Arcade Processes and Martingale Interpolation

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27 September 2022

Abstract

Arcade processes are a class of continuous stochastic processes that interpolate in a strong-sense, i.e., omega by omega, between zeros at fixed pre-specified times. Their additive randomization allows one to match any given random variable at the given dates on the whole probability space, and can be interpreted as a generalization of anticipative stochastic bridges. The filtrations generated by randomized arcade processes are utilized to construct a class of interpolating martingales between the same target random variables, the filtered arcade martingales (FAMs), which provide an extension to the paradigm of information-based asset pricing. FAMs are strong solutions of the martingale fitting problem, and can be connected to martingale optimal transport (MOT) by considering optimally-coupled target random variables. Another connection to optimal transport can be established by using FAMs to cast the information-based martingale optimal transport problem, a new problem that allows the introduction of noise in MOT, similar to how Schrödinger's problem introduces noise in optimal transport.

Keywords: Stochastic interpolation, martingale interpolation, martingale fitting, stochastic bridge, optimal transport, nonlinear stochastic filtering, information-based asset pricing.

1 Introduction

The problem of constructing martingales which match given marginales has been the subject of many works in probability theory and applications thereof. The origin of the problem dates back to Strassen [20]. He showed that a discrete-time martingale $(M_n)_{n\in\mathbb{N}}$ can match a sequence of real probability measures $(\mu_n)_{n\in\mathbb{N}}$ if and only if the measures $(\mu_n)_{n\in\mathbb{N}}$ are convexly ordered, i.e., $M_n \sim \mu_n$ is possible for all $n \in \mathbb{N}$ if and only if $\int f(x)\,\mathrm{d}\mu_n(x)$ is an increasing sequence in n for any positive convex function f. This result also extends to the continuous-time setting, see Kellerer's theorem [12].

Strassen's and Kellerer's theorems do not provide a method for constructing the matching martingales. Many techniques are dedicated to Kellerer's setting, where one is looking to construct a martingale that matches a given peacock [10]: Skorokhod's embedding problem, Brownian random time changes, local volatility models, scaling peacocks method, etc., see [14]. Since the target is a dynamical measure, the matching is necessarily weak, and the martingale is in continuous time. On the other hand, the matching in Strassen's setting can be weak or almost sure, depending on the nature of the targets (measures or random variables), and the martingale is in discrete time. A popular tool dedicated to this task is martingale optimal transport (MOT, [2], [3]). Inspired by Kantorovich's optimal transport theory ([11], [21], [4]), it provides a framework for selecting a meaningful martingale coupling by minimizing a cost functional over the set of all martingale couplings.

Our interest lies at the intersection of Strassen's and Kellerer's settings: how to construct a continuous martingale that matches, in law or almost surely, a given finite set of convexly ordered random variables, at pre-specified given dates? For example, this is of interest in a financial setting, where knowledge of the prices of vanilla call and put options provides an implied distribution for the underlying asset price at future dates under no arbitrage. This is what we call the martingale interpolation problem, since we are interpolation between the components of a discrete-time martingale without breaking the martingality condition. The latter is the tricky part of the problem. If we forget about the martingality condition, then bridging the gap between a coupling and an interpolating measure does not usually cost much. For instance, in Kantorovich's optimal transport theory, it is well known that there is a one-to-one correspondence between optimal couplings and optimal interpolating measures in Wasserstein spaces, called McCann's interpolations, under mild conditions. However, this property is not present in the martingale counterpart of optimal transport.
Solutions to the martingale interpolation problem exist. For example, the martingale Benamou-Brenier problem (MBBP) [1] aims to solve the optimization problem

$$\sup_{\mathcal{M}_t(\mu,\nu,B)} \mathbb{E}\left[\int_0^1 \sigma_t \,\mathrm{d}t\right]$$

over the set

$$\mathcal{M}_t(\mu,\nu,B) = \Bigg\{ (\sigma_t)_{t\geqslant 0} \in L^1(B_t) \,|\, M_t = M_0 + \int_0^t \sigma_s \,\mathrm{d}B_s \implies \\ M_0 \sim \mu, M_1 \sim \nu, (M_t) \text{ is a martingale} \Bigg\}, \tag{1.1}$$

where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion and (μ,ν) are convexly ordered probability measures with finite second moments. If (μ,ν) is irreducible in the sense of [2], Appendix A.1, the solutions to the MBBP are called (μ,ν)-stretched Brownian motions, and have the form $\mathbb{E}[F(B_1+Y_0)\,|\,\mathcal{F}_t]$, where $(\mathcal{F}_t)$ is the filtration generated by (B_t+Y_0). The function $F:\mathbb{R}\to\mathbb{R}$ and the random variable Y_0 must be selected such that $F(B_1+Y_0)\sim\nu$, $\mathbb{E}[F(B_1+Y_0)\,|\,Y_0]\sim\mu$ and $Y_0 \perp\!\!\!\perp (B_t)$. In the case where (μ,ν) have irreducible components, the solution to the MBBP is a stretched Brownian motion on each irreducible component, see Theorem 3.1 in [1]. Stretched Brownian motions are interpolating martingales between the given measures μ and ν. Another weak solution to the martingale interpolation problem is given by Schrödinger's volatility models (SVMs) [9]. Inspired by the entropic relaxation of optimal transport ([13], [17]), also known as Schrödinger's problem in particular cases [19], SVMs rely on a measure change. Consider two, possibly correlated, Brownian motions, $(B_t)_{t\geqslant 0}$ and $(\widetilde{B}_t)_{t\geqslant 0}$, under a measure $\mathbb{P}$, and the system of SDEs

$$\begin{aligned} \mathrm{d}M_t &= a_t\, M_t \,\mathrm{d}\widetilde{B}_t, \\ \mathrm{d}a_t &= b(a_t) + c\,(a_t)\ \mathrm{d}B_t, \end{aligned} \tag{1.2}$$

where the real functions b and c are given, and are such that a_t is an Itô process. The goal is to change the measure $\mathbb{P}$ by an equivalent one, such that the process $(M_t)_{0\leqslant t\leqslant 1}$ matches the given measures μ and ν at times zero and one respectively, while being a true martingale. If several equivalent measures that satisfy these constraints exist, one must select the one that is closest to $\mathbb{P}$ in terms of the Kullback-Leibler divergence. By Girsanov's theorem, changing $\mathbb{P}$ by an equivalent measure $\mathbb{Q}$ transforms the system of SDEs as follows:

$$\begin{aligned} \mathrm{d}M_t &= a_t M_t(\,\mathrm{d}\widetilde{B}_t + \widetilde{\lambda}_t \,\mathrm{d}t), \\ \mathrm{d}a_t &= b(a_t) + c\,(a_t)\,(\,\mathrm{d}B_t + \lambda_t \,\mathrm{d}t). \end{aligned} \tag{1.3}$$

One must then select the drifts $(\tilde{\lambda}_t)$ and (λ_t), generated by the same measure change, that minimize the functional $\mathbb{E}_{\mathbb{Q}}\left[\int_{T_0}^{T_1} \widetilde{\lambda}_s^2 + \lambda_s^2 \,\mathrm{d}s\right]$ while satisfying the constraints $M_0 \sim \mu$, $M_1 \sim \nu$ and (M_t) is a martingale.

In this paper, we construct a class of martingales that can match, almost surely, any set of convexly ordered random variables, at an arbitrary set of fixed times, inspired by the information-based asset pricing approach ([6],[7]). These interpolating martingales, called filtered arcade martingales (FAMs), need two main ingredients: a discrete-time martingale (for example, a solution to an MOT problem) and another type of interpolating process that we call arcade process (AP).

The second section is dedicated to APs. Defined as a functional of a given stochastic process called the driver, APs are sample-continuous stochastic processes that interpolate between zeros on the whole probability space, i.e., omega by omega. To execute the interpolation, they rely on deterministic functions called interpolating coefficients. APs may be viewed as multi-period anticipative stochastic bridges. We study their properties and focus on Gaussian APs, which will play an important role in the definition of FAMs. Starting from a Gauss-Markov driver, we show that it is always possible to construct a Markovian AP by utilizing the covariance structure of the driver. The resulting AP from this procedure is called standard AP, but is not unique, since there are infinitely many Markovian APs driven by the same Gauss-Markov process.

The third section treats the additive randomization of APs, that is, the sum of a stochastic process that is interpolates deterministically between random points, and an AP. Such processes are called randomized arcade processes (RAPs), and can be thought as a sum between a signal function and a noise process. By construction, a RAP can match any random variables on the whole probability space at any given time, i.e, it is a stochastic interpolator between target random variables in the strong sense. The notion of Markovianity does not suit RAPs in general, since their filtrations contain the σ-algebras generated by each previously matched target random variable. For that reason, we introduce a counterpart notion, the nearly Markov processes, and show under which conditions a RAP is nearly Markov.

In the fourth and final section, we introduce the FAMs. A FAM is defined as the conditional expectation of the final target random variable, given the information generated by a RAP and, hence, inherits the filtering framework from information-based asset pricing: the signal is the final target and it can only be observed through a noisy version, the RAP. FAMs are tractable thanks to the nearly-Markov property of the underlying RAPs, and can be simulated using Bayes formula. Applying Itô's lemma, we derive the stochastic differential equations satisfied by FAMs. Finally, we introduce the information-based martingale optimal transport (IBMOT) problem, a similar problem to martingale optimal transport, that incorporates noise in the optimization process, inspired by the entropic regularization of optimal transport and Schrödinger's problem. IBMOT selects the martingale coupling that maximizes the expectation of the weighted squared error between a target random variable and its associated FAM for a given RAP.

In what follows, we consider the collection of fixed dates $\{T_i \in \mathbb{R} \,|\, n \in \mathbb{N}_0 \text{ and } i = 0, 1, \ldots, n\}$ such that $0 \leqslant T_0 < T_1 < T_2 < \ldots < T_n < \infty$. We introduce the ordered sets $\{T_0, T_n\}_* = \{T_0, T_1, \ldots, T_n\}$ and $(T_0, T_n)_* = \bigcup_{i=0}^{n-1} (T_i, T_{i+1})$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(D_t)_{t\in[T_0,T_n]}$ a sample-continuous stochastic process such that $\mathbb{P}[D_t = 0] < 1$ whenever $t \in (T_0, T_n)_*$, and an $\mathbb{R}^{n+1}$-valued random vector X independent of (D_t).

2 Arcade processes

We construct stochastic processes on $[T_0, T_n]$, as a functional of (D_t), which match exactly 0 (for all $\omega \in \Omega$) at the given times $\{T_0, T_n\}_*$. The first step is to introduce deterministic functions called interpolating coefficients.

Definition 2.1. *The functions $f_0, f_1, \dots, f_n$ are interpolating coefficients on $\{T_0, T_n\}_*$ if $f_0, f_1, \dots, f_n \in C^0\left([T_0, T_n], \mathbb{R}\right)$, $f_i(T_i) = 1$, and $f_i(T_j) = 0$ for $i, j = 0, \dots, n$, $i \neq j$.*

We can now give the definition of what we call an *arcade process.*

Definition 2.2. *An arcade process (AP), denoted $(A_t^{(n)})_{t\in[T_0,T_n]}$, on the partition $\{T_0, T_n\}_*$ is a stochastic process of the form*

$$A_t^{(n)} := D_t - \sum_{i=0}^{n} f_i(t) D_{T_i}, \tag{2.1}$$

where $f_0, \dots, f_n$ are interpolating coefficients on $\{T_0, T_n\}_*$. The process $(D_t)_{t\in[T_0,T_n]}$ is the driver of the AP. We denote by $(\mathcal{F}_t^A)_{t\in[T_0,T_n]}$ the filtration generated by $(A_t^{(n)})_{t\in[T_0,T_n]}$.

We observe that $A_{T_0}^{(n)} = A_{T_1}^{(n)} = \ldots = A_{T_n}^{(n)} = 0$ by construction, for all $\omega \in \Omega$.

Example 2.3. *For $n = 1$, $f_0(t) = \dfrac{t - T_1}{T_0 - T_1}$, $f_1(t) = \dfrac{t - T_0}{T_1 - T_0}$, the AP driven by a standard Brownian motion $(B_t)_{t\geqslant 0}$ is the anticipative Brownian bridge on $[T_0, T_1]$,*

$$A_t^{(1)} = B_t - \frac{T_1 - t}{T_1 - T_0} B_{T_0} - \frac{t - T_0}{T_1 - T_0} B_{T_1}. \tag{2.2}$$

Example 2.4. *For $n > 1$, we can generalize the anticipative Brownian bridge by taking*

$$f_0(t) = \frac{T_1 - t}{T_1 - T_0} \mathbb{1}_{[T_0,T_1]}(t), \quad f_n(t) = \frac{t - T_{n-1}}{T_n - T_{n-1}} \mathbb{1}_{(T_{n-1},T_n]}(t), \tag{2.3}$$

and

$$f_i(t) = \frac{t - T_{i-1}}{T_i - T_{i-1}} \mathbb{1}_{(T_{i-1},T_i]}(t) + \frac{T_{i+1} - t}{T_{i+1} - T_i} \mathbb{1}_{(T_i,T_{i+1}]}(t), \quad \text{for } i = 1, \dots, n-1. \tag{2.4}$$

We call this AP the stitched Brownian AP for it can be written as

$$A_t^{(n)} = \begin{cases} B_t - \frac{T_1 - t}{T_1 - T_0} B_{T_0} - \frac{t - T_0}{T_1 - T_0} B_{T_1}, & \text{if } t \in [T_0, T_1), \\ B_t - \frac{T_2 - t}{T_2 - T_1} B_{T_1} - \frac{t - T_1}{T_2 - T_1} B_{T_2}, & \text{if } t \in [T_1, T_2), \\ \vdots & \\ B_t - \frac{T_n - t}{T_n - T_{n-1}} B_{T_{n-1}} - \frac{t - T_{n-1}}{T_n - T_{n-1}} B_{T_n}, & \text{if } t \in [T_{n-1}, T_n]. \end{cases} \tag{2.5}$$

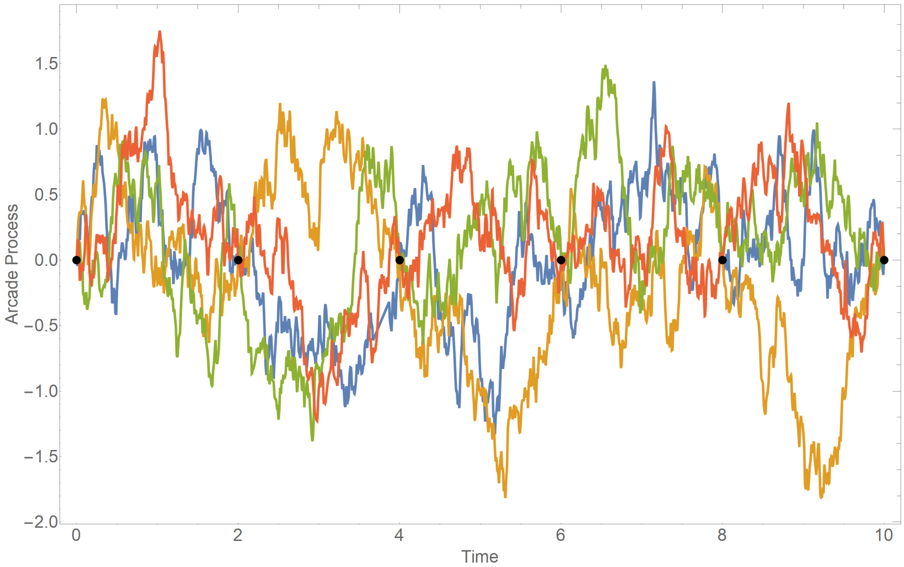

Figure 1: Paths simulation of a stitched Brownian AP with $n = 5$, using the equidistant partition $\{T_i = 2i \,|\, i = 0, 1, \dots, 5\}$.

Another way of generalizing the anticipative Brownian bridge to obtain an AP driven by Brownian motion is by using Lagrange's polynomial interpolation, that is

$$f_i(t) = \prod_{k=0,k\neq i}^{n} \frac{T_k - t}{T_k - T_i} \quad \text{for } i = 0, \dots, n. \tag{2.6}$$

We may call the resulting AP the Lagrange-Brownian AP, which has the form

$$A_t^{(n)} = B_t - \sum_{i=0}^{n} \prod_{k=0,k\neq i}^{n} \frac{T_k - t}{T_k - T_i} B_{T_i}. \tag{2.7}$$

More generally, the Lagrange AP driven by a stochastic process (D_t) has the form

$$A_t^{(n)} = D_t - \sum_{i=0}^{n} \prod_{k=0,k\neq i}^{n} \frac{T_k - t}{T_k - T_i} D_{T_i}. \tag{2.8}$$

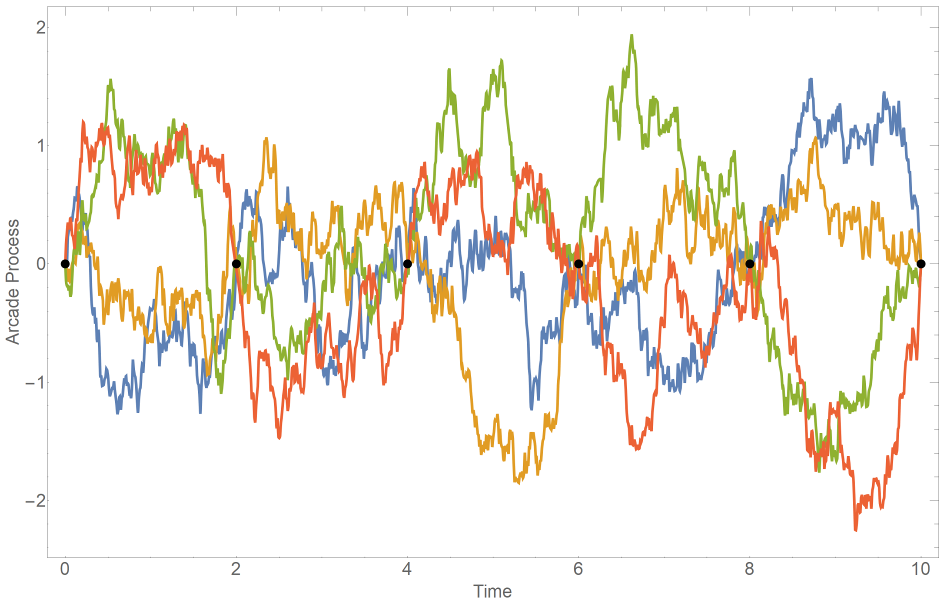

Figure 2: Paths simulation of a Lagrange-Brownian AP with $n = 5$, using the equidistant partition $\{T_i = 2i \,|\, i = 0, 1, \dots, 5\}$.

The Lagrange APs inherit Runge's phenomenon from their interpolation coefficients: when n is big, the AP oscillates around the edges of the interval.

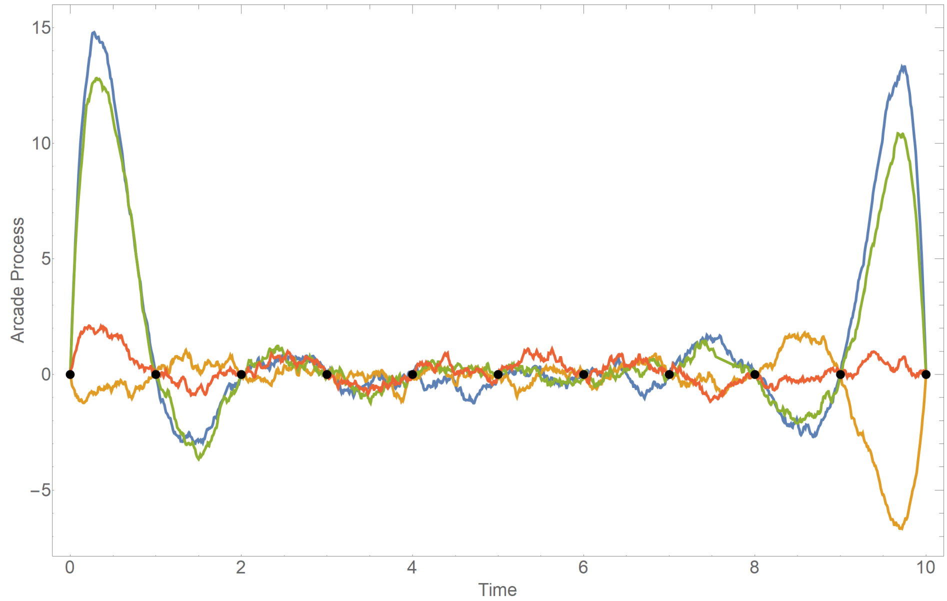

Figure 3: Paths simulation of a Lagrange-Brownian AP with $n = 10$, using the equidistant partition $\{T_i = i \,|\, i = 0, 1, \ldots, 10\}$, illustrating Runge's phenomenon.

One can negate this effect by applying a transformation to the interpolating coefficients. For instance, the map $x \mapsto |x|^{2(1-|x|)}$ applied to each interpolating coefficients $f_i(t) = \prod_{k=0,k\neq i}^{n} \frac{T_k - t}{T_k - T_i}$ for $i = 0, \ldots, n$, yields another set of interpolating coefficients $\tilde{f}_0, \ldots, \tilde{f}_n$ that do no suffer from Runge's phenomenon.

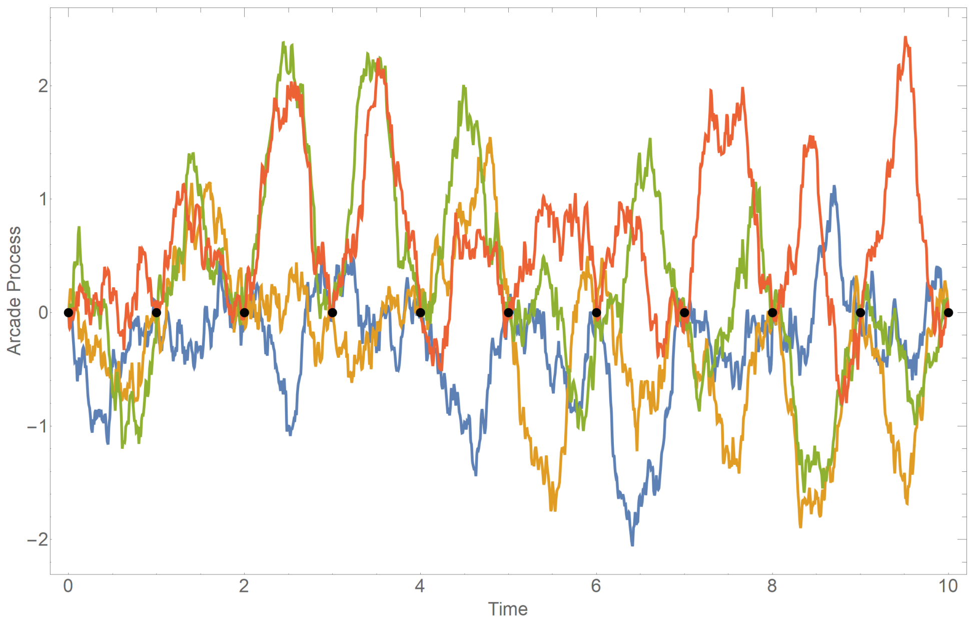

Figure 4: Paths simulation of a Brownian AP with $n = 10$ and interpolating coefficients $\tilde{f}_0, \ldots, \tilde{f}_{10}$, using the equidistant partition $\{T_i = i \,|\, i = 0, 1, \ldots, 10\}$.

Example 2.5. *Elliptic APs have interpolating coefficients given by*

$$f_0(t) = \sqrt{1 - \left(\frac{t - T_0}{T_1 - T_0}\right)^2} \mathbb{1}_{[T_0,T_1]}(t), \quad f_n(t) = \sqrt{1 - \left(\frac{t - T_n}{T_n - T_{n-1}}\right)^2} \mathbb{1}_{(T_{n-1},T_n]}(t), \tag{2.9}$$

$$f_i(t) = \sqrt{1 - \left(\frac{t - T_i}{T_i - T_{i-1}}\right)^2} \mathbb{1}_{(T_{i-1}, T_i]}(t) + \sqrt{1 - \left(\frac{t - T_i}{T_{i+1} - T_i}\right)^2} \mathbb{1}_{(T_i, T_{i+1}]}(t) \quad \textit{for } i = 1, \dots, n-1. \tag{2.10}$$

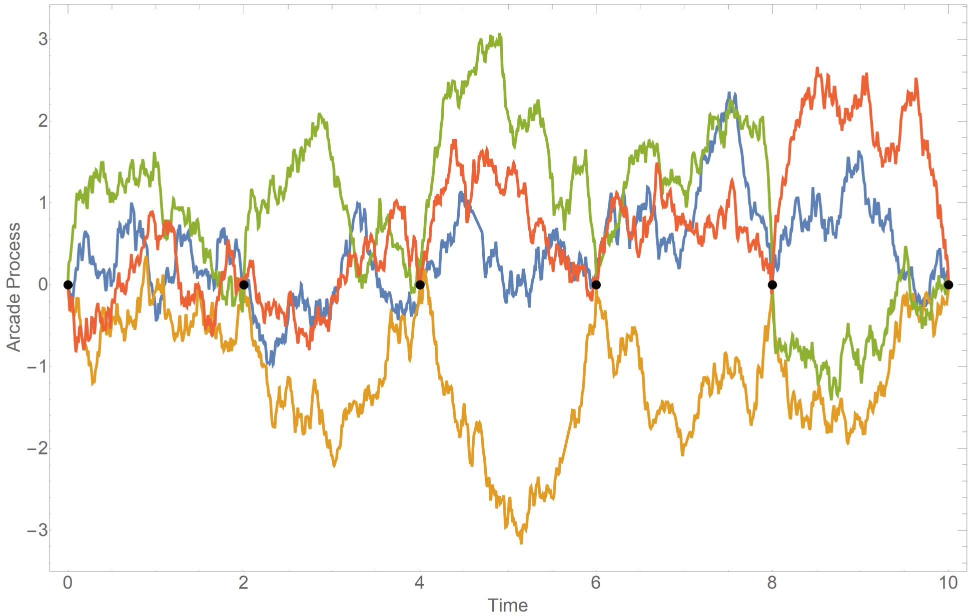

Figure 5: Paths simulation of an elliptic-Brownian AP with $n = 5$, using the equidistant partition $\{T_i = 2i \,|\, i = 0, 1, \dots, 5\}$.

The expectation and the covariance of an AP, when they exist, are fully characterised by the driver and the interpolating coefficients.

Proposition 2.6. *If the driver (D_t) has a mean function μ_D, a variance function σ_D^2, and an covariance function K_D, we have:*

$$\mu_A(t) := \mathbb{E}[A_t^{(n)}] = \mu_D(t) - \sum_{i=0}^{n} f_i(t)\mu_D(T_i), \tag{2.11}$$

$$\begin{aligned} \sigma_A^2(t) := \mathrm{Var}[A_t^{(n)}] = \sigma_D^2(t) &+ \sum_{i=0}^{n} f_i^2(t)\sigma_D^2(T_i) - 2f_i(t)K_D(t, T_i) \\ &+ 2\sum_{i=0}^{n}\sum_{j=i+1}^{n} f_i(t)f_j(t)K_D(T_i, T_j), \end{aligned} \tag{2.12}$$

$$\begin{aligned} K_A(s,t) := \mathrm{Cov}[A_s^{(n)}, A_t^{(n)}] = K_D(s,t) &- \sum_{i=0}^{n} f_i(t)K_D(s, T_i) + f_i(s)K_D(t, T_i) \\ &+ \sum_{i=0}^{n}\sum_{j=0}^{n} f_i(s)f_j(t)K_D(T_i, T_j). \end{aligned} \tag{2.13}$$

Proof. Eq. 2.11 follows from the linearity of the expectation and Eq. 2.12 follows from Eq. 2.13. Hence, it is enough to show Eq. 2.13.

$$
\begin{aligned}
\mathrm{Cov}[A_s^{(n)}, A_t^{(n)}] &= \mathrm{Cov}\left[D_s - \sum_{i=0}^{n} f_i(s) D_{T_i}, D_t - \sum_{i=0}^{n} f_i(t) D_{T_i}\right] && (2.14)\\
&= K_D(s,t) - \mathrm{Cov}\left[D_s, \sum_{i=0}^{n} f_i(t) D_{T_i}\right] - \mathrm{Cov}\left[\sum_{i=0}^{n} f_i(s) D_{T_i}, D_t\right]\\
&\quad + \mathrm{Cov}\left[\sum_{i=0}^{n} f_i(s) D_{T_i}, \sum_{i=0}^{n} f_i(t) D_{T_i}\right] && (2.15)\\
&= K_D(s,t) - \sum_{i=0}^{n} f_i(t) K_D(s, T_i) - \sum_{i=0}^{n} f_i(s) K_D(T_i, t)\\
&\quad + \sum_{i=0}^{n}\sum_{j=0}^{n} f_i(s) f_j(t) K_D(T_i, T_j) && (2.16)\\
&= K_D(s,t) - \sum_{i=0}^{n} f_i(t) K_D(s, T_i) + f_i(s) K_D(t, T_i)\\
&\quad + \sum_{i=0}^{n}\sum_{j=0}^{n} f_i(s) f_j(t) K_D(T_i, T_j). && (2.17)
\end{aligned}
$$

□

Markov APs will play a crucial role in the construction of FAMs. An immediate property of these processes is the following:

Proposition 2.7. *Let $(A_t^{(n)})_{t\in[T_0,T_n]}$ be an AP on $\{T_0, T_n\}_*$ that is Markov with respect to $(\mathcal{F}_t^A)_{t\in[T_0,T_n]}$. If $t > T_i$ for $T_i \in \{T_0, T_n\}_*$, then $A_t^{(n)} \perp\!\!\!\perp \mathcal{F}_{T_i}^A$.*

In particular, if $(A_t^{(n)})$ is Markov and has a covariance function, then $\mathrm{Cov}(A_s^{(n)}, A_t^{(n)}) = 0$ whenever $s \in [T_i, T_{i+1}]$, $t \in [T_j, T_{j+1}]$ and $j \neq i$.

If we think about APs as noise processes, the most natural subclass to study is the Gaussian subclass. But instead of looking at all APs that are also Gaussian processes, we restrict to APs driven by Gaussian processes.

Definition 2.8. *An AP $(A_t^{(n)})_{t\in[T_0,T_n]}$ is said to be a Gaussian AP if its driver (D_t) is a Gaussian stochastic process.*

Notice that not all APs that are Gaussian processes are Gaussian APs. But all Gaussian APs are Gaussian processes: Take $D_t = B_t + tY$, where (B_t) is a Gaussian process and Y a non-Gaussian random variable, and $A_t = D_t - (1-t)D_0 - tD_1$. The driver is not Gaussian while the AP is a Gaussian process.

We give a first result about the Markov property of Gaussian APs.

Theorem 2.9. *Let $(A_t^{(n)})_{t\in[T_0,T_n]}$ be a Gaussian AP on $\{T_0, T_n\}_*$ with covariance function K_A. Then $(A_t^{(n)})$ is Markov with respect to its own filtration if and only if $\forall\,(r,s,t) \in (T_0, T_n)_*^3$ such that $r \leqslant s < t$, there exists $a(s,t) \in \mathbb{R}_0$ such that*

$$
K_A(r,t) = \begin{cases} 0, & \text{if } t \in [T_i, T_{i+1}], r \in [T_j, T_{j+1}], i \neq j, \\ K_A(r,s)\,a(s,t), & \text{otherwise.} \end{cases} \tag{2.18}
$$

Proof. Suppose K_A is of the form 2.18, we shall show that $(A_t^{(n)})$ is Markov. Let $k > 1$ and $(s_1, s_2, \dots, s_k, t) \in [T_0, T_n]^{k+1}$ such that $s_1 < s_2 < \dots < s_k < t$. Then, $(A_t^{(n)})$ is Markov if and only if

$$\mathbb{P}\left[A_t^{(n)} \in \cdot \mid A_{s_1}^{(n)}, \dots, A_{s_k}^{(n)}\right] = \mathbb{P}\left[A_t^{(n)} \in \cdot \mid A_{s_k}^{(n)}\right]. \tag{2.19}$$

Since $K_A(s,t) = 0$ unless s and t are in the same sub-interval, we can assume without loss of generality that $(s_1, s_2, \dots, s_k, t) \in (T_m, T_{m+1})^{k+1}$ for a fixed $m \in \{0, \dots, n-1\}$. Define

$$\Delta_q = \sum_{i=1}^{k} c_{i,q} A_{s_i}^{(n)}, \quad q = 1, \dots, k-1, \tag{2.20}$$

where the coefficients $(c_{i,q})$ are chosen such that

$$\sum_{i=1}^{k} c_{i,q} K_A(s_i, t) = 0 \quad \text{and} \quad \det \begin{pmatrix} c_{1,1} & \dots & c_{1,k-1} \\ \vdots & & \vdots \\ c_{k-1,1} & \dots & c_{k-1,k-1} \end{pmatrix} \neq 0. \tag{2.21}$$

We notice that

$$\sigma(A_{s_1}^{(n)}, \dots, A_{s_k}^{(n)}) = \sigma(\Delta_1, \dots, \Delta_{k-1}, A_{s_k}^{(n)}) \iff \det \begin{pmatrix} c_{1,1} & \dots & c_{1,k-1} \\ \vdots & & \vdots \\ c_{k-1,1} & \dots & c_{k-1,k-1} \end{pmatrix} \neq 0. \tag{2.22}$$

It remains to be shown that

$$A_t^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}) \text{ and } A_{s_k}^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}). \tag{2.23}$$

Equivalently, because we are treating the Gaussian case, we shall show that

$$\mathrm{Cov}(A_t^{(n)}, \Delta_q) = 0 \quad \text{and} \quad \mathrm{Cov}(A_{s_k}^{(n)}, \Delta_q) = 0 \quad \forall q = 1, \dots, k-1. \tag{2.24}$$

Expanding these covariances, we get

$$\mathrm{Cov}(A_t^{(n)}, \Delta_q) = \sum_{i=1}^{k} c_{i,q} K_A(s_i, t) = 0, \tag{2.25}$$

which is guaranteed by the choice of the coefficients $(c_{i,q})$, and

$$\mathrm{Cov}(A_{s_k}^{(n)}, \Delta_q) = \sum_{i=1}^{k} c_{i,q} K_A(s_k, s_i) = 0. \tag{2.26}$$

The first equation implies the second because $K_A(s_i, t) = 0 \implies K_A(s_k, s_i) = 0$ by 2.18, and, when $K_A(s_i, t) \neq 0$, we have $K_A(s_i, t) = a(s_k, t) K_A(s_k, s_i)$, which means

$$\sum_{i=1}^{k} c_{i,q} K_A(s_i, t) = 0 \implies \sum_{i=1}^{k} c_{i,q} a(s_k, t) K_A(s_k, s_i) = 0 \implies \sum_{i=1}^{k} c_{i,q} K_A(s_k, s_i) = 0. \tag{2.27}$$

This concludes one implication.

For the converse, suppose without lose of generality that the driver has mean 0. We observe that $K_A(x, y) = 0$ if $x \in [T_i, T_{i+1}], y \in [T_j, T_{j+1}], i \neq j$. Let $(r, s, t) \in (T_i, T_{i+1})^3$ such that $r \leqslant s < t$. Since $(A_t^{(n)})$ is Gaussian, we have

$$\mathbb{E}[A_t^{(n)} \mid A_s^{(n)}] = \frac{K_A(s,t)}{K_A(s,s)} A_s^{(n)}. \tag{2.28}$$

Using the Markov property of $(A_t^{(n)})$, we get

$$K_A(r,t) = \mathbb{E}[A_r^{(n)} A_t^{(n)}] = \mathbb{E}[\mathbb{E}[A_r^{(n)} A_t^{(n)} \mid A_s^{(n)}]] = \mathbb{E}[\mathbb{E}[A_r^{(n)} \mid A_s^{(n)}]\mathbb{E}[A_t^{(n)} \mid A_s^{(n)}]], \tag{2.29}$$

which implies

$$K_A(r,t) = \mathbb{E}\left[\frac{K_A(s,t)}{K_A(s,s)} A_s^{(n)} \frac{K_A(r,s)}{K_A(s,s)} A_s^{(n)}\right] = \frac{K_A(s,t)K_A(r,s)}{K_A(s,s)}. \tag{2.30}$$

Hence $a(s,t) = \dfrac{K_A(s,t)}{K_A(s,s)}$.

□

We can simplify the statement of Theorem 2.9 in the following way.

Corollary 2.10. *A Gaussian AP $(A_t^{(n)})_{t\in[T_0,T_n]}$ is Markov with respect to its own filtration if and only if there exist real functions $A_1 : [T_0, T_n] \to \mathbb{R}$, and $A_2 : [T_0, T_n] \to \mathbb{R}$, such that*

$$K_A(s,t) = \sum_{i=0}^{n-1} A_1(\min(s,t)) A_2(\max(s,t)) \mathbb{1}_{(T_i,T_{i+1})}(s,t). \tag{2.31}$$

Proof. If

$$K_A(s,t) = \sum_{i=0}^{n-1} A_1(\min(s,t)) A_2(\max(s,t)) \mathbb{1}_{(T_i,T_{i+1})}(s,t), \tag{2.32}$$

then $\forall\,(r,s,t) \in (T_0,T_n)^3_*$ such that $r \leqslant s < t$,

$$K_A(r,t) = \begin{cases} 0, & \text{if } t \in [T_i, T_{i+1}], r \in [T_j, T_{j+1}], i \neq j, \\ K_A(r,s)\frac{A_2(t)}{A_2(s)}, & \text{otherwise.} \end{cases} \tag{2.33}$$

Hence, $(A_t^{(n)})$ is Markov.

For the converse, suppose that $(A_t^{(n)})$ is Markov. Let $T_m \in \{T_0, T_{n-1}\}_*$. Then,

$$\frac{K_A(r,t)}{K_A\left(r, \frac{T_m+T_{m+1}}{2}\right)} = \frac{K_A\left(\frac{T_m+T_{m+1}}{2}, t\right)}{K_A\left(\frac{T_m+T_{m+1}}{2}, \frac{T_m+T_{m+1}}{2}\right)} \tag{2.34}$$

for any $(r,t) \in (T_m, T_{m+1})^2$ such that $r < \frac{T_m+T_{m+1}}{2} < t$ by Theorem 2.9. Hence, if

$$A_1(x) = \sum_{i=0}^{n-1} K_A\left(x, \frac{T_i + T_{i+1}}{2}\right) \mathbb{1}_{(T_i,T_{i+1})}(x), \tag{2.35}$$

and

$$A_2(x) = \sum_{i=0}^{n-1} \frac{K_A\left(\frac{T_i+T_{i+1}}{2}, x\right)}{K_A\left(\frac{T_i+T_{i+1}}{2}, \frac{T_i+T_{i+1}}{2}\right)} \mathbb{1}_{(T_i,T_{i+1})}(x), \tag{2.36}$$

we have

$$K_A(s,t) = \sum_{i=0}^{n-1} A_1(\min(s,t)) A_2(\max(s,t)) \mathbb{1}_{(T_i,T_{i+1})}(s,t). \tag{2.37}$$

$\square$

If $T_i \in \{T_0, T_{n-1}\}_*$ and $(s,t) \in (T_i, T_{i+1})^2$ such that $s < t$, then $\lim_{s\to T_i} A_1(s) = 0$ or $\lim_{t\to T_{i+1}} A_2(t) = 0$ by continuity of K_A, and $\frac{A_1}{A_2}(t)$ is positive and non-decreasing on each interval (T_i, T_{i+1}), since K_A is a covariance function.

Starting from a Gauss-Markov driver (D_t), it is always possible to construct a Markovian $(A_t^{(n)})$ by applying the following procedure.

Theorem 2.11. *For any Gauss-Markov driver (D_t), there exists an AP $(A_t^{(n)})_{t\in[T_0,T_n]}$, driven by (D_t), that is Markovian.*

Proof. Let $T_m \in \{T_0, T_{n-1}\}_*$ and $(s,t) \in (T_m, T_{m+1})^2$ such that $s < t$. Choose the interpolating coefficients $f_0, \dots, f_n$ according to

$$\begin{pmatrix} K_D(T_0,T_0) & \dots & K_D(T_0,T_n) \\ \vdots & \ddots & \vdots \\ K_D(T_n,T_0) & \dots & K_D(T_n,T_n) \end{pmatrix} \begin{pmatrix} f_0(\cdot) \\ \vdots \\ f_n(\cdot) \end{pmatrix} = \begin{pmatrix} K_D(\cdot,T_0) \\ \vdots \\ K_D(\cdot,T_n) \end{pmatrix}. \tag{2.38}$$

Then,

$$\sum_{j=0}^{n} f_j(\cdot) K_D(T_i,T_j) - K_D(\cdot,T_i) = 0, \quad \forall i = 0,\dots,n, \tag{2.39}$$

which implies

$$K_A(x,y) = K_D(x,y) - \sum_{i=0}^{n} f_i(x) K_D(y,T_i) = K_D(x,y) - \sum_{i=0}^{n} f_i(y) K_D(x,T_i), \quad \forall (x,y) \in [T_0,T_n]^2. \tag{2.40}$$

Let $T_m \in \{T_0, T_{n-1}\}_*$ and $(s,t) \in (T_m, T_{m+1})^2$ such that $s < t$. Recalling that (D_t) is Gauss-Markov, there exists two functions, $H_1 : (T_m, T_{m+1}) \to \mathbb{R}$ and $H_2 : (T_m, T_{m+1}) \to \mathbb{R}$, such that $K_D(s,t) = H_1(s)H_2(t)$. Using this fact, we can write

$$K_A(s,t) = H_1(s)H_2(t) - \sum_{i=0}^{m} f_i(s)H_1(T_i)H_2(t) - \sum_{i=m+1}^{n} f_i(s)H_1(t)H_2(T_i) \tag{2.41}$$

$$= H_2(t)\left(H_1(s) - \sum_{i=0}^{m} f_i(s)H_1(T_i)\right) - H_1(t)\sum_{i=m+1}^{n} f_i(s)H_2(T_i) \tag{2.42}$$

$$= \left(\sum_{i=m+1}^{n} f_i(s)H_2(T_i)\right)\big(\lambda H_2(t) - H_1(t)\big) \tag{2.43}$$

for some $\lambda \in \mathbb{R}$, where we used Eq. 2.39 with $i = m$. Hence,

$$A_1(x)\mathbb{1}_{(T_m,T_{m+1})}(x) = \left(\sum_{i=m+1}^{n} f_i(x)H_2(T_i)\right)\mathbb{1}_{(T_m,T_{m+1})}(x), \tag{2.44}$$

$$A_2(x)\mathbb{1}_{(T_m,T_{m+1})}(x) = \big(\lambda H_2(x) - H_1(x)\big)\mathbb{1}_{(T_m,T_{m+1})}(x). \tag{2.45}$$

$(A_t^{(n)})$ is Markov by Corollary 2.10. $\square$

Recall that we imposed on all drivers (D_t) of APs the property that $\mathbb{P}[D_t = 0] < 1$ whenever $t \in (T_0, T_n)_*$. If we extend this property to $[T_0, T_n]$ instead, then the above construction of a Markovian AP becomes explicit.

Corollary 2.12. *If (D_t) is a Gauss-Markov process such that $\mathbb{P}[D_t = 0] < 1$ whenever $t \in [T_0, T_n]$, with $K_D(s,t) = H_1(\min(s,t))H_2(\max(s,t))$ for all $(s,t) \in [T_0, T_n]^2$ and for some real functions H_1 and H_2, then the solution to Eq. 2.39 is given by*

$$f_0(x) = \frac{H_1(T_1)H_2(x) - H_1(x)H_2(T_1)}{H_1(T_1)H_2(T_0) - H_1(T_0)H_2(T_1)}\mathbb{1}_{[T_0,T_1]}(x), \tag{2.46}$$

$$\begin{aligned} f_i(x) = &\frac{H_1(x)H_2(T_{i-1}) - H_1(T_{i-1})H_2(x)}{H_1(T_i)H_2(T_{i-1}) - H_1(T_{i-1})H_2(T_i)}\mathbb{1}_{[T_{i-1},T_i]}(x) \\ &+ \frac{H_1(T_{i+1})H_2(x) - H_1(x)H_2(T_{i+1})}{H_1(T_{i+1})H_2(T_i) - H_1(T_i)H_2(T_{i+1})}\mathbb{1}_{(T_i,T_{i+1}]}(x), \quad \text{for } i = 1, \dots, n-1, \end{aligned} \tag{2.47}$$

$$f_n(x) = \frac{H_1(x)H_2(T_{n-1}) - H_1(T_{n-1})H_2(x)}{H_1(T_n)H_2(T_{n-1}) - H_1(T_{n-1})H_2(T_n)}\mathbb{1}_{(T_{n-1},T_n]}(x). \tag{2.48}$$

Proof. Let $(T_{m^-}, T_m, T_{m^+}) \in \{T_0, T_{n-1}\}_*^3$ such that $T_{m^-} \leqslant T_m < T_{m^+}$, and $x \in (T_m, T_{m+1})$. Then,

$$\sum_{j=0}^{n} f_j(x)K_D(T_{m^-}, T_j) = f_m(x)H_1(T_{m^-})H_2(T_m) + f_{m+1}(x)H_1(T_{m^-})H_2(T_m) \tag{2.49}$$

$$\begin{aligned} &= \frac{H_1(T_{m+1})H_2(x) - H_1(x)H_2(T_{m+1})}{H_1(T_{m+1})H_2(T_m) - H_1(T_m)H_2(T_{m+1})}H_1(T_{m^-})H_2(T_m) \\ &\quad + \frac{H_1(x)H_2(T_m) - H_1(T_m)H_2(x)}{H_1(T_{m+1})H_2(T_m) - H_1(T_m)H_2(T_{m+1})}H_1(T_{m^-})H_2(T_{m+1}) \end{aligned} \tag{2.50}$$

$$\begin{aligned} &= \frac{H_1(x)(H_1(T_{m^-})H_2(T_m)H_2(T_{m+1}) - H_1(T_{m^-})H_2(T_m)H_2(T_{m+1}))}{H_1(T_{m+1})H_2(T_m) - H_1(T_m)H_2(T_{m+1})} \\ &\quad + \frac{H_2(x)(H_1(T_{m+1})H_1(T_{m^-})H_2(T_m) - H_1(T_m)H_1(T_{m^-})H_2(T_{m+1}))}{H_1(T_{m+1})H_2(T_m) - H_1(T_m)H_2(T_{m+1})} \end{aligned} \tag{2.51}$$

$$= H_1(T_{m^-})H_2(x) = K_D(T_{m^-}, x) \tag{2.52}$$

The same argument applies to show $\sum_{j=0}^{n} f_j(x)K_D(T_{m^+}, T_j) = K_D(x, T_{m^+})$. Hence,

$$\begin{pmatrix} K_D(T_0,T_0) & \dots & K_D(T_0,T_n) \\ \vdots & \ddots & \vdots \\ K_D(T_n,T_0) & \dots & K_D(T_n,T_n) \end{pmatrix} \begin{pmatrix} f_0(\cdot) \\ \vdots \\ f_n(\cdot) \end{pmatrix} = \begin{pmatrix} K_D(\cdot,T_0) \\ \vdots \\ K_D(\cdot,T_n) \end{pmatrix}. \tag{2.53}$$

□

Remark 2.13. *Notice that if (D_t) does not satisfy $\mathbb{P}[D_t = 0] < 1$ for some $t \in \{T_0, T_n\}_*$, it is still straightforward to construct the above AP by removing all the rows and columns of zeros in the matrix*

$$\begin{pmatrix} K_D(T_0,T_0) & \dots & K_D(T_0,T_n) \\ \vdots & \ddots & \vdots \\ K_D(T_n,T_0) & \dots & K_D(T_n,T_n) \end{pmatrix}. \tag{2.54}$$

Since the driver already matches 0 at $t \in \{T_0, T_n\}_*$, we will not need to find the corresponding interpolating coefficients because they will not appear in the AP expression. This is illustrated by taking the Brownian motion as a driver with $T_0 = 0$. Then f_0 does not matter since it is multiplied by 0 in the AP expression.

Remark 2.14. *For the choice of coefficients in Corollary 2.12, we can simplify Eqs. 2.44 and 2.45:*

$$A_1(x)\mathbb{1}_{(T_m,T_{m+1})}(x) = f_{m+1}(x)H_2(T_{m+1})\mathbb{1}_{(T_m,T_{m+1})}(x), \tag{2.55}$$

$$A_2(x)\mathbb{1}_{(T_m,T_{m+1})}(x) = \left(\frac{H_1(T_{m+1})}{H_2(T_{m+1})}H_2(x) - H_1(x)\right)\mathbb{1}_{(T_m,T_{m+1})}(x). \tag{2.56}$$

This method of producing Markovian APs is not unique, but certainly feels natural. The resulting APs are called standard.

Definition 2.15. *A standard AP $(A_t^{(n)})_{t\in[T_0,T_n]}$ is an AP driven by of a Gauss-Markov process (D_t), with $K_D(s,t) = H_1(\min(s,t))H_2(\max(s,t))\,\forall s,t \in [T_0,T_n]$ for some real functions H_1 and H_2, of the form*

$$A_t^{(n)} = \begin{cases} D_t - \frac{H_1(T_1)H_2(t)-H_1(t)H_2(T_1)}{H_1(T_1)H_2(T_0)-H_1(T_0)H_2(T_1)}D_{T_0} - \frac{H_1(t)H_2(T_0)-H_1(T_0)H_2(t)}{H_1(T_1)H_2(T_0)-H_1(T_0)H_2(T_1)}D_{T_1} & \text{if } t \in [T_0,T_1), \\ D_t - \frac{H_1(T_2)H_2(t)-H_1(t)H_2(T_2)}{H_1(T_2)H_2(T_1)-H_1(T_1)H_2(T_2)}D_{T_1} - \frac{H_1(t)H_2(T_1)-H_1(T_1)H_2(t)}{H_1(T_2)H_2(T_1)-H_1(T_1)H_2(T_2)}D_{T_2} & \text{if } t \in [T_1,T_2), \\ \vdots & \\ D_t - \frac{H_1(T_n)H_2(t)-H_1(t)H_2(T_n)}{H_1(T_n)H_2(T_{n-1})-H_1(T_{n-1})H_2(T_n)}D_{T_{n-1}} & \\ \qquad - \frac{H_1(t)H_2(T_{n-1})-H_1(T_{n-1})H_2(t)}{H_1(T_n)H_2(T_{n-1})-H_1(T_{n-1})H_2(T_n)}D_{T_n} & \text{if } t \in [T_{n-1},T_n]. \end{cases} \tag{2.57}$$

Example 2.16. *If (D_t) is an Ornstein-Uhlenbeck process with parameters $\theta > 0, \sigma > 0, \mu \in \mathbb{R}$ and starting value $d_0 \in \mathbb{R}$, that is, the solution to*

$$\mathrm{d}D_t = \theta\,(\mu - D_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad D_0 = d_0, \tag{2.58}$$

then $K_D(s,t) = \dfrac{\sigma^2}{2\theta}\mathrm{e}^{\theta\min(s,t)}\mathrm{e}^{-\theta\max(s,t)}$. The standard AP driven by (D_t) is

$$A_t^{(n)} = \begin{cases} D_t - \frac{\mathrm{e}^{\theta(T_1-t)}-\mathrm{e}^{-\theta(T_1-t)}}{\mathrm{e}^{\theta(T_1-T_0)}-\mathrm{e}^{-\theta(T_1-T_0)}}D_{T_0} - \frac{\mathrm{e}^{\theta(t-T_0)}-\mathrm{e}^{-\theta(t-T_0)}}{\mathrm{e}^{\theta(T_1-T_0)}-\mathrm{e}^{-\theta(T_1-T_0)}}D_{T_1} & \text{if } t \in [T_0,T_1), \\ D_t - \frac{\mathrm{e}^{\theta(T_2-t)}-\mathrm{e}^{-\theta(T_2-t)}}{\mathrm{e}^{\theta(T_2-T_1)}-\mathrm{e}^{-\theta(T_2-T_1)}}D_{T_1} - \frac{\mathrm{e}^{\theta(t-T_1)}-\mathrm{e}^{-\theta(t-T_1)}}{\mathrm{e}^{\theta(T_2-T_1)}-\mathrm{e}^{-\theta(T_2-T_1)}}D_{T_2} & \text{if } t \in [T_1,T_2), \\ \vdots & \\ D_t - \frac{\mathrm{e}^{\theta(T_n-t)}-\mathrm{e}^{-\theta(T_n-t)}}{\mathrm{e}^{\theta(T_n-T_{n-1})}-\mathrm{e}^{-\theta(T_n-T_{n-1})}}D_{T_{n-1}} - \frac{\mathrm{e}^{\theta(t-T_{n-1})}-\mathrm{e}^{-\theta(t-T_{n-1})}}{\mathrm{e}^{\theta(T_n-T_{n-1})}-\mathrm{e}^{-\theta(T_n-T_{n-1})}}D_{T_n} & \text{if } t \in [T_{n-1},T_n]. \end{cases} \tag{2.59}$$

There are infinitely many Markovian APs driven by the same Gauss-Markov driver. In general, when $T_m \in \{T_0, T_{n-1}\}_*$ and $(s,t) \in (T_m, T_{m+1})^2$ with $s < t$, we have

$$
\begin{aligned}
K_A(s,t) = &\left(H_1(s) - \sum_{i=0}^{m} f_i(s) H_1(T_i)\right)\left(H_2(t) - \sum_{i=m+1}^{n} f_i(t) H_2(T_i)\right) \\
&- \left(\sum_{i=m+1}^{n} f_i(s) H_2(T_i)\right)\left(H_1(t) - \sum_{i=0}^{m+1} f_i(t) H_1(T_i)\right) \\
&+ \left(\sum_{i=m+1}^{n} f_i(s) H_1(T_i)\right)\sum_{i=m+2}^{n} f_i(t) H_2(T_i) \\
&- \left(H_2(s) - \sum_{i=0}^{m} f_i(s) H_2(T_i)\right)\sum_{i=0}^{m} f_i(t) H_1(T_i), \qquad (2.60)
\end{aligned}
$$

where we use the convention that an empty sum is equal to zero. There are as many Markovian APs driven by (D_t) as there are ways to separate the variable of the above expression of K_A.

Example 2.17. *If (D_t) is a standard Brownian motion, applying Eq. 2.39 to find appropriate interpolating coefficients yields the stitched Brownian AP. But there are other Markovian APs driven by standard Brownian motion. For instance, in the two-period case, we may choose*

$$
f_0(t) = \frac{T_1 - t}{T_1 - T_0}\mathbb{1}_{\left[T_0, \frac{T_1+T_2}{2}\right]}(t) - \frac{T_2 - t}{T_1 - T_0}\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(t), \qquad (2.61)
$$

$$
f_1(t) = \frac{t - T_0}{T_1 - T_0}\mathbb{1}_{[T_0,T_1]}(t) + \frac{T_2 - t}{T_2 - T_1}\mathbb{1}_{(T_1,T_2]}(t), \qquad (2.62)
$$

$$
f_2(t) = \frac{t - T_1}{T_2 - T_1}\mathbb{1}_{[T_1,T_2]}(t). \qquad (2.63)
$$

It is straightforward to verify that these are interpolating coefficients. Let $(A_t^{(2)})_{t\in[T_0,T_2]}$ be the AP with these interpolating coefficients driven by a standard Brownian motion. Then,

$$
\begin{aligned}
&K_A(s,t) \\
&= \frac{(\min(s,t) - T_0)(T_1 - \max(s,t))}{T_1 - T_0}\mathbb{1}_{(T_0,T_1)}(s,t) \\
&+ (\min(s,t) - T_1)\left(\frac{T_0(\max(s,t) + T_0) - 3T_0T_1 + T_1^2}{(T_1 - T_0)^2} + \frac{T_1 - \max(s,t)}{T_2 - T_1}\right)\mathbb{1}_{\left(T_1, \frac{T_1+T_2}{2}\right]}(s,t) \\
&+ \frac{(\min(s,t) - T_1)(T_2 - \max(s,t))}{T_2 - T_1}\frac{T_0^2 + T_1^2 + T_0(T_2 - 3T_1)}{(T_1 - T_0)^2} \\
&\qquad \times \mathbb{1}_{\left(T_1, \frac{T_1+T_2}{2}\right]}(\min(s,t))\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(\max(s,t)) \\
&+ \frac{\min(s,t)(T_0^2 + T_1^2 - T_0(T_1 + T_2)) - (T_1 - T_0)^2 T_1 + T_0T_2(T_2 - T_1)}{(T_1 - T_0)^2}\frac{T_2 - \max(s,t)}{T_2 - T_1} \\
&\qquad \times \mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(s,t). \qquad (2.64)
\end{aligned}
$$

Thus,

$$A_1(x) = (x - T_0)\mathbb{1}_{[T_0,T_1]}(x) + (x - T_1)\mathbb{1}_{\left(T_1, \frac{T_1+T_2}{2}\right]}(x) + \frac{x(T_0^2 + T_1^2 - T_0(T_1 + T_2)) - (T_1 - T_0)^2 T_1 + T_0 T_2(T_2 - T_1)}{T_0^2 + T_1^2 + T_0(T_2 - 3T_1)}\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(x), \tag{2.65}$$

$$A_2(x) = \frac{T_1 - x}{T_1 - T_0}\mathbb{1}_{[T_0,T_1]}(x) + \left(\frac{T_0(x + T_0) - 3T_0T_1 + T_1^2}{(T_1 - T_0)^2} + \frac{T_1 - x}{T_2 - T_1}\right)\mathbb{1}_{\left(T_1, \frac{T_1+T_2}{2}\right]}(x) \tag{2.66}$$

$$+\frac{(T_2 - x)}{T_2 - T_1}\frac{T_0^2 + T_1^2 + T_0(T_2 - 3T_1)}{(T_1 - T_0)^2}\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(x).$$

Hence, recalling Theorem 3.8, this AP is Markovian. This AP is a slight modification of the stitched Brownian AP, where f_0 is not 0 on (T_1, T_2). Hence, B_{T_0} still has an influence (a negative one since f_0 is negative on (T_1, T_2)) on the paths of the AP on (T_1, T_2). Similarly, we can modify the stitched Brownian arcade by making f_2 not 0 on (T_0, T_1):

$$f_0(t) = \frac{T_1 - t}{T_1 - T_0}\mathbb{1}_{[T_0,T_1]}(t), \tag{2.67}$$

$$f_1(t) = \frac{t - T_0}{T_1 - T_0}\mathbb{1}_{[T_0,T_1]}(t) + \frac{T_2 - t}{T_2 - T_1}\mathbb{1}_{(T_1,T_2]}(t), \tag{2.68}$$

$$f_2(t) = \frac{T_0 - t}{T_2 - T_1}\mathbb{1}_{\left[T_0, \frac{T_0+T_1}{2}\right]}(t) + \frac{t - T_1}{T_2 - T_1}\mathbb{1}_{\left(\frac{T_0+T_1}{2}, T_2\right]}(t). \tag{2.69}$$

For this choice of interpolating coefficients, B_{T_2} has an influence on the paths of the AP on (T_0, T_1). Combining the interpolating coefficients f_0 from Eq. 2.61 and f_2 from Eq. 2.69, we can find an interpolating coefficient f_1, such that a Brownian AP with these interpolating coefficients is Markovian:

$$f_0(t) = \frac{T_1 - t}{T_1 - T_0}\mathbb{1}_{\left[T_0, \frac{T_1+T_2}{2}\right]}(t) + \frac{t - T_2}{T_1 - T_0}\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(t), \tag{2.70}$$

$$f_1(t) = \frac{(t - T_0)(T_2 - T_0)}{(T_2 - T_1)(T_1 - T_0)}\mathbb{1}_{\left[T_0, \frac{T_0+T_1}{2}\right]}(t) + \frac{T_1^2 - T_0T_2 + t(T_0 - 2T_1 + T_2)}{(T_1 - T_0)(T_2 - T_1)}\mathbb{1}_{\left(\frac{T_0+T_1}{2}, T_1\right]}(t)$$

$$+ \frac{T_1(T_0(T_1 - 2T_2) + T_1T_2) + t(T_0T_2 - T_1^2)}{(T_2 - T_1)(T_1 - T_0)T_1}\mathbb{1}_{\left(T_1, \frac{T_1+T_2}{2}\right]}(t)$$

$$+ \frac{(T_2 - t)(T_1^2 + T_0(T_2 - 2T_1))}{(T_2 - T_1)(T_1 - T_0)T_1}\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(t), \tag{2.71}$$

$$f_2(t) = \frac{T_0 - t}{T_2 - T_1}\mathbb{1}_{\left[T_0, \frac{T_0+T_1}{2}\right]}(t) + \frac{t - T_1}{T_2 - T_1}\mathbb{1}_{\left(\frac{T_0+T_1}{2}, T_2\right]}(t). \tag{2.72}$$

The key to building non-standard APs is to break each sub-interval into several pieces, and to define the interpolating coefficients by parts on these pieces while making sure that they remain continuous, and that the expression of K_A in Eq. 2.60 has separable variables.

3 Randomized arcade processes

We extend the idea of arcade processes to interpolate between the components of the random vector X instead of interpolating between zeros. Two sets $\{f_0, \dots, f_n\}$ and $\{g_0, \dots, g_n\}$ of interpolating coefficients (see Def. 2.1) are needed to ensure the matching of the target random variables. We recall that the $\mathbb{R}^{n+1}$-valued random vector $X = (X_0, \dots, X_n)$ is independent of the stochastic driver (D_t), while the random variables $X_0, \dots, X_n$ may be mutually dependent.

Definition 3.1. *An X-randomized arcade process (X-RAP) $(I_t^{(n)})_{t\in[T_0,T_n]}$ on the partition $\{T_0, T_n\}_*$ is a stochastic process of the form*

$$I_t^{(n)} := S_t^{(n)} + A_t^{(n)} = D_t - \sum_{i=0}^{n} \left(f_i(t) D_{T_i} - g_i(t) X_i \right), \tag{3.1}$$

where $f_0, \dots, f_n$ and $g_0, \dots, g_n$ are interpolating coefficients on $\{T_0, T_n\}_*$. We refer to

$$S_t^{(n)} = \sum_{i=0}^{n} g_i(t) X_i \tag{3.2}$$

as the signal function of $I_t^{(n)}$ and to

$$A_t^{(n)} = D_t - \sum_{i=0}^{n} f_i(t) D_{T_i} \tag{3.3}$$

as the noise process of $I_t^{(n)}$. We denote by $(\mathcal{F}_t^I)_{t\in[T_0,T_n]}$ the filtration generated by $(I_t^{(n)})$.

We notice that $I_{T_0}^{(n)} = X_0, \dots, I_{T_n}^{(n)} = X_n$, so $(I_t^{(n)})$ is a stochastic interpolator between the random variables $X_0, \dots, X_n$. We have that $(S_t^{(n)}) \perp\!\!\!\perp (A_t^{(n)})$ since $X \perp\!\!\!\perp (D_t)$.

Remark 3.2. *A related class of processes, introduced in [15], are known as the random n-bridges. These processes, defined weakly, match given probability measures instead, and are designed in the same way as a randomized stochastic bridge: by conditioning a stochastic process to match certain probability measures at given times. In particular cases, the law of a RAP satisfies the conditions for the RAP to be a random n-bridge. For instance, the RAP obtained by randomizing the stitched Brownian AP, using the same interpolating coefficients for the signal function as the ones used in the noise process, has a law that satisfies the conditions for the RAP to be a random n-bridge. But any other RAP driven by Brownian motion is not a random n-bridge. Similarly, certain random n-bridges cannot have the same law as a RAP. In this paper, we did not allow the driver to be have jumps, such as a non-continuous Lévy process, by choice. But if we did just for the sack of comparison, a random n-bridge built using a gamma process will never match the law of a RAP driven by a gamma process, since APs are sums, not products.*

The paths of an X-RAP will depend on the coupling of X, not only on its marginal distributions. This property is illustrated in the following example.

Example 3.3. *Let $X = (X_0, \dots, X_5)$ be a vector of independent $Unif(\{-1, 1\})$ random variables, and $Y = (Y_0, \dots, Y_5)$ be another vector of random variables such that $Y_0 \sim Unif(\{-1, 1\})$, $Y_i = -Y_{i-1}$ for $i = 1, \dots, 5$. Let $A_t^{(5)}$ be an AP with elliptic interpolation coefficients driven by Brownian motion multiplied by 0.2, $g_i = f_i$ for $i = 0, \dots, 5$, and $I_t^{(5)}$, $\tilde{I}_t^{(5)}$ its associated X-RAP and Y-RAP respectively. Although we are using the same driver and interpolating coefficients for both RAPs, and that the vectors X and Y have the same marginal distributions, the paths of $I_t^{(5)}$ and $\tilde{I}_t^{(5)}$, shown below, are very different.*

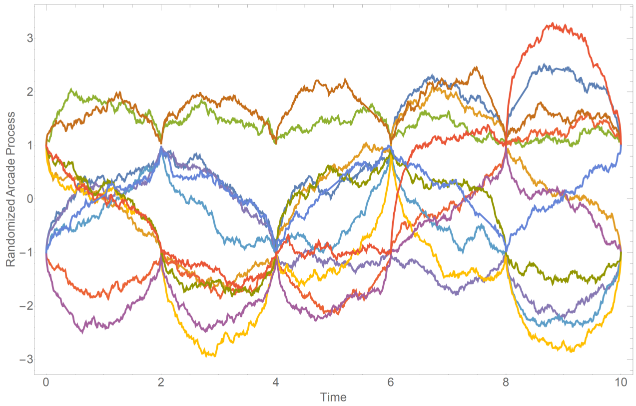

Figure 6: Paths simulation of $I_t^{(5)}$ on $[0,10]$ using the equidistant partition $\{T_i = 2i \,|\, i = 0,1,\dots,5\}$.

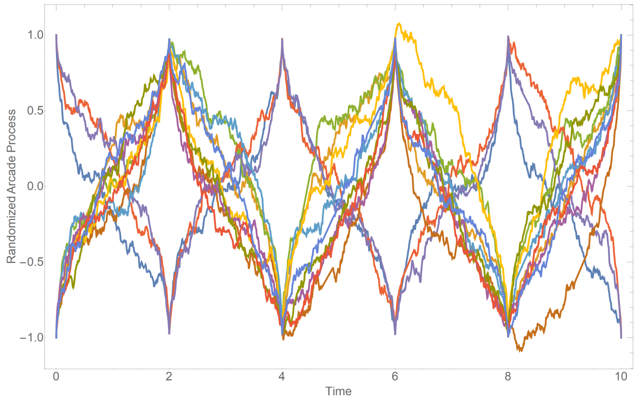

Figure 7: Paths simulation of $\tilde{I}_t^{(5)}$ on $[0,10]$ using the equidistant partition $\{T_i = 2i \,|\, i = 0,1,\dots,5\}$.

Besides its main purpose of interpolating in the strong sense, a RAP can also be used to mimic another stochastic process. Let $(Y_t)_{t\in[T_0,T_n]}$ be a sample-continuous stochastic process. For instance, if $\{T_0,T_n\}_*$ is the equidistant partition of an interval $[a,b]$, $X = (Y_{T_0},\dots,Y_{T_n})$, and $\{f_0,\dots,f_n\} = \{g_0,\dots,g_n\}$ are the piecewise linear interpolating coefficients, then for nearly all $\omega \in \Omega$, $\sup_{t\in[a,b]} A_t^{(n)}(\omega) \to 0$, and $\sup_{t\in[a,b]} S_t^{(n)}(\omega) \to Y_t(\omega)$ as $n \to \infty$. Hence, in that case,

$$\sup_{t\in[a,b]} \left| I_t^{(n)} - Y_t \right| = \sup_{t\in[a,b]} \left| S_t^{(n)} + A_t^{(n)} - Y_t \right| \to 0, \tag{3.4}$$

with probability one.

Example 3.4. *Let $[a,b] = [0,10]$, and $\{T_0,T_n\}_*$ its equidistant partition. If (Y_t) is a fractional Brownian motion and $X = (Y_{T_0},\dots,Y_{T_n})$, then the paths of an X-randomized stitched Brownian arcade will be similar to the one of (Y_t) when n is large enough.*

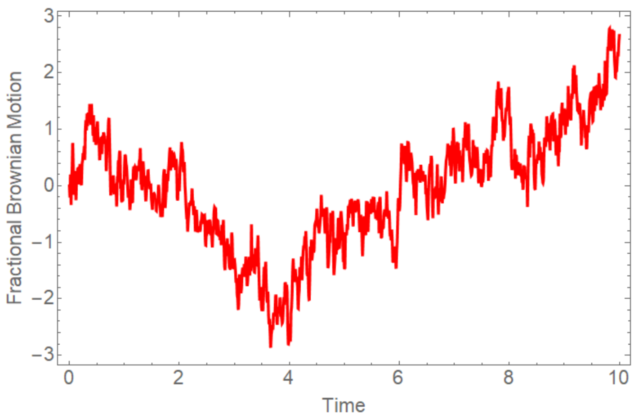

Figure 8: Path of the fractional Brownian motion (Y_t).

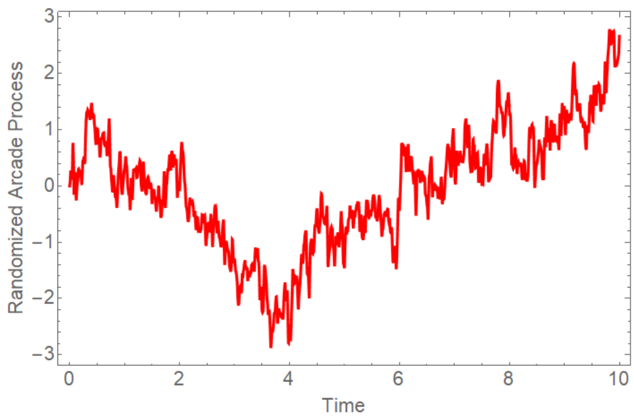

Figure 9: Path of an X-RAP mimicking the same fractional Brownian motion (Y_t).

Proposition 3.5. *Let $(S_t^{(n)})$ and $(A_t^{(n)})$ have mean functions μ_S, μ_A, variance functions σ_S^2, σ_A^2 and covariance functions K_S, K_A, respectively. Then*

$$\mu_I(t) := \mathbb{E}[I_t^{(n)}] = \mu_S(t) + \mu_A(t), \tag{3.5}$$

$$\sigma_I^2(t) := \mathrm{Var}[I_t^{(n)}] = \sigma_S^2(t) + \sigma_A^2(t), \tag{3.6}$$

$$K_I(s,t) := \mathrm{Cov}[I_s^{(n)}, I_t^{(n)}] = K_S(s,t) + K_A(s,t) \tag{3.7}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n} g_i(t)g_j(s)\mathrm{Cov}(X_i, X_j) + K_A(s,t). \tag{3.8}$$

We introduce terminologies similar to the ones from the previous section.

Definition 3.6. *Let $(I_t^{(n)})$ be a RAP.*

1. *$(I_t^{(n)})$ is said to be a Gaussian RAP if its stochastic driver (D_t) is a Gaussian process.*
2. *$(I_t^{(n)})$ is said to be a standard RAP if its noise process $(A_t^{(n)})$ is a standard AP, $g_j(x)\mathbb{1}_{[T_{j-1},T_j]}(x) = f_j(x)\mathbb{1}_{[T_{j-1},T_j]}(x)$, and $g_j(x)\mathbb{1}_{[T_0,T_{j-1}]}(x) = 0$, for all $x \in [T_0, T_n]$ and $j = 1, \ldots, n$.*

We give a similar Markovianity result for Gaussian RAPs to the one of AP.

Definition 3.7. *Let $\mathcal{I} \subseteq \mathbb{R}^+$ be a real interval, and $\tau_0 < \tau_1 < \ldots < \infty$ such that $\tau = \{\tau_0, \tau_1, \ldots\} \subset \mathcal{I}$. The set τ may be finite, i.e., there exists a maximal element $\tau_n \in \tau$, or contain infinitely many elements. A stochastic process $(Y_t)_{t\in\mathcal{I}}$ is called τ-nearly Markov if*

$$\mathbb{P}\left[Y_t \in \cdot \mid \mathcal{F}_s^Y\right] = \mathbb{P}\left[Y_t \in \cdot \mid Y_{\tau_0}, \ldots, Y_{\tau_{m(s)}}, Y_s\right] \tag{3.9}$$

for any $(s,t) \in \mathcal{I}^2$ such that $s \leqslant t$, and $\tau_{m(s)} = \max_{i\in\mathbb{N}}\{\tau_i \mid \tau_i \leqslant s\}$.

Theorem 3.8. *Let $(I_t^{(n)})_{t\in[T_0,T_n]} = (S_t^{(n)} + A_t^{(n)})_{t\in[T_0,T_n]}$ be a Gaussian X-RAP on $\{T_0, T_n\}_*$. Then $(I_t^{(n)})$ is $\{T_0, T_n\}_*$-nearly Markov if the following conditions are all satisfied:*

1. *The AP $(A_t^{(n)})$ is Markov, i.e., $K_A(s,t) = \sum_{i=0}^{n-1} A_1(\min(s,t))A_2(\max(s,t))\mathbb{1}_{(T_i,T_{i+1})}(s,t)$.*
2. *For all $j = 1, \ldots, n$, and for all $x \in [T_0, T_n]$,*

$$g_j(x)\mathbb{1}_{[T_0,T_{j-1}]}(x) = 0, \tag{3.10}$$

$$g_j(x)A_1(T_j)\mathbb{1}_{[T_{j-1},T_j]}(x) = A_1(x)\mathbb{1}_{[T_{j-1},T_j]}(x). \tag{3.11}$$

Proof. Let $k > 1$ and $(s_1, s_2, \ldots, s_k, t) \in [T_0, T_n]^{k+1}$ such that $s_1 < s_2 < \ldots < s_k < t$. Then, $(I_t^{(n)})$ is $\{T_0, T_n\}_*$-nearly Markov if and only if

$$\mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \ldots, X_{m(s_k)}, I_{s_1}^{(n)}, \ldots, I_{s_k}^{(n)}\right] = \mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \ldots, X_{m(s_k)}, I_{s_k}^{(n)}\right]. \tag{3.12}$$

where $m(s_k) := \max\{i \in \mathbb{N} \mid T_i \leqslant s_k\}$. In the following, we will refer to $m(s_k)$ by m since s_k is fixed.

We first show that $s_1, \ldots, s_k$ can be picked to all be in the sub-interval (T_m, T_{m+1}). To see this, assume there is an integer $j \in \{1, \ldots, k\}$ such that $s_j < T_m$ and $T_m < s_{j+1}$. Then

$$\sigma(X_0, \ldots, X_m, I_{s_1}^{(n)}, \ldots, I_{s_k}^{(n)}) = \sigma(X_0, \ldots, X_m, A_{s_1}^{(n)}, \ldots, A_{s_j}^{(n)}, I_{s_{j+1}}^{(n)}, \ldots, I_{s_k}^{(n)}) \tag{3.13}$$

by Eq. 3.10. We also know that $(A_{s_1}^{(n)}, \ldots, A_{s_j}^{(n)}) \perp\!\!\!\perp (X_0, \ldots, X_m)$ by the definition of the X-RAP, and $(A_{s_1}^{(n)}, \ldots, A_{s_j}^{(n)}) \perp\!\!\!\perp (I_{s_{j+1}}^{(n)}, \ldots, I_{s_k}^{(n)}, I_t^{(n)})$ since $(A_t^{(n)})$ is Markov. We conclude that

$$\mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \ldots, X_m, I_{s_1}^{(n)}, \ldots, I_{s_k}^{(n)}\right] = \mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \ldots, X_m, I_{s_{j+1}}^{(n)}, \ldots, I_{s_k}^{(n)}\right], \tag{3.14}$$

which means we can assume that $s_1, \dots, s_k$ are all in the same sub-interval (T_m, T_{m+1}).

Let us define $a_m(\cdot) := \sum_{i=0}^{m} g_i(\cdot) X_i$, and

$$\Delta_q := \sum_{i=1}^{k} c_{i,q} I_{s_i}^{(n)} = \sum_{i=1}^{k} c_{i,q} a_m(s_i) + \sum_{i=1}^{k} c_{i,q} g_{m+1}(s_i) X_{m+1} + \sum_{i=1}^{k} c_{i,q} A_{s_i}^{(n)}, \quad q = 1, \dots, k-1, \tag{3.15}$$

where the coefficients $(c_{i,q})$ are chosen such that

$$\sum_{i=1}^{k} c_{i,q} K_A(s_i, t) = 0 \quad \text{and} \quad \det \begin{pmatrix} c_{1,1} & \cdots & c_{1,k-1} \\ \vdots & & \vdots \\ c_{k-1,1} & \cdots & c_{k-1,k-1} \end{pmatrix} \neq 0. \tag{3.16}$$

This guarantees the following (where the notation "$\,|\,(X_0, \dots, X_m)$" means conditionally on $(X_0, \dots, X_m)$):

1. $\mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \dots, X_m, I_{s_1}^{(n)}, \dots, I_{s_k}^{(n)}\right] = \mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \dots, X_m, \Delta_1, \dots, \Delta_{k-1}, I_{s_k}^{(n)}\right]$.
2. $(\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m)$ is a Gaussian vector. To see this, we observe that $\forall q = 1, \dots k-1$,
$$\sum_{i=1}^{k} c_{i,q} K_A(s_i, t) = 0 \implies \sum_{i=1}^{k} c_{i,q} g_{m+1}(s_i) = 0, \tag{3.17}$$
where we used Eq. 3.11. Hence, $\Delta_q = \sum_{i=1}^{k} c_{i,q} I_{s_i}^{(n)} = \sum_{i=1}^{k} c_{i,q} a_m(s_i) + \sum_{i=1}^{k} c_{i,q} A_{s_i}^{(n)}$ for all $q = 1, \dots, k-1$, which implies that $(\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m)$ is a Gaussian vector.
3. $A_t^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m)$, since $\sum_{i=1}^{k} c_{i,q} K_A(s_i, t) = 0$.
4. $A_{s_k}^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m)$, since $(A_t^{(n)})$ is Markov.

To conclude, we need to show

$$I_t^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m) \text{ and } I_{s_k}^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m). \tag{3.18}$$

Since $(I_t^{(n)}) = (S_t^{(n)} + A_t^{(n)})$, and $(S_t^{(n)}) \perp\!\!\!\perp (A_t^{(n)})$, we have 3.18 if

$$A_t^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m) \text{ and } A_{s_k}^{(n)} \perp\!\!\!\perp (\Delta_1, \dots, \Delta_{k-1}) \,|\, (X_0, \dots, X_m), \tag{3.19}$$

which is guaranteed by conditions 3.16. □

Remark 3.9. *If $(A_t^{(n)})$ is standard (see Def. 2.15), then Eq. 3.11 is equivalent to*

$$g_j(x) \mathbb{1}_{[T_{j-1}, T_j]}(x) = f_j(x) \mathbb{1}_{[T_{j-1}, T_j]}(x). \tag{3.20}$$

This makes standard RAPs automatically nearly-Markov.

Remark 3.10. *Depending on the coupling of the vector X, $\mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \dots, X_{m(s)}, I_s^{(n)}\right]$ might simplify further. For instance, if X has continuous marginals and is distributed according to Kantorovich's coupling, then*

$$\mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, \dots, X_{m(s)}, I_s^{(n)}\right] = \mathbb{P}\left[I_t^{(n)} \in \cdot \mid X_0, I_s^{(n)}\right] \tag{3.21}$$

because $X_1, \dots, X_{m(s)}$ are all deterministic functions of X_0.

Remark 3.11. *It is important to notice that the nearly-Markov property is not symmetric in time. Define* $\mathcal{G}^I_t = \sigma(I^{(n)}_u \,|\, t \leqslant u \leqslant T_n)$. *Let* $s < t$ *in* $[T_0, T_n]$. *Then, to get*

$$\mathbb{P}\left[I^{(n)}_s \in \cdot \mid \mathcal{G}^I_t\right] = \mathbb{P}\left[I^{(n)}_s \in \cdot \mid I^{(n)}_t, X_{k(t)}, X_{k(t)+1}, \dots, X_n\right] \tag{3.22}$$

where $k(t) = \min\{i \in \mathbb{N} \,|\, T_i \geqslant t\}$, *one needs to replace Cond. 3.10 by*

$$g_j(x)\mathbb{1}_{[T_{j+1},T_n]}(x) = 0. \tag{3.23}$$

Example 3.12. *We give an example of a non-standard* X*-RAP on* $[T_0, T_2]$ *that is* $\{T_0, T_2\}_*$*-nearly Markov, where* $X_0 \overset{\mathcal{L}}{=} X_1 \overset{\mathcal{L}}{=} X_2 \overset{\mathcal{L}}{=} Unif(\{-1, 1\})$ *are pairwise independent. Consider the interpolating coefficients*

$$f_0(t) = \frac{T_1 - t}{T_1 - T_0}\mathbb{1}_{\left[T_0, \frac{T_1+T_2}{2}\right]}(t) - \frac{T_2 - t}{T_1 - T_0}\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(t), \tag{3.24}$$

$$f_1(t) = \frac{t - T_0}{T_1 - T_0}\mathbb{1}_{[T_0,T_1]}(t) + \frac{T_2 - t}{T_2 - T_1}\mathbb{1}_{(T_1,T_2]}(t), \tag{3.25}$$

$$f_2(t) = \frac{t - T_1}{T_2 - T_1}\mathbb{1}_{[T_1,T_2]}(t). \tag{3.26}$$

Let $(A^{(2)}_t)_{t\in[T_0,T_2]}$ *be the arcade process with these interpolating coefficients driven by a standard Brownian motion. As shown in Ex. 2.17,* $(A^{(2)}_t)$ *is Markov. For Eq. 3.10 to be fulfilled, one only need to impose* $g_2(t)\mathbb{1}_{[T_0,T_1]}(t) = 0$. *For Eq. 3.11 to be fulfilled, one requires that*

$$g_1(t)\mathbb{1}_{[T_0,T_1]}(t) = \frac{t - T_0}{T_1 - T_0}\mathbb{1}_{[T_0,T_1]}(t), \tag{3.27}$$

$$g_2(t)\mathbb{1}_{\left[T_1, \frac{T_1+T_2}{2}\right]}(t) = \left(\frac{t - T_1}{T_2 - T_1} + \frac{(t - T_1)T_0}{(T_1 - T_0)^2}\right)\mathbb{1}_{\left[T_1, \frac{T_1+T_2}{2}\right]}(t), \tag{3.28}$$

$$g_2(t)\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(t) = \left(\frac{t - T_1}{T_2 - T_1} + \frac{(T_2 - t)T_0}{(T_1 - T_0)^2}\right)\mathbb{1}_{\left(\frac{T_1+T_2}{2}, T_2\right]}(t). \tag{3.29}$$

Outside of the considered intervals, the functions g_i *can take any values as long as they remain interpolating coefficients. Notice that the theorem does not impose a condition on* g_0. *For example, we could choose* $g_i = f_i$ *outside the above intervals. Hence, all three conditions are fulfilled and this* X*-RAP is* $\{T_0, T_2\}_*$*-nearly Markov. As we can see from the paths simulation below, this process is visually different from a randomized stitched Brownian arcade on the second arc (the noise has been diminished to make the paths more informative). Simulating the signal function by itself highlights the following:* X_0 *will determine the fate of the signal function on* $[T_1, T_2]$ *since this RAP is not forgetting about previously matched random variables when changing arc. On the first arc, where the process is simply a randomized Brownian bridge: to go from* $X_0 = -1$ *to* $X_1 = -1$ *for instance, there is only one way, a straight line. On the second arc, to go from* $X_1 = -1$ *to* $X_2 = -1$, *there are two ways. The signal function will choose which way to use based on the value of* X_0. *This is illustrated by the paths of the signal function below: the blue path and the green path both take value* -1 *at* T_1 *and value* 1 *at* T_2, *but have different values in* T_0. *Hence they differ on* $[T_1, T_2]$, *as observed.*

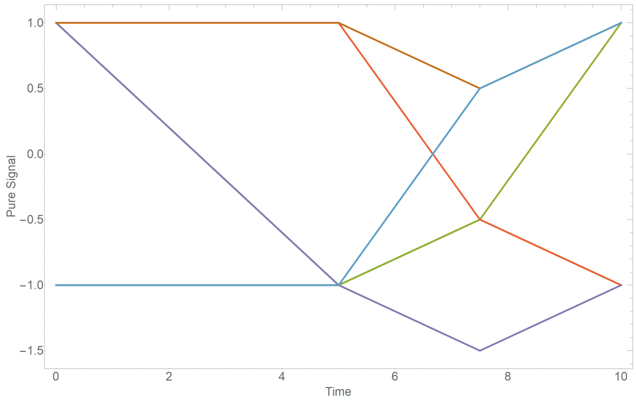

Figure 10: Paths simulation of the signal function of a non-standard X-RAP, where $\{T_0, T_2\}_* = \{0, 5, 10\}$.

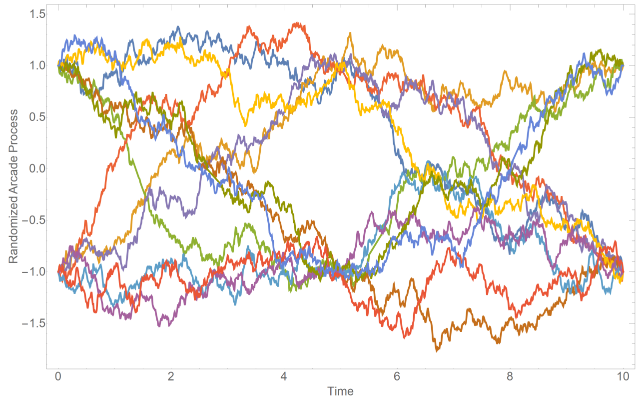

Figure 11: Paths simulation of a non-standard X-RAP, where the noise process was rescaled by 0.3, and $\{T_0, T_2\}_* = \{0, 5, 10\}$.

4 Filtered arcade martingales

In this section, we construct martingales with respect to the filtration generated by an X-RAP, which interpolate between the components of $X = (X_0, \ldots, X_n)$. These martingales solve an underlying stochastic filtering problem, and extend the martingale class constructed within the infomation-based theory of BHM. We call such martingales *filtered arcade martingales* (FAMs).

4.1 The one-arc FAM

Given a random vector $X = (X_0, X_1)$ with integrable, convexly ordered components $X_0 \leqslant_{\mathrm{cx}} X_1$, distributed according to the real probability measures μ_0 and μ_1, respectively, and an X-RAP $(I_t^{(1)})_{t\in[T_0,T_1]}$ on the partition $\{T_0, T_1\}_*$, we would like to construct a martingale $(M_t)_{t\in[T_0,T_1]}$ with respect to $(\mathcal{F}_t^I)$ such that $M_{T_0} \overset{a.s.}{=} X_0$ and $M_{T_1} \overset{a.s.}{=} X_1$. An equivalent claim to $X_0 \leqslant_{\mathrm{cx}} X_1$ is that their joint distribution π^X is in the set of martingale couplings, that is $\mathcal{M}(\mu_0, \mu_1) = \{\pi \in \Pi(\mu_0, \mu_1) \cap \mathcal{P}_1(\mathbb{R}^2) \,|\, (X_0, X_1) \sim \pi \implies \mathbb{E}[X_1 \,|\, X_0] \overset{a.s.}{=} X_0\}$. The BHM framework developed in [6] is recovered when $X_0 \overset{a.s.}{=} \mathbb{E}[X_1]$.

Definition 4.1. *Given an X-RAP $(I_t^{(1)})$ and a martingale coupling π^X, a one-arc FAM for X on $[T_0, T_1]$ is a stochastic process of the form $M_t = \mathbb{E}[X_1 \,|\, \mathcal{F}_t^I]$.*

Proposition 4.2. *The FAM $(M_t)_{t\in[T_0,T_1]}$ is an $(\mathcal{F}_t^I)$-martingale that interpolates between X_0 and X_1.*

Proof. 1. $\mathbb{E}[|M_t|] < +\infty$ for all $t \in [T_0, T_1]$ by Jensen inequality, since $\mathbb{E}[|X_1|] < +\infty$.

2. For $s < t$, $\mathbb{E}[M_t \,|\, \mathcal{F}_s^I] \overset{a.s.}{=} M_s$ by the tower property of the conditional expectation.

3. $M_{T_0} = \mathbb{E}[X_1 \,|\, X_0] \overset{a.s.}{=} X_0$, since $\pi^X \in \mathcal{M}(\mu_0, \mu_1)$ and $I_{T_0}^{(1)} = X_0$.

4. $M_{T_1} = \mathbb{E}[X_1 \,|\, \mathcal{F}_{T_1}^I] \overset{a.s.}{=} X_1$ by construction of $(I_t^{(1)})$.

Hence, this (M_t) is a martingale with respect to $(\mathcal{F}_t^I)$ that interpolates between X_0 and X_1 on $[T_0, T_1]$.

□

Remark 4.3. *The process (M_t) is also a martingale with regard to its own filtration, denoted $(\mathcal{F}_t^M)$: for $s < t$,*

$$\mathbb{E}[M_t \,|\, \mathcal{F}_s^M] = \mathbb{E}[\mathbb{E}[M_t \,|\, \mathcal{F}_s^M] \,|\, \mathcal{F}_s^I] = \mathbb{E}[\mathbb{E}[M_t \,|\, \mathcal{F}_s^I] \,|\, \mathcal{F}_s^M] = \mathbb{E}[M_s \,|\, \mathcal{F}_s^M] = M_s. \tag{4.1}$$

A notable relationship between the RAP $(I_t^{(1)})$, its noise process $(A_t^{(1)})$, and the associated FAM (M_t) is the following:

Proposition 4.4. *If $I_t^{(1)} = g_0(t)X_0 + g_1(t)X_1 + A_t^{(1)}$ and $M_t = \mathbb{E}[X_1 \,|\, \mathcal{F}_t^I]$, then*

$$\mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_s^I] = g_0(t)X_0 + g_1(t)M_s + \mathbb{E}[\mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_s^A] \,|\, \mathcal{F}_s^I] \tag{4.2}$$

for any pair $(s,t) \in [T_0, T_1]^2$ such that $s \leqslant t$. Furthermore, if $(A_t^{(1)})$ is Gauss-Markov with $K_A(x,y) = A_1(\min(x,y))A_2(\max(x,y))$ (see Coro. 2.10), then

$$\begin{aligned}\mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_s^I] = &\left(g_0(t) - \frac{A_2(t)}{A_2(s)}g_0(s)\right) X_0 + \left(g_1(t) - \frac{A_2(t)}{A_2(s)}g_1(s)\right) M_s \\ &+ \frac{A_2(t)}{A_2(s)}I_s + \mu_A(t) + \frac{A_2(t)}{A_2(s)}\mu_A(s)\end{aligned} \tag{4.3}$$

for any $(s,t) \in [T_0, T_1]^2$ such that $s \leqslant t$.

Proof. Let $(s,t) \in [T_0, T_1]^2$ such that $s \leqslant t$. Notice that

$$\begin{aligned}\mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_s^I] &= g_0(t)X_0 + g_1(t)M_s + \mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_s^I], &(4.4)\\ &= g_0(t)X_0 + g_1(t)M_s + \mathbb{E}[\mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_s^A] \,|\, \mathcal{F}_s^I], &(4.5)\end{aligned}$$

since $\mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_s^I] = \mathbb{E}[\mathbb{E}[A_t^{(1)} \,|\, X_0, X_1, \mathcal{F}_s^A] \,|\, \mathcal{F}_s^I] = \mathbb{E}[\mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_s^A] \,|\, \mathcal{F}_s^I]$, where we used the fact that $(A_t^{(1)}) \perp\!\!\!\perp (X_0, X_1)$, see Def. 3.1. If $(A_t^{(1)})$ is Gauss-Markov, then

$$\mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_s^A] = \mathbb{E}[A_t^{(1)} \,|\, A_s^{(1)}] = \mu_A(t) + \frac{K_A(s,t)}{\sigma_A^2(s)}\left(A_s^{(1)} - \mu_A(s)\right). \tag{4.6}$$

Hence, by linearity of conditional expectation,

$$\mathbb{E}[\mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_s^A] \,|\, \mathcal{F}_s^I] = \mu_A(t) + \frac{K_A(s,t)}{\sigma_A^2(s)}\left(\mathbb{E}[A_s^{(1)} \,|\, \mathcal{F}_s^I] - \mu_A(s)\right). \tag{4.7}$$

Notice that

$$I_t^{(1)} = \mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_t^I] = \mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_t^I] + g_0(t)X_0 + g_1(t)M_t, \tag{4.8}$$

which implies

$$\mathbb{E}[A_t^{(1)} \,|\, \mathcal{F}_t^I] = I_t^{(1)} - g_0(t)X_0 - g_1(t)M_t. \tag{4.9}$$

Then, plugging Eq. 4.9 in Eq. 4.7 and recalling that $K_A(s,t) = A_1(s)A_2(t)$ yields

$$\begin{aligned}\mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_s^I] = &\left(g_0(t) - \frac{A_2(t)}{A_2(s)}g_0(s)\right) X_0 + \left(g_1(t) - \frac{A_2(t)}{A_2(s)}g_1(s)\right) M_s \\ &+ \frac{A_2(t)}{A_2(s)}I_s + \mu_A(t) + \frac{A_2(t)}{A_2(s)}\mu_A(s).\end{aligned} \tag{4.10}$$

$\square$

If $(I_t^{(1)})$ is $\{T_0, T_1\}$-nearly Markov, then $M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}]$. We can then derive the dynamics of (M_t) using Bayes' rule and Itô's lemma under mild assumptions. In what follows, we assume that the driver of $(I_t^{(1)})$ has a density function.

Proposition 4.5. *Let $M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}]$ be a one-arc FAM restricted to $t \in (T_0, T_1)$. Then*

$$M_t = \frac{\int_{\mathbb{R}} y f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)}{\int_{\mathbb{R}} f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)}, \tag{4.11}$$

where $F^{X_1 \,|\, X_0}$ is the distribution function of X_1 given X_0. In particular,

1. *If (X_0, X_1) is a continuous random vector, then*

$$M_t = \frac{\int_{\mathbb{R}} y f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) f^{X_1 \,|\, X_0}(y) \, \mathrm{d}y}{\int_{\mathbb{R}} f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) f^{X_1 \,|\, X_0}(y) \, \mathrm{d}y} = \frac{\int_{\mathbb{R}} y f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) f^{(X_0, X_1)}(X_0, y) \, \mathrm{d}y}{\int_{\mathbb{R}} f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) f^{(X_0, X_1)}(X_0, y) \, \mathrm{d}y}. \tag{4.12}$$

2. *If (X_0, X_1) is a discrete random vector, then*

$$M_t = \frac{\sum_y y f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \mathbb{P}[X_1 = y \,|\, X_0]}{\sum_y f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \mathbb{P}[X_1 = y \,|\, X_0]}. \tag{4.13}$$

Proof. By Bayes rule,

$$\begin{aligned}
\mathbb{P}[X_1 \leqslant y \,|\, X_0, z \leqslant I_t^{(1)} \leqslant z + \epsilon] &= \frac{\mathbb{P}[z \leqslant I_t^{(1)} \leqslant z + \epsilon \,|\, X_0, X_1 \leqslant y] \mathbb{P}[X_1 \leqslant y \,|\, X_0]}{\mathbb{P}[z \leqslant I_t^{(1)} \leqslant z + \epsilon \,|\, X_0]}, &(4.14)\\
&= \frac{\mathbb{P}[z \leqslant I_t^{(1)} \leqslant z + \epsilon \,|\, X_0, X_1 \leqslant y] \mathbb{P}[X_1 \leqslant y \,|\, X_0]}{\int_{\mathbb{R}} \mathbb{P}[z \leqslant I_t^{(1)} \leqslant z + \epsilon \,|\, X_0, X_1 = y] \, \mathrm{d}F^{X_1 \,|\, X_0}(y)} &(4.15)\\
&= \frac{\mathbb{P}[z \leqslant I_t^{(1)} \leqslant z + \epsilon, X_1 \leqslant y \,|\, X_0]}{\int_{\mathbb{R}} \mathbb{P}[z \leqslant I_t^{(1)} \leqslant z + \epsilon \,|\, X_0, X_1 = y] \, \mathrm{d}F^{X_1 \,|\, X_0}(y)}. &(4.16)
\end{aligned}$$

This means that, by taking the limit when $\epsilon \to 0$,

$$F^{X_1 \,|\, X_0, I_t^{(1)} = z}(y) = \frac{\frac{\mathrm{d}}{\mathrm{d}z} \mathbb{P}[I_t^{(1)} \leqslant z \,|\, X_0, X_1 \leqslant y] F^{X_1 \,|\, X_0}(y)}{\int_{\mathbb{R}} f^{I_t^{(1)} \,|\, X_0, X_1 = y}(z) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)}, \tag{4.17}$$

which implies

$$\mathrm{d}F^{X_1 \,|\, X_0, I_t^{(1)} = z}(y) = \frac{f^{I_t^{(1)} \,|\, X_0, X_1 = y}(z) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)}{\int_{\mathbb{R}} f^{I_t^{(1)} \,|\, X_0, X_1 = y}(z) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)}. \tag{4.18}$$

Inserting the expression for $\mathrm{d}F^{X_1 \,|\, X_0, I_t^{(1)} = z}(y)$ into M_t, we obtain

$$M_t = \int_{\mathbb{R}} y \, \mathrm{d}F^{X_1 \,|\, X_0, I_t^{(1)}}(y) = \frac{\int_{\mathbb{R}} y f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)}{\int_{\mathbb{R}} f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)} \tag{4.19}$$

□

Example 4.6. *Let $X_0 \sim \mathcal{U}([-1, 1])$, and $X_1 \sim \mathcal{U}([-2, 2])$. These random variables are convexly ordered, i.e., $X_0 \leqslant_{\mathrm{cx}} X_1$, hence there exists at least one martingale coupling for (X_0, X_1). We choose the coupling defined by*

$$X_1 \,|\, X_0 = \begin{cases} \frac{3}{2} X_0 + \frac{1}{2} & \text{with probability } \frac{3}{4}, \\[2ex] -\frac{1}{2} X_0 - \frac{3}{2} & \text{with probability } \frac{1}{4}. \end{cases} \tag{4.20}$$

This is a martingale coupling since $\mathbb{E}[X_1 \,|\, X_0] \overset{a.s.}{=} X_0$. *In fact, it can be shown that this coupling is the solution to a martingale optimal transport problem, see [18]. For any* $\{T_0, T_1\}$*-nearly Markov* X*-RAP* $(I_t^{(1)})$*, we have*

$$M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}] = \frac{(9X_0+3)f^{I_t^{(1)} \,|\, X_0, X_1 = \frac{3}{2}X_0 + \frac{1}{2}}(I_t^{(1)}) - (X_0+3)f^{I_t^{(1)} \,|\, X_0, X_1 = -\frac{1}{2}X_0 - \frac{3}{2}}(I_t^{(1)})}{6f^{I_t^{(1)} \,|\, X_0, X_1 = \frac{3}{2}X_0 + \frac{1}{2}}(I_t^{(1)}) + 2f^{I_t^{(1)} \,|\, X_0, X_1 = -\frac{1}{2}X_0 - \frac{3}{2}}(I_t^{(1)})}. \tag{4.21}$$

Example 4.7. *Let* $X_0 \sim \mathcal{N}(0,1)$*, and* $X_1 \sim \mathcal{N}(0,2)$*, where* $X_1 \,|\, X_0 \sim \mathcal{N}(X_0, 1)$*. For any* $\{T_0, T_1\}$*-nearly Markov* X*-RAP* $(I_t^{(1)})$*, we have*

$$M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}] = \frac{\int_{\mathbb{R}} y f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \mathrm{e}^{\frac{-(y-X_0)^2}{2}} \, \mathrm{d}y}{\int_{\mathbb{R}} f^{I_t^{(1)} \,|\, X_0, X_1 = y}(I_t^{(1)}) \mathrm{e}^{\frac{-(y-X_0)^2}{2}} \, \mathrm{d}y}. \tag{4.22}$$

To simplify expressions, we introduce the following notation:

1. $u(t,z,X_0,y) = f^{I_t^{(1)} \,|\, X_0, X_1 = y}(z)$,
2. $u_t(t,z,X_0,y) = \frac{\partial u}{\partial t}(t,z,X_0,y)$,
3. $u_z(t,z,X_0,y) = \frac{\partial u}{\partial z}(t,z,X_0,y)$,
4. $u_{zz}(t,z,X_0,y) = \frac{\partial^2 u}{\partial z^2}(t,z,X_0,y)$,
5. $K_{\cdot}(t,z,X_0) = \int_{\mathbb{R}} u_{\cdot}(t,z,X_0,y) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)$,
6. $V_{\cdot}(t,z,X_0) = \int_{\mathbb{R}} y u_{\cdot}(t,z,X_0,y) \, \mathrm{d}F^{X_1 \,|\, X_0}(y)$.

Thus, under the conditions in Prop. 4.5, we may write Eq. 4.11 as

$$M_t = \frac{V(t, I_t^{(1)}, X_0)}{K(t, I_t^{(1)}, X_0)}. \tag{4.23}$$

Proposition 4.8. *If* $(I_t^{(1)})$ *is a semimartingale such that* $(t,x) \to \frac{V(t,x,X_0)}{K(t,x,X_0)}$ *is* $C^2(((T_0,T_1) \setminus N) \times Im(I^{(1)}))$ *where* $N \subset (T_0,T_1)$ *contains finitely many elements, and* $M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}]$ *is a one-arc FAM, then*

$$\begin{aligned}
\mathrm{d}M_t =& \frac{V_t(t, I_t^{(1)}, X_0) - M_t K_t(t, I_t^{(1)}, X_0)}{K(t, I_t^{(1)}, X_0)} \mathrm{d}t + \frac{V_z(t, I_t^{(1)}, X_0) - M_t K_z(t, I_t^{(1)}, X_0)}{K(t, I_t^{(1)}, X_0)} \mathrm{d}I_t^{(1)} \\
&+ \left(\frac{M_t K_z^2(t, I_t^{(1)}, X_0) - K_z(t, I_t^{(1)}, X_0) V_z(t, I_t^{(1)}, X_0)}{K^2(t, I_t^{(1)}, X_0)} \right. \\
&\qquad \left. + \frac{V_{zz}(t, I_t^{(1)}, X_0) - M_t K_{zz}(t, I_t^{(1)}, X_0)}{2K(t, I_t^{(1)}, X_0)} \right) \mathrm{d}[I^{(1)}]_t.
\end{aligned} \tag{4.24}$$

for $t \in (T_0, T_1)$.

Proof. This is verified by a straightforward application of Itô's lemma. □

Remark 4.9. *The Itô condition, i.e.,* $(t,x) \to \frac{V(t,x,X_0)}{K(t,x,X_0)}$ *is* $C^2(((T_0,T_1) \setminus N) \times Im(I^{(1)}))$ *where* $N \subset (T_0,T_1)$ *contains finitely many elements, imposes implicit integrability conditions on* (X_0, X_1).

Since we studied in detail Gaussian RAPs, we can specialize Prop. 4.8 to this particular subclass.

Corollary 4.10. *Under the conditions in Prop. 4.8, if the conditional probability distribution of* $(I_t^{(1)})$ *given* (X_0, X_1) *is* $\mathcal{N}(g_0(t)X_0 + g_1(t)X_1 + \mu_A(t), \sigma_A^2(t))$*, we have*

$$\begin{aligned}
\frac{V_t(t, I_t^{(1)}, X_0) - M_t K_t(t, I_t^{(1)}, X_0)}{K(t, I_t^{(1)}, X_0)} &= -\frac{(\mathbb{E}[X_1^3 \mid X_0, I_t^{(1)}] - M_t^3) g_1(t) \left(\frac{g_1(t)}{\sigma_A(t)}\right)'}{\sigma_A(t)} \\
&+ \left(\frac{M_t g_1(t) \left(\frac{g_1(t)}{\sigma_A(t)}\right)' + (I_t^{(1)} - g_0(t)X_0 - \mu_A(t)) \left(\left(\frac{g_1(t)}{\sigma_A(t)}\right)' + g_1(t)\left(\frac{1}{\sigma_A(t)}\right)'\right)}{\sigma_A(t)}\right. \\
&\left. - \frac{(X_0 g_0'(t) + \mu_A'(t)) g_1(t)}{\sigma_A^2(t)}\right) \times \mathrm{Var}[X_1 \mid X_0, I_t^{(1)}], \qquad (4.25)
\end{aligned}$$

$$\frac{V_z(t, I_t^{(1)}, X_0) - M_t K_z(t, I_t^{(1)}, X_0)}{K(t, I_t^{(1)}, X_0)} = \frac{g_1(t)}{\sigma_A^2(t)} \mathrm{Var}[X_1 \mid X_0, I_t^{(1)}], \qquad (4.26)$$

$$\begin{aligned}
\frac{M_t K_z^2(t, I_t^{(1)}, X_0) - K_z(t, I_t^{(1)}, X_0) V_z(t, I_t^{(1)}, X_0)}{K^2(t, I_t^{(1)}, X_0)} &= \frac{I_t^{(1)} - g_0(t)X_0 - \mu_A(t) - g_1(t)M_t}{\sigma_A^4(t)} g_1(t) \\
&\times \mathrm{Var}[X_1 \mid X_0, I_t^{(1)}], \qquad (4.27)
\end{aligned}$$

$$\begin{aligned}
\frac{V_{zz}(t, I_t^{(1)}, X_0) - M_t K_{zz}(t, I_t^{(1)}, X_0)}{2K(t, I_t^{(1)}, X_0)} &= \frac{\mathbb{E}[X_1^3 \mid X_0, I_t] - M_t^3}{2\sigma_A^4(t)} g_1^2(t) \\
&- \frac{2(I_t^{(1)} - g_0(t)X_0 - \mu_A(t)) + g_1(t)M_t}{2\sigma_A^4(t)} g_1(t) \mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]. \qquad (4.28)
\end{aligned}$$

Proof. Denoting $Z_t = I_t^{(1)} - g_0(t)X_0 - \mu_A(t)$, and $J_t = X_0 g_0'(t) + \mu_A'(t)$, the result follows from the following computations:

1.

$$\begin{aligned}
V_t(t, I_t^{(1)}, X_0) &= \frac{g_1(t)\sigma_A'(t) - g_1'(t)\sigma_A(t)}{\sigma_A^3(t)} g_1(t) K(t, I_t^{(1)}, X_0) \mathbb{E}[X_1^3 \mid X_0, I_t^{(1)}] \\
&+ \frac{Z_t J_t \sigma_A(t) + \sigma_A'(t)(Z_t^2 - \sigma_A^2(t))}{\sigma_A^3(t)} V(t, I_t^{(1)}, X_0) \\
&- \frac{g_1(t)(J_t \sigma_A(t) + 2Z_t \sigma_A'(t)) - Z_t \sigma_A(t) g_1'(t)}{\sigma_A^3(t)} K(t, I_t^{(1)}, X_0) \mathbb{E}[X_1^2 \mid X_0, I_t^{(1)}], \qquad (4.29)
\end{aligned}$$

2.

$$\begin{aligned}
K_t(t, I_t^{(1)}, X_0) &= \frac{g_1(t)\sigma_A'(t) - g_1'(t)\sigma_A(t)}{\sigma_A^3(t)} g_1(t) K(t, I_t^{(1)}, X_0) \mathbb{E}[X_1^2 \mid X_0, I_t^{(1)}] \\
&+ \frac{Z_t J_t \sigma_A(t) + \sigma_A'(t)(Z_t^2 - \sigma_A^2(t))}{\sigma_A^3(t)} K(t, I_t^{(1)}, X_0) \\
&- \frac{g_1(t)(J_t \sigma_A(t) + 2Z_t \sigma_A'(t)) - Z_t \sigma_A(t) g_1'(t)}{\sigma_A^3(t)} K(t, I_t^{(1)}, X_0) M_t, \qquad (4.30)
\end{aligned}$$

3.
$$V_z(t, I_t^{(1)}, X_0) = \frac{g_1(t)\mathbb{E}[X_1^2 \,|\, X_0, I_t^{(1)}]K(t, I_t^{(1)}, X_0) - Z_t V(t, I_t^{(1)}, X_0)}{\sigma_A^2(t)}, \tag{4.31}$$

4.
$$K_z(t, I_t^{(1)}, X_0) = \frac{g_1(t)V(t, I_t^{(1)}, X_0) - Z_t K(t, I_t^{(1)}, X_0)}{\sigma_A^2(t)}, \tag{4.32}$$

5.
$$\begin{aligned} V_{zz}(t, I_t^{(1)}, X_0) = &\frac{g_1^2(t)E[X_1^3 \,|\, X_0, I_t^{(1)}] - 2Z_t g_1(t)\mathbb{E}[X_1^2 \,|\, X_0, I_t^{(1)}]}{\sigma_A^4(t)} K(t, I_t^{(1)}, X_0) \\ &+ \frac{Z_t^2 - \sigma_A^2(t)}{\sigma_A^4(t)} V(t, I_t^{(1)}, X_0), \end{aligned} \tag{4.33}$$

6.
$$\begin{aligned} K_{zz}(t, I_t^{(1)}, X_0) = &\frac{g_1^2(t)E[X_1^2 \,|\, X_0, I_t^{(1)}]K(t, I_t^{(1)}, X_0) - 2Z_t g_1(t)V(t, I_t^{(1)}, X_0)}{\sigma_A^4(t)} K(t, I_t^{(1)}, X_0) \\ &+ \frac{Z_t^2 - \sigma_A^2(t)}{\sigma_A^4(t)} K(t, I_t^{(1)}, X_0). \end{aligned} \tag{4.34}$$

□

Keeping the notations $Z_t = I_t^{(1)} - g_0(t)X_0 - \mu_A(t)$, $J_t = X_0 g_0'(t) + \mu_A'(t)$ from the previous proof, and introducing $U_t = \mathbb{E}[X_1^3 \,|\, X_0, I_t] - M_t^3$, the SDE for (M_t) can be rewritten as

$$\mathrm{d}M_t = U_t \left(\frac{g_1^2(t)}{2\sigma_A^4(t)} \,\mathrm{d}[I^{(1)}]_t - \frac{g_1(t)\left(\frac{g_1(t)}{\sigma_A(t)}\right)'}{\sigma_A(t)} \,\mathrm{d}t \right) + \frac{g_1(t)}{\sigma_A^2(t)} \mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}] \tag{4.35}$$

$$\begin{aligned} \times \Bigg(\Bigg(M_t \sigma_A(t) \left(\frac{g_1(t)}{\sigma_A(t)}\right)' + Z_t \frac{\sigma_A(t)}{g_1(t)} \left(\left(\frac{g_1(t)}{\sigma_A(t)}\right)' + g_1(t)\left(\frac{1}{\sigma_A(t)}\right)' \right) - J_t \Bigg) \,\mathrm{d}t \\ + \frac{-3g_1(t)M_t}{2\sigma_A^2(t)} \,\mathrm{d}[I^{(1)}]_t + \mathrm{d}I_t^{(1)} \Bigg). \end{aligned} \tag{4.36}$$

If, furthermore, $(I_t^{(1)})$ is a standard RAP, its driver (D_t) is Gauss-Markov with $K_D(x, y) = H_1(\min(x, y))H_2(\max(x, y))$, where H_1 and H_2 are continuous functions on $[T_0, T_1]$ such that H_1/H_2 is positive and non-decreasing on $[T_0, T_1)$. Then, as shown below,

$$[I^{(1)}]_t = [D]_t = \int_{T_0}^t H_2(s) \,\mathrm{d}H_1(s) - \int_{T_0}^t H_1(s) \,\mathrm{d}H_2(s), \tag{4.37}$$

where the RHS is interpreted as a difference of Riemann-Stieltjes integrals. The RHS exists since

$$\int_{T_0}^t H_2(s) \,\mathrm{d}H_1(s) - \int_{T_0}^t H_1(s) \,\mathrm{d}H_2(s) = \int_{T_0}^t H_2^2(s) \,\mathrm{d}\left(\frac{H_1}{H_2}\right)(s), \tag{4.38}$$

and H_1/H_2 is monotone and so of bounded variation and differentiable almost everywhere.

Proposition 4.11. *If the driver (D_t) is a Gauss-Markov semimartingale with $K_D(x, y) = H_1(\min(x, y))H_2(\max(x, y))$, then*

$$[D]_t = \int_{T_0}^t H_2(s) \,\mathrm{d}H_1(s) - \int_{T_0}^t H_1(s) \,\mathrm{d}H_2(s). \tag{4.39}$$

Proof. Let $t \in [T_0, T_1]$, $T_0 = t_0 < t_1 < \ldots < t_n = t$ be a partition of $[T_0, t]$, and $\Delta_j = D_{t_{j+1}} - D_{t_j}$. Then,

$$\mathbb{E}\left[\sum_{j=0}^{n-1} \Delta_j^2\right] = \sum_{j=0}^{n-1} \mathbb{E}[D_{t_{j+1}}^2] + \mathbb{E}[D_{t_j}^2] - 2\mathbb{E}[D_{t_{j+1}} D_{t_j}], \tag{4.40}$$

$$= \sum_{j=0}^{n-1} H_1(t_{j+1})H_2(t_{j+1}) + H_1(t_j)H_2(t_j) - 2H_1(t_j)H_2(t_{j+1}) + \left(\mu_D(t_{j+1}) - \mu_D(t_j)\right)^2, \tag{4.41}$$

$$= \sum_{j=0}^{n-1} H_2(t_{j+1})(H_1(t_{j+1}) - H_1(t_j)) - H_1(t_j)(H_2(t_{j+1}) - H_2(t_j)) + \left(\mu_D(t_{j+1}) - \mu_D(t_j)\right)^2. \tag{4.42}$$

Since μ_D is continuous and of bounded variation (by the semimartingality of (D_t)),

$$\sum_{j=0}^{n-1} \left(\mu_D(t_{j+1}) - \mu_D(t_j)\right)^2 \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} 0. \tag{4.43}$$

This shows that

$$\mathbb{E}\left[\sum_{j=0}^{n-1} \Delta_j^2\right] \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} \int_{T_0}^{t} H_2(s)\,\mathrm{d}H_1(s) - \int_{T_0}^{t} H_1(s)\,\mathrm{d}H_2(s). \tag{4.44}$$

Furthermore,

$$\operatorname{Var}\left[\sum_{j=0}^{n-1} \Delta_j^2\right] = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \operatorname{Cov}\left(\Delta_i^2, \Delta_j^2\right), \tag{4.45}$$

$$= 2\sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \operatorname{Cov}^2\left(\Delta_i, \Delta_j\right) + 4\sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \mathbb{E}[\Delta_i]\mathbb{E}[\Delta_j]\operatorname{Cov}\left(\Delta_i, \Delta_j\right), \tag{4.46}$$

and

$$\operatorname{Cov}\left(\Delta_i, \Delta_j\right) = \begin{cases} (H_1(t_{i+1}) - H_1(t_i))(H_2(t_{j+1}) - H_2(t_j)) & \text{if } i < j, \\ (H_1(t_{j+1}) - H_1(t_j))(H_2(t_{i+1}) - H_2(t_i)) & \text{if } j < i, \\ H_2(t_{j+1})(H_1(t_{j+1}) - H_1(t_j)) - H_1(t_j)(H_2(t_{j+1}) - H_2(t_j)) & \text{if } i = j. \end{cases} \tag{4.47}$$

We split Eq. 4.46 into the cases $i < j$, $j < i$, and $i = j$. Considering the case $i = j$ first, we get

$$\sum_{j=0}^{n-1} 2\operatorname{Var}^2[\Delta_j] + 4\mathbb{E}^2[\Delta_j]\operatorname{Var}[\Delta_j] = \sum_{j=0}^{n-1} \operatorname{Var}[\Delta_j]\left(2\operatorname{Var}[\Delta_j] + 4\left(\mu_D(t_{j+1}) - \mu_D(t_j)\right)^2\right) \tag{4.48}$$

$$\leqslant \sum_{j=0}^{n-1} \left(2\operatorname{Var}[\Delta_j] + 4\left(\mu_D(t_{j+1}) - \mu_D(t_j)\right)^2\right) \times \max_{k\in\{0,1,\ldots,n-1\}} \operatorname{Var}[\Delta_k]. \tag{4.49}$$

Since $\max\limits_{k\in\{0,1,\ldots,n-1\}} \mathrm{Var}[\Delta_k] \xrightarrow[\max\left|t_{j+1}-t_j\right|\to 0]{} 0$ by uniform continuity of H_1 and H_2, and

$$\lim_{\max\left|t_{j+1}-t_j\right|\to 0} \sum_{j=0}^{n-1} \left(2\mathrm{Var}[\Delta_j] + 4\left(\mu_D(t_{j+1}) - \mu_D(t_j)\right)^2\right) < +\infty, \tag{4.50}$$

we have $\sum\limits_{j=0}^{n-1} 2\mathrm{Var}^2[\Delta_j] + 4\mathbb{E}^2[\Delta_j]\mathrm{Var}[\Delta_j] \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} 0.$

Now, for $i < j$, we have

$$\begin{aligned}
&2\sum_{j=0}^{n-1}\sum_{i=0}^{j-1}(H_1(t_{i+1}) - H_1(t_i))^2(H_2(t_{j+1}) - H_2(t_j))^2 \\
&+ 4\sum_{j=0}^{n-1}\sum_{i=0}^{j-1}(\mu_D(t_{i+1}) - \mu_D(t_i))(\mu_D(t_{j+1}) - \mu_D(t_j))(H_1(t_{i+1}) - H_1(t_i))(H_2(t_{j+1}) - H_2(t_j))
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
=&2\sum_{j=0}^{n-1}(H_2(t_{j+1}) - H_2(t_j))^2 \left(\sum_{i=0}^{j-1}(H_1(t_{i+1}) - H_1(t_i))^2\right) \\
&+ 4\sum_{j=0}^{n-1}(\mu_D(t_{j+1}) - \mu_D(t_j))(H_2(t_{j+1}) - H_2(t_j)) \left(\sum_{i=0}^{j-1}(\mu_D(t_{i+1}) - \mu_D(t_i))(H_1(t_{i+1}) - H_1(t_i))\right) \\
&\xrightarrow[\max|t_{u+1}-t_u|\to 0]{} 0,
\end{aligned} \tag{4.52}$$

since

1. $\sum\limits_{i=0}^{j-1}(H_1(t_{i+1}) - H_1(t_i))^2 \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} [H_1]_{t_j} = 0,$
2. $\sum\limits_{j=0}^{n-1}(H_2(t_{j+1}) - H_2(t_j))^2 \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} [H_2]_t = 0,$
3. $\sum\limits_{i=0}^{j-1}(\mu_D(t_{i+1}) - \mu_D(t_i))(H_1(t_{i+1}) - H_1(t_i)) \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} [\mu_D, H_1]_{t_j} = 0,$
4. $\sum\limits_{j=0}^{n-1}(\mu_D(t_{j+1}) - \mu_D(t_j))(H_2(t_{j+1}) - H_2(t_j)) \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} [\mu_D, H_2]_t = 0.$

The same argument can be applied to the case $j < i$. Hence,

$$\mathrm{Var}\left[\sum_{j=0}^{n-1}\Delta_j^2\right] \xrightarrow[\max|t_{u+1}-t_u|\to 0]{} 0, \tag{4.53}$$

which means

$$\sum_{j=0}^{n-1}\Delta_j^2 \xrightarrow[\max|t_{u+1}-t_u|\to 0]{L^2} \int_{T_0}^t H_2(s)\,\mathrm{d}H_1(s) - \int_{T_0}^t H_1(s)\,\mathrm{d}H_2(s). \tag{4.54}$$

$\square$

Recalling that, in the standard RAP case, we have $g_1(x) = \dfrac{H_1(x)H_2(T_0) - H_1(T_0)H_2(x)}{H_1(T_1)H_2(T_0) - H_1(T_0)H_2(T_1)}$, $A_1(x) = g_1(x)H_2(T_1)$, $A_2(x) = \dfrac{H_1(T_1)}{H_2(T_1)}H_2(x) - H_1(x)$, and that $\sigma_A^2(x) = A_1(x)A_2(x)$, we get the following expression for the SDE of (M_t):

Corollary 4.12. *Under the conditions in Prop. 4.8, if $(I_t^{(1)})$ is a standard X-RAP, with driver covariance $K_D(x,y) = H_1(\min(x,y))H_2(\max(x,y))$, then*

$$\begin{aligned}\mathrm{d}M_t = {} & \frac{\mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]}{H_1(T_1)H_2(t) - H_1(t)H_2(T_1)} \\ & \times \left(\left(\frac{Z_t(H_1'(t)H_2(T_1) - H_1(T_1)H_2'(t)) - M_t(H_1'(t)H_2(t) - H_1(t)H_2'(t))}{H_1(T_1)H_2(t) - H_1(t)H_2(T_1)} - J_t \right) \mathrm{d}t + \mathrm{d}I_t^{(1)} \right)\end{aligned} \tag{4.55}$$

where $Z_t = I_t^{(1)} - g_0(t)X_0 - \mu_A(t)$ and $J_t = X_0 g_0'(t) + \mu_A'(t)$.

Proof. The result follows from the following computations:

1.
$$\frac{g_1^2(t)(H_1'(t)H_2(t) - H_1(t)H_2'(t))}{2\sigma_A^4(t)} - \frac{g_1(t)\left(\frac{g_1(t)}{\sigma_A(t)}\right)'}{\sigma_A(t)} = 0, \tag{4.56}$$

2.
$$\begin{aligned} M_t\sigma_A(t)\left(\frac{g_1(t)}{\sigma_A(t)}\right)' + Z_t\frac{\sigma_A(t)}{g_1(t)}\left(\left(\frac{g_1(t)}{\sigma_A(t)}\right)' + g_1(t)\left(\frac{1}{\sigma_A(t)}\right)'\right) & \\ - \frac{-3g_1(t)M_t(H_1'(t)H_2(t) - H_1(t)H_2'(t))}{2\sigma_A^2(t)} & \\ = \frac{Z_t(H_1'(t)H_2(T_1) - H_1(T_1)H_2'(t)) - M_t(H_1'(t)H_2(t) - H_1(t)H_2'(t))}{H_1(T_1)H_2(t) - H_1(t)H_2(T_1)} & \end{aligned} \tag{4.57}$$

□

Theorem 4.13. *Under the conditions in Prop. 4.8, the process $(N_t)_{t\in[T_0,T_1]}$ defined by*

$$\mathrm{d}N_t = \left(\frac{Z_t(H_1'(t)H_2(T_1) - H_1(T_1)H_2'(t)) - M_t(H_1'(t)H_2(t) - H_1(t)H_2'(t))}{H_1(T_1)H_2(t) - H_1(t)H_2(T_1)} - J_t \right) \mathrm{d}t + \mathrm{d}I_t^{(1)} \tag{4.58}$$

is a martingale with respect to $(\mathcal{F}_t^I)$.

Proof. We introduce the following notations:

$$h_1(t) = H_1'(t)H_2(T_1) - H_1(T_1)H_2'(t), \tag{4.59}$$
$$h_2(t) = H_1'(t)H_2(t) - H_1(t)H_2'(t), \tag{4.60}$$
$$h_3(t) = H_1(T_1)H_2(t) - H_1(t)H_2(T_1), \tag{4.61}$$
$$S(X_0, T_0, t) = \int_{T_0}^{t} \frac{(g_0(u)X_0 + \mu_A(u))h_1(u)}{h_3(u)}\,\mathrm{d}u. \tag{4.62}$$

Then,

$$N_t = \int_{T_0}^{t} \frac{I_u^{(1)} h_1(u) - M_u h_2(u)}{h_3(u)}\, \mathrm{d}u - S(X_0, T_0, t) - g_0(t) X_0 - \mu_A(t) + I_t^{(1)}. \tag{4.63}$$

Let $(s,t) \in [T_0, T_1]^2$ such that $s < t$. We shall show that $\mathbb{E}[N_t \,|\, \mathcal{F}_s^I] = N_s$, the other conditions for (N_t) to be a martingale are immediate. By the linearity of the conditional expectation, we have

$$\begin{aligned} \mathbb{E}[N_t \,|\, \mathcal{F}_s^I] = &\int_{T_0}^{s} \frac{I_u^{(1)} h_1(u) - M_u h_2(u)}{h_3(u)}\, \mathrm{d}u + \int_{s}^{t} \frac{\mathbb{E}[I_u^{(1)} \,|\, \mathcal{F}_s^I] h_1(u)}{h_3(u)}\, \mathrm{d}u - M_s \int_s^t \frac{h_2(u)}{h_3(u)}\, \mathrm{d}u \\ &- (S(X_0, T_0, s) + S(X_0, s, t)) - g_0(t) X_0 - \mu_A(t) + \mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_s^I]. \end{aligned} \tag{4.64}$$

By Prop. 4.4, the last and the second terms in the above expression of $\mathbb{E}[N_t \,|\, \mathcal{F}_s^I]$ can be expressed as

$$\begin{aligned} \mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_s^I] = &\left(g_0(t) - \frac{A_2(t)}{A_2(s)} g_0(s) \right) X_0 + \left(g_1(t) - \frac{A_2(t)}{A_2(s)} g_1(s) \right) M_s \\ &+ \frac{A_2(t)}{A_2(s)} I_s + \mu_A(t) + \frac{A_2(t)}{A_2(s)} \mu_A(s), \end{aligned} \tag{4.65}$$

$$\begin{aligned} \int_s^t \frac{\mathbb{E}[I_u^{(1)} \,|\, \mathcal{F}_s^I] h_1(u)}{h_3(u)}\, \mathrm{d}u = X_0 &\int_s^t \frac{g_0(u) h_1(u)}{h_3(u)}\, \mathrm{d}u + M_s \int_s^t \frac{g_1(u) h_1(u)}{h_3(u)}\, \mathrm{d}u + \int_s^t \frac{\mu_A(u) h_1(u)}{h_3(u)}\, \mathrm{d}u \\ &+ \big(I_s - g_1(s) M_s - g_0(s) X_0 + \mu_A(s)\big) \frac{1}{A_2(s)} \int_s^t \frac{A_2(u) h_1(u)}{h_3(u)}\, \mathrm{d}u. \end{aligned} \tag{4.66}$$

Notice that

$$\frac{1}{A_2(s)} \int_s^t \frac{A_2(u) h_1(u)}{h_3(u)}\, \mathrm{d}u = \frac{H_1(t) - H_1(s) - \frac{H_1(T_1)}{H_2(T_1)} (H_2(t) - H_2(s))}{A_2(s)}, \tag{4.67}$$

$$= 1 - \frac{A_2(t)}{A_2(s)}. \tag{4.68}$$

Hence,

$$\begin{aligned} E[I_t^{(1)} \,|\, \mathcal{F}_s^I] + \int_s^t \frac{\mathbb{E}[I_u^{(1)} \,|\, \mathcal{F}_s^I] h_1(u)}{h_3(u)}\, \mathrm{d}u &= (g_0(t) - g_0(s)) X_0 + (g_1(t) - g_1(s)) M_s \\ &\quad + X_0 \int_s^t \frac{g_0(u) h_1(u)}{h_3(u)}\, \mathrm{d}u + M_s \int_s^t \frac{g_1(u) h_1(u)}{h_3(u)}\, \mathrm{d}u \\ &\quad + \int_s^t \frac{\mu_A(u) h_1(u)}{h_3(u)}\, \mathrm{d}u + I_s + \mu_A(s) + \mu_A(t), \qquad (4.69) \\ &= (g_0(t) - g_0(s)) X_0 + (g_1(t) - g_1(s)) M_s \\ &\quad + S(X_0, s, t) + M_s \int_s^t \frac{g_1(u) h_1(u)}{h_3(u)}\, \mathrm{d}u \\ &\quad + I_s + \mu_A(s) + \mu_A(t). \qquad (4.70) \end{aligned}$$

Observe that

$$\int_s^t \frac{g_1(u)h_1(u) - h_2(u)}{h_3(u)}\,\mathrm{d}u = \int_s^t \frac{H_1'(u)H_2(T_0) - H_1(T_0)H_2'(u)}{H_1(T_1)H_2(T_0) - H_1(T_0)H_2(T_1)}\,\mathrm{d}u, \tag{4.71}$$

$$= (g_1(t) - g_1(s)), \tag{4.72}$$

which allows us to write

$$E[I_t^{(1)} \,|\, \mathcal{F}_s^I] + \int_s^t \frac{\mathbb{E}[I_u^{(1)} \,|\, \mathcal{F}_s^I]h_1(u)}{h_3(u)}\,\mathrm{d}u = (g_0(t) - g_0(s))X_0 + M_s \int_s^t \frac{h_2(u)}{h_3(u)}\,\mathrm{d}u M_s$$
$$+ S(X_0, s, t) + I_s + \mu_A(s) + \mu_A(t). \tag{4.73}$$

Plugging Eq. 4.73 in Eq. 4.64, we get

$$\mathbb{E}[N_t \,|\, \mathcal{F}_s^I] = \int_{T_0}^s \frac{I_u^{(1)}h_1(u) - M_u h_2(u)}{h_3(u)}\,\mathrm{d}u - S(X_0, T_0, s) - g_0(s)X_0 - \mu_A(s) + I_s^{(1)} = N_s. \tag{4.74}$$

□

We can use the process (N_t) to construct a Brownian motion adapted to $(\mathcal{F}_t^I)$.

Corollary 4.14. *Under the conditions in Prop. 4.8, the stochastic process* $W_t = \int_{T_0}^t \frac{1}{\sqrt{h_2(s)}}\,\mathrm{d}N_s$ *is a standard Brownian motion on* $[T_0, T_1]$ *(i.e, there exists a standard Brownian motion* $(\hat{W}_t)_{t\geqslant 0}$ *such that* $W_t = \hat{W}_{t-T_0}$ *for all* $t \in [T_0, T_1]$*) adapted to* $(\mathcal{F}_t^I)$*.*

Proof. We compute the quadratic variation of (W_t):

$$[W]_t = \int_{T_0}^t \frac{1}{h_2(s)}\,\mathrm{d}[N]_s = \int_{T_0}^t \frac{1}{h_2(s)}\,\mathrm{d}[I^{(1)}]_s = \int_{T_0}^t \frac{h_2(s)}{h_2(s)}\,\mathrm{d}s = t - T_0. \tag{4.75}$$

Furthermore, since (N_t) is an $(\mathcal{F}_t^I)$-martingale and $\mathbb{E}[[W]_t] < \infty$, (W_t) is an $(\mathcal{F}_t^I)$-martingale. Hence, by Lévy characterization theorem, (W_t) is an $(\mathcal{F}_t^I)$-adapted standard Brownian motion on $[T_0, T_1]$. □

This means that we can write (M_t) as an integral with respect to a Brownian motion:

$$\mathrm{d}M_t = \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]\sqrt{H_1'(t)H_2(t) - H_1(t)H_2'(t)}}{H_1(T_1)H_2(t) - H_1(t)H_2(T_1)}\,\mathrm{d}W_t, \tag{4.76}$$

as long as this expression makes sense, i.e, under the conditions in Prop. 4.8. When $(I_t^{(1)})$ is the randomized anticipative Brownian bridge on $[T_0, T_1]$, the expressions become significantly simpler:

Corollary 4.15. *If* $(I_t^{(1)})_{t\in[T_0,T_1]}$ *is an X-randomized anticipative Brownian bridge on* $[T_0, T_1]$*, then*

1.
$$\mathrm{d}M_t = \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{T_1 - t}\left(\frac{I_t^{(1)} - M_t}{T_1 - t}\,\mathrm{d}t + \mathrm{d}I_t^{(1)}\right). \tag{4.77}$$

2. *For any* $(s, t) \in [T_0, T_1]^2$ *such that* $s \leqslant t$*,*
$$\mathbb{E}[I_t^{(1)} \,|\, \mathcal{F}_s^I] = \frac{(T_1 - t)I_s^{(1)} + (t - s)M_s}{T_1 - s}. \tag{4.78}$$

3. The process $(W_t)_{t\in[T_0,T_1]}$, defined by

$$W_t := \int_{T_0}^{t} \frac{I_u^{(1)} - M_u}{T_1 - u}\, \mathrm{d}u + I_t^{(1)} - X_0, \tag{4.79}$$

is a standard Brownian motion on $[T_0, T_1]$ adapted to $(\mathcal{F}_t^I)$.

We repeat the proof for this particular case for the sack of comparison.

Proof. The first two points are direct consequences of Prop. 4.8 and Prop. 4.4, respectively. To see that (W_t) is a Brownian motion adapted to $(\mathcal{F}_t^I)$, we notice that $[W]_t = t - T_0$ and that

$$\mathbb{E}[W_t \,|\, \mathcal{F}_s^I] = W_s - I_s^{(1)} - M_s \int_s^t \frac{1}{T_1 - u}\, \mathrm{d}u + \mathbb{E}\left[I_t^{(1)} + \int_s^t \frac{I_u^{(1)}}{T_1 - u}\, \mathrm{d}u \,|\, \mathcal{F}_s^I \right]. \tag{4.80}$$

By Eq. 4.78,

$$\begin{aligned}
\mathbb{E}\left[I_t^{(1)} + \int_s^t \frac{I_u^{(1)}}{T_1 - u}\, \mathrm{d}u \,|\, \mathcal{F}_s^I \right] &= \frac{(T_1 - t)I_s^{(1)} + (t-s)M_s}{T_1 - s} + \int_s^t \frac{(T_1 - u)I_s^{(1)} + (u-s)M_s}{(T_1 - s)(T_1 - u)}\, \mathrm{d}u \\
&= \frac{(T_1 - t)I_s^{(1)} + (t-s)M_s}{T_1 - s} + \frac{t-s}{T_1 - s} I_s^{(1)} \\
&\qquad - \frac{t - s - (T_1 - s)\int_s^t \frac{1}{T_1 - u}\, \mathrm{d}u}{T_1 - s} M_s \\
&= I_s^{(1)} + M_s \int_s^t \frac{1}{T_1 - u}\, \mathrm{d}u.
\end{aligned} \tag{4.81}$$

Hence,

$$\mathbb{E}[W_t \,|\, \mathcal{F}_s^I] = W_s, \tag{4.82}$$

and (W_t) is a Brownian motion adapted to $(\mathcal{F}_t^I)$.

□

It follows that

$$M_t = X_0 + \int_{T_0}^{t} \frac{\mathrm{Var}[X_1 \,|\, X_0, I_s^{(1)}]}{T_1 - s}\, \mathrm{d}W_s, \tag{4.83}$$

which is exactly what we would have obtained by putting $H_1(x) = x$ and $H_2(x) = 1$ in the general expression

$$M_t = X_0 + \int_{T_0}^{t} \frac{\mathrm{Var}[X_1 \,|\, X_0, I_s^{(1)}]\sqrt{H_1'(s)H_2(s) - H_1(s)H_2'(s)}}{H_1(T_1)H_2(s) - H_1(s)H_2(T_1)}\, \mathrm{d}W_s. \tag{4.84}$$

Note that, when $X_0 = 0$, X_1 is centered in 0, and $T_0 = 0$, this particular case of the randomized Brownian bridge yields the martingale developed in [6], i.e, $M_t = \int_0^t \frac{\mathrm{Var}[X_1 \,|\, I_s^{(1)}]}{T_1 - s}\, \mathrm{d}W_s$.

Using Eq. 4.84, we give examples of FAMs where the underlying RAP is standard.

Example 4.16. *Let $D_t = tB_t$ where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion. Then, $K_D(x,y) = \min(x,y)^2 \max(x,y)$, and so $H_1(x) = x^2$, $H_2(x) = x$. The standard AP driven by (D_t) is given by*

$$A_t^{(1)} = D_t - \frac{T_1 t - t^2}{T_1 T_0 - T_0^2} D_{T_0} - \frac{t^2 - t}{T_1^2 - T_1 T_0} D_{T_1}. \tag{4.85}$$

We have

$$K_A(x,y) = \frac{\min(x,y)(\min(x,y)-T_0)\max(x,y)(T_1-\max(x,y)}{T_1-T_0}, \tag{4.86}$$

hence $A_1(x) = \frac{x(x-T_0)}{T_1-T_0}$ *and* $A_2(x) = x(T_1-t)$. *A standard X-RAP with noise process* $(A_t^{(1)})$ *is*

$$I_t^{(1)} = D_t - \frac{T_1 t - t^2}{T_1T_0 - T_0^2}(D_{T_0} - X_0) - \frac{t^2-t}{T_1^2 - T_1T_0}(D_{T_1} - X_1). \tag{4.87}$$

Remember that we could have chosen a different interpolating coefficient for X_0 *without disrupting the standard property of* $(I_t^{(1)})$. *The quadratic variation of* $(I_t^{(1)})$ *is given by* $\mathrm{d}[I^{(1)}]_t = H_1'(t)H_2(t) - H_1(t)H_2'(t)\,\mathrm{d}t = t^2\,\mathrm{d}t$. *Hence,*

$$M_t = X_0 + \int_{T_0}^t \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]s}{T_1^2 s - s^2 T_1}\,\mathrm{d}W_s = X_0 + \int_{T_0}^t \frac{\mathrm{Var}[X_1 \,|\, X_0, I_s^{(1)}]}{T_1^2 - sT_1}\,\mathrm{d}W_s. \tag{4.88}$$

Example 4.17. *Let* (D_t) *be an Ornstein-Uhlenbeck process with parameters* $\theta > 0, \sigma > 0, \mu \in \mathbb{R}$ *and starting value* $d_0 \in \mathbb{R}$, *that is, the solution to*

$$\mathrm{d}D_t = \theta\,(\mu - D_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad D_0 = d_0. \tag{4.89}$$

Then $K_D(s,t) = \dfrac{\sigma^2}{2\theta}\mathrm{e}^{\theta\min(s,t)}\mathrm{e}^{-\theta\max(s,t)}$, *and so* $H_1(x) = \frac{\sigma^2}{2\theta}\mathrm{e}^{\theta x}$, $H_2(x) = \mathrm{e}^{-\theta x}$. *The standard AP driven by* (D_t) *is*

$$A_t^{(1)} = \frac{\mathrm{e}^{\theta(T_1-t)} - \mathrm{e}^{-\theta(T_1-t)}}{\mathrm{e}^{\theta(T_1-T_0)} - \mathrm{e}^{-\theta(T_1-T_0)}}D_{T_0} - \frac{\mathrm{e}^{\theta(t-T_0)} - \mathrm{e}^{-\theta(t-T_0)}}{\mathrm{e}^{\theta(T_1-T_0)} - \mathrm{e}^{-\theta(T_1-T_0)}}D_{T_1}. \tag{4.90}$$

A standard X-RAP with noise process $(A_t^{(1)})$ *is*

$$I_t^{(1)} = D_t - \frac{\mathrm{e}^{\theta(T_1-t)} - \mathrm{e}^{-\theta(T_1-t)}}{\mathrm{e}^{\theta(T_1-T_0)} - \mathrm{e}^{-\theta(T_1-T_0)}}(D_{T_0} - X_0) - \frac{\mathrm{e}^{\theta(t-T_0)} - \mathrm{e}^{-\theta(t-T_0)}}{\mathrm{e}^{\theta(T_1-T_0)} - \mathrm{e}^{-\theta(T_1-T_0)}}(D_{T_1} - X_1). \tag{4.91}$$

The quadratic variation of $(I_t^{(1)})$ *is given by* $\mathrm{d}[I^{(1)}]_t = H_1'(t)H_2(t) - H_1(t)H_2'(t)\,\mathrm{d}t = \sigma^2\,\mathrm{d}t$. *Hence,*

$$M_t = X_0 + \int_{T_0}^t \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]\sigma}{\frac{\sigma^2}{2\theta}\left(\mathrm{e}^{\theta(T_1-s)} - \mathrm{e}^{\theta(s-T_1)}\right)}\,\mathrm{d}W_s = X_0 + \frac{2\theta}{\sigma}\int_{T_0}^t \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{\mathrm{e}^{\theta(T_1-s)} - \mathrm{e}^{\theta(s-T_1)}}\,\mathrm{d}W_s. \tag{4.92}$$

For a given coupling π^X, it is usually unlikely that $M_t = \mathbb{E}[X_1 \,|\, \mathcal{F}_t^I]$ has an explicit analytical expression, even in the case $M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}]$. We give an example where $M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}]$ and its SDE are explicit.

Example 4.18. *Let* $X_0 \sim \mathit{Unif}(\{-1,1\})$ *and* $X_1 \sim \mathit{Unif}(\{-2,0,2\})$ *such that*

$$X_1 \,|\, X_0 = \begin{cases} X_0 + 1 & \textit{with probability } \frac{1}{2}, \\[2ex] X_0 - 1 & \textit{with probability } \frac{1}{2}. \end{cases} \tag{4.93}$$

Clearly, $\mathbb{E}[X_1 \,|\, X_0] = X_0$. *Let* $(B_t)_{t \geqslant 0}$ *be a standard Brownian motion and*

$$I_t^{(1)} = B_t - \frac{T_1 - t}{T_1 - T_0}(B_{T_0} - X_0) - \frac{t - T_0}{T_1 - T_0}(B_{T_1} - X_1). \tag{4.94}$$

Denoting by ϕ the density function of a $\mathcal{N}\left(0, \frac{(T_1-t)(t-T_1)}{T_1-T_0}\right)$, $g_1(x) = \frac{t-T_0}{T_1-T_0}$, and using Prop. 4.5, we get

$$M_t = \frac{(X_0+1)\phi(I_t^{(1)} - X_0 - g_1(t))\frac{1}{2} + (X_0-1)\phi(I_t^{(1)} - X_0 + g_1(t))\frac{1}{2}}{\phi(I_t^{(1)} - X_0 - g_1(t))\frac{1}{2} + \phi(I_t^{(1)} - X_0 + g_1(t))\frac{1}{2}}, \tag{4.95}$$

$$= X_0 + \frac{\phi(I_t^{(1)} - X_0 - g_1(t)) - \phi(I_t^{(1)} - X_0 + g_1(t))}{\phi(I_t^{(1)} - X_0 - g_1(t)) + \phi(I_t^{(1)} - X_0 + g_1(t))}, \tag{4.96}$$

$$= X_0 + \tanh\left(\frac{I_t^{(1)} - X_0}{T_1 - t}\right). \tag{4.97}$$

In SDE form, we get

$$M_t = X_0 + \int_{T_0}^{t} \frac{\operatorname{sech}^2\left(\frac{I_s - X_0}{T_1 - s}\right)}{T_1 - s}\, \mathrm{d}W_s. \tag{4.98}$$

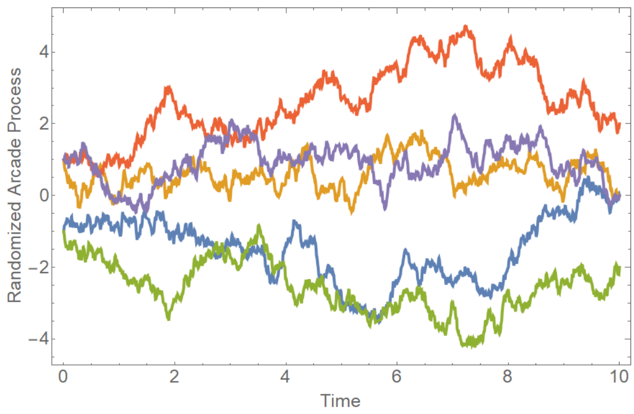

Figure 12: Paths of the X-RAP $(I_t^{(1)})$.

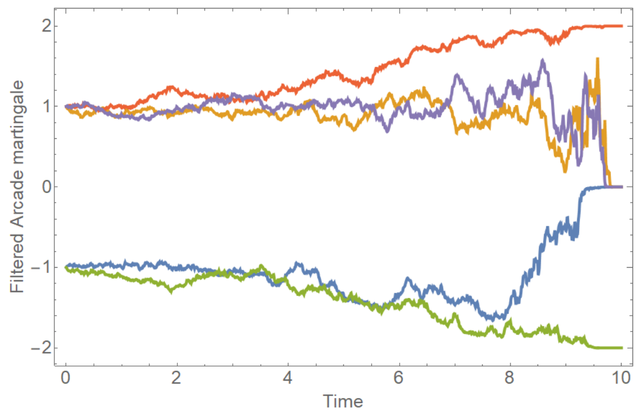

Figure 13: Paths of the associated FAM (M_t).

4.2 The n-arc FAM

Given a random vector $X = (X_0, \ldots, X_n)$ of convexly ordered components, i.e., $X_i \leqslant_{\text{cx}} X_{i+1}$ for $i = 0, \ldots, n-1$, and an X-RAP $(I_t^{(n)})_{t\in[T_0,T_n]}$ on the partition $\{T_0, T_n\}_*$, we construct a martingale $(M_t)_{t\in[T_0,T_n]}$ with respect to $(\mathcal{F}_t^I)$ such that $M_{T_i} \overset{a.s.}{=} X_i$, for $i = 0, \ldots, n$. A major difference in the n-arc case is that we need $(I_t^{(n)})$ to be $\{T_0, T_n\}_*$-nearly Markov right away.

Definition 4.19. *Given an X-RAP $(I_t^{(n)})_{t\in[T_0,T_n]}$ on $\{T_0, T_n\}_*$ that is $\{T_0, T_n\}_*$-nearly Markov, an n-arc FAM for X is a stochastic process of the form $M_t = \mathbb{E}[X_n \,|\, X_0, \ldots, X_{m(t)}, I_t^{(n)}]$ where $m(t) = \max\{i \in \mathbb{N} \,|\, T_i \leqslant t\}$.*

By the tower property of conditional expectation, $(M_t)_{t\in[T_0,T_1]}$ is a martingale with respect to $(\mathcal{F}_t^I)$. Moreover, $M_{T_i} = \mathbb{E}[X_n \,|\, X_0, \ldots, X_i] \overset{a.s.}{=} X_i$ by the convex ordering property. Hence (M_t) is an interpolating martingale with respect to $(\mathcal{F}_t^I)$. Like in the $n = 1$ case, we assume that the driver of $(I_t^{(n)})$ has a density function in what follows.

Proposition 4.20. *Let $M_t = \mathbb{E}[X_n \,|\, X_0, \ldots, X_{m(t)}, I_t^{(n)}]$ be an n-arc FAM restricted to $(T_0, T_n)_*$. Then,*

$$M_t = \frac{\int_{\mathbb{R}} y f^{I_t^{(n)} \mid X_0,\ldots,X_{m(t)},X_n=y}(I_t^{(n)})\, \mathrm{d}F^{X_n \mid X_0,\ldots,X_{m(t)}}}{\int_{\mathbb{R}} f^{I_t^{(n)} \mid X_0,\ldots,X_{m(t)},X_n=y}(I_t^{(n)})\, \mathrm{d}F^{X_n \mid X_0,\ldots,X_{m(t)}}}, \tag{4.99}$$

where $F^{X_n \mid X_0,\ldots,X_{m(t)}}$ is the distribution function of X_n conditional on $X_0, \ldots, X_{m(t)}$. In particular,

1. *if $(X_0, \ldots, X_{m(t)}, X_n)$ is a continuous random variable, with density function $f^{X_n \mid X_0,\ldots,X_{m(t)}}$, then*
$$M_t = \frac{\int_{\mathbb{R}} y f^{I_t^{(n)} \mid X_0,\ldots,X_{m(t)},X_n=y}(I_t^{(n)}) f^{X_n \mid X_0,\ldots,X_{m(t)}}(y)\, \mathrm{d}y}{\int_{\mathbb{R}} f^{I_t^{(n)} \mid X_0,\ldots,X_{m(t)},X_n=y}(I_t^{(n)}) f^{X_n \mid X_0,\ldots,X_{m(t)}}(y)\, \mathrm{d}y}. \tag{4.100}$$

2. *if $(X_0, \ldots, X_{m(t)}, X_n)$ is a discrete random variable, then*
$$M_t = \frac{\sum_y y f^{I_t^{(n)} \mid X_0,\ldots,X_{m(t)},X_n=y}(I_t^{(n)}) \mathbb{P}[X_n = y \mid X_0, \ldots, X_{m(t)}]}{\sum_y f^{I_t^{(n)} \mid X_0,\ldots,X_{m(t)},X_n=y}(I_t^{(n)}) \mathbb{P}[X_n = y \mid X_0, \ldots, X_{m(t)}]}. \tag{4.101}$$

The proof follows immediately from the one-arc case. Introducing the notation

$$u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y) = f^{I_t^{(n)} \mid X_0,\ldots,X_{m(t)},X_{m(t)+1}=x_{m+1},\ldots,X_{n-1}=x_{n-1},X_n=y}(I_t^{(n)}), \tag{4.102}$$

it is often more convenient to write

$$M_t = \frac{\int_{\mathbb{R}^{n-m(t)}} y u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}F^{X_{m(t)+1},\ldots,X_{n-1},X_n \mid X_0,\ldots,X_{m(t)}}(x_{m+1}, \ldots, x_{n-1}, y)}{\int_{\mathbb{R}^{n-m(t)}} u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}F^{X_{m(t)+1},\ldots,X_{n-1},X_n \mid X_0,\ldots,X_{m(t)}}(x_{m+1}, \ldots, x_{n-1}, y)}. \tag{4.103}$$

We can apply Itô's lemma in the same fashion as we did in the one-arc case. An interesting pattern appears if $(I_t^{(n)})$ satisfies $g_j(x)\mathbb{1}_{[T_0,T_{j-1}]}(x) = 0$ for all $j = 1, \ldots, n$, and for all $x \in [T_0, T_n]$, that simplifies the n-arc case significantly.

Theorem 4.21. *Let $M_t = \mathbb{E}[X_n \mid X_0, \ldots, X_{m(t)}, I_t^{(n)}]$ be an n-arc FAM, where $(I_t^{(n)})$ satisfies $g_j(x)\mathbb{1}_{[T_0,T_{j-1}]}(x) = 0$ for all $j = 1, \ldots, n$, and for all $x \in [T_0, T_n]$. Then $M_t = \mathbb{E}[X_{m(t)+1} \mid X_0, \ldots, X_{m(t)}, I_t^{(n)}]$.*

Proof. The result is trivial for $t \in \{T_0, T_n\}_*$, so we treat the case $t \in (T_0, T_n)_*$.
Let φ be the density function of $(A_t^{(n)})$, the noise process of $(I_t^{(n)})$. Then,

$$u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y) = \varphi\left(I_t^{(n)} - \sum_{i=0}^{m(t)} g_i(t) X_i - g_{m(t)+1}(t) x_{m+1}\right), \tag{4.104}$$

since $g_j(x)\mathbb{1}_{[T_0,T_{j-1}]}(x) = 0$ for all $j = 1, \ldots, n$, and for all $x \in [T_0, T_n]$. Denoting by $\tilde{F}$ the conditional distribution of $(X_{m(t)+1}, \ldots, X_n)$ given $X_0, \ldots, X_{m(t)}$, i.e,

$$\tilde{F}(x_{m+1}, \ldots, x_n) = F^{X_{m(t)+1},\ldots,X_n|X_0,\ldots,X_{m(t)}}(x_{m+1}, \ldots, x_n), \tag{4.105}$$

this implies

$$\int_{\mathbb{R}^{n-m(t)}} u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}\tilde{F}(x_{m+1}, \ldots, x_{n-1}, y)$$
$$= \int_{\mathbb{R}} u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}F^{X_{m(t)+1}|X_0,\ldots,X_{m(t)}}(x_{m+1}). \tag{4.106}$$

We also have

$$\int_{\mathbb{R}^{n-m(t)}} y\, \mathrm{d}\tilde{F}(x_{m+1}, \ldots, x_n)$$
$$= \int_{\mathbb{R}^{n-m(t)}} y\, \mathrm{d}F^{X_n|X_0,\ldots,X_{m(t)},X_{m(t)+1}=x_{m+1},\ldots,X_{n-1}=x_{n-1}}(y)$$
$$\mathrm{d}F^{X_{m(t)+1},\ldots,X_{n-1}|X_0,\ldots,X_{m(t)}}(x_{m+1}, \ldots, x_{n-1}), \tag{4.107}$$
$$= \int_{\mathbb{R}^{n-m(t)-1}} \mathbb{E}[X_n \mid X_0, \ldots, X_{m(t)}, X_{m(t)+1} = x_{m+1}, \ldots, X_{n-1} = x_{n-1}]$$
$$\mathrm{d}F^{X_{m(t)+1},\ldots,X_{n-1}|X_0,\ldots,X_{m(t)}}(x_{m+1}, \ldots, x_{n-1}), \tag{4.108}$$
$$= \int_{\mathbb{R}^{n-m(t)-1}} x_{n-1}\, \mathrm{d}F^{X_{m(t)+1},\ldots,X_{n-1}|X_0,\ldots,X_{m(t)}}(x_{m+1}, \ldots, x_{n-1}), \tag{4.109}$$

where we used the martingale property. Applying the same argument $m(t)$ times, we get

$$\int_{\mathbb{R}^{n-m(t)}} y\, \mathrm{d}\tilde{F}(x_{m+1}, \ldots, x_n) = \int_{\mathbb{R}} x_{m+1}\, \mathrm{d}F^{X_{m(t)+1}|X_0,\ldots,X_{m(t)}}(x_{m+1}), \tag{4.110}$$

which means that

$$\int_{\mathbb{R}^{n-m(t)}} y u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}\tilde{F}(x_{m+1}, \ldots, x_{n-1}, y) \tag{4.111}$$
$$= \int_{\mathbb{R}} x_{m+1} u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y) dF^{X_{m(t)+1}|X_0,\ldots,X_{m(t)}}(x_{m+1}). \tag{4.112}$$

Finally,

$$M_t = \frac{\int_{\mathbb{R}^{n-m(t)}} y u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}\tilde{F}(x_{m+1}, \ldots, x_{n-1}, y)}{\int_{\mathbb{R}^{n-m(t)}} u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}\tilde{F}(x_{m+1}, \ldots, x_{n-1}, y)}, \tag{4.113}$$

$$= \frac{\int_{\mathbb{R}} x_{m+1} u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}F^{X_{m(t)+1}|X_0,\ldots,X_{m(t)}}(x_{m+1})}{\int_{\mathbb{R}} u(I_t^{(n)}, t, x_{m+1}, \ldots, x_{n-1}, y)\, \mathrm{d}F^{X_{m(t)+1}|X_0,\ldots,X_{m(t)}}(x_{m+1})}, \tag{4.114}$$

$$= \int_{\mathbb{R}} x_{m+1}\, \mathrm{d}F^{X_{m(t)+1}|X_0,\ldots,X_{m(t)},I_t^{(n)}}(x_{m+1}) \tag{4.115}$$

$$= \mathbb{E}[X_{m(t)+1} \mid X_0, \ldots, X_{m(t)}, I_t^{(n)}], \tag{4.116}$$

where we used Bayes rule.

□

This allows one to use the one-arc case to derive the SDE in the n-arc case without much effort.

Proposition 4.22. *Let $(I_t^{(n)})$ be a semimartingale standard RAP, with driver covariance $K_D(x,y) = H_1(\min(x,y))H_2(\max(x,y))$, such that $(t,x) \to \mathbb{E}[X_n \,|\, X_0,\dots,X_{m(t)}, I_t^{(n)} = x]$ is $C^2(((T_0,T_n)_* \setminus N) \times Im(I_t^{(n)}))$ where $N \subset (T_0,T_n)_*$ contains finitely many elements. Let $M_t = \mathbb{E}[X_n \,|\, X_0,\dots,X_{m(t)}, I_t^{(n)}]$ be an n-arc FAM. Then,*

$$\begin{aligned}\mathrm{d}M_t = &\frac{\mathrm{Var}[X_{m(t)+1} \,|\, X_0,\dots,X_{m(t)}, I_t^{(n)}]}{H_1(T_{m(t)+1})H_2(t) - H_1(t)H_2(T_{m(t)+1})} \\ &\times \Bigg[\left(\frac{Z_t(H_1'(t)H_2(T_{m(t)+1}) - H_1(T_{m(t)+1})H_2'(t)) - M_t(H_1'(t)H_2(t) - H_1(t)H_2'(t))}{H_1(T_{m(t)+1})H_2(t) - H_1(t)H_2(T_{m(t)+1})} - J_t\right)\mathrm{d}t \\ &+ \mathrm{d}I_t^{(1)}\Bigg]\end{aligned} \tag{4.117}$$

where $Z_t = I_t^{(1)} - \sum_{i=0}^{m(t)} g_i(t)X_i - \mu_A(t)$ and $J_t = \sum_{i=0}^{m(t)} g_i'(t)X_i + \mu_A'(t)$.

The proof follows immediately from the one-arc case. Again, denoting

$$\begin{aligned}\mathrm{d}N_t = &\left(\frac{Z_t(H_1'(t)H_2(T_{m(t)+1}) - H_1(T_{m(t)+1})H_2'(t)) - M_t(H_1'(t)H_2(t) - H_1(t)H_2'(t))}{H_1(T_{m(t)+1})H_2(t) - H_1(t)H_2(T_{m(t)+1})} - J_t\right)\mathrm{d}t \\ &+ \mathrm{d}I_t^{(1)},\end{aligned} \tag{4.118}$$

$$\mathrm{d}W_t = \frac{\mathrm{d}N_t}{\sqrt{H_1'(t)H_2(t) - H_1(t)H_2'(t)}}, \tag{4.119}$$

we have that (W_t) is a standard Brownian motion on $[T_0,T_n]$ and adapted to $(\mathcal{F}_t^I)$, and

$$M_t = X_0 + \int_{T_0}^t \frac{\mathrm{Var}[X_{m(s)+1} \,|\, X_0,\dots,X_{m(s)}, I_t^{(n)}]\sqrt{H_1'(s)H_2(s) - H_1(s)H_2'(s)}}{H_1(T_{m(s)+1})H_2(s) - H_1(s)H_2(T_{m(s)+1})}\,\mathrm{d}W_s, \tag{4.120}$$

$$= X_{m(t)} + \int_{T_{m(t)}}^t \frac{\mathrm{Var}[X_{m(s)+1} \,|\, X_0,\dots,X_{m(s)}, I_t^{(n)}]\sqrt{H_1'(s)H_2(s) - H_1(s)H_2'(s)}}{H_1(T_{m(s)+1})H_2(s) - H_1(s)H_2(T_{m(s)+1})}\,\mathrm{d}W_s. \tag{4.121}$$

4.3 Filtered arcade martingales and optimal transport

Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion. For simplicity of notation, this section is focused on the one-arc case where $(I_t^{(1)})$ is given by

$$I_t^{(1)} = S_t^{(1)} + A_t^{(1)} = B_t - \frac{T_1 - t}{T_1 - T_0}(B_{T_0} - X_0) - \frac{t - T_0}{T_1 - T_0}(B_{T_1} - X_1). \tag{4.122}$$

The n-arc case with general standard RAP follows without much effort. Consider the FAM

$$M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}] = X_0 + \int_{T_0}^t \frac{\mathrm{Var}[X_1 \,|\, X_0, I_s^{(1)}]}{T_1 - s}\,\mathrm{d}W_s, \tag{4.123}$$

where (W_t) is a standard Brownian motion restricted to $[T_0,T_1]$ and adapted to $(\mathcal{F}_t^I)$. The FAM (M_t) depends heavily on the coupling π^X of (X_0,X_1). In real life problems, the coupling is usually not observed directly, only its marginals are. Choosing a coupling is then part of the modelling of the underlying problem. Another approach is to go "model-free" by utilizing a least action principle, such as martingale optimal transport.

Optimal transport dates back to Gaspard Monge in 1781 [16], with significant advancements by Leonid Kantorovich in 1942 [11] and Yann Brenier in 1987 [5]. It provides a way of comparing two measures, μ and ν, defined on the Borel sets of topological Hausdorff spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. The mental image of optimal transport is the one of a pile of sand, modelled by a measure μ, and a hole, modelled by another measure ν. One wishes to fill the hole with the sand at one's disposal in an optimal manner, by exercising the least amount of effort. To make this statement more precise, one needs a cost function $c : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ that measures the cost of transporting a unit mass from $x \in \mathcal{X}$ to $y \in \mathcal{Y}$. The optimal transport problem is how to transport μ to ν whilst minimizing the cost of transportation: given $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$,

$$\inf_{\pi\in\Pi(\mu,\nu)} \mathbf{K}(\pi) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\,\mathrm{d}\pi(x,y). \tag{4.124}$$

This problem enjoys many interesting properties. For instance, when $\mathcal{X} = \mathcal{Y}$ is a Polish space, and $(\mathcal{X}, c)$ is a metric space,

$$W_p(\mu,\nu) := \left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{X}} c(x,y)^p \mathrm{d}\pi(x,y) \right)^{1/p} \tag{4.125}$$

is a metric, for any $p \geqslant 1$, called the Wasserstein pth metric, on the space $\mathcal{P}_p(\mathcal{X})$ of probability measures on $\mathcal{X}$ with finite pth moment. Furthermore, if $\mathcal{X}$ is Euclidean, and $c(x,y) = \|x-y\|$, there is a one-to-one correspondence between the minimizers π^* of $\inf\limits_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)^p\,\mathrm{d}\pi(x,y)$, and the geodesics in $(\mathcal{P}_p(\mathcal{X}), W_p)$: if $(X_0, X_1) \sim \pi^*$, the law of the process

$$\left(\frac{T_1 - t}{T_1 - T_0} X_0 + \frac{t - T_0}{T_1 - T_0} X_1 \right)_{t\in[T_0,T_1]} \tag{4.126}$$

is the shortest paths from μ to ν in $(\mathcal{P}_p(\mathcal{X}), W_p)$. This links optimal transport to interpolation on the space of random variables. The interpolating process is very basic, since it is fully deterministic conditionally on (X_0, X_1). A simple solution to get a real stochastic process would be to add an independent Brownian bridge to $\frac{T_1-t}{T_1-T_0}X_0 + \frac{t-T_0}{T_1-T_0}X_1$. This is trivial in a way, since the "noise" term has nothing to do with the initial optimization problem. If one desires to interpolate using a true stochastic process without abandoning optimal transport, the entropic regularization of optimal transport provides an answer: given $\mu \in \mathcal{P}(\mathcal{X})$, $\nu \in \mathcal{P}(\mathcal{Y})$, and $\varepsilon > 0$,

$$\inf_{\pi_\varepsilon\in\Pi(\mu,\nu)} \mathbf{K}_\varepsilon(\pi_\varepsilon) := \inf_{\pi_\varepsilon\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\,\mathrm{d}\pi_\varepsilon(x,y) + \varepsilon\ \mathrm{KL}(\pi_\varepsilon \,|\, \mu\otimes\nu) \tag{4.127}$$

where KL designates the Kullback-Leibler divergence

$$\mathrm{KL}(\pi_\varepsilon \,|\, \mu\otimes\nu) = \begin{cases} \displaystyle\int_{\mathcal{X}\times\mathcal{Y}} \log\left(\frac{\mathrm{d}\pi_\varepsilon}{\mathrm{d}\mu\,\mathrm{d}\nu}(x,y) \right) \mathrm{d}\pi_\varepsilon(x,y), & \text{if } \pi_\varepsilon << \mu\otimes\nu, \\ \infty & \text{otherwise.} \end{cases} \tag{4.128}$$

When $\mathcal{X} = \mathcal{Y}$ is Euclidean, $c(x,y) = \|x-y\|^2$, and $\varepsilon = 2(T_1 - T_0)$, this problem is equivalent to Schrödinger's problem. If π_S^* is the solution to Schrödinger's problem, any stochastic process with distribution function, evaluated at $z \in \mathbb{R}$, equal to

$$\int_{\mathcal{X}} \int_{\mathcal{X}} F(x,y,z)\,\mathrm{d}\pi_S^*(x,y), \tag{4.129}$$

where $F(x, y, z)$ is the distribution function, evaluated at z, of a Brownian bridge starting at time T_0 in value x and ending at time T_1 in value y, is called Schrödinger's bridge. Hence, $(I_t^{(1)})$ is an anticipative representation of Schrödinger's bridge (in dimension one since we did not introduce a definition of RAPs in higher dimensions) as long as $(X_0, X_1) \sim \pi_S^*$, and it is precisely the replacement that we were looking for. To recap the three interpolators built using optimal transport in dimension one, we have:

1. The default OT interpolator, or, the shortest path interpolator on $[T_0, T_1]$: $\frac{T_1-t}{T_1-T_0}X_0 + \frac{t-T_0}{T_1-T_0}X_1$ where $(X_0, X_1) \sim \pi^*$.
2. The artificially noisy OT interpolator on $[T_0, T_1]$: $(I_t^{(1)})$, where $(X_0, X_1) \sim \pi^*$.
3. The truly noisy OT interpolator, or, Schrödinger's bridge on $[T_0, T_1]$: $(I_t^{(1)})$, where $(X_0, X_1) \sim \pi_S^*$.

Fix $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, since this is the setting of FAMs. We can adapt classical optimal transport to yield a martingale coupling instead: given $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ in convex order,

$$\inf_{\pi \in \mathcal{M}(\mu,\nu)} \mathbf{K}(\pi) = \inf_{\pi \in \mathcal{M}(\mu,\nu)} \int_{\mathbb{R}^2} c(x, y)\, \mathrm{d}\pi(x, y), \tag{4.130}$$

We denote the minimizers of this problem by π_m^*. There are many differences between optimal transport and its martingale counterpart. For instance, the most popular cost function $c(x, y) = (x - y)^2$ for optimal transport cannot be used in the martingale context:

$$\begin{aligned}\int_{\mathbb{R}^2} (x-y)^2\, \mathrm{d}\pi(x, y) &= \int_{\mathbb{R}^2} x^2 + y^2 - 2xy\, \mathrm{d}\pi(x, y) &(4.131)\\ &= \int_{\mathbb{R}^2} x^2\, \mathrm{d}\mu(x) + \int_{\mathbb{R}^2} y^2\, \mathrm{d}\nu(y) - 2\int_{\mathbb{R}^2} y^2\, \mathrm{d}\nu(y) &(4.132)\\ &= \int_{\mathbb{R}^2} x^2\, \mathrm{d}\mu(x) - \int_{\mathbb{R}^2} y^2\, \mathrm{d}\nu(y), &(4.133)\end{aligned}$$

which does not depend on π. Another difference is that we loose the geodesic interpretation since there is no counterpart to the Wasserstein distance in the martingale context. So what would the martingale counterparts to the default, artificially noisy and truly noisy OT interpolators be? This is not trivial, since the interpolators must be martingales, and $\frac{T_1-t}{T_1-T_0}X_0 + \frac{t-T_0}{T_1-T_0}X_1$ is not a martingale, regardless of the coupling of (X_0, X_1).

Since there is no such thing as the "shortest path" interpolator anymore, we expect it to be "broken" in the martingale context, i.e, not continuous. We propose the process that is equal to X_0 in T_0 and equal to X_1 for $t \in (T_0, T_1]$. Now, for the artificially noisy interpolator, the FAM (M_t) with $(X_0, X_1) \sim \pi_m^*$ is a candidate. Indeed, it is a martingale, and its noise process was not taken into account in the selection of the optimal coupling π_m^*. Furthermore, if we remove the artificial noise, i.e, we put $A_t^{(1)} = 0$ in the expression of (M_t), and denote by $(\mathcal{G}_t)_{t\in[T_0,T_1]}$ the filtration generated by $\left(\frac{T_1-t}{T_1-T_0}X_0 + \frac{t-T_0}{T_1-T_0}X_1\right)_{t\in[T_0,T_1]}$, we get

$$M_t = \mathbb{E}[X_1 \,|\, \mathcal{G}_t], \tag{4.134}$$

$$= \mathbb{E}\left[X_1 \,|\, X_0, \frac{T_1 - t}{T_1 - T_0} X_0 + \frac{t - T_0}{T_1 - T_0} X_1\right], \tag{4.135}$$

$$= \begin{cases} X_0 & \text{if } t = T_0, \\ X_1 & \text{if } t \in (T_0, T_1], \end{cases} \tag{4.136}$$

retrieving the "broken" default interpolator. For a truly noisy MOT interpolator, i.e, a sort of martingale counterpart to Schrödinger's bridge, we propose a new problem, called the information-based martingale optimal transport problem.

Definition 4.23. *Let $X_0 \sim \mu$ and $X_1 \sim \nu$, where μ and ν are L^2 probability measures on $\mathbb{R}$, in convex order. Let $M_t = \mathbb{E}[X_1 \,|\, X_0, I_t^{(1)}]$ be a one-arc FAM. The information-based martingale optimal transport (IBMOT) problem associated with the randomized Brownian bridge $(I_t^{(1)})$ is*

$$\sup_{\pi \in \mathcal{M}(\mu,\nu)} \mathbf{K}_I(\pi) := \sup_{\pi \in \mathcal{M}(\mu,\nu)} \mathbb{E}\left[\int_{T_0}^{T_1} \frac{(X_1 - M_t)^2}{T_1 - t} \,\mathrm{d}t\right]. \tag{4.137}$$

To make sure that Eq. 4.137 makes sense, notice that, by the definition of FAM, and Itô's isometry,

$$\mathbb{E}[(X_1 - X_0)^2] = \mathbb{E}[(M_{T_1} - X_0)^2], \tag{4.138}$$

$$= \mathbb{E}\left[\left(\int_{T_0}^{T_1} \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{T_1 - t} \,\mathrm{d}W_t\right)^2\right], \tag{4.139}$$

$$= \mathbb{E}\left[\int_{T_0}^{T_1} \left(\frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{T_1 - t}\right)^2 \mathrm{d}t\right]. \tag{4.140}$$

Since $\mathbb{E}[(X_1 - X_0)^2] = \mathbb{E}[X_1^2] - \mathbb{E}[X_0^2] < \infty$, we have that $\mathbb{E}\left[\int_{T_0}^{T_1} \left(\frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{T_1 - t}\right)^2 \mathrm{d}t\right] < \infty$, and because the product space $\Omega \times [T_0, T_1]$ is of finite measure, we also have $\mathbb{E}\left[\int_{T_0}^{T_1} \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{T_1 - t} \,\mathrm{d}t\right] < \infty$ by Hölder's inequality. Using the martingale property, we see that

$$\begin{aligned}
\mathbb{E}[\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]] &= \mathbb{E}[\mathbb{E}[X_1^2 \,|\, X_0, I_t^{(1)}] - M_t^2], && (4.141)\\
&= \mathbb{E}[X_1^2] - \mathbb{E}[M_t^2], && (4.142)\\
&= \mathbb{E}[X_1^2 + M_t^2] - 2\mathbb{E}[M_t \mathbb{E}[X_1 \,|\, \mathcal{F}_t^I]], && (4.143)\\
&= \mathbb{E}[X_1^2 + M_t^2] - 2\mathbb{E}[\mathbb{E}[X_1 M_t \,|\, \mathcal{F}_t^I]], && (4.144)\\
&= \mathbb{E}[X_1^2 + M_t^2 - 2X_1 M_t], && (4.145)\\
&= \mathbb{E}[(X_1 - M_t)^2]. && (4.146)
\end{aligned}$$

By Tonelli's theorem, since $\mathbb{E}\left[\int_{T_0}^{T_1} \frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{T_1 - t} \,\mathrm{d}t\right] < \infty$ and $\frac{\mathrm{Var}[X_1 \,|\, X_0, I_t^{(1)}]}{T_1 - t} \geqslant 0$, we get

$$\mathbb{E}\left[\int_{T_0}^{T_1} \frac{\mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]}{T_1 - t}\, \mathrm{d}t\right] = \int_{T_0}^{T_1} \frac{\mathbb{E}\left[\mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]\right]}{T_1 - t}\, \mathrm{d}t, \tag{4.147}$$

$$= \int_{T_0}^{T_1} \frac{\mathbb{E}[(X_1 - M_t)^2]}{T_1 - t}\, \mathrm{d}t, \tag{4.148}$$

$$= \mathbf{K}_I(\pi). \tag{4.149}$$

Hence, $\sup\limits_{\pi\in\mathcal{M}(\mu,\nu)} \mathbf{K}_I(\pi) = \sup\limits_{\pi\in\mathcal{M}(\mu,\nu)} \mathbb{E}\left[\int_{T_0}^{T_1} \frac{\mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]}{T_1-t}\, \mathrm{d}t\right] < \infty$. Since $\frac{\mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]}{T_1-t}$ is exactly the volatility process of (M_t), the IBMOT problem can be seen as the martingale Benamou-Brenier problem [1], under the extra constraint that the volatility must be of the form $\frac{\mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]}{T_1-t}$.

One can rewrite Eq. 4.137 using the integration by parts formula, making the innovation process $W_t = \int_{T_0}^{t} \frac{I_u^{(1)} - M_u}{T_1-u}\, \mathrm{d}u + I_t^{(1)} - X_0$, which is a standard Brownian motion on $[T_0, T_1]$ adapted to $(\mathcal{F}^I)_{t\in[T_0,T_1]}$, appear in the expression when evaluated at time T_1. Since we have that

$$\mathbb{E}\left[\int_{T_0}^{T_1} \frac{\mathrm{Var}[X_1 \mid X_0, I_t^{(1)}]}{T_1 - t}\, \mathrm{d}t\right] = \mathbb{E}[M_{T_1}W_{T_1} - M_{T_0}W_{T_0}] = \mathbb{E}[X_1 W_{T_1}], \tag{4.150}$$

we get $\sup\limits_{\pi\in\mathcal{M}(\mu,\nu)} \mathbf{K}_I(\pi) = \sup\limits_{\pi\in\mathcal{M}(\mu,\nu)} \mathbb{E}[X_1 W_{T_1}]$. Since W_{T_1} is a $\mathcal{N}(0, T_1 - T_0)$ regardless of the coupling π, we can complete the square to get an equivalent problem to 4.137:

$$\mathbb{E}[X_1^2] - 2 \sup_{\pi\in\mathcal{M}(\mu,\nu)} \mathbb{E}[X_1 W_{T_1}] + \mathbb{E}[W_{T_1}^2] = \inf_{\pi\in\mathcal{M}(\mu,\nu)} \mathbb{E}[X_1^2] - 2\mathbb{E}[X_1 W_{T_1}] + \mathbb{E}[W_{T_1}^2], \tag{4.151}$$

$$= \inf_{\pi\in\mathcal{M}(\mu,\nu)} \mathbb{E}[(X_1 - W_{T_1})^2]. \tag{4.152}$$

Example 4.24. *Let $T = T_1 - T_0, \sigma > 0, X_0 \sim \mathcal{N}(0, \sigma^2), X_1 \sim \mathcal{N}(0, \sigma^2 + T)$. We show that the Brownian coupling $\mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & \sigma^2\\ \sigma^2 & \sigma^2+T\end{pmatrix}\right)$ is a solution for the IBMOT problem. Recall that $\sup\limits_{\pi\in\mathcal{M}(\mu,\nu)} \mathbf{K}_I(\pi) \leqslant \mathbb{E}[X_1^2] - \mathbb{E}[X_0^2] = T$. Assuming $(X_0, X_1) \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & \sigma^2\\ \sigma^2 & \sigma^2+T\end{pmatrix}\right)$, we get that*

1. *$(X_1, X_0, I_s^{(1)})$ is Gaussian for all $s \in [T_0, T_1]$ and $(I_t^{(1)})$ is itself a Brownian motion on $[T_0, T_1]$,*
2. *$\mathrm{Cov}(X_0, I_s^{(1)}) = \sigma^2$,*
3. *$\mathrm{Cov}(X_1, I_s^{(1)}) = \frac{T_1-s}{T}\sigma^2 + \frac{s-T_0}{T}(\sigma^2+T) = \sigma^2 + s - T_0$,*
4.

$$\mathrm{Var}(I_s^{(1)}) = \frac{(T_1-s)^2}{T^2}\sigma^2 + \frac{(s-T_0)^2}{T^2}(\sigma^2+T) + 2\frac{(T_1-s)(s-T_0)}{T}\sigma^2 + \frac{(T_1-s)(s-T_0)}{T},$$

$$= (s-T_0)^2 + \sigma^2 T + \frac{(T_1-s)(s-T_0)}{T},$$

5.

$$
\begin{aligned}
M_s &= \begin{pmatrix} \sigma^2 & \sigma^2 + s - T_0 \end{pmatrix} \begin{pmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & (s-T_0)^2 + \sigma^2 T + \frac{(T_1-s)(s-T_0)}{T} \end{pmatrix}^{-1} \begin{pmatrix} X_0 \\ I_s^{(1)} \end{pmatrix}, \\
&= I_s^{(1)},
\end{aligned}
$$

6.

$$
\begin{aligned}
\mathbf{K}_I\left(\mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2+T\end{pmatrix}\right)\right) &= \mathbb{E}\left[\int_{T_0}^{T_1} \frac{(X_1 - I_s^{(1)})^2}{T_1 - s}\,\mathrm{d}s\right], \\
&= \mathbb{E}\left[\int_{T_0}^{T_1} \frac{\frac{(T_1-s)^2}{T^2}(X_1-X_0)^2 + \left(A_s^{(1)}\right)^2}{T_1 - s}\,\mathrm{d}s\right], \\
&= \int_{T_0}^{T_1} \frac{(T_1-s)^2 + (T_1-s)(s-T_0)}{T(T_1-s)}\,\mathrm{d}s, \\
&= \int_{T_0}^{T_1} \frac{(T_1-s)+(s-T_0)}{T}\,\mathrm{d}s, \\
&= T.
\end{aligned}
$$

Hence Brownian motion is the optimal FAM between X_0 and X_1 according to IBMOT. Equivalently, the Brownian coupling is the optimal martingale coupling for (X_0, X_1) according to IBMOT.

Acknowledgments

We are grateful for the many discussions on stochastic interpolation with C.A. Garcia Trillos, J. Sekine, especially in the context of information-based modelling and stochastic filtering. A.M. acknowledges a research visit at Osaka University supported by a Daiwa Research Grant. This project was supported by the Fields Institute for Research in Mathematical Sciences. Its contents are solely the responsibility of the authors and do not necessarily represent the official views of the Institute. Fields Institute for Research in Mathematical Sciences. G.K. acknowledges a Teaching Assistantship in the Department of Mathematics, UCL. Furthermore, we thank L.P. Hughston for useful discussions at the early stages of this work, B. Acciaio, G. W. Peters, and D. C. Schwarz for comments and suggestions as this work developed further.

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