



Extension operators for trimmed spline spaces

Erik Burman^a, Peter Hansbo^b, Mats G. Larson^c, Karl Larsson^{c,*}

^a *Mathematics, University College London, UK*

^b *Mechanical Engineering, Jönköping University, Sweden*

^c *Mathematics and Mathematical Statistics, Umeå University, Sweden*

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Abstract

We develop a discrete extension operator for trimmed spline spaces consisting of piecewise polynomial functions of degree p with k continuous derivatives. The construction is based on polynomial extension from neighboring elements together with projection back into the spline space. We prove stability and approximation results for the extension operator. Finally, we illustrate how we can use the extension operator to construct a stable cut isogeometric method for an elliptic model problem. © 2022 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

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1. Introduction

Contributions. Consider a function space of C^k piecewise polynomial splines defined on a background mesh. Let Ω be a domain embedded into the background mesh without requiring that boundary matches the mesh leading to so-called cut or trimmed elements in the vicinity of the boundary. The span of every basis function that has a support that intersects Ω form the so-called active spline space associated with Ω . Due to the presence of the cut elements, this basis is typically very ill-conditioned since the intersection of the domain and the support of some basis functions may be very small [1,2]. We, therefore, introduce an extension operator that expresses the ill-conditioned degrees of freedom in terms of the well-conditioned degrees of freedom in the interior of the domain in a stable and accurate way. Such a discrete extension operator has many uses, for instance, handling of trimmed elements [3] in isogeometric analysis [4] in a robust manner and as an alternative to adding stabilization terms to the weak form in cut and immersed finite elements, see [5–7].

To keep the presentation as simple as possible we consider C^k tensor product splines but the construction is very general and allows local refinement as well as splines on triangulations. The basic idea is to split the mesh into elements with a large respectively a small intersection with the domain Ω . To each small element, we associate a large element in a neighborhood with a size proportional to the local mesh parameter and we then use the polynomial extension from the associated large element to the small element. This preliminary extension manufactures a function, which may be discontinuous at the faces belonging to small elements. We then project back

* Corresponding author.

E-mail addresses: e.burman@ucl.ac.uk (E. Burman), peter.hansbo@ju.se (P. Hansbo), mats.larson@umu.se (M.G. Larson), karl.larsson@umu.se (K. Larsson).

to the spline space using an interpolation operator. This construction enables us to handle very general situations and to prove that the extension is stable and has optimal approximation properties in a systematic way.

Previous work. Extension operators for discontinuous spaces were constructed and analyzed in [8], for continuous spaces an early approach using cell merging in structured meshes was suggested in [9] and then a more general agglomeration approach was introduced in [10]. A framework for all nodal finite element spaces, including for instance the family of Hermite splines, was presented in [11]. Examples of applications of extension operators to cut finite element methods include [12–15]. Another approach using extension operators, where certain terms in the finite element formulation are evaluated at an extended polynomial, was proposed for Lagrange multiplier methods in [16] and in the context of splines using Nitsche type weak imposition of boundary conditions in [17]. Trimmed elements and immersed methods attract significant interest in isogeometric analysis, see for instance [3,18–21].

Outline. The paper is organized as follows: In Section 2 we introduce the spline spaces including assumptions on the basis functions, in Section 3 we construct an interpolation operator and establish stability and approximation properties, in Section 4 we develop the extension operator and establish its stability and approximation properties, in Section 5 we apply the extension to a cut isogeometric method for an elliptic model problem, and in Section 6 we present some numerical examples based on that method.

2. Spline spaces

- Let $\tilde{\mathcal{T}}_h$ be a uniform tensor product mesh on \mathbb{R}^d with mesh parameter $h \in (0, h_0]$. Let $\tilde{V}_{h,p,k} \subset C^k(\mathbb{R}^d)$ be a spline space of piecewise tensor product polynomials of order p with regularity parameter $0 \leq k \leq p - 1$. We also let $\tilde{V}_{h,p,-1}$ denote the space of discontinuous piecewise tensor product polynomials of order p .
- Let

$$\tilde{\mathcal{B}} = \{\tilde{\varphi}_i\}_{i \in \tilde{I}} \tag{1}$$

denote a basis in $\tilde{V}_{h,p,k}$ and define the sub mesh consisting of elements contained in the support of $\tilde{\varphi}_i$,

$$\tilde{\mathcal{T}}_{h,i} = \{T \in \tilde{\mathcal{T}}_h : T \subset \text{supp}(\tilde{\varphi}_i)\} \tag{2}$$

and the set of indices $\tilde{I}_T \subset \tilde{I}$ of the basis functions that contain element T ,

$$\tilde{I}_T = \{i \in \tilde{I} : T \subset \text{supp}(\tilde{\varphi}_i)\} \tag{3}$$

and the corresponding basis functions

$$\tilde{\mathcal{B}}_T = \{\tilde{\varphi}_i : i \in \tilde{I}_T\} \tag{4}$$

- Assume that the basis $\tilde{\mathcal{B}}$ satisfies:

A1. The basis functions are locally supported. There is a constant, such that

$$|\tilde{\mathcal{T}}_{h,i}| \lesssim 1 \tag{5}$$

A2. The restrictions of the basis functions $\tilde{\mathcal{B}}_T$ whose support contain T is a basis for $\mathbb{Q}_p(T)$, the tensor product of one-dimensional polynomials of degree p on T ,

$$\mathbb{Q}_p(T) = \text{span}\{\varphi_j : j \in \tilde{I}_T\} \tag{6}$$

and the basis $\tilde{\mathcal{B}}_T$ is stable in the sense that there are constants such that

$$h^d \|\hat{v}_T\|_{\mathbb{R}^{N_T}}^2 \sim \|v\|_T^2, \quad v \in \mathbb{Q}_p(T) \tag{7}$$

where $N_T = \dim(\mathbb{Q}_p(T))$ and

$$v = \sum_{i \in \tilde{I}_T} \hat{v}_{T,i} \varphi_i|_T \tag{8}$$

is the expansion of $v \in \mathbb{Q}_p(T)$ in the basis $\tilde{\mathcal{B}}_T$.

These two assumptions are standard and non-restrictive. Here and below the $a \lesssim b$ means $a \leq Cb$ where the constant C is independent of h and the intersection of the computational mesh and the domain boundary, but may depend on p and k .

Remark 2.1. The number of elements in $\mathcal{T}_{h,i}$ for the standard basis of B-splines with maximum regularity $k = p - 1$ is $(p + 1)^d$.

Remark 2.2. Our constructions and analysis extend to more general spline spaces including local refinements, splines on triangulations, and tensor products of splines of various order and regularity. The essential assumption is that we have a piecewise polynomial space and a basis that is local and the restriction of the basis functions to an element spans a suitable polynomial space. We have chosen the most common situation of uniform tensor product splines to keep the presentation simple.

- Given a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ we define the active mesh

$$\mathcal{T}_h = \{T \in \tilde{\mathcal{T}}_h : T \cap \Omega \neq \emptyset\} \tag{9}$$

and we let

$$\Omega_h = \cup_{T \in \mathcal{T}_h} T \tag{10}$$

We let the active basis $\mathcal{B} \subset \tilde{\mathcal{B}}$ consist of all basis functions that contain an active element in their support,

$$\mathcal{B} = \{\varphi \in \tilde{\mathcal{B}} : \exists T \in \mathcal{T}_h, T \subset \text{supp}(\varphi)\} = \cup_{T \in \mathcal{T}_h} \tilde{\mathcal{B}}_T \tag{11}$$

and we denote the index set of \mathcal{B} by I . The associated active spline space $V_{h,p,k}$ is defined by

$$V_{h,p,k} = \text{span}(\mathcal{B}) \tag{12}$$

For $T \in \mathcal{T}_h$ we use the notation $\tilde{\mathcal{B}}_T = \mathcal{B}_T$, and $\tilde{I}_T = I_T$.

3. Interpolation

In this section, we construct an interpolation operator $\pi_h : L^2(\Omega) \rightarrow V_{h,p,k}$. This operator is based on first extending the function outside of the domain, and then using standard interpolation operators for spline spaces. We establish basic stability and approximation properties. We also show an estimate of the error in the interpolant of a discontinuous piecewise polynomial function, where the bound is in terms of a jump operator measuring the size of the jumps in derivatives across faces. For background on spline (quasi-)interpolants we refer to [22–25]. The word *quasi* here refers to the fact that these operators are not necessarily interpolating in the classical sense, with a pointwise evaluation of data, and instead evaluate data in some smoother sense while still retaining other fundamental properties of interpolation operators.

3.1. Definitions

- There is a continuous extension operator

$$E_c : H^s(\Omega) \ni v \mapsto v^{E_c} \in H^s(\mathbb{R}^d) \tag{13}$$

independent of s , such that

$$\|v^{E_c}\|_{H^s(\mathbb{R}^d)} \lesssim \|v\|_{H^s(\Omega)} \tag{14}$$

This continuous extension exists given that the boundary $\partial\Omega$ satisfies some minimal smoothness properties, approximately equivalent to that of a Lipschitz domain, see [26]. When not needed for clarity we simply write $v^{E_c} = v$.

- For each element $T \in \mathcal{T}_h$ let $P_{T,p} : L^2(T) \rightarrow \mathbb{Q}_p(T)$ be the $L^2(T)$ projection. For $v \in H^s(\Omega)$ and each element $T \in \mathcal{T}_h$ we may expand the projection $P_{T,p}(v^{E_c}|_T) \in \mathbb{Q}_p(T)$ in the basis \mathcal{B}_T ,

$$P_{T,p}v|_T = \sum_{i \in I_T} \hat{v}_{T,i} \varphi_i|_T \tag{15}$$

Thus for each element $T \in \mathcal{T}_{h,i}$ we obtain a potential coefficient $\hat{v}_{T,i}$ multiplying basis function φ_i . We finally, take an average over the coefficients $\hat{v}_{T,i}$ obtained from each of the elements T in the support of φ_i to get the

final coefficient for φ_i . More precisely, we define

$$\pi_h v = \sum_{i \in I} \langle \widehat{v}_{T,i} \rangle_{T \in \mathcal{T}_{h,i}} \varphi_i \tag{16}$$

where the average is a convex combination

$$\langle \widehat{v}_{T,i} \rangle_{T \in \mathcal{T}_{h,i}} = \sum_{T \in \mathcal{T}_{h,i}} \kappa_{T,i} \widehat{v}_{T,i} \tag{17}$$

with arbitrary weights $\kappa_{T,i}$ such that $0 \leq \kappa_{T,i} \leq 1$ and $\sum_{T \in \mathcal{T}_{h,i}} \kappa_{T,i} = 1$. Note that the weights can be individually chosen for each basis function.

Remark 3.1. Let $\mathcal{B}_T^* = \{\varphi_{T,j}^* : j \in I_T\}$ be the dual basis to \mathcal{B}_T on T , characterized by

$$\varphi_{T,j}^*(\varphi_i|_T) = \delta_{ij} \tag{18}$$

Then there is $\chi_{T,i}^* \in \mathbb{Q}_p(T)$ such that $\varphi_{T,j}^*(w) = (\chi_{T,i}^*, w)_T$ for $w \in \mathbb{Q}_p(T)$ and we can extend the action of $\varphi_{i,T}^*$ to $L^2(T)$. We then have $\varphi_{T,j}^*(P_{T,p}v|_T) = \varphi_{T,j}^*(v|_T)$ and therefore

$$\varphi_{T,j}^*(v|_T) = \varphi_{T,j}^*(P_{T,p}v|_T) = \sum_{i \in I_T} \widehat{v}_{T,i} \varphi_{T,j}^*(\varphi_i|_T) = \widehat{v}_{T,j} \tag{19}$$

which allows us to express the interpolant in terms of the dual basis

$$\pi_h v = \sum_{i \in I} \langle \varphi_{T,i}^*(v|_T) \rangle_{T \in \mathcal{T}_{h,i}} \varphi_i \tag{20}$$

3.2. Properties

The spline space $V_{h,p,k}$ is invariant under the interpolation operator. To see this we consider an element T , clearly $P_{T,p}v = v$ on T . Expanding v in the spline basis \mathcal{B} and restricting to T ,

$$\sum_{i \in I} \widehat{v}_i \varphi_i|_T = v|_T = P_{T,p}v = \sum_{i \in I_T} \widehat{v}_{T,i} \varphi_i|_T \tag{21}$$

Since the expansions are unique we conclude that $\widehat{v}_{T,i} = \widehat{v}_i$, for all $T \in \mathcal{T}_{h,i}$, and therefore $\langle \widehat{v}_{T,i} \rangle_{T \in \mathcal{T}_{h,i}} = \widehat{v}_i$. It follows that $\pi_h v = v$ for $v \in V_{h,p,k}$. Furthermore, introducing the notation

$$\omega_{h,T} = \cup_{i \in I_T} \text{supp}(\varphi_i) \tag{22}$$

we note that the restriction of the interpolant $\pi_h v$ to T depends only on v restricted to $\omega_{h,T}$, and since $\text{diam}(\text{supp}(\varphi_i)) \lesssim h$ we have

$$\text{diam}(\omega_{h,T}) \lesssim h \tag{23}$$

We now proceed with some basic stability and approximation results for the interpolation operator.

Lemma 3.1. *There is a constant such that*

$$\|\pi_h v\|_T \lesssim \|v\|_{\omega_{h,T}}, \quad v \in L^2(\omega_{h,T}) \tag{24}$$

Proof. Starting from the definition (16) of the interpolant and the expression for the coefficients (17) we obtain

$$\|\pi_h v\|_T^2 \lesssim \sum_{i \in I_T} |\langle \widehat{v}_{T,i} \rangle_{T \in \mathcal{T}_{h,i}}|^2 h^d \lesssim \sum_{i \in I_T} \sum_{T' \in \mathcal{T}_{h,i}} \widehat{v}_{T',i}^2 h^d \tag{25}$$

$$\lesssim \sum_{i \in I_T} \sum_{T' \in \mathcal{T}_{h,i}} \|P_{T',p}(v^{Ec}|_{T'})\|_{T'}^2 \lesssim \sum_{i \in I_T} \sum_{T' \in \mathcal{T}_{h,i}} \|v^{Ec}\|_{T'}^2 \lesssim \|v^{Ec}\|_{\omega_{h,T}}^2 \tag{26}$$

where we used the equivalence (7). This completes the proof. \square

Lemma 3.2. *There is a constant such that for all $v \in H^r(\Omega)$*

$$\|v^{Ec} - \pi_h v\|_{H^m(\Omega_h)} \lesssim h^{s-m} \|v\|_{H^s(\Omega)}, \quad 0 \leq m \leq k \tag{27}$$

where $s = \min(r, p + 1)$.

Proof. We have the invariance

$$(\pi_h w)|_T = w, \quad w \in \mathbb{P}_p(\omega_T) \tag{28}$$

since the spline space is invariant under the action of π_h and polynomials can be represented exactly in the spline space. Therefore, for any $w \in \mathbb{P}_p(\omega_{h,T})$ we have,

$$\|v - \pi_h v\|_{H^m(T)} \leq \|v - w\|_{H^m(T)} + \|w - \pi_h v\|_{H^m(T)} \tag{29}$$

$$\lesssim \|v - w\|_{H^m(\omega_{h,T})} + h^{-m} \|w - \pi_h v\|_T \tag{30}$$

$$\lesssim \|v - w\|_{H^m(\omega_{h,T})} + h^{-m} \|\pi_h(w - v)\|_T \tag{31}$$

$$\lesssim \|v - w\|_{H^m(\omega_{h,T})} + h^{-m} \|w - v\|_{\omega_{h,T}} \tag{32}$$

$$\lesssim h^{s-m} \|v\|_{H^s(\omega_{h,T})} \tag{33}$$

where we first used the stability (24) and then choose w according to the Bramble–Hilbert lemma, see [27, Lemma 4.3.8]. Summing over the elements and using the stability (14) of the extension operator completes the proof. \square

Lemma 3.3. *There is a constant such that*

$$\|v - \pi_h v\|_{\Omega_h} \lesssim \|v\|_{j_h}, \quad v \in V_{h,p,-1} \tag{34}$$

with

$$\|v\|_{j_h}^2 = \sum_{l=0}^p h^{2l+1} \|[\nabla_n^l v]\|_{\mathcal{F}_h}^2 \tag{35}$$

where \mathcal{F}_h is the set of interior faces in \mathcal{T}_h and $[\nabla_n^l v]$ denotes the jump in the l :th normal derivate over a face.

Proof. Proceeding in a similar manner as in the proof of Lemma 3.2 we note that for any $w \in \mathbb{P}_p(\omega_{h,T})$ we have, using the L^2 stability of π_h ,

$$\|(I - \pi_h)v\|_T = \|(I - \pi_h)(v - w)\|_T \lesssim \|v - w\|_{\omega_{h,T}} \tag{36}$$

Finally, taking $w \in \mathbb{P}_p(\omega_{h,T})$ such that $w = v$ on T and using standard estimates, see [28,29], for face stabilization of higher order elements we get

$$\|v - w\|_{\omega_{h,T}}^2 \lesssim \sum_{l=0}^p h^{2l+1} \|[\nabla_n^l v]\|_{\mathcal{F}_{h,i}}^2 \tag{37}$$

where $\mathcal{F}_{h,i}$ is the set of interior faces in $\mathcal{T}_{h,i}$. \square

Remark 3.2. Note that the h -scaling in j_h is chosen such that we have the inverse inequality

$$\|v\|_{j_h} \lesssim \|v\|_{\mathcal{T}_h}, \quad v \in V_{h,p,-1} \cap C(\Omega_h) \tag{38}$$

Remark 3.3. The estimate (34) may be viewed as a generalization of estimates for Oswald interpolation introduced in [30], from piecewise linear spaces to spline spaces. This type of operator is also used for the analysis of interior penalty stabilized finite element methods [31,32]. In the case of high-order finite element spaces an hp -analysis was considered in [33]. In this context, Lemma 3.3 is instrumental for the analysis of the skeleton-based stabilized methods proposed in [18].

4. Extension

4.1. Definitions

- To define the extension operator we partition \mathcal{T}_h into the set of elements \mathcal{T}_h^L that have a large intersection with Ω , in the sense that

$$\gamma h^d \leq |T \cap \Omega| \tag{39}$$

for a parameter $\gamma \geq 0$, and the set of elements $\mathcal{T}_h^S = \mathcal{T}_h \setminus \mathcal{T}_h^L$ with a small intersection. We thus have

$$\mathcal{T}_h = \mathcal{T}_h^L \cup \mathcal{T}_h^S \tag{40}$$

We also define

$$\Omega_h^L = \cup_{T \in \mathcal{T}_h^L} T \subset \Omega_h \tag{41}$$

- Let $S_h : \mathcal{T}_h^S \rightarrow \mathcal{T}_h^L$ be a mapping that associates a large element to a small element such that

$$\text{diam}(S_h(T) \cup T) \lesssim h \tag{42}$$

According to Lemma 2.4 in [11] there is such a mapping for domains with Lipschitz boundary when the mesh is sufficiently fine. In general, such a mapping is non-unique.

- We let

$$\mathcal{B}^L = \cup_{T \in \mathcal{T}_h^L} \mathcal{B}_T \subset \mathcal{B}, \quad \mathcal{B}^S = \mathcal{B} \setminus \mathcal{B}^L \tag{43}$$

be the set of active basis functions that contain a large element in their support and the set of remaining active basis functions, respectively. We then have the (interior) direct sum

$$V_{h,p,k} = V_{h,p,k}^L \oplus V_{h,p,k}^S \tag{44}$$

where

$$V_{h,p,k}^L = \text{span}(\mathcal{B}^L), \quad V_{h,p,k}^S = \text{span}(\mathcal{B}^S) \tag{45}$$

The index sets of \mathcal{B}^L and \mathcal{B}^S are denoted by I^L and I^S , respectively.

- We define the extension operator

$$B_h : V_{h,p,k}^L \rightarrow V_{h,p,-1} \tag{46}$$

in such a way that

$$(B_h v)|_T = \begin{cases} (v|_{S_h(T)})^e|_T, & T \in \mathcal{T}_h^S \\ v|_T, & T \in \mathcal{T}_h^L \end{cases} \tag{47}$$

where v^e denotes the canonical extension of a polynomial v on T to a polynomial on \mathbb{R}^d .

- We define the extension operator

$$E_h : V_{h,p,k}^L \ni v \mapsto \pi_h B_h v \in V_{h,p,k}^E \subset V_{h,p,k} \tag{48}$$

where the extended finite element space $V_{h,p,k}^E$ is defined by

$$V_{h,p,k}^E = E_h V_{h,p,k}^L \tag{49}$$

- Extending the mapping S_h from \mathcal{T}_h^S to \mathcal{T}_h by setting $S_h(T) = T$, for $T \in \mathcal{T}_h^L$, we note that the set valued mapping $S_h^{-1} : \mathcal{T}_h^L \rightarrow \mathcal{T}_h$ induce a partition \mathcal{M}_h of \mathcal{T}_h into macro elements

$$M_T = \cup_{T' \in S_h^{-1}(T)} T', \quad T \in \mathcal{T}_h^L \tag{50}$$

that thanks to the property (42) satisfy

$$\text{diam}(M_T) \lesssim h \tag{51}$$

The macro elements M_T are invariant under S_h in the sense that if $T' \subset M_T$ then $S_h(T') \subset M_T$. Let $V_{h,p,-1}^M$ be the space of discontinuous tensor product polynomials of degree p on \mathcal{M}_h . Then it follows from the invariance of \mathcal{M}_h under S_h that $V_{h,p,-1}^M$ is invariant under B_h ,

$$\boxed{w = B_h w, \quad w \in V_{h,p,-1}^M} \tag{52}$$

- If we choose the weights $\kappa_{T,i}$ in the average (17) such that

$$\kappa_{T',i} = 0, \quad T' \in \mathcal{T}_h^S \tag{53}$$

for all basis functions $\varphi_i \in \mathcal{B}_T^L$, then the extension operator takes the form

$$\boxed{E_h : V_{h,p,k}^L \ni v^L \mapsto v^L \oplus (E_h v^L)^S \in V_{h,p,k}^E \subset V_{h,p,k}} \tag{54}$$

where $v^L = \sum_{i \in I^L} \widehat{v}_i \varphi_i \in V_{h,p,k}^L$ denotes the expansion of $v \in V_{h,p,k}$ in the large basis functions and $v^S = \sum_{i \in I^S} \widehat{v}_i \varphi_i \in V_{h,p,k}^S$ denotes the expansion of v in the small basis functions. This means that the component $(E_h v)^L$ of $E_h v$ in $V_{h,p,k}^L$ is identical to v^L and the extension operator determines a suitable component in $V_{h,p,k}^S$. In particular, we note that with weights satisfying (53) the extension operator does not change v on Ω_h^L ,

$$\boxed{(E_h v)|_{\Omega_h^L} = v|_{\Omega_h^L}} \tag{55}$$

4.2. Properties

Lemma 4.1 (Preservation of polynomials). *If $v \in \mathbb{Q}_p(\Omega)$, i.e., a tensor product of one-dimensional polynomials of degree p on Ω , then*

$$\boxed{E_h(v|_{\Omega_h^L}) = v} \tag{56}$$

Proof. It follows from the definition of B_h that $B_h(v|_{\Omega_h^L}) = v$ and $\pi_h v = v$. \square

Before proving a stability result for the extension operator we show the following technical lemma which provides a bound for the right-hand side of (34) for a function of the form $B_h v \in V_{h,p,-1}$ with $v \in V_{h,p,k}$.

Lemma 4.2. *There is a constant such that*

$$\boxed{\|B_h v\|_{j_h} \lesssim h^m \|\nabla^m v\|_{\Omega}, \quad 0 \leq m \leq k, \quad v \in V_{h,p,k}} \tag{57}$$

Proof. Consider a face F shared by elements $\mathcal{T}_h(F) = \{T_1, T_2\}$. Let ω_δ be a ball of radius δ such that

$$T_1 \cup S_h(T_1) \cup T_2 \cup S_h(T_2) \subset \omega_\delta \tag{58}$$

Since T_1 and T_2 share a face and S_h satisfies (42) we conclude that there is such a ball with radius $\delta \lesssim h$. For $w \in \mathbb{P}_p(\omega_\delta)$ we then have the estimates

$$h^{2l+1} \|\nabla_n^l B_h v\|_F^2 = h^{2l+1} \|\nabla_n^l (B_h v - w)\|_F^2 \lesssim h^{2l} \|\nabla_n^l (B_h v - w)\|_{\mathcal{T}_h(F)}^2 \tag{59}$$

$$\lesssim \|B_h v - w\|_{\mathcal{T}_h(F)}^2 \lesssim \|v - w\|_{S_h(\mathcal{T}_h(F))}^2 \lesssim \|v^E - w\|_{\omega_\delta}^2 \lesssim h^{2m} \|v^E\|_{H^m(\omega_\delta)}^2 \tag{60}$$

where we used inverse inequalities to pass from the face to elements and to remove derivatives, then we used the identity

$$(B_h v - w)|_{T_i} = (v|_{S_h(T_i)})^e - (w|_{S_h(T_i)})^e = ((v - w)|_{S_h(T_i)})^e \tag{61}$$

followed by stability

$$\|(q|_{S_h(T)})^e\|_T \lesssim \|q\|_{S_h(T)}, \quad q \in \mathbb{Q}_p(S_h(T)) \tag{62}$$

of polynomial extension, and finally, in the last inequality we choose w to be the $L^2(\omega_\delta)$ projection of the continuous extension v^{Ec} restricted to ω_δ and used a standard approximation result on the ball with diameter $\delta \lesssim h$. Note that we need the continuous extension in the last step since the ball ω_δ may not be contained in Ω . Summing over all faces and using the stability (14) of the continuous extension operator the desired estimate follows. \square

Lemma 4.3 (Stability). *There is a constant such that*

$$\|\nabla^m E_h v\|_{\Omega_h} \lesssim \|\nabla^m v\|_{\Omega}, \quad 0 \leq m \leq k, \quad v \in V_{h,p,k} \tag{63}$$

Proof. Using the stability (62) of polynomial extension we get

$$\|\nabla^m v^e\|_{M_T} \lesssim \|\nabla^m v\|_T, \quad v \in \mathbb{Q}_p(T), \quad T \in \mathcal{T}_h^L \tag{64}$$

where we recall that $M_T = \cup_{T \in \mathcal{S}_h^{-1}(T)} T$, and therefore

$$\|\nabla^m B_h v\|_{\mathcal{M}_h} \lesssim \|\nabla^m v\|_{\mathcal{T}_h^L} \tag{65}$$

Adding and subtracting $v \in V_{h,p,k}$ we have

$$\|\nabla^m E_h v\|_{\Omega_h} = \|\nabla^m \pi_h B_h v\|_{\Omega_h} \tag{66}$$

$$\lesssim \|\nabla^m (\pi_h - I) B_h v\|_{\mathcal{T}_h} + \|\nabla^m B_h v\|_{\mathcal{M}_h} \tag{67}$$

$$\lesssim h^{-m} \|(\pi_h - I) B_h v\|_{\mathcal{T}_h} + \|\nabla^m v\|_{\mathcal{T}_h^L} \tag{68}$$

where we used the stability (65) of B_h to estimate the second term. For the first term we employ Lemmas 3.3 and 4.2,

$$h^{-m} \|(\pi_h - I) B_h v\|_{\mathcal{T}_h} \lesssim h^{-m} \|B_h v\|_{j_h} \lesssim \|v\|_{H^m(\Omega)} \tag{69}$$

which completes the proof. \square

We now define an interpolant $\pi_h^E : L^2(\Omega) \rightarrow V_{h,p,k}$ by applying the discrete extension operator to the interpolation operator defined in (16), $\pi_h^E v := E_h \pi_h v$. This interpolation into the extended space satisfies the following approximation result.

Lemma 4.4 (Approximation). *There is a constant such that for all $v \in H^r(\Omega)$,*

$$\|v^{Ec} - \pi_h^E v\|_{H^m(\Omega_h)} \lesssim h^{s-m} \|v\|_{H^s(\Omega)}, \quad 0 \leq m \leq k, \quad s = \min(r, p + 1) \tag{70}$$

Proof. Let $\pi_h^M : L^2(\Omega_h) \rightarrow V_{h,p,-1}^M$ be the element-wise L^2 projection. We then have

$$\|v - \pi_h^M v\|_{\mathcal{M}_h} \lesssim h^s |v|_{H^s(\mathcal{M}_h)}, \quad s = \min(r, p + 1) \tag{71}$$

Adding and subtracting $\pi_h v$ and $\pi_h \pi_h^M v = \pi_h B_h \pi_h^M v$, where we used the invariance (52), we get the identity

$$v - \pi_h B_h (\pi_h v) = v - \pi_h v + \pi_h v - \pi_h \pi_h^M v + \pi_h \pi_h^M v - \pi_h B_h (\pi_h v) \tag{72}$$

$$= v - \pi_h v + \pi_h (v - \pi_h^M v) + \pi_h B_h (\pi_h^M v - \pi_h v) \tag{73}$$

Using the triangle inequality, inverse inequalities, the L^2 stability of π_h and B_h ,

$$\|v - \pi_h B_h (\pi_h v)\|_{H^m(\mathcal{T}_h)} \tag{74}$$

$$\leq \|v - \pi_h v\|_{H^m(\mathcal{T}_h)} + \|\pi_h (v - \pi_h^M v)\|_{H^m(\mathcal{T}_h)} + \|\pi_h B_h (\pi_h^M v - \pi_h v)\|_{H^m(\mathcal{T}_h)} \tag{75}$$

$$\lesssim \|v - \pi_h v\|_{H^m(\mathcal{T}_h)} + h^{-m} \|\pi_h (v - \pi_h^M v)\|_{\mathcal{T}_h} + h^{-m} \|\pi_h B_h (\pi_h^M v - \pi_h v)\|_{\mathcal{T}_h} \tag{76}$$

$$\lesssim \|v - \pi_h v\|_{H^m(\mathcal{T}_h)} + h^{-m} \|v - \pi_h^M v\|_{\mathcal{T}_h} + h^{-m} \|\pi_h^M v - \pi_h v\|_{\mathcal{T}_h^L} \tag{77}$$

$$\lesssim h^{s-m} \|v\|_{H^s(\Omega)} \tag{78}$$

Here we used the interpolation estimate for π_h in Lemma 3.2 and the estimate (71) for π_h^M . To estimate the third term in (77) we added and subtracted v and once again used the approximation of π_h^M and π_h . \square

We end this section with some results on the properties of the extended basis. To that end, let $I^L \subset I$ be the indices for the basis \mathcal{B}^L , and note that $\mathcal{B}^E = \{E_h \varphi_i : i \in I^L\}$ is a basis in $V_{h,p,k}^E$. Using the notation

$$\varphi_i^E = E_h \varphi_i \quad (79)$$

for the extended basis functions, we then have the expansion

$$v = \sum_{i=1}^{N^L} \widehat{v}_i \varphi_i^E \quad (80)$$

of $v \in V_{h,p,k}^E$. We next present a lemma that collects the basic properties of the extended basis functions and then we show an equivalence between the degrees of freedom norm and the L^2 norm. The latter result is crucial in the proof of bounds on the condition number of stiffness and mass matrices.

Lemma 4.5 (Properties of extended basis functions). *There are constants such that*

$$\text{diam}(\text{supp}(\varphi_i^E)) \lesssim h, \quad \|\varphi_i^E\|_{L^\infty(\Omega_h)} \lesssim 1 \quad (81)$$

and, with $\delta_{ij} = 1$ if $\text{supp}(\varphi_i^E) \cap \text{supp}(\varphi_j^E) \neq \emptyset$ and 0 otherwise,

$$\max_{i \in I^L} \sum_{j \in I^L} \delta_{ij} \lesssim 1 \quad (82)$$

which means that the number of nonempty intersections between the supports of an extended basis function and the neighboring extended basis functions is uniformly bounded.

Proof. We first note that if there is no element $T \in S_h(\mathcal{T}_h^S)$ in the support $\text{supp}(\varphi_i)$ then $\varphi_i^E = E_h \varphi_i = \varphi_i$. These are basis functions in the interior that are not affected by the extension and obviously satisfy the desired properties. If on the other hand there is $T \in \mathcal{T}_h^S$ such that $S_h(T) \subset \text{supp}(\varphi_i)$ then $\varphi_i^E \neq \varphi_i$. Using the definition (48) of the extension operator E_h we have $\varphi_i^E = \pi_h B_h \varphi_i$. We start by noting that the support of $B_h \varphi_i$ is given by

$$\text{supp}(B_h \varphi_i) = \cup_{T \in \mathcal{T}_h^L, T \subset \text{supp}(\varphi_i)} M_T \quad (83)$$

where M_T is the macro element defined in (50). Using the fact that $\text{diam}(\text{supp}(\varphi_i)) \lesssim h$ and $\text{diam}(M_T) \lesssim h$ we conclude that

$$\text{diam}(\text{supp}(B_h \varphi_i)) \lesssim h \quad (84)$$

Recalling the definition (22) and properties of the domain $\omega_{h,T}$ that influences $\pi_h v$ on T , we finally find that

$$\text{supp}(\varphi_i^E) \subset \cup_{\omega_{h,T} \cap \text{supp}(B_h \varphi_i) \neq \emptyset} \omega_{h,T} \quad (85)$$

which in particular means that

$$\text{diam}(\text{supp}(\varphi_i^E)) \lesssim h \quad (86)$$

since $\text{diam}(\omega_{h,T}) \lesssim h$ and $\text{diam}(\text{supp}(B_h \varphi_i)) \lesssim h$, see (23) and (84).

Next using an inverse estimate to pass from the max norm to the L^2 norm followed by the L^2 stability of π_h we have for any element $T \subset \text{supp}(\varphi_i^E)$,

$$\|\pi_h B_h \varphi_i\|_{L^\infty(T)}^2 \lesssim h^{-d} \|\pi_h B_h \varphi_i\|_T^2 \lesssim h^{-d} \|B_h \varphi_i\|_{\omega_{h,T}}^2 \lesssim \|B_h \varphi_i\|_{L^\infty(\omega_{h,T})}^2 \quad (87)$$

Using stability of polynomial extension we may estimate the right hand side as follows

$$\|B_h \varphi_i\|_{L^\infty(\omega_{h,T})}^2 \lesssim \|\varphi_i\|_{L^\infty(\text{supp}(\varphi_i) \cap \Omega_h^L)}^2 \lesssim \|\varphi_i\|_{L^\infty(\Omega_h)}^2 \lesssim 1 \quad (88)$$

Finally, (82) follows from the fact that $\text{supp}(\varphi_i^E)$, $i \in I^L$, is contained in a ball $B_\delta(x_i)$ with $\delta \lesssim h$ and x_i the midpoint of a unique element T_i and therefore $\|x_i - x_j\|_{\mathbb{R}^d} \gtrsim h$. Thus $\text{supp}(\varphi_i^E) \cap \text{supp}(\varphi_j^E) \neq \emptyset$ for a uniformly bounded number of indices $j \in I^L$. \square

Lemma 4.6 (Equivalence with the degrees of freedom norm). Assume that the weights $\kappa_{T',i}$ in (17) satisfy (53). Then there are constants such that

$$\|v\|_{\Omega}^2 \sim h^d \|\widehat{v}\|_{\mathbb{R}^N}^2 \tag{89}$$

Proof. We recall that since the weights satisfy (17) we have

$$(E_h v)|_{\Omega_h^L} = v|_{\Omega_h^L} \tag{90}$$

see (55), which means that the extension does not change v on Ω_h^L . Then we can apply the equivalence on the element level (7) on elements $T \in \mathcal{T}_h^L$,

$$h^d \|\widehat{v}\|_{\mathbb{R}^N}^2 \lesssim \sum_{T \in \mathcal{T}_h^L} h^d \|\widehat{v}\|_{\mathbb{R}^{N_T}}^2 \sim \sum_{T \in \mathcal{T}_h^L} \|v\|_T^2 \lesssim_{\gamma} \sum_{T \in \mathcal{T}_h^L} \|v\|_{T \cap \Omega}^2 = \|v\|_{\Omega}^2 \tag{91}$$

where we emphasized the dependency of γ in the last inequality. Conversely, using (81) and (82) it follows that

$$\|v\|_{\Omega}^2 = \sum_{i,j \in I^L} \widehat{v}_i \widehat{v}_j (\varphi_i^E, \varphi_j^E)_{\Omega} \lesssim \sum_{i,j \in I^L} |\widehat{v}_i| |\widehat{v}_j| h^d \delta_{ij} \tag{92}$$

$$\lesssim \sum_{i \in I^L} |\widehat{v}_i|^2 h^d \left(\sum_{j \in I^L} \delta_{ij} \right) \lesssim \sum_{i \in I^L} |\widehat{v}_i|^2 h^d \tag{93}$$

which completes the proof. \square

5. Application

To illustrate the application of the extension operator we consider an elliptic model problem with homogeneous boundary conditions in a domain Ω in \mathbb{R}^2 with smooth boundary, find $u : \Omega \rightarrow \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{94}$$

We use Nitsche’s method together with the extended spline space to discretize the problem. Since the below material is standard and only for illustration we only sketch the main arguments, for simplicity we assume $k = 1$. The method takes the form: find $u_h \in V_{h,p,k}^E$ such that

$$a_h(u_h, v) = (f, v)_{\Omega}, \quad \forall v \in V_{h,p,k}^E \tag{95}$$

where

$$a_h(v, w) = (\nabla v, \nabla w)_{\Omega} - (\nabla_n v, w)_{\partial\Omega} - (v, \nabla_n w)_{\partial\Omega} + \beta h^{-1} (v, w)_{\partial\Omega} \tag{96}$$

As is well known the key property required to show that a_h is coercive on the trial space $V_{h,p,k}^E$ is the inverse inequality

$$h \|\nabla_n v\|_{\partial\Omega}^2 \lesssim \|\nabla v\|_{\Omega}^2, \quad v \in V_h^E \tag{97}$$

We note that for $T \in \mathcal{T}_h$ we have

$$h \|\nabla_n v\|_{\partial\Omega \cap T}^2 \lesssim h |\partial\Omega \cap T| \|\nabla v\|_{L^{\infty}(T)}^2 \lesssim h |\partial\Omega \cap T| h^{-d} \|\nabla v\|_T^2 \tag{98}$$

Assuming, $|\partial\Omega \cap T| h^{-(d-1)} \lesssim 1$ which holds for instance if the boundary is Lipschitz, we get by summation over $T \in \mathcal{T}_h$,

$$h \|\nabla_n v\|_{\partial\Omega \cap T}^2 \lesssim \|\nabla v\|_{\Omega_h}^2 \tag{99}$$

which combined with the stability of the extension operator in Lemma 4.3 directly establish the desired estimate (97). It then follows using standard arguments that for β large enough (depending on the constant in (97)),

$$\|v\|^2 := \|\nabla v\|_{\Omega}^2 + \beta/h \|v\|_{\partial\Omega}^2 \lesssim a_h(v, v), \quad \forall v \in V_{h,p,k}^E \tag{100}$$

If we let $e_h = u_h - \pi_h^E u$ we immediately see that $\|u - u_h\| \leq \|u - \pi_h^E u\| + \|e_h\|$ and

$$\|e_h\|^2 \lesssim a_h(e_h, e_h) \tag{101}$$

By the consistency of Nitsche’s method $a_h(e_h, e_h) = a_h(u - \pi_h^E u, e_h)$. Using the Cauchy–Schwarz inequality and (97) we see that

$$a_h(u - \pi_h^E u, e_h) \lesssim (|||u - \pi_h^E u||| + \|h^{\frac{1}{2}} \nabla_n(u - \pi_h^E u)\|_{\partial\Omega}) |||e_h||| \tag{102}$$

Using a trace inequality (see for instance [34]) we see that

$$|||u - \pi_h^E u||| + \|h^{\frac{1}{2}} \nabla_n(u - \pi_h^E u)\|_{\partial\Omega} \lesssim \sum_{i=0}^2 h^{i-1} \|D^i(u - \pi_h^E u)\|_{\Omega_h} \tag{103}$$

where D^i is the standard multi-index notation for derivatives of order i . Then we use the approximation property in Lemma 4.4

$$\|\nabla(u - \pi_h^E u)\|_{\Omega} + h^{-1} \|u - \pi_h^E u\|_{\Omega} + h \|D^2(u - \pi_h^E u)\|_{\Omega} \lesssim h^{s-1} \|u\|_{H^s(\Omega)} \tag{104}$$

and obtain the optimal order a priori error estimate for $u \in H^r(\Omega)$,

$$|||u - u_h||| \lesssim h^{s-1} \|u\|_{H^s(\Omega)}, \quad s = \min(r, p + 1) \tag{105}$$

We can also show, following the ideas of [35], using Lemma 4.6 and the properties of the extension, that the condition number κ of the stiffness matrix associated with the form a_h satisfies the estimate

$$\kappa \lesssim h^{-2} \tag{106}$$

Remark 5.1. Observe that it is straightforward to design and analyze a similar method in the case of fourth-order elliptic PDE, indeed the ideas of [36] carry over verbatim if the ghost penalty terms are omitted.

6. Numerical experiments

In this section, we present a number of numerical experiments in 2D, where a high-resolution polygonal domain is immersed in a structured quadrilateral mesh equipped with full regularity B-spline basis functions. For details on implementation, see [37].

Extension choices. To construct the extension operator according to the above description, we must make the following three choices:

- *The value of the parameter $\gamma \geq 0$.* This parameter determines the partition of the mesh into large elements \mathcal{T}_h^L respectively small elements \mathcal{T}_h^S according to (39), which in turn gives the partition of the active basis functions into large basis functions \mathcal{B}^L respectively small basis functions \mathcal{B}^S . The partition of the elements in one specific example is illustrated in Fig. 1(b) and the corresponding partition of the basis functions in Fig. 2.
- *The small-to-large element mapping $S_h : \mathcal{T}^S \rightarrow \mathcal{T}^L$.* This mapping defines the extension operator B_h according to (47). In our experiments, we construct S_h by associating each small element T with the large element T' that minimizes the distance between the center of mass of $T \cap \Omega$ and the center of mass of $T' \cap \Omega$. This is illustrated by the arrows in Fig. 1(b).
- *The weights $\kappa_{T,i}$ in the definition of the interpolant.* In the extension $E_h v = \pi_h B_h v$ the weights determine the coefficient for φ_i , defining how much weight $(B_h v)|_T$ should be given for all T in the support of φ_i . For our numerical examples, we choose the weights associated with each basis function $\varphi_i \in \mathcal{B}$ as follows. If φ_i is a large basis function, $\varphi_i \in \mathcal{B}^L$, we set the weights for the elements its support, $T \in \mathcal{T}_{h,i}$, to

$$\kappa_{T,i} = \begin{cases} |T \cap \Omega|, & \text{if } T \in \mathcal{T}_h^L \\ 0, & \text{if } T \in \mathcal{T}_h^S \end{cases} \tag{107}$$

If φ_i is a small basis function, $\varphi_i \in \mathcal{B}^S$, we set the weights for the elements its support, $T \in \mathcal{T}_{h,i}$, to

$$\kappa_{T,i} = |T \cap \Omega| \tag{108}$$

We then normalize the weights associated with each basis function $\varphi_i \in \mathcal{B}$ such that $\sum_{T \in \mathcal{T}_{h,i}} \kappa_{T,i} = 1$. This choice of weights fulfills (53), which means that the coefficients for large basis functions are unaffected by the extension.

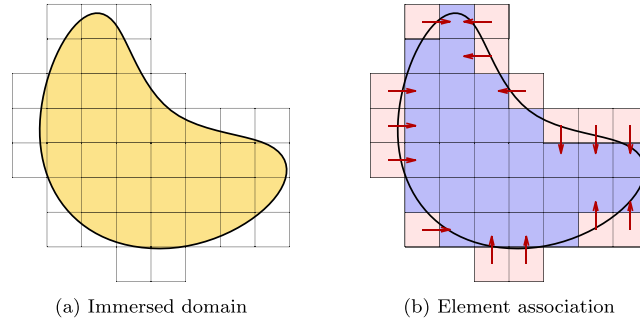


Fig. 1. Mesh and element association. (a) The domain Ω and the active mesh \mathcal{T}_h . (b) The case $\gamma = 0.5$ with corresponding large elements colored purple and small elements colored pink. The small-to-large element association S_h is illustrated by red arrows.

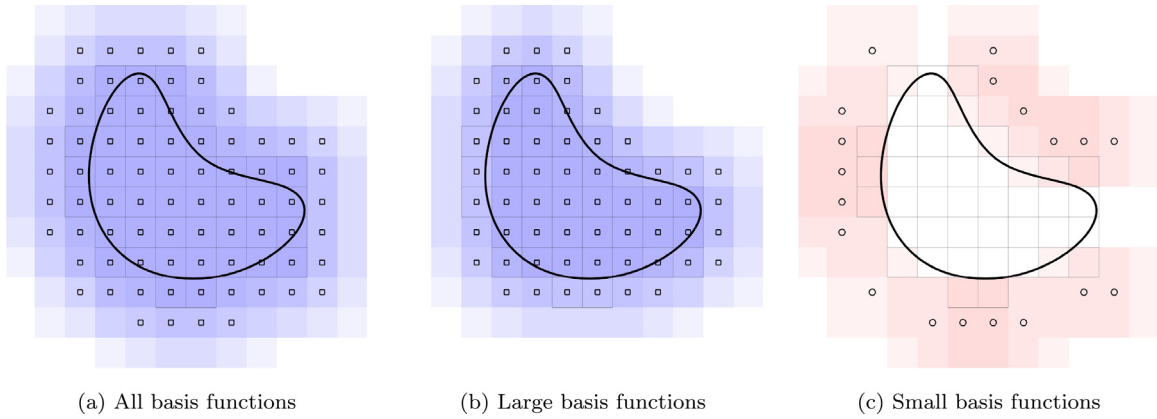


Fig. 2. Large and small basis functions. Partition of basis functions into large and small in the case $\gamma = 0.5$. Here full regularity B-spline basis functions of order $p = 2$ are used, and hence the support of a basis function will cover a 3×3 block of elements in the mesh. In the plots, each basis function is indicated by shading of its support and a marker at the center of its support.

As an illustration of how this realization of the extension couples small basis functions onto large basis functions, we present the supports of some extended basis functions in Fig. 3. Note that the above choices directly affect this outcome, where for instance an average with only a single non-zero entry per basis function would produce fewer couplings between any small basis functions and large basis functions.

Work flow. In practice the extension is applied to our method on a linear algebra level, by the following steps:

- Assemble the square linear system of equations for the original method

$$\widehat{A}\widehat{u} = \widehat{b} \tag{109}$$

where $\widehat{u} \in \mathbb{R}^{\dim(V_{h,p,k})}$ are coefficients for the full approximation space $V_{h,p,k}$.

- Given $\gamma \geq 0$, assemble the extension matrix $\widehat{E}_h \in \mathbb{R}^{\dim(V_{h,p,k}) \times \dim(V_{h,p,k}^L)}$ that maps the large degrees of freedom onto all degrees of freedom. An example pseudocode description of this assembly is provided in Fig. 4. In case $\gamma = 0$ the extension matrix \widehat{E}_h reduces to the identity matrix.
- Solve the reduced system

$$(\widehat{E}_h^T \widehat{A} \widehat{E}_h) \widehat{u}_\gamma = \widehat{E}_h^T \widehat{b} \tag{110}$$

where $\widehat{u}_\gamma \in \mathbb{R}^{\dim(V_{h,p,k}^L)}$ are coefficients for the extended space $V_{h,p,k}^E$.

- Expand \widehat{u}_γ in coefficients for the full approximation space $V_{h,p,k}$ via

$$\widehat{u}^E = \widehat{E}_h \widehat{u}_\gamma \tag{111}$$

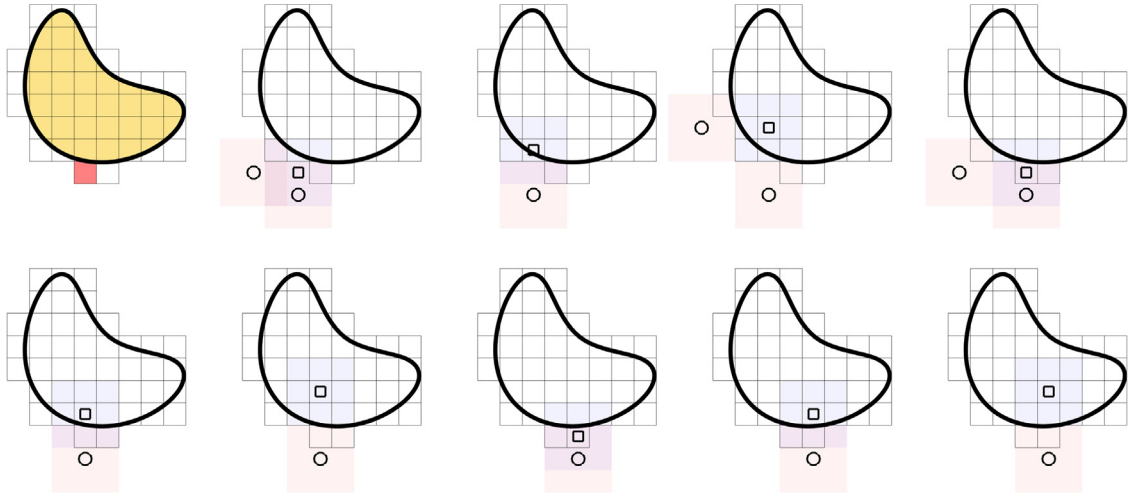


Fig. 3. *Extended basis support.* Illustrations of extended basis functions ($\gamma = 0.5$) which incorporate small basis functions with support in the element indicated in the upper left subplot. In each subplot the center of the original large basis function is indicated by a square and its support is shaded in light blue while the centers of associated small basis functions are indicated by circles and their supports are shaded in light pink. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Model problem. We consider the non-homogeneous version of the application examined in Section 5, i.e., we solve the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g_D \quad \text{on } \partial\Omega \tag{112}$$

using Nitsche’s method: find $u_h \in V_{h,p,k}^E$ such that

$$a_h(u_h, v) = l_h(v), \quad \forall v \in V_{h,p,k}^E \tag{113}$$

with a_h given by (96) and

$$l_h(w) = -(g_D, \nabla_n w)_{\partial\Omega} + \beta h^{-1} (g_D, w)_{\partial\Omega} \tag{114}$$

In all experiments, we use the Nitsche penalty parameter $\beta = 25p^2$. Note that this method does not include any additional stabilization terms for ensuring optimal stability properties regardless of how the domain cuts through the mesh, cf. [5,7,38].

In our quantitative experiments, we use the bean-shaped geometry in Fig. 1(a) as our domain Ω , whose boundary $\partial\Omega$ is constructed as the cubic spline interpolation of the periodic angular data points

$$\theta = \{0, -\pi/2, \pi/20, \pi/4, \pi/2, \pi, 3\pi/2, 0\} \tag{115}$$

$$x = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -0.8 \end{pmatrix}, \begin{pmatrix} 0.7 \\ -0.1 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}, \begin{pmatrix} -0.3 \\ 0.7 \end{pmatrix}, \begin{pmatrix} -0.8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -0.8 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \tag{116}$$

A problem with a known analytical solution is manufactured by deriving f and g_D from the ansatz $u = \frac{1}{10} (\sin(2x) + x \cos(3y))$, and we present a numerical solution to this problem in Fig. 5.

Worst case mesh. We embed the domain Ω in a structured quadrilateral mesh equipped with B-spline basis functions of order p and maximum regularity $k = p - 1$. To mitigate the effect of chance in the cut situations generated in our convergence and stability studies we for each mesh size h generate a sequence of 100 meshes by translating the original background mesh by $sh(\frac{1}{3})$, where s incrementally goes from 0 to 1, and then report the worst result from this sequence.

Convergence. In Fig. 6 we collect our convergence studies:

- Firstly, in (a)–(b) we consider convergence using $p = 2$ basis functions and extension for a wide range of $\gamma \in [0, 1]$. We note poor convergence results in the case without extension ($\gamma = 0$), likely due to loss of coercivity in the method in the worst cut situation. We also note that for larger γ the errors are initially somewhat higher, which seems reasonable since then more basis functions close to the boundary are extended.

procedure EXTENSIONMATRIX(γ)

$\mathcal{T}_h^L \leftarrow \emptyset, \mathcal{B}^L \leftarrow \emptyset$ ▷ Partition of elements and basis functions
for all $T \in \mathcal{T}_h$ **do**
 if $\gamma h^d \lesssim |T \cap \Omega|$ **then**
 $\mathcal{T}_h^L \leftarrow \mathcal{T}_h^L \cup T, \mathcal{B}^L \leftarrow \mathcal{B}^L \cup (\cup_{i=I_T} \varphi_i)$
 end if
end for
 $\mathcal{T}_h^S \leftarrow \mathcal{T}_h \setminus \mathcal{T}_h^L, \mathcal{B}^S \leftarrow \mathcal{B} \setminus \mathcal{B}^L$
for all $T \in \mathcal{T}_h^S$ **do** ▷ Small-to-large element association
 $S_h(T) \leftarrow \arg \min_{T' \in \mathcal{T}_h^L} \text{dist}(|T \cap \Omega|^{-1} \int_{T \cap \Omega} x \, dx, |T' \cap \Omega|^{-1} \int_{T' \cap \Omega} x \, dx)$
end for
 $\widehat{\mathcal{B}}_h \leftarrow \text{sparse}(|\mathcal{B}^{\text{dG}}|, |\mathcal{B}|)$ ▷ Preliminary extension from $V_{h,p,k}^L$ to a dG-space
for all $T \in \mathcal{T}_h$ **do**
 if $T \in \mathcal{T}_h^S$ **then**
 $T' \leftarrow S_h(T)$
 for $i \in I_T^{\text{dG}}$ **do**
 set $\widehat{c} \in \mathbb{R}^{1 \times |\mathcal{B}_{T'}|}$ such that $\varphi_i|_T = \sum_{j=1}^{|\mathcal{B}_{T'}|} \widehat{c}_j \varphi_{I_{T'}(j)}$
 $\widehat{B}_h(i, I_{T'}) \leftarrow \widehat{c}$
 end for
 else
 $\widehat{B}_h(I_T^{\text{dG}}, I_T) \leftarrow \text{Id}_{|\mathcal{B}_T|}$
 end if
end for
 $(\widehat{\mathcal{B}}_h)_{*, I^S} \leftarrow \emptyset$
 $\widehat{I}_h \leftarrow \text{sparse}(|\mathcal{B}|, |\mathcal{B}^{\text{dG}}|)$ ▷ Interpolation from the dG-space to $V_{h,k,p}$
for all $i \in I$ **do**
 for all $T \in \mathcal{T}_{h,i}$ **do**
 if $\varphi_i \in \mathcal{B}^L$ **and** $T \in \mathcal{T}_h^S$ **then**
 $\kappa_{T,i} \leftarrow 0$
 else
 $\kappa_{T,i} \leftarrow |T \cap \Omega|$
 end if
 end for
 normalize such that $\sum_{T \in \mathcal{T}_{h,i}} \kappa_{T,i} = 1$
 for all $T \in \mathcal{T}_{h,i}$ **do**
 find j such that $I_T(j) = i$
 $(\widehat{I}_h)_{i, I_T^{\text{dG}}(j)} \leftarrow \kappa_{T,i}$
 end for
end for
return $\widehat{E}_h \leftarrow \widehat{I}_h \widehat{\mathcal{B}}_h$ ▷ Final extension matrix
end procedure

Fig. 4. Extension matrix assembly. For a convenient description we here assume that B_h extends into a discontinuous Galerkin version of the space $V_{h,p,k}$, discontinuous between all elements. We let previously used notations extend to this dG-space, signified by superscript ‘dG’. The local ordering of the basis functions in the elementwise index sets is assumed fixed.

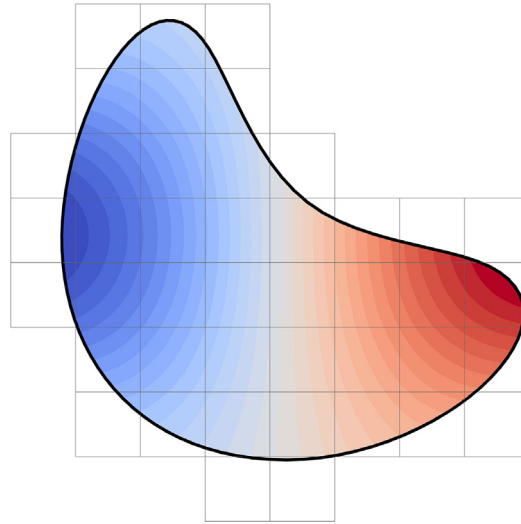


Fig. 5. Numerical solution. Model problem solution using extension with $\gamma = 0.5$ and full regularity B-spline basis function of order $p = 2$.

- Secondly, in (c)–(d) we consider convergence using $p = 1, 2, 3$ basis functions and extension with $\gamma = 1$ and note that optimal order convergence seems to be asymptotically obtained.

Condition number scaling. To ensure stable computation of the condition number we convert our sparse matrix A to a full storage matrix and employ dense linear algebra for the computation, i.e., using `cond(full(A))` in Matlab. In Fig. 7 we collect studies of how the condition number scales with h :

- Firstly, in (a)–(b) we consider the condition number h -scaling using $p = 2$ basis functions and extension for a wide range of $\gamma \in [0, 1]$. We note that without extension ($\gamma = 0$) the condition number seemingly can become unbounded. Also, for $\gamma > 0$ the size of γ seems to have the effect that for larger values, a smaller mesh size is required before entering the desired asymptotic h^{-2} scaling found in (106). This delay we attribute to the dependence of γ in the bound (91).
- Secondly, in (c) we consider condition number h -scaling using $p = 1, 2, 3$ basis functions and extension with $\gamma = 1$ and note that the desired h^{-2} scaling is plausibly asymptotically obtained in all cases, even though that stage is not quite reached in the case of $p = 3$.

In (d)–(f) we consider the same studies as in (a)–(c) albeit including simple preconditioning of the stiffness matrix using diagonal scaling.

The remaining h^{-2} dependence of the condition number we see in these studies is analogous to that of mesh-conforming methods and can be resolved by means of further multigrid preconditioning [39].

Isogeometric example on a surface. As an illustration that extension is directly applicable in an isogeometric setting [4], we consider the following problem posed on a curved surface. Let $\Gamma \subset \mathbb{R}^3$ be a surface described via the parametric mapping $\Phi : \Omega \rightarrow \Gamma \subset \mathbb{R}^3$ with induced Riemann metric $G \in \mathbb{R}^{2 \times 2}$, $G_{ij} = \partial_i \Phi \cdot \partial_j \Phi$. Using the lifting $v^\ell = v \circ \Phi^{-1}$ we can formulate the Dirichlet problem on Γ as seeking an unknown on the reference domain Ω ; find $u : \Omega \rightarrow \mathbb{R}$ such that

$$-\Delta_\Gamma u^\ell = f \quad \text{in } \Gamma, \quad u^\ell = g \quad \text{on } \partial\Gamma \tag{117}$$

where Δ_Γ is the Laplace–Beltrami operator on Γ . Transforming this problem back to Euclidean coordinates in Ω we can derive a Nitsche’s method for the problem defined by the forms

$$a_h(v, w) = (|G|^{1/2} \nabla v, G^{-1} \nabla w)_\Omega - (|G|^{1/2} n \cdot G^{-1} \nabla v, w)_{\partial\Omega} + (|G|^{1/2} v, \beta h^{-1} - n \cdot G^{-1} \nabla w)_{\partial\Omega} \tag{118}$$

$$l_h(v) = (|G|^{1/2} f \circ \Phi, v)_\Omega + (|G|^{1/2} g \circ \Phi, \beta h^{-1} v - n \cdot G^{-1} \nabla v)_{\partial\Omega} \tag{119}$$

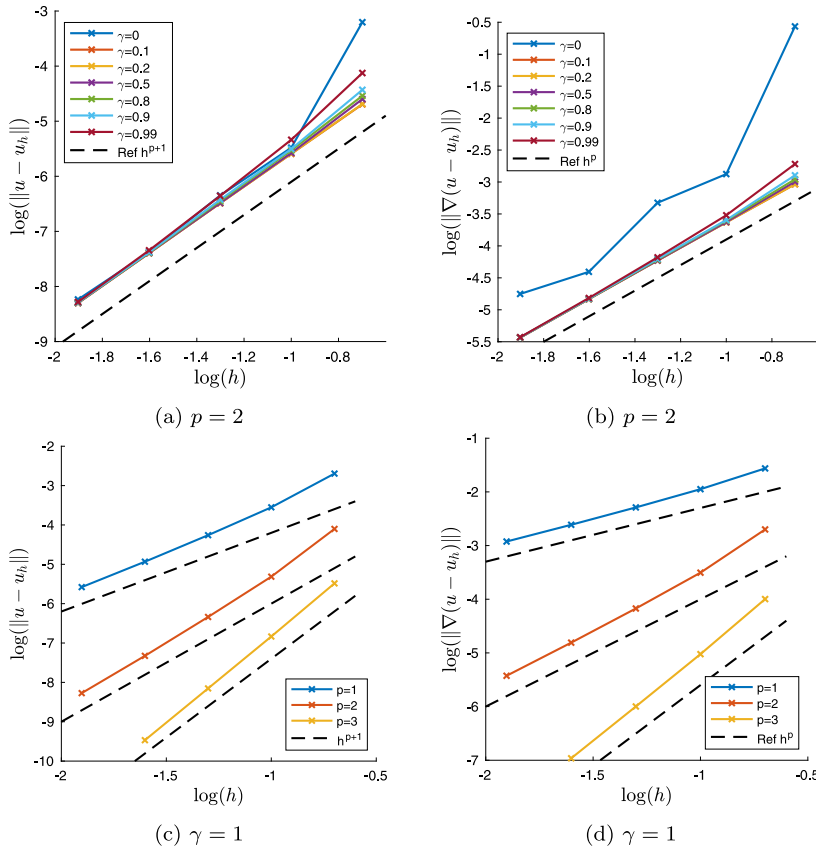


Fig. 6. Convergence. In these experiments, the largest error among 100 different cut situations is presented for each mesh size h . (a)–(b) Errors in L^2 -norm respectively in H^1 -seminorm when using extension for a wide range of $\gamma \in [0, 1]$ and full regularity B-spline basis functions of order $p = 2$. (c)–(d) Errors in L^2 -norm respectively in H^1 -seminorm using full regularity B-spline basis functions of orders $p = 1, 2, 3$ and extension with $\gamma = 1$.

Since the mesh \mathcal{T}_h and approximation space in this isogeometric setting are naturally defined on Ω , rather than on the curved surface Γ , we can directly apply the extension to the resulting system of equations. An example solution is presented in Fig. 8, where a circular reference domain is mapped onto a cone.

7. Conclusions

We provide a recipe for the construction of a family of extension operators for trimmed spline spaces that feature:

- *Proven approximation and stability properties.* The extension gives additional stabilization by eliminating the ill-conditioned ‘small’ degrees of freedom, expressing them in terms of well-conditioned ‘large’ degrees of freedom while maintaining optimal order accuracy for the extended space. This is reflected in a good performance in our numerical experiments.
- *Convenient implementation.* The extension describes a mapping from large degrees of freedom onto all degrees of freedom and is conveniently applied to the linear algebra formulation of the method. Also, since the extension induces the necessary stability, special stabilization terms or other forms of manipulation are not required to ensure robustness with respect to how the spline space is trimmed.
- *Natural choices in the construction.* The choices needed for the construction of the extension operator are quite natural; the parameter $\gamma \geq 0$ determining the partition into large and small elements that in turn gives the partition into large and small basis functions, the mapping S_h onto large elements, and the weights $w_{T,i}$ in the interpolation operator. During our numerical testing, we found no particular sensitivity in how these choices seem to affect performance.

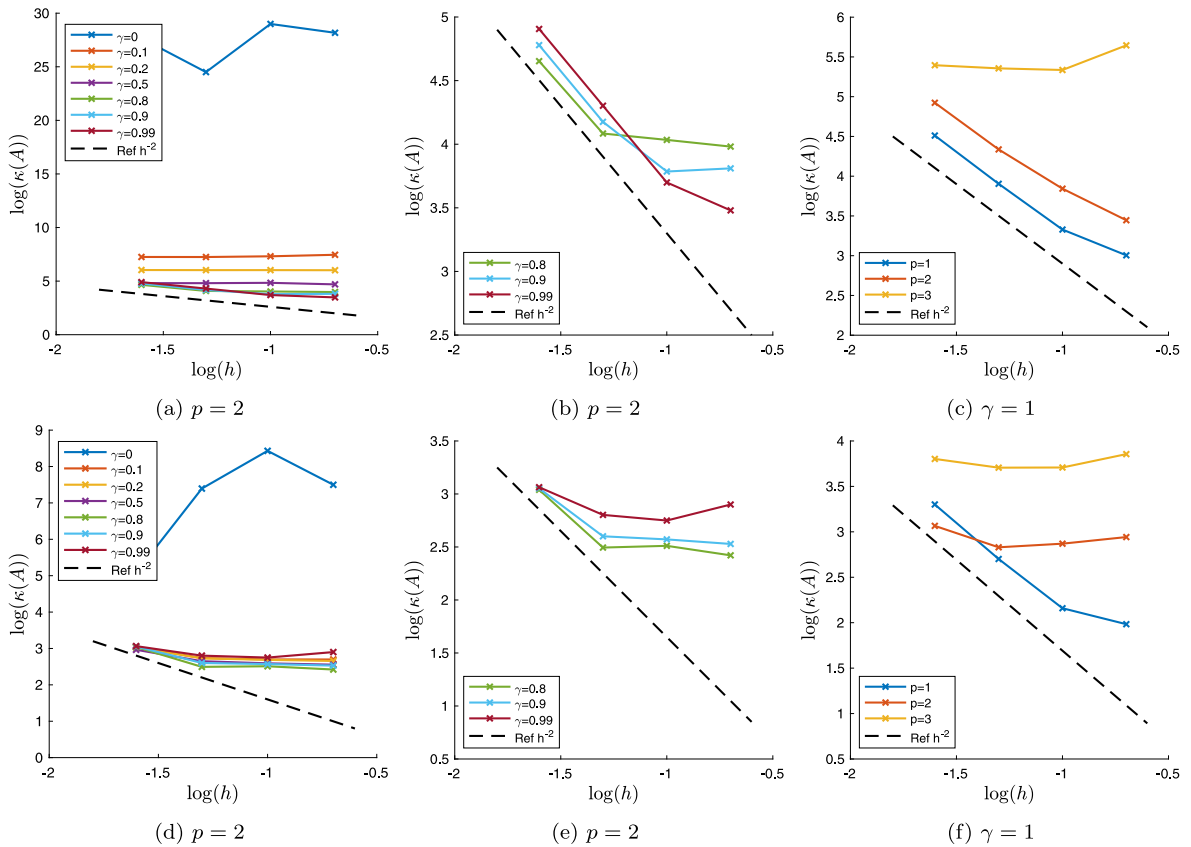


Fig. 7. Condition number h -scaling. In these experiments the worst stiffness matrix condition number among 100 different cut situations is presented for each mesh size h . (a) Condition numbers when using extension for a wide range of $\gamma \in [0, 1]$ and full regularity quadratic B-spline basis functions. (b) Detailed view of the study in (a) limited to the three largest values of γ . (c) Condition numbers using full regularity B-spline basis functions of orders $p = 1, 2, 3$ and extension with $\gamma = 1$. (d)–(f) Studies corresponding to (a)–(c) when the stiffness matrices are also preconditioned using diagonal scaling.

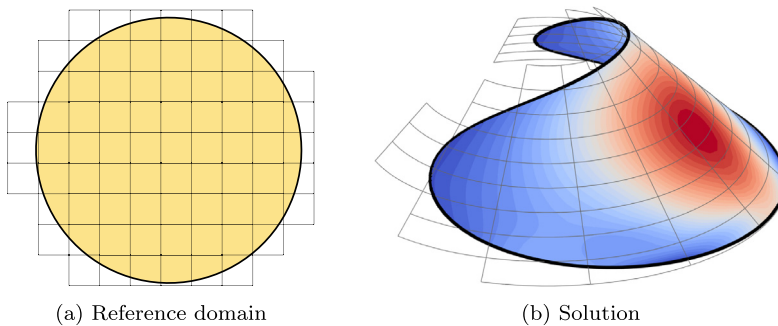


Fig. 8. Extension on a mapped domain. In this example, the domain is a curved surface Γ constructed as a map from the unit square onto the surface of a cone, and a circular trim curve in the reference domain Ω . (a) Reference domain Ω and the mesh on which the approximation space is defined. (b) Numerical solution for a Dirichlet problem using extension ($\gamma = 0.5$).

Overall, we find discrete extension a very attractive technology for stabilizing trimmed spline spaces, with solid performance and a most convenient implementation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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