Creating quantum correlations in generalized entanglement swapping

Pratapaditya Bej,[1,*] Arkaprabha Ghosal,[1,†] Arup Roy,[2,‡] Shiladitya Mal,[3,4,5,§] and Debarshi Das[6,7,‖]

[1]*Department of Physics and Center for Astroparticle Physics and Space Science, Bose Institute, EN Block, Sector V, Saltlake, Kolkata 700091, India*

[2]*Department of Physics, A B N Seal College, Cooch Behar, West Bengal 736 101, India*

[3]*Harish-Chandra Research Institute and HBNI, Chhatnag Road, Jhunsi, Allahabad 211019, India*

[4]*Department of Physics and Center for Quantum Frontiers of Research and Technology (QFort), National Cheng Kung University, Tainan 701, Taiwan*

[5]*Physics Division, National Center for Theoretical Sciences, Taipei 10617, Taiwan*

[6]*S. N. Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Kolkata 700 106, India*

[7]*Department of Physics and Astronomy, University College London, Gower Street, WC1E 6BT London, England, United Kingdom*

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We study how different types of quantum correlations can be established as the consequence of a generalized entanglement swapping protocol where, starting from two Bell pairs (1,2) and (3,4), a general quantum measurement [denoted by a positive operator-valued measure (POVM)] is performed on the pair (2,3), which results in creating quantum correlation in (1,4) shared between two spatially separated observers. Contingent upon using different kinds of POVMs, we show generation or destruction of different quantum correlations in the pairs (1,4), (1,2), and (3,4). This thus reflects nontrivial transfer of quantum correlations from the pairs (1,2) and (3,4) to the pair (1,4). As an offshoot, this paper provides an operational tool to generate different types of single parameter families of quantum correlated states [for example, entangled but not Einstein-Podolsky-Rosen (EPR) steerable, or EPR steerable but not Bell nonlocal, or Bell nonlocal] by choosing different quantum measurements in the basic entanglement swapping setup. We further extend our paper by taking mixed initial states shared by the pairs (1,2) and (3,4). Finally, we study network nonlocality in our scenario. Here, we find the appropriate POVM measurement for which the generated correlation demonstrates or does not demonstrate network nonlocality for the whole range of the measurement parameter.

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I. INTRODUCTION

Entanglement swapping is a protocol by which quantum systems that have never interacted in the past can become entangled [1]. This protocol can be described as follows: Alice and Bob share the pair (1,2) of two qubits in the Bell state. Bob and Charlie share another pair (3,4) of two qubits in the same Bell state. Bob performs projective measurement in the Bell basis on the pair (2,3) and communicates the outcome to Alice and Charlie. Alice and Charlie thus end up with a Bell pair (1,4). This protocol can be considered as the quantum teleportation [2] of a qubit that is maximally entangled with another qubit.

In the above standard entanglement swapping protocol (using two initial Bell states, measurements in the Bell basis, and classical communication), entanglement is completely destroyed in the pairs (1,2) and (3,4) and a maximally entangled state is created in the pair (1,4). In other words, quantum correlation (in the form of entanglement) is transferred completely from the pairs (1,2) and (3,4) to the pair (1,4). The entanglement swapping protocol can be generalized in various ways—by modifying the initial states, or by modifying the measurement performed by Bob, or by extending the number of parties [3,4]. These types of generalized entanglement swapping protocol have been studied in different contexts ranging from quantum networks [5,6] to quantum nonlocality [7,8] to information loss or gain [9,10]. Hence, the natural question that arises in this context is how to design different types of quantum correlation transfer from (1,2) and (3,4) to (1,4) by generalizing the standard entanglement swapping protocol, i.e., either by modifying the initial states, or by modifying the measurements, or both.

In the present paper, we consider a generalized entanglement swapping protocol, where each of the two pairs—Alice-Bob and Bob-Charlie—initially shares a two-qubit Bell state and Bob performs a general quantum measurement. A general quantum measurement is represented by a positive operator-valued measure (POVM) [11,12]. POVMs have been shown to possess operational advantages in many tasks over projective measurements, for example, in the context of distinguishing nonorthogonal quantum states [13], demonstrating hidden nonlocality [14,15], probing temporal correlations [16], sequential detection of quantum correlations by multiple

*pratap6906@gmail.com
†a.ghosal1993@gmail.com
‡arup145.roy@gmail.com
§shiladitya.27@gmail.com
‖dasdebarshi90@gmail.com

observers [17–21], recycling resources in information theoretic tasks [22–25], and so on. Here, our aim is to investigate how different types of quantum correlations are generated in the pair (1,4) and are deteriorated in the pairs (1,2) and (3,4) depending on the choice of the POVM by Bob.

In particular, we focus on the creation or destruction of the following types of quantum correlations: entanglement [26], Einstein-Podolsky-Rosen (EPR) steering [27], and Bell nonlocality [28]. In general, these three types of quantum correlations are inequivalent and there exists a hierarchy between these—Bell-nonlocal states form a strict subset of steerable states which again form a strict subset of entangled states [29,30]. All these three types of quantum correlations are the building blocks for different types of information theoretic and communication tasks. For example, entanglement is useful as resource in quantum teleportation [2], dense coding [31], and so on. Bell nonlocality is the resource for a number of device-independent tasks, e.g, quantum key distribution [32], randomness certification [33], randomness amplification [34], and Bayesian game theory [35]. EPR steering is the resource in one-sided device-independent quantum key distribution [36], secure quantum teleportation [37], quantum communication [37], and one-sided device-independent randomness generation [38]. Against the above backdrop, constructions of these quantum correlations are one of the basic requirements of modern quantum technology. The present paper provides some basic operational tools in the generalized entanglement swapping setup for generating these correlations in the pair (1,4) shared between two spatially separated observers that have never interacted before. Specifically, we report the generation of the following types of two-qubit states in (1,4) by choosing appropriate POVMs: (i) a single parameter class of mixed states that is Bell nonlocal for the whole range of the state parameter, (ii) a single parameter class of mixed states that is steerable but not Bell nonlocal for the whole range of the state parameter, (iii) a single parameter class of mixed states that is entangled but not EPR steerable for the whole range of the state parameter, and finally (iv) a single parameter class of mixed states that gradually shows all the above-mentioned quantum correlations as the state parameter is varied. We further extend our paper by probing the above aspects of creating or destroying quantum correlations in the pair (1,4) when mixed states are initially shared by the pairs (1,2) and (3,4) and appropriate POVMs are performed on (2,3). One important point to stress here is that while probing standard Bell nonlocality or EPR steering of the state shared by (1,4), we consider that the sources producing the two initial entangled states are not necessarily independent.

It is to be noted that Bell nonlocality has been studied in the context of the entanglement swapping scenario earlier [8]. However, contrary to the usual Bell scenario, the entanglement swapping scenario involves uncorrelated entangled states produced from several independent sources [39,40]. To characterize nonlocality in this context, local models with uncorrelated local variables are desired. This is the motivation behind introducing the concept of network nonlocality [39]. Network nonlocality in the entanglement swapping network scenario can be checked using what is known as the "bilocality" inequality [39–41]. The concepts of network nonlocality and bilocality inequality have attracted attention in recent years as witnessed by a series of works [39–45]. Recently, Gisin *et al.* [41] showed that the bilocality inequality can be violated when arbitrary pure entangled states are initially shared by the pairs (1,2) and (3,4) and when projective measurement in the Bell basis is performed on (2,3). Further, when two mixed states ρ_{12} and ρ_{34} are initially shared between the pairs (1,2) and (3,4) and projective measurement in the Bell basis is performed on (2,3), then violation of the bilocality inequality implies that either ρ_{12} or ρ_{34} or both are Bell nonlocal [41]. Motivated from these studies, we extend our paper for probing network nonlocality in the generalized entanglement swapping scenario where two two-qubit Bell states are initially shared by the pairs (1,2) and (3,4) and a POVM is performed on (2,3). Here we find the appropriate POVM for which the resulting measurement statistics violates or does not violate the bilocality inequality for the whole range of the measurement parameter. The present paper probes network nonlocality under POVMs.

The rest of the paper is organized as follows. In Sec. II, we present the mathematical tools that are useful for the present paper. In Sec. III, we describe in detail the generalized entanglement swapping scenario considered by us. Next, in Sec. IV, we demonstrate our main findings with different classes of POVMs. Finally, we conclude with a short discussion in Sec. V.

II. PRELIMINARIES

In this section, we discuss in brief the quantifications of Bell nonlocality, EPR steering, and entanglement pertaining to two-qubit states that will be used as tools in the present paper.

A. Quantification of Bell nonlocality

Whether an arbitrary two-qubit state ρ violates the Bell–Clauser-Horne-Shimony-Holt (CHSH) inequality [46,47] or not is determined by the function M_ρ defined as [48,49]

$$M_\rho = \max_{i>j}\{t_i + t_j\}. \tag{1}$$

Here t_i ($i \in \{1, 2, 3\}$) are the eigenvalues of the real symmetric matrix $T_\rho^T T_\rho$, where T_ρ is a real 3×3 matrix with elements $T_{ij} = \mathrm{Tr}[\rho\,(\sigma_i \otimes \sigma_j)]$ ($i, j \in \{1, 2, 3\}$); T_ρ^T is the transpose of T_ρ; σ_i with $i \in \{1, 2, 3\}$ are the Pauli matrices. The function M_ρ is related to the maximal mean value B_ρ of the Bell-CHSH operator through the relation [48,49] $B_\rho = 2\sqrt{M_\rho}$. Hence, $M_\rho > 1$ implies $B_\rho > 2$—the condition for quantum violation of the Bell-CHSH inequality by the state ρ. One can, therefore, quantify the degree of Bell nonlocality pertaining to the two-qubit state ρ using $\mathcal{N}_\rho \propto \max\{0, B_\rho - 2\}$ [50]. After simplifying and taking an appropriate normalization, we have

$$\mathcal{N}_\rho = \max\left\{0, \frac{\sqrt{M_\rho} - 1}{\sqrt{2} - 1}\right\}. \tag{2}$$

Consequently, $\mathcal{N}_\rho > 0$ implies that the Bell-CHSH inequality is violated by the two-qubit state ρ. In the present paper, Bell nonlocality of a two-qubit state is probed using the above measure.

B. Quantification of EPR steering

Similar to the case of Bell nonlocality, EPR steering of a two-qubit state ρ can be quantified [50] in terms of the maximum quantum violation of the n-setting linear steering inequality proposed in [51]. For $n = 2$, the analytical form of this quantifier is given by [50]

$$\mathcal{S}_\rho^{(2)} = \max\left\{0, \frac{\sqrt{\Lambda_\rho^{(2)}} - 1}{\sqrt{2} - 1}\right\}$$
$$\text{with} \quad \Lambda_\rho^{(2)} = \max_{i>j}\{t_i + t_j\}, \tag{3}$$

with t_i ($i \in \{1, 2, 3\}$) being defined earlier. Further, $\mathcal{S}_\rho^{(2)} > 0$ implies that the two-setting linear steering inequality [51] is violated by the two-qubit state ρ [50].

Next, $n = 3$; the analytical form of this quantifier is given by [50]

$$\mathcal{S}_\rho^{(3)} = \max\left\{0, \frac{\sqrt{\Lambda_\rho^{(3)}} - 1}{\sqrt{3} - 1}\right\}$$
$$\text{with} \quad \Lambda_\rho^{(3)} = t_1 + t_2 + t_3. \tag{4}$$

With this, $\mathcal{S}_\rho^{(3)} > 0$ implies that the three-setting linear steering inequality [51] is violated by the two-qubit state ρ [50].

One can easily check that, for any two-qubit state ρ, $\mathcal{S}_\rho^{(2)} = \mathcal{N}_\rho$ [52]. Also, a two-qubit state ρ violates the two-setting linear steering inequality if and only if the state violates the Bell-CHSH inequality [52]. Hence, nonequivalence of EPR steering and Bell nonlocality can be probed if we quantify Bell nonlocality by $\mathcal{N}_\rho$ and EPR steering by $\mathcal{S}_\rho^{(3)}$. This motivates us to use $\mathcal{S}_\rho^{(3)}$ as the quantifier of EPR steering pertaining to any two-qubit state ρ. In what follows, for simplicity, we will denote $\mathcal{S}_\rho^{(3)}$ by $\mathcal{S}_\rho$. We will determine whether a two-qubit state ρ is steerable or not depending on whether $\mathcal{S}_\rho > 0$ or $\mathcal{S}_\rho \leqslant 0$ respectively.

C. Quantification of entanglement

Entanglement of a two-qubit state ρ can be quantified using the concept of negativity defined as [53,54]

$$\mathcal{E}_\rho = \|\rho^{T_B}\|_1 - 1$$

where $\|\rho^{T_B}\|_1$ denotes the trace norm of the partial transpose of ρ. From the above definition, we have

$$\mathcal{E}_\rho = 2\max(0, -\mu_4) \tag{5}$$

where μ_4 is the minimum eigenvalue of ρ^{T_B}. A two-qubit state ρ is entangled if and only if $\mathcal{E}_\rho > 0$.

D. POVM

In general, any quantum measurement is represented by the POVM [11,12], which is a set of positive operators that add to identity, i.e., $E \equiv \{E_i | \sum E_i = \mathbb{I}, 0 < E_i \leqslant \mathbb{I}\}$. Every E_i can be decomposed as $E_i = M_i^\dagger M_i$. The effects (E_is) represent quantum events that may occur as outcomes of a measurement. For an arbitrary state ρ, the postmeasurement state after obtaining the outcome i is given by $\rho_{\text{PM}|i} = \frac{M_i \rho M_i^\dagger}{\text{Tr}(\rho E_i)} = \frac{U_i \sqrt{E_i}\rho\sqrt{E_i}^\dagger U_i^\dagger}{\text{Tr}(\rho E_i)}$ where U_i is an arbitrary unitary operator (specified by the actual physical realization of the POVM) and the probability of getting the ith outcome is given by $\text{Tr}(\rho E_i)$. As a special case, the postmeasurement state, when the ith outcome is obtained, can be determined using the generalized von Neumann–Lüders transformation rule [11,12] as follows: $\rho_{\text{PM}|i} = \frac{\sqrt{E_i}\,\rho\,\sqrt{E_i}^\dagger}{\text{Tr}(\rho E_i)}$. In the present paper, the postmeasurement states will be evaluated using the above-mentioned generalized von Neumann–Lüders transformation rule.

E. Network nonlocality

Next, we are going to describe in brief the basic concept and definition of bilocality and network nonlocality [39]. In a typical entanglement swapping scenario involving three observers, say, Alice, Bob, and Charlie, there are two independent sources S_1 and S_2. Bob shares two pairs of particles, one with Alice produced from S_1 and another with Charlie produced from S_2.

In order to capture nonlocality in the above network, let Alice, Bob, and Charlie receive inputs denoted by $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ respectively and return outputs denoted by $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ respectively. The produced correlation $P(\tilde{a}, \tilde{b}, \tilde{c}|\tilde{x}, \tilde{y}, \tilde{z})$ will exhibit network nonlocality if it cannot be described by the following bilocal model [39]:

$$P(\tilde{a}, \tilde{b}, \tilde{c}|\tilde{x}, \tilde{y}, \tilde{z}) = \int d\Lambda_1 d\Lambda_2 \rho(\Lambda_1, \Lambda_2) P(\tilde{a}|\tilde{x}, \Lambda_1) \times P(\tilde{b}|\tilde{y}, \Lambda_1, \Lambda_2) P(\tilde{c}|\tilde{z}, \Lambda_2)$$
$$\text{with} \quad \rho(\Lambda_1, \Lambda_2) = \rho_1(\Lambda_1)\rho_2(\Lambda_2), \tag{6}$$

where Λ_1 and Λ_2 are two hidden variables produced by two sources S_1 and S_2 respectively. Here the condition $\rho(\Lambda_1, \Lambda_2) = \rho_1(\Lambda_1)\rho_2(\Lambda_2)$ implies that the two sources S_1 and S_2 are independent.

The first approach towards deriving bilocal inequality was reported in [39] and later Branciard *et al.* derived more general bilocal inequalities [40]. Here, we will consider the latter bilocal inequalities. Consider the entanglement swapping scenario with each of Alice and Charlie performing a single-particle measurement with binary inputs $\tilde{x} \in \{0, 1\}$ and $\tilde{z} \in \{0, 1\}$ respectively and obtaining corresponding binary outputs $\tilde{a} \in \{0, 1\}$ and $\tilde{c} \in \{0, 1\}$ respectively. The middle party, Bob, performs the same measurement (and hence receives no input $\tilde{y}$) with four possible outcomes. Let us denote Bob's outcome by two bits: $\tilde{b}_0\tilde{b}_1 = 00$, 01, 10, or 11 (for details see [40,42]). Therefore, the tripartite correlations can be written as $P(\tilde{a}, \tilde{b}^0\tilde{b}^1, \tilde{c}|\tilde{x}, \tilde{z})$. With these, let us define the following [40]:

$$I = \langle (A_0 + A_1)B^0(C_0 + C_1)\rangle \tag{7}$$

and

$$J = \langle (A_0 - A_1)B^1(C_0 - C_1)\rangle, \tag{8}$$

where

$$\langle A_x B^y C_z\rangle = \sum_{\tilde{a},\tilde{b}^0\tilde{b}^1,c} (-1)^{\tilde{a}+\tilde{b}^y+\tilde{c}} P(\tilde{a}, \tilde{b}^0\tilde{b}^1, \tilde{c}|\tilde{x}, \tilde{z}). \tag{9}$$

Using these notations, the bilocality inequality [40] can be expressed as

$$\mathcal{B}_{\text{bilocal}} = \sqrt{|I|} + \sqrt{|J|} \leqslant 2. \tag{10}$$

When the above inequality is violated, then the tripartite measurement correlation is nonbilocal, i.e., network nonlocality is demonstrated [39–41].

III. SETTING UP THE SCENARIO

Specifically, we consider a general entanglement swapping protocol, where Alice shares a Bell pair $|\psi_1\rangle_{12} = (1/\sqrt{2})(|00\rangle + |11\rangle)$ denoted by (1,2) with Bob and Bob shares another Bell pair $|\psi_1\rangle_{34} = (1/\sqrt{2})(|00\rangle + |11\rangle)$ denoted by (3,4) with Charlie. Here, Alice, Bob, and Charlie are physically separated from each other. Bob performs a joint quantum measurement with four possible outcomes (which is a POVM $E \equiv \{E_1, E_2, E_3, E_4\}$) on (2,3) and discloses the outcome to both Alice and Charlie. Consequently, the pair (1,4) shared between Alice and Charlie may become correlated depending on the choice of the POVM by Bob. Similarly, the correlations present in the pairs (1,2) and (3,4) may get reduced. Within this framework, different types of POVM lead to different types of correlation transfer from the pairs (1,2) and (3,4) to the pair (1,4).

In the above scenario, the postmeasurement state [after Bob performs the POVM on (2,3) and discloses the outcome to Alice and Charlie] shared between Alice-Charlie, when the ith outcome ($i \in \{1, 2, 3, 4\}$) is obtained, is given by

$$\rho_{14|i} = \frac{\text{Tr}_{23}[(\mathbb{I}_2 \otimes \sqrt{E_i} \otimes \mathbb{I}_2)(|\psi_1\rangle\langle\psi_1|_{12} \otimes |\psi_1\rangle\langle\psi_1|_{34})(\mathbb{I}_2 \otimes \sqrt{E_i}^{\dagger} \otimes \mathbb{I}_2)]}{\text{Tr}[(\mathbb{I}_2 \otimes \sqrt{E_i} \otimes \mathbb{I}_2)(|\psi_1\rangle\langle\psi_1|_{12} \otimes |\psi_1\rangle\langle\psi_1|_{34})(\mathbb{I}_2 \otimes \sqrt{E_i}^{\dagger} \otimes \mathbb{I}_2)]}. \tag{11}$$

Similarly, the postmeasurement states shared between Alice-Bob and Bob-Charlie, when the ith outcome is obtained, are given by

$$\rho_{12|i} = \frac{\text{Tr}_{34}[(\mathbb{I}_2 \otimes \sqrt{E_i} \otimes \mathbb{I}_2)(|\psi_1\rangle\langle\psi_1|_{12} \otimes |\psi_1\rangle\langle\psi_1|_{34})(\mathbb{I}_2 \otimes \sqrt{E_i}^{\dagger} \otimes \mathbb{I}_2)]}{\text{Tr}[(\mathbb{I}_2 \otimes \sqrt{E_i} \otimes \mathbb{I}_2)(|\psi_1\rangle\langle\psi_1|_{12} \otimes |\psi_1\rangle\langle\psi_1|_{34})(\mathbb{I}_2 \otimes \sqrt{E_i}^{\dagger} \otimes \mathbb{I}_2)]} \tag{12}$$

and

$$\rho_{34|i} = \frac{\text{Tr}_{12}[(\mathbb{I}_2 \otimes \sqrt{E_i} \otimes \mathbb{I}_2)(|\psi_1\rangle\langle\psi_1|_{12} \otimes |\psi_1\rangle\langle\psi_1|_{34})(\mathbb{I}_2 \otimes \sqrt{E_i}^{\dagger} \otimes \mathbb{I}_2)]}{\text{Tr}[(\mathbb{I}_2 \otimes \sqrt{E_i} \otimes \mathbb{I}_2)(|\psi_1\rangle\langle\psi_1|_{12} \otimes |\psi_1\rangle\langle\psi_1|_{34})(\mathbb{I}_2 \otimes \sqrt{E_i}^{\dagger} \otimes \mathbb{I}_2)]} \tag{13}$$

respectively. Here, the postmeasurement states are determined using the Lüders state-update rule.

Let each E_i admit the following decomposition:

$$E_i = \sum_{k=0}^{3} q_{ik} |\Psi_{ik}\rangle\langle\Psi_{ik}|,$$

where $|\Psi_{ik}\rangle$ for $k = 0, 1, 2, 3$ form a complete orthonormal basis on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $q_{ik} \geqslant 0$ for all $k = 0, 1, 2, 3$. With this, we have [10]

$$\rho_{14|i} = \frac{E_i^*}{\text{Tr}(E_i)}$$
$$\text{with } E_i^* = \sum_{k=0}^{3} q_{ik} |\Psi_{ik}^*\rangle\langle\Psi_{ik}^*|, \tag{14}$$

with $|\Psi_{ik}^*\rangle$ being the complex conjugation of $|\Psi_{ik}\rangle$ in the computational basis.

However, obtaining similar compact expressions for (1,2) and (3, 4) seems difficult for an arbitrary POVM.

IV. RESULTS

We have analyzed with different classes of POVMs. Below, we present some of these results which seem interesting in the context of quantum correlation (e.g., Bell nonlocality, EPR steering, and entanglement) transfer from the pairs (1,2) and (3,4) to the pair (1,4).

A. Case I

At first, we consider a POVM $E \equiv \{E_1, E_2, E_3, E_4\}$ with the following effect operators:

$$E_i = \lambda |\psi_i\rangle\langle\psi_i| + \frac{1-\lambda}{4}\mathbb{I}_4, \tag{15}$$

where $\lambda \in [0, 1]$ is the sharpness parameter denoting the measurement strength; $(1 - \lambda)$ denotes the amount of white noise incorporated in the measurement; $|\psi_i\rangle$ for $i = 1, 2, 3, 4$ forms the Bell basis defined as

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\psi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

The above measurement becomes projective when $\lambda = 1$ and becomes trivial when $\lambda = 0$. This measurement is unentangled for $0 \leqslant \lambda \leqslant \frac{1}{3}$ and entangled for $\frac{1}{3} < \lambda \leqslant 1$.

For this POVM, each outcome i occurs with probability $p_i = \frac{1}{4}$ for all $i \in \{1, 2, 3, 4\}$.

After Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the above POVM and communicates the outcome to Alice and Charlie, the postmeasurement state shared between Alice and Charlie becomes

$$\rho_{14|i} = \lambda |\psi_i\rangle\langle\psi_i| + \frac{1-\lambda}{4}\mathbb{I}_4 \ \ \forall i \in \{1, 2, 3, 4\}. \tag{16}$$

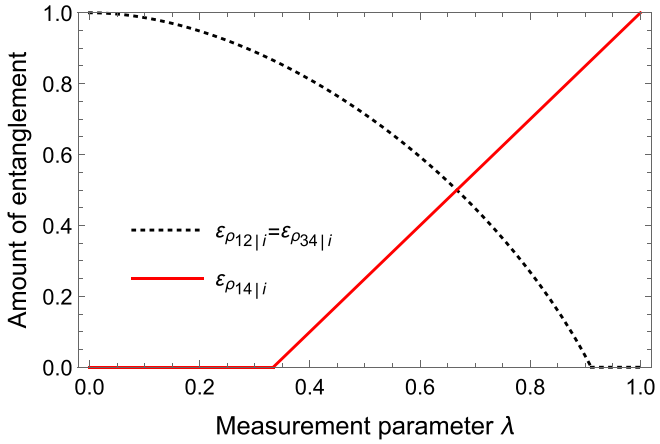

FIG. 1. Variation of negativity ($\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$) with the measurement parameter λ for the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (15) and communicates the outcome to Alice and Charlie. It is noteworthy that this plot is the same for all possible outcomes $i \in \{1, 2, 3, 4\}$ of Bob's measurement.

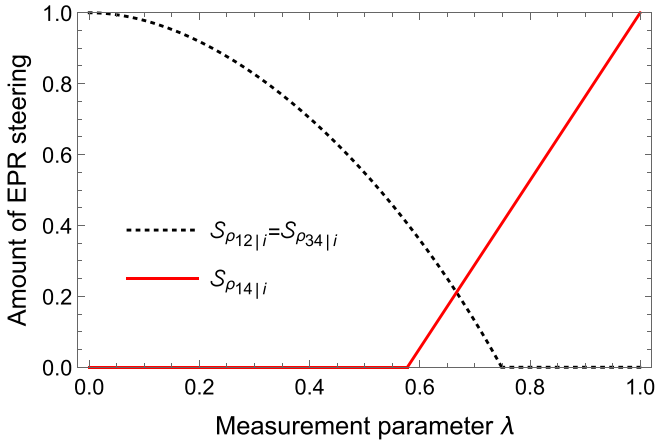

FIG. 2. Variation of the amount of EPR steering ($\mathcal{S}_{\rho_{14|i}}$, $\mathcal{S}_{\rho_{12|i}}$, and $\mathcal{S}_{\rho_{34|i}}$) with the measurement parameter λ for the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (15) and communicates the outcome to Alice and Charlie. This plot remains the same for all possible outcomes $i \in \{1, 2, 3, 4\}$ of Bob's measurement.

This is obtained from Eq. (14).

Similarly using Eqs. (12) and (13), we get the postmeasurement state shared between Alice-Bob and that between Bob-Charlie as written below:

$$\rho_{12|i} = \rho_{34|i} = s(\lambda)|\psi_1\rangle\langle\psi_1| + \frac{1 - s(\lambda)}{4}\mathbb{I}_4 \ \ \forall i \in \{1, 2, 3, 4\}$$
$$\text{with } s(\lambda) = \frac{1}{2}[1 - \lambda + \sqrt{(1 - \lambda)(1 + 3\lambda)}]. \tag{17}$$

Hence, each of $\rho_{14|i}$, $\rho_{12|i}$, and $\rho_{34|i}$ belongs to the Werner class of states for all $i \in \{1, 2, 3, 4\}$. Another important point to be stressed here is that $\rho_{14|i}$ depends on the outcome i, whereas $\rho_{12|i}$ and $\rho_{34|i}$ are independent of i.

1. Entanglement

Next, we evaluate the amount of entanglement of $\rho_{14|i}$, $\rho_{12|i}$, and $\rho_{34|i}$ for all $i \in \{1, 2, 3, 4\}$. From (5), we get

$$\mathcal{E}_{\rho_{14|i}} = \frac{3\lambda - 1}{2} \ \ \forall i \in \{1, 2, 3, 4\}. \tag{18}$$

Each $\rho_{14|i}$ is entangled if and only if $\frac{1}{3} < \lambda \leqslant 1$. Hence, it is clearly reflected that the state $\rho_{14|i}$ is entangled if and only if Bob's measurement is entangled.

In the case of $\rho_{12|i}$ or $\rho_{34|i}$, the amount of entanglement is given by

$$\mathcal{E}_{\rho_{12|i}} = \mathcal{E}_{\rho_{34|i}} = \frac{3s(\lambda) - 1}{2} \ \ \forall i \in \{1, 2, 3, 4\} \tag{19}$$

with $s(\lambda)$ being defined earlier. From Eq. (19), it follows that the states ρ_{12} and ρ_{34} are entangled if and only if $0 \leqslant \lambda < 0.91$.

We show the variation of $\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$ with λ in Fig. 1. From this figure, it is evident that all the resulting states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are simultaneously entangled when $\frac{1}{3} < \lambda < 0.91$. In other words, there exists a range of the measurement strength λ when the particle of Alice (Charlie) is simultaneously entangled with the particles of Bob and Charlie (Alice)—which is never possible in the usual entanglement swapping scenario with $\lambda = 1$. Moreover, when $\lambda = \frac{2}{3}$, all these three resulting states shared between different parties have equal amounts of entanglement—$\mathcal{E}_{\rho_{14|i}} = \mathcal{E}_{\rho_{12|i}} = \mathcal{E}_{\rho_{34|i}} = \frac{1}{2}$. These aspects are the same for all possible outcomes $i \in \{1, 2, 3, 4\}$.

2. EPR steering

We now move to analyze the EPR steering of all the postmeasurement states which we obtained earlier. For the state $\rho_{14|i}$, the eigenvalues of the matrix $T^T_{\rho_{14|i}} T_{\rho_{14|i}}$ are given by $t_1 = \lambda^2$, $t_2 = \lambda^2$, and $t_3 = \lambda^2$. Hence, using Eq. (4) we have

$$\mathcal{S}_{\rho_{14|i}} = \max\left\{0, \frac{\sqrt{3}\lambda - 1}{\sqrt{3} - 1}\right\} \ \ \forall i \in \{1, 2, 3, 4\}. \tag{20}$$

Hence, the state $\rho_{14|i}$ is steerable with respect to the quantum violation of the three-setting linear steering inequality [51] when $\frac{1}{\sqrt{3}} < \lambda \leqslant 1$. Next, let us focus on the states $\rho_{12|i}$ and $\rho_{34|i}$. For each of these two states, the eigenvalues of the matrix $T^T_{\rho_{mm+1|i}} T_{\rho_{mm+1|i}}$ (with $m \in \{1, 3\}$) are given by $t_1 = [s(\lambda)]^2$, $t_2 = [s(\lambda)]^2$, and $t_3 = [s(\lambda)]^2$, where $s(\lambda) = \frac{1}{2}[1 - \lambda + \sqrt{(1 - \lambda)(1 + 3\lambda)}]$. With these, we have

$$\mathcal{S}_{\rho_{12|i}} = \mathcal{S}_{\rho_{34|i}} = \max\left\{0, \frac{\sqrt{3}s(\lambda) - 1}{\sqrt{3} - 1}\right\} \ \ \forall i \in \{1, 2, 3, 4\}. \tag{21}$$

The states $\rho_{12|i}$ and $\rho_{34|i}$ are steerable when $s(\lambda) > \frac{1}{\sqrt{3}}$, i.e., when $0 \leqslant \lambda < 0.75$.

We have plotted the variation of $\mathcal{S}_{\rho_{14|i}}$, $\mathcal{S}_{\rho_{12|i}}$, and $\mathcal{S}_{\rho_{34|i}}$ with λ in Fig. 2. From this figure, it can be observed that all the three states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are simultaneously EPR steerable when $\frac{1}{\sqrt{3}} < \lambda < 0.75$. For $\lambda = \frac{2}{3}$, all these three resulting

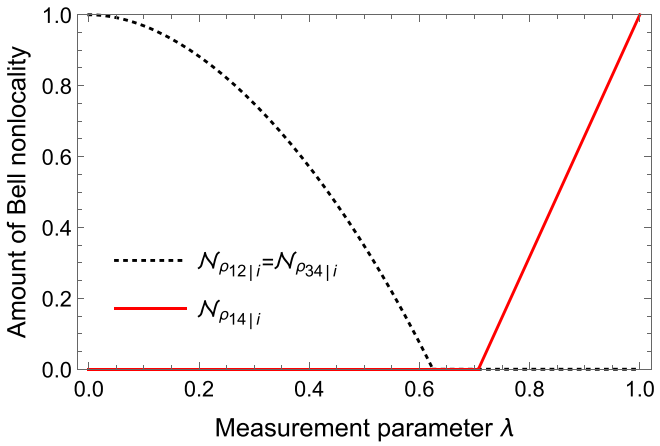

FIG. 3. Variation of Bell nonlocality ($\mathcal{N}_{\rho_{14|i}}$, $\mathcal{N}_{\rho_{12|i}}$, and $\mathcal{N}_{\rho_{34|i}}$) with the measurement parameter λ for the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (15) and communicates the outcome to Alice and Charlie. This plot remains the same for all possible outcomes $i \in \{1, 2, 3, 4\}$ of Bob's measurement.

states have the same amount of EPR steerability. Similar to the case of entanglement, here also, these features are the same for all possible outcomes $i \in \{1, 2, 3, 4\}$.

3. Bell nonlocality

Next, we will focus on Bell nonlocality of the postmeasurement states. Using the eigenvalues of the matrix $T^T_{\rho_{14|i}} T_{\rho_{14|i}}$ derived earlier, we get

$$\mathcal{N}_{\rho_{14|i}} = \max\left\{0, \frac{\sqrt{2}\lambda - 1}{\sqrt{2} - 1}\right\} \ \forall i \in \{1, 2, 3, 4\}. \tag{22}$$

Hence, $\rho_{14|i}$ is Bell nonlocal (with respect to the Bell-CHSH inequality) when $\frac{1}{\sqrt{2}} < \lambda \leqslant 1$. Similarly, for the states $\rho_{12|i}$ and $\rho_{34|i}$, we have

$$\mathcal{N}_{\rho_{12|i}} = \mathcal{N}_{\rho_{34|i}} = \max\left\{0, \frac{\sqrt{2}s(\lambda) - 1}{\sqrt{2} - 1}\right\} \ \forall i \in \{1, 2, 3, 4\}, \tag{23}$$

where $s(\lambda) = \frac{1}{2}[1 - \lambda + \sqrt{(1-\lambda)(1+3\lambda)}]$. These two states are Bell nonlocal when $s(\lambda) > \frac{1}{\sqrt{2}}$, i.e., when $0 \leqslant \lambda < 0.62$.

We have plotted the variation of $\mathcal{N}_{\rho_{14|i}}$, $\mathcal{N}_{\rho_{12|i}}$, and $\mathcal{N}_{\rho_{34|i}}$ with λ in Fig. 3. Unlike the case of entanglement and EPR steering, there does not exist any range of λ, where all the three states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are simultaneously Bell nonlocal with respect to quantum violation of the Bell-CHSH inequality. In particular, when $0.62 \leqslant \lambda \leqslant \frac{1}{\sqrt{2}}$, none of the three states $\rho_{14|i}$, $\rho_{12|i}$, and $\rho_{34|i}$ is Bell nonlocal.

From Table I, it can be observed that these analyses provide a tool for constructing different types of quantum correlations in different branches of a network by fine tuning the measurement strength λ. For example, if one chooses the POVM (15) with $\lambda \leqslant \frac{1}{\sqrt{2}}$, then after completion of the protocol the pair (1,4) becomes entangled without Bell nonlocality (with respect to the Bell-CHSH inequality). Similarly, if one chooses $\lambda \leqslant \frac{1}{\sqrt{3}}$, then the pair (1,4) is entangled without EPR steering (with respect to the three-setting steering inequality) and Bell nonlocality (with respect to the Bell-CHSH inequality). In a similar way, entanglement without steerability or entanglement without Bell nonlocality can be established between the pair (1,2) or (3,4) by appropriately choosing λ.

TABLE I. The ranges of the measurement strength λ for which the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are entangled, or EPR steerable, or Bell nonlocal for all possible $i \in \{1, 2, 3, 4\}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (15) and communicates the outcome to Alice and Charlie.

States	Entangled when	Steerable when	Bell nonlocal when
$\rho_{14\|i}$	$\frac{1}{3} < \lambda \leqslant 1$	$\frac{1}{\sqrt{3}} < \lambda \leqslant 1$	$\frac{1}{\sqrt{2}} < \lambda \leqslant 1$
$\rho_{12\|i}$	$0 \leqslant \lambda < 0.91$	$0 \leqslant \lambda < 0.75$	$0 \leqslant \lambda < 0.62$
$\rho_{34\|i}$	$0 \leqslant \lambda < 0.91$	$0 \leqslant \lambda < 0.75$	$0 \leqslant \lambda < 0.62$

B. Case II

We now consider another class of POVM $E \equiv \{E_1, E_2, E_3, E_4\}$ with

$$\begin{aligned}
E_1 &= x|\psi_1^{(\lambda)}\rangle\langle\psi_1^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_2^{(\lambda)}\rangle\langle\psi_2^{(\lambda)}| \\
&\quad + \frac{y_2(1-x)}{y_1+y_2}|\phi_3\rangle\langle\phi_3|, \\
E_2 &= x|\psi_2^{(\lambda)}\rangle\langle\psi_2^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_1^{(\lambda)}\rangle\langle\psi_1^{(\lambda)}| \\
&\quad + \frac{y_2(1-x)}{y_1+y_2}|\phi_4\rangle\langle\phi_4|, \\
E_3 &= x|\psi_3^{(\lambda)}\rangle\langle\psi_3^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_4^{(\lambda)}\rangle\langle\psi_4^{(\lambda)}| \\
&\quad + \frac{y_2(1-x)}{y_1+y_2}|\phi_1\rangle\langle\phi_1|, \\
E_4 &= x|\psi_4^{(\lambda)}\rangle\langle\psi_4^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_3^{(\lambda)}\rangle\langle\psi_3^{(\lambda)}| \\
&\quad + \frac{y_2(1-x)}{y_1+y_2}|\phi_2\rangle\langle\phi_2|,
\end{aligned} \tag{24}$$

where $x = 0.3$, $y_1 = \frac{2+2\sqrt{1-\lambda}-\lambda}{4}$, and $y_2 = \frac{\lambda}{4}$ with $0 \leqslant \lambda \leqslant 1$. We choose $|\psi_i^{(\lambda)}\rangle$ from the set $\{|\psi_i^{(\lambda)}\rangle, i = 1, \ldots, 4\}$ and $|\phi_i\rangle$ from the set $\{|\phi_i\rangle, i = 1, \ldots, 4\}$ that are given by

$$\begin{aligned}
|\psi_1^{(\lambda)}\rangle &= a|00\rangle - b|11\rangle, \quad |\psi_3^{(\lambda)}\rangle = a|01\rangle - b|10\rangle, \\
|\psi_2^{(\lambda)}\rangle &= b|00\rangle + a|11\rangle, \quad |\psi_4^{(\lambda)}\rangle = b|01\rangle + a|10\rangle,
\end{aligned}$$

with

$$a = \frac{\sqrt{1-\sqrt{1-\lambda}}}{\sqrt{2}}, \quad b = \frac{\sqrt{1+\sqrt{1-\lambda}}}{\sqrt{2}},$$

and

$$|\phi_1\rangle = |00\rangle, \quad |\phi_3\rangle = |01\rangle,$$
$$|\phi_2\rangle = |11\rangle, \quad |\phi_4\rangle = |10\rangle.$$

The above POVM is characterized by the single parameter λ.

In this case, it can be shown that the probability of getting the ith outcome is $p_i = \frac{1}{4}$ for all $i \in \{1, 2, 3, 4\}$.

Next, we will find out the postmeasurement states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ after Bob performs the above POVM on the pair (2,3) and communicates the outcome to Alice and Charlie. From Eq. (14), we get

$$\begin{aligned}
\rho_{14|1} =& x|\psi_1^{(\lambda)}\rangle\langle\psi_1^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_2^{(\lambda)}\rangle\langle\psi_2^{(\lambda)}| \\
&+ \frac{y_2(1-x)}{y_1+y_2}|\phi_3\rangle\langle\phi_3|, \\
\rho_{14|2} =& x|\psi_2^{(\lambda)}\rangle\langle\psi_2^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_1^{(\lambda)}\rangle\langle\psi_1^{(\lambda)}| \\
&+ \frac{y_2(1-x)}{y_1+y_2}|\phi_4\rangle\langle\phi_4|, \\
\rho_{14|3} =& x|\psi_3^{(\lambda)}\rangle\langle\psi_3^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_4^{(\lambda)}\rangle\langle\psi_4^{(\lambda)}| \\
&+ \frac{y_2(1-x)}{y_1+y_2}|\phi_1\rangle\langle\phi_1|, \\
\rho_{14|4} =& x|\psi_4^{(\lambda)}\rangle\langle\psi_4^{(\lambda)}| + \frac{y_1(1-x)}{y_1+y_2}|\psi_3^{(\lambda)}\rangle\langle\psi_3^{(\lambda)}| \\
&+ \frac{y_2(1-x)}{y_1+y_2}|\phi_2\rangle\langle\phi_2|. \qquad (25)
\end{aligned}$$

From Eqs. (12) and (13), we get

$$\begin{aligned}
\rho_{12|1} =& (e^2+f^2)|\xi_1\rangle\langle\xi_1| + h^2|\phi_3\rangle\langle\phi_3| + h^2|\phi_4\rangle\langle\phi_4| \\
&+ g^2|\phi_1\rangle\langle\phi_1|, \\
\rho_{34|1} =& (e^2+g^2)|\xi_2\rangle\langle\xi_2| + h^2|\phi_3\rangle\langle\phi_3| + h^2|\phi_4\rangle\langle\phi_4| \\
&+ f^2|\phi_2\rangle\langle\phi_2|,
\end{aligned}$$

$$\begin{aligned}
\text{with } |\xi_1\rangle =& \frac{e}{\sqrt{e^2+f^2}}|00\rangle + \frac{f}{\sqrt{e^2+f^2}}|11\rangle, \\
|\xi_2\rangle =& \frac{g}{\sqrt{e^2+g^2}}|00\rangle + \frac{e}{\sqrt{e^2+g^2}}|11\rangle, \\
e =& \sqrt{\frac{y_2(1-x)}{y_1+y_2}}, \\
f =& a^2\sqrt{\frac{y_1(1-x)}{y_1+y_2}} + b^2\sqrt{x}, \\
g =& b^2\sqrt{\frac{y_1(1-x)}{y_1+y_2}} + a^2\sqrt{x}, \\
h =& ab\sqrt{\frac{y_1(1-x)}{y_1+y_2}} - ab\sqrt{x}. \qquad (26)
\end{aligned}$$

Similarly, $\rho_{12|i}$ and $\rho_{34|i}$ for other values of i can be evaluated. Unlike the previous POVM, here the states $\rho_{12|i}$ and $\rho_{34|i}$ are different as the POVM (24) is asymmetric with respect to the two particles of the pair (2,3). This is due to the fact that the set $\{|\phi_i\rangle, i = 1, \ldots, 4\}$ is not symmetric. Note that each of $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ is a mixed state for $0 \leqslant \lambda \leqslant 1$.

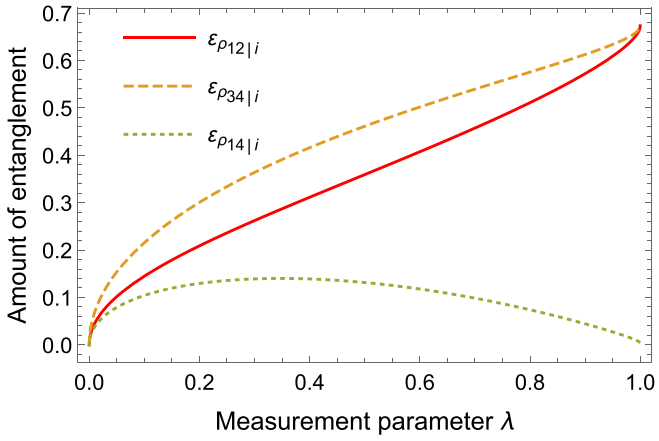

FIG. 4. Variation of negativity ($\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$) with the measurement parameter λ for the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (24) with $x = 0.3$ and communicates the outcome to Alice and Charlie. This plot is the same for all possible outcomes $i \in \{1, 2, 3, 4\}$ of Bob's measurement.

1. Entanglement

Let us first focus on the entanglement of the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$. The amount of entanglement of the state $\rho_{14|i}$ is given by

$$\begin{aligned}
\mathcal{E}_{\rho_{14|i}} &= \sqrt{4q^2+r^2} - r \\
\text{with } q &= \frac{ab[y_1(1-2x) - xy_2]}{y_1+y_2}, \\
r &= \frac{y_2(1-x)}{y_1+y_2} \quad \forall i \in \{1, 2, 3, 4\}. \qquad (27)
\end{aligned}$$

Next, the amount of entanglement of the states $\rho_{12|i}$ and $\rho_{34|i}$ is given by

$$\mathcal{E}_{\rho_{12|i}} = 2(ef - h^2) \ \ \forall i \in \{1, 2, 3, 4\} \qquad (28)$$

and

$$\mathcal{E}_{\rho_{34|i}} = 2(eg - h^2) \ \ \forall i \in \{1, 2, 3, 4\} \qquad (29)$$

respectively, where e, f, g, and h are defined earlier. Hence, though the structure of each of the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ is different for different i, each of $\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$ does not depend on i.

Setting $x = 0.3$, we have plotted the variation of $\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{14|i}}$, and $\mathcal{E}_{\rho_{14|i}}$ with λ in Fig. 4. It can be observed that each of the above three states is entangled for the whole range of λ except at $\lambda = 0$.

This result demonstrates the following interesting aspect of the generalized entanglement swapping protocol with the POVM (24). This protocol creates entanglement in the pair (1,4). However, it never completely destroys the entanglement content in the pairs (1,2) and (3,4) for

the whole range of the measurement parameter λ except at $\lambda = 0$.

2. EPR steering

Next, we will evaluate EPR steerability of different postmeasurement states. It can be checked that the eigenvalues of the matrix $T^T_{\rho_{14|i}} T_{\rho_{14|i}}$ for the state $\rho_{14|i}$ are given by

$$t_1 = \left(\frac{2ab\big[y_2 x + y_1(2x-1)\big]}{y_1 + y_2}\right)^2,$$
$$t_2 = \left(\frac{2ab\big[y_2 x + y_1(2x-1)\big]}{y_1 + y_2}\right)^2,$$
$$t_3 = \left(\frac{y_1 + y_2(2x-1)}{y_1 + y_2}\right)^2 \quad \forall i \in \{1,2,3,4\}. \tag{30}$$

The eigenvalues of the matrix $T^T_{\rho_{12|i}} T_{\rho_{12|i}}$ for the state $\rho_{12|i}$ are given by

$$t_1 = 4e^2 f^2$$
$$t_2 = 4e^2 f^2,$$
$$t_3 = (e^2 + f^2 + g^2 - 2h^2)^2 \quad \forall i \in \{1,2,3,4\}. \tag{31}$$

And finally, the eigenvalues of the matrix $T^T_{\rho_{34|i}} T_{\rho_{34|i}}$ for the state $\rho_{34|i}$ are given by

$$t_1 = 4e^2 g^2,$$
$$t_2 = 4e^2 g^2,$$
$$t_3 = (e^2 + f^2 + g^2 - 2h^2)^2 \quad \forall i \in \{1,2,3,4\}. \tag{32}$$

Using the above expressions, $\mathcal{S}_{\rho_{14|i}}$, $\mathcal{S}_{\rho_{12|i}}$, and $\mathcal{S}_{\rho_{34|i}}$ can be evaluated by setting $x = 0.3$. Interestingly, in this case, it can be shown that $\mathcal{S}_{\rho_{14|i}} = 0$ for the whole range of $\lambda \in [0, 1]$. On the other hand, $\mathcal{S}_{\rho_{12|i}} > 0$ and $\mathcal{S}_{\rho_{34|i}} > 0$ for the whole range of λ except at $\lambda = 0$.

3. Bell nonlocality

Using Eqs. (30), (31), and (32) and setting $x = 0.3$, it can be checked that each of the two states $\rho_{12|i}$ and $\rho_{34|i}$ is Bell nonlocal for the whole range of λ except at $\lambda = 0$. On the other hand, the state $\rho_{14|i}$ is not Bell nonlocal for the whole range of λ. In other words, $\mathcal{N}_{\rho_{14|i}} = 0$ for all $\lambda \in [0, 1]$, and $\mathcal{N}_{\rho_{12|i}} > 0$ and $\mathcal{N}_{\rho_{34|i}} > 0$ for all $\lambda \in (0, 1]$.

Therefore, the above results indicate that the resulting state $\rho_{14|i}$ (for all possible outcomes i) after completion of our generalized entanglement swapping protocol contingent upon using the POVM (24) with $x = 0.3$ is entangled, but does not violate the three-setting steering inequality [51] or the Bell-CHSH inequality. This is summarized in Table II. Interestingly, this holds for the whole range of λ except at $\lambda = 0$. Hence, this process leads to the generation of a single parameter (λ) family of mixed entangled states that does not violate the three-setting linear steering inequality or the Bell-CHSH inequality for the whole range of the state parameter, except for the case when the state parameter is equal to zero. This class of mixed entangled states has another interesting feature—entanglement increases with an increase in the state parameter λ whenever $\lambda \leqslant 0.34$. However, when $\lambda > 0.34$, then entanglement decreases with an increase in λ. Hence, it is implied that this class of states never violates the three-setting linear steering inequality or the Bell-CHSH inequality even when the entanglement content is varied (by varying λ).

TABLE II. The ranges of the measurement parameter λ for which the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are entangled, or EPR steerable, or Bell nonlocal for all possible $i \in \{1, 2, 3, 4\}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (24) with $x = 0.3$ and communicates the outcome to Alice and Charlie.

States	Entangled when	Steerable when	Bell nonlocal when
$\rho_{14\|i}$	$0 < \lambda \leqslant 1$	Never	Never
$\rho_{12\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$
$\rho_{34\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$

Similarly, from Table II, we see that the resulting states $\rho_{12|i}$ and $\rho_{34|i}$ (for all possible outcomes i) are Bell nonlocal for the whole range of λ except $\lambda = 0$. Thus, this generalized entanglement swapping protocol with the POVM (24) also provides a tool to generate a single parameter (λ) family of mixed entangled states that is Bell nonlocal for the whole range of the state parameter, except for $\lambda = 0$.

Finally, we would like to point out that this generalized entanglement swapping protocol generates entanglement in the pair (1,4), but never generates EPR steering (with respect to quantum violation of the three-setting linear steering inequality) or Bell nonlocality (with respect to quantum violation of the Bell-CHSH inequality) in (1,4). On the other hand, this protocol never completely destroys Bell nonlocality in each of the pairs (1,2) and (3,4). This is valid for the whole range of the measurement parameter λ, except at $\lambda = 0$.

C. Case III

Now, we consider the POVM mentioned in (24) with $x = 0.725$. For this POVM, we will now determine how different quantum correlations are created or destroyed in various pairs of particles.

1. Entanglement

Entanglement of each of the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ can be determined by setting $x = 0.725$ in Eqs. (27), (28), and (29). In this case, we get that $\mathcal{E}_{\rho_{14|i}} > 0$, $\mathcal{E}_{\rho_{12|i}} > 0$, and $\mathcal{E}_{\rho_{34|i}} > 0$ for the whole range of λ except at $\lambda = 0$. This is valid for any $i \in \{1, 2, 3, 4\}$. The variation of $\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$ with λ is depicted in Fig. 5.

Hence, similar to the previous case, this protocol creates entanglement in the pair $(1, 4)$ without completely destroying the entanglement content in the pairs $(1, 2)$ and $(3, 4)$. This feature persists for the whole range of the measurement parameter λ except at $\lambda = 0$.

2. EPR steering

Using the expressions of the eigenvalues of the matrices $T^T_{\rho_{14|i}} T_{\rho_{14|i}}$, $T^T_{\rho_{12|i}} T_{\rho_{12|i}}$, and $T^T_{\rho_{34|i}} T_{\rho_{34|i}}$ mentioned in Eqs. (30),

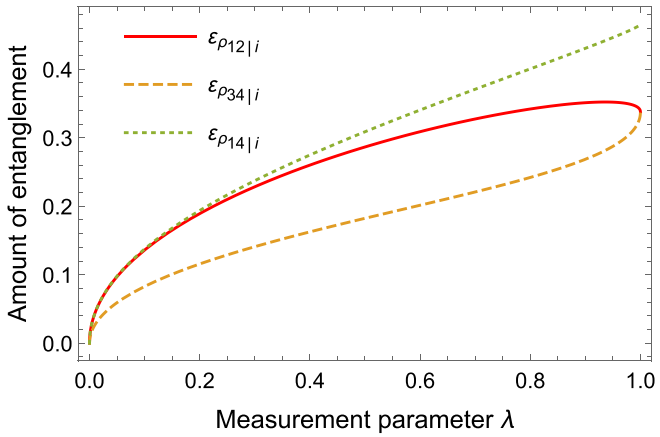

FIG. 5. Variation of negativity ($\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$) with the measurement parameter λ for the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (24) with $x = 0.725$ and communicates the outcome to Alice and Charlie. This plot is the same for all possible outcomes $i \in \{1, 2, 3, 4\}$ of Bob's measurement.

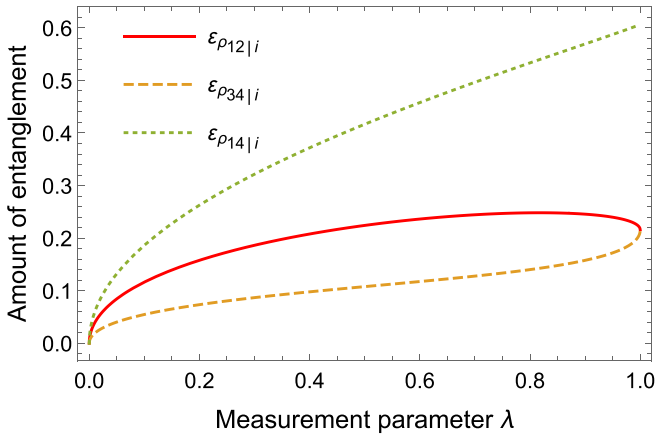

FIG. 6. Variation of negativity ($\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$) with the measurement parameter λ for the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (24) with $x = 0.8$ and communicates the outcome to Alice and Charlie. This plot is the same for all possible outcomes $i \in \{1, 2, 3, 4\}$ of Bob's measurement.

(31), and (32) and setting $x = 0.725$, we get the following results for all $i \in \{1, 2, 3, 4\}$: $\mathcal{S}_{\rho_{14|i}} > 0$ and $\mathcal{S}_{\rho_{12|i}} > 0$ for all $\lambda \in (0, 1]$, whereas $\mathcal{S}_{\rho_{34|i}} = 0$ for all $\lambda \in [0, 1]$.

3. Bell nonlocality

Similarly, using Eqs. (30), (31), and (32) and setting $x = 0.725$, we get the following results in the context of Bell nonlocality for all $i \in \{1, 2, 3, 4\}$: $\mathcal{N}_{\rho_{14|i}} = 0$, $\mathcal{N}_{\rho_{12|i}} = 0$, and $\mathcal{N}_{\rho_{34|i}} = 0$ for all $\lambda \in [0, 1]$.

All the above results are summarized in Table III. This table points out that each of the two resulting states $\rho_{14|i}$ and $\rho_{12|i}$ (for all possible outcomes i) after completion of our generalized entanglement swapping protocol contingent upon using the POVM (24) with $x = 0.725$ is entangled as well as steerable, but does not violate the Bell-CHSH inequality. This holds for the whole range of λ except at $\lambda = 0$. Hence, this protocol can be used as a tool to generate a single parameter family of EPR steerable mixed states (where the state parameter is λ) that does not violate the Bell-CHSH inequality for the whole range of the state parameter, except for the case when the state parameter is equal to zero. Further, this class of states never violates the Bell-CHSH inequality even when the amount of entanglement is varied by varying the state parameter λ.

TABLE III. The ranges of the measurement parameter λ for which the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are entangled, or EPR steerable, or Bell nonlocal for all possible $i \in \{1, 2, 3, 4\}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (24) with $x = 0.725$ and communicates the outcome to Alice and Charlie.

States	Entangled when	Steerable when	Bell nonlocal when
$\rho_{14\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	Never
$\rho_{12\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	Never
$\rho_{34\|i}$	$0 < \lambda \leqslant 1$	Never	Never

On the other hand, the resulting state $\rho_{34|i}$ (for all possible outcomes i) is entangled, but does not violate the three-setting steering inequality [51] or the Bell-CHSH inequality even when the entanglement of $\rho_{34|i}$ is varied by varying λ.

Finally, it should be noted that this generalized entanglement swapping protocol generates entanglement as well as EPR steering in the pair (1,4), but never generates Bell nonlocality in (1,4). This protocol never destroys EPR steering in (1,2), but completely destroys Bell nonlocality in (1,2). Also, this protocol never destroys entanglement in (3,4), but completely destroys EPR steering in (3,4). This is valid for the whole range of the measurement parameter λ, except at $\lambda = 0$. Here, destroying Bell nonlocality and EPR steering is probed with respect to the quantum violations of the Bell-CHSH inequality and the three-setting linear steering inequality respectively.

D. Case IV

Next, we again consider the POVM mentioned in (24), but with $x = 0.8$. For this POVM, let us now find out how different quantum correlations are created or destroyed in various pairs in the generalized entanglement swapping scheme.

1. Entanglement

Entanglement of each of the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ is determined by taking $x = 0.8$ in Eqs. (27), (28), and (29). In this case also, we get that $\mathcal{E}_{\rho_{14|i}} > 0$, $\mathcal{E}_{\rho_{12|i}} > 0$, and $\mathcal{E}_{\rho_{34|i}} > 0$ for the whole range of λ except at $\lambda = 0$. This holds for any $i \in \{1, 2, 3, 4\}$. The variation of $\mathcal{E}_{\rho_{14|i}}$, $\mathcal{E}_{\rho_{12|i}}$, and $\mathcal{E}_{\rho_{34|i}}$ with λ is depicted in Fig. 6. Hence, in this protocol also, entanglement is created in the pair (1,4), but the entanglement content is not destroyed completely in the pairs (1,2) and (3,4). This is valid

TABLE IV. The ranges of the measurement parameter λ for which the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are entangled, or EPR steerable, or Bell nonlocal for all possible $i \in \{1, 2, 3, 4\}$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (24) with $x = 0.8$ and communicates the outcome to Alice and Charlie.

States	Entangled when	Steerable when	Bell nonlocal when
$\rho_{14\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$
$\rho_{12\|i}$	$0 < \lambda \leqslant 1$	Never	Never
$\rho_{34\|i}$	$0 < \lambda \leqslant 1$	Never	Never

for the whole range of the measurement parameter λ except at $\lambda = 0$.

2. EPR steering

Using the expressions of the eigenvalues of the matrices $T^T_{\rho_{14|i}} T_{\rho_{14|i}}$, $T^T_{\rho_{12|i}} T_{\rho_{12|i}}$, and $T^T_{\rho_{34|i}} T_{\rho_{34|i}}$ mentioned earlier and setting $x = 0.8$, we get the following results for all $i \in \{1, 2, 3, 4\}$: $\mathcal{S}_{\rho_{14|i}} > 0$ for all $\lambda \in (0, 1]$, whereas $\mathcal{S}_{\rho_{34|i}} = \mathcal{S}_{\rho_{12|i}} = 0$ for all $\lambda \in [0, 1]$.

3. Bell nonlocality

Using Eqs. (30), (31), and (32) and setting $x = 0.8$, we get the following results in the context of Bell nonlocality for all $i \in \{1, 2, 3, 4\}$: $\mathcal{N}_{\rho_{14|i}} > 0$ for all $\lambda \in (0, 1]$, and $\mathcal{N}_{\rho_{12|i}} = 0$ and $\mathcal{N}_{\rho_{34|i}} = 0$ for all $\lambda \in [0, 1]$.

The above results are summarized in Table IV. It is evident from Table IV that the resulting state $\rho_{14|i}$ (for all possible outcomes i), generated after completion of our protocol using the POVM (24) with $x = 0.8$, is Bell nonlocal for the whole range of λ except at $\lambda = 0$. Thus this protocol is an operational tool to construct a single parameter family of mixed states (where the state parameter is λ) that is Bell nonlocal for the whole range of the state parameter, except for the case when the state parameter is equal to zero.

On the other hand, each of the states $\rho_{12|i}$ and $\rho_{34|i}$ (for all possible outcomes i) is entangled, but does not violate the three-setting steering inequality [51] or the Bell-CHSH inequality even if one varies the entanglement of it.

Finally, this generalized entanglement swapping protocol always generates Bell nonlocality (EPR steering) in (1,4) by completely destroying the Bell nonlocality (EPR steering) in (1,2) and (3,4) for the whole range of the measurement strength λ. On the other hand, entanglement in (1,4) is generated without completely destroying entanglement in (1,2) or (3,4). Importantly, this holds for the whole range of the measurement parameter λ, except at $\lambda = 0$. Here also, destroying Bell nonlocality and EPR steering is probed with respect to the quantum violations of the Bell-CHSH inequality and the three-setting linear steering inequality respectively.

E. When the initial states are mixed ones

So far we have studied quantum correlation transfer in the generalized entanglement swapping scenario when the initial states are maximally entangled ones. However, in realistic situations, such maximally entangled states are difficult to be prepared. That is why we next consider mixed initial states in order to incorporate experimental nonidealness. In what follows, we will show that similar types of quantum correlation transfer are possible in the generalized entanglement swapping scenario involving mixed initial states shared by the pairs (1,2) and (3,4) when the previously mentioned POVM given by Eq. (24) is performed on the pair of particles (2,3).

Here, instead of Bell states, Alice shares a mixed state $\rho_{12} = m|\phi^+\rangle\langle\phi^+| + (1-m)|\phi^-\rangle\langle\phi^-|$ with Bob and Bob shares another copy of the same state $\rho_{34} = m|\phi^+\rangle\langle\phi^+| + (1-m)|\phi^-\rangle\langle\phi^-|$ with Charlie where $\frac{1}{2} \leqslant m \leqslant 1$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$.

In this scenario, the postmeasurement state, when Bob gets the outcome 1 [associated with E_1 mentioned in Eq. (24)] and communicates the measurement outcome to Alice and Charlie, is given by

$$\rho_{14|1} = \frac{(2m-1)^2+1}{2}(g^2+h^2)|\eta_1\rangle\langle\eta_1| + \frac{1-(2m-1)^2}{2}(f^2+h^2)|\eta_2\rangle\langle\eta_2| + e^2|\phi_3\rangle\langle\phi_3|$$

$$\text{with } |\eta_1\rangle = \frac{g}{\sqrt{g^2+h^2}}|00\rangle + \frac{h}{\sqrt{g^2+h^2}}|11\rangle,$$

$$|\eta_2\rangle = \frac{h}{\sqrt{f^2+h^2}}|00\rangle - \frac{f}{\sqrt{f^2+h^2}}|11\rangle. \tag{33}$$

Similarly, the other postmeasurement states shared between Alice-Bob and Bob-Charlie are given by

$$\rho_{12|1} = m(e^2+f^2)|\zeta_1\rangle\langle\zeta_1| + (1-m)(e^2+f^2)|\zeta_2\rangle\langle\zeta_2| + g^2|\phi_1\rangle\langle\phi_1| + h^2|\phi_3\rangle\langle\phi_3| + h^2|\phi_4\rangle\langle\phi_4|$$

$$\text{with } |\zeta_1\rangle = \frac{e}{\sqrt{e^2+f^2}}|00\rangle + \frac{f}{\sqrt{e^2+f^2}}|11\rangle,$$

$$|\zeta_2\rangle = \frac{e}{\sqrt{e^2+f^2}}|00\rangle - \frac{f}{\sqrt{e^2+f^2}}|11\rangle, \tag{34}$$

and

$$\rho_{34|1} = m(e^2+g^2)|\chi_1\rangle\langle\chi_1| + (1-m)(e^2+g^2)|\chi_2\rangle\langle\chi_2| + f^2|\phi_2\rangle\langle\phi_2| + h^2|\phi_3\rangle\langle\phi_3| + h^2|\phi_4\rangle\langle\phi_4|$$

$$\text{with } |\chi_1\rangle = \frac{g}{\sqrt{e^2+g^2}}|00\rangle + \frac{e}{\sqrt{e^2+g^2}}|11\rangle,$$

$$|\chi_2\rangle = \frac{g}{\sqrt{e^2+g^2}}|00\rangle - \frac{e}{\sqrt{e^2+g^2}}|11\rangle, \tag{35}$$

where e, f, g, h, $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, and $|\phi_4\rangle$ are defined earlier. For a fixed value of m and x, all the postmeasurement states are single parameter families of mixed states.

Similarly, the postmeasurement states $\rho_{14|i}$, $\rho_{12|i}$, and $\rho_{34|i}$ for other values of $i \in \{2, 3, 4\}$ can be evaluated.

Furthermore, the probability of getting the ith outcome is $p_i = \frac{1}{4}$ for all $i \in \{1, 2, 3, 4\}$.

Now, it can be checked that one can generate all types of quantum correlated states (i.e., entangled but not steerable, steerable but not Bell nonlocal, and Bell nonlocal) that are mentioned in Tables II, III, and IV in the pairs (1,4), (1,2),

TABLE V. The ranges of the measurement parameter λ for which the states $\rho_{12|i}$, $\rho_{34|i}$, and $\rho_{14|i}$ are entangled, or EPR steerable, or Bell nonlocal for all possible $i \in \{1, 2, 3, 4\}$ when each of the initial states shared by the pairs (1,2) and (3,4) is $m|\phi^+\rangle\langle\phi^+| + (1 - m)|\phi^-\rangle\langle\phi^-|$ with $0.9 \leqslant m \leqslant 0.97$. Here $\rho_{\alpha\beta|i}$ denotes the postmeasurement state for the pair (α, β) after Bob gets the outcome i ($i \in \{1, 2, 3, 4\}$) contingent upon performing the POVM (24) with $x = 0.55$, 0.77, or 0.9.

x	States	Entangled when	Steerable when	Bell nonlocal when
$x = 0.55$	$\rho_{14\|i}$	$0 < \lambda \leqslant 1$	Never	Never
	$\rho_{12\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$
	$\rho_{34\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$
$x = 0.77$	$\rho_{14\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	Never
	$\rho_{12\|i}$	$0 < \lambda \leqslant 1$	Never	Never
	$\rho_{34\|i}$	$0 < \lambda \leqslant 1$	Never	Never
$x = 0.9$	$\rho_{14\|i}$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$	$0 < \lambda \leqslant 1$
	$\rho_{12\|i}$	$0 < \lambda \leqslant 1$	Never	Never
	$\rho_{34\|i}$	$0 < \lambda \leqslant 1$	Never	Never

and (3,4), when the initial states shared by (1,2) and (3,4) are the above-mentioned mixed states with $0.9 \leqslant m \leqslant 0.97$ and when Bob gets any outcome contingent upon performing the POVM (24) with $x = 0.55$, 0.77, or 0.9. We summarize the results in Table V.

F. Probing network nonlocality

So far, while studying Bell nonlocality and EPR steering, we have not taken into account the fact that the sources producing the two initial entangled states are in general independent. Now, in order to consider independence of sources, we will study network nonlocality in the scenario considered by us. Before considering the specific scenario, let us assume that two arbitrary two-qubit states are initially shared by the pairs Alice-Bob and Bob-Charlie. This will lead to a general condition for violating the bilocality inequality (10) in the generalized entanglement swapping scenario considered by us.

Let Alice and Bob share a two-qubit state

$$\rho_{AB} = \frac{1}{4}\left[\mathbb{I}_2 \otimes \mathbb{I}_2 + \mathbf{r}_A^{AB} \cdot \boldsymbol{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathbf{r}_B^{AB} \cdot \boldsymbol{\sigma} + \sum_{i,j=1}^{3} t_{ij}^{AB} \sigma_i \otimes \sigma_j\right] \tag{36}$$

produced by the source S_1. Similarly, Bob and Charlie share a two-qubit state

$$\rho_{BC} = \frac{1}{4}\left[\mathbb{I}_2 \otimes \mathbb{I}_2 + \mathbf{r}_B^{BC} \cdot \boldsymbol{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathbf{r}_C^{BC} \cdot \boldsymbol{\sigma} + \sum_{i,j=1}^{3} t_{ij}^{BC} \sigma_i \otimes \sigma_j\right] \tag{37}$$

produced by another source S_2. Here, $\mathbf{r}_A^{AB} = (r_{A|1}^{AB}, r_{A|2}^{AB}, r_{A|3}^{AB})$ and $\mathbf{r_B}^{AB} = (r_{B|1}^{AB}, r_{B|2}^{AB}, r_{B|3}^{AB})$ represent the Bloch vectors of Alice's reduced state and Bob's reduced state, respectively, in ρ_{AB}; $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector composed of Pauli matrices; while t_{ij}^{AB} are the terms corresponding to the correlation matrix of the state ρ_{AB} and similarly for the state ρ_{BC}. The correlation matrix of the initial states can be written in polar decomposition form $t^{AB} = U^{AB}R^{AB}$ where U^{AB} is a unitary matrix and $R^{AB} = \sqrt{t^{AB\dagger}t^{AB}}$. Let $\tau_1^{AB} \geqslant \tau_2^{AB} \geqslant \tau_3^{AB}$ be three non-negative eigenvalues of R^{AB} and let $\tau_1^{BC} \geqslant \tau_2^{BC} \geqslant \tau_3^{BC}$ be three non-negative eigenvalues of R^{BC}.

Now we find a criteria for violating the bilocality inequality (10) when Alice and Charlie measure single-qubit projective measurements but Bob does a POVM measurement on two qubits.

1. First type of POVM

At first, let us consider that Bob performs the POVM mentioned in Eq. (15). Also, consider that Alice performs measurements of the observables $\boldsymbol{\sigma} \cdot \mathbf{u}$ and $\boldsymbol{\sigma} \cdot \mathbf{u}'$ depending on whether Alice's input is $\tilde{x} = 0$ or 1 respectively. Similarly, Charlie also performs measurements of the observables $\boldsymbol{\sigma} \cdot \mathbf{v}$ and $\boldsymbol{\sigma} \cdot \mathbf{v}'$ depending on whether his input is $\tilde{z} = 0$ or 1 respectively. Here, $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{u}' = (u_1', u_2', u_3')$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{v}' = (v_1', v_2', v_3')$ are unit vectors. Now, using the relations given by Eqs. (7)–(9), it turns out that $B_0 = E_1 + E_2 - E_3 - E_4 = \lambda\sigma_3 \otimes \sigma_3$ and $B_1 = E_1 - E_2 + E_3 - E_4 = \lambda\sigma_1 \otimes \sigma_1$ where $0 \leqslant \lambda \leqslant 1$.

One important point to be mentioned here is that the bilocality inequality (10) is valid when $|\langle B_0\rangle| \leqslant 1$ and $|\langle B_1\rangle| \leqslant 1$ [40]. In the present case, since we have $B_0 = \lambda\sigma_3 \otimes \sigma_3$ and $B_1 = \lambda\sigma_1 \otimes \sigma_1$ with $0 \leqslant \lambda \leqslant 1$, the two conditions $|\langle B_0\rangle| \leqslant 1$ and $|\langle B_1\rangle| \leqslant 1$ are satisfied. Hence, we can safely use the bilocality inequality (10).

Now to get the maximum value of $\mathcal{B}_{\text{bilocal}}$ in the present context, we will follow the method mentioned in [41]. At first, let us calculate the quantity I as follows:

$$\begin{aligned} I &= \mathrm{Tr}[\{(\mathbf{u} + \mathbf{u}') \cdot \boldsymbol{\sigma} \otimes \lambda(\sigma_3 \otimes \sigma_3) \otimes (\mathbf{v} + \mathbf{v}') \cdot \boldsymbol{\sigma}\} \\ &\quad \times (\rho_{AB} \otimes \rho_{BC})] \\ &= \lambda\,\mathrm{Tr}[\{(\mathbf{u} + \mathbf{u}') \cdot \boldsymbol{\sigma} \otimes \sigma_3\}\rho_{AB}]\mathrm{Tr}[\{\sigma_3 \otimes (\mathbf{v} + \mathbf{v}') \cdot \boldsymbol{\sigma}\}\rho_{BC}] \\ &= \lambda\left(\sum_{i=1,2,3} (u_i + u_i')\, t_{i3}^{AB}\right)\left(\sum_{k=1,2,3} (v_k + v_k')\, t_{3k}^{BC}\right). \end{aligned} \tag{38}$$

Similarly, J can be expressed as

$$\begin{aligned} J &= \mathrm{Tr}[\{(\mathbf{u} - \mathbf{u}') \cdot \boldsymbol{\sigma} \otimes \lambda(\sigma_1 \otimes \sigma_1) \otimes (\mathbf{v} - \mathbf{v}') \cdot \boldsymbol{\sigma}\} \\ &\quad \times (\rho_{AB} \otimes \rho_{BC})] \\ &= \lambda\,\mathrm{Tr}[\{(\mathbf{u} - \mathbf{u}') \cdot \boldsymbol{\sigma} \otimes \sigma_1\}\rho_{AB}]\,\mathrm{Tr}[\{\sigma_1 \otimes (\mathbf{v} - \mathbf{v}') \cdot \boldsymbol{\sigma}\}\rho_{BC}] \\ &= \lambda\left(\sum_{i=1,2,3} (u_i - u_i')\, t_{i1}^{AB}\right)\left(\sum_{k=1,2,3} (v_k - v_k')\, t_{1k}^{BC}\right). \end{aligned} \tag{39}$$

Let the initial states given by Eqs. (36) and (37) be defined in such a way that the Z and X Bloch directions of ρ_{AB} are given by the eigenvectors of the matrix R^{AB} associated with the two largest eigenvalues- τ_1^{AB} and τ_2^{AB}. Similarly, the Z and X Bloch directions of ρ_{BC} are also given by the eigenvectors of the matrix R^{BC} associated with the two largest eigenvalues τ_1^{BC} and τ_2^{BC}.

Next, we want to maximize $\mathcal{B}_{\text{bilocal}}$ with respect to $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{v}$, and $\mathbf{v}'$. From Eqs. (38) and (39), it is clear that the unit vectors $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{v}$, and $\mathbf{v}'$ should lie within the two-dimensional subspace spanned by two eigenvectors associated with the two largest eigenvalues. Hence, let us consider that $\mathbf{u} = (\sin\alpha, 0, \cos\alpha)$, $\mathbf{u}' = (\sin\alpha', 0, \cos\alpha')$, $\mathbf{v} = (\sin\gamma, 0, \cos\gamma)$, and $\mathbf{v}' = (\sin\gamma', 0, \cos\gamma')$. With these, the maximum value of $\mathcal{B}_{\text{bilocal}}$ can be found by solving the following: $\partial_\alpha \mathcal{B}_{\text{bilocal}} = 0$, $\partial_{\alpha'} \mathcal{B}_{\text{bilocal}} = 0$, $\partial_\gamma \mathcal{B}_{\text{bilocal}} = 0$, and $\partial_{\gamma'} \mathcal{B}_{\text{bilocal}} = 0$. After doing the above partial differentiation, we get various solutions. It can be checked that among those solutions the following set gives the maximum of $\mathcal{B}_{\text{bilocal}}$: $\alpha = -\alpha'$, $\gamma = -\gamma'$, and $\cos\alpha = \cos\gamma = \sqrt{\frac{\lambda\tau_1^{AB}\tau_1^{BC}}{\lambda\tau_1^{AB}\tau_1^{BC}+\lambda\tau_2^{AB}\tau_2^{BC}}}$. And the maximal value of the left-hand side of the bilocality inequality (10) is given by

$$\mathcal{B}_{\text{bilocal}}^{\max} = 2\sqrt{\lambda\tau_1^{AB}\tau_1^{BC} + \lambda\tau_2^{AB}\tau_2^{BC}}. \tag{40}$$

The above relation is valid for any two arbitrary two-qubit states shared by Alice-Bob or Bob-Charlie and when Bob performs the two-qubit POVM given by (15). Hence, the bilocality inequality (10) will be violated if

$$\lambda\tau_1^{AB}\tau_1^{BC} + \lambda\tau_2^{AB}\tau_2^{BC} > 1. \tag{41}$$

In our scenario, we have considered maximally entangled Bell pairs: $\rho_{AB} = \rho_{BC} = |\phi^+\rangle\langle\phi^+|$. Hence, from (40), the bilocality inequality becomes

$$\mathcal{B}_{\text{bilocal}}^{\max} = 2\sqrt{2\lambda} \leqslant 2 \tag{42}$$

as $\tau_1^{AB} = \tau_2^{AB} = 1$ and $\tau_1^{BC} = \tau_2^{BC} = 1$ in the present case. Therefore, the bilocality inequality (10) is violated if $\lambda > 1/2$.

2. Second type of POVM

Now, we want to get a criteria to violate the bilocality inequality when Bob performs the POVM given by Eq. (24) with different values of x. Here also, Alice performs measurements of the observables $\boldsymbol{\sigma}\cdot\mathbf{u}$ and $\boldsymbol{\sigma}\cdot\mathbf{u}'$ depending on whether Alice's input is $\tilde{x} = 0$ or 1 respectively. Charlie performs measurements of the observables $\boldsymbol{\sigma}\cdot\mathbf{v}$ and $\boldsymbol{\sigma}\cdot\mathbf{v}'$ depending on whether his input is $\tilde{z} = 0$ or 1 respectively. Using the relations (7)–(9), it turns out in the present case that $B_1 = E_1 - E_2 + E_3 - E_4 = \Gamma_1\,\sigma_1\otimes\sigma_1 + \Gamma_2\,\sigma_3\otimes\mathbb{I}_2$ and $B_0 = E_1 + E_2 - E_3 - E_4 = \Gamma_3\,\sigma_3\otimes\sigma_3$, where $\Gamma_1 = \frac{2ab(y_1-2xy_1-xy_2)}{y_1+y_2}$, $\Gamma_2 = \frac{(1-2b^2)(2x-1)y_1+y_2(1-2b^2x)}{y_1+y_2}$, and $\Gamma_3 = \frac{y_1-y_2+2xy_2}{y_1+y_2}$ where a, b, y_1, and y_2 are defined earlier (case II) after Eq. (24). It can be checked that the maximum eigenvalue of B_0 and B_1 is less than or equal to unity for all $\lambda \in [0, 1]$ and for all $x \in [0, 1]$. Hence, the conditions $|\langle B_0\rangle| \leqslant 1$ and $|\langle B_1\rangle| \leqslant 1$ are satisfied and we can use the bilocality inequality (10).

Now, to find the maximum magnitude of $\mathcal{B}_{\text{bilocal}}$ mentioned in Eq. (10), we evaluate I and J following the process that was done in the case of the previous POVM. These can be written as

$$\begin{aligned} I &= \text{Tr}[\{(\mathbf{u}+\mathbf{u}')\cdot\boldsymbol{\sigma}\otimes\Gamma_3(\sigma_3\otimes\sigma_3) \\ &\quad\otimes(\mathbf{v}+\mathbf{v}')\cdot\boldsymbol{\sigma}\}(\rho_{AB}\otimes\rho_{BC})] \\ &= \Gamma_3\left(\sum_{i=1,2,3}(u_i+u_i')\,t_{i3}^{AB}\right)\left(\sum_{k=1,2,3}(v_k+v_k')\,t_{3k}^{BC}\right) \end{aligned} \tag{43}$$

and

$$\begin{aligned} J &= \text{Tr}[\{(\mathbf{u}-\mathbf{u}')\cdot\boldsymbol{\sigma}\otimes(\Gamma_2\,\sigma_3\otimes\mathbb{I}_2 \\ &\quad+\Gamma_1\,\sigma_1\otimes\sigma_1)\otimes(\mathbf{v}-\mathbf{v}')\cdot\boldsymbol{\sigma}\}(\rho_{AB}\otimes\rho_{BC})] \\ &= \Gamma_2\left(\sum_{i=1,2,3}(u_i-u_i')t_{i3}^{AB}\right)\left(\sum_{k=1,2,3}(v_k-v_k')r_{C|k}^{BC}\right) \\ &\quad+\Gamma_1\left(\sum_{i=1,2,3}(u_i-u_i')t_{i1}^{AB}\right)\left(\sum_{k=1,2,3}(v_k-v_k')t_{1k}^{BC}\right). \end{aligned} \tag{44}$$

Now, we have to maximize the bilocality inequality (10) with respect to the unit vectors $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{v}$, and $\mathbf{v}'$. Here, we choose the above unit vectors in three-dimensional space which is the most general case. Let $\mathbf{u}= (\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$, $\mathbf{u}' = (\sin\alpha'\cos\beta',\ \sin\alpha'\sin\beta',\ \cos\alpha')$, $\mathbf{v} = (\sin\gamma\cos\phi,\ \sin\gamma\sin\phi,\ \cos\gamma)$, and $\mathbf{v}' = (\sin\gamma'\cos\phi',\ \sin\gamma'\sin\phi',\ \cos\gamma')$. It can be checked that the the maximum value of $\mathcal{B}_{\text{bilocal}}$ is obtained at $\alpha = \alpha'$, $\beta = 0$, $\beta' = \pi$, $\gamma = \gamma', \phi = 0, \phi' = \pi$, and $\cos\alpha = \cos\gamma = \sqrt{\frac{|\Gamma_3\tau_1^{AB}\tau_1^{BC}|}{|\Gamma_1\tau_2^{AB}\tau_2^{BC}|+|\Gamma_3\tau_1^{AB}\tau_1^{BC}|}}$. And the maximum value is given by

$$\mathcal{B}_{\text{bilocal}}^{\max} = 2\sqrt{\left|\Gamma_1\tau_2^{AB}\tau_2^{BC}\right| + \left|\Gamma_3\tau_1^{AB}\tau_1^{BC}\right|}. \tag{45}$$

The above expression infers that when Alice-Bob and Bob-Charlie initially share two arbitrary two-qubit states between them produced from two independent sources and Bob performs the POVM given by (24) on his qubits, then network nonlocality is demonstrated if

$$\left|\Gamma_1\tau_2^{AB}\tau_2^{BC}\right| + \left|\Gamma_3\tau_1^{AB}\tau_1^{BC}\right| > 1. \tag{46}$$

In our scenario, we have considered maximally entangled Bell states: $\rho_{AB} = \rho_{BC} = |\phi^+\rangle\langle\phi^+|$. Hence, from (45), the bilocality inequality becomes

$$\mathcal{B}_{\text{bilocal}}^{\max} = 2\sqrt{|\Gamma_3| + |\Gamma_1|} \leqslant 2 \tag{47}$$

as $\tau_1^{AB} = \tau_2^{AB} = 1$ and $\tau_1^{BC} = \tau_2^{BC} = 1$ in the present case. Therefore, the bilocality inequality (10) is violated when $|\Gamma_{33}| + |\Gamma_{11}| > 1$.

Next, we find the condition under which bilocality inequality (10) is violated with the POVM given by Eq. (24) with different values of x using the expressions obtained above. The results obtained are presented in Table VI. In particular, we find different values of x for which the network is nonbilocal or bilocal with respect to the quantum violation of the bilocality inequality (10) for the whole range of the measurement parameter $\lambda \in [0, 1]$. Interestingly, we observe that for $x = 0.5$ the generated correlation does not violate the

TABLE VI. The ranges of the measurement parameter λ for which the network is nonbilocal or bilocal with respect to the quantum violation of the bilocality inequality (10) when the POVM (24) with different x is performed by Bob.

x	Nonbilocal when
$x = 0.3$	$0 < \lambda < 0.51$
$x = 0.5$	Never
$x = 0.725$	$0 < \lambda \leqslant 1$
$x = 0.8$	$0 < \lambda \leqslant 1$

bilocality inequality (10) for the whole range of the measurement parameter λ. However, the states shared between the pairs (1,4), (1,2), and (3,4) after completion of the generalized entanglement swapping protocol are entangled for the whole range of measurement parameter λ except at $\lambda = 0$ when the POVM (24) with $x = 0.5$ is performed by Bob. Also, for $x = 0.725$, the generated correlation demonstrates network nonlocality for the whole range of λ except at $\lambda = 0$, whereas the states shared by the pairs (1,4), (1,2), and (3,4) after completion of the protocol do not violate the Bell-CHSH inequality (see Table III) for any λ in this case. In other words, after a generalized entanglement swapping protocol, one may end up with states that seem to be (Bell) local, while, in fact, they are (network) nonlocal.

V. CONCLUDING DISCUSSIONS

In the present paper, we have considered a generalized entanglement swapping protocol, where Alice and Bob share a Bell pair (1,2) whereas Bob and Charlie share another Bell pair (3,4) in the same state. Bob performs a joint quantum measurement (POVM) on the pair (2,3) and discloses the outcome via classical communication to Alice and Charlie. In this scenario, our key findings are twofold. At first, using a generalized entanglement swapping protocol, we have created different kinds of quantum correlated single parameter families of two-qubit mixed states in (1,4). These states have the following features: (1) a single parameter class of mixed states that gradually shows entanglement, EPR steering, and Bell nonlocality as the state parameter is varied; (2) a single parameter class of mixed states that is entangled but not EPR steerable (hence, not Bell nonlocal) for the whole range of the state parameter; (3) a single parameter class of mixed entangled states that is steerable but not Bell nonlocal for the whole range of the state parameter; and (4) a single parameter class of mixed entangled states that is Bell nonlocal for the whole range of the state parameter.

Secondly, in the standard entanglement swapping protocol, the quantum correlation is completely transferred from the pairs (1,2) and (3,4) to the pair (1,4). Therefore, the natural question that arises in this context is how such transfer of quantum correlations changes when the standard entanglement swapping protocol is generalized by changing the initial states, or by changing the measurements, or both [3–8]. In the present paper, we have addressed such questions by demonstrating nontrivial transfers of quantum correlations (e.g., Bell nonlocality, EPR steering, and entanglement) from (1,2) and (3,4) to (1,4) by choosing different quantum measurements and/or initial states that are summarized below.

(1) For the first POVM considered by us, we have shown that different quantum correlations are generated gradually (i.e., entanglement is generated at first, followed by the generation of EPR steering and then Bell nonlocality) in (1,4) with an increase in the measurement parameter at the expense of gradual destruction of these three quantum correlations in (1,2) and (3,4).

(2) For the second class of POVM, entanglement is generated in the pair (1,4) although the entanglement in each of the pairs (1,2) and (3,4) is not destroyed completely. Further, Bell nonlocality or EPR steering in each of the pairs (1,2) and (3,4) is not completely lost and, consequently, these two correlations are not generated in (1,4).

(3) In the case of the third POVM, the pair (1,4) gains entanglement although the entanglement in each of the pairs (1,2) and (3,4) is not destroyed completely for the whole range of the measurement strength λ except at $\lambda = 0$. Further, Bell nonlocality is completely lost as a result of the protocol in each of the pairs (1,2) and (3,4) whereas Bell nonlocality is not generated in (1,4).

(4) For the fourth POVM, the pair (1,4) gains Bell nonlocality (and, hence, EPR steering as well as entanglement). On the other hand, each of the pairs (1,2) and (3,4) completely loses EPR steering and Bell nonlocality, but retains entanglement.

Hence, the present paper provides operational tools in the basic entanglement swapping setup for generating different kinds of correlated states between two spatially separated observers without any interaction between them. In each of the above cases, the whole class can be generated by varying the measurement parameter. Since the three types of quantum correlations, namely entanglement, EPR steering, and Bell nonlocality, act as resources in different kinds of tasks, each of the above-mentioned classes of mixed states has a distinct role as a resource depending on the type of task to be performed. Remarkably, our results point out that all these types of quantum correlated mixed states can be generated in a single setup just by choosing appropriate POVMs. We thus expect that our results will be helpful for experimental generation of different types of correlations in a single setup that can be employed in different information processing and communication tasks.

Note that experiments on entanglement swapping have been reported in several studies where maximally entangled initial states were produced with high fidelity [55–58]. However, in order to take into account realistic nonidealness, we have further extended our studies by taking mixed initial states shared by the pairs (1,2) and (3,4). In such cases also, we have shown that the aforementioned different kinds of quantum correlated single parameter families of two-qubit states can be generated in (1,4) by choosing appropriate POVM for Bob.

Here, we want to clarify that while studying standard Bell nonlocality or EPR steering of the states after completion of the generalized entanglement swapping protocol, we have not considered the possibility that the two sources producing the initial entangled states may be independent. That is why we have also studied network nonlocality in the generalized entanglement swapping scenario considered by us as this concept incorporates the fact of independence of sources that

is relevant in the context of entanglement swapping scenarios [39,42]. Here, we have found appropriate joint POVMs performed by Bob on the pair (2,3) for which the resulting statistics demonstrates or does not demonstrate network nonlocality for the whole range of the measurement parameter. Interestingly, we have shown an instance of a POVM for which the measurement statistics does not show network nonlocality, but all the states shared by (1,2), (3,4), and (1,4) become entangled after the generalized entanglement swapping for the whole range of the measurement parameter. We have also presented an example where all the states shared by the pairs (1,2), (3,4), and (1,4) after completion of the entanglement swapping protocol are Bell local if we consider that the sources producing the initial entangled states are not necessarily independent, while they show network nonlocality when the independence of sources has been taken into account. This may hold for EPR steering as well.

Before concluding we would like to mention some future research directions that we leave open. The multipartite entanglement swapping protocol [3] assisted with different kinds of POVM for generating different kinds of genuine multipartite quantum correlations is worth studying in the future. Also, probing network steering (incorporating statistical independence of the sources) in the context of our paper is another direction for future research. Finally, finding the amount of information loss or gain by different pairs [9,10] in our entanglement swapping setups is of fundamental interest.

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[1] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, "Event-Ready-Detectors" Bell Experiment Via Entanglement Swapping, Phys. Rev. Lett. **71**, 4287 (1993).

[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).

[3] S. Bose, V. Vedral, and P. L. Knight, Multiparticle generalization of entanglement swapping, Phys. Rev. A **57**, 822 (1998).

[4] G. Gour, and B. C. Sanders, Remote Preparation and Distribution of Bipartite Entangled States, Phys. Rev. Lett. **93**, 260501 (2004).

[5] S. Perseguers, C. I. Cirac, A. Acin, M. Lewenstein, and J. Wehr, Entanglement distribution in pure-state quantum networks, Phys. Rev. A **77**, 022308 (2008).

[6] S. Perseguers, G. J. Lapeyre, Jr., D. Cavalcanti, M. Lewenstein, and A. Acin, Distribution of entanglement in large-scale quantum networks, Rep. Prog. Phys. **76**, 096001 (2013).

[7] A. Wojcik, J. Modlawska, A. Grudka, and M. Czechlewski, Violation of Clauser-Horne-Shimony-Holt inequality for states resulting from entanglement swapping, Phys. Lett. A **374**, 4831 (2010).

[8] Waldemar Kłobus, W. Laskowski, M. Markiewicz, and A. Grudka, Nonlocality activation in entanglement-swapping chains, Phys. Rev. A **86**, 020302(R) (2012).

[9] C. Brukner, M. Aspelmeyer, and A. Zeilinger, Complementarity and information in delayed-choice for entanglement swapping, Found. Phys. **35**, 1909 (2005).

[10] P. Bej, A. Ghosal, D. Das, A. Roy, and S. Bandyopadhyay, Information-disturbance trade-off in generalized entanglement swapping, Phys. Rev. A **102**, 052416 (2020).

[11] P. Busch, P. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement*, 2nd ed. (Springer-Verlag, Berlin, 1996).

[12] P. Busch, M. Grabowski, and P. J. Lathi, *Operational Quantum Physics* (Springer-Verlag, Berlin, 1997).

[13] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).

[14] S. Popescu, Bell's Inequalities and Density Matrices: Revealing "Hidden" Nonlocality, Phys. Rev. Lett. **74**, 2619 (1995).

[15] N. Gisin, Hidden quantum nonlocality revealed by local filters, Phys. Lett. A **210**, 151 (1996).

[16] D. Das, S. Mal, and D. Home, Testing local-realism and macrorealism under generalized dichotomic measurements, Phys. Lett. A **382**, 1085 (2018).

[17] S. Mal, A. S. Majumdar, and D. Home, Sharing of nonlocality of a single member of an entangled pair of qubits is not possible by more than two unbiased observers on the other wing, Mathematics **4**, 48 (2015).

[18] D. Das, A. Ghosal, S. Sasmal, S. Mal, and A. S. Majumdar, Facets of bipartite nonlocality sharing by multiple observers via sequential measurements, Phys. Rev. A **99**, 022305 (2019).

[19] P. J. Brown, and R. Colbeck, Arbitrarily Many Independent Observers Can Share the Nonlocality of a Single Maximally Entangled Qubit Pair, Phys. Rev. Lett. **125**, 090401 (2020).

[20] S. Sasmal, D. Das, S. Mal, and A. S. Majumdar, Steering a single system sequentially by multiple observers, Phys. Rev. A **98**, 012305 (2018).

[21] A. Bera, S. Mal, A. Sen De, and U. Sen, Witnessing bipartite entanglement sequentially by multiple observers, Phys. Rev. A **98**, 062304 (2018).

[22] K. Mohan, A. Tavakoli, and N. Brunner, Sequential random access codes and self-testing of quantum measurement instruments, New J. Phys. **21**, 083034 (2019).

[23] S. Roy, A. Bera, S. Mal, A. Sen De, and U. Sen, Recycling the resource: Sequential usage of shared state in quantum teleportation with weak measurements, Phys. Lett. A **392**, 127143 (2021).

[24] D. Das, A. Ghosal, A. G. Maity, S. Kanjilal, and A. Roy, Ability of unbounded pairs of observers to achieve quantum advantage in random access codes with a single pair of qubits, Phys. Rev. A **104**, L060602 (2021).

[25] P. Halder, S. Mal, and A. Sen De, Measurement-based multipartite entanglement inflation, Phys. Rev. A **104**, 062412 (2021).
[26] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[27] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Guhne, Quantum steering, Rev. Mod. Phys. **92**, 015001 (2020).
[28] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).
[29] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, Phys. Rev. Lett. **98**, 140402 (2007).
[30] M. T. Quintino, T. Vertesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acin, and N. Brunner, Inequivalence of entanglement, steering, and Bell nonlocality for general measurements, Phys. Rev. A **92**, 032107 (2015).
[31] C. H. Bennett, and S. J. Wiesner, Communication via One- and Two-Particle Operators on Einstein-Podolsky-Rosen States, Phys. Rev. Lett. **69**, 2881 (1992).
[32] Antonio Acín, N. Gisin, and L. Masanes, From Bell's Theorem to Secure Quantum Key Distribution, Phys. Rev. Lett. **97**, 120405 (2006).
[33] S. Pironio, A. Acin, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, Nature (London) **464**, 1021 (2010).
[34] R. Colbeck, and R. Renner, Free randomness can be amplified, Nat. Phys. **8**, 450 (2012).
[35] N. Brunner, and N. Linden, Connection between Bell nonlocality and Bayesian game theory, Nat. Commun. **4**, 2057 (2013).
[36] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, Phys. Rev. A **85**, 010301(R) (2012).
[37] M. D. Reid, Signifying quantum benchmarks for qubit teleportation and secure quantum communication using Einstein-Podolsky-Rosen steering inequalities, Phys. Rev. A **88**, 062338 (2013).
[38] Y. Z. Law, L. P. Thinh, J. D. Bancal, and V. Scarani, Quantum randomness extraction for various levels of characterization of the devices, J. Phys. A: Math. Theor. **47**, 424028 (2014).
[39] C. Branciard, N. Gisin, and S. Pironio, Characterizing the Nonlocal Correlations Created via Entanglement Swapping, Phys. Rev. Lett. **104**, 170401 (2010).
[40] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus nonbilocal correlations in entanglement-swapping experiments, Phys. Rev. A **85**, 032119 (2012).
[41] N. Gisin, Q. Mei, A. Tavakoli, M. O. Renou, and N. Brunner, All entangled pure quantum states violate the bilocality inequality, Phys. Rev. A **96**, 020304(R) (2017).
[42] A. Tavakoli, A. P. Kerstjens, M. X. Luo, and M. O. Renou, Bell nonlocality in networks, Rep. Prog. Phys. **85**, 056001 (2022).
[43] F. Andreoli, G. Carvacho, L. Santodonato, R. Chaves, and F. Sciarrino, Maximal qubit violation of n-locality inequalities in a star-shaped quantum network, New J. Phys. **19**, 113020 (2017).
[44] S. Gupta, S. Datta, and A. S. Majumdar, Preservation of quantum nonbilocal correlations in noisy entanglement-swapping experiments using weak measurements, Phys. Rev. A **98**, 042322 (2018).
[45] A. Kundu, M. Kamal Molla, I. Chattopadhyay, and D. Sarkar, Maximal qubit violation of n-local inequalities in a quantum network, Phys. Rev. A **102**, 052222 (2020).
[46] J. S. Bell, On the einstein podolsky rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).
[47] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).
[48] R. Horodecki, P. Horodecki, and M. Horodecki, Violating Bell inequality by mixed spin-1/2 states: Necessary and sufficient condition, Phys. Lett. A **200**, 340 (1995).
[49] R. Horodecki, Two spin-1/2 mixture and Bell inequalities, Phys. Lett. A **210**, 223 (1996).
[50] A. C. S. Costa, and R. M. Angelo, Quantification of Einstein-Podoloski-Rosen steering for two qubit states, Phys. Rev. A **93**, 020103(R) (2016).
[51] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox, Phys. Rev. A **80**, 032112 (2009).
[52] P. Girdhar and E. G. Cavalcanti, All two-qubit states that are steerable via Clauser-Horne-Shimony-Holt-type correlations are Bell nonlocal, Phys. Rev. A **94**, 032317 (2016).
[53] G. Vidal, and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).
[54] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Volume of the set of separable states, Phys. Rev. A **58**, 883 (1998).
[55] J. Jin, M. Grimau Puigibert, L. Giner, J. A. Slater, M. R. E. Lamont, V. B. Verma, M. D. Shaw, F. Marsili, S. W. Nam, D. Oblak, and W. Tittel, Entanglement swapping with quantum-memory-compatible photons, Phys. Rev. A **92**, 012329 (2015).
[56] W. H. Zhang, G. Chen, X.-X. Peng, X. J. Ye, P. Yin, X.-Y. Xu, J.-S. Xu, C.-F. Li, and G.-C. Guo, Experimental Realization of Robust Self-Testing of Bell State Measurements, Phys. Rev. Lett. **122**, 090402 (2019).
[57] X. Liu, J. Hu, Z.-F. Li, X. Li, P.-Y. Li, P.-J. Liang, Z.-Q. Zhou, C.-F. Li, and G.-C. Guo, Heralded entanglement distribution between two absorptive quantum memories, Nature (London) **594**, 41 (2021).
[58] D. J. Saunders, A. J. Bennet, C. Branciard, and G. J. Pryde, Experimental demonstration of nonbilocal quantum correlations, Sci. Adv. **3**, e1602743 (2017).