\mathcal{U}(u, t, \epsilon) = \frac{\partial}{\partial x}\mathcal{U}(u, t, x), \\ (\mathfrak{s}\mathcal{H})(t) &= \epsilon^{-1}\mathcal{H}(t, \epsilon), & (\mathfrak{s}\mathcal{J})(t) &= \epsilon^{-1}\mathcal{J}(t, \epsilon).\end{aligned}$$

We define maps $(\mathcal{U} \bowtie) : \mathfrak{X}_r \rightarrow C_{r+2}(X_\infty)$ by

$$(\mathcal{U} \bowtie \mathcal{P})(u, t, x_1, \dots, x_r) = \text{pr} \left(v_{\mathcal{P}} + \sum_{i=1}^r \mathcal{U}(u, t, x_i) \right).$$

The next lemma says that $\mathcal{U} \bowtie$ is in some sense a chain homotopy between $\mathcal{H} \bowtie$ and $\mathcal{J} \bowtie$ (although these are themselves chain homotopies, not chain maps).

Lemma 18 *The map “ $\mathcal{U} \bowtie$ ” commutes with the action of μ_m . For every $\mathcal{P} \in \mathfrak{X}_r$ the following holds in $C_{r+1}(X_\infty)$:*

$$\mathcal{J} \bowtie \mathcal{P} - \mathcal{H} \bowtie \mathcal{P} = \partial(\mathcal{U} \bowtie \mathcal{P}) - \mathcal{U} \bowtie \partial \mathcal{P}.$$

Proof. By definition of the boundary map we have

$$\partial(\mathcal{U} \bowtie \mathcal{P}) = (\mathcal{J} \bowtie \mathcal{P} - \mathcal{H} \bowtie \mathcal{P}) - \text{degenerate cells} + \mathcal{U} \bowtie (\partial \mathcal{P}).$$

□

The crucial point in relating the two cocycles is the following formula.

Proposition 8 *Suppose that for every $d-1$ -cell \mathcal{P} in \mathfrak{X} , $\text{ord}_{0,X}(\mathcal{H} \bowtie \mathcal{P}^+)$ and $\text{ord}_{0,X}(\mathcal{J} \bowtie \mathcal{P}^+)$ are defined and for every $d-2$ -cell \mathcal{P} in \mathfrak{X} , $\text{ord}_{0,X}(\mathcal{U} \bowtie \mathcal{P}^+)$ is defined. Then so is $\text{ord}_{V,0}(\mathfrak{s}\mathcal{U} \cdot \mathfrak{s}\mathcal{G})$ and we have:*

$$\text{ord}_{X,0}(\mathcal{J} \bowtie \mathcal{G}^+) - \text{ord}_{X,0}(\mathcal{H} \bowtie \mathcal{G}^+) = \text{ord}_{V,0}(\mathfrak{s}\mathcal{U} \cdot \mathfrak{s}\mathcal{G}).$$

Proof. Applying lemma 18 to \mathcal{G} we obtain

$$\mathcal{J} \bowtie \mathcal{G} - \mathcal{H} \bowtie \mathcal{G} = \partial(\mathcal{U} \bowtie \mathcal{G}) - \mathcal{U} \bowtie \partial \mathcal{G}$$

Recall that \mathcal{G} is chosen to satisfy the relation $(1 - [\zeta])\mathcal{G} = \partial \mathcal{F}$. This implies $(1 - [\zeta])\partial \mathcal{G} = 0$. By the exact sequence (9), there is a $d-2$ -chain \mathcal{Q} in \mathfrak{X} such that

$$\partial \mathcal{G} = [\mu_m]\mathcal{Q}.$$

We therefore have

$$(\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G} = \partial(\mathcal{U} \bowtie \mathcal{G}) - [\mu_m](\mathcal{U} \bowtie \mathcal{Q}).$$

We cannot define the order at 0 of either side of the above equation. We therefore break \mathcal{G} and \mathcal{Q} into their singular and non-singular parts. This gives:

$$(\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^+ = \partial(\mathcal{U} \bowtie \mathcal{G}) - [\mu_m]\mathcal{U} \bowtie \mathcal{Q}^+ - [\mu_m]\mathcal{U} \bowtie \mathcal{Q}^\epsilon - (\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^\epsilon.$$

Note that $\text{ord}_0(\mathcal{U} \bowtie \mathcal{Q}^+)$ is defined, so we have by Lemma 4 $\text{ord}_0([\mu_m]\mathcal{U} \bowtie \mathcal{Q}^+) = 0$. Hence the following holds in \mathbb{Z}/m :

$$\text{ord}_{0,X}((\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^+) = -\text{ord}_{0,X}((\mathcal{U} \bowtie \partial \mathcal{G})^\epsilon) + (\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^\epsilon.$$

As every cell on the right hand side is contained in a small neighbourhood of 0, we may replace $\text{ord}_{0,X}$ by $\text{ord}_{0,V}$.

Now choose an r -cell \mathcal{P} in \mathfrak{X} . We have for $x_i \leq \epsilon$:

$$(\mathcal{U} \bowtie \mathcal{P})(u, t, x_1, \dots, x_r) = \sum_{i=1}^r \mathcal{U}(u, t, x_i) a_{\mathcal{P},i} = \mathfrak{s}\mathcal{U}(u, t) \sum_{i=1}^r x_i a_{\mathcal{P},i}.$$

Thus

$$(\mathcal{U} \bowtie \mathcal{P})^\epsilon = \mathfrak{s}\mathcal{U} \cdot (\mathcal{P}^\epsilon),$$

and similarly,

$$(\mathcal{H} \bowtie \mathcal{P})^\epsilon = \mathfrak{s}\mathcal{H} \cdot (\mathcal{P}^\epsilon), \quad (\mathcal{J} \bowtie \mathcal{P})^\epsilon = \mathfrak{s}\mathcal{J} \cdot (\mathcal{P}^\epsilon).$$

We therefore have

$$\text{ord}_{0,X}((\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^+) = -\text{ord}_{0,V}(\mathfrak{s}\mathcal{U} \cdot (\partial\mathcal{G})^\epsilon + (\mathfrak{s}\mathcal{J} - \mathfrak{s}\mathcal{H}) \cdot \mathcal{G}^\epsilon).$$

On the other hand, modulo degenerate cells, we have :

$$\partial\mathfrak{s}\mathcal{U} = \mathfrak{s}\mathcal{J} - \mathfrak{s}\mathcal{H}.$$

This implies

$$\text{ord}_{0,X}((\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^+) = -\text{ord}_{0,V}(\mathfrak{s}\mathcal{U} \cdot (\partial\mathcal{G})^\epsilon + (\partial\mathfrak{s}\mathcal{U}) \cdot (\mathcal{G}^\epsilon)).$$

Adding $\partial(\mathfrak{s}\mathcal{U} \cdot (\mathcal{G}^\epsilon))$ we obtain:

$$\text{ord}_{0,X}((\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^+) = -\text{ord}_{0,V}(\mathfrak{s}\mathcal{U} \cdot (\partial\mathcal{G})^\epsilon - \mathfrak{s}\mathcal{U} \cdot \partial(\mathcal{G}^\epsilon)).$$

Now by Lemma 17 we have:

$$\text{ord}_{0,X}((\mathcal{J} - \mathcal{H}) \bowtie \mathcal{G}^+) = \text{ord}_{0,V}(\epsilon\mathfrak{s}\mathcal{U} \cdot \mathfrak{s}\mathcal{G}).$$

The right hand side is however independent of ϵ so the result follows. \square

Corollary 2 *Suppose there is an $\epsilon > 0$ such that for all $0 < x < \epsilon$ we have $\mathcal{H}(t, x) = \mathcal{J}(t, x)$. Then $\text{ord}_0(\mathcal{H} \bowtie \mathcal{G}) = \text{ord}_0(\mathcal{J} \bowtie \mathcal{G})$. (By this we understand that if both sides are defined then they are equal).*

Proof. in this case we can choose \mathcal{U} so that $\mathfrak{s}\mathcal{U}$ is degenerate. \square

5 Deformation of fundamental functions.

Given a path \wp , we have constructed a function f^\wp on $X_\infty \setminus |\partial(\wp \bowtie \mathcal{P})|$, which is fundamental on all μ_m -orbits which do not intersect $|\partial(\wp \bowtie \mathcal{P})|$. In this chapter we describe a method for extending f^\wp to all but a finite number of points of X_∞ . Recall that we have fixed an ordered basis $\{b_1, \dots, b_r\}$ for \mathfrak{g} as a vector space over \mathbb{R} . The general idea is that if a point $x \in X_\infty$ is on the boundary of $\wp \bowtie \mathcal{P}$, then we move the path \wp a little in the direction of b_1 ; if x is still on the boundary then we move the path in the direction b_2 and so on. Thus, we define f^\wp as a limit of the form $\mathbf{lim}_{\epsilon \rightarrow 0^+}$ in the notation of §3.5.

We begin with some general results on the existence of such limits, and then prove the specific results which we need.

5.1 Existence of certain limits.

We shall call a subset $Z \subset \mathbb{R}^a$ a *small* subset of \mathbb{R}^a , if there is a $b \geq 0$ and a Zariski-closed subset $Z' \subset \mathbb{R}^{a+b}$ of codimension $\geq b + 1$, such that Z is contained in the archimedean closure of the projection of Z' in \mathbb{R}^a . If Z_1, \dots, Z_c are small subsets of \mathbb{R}^a then $Z_1 \cup \dots \cup Z_c$ is a small subset of \mathbb{R}^a . Note that if Z is small then it has codimension at least 1 in \mathbb{R}^a .

Lemma 19 *Let Z be a small subset of \mathbb{R}^a . Then for any locally constant function $\psi : \mathbb{R}^a \setminus Z \rightarrow \mathbb{Z}$ the following (ordered) limit exists:*

$$\lim_{\epsilon_1 \rightarrow 0^+} \lim_{\epsilon_2 \rightarrow 0^+} \dots \lim_{\epsilon_a \rightarrow 0^+} \psi(\epsilon_1, \dots, \epsilon_a).$$

Proof. We shall prove this by induction on a . When $a = 1$ the set $Z' \subset \mathbb{R}^{b+1}$ has codimension $\geq b + 1$, and is therefore finite. It follows that Z is finite, and it is clear that the limit exists in this case. Now suppose $a > 1$. We shall decompose the Zariski-closed set $Z' \subset \mathbb{R}^{a+b}$ into irreducible components:

$$Z' = Z'_1 \cup \dots \cup Z'_r.$$

We shall write Z_i for the archimedean closure of the projection of Z'_i in \mathbb{R}^a . Let $\{b_1, \dots, b_{a+b}\}$ be the standard basis of \mathbb{R}^{a+b} . Write H' for the hyperplane in \mathbb{R}^{a+b} spanned by $\{b_1, \dots, b_{a-1}, b_{a+1}, \dots, b_{a+b}\}$ and let $H =$

$\text{span}\{b_1, \dots, b_{a-1}\}$ be the projection of H in \mathbb{R}^a . For each component Z'_i of Z' , we define a subset $W'_i \subset H'$ by

$$W'_i = \begin{cases} Z'_i \cap H & \text{if } Z'_i \not\subset H', \\ \emptyset & \text{if } Z'_i \subset H'. \end{cases}$$

Thus $W' := W'_1 \cup \dots \cup W'_r$ is Zariski-closed in H' and has codimension $\geq b+1$ in H' . Let W be the archimedean closure of the projection of W' in H . Thus W is a small subset of H . By the inductive hypothesis, it is sufficient to show that the limit

$$\Psi(v) = \lim_{\epsilon_a \rightarrow 0^+} \psi(v + \epsilon_a b_a)$$

exists and is locally constant for $v \in H \setminus W$.

Let $v \in H \setminus W$. Choose a compact, connected, archimedean neighbourhood U of v in H such that $U \cap W = \emptyset$. We shall prove that the limit Ψ exists on U and is constant there. Let $w \in U$. For any i we either have $Z'_i \subset H$ or $w \notin Z'_i$. In either case there is a $\delta(w, i) > 0$ sufficiently small so that we have $w + \epsilon_a b_a \notin Z'_i$ whenever $0 < \epsilon_a < \delta(w, i)$. The $\delta(w, i)$ may be chosen to be continuous in w . As U is compact the $\delta(w, i)$ are bounded below by some positive δ . This means that the subset $\tilde{U} = U \times (0, \delta)b_a$ of \mathbb{R}^a does not intersect Z . The function ψ is therefore defined on \tilde{U} and since \tilde{U} is connected, ψ is equal to a constant c on \tilde{U} . It follows that $\Psi(w) = c$ for all $w \in U$. \square

Note that if the order of the limits is changed in the above Lemma, then the value of the limit may change.

5.2 Deformations of cells.

Given a d -chain $\mathcal{T} \in C_d(V_\infty)$ we will describe a method for defining $\text{ord}_v(\mathcal{T})$ for $v \in |\partial\mathcal{T}|$. The general idea is to replace \mathcal{T} by a map $\mathcal{T} : \mathbb{R}^b \rightarrow C_d(V_\infty)$ so that our original \mathcal{T} is $\mathcal{T}(0)$. We then have a function ψ , defined on part of \mathbb{R}^b by

$$\psi(\epsilon) = \text{ord}_x \mathcal{T}(\epsilon).$$

If the function \mathcal{T} is sufficiently nice then we may use Lemma 19 to define

$$\text{ord}_x(\mathcal{T}) = \lim_{\epsilon_1 \rightarrow 0^+} \lim_{\epsilon_2 \rightarrow 0^+} \dots \lim_{\epsilon_b \rightarrow 0^+} \psi(\epsilon_1, \dots, \epsilon_b).$$

In this section, we investigate the conditions, which we must impose on \mathcal{T} , in order for Lemma 19 to be applicable.

5.2.1 Deformable $d - 1$ -cells.

Let $\mathcal{T} : \mathbb{R}^b \times \mathbb{R}^{d-1} \rightarrow V_\infty$ be an algebraic function satisfying the following conditions:

- (A) For every $\underline{x} \in \mathbb{R}^{d-1}$, the map $\mathbb{R}^b \rightarrow V_\infty$ defined by $\epsilon \mapsto \mathcal{T}(\epsilon, \underline{x})$ is affine. We shall write $\text{rk}(\underline{x})$ for the rank of the linear part of this map.
- (B) For any $i = 1, \dots, d$, the set of $\underline{x} \in \mathbb{R}^{d-1}$ such that $\text{rk}(\underline{x}) = i$ has dimension $\leq i - 1$.

We shall call such a \mathcal{T} a *deformable $d - 1$ -cell*. If $\mathcal{T}(\epsilon, \underline{x})$ is constant for a certain \underline{x} then the value of the constant will be called a *vertex* of \mathcal{T} . For any $\epsilon \in \mathbb{R}^b$ we shall consider the $d - 1$ -cell $\mathcal{T}(\epsilon) : \mathbb{R}^{d-1} \rightarrow V_\infty$ defined by:

$$\left(\mathcal{T}(\epsilon)\right)(\underline{x}) := \mathcal{T}(\epsilon, \underline{x}).$$

Lemma 20 *Let \mathcal{T} be a deformable $d - 1$ -cell. Then, for any $v \in V_\infty$ which is not a vertex of \mathcal{T} , the set $\{\epsilon \in \mathbb{R}^b : v \in |\mathcal{T}(\epsilon)|\}$ is a small subset of \mathbb{R}^b .*

Proof. The above set is contained in the projection to \mathbb{R}^b of the following set:

$$Z' = \{(\epsilon, \underline{x}) \in \mathbb{R}^b \times \mathbb{R}^{d-1} : \mathcal{T}(\epsilon, \underline{x}) = v\}.$$

As Z' is algebraic, it remains only to show that Z' has codimension at least d in $\mathbb{R}^b \times \mathbb{R}^{d-1}$. To show this we shall write Z' as a finite union of subsets and bound the dimension of each of the subsets. Define

$$Z'_i = \{(\epsilon, \underline{x}) \in \mathbb{R}^b \times \mathbb{R}^{d-1} : \mathcal{T}(\epsilon, \underline{x}) = v, \text{rk}(\underline{x}) = i\} \quad (i = 0, 1, 2, \dots, d).$$

The set Z' is the union of the subsets Z'_i .

As we are assuming that v is not a vertex of \mathcal{T} , it follows that Z'_0 is empty. For $i > 0$, our assumption (B) on \mathcal{T} implies that the projection of Z'_i in \mathbb{R}^{d-1} has dimension $\leq i - 1$. However each fibre of this projection is an affine subspace with codimension i in \mathbb{R}^b . Therefore the dimension of Z'_i is $\leq b - 1$. This proves the lemma. \square

5.2.2 Deformable d -cells

Now suppose we have a continuous map $\mathcal{T} : \mathbb{R}^b \times I^d \rightarrow V_\infty$. Thus, for each $\epsilon \in \mathbb{R}^b$ we have a d -cell $\mathcal{T}(\epsilon)$. We shall call \mathcal{T} a *deformable d -cell* if the faces of \mathcal{T} are deformable $d - 1$ -cells. By a *vertex of \mathcal{T}* , we shall mean a vertex of a face of \mathcal{T} .

Lemma 21 *Let \mathcal{T} be a deformable d -cell. Then for any $v \in V_\infty$ which is not a vertex of \mathcal{T} , the following limit exists:*

$$\lim_{\epsilon_1 \rightarrow 0^+} \dots \lim_{\epsilon_b \rightarrow 0^+} \text{ord}_v(\mathcal{T}(\epsilon)).$$

If $v \notin |\partial\mathcal{T}(0)|$ then the limit is equal to $\text{ord}_v(\mathcal{T}(0))$.

Proof. We first prove the existence of the limit. Consider the set

$$Z = \{\epsilon \in \mathbb{R}^b : v \in |\partial\mathcal{T}(\epsilon)|\}.$$

Since $\mathcal{T}(\epsilon)$ tends uniformly to $\mathcal{T}(0)$, the set Z is archimedeanly closed in \mathbb{R}^b . On $\mathbb{R}^b \setminus Z$ we have a function

$$\psi(\epsilon) = \text{ord}_v(\mathcal{T}(\epsilon)).$$

To prove the existence of the limit, it is sufficient, by Lemma 19, to show that Z is small and ψ is locally constant on $\mathbb{R}^b \setminus Z$.

It follows from the previous lemma that Z is small. We shall show that ψ is locally constant on $\mathbb{R}^b \setminus Z$. Choose $\epsilon \notin Z$. As Z is closed there is a path connected neighbourhood U of ϵ in \mathbb{R}^b which does not intersect Z . We shall show that ψ is constant on U . Let $\epsilon' \in U$ and choose a path p from ϵ to ϵ' in U . Now consider the $(d + 1)$ -chain

$$\mathcal{V}(t, \underline{x}) = \mathcal{T}(p(t), \underline{x}).$$

The boundary of \mathcal{V} is

$$\partial\mathcal{V} = \mathcal{T}(\epsilon') - \mathcal{T}(\epsilon) - \sum_u \mathcal{V}_u,$$

where $\mathcal{V}_u(t, \underline{x}) = \mathcal{U}(p(t), \underline{x})$ and \mathcal{U} runs over the faces of \mathcal{T} . As $p(t) \notin Z$ we have $\mathcal{U}(p(t), \underline{x}) \neq v$. Therefore $v \notin |\mathcal{V}_u|$ so in $H_d(V_\infty, V_\infty \setminus \{v\})$ we have:

$$\mathcal{T}(\epsilon) = \mathcal{T}(\epsilon').$$

This implies $\psi(\epsilon) = \psi(\epsilon')$, so ψ is locally constant and we have proved the existence of the limit.

Now suppose that $v \notin |\partial\mathcal{J}(0)|$. This means that $0 \notin Z$. As Z is archimedeanly closed, there is a neighbourhood of 0 on which ψ is constant. We therefore have as required:

$$\lim_{\epsilon_1 \rightarrow 0^+} \dots \lim_{\epsilon_b \rightarrow 0^+} \psi(\epsilon) = \psi(0).$$

□

5.2.3 Deformations with respect to \mathfrak{g} .

We shall now specialize the above result to the case which we require. Recall that $\mathfrak{g} = \text{End}_{\mu_m}(V_\infty)$. Thus we have $\mathfrak{g} = M_s(\mathbb{Q}(\zeta) \otimes_{\mathbb{Q}} \mathbb{R})$. We shall write $S_\infty(\mathbb{Q}(\zeta))$ for the set of archimedean places of $\mathbb{Q}(\zeta)$. There is a decomposition:

$$\mathfrak{g} = \bigoplus_{v \in S_\infty(\mathbb{Q}(\zeta))} M_s(\mathbb{Q}(\zeta)_v).$$

As a \mathfrak{g} -module, V_∞ decomposes as a sum of simple modules:

$$V_\infty = \bigoplus_{v \in S_\infty(\mathbb{Q}(\zeta))} \mathbb{Q}(\zeta)_v^s.$$

For any subset $T \subseteq S_\infty(\mathbb{Q}(\zeta))$ we shall write V_T for the sum of the $\mathbb{Q}(\zeta)_v^s$ for $v \in T$. Every \mathfrak{g} -submodule of V_∞ is one of the submodules V_T .

We consider real-algebraic functions $\mathcal{J} : \mathfrak{g}^a \times I^{d-e} \rightarrow V_\infty$, which satisfy the following conditions:

(C) The map \mathcal{J} is of the form:

$$\mathcal{J}(\epsilon_1, \epsilon_2, \dots, \epsilon_a, \underline{x}) = \mathcal{J}(0, \underline{x}) + \sum_{i=1}^a \alpha_i \epsilon_i \phi_i(\underline{x}),$$

where the functions $\phi_i : \mathbb{R}^{d-1} \rightarrow V_\infty$ are algebraic and $\alpha_1, \dots, \alpha_a \in \text{GL}_n(k_\infty)$.

(D) For every non-empty subset $T \subseteq S_\infty(\mathbb{Q}(\zeta))$, the set

$$\{\underline{x} \in \mathbb{R}^{d-e} : \phi_1(\underline{x}), \dots, \phi_a(\underline{x}) \in V_T\},$$

has dimension $\leq \max\{\dim_{\mathbb{R}}(V_T) - e, 0\}$.

Such a function \mathcal{T} will be called a \mathfrak{g} -deformable $d - e$ -cell.

Lemma 22 *Let \mathcal{T} be a \mathfrak{g} -deformable $d - 1$ -cell. Then \mathcal{T} is a deformable $d - 1$ -cell when regarded as a function $\mathbb{R}^{ar} \times I^{d-1} \rightarrow V_\infty$. The vertices of \mathcal{T} are the points $\mathcal{T}(\underline{x})$, where \underline{x} is a solution to $\phi_1(\underline{x}) = \dots = \phi_i(\underline{x}) = 0$.*

Proof. The statement on the vertices is clear and condition (A) follows immediately from condition (C). It remains to verify condition (B). Let $Z_i = \{\underline{x} \in \mathbb{R}^{d-1} : \text{rk}(\underline{x}) = i\}$. We must show that for $i = 1, \dots, d$, the set Z_i has dimension $\leq i - 1$.

For any $\underline{x} \in \mathbb{R}^{d-1}$, the image of $\mathcal{T}(\underline{x})$ is a translation of a \mathfrak{g} -submodule of V_∞ . The \mathfrak{g} -submodules of V_∞ are of the form V_T for subsets T of $S_\infty(\mathbb{Q}(\zeta))$. If V_T is the submodule corresponding to \underline{x} , then we clearly have $\text{rk}(\underline{x}) = \dim(V_T)$.

Given $T \subseteq S_\infty(\mathbb{Q}(\zeta))$, let Z_T denote the set of \underline{x} , for which the corresponding submodule is V_T . With this notation we have:

$$Z_i = \bigcup_{T : \dim V_T = i} Z_T.$$

As this is a finite union, it is sufficient to show that for any non-empty T , the set Z_T has dimension $\leq \dim(V_T) - 1$.

For a particular \underline{x} , the corresponding submodule V_T is the \mathfrak{g} -span of the vectors $\phi_1(\underline{x}), \dots, \phi_a(\underline{x})$. Hence

$$Z_T \subseteq \{\underline{x} \in \mathbb{R}^{d-1} : \phi_i(\underline{x}) \in V_T\}.$$

The result now follows from condition (D). □

We have fixed an ordered basis $\{b_1, \dots, b_r\}$ for \mathfrak{g} as a vector space over \mathbb{R} . Let $\epsilon = \epsilon_1 b_1 + \dots + \epsilon_r b_r \in \mathfrak{g}$. Recall the abbreviation:

$$\text{“ } \mathbf{lim} \text{ ”} := \lim_{\epsilon \rightarrow 0^+} \lim_{\epsilon_1 \rightarrow 0^+} \dots \lim_{\epsilon_r \rightarrow 0^+} .$$

Consider a real-algebraic map $\mathcal{T} : \mathfrak{g}^a \times I^d \rightarrow V_\infty$. If the faces of \mathcal{T} satisfy (C) and (D) above then \mathcal{T} is a deformable d -cell. Hence by Lemma 21, the limit

$$\mathbf{lim}_{\epsilon_1 \rightarrow 0^+} \mathbf{lim}_{\epsilon_2 \rightarrow 0^+} \dots \mathbf{lim}_{\epsilon_a \rightarrow 0^+} \text{ord}_0 \mathcal{T}(\epsilon_1, \dots, \epsilon_r)$$

exists for all v apart from the vertices of \mathcal{T} .

Condition (D) above is rather technical. To be able to verify it in practice we shall use the following lemmata.

Lemma 23 *Let W be a $d - 1$ -dimensional \mathbb{Q} -subspace of V and let W_∞ be the closure of W in V_∞ . Then for any non-empty set T of archimedean places of $\mathbb{Q}(\zeta)$, we have*

$$\dim_{\mathbb{R}}(W_\infty \cap V_T) = \dim_{\mathbb{R}}(V_T) - 1.$$

Proof. As W_∞ is a hyperplane in V_∞ , it is sufficient to show that V_T is not contained in W_∞ . It is sufficient to prove this in the case that T consists of a single place v .

We have a non-degenerate \mathbb{Q} -bilinear form on V given by:

$$\langle v, w \rangle = \sum_{i=1}^s \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(v_i w_i);$$

here we are identifying V with $\mathbb{Q}(\zeta)^s$. Extending the form to V_∞ , the various subspaces V_v are orthogonal. The subspace W_∞ is the orthogonal complement of some $w \in V \setminus \{0\}$. As the coordinate of w in V_v is non-zero, it follows that w is not orthogonal to V_v . Therefore V_v is not a subspace of W_∞ and the result follows. \square

Lemma 24 *Let \mathcal{P} be a $d - 2$ -dimensional cell in \mathfrak{X} and let $W_{\mathcal{P}}$ be the \mathbb{R} -span of the vectors $a_{\mathcal{P},i}$. Then for any non-empty set T of archimedean places of $\mathbb{Q}(\zeta)$ we have*

$$\dim_{\mathbb{R}}(W_{\mathcal{P}} \cap V_T) = \max\{\dim_{\mathbb{R}}(V_T) - 2, 0\}.$$

More general statements than the above seem to be false.

Proof. As with the previous lemma, it is sufficient to prove this in the case $T = \{v\}$. If $m = 2$ then $V_v = V_\infty$ and there is nothing to prove. We therefore assume $m > 2$, so $\mathbb{Q}(\zeta_m)$ is totally complex.

The subspace $W_{\mathcal{P}}$ is the orthogonal complement of $\{v, w\}$ some $v, w \in V \setminus \{0\}$. If we show that the coordinates of v and w in V_v are linearly independent over \mathbb{R} , then the result follows.

Our strategy for finding the vectors v and w is as follows. The cell \mathcal{P} is a $d - 2$ -dimensional face of some d -dimensional cell \mathcal{Q} in \mathfrak{X} . There are two $d - 1$ -dimensional faces of \mathcal{Q} containing \mathcal{P} , each of which is obtained by removing one of the basis elements $\{a_{\mathcal{Q},i}\}$. For each $i = 1, \dots, d$ we shall find a non-zero vector $v(i)$, which is orthogonal to the vectors $\{a_{\mathcal{Q},j} : j \neq i\}$.

The vectors v, w may be taken to be $v(i), v(j)$, where $a_{\mathcal{Q},i}$ and $a_{\mathcal{Q},j}$ are the removed basis vectors.

We recall the construction of the basis $\{a_{\mathcal{Q},i}\}$. We begin with a basis $\{d_1, \dots, d_s\}$ for the lattice L over $\mathbb{Z}[\zeta]$. For each $i = 1, \dots, s$, we choose a set of representatives $\zeta^{a(i,1)}, \dots, \zeta^{a(i,m/p)}$ for μ_p -cosets in μ_m . Then the basis $\{a_{\mathcal{Q},i}\}$ is as follows:

$$\left\{ \frac{\rho^k}{1-\rho} \zeta^{a(i,j)} d_i : i = 1, \dots, s, j = 1, \dots, m/p, k = 1, \dots, p-1 \right\}.$$

To ease notation we shall use the index set:

$$\mathcal{I} = \{(i, j, k) : i = 1, \dots, s, j = 1, \dots, m/p, k = 1, \dots, p-1\}.$$

For $(i, j, k) \in \mathcal{I}$ we define $a_{i,j,k} = \rho^k \zeta^{a(i,j)} a_i$. Then the basis $\{a_{\mathcal{Q},i}\}$ is simply $\{\frac{1}{1-\rho} a_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$.

We shall use the following Hermitean form on V_∞ :

$$\left\langle \sum v_i d_i, \sum w_i d_i \right\rangle = \frac{p}{m} \sum_{i=1}^s \text{Tr}_{\mathbb{R}}^{\mathbb{Q}(\zeta)_\infty} (v_i \overline{w_i}),$$

where $\overline{w_i}$ denotes the complex conjugate of w_i in $\mathbb{Q}(\zeta)_\infty$. We shall write A for the $\mathcal{I} \times \mathcal{I}$ matrix $\langle a_{\mathbf{i}}, a_{\mathbf{j}} \rangle$. One can show that the entries of A are as follows:

$$\langle a_{i,j,k}, a_{i',j',k'} \rangle = \begin{cases} 0 & (i, j) \neq (i', j'), \\ -1 & (i, j) = (i', j'), k \neq k', \\ p-1 & (i, j, k) = (i', j', k'). \end{cases}$$

Now consider a vector $v = (1 - \bar{\rho}) \sum v_{\mathbf{i}} a_{\mathbf{i}}$ and let $[v]$ be the column vector of coefficients $v_{\mathbf{i}}$. The vector v is orthogonal to $\frac{1}{1-\rho} a_{\mathbf{i}}$ if and only if the \mathbf{i} -th row of $A[v]$ is zero. Fix an \mathbf{i} and suppose v is orthogonal to $\frac{1}{1-\rho} a_{\mathbf{j}}$ for all $\mathbf{j} \neq \mathbf{i}$. This means that all but the \mathbf{i} -th row of $A[v]$ is zero, so $[v]$ is a multiple of the \mathbf{i} -th column of A^{-1} . We shall write $v(\mathbf{i})$ for the element of V_∞ , for which $[v(\mathbf{i})]$ is the \mathbf{i} -th column of A^{-1} . To prove the theorem we need to show that for $\mathbf{i} \neq \mathbf{j}$ the coordinates of $v(\mathbf{i})$ and $v(\mathbf{j})$ in V_v are linearly independent over \mathbb{R} .

By finding A^{-1} one obtains:

$$v(i, j, k) = (1 - \bar{\rho})(\rho^k - 1) \zeta^{a(i,j)} d_i.$$

Let the place v correspond to the embedding $\iota : \mathbb{Q}(\zeta) \hookrightarrow \mathbb{C}$. We must show that for $(i, j, k) \neq (i', j', k')$ the vectors $\iota(v(i, j, k)), \iota(v(i', j', k')) \in \mathbb{C}^s$ are linearly independent over \mathbb{R} . If $i \neq i'$ then this is clearly the case as they are independent over \mathbb{C} . We therefore assume $i = i'$. We are reduced to showing that the complex number

$$z = \iota \left(\frac{(\rho^k - 1)\zeta^{a(i,j)}}{(\rho^{k'} - 1)\zeta^{a(i,j')}} \right).$$

is not real. We let $z_1 = \iota \left(\frac{\rho^k - 1}{\rho^{k'} - 1} \right)$ and $z_2 = \iota(\zeta)^{a(i,j) - a(i,j')}$. The argument of z_1 is of the form

$$\pi(k - k') \frac{r}{p}, \quad r \in \{1, 2, \dots, p-1\}.$$

Therefore $z_1^p \in \mathbb{R}$. If $j \neq j'$ then $z_2^p \notin \mathbb{R}$ so we are done. Finally, if $j = j'$ then $z_2 = 1$ but $k \neq k'$, so $z_1 \notin \mathbb{R}$ and again we are done. \square

Up until now, we have examined deformable cells in V_∞ . However, we need to define the order of a cell in X_∞ at points on its boundary. We call a map $\mathcal{T} : \mathbb{R}^b \times I^d \rightarrow X_\infty$ a deformable cell in X_∞ , if one, or equivalently all, of its lifts to V_∞ are deformable cells. We define the vertices of such a \mathcal{T} to be the projections in X_∞ of the vertices of a lift of \mathcal{T} .

Lemma 25 *Let $\mathcal{T} : \mathbb{R}^b \times I^d \rightarrow X_\infty$ be a deformable d -cell in X_∞ . If $x \in X_\infty$ is not a vertex of \mathcal{T} then the following limit exists:*

$$\lim_{\epsilon_1 \rightarrow 0^+} \dots \lim_{\epsilon_b \rightarrow 0^+} \text{ord}_x(\mathcal{T}(\epsilon)).$$

Proof. This follows from the previous results using the relation

$$\text{ord}_x(\mathcal{T}(\epsilon)) = \sum_{y \rightarrow x} \text{ord}_y(\tilde{\mathcal{T}}(\epsilon)),$$

where $\tilde{\mathcal{T}}$ is a lift of \mathcal{T} and the sum is over the preimages of x in V_∞ . As $|\tilde{\mathcal{T}}|$ is compact, the sum is in fact finite and hence commutes with the limits. \square

It will be more convenient to speak of *piecewise deformable cells*. We shall call a map $\mathbb{R}^b \times I^d \rightarrow X_\infty$ a piecewise deformable cell, if there is a subdivision of I^d , such that the restriction of \mathcal{T} to any of the pieces in the

subdivision is deformable. The reason for this is that our functions will be piecewise algebraic rather than algebraic. By a *piecewise deformable chain* we shall simply mean a formal sum of piecewise deformable cells. Thus a piecewise deformable chain will be a map $\mathbb{R}^b \rightarrow C_\bullet$. If x is not a vertex of a piecewise deformable d -chain \mathcal{T} then we define

$$\text{ord}_x(\mathcal{T}) := \lim_{\epsilon_1 \rightarrow 0^+} \dots \lim_{\epsilon_b \rightarrow 0^+} \text{ord}_x(\mathcal{T}(\epsilon)).$$

It follows from the above results that this limit exists. If \mathcal{T} is deformable then we shall call \mathcal{T} a *deformation* of $\mathcal{T}(0)$.

5.3 Deforming paths.

The function \bar{f} . We now apply Lemma 21 to the function f . Define a path $\wp(\epsilon)$ for $\epsilon \in \mathfrak{g}$ by

$$\wp(\epsilon) = [0, \frac{1}{2} + \epsilon] + [\frac{1}{2} + \epsilon, 1].$$

Proposition 9 *For any $d - 1$ or $d - 2$ -dimensional cell \mathcal{P} in \mathfrak{X} The map $\epsilon \mapsto \wp(\epsilon) \bowtie \mathcal{P}$ is piecewise \mathfrak{g} -deformable. Its vertices are in $\frac{1}{1-\rho}L$.*

Proof. Let \mathcal{P} be any $d - 1$ -cell in \mathfrak{X} . We first cut $\wp(\epsilon) \bowtie \mathcal{P}$ into its 2^{d-1} algebraic pieces and then prove that each piece is deformable. Thus for any subset $A \subseteq \{1, 2, \dots, d - 1\}$ we define

$$\mathcal{A}_A(\epsilon) = [v_{\mathcal{P}}] \diamond \prod_{i \in A} [0, (\frac{1}{2} + \epsilon)a_{\mathcal{P},i}] \diamond \prod_{i \notin A} [(\frac{1}{2} + \epsilon)a_{\mathcal{P},i}, a_{\mathcal{P},i}].$$

We shall show that each \mathcal{A}_A is deformable by verifying conditions (C) and (D) above. We have

$$\mathcal{A}(\epsilon, \underline{x}) = v_{\mathcal{P}} + \sum_{i \in A} (\frac{1}{2} + \epsilon)x_i a_{\mathcal{P},i} + \sum_{i \notin A} ((\frac{1}{2} + \epsilon) + (\frac{1}{2} - \epsilon)x_i) a_{\mathcal{P},i}.$$

This implies

$$\mathcal{A}(\epsilon, \underline{x}) = \mathcal{A}(0, \underline{x}) + \epsilon \left(\sum_{i \in A} x_i a_{\mathcal{P},i} + \sum_{i \notin A} (1 - x_i) a_{\mathcal{P},i} \right).$$

Therefore \mathcal{A} verifies condition (C) with

$$\phi(\underline{x}) = \sum_{i \in A} x_i a_{\mathcal{P},i} + \sum_{i \notin A} (1 - x_i) a_{\mathcal{P},i}.$$

To verify (D) we let $W_{\mathcal{P}}$ be the \mathbb{R} -span of the vectors $a_{\mathcal{P},i}$. Thus ϕ maps \mathbb{R}^{d-1} bijectively to $W_{\mathcal{P}}$. We must show that for any non-empty set of archimedean places T we have $\dim_{\mathbb{R}}(W_{\mathcal{P}} \cap V_T) \leq \dim_{\mathbb{R}}(V_T) - 1$. This follows from Lemma 23. It follows from the formula for ϕ that the only vertex of \mathcal{A} is $v_{\mathcal{P}} + \sum_{i \notin A} a_{\mathcal{P},i}$. By Lemma 8 we know that this is in $\frac{1}{1-\rho}L$. The case of a $d - 2$ -cell in \mathfrak{X} is similar except that one must use Lemma 24 instead of Lemma 23. \square

We may now define

$$\bar{f}(x) := \mathbf{lim}_{\epsilon \rightarrow 0^+} \text{ord}_x(\wp(\epsilon) \bowtie \mathcal{F}).$$

This limit exists for all x not in $\frac{1}{1-\rho}L$.

Proposition 10 *If $f(x)$ is defined then so is $\bar{f}(x)$ and they are equal. However $\bar{f}(x)$ is defined for all $x \notin \text{Vert}(\mathcal{P})$. Furthermore if the μ_m -orbit of x does not intersect $\text{Vert}(\mathcal{P})$ then we have*

$$\sum_{\zeta \in \mu_m} \bar{f}(\zeta x) = 1.$$

In particular the restriction of f to $X \setminus \{0\}$ is a fundamental function.

Proof. The first two assertions follow from Lemma 25. To prove the formula, we use the fact that finite sums commute with limits as follows:

$$\sum_{\zeta \in \mu_m} \bar{f}(\zeta x) = \sum_{\zeta \in \mu_m} \mathbf{lim}_{\epsilon \rightarrow 0^+} f^{(\wp(\epsilon))}(\zeta x) = \mathbf{lim}_{\epsilon \rightarrow 0^+} \sum_{\zeta \in \mu_m} f^{(\wp(\epsilon))}(\zeta x) = \mathbf{lim}_{\epsilon \rightarrow 0^+} 1 = 1.$$

\square

The function \bar{f}^α . Recall that for any $\alpha \in \text{GL}_n(k_\infty)$ we have a path \wp^α from 0 to 1 in \mathfrak{g} , defined by

$$\wp^\alpha = [0, \alpha] + [\alpha, 1].$$

More precisely, let

$$\wp^\alpha(x) = \begin{cases} 2x\alpha & \text{for } x \leq \frac{1}{2}, \\ \alpha + (2x - 1)(1 - \alpha) & x \geq \frac{1}{2}. \end{cases}$$

By a vertex of \wp^α we shall mean one of the following points of I :

$$\text{Vert}(\wp^\alpha) = \left\{ 0, \frac{1}{2}, \frac{m^2 + 1}{2m^2}, \frac{m^2 + 2}{2m^2}, \dots, 1 \right\}.$$

By a vertex of $\wp^\alpha \bowtie \mathcal{F}$ we shall mean a point $v \in V_\infty$ of the form

$$v = (\wp^\alpha \bowtie \mathcal{P})(\underline{x}), \quad \underline{x} \in \text{Vert}(\wp^\alpha)^{d-1},$$

where \mathcal{P} is a $d - 1$ -cell in \mathfrak{X} .

We shall write $\text{Vert}(\wp^\alpha \bowtie \mathcal{F})$ for the set of all vertices of $\wp^\alpha \bowtie \mathcal{F}$. The path \wp^α gives rise to a fundamental function f^α away from the boundary of $\wp^\alpha \bowtie \mathcal{P}$. We shall extend the definition of f^α to all points of X_∞ apart from the vertices of $\wp^\alpha \bowtie \mathcal{F}$.

Given $\epsilon, \nu \in \mathfrak{g}$ we define a new path $\wp^\alpha(\epsilon, \nu)$ by

$$\wp^\alpha(\epsilon, \nu, x) = \begin{cases} \alpha\wp(\epsilon, 2x) & \text{for } x \leq \frac{1}{2}, \\ \alpha + (2x - 1)(1 - \alpha) + \phi(2m^2x)\nu & x \geq \frac{1}{2}. \end{cases}$$

Here $\phi : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ is the \mathbb{Z} -periodic function defined on the interval I by

$$\phi(x) = \begin{cases} x & x \leq \frac{1}{2}, \\ 1 - x & x \geq \frac{1}{2}. \end{cases}$$

The path $\wp^\alpha(\epsilon, \nu)$ reduces to \wp^α when ϵ and ν are both zero. We extend our definition of f^α as follows:

$$\bar{f}^\alpha(x) = \lim_{\epsilon \rightarrow 0^+} \lim_{\nu \rightarrow 0^+} \text{ord}_x(\wp^\alpha(\epsilon, \nu) \bowtie \mathcal{F}).$$

Proposition 11 *Let \mathcal{P} be a $d - 1$ - or $d - 2$ -cell in \mathfrak{X} . The map $(\epsilon, \nu) \mapsto \wp^\alpha(\epsilon, \nu) \bowtie \mathcal{P}$ is piecewise \mathfrak{g} -deformable. It is a deformation of $\wp^\alpha \bowtie \mathcal{P}$. Its vertices are in $\text{Vert}(\wp^\alpha \bowtie \mathcal{P})$. Consequently the limit $\bar{f}^\alpha(x)$ exists for all $x \notin \text{Vert}(\wp^\alpha \bowtie \mathcal{F})$. The function f^α is fundamental on all μ_m -orbits which do not intersect $\text{Vert}(\wp^\alpha \bowtie \mathcal{F})$.*

Proof. Let \mathcal{P} be any $d - 1$ -cell in \mathfrak{X} . As before we must show that the map

$$\mathcal{T}(\epsilon, \nu, \underline{x}) = (\wp^\alpha(\epsilon, \nu) \bowtie \mathcal{P})(\underline{x})$$

is piecewise deformable.

We cut \mathcal{T} into its algebraic pieces. These pieces are translations by elements of $\text{Vert}(\wp^\alpha \bowtie \mathcal{P})$ of the pieces

$$\begin{aligned} \mathcal{A} = & \left(\alpha\left(\frac{1}{2} + \epsilon\right) \cdot \diamond_{i \in A} [0, a_{\mathcal{P}, i}] \right) \\ & \diamond \left(\alpha\left(\frac{1}{2} - \epsilon\right) \cdot \diamond_{i \in B} [0, a_{\mathcal{P}, i}] \right) \\ & \diamond \left(\left(\frac{1-\alpha}{2m^2} + \frac{\nu}{2}\right) \cdot \diamond_{i \in C} [0, a_{\mathcal{P}, i}] \right) \\ & \diamond \left(\left(\frac{1-\alpha}{2m^2} - \frac{\nu}{2}\right) \cdot \diamond_{i \in D} [0, a_{\mathcal{P}, i}] \right), \end{aligned}$$

where the sets A, B, C, D form a partition of $\{1, 2, \dots, d - 1\}$. The piece \mathcal{A} satisfies condition (C) above with

$$\phi_1(\underline{x}) = \left(\sum_{i \in A} - \sum_{i \in B} \right) x_i a_{\mathcal{P}, i}, \quad \phi_2(\underline{x}) = \left(\sum_{i \in C} - \sum_{i \in D} \right) x_i a_{\mathcal{P}, i}.$$

To prove condition (D) we apply lemma 23 to the subspace $W_{\mathcal{P}}$ spanned by $\{a_{\mathcal{P}, i}\}$. The case of a $d - 2$ -cell \mathcal{P} is similar but one must use Lemma 24 instead of Lemma 23.

As the $a_{\mathcal{P}, i}$ are linearly independent it follows that the only vertex of \mathcal{A} is 0, so the vertices of \mathcal{T} are the translations, which are in $\text{Vert}(\wp^\alpha \bowtie \mathcal{P})$. \square

The function $\bar{f}^{\alpha\beta, \alpha}$. Finally for $\alpha, \beta \in \text{GL}_n(k_\infty)$ we define a path $\wp^{\alpha\beta, \alpha}$ by

$$\wp^{\alpha\beta, \alpha} = [0, \alpha\beta] + [\alpha\beta, \alpha] + [\alpha, 1].$$

More precisely let

$$\wp^{\alpha\beta, \alpha}(x) = \begin{cases} 4x\alpha\beta & x \leq \frac{1}{4}, \\ \alpha\beta + (4x - 1)(\alpha - \alpha\beta) & \frac{1}{4} \leq x \leq \frac{1}{2}, \\ \alpha + (2x - 1)(1 - \alpha) & x \geq \frac{1}{2}. \end{cases}$$

We define

$$\text{Vert}(\wp(\alpha\beta, \alpha)) = \left\{ 0, \frac{1}{4}, \frac{m^2 + 1}{4m^2}, \dots, \frac{1}{2}, \frac{m^2 + 1}{2m^2}, \dots, 1 \right\},$$

$$\text{Vert}(\mathcal{P}^{\alpha\beta, \alpha}) = \{\mathcal{P}_i^{\alpha\beta, \alpha}(\underline{x}) : \underline{x} \in \text{Vert}(\wp(\alpha\beta, \alpha))^{d-1}\}.$$

We shall extend the definition of $f^{\alpha\beta, \alpha}$ to $X_\infty \setminus \text{Vert}(\mathcal{P}^{\alpha\beta, \alpha})$. To do this we define a deformation of $\wp^{\alpha\beta, \alpha}$ as follows:

$$\wp^{\alpha\beta, \alpha}(\epsilon, \nu, \xi, x) = \begin{cases} \alpha\wp^\beta(\epsilon, \nu, 2x) & x \leq \frac{1}{2}, \\ \wp^\alpha(0, \xi, x) & x \geq \frac{1}{2}. \end{cases}$$

Again, for any $x \in X_\infty$, which is not a vertex of $\mathcal{P}^{\alpha\beta, \alpha}$, we may define

$$\bar{f}^{\alpha\beta, \alpha}(x) = \lim_{\epsilon \rightarrow 0^+} \lim_{\nu \rightarrow 0^+} \lim_{\xi \rightarrow 0^+} \text{ord}_x(\wp^{\alpha\beta, \alpha}(\epsilon, \nu, \xi) \bowtie \mathcal{F}).$$

Proposition 12 *Let \mathcal{P} be a $d - 1$ - or $d - 2$ -cell in \mathfrak{X} . The map*

$$(\epsilon, \nu, \xi) \mapsto \wp^{\alpha\beta, \alpha}(\epsilon, \nu, \xi) \bowtie \mathcal{P}$$

is piecewise a \mathfrak{g} -deformation of $\wp^{\alpha\beta, \alpha} \bowtie \mathcal{P}$. Its vertices are in $\text{Vert}(\wp^{\alpha\beta, \alpha} \bowtie \mathcal{F})$. Consequently the limit $\bar{f}^{\alpha\beta, \alpha}(x)$ exists for all $x \notin \text{Vert}(\wp^{\alpha\beta, \alpha} \bowtie \mathcal{F})$ and is fundamental there.

This is proved in a similar way to Proposition 11.

5.4 Deformation of homotopies.

Again to take account of points on the boundary of $\mathcal{H} \bowtie \mathcal{G}$ we must construct a deformation of $\mathcal{H} \bowtie \mathcal{G}$.

Proposition 13 *Suppose that for $\epsilon_1, \dots, \epsilon_r, \nu_1, \dots, \nu_s \in \mathfrak{g}$ we have paths $\wp_1(\epsilon_1, \dots, \epsilon_r)$ and $\wp_2(\nu_1, \dots, \nu_s)$. Suppose further that for any $d - 1$ - or $d - 2$ -cell \mathcal{P} in \mathfrak{X} , the maps $\wp_1 \bowtie \mathcal{P}$ and $\wp_2 \bowtie \mathcal{P}$ are piecewise \mathfrak{g} -deformable. Define a homotopy $\mathcal{H}(\epsilon_1, \dots, \epsilon_r, \nu_1, \dots, \nu_s)$ from $\wp_1(\epsilon_1, \dots, \epsilon_r)$ to $\wp_2(\nu_1, \dots, \nu_s)$ by*

$$\mathcal{H}(t, x) = (1 - t)\wp_1(x) + t\wp_2(x).$$

Then $\partial(\mathcal{H} \bowtie \mathcal{G})$ is piecewise \mathfrak{g} -deformable.

Proof. Recall that $\mathcal{G} \in \mathfrak{X}_{d-1}$. We must therefore show that for any $d-1$ -cell \mathcal{P} in \mathfrak{X} , the $d-1$ -chain $\partial(\mathcal{H} \bowtie \mathcal{Q})$ is piecewise \mathfrak{g} -deformable. We have

$$\partial(\mathcal{H} \bowtie \mathcal{P}) = (\wp_2 - \wp_1) \bowtie \mathcal{P} + \mathcal{H} \bowtie \partial\mathcal{P}.$$

As we already know that $\wp_i \bowtie \mathcal{P}$ is piecewise \mathfrak{g} -deformable, it is sufficient to show that for any $d-2$ -cell \mathcal{Q} the cell $\mathcal{H} \bowtie \mathcal{Q}$ is piecewise \mathfrak{g} -deformable. We have

$$(\mathcal{H} \bowtie \mathcal{Q})(t, \underline{x}) = (1-t)(\wp_1 \bowtie \mathcal{Q})(\underline{x}) + t(\wp_2 \bowtie \mathcal{Q})(\underline{x}).$$

This implies

$$\mathrm{rk}_{\mathcal{H} \bowtie \mathcal{Q}}(t, \underline{x}) \geq \min\{\mathrm{rk}_{\wp_1 \bowtie \mathcal{Q}}(\underline{x}), \mathrm{rk}_{\wp_2 \bowtie \mathcal{Q}}(\underline{x})\}.$$

The result now follows as $\wp_i \bowtie \mathcal{Q}$ is piecewise \mathfrak{g} -deformable. \square

5.5 The limit defining Dec_∞ .

Recall that we have defined $\mathrm{Dec}_\infty(\alpha, \beta)$ in terms of the limit

$$\lim_{\epsilon_1 \rightarrow 0^+} \lim_{\epsilon_2 \rightarrow 0^+} \lim_{\epsilon_3 \rightarrow 0^+} \mathrm{ord}_{0,V} \left([1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)] \cdot \mathcal{E} \right).$$

To show that this limit exists we must prove:

Proposition 14 *The map*

$$(\epsilon_1, \epsilon_2, \epsilon_3) \mapsto [1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)] \cdot \mathcal{E}$$

is a \mathfrak{g} -deformation of $[1, \alpha, \alpha\beta] \cdot \mathcal{E}$ with no vertices.

Proof. We have $\mathcal{E} = \mathfrak{s}\mathcal{G}$, where \mathcal{G} is an element of \mathfrak{X}_{d-1} . We therefore fix a $d-1$ -cell \mathcal{P} in \mathfrak{X} and define

$$\mathcal{A} = [1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)] \cdot \mathfrak{s}\mathcal{P}.$$

We must show that the boundary of \mathcal{A} is deformable. As \mathfrak{s} anticommutes with ∂ we have:

$$\partial\mathcal{A} = \partial([1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)]) \cdot \mathfrak{s}\mathcal{P} - [1 + \epsilon_1, \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)] \cdot \mathfrak{s}\partial\mathcal{P}.$$

We must show that both the summands above are deformable. We shall show that all the cells in the above expression satisfy conditions (C) and (D) above and have no vertices.

We begin with the first summand. This is made up of cells of the form

$$[(1 + \epsilon_1), \alpha(1 + \epsilon_2)] \cdot \mathfrak{sP} : \Delta^1 \times \Delta^{d-2} \rightarrow V_\infty.$$

Condition (C) is satisfied with

$$\phi_i(\underline{x}, \underline{y}) = x_i \mathfrak{sP}(\underline{y}), \quad (\underline{x}, \underline{y}) \in \Delta^1 \times \Delta^{d-2}.$$

As $0 \notin |\mathfrak{sP}|$ and the x_i are not all zero it follows that there are no vertices. Let H be the hyperplane in $W_{\mathcal{P}}$ containing $|\mathfrak{sP}|$. To verify condition (D) we must show that $H \cap V_T$ has dimension $\leq \dim_{\mathbb{R}}(V_T) - 2$. This follows from Lemma 23 since H does not contain 0.

The second summand contains cells of the form:

$$[(1 + \epsilon_1), \alpha(1 + \epsilon_2), \alpha\beta(1 + \epsilon_3)] \cdot \mathfrak{sQ} : \Delta^2 \times \Delta^{d-3} \rightarrow V_\infty,$$

with \mathcal{Q} a $d - 2$ -cell of \mathfrak{X} . Again condition (C) is satisfied with

$$\phi_i(\underline{x}, \underline{y}) = x_i \mathfrak{sQ}(\underline{y}), \quad (\underline{x}, \underline{y}) \in \Delta^2 \times \Delta^{d-3}$$

As the x_i are never all zero and $0 \notin |\mathfrak{sQ}|$, it follows that the functions ϕ_i are never simultaneously all 0. This shows that there are no vertices.

The base set of \mathfrak{sQ} lies in a hyperplane H in $W_{\mathcal{Q}}$. To verify condition (D) we must show that for any set T of archimedean places of $\mathbb{Q}(\zeta)$ the dimension of $H \cap V_T$ is $\leq \dim_{\mathbb{R}}(V_T) - 3$. As H does not go through 0 this reduces to proving that $\dim_{\mathbb{R}}(W_{\mathcal{P}} \cap V_T) \leq \dim_{\mathbb{R}}(V_T) - 2$. However this follows from Lemma 24. \square

6 The relation between the arithmetic and geometric cocycles.

6.1 Main Results.

We now state our main results. Recall that we have a homotopy \mathcal{H}_β^1 from $\wp(1)$ to $\wp(\beta)$ defined by

$$\mathcal{H}_\beta^1(x, t) = t\wp(\beta)(x) + (1 - t)x.$$

From this we have constructed a homotopy $\mathcal{H}_{\alpha\beta,\alpha}^\alpha$ from $\wp(\alpha)$ to $\wp(\alpha\beta, \alpha)$ as follows:

$$\mathcal{H}_{\alpha\beta,\alpha}^\alpha(x, t) = \begin{cases} \alpha\mathcal{H}_\beta^1(2x, t) & x \leq \frac{1}{2}, \\ \wp(\alpha)(x) & x \geq \frac{1}{2}. \end{cases}$$

Finally we have a homotopy $\mathcal{H}_{\alpha\beta}^{\alpha\beta,\alpha}$ from $\wp'(\alpha\beta, \alpha)$ to $\wp(\alpha\beta)$ satisfying

$$\mathcal{H}_{\alpha\beta}^{\alpha\beta,\alpha}(x, t) = 2x\alpha\beta, \quad x \leq \frac{1}{2}.$$

and constructed by splitting the triangle $[\alpha\beta, \alpha, 1]$ into m^2 smaller triangles.

Recall that our geometric cocycle is given by the formula:

$$\text{Dec}_\infty^{(\mathfrak{s}\mathcal{F})}(\alpha, \beta) = \zeta^{\text{ord}_{0,V}}([1, \alpha, \alpha\beta] \cdot \mathfrak{s}\mathcal{G}).$$

By Proposition 8 we have

$$\text{Dec}_\infty^{(\mathfrak{s}\mathcal{F})}(\alpha, \beta) = \zeta^{\text{ord}_{0,X}} \left((\mathcal{H}_\alpha^1 + \mathcal{H}_{\alpha\beta,\alpha}^\alpha + \mathcal{H}_{\alpha\beta}^{\alpha\beta,\alpha} - \mathcal{H}_{\alpha\beta}^1) \rtimes \mathcal{G} \right). \quad (18)$$

One may check that this remains true even if the quantities are defined as limits in the sense of §5. Looked at from this point of view, it seems natural to divide out a certain coboundary from this cocycle. For $\alpha \in \Upsilon_{\mathfrak{f}}$ define

$$\tau(\alpha) = \zeta^{\{\mathcal{H}_\alpha^1 \rtimes \mathcal{G} | \alpha L\}}.$$

We shall prove the following.

Theorem 5 *For $\alpha, \beta \in \Upsilon_{\mathfrak{f}}$ we have*

$$\text{Dec}_\infty(\alpha, \beta) \text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)}.$$

We shall show that there is a continuous cocycle Dec_m on $\text{SL}_n(k_m)$ and an extension of τ to $\text{SL}_n(k)$ such that for $\alpha, \beta \in \text{SL}_n(k)$ we have

$$\text{Dec}_\infty(\alpha, \beta) \text{Dec}_m(\alpha, \beta) \text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)}. \quad (19)$$

The proof of Theorem 5 will be broken down into the following three lemmata.

Lemma 26 *Let $\alpha \in \Upsilon_{\mathfrak{f}}$. Then for any μ_m -invariant lattice $M \subset V_m$ containing L , we have*

$$\langle f\alpha^{-1} - f^\alpha | \alpha M - \alpha L \rangle_X = 1.$$

Lemma 27 *Let $\alpha \in G_{\mathfrak{f}}$ and $\beta \in \mathrm{GL}_n(k_\infty)$. Then for any μ_m -invariant lattice $M \subset V_m$ containing both L and $\alpha^{-1}L$, the following holds in \mathbb{Z}/m :*

$$\{\mathcal{H}_\beta^1 \rtimes \mathcal{G} | M\} = \{\mathcal{H}_{\alpha\beta, \alpha}^\alpha \rtimes \mathcal{G} | \alpha M\}.$$

Lemma 28 *Let $\alpha, \beta \in G_{\mathfrak{f}}$. For any μ_m -invariant lattice $M \subset V_m$ containing L , αL and $\alpha\beta L$, the following holds in \mathbb{Z}/m :*

$$\{\mathcal{H}_{\alpha\beta, \alpha}^{\alpha\beta} \rtimes \mathcal{G} | M\} = 0.$$

The proofs of these lemmata in §6.4-6.6 are essentially exercises in the geometry of numbers. In each case one is reduced to showing that the number of lattice points in a certain set is a multiple of m . One achieves this by cutting the set into m pieces which are translations of one another by elements of the lattice.

6.2 A formula for the Kubota symbol.

Assume in this section that k is totally complex. We shall describe the Kubota symbol on the group

$$\Gamma_{\mathfrak{f}} = \{\alpha \in \mathrm{SL}_n(k) : \alpha L = L, \alpha \equiv I_n \pmod{\mathfrak{f}}\} = \Upsilon_{\mathfrak{f}} \cap \mathrm{SL}_n(\mathfrak{o}).$$

Corollary 3 *For $\alpha \in \Gamma_{\mathfrak{f}}$ the Kubota symbol $\kappa_m(\alpha)$ is given by the formula:*

$$\kappa_m(\alpha) = \frac{\zeta^{\mathrm{ord}_{0,X}(\mathcal{H}_\alpha^1 \rtimes \mathcal{G})}}{h(w(\alpha))}.$$

Proof. We shall regard $\mathrm{SL}_n(k)$ as a dense subgroup of $\mathrm{SL}_n(\mathbb{A}_f)$, where \mathbb{A}_f denotes the ring of finite adèles of k . Let U be the closure of $\Gamma_{\mathfrak{f}}$ in $\mathrm{SL}_n(\mathbb{A}_f)$. This is a compact open subgroup of $\mathrm{SL}_n(\mathbb{A}_f)$. Consider the extension

$$1 \rightarrow \mu_m \rightarrow \widetilde{\mathrm{SL}}_n(\mathbb{A}_f) \rightarrow \mathrm{SL}_n(\mathbb{A}_f) \rightarrow 1,$$

corresponding to the cocycle $\text{Dec}_{\mathbb{A}(S)}\text{Dec}_m$. The cocycle $\text{Dec}_{\mathbb{A}(S)}\text{Dec}_m$ is 1 on $U \times U$. Therefore on U the map $\alpha \mapsto (\alpha, 1)$ is a splitting of the extension. On $\text{SL}_n(k)$ we have by (19) and Theorem 3:

$$\text{Dec}_{\mathbb{A}(S)}\text{Dec}_m(\alpha, \beta)\partial hw(\alpha, \beta) = \partial\tau(\alpha, \beta).$$

This implies that on $\text{SL}_n(k)$ the map $\alpha \mapsto (\alpha, \tau(\alpha)/hw(\alpha))$ splits the extension. As the Kubota symbol is the ratio of these two splittings, the result follows. \square

Remark 3 *Some authors speak of the “Kubota symbol on GL_n ”. By this they are in effect choosing an embedding of GL_n in SL_{n+r} and pulling back the Kubota symbol. As the above formula is valid for n arbitrarily large there is no need to have a separate formula for GL_n . It is worth mentioning that the formula of the above corollary gives a homomorphism*

$$\text{GL}_n(\mathfrak{o}, \mathfrak{f}) \rightarrow \bar{\mu}, \quad \alpha \mapsto \frac{\zeta^{\text{ord}_{0,X}}(\mathcal{H}_\alpha^1 \rtimes \mathcal{G})}{w(\alpha)},$$

which is a rather more canonical extension of the Kubota symbol on GL_n . Here $\text{GL}_n(\mathfrak{o}, \mathfrak{f})$ represents the principal congruence subgroup modulo \mathfrak{f} .

6.3 Proof of Theorem 5.

We shall deduce the theorem from lemmata 26, 27 and 28.

Let $\alpha, \beta \in \Upsilon_{\mathfrak{f}}$. We begin with the definition of $\text{Dec}_{\mathbb{A}(S)}$:

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \langle f - f\alpha | \beta L - L \rangle = \langle f\alpha^{-1} - f | \alpha\beta L - \alpha L \rangle.$$

By Lemma 26 we have:

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \langle f^\alpha - f | \alpha\beta L - \alpha L \rangle.$$

By Proposition 7 we have:

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \zeta \{ \mathcal{H}_1^\alpha \rtimes \mathcal{G} | \alpha\beta L - \alpha L \}.$$

This implies

$$\begin{aligned} \text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) &= \tau(\alpha) \zeta \{ \mathcal{H}_1^\alpha \rtimes \mathcal{G} | \alpha\beta L \} \\ &= \frac{\tau(\alpha)}{\tau(\alpha\beta)} \zeta \{ (\mathcal{H}_1^\alpha + \mathcal{H}_{\alpha\beta}^1) \rtimes \mathcal{G} | \alpha\beta L \}. \end{aligned}$$

On the other hand we have by Lemma 27:

$$\tau(\beta) = \zeta \{ \mathcal{H}_\beta^1 \bowtie \mathcal{G} | \beta L \} = \zeta \{ \mathcal{H}_{\alpha\beta, \alpha}^\alpha \bowtie \mathcal{G} | \alpha\beta L \}.$$

Taking the last two formulae together we obtain:

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)} \zeta \{ (\mathcal{H}_1^\alpha + \mathcal{H}_{\alpha\beta}^1 - \mathcal{H}_{\alpha\beta, \alpha}^\alpha) \bowtie \mathcal{G} | \alpha\beta L \}.$$

By Lemma 28 we have

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)} \zeta \{ (\mathcal{H}_1^\alpha + \mathcal{H}_{\alpha\beta}^1 + \mathcal{H}_{\alpha\beta, \alpha}^{\alpha\beta} + \mathcal{H}_\alpha^{\alpha\beta, \alpha}) \bowtie \mathcal{G} | \alpha\beta L \}.$$

This implies by (18):

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) \text{Dec}_\infty(\alpha, \beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)} \zeta \{ (\mathcal{H}_1^\alpha + \mathcal{H}_{\alpha\beta}^1 + \mathcal{H}_{\alpha\beta, \alpha}^{\alpha\beta} + \mathcal{H}_\alpha^{\alpha\beta, \alpha}) \bowtie \mathcal{G} | \alpha\beta L - L \}.$$

On the other hand by Proposition 7 the final term in the above is equal to:

$$\langle f^\alpha - f + f - f^{\alpha\beta} + f^{\alpha\beta} - f^{\alpha\beta, \alpha} + f^{\alpha\beta, \alpha} - f^\alpha | \alpha\beta L - L \rangle = 1.$$

□

6.4 Proof of Lemma 26.

We have by definition,

$$\langle f^\alpha - f\alpha^{-1} | \alpha M - \alpha L \rangle_X = \prod_{\zeta \in \mu_m} \zeta \sum_{x \in T} f^\alpha(x) f(\zeta \alpha^{-1} x),$$

where the sums are over all x in the finite subset T of X given by:

$$T = ((\alpha M)/L) \setminus ((\alpha L)/L).$$

The functions f and f^α are defined as limits of the functions $f^{(\epsilon)}$ and $f^{\alpha, (\epsilon, \nu)}$. As T is finite we may choose ϵ, ν small enough so that for every $x \in T$ we have $f(\alpha^{-1}x) = f^{(\epsilon)}(\alpha^{-1}x)$ and $f^\alpha(x) = f^{\alpha, (\epsilon, \nu)}(x)$. Fix $\zeta \neq 1$. It is sufficient to show that in \mathbb{Z}/m we have

$$\sum_{x \in T} f^{\alpha, (\epsilon, \nu)}(x) f^{(\epsilon)}(\zeta \alpha^{-1} x) = 0.$$

By definition we have

$$f^{\alpha,(\epsilon,\nu)}(x) = \text{ord}_x(\wp(\alpha, (\epsilon, \nu)) \bowtie \mathcal{F})$$

It is therefore sufficient to show that for any d -cell \mathcal{P} in \mathfrak{X} we have:

$$\sum_{x \in T} \text{ord}_x(\wp(\alpha, (\epsilon, \nu)) \bowtie \mathcal{P}) f^{(\epsilon)}(\zeta \alpha^{-1} x) = 0.$$

Fix such a \mathcal{P} . We cut $\wp(\alpha, (\epsilon, \nu)) \bowtie \mathcal{P}$ into 2^d smaller pieces. Define for each subset $A \subseteq \{1, 2, \dots, d\}$

$$\mathcal{A}_A(x_1, \dots, x_d) = \sum_{i \notin A} \alpha \wp(\epsilon, x_i) \cdot a_{\mathcal{P},i} + \sum_{i \in A} (\alpha + (1 - \alpha)x_i + \phi(m^2 x_i)\nu) a_{\mathcal{P},i}.$$

Here ϕ is as in §5.3. Then for $x \in T$ we have

$$\text{ord}_x(\wp(\alpha, (\epsilon, \nu)) \bowtie \mathcal{P}) = \sum_{A \subseteq \{1, \dots, d\}} \text{ord}_x(\mathcal{A}_A).$$

It is therefore sufficient to prove that for any A we have in \mathbb{Z}/m :

$$\sum_{x \in T} \text{ord}_x(\mathcal{A}_A) f^{(\epsilon)}(\zeta \alpha^{-1} x) = 0.$$

There are two cases which we must consider. In the case that A is empty, we have

$$\alpha^{-1} \mathcal{A}_A = \wp(\epsilon) \bowtie \mathcal{P}.$$

Therefore if x is in the base set of \mathcal{A}_A , then $\alpha^{-1}x$ (which is well defined as $\alpha \in \Upsilon$) is in the base set of $\wp(\epsilon) \bowtie \mathcal{P}$. If this is the case then as $\zeta \neq 1$, we have $f^{(\epsilon)}(\alpha^{-1}\zeta x) = 0$.

Next suppose that there is an index $j \in A$. Without loss of generality assume $1 \in A$. Then we have a decomposition $\mathcal{A}_A = \mathcal{B} \diamond \mathcal{C}$, where $\mathcal{B} : I \rightarrow X_\infty$ is the 1-cell given by

$$\mathcal{B}(x) = ((1 - \alpha)x + \phi(m^2 x)\nu) \cdot a_{\mathcal{P},1},$$

and \mathcal{C} is a $d - 1$ -cell. As the function ϕ is periodic, we may write \mathcal{B} as a sum of m^2 translations of

$$\mathcal{B}_1(x) = \left(\frac{1 - \alpha}{m^2} x + \phi(x)\nu \right) \cdot a_{\mathcal{P},1},$$

where the translations are by multiples of $\frac{1-\alpha}{m^2}a_{\mathcal{P},1}$. It follows from our congruence condition of α that these translations are in $\alpha L/L$.

Our expression for \mathcal{B} implies a similar expression for \mathcal{A}_A :

$$\mathcal{A}_A = \sum_{l=1}^{m^2} \mathcal{S}(l),$$

where each $\mathcal{S}(l)$ is a translation of $\mathcal{S}(1)$ by an element of $\alpha L/L$. As both the set T and the function $f^{(\epsilon)}(\zeta\alpha^{-1}x)$ are invariant under such translations, we have

$$\begin{aligned} \sum_{x \in T} \text{ord}_x(\mathcal{A}_A) f(\zeta\alpha^{-1}x) &= \sum_{l=1}^{m^2} \sum_{x \in T} \text{ord}_x(\mathcal{S}(l)) f(\zeta\alpha^{-1}x) \\ &= m^2 \sum_{x \in T} \text{ord}_x(\mathcal{S}(1)) f(\zeta\alpha^{-1}x) \equiv 0 \pmod{m}. \end{aligned}$$

□

6.5 Proof of Lemma 27.

Choose a lattice $M \subset V_m$ containing L and $\alpha^{-1}L$. It is sufficient to show that for every $d-1$ -cell \mathcal{P} in \mathfrak{X} we have in \mathbb{Z}/m :

$$\{\mathcal{H}_{\alpha\beta,\alpha}^\alpha \bowtie \mathcal{P} | \alpha M\} = \{\mathcal{H}_\beta^1 \bowtie \mathcal{P} | M\}.$$

To prove this, we cut $\mathcal{H}_{\alpha\beta,\alpha}^\alpha \bowtie \mathcal{P}$ into 2^{d-1} pieces. One of the pieces will be precisely $\alpha \cdot \mathcal{H}_\beta^1 \bowtie \mathcal{P}$; for the other pieces \mathcal{A} we will show that $\{\mathcal{A} | M\} = 0$.

The various pieces of $\mathcal{H}_{\alpha\beta,\alpha}^\alpha \bowtie \mathcal{P}$ will be indexed by the subsets $A \subseteq \{1, 2, \dots, d-1\}$. Recall that $\mathcal{H}_{\alpha\beta,\alpha}^\alpha \bowtie \mathcal{P}$ is defined by

$$(\mathcal{H}_{\alpha\beta,\alpha}^\alpha \bowtie \mathcal{P})(t, x_1, \dots, x_{d-1}) = v_{\mathcal{P}} + \sum_{i=1}^{d-1} \mathcal{H}_{\alpha\beta,\alpha}^\alpha(t, x_i) \cdot a_{\mathcal{P},i}.$$

For any subset $A \subset \{1, 2, \dots, d-1\}$ we define

$$\mathcal{A}_A(t, x_1, \dots, x_{d-1}) = v_{\mathcal{P}} + \sum_{i \in A} \mathcal{H}_{\alpha\beta,\alpha}^\alpha(t, x_i/2) \cdot a_{\mathcal{P},i} + \sum_{i \notin A} \mathcal{H}_{\alpha\beta,\alpha}^\alpha(t, (x_i+1)/2) \cdot a_{\mathcal{P},i}.$$

As no element of αM lies in any set $|\partial\mathcal{A}_A|$, we have:

$$\{\mathcal{H}_{\alpha\beta,\alpha}^\alpha \bowtie \mathcal{P} | \alpha M\} = \sum_A \{\mathcal{A}_A | \alpha M\}.$$

We now examine the pieces \mathcal{A}_A . First consider $\mathcal{A}_{\{1,2,\dots,d-1\}}$. For $t, x \in I$ we have

$$\alpha^{-1} \cdot \mathcal{H}_{\alpha\beta,\alpha}^\alpha(t, x/2) = \mathcal{H}_\beta^1(t, x).$$

This implies

$$\alpha^{-1} \cdot \mathcal{A}_{\{1,2,\dots,d-1\}} = \mathcal{H}_\beta^1 \bowtie \mathcal{P}.$$

Therefore

$$\{\mathcal{A}_{\{1,2,\dots,d-1\}} | \alpha M\} = \{\mathcal{H}_\beta^1 \bowtie \mathcal{P} | M\}.$$

The lemma will now follow when we show that for any proper subset $A \subset \{1, 2, \dots, d-1\}$ we have $\{\mathcal{A}_A | \alpha M\} = 0$.

Fix a proper subset $A \subset \{1, 2, \dots, d-1\}$. Without loss of generality we have $1 \notin A$. Note that for $t, x \in I$ we have

$$\mathcal{H}_{\alpha\beta,\alpha}^\alpha(t, (x+1)/2) = \mathcal{M}(x), \quad \text{where } \mathcal{M}(x) = \wp(\alpha, (x+1)/2).$$

We therefore have

$$\mathcal{A}_A = (\mathcal{M} \cdot a_{\mathcal{P},1}) \diamond \mathcal{B},$$

for some $d-1$ -cell \mathcal{B} . We shall now use this fact to cut \mathcal{A}_A into m pieces, which are translations of each other by elements of M . We note that from the definition of $\wp(\alpha)$ we have

$$\mathcal{M}\left(x + \frac{1}{m}\right) = \mathcal{M}(x) + \frac{1-\alpha}{m}, \quad 0 \leq x \leq 1 - \frac{1}{m}.$$

We define for $x \in I$:

$$\mathcal{N}(x) = \mathcal{M}\left(\frac{x}{m}\right) \cdot a_{\mathcal{P},1}.$$

With this notation we have

$$\{\mathcal{A}_A | \alpha M\} = \sum_{i=0}^{m-1} \left\{ \left[i \frac{1-\alpha}{m} a_{\mathcal{P},1} \right] \diamond \mathcal{N} \diamond \mathcal{B} \Big| \alpha M \right\}.$$

To prove the lemma it only remains to show that all the terms in the above sum are equal. We have

$$\left\{ \left[i \frac{1-\alpha}{m} a_{\mathcal{P},1} \right] \diamond \mathcal{N} \diamond \mathcal{B} \Big| \alpha M \right\} = \left\{ \mathcal{N} \diamond \mathcal{B} \Big| \alpha M - i \frac{1-\alpha}{m} a_{\mathcal{P},1} \right\},$$

so we need only show that the translation $\frac{1-\alpha}{m}a_{\mathcal{P},1}$ is an element of the lattice αM . This follows from the congruence condition on α , the condition on M and the fact that $(1-\rho)a_{\mathcal{P},1} \in L$. \square

6.6 Proof of Lemma 28.

We must show that $\{\mathcal{H}_{\alpha\beta}^{\alpha\beta,\alpha} \bowtie \mathcal{G}|M\} = 0$ in \mathbb{Z}/m . However by Corollary 2 it is sufficient to prove this formula with the homotopy $\mathcal{H}_{\alpha\beta}^{\alpha\beta,\alpha}$ replaced by another homotopy \mathcal{H} from $\wp(\alpha\beta, \alpha)$ to $\wp(\alpha, \beta)$ as long as we have for x close to 0: $\mathcal{H}(t, x) = 2\alpha\beta x$. We shall choose such a homotopy \mathcal{H} for which the calculation is easier.

The homotopy \mathcal{H} . Before beginning we shall fix a parametrization of the path $\wp(\alpha\beta, \alpha)$ as follows:

$$\wp(\alpha\beta, \alpha)(x) = \begin{cases} \alpha\beta\wp(\epsilon, 2x) & x \leq \frac{1}{2}, \\ \alpha\wp^\beta(0, \nu, 2x - \frac{1}{2}) & \frac{1}{2} \leq x \leq \frac{3}{4}, \\ \wp^\alpha(0, \xi, 2x - 1) & x \geq \frac{3}{4}. \end{cases}$$

This has the advantage that it agrees with $\wp^{\alpha\beta}(x)$ for $x \leq \frac{1}{2}$. We fix ϵ, ξ and ν sufficiently small for our purposes, and we forget about these variables for the rest of the proof.

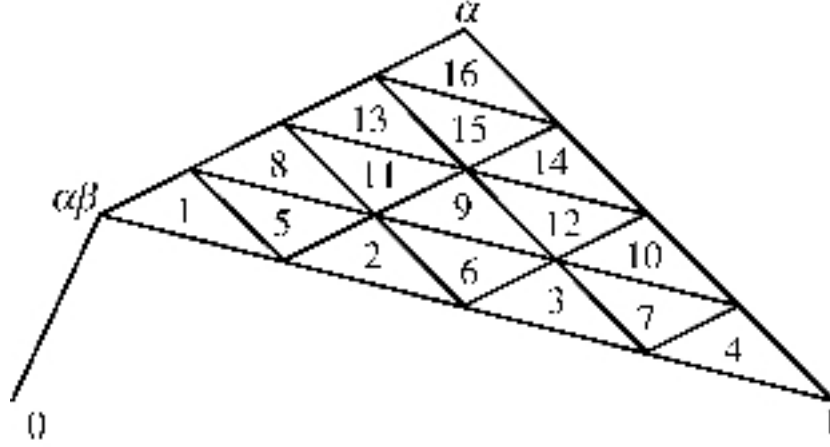
We introduce a sequence of auxiliary paths

$$\wp(\alpha\beta, \alpha) = \wp_0, \dots, \wp_{m^2} = \wp(\alpha\beta).$$

These are defined as follows. For any $i = 0, \dots, m^2$ we define

$$\wp_i(x) = 2x\alpha\beta, \quad x \leq \frac{1}{2}.$$

The triangle in \mathfrak{g} with vertices $\alpha\beta, \alpha$ and 1 is cut into m^2 similar triangles, or $4m^2$ if m is even. These similar triangles are numbered as in the following diagram.



We define φ_i to be the path from 0 to 1 in the diagram which passes below the triangles numbered $1, \dots, i$ and above the triangles numbered $i+1, \dots, m^2$. This means that paths φ_{i-1} and φ_i differ only by the i -th triangle. We shall parametrize these paths as follows. For any $x \in I$ which is not mapped into the edge of the i -th triangle we define $\varphi_{i-1}(x) = \varphi_i(x)$. Let $[a_i, b_i] \subset I$ be the subinterval mapped to the i -th triangle by both φ_i and φ_{i-1} . The interval $[a_i, b_i]$ will have length $\frac{1}{2m}$. One of the two paths will go around two edges of the triangle and the other will go around the third edge. We parametrize the path which goes around only one of the edges so that the path is affine there. The other path will be affine on the two subintervals $[a_i, a_i + \frac{1}{4m}]$ and $[a_i + \frac{1}{4m}, b_i]$, and will map $a_i + \frac{1}{4m}$ to the vertex of the triangle. We define homotopies $\mathcal{H}_1, \dots, \mathcal{H}_{m^2}$ as follows:

$$\mathcal{H}_i(x, t) = (1 - t)\varphi_{i-1}(x) + t\varphi_i(x).$$

Thus \mathcal{H}_i is a homotopy from φ_{i-1} to φ_i . Finally we put all these homotopies together to make the homotopy \mathcal{H} :

$$\mathcal{H}(x, t) = \mathcal{H}_{[mt+1]}(x, \{mt\}),$$

where $[\cdot]$ and $\{\cdot\}$ denote the integer part and fractional part respectively. This is a homotopy from $\varphi(\alpha\beta, \alpha)$ to $\varphi(\alpha\beta)$.

Calculation of $\{\mathcal{H} \rtimes \mathcal{G} | M\}$. Note that we have

$$\{\mathcal{H} \rtimes \mathcal{G} | M\} = \lim_{\epsilon \rightarrow 0^+} \lim_{\nu \rightarrow 0^+} \lim_{\xi \rightarrow 0^+} \lim_{\eta \rightarrow 0^+} \{\mathcal{H}(\epsilon, \nu, \xi, \eta) \rtimes \mathcal{G} | M\}.$$

We fix ϵ, ν, ξ, η so that all our functions are defined on M and equal to their limits.

It is sufficient to show that for any $d - 1$ cell \mathcal{P} in \mathfrak{X} we have in \mathbb{Z}/m :

$$\{\mathcal{H} \bowtie \mathcal{P}|M\} = 0.$$

We now fix such a \mathcal{P} .

Assume for a moment that m is odd. Recall that to construct the homotopy \mathcal{H} we used a sequence of paths \wp_0, \dots, \wp_{m^2} and homotopies $\mathcal{H}_1, \dots, \mathcal{H}_{m^2}$, where \mathcal{H}_i is a homotopy from \wp_{i-1} to \wp_i . We therefore have

$$\{\mathcal{H}|M\} = \sum_{i=1}^{m^2} \{\mathcal{H}_i \bowtie \mathcal{P}|M\}.$$

Each homotopy \mathcal{H}_i corresponds to one of the m^2 subtriangles T_1, \dots, T_{m^2} of the triangle with vertices $\alpha\beta, \alpha, 1$. Each of these triangles is either a translation of T_1 or a translation of T_2 . We shall prove that if T_i is a translation of T_j then we have

$$\{\mathcal{H}_i \bowtie \mathcal{P}|M\} \equiv \{\mathcal{H}_j \bowtie \mathcal{P}|M\} \pmod{m}. \quad (20)$$

As there are $\frac{m(m+1)}{2}$ triangles of type T_1 and $\frac{m(m-1)}{2}$ of type T_2 , this implies

$$\begin{aligned} \{\mathcal{H} \bowtie \mathcal{P}|M\} &\equiv \frac{m(m+1)}{2} \{\mathcal{H}_1 \bowtie \mathcal{P}|M\} + \frac{m(m-1)}{2} \{\mathcal{H}_2 \bowtie \mathcal{P}|M\} \\ &\equiv 0 \pmod{m}, \end{aligned}$$

which proves the result. This is the only place in which we need to assume that m is odd. In the case that m is even we must cut the large triangle into $4m^2$ subtriangles instead of just m^2 so $m(2m+1)$ of them are of type T_1 and $m(2m-1)$ are of type T_2 . This is why we need a slightly different congruence condition on α and β when m is even.

It remains only to prove the congruence (20). To do this we shall cut $\mathcal{H}_i \bowtie \mathcal{P}$ and $\mathcal{H}_j \bowtie \mathcal{P}$ into pieces. Some of the pieces of $\mathcal{H}_i \bowtie \mathcal{P}$ will be translated by elements of M to pieces of $\mathcal{H}_j \bowtie \mathcal{P}$, and so will cancel each other out. Any piece which does not cancel in this way will be the product of a line segment with length in $m \cdot M$ and a $d - 1$ -chain. Thus its contribution to (20) will vanish modulo m .

Without loss of generality we shall assume that T_i is a translation of T_1 . We begin by cutting the interval I onto four pieces. For x in the interval $[0, \frac{1}{2}]$ we have

$$\mathcal{H}_i(t, x) = \wp_i(x) = \alpha\beta\wp(\epsilon)(2x).$$

Let $[a_i, b_i]$ be the subinterval of $[\frac{1}{2}, 1]$ which is mapped by \wp_i and \wp_{i-1} to the triangle T_i . The other two pieces of I are $[\frac{1}{2}, a_i]$ and $[b_i, 1]$. It is possible that one of these two will be a single point. For x in the interval $[\frac{1}{2}, a_i]$ we have $\mathcal{H}_i(x, t) = \wp_i(x)$. In this region, \wp_i is a sum of line segments whose endpoints differ by $\frac{1-\alpha}{m}$, $\frac{1-\alpha\beta}{m}$ or $\frac{\alpha-\alpha\beta}{m}$. We define

$$\mathcal{U}_i(x) = \wp_i\left(\left(1-x\right)\frac{1}{2} + xa_i\right).$$

The region of \wp_i between b_i and 1 is similar and we define

$$\mathcal{U}'_i(x) = \wp_i((1-x)b_i + x).$$

Suppose $\{1, \dots, d-1\}$ is the disjoint union of the four sets A, B, C and D . We shall define $\mathcal{A}_i(A, B, C, D)$ to be the restriction of $\mathcal{H}_i \bowtie \mathcal{P}$ to the subset

$$I \times [0, \frac{1}{2}]^A \times [\frac{1}{2}, a_i]^B \times [a_i, b_i]^C \times [b_i, 1]^D \subset I^d.$$

In other words we have

$$\begin{aligned} \mathcal{A}_i(A, B, C, D)(t, x_1, \dots, x_{d-1}) &= v_{\mathcal{P}} + \sum_{j \in A} \alpha\beta\wp(\epsilon)(x_j) \cdot a_{\mathcal{P},j} \\ &\quad + \sum_{j \in B} \mathcal{U}_i(x_j) \cdot a_{\mathcal{P},j} \\ &\quad + \sum_{j \in C} \mathcal{H}_i((1-x_j)a_i + x_j b_i, t) \cdot a_{\mathcal{P},j} \\ &\quad + \sum_{j \in D} \mathcal{U}'_i(x_j) \cdot a_{\mathcal{P},j}. \end{aligned}$$

We have

$$\{\mathcal{H}_i \bowtie \mathcal{P} | M\} = \sum_{A, B, C, D} \{\mathcal{A}_i(A, B, C, D) | M\}.$$

It is sufficient to prove that for any choice of A, B, C and D we have

$$\{\mathcal{A}_i(A, B, C, D) | M\} = \{\mathcal{A}_1(A, B, C, D) | M\}.$$

To prove this we shall consider three cases.

Case 1. Suppose that B is non-empty and let $j \in B$. We can then decompose $\mathcal{A}_i(A, B, C, D)$ as

$$\mathcal{A}_i(A, B, C, D) = \mathcal{V} \diamond \mathcal{W},$$

where $\mathcal{V}(x) = \mathcal{U}_i(x) \cdot a_{\mathcal{F},j}$ and \mathcal{W} is a $d-1$ -chain. The cell \mathcal{V} is a sum of line segments whose length is in $m \cdot M$. It follows that $\{\mathcal{A}_i(A, B, C, D)|M\} \equiv 0 \pmod{m}$. Similarly $\{\mathcal{A}_1(A, B, C, D)|M\} \equiv 0 \pmod{m}$.

Case 2. Suppose that D is non-empty. We may reason as in case 1 to show that both $\{\mathcal{A}_i(A, B, C, D)|M\}$ and $\{\mathcal{A}_1(A, B, C, D)|M\}$ are congruent to 0 modulo m .

Case 3. Suppose B and D are empty. We shall prove that $\mathcal{A}_i(A, B, C, D)$ is a translation of $\mathcal{A}_1(A, B, C, D)$ by an element of M . We shall assume without loss of generality that $C = \{1, \dots, r\}$ and $A = \{r+1, \dots, d-1\}$. We may then decompose $\mathcal{A}_i(A, B, C, D)$ as follows:

$$\mathcal{A}_i(A, B, C, D) = \mathcal{B}_i \diamond \mathcal{C},$$

where $\mathcal{B}_i : I^{r+1} \rightarrow X_\infty$ is given by

$$\mathcal{B}_i(t, x_1, \dots, x_r) = \sum_{j=1}^r \mathcal{H}_i(t, (1-x_j)a_i + x_j b_i) \cdot a_{\mathcal{P},j}$$

and $\mathcal{C} : I^{d-r-1} \rightarrow X_\infty$ is given by:

$$\mathcal{C}(\underline{x}) = \sum_{j=1}^{d-r-1} \alpha \beta \varphi(\epsilon, x_j) \cdot a_{\mathcal{P},j}.$$

It is therefore sufficient to prove that \mathcal{B}_i is a translation of \mathcal{B}_1 by an element of M . Recall that the triangle T_i is a translation of T_1 by some vector $v \in \mathfrak{g}$. Furthermore v is of the form

$$v = r \frac{\alpha\beta - 1}{m} + s \frac{\alpha - 1}{m}, \quad r, s \in \mathbb{Z}.$$

It follows that the restriction of \mathcal{H}_i to $[a_i, b_i]$ and the restriction of \mathcal{H}_1 to $[a_1, b_1]$ also differ by the translation v . In other words we have for $x \in I$,

$$\mathcal{H}_i(t, (1-x)a_i + xb_i) = \mathcal{H}_1(t, (1-x)a_1 + xb_1) + v.$$

This implies

$$\mathcal{B}_i(t, x_1, \dots, x_r) = \mathcal{B}_1(t, x_1, \dots, x_r) + v \sum_{j=1}^r a_{\mathcal{P},j}.$$

It remains to show that the translations $v \cdot a_{\mathcal{P},j}$ are in M . This reduces to showing that $\frac{\alpha-1}{m}a_{\mathcal{P},j}$ and $\frac{\alpha\beta-1}{m}a_{\mathcal{P},j}$ are in M . However this is true by our congruence conditions on α and β , the fact that $(1-\rho)a_{\mathcal{P},j} \in L$ and the assumption that M contains L , αL and $\alpha\beta L$. \square

7 Extending the cocycle to $\mathrm{GL}_n(\mathbb{A})$.

7.1 The cocycle on $\mathrm{SL}_n(k_m)$.

Let k_m be the sum of the fields k_v for all finite places v dividing m . We therefore have $\mathbb{A} = \mathbb{A}(S) \oplus k_\infty \oplus k_m$. So far, we have a cocycle $\mathrm{Dec}_{\mathbb{A}(S)}$ on $\mathrm{GL}_n(\mathbb{A}(S))$ and a cocycle Dec_∞ on $\mathrm{GL}_n(k_\infty)$. We shall now find a continuous cocycle Dec_m on $\mathrm{SL}_n(k_m)$ so that $\mathrm{Dec}_{\mathbb{A}(S)}\mathrm{Dec}_\infty\mathrm{Dec}_m$ is metaplectic on $\mathrm{SL}_n(\mathbb{A})$.

Extending τ . Recall that $G_{\mathfrak{f}}$ is the subgroup of $\mathrm{GL}_n(k)$ consisting of matrices which are integral at all places dividing m and congruent to the identity modulo \mathfrak{f} . This is the subgroup generated by the semigroup $\Upsilon_{\mathfrak{f}}$. We have a function $\tau : \Upsilon_{\mathfrak{f}} \rightarrow \mu_m$ such that for $\alpha, \beta \in \Upsilon_{\mathfrak{f}}$ the following holds:

$$\mathrm{Dec}_{\mathbb{A}(S)}(\alpha, \beta)\mathrm{Dec}_\infty(\alpha, \beta) = \partial\tau(\alpha, \beta). \quad (21)$$

For any matrix $\alpha \in G_{\mathfrak{f}}$ there is a natural number N such that $\alpha/N \in \Upsilon_{\mathfrak{f}}$. We extend τ to a function on $G_{\mathfrak{f}}$ by defining

$$\tau(\alpha) = \frac{\mathrm{Dec}_{\mathbb{A}(S)}(N, N^{-1}\alpha)\mathrm{Dec}_\infty(N, N^{-1}\alpha)}{\mathrm{Dec}_{\mathbb{A}(S)}(N, N^{-1})\mathrm{Dec}_\infty(N, N^{-1})}\tau(N^{-1}\alpha), \quad \alpha \in G_{\mathfrak{f}},$$

where $N \in \mathbb{N}$ is chosen so that $N^{-1}\alpha \in \Upsilon_{\mathfrak{f}}$. It follows from the cocycle relation and the fact that $\tau(N^{-1}) = 1$ for $N \in \mathbb{N} \cap \Upsilon^{-1}$, that this does not depend on our choice of N . Furthermore it is easy to check that on the whole group $G_{\mathfrak{f}}$ the formula (21) still holds.

We extend τ to a function on $\mathrm{GL}_n(k)$. To do this we choose a set Rep of representatives for cosets $\mathrm{GL}_n(k)/G_{\mathfrak{f}}$. Thus every element of $\mathrm{GL}_n(k)$ may be uniquely expressed in the form $r\alpha$ with $r \in \mathrm{Rep}$ and $\alpha \in G_{\mathfrak{f}}$. We define

$$\tau(r\alpha) = \mathrm{Dec}_{\mathbb{A}(S)}(r, \alpha)\mathrm{Dec}_{\infty}(r, \alpha)\tau(\alpha).$$

The cocycle Dec_m . We define the cocycle Dec_m on the dense subgroup $\mathrm{SL}_n(k)$ of $\mathrm{SL}_n(k_m)$ by

$$\mathrm{Dec}_m(\alpha, \beta) = \frac{\partial\tau(\alpha, \beta)}{\mathrm{Dec}_{\mathbb{A}(S)}(\alpha, \beta)\mathrm{Dec}_{\infty}(\alpha, \beta)}. \quad (22)$$

We shall prove that Dec_m extends to a continuous cocycle on $\mathrm{SL}_n(k_m)$. We then define for $\alpha, \beta \in \mathrm{SL}_n(\mathbb{A})$

$$\mathrm{Dec}_{\mathbb{A}}(\alpha, \beta) = \mathrm{Dec}_{\mathbb{A}(S)}(\alpha, \beta)\mathrm{Dec}_{\infty}(\alpha, \beta)\mathrm{Dec}_m(\alpha, \beta).$$

It follows immediately from the definition (22) that the restriction of $\mathrm{Dec}_{\mathbb{A}}$ to $\mathrm{SL}_n(k)$ is $\partial\tau$. Therefore $\mathrm{Dec}_{\mathbb{A}}$ is metaplectic. It remains to prove the following.

Theorem 6 *The cocycle Dec_m on $\mathrm{SL}_n(k)$ extends by continuity to $\mathrm{SL}_n(k_m)$.*

Proof. It follows immediately from the definitions that for $\alpha, \beta \in \mathrm{SL}_n(k)$ and $\epsilon \in G_{\mathfrak{f}}$ we have

$$\mathrm{Dec}_m(\alpha, \beta\epsilon) = \mathrm{Dec}_m(\alpha, \beta).$$

It is therefore sufficient to prove that for any $\beta \in \mathrm{SL}_n(k)$, the function $\alpha \mapsto \mathrm{Dec}_m(\alpha, \beta)$ is continuous. We fix β . Note that for $\epsilon \in G_{\mathfrak{f}} \cap (\beta G_{\mathfrak{f}} \beta^{-1})$ we have from the cocycle relation:

$$\mathrm{Dec}_m(\alpha\epsilon, \beta) = \mathrm{Dec}_m(\alpha, \beta)\mathrm{Dec}_m(\epsilon, \beta).$$

In particular the map $\psi : G_{\mathfrak{f}} \cap \beta G_{\mathfrak{f}} \beta^{-1} \rightarrow \mu_m$ given by $\psi(\epsilon) = \mathrm{Dec}_m(\epsilon, \beta)$ is a homomorphism. To prove continuity we need only show that $\ker(\psi) \cap \mathrm{SL}_n(k)$ is open in the induced topology from $\mathrm{SL}_n(k_m)$. However this fact follows immediately from Lemma 29 below. \square

Remark 4 *It is worth noting that $\ker(\psi)$ is not open in the topology induced by $\mathrm{GL}_n(k_m)$ and Dec_m does not extend by continuity to $\mathrm{GL}_n(k_m)$.*

A weak form of the congruence subgroup problem. Let S_m be the set of finite primes in S and define $R = \{x \in k : \forall v \in S_m |x|_v \leq 1\}$. This is a subring of k whose primes correspond to the elements of S_m . The ring R is a dense subring of $\mathfrak{o}_m := \bigoplus_{v \in S_m} \mathfrak{o}_v$.

Lemma 29 *Every subgroup H of finite index in $\mathrm{SL}_n(R)$ is a congruence subgroup, ie. H contains all matrices congruent to the identity modulo some non-zero ideal of R . Equivalently $\mathrm{SL}_n(\mathfrak{o}_m)$ is the profinite completion of $\mathrm{SL}_n(R)$.*

Remark 5 *This is a very weak statement, which follows immediately from [4] (or [28] for $n = 2$). However since these papers effectively construct the universal metaplectic cover of SL_n , it is worth noting that the limited result required here can be obtained in an elementary way.*

Proof. Let H be a subgroup of $\mathrm{SL}_n(R)$ of index d . We shall assume without loss of generality that m divides d and that H is normal in $\mathrm{SL}_n(R)$. We shall show that any matrix congruent to the identity modulo d^2R is in H . First note that the d -th power of any element of $\mathrm{SL}_n(R)$ is in H . Therefore H contains the elementary matrices

$$I_n + \lambda d e_{i,j}, \quad \lambda \in R, \quad i \neq j.$$

Here $e_{i,j}$ denotes the matrix whose (i, j) -entry is 1 and whose other entries are all zero.

By a d -operation we shall mean an operation of the form “add λd times row i to row j ” ($i \neq j, \lambda \in R$). If a matrix can be reduced by d -operations to the identity matrix then that matrix must be in H since d -operations have the effect of multiplying on the left by $I_n + \lambda d e_{i,j}$.

Now let $A = (a_{i,j})$ be any matrix congruent to the identity modulo d^2R . We shall show that A may be reduced to the identity matrix by d -operations. Since m divides d^2 , the entries $a_{i,i}$ on the diagonal of A are units in R (they are congruent to 1 modulo every prime ideal of R). The entries off the diagonal are divisible by d^2 . Therefore we can reduce A by d^2 -operations to a diagonal matrix. Furthermore the diagonal matrix which we obtain will still be congruent to the identity modulo d^2 .

Now let A be diagonal. It remains to show how A can be reduced to the identity. We first describe a method for converting $a_{i,i}$ to 1. For $i < n$ we may add $da_{i,i}^{-1}$ times row i to row $i + 1$. This gives us a d in the $(i + 1, i)$

entry. Next subtracting $d^{\frac{a_{i,i}-1}{d^2}}$ times row $i + 1$ from row i we obtain a 1 in the (i, i) entry. After this we subtract d times row i from row $i + 1$ to obtain a zero there. Finally, subtracting a multiple of row $i + 1$ from row i we obtain a diagonal matrix with a 1 in the (i, i) position. In this process we have only changed the (i, i) and $(i + 1, i + 1)$ -entries. We may perform this process for $i = 1, 2, \dots, n - 1$ consecutively to obtain a diagonal matrix with $a_{i,i} = 1$ for $i = 1, 2, \dots, n - 1$. Since the resulting matrix has determinant 1, it follows that $a_{n,n}$ is also 1. \square

A product formula for $\text{Dec}_{\mathbb{A}}$. For any place v of k , let Dec_v be the restriction of $\text{Dec}_{\mathbb{A}}$ to $\text{SL}_n(k_v)$. It is known (see [10, 11]) that for almost all places v the cocycle Dec_v is trivial on $\text{GL}_n(\mathfrak{o}_v)$. Therefore the product $\prod_v \text{Dec}_v(\alpha, \beta)$ converges and we have up to a coboundary¹

$$\text{Dec}_{\mathbb{A}(S)}(\alpha, \beta) = \prod_{v \notin S} \text{Dec}_v(\alpha, \beta), \quad \alpha, \beta \in \text{SL}_n(\mathbb{A}(S)).$$

As the groups $\text{SL}_n(k_v)$ ($v \in S$) are perfect, the restriction maps give an isomorphism:

$$H^2(\text{SL}_n(k_S), \mu_m) \cong \bigoplus_{v \in S} H^2(\text{SL}_n(k_v), \mu_m).$$

(The corresponding statement for GL_n would be false). This implies up to a coboundary on $\text{SL}_n(\mathbb{A})$:

$$\text{Dec}_{\mathbb{A}} = \prod_v \text{Dec}_v.$$

7.2 Application : The power reciprocity law.

We shall now deduce the power reciprocity law from our results.

Consider the cocycle $\text{Dec}_{\mathbb{A}}$ on $\text{SL}_3(\mathbb{A})$. For $\alpha \in \mathbb{A}^\times$ we define

$$\varphi(\alpha) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi'(\alpha) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix},$$

¹A formula for this coboundary is given in [10].

One knows (see [10] Lemma 23) that for $\alpha, \beta \in \mathbb{A}^\times$ we have

$$[\varphi(\alpha), \varphi'(\beta)]_{\text{Dec}_{\mathbb{A}(S)}} = \prod_{v \notin S} (\alpha, \beta)_{v,m}.$$

On the other hand if $\alpha, \beta \in k^\times$, then since $\text{Dec}_{\mathbb{A}}$ is metaplectic we have

$$[\varphi(\alpha), \varphi'(\beta)]_{\text{Dec}_{\mathbb{A}}} = 1.$$

This implies for $\alpha, \beta \in k^\times$,

$$\prod_{v \notin S} (\alpha, \beta)_{v,m} \prod_{v \in S} [\varphi(\alpha), \varphi'(\beta)]_{\text{Dec}_v} = 1.$$

Fix a place $v \in S$ and consider the function $\psi : k_v^\times \times k_v^\times \rightarrow \mu_m$ defined by

$$\psi(\alpha, \beta) = [\varphi(\alpha), \varphi'(\beta)]_{\text{Dec}_v},$$

where Dec_v is the restriction of $\text{Dec}_{\mathbb{A}}$ to $\text{SL}_3(k_v)$. To prove the reciprocity law it remains to show that ψ is the m -th power Hilbert symbol.

For real v this is a consequence of Corollary 1 (§3.8). For complex v both ψ and the Hilbert symbol are trivial for topological reasons. Assume from now on that v is a non-archimedean place dividing m . The function ψ is bimultiplicative and continuous by the properties of commutators. Furthermore we have $\psi(\alpha, 1 - \alpha) = 1$ for all $\alpha \neq 1$ since this formula holds on the dense subset $k^\times \setminus \{0\}$. Therefore ψ is a continuous Steinberg symbol. As the Hilbert symbol is the universal continuous Steinberg symbol on k_v we have $\psi(\alpha, \beta) = (\alpha, \beta)_{m,v}^a$ for some fixed $a \in \mathbb{Z}/m\mathbb{Z}$. Substituting $\alpha = \zeta \in \mu_m$, $\beta \in \mathfrak{o}_v^\times$ we have (see for example [29] XIV Proposition 6):

$$(\zeta, \beta)_{m,v} = \zeta^{\frac{1 - N_{\mathbb{Q}_v}^{k_v}(\beta)}{m}}$$

Taking $\beta \in \mathfrak{o}$ close to 1 in the topology of \mathfrak{o}_w for all $w \in S \setminus \{v\}$ we have

$$\psi(\zeta, \beta) = [\varphi(\zeta), \varphi'(\beta)]_{\text{Dec}_{\mathbb{A}(S)}}^{-1}.$$

Proposition 2 of [10] implies

$$[\varphi(\zeta), \varphi'(\beta)]_{\text{Dec}_{\mathbb{A}(S)}}^{-1} = \zeta^{\frac{1 - N_{\mathbb{Q}}^k(\beta)}{m}} = (\zeta, \beta)_{m,v}.$$

With a suitable choice of ζ, β this shows that $a = 1$. □

As was mentioned in the introduction, it would be more satisfactory to have a local definition of Dec_v for $v \in S_m$ and a local proof that ψ is the Hilbert symbol.

7.3 Extending the cocycle to GL_n .

We now have a metaplectic cocycle $\text{Dec}_{\mathbb{A}}$ on $SL_n(\mathbb{A})$, whose restriction to $SL_n(\mathbb{A}(S) \oplus k_{\infty})$ extends naturally to $GL_n(\mathbb{A}(S) \oplus k_{\infty})$. One might ask whether $\text{Dec}_{\mathbb{A}(S)}\text{Dec}_{\infty}$ extends to a metaplectic cocycle on $GL_n(\mathbb{A})$. The answer to this depends on precisely how one poses the question. If one asks whether there is a cocycle Dec_m on $GL_n(k_m)$, which when multiplied together with $\text{Dec}_{\mathbb{A}(S)}$ and Dec_{∞} gives a metaplectic cocycle then the answer is “no”. However it is true that $\text{Dec}_{\mathbb{A}(S)}\text{Dec}_{\infty}$ is the restriction of a metaplectic cocycle $\text{Dec}_{\mathbb{A}}$ on $GL_n(\mathbb{A})$.

The change of base field property. By embedding the group GL_n in SL_{n+1} , one can obtain a perfectly good metaplectic extension of GL_n by restriction. In fact, this is the metaplectic extension which has been most studied. However, such extensions are badly behaved under change of base field (see [25]) compared with $\text{Dec}_{\mathbb{A}(S)}\text{Dec}_{\infty}$. For this reason, I shall extend $\text{Dec}_{\mathbb{A}(S)}\text{Dec}_{\infty}$ to $GL_n(\mathbb{A})$ to obtain an extension which is well behaved under change of base field.

More precisely, if l is a finite extension of k of degree d then by choosing a basis for l over k we may regard $GL_n(\mathbb{A}_l)$ as a subgroup of $GL_{nd}(\mathbb{A}_k)$. With this identification $GL_n(l)$ is a subgroup of $GL_{nd}(k)$. Thus metaplectic extensions of GL_{nd}/k restrict to metaplectic extensions of GL_n/l . We shall write R the restriction map from GL_{nd}/k to GL_n/l . The classes $\text{Dec}_{\mathbb{A}(S)}$ and Dec_{∞} have the following “change of base field property”:

$$R(\text{Dec}_{\mathbb{A}(S)}^{(k)}) = \text{Dec}_{\mathbb{A}(S)}^{(l)}, \quad R(\text{Dec}_{\infty}^{(k)}) = \text{Dec}_{\infty}^{(l)}.$$

This is clear since the base field never arises in the definitions of $\text{Dec}_{\mathbb{A}(S)}$ or Dec_{∞} . It is known that the class on GL_n obtained by restricting from SL_{n+1} does not have the change of base field property (see for example [25]). We shall extend $\text{Dec}_{\mathbb{A}(S)}\text{Dec}_{\infty}$ to $GL_n(\mathbb{A})$ in such a way that it does have this property. To achieve this it is clearly sufficient to treat the case $k = \mathbb{Q}(\mu_m)$. From now on we shall restrict ourselves to this case.

The metaplectic kernel of GL_n . The group GL_n is the semi-direct product of SL_n and GL_1 . The normal subgroups $SL_n(\mathbb{A})$ and $SL_n(k)$ are perfect, i.e. they are equal to their own commutator subgroups. Therefore the Hochschild–Serre spectral sequence shows that the restriction maps give

isomorphisms:

$$H^2(\mathrm{GL}_n(\mathbb{A}), \mu_m) \cong H^2(\mathrm{SL}_n(\mathbb{A}), \mu_m) \oplus H^2(\mathrm{GL}_1(\mathbb{A}), \mu_m),$$

$$H^2(\mathrm{GL}_n(k), \mu_m) \cong H^2(\mathrm{SL}_n(k), \mu_m) \oplus H^2(\mathrm{GL}_1(k), \mu_m).$$

For an algebraic group G we shall write $\mathcal{M}(G, \mu_m)$ for the kernel of the restriction map $H^2(G(\mathbb{A}), \mu_m) \rightarrow H^2(G(k), \mu_m)$. The above isomorphisms imply that we have

$$\mathcal{M}(\mathrm{GL}_n, \mu_m) \cong \mathcal{M}(\mathrm{SL}_n, \mu_m) \oplus \mathcal{M}(\mathrm{GL}_1, \mu_m).$$

We have already constructed a metaplectic extension of SL_n , so to show that our cocycle extends to a metaplectic cocycle of GL_n we need only show that its restriction to $\mathrm{GL}_1(\mathbb{A}(S) \oplus k_\infty)$ extends to a metaplectic cocycle on $\mathrm{GL}_1(\mathbb{A})$. However by Theorem 5 and [10], the restriction of $\mathrm{Dec}_{\mathbb{A}(S)} \mathrm{Dec}_\infty$ to GL_1 is simply the cocycle $\mathrm{Dec}_{\mathbb{A}(S)} \mathrm{Dec}_\infty$ constructed in the case $n = 1$.

We describe the group $H^2(\mathbb{A}^\times, \mu_m)$. For $\sigma \in H^2(\mathbb{A}^\times, \mu_m)$ the commutator of σ is a continuous bimultiplicative, skew symmetric function $\mathbb{A}^\times \times \mathbb{A}^\times \rightarrow \mu_m$. We therefore have a map

$$H^2(\mathbb{A}^\times, \mu_m) \rightarrow \mathrm{Hom}(\wedge^2 \mathbb{A}^\times, \mu_m).$$

This map is surjective. We write $H_{sym}^2(\mathbb{A}^\times, \mu_m)$ for its kernel. There is an isomorphism given by the restriction maps (see [16]):

$$H_{sym}^2(\mathbb{A}^\times, \mu_m) \cong \bigoplus_v H^2(\mu_m(k_v), \mu_m).$$

Each of the groups $H^2(\mu_m(k_v), \mu_m)$ is canonically isomorphic to \mathbb{Z}/m . We write $H_{asym}^2(\mathbb{A}^\times, \mu_m)$ for the kernel of the restriction map $H^2(\mathbb{A}^\times, \mu_m) \rightarrow \bigoplus_v H^2(\mu_m(k_v), \mu_m)$. Thus the commutator gives an isomorphism:

$$H_{asym}^2(\mathbb{A}^\times, \mu_m) \cong \mathrm{Hom}(\wedge^2 \mathbb{A}^\times, \mu_m),$$

and we have a decomposition

$$H^2(\mathbb{A}^\times, \mu_m) = H_{sym}^2(\mathbb{A}^\times, \mu_m) \oplus H_{asym}^2(\mathbb{A}^\times, \mu_m).$$

There is a similar decomposition of $H^2(k^\times, \mu_m)$:

$$\begin{aligned} H^2(k^\times, \mu_m) &= H_{sym}^2(k^\times, \mu_m) \oplus H_{asym}^2(k^\times, \mu_m), \\ H_{sym}^2(k^\times, \mu_m) &\cong H^2(\mu_m(k), \mu_m) \cong \mathbb{Z}/m\mathbb{Z}, \\ H_{asym}^2(k^\times, \mu_m) &\cong \mathrm{Hom}(\wedge^2 k^\times, \mu_m). \end{aligned}$$

Consider the restriction map $H^2(\mathbb{A}^\times, \mu_m) \rightarrow H^2(k^\times, \mu_m)$. Clearly the restriction to k^\times of a symmetric cocycle on \mathbb{A}^\times is symmetric². The resulting map $H_{sym}^2(\mathbb{A}^\times, \mu_m) \rightarrow H^2(k^\times, \mu_m)$ corresponds to the map $\bigoplus_v \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ given by

$$(a_v) \mapsto \sum_v a_v.$$

We now examine the commutator of $\text{Dec}_{\mathbb{A}(S)} \text{Dec}_\infty$. Under our conditions on k the group $\text{Hom}(\wedge^2 k_\infty^\times, \mu_m)$ is trivial, so Dec_∞ has trivial commutator. Therefore the commutator of $\text{Dec}_\infty \text{Dec}_{\mathbb{A}(S)}$ is the same as the commutator of $\text{Dec}_{\mathbb{A}(S)}$. This has been calculated in [10] (Theorems 4 and 5) and is given by

$$[\alpha, \beta]_{\mathbb{A}(S)} = \begin{cases} (-1)^{\frac{(|\alpha|_{\mathbb{A}(S)} - 1)(|\beta|_{\mathbb{A}(S)} - 1)}{m^2}} \prod_{v \notin S} (\alpha, \beta)_{v,m} & \text{if } m \text{ is even,} \\ \prod_{v \notin S} (\alpha, \beta)_{v,m} & \text{if } m \text{ is odd.} \end{cases}$$

Define a map $\chi_k : \mathbb{A}_k^\times / k^\times \rightarrow \mathbb{Z}_2^\times$ by

$$\chi_k(\alpha) = \chi_{\mathbb{Q}}(N_{\mathbb{Q}}^k \alpha), \quad \chi_{\mathbb{Q}}(\alpha) = \text{sign}(\alpha_\infty) \alpha_2 \prod_{p \text{ finite}} |\alpha|_p.$$

Define also a bilinear map $\psi_k : (\mathbb{A}^\times / k^\times) \times (\mathbb{A}^\times / k^\times) \rightarrow \mu_m$ by

$$\psi_k(\alpha, \beta) = (-1)^{\frac{(\chi_k(\alpha) - 1)(\chi_k(\beta) - 1)}{m^2}}.$$

The point of this is that for $\alpha, \beta \in \mathbb{A}(S)^\times$ we have

$$\psi_k(\alpha, \beta) = (-1)^{\frac{(|\alpha|_{\mathbb{A}(S)} - 1)(|\beta|_{\mathbb{A}(S)} - 1)}{m^2}}.$$

Theorem 7 *There is a unique class $\text{Dec}_{\mathbb{A}} \in \mathcal{M}(\text{GL}_1/\mathbb{Q}(\mu_m), \mu_m)$ with the following properties:*

- *the restriction of $\text{Dec}_{\mathbb{A}}$ to $\mathbb{A}(S)^\times$ is $\text{Dec}_{\mathbb{A}(S)}$;*
- *the restriction of $\text{Dec}_{\mathbb{A}}$ to k_∞^\times is Dec_∞ ;*

²However the restriction of an asymmetric cocycle is not necessarily asymmetric.

- The commutator of $\text{Dec}_{\mathbb{A}}$ is

$$[\alpha, \beta]_{\text{Dec}_{\mathbb{A}}} = \psi_{\mathbb{Q}(\mu_m)}(\alpha, \beta) \prod_v (\alpha, \beta)_{m,v}.$$

Proof. It follows from the above discussion that there is a unique class with the given commutator and restrictions to $\mathbb{A}(S)^\times$ and k_∞^\times and any given symmetric restriction $\mu_m(k_m)$. As m is a power of a prime, k_m is a field, so there is a unique choice of restriction to $\mu_m(k_m)$ for which the restriction to k is asymmetric. \square

References

- [1] E. Artin and J. Tate, *Class Field Theory*, W. A. Benjamin 1967.
- [2] M. F. Atiyah and C. T. C. Wall, “Cohomology of Groups” in *Algebraic Number Theory* ed. J.W.S.Cassels and A.Fröhlich, Academic Press 1967.
- [3] W. D. Banks, J. Levy, M. R. Sepanski, “Block-compatible metaplectic cocycles”, *J. Reine Angew. Math.* 507 (1999), 131-163.
- [4] H. Bass, J. Milnor and J.-P. Serre, “Solution of the congruence subgroup problem for $\text{SL}_n(n \geq 3)$ and $\text{Sp}_{2n}(n \geq 2)$ ”, *Publ. Math. I.H.E.S.* no 33 (1967) 59-137.
- [5] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics 55, Cambridge University Press 1996.
- [6] C. F. Gauß, *Disquisitiones Arithmeticae*, Art. 133, Werke Band I, S.101.
- [7] C. F. Gauß, “Zur Theorie der biquadratischen Reste”, *Werke*, Band 2, 313-385.
- [8] W. Habicht, “Ein elementarer Beweis des kubischen Reziprozitätsgesetzes”, *Math. Annalen* 139 (1959-60) 343-365.
- [9] R. Hill, “A geometric proof of a reciprocity law”, *Nagoya Math. Journ.* 137 (1995) 77-144.
- [10] R. Hill, “Space forms and Higher Metaplectic Groups”, *Math. Annalen.* 310 (1998) 735-775.

- [11] R. Hill, “Metaplectic Covers of GL_n and the Gauss-Schering Lemma” *Journal de Théorie des Nombres de Bordeaux* 13 (2001), 189-199.
- [12] A. J. Hahn and O. T. O’Meara, *The Classical Groups and K-Theory* Grundlehren der mathematischen Wissenschaften 291, Springer-Verlag Berlin Heidelberg New York 1989.
- [13] H. Ito, “Dedekind sums and quadratic residue symbols”, *Nagoya Math. J.* 118, 35-43 (1990).
- [14] D. Kazhdan and S. J. Patterson, “Metaplectic forms”, *Publ. Math. I.H.E.S.* 59 (1984), 35-142 + Corrigendum *Ibid* 62 (1985) 149.
- [15] D. Kazhdan, S. J. Patterson, “Towards a generalized Shimura Correspondence” *Publ. Math. I.H.E.S.*,
- [16] J. Klose, “Metaplektische Erweiterungen von Quaternionenschiefkörpern”, *Math. Zeit.* 193 (1986), 625-649.
- [17] J. Klose, “Kommutatoren und 2-Cohomologie \mathfrak{p} -adischer Quaternionenschiefkörper”, *Math. Zeit.* 191 (1986), 261-282.
- [18] T. Kubota, “Ein arithmetischer Satz über eine Matrizen­gruppe”, *J. für die reine u. angew. Math* 222 (1965) 55-57.
- [19] T. Kubota, “Geometry of numbers and class field theory”, *Japan J. Math.* vol.13, no.2 (1987) 235-275.
- [20] T. Kubota, “Foundations of Class Field Theory through properties of space figures”, (in Japanese) *Sugaka* 44 1 (1992) 1-12.
- [21] W. S. Massey, *Singular Homology Theory*, Graduate Texts in Mathematics, Springer-Verlag.
- [22] H. Matsumoto, “Sur les sous-groupes arithmétiques des groupes semi-simples déployés”, *Ann. scient., Ec. Norm. Sup.* (1969), 1-62.
- [23] J. Milnor, *Introduction to Algebraic K-Theory*, *Annals of Mathematics Studies* 72 (1971).
- [24] C. C. Moore, “Group extensions of p -adic and adelic linear groups”, *Publ. Math. I.H.E.S.* 35 (1969), 5-70.

- [25] S. J. Patterson, “Tori in metaplectic covers of GL_2 and applications to a formula of Loxton-Matthews”, *Math. Proc. Camb. Phil. Soc.* 100 (1986), 249-263.
- [26] G. Prasad and M. S. Raghunathan, “On the Congruence Subgroup Problem: Determination of the ‘Metaplectic Kernel’”, *Invent. Math.* 71 (1983), 21-42.
- [27] E. Schering, “Verallgemeinerung des Gaußischen Criterium für den quadratischen Rest-charakter einer Zahl in Bezug auf eine andere”, *Werke Band I*, 285-286.
- [28] J.-P. Serre, “Le problème des groupes de congruence pour SL_2 ”, *Ann. of Math.* 92 (1970) 489-527.
- [29] J.-P. Serre, *Local Fields*, Springer Verlag Graduate Texts in Mathematics 67, 1979.
- [30] C. Weibel, *An introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, Cambridge 1994.
- [31] A. Weil, “Sur certains groupes d’opérateurs unitaires”, *Acta Math.* 111 (1964) 143-211.
- [32] G. Zolotareff. Nouvelle démonstration de la loi de de réciprocité de Legendre. *Nouvelles Annales de Mathématiques, 2e série*, 11:354–362, 1872.