Non-associative, Non-commutative Multi-modal Linear Logic

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Abstract. Adding multi-modalities (called *subexponentials*) to linear logic enhances its power as a logical framework, which has been extensively used in the specification of *e.g.* proof systems, programming languages and bigraphs. Initially, subexponentials allowed for classical, linear, affine or relevant behaviors. Recently, this framework was enhanced so to allow for commutativity as well. In this work, we close the cycle by considering associativity. We show that the resulting system (acLL_Σ) admits the (multi)cut rule, and we prove two undecidability results for fragments/variations of acLL_Σ.

1 Introduction

Resource aware logics have been object of passionate study for quite some time now. The motivations for this passion vary: resource consciousness are adequate for modeling steps of computation; logics have interesting algebraic semantics; calculi have nice proof theoretic properties; multi-modalities allow for the specification of several behaviors; there are many interesting applications in linguistics, etc.

With this variety of subjects, applications and views, it is not surprising that different groups developed different systems based on different principles. For example, the Lambek calculus (L) [29] was introduced for mathematical modeling of natural language syntax, and it extends a basic categorial grammar [3,4] by a concatenation operator. Linear logic (LL) [16], originally discovered by Girard from a semantical analysis of the models of polymorphic λ-calculus, turned out to be a refinement of classical and intuitionistic logic, having the dualities of the former and constructive properties of the

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latter. The key point is the presence of the *modalities* $!, ?$, called *exponentials* in LL. In the intuitionistic version of LL, denoted by ILL, only the ! exponential is present.

L and LL were compared in [2], when Abrusci showed that Lambek calculus coincides with a variant of the non-commutative, multiplicative version of ILL [41]. This correspondence can be lifted for considering also the additive connectives: Full (multiplicative-additive) Lambek calculus FL relates to non-commutative multiplicative-additive version of ILL, here denoted by cLL.

In this paper we propose the sequent based system acLL_Σ, a conservative extension of cLL, where associativity is allowed only for formulas marked with a special kind of modality, determined by a *subexponential signature* Σ. The notation adopted is modular, uniform and scalable, in the sense that many well known systems will appear as fragments or special cases of acLL_Σ, by only modifying the signature Σ. The core fragment of acLL_Σ (*i.e.*, without the subexponentials) corresponds to the non-associative version of full Lambek calculus, FNL [8].[1]

The language of acLL_Σ consists of a denumerable infinite set of propositional variables $\{p, q, r, \ldots\}$, the unities $\{1, \top\}$, the binary connectives for additive conjunction and disjunction $\{\&, \oplus\}$, the non-commutative multiplicative conjunction $\otimes$, the non-commutative linear implications $\{\rightarrow, \leftarrow\}$, and the unary subexponentials $!^i$, with i belonging to a pre-ordered set of labels $(I, \preceq)$.

Roughly speaking, subexponentials [13] are substructural multi-modalities. In LL, $!A$ indicates that the linear formula A behaves *classically*, that is, it can be contracted *and* weakened. Labeling ! with indices allows moving one step further: The set I can be partitioned so that, in $!^i A$, A can be contracted *and/or* weakened. This allows for two other types of behavior (other than classical or linear): affine (only weakening) or relevant (only contraction). Pre-ordering the labels (together with an upward closeness requirement) guarantees cut-elimination [42]. But then, why consider only weakening and contraction? Why not also take into account other structural properties, like commutativity or associativity? In [20,21] commutativity was added to the picture, so that in $!^i A$, A can be contracted, weakened, classical or linear, but it may also commute with the neighbor formula. In this work we consider the last missing part: Associativity.

Smoothly extending cLL to allow consideration of the non-associative case is non trivial. This requires a structural recasting/reframing of sequents: we pass from sets/multisets to lists in the non-commutative case, onto trees in the case of non-associativity [28]. As a consequence, the inference rules should act deeply over formulas in tree-structured sequents, which can be tricky in the presence of modalities [17].

On the other side, the multi-modal Lambek calculus introduced in [35,45] and extended/compiled/implemented in [18,36–38][2] use different *families of connectives and contexts*, distinguished by means of indices, or *modes*. Contexts are indexed binary trees, with formulas built from the indexed adjoint connectives $\{\rightarrow_i, \leftarrow_i\}$ and $\otimes_i$ (*e.g.*

[1] The multiplicative fragment of acLL_Σ is the non-associative version of Lambek's calculus, NL, introduced by Lambek himself in [30]. Both the associative calculus L and the non-associative calculus NL have their advantages and disadvantages for the analysis of natural language syntax, as we discuss in more detail in Sect. 2.2.

[2] The Grail family of theorem provers [37] works with a variety of modern type-logical frameworks, including multimodal type-logical grammars.

$(A \rightarrow_i B, (C \otimes_j D, H)^k)^i)$. Each mode has its own set of logical rules (following the same rule scheme), and different structural features can be combined via the mode information on the formulas. This gives to the resulting system a multi-modal flavor, but it also results in a language of *binary connectives*, determined by the modes. This forces an unfortunate second level synchronization between implications and tensor, and modalities act over whole *sequents*, not on single *formulas*.

In order to attribute particular resource management properties to individual resources, in [27,33] explicit (classical) multi-modalities $\Diamond_i, \Box_i$ were proposed. While such unary modalities were inspired in LL exponentials, the resemblance stops there. First of all, the logical connectives come together with structural constructors for contexts, which turns $\Diamond_i, \Box_i$ into truncated forms of product and implication.

Second, $\Diamond_i, \Box_i$ have a *temporal behavior*, in the sense that $\Diamond\Box F \Rightarrow F$ and $F \Rightarrow \Box\Diamond F$, which are not provable in LL using the "natural interpretation" $\Diamond = ?, \Box = !$.

In this paper, multi-modality is *totally local*, given by the subexponentials. The signature Σ contains the pre-ordered set of labels, together with a function stating which axioms, among weakening, contraction, exchange and associativity, are assumed for each label. Sequents will have a *nested structure*, corresponding to trees of formulas. And rules will be applied deeply in such structures. This not only gives the LL based system a more modern presentation (based on nested systems, like *e.g.* in [10,15]), but it also brings the notation closer to the one adopted by the Lambek community, like in [25]. Finally, it also uniformly extends several LL based systems present in the literature, as Example 8 in the next section shows.

Designing a good system serves more than simple pure proof-theoretic interests: Well behaved, neat proof systems can be used in order to approach several important problems, such as interpolation, complexity and decidability. And decidability of extensions/variants/fragments of L and LL is a fascinating subject of study, since the presence or absence of substructural properties/connectives may completely change the outcome. Indeed, it is well known that LL is undecidable [32], but adding weakening (affine LL) turns the system decidable [24], while removing the additives (MELL – multiplicative, exponential LL) reaches the border of knowledge: It is a long standing open problem [50]. Non-associativity also alters decidability and complexity: L is NP-complete [47], while NL is decidable in polynomial time [1,6]. Finally, the number of subexponentials also plays a role in decision problems: MELL with two subexponentials is undecidable [9].

In this work, we will present two undecidability results, all orbiting (but not encompassing) MELL/FNL. First, we show that acLL_Σ containing the multiplicatives $\otimes, \rightarrow$, the additive $\oplus$ and one classical subexponential (allowing contraction and weakening) is undecidable. This is a refinement of the unpublished result by Tanaka [51], which states that FNL plus one fully-powered subexponential is undecidable.

In the second undecidability result, we keep two subexponentials, but with a minimalist configuration: the implicational fragment of the logic plus two subexponentials: the "main" one allowing for contraction, exchange, and associativity (weakening is optional), and an "auxiliary" one allowing only associativity. This is a variation of Chaudhuri's result (in the non-associative, non-commutative case), making use of fewer connectives (tensor is not needed) and less powerful subexponentials.

Table 1. Acronyms/decidability of systems mentioned in the paper.

Acronym	System	Decidable?
L	Lambek calculus	✓
LL	(propositional) linear logic	✗
ILL	intuitionistic LL	✗
MALL	multiplicative-additive LL	✓
iMALL	intuitionistic MALL	✓
FL	full (multiplicative-additive) L	✓
cLL	non-commutative iMALL	✓
acLL$_\Sigma$	non-commutative, non-associative ILL with subexponentials	–
NL	non-associative L	✓
FNL	full (multiplicative-additive) NL	✓
MELL	multiplicative-exponential LL	unknown
SDML	simply dependent multimodal linear logics	–
SMALC$_\Sigma$	FL with subexponentials	–

The rest of the paper is organized as follows: Sect. 2 presents the system acLL$_\Sigma$, showing that it has the cut-elimination property and presenting an example in linguistics; Sect. 3 shows the undecidability results; and Sect. 4 concludes the paper.

We have placed, in Table 1, the acronyms for and decidability of all considered systems. Decidability for the cases marked with "–" depends on the signature Σ.

2 A Nested System for Non-associativity

Similar to modal connectives, the exponential ! in ILL is not *canonical* [13], in the sense that if $i \neq j$ then $!^i F \not\equiv !^j F$. Intuitively, this means that we can mark the exponential with *labels* taken from a set I organized in a pre-order $\preceq$ (*i.e.*, reflexive and transitive), obtaining (possibly infinitely-many) exponentials ($!^i$ for $i \in I$). Also as in multi-modal systems, the pre-order determines the provability relation: for a general formula F, $!^b F$ *implies* $!^a F$ iff $a \preceq b$.

The algebraic structure of subexponentials, combined with their intrinsic structural property allow for the proposal of rich linear logic based frameworks. This opened a venue for proposing different multi-modal substructural logical systems, that encountered a number of different applications. Originally [42], subexponentials could assume only weakening and contraction axioms:

$$\mathsf{C}: \quad !^i F \rightarrow !^i F \otimes !^i F \qquad \mathsf{W}: \quad !^i F \rightarrow 1$$

This allows the specification of systems with multiple contexts, which may be represented by sets or multisets of formulas [44], as well as the specification and verification of concurrent systems [43], and biological systems [46]. In [20,21], non-commutative systems allowing commutative subexponentials were presented:

$$\mathsf{E}: \quad (!^i F) \otimes G \equiv G \otimes (!^i F)$$

and this has many applications, *e.g.*, in linguistics [21].

In this work, we will present a non-commutative, non-associative linear logic based system, and add the possibility of assuming associativity[3]

$$\mathsf{A1}: \quad !^i F \otimes (G \otimes H) \equiv (!^i F \otimes G) \otimes H \qquad \mathsf{A2}: \quad (G \otimes H) \otimes !^i F \equiv G \otimes (H \otimes !^i F)$$

as well as commutativity and other structural properties.

We start by presenting an adaption of simply dependent multimodal linear logics (SDML) appearing in [31] to the non-associative/commutative case.

The language of non-commutative SDML is that of (propositional intuitionistic) linear logic with subexponentials [21] supplied with the *left residual*; or similarly, that of FL with subexponentials. Non-associative contexts will be organized via binary trees, here called *structures*.

Definition 1 (Structured sequents). Structures *are formulas or pairs containing structures:*

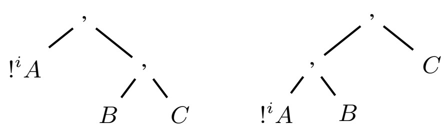
$$\Gamma, \Delta := F \mid (\Gamma, \Gamma)$$

where the constructors may be empty but never a singleton.

An n-ary context $\Gamma\left\{^1\right\}\ldots\left\{^n\right\}$ *is a context that contains n pairwise distinct numbered* holes $\{\,\}$ *wherever a formula may otherwise occur. Given n contexts $\Gamma_1, \ldots, \Gamma_n$, we write $\Gamma\{\Gamma_1\}\cdots\{\Gamma_n\}$ for the context where the k-th hole in $\Gamma\left\{^1\right\}\ldots\left\{^n\right\}$ has been replaced by Γ_k (for $1 \leq k \leq n$). If $\Gamma_k = \varnothing$ the hole is removed.*

A structured sequent *(or simply* sequent*) has the form $\Gamma \Rightarrow F$ where Γ is a structure and F is a formula.*

Example 2. Structures are binary trees, with formulas as leaves and commas as nodes. The structure $!^i A, (B, C)$ represents the tree below left, while $(!^i A, B), C$ represents the tree below right



Definition 3 (SDML). *Let $\mathcal{A}$ be a set of axioms. A (non-associative/commutative)* simply dependent multimodal logical system *(*SDML*) is given by a triple $\Sigma = (I, \preccurlyeq, f)$, where I is a set of indices, $(I, \preccurlyeq)$ is a pre-order, and f is a mapping from I to $2^{\mathcal{A}}$.*

If Σ is a SDML*, then the* logic described by Σ *has the modality $!^i$ for every $i \in I$, with the rules of* FNL *depicted in Fig. 1, together with rules for the axioms $f(i)$ and the interaction axioms $!^j A \rightarrow !^i A$ for every $i, j \in I$ with $i \preccurlyeq j$. Finally, every* SDML *is assumed to be upwardly closed w.r.t. $\preceq$, that is, if $i \preceq j$ then $f(i) \subseteq f(j)$ for all $i, j \in I$.*

[3] Note that the implemented rules in Fig. 2 reflect the left to right direction of such axioms only.

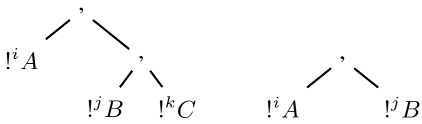
Figure 2 presents the structured system acLL_Σ, for the logic described by the SDML determined by Σ, with $\mathcal{A} = \{\mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}, \mathsf{E}\}$ where, in the subexponential rule for $\mathsf{S} \in \mathcal{A}$, the respective $s \in I$ is such that $\mathsf{S} \in f(s)$ (*e.g.* the subexponential symbol e indicates that $\mathsf{E} \in f(e)$). We will denote by $!^{\mathsf{Ax}}\Delta$ the fact that the structure Δ contains only banged formulas as leaves, each of them assuming the axiom Ax.

As an economic notation, we will write $\uparrow i$ for the *upset* of the index i, *i.e.*, the set $\{j \in I : i \preccurlyeq j\}$. We extend this notation to structures in the following way. Let Γ be a structure containing only banged formulas as leaves. If such formulas admit the multiset partition

$$\{!^j F \in \Gamma : i \preccurlyeq j\} \cup \{!^k F \in \Gamma : i \not\preccurlyeq k \text{ and } \mathsf{W} \in f(k)\}$$

then $\Gamma^{\uparrow i}$ is the structure obtained from Γ by easing the formulas in the second component of the partition (equivalently, the substructure of Γ formed with all and only formulas of the first component of the partition). Otherwise, $\Gamma^{\uparrow i}$ is undefined.

Example 4. Let $\Gamma = (!^i A, (!^j B, !^k C))$ be represented below left, $i \preceq j$ but $i \not\preceq k$, and $\mathsf{W} \in f(k)$. Then $\Gamma^{\uparrow i} = (!^i A, !^j B)$ is depicted below right



Observe that, if $\mathsf{W} \notin f(k)$, then $\Gamma^{\uparrow i}$ cannot be built. In this case, any derivation of $\Gamma \Rightarrow !^i(A \otimes B)$ cannot start with an application of the promotion rule $!^i R$ (similarly to how promotion in ILL cannot be applied in the presence of non-classical contexts). In this case, if A, B are atomic, this sequent would not be provable.

Example 5. The use of subexponentials to deal with associativity can be illustrated by the prefixing sequent $A \to B \Rightarrow (C \to A) \to (C \to B)$: It is not provable for an arbitrary formula C, but if $C = !^a C'$, then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{!^a C' \Rightarrow !^a C'}\ \mathsf{init} \qquad \dfrac{\dfrac{}{A \Rightarrow A}\ \mathsf{init} \quad \dfrac{}{B \Rightarrow B}\ \mathsf{init}}{(A, A \to B) \Rightarrow B}\ \to L}{((!^a C', (!^a C' \to A)), (A \to B)) \Rightarrow B}\ \to L}{(!^a C', ((!^a C' \to A), (A \to B))) \Rightarrow B}\ \mathsf{A1}}{((!^a C' \to A), (A \to B)) \Rightarrow !^a C' \to B}\ \to R}{A \to B \Rightarrow (!^a C' \to A) \to (!^a C' \to B)}\ \to R$$

2.1 Cut-Elimination

When it comes to the proof of cut-elimination for acLL_Σ, the cut reductions for the propositional connectives follow the standard steps for similar systems such as, *e.g.*, Moot and Retoré's system $\mathsf{NL}\diamond$ in [38, Chapter 5.2.2]. The case of structural rules, on the other hand, should be treated with care.

PROPOSITIONAL RULES

$$\frac{\Gamma\{(F,G)\} \Rightarrow H}{\Gamma\{F \otimes G\} \Rightarrow H}\ \otimes L \qquad \frac{\Gamma_1 \Rightarrow F \quad \Gamma_2 \Rightarrow G}{(\Gamma_1,\Gamma_2) \Rightarrow F \otimes G}\ \otimes R \qquad \frac{\Gamma\{F\} \Rightarrow H \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{F \oplus G\} \Rightarrow H}\ \oplus L$$

$$\frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \oplus F_2}\ \oplus R_i \qquad \frac{\Gamma\{F_i\} \Rightarrow G}{\Gamma\{F_1 \,\&\, F_2\} \Rightarrow G}\ \&L_i \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \,\&\, G}\ \&R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(\Delta, F \to G)\} \Rightarrow H}\ \to L \qquad \frac{(F,\Gamma) \Rightarrow G}{\Gamma \Rightarrow F \to G}\ \to R \qquad \frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(G \leftarrow F, \Delta)\} \Rightarrow H}\ \leftarrow L$$

$$\frac{(\Gamma,F) \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F}\ \leftarrow R \qquad \frac{\Gamma\{\,\} \Rightarrow F}{\Gamma\{1\} \Rightarrow F}\ 1L \qquad \frac{}{\Rightarrow 1}\ 1R \qquad \frac{}{\Gamma \Rightarrow \top}\ \top R$$

INITIAL AND CUT RULES

$$\frac{}{F \Rightarrow F}\ \mathsf{init} \qquad \frac{\Delta \Rightarrow F \quad \Gamma\left\{{}^{1}F\right\}\ldots\left\{{}^{n}F\right\} \Rightarrow G}{\Gamma\left\{{}^{1}\Delta\right\}\ldots\left\{{}^{n}\Delta\right\} \Rightarrow G}\ \mathsf{mcut}$$

Fig. 1. Structured system FNL for non-associative, full Lambek calculus.

SUBEXPONENTIAL RULES

$$\frac{\Gamma^{\uparrow i} \Rightarrow F}{\Gamma \Rightarrow\ !^{i}F}\ !^{i}R \qquad \frac{\Gamma\{F\} \Rightarrow G}{\Gamma\left\{!^{i}F\right\} \Rightarrow G}\ \mathsf{der}$$

STRUCTURAL RULES

$$\frac{\Gamma\{((!^{a}\Delta_1,\Delta_2),\Delta_3)\} \Rightarrow G}{\Gamma\{(!^{a}\Delta_1,(\Delta_2,\Delta_3))\} \Rightarrow G}\ \mathsf{A1} \qquad \frac{\Gamma\{(\Delta_1,(\Delta_2,!^{a}\Delta_3))\} \Rightarrow G}{\Gamma\{((\Delta_1,\Delta_2),!^{a}\Delta_3)\} \Rightarrow G}\ \mathsf{A2} \qquad \frac{\Gamma\{(\Delta_2,!^{e}\Delta_1)\} \Rightarrow G}{\Gamma\{(!^{e}\Delta_1,\Delta_2)\} \Rightarrow G}\ \mathsf{E1}$$

$$\frac{\Gamma\{(!^{e}\Delta_2,\Delta_1)\} \Rightarrow G}{\Gamma\{(\Delta_1,!^{e}\Delta_2)\} \Rightarrow G}\ \mathsf{E2} \qquad \frac{\Gamma\{\,\} \Rightarrow G}{\Gamma\{!^{w}\Delta\} \Rightarrow G}\ \mathsf{W} \qquad \frac{\Gamma\left\{{}^{1}!^{c}\Delta\right\}\ldots\left\{{}^{n}!^{c}\Delta\right\} \Rightarrow G}{\Gamma\left\{{}^{1}\right\}\ldots\left\{{}^{k}!^{c}\Delta\right\}\ldots\left\{{}^{n}\right\} \Rightarrow G}\ \mathsf{C}$$

Fig. 2. Structured system acLL_Σ for the logic described by Σ.

Theorem 6. *If the sequent $\Gamma \Rightarrow F$ is provable in acLL_Σ, then it has a proof with no instances of the rule* mcut.

Proof. The most representative cases of cut reductions involving subexponentials are detailed next. In order to simplify the notation, when possible, the mcut rule is presented in its simple form, with an 1-ary context.

Case $!^a$: Suppose that

$$\cfrac{\cfrac{\overset{\pi_1}{\Delta_1^{\uparrow a} \Rightarrow F}}{\Delta_1 \Rightarrow\ !^{a}F}\ !^{a}R \qquad \cfrac{\overset{\pi_2}{\Gamma\{((!^{a}F,\Delta_2),\Delta_3)\} \Rightarrow G}}{\Gamma\{(!^{a}F,(\Delta_2,\Delta_3))\} \Rightarrow G}\ \mathsf{A1}}{\Gamma\{(\Delta_1,(\Delta_2,\Delta_3))\} \Rightarrow G}\ \mathsf{mcut}$$

Since axioms are upwardly closed w.r.t. $\preceq$, it must be the case that $\Delta_1^{\uparrow a}$ contains only formulas marked with subexponentials allowing associativity. All the other formulas in Δ_1 can be weakened; this is guaranteed by the application of the rule $!^a R$ in π_1. Hence the derivation above reduces to

$$\dfrac{\dfrac{\overset{\pi_1}{\Delta_1^{\uparrow a} \Rightarrow F}}{\Delta_1^{\uparrow a} \Rightarrow !^a F}\ !^a R \quad \overset{\pi_2}{\Gamma\{(!^a F, \Delta_2), \Delta_3)\} \Rightarrow G}}{\dfrac{\Gamma\left\{((\Delta_1^{\uparrow a}, \Delta_2), \Delta_3)\right\} \Rightarrow G}{\dfrac{\Gamma\left\{(\Delta_1^{\uparrow a}, (\Delta_2, \Delta_3))\right\} \Rightarrow G}{\Gamma\{(\Delta_1, (\Delta_2, \Delta_3))\} \Rightarrow G}\ \mathsf{W}}\ \mathsf{A1}}\ \mathsf{mcut}$$

Case $!^c$: Suppose that

$$\dfrac{\dfrac{\overset{\pi_1}{\Delta^{\uparrow c} \Rightarrow F}}{\Delta \Rightarrow !^c F}\ !^c R \quad \dfrac{\overset{\pi_2}{\Gamma\{!^c F\} \ldots \{!^c F\} \ldots \{!^c F\} \Rightarrow G}}{\Gamma\{\,\} \ldots \{!^c F\} \ldots \{\,\} \Rightarrow G}\ \mathsf{C}}{\Gamma\{\,\} \ldots \{\Delta\} \ldots \{\,\} \Rightarrow G}\ \mathsf{mcut}$$

Since $\Delta^{\uparrow c}$ contains only formulas marked with subexponentials allowing contraction, the derivation above reduces to

$$\dfrac{\dfrac{\overset{\pi_1}{\Delta^{\uparrow c} \Rightarrow F}}{\Delta^{\uparrow c} \Rightarrow !^c F}\ !^c R \quad \overset{\pi_2}{\Gamma\{!^c F\} \ldots \{!^c F\} \ldots \{!^c F\} \Rightarrow G}}{\dfrac{\Gamma\{\Delta^{\uparrow c}\} \ldots \{\Delta^{\uparrow c}\} \ldots \{\Delta^{\uparrow c}\} \Rightarrow G}{\dfrac{\Gamma\{\,\} \ldots \{\Delta^{\uparrow c}\} \ldots \{\,\} \Rightarrow G}{\Gamma\{\,\} \ldots \{\Delta\} \ldots \{\,\} \Rightarrow G}\ \mathsf{W}}\ \mathsf{C}}\ \mathsf{mcut}$$

Observe that here, as usual, the multicut rule is needed in order to reduce the cut complexity.

Case $!^i R$: Suppose that

$$\dfrac{\dfrac{\overset{\pi_1}{\Delta^{\uparrow i} \Rightarrow F}}{\Delta \Rightarrow !^i F}\ !^i R \quad \dfrac{\overset{\pi_2}{\left(\Gamma\{!^i F\}\right)^{\uparrow j} \Rightarrow G}}{\Gamma\{!^i F\} \Rightarrow !^j G}\ !^j R}{\Gamma\{\Delta\} \Rightarrow !^j G}\ \mathsf{mcut}$$

If $j \not\preceq i$, then it should be the case that $\mathsf{W} \in f(i)$ and $\left(\Gamma\{!^i F\}\right)^{\uparrow j} = \Gamma\{\,\}^{\uparrow j}$, since $!^i F$ will be weakened in the application of rule $!^j R$. Hence, all formulas in Δ can be weakened as well and the reduction is

$$\dfrac{\dfrac{\overset{\pi_2}{\Gamma\{\,\}^{\uparrow j} \Rightarrow G}}{\Gamma\{\,\} \Rightarrow !^j G}\ !^j R}{\Gamma\{\Delta\} \Rightarrow !^j G}\ \mathsf{W}$$

On the other hand, if $j \preceq i$, by transitivity all the formulas in $\Delta^{\uparrow i}$ also have this property (implying that $\Delta^{\uparrow i}$ is a substructure of $\Delta^{\uparrow j}$), and the rest of formulas of Δ can be weakened. Hence the derivation above reduces to

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Delta^{\uparrow i} \Rightarrow F}}{\Delta^{\uparrow j} \Rightarrow !^i F}\ !^i R \qquad \overset{\pi_2}{\left(\Gamma\{!^i F\}\right)^{\uparrow j} \Rightarrow G}}{(\Gamma\{\Delta\})^{\uparrow j} \Rightarrow G}}{\Gamma\{\Delta\} \Rightarrow !^j G}\ !^j R$$

The other cases for subexponentials are similar or simpler. □

The next examples illustrate what we mean by acLL_Σ being a "conservative extension" of subsystems and variants. Indeed, although we remove structural properties of the core LL, subexponentials allow them to be added back, either locally or globally.

Example 7 (Structural variants of iMALL*).* Adding combinations of contraction C and / or weakening W for *arbitrary formulas* to additive-multiplicative intuitionistic linear logic (iMALL) yields, respectively, propositional intuitionistic logic $\mathsf{ILP} = \mathsf{iMALL} + \{\mathsf{C}, \mathsf{W}\}$, and the intuitionistic versions of affine linear logic $\mathsf{aLL} = \mathsf{iMALL} + \mathsf{W}$ and relevant logic $\mathsf{R} = \mathsf{iMALL} + \mathsf{C}$. For the sake of presentation we overload the notation and use the connectives of linear logic also for these logics. In order to embed the logics above into acLL_Σ, let $\alpha \in \{\mathsf{ILP}, \mathsf{aLL}, \mathsf{R}\}$ and consider modalities $!^\alpha$ with $f(\alpha) = \{\mathsf{E}, \mathsf{A1}, \mathsf{A2}\} \cup \mathcal{A}$ where $\mathcal{A} \subseteq \{\mathsf{C}, \mathsf{W}\}$ is the set of axioms whose corresponding rules are in α. The translation τ_α prefixes *every subformula* with the modality $!^\alpha$. For $\mathcal{L} \in \{\mathsf{ILP}, \mathsf{aLL}, \mathsf{R}\}$ it is then straightforward to show that a structured sequent S is cut-free derivable in $\mathcal{L}$ iff its translation $\tau_\alpha(S)$ is cut-free derivable in the logic described by $(\{\alpha\}, \preccurlyeq, f)$ with $\preccurlyeq$ the obvious relation, and f as given above.

Example 8 (Structural variants of FNL*).* Following the same script as above and starting from FNL:

- considering $f(\alpha) = \mathcal{A} \subseteq \{\mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$;
  - If $\mathcal{A} = \{\mathsf{A1}, \mathsf{A2}\}$, then we obtain the system FL;
  - If $\mathcal{A} = \{\mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$ then the resulting system corresponds to iMALL.
  - Adding C, W as options to $\mathcal{A}$ will result the affine/relevant versions of the systems above.
- in a pre-order $(I, \preceq)$, if $f(i) = \{\mathsf{A1}, \mathsf{A2}\} \cup \mathcal{A}_i$ where $\mathcal{A}_i \subseteq \{\mathsf{E}, \mathsf{C}, \mathsf{W}\}$ for each $i \in I$, then the resulting system corresponds to SMALC_Σ in [21] (that is, the extension of FL with subexponentials).

2.2 An Example in Linguistics

Since its inception, Lambek calculus [29] has been applied to the modeling of natural language syntax by means of categorial grammars. In a categorial grammar, each word is assigned one or several Lambek formulas, which serve as syntactic categories. For a simple example, *John* and *Mary* are assigned np ("noun phrase") and *loves* gets

$(np \to s) \leftarrow np$. Here s stands for "sentence", and *loves* is a transitive verb, which lacks noun phrases on both sides to become a sentence. Grammatical validity of *"John loves Mary"* is supported by derivability of the sequent $np, (np \to s) \leftarrow np, np \Rightarrow s$. Notice that this derivability keeps valid also in the non-associative setting, if the correct nested structure is provided: $(np, ((np \to s) \leftarrow np, np)) \Rightarrow s$.

The original Lambek calculus L is associative. In some cases, however, associativity leads to over-generation, *i.e.*, validation of grammatically incorrect sentences. Lambek himself realized this and proposed the non-associative calculus NL in [30]. We will illustrate this issue with the example given in [38, Sect. 4.2.2]. The syntactic category assignment is as follows (where n stands for "noun"):

Words	Types
the	$np \leftarrow n$
Hulk	n
is	$(np \to s) \leftarrow (n \leftarrow n)$
green, incredible	$n \leftarrow n$

With this assignment, sentences *"The Hulk is green"* and *"The Hulk is incredible"* are correctly marked as valid, by deriving the sequent

$$(np \leftarrow n, n), ((np \to s) \leftarrow (n \leftarrow n), n \leftarrow n) \Rightarrow s$$

However, in the associative setting the sequent for the phrase *"The Hulk is green incredible,"* which is grammatically incorrect, also becomes derivable:

$$np \leftarrow n, n, (np \to s) \leftarrow (n \leftarrow n), n \leftarrow n, n \leftarrow n \Rightarrow s,$$

essentially due to derivability of $n \leftarrow n, n \leftarrow n \Rightarrow n \leftarrow n$.

In other situations, however, associativity is useful. Standard examples include handling of dependent clauses, *e.g.*, *"the girl whom John loves,"* which is validated as a noun phrase by the following derivable sequent:

$$np \leftarrow n, n, (n \to n) \leftarrow (s \leftarrow np), np, (np \to s) \leftarrow np \Rightarrow np$$

Here $(n \to n) \leftarrow (s \leftarrow np)$ is the syntactic category for *who*.

Our subexponential extension of NL, however, handles this case using local associativity instead of the global one. Namely, the category for *whom* now becomes $(n \to n) \leftarrow (s \leftarrow !^a np)$, where $!^a$ is a subexponential which allows the A2 rule, and the following sequent is happily derivable:

$$np \leftarrow n, (n, ((n \to n) \leftarrow (s \leftarrow !^a np), (np, (np \to s) \leftarrow np))) \Rightarrow np$$

The necessity of this more fine-grained control of associativity, instead of a global associativity rule, is seen via a combination of these examples. Namely, we talk about sentences like *"The superhero whom Hawkeye killed was incredible"* and *"... was green"*. With $!^a$, each of them is handled in the same way as the previous examples:

$$(np \leftarrow n, (n, ((n \to n) \leftarrow (s \leftarrow !^a np), (np, (np \to s) \leftarrow np)))),$$
$$((np \to s) \leftarrow (n \leftarrow n), n \leftarrow n) \Rightarrow s.$$

On one hand, without $!^a$ this sequent cannot be derived in the non-associative system. On the other hand, if we make the system globally associative, it would validate incorrect sentences like *"The superhero whom Hawkeye killed was green incredible."*

3 Some Undecidability Results

Non-associativity makes a significant difference in decidability and complexity matters. For example, while L is NP-complete [47], NL is decidable in polynomial time [1,14].

For our system acLL_Σ, its decidability or undecidability depends on its signature Σ. In fact, we have a family of different systems acLL_Σ, with Σ as a parameter. Recall that the subexponential signature Σ controls not just the number of subexponentials and the preorder among them. More importantly, it dictates, for each subexponential, which structural rules this subexponential licenses. If for every $i \in I$ we have $\mathsf{C} \notin f(s)$, that is, no subexponential allows contraction, then acLL_Σ is clearly decidable, since the cut-free proof search space is finite. Therefore, for undecidability it is necessary to have at least one subexponential which allows contraction.

For a non-associative system with only one fully-powered exponential modality s (that is, $f(s) = \{\mathsf{E}, \mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}\}$), undecidability was proven in a preprint by Tanaka [51], based on Chvalovský's [11] result on undecidability of the finitary consequence relation in FNL.

In this section, we prove two undecidability results. The first one is a refinement of Tanaka's result: We establish undecidability with at least one subexponential which allows contraction and weakening (commutativity/associativity are optional), in a subsystem containing only the additive connective $\oplus$ and the multiplicatives $\otimes$ and $\to$.

The second undecidability result is for the minimalistic, purely multiplicative fragment, which includes only $\to$ (not even $\otimes$). As a trade-off, however, it requires two subexponentials: the "main" one, which allows contraction, exchange, and associativity (weakening is optional), and an "auxiliary" one, which allows only associativity.

It should be noted that this undecidability result is orthogonal to Tanaka's [51], and the proof technique is essentially different. Indeed, Chvalovský's undecidability theorem does not hold for the non-associative Lambek calculus without additives, where the consequence relation is decidable [7].

Finally, we observe that *if* the intersection of these systems is decidable (which is still an open question), then our two undecidability results are *incomparable:* we have two undecidable fragments of acLL_Σ, but their common part, which includes only divisions and one exponential, would be decidable.

3.1 Undecidability with Additives and One Subexponential

We are going to derive the next theorem from undecidability of the finitary consequence relation in FNL [11]. Recall that FNL is, in fact, the fragment of acLL_Σ without subexponentials (that is, with an empty I).

Theorem 9. *If there exists such $s \in I$ that $f(s) \supseteq \{\mathsf{C}, \mathsf{W}\}$, then the derivability problem in acLL_Σ is undecidable. Moreover, this holds for the fragment with only $\otimes$, $\to$, $\oplus$, $!^s$.*

In fact, using C and W, one can also derive A1, A2, E1, and E2. Therefore, if $f(s) \supseteq \{\mathsf{C}, \mathsf{W}\}$, then $!^s$ is actually a full-power exponential modality. (In the proof of Theorem 9 below, we use only W and C rules, in order to avoid confusion.) However, Theorem 9 does not directly follow from undecidability of propositional linear logic [32], because here the basic system is non-associative and non-commutative, while linear logic is both associative and commutative. Thus, we need a different encoding for undecidability.

Let $\mathbf{\Phi}$ be a finite set of FNL sequents. By FNL($\mathbf{\Phi}$) let us denote FNL extended by adding sequents from $\mathbf{\Phi}$ as additional (non-logical) axioms. In general, FNL($\mathbf{\Phi}$) does not enjoy cut-elimination, so mcut is kept as a rule of inference in FNL($\mathbf{\Phi}$). A sequent $\Gamma \Rightarrow F$ is called *a consequence of* $\mathbf{\Phi}$ if this sequent is derivable in FNL($\mathbf{\Phi}$).

Theorem 10 (Chvalovský [11]). *The consequence relation in* FNL *is undecidable, that is, there exists no algorithm which, given* $\mathbf{\Phi}$ *and* $\Gamma \Rightarrow F$*, determines whether* $\Gamma \Rightarrow F$ *is a consequence of* $\mathbf{\Phi}$*. Moreover, undecidability keeps valid when* $\mathbf{\Phi}$ *and* $\Gamma \Rightarrow F$ *are built from variables using only* $\otimes$ *and* $\oplus$*.*

Now, in order to prove Theorem 9, we internalize $\mathbf{\Phi}$ into the sequent using $!^s$, assuming $f(s) \supseteq \{\mathsf{C}, \mathsf{W}\}$.

First we notice that we may suppose, without loss of generality, that all sequents in $\mathbf{\Phi}$ are of the form $\Rightarrow A$, that is, have empty antecedents. Namely, each sequent of the form $\Pi \Rightarrow B$ can be replaced by $\Rightarrow (\bigotimes \Pi) \to B$, where $\bigotimes \Pi$ is obtained from Π by replacing each comma with $\otimes$. Indeed, these sequents are derivable from one another: from $\Pi \Rightarrow B$ to $\Rightarrow (\bigotimes \Pi) \to B$ we apply a sequence of $\otimes L$ followed by $\to R$, and for the other direction we apply a series of cuts, first with $(\bigotimes \Pi, (\bigotimes \Pi) \to B) \Rightarrow B$, and then with $(F, G) \Rightarrow F \otimes G$ several times, for the corresponding subformulas of $\bigotimes \Pi$. The following embedding lemma ("modalized deduction theorem") holds.

Lemma 11. *The sequent* $\Gamma \Rightarrow F$ *is a consequence of* $\mathbf{\Phi} = \{\Rightarrow A_1, \ldots, \Rightarrow A_n\}$ *if and only if the sequent* $\big((\ldots((!^s A_1, !^s A_2), !^s A_3), \ldots, !^s A_n), \Gamma\big) \Rightarrow F$ *is derivable in* acLL_Σ.

Proof. Let us denote $(\ldots((!^s A_1, !^s A_2), !^s A_3), \ldots, !^s A_n)$ by $!\Phi$. Notice that C and W can be applied to $!\Phi$ as a whole; this is easily proven by induction on n.

For the "only if" direction let us take the derivation of $\Gamma \Rightarrow F$ in FNL($\mathbf{\Phi}$) (with cuts) and replace each sequent of the form $\Delta \Rightarrow G$ in it with $(!\Phi, \Delta) \Rightarrow G$, and each sequent of the form $\Rightarrow G$ with $!\Phi \Rightarrow G$. The translations of non-logical axioms from $\mathbf{\Phi}$ are derived as follows:

$$\dfrac{\dfrac{\dfrac{}{A_i \Rightarrow A_i}\ \mathsf{init}}{!^s A_i \Rightarrow A_i}\ \mathsf{der}}{!\Phi \Rightarrow A_i}\ \mathsf{W},\ n-1 \text{ times}$$

Translations of axioms init and $1R$ are derived from the corresponding original axioms by W, n times; $\top R$ remains valid.

Rules $\otimes L$, $\oplus L$, $\oplus R_i$, $\& L_i$, $\& R$, and $1L$ remain valid. For $\to L$, $\leftarrow L$, and mcut we contract $!\Phi$ as a whole:

$$\dfrac{\dfrac{(!\Phi, \Delta) \Rightarrow F \quad (!\Phi, \Gamma\{G\}) \Rightarrow H}{(!\Phi, \Gamma\{((!\Phi, \Delta), F \to G)\}) \Rightarrow H} \to L}{(!\Phi, \Gamma\{(\Delta, F \to G)\}) \Rightarrow H} \mathsf{C} \qquad \dfrac{\dfrac{(!\Phi, \Delta) \Rightarrow F \quad (!\Phi, \Gamma\{F\} \ldots \{F\}) \Rightarrow C}{(!\Phi, \Gamma\{(!\Phi, \Delta)\} \ldots \{(!\Phi, \Delta)\}) \Rightarrow C} \mathsf{mcut}}{(!\Phi, \Gamma\{\Delta\} \ldots \{\Delta\}) \Rightarrow C} \mathsf{C}$$

For $\otimes R$, $\to R$, and $\leftarrow R$, we combine contraction and weakening:

$$\dfrac{\dfrac{\dfrac{(!\Phi, \Gamma_1) \Rightarrow F \quad (!\Phi, \Gamma_2) \Rightarrow G}{((!\Phi, \Gamma_1), (!\Phi, \Gamma_2)) \Rightarrow F \otimes G} \otimes R}{(!\Phi, ((!\Phi, \Gamma_1), (!\Phi, \Gamma_2))) \Rightarrow F \otimes G} \mathsf{W}}{(!\Phi, (\Gamma_1, \Gamma_2)) \Rightarrow F \otimes G} \mathsf{C} \qquad \dfrac{\dfrac{\dfrac{(!\Phi, (F, \Gamma)) \Rightarrow G}{(!\Phi, (F, (!\Phi, \Gamma)) \Rightarrow G} \mathsf{W}}{(F, (!\Phi, \Gamma)) \Rightarrow G} \mathsf{C}}{(!\Phi, \Gamma) \Rightarrow F \to G} \to R$$

Notice that our original derivation was in $\mathsf{FNL}(\mathbf{\Phi})$, so it does not include rules operating subexponentials.

For the "if" direction we take a cut-free proof of $(!\Phi, \Gamma) \Rightarrow F$ in acLL_Σ and erase all formulas which include the subexponential. In the resulting derivation tree all rules and axioms, except those which operate $!^s$, remain valid. Structural rules for $!^s$ trivialize (since the !-formula was erased). The $!^s R$ rule could not have been used, since we do not have positive occurrences of $!^s F$, and our proof is cut-free.

Finally, der translates into

$$\dfrac{\Gamma\{A_i\} \Rightarrow G}{\Gamma\{\} \Rightarrow G}$$

This is modeled by cut with one of the sequents from $\mathbf{\Phi}$:

$$\dfrac{\Rightarrow A_i \quad \Gamma\{A_i\} \Rightarrow G}{\Gamma\{\} \Rightarrow G} \mathsf{mcut}$$

Thus, we get a correct derivation in $\mathsf{FNL}(\mathbf{\Phi})$. □

Theorem 10 and Lemma 11 immediately yield Theorem 9.

3.2 Undecidability Without Additives and with Two Subexponentials

Theorem 12. *If there are $a, c \in I$ such that $f(a) = \{\mathsf{A1}, \mathsf{A2}\}$ and $f(c) \supseteq \{\mathsf{C}, \mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$, then the derivability problem in acLL_Σ is undecidable. Moreover, this holds for the fragment with only $\to$, $!^a$, and $!^c$.*

Remember from Example 8 that SMALC_Σ [21] denotes the extension of FL with subexponentials. The undecidability theorem above is proved by encoding the one-division fragment of SMALC_Σ containing one exponential c such that $f(c) \supseteq \{\mathsf{C}, \mathsf{E}\}$. It turns out that that such a system is undecidable.

Theorem 13 (Kanovich et al. [22,23]). *If there exists such $c \in I$ that $f(c) \supseteq \{\mathsf{C}, \mathsf{E}\}$, then the derivability problem in SMALC_Σ is undecidable. Moreover, this holds for the fragment with only $\to$ and $!^c$.*

Observe that SMALC_Σ can be obtained from acLL_Σ by adding "global" associativity rules:

$$\frac{\Gamma\{((\Delta_1, \Delta_2), \Delta_3)\} \Rightarrow G}{\Gamma\{(\Delta_1, (\Delta_2, \Delta_3))\} \Rightarrow G} \qquad \frac{\Gamma\{(\Delta_1, (\Delta_2, \Delta_3))\} \Rightarrow G}{\Gamma\{((\Delta_1, \Delta_2), \Delta_3)\} \Rightarrow G}$$

The usual formulation of SMALC_Σ, of course, uses sequences of formulas instead of nested structures as antecedents. The alternative formulation, however, would be more convenient for us now. It will be also convenient for us to regard all subexponentials in SMALC_Σ to be associative, that is, $f(s) \supseteq \{\mathsf{A1}, \mathsf{A2}\}$ for each $s \in I$.

In order to embed SMALC_Σ into acLL_Σ, we define two translations, $A^{!-}$ and $A^{!+}$, by mutual recursion:

$z^{!-} = !^a z$	$z^{!+} = z$	where z is a variable,1, or $\top$
$(A \to B)^{!-} = !^a(A^{!+} \to B^{!-})$	$(A \to B)^{!+} = A^{!-} \to B^{!+}$	
$(B \leftarrow A)^{!-} = !^a(B^{!-} \leftarrow A^{!+})$	$(B \leftarrow A)^{!+} = B^{!+} \leftarrow A^{!-}$	
$(A \circledast B)^{!-} = !^a(A^{!-} \circledast B^{!-})$	$(A \circledast B)^{!+} = A^{!+} \circledast B^{!+}$	where $\circledast \in \{\otimes, \oplus, \&\}$
$(!^s A)^{!-} = !^s(A^{!-})$	$(!^s A)^{!+} = !^s(A^{!+})$	

Informally, our translation adds a $!^a$ over any formula (not only over atoms) of negative polarity, unless this formula was already marked with a $!^s$. Thus, all formulae in antecedents would begin with either the new subexponential $!^a$ or one of the old subexponentials $!^s$, and all these subexponentials allow associativity rules A1 and A2.

Lemma 14. *A sequent $A_1, \ldots, A_n \Rightarrow B$ is derivable in* SMALC_Σ *if and only if its translation $(\ldots(A_1^{!-}, A_2^{!-}), \ldots, A_n^{!-}) \Rightarrow B^{!+}$ is derivable in* acLL_Σ.

Proof. For the "only if" part, let us first note that each formula $A_i^{!-}$ is of the form $!^s F$ and $\mathsf{A1}, \mathsf{A2} \in f(s)$. Indeed, either s is an "old" subexponential label (for which we added $\mathsf{A1}, \mathsf{A2}$) or $s = a$. Thus brackets can be freely rearranged in the antecedent.

Now we take a cut-free proof of $A_1, \ldots, A_n \Rightarrow B$ in SMALC_Σ and replace each sequent in it with its translation. Right rules for connectives other than subexponentials, i.e., $\otimes R$, $\oplus R_i$, $\& R$, $\to R$, and $\leftarrow R$, remain valid as they are, up to rearranging brackets in antecedents. For $!^i R$, we notice that the translation of a formula of the form $!^j F$, where $j \preceq i$, is also a formula of the form $!^j F'$. Thus, this rule also remains valid. The same holds for the dereliction rule der, because $(!^i F)^{!-}$ is exactly $!^i(F^{!-})$. Finally, the "old" structural rules (exchange, contraction, weakening) also remain valid (up to rearranging of brackets), since $!^i F$ gets translated into $!^i(F^{!-})$, which enjoys the same structural rules.

For the other left rules, we need to derelict $!^a$ first, and then perform the corresponding rule application. Rearrangement of brackets, if needed, is performed below dereliction or above the application of the rule in question.

The "if" part is easier. Given a derivation of $(\ldots(A_1^{!-}, A_2^{!-}), \ldots, A_n^{!-}) \Rightarrow B^{!+}$ in acLL_Σ, we erase $!^a$ everywhere, and consider it as a derivation in SMALC_Σ. Associativity rules for the erased $!^a$ (which are the only structural rules for this subexponential) keep valid, because now associativity is global. Dereliction and right introduction for $!^a$ trivialize. All other rules, which do not operate $!^a$, remain as they are. Thus, we get a derivation of $A_1, \ldots, A_n \Rightarrow B$ in SMALC_Σ, since erasing $!^a$ makes our translations just identical. □

4 Related Work and Conclusion

In this paper, we have presented acLL_Σ, a sequent-based system for non-associative, non-commutative linear logic with subexponentials. Starting form FNL, we modularly and uniformly added rules for exchange, associativity, weakening and contraction, which can be applied with the subexponentials having with the respective features. This allows for the application of structural rules locally, and it conservatively extends well known systems in the literature, continuing the path of controlling structural properties started by Girard himself [16].

Another approach to combining associative and non-associative behavior in Lambek-style grammars is the framework of *the Lambek calculus with brackets* by Morrill [39,40] and Moortgat [34]. The bracket approach is dual to ours: there the base system is associative, and brackets, which are controlled by bracket modalities, introduce local non-associativity. Both the associative Lambek calculus and the non-associative Lambek calculus can be embedded into the Lambek calculus with brackets: the former is just by design of the system and the latter was shown by Kurtonina [26] by constructing a translation.

From the point of view of generative power, however, the (associative) Lambek calculus with brackets is weaker than the non-associative system with subexponentials, which is presented in this paper. Namely, as shown by Kanazawa [19], grammars based on the Lambek calculus with brackets can generate only context-free languages. In contrast, grammars based on our system with subexponentials go beyond context-free languages, even when no subexponential allows contraction (subexponentials allowing contraction may lead to undecidability, as shown in the last section).

As a quick example, let us consider a subexponential $!^{ae}$ which allows both associativity (A1 and A2) and exchange (E). If we put this subexponential over any (sub)formula, the system becomes associative and commutative. Using this system, one can describe the non context-free language MIX_3, which contains all non-empty words over $\{a, b, c\}$, in which the numbers of a, b, and c are equal. Indeed, MIX_3 is the permutation closure of the language $\{(abc)^n \mid n \geq 1\}$. The latter is regular, therefore context-free, and therefore definable by a Lambek grammar. The ability of our system to go beyond context-free languages is important from the point of view of applications, since there are known linguistic phenomena which are essentially non-context-free [49].

Regarding decidability, let us compare our results with the more well-known associative non-commutative and associative commutative cases.

In the associative and commutative case the situation is as follows. In the presence of additives, the system is known to be undecidable with one exponential modality [32]. Without additives, we get MELL, the (un)decidability of which is a well-known open problem [50]. However, with two subexponentials MELL again becomes undecidable [9]. Thus, we have the same trade-off as in our non-associative non-commutative case: for undecidability one needs either additives, or two subexponentials.

Our results help to shed some light in the (un)decidability problem for the spectrum of logical systems surrounding MELL/FNL, allowing for a fine-grained analysis of the problem, specially the trade-offs on connectives and subexponentials for guaranteeing (un)decidability.

There is a lot to be done from now on. First of all, we would like to analyze better the minimalist fragment of acLL_Σ containing only implication and one fully-powered subexponential, as it seems to be crucial for understanding the lower bound of undecidability (or the upper bound of decidability). Second, one should definitely explore more the use of acLL_Σ in modeling natural language syntax. The examples in Sect. 2.2 show how to locally combine sentences with different grammatical characteristics, and the MIX_3 example above illustrates how that can be of importance. That is, it would be interesting to have a formal study about acLL_Σ and categorial grammars. Third, we plan to investigate the connections between our work and Adjoint logic [48] as well as with Display calculus [5,12]. Finally, we intend to study proof-theoretic properties of acLL_Σ, such as normalization of proofs (*e.g.* via focusing) and interpolation.

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