

Skier and loop-the-loop with friction

Dominik Kufel and Alan D. Sokal

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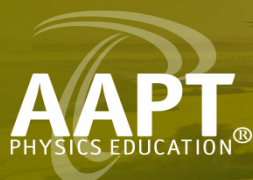
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Skier and loop-the-loop with friction

Dominik Kufel^{a)}

Department of Physics and Astronomy, University College London, London WC1E 6BT, United Kingdom

Alan D. Sokal^{b)}

Department of Mathematics, University College London, London WC1E 6BT, United Kingdom
and Department of Physics, New York University, New York, New York 10003

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We solve analytically the differential equations for a skier on a hemispherical hill and for a particle on a loop-the-loop track when the hill or track is endowed with a coefficient of kinetic friction μ . For each problem, we determine the exact “phase diagram” in the two-dimensional parameter plane. © 2022 All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).

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I. INTRODUCTION

Two classic homework exercises in an elementary mechanics course are the skier on a hemispherical hill (Fig. 1) and the particle on a loop-the-loop track (Fig. 2).¹ Both problems illustrate nicely the use of conservation of energy (to find the speed as a function of height) followed by $\mathbf{F} = m\mathbf{a}$ (to find the normal force).

It is interesting to consider what happens when the hill or track is endowed with a coefficient of kinetic friction μ . Somewhat surprisingly, the exact differential equations turn out to be analytically solvable.^{2–11} Our purpose here is to provide a unified treatment of the two problems, using only elementary methods that are easily accessible to undergraduates (e.g., linear first-order differential equations). Though most of our results have been obtained previously—as we shall document in detail—they are somewhat scattered in the literature. It may thus be of some modest value to have a complete elementary derivation collected in one place.

The skier and loop-the-loop problems give rise to very similar differential equations, which differ only by some sign changes. However, these sign changes lead to significant differences in the qualitative interpretation of the solutions. Since the skier problem turns out to be somewhat simpler, we treat it first and give a complete solution; in particular, we determine the exact phase diagram in the two-dimensional parameter plane. For the loop-the-loop, we solve the differential equations only up to the first time (if any) that the particle halts or completes one cycle of the loop, so we obtain only a partial phase diagram. The full phase diagram will (as we explain later) contain an infinite sequence of bifurcations, and we leave its computation to a reader who wishes to take up where we have left off.

II. SKIER ON A HEMISPHERICAL HILL

Consider a skier of mass m on a hemispherical hill of radius R (or more generally, any hill of circular cross section) and coefficient of kinetic friction μ , entering at the top with forward velocity v_0 ; let θ denote the angle from the vertical (Fig. 1). Then, the radial and tangential components of $\mathbf{F} = m\mathbf{a}$ are¹²

$$N - mg \cos \theta = -mR\dot{\theta}^2, \quad (1)$$

$$mg \sin \theta - \mu N \operatorname{sgn}(\dot{\theta}) = mR\ddot{\theta}. \quad (2)$$

This is a pair of coupled differential equations for the unknown functions $\theta(t)$ and $N(t)$. We stress, however, that these equations are valid only as long as $N \geq 0$; after that, the skier flies off the hill. Since it is clear that the skier will only go down the hill, not up, we have $\dot{\theta} \geq 0$ throughout the motion, and the factor $\operatorname{sgn}(\dot{\theta})$ in Eq. (2) can be dropped.¹³

Differentiating Eq. (1) with respect to time yields

$$\frac{dN}{dt} = -(mg \sin \theta + 2mR\ddot{\theta}) \dot{\theta}, \quad (3)$$

and inserting $\ddot{\theta}$ from Eq. (2) [with $\operatorname{sgn}(\dot{\theta}) = 1$] yields

$$\frac{dN}{dt} = -(3mg \sin \theta - 2\mu N) \dot{\theta}. \quad (4)$$

Using the chain rule $dN/dt = (dN/d\theta)(d\theta/dt)$, we can eliminate $\dot{\theta}$ from Eq. (4), leading to

$$\frac{dN}{d\theta} - 2\mu N = -3mg \sin \theta. \quad (5)$$

This is a first-order inhomogeneous linear differential equation with constant coefficients for the unknown function $N(\theta)$, and it can be solved by the method of integrating factors. Here, the integrating factor is $e^{-2\mu\theta}$, and the solution is¹⁴

$$N(\theta) = N_0 e^{2\mu\theta} - 3mg \frac{e^{2\mu\theta} - \cos \theta - 2\mu \sin \theta}{1 + 4\mu^2}, \quad (6)$$

where $N_0 = N(0)$. We again stress that this solution is valid only where $N(\theta) \geq 0$; at the first angle (if any) where $N(\theta)$ crosses zero to a negative value, the skier flies off the hill.

Evaluating Eq. (1) at $\theta = 0$, where the skier’s angular velocity is $\dot{\theta} = v_0/R$, we obtain $N_0 = mg - mv_0^2/R$. In particular, if the dimensionless parameter $\lambda \stackrel{\text{def}}{=} v_0^2/gR$ is ≥ 1 , then $N_0 \leq 0$ and the skier immediately flies off the hill; we, therefore, assume henceforth that $0 \leq \lambda < 1$. Inserting $N_0 = (1 - \lambda)mg$ in Eq. (6), we obtain

$$N(\theta) = (1 - \lambda)mg e^{2\mu\theta} - 3mg \frac{e^{2\mu\theta} - \cos \theta - 2\mu \sin \theta}{1 + 4\mu^2}, \quad (7)$$

which is the closed-form solution giving the normal force as a function of angle.



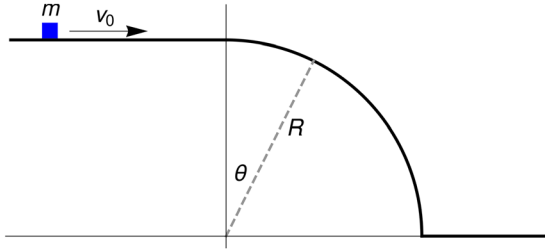


Fig. 1. Skier on a hill of quarter-circular cross section. The horizontal portion of the hill is frictionless; the circular portion has a coefficient of kinetic friction μ .

In the absence of friction ($\mu = 0$), Eq. (7) simplifies to

$$N(\theta) = (3 \cos \theta - 2 - \lambda)mg. \quad (8)$$

This is a decreasing function of θ , and skier flies off the hill when $N = 0$, i.e., when

$$\theta = \cos^{-1} \left(\frac{2 + \lambda}{3} \right). \quad (9)$$

In the usual textbook problem, one has also $v_0 = 0$ (i.e., $\lambda = 0$), and we obtain the standard answer that the skier flies off at angle $\theta = \cos^{-1}(2/3) \approx 48.19^\circ$.

When $\mu > 0$, by contrast, the normal force is no longer a decreasing function of θ , nor is it guaranteed to reach zero within the interval $0 \leq \theta \leq \pi/2$. Indeed, $dN/d\theta|_{\theta=0} = 2\mu(1-\lambda)mg > 0$, so the normal force is initially increasing.

We can also obtain the velocity as a function of angle. It is convenient to define the dimensionless quantity $\Lambda \stackrel{\text{def}}{=} v^2/gR = R\dot{\theta}^2/g$; its value at $\theta = 0$ is what we have called λ . Then, from Eq. (1) we have immediately

$$N = (\cos \theta - \Lambda)mg, \quad (10)$$

[which reduces to $N_0 = (1 - \lambda)mg$ when $\theta = 0$] or equivalently,

$$\Lambda = \cos \theta - \frac{N}{mg}. \quad (11)$$

In particular, from $N \geq 0$ we deduce that $\Lambda \leq \cos \theta$: This gives the maximum speed that the skier can have at any

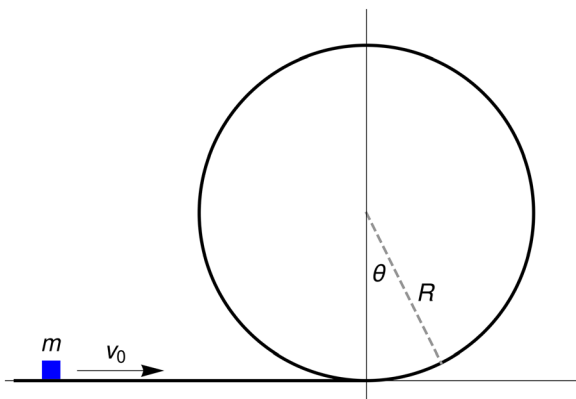


Fig. 2. Particle on a loop-the-loop. The horizontal portion of the track is frictionless; the circular portion has a coefficient of kinetic friction μ .

given angle if she is to avoid flying off the hill. Combining Eqs. (7) and (11) gives the closed-form solution for the speed as a function of angle as follows:¹⁵

$$\Lambda(\theta) = \cos \theta - (1 - \lambda)e^{2\mu\theta} + 3 \frac{e^{2\mu\theta} - \cos \theta - 2\mu \sin \theta}{1 + 4\mu^2}. \quad (12)$$

Note, however, that this solution is valid only where $\Lambda(\theta) \geq 0$; at the first angle (if any) where $\Lambda(\theta) = 0$, the skier comes to rest (perhaps only asymptotically as $t \rightarrow +\infty$). The solution (12) must therefore be supplemented by the two inequalities $0 \leq \Lambda(\theta) \leq \cos \theta$.

From Eqs. (5) and (10)/(11), we see that $\Lambda(\theta)$ satisfies the differential equation,¹⁶

$$\frac{d\Lambda}{d\theta} - 2\mu\Lambda = 2(\sin \theta - \mu \cos \theta). \quad (13)$$

The solution of this differential equation with the initial condition $\Lambda(0) = \lambda$ is of course Eq. (12).¹⁴

In the absence of friction ($\mu = 0$), Eq. (12) simplifies to

$$\Lambda(\theta) = \lambda + 2(1 - \cos \theta), \quad (14)$$

which is just the expression for conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + mgR(1 - \cos \theta)$. More generally, the kinetic energy plus gravitational potential energy is

$$\begin{aligned} E &= \frac{1}{2}mv^2 + mgR(\cos \theta - 1) \\ &= \frac{1}{2}mgR[\Lambda(\theta) + 2(\cos \theta - 1)], \end{aligned} \quad (15)$$

so that

$$\frac{dE}{dt} = \frac{1}{2}mgR \left[\frac{d\Lambda}{d\theta} - 2 \sin \theta \right] \dot{\theta}. \quad (16)$$

The work-energy theorem asserts that dE/dt must equal the rate of work done by friction, which is $-\mu NR\dot{\theta}$; and this equality is an immediate consequence of Eqs. (10), (13), and (16). Conversely, the differential equation (13) could alternatively be derived by combining the work-energy theorem with Eqs. (10) and (16).¹⁷ It may be useful for students to compare these two derivations: one directly from the Newtonian equations of motion, and the other from the work-energy theorem.

Finally, we can use Eq. (12) to obtain the time-dependence of the motion. From $\Lambda = R\dot{\theta}^2/g$, we have

$$\frac{d\theta}{dt} = \left[\frac{g}{R} \Lambda(\theta) \right]^{1/2}, \quad (17)$$

and hence

$$t(\theta) = \int_0^\theta \frac{d\theta'}{\left[\frac{g}{R} \Lambda(\theta') \right]^{1/2}}. \quad (18)$$

We can now analyze the qualitative behavior of the motion as a function of the two parameters $\mu \in [0, \infty)$ and $\lambda \in [0, 1)$. We have seen that the skier halts when $\Lambda(\theta) = 0$

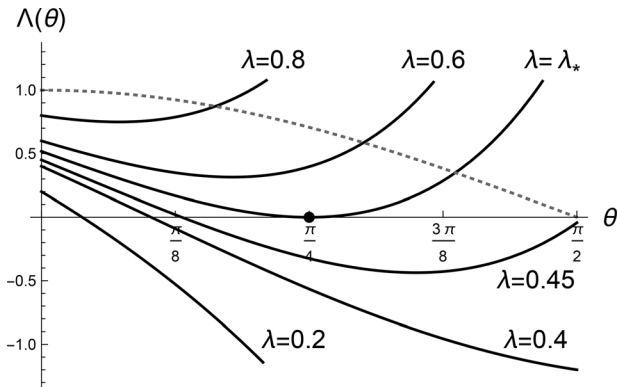


Fig. 3. The curves $\Lambda(\theta)$ for $\mu=1$ and $\lambda = 0.2, 0.4,$ and $0.45,$ and $\lambda_*(1) \approx 0.517594, 0.6,$ and 0.8 . The skier halts when $\Lambda(\theta) = 0$, or flies off the hill when $\Lambda(\theta) = \cos \theta$ (shown as a dotted curve), whichever happens first. The critical curve is $\lambda = \lambda_*$. The dot indicates the point $\theta = \theta_*$ (here $\theta_* = \arctan 1 = \pi/4$).

or flies off the hill when $\Lambda(\theta) = \cos \theta$, whichever happens first; if neither happens for $\theta < \pi/2$, then the skier reaches the bottom of the hill. (We will see later that this last case never occurs.) The critical solution that separates these two scenarios is given by the trajectory for which the skier halts at an angle θ_* (hence $\Lambda(\theta_*) = 0$) that also satisfies $\Lambda'(\theta_*) = 0$: See the curve marked $\lambda = \lambda_*$ in Fig. 3. Applying this condition in Eq. (13) leads immediately to¹⁸

$$\theta_*(\mu) \stackrel{\text{def}}{=} \arctan \mu. \quad (19)$$

Substituting this in Eq. (12), we obtain the relationship between the initial velocity and the friction coefficient that defines the phase boundary,¹⁹

$$\lambda_*(\mu) \stackrel{\text{def}}{=} \frac{4\mu^2 - 2 + 2e^{-2\mu \arctan \mu} \sqrt{1 + \mu^2}}{1 + 4\mu^2}. \quad (20)$$

Please observe that $\lambda_*(\mu)$ is an increasing function of μ that runs from 0 to 1 as μ runs from 0 to ∞ (see Fig. 4).

In this way, we have obtained a phase diagram that divides the (μ, λ) plane into three possible qualitative behaviors:

- For $0 \leq \lambda < \lambda_*(\mu)$, the skier halts after a finite time at some angle $\theta_{\text{halt}}(\mu, \lambda)$: This angle is an increasing function of λ that runs from 0 to $\arctan \mu$ as λ runs from 0 to $\lambda_*(\mu)$.
- For $\lambda = \lambda_*(\mu)$, the skier comes to rest asymptotically as $t \rightarrow +\infty$ at the angle $\theta = \arctan \mu$.²⁰

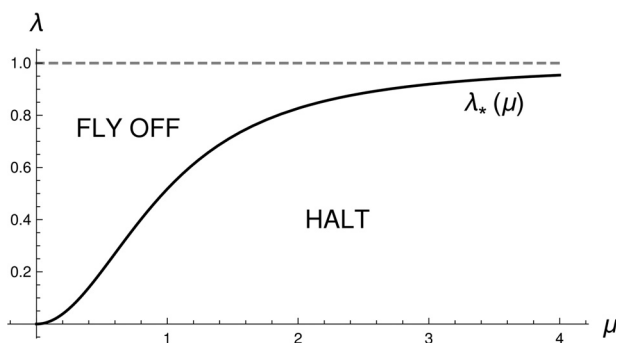


Fig. 4. The curve $\lambda_*(\mu)$ that forms the boundary between the “halt” phase and the “fly-off” phase.

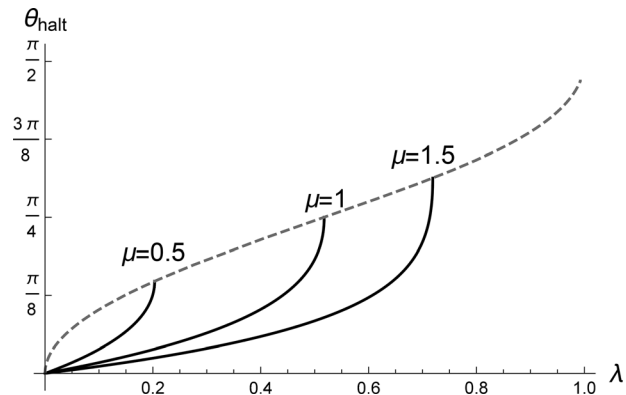


Fig. 5. θ_{halt} as a function of λ in the halt phase $0 \leq \lambda \leq \lambda_*(\mu)$, for $\mu = 0.5, 1,$ and 1.5 . The endpoints lie on the dashed curve, defined parametrically by $\lambda = \lambda_*(\mu)$ and $\theta = \arctan \mu$.

- For $\lambda_*(\mu) < \lambda < 1$, the skier flies off the hill at some angle $\theta_{\text{fly}}(\mu, \lambda)$: This angle is a decreasing function of λ that tends to 0 as $\lambda \rightarrow 1$.

The curve $\lambda_*(\mu)$ thus forms the boundary between the “halt” phase and the “fly-off” phase (see again Fig. 4).²¹ In particular, the skier always either halts or flies off; she never reaches angle $\pi/2$.

Some typical curves of $\Lambda(\theta)$ for all three scenarios are shown in Fig. 3. Note, in particular, that $\Lambda(\theta) = \Lambda'(\theta) = 0$ when $\lambda = \lambda_*(\mu)$ and $\theta = \theta_*(\mu)$; and note the fundamental qualitative difference between the curves for $\lambda < \lambda_*(\mu)$, which reach the $\Lambda = 0$ axis, and those for $\lambda > \lambda_*(\mu)$, which do not.²²

Some typical curves of $\theta_{\text{halt}}(\mu, \lambda)$ as a function of λ are shown in Fig. 5, and some typical curves of $\theta_{\text{fly}}(\mu, \lambda)$ as a function of λ are shown in Fig. 6. Please note the discontinuous change in behavior as the phase boundary $\lambda_*(\mu)$ is crossed: $\theta_{\text{fly}}(\mu, \lambda_*(\mu))$ (the dotted curve in Fig. 6) is much larger than $\theta_{\text{halt}}(\mu, \lambda_*(\mu))$ (the dashed curve in Fig. 5).²³ This is a very simple example of sensitive dependence to initial conditions, giving rise to a discontinuous phase transition—a phenomenon pointed out already by James Clerk Maxwell in 1876.²⁴

Since the proofs of all the previous claims involve some slightly intricate calculus, we relegate them to Appendix A in the supplementary material.²⁵

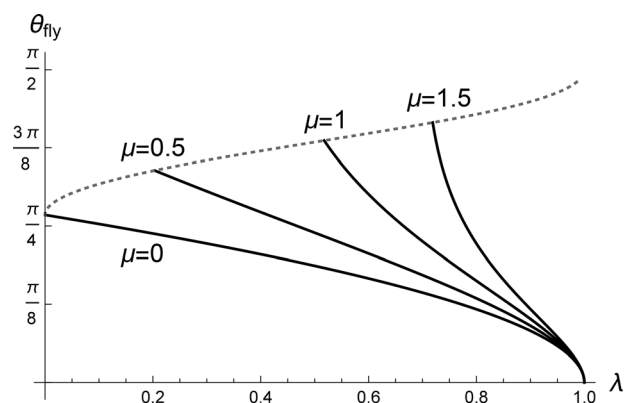


Fig. 6. θ_{fly} as a function of λ in the fly-off phase $\lambda_*(\mu) < \lambda \leq 1$, for $\mu = 0, 0.5, 1,$ and 1.5 . The endpoints lie on the dotted curve, corresponding to $\lambda \rightarrow \lambda_*(\mu)$ from above. For $\mu = 0$, we have the closed-form solution (9).

Let us remark, finally, that by the same methods one can study the more general problem in which the coefficient of kinetic friction is an arbitrary function $\mu(\theta)$ of the position along the hill: The Eq. (5) is still a first-order inhomogeneous linear differential equation for the unknown function $N(\theta)$ —albeit now one with nonconstant coefficients—so can still be solved by the method of integrating factors (though the result may not be analytically expressible in terms of elementary functions). We leave it to interested readers to pursue this generalization.

Some recent related articles are Refs. 10, 11, and 26, which study a particle sliding down an arbitrary curve in the presence of kinetic friction; Ref. 27, which uses the Lagrangian formalism with Lagrange multipliers to analyze a particle sliding without friction down an arbitrary concave curve; and Ref. 28, which studies a ball rolling (initially without slipping, later with sliding and kinetic friction) on an arbitrary curve in the presence of gravity, including an experimental realization.

III. PARTICLE ON LOOP-THE-LOOP TRACK

A block of mass m is injected with forward velocity v_0 into a loop-the-loop track of radius R and coefficient of kinetic friction μ ; let θ denote the angle up from the bottom, as shown in Fig. 2. (In one common version of the problem,¹ the block is released from rest at height h and slides to the bottom via a frictionless track; in this case, $v_0 = \sqrt{2gh}$.) The radial and tangential components of $\mathbf{F} = m\mathbf{a}$ are

$$mg \cos \theta - N = -mR\dot{\theta}^2, \quad (21)$$

$$-mg \sin \theta - \mu N \operatorname{sgn}(\dot{\theta}) = mR\ddot{\theta}. \quad (22)$$

As before, these equations are valid only as long as $N \geq 0$; after that, the block falls off the track.

The loop-the-loop problem is more complicated than the skier, for three reasons: The particle can cycle around the track; it can reverse direction; and it can halt due to static friction. Each time the particle reverses direction, we need to apply Eq. (22) with a new value for $\operatorname{sgn}(\dot{\theta})$; this repeated switching between different equations seems quite complicated, and probably needs to be handled by numerical solution.²⁹ To simplify matters, we will here follow the block only until it first reaches $\theta = 0$ or falls off the track; we, therefore, have $\dot{\theta} \geq 0$.

Proceeding as in Eqs. (3)–(5) leads to the differential equation,

$$\frac{dN}{d\theta} + 2\mu N = -3mg \sin \theta, \quad (23)$$

for the unknown function $N(\theta)$; this equation differs from Eq. (5) only by the replacement $\mu \rightarrow -\mu$. The solution is, therefore,

$$N(\theta) = N_0 e^{-2\mu\theta} - 3mg \frac{e^{-2\mu\theta} - \cos \theta + 2\mu \sin \theta}{1 + 4\mu^2}, \quad (24)$$

where $N_0 = N(0)$. Applying Eq. (21) at $\theta = 0$, where the block's angular velocity is $\dot{\theta} = v_0/R$, we see that $N_0 = mg + mv_0^2/R$. Using again the dimensionless parameter $\lambda \stackrel{\text{def}}{=} v_0^2/gR$, we have $N_0 = (1 + \lambda)mg$ and hence

$$N(\theta) = (1 + \lambda)mg e^{-2\mu\theta} - 3mg \frac{e^{-2\mu\theta} - \cos \theta + 2\mu \sin \theta}{1 + 4\mu^2}. \quad (25)$$

To obtain the velocity as a function of angle, we define once again the dimensionless quantity $\Lambda \stackrel{\text{def}}{=} v^2/gR = R\dot{\theta}^2/g$, which takes the value λ at $\theta = 0$. Then, from Eq. (21) we have

$$N = (\cos \theta + \Lambda)mg, \quad (26)$$

[which reduces to $N_0 = (1 + \lambda)mg$ when $\theta = 0$] and therefore³⁰

$$\Lambda(\theta) = -\cos \theta + (1 + \lambda) e^{-2\mu\theta} - 3 \frac{e^{-2\mu\theta} - \cos \theta + 2\mu \sin \theta}{1 + 4\mu^2}. \quad (27)$$

Since $\Lambda \geq 0$, we must have $N \geq mg \cos \theta$; and when $N = mg \cos \theta$, the block comes instantaneously to rest. After that, the particle might either reverse direction or halt due to static friction. As mentioned earlier, we refrain from following the particle beyond the first time it comes instantaneously to rest.

The solution (25) must, therefore, be supplemented by the two inequalities $N(\theta) \geq 0$ and $N(\theta) \geq mg \cos \theta$. (Please note that, unlike in the skier problem, both of these inequalities point in the *same* direction; this radically changes the nature of the qualitative analysis.) The block comes instantaneously to rest when $N(\theta) = mg \cos \theta$ or falls off the track when $N(\theta) = 0$, whichever happens first; if neither happens for $\theta < 2\pi$, then the block completes one full cycle of the loop-the-loop. Now, the inequality $N(\theta) \geq mg \cos \theta$ is the more stringent one in the lower half of the loop-the-loop (that is, $-\pi/2 \leq \theta \leq \pi/2$ modulo 2π), while the inequality $N(\theta) \geq 0$ is the more stringent one in the upper half of the loop-the-loop (that is, $\pi/2 \leq \theta \leq 3\pi/2$ modulo 2π). Therefore, the block can come instantaneously to rest only in the lower half of the loop-the-loop, and it can fall off the track only in the upper half of the loop-the-loop.

In the absence of friction ($\mu = 0$), Eq. (25) simplifies to

$$N(\theta) = (\lambda - 2 + 3 \cos \theta)mg. \quad (28)$$

If $\lambda \leq 2$, then the block reverses direction at

$$\theta = \theta_{\max} \stackrel{\text{def}}{=} \cos^{-1} \left(\frac{2 - \lambda}{2} \right), \quad (29)$$

(a value that follows immediately from conservation of energy) and oscillates forever between $-\theta_{\max}$ and θ_{\max} ; if $2 < \lambda < 5$, then the block falls off the track at

$$\theta = \theta_{\text{fall}} \stackrel{\text{def}}{=} \cos^{-1} \left(\frac{2 - \lambda}{3} \right), \quad (30)$$

which lies between $\pi/2$ and π ; if $\lambda = 5$, then the block asymptotically approaches $\theta = \pi$ as $t \rightarrow +\infty$; if $\lambda > 5$, then the block cycles forever around the track without loss of energy.

In the presence of friction ($\mu > 0$), the analysis proceeds as follows:

- (1) The first step is to determine the conditions under which the particle halts in the first quadrant ($0 \leq \theta \leq \pi/2$). The particle halts at angle θ when $\Lambda(\theta) = 0$, i.e., in case the initial velocity satisfies

$$\lambda = \lambda_{\text{halt}}(\theta, \mu) \stackrel{\text{def}}{=} \frac{2}{1 + 4\mu^2} [(1 - 2\mu^2) + e^{2\mu\theta} [3\mu \sin \theta - (1 - 2\mu^2) \cos \theta]]. \quad (31)$$

Since

$$\frac{\partial \lambda_{\text{halt}}(\theta, \mu)}{\partial \theta} = 2e^{2\mu\theta} (\mu \cos \theta + \sin \theta), \quad (32)$$

$\lambda_{\text{halt}}(\theta, \mu)$ is an increasing function of θ in the interval $0 \leq \theta \leq \pi/2$ (as is intuitively clear: To reach a larger angle, more initial velocity is needed). In particular, the particle reaches $\theta = \pi/2$ with $\dot{\theta} > 0$ if and only if

$$\lambda > \lambda_{\text{halt}}(\pi/2, \mu) \stackrel{\text{def}}{=} \frac{2 - 4\mu^2 + 6\mu e^{\pi\mu}}{1 + 4\mu^2}. \quad (33)$$

- (2) If the particle reaches angle $\pi/2$ without halting, the next step is to determine the conditions under which the particle flies off in the second or third quadrant ($\pi/2 \leq \theta \leq 3\pi/2$). The particle flies off at angle θ when $N(\theta) = 0$, i.e., in case the initial velocity satisfies

$$\lambda = \lambda_{\text{fly}}(\theta, \mu) \stackrel{\text{def}}{=} \frac{2 - 4\mu^2 + 3e^{2\mu\theta} (2\mu \sin \theta - \cos \theta)}{1 + 4\mu^2}. \quad (34)$$

Note that $\lambda_{\text{fly}}(\pi/2, \mu) = \lambda_{\text{halt}}(\pi/2, \mu)$. Since

$$\frac{\partial \lambda_{\text{fly}}(\theta, \mu)}{\partial \theta} = 3e^{2\mu\theta} \sin \theta, \quad (35)$$

we see that $\lambda_{\text{halt}}(\theta, \mu)$ is an increasing function of θ in the interval from $\pi/2$ to π and then a decreasing function in the interval from π to $3\pi/2$. The first of these facts is again intuitively clear: To survive to a larger angle without flying off, more initial velocity is needed. The second fact implies that if the particle reaches angle π without flying off—that is, if

$$\lambda \geq \lambda_{\text{fly}}(\pi, \mu) \stackrel{\text{def}}{=} \frac{2 - 4\mu^2 + 3e^{2\pi\mu}}{1 + 4\mu^2}, \quad (36)$$

then it also reaches angle $3\pi/2$ without flying off. This is intuitively clear when there is no friction, but not so obvious in the presence of friction. This implies—analogously to what happens in the skier problem—a discontinuous change of behavior as λ passes through $\lambda_{\text{fly}}(\pi, \mu)$. See Fig. 7 for plots of $\lambda_{\text{halt}}(\theta, \mu)$ and $\lambda_{\text{fly}}(\theta, \mu)$ versus θ for some selected values of μ .

- (3) If the particle reaches angle π (and hence also angle $3\pi/2$) without halting or flying off, the next step is to determine what happens in the fourth quadrant ($3\pi/2 < \theta < 2\pi$). The particle halts at angle θ in case λ equals the quantity $\lambda_{\text{halt}}(\theta, \mu)$ defined in Eq. (31). From Eq. (32), we see that $\partial \lambda_{\text{halt}}(\theta, \mu) / \partial \theta$ is negative at $\theta = 3\pi/2$ and positive at $\theta = 2\pi$, with a unique zero at $\theta = 2\pi - \arctan \mu$. So $\lambda_{\text{halt}}(\theta, \mu)$ is decreasing in the interval $3\pi/2 \leq \theta \leq 2\pi - \arctan \mu$ and increasing in the interval $2\pi - \arctan \mu \leq \theta \leq 2\pi$. Its maximum value in the interval $[3\pi/2, 2\pi]$, therefore, lies either at $\theta = 3\pi/2$ or at $\theta = 2\pi$. Since we are in the situation $\lambda \geq \lambda_{\text{fly}}(\pi, \mu) > \lambda_{\text{fly}}(3\pi/2, \mu) = \lambda_{\text{halt}}(3\pi/2, \mu)$, the only relevant

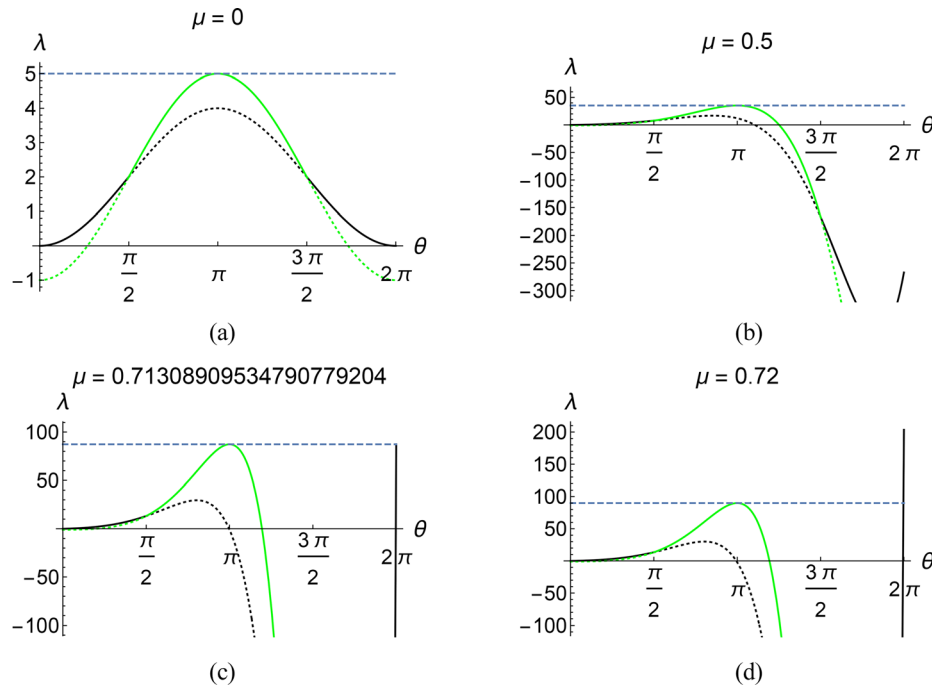


Fig. 7. The functions λ_{halt} (black) and λ_{fly} (green or gray) vs θ for some selected values of μ . The dominant (respectively, subdominant) condition is shown as a solid (respectively, dotted) curve. A horizontal dashed line is shown at $\lambda_{\text{fly}}(\pi, \mu)$. The curve in the bottom-left panel corresponds to the value $\mu = \mu_{\text{crit}} \approx 0.713089$ where $\lambda_{\text{fly}}(\pi, \mu) = \lambda_{\text{halt}}(2\pi, \mu)$. From Eq. (32), we see that λ_{halt} is increasing for $0 \leq \theta \leq \pi - \arctan \mu$, decreasing for $\pi - \arctan \mu \leq \theta \leq 2\pi - \arctan \mu$, and increasing for $2\pi - \arctan \mu \leq \theta \leq 2\pi$. From Eq. (35), we see that λ_{fly} is increasing for $0 \leq \theta \leq \pi$ and decreasing for $\pi \leq \theta \leq 2\pi$. The two curves cross at $\pi/2$ and $3\pi/2$.

question is whether λ is larger than $\lambda_{\text{halt}}(2\pi, \mu)$ or not. If it is, then the particle reaches angle 2π without halting. If it is not, then the particle halts at some angle in the interval $(2\pi - \arctan\mu, 2\pi]$, namely, the unique angle where $\lambda = \lambda_{\text{halt}}(\theta, \mu)$. The first of these cases always occurs when $\lambda_{\text{fly}}(\pi, \mu) > \lambda_{\text{halt}}(2\pi, \mu)$, i.e., when $0 \leq \mu < \mu_{\text{crit}} \approx 0.713\,089$. (See Appendix B in the supplementary material²⁵ for the proof that there is a unique such value μ_{crit} .) When $\mu \geq \mu_{\text{crit}}$, then there is a “halt in fourth quadrant” phase at $\lambda_{\text{fly}}(\pi, \mu) \leq \lambda \leq \lambda_{\text{halt}}(2\pi, \mu)$ and a “survive to angle 2π ” phase at $\lambda \geq \lambda_{\text{halt}}(2\pi, \mu)$. We record the formula

$$\lambda_{\text{halt}}(2\pi, \mu) \stackrel{\text{def}}{=} \frac{(4\mu^2 - 2)(e^{4\pi\mu} - 1)}{1 + 4\mu^2}. \quad (37)$$

(4) If the particle survives to angle 2π , then it has there a forward velocity corresponding to a value,

$$\lambda_{\text{new}} \stackrel{\text{def}}{=} \Lambda(2\pi) = \lambda e^{-4\pi\mu} + \frac{(2 - 4\mu^2)(1 - e^{-4\pi\mu})}{1 + 4\mu^2}, \quad (38a)$$

$$= e^{-4\pi\mu}[\lambda - \lambda_{\text{halt}}(2\pi, \mu)], \quad (38b)$$

$$\geq 0. \quad (38c)$$

Since $\lambda_{\text{halt}}(2\pi, \mu) > 0$ in the survive to angle 2π phase, we have $\lambda_{\text{new}} < e^{-4\pi\mu}\lambda$: Thus, the kinetic energy is reduced by at least a factor $e^{-4\pi\mu}$ at each revolution. The subsequent motion can then be found by repeating the foregoing analysis with λ replaced by λ_{new} .

The resulting phase diagram is shown in Fig. 8. Since $\lambda_{\text{halt}}(2\pi, \mu)$ grows extremely rapidly with μ , we have used $\sqrt{\lambda}$ instead of λ on the vertical axis, to compress the plot. This phase diagram agrees with the one found by Klobus (Ref. 8, Fig. 2); the value of $\lambda_{\text{halt}}(2\pi, 1)$ also agrees with his. All three phase boundaries are increasing functions of μ : See Appendices B1–B3 in the supplementary material.²⁵

Of course, this phase diagram only follows the particle up to the first time that it reaches $\theta = 0$ or $\theta = 2\pi$. A more

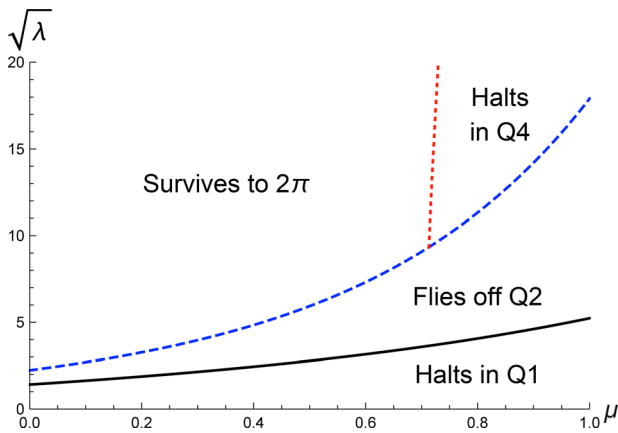


Fig. 8. Phase diagram for the loop-the-loop problem, up to the first time that the particle reaches $\theta = 0$ or $\theta = 2\pi$. The vertical axis shows $\sqrt{\lambda}$. The three boundary curves are, from bottom to top, $\lambda_{\text{halt}}(\pi/2, \mu)$, $\lambda_{\text{fly}}(\pi, \mu)$, and $\lambda_{\text{halt}}(2\pi, \mu)$, shown respectively in solid black, dashed blue, and dotted red. The particle either halts in the first quadrant (Q1), flies off the second quadrant (Q2), halts in the fourth quadrant (Q4), or survives to angle 2π .

complete analysis would show that the phase “survives to angle 2π ” is itself divided into sub-phases “halts in the first quadrant” ($2\pi < \theta < 5\pi/2$), “flies off the second quadrant” ($5\pi/2 < \theta < 3\pi$), “halts in the fourth quadrant” ($7\pi/2 < \theta < 4\pi$), and “survives to angle 4π ”; and this latter phase is further divided into sub-phases; and so on infinitely. We leave it to interested readers to work out the details of this infinite sequence of bifurcations.

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^aElectronic mail: dominik.kufel.17@ucl.ac.uk

^bElectronic mail: sokal@nyu.edu

¹See e.g. D. Kleppner and R. Kolenkow, *An Introduction to Mechanics*, 2nd ed. (Cambridge U. P., Cambridge, 2014), Problems 5.1 (loop-the-loop) and 5.6 (block sliding down a sphere); D. Morin, *Introduction to Classical Mechanics* (Cambridge U. P., New York, 2008), Exercises 5.39 (loop-the-loop) and 5.53 (skier on a frictionless hemisphere of finite mass M , which is considerably more difficult than the usual case $M = \infty$).

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¹²See Ref. 4, Eqs. (1) and (3). See also Ref. 10 for a generalization to an arbitrary curve in the vertical plane, using the Frenet–Serret formalism.

¹³Taking literally the Eqs. (1) and (2), the skier would reverse direction when $\theta = 0$ and begin climbing back up the hill, but this is not, of course, what actually happens. Rather, when the skier halts, static friction takes over, and the skier remains forever at rest; this occurs because static friction is governed by the inequality $|F_s| \leq \mu N$, not the equality $|F_s| = \mu N$.

¹⁴See also Ref. 4, Appendix A for an alternate approach to solving Eqs. (5) and (13), which does not require the student to be familiar with the method of integrating factors.

¹⁵See Ref. 4, Eq. (A3); Ref. 6, Eq. (25); Ref. 7, Eq. (2); Ref. 10, Eq. (23), or Eq. (24) in the arXiv version.

¹⁶See Ref. 4, Eq. (5); Ref. 6, Eq. (17); and Ref. 7, Eq. (1). See also Ref. 10, Eq. (9) for a generalization to an arbitrary curve in the vertical plane.

¹⁷This approach is taken, for instance, in Ref. 6, Eqs. (9) and (17); in Ref. 7, Eq. (1); and in Ref. 11 for an arbitrary curve in the vertical plane.

¹⁸See Ref. 7; Ref. 10 (arXiv version), Eq. (59) ff.

¹⁹See Ref. 7, Eq. (5); Ref. 10 (arXiv version), Eq. (62). This latter paper also gives analogous formulas for the parabola, cycloid, catenary, and ellipse.

²⁰It can be seen directly from the equations of motion (1) and (2) that this “asymptotic equilibrium” position can only be $\theta = \arctan\mu$. To see this, observe that $\text{sgn}(\dot{\theta}) = +1$ for all $t \geq 0$, and that $\dot{\theta}, \ddot{\theta} \rightarrow 0$ as $t \rightarrow +\infty$. Combining Eqs. (1) and (2) as $t \rightarrow +\infty$ then yields $N = mg \cos\theta$ and $\theta = \arctan\mu$. This argument does *not* apply to the “subcritical” trajectories in which the skier halts after a finite time, since in these trajectories $\dot{\theta} = 0$ but $\ddot{\theta} \neq 0$ at the halting time.

²¹See Ref. 7, Fig. 1; Ref. 10 (arXiv version), Fig. 13. This latter figure also shows the phase diagram for the parabola, cycloid, catenary, and ellipse.

²²Compare Ref. 4, Fig. 2; Ref. 10, Fig. 3, or Fig. 4 in the arXiv version. The latter plot shows different values of μ for the same $\lambda = 0.6$, which is complementary to our Fig. 3.

²³See Ref. 7, Fig. 2 for a superposed version of our Figs. 5 and 6 that highlight this discontinuity.

²⁴J. C. Maxwell, *Matter and Motion* (Society for Promoting Christian Knowledge, London, 1876); reprinted by Dover, New York, 1952 and Cambridge University Press, Cambridge, 2010. After stating (p. 20) what he sees as “the general maxim of physical science”—namely, “the same causes will always produce the same effects”—Maxwell goes on to observe (p. 21) that

There is another maxim which must not be confounded with [this one], which asserts “That like causes produce like effects.”

This is only true when small variations in the initial circumstances produce only small variations in the final state of the system. In a great many physical phenomena, this condition is satisfied, but there are other cases in which a small initial variation may produce a very great change in the final state of the system, as when the displacement of the “points” causes a

railway train to run into another instead of keeping its proper course.

²⁵See supplementary material at <https://www.scitation.org/doi/suppl/10.1119/5.0095150> for proofs of these claims.

²⁶V. Ivchenko, “Sliding down a rough curved hill,” *Eur. J. Phys.* **42**, 025005 (2021).

²⁷L. Balart and S. Belmar-Herrera, “Particle sliding down an arbitrary concave curve in the Lagrangian formalism,” *Am. J. Phys.* **87**, 982–985 (2019).

²⁸G. M. Mejía, J. M. Betancourt, C. D. Forero, N. Avilán, F. J. Rodríguez, L. Quiroga, and N. F. Johnson, “Dynamics of a round object moving along curved surfaces with friction,” *Am. J. Phys.* **88**, 229–237 (2020).

²⁹The qualitative behavior after the particle reverses direction is, however, very simple. As will be seen below, the particle can come instantaneously to rest only in the lower half of the loop-the-loop ($-\pi/2 \leq \theta \leq \pi/2$ modulo 2π). After this happens, the particle simply oscillates back and forth, with constant amplitude if $\mu = 0$ and with decreasing amplitude if $\mu > 0$.

³⁰See Ref. 2, Eq. (12); Ref. 3, Eq. (6); Ref. 8, Eq. (6). See also Ref. 3 for a generalization that includes a viscous drag force $-\beta v^2$.



Plane Mirrors at an Angle

Some demonstrations help the student to understand basic physical phenomena, while others not only do this but are also attractive. These mirrors are hinged so that they can be set at angles of $360/n$ degrees with respect to each other, where n is a small integer. In this case n is six, so that the angle of inclination is sixty degrees. The real space is the sixty degrees between the mirrors, while there are five *virtual* spaces with the same included angle. See: Thomas B. Greenslade, Jr., “Virtual Mirrors”, *Phys. Teach.* **48**, 26-27 (2010) (Amherst College photograph; text by Thomas B. Greenslade, Jr., Kenyon College)