# Strongly convergent unitary representations of limit groups

Larsen Louder and Michael Magee
With appendix by Will Hide and Michael Magee

**Abstract**

We prove that all finitely generated fully residually free groups (limit groups) have a sequence of finite dimensional unitary representations that 'strongly converge' to the regular representation of the group. The corresponding statement for finitely generated free groups was proved by Haagerup and Thorbjørnsen in 2005. In fact, we can take the unitary representations to arise from representations of the group by permutation matrices, as was proved for free groups by Bordenave and Collins.

As for Haagerup and Thorbjørnsen, the existence of such representations implies that for any non-abelian limit group, the Ext-invariant of the reduced $C^*$-algebra is not a group (has non-invertible elements).

An important special case of our main theorem is in application to the fundamental groups of closed orientable surfaces of genus at least two. In this case, our results can be used as an input to the methods previously developed by the authors of the appendix. The output is a variation of our previous proof of Buser's 1984 conjecture that there exist a sequence of closed hyperbolic surfaces with genera tending to infinity and first eigenvalue of the Laplacian tending to $\frac{1}{4}$. In this variation of the proof, the systoles of the surfaces are bounded away from zero and the surfaces can be taken to be arithmetic.

# Contents



# 1 Introduction

A discrete group $\Gamma$ is *fully residually free* (FRF) if for any finite set $S \subset \Gamma$, there exists a homomorphism $\Gamma \to \mathbf{F}$ that is injective on $S$ where $\mathbf{F}$ is a free group. Finitely generated FRF groups are known to coincide with Sela's *limit groups* [Sel01], so we use these two notions interchangeably in the sequel.

For $N \in \mathbf{N}$ let $\mathbf{U}(N)$ denote the group of $N \times N$ complex unitary matrices. For a discrete group $\Gamma$, $\lambda_\Gamma : \Gamma \to \mathrm{End}(\ell^2(\Gamma))$ is the left regular representation. It was an open problem for some years, popularized by Voiculescu in [Voi93, Qu. 5.12], whether for a finitely generated free group $\mathbf{F}$, there exists a sequence of unitary representations $\{\rho_i : \mathbf{F} \to \mathbf{U}(N_i)\}_{i=1}^\infty$ such that for any element $z \in \mathbf{C}[\mathbf{F}]$,

$$\limsup_{i\to\infty} \|\rho_i(z)\| \leq \|\lambda_{\mathbf{F}}(z)\|.$$

The norm on the left is the operator norm on $\mathbf{C}^{N_i}$ with respect to the standard Hermitian metric, and the norm on the right is the operator norm on $\ell^2(\Gamma)$. This problem was solved in the affirmative in a huge breakthrough by Haagerup and Thorbjørnsen [HT05].

In fact, following [Voi93], given that the reduced $C^*$-algebra of $\mathbf{F}$ is simple by a result of Powers [Pow75], the inequality above can be improved automatically[1] to

$$\lim_{i\to\infty} \|\rho_i(z)\| = \|\lambda_{\mathbf{F}}(z)\| \quad \forall z \in \mathbf{C}[\mathbf{F}]. \tag{1.1}$$

This notion of convergence of a sequence of finite dimensional unitary representations given by (1.1) applies equally as well to any discrete group $\Gamma$ and we refer to this as *strong convergence.*

**Theorem 1.1.** *Any limit group $\Gamma$ has a sequence of finite dimensional unitary representations that strongly converge to the regular representation of $\Gamma$. In fact, these unitary representations can be taken to factor through*

$$\Gamma \to S_N \xrightarrow{\mathrm{std}} \mathbf{U}(N-1) \tag{1.2}$$

*for some varying $N$, where $S_N$ is the group of permutations of $N$ letters, and* std *is the $N-1$ dimensional irreducible component of the representation of $S_N$ by 0-1 matrices.*

It was proved by G. Baumslag in [Bau62] that the fundamental groups $\Lambda_g$ of closed orientable surfaces are FRF, and it is also known [Bau67, pp. 414-415] that the fundamental groups of non-orientable surfaces $S$ with $\chi(S) \leq -2$ are FRF. This gives the following corollary of Theorem 1.1.

**Corollary 1.2.** *Let $\Gamma$ denote the fundamental group of a connected closed surface $S$ that is either orientable with no constraint on $\chi(S)$, or non-orientable with $\chi(S) \leq -2$. Then $\Gamma$ has a sequence of finite dimensional unitary representations that strongly converge to the regular representation. Moreover, they can be taken to be of the form (1.2) for some varying $N$.*

Corollary 1.2 leaves open the cases of connected non-orientable surfaces with $\chi = 1$ ($\mathbf{R}P^2$), $\chi = 0$ (the Klein bottle), and $\chi = -1$. In all these cases the corresponding fundamental groups are not FRF[2]. The fundamental group of $\mathbf{R}P^2$ is $\mathbf{Z}/2\mathbf{Z}$, and its regular representation is finite dimensional. We prove the following.

[1] See proof of Theorem 1.1 below for details.

[2] The first two cases are easy to check, and the case of $\chi = -1$ is due to Lyndon [Lyn59].

**Proposition 1.3.** *The Klein bottle group $\langle\, a, b \,|\, b^{-1} = aba^{-1} \,\rangle$ has a sequence of finite dimensional unitary representations that strongly converge to its regular representation.*

This leaves open the seemingly significant problem of extending Theorem 1.1 to the fundamental group of the connected sum of three copies of the real projective plane.

The proof of Theorem 1.1 revolves around the following potential property of discrete groups that we introduce here.

**Definition 1.4.** A discrete group $\Gamma$ is *$C^*$-residually free* if for any finite set $S$ and $\epsilon > 0$, there is a homomorphism $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(z))\| \leq \|\lambda_\Gamma(z)\| + \epsilon.$$

for all $z \in \mathbf{C}[\Gamma]$ supported on $S$ with unit $\ell^1$ norm.

**Example 1.5.** Any extension $N \to G \xrightarrow{\phi} \mathbf{F}$ of a free group by an amenable group $N$ is $C^*$-residually free. Indeed, since $N$ is amenable 1 is weakly contained in the regular representation of $N$. Then by Fell's continuity of induction ([Fel62], [BdlHV08, Thm. F.3.5]) we have that the quasi-regular representation of $G$ on $\ell^2(G/N)$ is weakly contained in the regular representation $G$, hence by [BdlHV08, Thm. F.4.4] for any $z \in \mathbf{C}[\mathbf{F}]$

$$\|\lambda_{G/N}(zN)\| = \|\lambda_{\mathbf{F}}(\phi(z))\| \leq \|\lambda_G(z)\|.$$

Here we prove the following.

**Theorem 1.6.** *Limit groups are $C^*$-residually free.*

The converse to Theorem 1.6 does not hold: Example 1.5 shows that $\mathbf{Z} \times \mathbf{F}$ is $C^*$-residually free, but it is easy to see that it is not FRF. It is, however, also easy to see that it is residually free[3].

Given a free group $\mathbf{F}$, and a basis $X$ of $\mathbf{F}$, we write $|f|_X$ for the word length of $f$ in the basis $X$. In any discrete group $\Gamma$ with generating set we write $B_Y(r)$ for the elements of $\Gamma$ that can be written a product of as most $r$ elements of $Y \cup Y^{-1}$. The proof of Theorem 1.6 relies on the following key proposition.

[3]It is an interesting question, not pursued here, to give an alternative characterization of a group being $C^*$-residually free.

**Proposition 1.7.** *Let $\Gamma$ be a limit group with a fixed finite generating set $Y$. There is $D = D(\Gamma, Y) > 0$ and $C = C(\Gamma, Y) > 0$ such that for any $r > 0$ there is an epimorphism $f : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free, which is injective on $B_Y(r)$, and a basis $X$ of $\mathbf{F}$ such that*

$$\max_{g \in B_Y(r)} |f(g)|_X \leq C r^D.$$

## 1.1 Further consequences I: Spectral gaps

A hyperbolic surface is a complete Riemannian surface (without boundary) of constant curvature -1 . Given a hyperbolic surface $X$, we write $\Delta_X$ for the Laplace-Beltrami operator on $L^2(X)$. If $X$ is closed this operator's spectrum $\mathrm{spec}(\Delta_X)$ consists of eigenvalues $0 = \lambda_0(X) \leq \lambda_1(X) \leq \cdots \leq \lambda_k(X) \leq \cdots$ with $\lambda_k(X) \to \infty$ as $k \to \infty$. It was a conjecture of Buser [Bus84] whether there exist a sequence of closed hyperbolic surfaces $X_i$ with genera tending to infinity and with

$$\lambda_1(X_i) \to \frac{1}{4}$$

where $\lambda_1$ denotes the first non-zero eigenvalue of the Laplacian. The value $\frac{1}{4}$ is the asymptotically optimal one by a result of Huber [Hub74]. See [HM, Introduction] for an overview of the rich history of this problem. Buser's conjecture was settled in [HM]. One interesting feature of the proof therein is that the surfaces constructed may have very short closed curves, so there is no control on the systole: the shortest closed curve on the surface. The results of this work in conjunction with the ideas in [HM] allow us, along with Hide, to prove:

**Theorem 1.8.** *There exists a sequence of closed hyperbolic surfaces $\{X_i\}_{i \in \mathbb{N}}$ with $g(X_i) \to \infty$, systoles uniformly bounded away from zero, and with*

$$\lambda_1(X_i) \to \frac{1}{4}.$$

*In fact the $X_i$ can be taken to be covering spaces of a fixed hyperbolic surface $X$. This base surface $X$, and hence all the $X_i$, can be taken to be arithmetic.*

Theorem 1.8 is proved in the Appendix[4] by the second named author (MM) and Hide, as a consequence of the following corollary of Theorem 1.1.

[4] In fact, the Appendix proves a more general statement about coverings of *any* hyperbolic surface; see Theorem A.1.

**Corollary 1.9** (Matrix coefficients version of Theorem 1.1)**.** *Let $\Gamma$ be a non-abelian limit group. There exist a sequence of finite dimensional representations $\rho_i$ of the form (1.2) such that for any $r \in \mathbf{N}$ and finitely supported map $a : \Gamma \to \mathrm{Mat}_{r\times r}(\mathbf{C})$, we have*

$$\limsup_{i\to\infty} \|\sum_{\gamma\in\Gamma} a(\gamma) \otimes \rho_i(\gamma)\| \leq \|\sum_{\gamma\in\Gamma} a(\gamma) \otimes \lambda(\gamma)\|.$$

*The norm on the left hand side is the operator norm for the tensor product of ($r$ and $N_i$-dimensional) $\ell^2$ norms. The norm on the right is the operator norm for the tensor product of $\ell^2$ and the inner product on $\ell^2(\Gamma)$.*

The proof of Corollary 1.9 from Theorem 1.1 is explained in §7.

## 1.2 Further consequences II: $\mathrm{Ext}(\Gamma)$ is not a group

In [BDF73, BDF77], Brown, Douglas, and Fillmore introduced and studied a homological/K-theoretic invariant $\mathrm{Ext}(\mathcal{A})$ of a unital separable $C^*$-algebra $\mathcal{A}$. By definition, $\mathrm{Ext}(\mathcal{A})$ is the collection of $*$-homomorphisms

$$\pi : \mathcal{A} \to B(\ell^2(\mathbf{N}))/\mathcal{K}$$

modulo conjugation of unitary operators on $\ell^2(\mathbf{N})$, where $B(\ell^2(\mathbf{N}))$ is the bounded operators on $\ell^2(\mathbf{N})$ and $\mathcal{K}$ is the ideal of compact operators therein. This is naturally a semigroup with multiplication arising from $(\pi_1, \pi_2) \mapsto \pi_1 \oplus \pi_2$ composed with an isomorphism $\ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N}) \cong \ell^2(\mathbf{N})$.

One of the motivations of the work of Haagerup and Thorbjørnsen [HT05] was to prove that there are non-invertible elements of $\mathrm{Ext}(C_r^*(\mathbf{F}))$ when $\mathbf{F}$ is a finitely generated non-abelian free group, i.e., $\mathrm{Ext}(C_r^*(\mathbf{F}))$ is not a group.

The passage from the existence of strongly convergent unitary representations of $\mathbf{F}$ to this statement uses the following result proved by Voiculescu in [Voi93, §§5.14] (see [HT05, Rmk. 8.6] for another exposition).

**Proposition 1.10.** *If $\Gamma$ is a discrete, countable, non-amenable group with a sequence of finite dimensional unitary representations that strongly converge to the regular representation of $\Gamma$, then $\mathrm{Ext}(C_r^*(\Gamma))$ is not a group.*

Since non-abelian limit groups $\Gamma$ are $C^*$-simple (Lemma 5.2), they are non-amenable. Indeed, an amenable group $\Gamma$ has a $C^*$-algebra morphism $C_r^*(\Gamma) \to \mathbf{C}$ by [BdlHV08, Thm. F.4.4] whose kernel contradicts simplicity. Hence combining Theorem 1.1 with Proposition 1.10 we obtain the following extension of '$\mathrm{Ext}(C_r^*(\mathbf{F}))$ is not a group':

**Corollary 1.11.** *If* $\Gamma$ *is a non-abelian limit group, then* $\mathrm{Ext}(C_r^*(\Gamma))$ *is not a group.*

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# 2 Background

### Groups

We write $e$ for the identity in any group. For any group $\Gamma$, $\mathbf{C}[\Gamma]$ denotes the group algebra of $\Gamma$ with complex coefficients. For a free group $\mathbf{F}$ with a fixed set of generators $X$, for each $h \in \mathbf{F}$, we write $|h|_X$ for the reduced word length of $h$ with respect to $X$. If $Y$ is a symmetric generating set of any group $\Gamma$, we write $B_Y(r) \subset \Gamma$ for the elements of $\Gamma$ that can be written as the product of at most $r$ elements of $Y$.

### Analysis

Given a discrete group $\Gamma$, $\lambda_\Gamma : \Gamma \to \mathrm{End}(\ell^2(\Gamma))$ is the *left regular representation*

$$\lambda_\Gamma(g)[f](h) \stackrel{\mathrm{def}}{=} f(g^{-1}h).$$

This representation extends by linearity to one of the convolution algebra $\ell^1(\Gamma)$. For $\psi \in \ell^1(\Gamma)$, since $\lambda$ is unitary we have the basic inequality

$$\|\lambda(\psi)\| \leq \|\psi\|_{\ell^1} \tag{2.1}$$

where the norm on the left is operator norm. The *reduced* $C^*$*-algebra* of $\Gamma$, denoted $C_r^*(\Gamma)$, is the closure of $\lambda(\ell^1(\Gamma))$ with respect to the operator norm topology. A *tracial state* on a unital $C^*$ algebra $\mathcal{A}$ is a linear functional $\tau$ such that $\tau(1) = 1$, $\tau(a^*a) \geq 0$ (in particular, is real) for all $a \in \mathcal{A}$ , and $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$.

An important inequality due to Haagerup [Haa79] links the operator norm in $\mathrm{End}(\ell^2(\mathbf{F}))$ and the $\ell^2$ norm in $\mathbf{C}[\mathbf{F}]$.

**Lemma 2.1** (Haagerup)**.** *Let $X$ denote a finite generating set for a free group $\mathbf{F}$. Suppose that $a \in \mathbf{C}[\mathbf{F}]$ is supported on $B_X(r)$. Then*

$$\|\lambda_{\mathbf{F}}(a)\| \leq (r+1)^{\frac{3}{2}} \|a\|_{\ell^2}.$$

*Proof.* Haagerup in [Haa79, Lemma 1.4] proved that

$$\|\lambda_{\mathbf{F}}(a)\| \leq \sum_{i=0}^{\infty} (i+1) \|a_i\|_{\ell^2}$$

where $a_i$ is the function $a$ multiplied pointwise by the indicator function of $B_X(i) \backslash B_X(i-1)$, i.e. the sphere of radius $i$. If $a$ is supported on $B_X(r)$ then using Cauchy-Schwarz above gives the result, since $\sum_{i=0}^{r} \|a_i\|_{\ell^2}^2 = \|a\|_{\ell^2}^2$. ☐

There is also a more basic inequality in the reverse direction that holds for arbitrary discrete groups. Suppose that $\Gamma$ is a discrete group. Then let $\delta_e \in \ell^2(\Gamma)$ denote the indicator function of the identity. We have for $a \in \mathbf{C}[\Gamma]$

$$\|a\|_{\ell^2}^2 = \langle \lambda(a)\delta_e, \lambda(a)\delta_e \rangle \leq \|\lambda_\Gamma(a)\|^2. \tag{2.2}$$

# 3 Proof of Proposition 1.7

The proof of Proposition 1.7 relies on the deep fact that any limit group embeds in an iterated extension of centralizers of a free group, and quantified versions of theorems of Gilbert and Benjamin Baumslag.

**Definition 3.1.** Let $\Gamma$ be a limit group, $A < \Gamma$ a maximal abelian subgroup. A group $\Gamma' = \Gamma *_A B$, $B = A \times \langle t \rangle$ is an *extension of centralizers* of $\Gamma$. A group $\Gamma$ is an *iterated extension of centralizers* if there is a chain of subgroups

$$\mathbf{F} = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_n = \Gamma$$

such that $\Gamma_{i+1}$ is an extension of centralizers of $\Gamma_i$. The *height* of the extension is $n$.

Any iterated extension of centralizers is fully residually free, and so are their finitely generated subgroups, hence such subgroups are limit groups. Amazingly, the converse holds: any limit group actually embeds in a (finitely) iterated extension of centralizers. This was first claimed by Kharlampovich and Myasnikov in their papers on the Tarski problem [KM98, Theorem 4]. For a proof following Sela see [CG05, Theorem 4.2]. The forward implication seems to be contained in Lyndon's original paper on his free exponential group [Lyn60, last two paragraphs, page 533], which is the direct limit over the family of all iterated extensions of centralizers of $\mathbf{F}$, ordered by inclusion. See also [BMR02, Theorem C1].

Let $\Gamma$ be a limit group with some fixed generating set $Y$. The *distortion function of* $\Gamma$ *with respect to* $Y$ is the function

$$d_Y(r) = \min_{\substack{f\colon \Gamma\to\mathbf{F}\\ X\subset\mathbf{F}}} \max_{g\in B_Y(r)} |f(g)|_X \,,$$

where the minimum is over all $f\colon \Gamma \to \mathbf{F}$ which are injective on $B_Y(r)$ and $X$ which are bases of $\mathbf{F}$. The proof of Proposition 1.7 is a recapitulation of the proof that an iterated extension of centralizers is fully residually free in a way that lets us bound the distortion function by a polynomial whose degree depends on the height. We start with an improvement of Baumslag's power lemma.

**Lemma 3.2** (cf. [Bau62, Proposition 1]). *Let $u, b_1, \ldots, b_n$, reduced words in $\mathbf{F}$, with $u$ also cyclically reduced, nontrivial, and not a proper power of another element. If*

$$w = \prod_{i=0}^{n} u^{k_i} b_i = e \tag{3.1}$$

*for*

$$\min_{i>0}\{|k_i|\} > (8n+2)\cdot \max_{i\geq 0}\{1, |b_i|/|u|\}\,,$$

*then $[u, b_i] = e$ for some $i$.*

G. Baumslag proved the same thing if $w = e$ for infinitely many integral values of each of the $k_i$. See also the proof of [Wil09, Lemma 4.13], which has, implicitly, an effective version of Lemma 3.2 in it.

*Proof.* The proof is by induction on $n$. Clearly for $n = 0$, if $u^{k_0} b_0 = e$ then $b_0$ is a power of $u$ and hence commutes with $u$.

We begin by manipulating our hypothesis to a more convenient form for the induction. If

$$\min_{i>0}\{|k_i|\} > (8n+2)\cdot \max_{i\geq 0}\{1, |b_i|/|u|\}$$

then

$$|w| > |u|\cdot \sum_{i>0} |k_i| > (8n+2)\cdot \max_{i\geq 0}\{|u|, |b_i|\}. \tag{3.2}$$

(Here $|w|$ is the non-reduced length of $w$.)

Let $T$ be the tree of cancellations for $w$. A vertex of $T$ is *special* if it either corresponds to an endpoint of one of the subwords $u^{k_i}$, one of the $b_i$, or has valence at least three. An embedded segment in $T$ with special endpoints and no special vertices in its interior is a *long edge*.

Every valence one vertex in $T$ is special, so there are at most $2n+2$ of them. We now work out the maximal number of long edges in a tree with at most $2n+2$ valence one vertices, which will happen when the number $q_{\geq 3}$ of vertices of valence at least 3 is maximized. Let $q_m$ be the number of valence $m$ vertices in $T$. Then

$$1 = \chi(T) = \sum q_m(1 - m/2) \leq \frac{2n+2}{2} - \frac{1}{2} q_{\geq 3}$$

implies $q_{\geq 3} \leq 2n$, there are at most $4n+2$ special vertices, and there are at most $4n+1$ long edges.

The sum of the lengths of the long edges is $|w|/2$, so there is a long edge of length at least $|w|/(8n+2)$, which from (3.2) is at least

$$\max_{i \geq 0} \{|u|, |b_i| + 1\} .$$

If this is the case, since the endpoints of the $b_i$ are special, the long edge is covered only by subsegments of powers of $u$. Because $u$ is not a proper power, the segment (with a fixed direction) corresponds to a unique reduced expression of the form $u_0 u^a u_1$ where $u_0$ and $u_1$ are proper subwords of $u$ and $a > 0$. (Otherwise, one is led to the conclusion that $u$ can be written as a reduced product of reduced words $u = pq = qp$, and by [Raz14, Lemma 2.2], this contradicts $u$ being a proper power.) Let us now fix the direction of the long edge so $a > 0$.

The upshot of this unique expression is that the term $u_0$ corresponds to a terminal subsegment of a $u$ as written in (3.1) (part of a $u^{k_i}$ with $k_i > 0$), for each time the long edge is traversed in its given direction. If the long edge is traversed in the other way by the path of $w$, then the $u_0$ segment corresponds to an initial subsegment of a $u^{-1}$ in a $u^{k_i}$ with $k_i < 0$.

Fix an endpoint $v$ of the $u_0$ segment in the long edge. Consider the subpaths of the path of $w$ punctuated by returns to $v$. After cutting the tree at $v$, there must be at least one $b_i$ subpath on either half of the resulting forest. So there must be some closed subpath of $w$ beginning and ending at $v$ and corresponding, possibly after cyclic rotation of $w$, to a subsequence

$$u^{k_0} b_0 u^{k_1} b_1 \cdots u^{k_j - a} \underbrace{u^a b_j u^{k_{j+1}} \cdots u^{k_l} b_l u^c}_{\downarrow} u^{k_{l+1} - c} b_{l+1} \cdots u^{k_n} b_n$$

with $l - j < n$, and

$$u^a b_j u^{k_{j+1}} \cdots u^{k_l} b_l u^c = e\,,$$

which implies

$$u^{a+c} b_j u^{k_{j+1}} \cdots u^{k_l} b_l = e.$$

Reducing $a + c$, we can use the inductive hypothesis to conclude that for some $j$, $[u, b_j] = e$. (Note that this is where the minimum of $k_i$ only over $i > 0$ is useful in the induction; $a + c$ could in principle be very small.) $\square$

A similar result holds when $u$ is not necessarily cyclically reduced and is a power, e.g., $u = ps^l p^{-1}$, with $s$ cyclically reduced and $|u| = 2|p| + l|s|$. Rewrite the expression for $w$ as

$$e = w = \prod ps^{lk_i} p^{-1} b_i$$

conjugate by $p^{-1}$, and absorb the $p$'s into the $b$'s to get

$$e = w' = \prod s^{lk_i} b'_i\,.$$

Then the same conclusion clearly holds when

$$l \min_{i \geq 0}\{|k_i|\} > (8n + 2) \cdot \max_{i \geq 0}\{1, (|b_i| + 2|p|)/|s|\}\,.$$

For the applications, since $|u| = l|s| + 2|p| > 2|p|$ so we can use instead the easier to use yet still sufficient inequality

$$\min_{i \geq 0}\{|k_i|\} \geq (8n + 2) \cdot \max_{i \geq 0}\{|b_i| + |u|\}\,, \tag{3.3}$$

which gives the same conclusion. Note that this minimum of $k_i$ is now over all $i$, and not just $i > 0$. (The latter was just more convenient for the previous induction.)

In what follows, $\Gamma$ is a limit group with a fixed finite generating set $Y$, $A$ is a maximal abelian subgroup in $\Gamma$, and $\Gamma'$ is the extension of centralizers $\Gamma' = \Gamma *_A B$, where $B = A \times \langle t \rangle$. Let $Y' = Y \cup t$. The standard fact about amalgamated products lets us write any element of $\Gamma'$ in normal form:

$$\prod_{i=0}^{n} \beta_i \gamma_i$$

with $\gamma_i \in \Gamma$, $\beta_i \in B$, so that the only elements which are allowed to be trivial are $\beta_0$ and $\gamma_n$, and if any of them are in $A$ then the expression has length one – if, say, $\beta_i \in A$, then $\gamma_i \beta_i \gamma_{i+1} \in \Gamma$ and we group these together. Normal forms are unique (up to insertion of $\alpha\alpha^{-1}$ pairs), but we will not use this fact. Given an element as above, write each $\beta_i$ (uniquely) as $t^{n_i}\alpha_i$ with $\alpha_i \in A$

$$\prod_{i=0}^{n} t^{n_i} \alpha_i \gamma_i$$

and absorb the $\alpha_i$ into an adjacent $\gamma_i$ or $\gamma_{i+1}$ to get

$$\prod_{i=0}^{n} t^{n_i} v_i$$

with each $v_i \in \Gamma$. If this isn't possible, leave it alone. In this case the word is of the form

$$t^n \alpha\,,$$

for some $\alpha \in A$. For the purposes of this argument, a word is in normal form if it is of either of these two types.

**Lemma 3.3** (QI lemma)**.** *There is a constant $K$ such that if $w$ is a word in $Y'$, then there is a word $w' =_\Gamma w$ in normal form such that $|w'| \leq K \cdot |w|$.*

*Proof.* The word $w$ can be put in normal form by replacing subwords which are elements of $A$ in one step. Since $A$ is maximal abelian in $\Gamma$ it is quasiconvex by [Ali05, Theorem 3.4], and the rewritten word can only increase in length by a factor of $K$, where $1/K$ is the shrinking factor of the embedding $A \hookrightarrow \Gamma'$. ☐

**Lemma 3.4** (cf. [Bau67, Lemma 7, Theorem 8])**.** *Let $K$ be the constant from Lemma 3.3, and fix $a \in A \backslash \{e\}$. Then*

$$d_{Y'}(r/K) \leq (8r^2 + 4r) d_Y(2(r + |a|))^2\,.$$

*If $d_Y(r)$ is a polynomial of degree $D$ then $d_{Y'}(r)$ is bounded above by a polynomial of degree $2D + 2$.*

This is essentially a version of B. Baumslag's generalization of G. Baumslag's version of Lemma 3.2 from free groups to limit groups, where we keep track of the constants and avoid the phrases "sufficiently large" and "as large as we like."

*Proof.* Let $\pi\colon \Gamma' \to \Gamma$ be the retraction to $\Gamma$ defined by $\pi(t) = e$, let $\tau$ be the automorphism of $\Gamma'$ fixing $\Gamma$ with

$$\tau(t) \stackrel{\text{def}}{=} ta.$$

We will find an $h\colon \Gamma' \to \mathbf{F}$ which is injective on normal forms in $Y'$ of length at most $r$ and doesn't stretch too much.

Suppose first that $g$ has the normal form

$$\prod t^{n_i} v_i\,,$$

which is a product of at most $\lfloor r/2 \rfloor$ terms $t^{n_i} v_i$. Then

$$f \circ \pi \circ \tau^m(g) = \prod f(a)^{mn_i} f(v_i)\,.$$

In order to use Lemma 3.2 we need to choose $f$ so that $f([a, v_i]) \neq e$. In the worst case the commutator $[a, v_i]$ has length at most

$$L \stackrel{\text{def}}{=} 2(r + |a|).$$

Choose $f\colon \Gamma \to \mathbf{F}$ and a basis $X$ of $\mathbf{F}$ such that

$$d_Y(L) = \max_{g \in B_Y(L)} |f(g)|_X$$

and $f$ embeds $B_Y(L) \subset \Gamma$. By (3.3), with $k_i = mn_i$, $n = \lfloor r/2 \rfloor$, $u = f(a)$, and $b_i = f(v_i)$, as long as

$$\min\{|mn_i|\} \geq (4r + 2) \cdot \max\{|f(v_i)|_X + |f(a)|_X\}$$

$f \circ \pi \circ \tau^m(g)$ is nontrivial. In the worst case $n_i = 1$ for all $i$ and $f(a)$ and $f(v_i)$ have length $d_Y(L)$, so choose $m = m(r, |a|) = (4r + 2) \cdot 2d_Y(L)$ and let

$$h = f \circ \pi \circ \tau^m.$$

We continue to use the same basis $X$ for $\mathbf{F}$. Now overestimate the length of $h(g)$: the normal form which can be expanded the most is $t^r$, so we have $r \cdot m$ terms whose images have length at most $d_Y(L)$, and therefore

$$|h(g)|_X \leq r \cdot \underbrace{(8r + 4) \cdot d_Y(L)}_{m} \cdot d_Y(L) = (8r^2 + 4r) d_Y(L)^2\,.$$

If $g$ is of the form $t^n \alpha$ then the worst that can happen is $n = 1$ and $\alpha$ has length at most $r - 1$, $h(g) = f(a)^m f(\alpha)$, but in this case

$$m \cdot d_Y(L) \geq m \cdot |f(a)|_X \geq |h(t)|_X \geq m > d_Y(r - 1) \geq |f(\alpha)|_X\,,$$

so $h(g)$ is nontrivial – $h(t)$ and $f(\alpha)$ cannot fully cancel since $m > d_Y(r-1)$, and is not longer than $m \cdot d_Y(L) + d_Y(r-1)$, which is less than $r \cdot m \cdot d_Y(L)$.

Now by Lemma 3.3 if $g \in B_{Y'}(r/K)$, it has a normal form of length at most $r$ and $h(g) \neq e$, so $h$ embeds $B_{Y'}(r/K)$, $|h(g)|_X \leq (8r^2 + 4r)d_Y(L)^2$, and the first part of the lemma follows.

The statement about degrees is obvious since $L$ is linear in $r$. ☐

**Corollary 3.5.** *Let $\Gamma$ be a limit group, and suppose $\Gamma$ embeds in an extension of centralizers of height $n$. Then $d_Y(r)$ is bounded above by a polynomial in $r$ of degree*

$$D(n) = 2^{n+2} - 2^n - 2\,.$$

*Proof.* For height 0, the distortion function is just $r$. Clearly by Lemma 3.4 and induction a polynomial of degree $D(n)$ suffices. Now embed $\Gamma$ in an iterated extension of centralizers of height $n$:

$$\Gamma \hookrightarrow \Gamma_n > \Gamma_{n-1} > \cdots > \Gamma_1 > \mathbf{F}\,.$$

Since the embedding $\Gamma \hookrightarrow \Gamma_n$ expands lengths at most linearly, $\Gamma$ has distortion function bounded above by a polynomial of degree $D(n)$ as well. ☐

Proposition 1.7 follows immediately.

## 4 Proof of Theorem 1.6

*Proof of Theorem 1.6.* Fix a set $Y$ of generators of $\Gamma$. It suffices to prove the theorem for the finite set $B_Y(R)$ for arbitrary $R > 0$. We are given $\epsilon > 0$. Let $S_Y(R) \subset \mathbf{C}[\Gamma]$ denote the $\ell^1$-unit sphere of the elements supported on $B_Y(R)$. Our task is to prove that there is a homomorphism $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(a))\| \leq \|\lambda_\Gamma(a)\| + \epsilon. \tag{4.1}$$

for all $a \in S_Y(R)$. The set $S_Y(R)$ is compact with respect to the $\ell^1$ norm. Take a finite $\frac{\epsilon}{3}$-net $\{a_i\}_{i \in \mathcal{I}}$ for $S_Y(R)$ w.r.t. the $\ell^1$ norm.

Due to the inequality (2.1) and triangle inequality, the functions $a \mapsto \|\lambda_{\mathbf{F}}(a)\|$ and $a \mapsto \|\lambda_\Gamma(a)\|$ are 1-Lipschitz on $S_Y(R)$ with respect to the $\ell^1$ norm and hence if we can prove the existence of $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(a_i))\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3} \tag{4.2}$$

for all $i \in \mathcal{I}$ then (4.1) will follow for all $a \in S_Y(R)$ as required. So we now set out to prove (4.2).

Let $C$ and $D$ be the constants from Proposition 1.7. Choose $m = m(\epsilon) \in \mathbf{N}$ large enough so that

$$[C(2mR)^D]^{\frac{3}{4m}} \leq 1 + \frac{\epsilon}{3}. \tag{4.3}$$

We apply Proposition 1.7 with $r = 2mR$ to get an epimorphism $\phi : \Gamma \to \mathbf{F}$ injective on $B_Y(2mR)$, and a generating set $X$ of $\mathbf{F}$ such that

$$\phi(B_Y(2mR)) \subset B_X\left(C(2mR)^D\right). \tag{4.4}$$

Let $b_i \overset{\text{def}}{=} \phi(a_i)$ for each $i \in \mathcal{I}$.

Note that

$$\|\lambda_\Gamma(a_i)\|^{2m} = \|\lambda_\Gamma\left(a_i^* a_i\right)\|^m = \|\lambda_\Gamma\left(a_i^* a_i\right)^m\|$$

and similarly, $\|\lambda_{\mathbf{F}}(b_i)\|^{2m} = \|\lambda_{\mathbf{F}}\left(b_i^* b_i\right)^m\|$. Each $\left(b_i^* b_i\right)^m$ is supported on $B_X(C(2mR)^D)$ by (4.4), hence by Haagerup's inequality (Lemma 2.1) we have

$$\begin{aligned}
\|\lambda_{\mathbf{F}}(b_i)\|^{2m} &= \|\lambda_{\mathbf{F}}\left(b_i^* b_i\right)^m\| \\
&\leq [C(2mR)^D]^{\frac{3}{2}}\|\left(b_i^* b_i\right)^m\|_{\ell^2} \\
&= [C(2mR)^D]^{\frac{3}{2}}\|\left(a_i^* a_i\right)^m\|_{\ell^2} \\
&\leq [C(2mR)^D]^{\frac{3}{2}}\|\lambda_\Gamma(a_i)\|^{2m}.
\end{aligned}$$

The equality on the third line used that $\phi$ is injective on $B_Y(2mR)$, and the final inequality used (2.2). Hence

$$\begin{aligned}
\|\lambda_{\mathbf{F}}(b_i)\| &\leq [C(2mR)^D]^{\frac{3}{4m}}\|\lambda_\Gamma(a_i)\| \\
&\leq \left(1 + \frac{\epsilon}{3}\right)\|\lambda_\Gamma(a_i)\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3}
\end{aligned}$$

by our choice of $m$ in (4.3); the last inequality used that $\|a_i\|_{\ell^1} = 1$ and (2.1). ☐

# 5 Proof of Theorem 1.1

Here we split into cases when $\Gamma$ is abelian or not. Limit groups cannot have torsion, so abelian limit groups are of the form $\mathbf{Z}^r$ for some $r \in \mathbf{N}$.

## 5.1 Proof when $\Gamma = \mathbf{Z}^r$

The case when $\Gamma = \mathbf{Z}^r$ must be dealt with by hand here.

**Lemma 5.1.** *Theorem 1.1 holds when* $\Gamma = \mathbf{Z}^r$.

*Proof.* Let $T^r \stackrel{\text{def}}{=} (S^1)^r$ be the standard $r$-dimensional flat torus. The Fourier transform gives an isomorphism of $C^*$-algebras

$$\mathcal{F}\colon C(T^r) \to C^*_r(\mathbf{Z}^r).$$

For $q \in \mathbf{N}$ let $T^r_q$ denote the subtorus $(\mathbf{Z}/q\mathbf{Z})^r \subset T^r$. We obtain, via restriction and Fourier transform, a finite dimensional representation

$$C^*_r(\mathbf{Z}^r) \xrightarrow{\rho_q} C(T^r_q)$$

that restricts to finite dimensional unitary representation of $\mathbf{Z}^r$. For any $z \in \mathbf{C}[\mathbf{Z}^r]$ we have

$$\|\rho_q(z)\| = \max_{x \in T^r_q} |\mathcal{F}^{-1}[z](x)| \to \max_{x \in T^r} |\mathcal{F}^{-1}[z](x)| = \|\lambda_{\mathbf{Z}^r}(z)\|$$

as $q \to \infty$. We have only used here the fact that $T^r_q$ Hausdorff converges to $T^r$ as $q \to \infty$. □

## 5.2 Proof for non-abelian limit groups

In the following, $\mathbf{F}$ will always denote some (not always the same) free group, and $\Gamma$ will be a fixed limit group.

**Lemma 5.2.** *If $\Gamma$ is a non-abelian limit group, then the reduced $C^*$ algebra of $\Gamma$ is simple (has no non-trivial closed ideals) and has a unique tracial state.*

*Proof.* We claim that any non-abelian FRF group $\Gamma$ has the $P_{\text{nai}}$ property of Bekka, Cowling, and de la Harpe [BCdlH94, Def. 4]. This states that for any finite set $S \subset \Gamma \backslash \{e\}$, there is $y \in \Gamma$ of infinite order such that for every $x \in S$, $x$ and $y$ are free generators of a free rank 2 subgroup of $\Gamma$.

*Proof of Claim.* It is easy to check that since $\Gamma$ is FRF, two elements $x$ and $y$ are free generators of a free rank 2 subgroup of $\Gamma$ if and only if they do not commute. So to check property $P_{\text{nai}}$ above, it remains to check that given any finite subset $S \subset \Gamma \backslash \{e\}$, there is an infinite order $y$ not commuting with any element of $S$.

Because $\Gamma$ is non-abelian, there are two elements $a, b \in \Gamma$ with $[a, b] \neq e$. By the FRF condition, there is a epimorphism $\phi : \Gamma \to \mathbf{F}$ that is an injection on $S \cup \{e\} \cup \{[a, b]\}$. In particular, the rank of $\mathbf{F}$ must be at least 2. Since $\phi(S)$ is a finite subset of $\mathbf{F}$ not containing the identity, there is an (necessarily infinite order) element $f$ not commuting with any element of $\phi(S)$. Then any preimage of $f$, say $y$, is infinite order and does not commute with any element of $S$. *This ends the proof of the claim.*

The proof of Lemma 5.2 now concludes by using [BCdlH94, Lemmas 2.1 and 2.2]. □

*Proof of Theorem 1.1.* The upshot of Lemma 5.2 is that proving the existence of a sequence of unitary representations $\{\rho_i : \Gamma \to \mathbf{U}(N_i)\}_{i=1}^{\infty}$ strongly converging to the regular representation reduces to proving the existence of a sequence with

$$\limsup_{i\to\infty} \|\rho_i(z)\| \leq \|\lambda_\Gamma(z)\| \tag{5.1}$$

for all $z \in \mathbf{C}[\Gamma]$ of unit $\ell^1$ norm. We give a proof of this passage that was also mentioned in the Introduction.

Suppose (5.1) holds. Then for any non-principal ultrafilter $\mathcal{F}$, we form the ultraproduct[5] $C^*$-algebra $\mathcal{U} \stackrel{\text{def}}{=} \prod_{\mathcal{F}} \rho_i(\mathbf{C}[\Gamma])$. There is a natural $*$-algebra map $\iota : \mathbf{C}[\Gamma] \to \mathcal{U}$. The inequality (5.1) implies

$$\|\iota(z)\|_{\mathcal{U}} \leq \|\lambda_\Gamma(z)\| \tag{5.2}$$

for all $z \in \mathbf{C}[\Gamma]$. If $\mathcal{U}_1$ denotes the closure of $\iota(\mathbf{C}[\Gamma])$ in $\mathcal{U}$, then inequality (5.2) implies that the map $\iota$ extends continuously to $C^*$-algebra map from $C^*_r(\Gamma)$ to $\mathcal{U}_1$. But since we know $C^*_r(\Gamma)$ is simple by Lemma 5.2, this map must be injective. But injective $C^*$-algebra maps are isometries (to their images), so we have for all $z \in \mathbf{C}[\Gamma]$

$$\|\lambda_\Gamma(z)\| = \|\iota(z)\|_{\mathcal{U}} = \lim_{i\to\mathcal{U}} \|\rho_i(z)\|.$$

Since this holds for arbitrary non-principal ultrafilters, it holds also that $\|\lambda_\Gamma(z)\| = \lim_{i\to\infty} \|\rho_i(z)\|$.

This reduces our task to proving 5.1, which we begin now. Given $\epsilon > 0$ we will prove that there is a unitary representation $\rho = \rho(U, \epsilon) : \Gamma \to \mathbf{U}(N)$ with $N = N(U, \epsilon)$ such that

$$\|\rho(z)\| \leq \|\lambda_\Gamma(z)\| + \epsilon$$

[5] For background on ultrafilters and ultraproducts, see [BO08, Appendix A].

for $z \in \mathbf{C}[\Gamma]$ with support in $B\left(\frac{1}{\epsilon}\right)$ and $\|z\|_{\ell^1} = 1$. By taking $\epsilon \to 0$, this will imply the existence of a sequence $\rho_i$ satisfying (5.1) for any $z$.

As in the proof of Theorem 1.6 (§4), by taking an $\frac{\epsilon}{3}$-net of the unit $\ell^1$ sphere of the elements in $\mathbf{C}[\Gamma]$ supported on $B\left(\frac{1}{\epsilon}\right)$, it suffices to prove

$$\|\rho(a_i)\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3}$$

for a finite collection $\{a_i\}_{i\in\mathcal{I}}$ of elements of $\mathbf{C}[\Gamma]$ with $\|a_i\|_{\ell^1} = 1$.

We apply Theorem 1.6 with $S = B\left(\frac{1}{\epsilon}\right)$ to obtain a homomorphism $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(a_i))\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{6}. \tag{5.3}$$

for all $i \in \mathcal{I}$. Let $b_i \overset{\text{def}}{=} \phi(a_i) \in \mathbf{C}[\mathbf{F}]$.

The remainder of the proof splits into three cases.

**A.** If $\mathbf{F}$ is rank 1, i.e. $\mathbf{F} = \mathbf{Z}$ then Lemma 5.1 tells that there is a finite dimensional unitary representation $\pi$ of $\mathbf{F}$ such that

$$\|\pi(b_i)\| \leq \|\lambda_{\mathbf{F}}(b_i)\| + \frac{\epsilon}{6} \tag{5.4}$$

for all $i \in \mathcal{I}$.

**B.** Otherwise, if one only wants unitary representations in Theorem 1.1, then by Haagerup and Thorbjørnsen [HT05, Thm. B] there is a finite dimensional unitary representation $\pi$ of $\mathbf{F}$ such that (5.4) holds for all $i \in \mathcal{I}$.

**C.** If one wants the full strength of Theorem 1.1 and $\mathbf{F}$ has rank at least 2, then unitary representations satisfying 5.4 for all $i \in \mathcal{I}$ exist by the work of Bordenave and Collins [BC19].

Then let $\rho \overset{\text{def}}{=} \pi \circ \phi$, a finite dimensional unitary representation of $\Gamma$. Since $\rho(a_i) = \pi(b_i)$, using (5.3) we obtain

$$\|\rho(a_i)\| = \|\pi(b_i)\| \leq \|\lambda_{\mathbf{F}}(b_i)\| + \frac{\epsilon}{6} \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3}$$

for all $i \in \mathcal{I}$ as required. □

# 6 Proof of Proposition 1.3

We now turn to the Klein bottle group, $K = \langle t, a \mid tat^{-1} = a^{-1}\rangle$. Let $A < K$ be the rank two free abelian index two subgroup of $K$ generated by $a$ and $t^2$, and $T^2 = \hat{A}$ (a two-torus) the Pontryagin dual of $A$. By [Tay89, Proposition 1], $C_r^*(K)$ is isomorphic (via an inverse fourier transform $\hat{\mathcal{F}}$) to a $C^*$-subalgebra of $\mathrm{Mat}_{2\times 2}(C(T^2)) \cong C(T^2; \mathrm{Mat}_{2\times 2}(\mathbf{C}))$ with norm defined by

$$\|f\|_\infty \stackrel{\text{def}}{=} \sup_{x \in T^2} \|f(x)\|_{\text{op}}.$$

The norm on the right hand side is the operator norm on $\mathrm{Mat}_{2\times 2}(\mathbf{C})$ coming from the standard Hermitian form on $\mathbf{C}^2$. As in the proof of Lemma 5.1, for $q \in \mathbf{N}$ let $T_q^2 \stackrel{\text{def}}{=} \mathbf{Z}/q\mathbf{Z}$. Restriction gives a $C^*$-algebra map

$$C_r^*(K) \xrightarrow{\rho_q} C(T_q^2; \mathrm{Mat}_{2\times 2}(\mathbf{C}))$$

with finite dimensional right hand side. The restriction to $K$ is unitary and for any $z \in \mathbf{C}[K]$ we have as $q \to \infty$

$$\|\rho_q(z)\| = \max_{x \in T_q^2} \|\mathcal{F}[z](x)\|_{\text{op}} \to \max_{x \in T^2} \|\mathcal{F}[z](x)\|_{\text{op}} = \|\lambda_K(z)\|.$$

# 7 Proof of Corollary 1.9

We wish to appeal to results in the literature to deduce Corollary 1.9 from Theorem 1.1. To do so, we first establish the following.

**Lemma 7.1** (Strong convergence implies weak convergence)**.** *Let $\Gamma$ be a finitely generated discrete group such that $C_r^*(\Gamma)$ has a unique tracial state. If $\{\rho_i : \Gamma \to \mathbf{U}(N_i)\}_{i=1}^\infty$ are a sequence of finite dimensional unitary representations that strongly converge to the regular representation of $\Gamma$, then for any $z \in \mathbf{C}[\Gamma]$*

$$\lim_{i \to \infty} \frac{\mathrm{tr}(\rho_i(z))}{N_i} = \tau(z),$$

*where $\tau$ is the unique tracial state on $C_r^*(\Gamma)$.* tr *denotes the usual matrix trace on $\mathbf{U}(N_i)$ extended linearly to $\mathbf{C}[\mathbf{U}(N_i)]$.*

We heard this lemma stated by Benoît Collins in a talk in Northwestern University in June 2022. The proof is to our knowledge not in the literature so we give it here.

*Proof of Lemma 7.1.* Consider any non-principal ultrafilter $\mathcal{F}$ on $\mathbf{N}$, and form the ultraproduct $C^*$-algebra $\mathcal{U} \stackrel{\text{def}}{=} \prod_{\mathcal{F}} \rho_i(\mathbf{C}[\Gamma])$. Let $\mathcal{U}_1$ denote the $C^*$-subalgebra in $\mathcal{U}$ generated by the images $\hat{\gamma}_i$ in $\mathcal{U}$ of the generators $\gamma_i$ of $\Gamma$. Strong convergence implies that the natural map from $\mathbf{C}[\Gamma]$ to $\mathcal{U}_1$ is an isometric embedding with respect to the norm on $\mathbf{C}[\Gamma]$ coming from $C_r^*(\Gamma)$, and hence extends to an isomorphism between $C_r^*(\Gamma)$ and $\mathcal{U}_1$. On the other hand,

$$\lim_{i\to\mathcal{F}} \frac{\mathrm{tr}\circ\rho_i}{N_i}$$

defines a tracial state on $\mathcal{U}_1$, and when transferred to $C_r^*(\Gamma)$ must coincide with the unique tracial state there. Since the convergence holds for all non-principal ultrafilters, the convergence must hold in general. $\square$

*Proof of Corollary 1.9.* Since non-abelian limit groups $\Gamma$ have unique tracial states on $C_r^*(\Gamma)$ by Lemma 5.2, Lemma 7.1 implies that

$$\lim_{i\to\infty} \frac{\mathrm{tr}(\rho_i(z))}{N_i} = \tau(z) \tag{7.1}$$

where $\tau(z)$ is the unique tracial state on $C_r^*(\Gamma)$ (induced by $\tau(e) = 1$ and $\tau(g) = 0$ for $e \neq g \in \Gamma$).

Note that for unitary matrices, $u_1, \ldots, u_r$, any non-commutative polynomial (possibly with matrix coefficients) in the $u_i$ is another polynomial (possibly with matrix coefficients) in the Hermitian matrices $u_i + u_i^*$ and $i(u_i - u_i^*)$, and vice versa.

Given this observation, (7.1) together with strong convergence of the sequence $\rho_i$ provided by Theorem 1.1 used as inputs to [Mal12, Prop. 7.3] yield Corollary 1.9. $\square$

# A Spectral gaps of hyperbolic surfaces

The purpose of this appendix is to explain how the following theorem can be deduced from Corollary 1.9.

**Theorem A.1.** *Let $X$ be a compact hyperbolic surface. There exists a sequence of Riemannian covers $\{X_i\}_{i\in\mathbb{N}}$ of $X$ with genera $g(i) \to \infty$ as $i \to \infty$ such that for any $\epsilon > 0$, for $i$ large enough depending on $\epsilon$,*

$$\mathrm{spec}\,(\Delta_{X_i}) \cap \left[0, \frac{1}{4} - \epsilon\right) = \mathrm{spec}\,(\Delta_X) \cap \left[0, \frac{1}{4} - \epsilon\right),$$

*where the multiplicities are the same on either side.*

To see that we can take all surfaces to be arithmetic we use the following argument. Let

$$\Gamma_0(15) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \,:\, c \equiv 0 \bmod 15 \right\}.$$

The cusped hyperbolic surface $Y_0(15) \stackrel{\text{def}}{=} \Gamma_0(15)\backslash\mathbb{H}$ has no spectrum in $(0, \frac{1}{4})$ by a result of Huxley [Hux85, Thm., pg. 250]. Let $D_{3,5}$ denote the quaternion algebra over $\mathbf{Q}$ generated by $i, j, k$ such that

$$i^2 = 3,\ j^2 = 5,\ ij = -ji = k.$$

Then $D_{3,5}$ is a division algebra with discriminant 15 [Ber16, Ex. 8.27]. Let $\mathcal{O}$ denote a maximal order[6] in $D_{3,5}$ and $\mathcal{O}^1$ the elements of norm 1 in $\mathcal{O}$. Then $\mathcal{O}^1$ embeds as a cocompact subgroup of $\mathrm{PSL}_2(\mathbf{R})$; let $X = \mathcal{O}^1\backslash\mathbb{H}$. By the work of Jacquet and Langlands [JL70] (see [Ber16, Thm. 8.18] for a convenient concise reference) every eigenvalue of $X$ is an eigenvalue of $Y_0(15)$ and hence $X$ has no eigenvalues in $(0, \frac{1}{4})$.

Taking this $X$ in Theorem A.1, one obtains a different proof of [HM, Corollary 1.3] with a slightly stronger conclusion, i.e. there exists a sequence of compact *arithmetic* hyperbolic surfaces $\{X_i\}_{i\in\mathbb{N}}$ with genera $g(X_i) \to \infty$ and $\lambda_1(X_i) \to \frac{1}{4}$. Such a sequence of covering surfaces also have systoles uniformly bounded away from 0, also in contrast to the proof of [HM, Corollary 1.3] (this conclusion on the systole is independent of arithmeticity).

## A.1 Set up

For any $n \in \mathbb{N}$, let $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ and $S_n$ denote the group of permutations of $[n]$. Let $X$ be a fixed compact hyperbolic surface with genus $g \geqslant 2$. We view $X$ as

$$X = \Gamma\backslash\mathbb{H},$$

where $\Gamma$ is a discrete, torsion free subgroup of $\mathrm{PSL}_2(\mathbb{R})$, isomorphic to the surface group $\Lambda_g$. Given any $\phi \in \mathrm{Hom}(\Gamma, S_n)$ we define an action of $\Gamma$ on $\mathbb{H} \times [n]$ by

$$\gamma(z, x) \stackrel{\text{def}}{=} (\gamma z, \phi(\gamma)[x]).$$

Then we obtain a degree $n$ covering space $X_\phi$ of $X$ by

$$X_\phi \stackrel{\text{def}}{=} \Gamma\backslash_\phi(\mathbb{H} \times [n]). \tag{A.1}$$

Let $V_n \stackrel{\text{def}}{=} \ell^2([n])$ and $V_n^0 \subset V_n$ the subspace of functions with zero mean. Then $S_n$ acts on $V_n$ via std, the standard representation by 0-1 matrices, and $V_n^0$ is the $n-1$ dimensional irreducible component. Throughout this

[6] See [Ber16, Ex. 8.27] for an explicit maximal order.

appendix, we let $\{\rho_i\}_{i\in\mathbb{N}}$ be a sequence of $N_i$-dimensional unitary representations of $\Gamma$ that factor through $S_{N_i}$ by

$$\Gamma \xrightarrow{\phi_i} S_{N_i} \xrightarrow{\text{std}} \operatorname{End}\left(V_{N_i}^0\right), \tag{A.2}$$

such that for any $r \in \mathbf{N}$ and finitely supported map $a : \Gamma \to \mathrm{Mat}_{r\times r}(\mathbf{C})$, we have

$$\limsup_{i\to\infty} \|\sum_{\gamma\in\Gamma} a(\gamma) \otimes \rho_i(\gamma)\| \leq \|\sum_{\gamma\in\Gamma} a(\gamma) \otimes \lambda(\gamma)\|, \tag{A.3}$$

as provided by Corollary 1.9. Note that by approximation by finite-rank operators on either side (as in [HM, Proof of Prop. 6.3]) the property in (A.3) extends easily to the case of

$$a : \Gamma \to \mathcal{K}$$

where $\mathcal{K}$ are the compact operators on a separable Hilbert space. We use this extension in the sequel.

Then through $\{\rho_i\}_{i\in\mathbb{N}}$, we obtain a sequence of degree-$N_i$ covering surfaces $\{X_i\}_{i\in\mathbb{N}}$ from (A.1).

## A.2 Function spaces

For the convenience of the reader we recall the following function spaces from [HM, Section 2.2]. We define $L^2_{\text{new}}(X_i)$ to be the space of $L^2$ functions on $X_i$ orthogonal to all lifts of $L^2$ functions from $X$. Then

$$L^2(X_i) \cong L^2_{\text{new}}(X) \oplus L^2(X).$$

We fix $F$ to be a Dirichlet fundamental domain for $X$. Let $C^\infty\left(\mathbb{H}; V_{N_i}^0\right)$ denote the smooth $V_{N_i}^0$-valued functions on $\mathbb{H}$. There is an isometric linear isomorphism between

$$C^\infty(X_i) \cap L^2_{\text{new}}(X_i),$$

and the space of smooth $V_{N_i}^0$-valued functions on $\mathbb{H}$ satisfying

$$f(\gamma z) = \rho_i(\gamma) f(z),$$

for all $\gamma \in \Gamma$, with finite norm

$$\|f\|^2_{L^2(F)} \stackrel{\text{def}}{=} \int_F \|f(z)\|^2_{V_{N_i}^0} d\mu_{\mathbb{H}}(z) < \infty.$$

We denote the space of such functions by $C^{\infty}_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$. The completion of $C^{\infty}_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ with respect to $\| \bullet \|_{L^2(F)}$ is denoted by $L^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$; the isomorphism above extends to one between $L^2_{\mathrm{new}}\left(X_i\right)$ and $L^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$.

We introduce the following Sobolev spaces. Let $H^2\left(\mathbb{H}\right)$ denote the completion of $C^{\infty}_c\left(\mathbb{H}\right)$ with respect to the norm

$$\|f\|^2_{H^2(\mathbb{H})} \stackrel{\mathrm{def}}{=} \|f\|^2_{L^2(\mathbb{H})} + \|\Delta f\|^2_{L^2(\mathbb{H})}.$$

Let $C^{\infty}_{c,\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ denote the subset of $C^{\infty}_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ consisting of functions which are compactly supported modulo $\Gamma$. We let $H^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ denote the completion of $C^{\infty}_{c,\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ with respect to the norm

$$\|f\|^2_{H^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)} \stackrel{\mathrm{def}}{=} \|f\|^2_{L^2(F)} + \|\Delta f\|^2_{L^2(F)}.$$

We let $H^2\left(X_i\right)$ denote the completion of $C^{\infty}_c\left(X_i\right)$ with respect to the norm

$$\|f\|^2_{H^2(X_i)} \stackrel{\mathrm{def}}{=} \|f\|^2_{L^2(X_i)} + \|\Delta f\|^2_{L^2(X_i)}.$$

Viewing $H^2\left(X_i\right)$ as a subspace of $L^2\left(X_{\phi_i}\right)$, we let

$$H^2_{\mathrm{new}}\left(X_i\right) \stackrel{\mathrm{def}}{=} H^2\left(X_i\right) \cap L^2_{\mathrm{new}}\left(X_{\phi_i}\right).$$

There is an isometric isomorphism between $H^2_{\mathrm{new}}\left(X_i\right)$ and $H^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ that intertwines the two relevant Laplacian operators.

### A.3 Operators on $\mathbb{H}$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$, let

$$\begin{aligned} &R_{\mathbb{H}}(s) : L^2\left(\mathbb{H}\right) \to L^2(\mathbb{H}), \\ &R_{\mathbb{H}}(s) \stackrel{\mathrm{def}}{=} \left(\Delta_{\mathbb{H}} - s(1-s)\right)^{-1}, \end{aligned}$$

be the resolvent on the upper half plane. Then $R_{\mathbb{H}}(s)$ is an integral operator with radial kernel $R_{\mathbb{H}}(s; r)$. Let $\chi_0 : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\chi_0\left(t\right) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ 1 & \text{if } t \geqslant 1. \end{cases}.$$

For $T > 0$, we define a smooth cutoff function $\chi_T$ by $\chi_T(t) \stackrel{\text{def}}{=} \chi_0(t - T)$.We then define the operator $R_{\mathbb{H}}^{(T)}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ to be the integral operator with radial kernel

$$R_{\mathbb{H}}^{(T)}(s;r) \stackrel{\text{def}}{=} \chi_T(r) R_{\mathbb{H}}(s;r).$$

Following [HM, Section 5.2] we define $L_{\mathbb{H}}^{(T)}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ to be the integral operator with radial kernel

$$\mathbb{L}^{(T)}(s;r) \stackrel{\text{def}}{=} \left(-\frac{\partial^2}{\partial r^2}[\chi_T] - \frac{1}{\tanh r}\frac{\partial}{\partial r}[\chi_T]\right) R_{\mathbb{H}}(s;r) - 2\frac{\partial}{\partial r}[\chi_T]\frac{\partial R_{\mathbb{H}}}{\partial r}(s;r).$$

It is proved in [HM, Lemma 5.3] that for any $f \in C_c^{\infty}(\mathbb{H})$and $s \in \left[\frac{1}{2}, 1\right]$, we have

1. $R_{\mathbb{H}}^{(T)}(s) f \in H^2(\mathbb{H})$.
2. $(\Delta - s(1-s)) R_{\mathbb{H}}^{(T)}(s) f = f + \mathbb{L}_{\mathbb{H}}^{(T)}(s) f$ as equivalence classes of $L^2$ functions.

It is also proved, as a consequence of [HM, Lemma 5.2], that for any $s_0 > \frac{1}{2}$ we can choose a $T = T(s_0)$ such that for all $s \in [s_0, 1]$ we have

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2} \leq \frac{1}{4}. \tag{A.4}$$

## A.4 Proof of Theorem A.1

Recall that $\{\rho_i\}_{i \in \mathbb{N}}$ is a sequence of strongly convergent representations of the form (A.2) that satisfy (A.3) as guaranteed by Corollary 1.9. As in [HM, Section 5.3], we define

$$R_{\mathbb{H},i}^{(T)}(s;x,y) \stackrel{\text{def}}{=} R_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_{N_i}^0},$$
$$\mathbb{L}_{\mathbb{H},i}^{(T)}(s;x,y) \stackrel{\text{def}}{=} \mathbb{L}_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_{N_i}^0}.$$

We define $R_{\mathbb{H},i}^{(T)}(s)$, $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ to be the corresponding integral operators. We have the following analogue of [HM, Lemma 5.5].

**Lemma A.2.** *For all* $s \in \left[\frac{1}{2}, 1\right]$,

1. *The integral operator* $R_{\mathbb{H},i}^{(T)}(s)$ *is well-defined on* $C_{c,\phi_i}^{\infty}\left(\mathbb{H}; V_{N_i}^0\right)$ *and extends to a bounded operator*

$$R_{\mathbb{H},i}^{(T)}(s) : L_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right) \to H_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right).$$

2. *The integral operator* $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ *is well-defined on* $C_{c,\phi_i}^{\infty}\left(\mathbb{H}; V_{N_i}^0\right)$ *and and extends to a bounded operator on* $L_\phi^2\left(\mathbb{H}; V_{N_i}^0\right)$.

3. *We have*
$$[\Delta - s(1-s)]\, R_{\mathbb{H},i}^{(T)}(s) = 1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s)$$
*as an identity of operators on* $L_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right)$.

The proof of Lemma A.2 easily follows from the proof of [HM, Lemma 5.5], simplified in places by the compactness of the fundamental domain $F$ in the current setting.

We have an isomorphism of Hilbert spaces

$$L_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right) \cong L^2(F) \otimes V_{N_i}^0,$$

given by

$$f \mapsto \sum_{e_i} \langle f|_F, e_i\rangle \otimes e_i,$$

where $\{e_j\}_{j=1}^{N_i-1}$ is some choice of basis for $V_{N_i}^0$. After conjugation by this isomorphism, the operator $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ becomes

$$\mathbb{L}_{\mathbb{H},i}^{(T)}(s) \cong \sum_{\gamma \in S} a_\gamma^{(T)}(s) \otimes \rho_i\left(\gamma^{-1}\right), \tag{A.5}$$

where

$$\begin{aligned} a_\gamma^{(T)}(s) &: L^2(F) \to L^2(F), \\ a_\gamma^{(T)}(s)[f](x) &\stackrel{\text{def}}{=} \int_{y \in F} \mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)\, f(y)\, d\mu_{\mathbb{H}}(y). \end{aligned}$$

Since $\mathbb{L}_{\mathbb{H}}^{(T)}(s, \gamma x, y)$ is only non-zero when $d(\gamma x, y) \leqslant T+1$, in (A.5) one can take $S = S(T) \subset \Gamma$ to be finite. Since $\mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)$ is smooth and bounded it follows that the operators $a_\gamma^{(T)}(s)$ are Hilbert-Schmidt and therefore compact. We define

$$\mathcal{L}_{s,\infty}^{(T)} \stackrel{\text{def}}{=} \sum_{\gamma \in S} a_\gamma^{(T)}(s) \otimes \lambda\left(\gamma^{-1}\right).$$

Under the isomorphism

$$\begin{aligned} L^2(F) \otimes \ell^2(\Gamma) &\cong L^2(\mathbb{H}), \\ f \otimes \delta_\gamma &\mapsto f \circ \gamma^{-1}, \end{aligned}$$

(with $f \circ \gamma^{-1}$ extended by zero from a function on $\gamma F$) the operator $\mathcal{L}_{s,\infty}^{(T)}$ is conjugated to

$$\mathbb{L}_{\mathbb{H}}^{(T)}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H}).$$

To prove Theorem A.1, we need to replace the probabilistic bound [HM, Lemma 6.3] by a deterministic one.

**Proposition A.3.** *For any $s_0 > \frac{1}{2}$ there is a $T = T(s_0) > 0$ such that for any fixed $s \in [s_0, 1]$ there is an $I(s_0, s)$ with*

$$\|\mathcal{L}_{s,\phi_i}^{(T)}\|_{L^2(F)\otimes V_{N_i}^0} \leq \frac{1}{4},$$

*for all $i \geqslant I$.*

*Proof.* Let $s_0 > \frac{1}{2}$ and a fixed $s \in [s_0, 1]$ be given. By (A.4) we can find a $T(s_0)$ such that

$$\|\mathcal{L}_{s,\infty}^{(T)}\|_{L^2(F)\otimes \ell^2(\Gamma)} \leq \frac{1}{8}. \tag{A.6}$$

Recall that the coefficients $a_\gamma(s)$ are supported on a finite set $S = S(T) \subset \Gamma$. Because the $a_\gamma(s)$ are compact, we apply (A.3) (and the following remark) to the operators $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ to find that there is $I \in \mathbf{N}$ such that for all $i \geq I$

$$\|\mathcal{L}_{s,\phi_i}^{(T)}\|_{L^2(F)\otimes V_{N_i}^0} \leq \|\mathcal{L}_{s,\infty}^{(T)}\|_{L^2(F)\otimes \ell^2(\Gamma)} + \frac{1}{8} \leq \frac{1}{4}.$$

□

We can now prove Theorem A.1.

*Proof of Theorem A.1.* Given $\epsilon > 0$ let $s_0 = \frac{1}{2} + \sqrt{\epsilon}$ so that $s_0(1 - s_0) = \frac{1}{4} - \epsilon$. Let $T = T(s_0)$ be the value provided by Proposition A.3 for this $s_0$.

We use a finite net to control all values of $s \in [s_0, 1]$. Using [HM, Lemma 6.1] as in [HM, Proof of Thm. 1.1] tells us that there is a finite set $Y = Y(s_0)$ of points in $[s_0, 1]$ such that for any $s \in [s_0, 1]$ there is $s' \in Y$ with

$$\|\mathcal{L}_{s,\phi_i}^{(T)} - \mathcal{L}_{s',\phi_i}^{(T)}\| \leq \frac{1}{4} \tag{A.7}$$

for all $i$.

Combining (A.7) with Proposition A.3 applied to $\mathcal{L}_{s,\phi_i}^{(T)}$ for every $s \in Y$ we find that there is an $I(s_0)$ such that for all $s \in [s_0, 1]$ and $i \geq I(s_0)$

$$\|\mathbb{L}_{\mathbb{H},i}^{(T)}(s)\|_{L^2_{\mathrm{new}}(X_i)} = \|\mathcal{L}_{s,\phi_i}^{(T)}\|_{L^2(F)\otimes V_{N_i}^0} \leq \frac{1}{2}. \tag{A.8}$$

By Lemma A.2, for $s > \frac{1}{2}$ $R_{\mathbb{H},i}^{(T)}(s)$ is a bounded operator from $L^2_{\text{new}}(X_i)$ to $H^2_{\text{new}}(X_i)$. By Lemma A.2 we have that

$$\left(\Delta_{X_\phi} - s(1-s)\right) R_{\mathbb{H},i}^{(T)}(s) = 1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s),$$

on $L^2_{\text{new}}(X_i)$. From (A.8) for all $i \geqslant I(s_0)$ and $s \in [s_0, 1]$ $\left(1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s)\right)^{-1}$ exists as a bounded operator on $L^2_{\text{new}}(X_i)$. We now get that for all $i \geqslant I(s_0)$ and all $s \in [s_0, 1]$,

$$(\Delta_{X_i} - s(1-s)) R_{\mathbb{H},i}^{(T)}(s) \left(1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s)\right)^{-1} = 1,$$

and we conclude that $(\Delta_{X_i} - s(1-s))$ has a bounded right inverse from $L^2_{\text{new}}(X_i)$ to $H^2_{\text{new}}(X_i)$, implying that for $i \geqslant I(s_0)$, $\Delta_{X_i}$ has no new eigenvalues $\lambda$ with $\lambda \leq s_0(1-s_0) = \frac{1}{4} - \epsilon$. ☐

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Will Hide,
Department of Mathematical Sciences,
Durham University,
Lower Mountjoy, DH1 3LE Durham,
United Kingdom
william.hide@durham.ac.uk

Larsen Louder,
Department of Mathematics,
University College London,
Gower Street, London WC1E 6BT,
United Kingdom
l.louder@ucl.ac.uk

Michael Magee,
Department of Mathematical Sciences,
Durham University,
Lower Mountjoy, DH1 3LE Durham,
United Kingdom
michael.r.magee@durham.ac.uk