Models of Hyperelliptic Curves Over p-adic Fields

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I, Sarah Catherine Nowell, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work. Chapters 8 and 9 are the result of joint work with Omri Faraggi.

Abstract

Let $C : y^2 = f(x)$ be a hyperelliptic curve, with tame potentially semistable reduction, over a local field with algebraically closed residue field. The p-adic distances between the roots of $f(x)$ can be described by a purely combinatorial object known as a cluster picture. We show that the cluster picture of C, along with the leading coefficient of f, completely determines the dual graph of the special fibre of the minimal strict normal crossings (SNC) model of C. In particular, we give an explicit description of the special fibre in terms of this data. Further to this, we define open quotient BY trees, showing there is a one-to-one correspondence between these and cluster pictures of hyperelliptic curves with tame reduction. Using these trees we introduce a way of classifying reduction types of hyperelliptic curves. As a demonstration of our results we give a complete classification in genus 2 using cluster pictures and open quotient BY trees.

Impact Statement

Elliptic curves are well understood, and many people are now studying the arithmetic of higher genus curves. In particular, the theory surrounding elliptic curves is increasingly being generalised to hyperelliptic curves. Models of hyperelliptic curves are invaluable objects which can be used to deduce a huge amount of arithmetic information. The p-adic data that features in the statement of the famous Birch–Swinnerton-Dyer conjecture relies heavily on models and the data that may be extracted from them, such as Tamagawa numbers. A significant amount of work has already been done on computing the special fibres of regular models. For the genus 1 case, Tate's letter to Cassels describes an algorithm, known as Tate's algorithm. This outputs the minimal regular model of an elliptic curve E, classifying the type of reduction of E at a prime p. There is also a full account of this in Silverman's book 'Advanced Topics in the Arithmetic of Elliptic Curves', which has become one of the most standard references for number theory and algebraic geometry. For genus 2 curves Namikawa and Ueno give a classification of all possible minimal regular models. However for genus > 2 not so much was known.

This thesis extends existing results due to Dokchitser, Dokchitser, Maistret and Morgan. We study models of hyperelliptic curves with using cluster pictures, purely combinatorial objects defined by the root configurations of defining polynomials. Although relatively new objects of interest, cluster pictures have already proved hugely advantageous in studying arithmetic of hyperelliptic curves, and have been added to LMFDB, a huge database of mathematical objects. The results laid out in this thesis make it much easier to work with models of hyperelliptic curves, describing how to easily check whether two hyperelliptic curves have the same reduction type, find their special fibres, and move between different models of any given hyperelliptic curve.

Hyperelliptic curves also have a direct application to cryptography. Whilst this is not the purpose of this thesis, nor is it directly related, better understanding the arithmetic of hyperelliptic curves could prove to have useful applications in the future.

Acknowledgements

Firstly, I would like to thank Vladimir Dokchitser not only for suggesting the problems discussed in this thesis, but for his extensive support and guidance, without whom this work would not have been possible.

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Contents

Chapter 1

Introduction

1.1 Motivation

Models of curves are invaluable objects which can be used to deduce a large amount of arithmetic information more easily than would otherwise be possible. For example, the p-adic data that features in the statement of the famous Birch–Swinnerton-Dyer conjecture relies heavily on models, and the data that may be extracted from them such as Tamagawa numbers. Checking for rational points is another common problem in number theory. Models can be used to check for p-adic points, which helps us check for these rational points.

In this thesis we study hyperelliptic curves, with the two main goals of giving a description of their minimal strict normal crossing (SNC) models, and providing a way to classify all possible reduction types. Not only will we give a classification but also a practical way to use it. We are interested in studying these over p-adic fields, however the arguments work more generally and the main results will be stated over general local fields. We use *cluster pictures*, a relatively new innovation which have already proved advantageous in studying the arithmetic of hyperelliptic curves. In particular, cluster pictures have been used to calculate semistable models, conductors, minimal discriminants and Galois representations in [DDMM18], Tamagawa numbers in [Bet18], root numbers in [Bis19], and differentials in [Kun19]. More recent papers which make use of cluster pictures are [Mus20], where the author constructs the minimal regular model with normal crossings of hyperelliptic curves and determines a basis of integral differentials, and [BBB$^+$20], where many of the numerous papers using cluster pictures are summarised and complemented by examples. Cluster pictures have also been added to LMFDB, a huge database of mathematical objects.

1.2 Setup

Let K be a field complete with respect to a discrete valuation v_K, with algebraically closed residue field k of characteristic $p > 2$. Write $G_K = \mathrm{Gal}(\bar{K}/K)$, the absolute Galois group. Let C/K be a hyperelliptic curve given by Weierstrass equation $y^2 = f(x)$, with genus $g = g(C)$. Unless explicitly mentioned otherwise, we assume $g \geq 2$ throughout the thesis. We write $\mathcal{R}$ for the set of roots of $f(x)$ in the algebraic closure $\bar{K}$ of K and c_f for the leading coefficient of f. So,

$$f(x) = c_f \prod_{r \in \mathcal{R}} (x - r),$$

and $|\mathcal{R}| \in \{2g+1, 2g+2\}$. Following [DDMM18] we associate to C a cluster picture, defined by the combinatorics of the root configuration of f.

We extend existing results about reduction types and models of hyperelliptic curves to the more general case where C has tame (potentially semistable) reduction over K. That is, there exists some finite extension L/K such that C has semistable reduction over L, and $[L : K]$ is coprime to p. This is equivalent to f having tame splitting field, since C will be semistable over the splitting field of f or a quadratic extension of the splitting field of f. It is important to note that our theorems do not apply in the case where a wild extension is required for semistability. However this condition is not too strong since for large enough p, every curve of genus g has tame reduction.

Using cluster pictures, the collaborative work in [FN20] with Omri Faraggi, allows us to calculate a combinatorial description of the minimal SNC model $\mathscr{X}$ of C/K: a model whose singularities on the special fibre $\mathscr{X}_k$ are normal crossings (i.e. locally they look like the union of two axes), and where blowing down any exceptional component of $\mathscr{X}_k$ would result in a worse singularity. Such models can be used to calculate arithmetic invariants, to study the Galois representation, and (in more general settings) to deduce the existence of K-rational points of C. For the case of elliptic curves, Tate's algorithm [Sil94] is sufficient to calculate the minimal SNC model of a given curve. For hyperelliptic curves, [DDMM18] the authors calculate the SNC model when C has semistable reduction, and in [Dok18] when C has a particularly nice cluster picture. In fact, the methods of [Dok18] work for a much larger class of smooth projective curves, but we restrict our attention to its applications for hyperelliptic curves. Similar work has also been done on models of different classes of curves and the applications of these models — such as [BW17] on stable models of superelliptic curves and [LLLGR18] and [BCK$^+$20] on non-hyperelliptic genus 3 curves. Other work on hyperelliptic invariants has also been done in [OS19], where the authors prove a conductor-discriminant inequality for hyperelliptic curves.

Most of the information required to classify the reduction types of hyperelliptic curves, or deduce the special fibre of $\mathscr{X}$ of a hyperelliptic curve C/K, is contained in the cluster picture.

Definition 1.2.1. A *cluster* is a non-empty subset $\mathfrak{s} \subseteq \mathcal{R}$ of the form $\mathfrak{s} = D \cap \mathcal{R}$ for some disc $D = z + \pi_K^n \mathcal{O}_{\overline{K}}$, where $z \in \overline{K}$, $n \in \mathbb{Q}$ and π_K is a uniformiser of K. If $\mathfrak{s}$ is a cluster and $|\mathfrak{s}| > 1$, we say that $\mathfrak{s}$ is a *proper cluster*. For a proper cluster $\mathfrak{s}$ we define its *depth* $d_{\mathfrak{s}}$ to be

$$d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v_K(r - r').$$

The *cluster picture* $\Sigma_{C/K}$ of C is the collection of all clusters of the roots of f. When there is no risk of confusion, we may simplify this to Σ_C. We refer to $\mathcal{R}$ as the *top cluster*.

The cluster picture $\Sigma_{C/K}$ comes with a natural action of $\mathrm{G}_K = \mathrm{Gal}(\overline{K}/K)$.

Example 1.2.2. Take the polynomial $f(x) = (x-1)(x^3 - p)(x^2 - p^4)$ over $\mathbb{Q}_p$ for $p > 3$, with roots $\mathcal{R} = \{1, p^{\frac{1}{3}}, \zeta_3 p^{\frac{1}{3}}, \zeta_3^2 p^{\frac{1}{3}}, p^2, -p^2\}$, where ζ_3 is a third root of unity. Consider the p-adic valuations of the differences between pairs of roots:

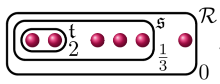
$$\begin{aligned} v(1 - r) &= 0 && \text{for } 1 \neq r \in \mathcal{R}, \\ v(\zeta_3^i p^{\frac{1}{3}} - \zeta_3^j p^{\frac{1}{3}}) &= \frac{1}{3} && \text{for } i, j \in \{0,1,2\}\ i \neq j, \\ v(\zeta_i p^{\frac{1}{3}} \pm p^2) &= \frac{1}{3} && \text{for } i \in \{0,1,2\}, \\ v(p^2 + p^2) &= 2. \end{aligned}$$

This gives the list of all clusters as:

$$\mathcal{R}, \quad \{1\}, \quad \{p^{\frac{1}{3}}\}, \quad \{\zeta_3 p^{\frac{1}{3}}\}, \quad \{\zeta_3^2 p^{\frac{1}{3}}\}, \quad \{p^2\}, \quad \{-p^2\},$$

$$\mathfrak{s} = \{p^{\frac{1}{3}}, \zeta_3 p^{\frac{1}{3}}, \zeta_3^2 p^{\frac{1}{3}}, p^2, -p^2\},$$

$$\mathfrak{t} = \{p^2, -p^2\}.$$

We draw the cluster picture by drawing a node for each root and drawing circles around them to indicate how p-adically close to each other they are. These circles represent p-adic discs, and their depths are indicated next to them. So, the cluster picture of $C : y^2 = f(x)/\mathbb{Q}_p$ is $\Sigma_C =$ 

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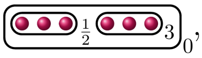
1.3 Minimal SNC Models

It turns out that, along with the valuation of the leading coefficient $v_K(c_f)$, $\Sigma_{C/K}$ with its natural action of G_K is all we need to calculate a combinatorial description of the minimal SNC model of C/K.

Theorem 1.3.1. *[Theorem 9.2.1] Let K be a complete discretely valued field with algebraically closed residue field of characteristic $p > 2$. Let $C : y^2 = f(x)$ be a hyperelliptic curve over K with tame reduction. Then the structure (i.e. the dual graph) of the special fibre of the minimal SNC model of C/K, with genus and multiplicity, is completely determined by $\Sigma_{C/K}$ (with depths), the valuation of the leading coefficient $v_K(c_f)$ of f, and the action of G_K.*

Example 1.3.2. Consider the two curves $C_1 : y^2 = (x^2-p)(x-p^4)((x-1)^3-p^9)$ and $C_2 : y^2 = (x-1)((x-1)^2-p)(x-p^3)(x-p^5)(x+p^3-p^4)$ over $\mathbb{Q}_p^{\mathrm{ur}}$, $p > 3$. Both C_1 and C_2 have Namikawa-Ueno type I_0^*-III-1. Note that C_1 and C_2 both have cluster picture , and their defining polynomials have equal leading coefficients. This illustrates Theorem 1.3.1, that the reduction type is completely determined by the cluster picture and leading coefficient.

Another useful example to consider is the following summary of the case for elliptic curves.

Example 1.3.3. The following table shows the special fibre $\mathscr{X}_k$, of the minimal SNC model $\mathscr{X}$ for the different Kodaira-Néron types of elliptic curves with tame reduction (for which it is sufficient to take $p \geq 5$). Every elliptic curve can be put into the form $y^2 = x^3 + ax + b$. After allowing for shifts and scalings we can present each elliptic curve in precisely one of the following ways.

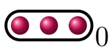
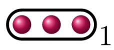
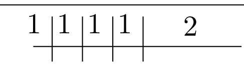
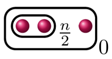
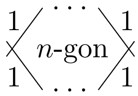
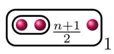
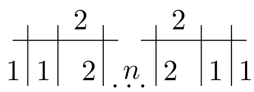
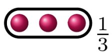
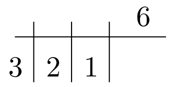
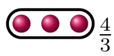
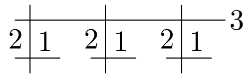
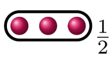
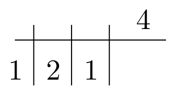
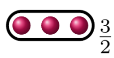
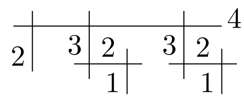
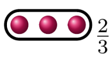
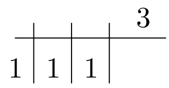
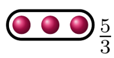
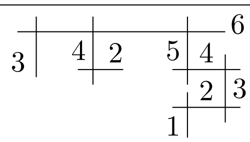
Type	Σ	$\mathscr{X}_k$	Type	Σ	$\mathscr{X}_k$
I_0			I_0^*		
I_n			I_n^*		
II			IV^*		
III			III^*		
IV			II^*		

Table 1.1: Kodaira-Néron types of elliptic curves with $p \geq 5$.

Here, the $g1$ labeling represents a component of genus 1. This differs from the table found in [Sil94, p 365], where instead the special fibres of the *minimal regular models* for the different types of elliptic curves are shown. This makes a difference for type II, III or IV elliptic curves, whereas for all the other types the minimal regular model is SNC. These special fibres can be read off the cluster pictures. Roughly, we can apply Theorems 1.3.6 and 1.3.13. It is worth noting that technically these theorems only apply for genus > 1. However, looking at the genus 1 case is helpful in familiarising oneself with this way of studying special fibres. In [Sil94] the special fibres are presented alongside the discriminants and j-invariants, the inputs required for Tate's algorithm. Knowing the discriminant and j-invariant of an elliptic curve is equivalent to understanding the p-adic distances between roots. By instead taking the approach of cluster pictures, one can quite naturally read off the special fibre, removing the need to follow a lengthy algorithm.

We will make use of the following formal definitions from [DDMM18].

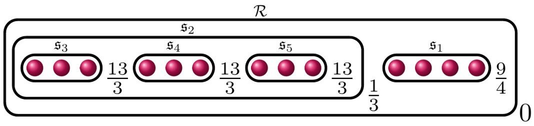
Definition 1.3.4. A maximal subcluster $\mathfrak{s}'$ of a cluster $\mathfrak{s}$ is called a *child* of $\mathfrak{s}$, denoted $\mathfrak{s}' < \mathfrak{s}$, and $\mathfrak{s}$ is the *parent* of $\mathfrak{s}'$, denoted $P(\mathfrak{s}')$. We say $\mathfrak{s}$ is *odd* (resp. *even*) if $|\mathfrak{s}|$ is odd (resp. even) Furthermore, $\mathfrak{s}$ is a *twin* if $|\mathfrak{s}| = 2$, and $\mathfrak{s}$ is *übereven* if $\mathfrak{s}$ has only even children. A cluster $\mathfrak{s}$ is *principal* if $|\mathfrak{s}| \geq 3$, unless it has a child of size $2g(C)$, or if $\mathfrak{s} = \mathcal{R}$ is even and has exactly two children, in which case $\mathfrak{s}$ is *not principal*.

Chapters 8 and 9 are dedicated to explicitly describing the structure, multiplicities and genera of components of the special fibre $\mathscr{X}_k$ of the minimal SNC model. Before we give a precise statement let us illustrate the main result of these chapters, along with the definitions of linking chains and central components, via an example.

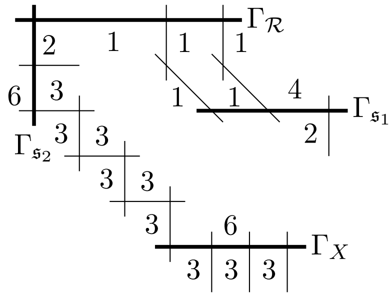
Example 1.3.5. Let $K = \mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 5$, and C/K be the hyperelliptic curve of genus 3 given by

$$C : y^2 = ((x^3 - p)^3 - p^{15})((x-1)^4 - p^9).$$

The cluster picture of C/K is shown in Figure 1.1a and the special fibre $\mathscr{X}_k$ of the minimal SNC model of C/K is shown in Figure 1.1b. The principal clusters in $\Sigma_{C/K}$ are $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5$, and $\mathcal{R}$, as labeled in Figure 1.1a. Note that $\mathfrak{s}_3$, $\mathfrak{s}_4$ and $\mathfrak{s}_5$ are permuted by G_K and denote their orbit by X. None of the principal clusters in this example are übereven so, as we will see in Theorem 1.3.6, each orbit of principal clusters gives rise to one central component, shown in bold and labeled in Figure 1.1b. Clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are children of $\mathcal{R}$, and contribute chains of rational curves linking $\Gamma_{\mathcal{R}}$ and $\Gamma_{\mathfrak{s}_i}$ for $i = 1, 2$. In this case, the chain from $\Gamma_{\mathcal{R}}$ to $\Gamma_{\mathfrak{s}_2}$ is trivial so they intersect, and we get two identical chains from $\Gamma_{\mathcal{R}}$ to $\Gamma_{\mathfrak{s}_1}$. Similarly, the elements of X are children of $\mathfrak{s}_2$ and contribute a chain of rational curves between $\Gamma_{\mathfrak{s}_2}$ and Γ_X. How one determines the number and



(a) Cluster picture $\Sigma_{C/K}$.

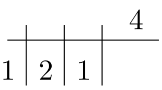


(b) Special fibre of the minimal SNC model of C/K.

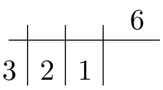
Figure 1.1: $C : y^2 = ((x^3 - p)^3 - p^{15})((x-1)^4 - p^9)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

length of the chains linking the central components is discussed in Theorem 1.3.13. Each of $\Gamma_{\mathfrak{s}_1}, \Gamma_{\mathfrak{s}_2}$, and Γ_X are also intersected by a few other components, again this is discussed in Theorem 1.3.13.

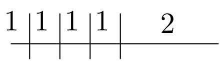
In this example, we can compare the chains intersecting some of the central components in $\mathscr{X}_k$ to those appearing in the minimal SNC models of related elliptic curves, seen in Table 1.1, but pictured again in Figure 1.2 below. The component $\Gamma_{\mathfrak{s}_1}$, and those intersecting $\Gamma_{\mathfrak{s}_1}$, look much like a type III elliptic curve. Similarly type II for $\mathfrak{s}_2$, and type I_0^* for X (but with multiplicities multiplied by $|X| = 3$).



(a) Type III



(b) Type II



(c) Type I_0^*

Figure 1.2: Special fibres of elliptic curves appearing as "submodels" of $\mathscr{X}_k$.

This example illustrates the main idea that every Galois orbit of principal clusters X contributes components to the special fibre $\mathscr{X}_k$. More precisely: orbits of principal, übereven clusters contribute either one or two components and orbits of principal, non-übereven clusters contribute one component. We call these components *central components*, and they are linked by either one or two chains of rational curves which we call *linking chains*. The central components of two orbits X and X' are linked by a chain (or chains) of rational curves if and only if there exits some $\mathfrak{s} \in X$ and $\mathfrak{s}' \in X'$ such that $\mathfrak{s}' < \mathfrak{s}$. An orbits of twins gives rise to a chain of rational curves, which intersects the component(s) arising from their parent's orbit. Some central components are also intersected by other chains of rational curves: *loops*, *tails* and *crossed tails*. Loops are chains from a component to itself; tails are chains which intersect the rest of the special fibre in only one place; crossed tails are similar to tails but with two additional components, called *crosses*, intersecting the final component of the chain. Figures 1.3 and 1.4 give pictorial descriptions of the different chains of rational curves that can occur, where the dashed lines illustrate all the components of $\mathscr{X}_k$ that are intersected by the chain.

(a) Linking chain **(b)** Loop

Figure 1.3: Pictorial description of linking chains and loops.

(a) Tail **(b)** Crossed tail

Figure 1.4: Pictorial description of tails and crossed tails.

We write such chains of rational curves as $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$, where E_i intersects E_{i+1} exactly once for all $1 \leq i < \lambda$, and intersects no other components of $\mathcal{C}$. E_1 will intersect the rest of the special fibre, say at component Γ_1, exactly once. If $\mathcal{C}$ is a linking chain then E_λ will also intersect the rest of the special fibre, say at component Γ_2, exactly once. In this case we say that $\mathcal{C}$ is a linking chain from Γ_1 to Γ_2.

The theorems given in this section of the introduction assume that $\mathcal{R}$ is principal. Full theorems including the case when $\mathcal{R}$ is not principal are given in Chapter 9. Here we give an abridged version of the description of the structure of the special fibre, given in full in Theorem 9.2.3.

Theorem 1.3.6 (Structure of SNC model)**.** *Let K be a complete discretely valued field with algebraically closed residue field of characteristic $p > 2$. Let C/K be a hyperelliptic curve with tame reduction, and with $\mathcal{R}$ principal. If X*

is a Galois orbit of even clusters with $\mathfrak{s} \in X$, define

$$\epsilon_X = (-1)^{|X|\left(v_K(c_f)+\sum_{r\notin\mathfrak{s}} v_K(r_{\mathfrak{s}}-r)\right)},$$

where $r_{\mathfrak{s}}$ is any root of f in $\mathfrak{s}$. Then the special fibre of its minimal SNC model is structured as follows. Every Galois orbit of principal clusters X contributes one component Γ_X, unless X is übereven with $\epsilon_X = 1$, in which case X contributes two components Γ_X^+ and Γ_X^-.

These components are linked by chains of rational curves in the following cases (where, for any orbit Y, we write $\Gamma_Y^+ = \Gamma_Y^- = \Gamma_Y$ if Y contributes only one central component):

Name	*From*	*To*	*Condition*
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	*$X' < X$ both principal, X' odd*
$L^+_{X,X'}$	Γ^+_X	$\Gamma^+_{X'}$	*$X' < X$ both principal, X' even with $\epsilon_{X'} = 1$*
$L^-_{X,X'}$	Γ^-_X	$\Gamma^-_{X'}$	*$X' < X$ both principal, X' even with $\epsilon_{X'} = 1$*
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	*$X' < X$ both principal, X' even with $\epsilon_{X'} = -1$*
$L_{X'}$	Γ^-_X	Γ^+_X	*X principal, $X' < X$ orbit of twins, $\epsilon_{X'} = 1$*
$T_{X'}$	Γ_X	-	*X principal, $X' \leq X$ orbit of twins, $\epsilon_{X'} = -1$*

Chains where the "To" column has been left blank are crossed tails. Some central components Γ_X are also intersected transversally by tails. These are explicitly described in Theorem 1.3.13.

The case when $\mathcal{R}$ is not principal is described in Theorem 9.2.3. We do not give explicit equations for the components in the special fibre. However, these could be calculated using the method laid out in this thesis if desired (see Remark 9.2.4).

The linking chains, tails, and the multiplicities and genera of the components in the special fibre are given explicitly in Theorem 1.3.13 below. In order to describe the chains of rational curves in detail, we introduce the notion of sloped chains of rational curves. We also need a few other numerical invariants associated to clusters.

Definition 1.3.7. Fix $\mu \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{Q}$ with $t_1 > t_2$. Then we can find $\lambda \in \mathbb{N}$ with λ minimal, and $m_i \in \mathbb{Z}$, $d_i \in \mathbb{Z}_{>0}$ such that

$$\mu t_1 = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \cdots > \frac{m_\lambda}{d_\lambda} > \frac{m_{\lambda+1}}{d_{\lambda+1}} = \mu t_2, \text{ and } \begin{vmatrix} m_i & m_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1.$$

Suppose $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ is a chain of rational curves where E_i has multiplicity μd_i. Then $\mathcal{C}$ is a *sloped chain of rational curves* with parameters (t_2, t_1, μ).

In practice there is an easy way to find such integers λ, m_i and d_i by taking all numbers in $[\mu t_2, \mu t_1] \cap \mathbb{Q}$ of denominator $\leq \max\{\mathrm{denom}(\mu t_1), \mathrm{denom}(\mu t_2)\}$, where denom denotes the denominator. This is discussed in Remark 7.2.12. It is helpful to see a worked example such as in Example 1.4.22.

Notation 1.3.8. Write $\tilde{\mathfrak{s}}$ for the set of odd children of $\mathfrak{s}$, and $\mathfrak{s}_{\mathrm{sing}}$ for the set of size 1 children of $\mathfrak{s}$.

Definition 1.3.9. We define the following invariants for a cluster $\mathfrak{s}$:

$$\nu_{\mathfrak{s}} = v_K(c_f) + \sum_{r \in \mathcal{R}} d_{\mathfrak{s} \wedge r}, \quad \lambda_{\mathfrak{s}} = \frac{\nu_{\mathfrak{s}}}{2} - d_{\mathfrak{s}} \sum_{\mathfrak{s}' < \mathfrak{s}} \left\lfloor \frac{|\mathfrak{s}'|}{2} \right\rfloor.$$

Definition 1.3.10. Let $\mathfrak{s}$ be a cluster, and write $d_{\mathfrak{s}} = \frac{a_{\mathfrak{s}}}{b_{\mathfrak{s}}}$, where $(a_{\mathfrak{s}}, b_{\mathfrak{s}}) = 1$. The *semistable genus of* $\mathfrak{s}$, $g_{\mathrm{ss}}(\mathfrak{s})$, is given by

$$|\tilde{\mathfrak{s}}| = 2g_{\mathrm{ss}}(\mathfrak{s}) + 1 \text{ or } 2g_{\mathrm{ss}}(\mathfrak{s}) + 2,$$

or $g_{\mathrm{ss}}(\mathfrak{s}) = 0$ if $\mathfrak{s}$ is übereven. If X is a G_K-orbit of clusters with $\mathfrak{s} \in X$, the *semistable genus of* X is $g_{\mathrm{ss}}(X) = g_{\mathrm{ss}}(\mathfrak{s})$. From this we define the *genus* $g(X)$ of X. If $X = \{\mathfrak{s}\}$ is a trivial orbit $g(\mathfrak{s})$ is given by

$$g(X) = g(\mathfrak{s}) = \begin{cases} \lfloor \frac{g_{\mathrm{ss}}(\mathfrak{s})}{b_{\mathfrak{s}}} \rfloor & \lambda_{\mathfrak{s}} \in \mathbb{Z}, \\ \lfloor \frac{g_{\mathrm{ss}}(\mathfrak{s})}{b_{\mathfrak{s}}} + \frac{1}{2} \rfloor & \lambda_{\mathfrak{s}} \notin \mathbb{Z}, b_{\mathfrak{s}} \text{ even}, \\ 0 & \lambda_{\mathfrak{s}} \notin \mathbb{Z}, b_{\mathfrak{s}} \text{ odd}. \end{cases}$$

For a general orbit X, define $g(X) = g(\mathfrak{s})$ for $\mathfrak{s} \in X$, where $\mathfrak{s}$ is considered as a cluster in Σ_{C/K_X}, and K_X is the unique extension of K of degree $|X|$ (uniqueness follows from k being algebraically closed).

Definition 1.3.11. Let X be a G_K-orbit of clusters with $\mathfrak{s} \in X$, and $r_{\mathfrak{s}}$ any root in $\mathfrak{s}$. Define e_X to be the minimal positive integer such that $e_X|X|d_{\mathfrak{s}} \in \mathbb{Z}$ and $e_X|X|\nu_{\mathfrak{s}} \in 2\mathbb{Z}$ for all $\mathfrak{s} \in X$. The orbit X also has the following invariants:

$$d_X = d_{\mathfrak{s}}, \quad b_X = b_{\mathfrak{s}}, \quad \lambda_X = \lambda_{\mathfrak{s}} \text{ and } \delta_X = \begin{cases} d_{\mathcal{R}} & \text{if } \mathfrak{s} = \mathcal{R}, \\ d_{\mathfrak{s}} - d_{P(\mathfrak{s})} & \text{otherwise}, \end{cases}.$$

Definition 1.3.12. A child $\mathfrak{s}' < \mathfrak{s}$ is *stable* if it has the same G_K-stabiliser as $\mathfrak{s}$, and a G_K-orbit of clusters is *stable* if all (or equivalently any) of its elements are stable.

Theorem 1.3.13. *Let K and C/K be as in Theorem 1.3.6. Let X be a principal orbit of clusters in the cluster picture of C/K, and suppose that $\mathcal{R}$ is principal. Then $\Gamma_X^{\pm}$ has genus $g(X)$ and multiplicity $|X|e_X$. Suppose further that $e_X > 1$, and choose some $\mathfrak{s} \in X$. Then the central component(s) associated to X are intersected transversely by the following tails, which are sloped chains with parameters $(\frac{1}{\mu}\lfloor \mu t_1 - 1 \rfloor, t_1, \mu)$ (writing $\Gamma_X = \Gamma_X^+ = \Gamma_X^-$ if X contributes only one central component):*

Name	*From*	*Number*	t_1	μ	*Condition*
T_∞	Γ_X	1	$(g+1)d_\mathcal{R} - \lambda_\mathcal{R}$	1	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *odd*
$T_\infty^{\pm}$	$\Gamma_X^{\pm}$	2	$-d_\mathcal{R}$	1	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *even*, $\epsilon_\mathcal{R} = 1$
T_∞	Γ_X	1	$-d_\mathcal{R}$	2	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *even*, $e_\mathcal{R} > 2$, $\epsilon_\mathcal{R} = -1$
$T_{y_\mathfrak{s}=0}$	Γ_X	$\lfloor \frac{\lvert\mathfrak{s}_{\mathrm{sing}}\rvert \lvert X\rvert}{b_X} \rfloor$	$-\lambda_X$	b_X	$\lvert\mathfrak{s}_{\mathrm{sing}}\rvert \geq 2$, *and* $e_X > b_X/\lvert X\rvert$
$T_{x_\mathfrak{s}=0}$	Γ_X	1	$-d_X$	$2\lvert X\rvert$	*X has no stable child,* $\lambda_X \notin \mathbb{Z}$, $e_X > 2$, *and either* $g_{\mathrm{ss}}(X) > 0$ *or X is übereven*
$T_{x_\mathfrak{s}=0}^{\pm}$	$\Gamma_X^{\pm}$	2	$-d_X$	$\lvert X\rvert$	*X has no stable child,* $\lambda_X \in \mathbb{Z}$, *and either* $g_{\mathrm{ss}}(X) > 0$ *or X is übereven*
$T_{(0,0)}$	Γ_X	1	$-\lambda_X$	$\lvert X\rvert$	*X has a stable singleton, or* $g_{\mathrm{ss}}(X) = 0$, *X is not übereven and X has no proper stable odd child*

Furthermore, regardless of whether $e_X > 1$ or not, for $X' < X$ an orbit of clusters, the central components are intersected by the following sloped chains of rational curves with parameters $(t_1 - \delta, t_1, \mu)$:

Name	t_1	δ	μ	*Condition*
$L_{X,X'}$	$-\lambda_X$	$\delta_{X'}/2$	$\lvert X'\rvert$	X', X *principal,* X' *odd*
$L_{X,X'}^+$	$-d_X$	$\delta_{X'}$	$\lvert X'\rvert$	X', X *principal,* X' *even,* $\epsilon_{X'} = 1$
$L_{X,X'}^-$	$-d_X$	$\delta_{X'}$	$\lvert X'\rvert$	X', X *principal,* X' *even,* $\epsilon_{X'} = 1$
$L_{X,X'}$	$-d_X$	$\delta_{X'}$	$2\lvert X'\rvert$	X', X *principal,* X' *even,* $\epsilon_{X'} = -1$
$L_{X'}$	$-d_X$	$2\delta_{X'}$	$\lvert X'\rvert$	X *principal,* X' *orbit of twins,* $\epsilon_{X'} = 1$
$T_{X'}$	$-d_X$	$\delta_{X'} + \frac{1}{\mu}$	$2\lvert X'\rvert$	X *principal,* X' *orbit of twins,* $\epsilon_{X'} = -1$

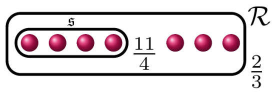
Note that the names indicate the components which each chain intersects, as explicitly written in the second table of Theorem 1.3.6. Finally, the crosses of any crossed tail have multiplicity $\frac{\mu}{2}$.

This is proved in Theorem 9.2.3. In practice, there is a one-to-one correspondence between the chains intersecting a central component Γ_X and the tails of the unique central component of the minimal SNC model of a related curve. Roughly, for some $\mathfrak{s} \in X$ we can construct a hyperelliptic curve over K_X (an extension of K of degree $|X|$) whose set of roots consists of one root of each odd child of $\mathfrak{s}$. This allows us to construct minimal SNC models in terms of simpler models. This idea was briefly explored in Example 1.3.5, comparing parts of the special fibre to the special fibres of minimal SNC models of certain elliptic curves. We now have a closer look at this idea in the following example.

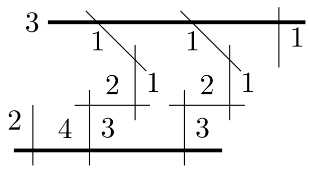
Example 1.3.14. Let C over $K = \mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 5$ be the hyperelliptic curve given by

$$C : y^2 = f(x) = (x^3 - p^2)(x^4 - p^{11}).$$

The cluster picture of C/K consists of two proper clusters $\mathcal{R}$ and $\mathfrak{s}$, shown in Figure 1.5a. The special fibre $\mathscr{X}_k$ of the minimal SNC model $\mathscr{X}$ of C/K is shown in Figure 1.5b.



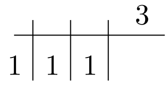
(a) Cluster picture $\Sigma_{C/K}$.



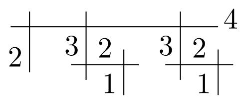
(b) Special fibre of the minimal SNC model of C/K.

Figure 1.5: $C : y^2 = (x^3 - p^2)(x^4 - p^{11})$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

Define elliptic curves C_1 and C_2 over K by $C_1 : y^2 = f_1(x) = x^3 - p^2$ and $C_2 : y^2 = p^2 f_2(x) = p^2(x^4 - p^{11})$ respectively. Note that $f(x) = f_1(x) \cdot f_2(x)$. The roots of $f_1(x)$ contribute the roots in $\mathcal{R} \backslash \mathfrak{s}$, and the roots of $f_2(x)$ contribute the roots in $\mathfrak{s}$. The coefficient in the defining equation of C_2 is chosen to somehow "see" the roots of f_1. It is interesting to compare the minimal SNC models of C_i to that of C for $i = 1, 2$. Note that C_1 and C_2 are type IV and type III* elliptic curves respectively, as shown in Figure 1.6.



(a) Type IV



(b) Type III*

Figure 1.6: Special fibres of minimal SNC models of C_1 and C_2.

It appears that the roots of f_1 and f_2 are making their own contributions to $\mathscr{X}_k$, as both the special fibres of the minimal SNC models of C_i can be seen as “submodels” of $\mathscr{X}_k$ for $i = 1, 2$. This shows how $\mathcal{R}$ and $\mathfrak{s}$ each make their own contribution to $\mathscr{X}_k$. Since $\mathfrak{s}$ is an even child of $\mathcal{R}$, and $\epsilon_{\mathfrak{s}} = 1$, there are two linking chains between their contributions in $\mathscr{X}_k$.

1.4 Classification

Theorems 1.3.6 and 1.3.13 tell us how to construct special fibres from cluster pictures. The remainder of the work presented in this thesis is concerned with classifying these special fibres. Example 1.3.3 demonstrates that cluster pictures determine the special fibres in the genus 1 setting. It is less clear how one should produce a similar classification in higher genus. It turns out to be useful to introduce a notion of *quotient BY trees*, combinatorial objects which will enable us to classify cluster pictures of hyperelliptic curves in higher genus settings. Quotient BY trees are so named because they are quotients of *BY trees*, similar objects introduced in [DDMM17] as a way to study semistable hyperelliptic curves.

In [Bis19] the author describes the possible cluster pictures which can arise from hyperelliptic curves with tame reduction. Furthermore, it is shown that the Galois action is determined by the cluster picture (with depths). In combination with the work laid out in this thesis, this can be used to give a complete classification of the reduction types of hyperelliptic curves with tame reduction. Our approach for classifying such hyperelliptic curves generalises processes described for semistable hyperelliptic curves and BY trees in [DDMM17]. We demonstrate how open quotient BY trees can be used in genus 1 in Example 1.4.18, and give the full classification for genus 2 hyperelliptic curves in Appendix A.2. A similar classification of genus 2 reduction types is given by Namikawa and Ueno in [NU73]. Their classification presents all possible special fibres of minimal regular models of genus 2 curves, rather than minimal SNC models. We make reference to their type naming convention in Appendix A, however we also present a naming convention, set out in [DDMM17] which can easily be used in higher genus settings. It is worth noting that [NU73] is able to deal with wild reduction and $p = 2$. However, our way of classifying reduction types of hyperelliptic curves via quotient BY trees is particularly useful, as given any hyperelliptic curve of arbitrary genus, we may not only compute the special fibre of its minimal SNC model, but we can also provide a complete list of all cluster pictures of hyperelliptic curves of this reduction type. This classifies all the defining equations for hyperelliptic curves of this type. In contrast, [NU73] do not even provide an algorithm for checking the reduction type. Our genus 2 classification presented in Appendix

A.2, provides a way of quickly reading off the special fibre of any genus 2 curve with minimal calculation. More generally, for $g \geq 2$ open quotient BY trees provide a way of easily checking whether two hyperelliptic curves with tame reduction have the same reduction type. It would also be possible to produce complete classifications in $g > 2$ using the work set out in this thesis. As with cluster pictures, useful arithmetic invariants such as the valuation of the discriminant can also be read off quotient BY trees.

Definition 1.4.1. An *open quotient BY tree* is a finite tree T with a unique open edge ε (an edge with only one defined end point), a *marked point* m which lies on the closure of the open edge $\bar{\varepsilon}$, a genus function $g : V(T) \to \mathbb{Z}_{\geq 0}$ on vertices, a multiplicity function $M : V(T) \cup E(T) \to \mathbb{Z}_{>0}$, and a 2-colouring blue/yellow on vertices and edges such that:

(i) If v is a fixed yellow vertex, then v has genus $g(v) = 0$, all edges incident to v are yellow, and

$$\sum_{e \text{ edge incident to } v} \frac{M(e)}{M(v)} \geq 3.$$

(ii) Let v_0 be the unique vertex incident to ε. Then the embedded path from v_0 to any vertex v has non-decreasing multiplicities.

(iii) Let $v \in V(T)$ be any vertex, then there exists some $n \in \mathbb{Z}_{>0}$ such that either 1 or 2 edges incident to v have multiplicity $M(v)$ and all remaining incident edges have multiplicity $nM(v)$. Furthermore, $M(\varepsilon) = 1$.

(iv) If v is blue then the genus of v is such that:

- If only one incident edge, say e, has multiplicity $M(v)$ and all other incident edges have multiplicity $nM(v)$ for $n \in \mathbb{Z}_{>0}$, where $e = \varepsilon$ if $v = v_0$ (the unique vertex incident to ε), then

$$n \mid 2g(v) + 1 \text{ or } 2g(v) \text{ if } e \text{ is blue,}$$
$$n \mid 2g(v) + 2 \text{ or } 2g(v) + 1 \text{ if } e \text{ is yellow.}$$

- If two incident edges, say e_1 and e_2, have multiplicity $M(v)$ and all other incident edges have multiplicity $nM(v)$ for $n \in \mathbb{Z}_{>0}$, where

$\varepsilon \in \{e_1, e_2\}$ if $v = v_0$, then

$$n \mid 2g(v) \text{ if } e_1 \text{ and } e_2 \text{ are both blue,}$$
$$n \mid 2g(v) + 2 \text{ if } e_1 \text{ and } e_2 \text{ are both yellow,}$$
$$n \mid 2g(v) + 1 \text{ if } e_1 \text{ and } e_2 \text{ are different colours.}$$

Note that when $n = 1$ there is no constraint on the values of $g(v)$.

(v) Blue vertices of genus 0 have at least one yellow incident edge.

(vi) For every blue vertex $v \in V(T)$, $2g(v) + 2 \geq \sum_{\substack{e \in E(T), \text{ blue} \\ \text{incident to } v}} \frac{M(e)}{M(v)}$.

We will put metrics $d : T \times T \to \mathbb{Q}_{\geq 0}$ on open quotient BY trees (as topological spaces), to allow us to move between open quotient BY trees and cluster pictures of hyperelliptic curves. There are several constraints on which metrics we can allow, the details of this can be found in Definition 4.1.13.

Theorem 1.4.2 (Theorem 5.1.2)**.** *There is a one-to-one correspondence between metric open quotient BY trees and metric cluster pictures of hyperelliptic curves with tame reduction and top cluster depth $d_{\mathcal{R}} \geq 0$.*

To pass between the two we define a metric open quotient BY tree from the cluster picture of such a hyperelliptic curve and vice versa.

Definition 1.4.3. Let $C : y^2 = f(x)$ be a hyperelliptic curve over K with tame reduction such that $d_{\mathcal{R}} \geq 0$. Define the *open quotient BY tree associated to C*, $T = \underline{\mathrm{T}}(\Sigma_{C/K})$ as follows. The tree T is finite and is equipped with a genus marking $g : V(T) \to \mathbb{Z}_{\geq 0}$ on vertices, a multiplicity function $M : V(T) \cup E(T) \to \mathbb{Z}_{>0}$, and a 2-colouring blue/yellow on vertices and edges. There is one vertex v_X of T for every Galois orbit X of proper clusters in Σ, coloured yellow if X is übereven and blue otherwise. For X and X' both proper orbits, with $X' < X$, T has an edge between v_X and $v_{X'}$ coloured yellow if X' is even, and blue otherwise. One additional open edge is added to $v_{\mathcal{R}}$, of multiplicity 1, coloured yellow if $\mathcal{R}$ is even, and blue otherwise.

The genus of a vertex v_X is defined to be the semistable genus of any cluster $\mathfrak{s} \in X$, as in Definition 2.1.14. The multiplicity of a vertex $v_{X'}$ or an edge between v_X and $v_{X'}$, where $X' < X$ is defined to be $|X'|$. Note that this means that $M(v_X)$ is the minimum of $M(e)$ over all incident edges e, and if e is incident to v_1 and v_2, then $M(e) = \max\{M(v_1), M(v_2)\}$. For this reason, we can omit writing the multiplicity of edges when we draw T, as they can be deduced from the multiplicities of the vertices.

Furthermore, we can define a metric on T, by defining the length of a closed edge e between v_X and v'_X with $X' < X$ to be $\delta_{X'}$, and a *marked point* m lies distance $d_{\mathcal{R}}$ along the open edge. We mark m with a cross on the open edge.

The construction of a cluster picture from an open quotient BY tree simply reverses this construction, this is described formally in 5.1.9, and completes our one-to-one correspondence.

It is useful, in order to produce a usable classification, to have a formula for the genus of a quotient BY tree. This gives a way of listing all quotient BY trees corresponding to hyperelliptic curves of a given genus.

Definition 1.4.4. Let T be an open quotient BY tree and let $B_1, \ldots, B_n$ be the connected components of T_b, the blue part of T, then the genus of T is

$$g(T) = \left(\sum_{i=1}^{n} \min_{w \in V(B_i)} \{M(w)\} \right) - 1 + \sum_{v \in V(T)} g(v)M(v).$$

Proposition 1.4.5 (Proposition 6.2.5)**.** *Let C be a hyperelliptic curve with associated open quotient BY tree T. Then $g(T) = g(C)$.*

Example 1.4.6. There are 56 different non-metric open quotient BY trees of genus 2, but we will not list them all here. There are 8 non-metric open quotient BY trees of genus 1. These are shown in Figure 1.7. Here the labelings represent the genera and multiplicities of vertices. For example $g0\ M1$ represents a genus 0 multiplicity 1 vertex. Marked points shown by crosses.

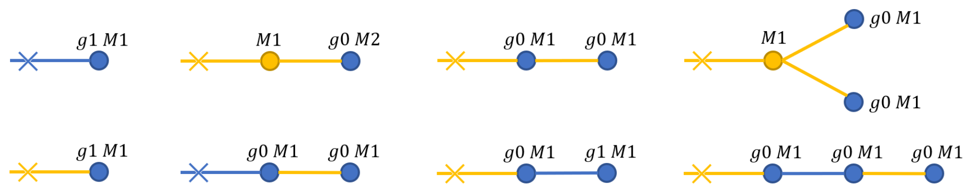


Figure 1.7: All open quotient BY trees of genus 1.

The corresponding cluster pictures are shown in Figure 1.8 respectively, where a line drawn between two proper clusters represents a Galois orbit.

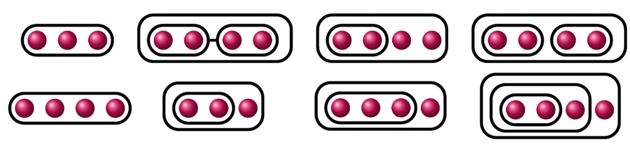
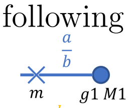
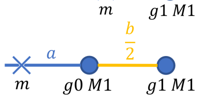


Figure 1.8: All non-metric cluster pictures of genus 1.

Note that in Figure 1.7, there only two open quotient BY trees with a blue open edge. This colouring means these open quotient BY trees correspond to genus 1 curves whose defining equations have 3 roots, whereas those with yellow open edges correspond to genus 1 curves whose defining equations have 4 roots. As such, if we wanted to verify that we can recover the Kodaira-Néron types shown in Table 1.1 using open quotient BY trees we only want to consider the open quotient BY trees with a blue open edge. We can give open quotient BY trees different metrics, as defined in Definition 4.1.13, in particular we can take any of the following metrics:



for $(a, b) = 1$, $a \in \mathbb{Z}_{\geq 0}$ and $b \in \{1, 2, 3\}$,



for $a, b \in \mathbb{Z}$, $a \geq 0$, $b > 0$.

Let $E : y^2 = c_f(x - r_1)(x - r_2)(x - r_3)$ be an elliptic curve over K. Note that under a substitution $x = px'$, $y = py'$ we get a change of model

$$y'^2 = pc_f \left(x' - \frac{r_1}{p} \right) \left(x' - \frac{r_2}{p} \right) \left(x' - \frac{r_3}{p} \right).$$

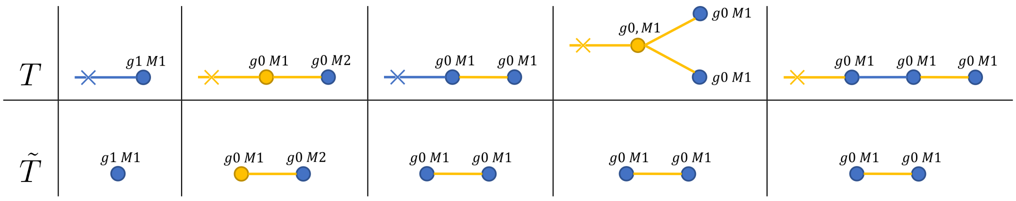
This certainly has the same reduction type, although note that the leading coefficient has been changed, and as a result all depths in the cluster picture are decreased by 1. In terms of quotient BY trees, provided the top clusters have depth ≥ 0 we can think of the marked point having moved distance 1 along the open edge.

So, in order to produce a classification, we need to consider isomorphisms of curves and how these affect cluster pictures and quotient BY trees. Again, this turns out be possible to determine in a completely combinatorial way. This motivates the need for a concept of equivalence of open quotient BY trees that will enable us to classify the reduction types, along with a criterion for how the leading coefficients are affected. The following technical details help us achieve this. In general, the principle is that there is some subtree (the *core*) that must remain unchanged, and the marked point is allowed to move by an integer.

Definition 1.4.7. Let T be an open quotient BY tree. Then the *core* $\tilde{T}$ of T is the tree obtained from T by deleting the open edge and then possibly one of the following: Viewing the unique vertex v_0 incident to the open edge in T as a point on an edge in $\tilde{T}$, provided $g(v_0) = 0$ and there are precisely two closed edges incident to v_0 in T, both of which are coloured the same as v_0 and have multiplicity 1; Deleting v_0 along with a unique incident closed edge, provided

v_0 is blue, $g(v_0) = 0$, and the unique incident closed edge has multiplicity 1 and is coloured blue.

Example 1.4.8. The following are some simple examples of open quotient BY trees along with their cores.



Notation 1.4.9. Let S be an open quotient BY tree or the core of an open quotient BY tree, then we denote by S^1 the subtree consisting of all multiplicity 1 edges and vertices.

Definition 1.4.10. Let T be an open quotient BY tree and $v \in V(T)$. Define

$$s(v,T) = \begin{cases} 2g(v) + 2 - \sum\limits_{\substack{e \in E(T),\ \text{blue} \\ \text{incident to } v}} \frac{M(e)}{M(v)} & \text{if } v \text{ is blue,} \\ 0 & \text{if } v \text{ is yellow.} \end{cases}$$

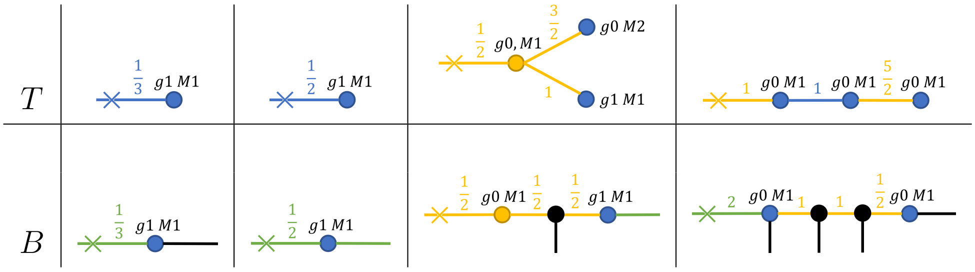
This is the number of singletons of a cluster corresponding to v in the cluster picture associated to T, which is proved in Proposition 5.1.12.

Construction 1.4.11. Let T be a metric open quotient BY tree and v_0 the unique vertex incident to the open edge. Then we create an extended tree B from T^1 as follows:

- If $\tilde{T}$ is obtained from T by deleting just the open edge: change the colour of the open edge of T^1 to green if it was previously blue;
- If $\tilde{T}$ is obtained from T by deleting 'open yellow edge $\varepsilon \to$ genus 0, multiplicity 1, blue vertex $v_0 \to$ closed multiplicity 1 blue edge e' from a vertex $v_1 \in V(T)$: colour ε, v_0 and e green and view v_0 as a point on the open edge rather than a vertex.

For every blue vertex $v \in V(\tilde{T}^1)$ if the denominator $\text{denom}(d(v,m)) \nmid s(v,T)$ then add a green open edge, to v. Next, for any point P on $\tilde{T}^1$ if $d(m,P) \in \mathbb{Z}$ then add a black open edge to P (creating a black vertex at P if P was not already a vertex). Call the tree resulting from these moves so far A. Finally, for every leaf $v \in V(A)$ add a black open edge to v.

Example 1.4.12. The following are some simple examples of open quotient BY trees T along with their extended trees B.



For an open quotient BY tree T we describe how to obtain a new open quotient BY tree T^* with a yellow open edge and marked point as close to the *centre* of the core as possible. This construction is described fully in 4.5.5. Two open quotient BY trees T_1 and T_2 are *equivalent* if T_1^* and T_2^* are isomorphic.

Definition 1.4.13. Let T be an open quotient BY tree with core $\tilde{T}$. The *centre* c of $\tilde{T}$ is the vertex or the midpoint of the edge between two vertices in $V(T^1)$ minimising the value of ϕ, where for $v \in V(T)$:

$$\phi(v) = \max\{w(T') \mid T' \text{ is a connected component of } T \setminus \{v\}\},$$

where

$$w(T') = \frac{1}{\min_{v' \in T'}\{M(v')\}} \sum_{v \in T'} M(v)w(v),$$

$$w(v) = \begin{cases} 0 & \text{if } v \text{ is yellow}, \\ 2g(v) + 2 - \sum_{\substack{e \in E(T_b),\\ \text{incident to } v}} \frac{M(e)}{M(v)} & \text{if } v \text{ is blue}. \end{cases}$$

Remark 1.4.14. There is indeed either a unique minimising vertex or precisely two minimising vertices, in which case they are adjacent. This is proved in Lemma 4.4.4.

An example of this centre calculation is included in Example 1.4.22.

Construction 1.4.15 (T^*)**.** Let T be a metric open quotient BY tree with extended tree B. Let m' be a point on B such that $d(c, m')$ is minimal subject to $d(m', m) \in \mathbb{Z}$. Denote by T^*, a tree obtained in the following way. If m' is green add "open yellow edge $\to$ genus 0, multiplicity 1 blue vertex $\to$ closed blue edge" to the closest vertex of $\tilde{T}$ to m'. Otherwise, add an open yellow edge to the closest point of $\tilde{T}$ to m' creating a vertex there, coloured the same as the edge it lies on, if it is not already a vertex.

Remark 1.4.16. Construction 1.4.15 only results in a unique tree when $\mathrm{denom}(d(m,c)) \neq 2$. For now we will not worry about this but it is dealt with in depth in Sections 4.5 and 4.6.

Definition 1.4.17. Two metric open quotient BY trees T_1 and T_2 are *equivalent* if $T_1^* \cong T_2^*$.

Example 1.4.18. We can use this equivalence to reproduce the classification for elliptic curves with tame reduction in Example 1.3.3. Considering the equivalence classes of metric open quotient BY trees of genus 1, we are able to produce a classification, choosing a representative for each equivalence class of metric open quotient BY trees, as shown in Table 1.2. There are two choices of leading coefficient for each representative, giving ten reduction types.

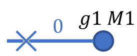
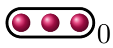
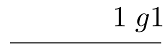
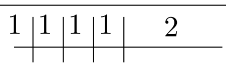
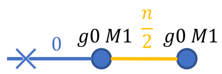
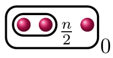
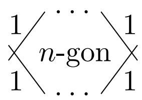
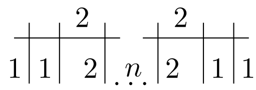
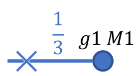
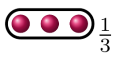
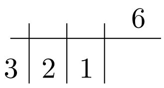
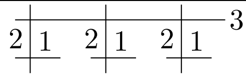
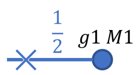
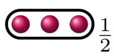
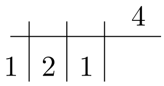
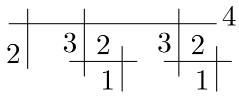
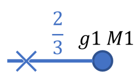
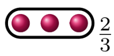
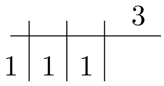
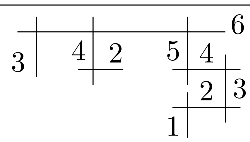
T	Σ	$v(c_f)$ mod 2	Type	$\mathscr{X}_k$
		0	I_0	
		1	I_0^*	
		0	I_n	
		1	I_n^*	
		0	II	
		1	IV*	
		0	III	
		1	III*	
		0	IV	
		1	II*	

Table 1.2: Kodaira-Néron types of elliptic curves with $p \geq 5$.

Theorem 1.4.19. *Let C and C' be two isomorphic hyperelliptic curves over K, with associated metric open quotient BY trees T and T' respectively. Then T and T' are equivalent. Conversely, let T'' be a metric open quotient BY tree equivalent to T. Then there exists a hyperelliptic curve curve C'' which is isomorphic to C over K, such that the metric open quotient BY tree of C'' is isomorphic to T''.*

For a metric open quotient BY tree T, every equivalent metric open quotient BY tree T' can be obtained by taking a new marked point m' to be any point on the extended tree of T integer distance distance from m, and adding either 'open yellow edge', 'open blue edge', or 'closed blue multiplicity 1 edge $\rightarrow$ genus 0 multiplicity 1 blue vertex $\rightarrow$ open yellow edge' to $\tilde{T}$ at the point closest to m' (creating a vertex at this point the same colour as the edge it lies on if it is not already a vertex of $\tilde{T}$). Moreover, we give a full description of the equivalence class in 4.6.4.

Note that for our classification of elliptic curves in Example 1.4.18 we choose a representative of each class with a blue open edge. However, more generally, for a metric open quotient BY tree T we will choose T^* as the canonical representative of the equivalence class. We will see in Section 5.4 that the equivalence class of a metric open quotient BY tree encodes the effect of Möbius transformations on the roots of the associated cluster picture. These will affect the leading coefficient, and by Theorem 1.3.1, this is something we need to keep track of. The following theorem provides an easy way to do this using metric open quotient BY trees.

Theorem 1.4.20. *Let $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ be hyperelliptic curves of genus g over K, with cluster pictures Σ and Σ' respectively and metric open quotient BY trees $T = \underline{T}(\Sigma)$ and $T' = \underline{T}(\Sigma')$. Suppose that the sets of roots $\mathcal{R}$ and $\mathcal{R}'$ of f and f' respectively are such that $d_{\mathcal{R}}, d_{\mathcal{R}'} \geq 0$. Then the dual graphs of the special fibres of the minimal SNC models of C and C' are isomorphic if T and T' are equivalent and the leading coefficients c_f and $c_{f'}$ of f and f' are such that:*

- *if g is even: if*

$$v\left(\mathrm{disc}\left(\frac{f}{c_f}\right)\right) - v\left(\mathrm{disc}\left(\frac{f'}{c_{f'}}\right)\right) \equiv 2(g+1)(2g+1)d(m,m') \mod 4(2g+1)$$

then $v(c_f) \equiv v(c_{f'}) \mod 2$, else $v(c_f) \not\equiv v(c_{f'}) \mod 2$

- *if g is odd: then*

$$\frac{v\left(\mathrm{disc}\left(\frac{1}{c_f}f\right)\right) - v\left(\mathrm{disc}\left(\frac{1}{c_{f'}}f'\right)\right)}{2(2g+1)} \equiv v(c_f) - v(c_{f'}) \mod 2$$

If either $d_{\mathcal{R}}, d_{\mathcal{R}'} < 0$ then note that a simple scaling gives us a change of model, and will allow us to transform the cluster picture into something with non-negative top cluster depth. We will discuss how such a transformation affects the leading coefficient later. Piecing this together with the theorem above will allow us to handle changes in leading coefficients regardless of what the value of the top cluster depths are.

We provide a way of reading off the discriminant of f from its associated metric open quotient BY tree. This is discussed in more detail in Section 6.3.1, where the following result is proved.

Theorem 1.4.21. *Let $C : y^2 = f(x)$ be a hyperelliptic curve with tame reduction, with metric open quotient BY tree T. Denote the marked point of T by m, and define a partial order on the vertices of T by setting $v' \preceq v$ if v lies on the embedded path from m to v'. Then*

$$\begin{aligned} v(\Delta_C) &= v(c_f)(4g+2) + v\left(\mathrm{disc}\left(\frac{1}{c_f}f\right)\right) \\ &= v(c_f)(4g+2) + \sum_{v \in V(T)} M(v)d(v,m)\left(|v|^2 - \sum_{v' < v} |v'|^2 \tfrac{M(v')}{M(v)} - s(v,T)\right), \\ &= v(c_f)(4g+2) + \sum_{v \in V(T)} M(v)\delta_v |v| \left(|v| - 1\right), \end{aligned}$$

where

$$|v| = \sum_{v' \preceq v} s(v',T)\frac{M(v')}{M(v)},$$

$\delta_v = length(e_v)$, the length of the edge incident to v lying on the embedded path between v and m, and $v' < v$ if $v' \neq v$ is adjacent to v and $v' \preceq v$. If $v = v_0$, the unique vertex incident to the open edge, then we take $\delta_{v_0} = d(v_0, m)$.

This allows us to give a complete classification in higher genus cases using metric open quotient BY trees. We do so for genus 2 in Appendix A.2, following a proposed naming convention for open quotient BY trees in Appendix A.1. In particular we can see that the reduction type is determined by the cluster picture, and we are able to read off the reduction type of any genus 2 curve from this classification. To demonstrate the power of the work laid out in this thesis we present a longer worked example that spans all our main results.

Example 1.4.22. Consider the hyperelliptic curve $C : y^2 = (x^2 - p^2)(x^4 - p^{11})$ over $\mathbb{Q}_p^{\mathrm{ur}}$. Suppose that we wish to find the special fibre of the minimal SNC model of $C/\mathbb{Q}_p^{\mathrm{ur}}$, and classify all hyperelliptic curves with the same reduction type. Here we lay out a process to answer both of these using cluster pictures and metric open quotient BY trees. The cluster picture Σ and metric open quotient BY tree T of $C/\mathbb{Q}_p^{\mathrm{ur}}$ are shown in Figure 1.9.

(a) Cluster picture Σ of $C/\mathbb{Q}_p^{\mathrm{ur}}$ **(b)** Metric open quotient BY tree T

Figure 1.9: Cluster picture and open quotient BY tree associated to $C/\mathbb{Q}_p^{\mathrm{ur}}$.

Let us start by using the cluster picture and valuation of the leading coefficient of f, which in this case is $c_f = 1$, to determine the special fibre of the minimal SNC model, as described in Theorem 1.3.13. The cluster picture Σ consists of just two proper clusters, $\mathcal{R}$, and the cluster of size 4 which we will label $\mathfrak{s}$. We must first calculate the following arithmetic invariants:

$$\begin{aligned} \nu_{\mathcal{R}} &= v_{\mathbb{Q}_p^{\mathrm{ur}}}(c_f) + \sum_{r \in \mathcal{R}} d_{\mathcal{R} \wedge r}, \\ &= 6, \\ \lambda_{\mathcal{R}} &= \frac{\nu_{\mathcal{R}}}{2} - d_{\mathcal{R}} \sum_{\mathfrak{t} < \mathcal{R}} \left\lfloor \frac{|\mathfrak{t}|}{2} \right\rfloor, \\ &= 1, \end{aligned} \qquad \begin{aligned} \nu_{\mathfrak{s}} &= v_{\mathbb{Q}_p^{\mathrm{ur}}}(c_f) + \sum_{r \in \mathfrak{s}} d_{\mathfrak{s} \wedge r}, \\ &= 13, \\ \lambda_{\mathfrak{s}} &= \frac{\nu_{\mathfrak{s}}}{2} - d_{\mathfrak{s}} \sum_{\mathfrak{s}' < \mathfrak{s}} \left\lfloor \frac{|\mathfrak{s}'|}{2} \right\rfloor, \\ &= \frac{13}{2}. \end{aligned}$$

Recall that the genus of a cluster $\mathfrak{t}$ is given by

$$g(\mathfrak{t}) = \begin{cases} \lfloor \frac{g_{ss}(\mathfrak{t})}{b_{\mathfrak{t}}} \rfloor & \lambda_{\mathfrak{t}} \in \mathbb{Z}, \\ \lfloor \frac{g_{ss}(\mathfrak{t})}{b_{\mathfrak{t}}} + \frac{1}{2} \rfloor & \lambda_{\mathfrak{t}} \notin \mathbb{Z}, b_{\mathfrak{t}} \text{ even}, \\ 0 & \lambda_{\mathfrak{t}} \notin \mathbb{Z}, b_{\mathfrak{t}} \text{ odd}. \end{cases}$$

So, since $g_{ss}(\mathcal{R}) = 0$ we have $g(\mathcal{R}) = 0$, and since $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$, $b_{\mathfrak{s}} = 4 \in 2\mathbb{Z}$, we have $g(\mathfrak{s}) = \lfloor \frac{g_{ss}(\mathfrak{s})}{b_{\mathfrak{s}}} + \frac{1}{2} \rfloor = \lfloor \frac{1}{4} + \frac{1}{2} \rfloor = 0$. Recall also that for a proper cluster $\mathfrak{t}$, in a Galois orbit X, $e_{\mathfrak{t}} \in \mathbb{Z}_{>0}$ is minimal such that $e_{\mathfrak{t}}|X|d_{\mathfrak{t}} \in \mathbb{Z}$ and $e_{\mathfrak{t}}|X|\nu_{\mathfrak{t}} \in 2\mathbb{Z}$. Both $\mathcal{R}$ and $\mathfrak{s}$ are in trivial Galois orbits, so we find that $e_{\mathcal{R}} = 1$, and $e_{\mathfrak{s}} = 4$. So, by Theorem 1.3.13, $\mathcal{R}$ contributes one component of multiplicity 1 and genus 0, and $\mathfrak{s}$ contributes one component of multiplicity 4 and genus 0. It remains to calculate any linking chains and tails/crossed tails. Since $e_{\mathcal{R}} = 1$ there are no sloped tails intersecting $\Gamma_{\mathcal{R}}$. However $\mathfrak{s} < \mathcal{R}$ is an even child of $\mathcal{R}$ and $\epsilon_{\mathfrak{s}} = (-1)^{\nu_{\mathfrak{s}} - |\mathfrak{s}| d_{\mathfrak{s}}} = (-1)^2 = 1$, so we get two chains $L^+_{\mathcal{R},\mathfrak{s}}$, and $L^-_{\mathcal{R},\mathfrak{s}}$

intersecting $\Gamma_{\mathcal{R}}$. These have parameters $t_1 = -d_{\mathcal{R}} = -1$, $t_2 = t_1 - \delta_{\mathfrak{s}} = -\frac{11}{4}$, and $\mu = 1$. Notice that the following inequalities satisfy Definition 1.3.7:

$$\mu t_1 = \frac{-1}{1} > \frac{-2}{1} > \frac{-5}{2} > \frac{-8}{3} > \frac{-11}{4} = \mu t_2.$$

In practice, finding such chains is very straightforward and is discussed in more detail in Remark 7.2.12. This gives us two chains of rational curves from $\Gamma_{\mathcal{R}}$ to $\Gamma_{\mathfrak{s}}$, each with three components of multiplicity $3, 2$ and 1. There are no further sloped chains intersecting $\Gamma_{\mathcal{R}}$. Finally, we need to check for sloped chains intersecting $\Gamma_{\mathfrak{s}}$. Note that $\mathfrak{s}$ has no stable child (all four roots are in an orbit of size 4) $\lambda_{\mathfrak{s}} = \frac{13}{2} \notin \mathbb{Z}$, $e_{\mathfrak{s}} = 4 > 2$ and $g_{\mathrm{ss}}(\mathfrak{s}) = 1 > 0$, therefore we get one $T_{x_{\mathfrak{s}}=0}$ tail intersecting $\Gamma_{\mathfrak{s}}$. This has parameters $t_1 = -d_{\mathfrak{s}} = -\frac{11}{4}$, $\mu = 2$, and $t_2 = \frac{1}{\mu}\lfloor \mu t_1 - 1 \rfloor = -\frac{7}{2}$. The following inequalities satisfy Definition 1.3.7:

$$\mu t_1 = \frac{-11}{2} > \frac{-6}{1} > \frac{-7}{1} = \mu t_2.$$

So, $T_{x_{\mathfrak{s}}=0}$ has length 1 and its only component has multiplicity $\mu \cdot 1 = 2$. There are no further sloped chains, and the special fibre is as pictured in Figure 1.10. Indeed, the Namikawa-Ueno type of $C/\mathbb{Q}_p^{\mathrm{ur}}$ is III*-II$_3$, so we are in this case

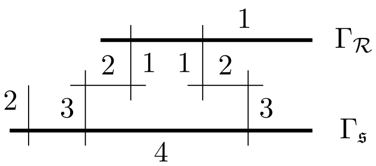


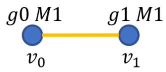
Figure 1.10: Special fibre of the minimal SNC model of $C/\mathbb{Q}_p^{\mathrm{ur}}$.

able to verify our construction.

If we wish to produce useable classifications we need to know exactly which other curves have this reduction type. So, we turn to open quotient BY trees. The core $\tilde{T}$ of T and the extended tree, constructed from $T^1 \cong T$ by adding open edges to the leaves and points which are integer distance from the marked point m, are shown in Figure 1.11. It is convenient to select a

Figure 1.11: The core $\tilde{T}$ of T and the extended tree B.

canonical representative of the equivalence class as this provides an easy way of checking whether or not two metric open quotient BY trees are equivalent without producing the full equivalence class. To do this we calculate the centre of $\tilde{T}$, take a marked point m' as close to the centre as possible, and add an open yellow edge to $\tilde{T}$ at the closest point (which is not necessarily a vertex) to m'. In particular, for this example, the process is carried out as follows. To calculate the centre we calculate $\phi(v)$ for all $v \in V(T^1)$. Note that here $T^1 \cong T$, and the metric has no effect on the centre calculation. Let us label the vertices of $\tilde{T}$ as follows:

$\tilde{T} =$ 

Then $w(v_0) = 2g(v_0)+2 = 2$, and $w(v_1) = 2g(v_1)+2 = 4$. So, $\phi(v_0) = w(v_1) = 4$, $\phi(v_1) = w(v_0) = 2$. That is v_1 is the minimising vertex and the centre of $\tilde{T}$ is $c = v_1$. We can view c on B and find the closest point m' of B to c which is integer distance from m. This is pictured in Figure 1.12. In this case c is the

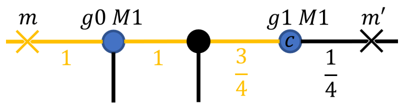
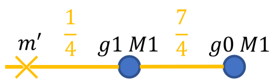


Figure 1.12: Extended tree B showing c and m'.

closest point of $\tilde{T}$ to m'. So, to construct the canonical representative we take the marked point to be m' and add an open yellow edge to $\tilde{T}$ at $c = v_1$. This results in the following metric open quotient BY tree:

$T^* =$ 

We can also use the extended tree and the core to construct the full equivalence class of metric open quotient BY trees. In particular, each equivalent metric open quotient BY tree is obtained by taking a new marked point m' to be any point on the extended tree B integer distance distance from m, and adding either 'open yellow edge', 'open blue edge', or 'closed blue multiplicity 1 edge $\rightarrow$ genus 0 multiplicity 1 blue vertex $\rightarrow$ open yellow edge' to $\tilde{T}$ at the point closest to m' (creating a vertex at this point the same colour as the edge it lies on if it is not already a vertex of $\tilde{T}$). Any such move which results in a metric open quotient BY tree is equivalent to T, and every equivalent metric open quotient BY tree can be obtained by one of these moves. We give a more precise description of exactly which moves result in metric open quotient BY trees in 4.6.4. The full equivalence class is shown in Figure 1.13. We have a one-to-one correspondence between metric open quotient BY trees and cluster pictures of hyperelliptic curves with tame reduction, whose top cluster has

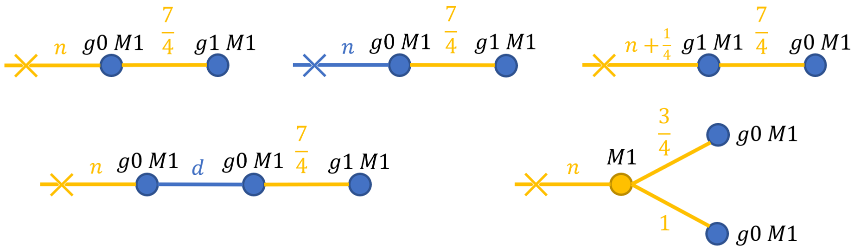


Figure 1.13: Full equivalence class of T with $n, d \in \mathbb{Z}$, $n \geq 0$, $d > 0$.

depth ≥ 0. In particular the cluster pictures corresponding to the metric open quotient BY trees in our equivalence class are pictured in Figure 1.14. Note

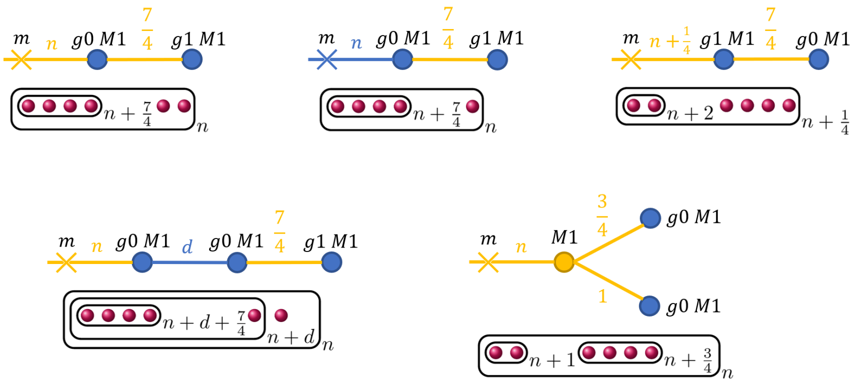
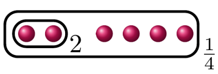


Figure 1.14: Full equivalence class of T and their corresponding cluster pictures with $n, d \in \mathbb{Z}$, $n \geq 0$, $d > 0$.

that applying a Möbius transformation $z \mapsto pz$ to a set of roots increases the depths of all clusters in a cluster picture by 1. So, we can use our equivalence class of metric open quotient BY trees to list all cluster pictures of hyperelliptic curves with tame reduction (even those with top cluster depth < 0) and we just then drop the condition that $n \geq 0$.

The cluster picture, along with the leading coefficient completely determine the special fibre of the minimal SNC model. For this reason we also need to determine what leading coefficient a member of the equivalence class can take to ensure that the reduction type is the same. Note that this is not necessarily the only option, just one that certainly does ensure this. For example, consider $C' : y^2 = f'(x)/\mathbb{Q}_p^{\mathrm{ur}}$, where $f'(x) = c_{f'}(x^4 - p)(x^2 - p^4)$. This has cluster picture

$\Sigma' =$ 

and metric open quotient BY tree T^*. We want to know what valuation of c'_f will ensure that C' has the same reduction type as C over $\mathbb{Q}_p^{\mathrm{ur}}$. By Theorem

1.4.20, since $g = 2$ is even we simply need to check whether or not

$$v\left(\operatorname{disc}\left(\frac{f}{c_f}\right)\right) - v\left(\operatorname{disc}\left(\frac{f'}{c_{f'}}\right)\right) \equiv 2(g+1)(2g+1)d(m,m') \mod 4(2g+1).$$

In this case, $2(g+1)(2g+1) = 30$, $4(2g+1) = 20$, and we can see from Figure 1.12 that $d(m,m') = 3$. By Theorem 1.4.21, we are able to read the valuations of the discriminants of $\frac{1}{c_f}f(x)$ and $\frac{1}{c_{f'}}f'(x)$ off the metric open quotient BY trees T and T^* respectively. In particular, we find that

$$v\left(\operatorname{disc}\left(\frac{1}{c_f}f(x)\right)\right) = 51,$$

$$v\left(\operatorname{disc}\left(\frac{1}{c_{f'}}f'(x)\right)\right) = 11.$$

Therefore,

$$v\left(\operatorname{disc}\left(\frac{f}{c_f}\right)\right) - v\left(\operatorname{disc}\left(\frac{f'}{c_{f'}}\right)\right) \equiv 40 \not\equiv 90 \mod 20,$$

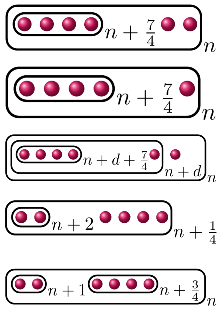
and by Theorem 1.4.21, we will take $v(c_{f'}) \equiv 1 \mod 2$.

As a check, we can use Theorem 1.3.13 to determine the special fibre of the minimal SNC model of $C' : y^2 = p(x^4 - p)(x^2 - p^4)$. Let $\mathcal{R}'$ be the top cluster of Σ', and $\mathfrak{s}'$ the child of size 2. $\mathcal{R}'$ is the only principal cluster in Σ. We find that

$$\nu_{\mathcal{R}'} = \frac{5}{2}, \lambda_{\mathcal{R}'} = 1, e_{\mathcal{R}'} = 4, g(\mathcal{R}') = 0, \text{ and } \epsilon_{\mathcal{R}'} = -1.$$

So, we have one component $\Gamma_{\mathcal{R}'}$ of multiplicity 4 and genus 0. Using the tables in Theorem 1.3.13 we find that $\Gamma_{\mathcal{R}'}$ has one T_∞ tail with parameters $t_1 = -\frac{1}{4}$, $\mu = 2$, and $t_2 = -1$. This gives a tail of length 1 whose only component has multiplicity 2. Since $\mathfrak{s}'$ is a twin and $\epsilon_{\mathfrak{s}'} = -1$, we get a $L_{\mathfrak{s}'}$ loop off $\Gamma_{\mathcal{R}'}$ with parameters $t_1 = -\frac{1}{4}$, $t_2 = -\frac{15}{4}$, $\mu = 1$. This gives that $L_{\mathfrak{s}'}$ has 7 components of multiplicities $3, 2, 1, 1, 1, 2$ and 3 in order. There are no further components in the special fibre. That is, we have shown that C' has the same special fibre as C over $\mathbb{Q}_p^{\mathrm{ur}}$, as shown in Figure 1.10.

We can do the same for any hyperelliptic curve whose metric open quotient BY tree lies in the equivalence class of T and, provided the leading coefficient has been selected appropriately we will always find that Theorem 1.3.13 produces isomorphic special fibres. In particular, using metric open quotient BY trees, Theorem 1.4.20, and accounting for how leading coefficients change under scaling we obtain the following complete list of cluster pictures and valuations of leading coefficients which result in the same special fibre:



$$v(c_g) \equiv 0 \mod 2,$$
$$v(c_g) \equiv n \mod 2,$$
$$v(c_g) \equiv d \mod 2,$$
$$v(c_g) \equiv 1 \mod 2,$$
$$v(c_g) \equiv 0 \mod 2.$$

So, if $y^2 = g(x)$ is a hyperelliptic curve over $\mathbb{Q}_p^{\mathrm{ur}}$ with leading coefficient c_g, which has one of these cluster picture and leading coefficient pairs, then it has the same reduction type as $C/\mathbb{Q}_p^{\mathrm{ur}}$. That is the Namikawa-Ueno type of any such hyperelliptic curve is III^*-III_3.

1.5 Structure of Thesis

The thesis is structured as follows. In Chapters 2 to 6 we focus on open quotient BY trees and how we can use them to classify the reduction types of hyperelliptic curves with tame reduction. Specifically, in Chapter 2 we start with a brief introduction to cluster pictures and BY trees. This is work taken from literature which motivates the discussion for what approach we should take for hyperelliptic curves with tame reduction in Chapter 3. In Chapter 4 we take a purely combinatorial approach, defining open quotient BY trees, an equivalence relation and a choice of canonical representative. In Chapter 5 we relate open quotient BY trees to polynomials with tame splitting fields, associating open quotient BY trees to their cluster pictures. We go on to prove that there is a Möbius transformation between any two equivalent open quotient BY trees, and applying any Möbius transformation always results in an equivalent open quotient BY tree. Finally in Chapter 6 we relate Chapters 4 and 5 to hyperelliptic curves including discussions on leading coefficients and discriminant.

In Chapters 7 to 9 we turn our attention to using cluster pictures to study special fibres of minimal SNC models. These chapters present a discussion of relevant background and work of the author and Omri Faraggi as presented in [FN20]. In Chapter 7, we restate key definitions and theorems from literature, which we will make use of in the remainder of the thesis. In Chapter 8, we calculate the minimal SNC model for two special cases. The first of these special cases, Section 8.1, is where C has *tame potentially good reduction* - that is, it has a smooth model over a tame extension of K. This will act as a base case for our eventual proof by induction. The second of these cases, Section 8.2, examines curves C with a cluster picture which consists of exactly two proper clusters $\mathfrak{s} < \mathcal{R}$. Curves with such cluster pictures are used to deduce the linking chains between central components in the main theorems. These main theorems are stated and proved in Section 9. In Appendix A we propose a naming convention for open quotient BY trees and give a complete classification of the special fibres of genus 2 curves with tame reduction, afforded by all of the work in this thesis.

1.6 Notation

For the convenience of the reader, the following two tables collate the general notation and terminology which we make use of throughout the thesis. Table 1.3 lists the general notation associated to fields, hyperelliptic curves, and models. Table 1.4 lists the notation and terminology associated to BY trees, cluster pictures and Newton polytopes.

K	non-archimedean field	v_K	discrete valuation
$\mathcal{O}_K$	ring of integers	π_K	uniformiser of K
k	residue field of K	$\overline{K}$	algebraic closure of K
C	hyperelliptic curve over K given by $y^2 = f(x)$	L	field extension of K over which C_L is semistable
$g(C)$	genus of C, also denoted g	$\mathcal{R}$	set of roots of $f(x)$ in $\overline{K}$
e	degree of L/K for such L	mod $\mathfrak{m}$	reduction to the residue field
X	Galois orbit of clusters	$\mathscr{X}$	minimal SNC model of C/K
$\mathscr{X}_k$	special fibre of $\mathscr{X}$	$\Gamma^{\pm}_{X,K}$	component(s) from X in $\mathscr{X}_k$
$\mathscr{Y}$	minimal SNC model of C/L	$\mathscr{Y}_k$	special fibre of $\mathscr{Y}$
$\Gamma^{\pm}_{\mathfrak{s},L}$	component(s) from $\mathfrak{s}$ in $\mathscr{Y}_k$	$\mathbb{Q}_p^{\mathrm{ur}}$	maximal unramified extension

Table 1.3: General notation associated to fields, hyperelliptic curves, and models

$\Sigma_{C/K}$	(1.2.1)	ε, m	(4.1.3)	$\nu_{\mathfrak{s}}$	(7.1.12)
$\mathfrak{s}$	(1.2.1)	v_0	(4.1.3)	χ	(7.1.14)
$d_{\mathfrak{s}}$	(1.2.1)	T_b	(4.1.11)	$\lambda_{\mathfrak{s}}$	(7.1.14)
$a_{\mathfrak{s}}, b_{\mathfrak{s}}$	(2.1.1)	$l(e)$	(4.1.13)	$\alpha_{\mathfrak{s}}$	(7.1.14)
odd cluster	(2.1.7)	$\tilde{T}$	(4.3.5)	$\beta_{\mathfrak{s}}$	(7.1.14)
even cluster	(2.1.7)	$w(v)$	(4.4.1)	$\gamma_{\mathfrak{s}}$	(7.1.14)
twin	(2.1.7)	$\phi(v)$	(4.4.1)	$\theta_{\mathfrak{s}}$	(7.1.14)

$\mathfrak{s}' < \mathfrak{s}$	(2.1.8)	T^1	(4.4.3)	$\epsilon_{\mathfrak{s}}$	(7.1.14)
$P(\mathfrak{s})$	(2.1.8)	c	(4.4.5)	$c_{\mathfrak{s}}$	(7.1.17)
$\widehat{\mathfrak{s}}, \widetilde{\mathfrak{s}}$	(2.1.8)	B	(4.5.1)	$\mathrm{red}_{\mathfrak{s}}$	(7.1.17)
cotwin	(2.1.8)	T^*	(4.5.5)	$\Delta(C)$	(7.2.1)
übereven	(2.1.8)	$(T^*)^{\pm}$	(4.5.9)	$\Delta_v(C)$	(7.2.1)
$z_{\mathfrak{s}}$	(2.1.9)	$\underline{T}(\Sigma)$	(5.1.5)	v_{Δ}	(7.2.1)
$\mathfrak{s} \wedge \mathfrak{s}'$	(2.1.10)	$v_{X'}$	(5.1.5)	L, F	(7.2.2)
$\delta_{\mathfrak{s}}$	(2.1.11)	$\underline{\Sigma}(T)$	(5.1.9)	$\Delta(\mathbb{Z}), L(\mathbb{Z}), F(\mathbb{Z})$	(7.2.3)
$\delta(\mathfrak{s}, \mathfrak{s}')$	(2.1.11)	$\mathfrak{s}_{v,i}$	(5.1.9)	$\overline{\Delta}(\mathbb{Z})\overline{L}(\mathbb{Z}), \overline{F}(\mathbb{Z})$	(7.2.3)
principal	(2.1.12)	orphan	(5.1.15)	δ_{λ}	(7.2.4)
$\mathfrak{s}^*$	(2.1.13)	$D(\mathfrak{s})$	(5.2.2)	s_1^L, s_2^L	(7.2.8)
$g_{\mathrm{ss}}(\mathfrak{s})$	(2.1.14)	$\mathscr{T}_{K(\mathcal{R})}$	(5.2.3)	$g(\mathfrak{s})$	(8.1.22)
singleton	(2.1.19)	$\mathscr{T}(f/K)$	(5.2.6)	principal orbit	(5.1.3)
$\mathfrak{s}_{\mathrm{sing}}$	(2.1.19)	$\tilde{\mathscr{T}}(f/K)$	(5.2.12)	λ_X	(9.1.4)
$M(v), M(e)$	(4.1.1)	$\mathcal{R}^+$	(5.2.13)	K_X	(9.1.3)
$g(v)$	(4.1.1)	$g(T)$	(6.2.2)	e_X	(9.1.7)
$s(v,T), s(v)$	(4.1.1)	$\lvert v \rvert$	(6.3.8)	$g(X)$	(9.1.7)

Table 1.4: Notation for cluster pictures, quotient BY trees and Newton polytopes

Throughout this thesis, the word graph refers to a topological space G homeomorphic to a finite (combinatorial) graph. It comes with a set of vertices $V(G)$ and edges $E(G)$. Graph isomorphisms are homotopy classes of homeomorphisms that preserve vertices and edges. With the exception of dual graphs, we will only be discussing trees where loops and multiple edges are not allowed. By a metric graph we mean a topological graph G along with a function $l : E(G) \to \mathbb{R}_{\geq 0}$ which assigns a length to each edge. This can be extended to a metric on all of G. We will write $d(v, v')$ for the shortest distance between two vertices $v, v' \in V(G)$. For metric graphs, isomorphisms and automorphisms must preserve lengths.

1.6.1 BY Trees

In any figure showing a BY tree or quotient BY tree, genera of vertices are preceded by g and multiplicities of vertices or edges (in the quotient case) are preceded by M. So, $g2, M1$ written next to a vertex indicates that it has genus 2 and multiplicity 1. For quotient BY trees, marked points are drawn as crosses.

1.6.2 Cluster Pictures

Roots in cluster pictures are drawn as nodes, and circles are drawn to represent the proper clusters, indicating how p-adically close roots are to each other. In the metric case the depths of clusters are included on the cluster pictures. It is worth pointing out that this is not the same as writing relative depths on cluster pictures, the convention used in [DDMM18].

1.6.3 Tame Reduction

A hyperelliptic curve $C : y^2 = f(x)$ has tame potentially semistable reduction over K if there exists some finite extension L/K such that C has semistable reduction over L, and $[L : K]$ is coprime to p. This is equivalent to f having tame splitting field. We refer to this as *tame reduction*.

1.6.4 Special Fibres

Whenever a component in a figure of a special fibre is drawn in bold it is a central component. In any figure describing the special fibre of a model, numbers indicate multiplicities, except those preceded by g, which indicate the genus of a component. So 2 indicates a rational curve of multiplicity 2 and $2g1$ indicates a genus 1 curve of multiplicity 2.

Chapter 2

Background - Cluster Pictures and BY Trees

2.1 Cluster Pictures

Let C/K be a hyperelliptic curve given by Weierstrass equation $y^2 = f(x)$, with genus $g(C) \geq 1$. Let $\mathcal{R}$ denote the set of roots of $f(x)$ in $\bar{K}$. The p-adic distances between the roots contain a large amount of useful information. To visualise these p-adic distances we use *cluster pictures*, as described in [DDMM18]. In this section we outline the key definitions required for this thesis concerning cluster pictures.

Definition 2.1.1. A *cluster* is a non-empty subset $\mathfrak{s} \subseteq \mathcal{R}$ of the form $\mathfrak{s} = D \cap \mathcal{R}$ for some disc $D = z + \pi_K^n \mathcal{O}_{\overline{K}}$, where $z \in \overline{K}$, $n \in \mathbb{Q}$ and π_K is a uniformiser of K. If $\mathfrak{s}$ is a cluster and $|\mathfrak{s}| > 1$, we say that $\mathfrak{s}$ is a *proper cluster*. For a proper cluster $\mathfrak{s}$ we define its *depth* $d_\mathfrak{s}$ to be

$$d_\mathfrak{s} = \min_{r,r' \in \mathfrak{s}} v_K(r - r').$$

We write $d_\mathfrak{s} = \frac{a_\mathfrak{s}}{b_\mathfrak{s}}$ with $a_\mathfrak{s}, b_\mathfrak{s}$ coprime. The *cluster picture* $\Sigma_{C/K} = (\mathcal{R}, \Sigma, d)$ of C is the collection of all clusters of the roots of f. When there is no risk of confusion, we may simplify this to Σ_C.

The cluster picture Σ_C is a way of visualising which roots of f are p-adically close. In a non-archimedean algebra, two discs either have a non empty intersection or one is contained in the other. So Definition 1.2.1 gives us that any two clusters are either disjoint or one is contained in the other. Moreover $d_{\mathfrak{s}'} > d_\mathfrak{s}$ if $\mathfrak{s}' \subsetneq \mathfrak{s}$. Every root is a cluster, that is $\{r\} \in \Sigma_C$ for every $r \in \mathcal{R}$, and $\mathcal{R} \in \Sigma_C$. It is also possible to describe cluster pictures as purely combinatorial objects.

Definition 2.1.2. Let X be a finite set and $\Sigma \subset \mathcal{P}(X)$ be a collection of non-empty subsets of X. Elements of Σ are called *clusters*. Σ (or (X, Σ)) is a *cluster picture* if

(i) Every singleton ('root') of X is a cluster, and X itself is a cluster,

(ii) Two clusters are either disjoint or one is contained in the other.

We refer to X as the *top cluster*.

Remark 2.1.3. Note that if C/K is a hyperelliptic curve then its cluster picture Σ_C is still a cluster picture in the sense of Definition 2.1.2. The set of roots $\mathcal{R}$ is the top cluster of Σ_C.

In order to work with clusters we need a significant amount of terminology from [DDMM18] which we describe here.

Definition 2.1.4. Two cluster pictures (X, Σ) and (X', Σ') are *isomorphic* if there is a bijection $X \to X'$ that takes Σ to Σ'.

Definition 2.1.5. A cluster picture (X, Σ) is *metric* if every proper cluster $\mathfrak{s}$ has a depth $d_\mathfrak{s} \in \mathbb{Q}$ assigned to it, and $d_{\mathfrak{s}'} > d_\mathfrak{s}$ if $\mathfrak{s}' \subsetneq \mathfrak{s}$. We may denote the cluster picture by (X, Σ, d) rather than Σ. An *isomorphism of metric cluster pictures* is an isomorphism that preserves these depths. That is, (X, Σ, d) and (X', Σ', d') are isomorphic if there is a bijection $\phi : X \to X'$ taking Σ to Σ' such that $d_\mathfrak{s} = d'_{\phi(\mathfrak{s})}$.

Definition 2.1.6. Let $(\mathcal{R}, \Sigma)$ be a cluster picture. Then the genus of Σ is such that

$$|\mathcal{R}| = 2g(\Sigma) + 1 \text{ or } 2g(\Sigma) + 2.$$

Definition 2.1.7. A cluster $\mathfrak{s}$ is *even* (resp. *odd*) if $|\mathfrak{s}|$ is even (resp. odd). Furthermore $\mathfrak{s}$ is a *twin* if $|\mathfrak{s}| = 2$.

Definition 2.1.8. Let $\mathfrak{s}$ be a cluster. If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster of $\mathfrak{s}$ then $\mathfrak{s}'$ is a *child* of $\mathfrak{s}$ and $\mathfrak{s}$ is a *parent* of $\mathfrak{s}'$. We write $\mathfrak{s}' < \mathfrak{s}$, and $P(\mathfrak{s}') = \mathfrak{s}$. Denote by $\widehat{\mathfrak{s}}$ the set of all children of $\mathfrak{s}$, and by $\widetilde{\mathfrak{s}}$ the set of all odd children. A cluster is *übereven* if it only has even children. A cluster $\mathfrak{s}$ is a *cotwin* if it has a child of size $2g$ whose complement is not a twin.

Definition 2.1.9. A *centre* $z_\mathfrak{s}$ of a proper cluster $\mathfrak{s}$ is any element $z_\mathfrak{s} \in \overline{K}$ such that $v_K(z_\mathfrak{s} - r) \geq d_\mathfrak{s}$ for all $r \in \mathfrak{s}$. Equivalently, $z_\mathfrak{s}$ is a centre of $\mathfrak{s}$ if $\mathfrak{s}$ can be written as $D \cap \mathcal{R}$, where $D = z_\mathfrak{s} + \pi^{d_\mathfrak{s}} \mathcal{O}_{\overline{K}}$. Note that any root $r \in \mathfrak{s}$ can be chosen as a centre, and if $\mathfrak{s} = \{r\}$ then the only centre is $z_\mathfrak{s} = r$.

Definition 2.1.10. For clusters $\mathfrak{s}$ and $\mathfrak{s}'$, write $\mathfrak{s} \wedge \mathfrak{s}'$ for the smallest cluster containing $\mathfrak{s}$ and $\mathfrak{s}'$.

Definition 2.1.11. If $\mathfrak{s}$ and $\mathfrak{s}'$ are two clusters then the *distance* between them is $\delta(\mathfrak{s},\mathfrak{s}') = d_{\mathfrak{s}} + d_{\mathfrak{s}'} - 2d_{\mathfrak{s}\wedge\mathfrak{s}'}$. For a proper cluster $\mathfrak{s} \neq \mathcal{R}$ define the *relative depth* to be $\delta_{\mathfrak{s}} = \delta(\mathfrak{s}, P(\mathfrak{s})) = d_{\mathfrak{s}} - d_{P(\mathfrak{s})}$.

Definition 2.1.12. A cluster $\mathfrak{s}$ is *principal* if $|\mathfrak{s}| \geq 3$ except if either $\mathfrak{s} = \mathcal{R}$ is even and has exactly two children, or if $\mathfrak{s}$ has a child of size $2g$.

We will see later that principal clusters form an important class of clusters. Roughly, if C/K is a hyperelliptic curve, every orbit of principal clusters in $\Sigma_{C/K}$ makes a contribution to the minimal SNC model of C over K.

Definition 2.1.13. For a cluster $\mathfrak{s}$ that is not a cotwin we write $\mathfrak{s}^*$ for the smallest cluster containing $\mathfrak{s}$ such that the parent of $\mathfrak{s}^*$ is not übereven. If no such cluster exists we write $\mathfrak{s}^* = \mathcal{R}$. If $\mathfrak{s}$ is a cotwin, we write $\mathfrak{s}^*$ for its child of size $2g$.

Definition 2.1.14. For a proper cluster $\mathfrak{s}$ we write $g_{\mathrm{ss}}(\mathfrak{s})$ for the *semistable genus of* $\mathfrak{s}$. If $\mathfrak{s}$ is übereven, we set $g_{\mathrm{ss}}(\mathfrak{s}) = 0$. Otherwise, if $\mathfrak{s}$ is not übereven the semistable genus is determined by

$$|\tilde{\mathfrak{s}}| = 2g_{\mathrm{ss}}(\mathfrak{s}) + 1, \text{ or } 2g_{\mathrm{ss}}(\mathfrak{s}) + 2.$$

It is important to note that $g_{\mathrm{ss}}(\mathcal{R})$ is not necessarily the same as $g(C)$. In fact, they will only be the same when $\mathcal{R}$ has no proper children. If C has semistable reduction over L and $\mathfrak{s} \in \Sigma_{C/K}$ is principal, the semistable genus of $\mathfrak{s}$ represents the genus of the contribution of $\mathfrak{s}$ to the special fibre of the minimal semistable model of C over L.

We also need some new terminology, and the remainder of the definitions in this section are not given in [DDMM18].

Definition 2.1.15. A cluster picture Σ is *nested* if for all proper clusters $\mathfrak{s}, \mathfrak{s}' \in \Sigma$ either $\mathfrak{s} \subseteq \mathfrak{s}'$, or $\mathfrak{s}' \subseteq \mathfrak{s}$. If C is a hyperelliptic curve, we say C is *nested* if Σ_C is nested.

Since the elements of $\mathcal{R}$ lie in $\overline{K}$, there is a natural action of G_K on $\mathcal{R}$, hence also on Σ_C. Since K has algebraically closed residue field, $\mathrm{G}_K = I_K$ where I_K is the inertia subgroup of G_K. It will be important later to know exactly how G_K acts on the clusters of Σ_C. The following lemma is useful for this purpose.

Lemma 2.1.16. *Let Σ_C be such that $K(\mathcal{R})/K$ is a tame extension, and let $\mathfrak{s} \in \Sigma_C$ be a proper cluster fixed by G_K.*

(i) There exists a centre $z_{\mathfrak{s}}$ *of* $\mathfrak{s}$ *such that* $z_{\mathfrak{s}} \in K$.

(ii) Any child $\mathfrak{s}' < \mathfrak{s}$ *is in an orbit of size* $b_{\mathfrak{s}}$*, except possibly for one child* $\mathfrak{s}_f$*, where we can choose* $z_{\mathfrak{s}_f}$ *such that* $v_K(z_{\mathfrak{s}_f} - z_{\mathfrak{s}}) > d_{\mathfrak{s}}$*, which is fixed by* G_K.

Proof. For (i) see [DDMM18, Lemma B.1], and (ii) [Bis19, Theorem 1.3]. □

Definition 2.1.17. Let $\mathfrak{s}' < \mathfrak{s}$ be clusters in Σ_C. Then $\mathfrak{s}'$ is a *stable child* of $\mathfrak{s}$ if the stabiliser of $\mathfrak{s}$ also stabilises $\mathfrak{s}'$. Otherwise $\mathfrak{s}'$ is an *unstable child* of $\mathfrak{s}$.

Remark 2.1.18. Let $\mathfrak{s} \in \Sigma_C$ be fixed by G_K. If $\mathfrak{s}$ has depth $d_{\mathfrak{s}}$ with denominator > 1 then, by Lemma 2.1.16 ii), $\mathfrak{s}$ has at most one stable child.

Definition 2.1.19. If $r \in \mathfrak{s}$ is a root which is not contained in a proper child of $\mathfrak{s}$ then we call r a *singleton* of $\mathfrak{s}$. Define $\mathfrak{s}_{\mathrm{sing}}$ to be the set of all singletons of $\mathfrak{s}$. In other words $\mathfrak{s}_{\mathrm{sing}}$ is the set of all children of size 1 of $\mathfrak{s}$.

2.2 BY Trees

Two different presentations $y^2 = f(x)$ of the same hyperelliptic curve may have different cluster pictures. In the semistable setting, an equivalence relation is defined on cluster pictures by [DDMM17, §3.3]. By [DDMM18, § 14], at least in the semistable setting, this equivalence relation respects isomorphisms between hyperelliptic curves. In particular, isomorphic curves have equivalent cluster pictures, and conversely every cluster picture in the equivalence class is realised by some curve over $\bar{K}$. When producing classifications it is useful to choose a canonical representative.

Given a cluster picture of a semistable hyperelliptic curve, the method for finding the canonical representative involves passing to something called an open BY tree (a combinatorial object easily obtained from the cluster picture). We will generalise this approach in later chapters, so here we collate some useful definitions from [DDMM17, §3].

Definition 2.2.1 (BY tree)**.** A *BY tree* is a finite tree T with a genus function $g : V(T) \to \mathbb{Z}_{\geq 0}$ on vertices and a 2-colouring blue/yellow on vertices and edges such that

(1) yellow vertices have genus 0, degree ≥ 3, and only yellow edges;

(2) blue vertices of genus 0 have at least one yellow edge;

(3) at every vertex, $2g(v) + 2 \geq \#\{\text{blue edges incident to } v\}$.

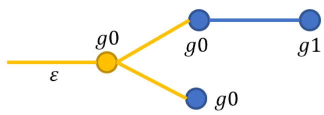
Note that all leaves are blue.

Notation 2.2.2. As a topological space (with the graph topology), a BY tree T can be written as $T = T_b \sqcup T_y$, with T_b the blue part, and T_y the yellow part. Thus $T_b \subset T$ is a closed subset.

Definition 2.2.3. An *open BY tree* T is a finite tree with a unique open edge, that is an edge with only one end vertex, a genus function $g : V(T) \to \mathbb{Z}_{\geq 0}$ on vertices and a 2-colouring blue/yellow on vertices and edges, satisfying conditions (1), (2) and (3) of Definition 2.2.1.

An open BY tree can be thought of as a BY tree with one "missing" vertex, that we refer to as ∞. Sometimes we may refer to BY trees as in Definition 2.2.1 as *closed*, to distinguish them from open BY trees. [DDMM17]

Example 2.2.4. The following is an example of an open BY tree:



In this instance the open edge is labeled ε, and its open end is where the "missing vertex is" which we refer to as ∞.

Definition 2.2.5. Two (closed or open) BY trees are *isomorphic* if there is a homeomorphism between them that preserves their defining data (vertices, edges, genus markings, colouring).

Definition 2.2.6. A *(closed) BY subtree* of a (closed or open) BY tree T is a (closed) BY tree T' such that:

- As a topological space, T' is a union of vertices and edges of T, and is closed in T.
- The vertices of T' are exactly those vertices of T that are in T' (as a topological space) except for those of genus 0 that in T' have degree 2 and incident edges of the same colour as the vertex. These exceptional vertices become points on the edges of T' rather than vertices.
- The genus of a vertex of T' is the same as its genus in T.

The *core* $\tilde{T}$ of an open BY tree T is its maximal closed BY subtree.

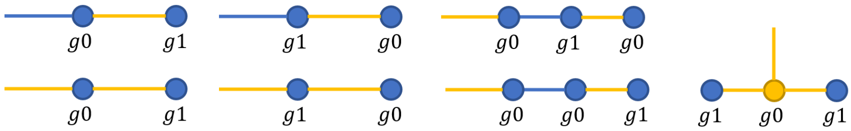
Remark 2.2.7. In [DDMM17, Proposition 5.7] the authors show that the core of a BY tree T is unique and is obtained from T by removing a few vertices and edges 'near' ∞.

Definition 2.2.8. We say that two open BY trees T and T' are *equivalent* if they have isomorphic cores, and write $T \sim T'$.

The following example is taken from [DDMM17, Example 3.28]. Take $\tilde{T}$ to be the following closed BY tree:

$\tilde{T} =$ 

Up to isomorphism, there are seven open BY trees which have $\tilde{T}$ as their core:



Definition 2.2.9. A *metric (open or closed) BY tree* is a BY tree with a length function on the edges (excluding the open edge), $\delta : E(T) \to \mathbb{R}_{>0}$. We denote by $\delta(v, v')$ the distance between $v, v' \in V(T)$, and we require isomorphisms/automorphisms of metric trees to preserve δ. Similarly, we say that two open metric BY trees are equivalent if there is an isomorphism between their cores which preserves distance.

There is in fact a one-to-one, genus preserving, correspondence between isomorphism classes of (either metric or not) cluster pictures and open BY trees. The proof of this can be seen in [DDMM17, §4.2]. Here we simply state how to construct the corresponding open BY tree from a cluster picture, following Construction 4.13 in [DDMM17].

Construction 2.2.10. Let Σ be a cluster picture with set of roots $\mathcal{R}$. Then the corresponding open BY tree $\underline{T}(\Sigma)$ has the following vertices:

- one vertex $v_{\mathfrak{s}}$ for every proper cluster $\mathfrak{s}$ that is not a twin, coloured yellow if $\mathfrak{s}$ is übereven and blue otherwise,
- one blue vertex (a leaf) $v_{\mathfrak{t}}$ for every twin $\mathfrak{t}$,

and edges:

- for every pair $\mathfrak{s}' < \mathfrak{s}$ with $\mathfrak{s}'$ proper, $v_{\mathfrak{s}'}$ and $v_{\mathfrak{s}}$ are linked by an edge, coloured yellow if $\mathfrak{s}'$ is even and blue otherwise,
- add one open edge from $v_{\mathcal{R}}$, coloured yellow if $\mathcal{R}$ is even and blue otherwise.

For the metric version set the length to be $\delta(\mathfrak{s}, \mathfrak{s}')$ for blue edges and $2\delta(\mathfrak{s}, \mathfrak{s}')$ for yellow edges. Finally, define the genus of a vertex $v_{\mathfrak{s}}$ to be the semistable genus $g_{\mathrm{ss}}(\mathfrak{s})$ of the cluster $\mathfrak{s}$ as in Definition 2.1.14.

The construction in the opposite direction can be found in [DDMM17, Construction 4.15] but we will not repeat it here. In practice this correspondence is easy to use and examples can be found in [DDMM17, §4.2].

Here we are interested in equivalence classes of cluster pictures, and finding a canonical representative of each class. As defined in Definition 2.2.8 two BY trees are equivalent if they have isomorphic cores. By [DDMM17, Theorem 5.1], this can be translated to an equivalence relation on cluster pictures. In particular, in [DDMM17], they define cluster pictures to be equivalent if the cores of their open BY trees are isomorphic. Therefore, it is important to know how to easily move between open BY trees and their cores. Corollary 5.10 of [DDMM17] tells us when an open BY tree has core $\tilde{T}$.

Corollary 2.2.11. *Let $\tilde{T}$ be a closed BY tree. Then an open BY tree T has core $\tilde{T}$ if and only if it is obtained from $\tilde{T}$ in one of the following ways:*

- *declaring a point on an edge of $\tilde{T}$ to be a vertex of genus 0 (and the same colour as the edge) and adding a yellow open edge at this vertex,*
- *adding a yellow open edge to a vertex of $\tilde{T}$*
- *adding a blue open edge to a blue vertex v of $\tilde{T}$ which has $2g(v) + 2 > \#\{\text{blue edges incident to } v\}$,*
- *adding 'closed blue edge $\to$ genus 0 blue vertex $\to$ open yellow edge' to a blue vertex v of $\tilde{T}$ which has $2g(v) + 2 > \#\{\text{blue edges incident to } v\}$.*

This corollary can be thought of as describing the equivalence class of open BY trees arising from semistable hyperelliptic curves with core (isomorphic to) $\tilde{T}$. In [DDMM17] the authors choose a canonical representative in each equivalence class, something which we hope to emulate for the non-semistable case. To do this, they first define a canonical 'centre' (either a vertex or an edge) for a closed BY tree $\tilde{T}$. Glueing on an open yellow edge to the centre gives the canonical representative of the equivalence class of open BY trees with centre $\tilde{T}$.

Lemma 2.2.12. *Let T be a finite connected tree and $w : V(T) \to \mathbb{R}_{\geq 0}$ be a 'weight' function on the vertices of T such that each vertex of degree one or two has positive weight. For a subtree $T' \leq T$, set $w(T') = \sum_{v \in T'} w(v)$ and for each $v \in T$, define*

$$\phi(v) = \max\{w(T') \mid T' \text{ is a connected component of } T \setminus \{v\}\}.$$

Then either

(1) $\min_{v\in T}\phi(v) < \frac{1}{2}w(T)$*, in which case the minimum is attained at a unique vertex of* T*, and all other vertices have* $\phi(v) > \frac{1}{2}\phi(T)$*,*

(2) or $\min_{v\in T}\phi(v) = \frac{1}{2}w(T)$*, in which case the minimum is attained at precisely two vertices of* T*, and these vertices are adjacent.*

In case (1) we call the minimising vertex the centre of T *with respect to the weighting* ϕ*. In case (2), we define the centre to be the midpoint of the edge joining the two minimising vertices.*

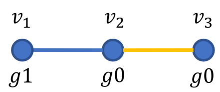
Remark 2.2.13. Actually the authors in [DDMM17] take the centre in case (2) to be the edge joining the two minimising vertices. However, taking the midpoint of the edge to be the centre does not change any of their results. Indeed in [BBB$^+$20, Definition 18.2] they take the centre to be the midpoint rather than the whole edge. In fact, this is taken one step further in [BBB$^+$20] as an extra genus 0 vertex is added at the centre.

Definition 2.2.14. Let T be a closed BY tree. We define its centre to be the vertex or edge afforded by Lemma 2.2.12 applied to the weight function $w : V(T) \to \mathbb{Z}_{\geq 0}$ given by

$$w(v) = \begin{cases} 0 & v \text{ yellow}, \\ 2g(v) + 2 - \deg_{T_b}(v) & v \text{ blue}, \end{cases}$$

where $\deg_{T_b}(v)$ denotes the number of blue edges at v. Note that as w is invariant under all automorphisms of T, the centre of T is also.

Example 2.2.15. Let us consider the following closed BY tree T:



Using the weight function defined in Definition 2.2.14, we can calculate the centre of T as follows. Note that $w(v_1) = 3, w(v_2) = 1$, and $w(v_3) = 2$. So, we can calculate the following:

$$\begin{aligned} \phi(v_1) &= w(v_2) + w(v_3) = 3, \\ \phi(v_2) &= \max\{w(v_1), w(v_3)\} = 3, \\ \phi(v_3) &= w(v_1) + w(v_2) = 4. \end{aligned}$$

Therefore, the minimum is attained at both v_1 and v_2 and we take the centre to be the midpoint of the edge between them.

In [DDMM17, Remark 5.15], they select their canonical representative for each equivalence class of open BY trees to be the one obtained by glueing on an open yellow edge to the centre of the core. As stated in [DDMM17, Lemma 5.25], it turns out that every cluster picture is equivalent to a unique (up to isomorphism) “balanced” cluster picture. Furthermore, this “balanced” cluster picture corresponds to this canonical representative of the equivalence class of the associated open BY tree.

Chapter 3

Hyperelliptic Curves with Tame Reduction

3.1 Equivalence and BY Trees

One of the primary aims of this thesis is to create a notion of equivalence class of cluster pictures for hyperelliptic curves with tame reduction and define a canonical representative. This would allow us to classify the reduction types in the tame situation.

Definition 3.1.1. By the reduction type of a hyperelliptic curve of genus ≥ 2 over a non-archimedean local field we mean (the isomorphism class of) the dual graph of the special fibre of its minimal SNC model with a genus and multiplicity associated to every vertex.

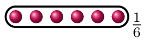
In the semistable situation, as discussed in the previous section, a notion of equivalence is given to cluster pictures by [DDMM17, §3.3] and a "balanced" cluster picture is selected as the canonical representative. Unfortunately, there are a few things to note that make the tame case more complicated. This means that the notion of a "balanced" cluster picture as the canonical representative does not trivially extend. Instead we will seek to define our own equivalence relation. First let us take a look at the complications that the tame setting presents:

(i) It is no longer always possible to choose a model whose cluster picture has the depth of the top cluster being 0;

(ii) It is not always possible to choose a model whose cluster picture has no clusters (other than $\mathcal{R}$) of size $> g+1$;

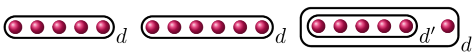
(iii) There may be situations where we have to choose a model whose cluster picture either has exactly one cluster of size $g+1$, or has two clusters of size $g+1$ where their depths are not equal.

We illustrate these complications with an example for each of these situations:

Example 3.1.2. Let $C/\mathbb{Q}_p^{\mathrm{ur}}$ be the hyperelliptic curve defined by $C : y^2 = x^6 - p$, for $p \geq 5$. This has cluster picture  and Namikawa-Ueno Type V. To obtain a cluster picture $(\Sigma', \mathcal{R}')$, with $d_{\mathcal{R}'} = 0$ we could decrease the depth of all clusters by $\frac{1}{6}$. This would result in Namikawa-Ueno type I_{0-0-0} or I^*_{0-0-0}, depending on the leading coefficient. It is therefore clear why, in our situation, we do not want to say that these are equivalent. This demonstrates complication (i).

An alternative way of thinking about this is as follows. We still want our classification to apply in semistable cases. That is, whatever we define to be our equivalence relation should reduce nicely to the semistable situation. As such, any moves we make over the ground field to obtain something equivalent should equate to moves upstairs after taking a field extension so that we become semistable. So, we can restrict ourselves to being able to make (at most) combinations of the moves we can make in the semistable setting. What this also tells us is that, if C/K is a hyperelliptic curve and L/K is a finite extension such that C/L is semistable, we want to take the equivalence class of cluster pictures of $\Sigma_{C/K}$ to be a subset of elements in the equivalence class of $\Sigma_{C/L}$ after taking the quotient by a degree $[L : K]$ action. Some of these quotients simply won't make sense. Our notion of equivalence should also preserve non-trivial orbits of roots, else certainly we will not get the detail from a classification arising from equivalence classes of cluster pictures that we are looking for.

Example 3.1.3. We can return to Example 3.1.2 to illustrate this. After a field extension $L/\mathbb{Q}_p^{\mathrm{ur}}$ of degree 6 so that C/L is semistable, using [DDMM18, Table 7], we can find all cluster pictures equivalent to $\Sigma_{C/L}$. These are clusters with Namikawa-Ueno type I_{0-0-0} or I^*_{0-0-0}, depending on the leading coefficient. In this case the equivalence class of $\Sigma_{C/L}$ is:



for $d, d' \in \mathbb{Z}$.

So, we only have two other cluster pictures in the equivalence class. The quotient of either of these by a degree 6 action is not "valid" since, by 5.1.18, all but at most one child of any given cluster must lie in orbits of the same size. This suggests that our equivalence relation should not produce any equivalent cluster pictures (up to scaling the depth by an integer).

Example 3.1.4. Let $C/\mathbb{Q}_p^{\mathrm{ur}}$ be the hyperelliptic curve defined by $C : y^2 = x(x^4 - p)(x - 1)$. Σ_C consists of two proper clusters $\mathcal{R}$ and $\mathfrak{s}$, with $|\mathcal{R}| = 6$, $|\mathfrak{s}| = 5$, $d_{\mathcal{R}} = 0$, and $d_{\mathfrak{s}} = \frac{1}{4}$. As in the previous example, in this case we only have 2 other possibilities for cluster pictures that lie in this equivalence class.

Namely, a quotient by degree 4 of clusters with Namikawa-Ueno type I_{0-0-0} as pictured in the previous example. As in the semistable setting in [DDMM18], we want our equivalence class to have a canonical representative with an even top cluster. This leaves us with only two choices, the cluster picture that we started with, or the degree 4 quotient of a cluster picture of size 6 and no proper clusters $\neq \mathcal{R}$. By [Bis19, Theorem 1.3], we know that the children of a cluster all lie in inertia orbits of the same size, except possibly for one child that is fixed by inertia. This means that the latter of these two choices is not a "valid" cluster picture since it results in two fixed children and four children in an orbit of size 4. So, we do not want to be able to obtain a cluster picture equivalent to $\Sigma_{C/\mathbb{Q}_p^{\mathrm{ur}}}$ with even size without a cluster $\mathfrak{s} \neq \mathcal{R}$ of size $> g+1$. This demonstrates (ii).

Example 3.1.5. Let $C/\mathbb{Q}_p^{\mathrm{ur}}$ be the hyperelliptic curve defined by $C : y^2 = x(x^2-p)(x-1)((x-1)^2-p^3)$. Σ_C has $|\mathcal{R}| = 6$, with $d_{\mathcal{R}} = 0$, and two proper children $\mathfrak{s}_1$ and $\mathfrak{s}_2$ with $|\mathfrak{s}_i| = 3$ for $i = 1, 2$, $d_{\mathfrak{s}_1} = \frac{1}{2}$, and $d_{\mathfrak{s}_2} = \frac{3}{2}$. If we were to try and re-balance the depths to give $\mathfrak{s}_1$ and $\mathfrak{s}_2$ equal depths then we would need to give them both depth 1, since the distance between them must remain fixed. However, this would mean eliminating our orbits of size 2, as the denominators of the depths reflect the size of the orbits of children by [Bis19, Theorem 1.3]. So, we do not to be able to choose an equivalent cluster picture with two clusters of size $g+1$ of equal depths. This demonstrates (iii).

Because of these more complicated situations, it is now clear that extending the notion of equivalence from [DDMM18] is not as straightforward as one might initially hope. Instead we will use, but adapt, their method of passing to the corresponding BY tree of a cluster picture, and calculating the centre of the core to establish an equivalence relation.

It is important to note that the core, as they defined it, no longer completely determines what we would like the equivalence classes of open quotient BY trees to be (and therefore what we would like the equivalence class of cluster pictures to be), as it did in the semistable case. It is now possible for two cluster pictures which we would like to define to be non-equivalent, to have open BY trees with the same core. This is illustrated in the following example.

Example 3.1.6. Let C_1 and C_2 be hyperelliptic curves over $\mathbb{Q}_p$ defined by equations

$$C_1 : y^2 = px(x^2-p^5)(x-1)((x-1)^2-p^3),$$
$$C_2 : y^2 = (x^3-p^9)((x-1)^3-p^3).$$

Note that C_2 is semistable, whereas C_1 is not. Therefore, we want to define an equivalence relation which gives the cluster pictures of C_1 and C_2, pictured in Figure 3.1 to be non-equivalent. It is not hard to check that C_1 and C_2 have

(a) Σ_{C_1} **(b)** Σ_{C_2}

Figure 3.1: Cluster pictures of C_1 and C_2.

open BY trees T_1 and T_2 respectively as shown in Figure 3.2. The cores $\tilde{T}_1$ and $\tilde{T}_2$ turn out to be isomorphic, as shown in Figure 3.3.

(a) Open BY tree of C_1 **(b)** Open BY tree of C_2

Figure 3.2: The open BY trees of C_1 and C_2.

Figure 3.3: Core of the open BY trees of C_1 and C_2

This demonstrates that the core of the associated open BY tree is no longer enough to completely determine equivalence classes with our desired properties. Something extra is needed. The aim for this section is to illustrate how we need to adapt BY trees. As a starting point, note that open BY trees only encode the depths of proper clusters $\mathfrak{s} \neq \mathcal{R}$, that is the depth of $\mathcal{R}$ cannot be reconstructed from the open BY tree.

Example 3.1.7. Let C_1 and C_2 be hyperelliptic curves over $\mathbb{Q}_p$, with $p > 3$, defined by equations

$$C_1 : y^2 = (x^2 - p)^3 - p^4$$
$$C_2 : y^2 = (x^3 - p)((x-1)^3 - p).$$

The cluster pictures of these two curves are shown in Figure 3.4 below. Since these are both genus 2 curves, we can check their reduction types using Sage and find that C_1 has Namikawa-Ueno type $2\mathrm{IV} - 0$ and C_2 has type $\mathrm{II} - \mathrm{II} - 0$. Therefore, we do not want C_1 and C_2 to have equivalent cluster pictures. It is not hard to check that both C_1 and C_2 have open BY tree T as shown in Figure 3.5. Given T, we have no way of knowing which of these two cluster

(a) Σ_{C_1} **(b)** Σ_{C_2}

Figure 3.4: Cluster pictures of the hyperelliptic curves C_1, and C_2.

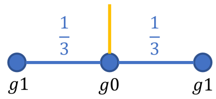


Figure 3.5: Open BY tree T of both C_1 and C_2

pictures T came from. In particular we have no way of knowing whether or not the top cluster had integer depth.

3.2 Marked BY Trees

To solve the problem discussed at the end of the previous section we introduce a *marked point* on the open edge which gives us the depth of the top cluster. We also slightly adjust the lengths of yellow edges.

Definition 3.2.1. Let Σ be a cluster picture with top cluster $\mathcal{R}$ and depth $d_{\mathcal{R}} \geq 0$, and $T = \underline{T}(\Sigma)$ its associated open BY tree as described in Construction 2.2.10, but with all edges (except the open edge) now assigned length $\delta_{\mathfrak{s}}$. We define the *marked point* of T to be the point on the open edge distance $d_{\mathcal{R}}$ from $v_{\mathcal{R}}$, the vertex corresponding to $\mathcal{R}$. A *marked BY tree* is an open BY tree with marked point.

It is important to emphasise that the Galois action on the cluster picture can be attached to the associated marked BY tree, and this is certainly how we should be viewing marked BY trees. To illustrate how the addition of the marked point is useful let us go back to Example 3.1.7.

Example 3.2.2. Let C_1 and C_2 be as in Example 3.1.7. Their open BY trees are the same, however we are able to distinguish between their marked BY trees due to the marked points. These are shown in Figure 3.6 below.

(a) Marked BY tree of C_1 **(b)** Marked BY tree of C_2

Figure 3.6: The marked BY trees of $C_1 : y^2 = (x^2 - p)^3 - p^4$, and $C_2 : y^2 = (x^3 - p)((x-1)^3 - p)$.

Unfortunately, when we pass to the core we eliminate the open edge, losing the key information that the marked point adds. In passing to the core we also, therefore, come up with the same problem that we had before. Two marked BY trees, which we would like to not be equivalent, have the same core. For instance T_1 and T_2 in Example 3.2.2 have the same core. It is also important to note that, unlike in the semistable case, the open edge may not be always moved by an integer amount in all directions. This is due to the open edge of a cluster picture always being in a trivial inertia orbit so, for instance, we cannot move it to a vertex which itself is in a non-trivial inertia orbit. To solve this and make things easier we turn to quotient trees, objects that we will see later are in fact quotients of marked BY trees by their Galois action.

Chapter 4

Open Quotient BY Trees as Combinatorial Objects

4.1 Open Quotient BY Trees

In this section we introduce open quotient BY trees. As with BY trees in [DDMM17], these can be defined as standalone objects as well as objects associated to cluster pictures. For now we will focus on them as purely combinatorial objects, and will not link them back to cluster pictures until Chapter 5. This is to simply emphasise that the work done this chapter does not rely on curves or indeed even on polynomials. Of course, for this to be practically useful for our aims we do need to make this link later. After we make this link, any equivalence relation on open quotient BY trees can be translated to an equivalence relation on cluster pictures. For now let us just say that the open quotient BY trees we define in this section will later be shown to be quotients of marked BY trees associated to hyperelliptic curves by a Galois action. In particular, we will prove in Section 5.1 that the open quotient BY trees defined here are in one-to-one correspondence with objects associated to cluster pictures (once we restrict ourselves to the situation where $d_{\mathcal{R}} \geq 0$). Here edges and vertices are assigned multiplicities which will turn out to be in accordance with the Galois orbits of clusters. We will make use of the following notation.

Notation 4.1.1. Let T be an (open or closed) tree with a two colouring, blue and yellow on vertices and edges, equipped with a multiplicity function $M : V(T) \cup E(T) \to \mathbb{Z}_{>0}$ and a genus function $g : V(T) \to \mathbb{Z}_{\geq 0}$. For every vertex $v \in V(T)$ define

$$s(v,T) = \begin{cases} 2g(v) + 2 - \sum\limits_{\substack{e \in E(T),\text{ blue} \\ \text{incident to } v}} \frac{M(e)}{M(v)} & \text{if } v \text{ is blue,} \\ \\ 0 & \text{if } v \text{ is yellow.} \end{cases}$$

When there is no risk of confusion we may shorten this notation to $s(v)$.

Remark 4.1.2. We will see later, in Construction 5.1.9 where we associate a cluster picture to an open quotient BY tree T, that the notation $s(v, T)$ refers to the number of singletons which lie in each cluster arising from a blue vertex $v \in V(T)$. Likewise if Σ is a cluster picture, and X an orbit of clusters then when we later define an open quotient BY tree associated to Σ, we will see that every $\mathfrak{s} \in X$ has $s(v_X, T)$ singletons where $v_X \in V(T)$ arises from X. It is for this reason that, if T is an open quotient BY tree, in Definition 4.1.3 (vi), we specify that $s(v, T)$ must be non-negative for every blue vertex $v \in V(T)$. It is also worth noting that $s(v, T)$ was not only defined for open quotient BY trees, so we will be able to make use this notation later when we discuss closed quotient BY trees and cores in Section 4.3.

Definition 4.1.3. An *open quotient BY tree* is a finite tree T with a unique open edge ε, a *marked point* m which lies on the closure of the open edge, a genus function $g : V(T) \to \mathbb{Z}_{\geq 0}$, a multiplicity function $M : V(T) \cup E(T) \to \mathbb{Z}_{>0}$, and a 2-colouring blue/yellow on vertices and edges such that:

(i) If v is a yellow vertex, then v has genus $g(v) = 0$, all edges incident to v are yellow, and
$$\sum_{e \text{ edge incident to } v} \frac{M(e)}{M(v)} \geq 3.$$

(ii) Let v_0 be the unique vertex incident to ε. Then the embedded path from v_0 to any vertex v has non-decreasing multiplicities.

(iii) Let $v \in V(T)$ be any vertex, then there exists some $n \in \mathbb{Z}_{>0}$ such that either 1 or 2 edges incident to v have multiplicity $M(v)$ and all remaining incident edges have multiplicity $nM(v)$. Furthermore, $M(\varepsilon) = 1$.

(iv) If v is blue then the genus of v is such that:

- If only one incident edge, say e, has multiplicity $M(v)$ and all other incident edges have multiplicity $nM(v)$ for $n \in \mathbb{Z}_{>0}$, where $e = \varepsilon$ if $v = v_0$, then
$$n \mid 2g(v) + 1 \text{ or } 2g(v) \text{ if } e \text{ is blue},$$
$$n \mid 2g(v) + 2 \text{ or } 2g(v) + 1 \text{ if } e \text{ is yellow}.$$

- If two incident edges, say e_1 and e_2, have multiplicity $M(v)$ and all other incident edges have multiplicity $nM(v)$ for $n \in \mathbb{Z}_{>0}$, where

$\varepsilon \in \{e_1, e_2\}$ if $v = v_0$, then

$$n \mid 2g(v) \text{ if } e_1 \text{ and } e_2 \text{ are both blue,}$$
$$n \mid 2g(v) + 2 \text{ if } e_1 \text{ and } e_2 \text{ are both yellow,}$$
$$n \mid 2g(v) + 1 \text{ if } e_1 \text{ and } e_2 \text{ are different colours.}$$

Note that when $n = 1$ this means that there is no constraint on what values $g(v)$ can take.

(v) Blue vertices of genus 0 have at least one yellow incident edge.

(vi) For every vertex $v \in V(T)$, $s(v, T) \geq 0$.

As in the case of open BY trees in Section 2.2, an open quotient BY tree T has a unique open edge which is "missing" one vertex. We refer to this "missing" vertex as ∞.

Remark 4.1.4. Note that condition (iii) means that for a vertex v, if $\deg(v) = 1$ the only edge incident to v will have multiplicity $M(v)$, and if $\deg(v) = 2$ then it is also possible that both incident edges have multiplicity $M(v)$. However, in every other situation there will be at least one edge of multiplicity $nM(v)$, although it is possible that this is equal to $M(v)$, in which case all edges incident to v have equal multiplicity.

Lemma 4.1.5. *Let T be an open quotient BY tree, and $e \in E(T)$ a closed edge between two vertices v_1 and v_2. Then $M(e) = \max\{M(v_1), M(v_2)\}$.*

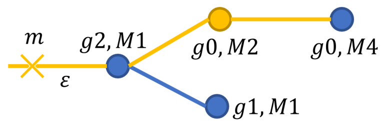
Proof. Note that, for any vertex v,

$$M(v) = \min_{\substack{\text{incident} \\ \text{edges, } e}} \{M(e)\}.$$

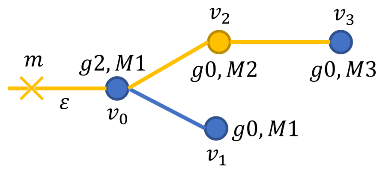
So, $M(v_i) \leq M(e)$ for $i = 1, 2$. Suppose that $M(v_1) \geq M(v_2)$. Condition (ii) tells us that $M(v_1) \geq M(e) \geq M(v_2)$. So, we obtain the inequality $M(e) \geq M(v_1) \geq M(e) \geq M(v_2)$, meaning we must have $M(e) = M(v_1)$. Therefore $M(e) = \max\{M(v_1), M(v_2)\}$. $\square$

Remark 4.1.6. As a result of this lemma, the multiplicities of edges can easily be recovered from the multiplicities of vertices. So, we omit the multiplicities of the edges when drawing open quotient BY trees.

Example 4.1.7. The following is an example of an open quotient BY tree:



Example 4.1.8. However, if we instead consider the following, then we find it is not an example of an open quotient BY tree:



In particular, v_1 is a blue vertex of genus 0 but does not have any incident yellow edges, therefore does not satisfy condition (v). Furthermore, by condition (iii), v_3 would require the edge $[v_2, v_3]$ to have multiplicity $M(v_3) = 3$, whereas v_2 would require the edge $[v_2, v_3]$ to have multiplicity divisible by $M(v_2) = 2$. So, condition (iii) is also not satisfied.

Remark 4.1.9. From now on, we will omit writing the genera of yellow vertices and only write their multiplicities, since every yellow vertex in an open quotient BY tree will have genus 0.

Lemma 4.1.10. *Let T be an open quotient BY tree. The union of all multiplicity 1 edges and vertices, T^1, of T is always non-empty and connected.*

Proof. Label the open edge of T by ε. By condition Definition 4.1.3 (iii), $M(\varepsilon) = 1$. Furthermore, by (iii), for every vertex v the embedded path from v to v_0, the unique vertex incident to ε, has non-decreasing multiplicities. □

Notation 4.1.11. Like BY trees (see Section 2.2), as a topological space, an open quotient BY tree T can be written as $T = T_b \sqcup T_y$, with T_b the blue part, and T_y the yellow part. Note that $T_b \subset T$ is a closed subset.

Definition 4.1.12. Two open quotient BY trees are *isomorphic* if there is a homeomorphism between them that preserves their defining data (vertices, edges, genus markings, multiplicities, and colouring).

We can also define a metric on open quotient BY trees as follows.

Definition 4.1.13. A *metric open quotient BY tree* is an open quotient BY tree T, with open edge ε and *marked point* m, along with a distance function $d : T \times T \to \mathbb{Q}_{\geq 0}$ (on T as a topological space) such that, for all $v \in V(T)$:

(i) If $\deg(v) = 1$ then

$$\mathrm{denom}(d(v,m)) \mid M(v)(2g(v) + 2 - \#\{\text{blue edges incident to } v\}) \text{ or}$$
$$\mathrm{denom}(d(v,m)) \mid M(v)(2g(v) + 1 - \#\{\text{blue edges incident to } v\}).$$

(ii) If $\deg(v) = 2$ and both edges incident to v have equal multiplicity then

$$\mathrm{denom}(d(v,m)) \mid M(v)(2g(v) + 2 - \#\{\text{blue edges incident to } v\}).$$

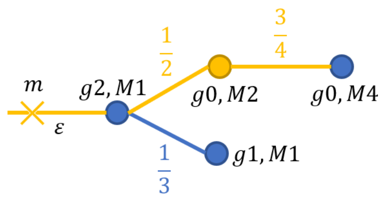
(iii) Otherwise, v has either one or two incident edges of multiplicity $M(v)$ and all others have multiplicity $nM(v) \geq M(v)$, and $d(v,m)$ is such that $\mathrm{lcm}(M(v), \mathrm{denom}(d(v,m))) = nM(v)$.

We write $l(e)$ for the length of an edge $e \in E(T)$.

Remark 4.1.14. For $v, v' \in V(T)$ then $d(v,v') = d(v,m) + d(v',m) - 2d(w,m)$, where w is the closest vertex to v and v' that lies on both the embedded paths between v and m, and v' and m.

Remark 4.1.15. Alternatively, to define a metric on an open quotient BY tree we could simply have asked that for every vertex $v \in V(T)$ we can take $n = \frac{\mathrm{denom}(d(v,m))}{\gcd(\mathrm{denom}(d(v,m)), M(v))}$ in Definition 4.1.3.

Example 4.1.16. It is possible to put the following metric on the open quotient BY tree from Example 4.1.8:



It is worth noting that this is not the only metric we can give to this tree. For example we could give it any metric where the edge lengths had denominators as in the above metric.

It is worth noting the following, as we will make use of this later.

Proposition 4.1.17. *Let T be a metric open quotient BY tree with marked point m. For every $v \in V(T)$, $\frac{\mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v))}{M(v)} \mid s(v,T)$ or $s(v,T) - 1$.*

Proof. Suppose first that $\deg(v) = 1$. By Definition 4.1.13, we have that

$$\mathrm{denom}(d(v,m)) \mid M(v)(2g(v) + 2 - i - \#\{\text{blue edges incident to } v\}),$$

for $i = 0$ or 1. Since v has only one incident edge, which has multiplicity $M(v)$ by construction, we have

$$\sum_{\substack{e,\text{ blue edge}\\ \text{incident to } v}} \frac{M(e)}{M(v)} = \begin{cases} 0 & \text{if the incident edge is yellow,} \\ 1 & \text{if the incident edge is blue.} \end{cases}$$

In both cases we get

$$\sum_{\substack{e,\text{ blue edge}\\ \text{incident to } v}} \frac{M(e)}{M(v)} = \#\{\text{blue edges incident to } v\}.$$

Therefore, we have that $M(v)(2g(v) + 2 - \#\{\text{blue edges incident to } v\}) = M(v)s(v,T)$. This gives $\mathrm{denom}(d(v,m)) \mid M(v)s(v,T)$ or $M(v)(s(v,T) - 1)$, and so certainly

$$\frac{\mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v))}{M(v)} \mid s(v,T) \text{ or } s(v,T) - 1.$$

Suppose instead that $\deg(v) = 2$ and both incident edges have equal multiplicity. Definition 4.1.3 (iii) tells us that both these edges have multiplicity $M(v)$ and a similar argument to above works.

In all other cases, Definition 4.1.13 tells us that

$$\mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v)) = \max\{M(e) \mid e \text{ edge incident to } v\}.$$

Suppose that v has only one incident edge of multiplicity $M(v)$, which is coloured blue, and all other incident edges have multiplicity $\max\{M(e) \mid e \text{ edge incident to } v\} \geq M(v)$. Then Definition 4.1.3 (iv) tells us that

$$\frac{\mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v))}{M(v)} \mid 2g(v) + 1 \text{ or } 2g(v).$$

Note that in this situation

$$\begin{aligned} s(v,T) &= 2g(v) + 2 - \sum_{\substack{e,\text{ blue edge}\\ \text{incident to } v}} \frac{M(e)}{M(v)}, \\ &= 2g(v) + 1 - (\#\{e \in E(T_b) \text{ incident to } v\} - 1)\tfrac{\mathrm{lcm}(\mathrm{denom}(d(v,m)),M(v))}{M(v)}. \end{aligned}$$

Therefore,

$$\frac{\mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v))}{M(v)} \mid s(v,T) \text{ or } s(v,T) - 1.$$

The remaining cases can be dealt with in a similar way by referring to Definition 4.1.3 (iv). □

4.2 Moving Between Open Quotient BY Trees and Open BY Trees

The definition of an open quotient BY tree has been carefully constructed to coincide with quotients of open BY trees. In particular, let C be a hyperelliptic curve with tame reduction over K, and L/K be a tame field extension such that C/L is semistable. We will see later in Section 5.1 how to associate open quotient BY trees, say T to C/K, and T' to C'/L. The tree T is defined carefully, so that it is the quotient of T' by the action induced by Galois. This is proved explicitly in Proposition 6.1.3, once we have made the link between open quotient BY trees and cluster pictures. For now, it will be useful in proofs to be able to explicitly pass between open quotient BY trees and open BY trees, so in this section we make the quotient map and its inverse precise. First let us construct an open BY tree from an open quotient BY tree.

Definition 4.2.1. Let T' be an open BY tree with a cyclic group of automorphisms acting on the vertices, then $q(T')$ is the quotient of T' by this action, and is itself a tree. Of course, $q(T')$ depends on the given cyclic group of automorphisms, however in practice this group is always clear from context so the notation $q(T')$ does not show this.

Conversely, let T be an open quotient BY tree. Then we write $q^{-1}(T)$ for the unique tree obtained from T by letting every vertex v and edge e of T give rise to $M(v)$ vertices and $M(e)$ edges respectively, in the way one would expect.

That, for an open quotient BY tree T, $q^{-1}(T)$ exists and is unique can be seen clearly from the following precise construction.

Construction 4.2.2 ($q^{-1}(T)$)**.** Let T be an open quotient BY tree, then we construct an open BY tree $q^{-1}(T)$ such that every vertex $v \in V(T)$ gives $M(v)$ vertices $q^{-1}(v)_1, \dots q^{-1}(v)_{M(v)}$ in $q^{-1}(T)$, all coloured the same as v and with $g(q^{-1}(v)_i) = g(v)$ for all $1 \leq i \leq M(v)$. The edges of $q^{-1}(T)$ are as follows:

- if $v, v' \in V(T)$ are adjacent vertices with $M(v) \leq M(v')$, then there are edges in $q^{-1}(T)$ between $q^{-1}(v)_i$ and $q^{-1}(v')_{(i-1)\frac{M(v')}{M(v)}+1}, \dots, q^{-1}(v')_{i\frac{M(v')}{M(v)}}$, for $1 \leq i \leq M(v)$. These edges are all coloured the same as the edge between v and v',

- if v_0 is the unique vertex in T adjacent to the open edge ε, then $M(v_0) = 1$ so there is just one vertex $q^{-1}(v_0)_1$ in $q^{-1}(T)$ arising from v_0. We attach an open edge ε' to $q^{-1}(v_0)_1$ and colour it the same as ε.

Note that this means every edge $e \in E(T)$ gives rise to $M(e)$ edges $\{q^{-1}(e)_1, \ldots, q^{-1}(e)_{M(e)}\}$ in $q^{-1}(T)$. Furthermore, $q^{-1}(T)$ comes with a natural automorphism. In particular, for $e \in E(T)$ the edges $\{q^{-1}(e)_1, \ldots, q^{-1}(e)_{M(e)}\}$ in $q^{-1}(T)$ are in an orbit of size $M(e)$, and for $v \in V(T)$ the vertices $\{q^{-1}(v)_1, \ldots, q^{-1}(v)_{M(v)}\}$ are in an orbit of size $M(v)$. We can use this to define a surjective graph morphism $q : q^{-1}(T) \twoheadrightarrow T$. We say that T is the *quotient* of $T' = q^{-1}(T)$ and write $T = q(T')$.

Example 4.2.3. Take T to be the open quotient BY tree as in Example 4.1.8, shown again below. Then $q^{-1}(T)$ can be constructed following Construction 4.2.2, to give the tree pictured below. Note that in this example $q^{-1}(T)$ is indeed an open BY tree. This is proved in general shortly. Assume instead

(a) Open quotient BY tree T

(b) $q^{-1}(T)$

Figure 4.1: A example of the construction of $q^{-1}(T)$.

that we were given the open BY tree $q^{-1}(T)$, but not T, with a cyclic group of automorphisms acting on $q^{-1}(T)$ such that the four genus 0 blue vertices were in an orbit of size 4 and the two yellow vertices were in an orbit of size 2. Then we can recover T by taking the quotient of $q^{-1}(T)$, provided we keep track of the marked point.

Remark 4.2.4. Should T or T' be metric trees, then these metrics can be given to $q^{-1}(T)$ and $q(T')$, respectively. At various points we will want to refer to different metrics. For our current purposes a non-metric version is all we need. However, later in Section 5.2 we describe the metrics so that for $e \in E(T)$ the length of each $q^{-1}(e)_i$ is $l(q^{-1}(e)_i) = l(e)$, and for each $e' \in E(T')$, $l(q(e')) = l(e')$. In Section 6.1 we define a metric that allows us to compare open quotient BY trees with open BY trees after taking field extensions.

Notation 4.2.5. If there is only one vertex in the preimage of a vertex $v \in V(T)$, i.e. $M(v) = 1$, then we denote this unique vertex in $V(q^{-1}(T))$ by $q^{-1}(v) = q^{-1}(v)_1$.

Theorem 4.2.6. *Let T be an open quotient BY tree. Then $q^{-1}(T)$ is an open BY tree.*

Proof. One can verify that conditions (1), (2), and (3) of Definition 2.2.1 are satisfied as follows:

(1) A vertex $v \in q^{-1}(T)$ is yellow if and only if $q(v)$ is. Since every yellow vertex in T has genus 0, only yellow edges, and $\sum_{v' \text{ incident to } q(v)} M(v') \geq 3$ we conclude that yellow vertices in $q^{-1}(T)$ have genus 0, degree ≥ 3, and only yellow edges;

(2) Blue vertices in $V(T)$ of genus 0 have at least one yellow edge, therefore the same is true in $q^{-1}(T)$;

(3) Let $v \in V(q^{-1}(T))$. Then,

$$2g(q(v)) + 2 \geq \sum_{\substack{e, \text{ blue edge} \\ \text{incident to } q(v)}} \frac{M(e)}{M(q(v))}.$$

Each edge e incident to $q(v)$ gives rise to $\frac{M(e)}{M(q(v))}$ edges incident to v, so we can conclude that

$$2g(v) + 2 = 2g(q(v)) + 2 \geq \#\{\text{blue edges incident to } v\}.$$

□

Proposition 4.2.7. *Let T be an open quotient BY tree, then $q(q^{-1}(T)) \cong T$, where the quotient action on $q^{-1}(T)$ is the natural action arising from T.*

Proof. The proof of this follows trivially from the proof of Lemma 4.4.4. □

4.3 Closed Quotient BY Trees and Cores

Recall that the classification of open BY trees relies on considering their cores, which are maximal closed BY subtrees. In this section we look to define a similar notion for open quotient BY trees. Unfortunately, closed quotient BY trees turn out to be difficult to define in a way that is as practically useful as closed BY trees. However, we instead make a slight adjustment in our approach in later sections, which still allows us to use cores of open quotient BY trees to define an equivalence relation.

Before we can define a core of an open quotient BY tree, we first need to define closed quotient BY trees and subtrees. However, it is worth highlighting that these are only used for the purpose of obtaining the core. We will later see that Proposition 4.3.9 gives a better way of thinking about how to obtain the core. For this reason one should not be overly concerned with the specifics or remembering the criteria in Definitions 4.3.1 and 4.3.2.

Definition 4.3.1. A *closed quotient BY tree* is a finite tree T with a genus function $g : V(T) \to \mathbb{Z}_{\geq 0}$ on vertices, a multiplicity function $M : V(T) \cup E(T) \to \mathbb{Z}_{>0}$ on vertices and edges, and a 2-colouring blue/yellow on vertices and edges such that:

(i) yellow vertices have genus 0, only yellow edges, and if v is a yellow vertex of degree $\deg(v) \geq 2$ then

$$\sum_{\substack{e,\text{ edge} \\ \text{incident to } v}} \frac{M(e)}{M(v)} \geq 3;$$

(ii) blue vertices of genus 0 have at least one yellow edge, or a blue edge of multiplicity 2;

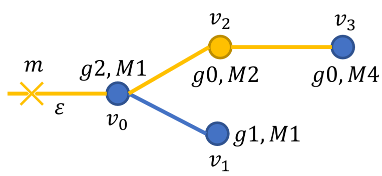
(iii) every vertex $v \in V(T)$ has

$$2g(v) + 2 \geq \sum_{\substack{v',\text{ blue vertex} \\ \text{adjacent to } v \text{ in } T_b}} \frac{M(v')}{M(v)}.$$

Definition 4.3.2. A *closed quotient BY subtree* of a (closed or open) quotient BY tree T is a closed quotient BY tree T' such that:

- As a topological space, T' is a union of vertices and edges of T, and is closed in T.
- The vertices of T' are exactly those vertices in T that are in T' as a topological space, except for those of genus 0 that have degree equal to 2 and both incident edges have multiplicity 1 and are the same colour as the vertex. These exceptional vertices become points on the edges of T'.
- The genus of a vertex of T', and the multiplicities of the vertices and edges of T' are the same as in T. Note that any two edges of T that were 'combined' by removing an exceptional vertex both had multiplicity 1 in T, so this does not cause any problems in defining the multiplicities of edges in T'.

Example 4.3.3. Let us consider again the open quotient BY tree T as in Example 4.1.8:



Here we give both an example and a non-example of closed quotient BY subtrees of T. Let T_1 and T_2 be as shown below in Figure 4.2 below. Then T_1 is

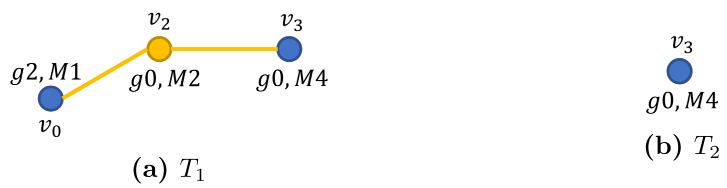


Figure 4.2: An example and a non-example of a closed quotient BY subtree of T.

a closed quotient BY subtree of T. However, T_2 is not a closed quotient BY subtree of T since it has a genus 0 blue vertex which has no incident edges in T_2. So, $v_3 \in V(T_2)$ does not satisfy condition (ii) of Definition 4.3.1 and is not a closed quotient BY tree, thus cannot be a closed quotient BY subtree of T.

Definition 4.3.4. Two closed quotient BY trees are *isomorphic* if there is a homeomorphism between them that preserves their defining data (vertices, edges, genus markings, multiplicities, and colouring).

Closed quotient BY trees will not be used in the same way as the closed BY trees were in the semistable situation, where equivalence classes were classified by closed BY trees. Closed quotient BY trees do not have anywhere near enough conditions on them to make them practically useful as standalone objects. Trying to define additional conditions which would enable them to be used in such a way is hard, as closed quotient BY trees do not have a defined marked point (which we have already seen will be key for determining equivalence classes). Instead, we simplify the definition and will only use closed quotient BY trees as subtrees of open quotient BY trees (or subtrees of closed quotient BY trees which are themselves subtrees of an open quotient BY tree). This allows the conditions that open quotient BY trees satisfy to be inherited by any closed subtrees that we make use of.

Definition 4.3.5. The *core* $\tilde{T}$ of an open quotient BY tree T is its maximal closed quotient BY subtree.

Remark 4.3.6. Note that a metric on an open quotient BY tree induces a metric on all closed quotient BY subtrees, in particular on the core. We require isomorphisms of metric trees to preserve distance.

Example 4.3.7. Let T be the metric open quotient BY tree shown in Figure 4.3. The core of T is the maximal closed quotient BY subtree and is shown in Figure 4.4.

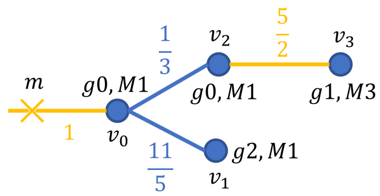


Figure 4.3: Metric open quotient BY tree T.

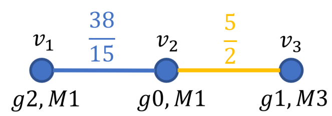
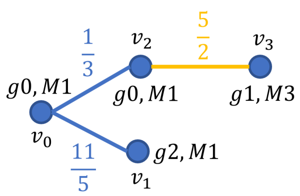
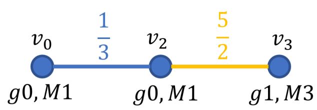


Figure 4.4: Core $\tilde{T}$ of T.

One can easily verify that $\tilde{T}$ is indeed a closed quotient BY subtree. To see that $\tilde{T}$ is the core we must check that it is maximal. Let T_1 be the closed tree obtained by deleting just the open edge of T, as shown in Figure 4.5a. Note that, as a topological space, T_1 is the maximal closed subspace of T. However, T_1 is not a closed BY subtree of T, since v_0 has genus 0 and degree 2 in T_1 with both incident edges of multiplicity 1. So, by Definition 4.3.2, v_0 should become a point on and edge, giving $\tilde{T}$. Thus $\tilde{T}$ is indeed the core of T. Any closed quotient BY subtree of T will be contained in $\tilde{T}$, for example the closed tree T_2 shown in Figure 4.5b is a closed quotient BY subtree and is contained in $\tilde{T}$, as a topological space.



(a) T_1



(b) T_2

Figure 4.5: Subtrees of T which are not the core.

Remark 4.3.8. The three criteria in Definition 4.3.1 nearly mirror the criteria for a finite tree to be a closed BY tree, given in Definition 2.2.1, but with some minor adjustments and the inclusion of multiplicities. The main difference is we only require yellow vertices to satisfy

$$\sum_{\substack{e,\text{ edge} \\ \text{incident to } v}} \frac{M(e)}{M(v)} \geq 3$$

if they have degree ≥ 2. For BY trees, we required all yellow vertices to have degree ≥ 3. This adjustment will ensure that, when $\mathcal{R}$ is even and $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $X = \{\mathfrak{s}_1, \mathfrak{s}_2\}$ a Galois orbit, $v_{\mathcal{R}}$ is a vertex of the core. If $v_{\mathcal{R}}$ did not lie on the core $\tilde{T}$, then $\tilde{T}$ would have no multiplicity 1 components, and "undoing the quotient" would result in a disconnected graph. Another way to see why we wish to include $v_{\mathcal{R}}$ in the core is to consider $q^{-1}(T)$. Here, when passing to the core of $q^{-1}(T)$, $q^{-1}(v_{\mathcal{R}})$ would be considered as a point on an edge of $\widetilde{q^{-1}(T)}$. Taking the quotient of $\widetilde{q^{-1}(T)}$ would result in the edge between $v_{\mathfrak{s}_1} = q^{-1}(v_X)_1$ and $v_{\mathfrak{s}_2} = q^{-1}(v_X)_2$ being "folded in half on top of itself", thus becoming an edge with only one defined end point. Adding $v_{\mathcal{R}}$ in as a vertex makes this process make sense. For more clarity see Proposition 4.3.15, and the construction preceding it giving the quotient map on closed BY trees.

Proposition 4.3.9. *Let T be an open quotient BY tree. Then the core $\tilde{T}$ is unique and obtained from T by removing a few vertices and edges near ∞. In particular, $\tilde{T}$ is obtained from T in one of the following ways:*

(i) by deleting the open edge,

(ii) by deleting the open edge and viewing v_0 as a point on an edge, provided that $g(v_0) = 0$, v_0 has exactly two incident closed edges e and e', $M(e') = M(e) = 1$, and v_0, e, and e' are coloured the same,

(iii) by deleting the open edge, along with v_0 and a unique blue closed edge e incident to v_0, provided v_0 is blue, $g(v_0) = 0$, $\deg_T(v_0) = 2$, and $M(e) = 1$.

Proof. Let v_0 be the unique vertex which is incident to the open edge. To get to $\tilde{T}$ from T, the open edge certainly needs to be removed. If after removing the open edge, v_0 satisfies Definition 4.3.1 we are done. Otherwise, v_0 must violate Definition 4.3.1. Note that v_0 must satisfy condition (iii) of Definition 4.3.1 since, in T, v_0 satisfies condition (vi) of Definition 4.1.3, and the right hand side of each of these conditions is either decreased or remains constant by removing the open edge from v_0. So, v_0 must violate condition (i) or (ii) of Definition 4.3.1.

If v_0 violates (i) then v_0 must be yellow and have degree ≥ 2 in $T \setminus \{\text{the open edge}\}$ with

$$\sum_{\substack{e,\text{ closed edge}\\ \text{incident to } v_0}} \frac{M(e)}{M(v_0)} = \sum_{\substack{e,\text{ closed edge}\\ \text{incident to } v_0}} M(e) < 3.$$

This is only possible if v_0 has degree 2 in $T \setminus \{\text{the open edge}\}$, and both incident closed edges have multiplicity 1. Declaring v_0 to not be a vertex, and be a point on an edge gives a closed BY tree.

If v_0 violates (ii) then (since v_0 satisfies Definition 4.1.3 (vi) in T) v_0 must have degree 1 or 2 in $T \setminus \{\text{the open edge}\}$, be blue of genus 0, with no yellow edges and no blue edge of multiplicity 2. In particular, either v_0 has either one or two closed, blue, incident edges of multiplicity 1 and no other closed incident edges. If v_0 has one incident blue edge of multiplicity 1, then removing v_0 and its incident blue edge results in a closed BY tree. If instead v_0 has two incident blue edges of multiplicity 1, then declaring v_0 to not be a vertex gives a closed BY tree. □

Proposition 4.3.10. *Let T be an open quotient BY tree. Then the core $\tilde{T}$ contains at least one multiplicity 1 vertex.*

Proof. This follows as a direct consequence of the proof of Proposition 4.3.9. In particular, as stated in Proposition 4.3.9, we noted that at most one vertex, namely v_0, is removed when passing from T to $\tilde{T}$, be that by either deleting v_0 or viewing it as a point on an edge. However, if v_0 is the only multiplicity 1 vertex of T then, in the proof of Proposition 4.3.9, v_0 never gets deleted and we see that v_0 must lie on the core. □

Remark 4.3.11. It is also useful to note that, whilst the relationship is not quite as strong in the closed case, we can apply the quotient map and inverse constructed in Section 4.2 to closed quotient BY trees and closed BY trees. However, a slight tweak is needed given how we remove vertices and edges near ∞ as discussed in Remark 4.3.8.

First, as hinted at above, we are able to extend our construction of the quotient map and its inverse given in Construction 4.2.2.

Construction 4.3.12 ($q^{-1}(T')$)**.** Let T' be a closed subtree of an open quotient BY tree T. We say that a vertex $v \in V(T')$ with genus 0, multiplicity 1, degree 1 in T', and such that its only closed incident edge has multiplicity 2 and is coloured the same as v, is *exceptional*. We construct a graph $q^{-1}(T')$ in the following way. Every vertex $v \in V(T')$ that is not exceptional gives $M(v)$ vertices $q^{-1}(v)_1, \dots q^{-1}(v)_{M(v)}$ in $q^{-1}(T')$, all coloured the same as v and with $g(q^{-1}(v)_i) = g(v)$ for all $1 \leq i \leq M(v)$. The edges of $q^{-1}(T')$ are as follows:

- If $v, v' \in V(T')$ are non-exceptional and adjacent with $M(v) \leq M(v')$, then there are edges in $q^{-1}(T)$ between $q^{-1}(v)_i$ and $q^{-1}(v')_{(i-1)\frac{M(v')}{M(v)}+1}$, $\dots, q^{-1}(v')_{i\frac{M(v')}{M(v)}}$, for $1 \leq i \leq M(v)$. These edges are all coloured the same as the edge between v and v'.

- If v is an exceptional vertex and the only vertex adjacent to v is v', then $M(v') = 2$, and we have an edge between $q^{-1}(v')_1$ and $q^{-1}(v')_2$. We call such edges of $q^{-1}(T)$ *exceptional.*

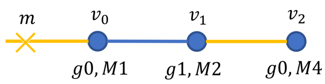
- If v_0 is the unique vertex in T adjacent to the open edge ε, then $M(v_0) = 1$ so there is just one vertex $q^{-1}(v_0)_1$ in $q^{-1}(T)$ arising from v_0. We attach an open edge ε' to $q^{-1}(v_0)_1$ and colour it the same as ε.

We can use this to define a surjective graph morphism $q : q^{-1}(T') \twoheadrightarrow T'$. We say that T' is the *quotient* of $q^{-1}(T')$.

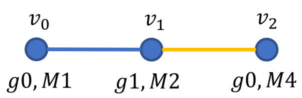
Note that the use of the term exceptional here, although it may seem contradictory to the use of the term in Definition 4.3.2, does actually coincide. If one were to instead precisely construct the "undone quotient" of T', where every vertex v contributes $M(v)$ vertices, even if v is exceptional, then the vertex arising from an exceptional vertex of T' would be an exceptional vertex in the sense of Definition 4.3.2.

Remark 4.3.13. Let T' be a closed subtree of an open quotient BY tree T. Then $q^{-1}(T')$ is a tree if and only if T' contains a multiplicity 1 vertex.

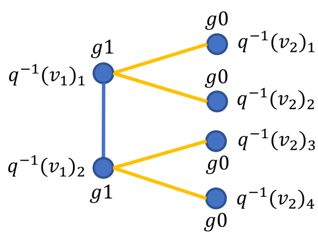
Example 4.3.14. Let T be the following open quotient BY tree:



Two subtrees T_1 and T_2 of T are shown in Figures 4.6a and 4.7a respectively.

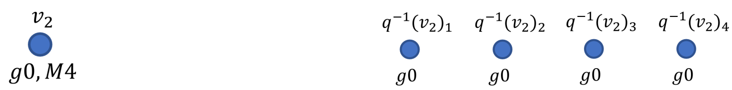


(a) T_1



(b) $q^{-1}(T_1)$

Figure 4.6: Subtree T_1 of T and $q^{-1}(T_1)$



(a) T_2 **(b)** $q^{-1}(T_2)$

Figure 4.7: Subtree of T_2 of T and $q^{-1}(T_2)$

Using Construction 4.3.12, we can construct both $q^{-1}(T_1)$ and $q^{-1}(T_2)$, pictured in Figures 4.6 and 4.7 respectively. We see that T_1 contains a multiplicity 1 vertex, thus $q^{-1}(T_1)$ is connected, whereas T_2 does not and $q^{-1}(T_2)$ is not a tree. We can further see that, in T_1, v_0 is an exceptional vertex and v_1 is the unique vertex adjacent to v_0, with $M(v_1) = 2$. Therefore, v_0 does not contribute a vertex to $q^{-1}(T_1)$, instead we obtain an edge between $q^{-1}(v_1)_1$ and $q^{-1}(v_1)_2$.

Proposition 4.3.15. *Let T be an open quotient BY tree. Then $q^{-1}(\tilde{T})$ and $\widetilde{q^{-1}(T)}$ are isomorphic as trees.*

Proof. By Proposition 4.3.9, we know that $\tilde{T}$ is obtained from T by either, removing ε, removing ε and viewing v_0 as a point on an edge, or removing '$\varepsilon \to$ genus 0, multiplicity 1 blue vertex, $v_0 \to$ unique closed edge incident to v_0, coloured blue and with multiplicity 1'. We will consider each of these cases separately. Note that, the preimage of v_0 under q consists of exactly one vertex, denoted $q^{-1}(v_0)$, which is incident to the open edge of $q^{-1}(T)$.

Suppose first that $\tilde{T}$ is obtained by removing ε and viewing v_0 as a point on an edge. Then, by Proposition 4.3.9, we know that v_0 must be a multiplicity 1 vertex, with precisely 2 incident closed edges, each coloured the same as v_0 and with multiplicity 1. Therefore, $q^{-1}(v_0)$ has exactly 2 incident closed edges, each coloured the same as $q^{-1}(v_0)$. By [DDMM17, Proposition 5.7], we know that $\widetilde{q^{-1}(T)}$ is obtained from $q^{-1}(T)$ by deleting the open edge and viewing $q^{-1}(v_0)$ as a point on an edge. Thus, $q^{-1}(\tilde{T})$ and $\widetilde{q^{-1}(T)}$ are isomorphic as graphs.

Supposed instead that $\tilde{T}$ is obtained from T by removing '$\varepsilon \to$ genus 0, multiplicity 1 blue vertex, $v_0 \to$ unique closed edge incident to v_0, coloured blue and with multiplicity 1'. Then, $q^{-1}(v_0)$ is a genus 0 blue vertex and has precisely one incident closed edge, which is blue. By [DDMM17, Proposition 5.7], removing the open edge of $q^{-1}(T)$ followed by $q^{-1}(v_0)$ and its unique incident closed edge, results in the core $\widetilde{q^{-1}(T)}$. So, again $q^{-1}(\tilde{T})$ and $\widetilde{q^{-1}(T)}$ are isomorphic as graphs.

Finally, let us suppose that $\tilde{T}$ is obtained from T by removing just ε. Then, by Proposition 4.3.9, v_0 must satisfy Definition 4.3.1. We consider the two different cases for the colouring of v_0.

If v_0 is blue, then either $g(v_0) > 0$, or $g(v_0) = 0$ and v_0 has at least one yellow incident closed edge, or a blue incident closed edge of multiplicity 2. If $g(v_0) > 0$ then $g(q^{-1}(v_0)) > 0$, so $q^{-1}(v_0)$ is certainly not removed when passing from $q^{-1}(T)$ to $\widetilde{q^{-1}(T)}$. If $g(v_0) = 0$ and and v_0 has at least one yellow incident closed edge, then $q^{-1}(v_0)$ has at least one yellow incident closed edge. So again, $q^{-1}(v_0)$ is not removed when passing from $q^{-1}(T)$ to $\widetilde{q^{-1}(T)}$. If $g(v_0) = 0$ and v_0 has no incident yellow closed edge, then v_0 must have a blue incident closed edge of multiplicity 2, and no other incident edges (else $g(v_0) \neq 0$). So, $q^{-1}(v_0)$ has exactly two incident closed edges, both of which are coloured blue. So, by [DDMM17, Proposition 5.7], removing the open edge of $q^{-1}(T)$ and viewing $q^{-1}(v_0)$ as a point on an edge, results in the core $\widetilde{q^{-1}(T)}$. Taking the quotient $q(\widetilde{q^{-1}(T)})$ of this we see that the two end points of the edge in $\widetilde{q^{-1}(T)}$ on which $q^{-1}(v_0)$ lies are mapped to the same vertex in $q(\widetilde{q^{-1}(T)})$. So, $q^{-1}(\tilde{T})$ and $\widetilde{q^{-1}(T)}$ are isomorphic as graphs.

If v_0 is yellow then either

$$\sum_{\substack{e \in E(\tilde{T}) \\ \text{incident to } v_0}} \frac{M(e)}{M(v_0)} \geq 3,$$

or

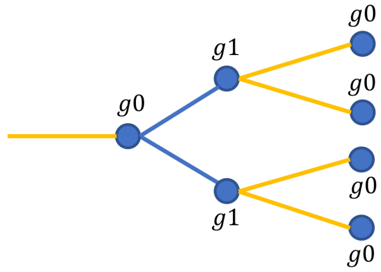
$$\deg_{\tilde{T}}(v_0) \in \{0, 1\} \quad \text{and} \quad \sum_{\substack{e \in E(\tilde{T}) \\ \text{incident to } v_0}} \frac{M(e)}{M(v)} < 3.$$

In the first instance, by [DDMM17, Proposition 5.7], simply removing the open edge of $q^{-1}(T)$ results in the core $\widetilde{q^{-1}(T)}$, so we are done. In the second instance, since v_0 must satisfy condition (i) of Definition 4.1.3 and only the open edge has been deleted from v_0 to obtain $\tilde{T}$, we must have that

$$\deg_{\tilde{T}}(v_0) = 1 \quad \text{and} \quad \sum_{\substack{e \in E(\tilde{T}) \\ \text{incident to } v}} \frac{M(e)}{M(v)} = 2.$$

So, in $q^{-1}(T) \setminus \{\varepsilon\}$, $q^{-1}(v_0)$ is yellow and has exactly two incident edges, which are both coloured yellow. Therefore, by [DDMM17, Proposition 5.7], removing the open edge of $q^{-1}(T)$ and viewing $q^{-1}(v_0)$ as a point on an edge, results in the core $\widetilde{q^{-1}(T)}$. So, again $q^{-1}(\tilde{T})$ and $\widetilde{q^{-1}(T)}$ are isomorphic as graphs. □

Example 4.3.16. Let T be the open quotient BY tree as in Example 4.3.14. Note that the subtree denoted by T_1, pictured alongside $q^{-1}(T_1)$ in Figure 4.6 is actually the core $\tilde{T}$ of T. Using Construction 4.2.2, we can calculate $q^{-1}(T)$:

$$q^{-1}(T) =$$ 

So, we do indeed have that $\widetilde{q^{-1}(T)} \cong q^{-1}(\tilde{T})$.

4.4 Centres

Much like for BY trees described in Section 2.2, for an open quotient BY tree T, we can define a centre of the core $\tilde{T}$. Doing so will allow us to construct an open quotient BY tree from T in Section 4.5 with marked point as close to the centre of $\tilde{T}$ as possible.

Definition 4.4.1. Let T be a closed quotient BY tree then we define the following weight function $w : V(T) \to \mathbb{Z}_{\geq 0}$ on the vertices of T:

$$w(v) = s(v,T) = \begin{cases} 0 & \text{if } v \text{ is yellow,} \\ 2g(v) + 2 - \sum_{\substack{e \in E(T_b), \\ \text{incident to } v}} \frac{M(e)}{M(v)} & \text{if } v \text{ is blue.} \end{cases}$$

If T' is a closed quotient BY subtree of T then we define

$$w(T') = \frac{1}{\min_{v' \in T'}\{M(v')\}} \sum_{v \in T'} M(v)w(v).$$

Furthermore, for $v \in V(T)$ we define

$$\phi(v) = \max\{w(T') \mid T' \text{ is a connected component of } T \setminus \{v\}\}.$$

Remark 4.4.2. In Remark 5.1.7 we discuss how we can view open BY trees as open quotient BY trees with all multiplicities equal to 1. Indeed this is proved formally in Proposition 6.1.3. Since the above definition is not dependent on the metric, this weight function reduces to the open BY tree formula when all multiplicities are taken to equal 1.

Notation 4.4.3. For an open or closed quotient BY tree T we write T^1 to be the subtree of T consisting of all multiplicity 1 vertices and edges.

Note that, by construction, for any open quotient BY tree T, T^1 is connected and closed, therefore is a closed subtree of T. However, T^1 is not necessarily a closed quotient BY subtree.

It will be useful to have defined a partial order on the vertices of an open quotient BY tree T by setting $v' \preceq v$ if v lies on the embedded path from m to v'.

Lemma 4.4.4. *Let T be an open quotient BY tree with core $\tilde{T}$. Then, with w and ϕ defined as above in Definition 4.4.1, either*

(i) $\min_{v\in\tilde{T}^1} \phi(v) < \frac{1}{2}w(\tilde{T})$, *in which case the minimum is attained at a unique vertex of $\tilde{T}^1$, and all other vertices of $\tilde{T}^1$ have $\phi(v) > \frac{1}{2}w(\tilde{T})$, or*

(ii) $\min_{v\in\tilde{T}^1} \phi(v) = \frac{1}{2}w(\tilde{T})$, *in which case the minimum is attained at either a unique exceptional vertex of $\tilde{T}$, or at precisely two vertices of $\tilde{T}^1$, and these vertices are adjacent.*

Proof. Let T be an open quotient BY tree with core $\tilde{T}$. Write $T' = q^{-1}(T)$. We have already proven in Theorem 4.2.6 that T' is an open BY tree. For the first step in this proof we note that the centre of $\tilde{T}'$ arises from a multiplicity 1 vertex or a point on a multiplicity 1 edge of T. By [DDMM17, Definition 5.13], the centre of $\tilde{T}'$ is invariant under all automorphisms of T'. It is possible to describe an automorphism on T' arising from q. Note that, if $v \in V(T)$ has $M(v) \geq 2$ then v and all $v' \preceq v$ gives rise to $M(v)$ identical branches in T'. The natural automorphism on T', as described in Construction 4.2.2, permutes these branches and fixes any elements of T' that arose from multiplicity 1 elements of T. Any vertex $v' \in V(T')$ has degree $M(q(v'))$. By Proposition 4.3.15, $q^{-1}(\tilde{T}) \cong \widetilde{q^{-1}(T)} = \tilde{T}'$, and we know that only multiplicity 1 edges and vertices are removed when passing from T to $\tilde{T}$. So, only edges and vertices arising from multiplicity 1 edges and vertices of T are removed when passing from T' to $\tilde{T}'$. This also gives an automorphism on $\tilde{T}'$ which fixes everything arising from the multiplicity 1 component of $\tilde{T}$. Denote the center of $\tilde{T}'$ by c'. Then we must have $c' = q^{-1}(P)$, where P is a multiplicity 1 vertex, or a point on a multiplicity 1 edge of $\tilde{T}$. Recall that c' is either a vertex of $\tilde{T}'$ or the mid point of an edge $e' \in E(\tilde{T}')$. Suppose first that the centre c' of $\tilde{T}'$ is the mid point of an edge e'. Then e' is either an edge in T', in which case it arises from a multiplicity 1 edge in T, or e' is not an edge in T'.

If e' is an edge in T', then let $e \in E(T)$ be such that $M(e) = 1$ and $q^{-1}(e) = e'$. Note that the two end points, say v_1 and v_2, of e must both have multiplicity 1, and the two end points of e' arise from these vertices.

If e' is not an edge in T' then we have an exceptional vertex v' in T' which becomes a point on e' when passing to the core. Let $w_1', w_2' \in V(\tilde{T}')$ be the two end points of e'. Note that, since the centre of $\tilde{T'}$ is fixed by the quotient action, we have just two options, either the endpoints w'_1 and w'_2 of e' are fixed, or w'_1 and w'_2 are in an orbit of size 2 (induced by the quotient action on T'). If w'_1 and w'_2 are permuted by the quotient action, then v' must be equidistant between them, i.e. v' is the centre of $\tilde{T'}$, and in T there is a multiplicity 1 vertex v with $q^{-1}(v) = v'$, and a multiplicity 2 vertex w with $q^{-1}(w)_1 = w'_1$ and $q^{-1}(w)_2 = w'_2$. In this situation v is a vertex of $\tilde{T}$. Otherwise, if w'_1 and w'_2 are fixed by the quotient action, then v' is also. So again, in T, there is a multiplicity 1 vertex v with $q^{-1}(v) = v'$. Furthermore, there are vertices $w_1, w_2 \in V(T)$ with multiplicity 1 and $q^{-1}(w_i) = w'_i$, $i = 1, 2$. The centre of $\tilde{T'}$ either lies at v' or between v' and w'_i for $i = 1$ or 2. That is, in all possible situations above, the centre of $\tilde{T'}$ lies at a vertex or on an edge of T' that arises from a multiplicity one vertex or edge of T. Since all multiplicity 1 edges of T have end points being multiplicity 1 vertices, we can conclude that the minimising vertex, or vertices, of $\tilde{T'}$ for ϕ arise from multiplicity 1 vertices of T. This last sentence also holds in the case when c' is a vertex of $\tilde{T'}$.

Therefore, in all of these cases above, we have shown that, if $q^{-1}(T^1) \cap V(\tilde{T'}) \neq \emptyset$, then

$$\min_{v \in V(\tilde{T'})} \phi(v) = \min_{v \in q^{-1}(T^1) \cap V(\tilde{T'})} \phi(v),$$

and the minimising vertex, or vertices of $\tilde{T'}$ will give the centre. If $q^{-1}(T^1) \cap V(\tilde{T'}) = \emptyset$ then $|v(\tilde{T}^1)| = 1$ and the single vertex of $v(\tilde{T}^1)$ must be exceptional. In fact we can say in more generality, that if $v(\tilde{T}^1) = \{u\}$ then, regardless of whether u is exceptional or not, $q^{-1}(u)$ is the centre of $\tilde{T'}$.

Recall that, by Construction 4.3.12, that a vertex $v \in \tilde{T}$ corresponds to $M(v)$ vertices in $\tilde{T'}$ except when v is exceptional, i.e. has genus 0, and exactly one closed incident edge which is coloured the same as v, and has multiplicity 2. When v is exceptional, $q^{-1}(v)$ is a point on an edge in $\tilde{T'}$. Regardless of the colouring of v, $w(v) = 0$.

Choose some vertex u of $\tilde{T}^1$ (by Proposition 4.3.10 we know that such a vertex will always exist). Suppose first that u is a multiplicity 1 vertex of $\tilde{T}$ such that $q^{-1}(u)$ is a vertex of $\tilde{T'}$. That is, u is not exceptional. Let $T_1, \dots T_n$ be the connected components of $\tilde{T} \setminus \{u\}$. So, we get

$$\phi(u) = \max\{w(T_1), \dots, w(T_n)\}.$$

Note that, since u has multiplicity 1, $q^{-1}(\tilde{T} \setminus \{u\}) = q^{-1}(\tilde{T}) \setminus \{q^{-1}(u)\}$. The connected components of $\tilde{T'} \setminus \{q^{-1}(u)\}$ will be the connected components of

$\cup_{i=1}^{n} q^{-1}(T_i)$. There are $n_i := \min_{v \in T_i}\{M(v)\}$ connected components of $q^{-1}(T_i)$, which we label $q^{-1}(T_i)_1, \ldots, q^{-1}(T_i)_{n_i}$. Note that if T_i has a multiplicity 1 component, this means that $q^{-1}(T_i)$ is connected. For every non-exceptional vertex $v \in T_i$ there will be $\frac{M(v)}{n_i}$ vertices $q^{-1}(v)_{(k-1)\frac{M(v)}{n_i}+1}, \ldots, q^{-1}(v)_{k\frac{M(v)}{n_i}}$ in each of $q^{-1}(T_i)_k$, $1 \leq k \leq n_i$. Each blue edge e incident to a non-exceptional vertex v gives rise to $\frac{M(e)}{M(v)}$ blue edges incident to $q^{-1}(v)_j$ for all $1 \leq j \leq M(v)$. In particular, every non-exceptional vertex v in $\tilde{T}$ is such that $w(v) = w(q^{-1}(v)_j)$ for all $1 \leq j \leq M(v)$. If v is exceptional then, as we showed above, we have $w(v) = 0$. So we get

$$
\begin{aligned}
w(T_i) &= \sum_{v \in T_i} \frac{M(v)w(v)}{\min_{v' \in T_i}\{M(v')\}}, \\
&= \sum_{\substack{v \in T_i,\ \text{not} \\ \text{exceptional}}} \frac{M(v)w(v)}{\min_{v' \in T_i}\{M(v')\}}, \\
&= \sum_{\substack{v \in T_i,\ \text{not} \\ \text{exceptional}}} \frac{M(v)w(v)}{n_i}, \\
&= \sum_{v \in q^{-1}(T_i)_j} w(v), \quad \text{for } 1 \leq j \leq n_i, \\
&= w(q^{-1}(T_i)_j), \quad \text{for } 1 \leq j \leq n_i.
\end{aligned}
$$

Thus, for every non-exceptional vertex $u \in V(\tilde{T})$, we can conclude that

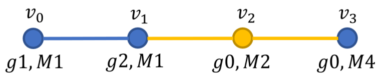
$$\phi(u) = \phi(q^{-1}(u)).$$

Suppose instead that $u \in V(\tilde{T})$ is exceptional. So, $q^{-1}(u)$ is not a vertex of $\tilde{T}'$. Let v be the unique vertex incident to u in $\tilde{T}$. So, $M(v) = 2$, and the two vertices in $q^{-1}(v)$ are $v_1' = q^{-1}(v)_1$ and $v_2' = q^{-1}(v)_2$. Note that $q^{-1}(u)$ is the mid point of the edge e between v_1' and v_2' in $\tilde{T}'$. By construction, $q^{-1}(u)$ is the centre of $\tilde{T}'$, $\phi(v_1') = \phi(v_2')$, and $q^{-1}(\tilde{T} \setminus \{u\}) \cong \tilde{T}' \setminus \{e\}$. In particular, $\phi(v_1') = w(T_2')$ and $\phi(v_2') = w(T_1')$, where T_2' is the connected component of $\tilde{T}' \setminus \{v_1'\}$ which contains v_2', and T_1' is the connected component of $\tilde{T}' \setminus \{v_2'\}$ which contains v_1'. Note that $\tilde{T}' \setminus \{e\}$ has two connected components, which are T_1' and T_2'. Therefore, by construction and using a similar proof to above, one can easily show that $\phi(u) = w(T_i')$ for $i = 1, 2$. In particular, $\phi(u) = \phi(v_i')$. Since u is exceptional, it is the only multiplicity 1 vertex of $\tilde{T}$, so we have $\min_{v \in V(\tilde{T}^1)} \phi(v) = \phi(u) = \phi(v_1') = \phi(v_2') = \min_{v' \in V(\tilde{T})} \phi(v') = \frac{1}{2}w(\tilde{T}')$. Furthermore, this minimum is not obtained elsewhere since $V(\tilde{T}^1) = \{u\}$.

Finally, by a similar argument to above, and excluding any exceptional vertices from the sum (since if v is exceptional $w(v) = 0$), one can show that $w(\tilde{T}) = w(\tilde{T}')$. Applying [DDMM17, Lemma 5.12] gives the desired result. $\square$

Definition 4.4.5. For an open quotient BY tree T with core $\tilde{T}$, we define the *centre*, c, of $\tilde{T}$ to be the minimising vertex or edge described in Lemma 4.4.4.

Example 4.4.6. Let T' be the following closed quotient BY tree:

$T' =$ 

Note that T' is in fact the core of the open quotient BY tree T in Example 4.1.8. To calculate the centre of T' let us first calculate the weight function, as defined in Definition 4.4.1, for each of the vertices. We find that

$$w(v_0) = 2g(v_0) + 2 - \sum_{\substack{e \in E(T'_b) \\ \text{incident to } v_0}} \frac{M(e)}{M(v_0)} = 3.$$

Similarly $w(v_1) = 5$ and $w(v_3) = 2$, and, since v_2 is yellow, $w(v_2) = 0$. To find the centre we must calculate $\phi(v)$ for all $v \in V((T')^1)$, where ϕ is as defined in Definition 4.4.1. That is,

$$\phi(v) = \max\{w(S) \mid S \text{ is a connected component of } T' \setminus \{v\}\}.$$

Note that $V((T')^1) = \{v_0, v_1\}$. We find that:

$$\begin{aligned}
\phi(v_0) &= M(v_1)w(v_1) + M(v_2)w(v_2) + M(v_3)w(v_3), \\
&= 13, \\
\phi(v_1) &= \max\{M(v_0)w(v_0), \frac{1}{M(v_2)}(M(v_2)w(v_2) + M(v_3)w(v_3))\}, \\
&= \max\{3, 4\}, \\
&= 4.
\end{aligned}$$

Therefore, $\min_{v \in V((T')^1)} \phi(v)$ is attained at just v_1, and v_1 is the centre of T'.

Proposition 4.4.7. *Let T be an open quotient BY tree, and let c be the centre of the core of T. Then $q^{-1}(c)$ is the centre of the core of $q^{-1}(T)$, and $q(q^{-1}(c)) = c$.*

Proof. The proof of this follows trivially from the proof of Lemma 4.4.4. $\square$

4.5 A Canonical Representative

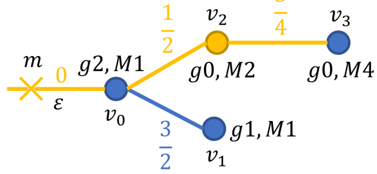
In this section we define an equivalence relation on metric open quotient BY trees and explain a method for finding a canonical representative. Once we have established a relation between cluster pictures and open quotient BY trees in Section 5.1, if desired, one can then translate this canonical representative back into the language of cluster pictures. Roughly, the construction of the canonical representative of an equivalence class of open quotient BY trees follows that of open BY trees. We calculate the centre of the core and attach an open yellow edge as close to the centre as possible. The main difference is that we also need to keep track of our marked point, and must be more careful about where we attach the open edge. In particular it is not always possible to attach an open yellow edge directly to the centre, as it is in the semistable situation. To enable a description of how the marked point can be selected, we construct the following tree.

Construction 4.5.1 (Extended tree, B)**.** Let T be a metric open quotient BY tree with open edge ε, marked point m, and core $\tilde{T}$.

Define the *extended tree* B as follows. Perform the following moves to T^1, and call the resulting tree A. For every vertex $v \in V((T_b)^1)$ if $\mathrm{denom}(d(v,m)) \nmid s(v,T)$ then add a green open edge to T^1 at v. If the open edge ε of T is blue then change the colouring of the open edge in T^1 to green. Furthermore, if v_0, the unique vertex incident to ε, is blue, of genus 0, has only one closed incident edge in T, say e, and e has multiplicity 1, then colour ε, v_0 and e green and view v_0 as a point on the open edge of A rather than a vertex.

For any leaf $v \in V(A)$ if $d(v,m) \notin \mathbb{Z}$ add a black open edge to v. Finally, at every point P on $A \setminus \{\text{open edges of } A\}$ with $d(P,m) \in \mathbb{Z}$, create a vertex at P if it was not already a vertex, and add an open black edge there. This resulting tree is B and a metric is induced by the metric on T^1 with any added open edges being thought of as intervals $[0,\infty)$.

Example 4.5.2. Let us consider the following metric open quotient BY tree:

$T =$ 

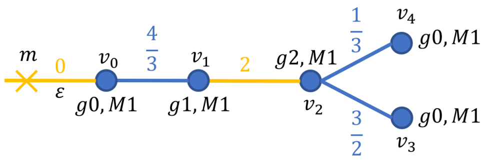
To find the extended tree B of T we first note that the only multiplicity 1 vertices and edges of T are the open edge ε, v_0 and v_1, and the edge between them. That is, T^1 is as pictured in Figure 4.8a. Since $\mathrm{denom}(d(v_1,m)) =$

(a) T^1 **(b)** Extended tree B of T

Figure 4.8: Computing the extended tree B of T.

denom $\left(\frac{3}{2}\right) = 2$, and $s(v_1, T) = 3$, we have that $\text{denom}(d(v_1, m)) \nmid s(v_1, T)$ and therefore add an open edge coloured green to T^1 at v_1. Note that $\text{denom}(d(v_0, m)) = 1$, so certainly $\text{denom}(d(v_0, m)) \mid s(v_0, T)$, and we do not add an open green edge to v_0. Finally, we add an open black edge to v_0, since it is integer distance from m, and create a black vertex and black open edge at the point on the blue edge distance 1 from m. This gives us the extended tree B pictured in 4.8b.

Example 4.5.3. Let us consider the following metric open quotient BY tree:

$T =$ 

To find the extended tree B of T we first note that in this case $T^1 = T$. In this example v_0 is a genus 0 blue vertex and only has one incident closed edge in T which is blue and has multiplicity 1. As such we colour the open edge ε, v_0, and the unique closed edge incident to v_0, green and view v_0 as a point on the open edge. Note that this means we view the marked point as being distance $\frac{4}{3}$ along the open edge from v_1. As in the previous example, for each vertex $v \in V(T^1)$ we check whether $\text{denom}(d(v, m)) \mid s(v, T)$. Here, we find that $\text{denom}(d(v_3, m)) = \text{denom}\left(\frac{29}{6}\right) = 6 \nmid s(v_3, T) = 3$, thus add an open green edge to v_3. All other vertices $v \in V(T^1) \setminus \{v_3\}$ have $\text{denom}(d(v, m)) \mid s(v, T)$, so we do not add any additional open green edges. Note that v_4 is a leaf in this new tree, so we add an open black edge to v_4. Finally, we add open black edges at every point on $\tilde{T}^1$ (provided that it is not a leaf) integer distance from m, creating a black vertex at any such point which was not already a vertex in $V(T^1)$. This gives us the extended tree B of T as pictured in Figure 4.9.

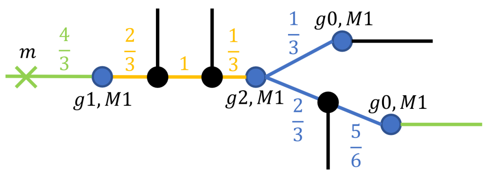


Figure 4.9: Extended tree B of T.

For an open quotient BY tree T, recall that the centre c of $\tilde{T}$ is either a multiplicity 1 vertex, or the midpoint of a multiplicity 1 edge of $\tilde{T}$. So, we can view c as a point on T^1, the subtree of T containing only the multiplicity 1 edges and vertices.

Lemma 4.5.4. *Let T be a metric open quotient BY tree with marked point m, and core $\tilde{T}$ with centre c. Take B to be the tree as described in Construction 4.5.1. Let m' be a point on B with $d(c, m')$ minimal such that $d(m', m) \in \mathbb{Z}$.*

(i) If m' is green then, m' does not lie on $\tilde{T}$, and the closest point of $\tilde{T}$ to m' in B is a vertex of $\tilde{T}^1$.

(ii) If m' does not lie on $\tilde{T}^1$ and m' is not green, then, when viewed in B, the (necessarily unique) point of $\tilde{T}^1$ which is closest to m' is a vertex of $\tilde{T}$. In this situation, m' either lies on a black open edge, or lies on the open edge of T. Furthermore, if m' lies on a black open edge then the closest vertex of $\tilde{T}^1$ to m' is a leaf of $\tilde{T}^1$.

Proof. Suppose first that m' is green. Then m' either lies on T^1, or it lies on part of $B \setminus T^1$. Suppose that m' lies on T^1, then since m' is green, by construction we must have that: either v_0 is blue, of genus 0, and the only closed edge incident to v_0 is blue and has multiplicity 1; or the open edge of T is blue. In the first instance, by Proposition 4.3.9, we know that $\tilde{T}$ is obtained from T by deleting the open edge (which must have been yellow), v_0, and the unique closed edge incident to v_0. In the second instance, by Proposition 4.3.9, we know that $\tilde{T}$ is obtained from T by deleting the open edge. That is, in both cases m' does not lie on $\tilde{T}$. By construction, in both cases, the closest point of $\tilde{T}$ to m' in B is a vertex of $\tilde{T}$.

Suppose instead that m' is green and does not lie on T^1. Then certainly m' does not lie on $\tilde{T}^1$. In this case m' must lie on a green edge that has been added to T^1 to create B. In particular this green open edge has been added to a vertex say v of T^1, with $\mathrm{denom}(d(v, m)) \nmid s(v, T)$. So, $d(v, m) \notin \mathbb{Z}$. The centre of $\tilde{T}$ certainly lies on T^1. Therefore, the closest point to c on this open green edge that has been added to v which is integer distance from m is, the point distance $\lceil d(v, m) \rceil - d(v, m)$ from v. So, $v \in V(T)$ is the closest vertex of T^1 to m' in B and is unique in this way. Suppose that $s(v, T) = 2$, then (since $M(v) = 1$), by Proposition 4.1.17, $\mathrm{denom}(d(v, m)) \mid s(v, T) - 1 = 1$. This gives that $d(v, m) \in \mathbb{Z}$ which is a contradiction, since then we would have $\mathrm{denom}(d(v, m)) \mid s(v, T) = 2$. So, $s(v, T) > 2$, or $s(v, T) = 1$. If $s(v, T) > 2$, then $g(v) > 0$. By Proposition 4.3.9, v is not deleted from T when passing to

$\tilde{T}$. That is, $v \in V(\tilde{T}^1)$. If $s(v,T) = 1$ there is at least one edge of multiplicity > 2 incident to v in T. Again, by Proposition 4.3.9, v is not deleted from T when passing to $\tilde{T}$. This completes the proof of (i).

Now, suppose that m' does not lie on $\tilde{T}^1$, and m' is not green. Then m' either lies on a black part of B, or on a part of T^1 that is deleted from T when passing to the core. If m' lies on a black part of B then, since $d(m,m') \in \mathbb{Z}$, by construction, we must have that m' lies on a black open edge attached to a leaf of T^1. Let v be the leaf of T^1 which by construction is the only vertex of B adjacent to m'. That is, in T, v has exactly one incident edge of multiplicity 1 (and perhaps other incident edges of higher multiplicity). Note that m' will be distance $\lceil d(v,m) \rceil - d(v,m)$ from v. So, by Proposition 4.3.9, we know that v is not removed when passing to the core. Therefore, the unique point of $\tilde{T}^1$ which is closest to m' in B, is a vertex of $\tilde{T}$. Suppose instead that m' lies on a part of T^1 that is deleted from T when passing to the core. Then, since m' is not green we must have, by Proposition 4.3.9, that $\tilde{T}$ is obtained from T by deleting a yellow open edge (and possibly viewing v_0 as a point on an edge rather than a vertex). That is, m' must lie on the open edge of T, and $m' \neq v_0$. Then v_0 is the closest point on $\tilde{T}^1$ to m'. If v_0 is a vertex of $\tilde{T}$ then we are done. So, suppose that v_0 is not a vertex of $\tilde{T}$, that is, v_0 is viewed as a point on an edge of $\tilde{T}$. Since the center of $\tilde{T}$ lies on $\tilde{T}$, and m' lies on the open edge of T, $d(v_0,c) < d(m',c)$. Since m' is the closest point to c which is integer distance from m, we must have that $d(v_0,m) \notin \mathbb{Z}$. Therefore, v_0 can have at most two incident edges of multiplicity 1. In particular, v_0 does not satisfy the requirements to view v_0 as a point on an edge of $\tilde{T}$ rather than a vertex. So, v_0 is a vertex of $\tilde{T}$ which completes the proof of (ii). □

Construction 4.5.5 (T^*)**.** Let T be a metric open quotient BY tree with extended tree B, and let m' be a point on B such that $d(c,m')$ is minimal subject to $d(m',m) \in \mathbb{Z}$. We create a new tree from T and m' as follows. Either m' lies on $(\tilde{T})^1$ or it does not. We deal with these two cases separately.

Case 1. *m' lies on $(\tilde{T})^1$ (so, by Lemma 4.5.4, m' is not green):*

(a) If m' is a vertex of $(\tilde{T})^1$ then we attach an open yellow edge to $\tilde{T}$ at m'.

(b) Otherwise, m' is a point on the edge of $(\tilde{T})^1$ in which case we create a new genus 0, multiplicity 1 vertex of $\tilde{T}$ at m', coloured the same as the edge m' lies on, and attach an open yellow edge there.

Note that in both situations m' is distance 0 along this new open edge.

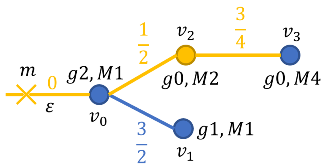
Case 2. *m' does not lie on $(\tilde{T})^1$:*

(a) If m' is green then, by Lemma 4.5.4, the closest point of $\tilde{T}$ to m' on B is a vertex $v_c \in V(\tilde{T})$. Add 'open yellow edge $\to$ genus 0 blue vertex of multiplicity 1 $\to$ blue edge of length $d(m', v_c)$' to $\tilde{T}$ at v_c. Note that m' is distance 0 along this new open edge.

(b) If m' is not green then let v_c be the (necessarily unique) vertex of B closest to m'. Then add an open yellow edge to $\tilde{T}$ at v_c, with m' distance $d(v_c, m')$ along the open edge.

Denote by T^*, a tree with marked point m' obtained by an addition of an open edge (and extra vertex and closed edge where necessary) to $\tilde{T}$, as described above.

Example 4.5.6. Let T be the metric open quotient BY tree in Example 4.5.2.

$T =$ 

By Example 4.4.6, the centre of $\tilde{T}$ is v_0 (note the change in labeling of vertices). So, there is a unique tree T^*, which turns out to be isomorphic to T.

Proposition 4.5.7. *Let T be a metric open quotient BY tree with marked point m, core $\tilde{T}$ and centre c. The description in Construction 4.5.5 yields a unique tree, except when* $\mathrm{denom}(d(m, c)) = 2$, *in which case Construction 4.5.5 yields at most two different trees, up to isomorphism.*

Proof. Recall that c is either a multiplicity 1 vertex, or the midpoint of a multiplicity 1 edge of $\tilde{T}$. If $d(c, m) \in \mathbb{Z}$ then we are done so suppose otherwise. Certainly, by construction, there exists at least one point on B distance $\leq \frac{1}{2}$ from c and integer distance from m. Suppose there are two such points, P and P'. Take Q to be the point on B which is the centre of a (possibly degenerate) tripod between P, P', and c. If $c = Q$ then we must have $d(P, P') = 1$ so $d(c, P) = d(c, P') = \frac{1}{2}$. Otherwise, if $c \neq Q$ then c does not lie on the path between P and P', and Q is a degree ≥ 3 vertex on B. That is, in T^1, Q has degree ≥ 2, in particular by Definition 4.1.3 (iv) $\mathrm{denom}(d(m, Q)) \mid s(Q, T)$. So, by Construction 4.5.1, $d(m, Q) \in \mathbb{Z}$, or no edges are added to T^1 at Q to create B. That is, if $d(m, Q) \notin \mathbb{Z}$, $\deg(Q)_{T^1} = \deg(Q)_B$, so $\deg(Q)_{T^1} \geq 3$ and $d(m, Q) \in \mathbb{Z}$. Therefore, $P = P' = Q$ and there is a unique closest point to c on B integer distance from m except in the case when $\mathrm{denom}(d(c, m)) =$

2. It remains to show that there are exactly 2 such points P and P' when $\mathrm{denom}(d(c,m)) = 2$. However, this is straightforward as assuming there are three distinct such points P, P' and P'' all distance $\frac{1}{2}$ from c leads to the same conclusion as before. That is, there exists a point Q on B which is the centre of the tripod between P, P', and P''. If the tripod is non-degenerate then $d(Q,m) \in \mathbb{Z}$, which provides a contradiction that P, P', and P'' are all distinct. Finally, note that by construction of B, if $\mathrm{denom}(d(c,m)) = 2$ there are always two points on B distance $\frac{1}{2}$ from c.

So, we have shown that there is a unique choice for m' in Construction 4.5.5 except in the case when $\mathrm{denom}(d(c,m)) = 2$ when there are precisely two options. It is however possible that the two choices for m' result in isomorphic trees in Construction 4.5.5. □

Proposition 4.5.8. *Let T be a metric open quotient BY tree, then a tree T^* constructed above in Construction 4.5.5 is itself a metric open quotient BY tree.*

Proof. This is proved later in Section 4.6, and restated as Corollary 4.6.10. □

Notation 4.5.9. By Proposition 4.5.7, the notation T^* either refers to a unique tree, or one of two possible trees. We distinguish between the two possibilities when $\mathrm{denom}(d(m,c)) = 2$, referring to these two trees as $(T^*)^+$ and $(T^*)^-$ (in no particular order), and T^* can refer to either of these two trees. If $\mathrm{denom}(d(m,c)) \neq 2$, then we write $(T^*)^+ = (T^*)^- = T^*$.

4.6 Equivalence Classes of Open Quotient BY Trees

Recall that Corollary 2.2.11 gives criteria for when an open BY tree T has core $\tilde{T}$ and vice versa. In a similar way, it is important for us to know what "moves" we can make to a closed quotient BY tree $\tilde{T}$, in order to construct an open quotient BY tree with this as its core. Recall that, just because two open quotient BY trees have isomorphic cores, it does not follow that they have the same reduction type (as defined in Definition 3.1.1). Thus, this is not the notion of equivalence that we should take. Instead, given a metric open quotient BY tree T, we want to describe a set of moves which will allow us to obtain a complete equivalence class of T from $\tilde{T}$.

Definition 4.6.1. We say that two metric open quotient BY trees T_1 and T_2 are *equivalent* if $(T_1^*)^{\pm}$ and $(T_2^*)^{\pm}$ are isomorphic metric open quotient

BY trees. By this we mean that $(T_1^*)^+ \cong (T_2^*)^+$ and $(T_1^*)^- \cong (T_2^*)^-$, or $(T_1^*)^+ \cong (T_2^*)^-$ and $(T_1^*)^- \cong (T_2^*)^+$. We write $T_1 \sim T_2$.

Remark 4.6.2. We take T^* given by Construction 4.5.5 to be our canonical representative, noting that when $\mathrm{denom}(d(m,c)) = 2$ this is not always uniquely defined. In practice when there is a choice of two metric open quotient BY trees produced by Construction 4.5.5 it can be helpful to list the set of both as the 'canonical representative' of the equivalence class.

So, our work in the previous section describes how to construct a canonical representative of the equivalence class of a metric open quotient BY tree T. We did this by extending T^1, and taking the closest possible point to the centre of the core. In this section, we list the moves that can be made in order to obtain every equivalent metric open quotient BY tree. Suppose that T has marked point m. Throughout this section we extend our tree in the same way as described in Construction 4.5.5. We let m' be any point on the extended tree which is integer distance from m. That is, we allow our new marked point to be as far from the centre of $\tilde{T}$ as we wish, rather than selecting the closest possible point. As we are no longer describing one canonical representative, there will sometimes be multiple ways in which we can attach this new marked point m' to $\tilde{T}$, via an open edge, to obtain a metric open quotient BY tree equivalent to T. We will also show that every equivalent metric open quotient BY tree can be constructed in this way. Before we get into the description of the equivalence class we need to note the following.

Proposition 4.6.3. *Let T be a metric open quotient BY tree with marked point m. Let B be the tree created from T described in Construction 4.5.1. Let m' be any point on B which is an integer distance from m. Then, for any point P on B, we have* $\mathrm{denom}(d(P,m)) = \mathrm{denom}(d(P,m'))$.

Proof. Note that the paths between m, m' and P will always form a (possibly degenerate) tripod, as shown in Figure 4.10, with central point Q. Let us deal

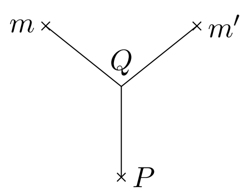


Figure 4.10: The (possibly degenerate) tripod connecting m, m' and P.

with the degenerate cases first, that is when Q coincides with m, m', or P.

- If $Q = m$ then $\text{denom}(d(P, m')) = \text{denom}(d(P, m) + d(m, m')) = \text{denom}(d(P, m))$.
- If $Q = m'$ then $\text{denom}(d(P, m')) = \text{denom}(d(P, m) - d(m, m')) = \text{denom}(d(P, m))$.
- If $Q = P$ then $\text{denom}(d(P, m')) = \text{denom}(d(m, m') - d(P, m)) = \text{denom}(d(P, m))$.

Now let us assume we are in the non-degenerate case, that is $Q \neq m, m'$ or P. Note that Q must be a vertex of B. Note further that, since $d(m, m') \in \mathbb{Z}$, $d(Q, m) \in \mathbb{Z}$ if and only if $d(Q, m') \in \mathbb{Z}$. If $d(Q, m), d(Q, m') \in \mathbb{Z}$ then

$$\begin{aligned} \text{denom}(d(P, m')) &= \text{denom}(d(P, Q) + d(Q, m')), \\ &= \text{denom}(d(P, Q)), \\ &= \text{denom}(d(P, Q) + d(Q, m)), \\ &= \text{denom}(d(P, m)). \end{aligned}$$

So, suppose that $d(Q, m), d(Q, m') \notin \mathbb{Z}$. If $Q \notin V(T)$ then, by Construction 4.5.1, we must have $d(Q, m) \in \mathbb{Z}$, which contradicts our assumption. So $Q \in V(T)$. Since we are in the non-degenerate case, $\deg_B(Q) \geq 3$. Note that either $\deg_{T^1}(Q) = \deg_B(Q)$ or $\deg_{T^1}(Q) = \deg_B(Q) - 1$. If $\deg_{T^1}(Q) \geq 3$ then we have $d(Q, m) \in \mathbb{Z}$, by Definition 4.1.13 (iii), again contradicting our assumption. So we have that $\deg_B(Q) = 3$ and $\deg_{T^1}(Q) = 2$, and by Construction 4.5.1 (since we are assuming $d(Q, m) \notin \mathbb{Z}$) we must have added an open green edge to Q when creating B from T. That is, $\text{denom}(d(Q, m)) \nmid s(Q, T)$. However Definition 4.1.3 (iv) (along with Remark 4.1.15) tells us that $\text{denom}(d(Q, m)) \mid s(Q, T)$. This gives a contradiction. So, $d(Q, m), d(Q, m') \in \mathbb{Z}$. □

The construction of a representative in the equivalence class with new marked point integer distance from m in B, say m', won't always be unique. There may well be multiple ways of attaching an open edge to $\tilde{T}$ all of which result in a metric open quotient BY tree which is equivalent to T. Here we describe in what ways this attachment can be carried out.

Theorem 4.6.4. *Let T be a metric open quotient BY tree with marked point m, core $\tilde{T}$ and centre c. Let B be the tree constructed from T as described in Construction 4.5.1. For each possible choice of a point m' (not necessarily a vertex) of B such that $d(m, m') \in \mathbb{Z}$, a metric open quotient BY tree T' with marked point m' can be constructed from $\tilde{T}$ in any of the following ways, provided the specified conditions hold. Moreover T' is equivalent to T. Conversely,* *any metric open quotient* BY *tree equivalent to* T *can be constructed in one of these ways for some choice of* m'.

(A) *If* m' *lies on* $\tilde{T}$ *and is a vertex of* $\tilde{T}$ *(note that it is important that* m' *is a vertex of* $\tilde{T}$ *not just a vertex of* B*) then we can create* T' *from* $\tilde{T}$ *in the following ways:*

- (A1) *Attach an open yellow edge at* m' *(so the marked point of* T' *is distance* 0 *along the open edge);*
- (A2) *Attach an open blue edge at* m' *(so the marked point of* T' *is distance* 0 *along the open edge), provided* m' *is blue,*

$$2g(m') + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } m'}} M(e),$$

and if $g(m') = 0$ *then* m' *has at least one incident yellow edge in* $\tilde{T}$.

(B) *If* m' *lies on an edge* e *of* $\tilde{T}$ *(note that* m' *may be a vertex of* B *but not of* $\tilde{T}$*) then:*

- (B1) *Create a genus* 0*, multiplicity* 1 *vertex at* m'*, the same colour as the edge* e*, and add an open yellow edge at* m'*;*

(C) *If* m' *does not lie on* $\tilde{T}$ *and* m' *lies on a green part of* B *then:*

- (C1) *Let* v_c *be the closest vertex of* $\tilde{T}$ *to* m' *in* B*. Create a blue genus* 0*, multiplicity* 1 *vertex* v_0'*, and attach* v_0' *to* $\tilde{T}$ *at* v_c *via a blue multiplicity* 1 *closed edge and add a yellow open edge at* v_0'*. Finally,* v_0' *must be such that* $d(m', v_0') \in \mathbb{Z}$ *and* $0 < d(v_0', v_c) \leq d(m', v_c)$*;*
- (C2) *Let* v_c *be the closest vertex of* $\tilde{T}$ *to* m' *in* B*. Add a blue open edge to* v_c*, the closest vertex of* $\tilde{T}$ *to* m'*, with* m' *lying distance* $d(m', v_c)$ *along this open edge.*

(D) *If* m' *does not lie on* $\tilde{T}$*,* m' *is not green, and the closest point on* $\tilde{T}$ *to* m' *is not a vertex of* $\tilde{T}$ *then:*

- (D1) *Create a genus* 0*, multiplicity* 1 *vertex at the closest point on* $\tilde{T}$ *to* m'*, coloured the same as the edge that the point lies on in* $\tilde{T}$*, and attach an open yellow edge there.*

(E) *If* m' *does not lie on* $\tilde{T}$*,* m' *is not green, and the closest point on* $\tilde{T}$ *to* m' *is a vertex, say* v_c*, of* $\tilde{T}$ *then:*

(E1) *Add an open yellow edge to* v_c*, provided that if* v_c *has at least two incident edges of multiplicity* 1 *in* T*, and* $\#\{e \in E(T_b)$ *incident to* $v_c\} \neq \#\{e \in E(T_b')$ *incident to* $v_c\}$ *then we have* $d(v_c, m') \in \mathbb{Z}$*;*

(E2) *Add an open blue edge to* v_c*, provided* v_c *is blue,*

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e),$$

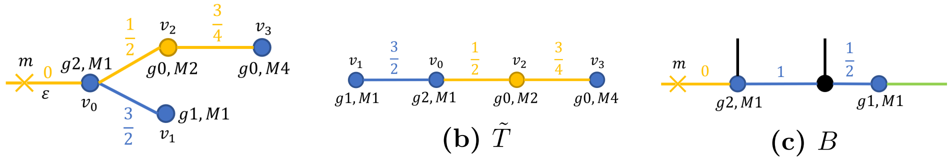
and if $g(v_c) = 0$ *then* v_c *has at least one incident yellow edge in* $\tilde{T}$*. Finally, we require that if* $\#\{$*incident blue edges to* v_c *in* $T\} \neq \#\{$*incident blue edges to* v_c *in* $T'\}$*, or if* $\deg_{\tilde{T}^1}(v_c) = 0$ *(i.e.* v_c *has no closed incident edges of multiplicity* 1*), then* $d(m', v_c) \in \mathbb{Z}$*;*

(E3) *Create a blue genus* 0*, multiplicity* 1 *vertex* v_0' *with* $d(m', v_0') \in \mathbb{Z}$ *and* $0 < d(v_0', v_c) \leq d(m', v_c)$*. Attach* v_0' *to* v_c *via a blue multiplicity* 1 *closed edge, and add a yellow open edge at* v_0'*, provided* v_c *is blue,*

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e),$$

and if $g(v_c) = 0$ *then* v_c *has at least one incident yellow edge in* $\tilde{T}$*. Finally, we require that if* $\#\{$*incident blue edges to* v_c *in* $T\} \neq \#\{$*incident blue edges to* v_c *in* $T'\}$*, or if* $\deg_{\tilde{T}^1}(v_c) = 0$ *(i.e.* v_c *has no closed incident edges of multiplicity* 1*), then* $d(m', v_c) \in \mathbb{Z}$*.*

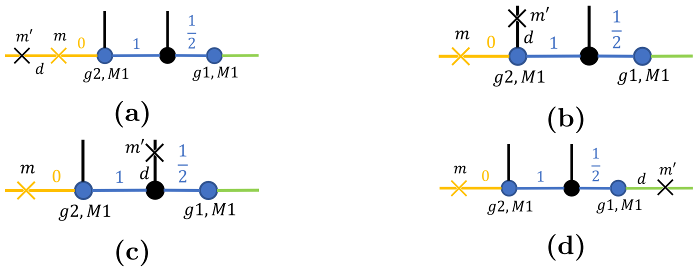
Example 4.6.5. Let us consider the metric open quotient BY tree T as in Example 4.5.2, where we constructed the extended tree B of T. Recall that T and B are as shown in Figure 4.11 below, where the core $\tilde{T}$ of T is also shown. Let m' be any point on B integer distance from m. Let $d \geq 0$ denote



(a) T (b) $\tilde{T}$ (c) B

Figure 4.11: Metric open quotient BY tree T and its extended tree B.

the distance from m' to the nearest vertex of B. We have the following four possibilities for where m' can lie: In Figure 4.12a we can see that m' lies either at v_0 (i.e. m' is distance 0 from m), or m' lies on the open edge of T. Let us



(a) (b) (c) (d)

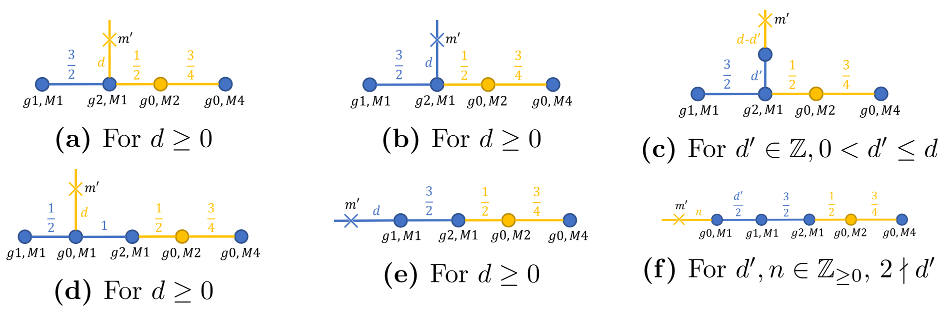
Figure 4.12: Possible placement of m' on the extended tree B

first consider the case when $m' = v_0$, that is $d = 0$. In this situation, m' is a vertex of $\tilde{T}$ so we are in case (A) of Theorem 4.6.4. We certainly satisfy the conditions for (A1), so can add an open yellow edge to $\tilde{T}$ at v_0, and note that the resulting tree is isomorphic to T. Now let us check whether we satisfy the conditions for (A2). Note that $m' = v_0$ is blue, and

$$2g(m') + 2 = 4 > 1 = \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } m'}} M(e).$$

So, we can also add an open blue edge to $\tilde{T}$ at v_0 as described in (A2). Let us instead now assume that $d > 0$, and note that this means we are in case (E) of Theorem 4.6.4. It is straightforward to check that we satisfy the conditions of (E1), (E2) and (E3).

We can assess the other three options for the placement of m' similarly. In particular, we find the following. Figure 4.12b turns out to satisfy the same conditions as Figure 4.12a, so we do not need to consider this again. In Figure 4.12c, when $d = 0$ we are in case (B) and satisfy the conditions of (B1), and when $d > 0$ we are in case (D) and satisfy the conditions of (D1). Finally, in Figure 4.12d note that $d \neq 0$, since $d(m, m') \in \mathbb{Z}$, so we are in case (C) and satisfy the conditions for both (C1) and (C2). This gives the following complete list of trees afforded by Theorem 4.6.4:



(a) For $d \geq 0$ **(b)** For $d \geq 0$ **(c)** For $d' \in \mathbb{Z}, 0 < d' \leq d$

(d) For $d \geq 0$ **(e)** For $d \geq 0$ **(f)** For $d', n \in \mathbb{Z}_{\geq 0}$, $2 \nmid d'$

Figure 4.13: A complete list of metric open quotient BY trees constructed from T afforded by Theorem 4.6.4

It is straightforward to check that each of the trees pictured in Figure 4.13 are indeed metric open quotient BY trees.

Remark 4.6.6. It may be that, for instance, m' is on a green part of B and an equivalent metric open quotient BY tree to T can be achieved by adding an open yellow edge to the closest vertex on $\tilde{T}$ to m', however we do not include this construction in this case, instead one will find an isomorphic tree can be constructed by a different choice of new marked point which is not green.

Before we can prove Theorem 4.6.4 we prove the following Lemma, which will form an integral part of the proof.

Lemma 4.6.7. *Let T be a metric open quotient BY tree with marked point m, and let m' be any point on B integer distance from m. Take T' to be any tree with marked point m' as described in Theorem 4.6.4. Then* $\mathrm{lcm}(\mathrm{denom}(d(m,v)), M(v)) = \mathrm{lcm}(\mathrm{denom}(d(m',v)), M(v))$ *for every vertex $v \in V(\tilde{T})$.*

Proof. If $M(v) = 1$ then we are done by Proposition 4.6.3. So, suppose that $M(v) > 1$. Consider $T \cup B$, and note that v, m, m' can all be considered as points on $T \cup B$. By Proposition 4.3.10, we know that there exists at least one multiplicity 1 vertex in $\tilde{T}$. In particular this means there exists some vertex $w \in V(\tilde{T})$ with $M(w) = 1$ such that, in $T \cup B$, $d(m,v) = d(m,w) + d(w,v)$ and $d(m',v) = d(m',w) + d(w,v)$. By Definition 4.1.3 (ii) and (iii), and Definition 4.1.13, since T is a metric open quotient BY tree, we must have that $\mathrm{denom}(d(m,w)) \mid M(v)$. So, $M(v)d(m,w) \in \mathbb{Z}$. Furthermore, since $M(w) = 1$, we have already shown that $\mathrm{denom}(d(m,w)) = \mathrm{denom}(d(m',w))$, giving that $M(v)d(m',w) \in \mathbb{Z}$ also. Now, $M(v)d(m,v) = M(v)d(m,w) + M(v)d(w,v)$ and $M(v)d(m',v) = M(v)d(m',w) + M(v)d(w,v)$, so

$$\mathrm{denom}(M(v)d(m,v)) = \mathrm{denom}(M(v)d(m',v)).$$

$$\text{Note that}\quad \mathrm{denom}(M(v)d(m,v)) = \frac{\mathrm{denom}(d(m,v))}{\gcd(\mathrm{denom}(d(m,v)), M(v))}, \\ = \frac{\mathrm{lcm}(\mathrm{denom}(d(m,v)), M(v))}{M(v)}.$$

Similarly for $\mathrm{denom}(M(v)d(m',v))$, so

$$\mathrm{lcm}(\mathrm{denom}(d(m,v)), M(v)) = \mathrm{lcm}(\mathrm{denom}(d(m',v)), M(v)).$$

□

Proof of Theorem 4.6.4. Throughout this proof we will use the notation ε to refer to the open edge of T, and v_0 to refer to the unique vertex incident to ε. Similarly we will use the notation ε' to refer to the open edge of T', and v_0' to refer to the unique vertex incident to ε'. Before we prove each individual case of Theorem 4.6.4, it is important to note that the following aspects of Definition 4.1.3 are preserved by either the adding or removing of certain edges and vertices.

(i) Genera of vertices that lie in both T' and T are the same. If a yellow vertex was added (along an edge of $\tilde{T}$) to create T' it was given genus 0. So, all yellow vertices in T' have genus 0. We have also been careful to never add a blue edge to a yellow vertex when creating T' from $\tilde{T}$. In passing from T to $\tilde{T}$ removing edges and vertices does not alter the fact that yellow vertices have only yellow incident edges. So, every yellow vertex in T' has genus 0 and only yellow incident edges.

So, in each case we will study below, to prove that Definition 4.1.3 (i) holds, it remains to show that each yellow vertex v' of T' is such that

$$\sum_{\substack{e\in E(T') \\ \text{incident to } v'}} \frac{M(e)}{M(v')} \geq 3.$$

The only yellow vertex we ever add to $\tilde{T}$ to obtain T' (i.e. the only time when a yellow vertex of T' is not also a yellow vertex of $\tilde{T}$) is when we create a yellow vertex on a yellow edge of $\tilde{T}$. In this case, we only ever add an open yellow edge to this vertex, meaning it has degree 3 in T' and satisfies Definition 4.1.3 (i). All other yellow vertices of T' are vertices of $\tilde{T}$, which in turn are vertices of T. So, let v' be a yellow vertex of T' which also lies in $\tilde{T}$. Looking at our cases, we can see that we only ever remove an edge (which is always yellow) from v' when passing from T to $\tilde{T}$ if $v' = v_0$. So, if $v' \neq v_0$ then certainly v' satisfies Definition 4.1.3 (i) in $\tilde{T}$, as the number, multiplicity and colouring of edges incident to v' in $\tilde{T}$ was not altered when compared to what they were in T. If $v' = v_0$ and doesn't already satisfy Definition 4.1.3 (i) when considered in $\tilde{T}$, then v_0 is yellow and has

$$\sum_{\substack{e\in E(\tilde{T}) \\ \text{incident to } v_0}} \frac{M(e)}{M(v_0)} < 3.$$

We know that in T, v_0 satisfies Definition 4.1.3 (i). So, we must have

that
$$\sum_{\substack{e\in E(\tilde{T})\\ \text{incident to } v_0}} \frac{M(e)}{M(v_0)} = 2.$$
In particular, v_0 must have one incident edge which is yellow of multiplicity 2 (else it has two incident yellow edges in $\tilde{T}$ of multiplicity 1 which is a contradiction, as then v_0 would have been removed when passing from T to $\tilde{T}$). This means that v_0 is the only vertex of multiplicity 1 in $\tilde{T}$ (since T satisfies Definition 4.1.3 (ii)). So, v_0 is where we add to $\tilde{T}$ to create T'. Looking at our possible cases, it is easy to see that we only ever add an open yellow edge to v_0 to create T'. Therefore, in T' every yellow vertex satisfies Definition 4.1.3 (i).

(ii) In all instances above, an open edge ε' is attached to $\tilde{T}$, in such a way that the unique vertex, v_0', incident to ε' in T' is always a multiplicity 1 vertex of T' and is such that the trees formed by collapsing the multiplicity 1 components to points are isomorphic, i.e. $T'/(T')^1 \cong T/T^1$. So, the embedded path from v_0' to any vertex of T' has increasing multiplicities, that is Definition 4.1.3 (ii) holds for T' as it did for T.

(iii) Recall that $\operatorname{lcm}(\operatorname{denom}(d(v,m)), M(v)) = \operatorname{lcm}(\operatorname{denom}(d(v,m')), M(v))$, by Lemma 4.6.7. So, given we have only removed edges and vertices from T to obtain $\tilde{T}$, and then added multiplicity 1 vertices and edges to a multiplicity 1 part of $\tilde{T}$ to obtain T', we have that all edges incident to a vertex $v \in V(T')$ have multiplicity equal to $M(v)$ or $\operatorname{lcm}(M(v), \operatorname{denom}(d(v,m')))$ (since T satisfies Definition 4.1.3 (iii) and Definition 4.1.13). This gets us part of the way to showing that Definition 4.1.3 (iii) holds for T', and we finish proving this for each case individually below.

(iv) Similarly we inherit a large amount of the information we need to prove that Definition 4.1.3 (iv) holds for T' from the fact that it holds for T, but we will prove this case by case below.

(v) By Proposition 4.3.9, we only remove a yellow edge from a blue vertex of T to obtain $\tilde{T}$ when ε is yellow, v_0 is blue and v_0 doesn't get deleted when passing to $\tilde{T}$. So, assume this is the case. If v_0 has an incident yellow closed edge then no vertex of $\tilde{T}$ violates Definition 4.1.3 (v). The addition of any combination of edges and vertices to create T' from $\tilde{T}$ maintains this, meaning T' will also satisfy Definition 4.1.3 (v). In order to violate

Definition 4.1.3 (v), v_0 must have genus 0 and no incident yellow edges in $\tilde{T}$. Since v_0 satisfies Definition 4.1.3 (vi) in T, v_0 must have either exactly one incident closed blue edge of multiplicity 2, or either exactly 1 or 2 blue incident edges of multiplicity 1. However, if v_0 has exactly 1 or 2 blue incident edges of multiplicity 1 in $T \setminus \{\varepsilon\}$ (and no other incident edges), then v_0 would not be a vertex of $\tilde{T}$, contradicting our assumption. So, the only case left to consider is when v_0 is genus 0, blue, and has one incident edge in $T \setminus \{\varepsilon\}$ which is blue and of multiplicity 2. In this case, $\tilde{T}^1 = \{v_0\}$, so we only ever add to v_0 to create T'. In particular, we only ever add a yellow open edge. This is clear in cases (A) and (E). Cases (B), (C) and (D) do not apply to this situation (it is easy to check they do not coincide with v_0 being blue, having degree 1 in $\tilde{T}$ with the only incident edge in $\tilde{T}$ being blue of multiplicity 2). So, any construction of T' described in the statement of Theorem 4.6.4 satisfies Definition 4.1.3 (v). It is important to note that, if v_0 is genus 0, blue, and has one incident edge in $T \setminus \{\varepsilon\}$ which is blue and of multiplicity 2, then in order to create a metric open quotient BY tree T' equivalent to T, we must attach a yellow edge to v_0 when creating T'. By Proposition 4.3.9 we know that the only way of doing this is to add an open yellow edge to v_0. So, there are no ways other than those we have already described, namely (A) and (E) of creating such a T' from T.

(vi) Note that, for every $v \in \tilde{T}$ removing edges and vertices from T to obtain $\tilde{T}$ only decreases any sum over incident blue edges. In particular, for every $v \in V(\tilde{T})$,

$$\sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v}} \frac{M(e)}{M(v)} \leq \sum_{\substack{e \in E(T) \text{ blue,} \\ \text{incident to } v}} \frac{M(e)}{M(v)}.$$

So certainly in $\tilde{T}$, every vertex satisfies Definition 4.1.3 (vi), as the genus of a vertex in $\tilde{T}$ is equal to that in T.

Adding in a yellow open edge to a vertex of $\tilde{T}$ does not change this, so if T' is created from $\tilde{T}$ in this way then T' certainly satisfies Definition 4.1.3 (vi). That is, Definition 4.1.3 (vi) is certainly satisfied in cases (A1) and (E1).

If a vertex v_0' is added along an edge e of $\tilde{T}$ when constructing T', then we are in case (B1) or (D1). If e is yellow, then v_0' is yellow and we always add an open yellow edge to v_0'. So in this case v_0' has no incident blue edges. If instead, e is blue, then v_0' is blue and is given genus 0. An open yellow edge is always added to v_0' when creating T'. So, v_0' has exactly two incident blue edges, each of multiplicity 1, in T', arising from the edge on which it lay in $\tilde{T}$. Therefore v_0' satisfies Definition 4.1.3 (vi). In both cases (e being yellow or blue), all vertices of $\tilde{T}$ have not had any edges added to them in passing from $\tilde{T}$ to T'. Therefore, Definition 4.1.3 (vi) is always satisfied in cases (B1) and (D1).

A blue open edge is added directly to a vertex v_0' of $\tilde{T}$ to create T' in cases (A2), (C2) and (E2). In cases (A2) and (E2) we have that

$$2g(v_0') + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_0'}} M(e).$$

Since $M(v_0') = 1$, and given that ε' has multiplicity 1, T' satisfies Definition 4.1.3 (vi) in cases (A2) and (E2). In case (C2), m' lies on a green part of B, so either v_0' is blue with $\operatorname{denom}(d(v_0', m)) \nmid s(v_0', T)$ in T, or v_0' had a blue edge removed from it when passing from T to $\tilde{T}$. In this second instance, certainly after adding in a blue open edge at v_0, v_0 still satisfies Definition 4.1.3 (vi) in T'. So, suppose that v_0' is blue with $\operatorname{denom}(d(v_0', m)) \nmid s(v_0', T)$ in T. So, we must have that

$$\operatorname{denom}(d(v_0', m)) \nmid 2g(v_0') + 2 - \sum_{\substack{e \in E(T) \text{ blue,} \\ \text{incident to } v_0'}} \frac{M(e)}{M(v_0')}.$$

Note that this means that

$$2g(v_0') + 2 > \sum_{\substack{e \in E(T) \text{ blue,} \\ \text{incident to } v_0'}} \frac{M(e)}{M(v_0')} > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_0'}} \frac{M(e)}{M(v_0')}.$$

So, by the same argument that we used for (A2) and (E2), we get that in case (C2) T' satisfies Definition 4.1.3 (vi) also.

In our two remaining cases, (C1) and (E3), we add 'blue multiplicity 1 closed edge $\to$ blue genus 0, multiplicity 1 vertex, v_0' $\to$ blue open edge' to a vertex v_c of $\tilde{T}$ to create T'. In case (C1) we can show by a similar argument to above, that

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e).$$

Note that this condition is already an assumption in (E3), and in both cases $M(v_c) = 1$. So, by the above argument, v_c satisfies Definition 4.1.3 (vi) in T' in both (C1) and (E3). It remains to check that v_0' also satisfies Definition 4.1.3 (vi), but this is clear, so T' satisfies Definition 4.1.3 (vi) in both (C1) and (E3).

Now let us prove the remaining parts of Definition 4.1.3 hold on a case by case basis. Note that for Definition 4.1.3, we have shown above that T' satisfies (i), (ii), (v) and (vi) so it remains finish proving that (iii) and (iv) hold. It is also important to note that throughout the proofs of these we take $n = \frac{\mathrm{denom}(d(v,m))}{\gcd(\mathrm{denom}(d(v,m),M(v))}$ in Definition 4.1.3. By Lemma 4.6.7, we have that $\mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v)) = \mathrm{lcm}(\mathrm{denom}(d(v,m')), M(v))$. So, due to Remark 4.1.15, the proof that Definition 4.1.13 holds follows once the proof of Definition 4.1.3 is complete with $n = \frac{\mathrm{denom}(d(v,m))}{\gcd(\mathrm{denom}(d(v,m),M(v))}$.

(A1) Suppose m' is a vertex of $\tilde{T}$ and we have attached an open yellow edge at m':

(iii) We noted above that every edge incident to a vertex $v' \in V(T')$ has multiplicity $M(v')$, or $\mathrm{lcm}(M(v'), \mathrm{denom}(d(v', m')))$. It remains to show that exactly one or two edges incident to v' has multiplicity $M(v')$, and all the rest have multiplicity $\mathrm{lcm}(M(v'), \mathrm{denom}(d(v', m')))$.

Note that in $\tilde{T}$, since this is obtained by removing edges and vertices, every vertex v has at most two incident edges of multiplicity $M(v)$ (potentially none), and all other incident edges have multiplicity $\mathrm{lcm}(M(v), \mathrm{denom}(d(v, m')))$. Recall from Proposition 4.3.9 that $\tilde{T}$ is obtained by either removing just the open edge ε of T, removing ε and viewing v_0 as a point on an edge, or removing ε, v_0 and a unique closed blue edge of multiplicity 1 incident to v_0. In each of these cases there is at most one vertex $v' \in \tilde{T}$ with no incident edges of multiplicity $M(v')$, and this vertex has $M(v') = 1$.

If there is such a vertex, then v' is the only multiplicity 1 vertex of $\tilde{T}$ (by Definition 4.1.3 (iii)). By construction m' has multiplicity 1, so we must have $m' = v'$, and the open edge w e attached here to create T' ensures that m' satisfied Definition 4.1.3 (iii) in T'. All other vertices of T' in this situation have their incident edges unchanged in the transition from T to T'. So, because they satisfied Definition 4.1.3 (iii) in T, they still do in T'.

Otherwise, every vertex v in $\tilde{T}$ has an incident edge of multiplicity $M(v)$. Note that since m' is a vertex of $\tilde{T}$, and m' is integer distance from m, that every edge incident to m' in $\tilde{T}$ has multiplicity 1. The addition of an open edge at m' to create T' means m' still satisfies Definition 4.1.3 (iii) in T'. Again, all other vertices of T' in this situation had their incident edges unchanged from T, so because they satisfied Definition 4.1.3 (iii) in T, they still do in T'.

So T' certainly satisfies Definition 4.1.3 (iii).

(iv) Again, it is important to note that there is at most one vertex of $\tilde{T}$, where an incident edge to it in T has been deleted in passing to the core (be that v_0 or the unique vertex adjacent to v_0 if v_0 was deleted along with a closed blue edge).

If such a vertex exists, call it v'. If v' has no incident edges of multiplicity 1 in $\tilde{T}$ then, as already discussed above in the proof of (A1)(iii), this means $m' = v'$. So, the open edge added to produce T' is attached to $\tilde{T}$ at v'. In particular, this means that $M(v') = \mathrm{denom}(d(v', m')) = 1$, so after the addition of the multiplicity 1 open edge to give T' certainly v' satisfies Definition 4.1.3 (iv), since $1 \mid N$ for any integer N. All other vertices of T' lie on T and have not had their incident edges altered in the process of creating T' from T, so all still satisfy Definition 4.1.3 (iv).

Otherwise, every vertex of $\tilde{T}$ has had the number of incident edges, their colouring and multiplicities unaltered, so in $\tilde{T}$ every vertex satisfies Definition 4.1.3 (iv). The addition of an open edge at m', does not change this since $M(m') = \mathrm{denom}(d(m', m')) = 1$, and $1 \mid n$ for any integer n.

So T' satisfies Definition 4.1.3 (iv).

(A2) Let m' be a vertex of $\tilde{T}$ such that m' is blue,

$$2g(m') + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } m'}} M(e),$$

and if $g(m') = 0$ then m' has at least one incident yellow edge in $\tilde{T}$. Suppose that we have attached an open blue edge at m':

(iii) The proof of (A1)(iii) did not rely on the colouring of the open edge at any point. So, the same proof as above can be applied to this case where the open edge is instead blue, hence T' satisfies Definition 4.1.3 (iii).

(iv) Again, the proof of (A1)(iv) did not actually use that the open edge was coloured yellow. So, the same proof as above can be applied to this case where the open edge is instead blue, hence T' satisfies Definition 4.1.3 (iv).

(B1) Suppose that m' lies on $\tilde{T}$ but is not a vertex of $\tilde{T}$, (i.e. m' lies on an edge e of $\tilde{T}$), and that we have created a multiplicity 1, genus 0 vertex at m' coloured the same as e, and added an open yellow edge at m':

(iii) By construction of B, m' must lie on a multiplicity 1 edge of $\tilde{T}$. As explained in case (A1)(iii) above, there is at most one vertex in $\tilde{T}$ which has had an edge (of multiplicity 1) removed from it when passing from T to $\tilde{T}$. Suppose such a vertex $v' \in \tilde{T}$ exists (i.e. where its degree in $\tilde{T}$ is one less than its degree in T). Then, as in (A1)(iii), $M(v') = 1$. If v' violates Definition 4.1.3 (iii) in $\tilde{T}$, then it has no incident edges of multiplicity 1. However, this would mean that $\tilde{T}^1 = \{v'\}$, contradicting m' lying on a multiplicity 1 edge of $\tilde{T}$. So, v' satisfies Definition 4.1.3 (iii) in $\tilde{T}$, and therefore in T'. It only remains to check that m' satisfies Definition 4.1.3 (iii) in T' (all other vertices and their incident edges of T' have remained unchanged from T to T', so certainly satisfy Definition 4.1.3 (iii)). It is, however, clear that m' satisfies Definition 4.1.3 (iii), since all its incident edges have multiplicity 1. So T' satisfies Definition 4.1.3 (iii).

(iv) Again, there is at most one vertex in $\tilde{T}$ which has had an edge (of multiplicity 1) removed from it when compared with in T. Suppose such a vertex $v' \in \tilde{T}$ exists. Then, as above, v' still has a multiplicity 1 incident edge, and all other edges have multiplicity $\operatorname{denom}(d(m', v'))$. Since a multiplicity 1 edge was removed from v' in passing from T to $\tilde{T}$, in T v' had (at least) two incident edges of multiplicity 1. Since v' satisfies Definition 4.1.3 (iv) in T, we can see that, regardless of the colouring of the multiplicity 1 incident edge that remains in $\tilde{T}$, v' still satisfies Definition 4.1.3 (iv) in $\tilde{T}$, and therefore in T'. It remains to check that if m' is a blue vertex of T' then m' satisfies Definition 4.1.3 (iv) (all other vertices and their incident edges of T' have remained unchanged from T to T', so certainly satisfy Definition 4.1.3 (iv)). However this is clear since $d(m', m') = 0$. So T' satisfies Definition 4.1.3 (iv).

(C1) Suppose that m' does not lie on $\tilde{T}$ and m' lies on a green part of B. Suppose also that we have created a blue genus 0, multiplicity 1 vertex v_0', and attached v_0' to $\tilde{T}$ at v_c via a blue multiplicity 1 closed edge and added a yellow open edge at v_0'. Note that, v_0' must be such that $d(m', v_0') \in \mathbb{Z}$ and $0 < d(v_0', v_c) \leq d(m', v_c)$.

(iii) Again, there is at most one vertex in $\tilde{T}$ which has had an edge (of multiplicity 1) removed from it when compared with in T. Suppose such a vertex $v' \in \tilde{T}$ exists (i.e. where its degree in $\tilde{T}$ is one less than its degree in T), and recall that we must have $M(v') = 1$. If v' violates (iii) in $\tilde{T}$, then it has no incident edges of multiplicity 1. However, this would mean that $\tilde{T}^1 = \{v'\}$, in particular $v' = v_c$, so we add a multiplicity 1 blue closed edge to v' to create T', so v' satisfies (iii) in T'. All other vertices of $\tilde{T}$ and their incident edges remain unchanged from that in T, so satisfy (iii) in $\tilde{T}$. Certainly v_0' satisfies (iii) in T', so it remains to check that v_c does in the case when no vertex of $\tilde{T}$ violates (iii). If $\mathrm{denom}(d(v_c, m')) = 1$, or if v_c had exactly one incident edge of multiplicity 1 in $\tilde{T}$ then it is clear that the addition of a closed multiplicity 1 edge to v_c in T' does not change that v_c satisfies (iii). So, the only case left to consider is when v_c has exactly two incident edges e_1 and e_2 of multiplicity 1 in $\tilde{T}$, and $\mathrm{denom}(d(v_c, m')) > 1$. Since m' is green, either a multiplicity 1 edge was removed from v_c when passing from T to $\tilde{T}$, or $\mathrm{denom}(d(v_c, m')) \nmid s(v_c, T)$. In the first instance v_c had three incident edges of multiplicity 1 in T, so certainly still satisfies (iv) in T'. So, suppose that $\deg_T(v_c) = \deg_{\tilde{T}}(v_c)$ and $\mathrm{denom}(d(v_c, m')) \nmid s(v_c, T)$. Note that every edge incident to v_c except e_1 and e_2 has multiplicity $\mathrm{denom}(d(v_c, m'))$. Suppose e_1 and e_2 are both yellow. Then, in T, v_c satisfies (iv), so $\mathrm{denom}(d(v_c, m')) \mid 2g(v_c) + 2$. This gives that $\mathrm{denom}(d(v_c, m')) \mid s(v_c, T)$, which is a contradiction. We can obtain a contradiction in a similar if e_1 and e_2 are both blue, or different colours to each other. In particular, this shows that this case cannot happen so we are done.

(iv) Again, there is at most one vertex $v' \in \tilde{T}$ which has had an edge (of multiplicity 1) removed from it when compared with in T. If such a vertex v' exists then we know, by above, that it has at least one incident edge of multiplicity 1 in $\tilde{T}$, or $v' = v_c$. In the first case, v' must have had two incident edges of multiplicity 1 in T, where it satisfied (iv), so it is not hard to see that (iv) still holds in $\tilde{T}$. Otherwise if $v' = v_c$ and v_c has no incident edges of multiplicity 1 in $\tilde{T}$. In this case, the one edge of multiplicity 1 in T incident to v_c, say e, must have been blue, or $\mathrm{denom}(d(v_c, m')) \nmid s(v_c, T)$. If e was blue then we are done. So suppose e is yellow and $\mathrm{denom}(d(v_c, m')) \nmid s(v_c, T)$. Since e is yellow we must have $\mathrm{denom}(d(v_c, m')) \mid 2g(v_c) + 1$. So, once we have added a closed blue edge to v_c to create T' we satisfy (iv). A similar argument can be applied even if v_c did not have higher degree in T that in $\tilde{T}$, since then we must have that $\mathrm{denom}(d(v_c, m')) \nmid s(v_c, T)$ and there is a unique edge of multiplicity 1 incident to v_c in T which does not get deleted when passing to $\tilde{T}$. If e is yellow we must have that $\mathrm{denom}(d(v_c, m')) \mid 2g(v_c)+1$, and if e is blue that $\mathrm{denom}(d(v_c, m')) \mid 2g(v_c)$. So v_c satisfies (iv) in T'. The vertex v_0' in T' clearly satisfies (iv). All other vertices of T' are vertices of $\tilde{T}$, different from v_c and whose degrees in T equal their degrees in $\tilde{T}$, that is their incident edges have not be altered in the move from T to T', so they certainly still satisfy (iv). So T' satisfies (iv).

(C2) Suppose that m' does not lie on $\tilde{T}$, m' is green and we have added an open blue edge to the closest vertex of $\tilde{T}$ to m', say v_c. In order to prove this case, we simply note that adding a blue open edge to v_c, rather than 'closed blue edge $\to$ genus 0 multiplicity 1 blue vertex $\to$ open yellow edge' as we did in case (C1), does not change that all the required conditions are satisfied for every vertex of T (note that here $V(\tilde{T}) = V(T')$). So certainly in this case T' is a metric open quotient BY tree.

(D1) Suppose that m' does not lie on $\tilde{T}$, m' is not green, and the closest point on $\tilde{T}$ to m' is not a vertex of $\tilde{T}$ but lies on an edge $e \in E(\tilde{T})$. Suppose also that we have created a genus 0, multiplicity 1 vertex, say v_c at the closest point on $\tilde{T}$ to m', coloured the same as the edge e and attached an open yellow edge to v_c. Note that in T', by Construction 4.5.1 $d(v_c, m') \in \mathbb{Z}$. with this in mind it is easy to see that the same method of proof that we used for case (B1) (i.e. when m' lay on $\tilde{T}$ but was not a vertex of $\tilde{T}$) can be applied here. To see this more clearly, simply replace any instances of m' in the proof of (B1)(iii) with v_c, and replace the use of $d(m', m') = 0$

in the proof of case (B1)(iv) by instead using that $d(v_c, m') \in \mathbb{Z}$.

(E1) Suppose m' does not lie on $\tilde{T}$, m' is not green, the closest point on $\tilde{T}$ to m' is a vertex, say v_c of $\tilde{T}$. Suppose further that, if v_c has two incident edges of multiplicity 1 in T and $\#\{\text{blue edges incident to } v_c \text{ in } T\} \neq \#\{\text{blue edges incident to } v_c \text{ in } T'\}$, then we have $d(v_c, m') \in \mathbb{Z}$. Let us assume that we have added an open yellow edge to v_c. Note that, this means $v_c = v_0'$.

(iii) The proof that (iii) holds in this case follows by the similar method to case (A1), when m' was a vertex of $\tilde{T}$ and we added an open yellow edge at m'. Instead, we note that if there exists a vertex v' of $\tilde{T}$ with no incident edges of multiplicity $M(v')$, then $v' = v_0'$. So, in T', v' satisfies (iii). Else, if v_0' has either exactly one incident edge or > 2 incident edges of multiplicity 1, in $\tilde{T}$, the addition of the open edge to v_0' does not change that v_0' still satisfies (iii) in T'. The only case left to consider is when v_0' has exactly two incident edges in $\tilde{T}$, both of multiplicity 1. However we can then simply note that by construction of B we must have $d(m', v_0') \in \mathbb{Z}$, so again even after adding an open edge at v_0', v_0' satisfies (iii). All other vertices of T' satisfy (iii) in T' because they did in T.

(iv) As we've done on many occasions above, note that at most one blue vertex of $\tilde{T}$ has had an edge deleted from it in passing from T to $\tilde{T}$. Suppose there exists such a vertex v' in $\tilde{T}$ (i.e. the degree of v' in T was one larger than its degree in $\tilde{T}$). Then $M(v') = 1$. If v' has no incident edges in $\tilde{T}$ of multiplicity 1 then $v' = v_0'$ so we want to show that $\text{denom}(d(v', m')) \mid 2g(v') + 2$ or $2g(v') + 1$. If v' had an incident yellow edge of multiplicity 1 in T then we are done, so suppose in T the only edge incident to v' of multiplicity 1 was blue. So we have $\text{denom}(d(v', m')) \mid 2g(v') + 1$ or $2g(v')$, however since m' is not green, we have that $\text{denom}(d(v', m')) \mid s(v', T)$, in particular this means $\text{denom}(d(v', m')) \mid 2g(v') + 1$ so we are done.

Otherwise, v' has an incident edge in $\tilde{T}$ of multiplicity 1. If $v' \neq v_0'$ then certainly we are done. So, suppose that $v' = v_0'$. If $\#\{e \in E(T_b) \text{ incident to } v_0'\} = \#\{e \in E(T_b') \text{ incident to } v_0' T'\}$ then we are done as ε must have been yellow. Otherwise, $\#\{e \in E(T_b) \text{ incident to } v_0'\} \neq \#\{e \in E(T_b') \text{ incident to } v_0' T'\}$ then we have assumed that $d(v_0', m') \in \mathbb{Z}$ so we are done. Note that this assumption is indeed necessary to ensure that we satisfy (iv).

Now suppose that v_0' has the same degree in $\tilde{T}$ as it did in T (i.e. no edge was removed from v_0' in passing from T to $\tilde{T}$. Then v_0' must have at least one incident edge of multiplicity 1 in $\tilde{T}$ (else v_0' would be the only vertex of multiplicity 1 in $\tilde{T}$, contradicting that v_0' has the same degree in $\tilde{T}$ as it did in T). If v_0' had 2 or more incident edges of multiplicity 1 in $\tilde{T}$ then, since $v_0' = v_c$ (and m' does not lie on $\tilde{T}$) we must have added an open edge at v_0' in the process of obtaining B from T^1. So, by construction of B, $d(v_0', m') \in \mathbb{Z}$, and we are done. Otherwise, v_0' has exactly one incident edge of multiplicity 1, say e, in $\tilde{T}$. In this case, since m' is not green (so $\text{denom}(d(v_0', m')) \mid s(v_0', T)$), if e is blue we have $\text{denom}(d(v_0', m')) \mid 2g(v_0') + 1$, and if e is yellow we have $\text{denom}(d(v_0', m')) \mid 2g(v_0') + 2$. In either case we are done.

All other vertices of T' have not had any edges added or removed from them in the process of passing from T to $\tilde{T}$ to T'. Therefore all remaining vertices of T' satisfy (iv) as they did in T. This completes the proof that T' satisfies (iv).

(E2) Suppose that m' does not lie on $\tilde{T}$, m' is not green, the closest point on $\tilde{T}$ to m' is a vertex, say v_c of $\tilde{T}$, v_c is blue,

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e),$$

and if $g(v_c) = 0$ then m' has at least one incident yellow edge in $\tilde{T}$. Furthermore suppose that if $\#\{\text{incident blue edges to } v_c \text{ in } T\} \neq \#\{\text{incident blue edges to } v_c \text{ in } T'\}$, then $d(m', v_c) \in \mathbb{Z}$. Assume that we have added an open blue edge to v_c (So, $v_0' = v_c$).

(iii) Note that, although our assumptions here are slightly different, the proof that (iii) holds in case (E1), when instead we added an open yellow edge at m' did not make any reference to the colouring of the open edge, so we can apply the same proof here.

(iv) Again, we know that at most one vertex, $v' \in \tilde{T}$ has had an edge (of multiplicity 1) removed from it when compared with in T. Suppose such a vertex v' exists. If v' has no incident edges in $\tilde{T}$ of multiplicity 1, then we must have $v' = v_c$, as in this case $\tilde{T}^1 = \{v'\}$. If in T the unique edge of multiplicity 1 adjacent to v_c was blue then we are done. Otherwise this unique incident edge of multiplicity 1 in T was yellow (in particular it must have been ε). Under our assumption, $d(v_c, m') \in \mathbb{Z}$, so again we are done. Note that if we had not assumed that $d(v_c, m') \in \mathbb{Z}$ then we would not be able to satisfy (iv) as, since m' is not green, we have that $\text{denom}(d(v_c, m')) \mid s(v_c, T)$, in particular $\text{denom}(d(v_c, m')) \mid 2g(v_c)+2$. So, if $d(v_c, m') \notin \mathbb{Z}$, adding an open blue edge here would violate (iv).

Otherwise v' has an incident edge in $\tilde{T}$ of multiplicity 1. So, in T, v' had two incident edges of multiplicity 1 and certainly still satisfies (iv) in $\tilde{T}$. If $v' \neq v_c$ then we are done since nothing is added to v' in passing from $\tilde{T}$ to T' so it certainly still satisfies (iv) in T'. Otherwise, $v' = v_c$. If the multiplicity 1 edge that was removed from v' in passing from T to $\tilde{T}$ was blue then we are done. If instead there was a yellow edge removed from v' on passing to $\tilde{T}$ from T then this edge must have been ε, so under our assumption $\text{denom}(d(v_c, m')) = 1$ and we are done.

Finally, we need to consider the case when v_c has not been altered in passing from T to $\tilde{T}$. If v_c has at least two incident edges of multiplicity 1 in T then, by construction of B, $\text{denom}(d(v_c, m')) = 1$, so we are done. Otherwise, we must have that v_c has exactly one incident edge, e of multiplicity 1 in T (otherwise v_c would be the only multiplicity 1 vertex in $\tilde{T}$, which would contradict our assumption that v_c has not had any edges deleted from it in passing from T to $\tilde{T}$). So, our assumption tells us that $\text{denom}(d(v_c, m')) = 1$, and we are done. It is also worth noting that since m' is not green, we have $\text{denom}(d(v_c, m')) \mid s(v_c, T)$. So, if e is blue we have that $\text{denom}(d(v_c, m')) \mid 2g(v_c) + 1$, and if e is yellow we have that $\text{denom}(d(v_c, m')) \mid 2g(v_c) + 2$ so our assumption that $\text{denom}(d(v_c, m')) = 1$, is indeed necessary in order to satisfy (iv).

In all other cases a vertex of T' has not had any edges added or removed from it in passing from T to $\tilde{T}$ and then to T'. In particular, all other vertices of T' certainly satisfy (iv), so T' satisfies (iv).

(E3) Suppose that m' does not lie on $\tilde{T}$, m' is not green, the closest point on $\tilde{T}$ to m' is a blue vertex, say v_c of $\tilde{T}$, with

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e),$$

and if $g(v_c) = 0$ then m' has at least one incident yellow edge in $\tilde{T}$. Furthermore, suppose that if $\#\{\text{incident blue edges to } v_c \text{ in } T\} \neq \#\{\text{incident blue edges to } v_c \text{ in } T'\}$, then $d(m', v_c) \in \mathbb{Z}$.

a blue genus 0, multiplicity 1 vertex v_0' with $d(m', v_0') \in \mathbb{Z}$ and $0 < d(v_0', v_c) \leq d(m', v_c)$. Attach v_0' to v_c via a blue multiplicity 1 closed edge, and add a yellow open edge at v_0'

Let us have created a blue genus 0, multiplicity 1 vertex v_0' with $d(m', v_0') \in \mathbb{Z}$ and $0 < d(v_0', v_c) \leq d(m', v_c)$. Attach v_0' to v_c via a blue multiplicity 1 closed edge, and add a yellow open edge at v_0'. It is not hard to see that the vertex created at v_0' satisfies all required conditions. We can complete the proof by referencing the previous case (E2) and noting that, for all vertices $v' \neq v_0'$ of T', v' satisfies all required conditions as the same proof can be applied here.

This proves that any such T' is indeed a metric open quotient BY tree. It remains to prove that T' is equivalent to T and that the converse of the statement also holds. This is proved in Theorem 4.6.8. □

Theorem 4.6.8. *Let T be a metric open quotient BY tree. Then any T' constructed from $\tilde{T}$ by one of the ways described in Theorem 4.6.4 is equivalent to T. Moreover, any metric open quotient BY tree which is equivalent to T can be described in one of these ways. That is to say, the moves described in Theorem 4.6.4 completely describe the equivalence class of T.*

Proof. Since T' is an open quotient BY tree (we proved this in Theorem 4.6.4), we know by Proposition 4.3.9 that the core $\tilde{T}'$ is obtained by removing a few vertices and edges near ∞. After removing the open edge from $\tilde{T}'$, just as in the proof of Proposition 4.3.9, we assess whether or not we keep the unique vertex v_0' incident to the open edge or either delete it (along with the unique edge incident to it in this case) or declare it to be a point on an edge of $\tilde{T}'$. When creating T' from $\tilde{T}$ we either:

(a) added an open yellow edge to a vertex of $\tilde{T}$,

(b) added an open blue edge to a vertex of $\tilde{T}$,

(c) created a genus 0 vertex on a multiplicity 1 edge e of $\tilde{T}$, coloured the same as e, and added an open yellow edge to this new vertex,

(d) or added 'closed multiplicity 1 blue edge $\to$ blue genus 0, multiplicity 1 vertex $\to$ open yellow edge' to a multiplicity 1 vertex of $\tilde{T}$.

Call the unique vertex incident to the open edge in T', v_0'. In cases (a) and (b) after removing the open edge we are left with $\tilde{T}$, which must therefore be the core or T', else this would contradict it being the core of T. In case (c) after removing the open edge we find that v_0' has exactly two incident edges each of multiplicity 1 and coloured the same as v_0'. If v_0' is yellow then we find v_0' violates condition (i) of Definition 4.3.1. So, as in the proof of Proposition 4.3.9, we declare v_0' to be a point on an edge. This gives us $\tilde{T}'$ being isomorphic to $\tilde{T}$. A similar argument applied when v_0' is blue, as then v_0' will violate condition (ii) of Definition 4.3.1 in $T' \setminus \{\varepsilon'\}$. Finally, in case (d), after removing the open edge we again find that v_0' violates condition (ii) of Definition 4.3.1. So, by Proposition 4.3.9, we remove v_0' along with the incident blue edge. This again gives us $\tilde{T}'$ being isomorphic to $\tilde{T}$.

Since $\tilde{T}' \cong \tilde{T}$ they have the same centre. Let B be the tree we obtain by extending T according to Construction 4.5.1 and B' be the extended tree of T'. By Lemma 4.6.7, under this isomorphism, the closest point(s) of B to the centre of $\tilde{T}$ which are integer distance from m are mapped to the closest point(s) of B' to the centre of $\tilde{T}'$ which are integer distance from m'. Suppose there is a unique such closest point of B, say P, to the centre of $\tilde{T}$ which is integer distance from m. That is P will be taken to be the marked point of T^*. By Lemma 4.6.7 there is then also a unique closest point Q to the centre of $\tilde{T}'$. Similarly Q will be taken to be the marked point of $(T')^*$. Certainly when P lies on the core, Q does also, and $d(P, Q) = 0$. So, T^* and $(T')^*$ will be isomorphic (an open yellow edge will be added at P and a vertex created if P was not already a vertex of $\tilde{T}$ to create both T^* and $(T')^*$). So, suppose that P does not lie on the core. Then certainly Q does not either and $d(c, P) = d(c, Q)$ where c is the centre of $\tilde{T} \cong \tilde{T}'$. It remains to show that P and Q are both either green or not green on B and B' respectively. So, suppose that P does not lie on a green part of B. Denote the unique vertex of $\tilde{T}$ closest to P by v_c. That is either P lies on the open edge of T which is coloured yellow (so $v_0 = v_c$), or v_c is a leaf of $\tilde{T}^1$ such that $\operatorname{denom}(d(v_c, m)) \mid s(v_c, T)$.

Note that if v_c is yellow, no green edges are ever going to be attached to v_c when constructing B or B' so certainly a yellow open edge is simply attached at v_c to create both T^* and $(T')^*$. So $T^* \cong (T')^*$.

So, it remains to deal with the case when v_c is blue. If v_c has more than one closed incident edge of multiplicity 1 in T^* then in $d(m, v_c) \in \mathbb{Z}$ which would contradict our assumption that P does not lie on the core, since v_c is on the path between P and m. So v_c has either 1 or no incident closed edges of multiplicity 1 in T^*. Suppose first that P is not green. That is, T^* is obtained by attaching an open yellow edge to v_c.

If v_c has no incident closed edges of multiplicity 1 in T^* then $V(\tilde{T}^1) = \{v_c\}$ and v_c is the centre. Note that we then must be in case (C) or (E) when we are creating T'. Suppose we are in case (E). Then if a blue edge is added to v_c when creating T' from $\tilde{T}$ then we must have that $d(m', v_c) \in \mathbb{Z}$, since $\deg_{\tilde{T}^1}(v_c) = 0$, which gives a contradiction. So T' has a yellow open edge attached at v_c. This gives us that if there is a green edge attached to v_c in B then there is also a green edge attached to v_c in B'. Furthermore, if a green edge is attached to v_c in B, since T' was created under case (E), m' does not lie on this green edge (as m' is not green). So, m' lies on ε, as does P. Therefore P and Q lie on the same branch and an open yellow edge is added to $\tilde{T}$ at v_c to create $(T')^*$. Thus $T^* \cong (T')^*$. Case (C) can be dealt with similarly.

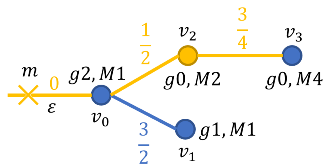
If v_c has one incident closed edge in $(T^*)^1$ then $\text{denom}(d(v_c, m)) \mid s(v, T^*)$. So, in $(T')^1$, v_c is either a leaf or has an edge removed from it when passing from T' to $\tilde{T}'$. If v_c is a leaf in $(T')^1$ then $s(v_c, T^*) = s(v_c, T')$, so $\text{denom}(d(v_c, m)) \mid s(v, T')$ and the open edge added to v_c that Q lies on is black. The same holds if a yellow open edge is attached at v_c to create T' from $\tilde{T}$. Otherwise, if a blue edge is removed from v_c when passing from T' to $\tilde{T}'$, $s(v, T') = s(v, T^*) - 1$. However we must have that $\text{denom}(d(v_c, m)) \mid s(v, T')$, since v_c has two incident multiplicity 1 edges in T'. So, since $\text{denom}(d(v_c, m)) \mid s(v, T^*) = s(v, T') + 1$ we must have that $\text{denom}(d(v_c, m)) \in \mathbb{Z}$ which contradicts that the closest point integer distance to the centre c does not lie on the core. So, $T^* \cong (T')^*$.

If P is green a similar proof follows and we find that $T^* \cong (T')^*$.

Should there be two points equidistant from the center we can follow the exact method of the above proof, just accounting for the fact that there will be two "canonical" representatives.

Suppose that T'' is a metric open quotient BY tree with marked point m'' that is equivalent to T. Then by Proposition 4.3.9 as above we know that T'' must be obtained from $\tilde{T}$ by one of the four moves (a)-(d). We note throughout the proof of Theorem 4.6.4 where the assumptions we have made are necessary to ensure the resulting open tree is indeed a metric open quotient BY tree. There are just a couple of instances where it might sometimes be possible to add to $\tilde{T}$ and still obtain a metric open quotient BY tree that is not obtained by one of the moves in Theorem 4.6.4. In particular, it is possible that dropping the condition in both (E2) and (E3) that if $\deg^1_{\tilde{T}}(v_c) = 0$ then $d(v_c, m) = 0$ still results in a metric open quotient BY tree, or that in case (C) we could have instead added an open yellow edge at the closest vertex v_c of $\tilde{T}$ to m'. However, dropping this condition in (E2) and (E3) will not result in an equivalent tree. This is proved very similarly to the earlier part of this proof. Similarly, adding an open yellow edge in case (C) will only result in an equivalent metric open quotient BY tree when $d(m, v_c) \in \mathbb{Z}$, in which case this is covered by case (E1). From this, we conclude that if T'' was obtained by adding an open edge to $\tilde{T}$ via one of (a)-(d), but was not described in Theorem 4.6.4, then T'' cannot be a metric open quotient BY tree equivalent to T. $\square$

Example 4.6.9. Let T be the metric open quotient BY tree as in Example 4.6.5.

$T =$ 

It is straightforward to check that each of the trees pictured in Figure 4.13 is indeed equivalent to T. Furthermore, any of the possible ways described by Proposition 4.3.9 to create a metric open quotient BY tree from $\tilde{T}$ are shown in Figure 4.13. Thus Figure 4.13 depicts the full equivalence class of T. By Example 4.5.6, the canonical representative T^* is isomorphic to T and thus in its equivalence class.

Corollary 4.6.10. *Let T be a metric open quotient BY tree, then a tree T^* as constructed in Construction 4.5.5 is itself a metric open quotient BY tree. Furthermore, T^* is equivalent to T.*

Proof. In Case 1 (a), when m' is a vertex of $\tilde{T}^1$, and we attach an open yellow edge to $\tilde{T}$ at m', then we lie in case (A1) of Theorem 4.6.4. So, T^* is a metric open quotient BY tree.

In Case 1 (b), m' lies on an edge e of $\tilde{T}$, and we create a genus 0 vertex there, coloured the same as e, and attach an open yellow edge at m'. That is, we lie in case (B1) of Theorem 4.6.4, and T^* is a metric open quotient BY tree.

In Case 2 (a), m' does not lie on $\tilde{T}$ and is coloured green. We attach 'open yellow edge $\rightarrow$ genus 0, multiplicity 1, blue vertex $\rightarrow$ closed blue multiplicity 1 edge' to the closest vertex v_c of $\tilde{T}$ to m'. That is, we lie in case (C1) of Theorem 4.6.4, and T^* is indeed a metric open quotient BY tree.

In Case 2 (b), m' does not lie on $\tilde{T}$ and is not coloured green. We attach an open yellow edge to the closest vertex v_c of $\tilde{T}$ to m'. By Lemma 4.5.4 (ii), v_c is the closest point on $\tilde{T}$ to m'. Furthermore, by Lemma 4.5.4, $v_c = v_0$, or v_c is a leaf of T^1 with $\mathrm{denom}(d(v_c, m)) \mid s(v_c, T)$. If v_c is a leaf of T^1, then certainly there is at most one closed edge of multiplicity 1 incident to v_c in

T. So, we satisfy the conditions in (E1) and can attach an open yellow edge at v_c to obtain a metric open quotient BY tree. If $v_c = v_0 \in V(\tilde{T})$ then note that, $d(m', v_c) > 0$, and v_c lies on the path between m' and c. Since m' is the closest point of B to c which is also integer distance from m, we must have that $d(m', v_c) \notin \mathbb{Z}$. So, v_c either has one or two multiplicity 1 incident edges in T (including the open edge). If ϵ is blue, and $v_c = v_0$ has two incident edges of multiplicity 1 in T, then by Theorem 4.6.4, we cannot add an open yellow edge at v_c. However, this case cannot occur since if v_c has two incident edges of multiplicity 1 in T, and $d(m, v_c) \notin \mathbb{Z}$ then no black edge is added which means m' lies on the open edge which is coloured green in B. This is a contradiction since m' is not green. In all other cases we do lie in Case $(E1)$ so adding an open yellow edge at v_c to create T^* does indeed result in a metric open quotient BY tree.

It follows that, by Theorem 4.6.8, T^* is always equivalent to T. □

We will see shortly, that we are able to embed the extended tree described in Construction 4.5.1 into a larger tree, whose vertices are p-adic discs. This embedding will allow us to view the possible marked points m', as described in Theorem 4.6.4, as the vertices in this tree which are discs with centre in K and integer radius. First, let us make the link between open quotient BY trees and cluster pictures, thus allowing the equivalence relation and complete description of the equivalence class to be translated to metric cluster pictures.

Chapter 5

Polynomials and Open Quotient BY Trees

5.1 Open Quotient BY Trees Associated to Cluster Pictures

In Section 4.1 we defined open quotient BY trees. Here we instead construct quotient BY trees associated to cluster pictures of what will turn out to be a tame hyperelliptic curve (but for now we will be thinking purely in terms of polynomials rather than curves), before showing there is a one-to-one correspondence between these objects. Throughout Chapters 5 and 6, unless otherwise stated, we restrict ourselves to the situation where a cluster picture $(\mathcal{R}, \Sigma, d)$ has $d_{\mathcal{R}} \geq 0$. However, it is worth noting that this will not limit our final results. If $d_{\mathcal{R}} < 0$, a simple scaling allows us to increase the depths of all clusters by an integer. This is discussed in more detail in Section 5.3.

For our purposes, a cluster picture will only be "valid" if there is a tame hyperelliptic curve with an isomorphic cluster picture. We use the following (slightly restated) definition from [Bis19] to talk about when this is the case.

Definition 5.1.1. Let Σ be a metric cluster picture. Then Σ is of *polynomial type* over K if there exists a square-free polynomial $f \in K[x]$ whose splitting field is tamely ramified, such that if $\mathcal{R}$ is the set of roots of f in $\bar{K}$ then $\Sigma' = (\mathcal{R}, \Sigma', d)$ is isomorphic to Σ.

Note that whilst this definition doesn't explicitly mention hyperelliptic curves, it does mean that Σ is of *polynomial type* over K if there exists a hyperelliptic curve $C : y^2 = f(x)$ with tame reduction, such that Σ is isomorphic to Σ_C. However, for the purposes of this section it is only necessary to consider such polynomials rather than hyperelliptic curves.

Theorem 5.1.2. *There is a one-to-one correspondence between metric open quotient BY trees and metric cluster pictures* $(\Sigma, \mathcal{R}, d)$ *of polynomial type with* $d_{\mathcal{R}} \geq 0$.

Proof. This is a direct consequence of Theorems 5.1.19 and 5.1.20 and Proposition 5.1.12. □

The construction below makes use of Galois orbits of clusters, so we first make the following definitions.

Definition 5.1.3. Let X be a Galois orbit of clusters. Then X is *übereven* if for all $\mathfrak{s} \in X$, $\mathfrak{s}$ is übereven. We define an orbit X to be *odd*, *even*, and *principal* similarly.

Definition 5.1.4. Let X and X' be Galois orbits of clusters. We say that X' is a *child* of X, written $X' < X$, if for every $\mathfrak{s}' \in X'$ there exists some $\mathfrak{s} \in X$ such that $\mathfrak{s}' < \mathfrak{s}$. Define $\delta_{X'} = \delta_{\mathfrak{s}'}$ for some $\mathfrak{s}' \in X'$.

Construction 5.1.5 ($\underline{T}(\Sigma)$)**.** Let $(\mathcal{R}, \Sigma)$ be a cluster picture of polynomial type, with $d_{\mathcal{R}} \geq 0$. We define $T = \underline{T}(\Sigma)$, the *open quotient BY tree associated to* Σ as follows. Let T be a finite tree, equipped with a genus marking $g : V(T) \to \mathbb{Z}_{\geq 0}$ on vertices, a multiplicity function $M : V(T) \cup E(T) \to \mathbb{Z}_{>0}$, and a 2-colouring blue/yellow on vertices and edges. T has one vertex v_X for every Galois orbit X of proper clusters in Σ, coloured yellow if X is übereven and blue otherwise. For X and X' both proper orbits, with $X' < X$, T has an edge between v_X and $v_{X'}$ coloured yellow if X' is even, and blue otherwise. One additional open edge is added to $v_{\mathcal{R}}$, coloured yellow if $\mathcal{R}$ is even, and blue otherwise.

The genus of a vertex v_X is defined to be the semistable genus $g_{\mathrm{ss}}(\mathfrak{s})$ of any cluster $\mathfrak{s} \in X$, as in Definition 2.1.14. The multiplicity of a vertex $v_{X'}$ or an edge between v_X and $v_{X'}$, where $X' < X$ is defined to be $|X'|$. Note that this means that $M(v_X)$ is the minimum of $M(e)$ over all incident edges e, and if e is incident to v_1 and v_2, then $M(e) = \max\{M(v_1), M(v_2)\}$. For this reason, we can omit writing the multiplicity of edges when we draw T, as they can be deduced from the multiplicities of the vertices.

Furthermore, we can define a metric on T, by defining the length of a closed edge e between v_X and v_X' with $X' < X$ to be $\delta_{X'}$, and a *marked point* m lies distance $d_{\mathcal{R}}$ along the open edge. We mark m with a cross.

Example 5.1.6. Let C be the hyperelliptic curve over $\mathbb{Q}_p^{\mathrm{ur}}$, for $p \geq 5$, defined by

$$C : y^2 = (x^2 - p^2)((x^3 - p)^2 - p^5)((x-1)^4 - p^8)((x-2)^3 - p^2).$$

Then the cluster picture Σ is shown in Figure 5.1 and the open quotient BY tree of Σ is shown in Figure 5.2 below.

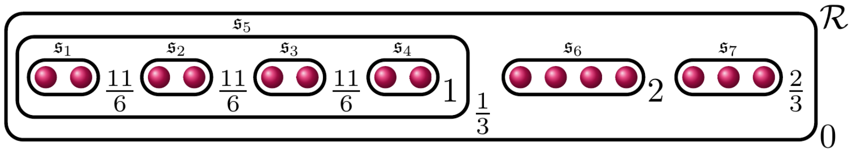


Figure 5.1: Cluster picture Σ of $C/\mathbb{Q}_p^{\mathrm{ur}}$.

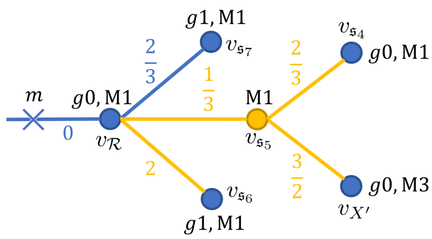
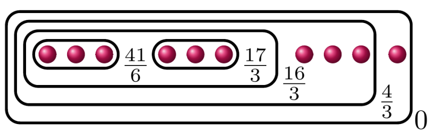


Figure 5.2: Open quotient BY tree of Σ

Remark 5.1.7. The idea, of open quotient BY trees associated to cluster pictures, is to generalise open BY trees in a way that is practically useful even in the non-semistable situation. One slight subtlety to note is the difference in convention between edge lengths on BY trees and open quotient BY trees. For BY trees, a yellow edge between $v_{\mathfrak{s}'}$ and $v_{\mathfrak{s}}$ with $v_{\mathfrak{s}'} \preceq v_{\mathfrak{s}}$ (recall, $v' \preceq v$ if v lies on the embedded path from m to v') would be assigned length $2\delta_{\mathfrak{s}'}$ rather than $\delta_{\mathfrak{s}'}$, that is, our yellow edges have half the length of those in BY trees. Although this notation $\underline{T}(\Sigma)$ is also used in the semistable setting, we do not need to worry about this being confusing. One can simply think of open BY trees as being open quotient BY trees with all edges and vertices having multiplicity 1 (and all yellow edges having the edge lengths adjusted accordingly). Indeed this is proved explicitly in Proposition 6.1.3. As such, all statements relating to BY trees in [DDMM17] and [DDMM18] can be applied to the open quotient BY tree associated to the cluster picture of a semistable hyperelliptic curve. So, whilst on occasion we will refer back to BY trees and use the same notation, one is not to worry about it being confusing as there is no conflict between them (other than the convention for lengths of yellow edges). The quotient case simply encapsulates slightly more information, allowing it to be used more generally.

Example 5.1.8. Consider the following cluster picture of polynomial type:

$\Sigma =$ 

The associated open BY and open quotient BY trees are shown in Figure 5.3.

(a) Open BY tree associated to Σ **(b)** Open quotient BY tree associated to Σ

Figure 5.3: Open BY and open quotient BY trees associated to Σ.

The aim for the rest of this section is to prove Theorem 5.1.2. In particular, we will show that the open quotient BY tree of a cluster picture of polynomial type is an open quotient BY tree, and that the converse is also true. That is, every open quotient BY tree corresponds to a cluster picture of polynomial type. This will give us a one-to-one correspondence (up to isomorphism) and allow us to work with open quotient BY trees instead of cluster pictures. First, we describe how to construct a cluster picture from an open quotient BY tree. We will then show that this construction actually gives a "valid" cluster picture (i.e. it is of polynomial type), as well as showing that the open quotient BY tree associated to a cluster picture is indeed an open quotient BY tree in the sense of Definition 4.1.3. This construction is similar to that of Construction 4.15 in [DDMM17], bar the lengths of yellow edges differing by a factor of 2.

Construction 5.1.9 ($\underline{\Sigma}(T)$)**.** Let T be an open quotient BY tree as defined in Definition 4.1.3 with marked point m. Then we define the *associated cluster picture* $\underline{\Sigma}(T)$ as follows. Define a partial order on the vertices of T by setting $v' \preceq v$ if v lies on the embedded path from m to v'.

Define $M(v)$ sets of singletons:

$$R_i^v = \{x_{i,1}^v, \dots x_{i,s(v)}^v\}, \quad \text{for } 1 \leq i \leq M(v).$$

Now take

$$R = \bigcup_{\substack{v \in V(T), \\ \text{blue}}} R_1^v \cup \cdots \cup R_{M(v)}^v.$$

Furthermore, for any vertex $v \in V(T)$ (blue or yellow), for $1 \leq i \leq M(v)$ define

$$\mathfrak{s}_{v,i} = \bigcup_{\substack{v' \preceq v, \\ \text{blue}}} R^{v'}_{(i-1)\frac{M(v')}{M(v)}+1} \cup \cdots \cup R^{v'}_{i\frac{M(v')}{M(v)}}.$$

Note that if $v' \preceq v$ then by Definition 4.1.3 (iv) $M(v) \mid M(v')$, so $\frac{M(v')}{M(v)}$ is always an integer. Now define $\Sigma \subseteq \mathcal{P}(R)$, a subset of the power set of R, by

$$\Sigma = \left(\bigcup_{v \in V(T)} \left(\mathfrak{s}_{v,1} \cup \cdots \cup \mathfrak{s}_{v,M(v)} \right) \right) \cup \left(\bigcup_{x \in R} \{x\} \right),$$
$$= \bigcup_{v \in V(T)} \bigcup_{i=1}^{M(v)} \mathfrak{s}_{v,i} \cup \bigcup_{x \in R} \{x\}.$$

We define the *(non-metric) cluster picture associated to* T to be $\underline{\Sigma}(T) = (\Sigma, R)$. We may also define a metric on Σ in the following way. If $v, w \in V(T)$ then, for all $1 \leq i \leq M(v)$ and $1 \leq j \leq M(w)$, we define $d_{\mathfrak{s}_{v,i}} = d(v, m)$ and $\delta(\mathfrak{s}_{v,i}, \mathfrak{s}_{w,j}) = d_{\mathfrak{s}_{v,i}} + d_{\mathfrak{s}_{w,j}} - 2d_{(\mathfrak{s}_{v,i} \wedge \mathfrak{s}_{w,j})}$, where $\mathfrak{s}_{v,i} \wedge \mathfrak{s}_{w,j}$ is the smallest cluster containing both $\mathfrak{s}_{v,i}$ and $\mathfrak{s}_{w,j}$, and where $d(v, w)$ is the length of the shortest path between v and w. Define the *(metric) cluster picture associated to* T to be $\underline{\Sigma}(T) = (R, \Sigma, d)$.

Remark 5.1.10. Again, as in Remark 5.1.7, this construction agrees with that in [DDMM17, Construction 4.15] (after allowing for the difference in convention between edge lengths of BY trees and open quotient BY trees).

Proposition 5.1.11. *Let* T *be an open quotient BY tree, then* $\underline{\Sigma}(T)$ *is of polynomial type. Similarly, let* $(\mathcal{R}, \Sigma, d)$ *be a cluster picture of polynomial type, with* $d_{\mathcal{R}} \geq 0$*. Then* $\underline{T}(\Sigma)$ *is an open quotient BY tree.*

Proof. Theorem 5.1.19 proves that $\underline{\Sigma}(T)$ is of polynomial type, and Theorem 5.1.20 proves that $\underline{T}(\Sigma)$ is an open quotient BY tree, in the sense of Definition 4.1.3. □

Before we move on to discussing the details of this proof, let us first show that the construction above works as we would hope and recovers the starting cluster picture or open quotient BY tree.

Proposition 5.1.12. *Let* T *be a metric open quotient BY tree, then* $\underline{T}(\underline{\Sigma}(T)) \cong T$*. Similarly, let* $(\mathcal{R}, \Sigma)$ *be the cluster picture of polynomial type with* $d_{\mathcal{R}} \geq 0$*, then* $\underline{\Sigma}(\underline{T}(\Sigma)) \cong \Sigma$*.*

Proof. First let us prove that, for a cluster picture Σ of polynomial type, $\underline{\Sigma}(\underline{T}(\Sigma)) \cong \Sigma$. Certainly Σ and $\underline{\Sigma}(T)$ have isomorphic clusters. To see this note that for every Galois orbit of clusters X, $\underline{T}(\Sigma)$ has a vertex v_X with multiplicity $M(v_X) = |X|$. So, $\underline{\Sigma}(\underline{T}(\Sigma))$ has precisely $|X|$ clusters $\mathfrak{s}_{v_X,i}$, $1 \leq i \leq |X|$, arising from v_X.

Let X' be an orbit of proper clusters with $X' < X$, then by definition we must have precisely $\frac{|X'|}{|X|}$ clusters in X' that are children of any $\mathfrak{s} \in X$. Note that every proper child of $\mathfrak{s}$ will lie in such an orbit. X' gives rise to a vertex $v_{X'}$, coloured yellow if X' is übereven and blue otherwise, in $\underline{T}(\Sigma)$ which is adjacent to v_X. The edge between them is coloured blue if X' is odd and yellow otherwise. By construction, all edges incident to v_X, except the unique edge on the path between v_X and m (when $X = \mathcal{R}$ this will be the open edge), arise in this way. That is, orbits $X' < X$ each contribute a vertex $v_{X'}$, which correspond to precisely the vertices $v \in V(\underline{T}(\Sigma))$ which are adjacent to v_X, with $v \preceq v_X$, i.e. $v < v_X$. Each such vertex gives rise to $|X'|$ clusters

$$\mathfrak{s}_{v_{X'},i} = \bigcup_{\substack{v' \preceq v_{X'},\\ \text{blue}}} R^{v'}_{(i-1)\frac{M(v')}{M(v_{X'})}+1} \cup \cdots \cup R^{v'}_{i\frac{M(v')}{M(v_{X'})}}.$$

Note that, since $\mathfrak{s}_{v_X,i} = \bigcup_{\substack{v' \preceq v_X,\\ \text{blue}}} R^{v'}_{(i-1)\frac{M(v')}{M(v_X)}+1} \cup \cdots \cup R^{v'}_{i\frac{M(v')}{M(v_X)}}$, we can write

$$\mathfrak{s}_{v_X,i} = \begin{cases} \displaystyle\bigcup_{v_{X'} < v_X} \left(\bigcup_{\substack{v' \preceq v_{X'},\\ \text{blue}}} R^{v'}_{(i-1)\frac{M(v')}{M(v_X)}+1} \cup \cdots \cup R^{v'}_{i\frac{M(v')}{M(v_X)}} \right), & X \text{ übereven}, \\ \displaystyle R^{v_X}_i \cup \bigcup_{v_{X'} < v_X} \left(\bigcup_{\substack{v' \preceq v_{X'},\\ \text{blue}}} R^{v'}_{(i-1)\frac{M(v')}{M(v_X)}+1} \cup \cdots \cup R^{v'}_{i\frac{M(v')}{M(v_X)}} \right), & \text{otherwise.} \end{cases}$$

Note that the set $\left\{(i-1)\frac{M(v')}{M(v_X)} + 1, (i-1)\frac{M(v')}{M(v_X)} + 2, \ldots, i\frac{M(v')}{M(v_X)}\right\}$ can be broken up in the following way:

$$\bigcup_{k=1}^{\frac{M(v_{X'})}{M(v_X)}} \bigcup_{j=1}^{\frac{M(v')}{M(v_{X'})}} \left\{(i-1)\frac{M(v')}{M(v_X)} + (k-1)\frac{M(v')}{M(v_{X'})} + j\right\}.$$

So, if X is übereven, we have that

$$\mathfrak{s}_{v_X,i} = \bigcup_{v_{X'} < v_X} \mathfrak{s}_{v_{X'},(i-1)\frac{M(v_{X'})}{M(v_X)}+1} \cup \cdots \cup \mathfrak{s}_{v_{X'},i\frac{M(v_{X'})}{M(v_X)}},$$

and otherwise we have

$$\mathfrak{s}_{v_X,i} = R^{v_X}_i \cup \bigcup_{v_{X'} < v_X} \mathfrak{s}_{v_{X'},(i-1)\frac{M(v_{X'})}{M(v_X)}+1} \cup \cdots \cup \mathfrak{s}_{v_{X'},i\frac{M(v_{X'})}{M(v_X)}}.$$

Therefore, the only proper children of $\mathfrak{s}_{v_X,i}$ arise from $X' < X$ and are $\mathfrak{s}_{v_{X'},(i-1)\frac{M(v_{X'})}{M(v_X)}+k}$ for $1 \leq k \leq \frac{M(v_{X'})}{M(v_X)} = \frac{|X'|}{|X|}$.

The only other possible children of $\mathfrak{s}_{v_X,i}$ will be singletons. Suppose that X is an orbit of übereven clusters. So every $\mathfrak{s} \in X$ has only even children. There are no elements of R contained in $\mathfrak{s}_{v_X,i}$ that aren't also contained in another cluster in $\underline{\Sigma}(\underline{T}(\Sigma))$, that is $\mathfrak{s}_{v_X,i}$ has no singletons.

Otherwise, X is an orbit of non-übereven clusters, so v_X is coloured blue. The singletons of $\mathfrak{s}_{v_X,i}$ are precisely the elements of the set $R_i^{v_X}$, as defined in Construction 5.1.9. That is, $\mathfrak{s}_{v_X,i}$ has precisely $|R_i^{v_X}| = s(v_X, \underline{T}(\Sigma))$ singletons. By definition, for $\mathfrak{s} \in X$, $\#\{\text{odd children of } \mathfrak{s}\} \in \{2g_{\text{ss}}(\mathfrak{s}) + 1, 2g_{\text{ss}}(\mathfrak{s}) + 2\}$. If X is an even orbit, then we must have $\#\{\text{odd children of } \mathfrak{s}\} = 2g_{\text{ss}}(\mathfrak{s}) + 2$. In $\underline{T}(\Sigma)$ the edge incident to v_X which lies on the path between v_X and m must therefore be coloured yellow. Let X' be an orbit of odd clusters with $X' < X$, then by definition we must have precisely $\frac{|X'|}{|X|}$ clusters in X' that are children of any $\mathfrak{s} \in X$. Note that every proper odd child of $\mathfrak{s}$ will lie in such an orbit. X' gives rise to a vertex $v_{X'}$ in $\underline{T}(\Sigma)$ which is adjacent to v_X, and the edge between them is coloured blue. By construction, all blue edges incident to v_X arise in this way. So,

$$\begin{aligned} 2g_{\text{ss}}(\mathfrak{s}) + 2 &= \#\{\text{odd children of } \mathfrak{s}\}, \\ &= \#\{\text{proper odd children of } \mathfrak{s}\} + \#\{\text{singletons of } \mathfrak{s}\}, \\ &= \sum_{\substack{e \in V(\underline{T}(\Sigma)_b) \\ \text{incident to } v_X}} \frac{M(e)}{M(v_X)} + \#\{\text{singletons of } \mathfrak{s}\}. \end{aligned}$$

So, $\mathfrak{s}$ has $s(v_X, \underline{T}(\Sigma))$ singletons, that is the same number of singletons as each $\mathfrak{s}_{v_X,i}$ has. A similar argument follows when $\mathfrak{s}$ is odd, bearing in mind that in this situation $\#\{\text{odd children of } \mathfrak{s}\} = 2g_{\text{ss}}(\mathfrak{s}) + 1$, however the edge incident to v_X, lying between v_X and m is now coloured blue. So again, we get that $\mathfrak{s}$ and $\mathfrak{s}_{v_X,i}$ have the same number of singletons.

So far we have shown that there is a one-to-one correspondence between the proper children of $\mathfrak{s}$ and $\mathfrak{s}_{v_X,i}$, and, if X is not übereven, the singletons of $\mathfrak{s}_{v_X,i}$. It remains to show that $|\mathfrak{s}| = |\mathfrak{s}_{v_X,i}|$, their children have the same sizes, $g_{\text{ss}}(\mathfrak{s}) = g_{\text{ss}}(\mathfrak{s}_{v_X,i})$ and $d_{\mathfrak{s}} = d_{\mathfrak{s}_{v_X,i}}$.

It follows, by an inductive argument starting at the leaves of $\underline{T}(\Sigma)$, that every cluster $\mathfrak{s}_{v_X,i}$ has the same size as the clusters in X. Since this argument works for all orbits of clusters in Σ, we also have that every proper child of $\mathfrak{s}$ corresponds to a child of $\mathfrak{s}_{v_X,i}$ of the same size. It then also follows that $g_{\text{ss}}(\mathfrak{s}) =$

$g_{\mathrm{ss}}(\mathfrak{s}_{v_X,i})$. Again, an inductive argument starting at the leaves of $\underline{T}(\Sigma)$ gives us an isomorphism between the non-metric versions of the two cluster pictures Σ and $\underline{\Sigma}(\underline{T}(\Sigma))$. Finally, by construction, we have $d_{\mathfrak{s}_{v_X,i}} = d(v_X, m) = d_X = d_{\mathfrak{s}}$.

Now let us prove that, for a metric open quotient BY tree T, $\underline{T}(\underline{\Sigma}(T)) \cong T$. This can be proved in a similar way. However, one can also note that if $\underline{T}(\underline{\Sigma}(T)) \ncong T$ then, by construction of $\underline{\Sigma}$, $\underline{\Sigma}(\underline{T}(\underline{\Sigma}(T))) \ncong \underline{\Sigma}(T)$. We will see later in Theorem 5.1.19 that $\underline{\Sigma}(T)$ is a cluster picture of polynomial type, so this would contradict the first part of this proof since we should have $\underline{\Sigma}(\underline{T}(\underline{\Sigma}(T))) \cong \underline{\Sigma}(T)$. Therefore, $\underline{T}(\underline{\Sigma}(T)) \cong T$. □

So indeed the constructions of $\underline{T}(\Sigma)$ and $\underline{\Sigma}(T)$ act as we hoped.

Remark 5.1.13. Note that if $v \in V(T)$ has either degree 1, or degree 2 and both incident edges have equal multiplicity, then each of the clusters arising from v, in the construction of $\underline{\Sigma}(T)$, have either no proper child, or exactly one proper child respectively. In these cases the only possible non-trivial orbits that can occur are completely determined by the number of singletons each cluster $\mathfrak{s}_{v,i}$ has, and $M(v)$ (the size of the orbit of $\mathfrak{s}_{v,i}$ itself). In all other cases the construction of $\underline{\Sigma}(T)$ results in more than one proper child of each of $\mathfrak{s}_{v,i}$. In this case, the size of the orbits of children is determined by $M(v)$ and the maximal multiplicity of edges incident to v. We will discuss in more detail below that, by [Bis19, Theorem 1.3], for a hyperelliptic curve C all children of a cluster $\mathfrak{s} \in \Sigma_C$, except for possibly one child, must be in orbits of the same size. So, any singletons of $\mathfrak{s}_{v,i}$ (except for possibly one lone singleton) must lie in orbits of the same size. Indeed, we saw in Proposition 4.1.17 that $\mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v))$ divides $M(v)s(v,T)$ or $M(v)(s(v,T)-1)$, which corresponds to the case when all singletons are in orbits of the same size, and the case where there is one singleton in a smaller orbit.

An open quotient BY tree corresponds to the cluster picture of a hyperelliptic curve with tame reduction, if and only if it corresponds to a cluster picture of polynomial type. So, for an open quotient BY tree T, we need $\underline{\Sigma}(T)$ to be of polynomial type over K. The following hypothesis and theorem from [Bis19] provide a useful way to check if a cluster picture is of polynomial type. For a hyperelliptic curve C, by [Bis19, Theorem 1.3], all children of a cluster $\mathfrak{s} \in \Sigma_C$, except for possibly one child must be in orbits of the same size. Here we make this more precise by restating this result, along with the following notation and definition, from [Bis19].

Notation 5.1.14. For a cluster $\mathfrak{s}$, we denote by $G_{\mathfrak{s}}$ the stabiliser of $\mathfrak{s}$ under the Galois group G.

Definition 5.1.15. For a cluster picture (R, Σ) an *orphan* of a cluster $\mathfrak{s} \in \Sigma$ is a unique fixed child of $\mathfrak{s}$ under the action of $G_{\mathfrak{s}}$

One can also replace the Galois group by a subgroup of $\mathrm{Sym}(R)$ for an abstract definition of orphans as follows.

Definition 5.1.16. Let (R, Σ, d) be a metric cluster picture and $h \in \mathrm{Sym}(R)$ which induces an automorphism of Σ. Write $H = \langle h \rangle$. Then an *orphan (with respect to H)* of a cluster $\mathfrak{s} \in \Sigma$ is a child $\mathfrak{s}'$ of $\mathfrak{s}$ such that $\mathfrak{s}'$ is fixed under the stabiliser $\mathrm{Stab}_H(\mathfrak{s})$ of $\mathfrak{s}$, and $\mathfrak{s}'$ is unique in this respect.

Hypothesis H. *Let (R, Σ, d) be a cluster picture. Then we say Σ satisfies Hypothesis H if there exists a $h \in \mathrm{Sym}(R)$ which induces an automorphism of Σ such that, if we write $H = \langle h \rangle$:*

- *The orbits of non-orphan children of a proper cluster $\mathfrak{s}$ all have length equal to* $\mathrm{denom}(d_{\mathfrak{s}}[H : \mathrm{Stab}_H(\mathfrak{s})])$ *under* $\mathrm{Stab}_H(\mathfrak{s})$*;*
- *Let $R \neq \mathfrak{s} \in \Sigma$ then*

$$[H : \mathrm{Stab}_H(\mathfrak{s})] = \mathrm{lcm}_{\mathfrak{s} \subsetneq \mathfrak{s}'} \mathrm{denom}(d^*_{\mathfrak{s}'}),$$

where for a cluster $\mathfrak{s}' \supsetneq \mathfrak{s}$,

$$d^*_{\mathfrak{s}'} = \begin{cases} 1 & \textit{if the child of } \mathfrak{s}' \textit{ containing } \mathfrak{s} \textit{ is an orphan,} \\ d_{\mathfrak{s}'} & \textit{else.} \end{cases}$$

Remark 5.1.17. The above conditions imply that $|H| = \mathrm{lcm}_{\mathfrak{s} \in \Sigma} d_{\mathfrak{s}}$ (where the lcm runs over all proper clusters). This gives a useful preliminary criterion to check if a cluster picture satisfies Hypothesis H. [Bis19, Remark 2.3]

Theorem 5.1.18 ([Bis19, Theorem 2.4])**.** *Let (Σ, R, d) be a cluster picture and suppose that $p > |R|$. Then Σ is of polynomial type over K if and only if Σ satisfies Hypothesis H.*

We will now prove that for an open quotient BY tree T, $\underline{\Sigma}(T)$ is of polynomial type. The approach is to construct an action on $\underline{\Sigma}(T)$ that satisfies Hypothesis H.

Theorem 5.1.19. *Let T be an open quotient BY tree, then $\underline{\Sigma}(T)$ is of polynomial type over K (for p sufficiently large).*

Proof. We wish to construct an action of a cyclic group H on the clusters in $\underline{\Sigma}(T)$. Recall, as described in Construction 5.1.9, each vertex $v \in V(T)$ contributes $M(v)$ proper clusters $\mathfrak{s}_{v,1}, \ldots, \mathfrak{s}_{v,M(v)}$, each containing $s(v)$ singletons of $\underline{\Sigma}(T)$. To make life easier for the sake of this proof, we note that we can relabel the clusters in $\underline{\Sigma}(T)$ inductively as follows: Label the one proper cluster arising from v_0 as $\mathfrak{s}_{v_0}$. For any vertex $v \in V(T)$ assume proper clusters arising from v have been labeled $\mathfrak{s}_v, \sigma(\mathfrak{s}_v), \ldots, \sigma^{M(v)-1}(\mathfrak{s}_v)$. We also write $\sigma^{M(v)}(\mathfrak{s}_v) = \mathfrak{s}_v$, giving us an action of degree $M(v)$ on the set of clusters arising from v. Assume further, that these clusters have been labeled in such a way that for any $v, v' \in V(T)$ with $v' \preceq v$, $\sigma^i(\mathfrak{s}_{v'})$ is contained in $\sigma^j(\mathfrak{s}_v)$ if and only if the coset $\sigma^i \mathrm{Stab}_H(\mathfrak{s}_{v'}) = \sigma^i \langle \sigma^{M(v')} \rangle$ is contained in $\sigma^j \mathrm{Stab}_H(\mathfrak{s}_v) = \sigma^j \langle \sigma^{M(v)} \rangle$, where σ has order $\mathrm{lcm}_{v \in V(T)} M(v)$ and $H = \langle \sigma \rangle$. Note that such a labeling is indeed possible as by construction if $v' \preceq v$ then there are precisely $\frac{M(v')}{M(v)}$ clusters arising from v' that are contained in any given cluster arising from v (of which there are $M(v)$ such clusters).

For any $v \in V(T)$ write $s(v) = s(v,T)$ and let $d = \mathrm{denom}(d(v,m))$, where m is the marked point of T. Recall that every cluster arising from v has $s(v)$ singletons, and

$$\frac{\mathrm{lcm}(d, M(v))}{M(v)} \mid s(v) \text{ or } (s(v)-1).$$

If $\frac{\mathrm{lcm}(d,M(v))}{M(v)} \mid s(v)$ then let us relabel the $s(v)M(v)$ singletons arising from v as

$$r_{v,i}, \sigma(r_{v,i}), \ldots, \sigma^{\mathrm{lcm}(d,M(v))-1}(r_{v,i}), \text{ for } 1 \leq i \leq \frac{s(v)M(v)}{\mathrm{lcm}(d, M(v))},$$

in such a way that $\sigma^i(r_{v,\alpha})$ is contained in $\sigma^j(\mathfrak{s}_v)$ if and only if the coset $\sigma^i \mathrm{Stab}_H(r_{v,\alpha}) = \sigma^i \langle \sigma^{\mathrm{lcm}(d,M(v))} \rangle$ is contained in $\sigma^j \mathrm{Stab}_H(\mathfrak{s}_v) = \sigma^j \langle \sigma^{M(v)} \rangle$. Note that this means there are precisely $s(v)$ singletons arising from v in each of $\sigma^j(\mathfrak{s}_v)$ for $0 \leq j \leq M(v)-1$ (where $\mathfrak{s}_v = \sigma^0(\mathfrak{s}_v)$). Otherwise, if $\frac{\mathrm{lcm}(d,M(v))}{M(v)} \mid (s(v)-1)$, then let us relabel $(s(v)-1)M(v)$ of the $s(v)M(v)$ singletons arising from v as

$$r_{v,i}, \sigma(r_{v,i}), \ldots, \sigma^{\mathrm{lcm}(d,M(v))-1}(r_{v,i}), \text{ for } 1 \leq i \leq \frac{(s(v)-1)M(v)}{\mathrm{lcm}(d, M(v))},$$

and the remaining $M(v)$ singletons arsing from v as

$$r_v, \ldots, \sigma^{M(v)-1}(r_v),$$

in such a way that $\sigma^i(r_{v,\alpha})$ is contained in $\sigma^j(\mathfrak{s}_v)$ if and only if the coset $\sigma^i \mathrm{Stab}_H(r_{v,\alpha}) = \sigma^i \langle \sigma^{\mathrm{lcm}(d,M(v))} \rangle$ is contained in $\sigma^j \mathrm{Stab}_H(\mathfrak{s}_v) = \sigma^j \langle \sigma^{M(v)} \rangle$, and

$\sigma^i(r_v)$ is contained in $\sigma^j(\mathfrak{s}_v)$ if and only if $i = j$ (where σ^0 is the identity map). Note that this means there are precisely $s(v)$ singletons arising from v in each of $\sigma^j(\mathfrak{s}_v)$ for $0 \leq j \leq M(v)-1$. In either situation, it is clear that if $r \in \sigma^i(\mathfrak{s}_v)$ then $\sigma^{M(v)}(r)$ is also, and no lower powers of σ will satisfy this so labeling in this way is indeed consistent with us having $\mathrm{Stab}(\sigma^i(\mathfrak{s}_v)) = \langle \sigma^{M(v)} \rangle$.

It is immediate, after relabeling in this way, that σ is an automorphism of $\underline{\Sigma}(T)$, and a bijection on R.

In order to verify Hypothesis H, we need to check that the orbits of non-orphan children of a proper cluster $\sigma^i(\mathfrak{s}_v)$ all have length equal to $\mathrm{denom}(d_{\sigma^i(\mathfrak{s}_v)}[H : \mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v))])$ under $\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v)) = \langle \sigma^{M(v)} \rangle$, where $d_{\sigma^i(\mathfrak{s}_v)} = d(v,m)$. To do this let us calculate the orbit sizes of children of $\sigma^i(\mathfrak{s}_v)$ under $\sigma^{M(v)}$.

Again, let us write $d = \mathrm{denom}(d_{\sigma^i(\mathfrak{s}_v)}) = \mathrm{denom}(d(v,m))$. Let $v' \preceq v$ be a vertex which is adjacent to v. Recall from Definitions 4.1.3 and 4.1.13 that $M(v') = M(v)$, or $M(v') = \mathrm{lcm}(d, M(v))$. It is easy to read off this new labeling that, under $\sigma^{M(v)}$, clusters $\sigma^j(\mathfrak{s}_{v'})$ have orbit size $\frac{M(v')}{M(v)}$. So, if $M(v') = \mathrm{lcm}(d, M(v))$ then under $\sigma^{M(v)}$, clusters $\sigma^j(\mathfrak{s}_{v'})$ have orbit size $\frac{\mathrm{lcm}(d,M(v))}{M(v)}$. If instead, $M(v') = M(v)$ then the orbit of $\sigma^j(\mathfrak{s}_{v'})$ under $\sigma^{M(v)}$ has size 1. It remains to consider the orbit sizes of the singletons of $\sigma^i(\mathfrak{s}_v)$. Similarly, from the labeling it follows that under $\sigma^{M(v)}$ any root $\sigma^j(r_{v,\alpha})$ has orbit size $\frac{\mathrm{lcm}(d,M(v))}{M(v)}$. If $\frac{\mathrm{lcm}(d,M(v))}{M(v)} \mid (s(v)-1)$, then the remaining roots labeled, $\sigma^j(r_v)$ have orbit size 1 under $\sigma^{M(v)}$. Note that

$$\begin{aligned}
\frac{\mathrm{lcm}(d, M(v))}{M(v)} &= \frac{d\mathrm{lcm}(d, M(v))}{dM(v)}, \\
&= \frac{d}{\gcd(M(v), d)}, \\
&= \mathrm{denom}(d_{\sigma^i(\mathfrak{s}_v)} M(v)), \\
&= \mathrm{denom}(d_{\sigma^i(\mathfrak{s}_v)}[H : \mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v))]).
\end{aligned}$$

By Definition 4.1.3 (iii) either every $v' \preceq v$ incident to v has multiplicity $\mathrm{lcm}(d, M(v))$, or all but one $v' \preceq v$ incident to v has multiplicity $\mathrm{lcm}(d, M(v)) \neq M(v)$ and the other one has multiplicity $M(v)$. Furthermore, if every $v' \preceq v$ incident to v has multiplicity $\mathrm{lcm}(d, M(v))$ then $\frac{\mathrm{lcm}(d,M(v))}{M(v)}$ can divide $s(v)$ or $s(v) - 1$, otherwise if there exists a $v' \preceq v$ adjacent to v with multiplicity $M(v') = M(v) \neq \mathrm{lcm}(d, M(v))$, then $\frac{\mathrm{lcm}(d,M(v))}{M(v)}$ divides $s(v)$, by Definition 4.1.3 (iv). So, in either case every child of $\sigma^i(\mathfrak{s}_v)$ has orbit size $\mathrm{denom}(d_{\sigma^i(\mathfrak{s}_v)}[H : \mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v))])$ under $\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v))$, except possibly for one child which is fixed by $\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v))$ (in which case this child is an orphan by definition). So, every non-orphan child has orbit size $\mathrm{denom}(d_{\sigma^i(\mathfrak{s}_v)}[H:\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v))])$ under $\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_v))$

It remains to prove the second part of Hypothesis H. That is, that for $R \neq \mathfrak{s} \in \Sigma$,

$$[H:\mathrm{Stab}_H(\mathfrak{s})] = \mathrm{lcm}_{\mathfrak{s}\subsetneq\mathfrak{s}'}\mathrm{denom}(d^*_{\mathfrak{s}'}),$$

where for a cluster $\mathfrak{s}' \supsetneq \mathfrak{s}$,

$$d^*_{\mathfrak{s}'} = \begin{cases} 1 & \text{if the child of } \mathfrak{s}' \text{ containing } \mathfrak{s} \text{ is an orphan,} \\ d_{\mathfrak{s}'} & \text{otherwise.} \end{cases}$$

We will prove this by induction, but first let us note the following. By the previous part of this proof we know that if $v' \preceq v$ are adjacent vertices, then v' gives rise to an orphan of $\sigma^j(\mathfrak{s}_v)$ if and only if $M(v') = M(v) \neq \mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v))$, and otherwise we have $M(v') = \mathrm{lcm}(\mathrm{denom}(d(v,m)), M(v))$. Recall that $[H:\mathrm{Stab}_H(\sigma^j(\mathfrak{s}_v))] = M(v)$.

For the base case let us assume that $v = v_0$. So, $\sigma^i(\mathfrak{s}_{v'}) < R$. Note that $M(v') = M(v_0) \neq \mathrm{lcm}(\mathrm{denom}(d(v_0,m)), M(v_0))$ if and only if $\sigma^i(\mathfrak{s}_{v'})$ is an orphan of R, and otherwise $M(v') = \mathrm{lcm}(\mathrm{denom}(d(v_0,m)), M(v_0))$. By definition $M(v_0) = 1$, $d(v_0,m) = d_R$. So, $M(v') = 1 \neq \mathrm{denom}(d_R)$ if $\mathfrak{s}_{v'} = \sigma^i(\mathfrak{s}_{v'})$ is an orphan of R, and otherwise $M(v') = \mathrm{denom}(d_R)$. Therefore $[H:\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_{v'}))] = \mathrm{denom}(d^*_R)$.

For the inductive step let $v' \preceq v$ be adjacent vertices and assume that for every $1 \leq j \leq M(v)$, $[H:\mathrm{Stab}_H(\sigma^j(\mathfrak{s}_v))] = \mathrm{lcm}_{\sigma^j(\mathfrak{s}_v)\subsetneq\mathfrak{s}'}\mathrm{denom}(d^*_{\mathfrak{s}'})$. Let $\sigma^i(\mathfrak{s}_{v'})$ be a child of $\sigma^j(\mathfrak{s}_v)$. If $\sigma^i(\mathfrak{s}_{v'})$ is an orphan of $\sigma^j(\mathfrak{s}_v)$ then $M(v') = M(v) \neq \mathrm{lcm}(\mathrm{denom}(d_{\sigma^j(\mathfrak{s}_v)}), M(v))$, otherwise we have $M(v') = \mathrm{lcm}(\mathrm{denom}(d_{\sigma^j(\mathfrak{s}_v)}), M(v))$. Therefore

$$[H:\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_{v'}))] = \begin{cases} M(v) & \sigma^i(\mathfrak{s}_{v'}) \text{ orphan of } \sigma^j(\mathfrak{s}_v), \\ \mathrm{lcm}(\mathrm{denom}(d_{\sigma^j(\mathfrak{s}_v)}), M(v)) & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \mathrm{lcm}_{\sigma^j(\mathfrak{s}_v)\subsetneq\mathfrak{s}'}\mathrm{denom}(d^*_{\mathfrak{s}'}) & \sigma^i(\mathfrak{s}_{v'}) \text{ orphan of } \sigma^j(\mathfrak{s}_v), \\ \mathrm{lcm}(\mathrm{denom}(d_{\sigma^j(\mathfrak{s}_v)}), \mathrm{lcm}_{\sigma^j(\mathfrak{s}_v)\subsetneq\mathfrak{s}'}\mathrm{denom}(d^*_{\mathfrak{s}'})) & \text{otherwise.} \end{cases}$$

This can be simplified to $[H:\mathrm{Stab}_H(\sigma^i(\mathfrak{s}_{v'}))] = \mathrm{lcm}_{\sigma^i(\mathfrak{s}_{v'})\subsetneq\mathfrak{s}}\mathrm{denom}(d^*_{\mathfrak{s}})$ which concludes our proof. In particular, we have shown that for an open quotient BY tree T, $\underline{\Sigma}(T)$ satisfies Hypothesis H and is therefore of polynomial type. $\square$

Now let us prove that, when Σ is a cluster picture of polynomial type, $\underline{T}(\Sigma)$ is an open quotient BY tree.

Theorem 5.1.20. *Let $(\mathcal{R}, \Sigma, d)$ be a cluster picture of polynomial type, with $d_{\mathcal{R}} \geq 0$. Then the open quotient BY tree $\underline{T}(\Sigma)$ associated to Σ is an open quotient BY tree in the sense of Definition 4.1.3.*

Proof. Since Σ is of polynomial type there exists a square free polynomial $f \in K[x]$ whose splitting field is tamely ramified, such that $\mathcal{R}$ is the set of roots of f in $\bar{K}$. By Theorem 5.1.18, provided p is sufficiently large, Σ satisfies Hypothesis H. In particular H can be taken to be the Galois group G of the splitting field of f (see the proof of Theorem 2.4 in [Bis19, p. 5]). Let's go through the criteria in Definition 4.1.3 and check they are all satisfied.

(i) By Construction 5.1.5 a vertex in $\underline{T}(\Sigma)$ is coloured yellow if and only if it arose from an orbit of übereven clusters, say X. Since every such $\mathfrak{s} \in X$ is even, the edge from the vertex arising from the orbit of $P(\mathfrak{s})$ to v_X (or the open edge if $X = \{\mathcal{R}\}$) will be coloured yellow. All other edges incident to v_X arise from orbits $X' < X$. That is, they arise from the orbits X' of even children. Therefore all incident edges must be yellow. Finally by Definition 2.1.14 every such v_X has genus 0.

(ii) Let v_X be any vertex in $\underline{T}(\Sigma)$, corresponding to an orbit X of clusters in Σ. Let $P(X)$ be the orbit of the parents of clusters in X. Then for any orbit $X' < X$ we have that $|X'| \geq |X|$. By construction, the edge from $v_{P(X)}$ to v_X has multiplicity $|X|$, whereas the edge from v_X to the vertex $v_{X'}$ arising from the Galois orbit X' of $\mathfrak{s}'$ has multiplicity $|X'|$. So, the path from m to any vertex in $\underline{T}(\Sigma)$ has increasing multiplicities.

(iii) Let X be an orbit of clusters in Σ. Then, by construction, $d(v_X, m) = d_X$. Hypothesis H says that, for any cluster $\mathfrak{s} \in X$, $|X| = [G : G_{\mathfrak{s}}] = \mathrm{lcm}_{\mathfrak{s} \subsetneq \mathfrak{s}'} \mathrm{denom}(d^*_{\mathfrak{s}'})$, where $d^*_{\mathfrak{s}'}$ is defined in Hypothesis H. So, for $\mathfrak{s}_2 < \mathfrak{s}_1 \in X$ we have that

$$[G : G_{\mathfrak{s}_2}] = \begin{cases} [G : G_{\mathfrak{s}_1}], & \text{if } s_2 \text{ is an orphan,} \\ \mathrm{lcm}([G : G_{\mathfrak{s}_1}], \mathrm{denom}(d_{\mathfrak{s}_1})), & \text{otherwise.} \end{cases}$$

Since the edge from $v_{\mathfrak{s}_1}$ to $v_{P(\mathfrak{s}_1)}$ always has multiplicity $|X| = [G : G_{\mathfrak{s}_1}]$, and $\mathfrak{s}_1$ can have at most one orphan we can conclude that either one or two edges adjacent to $v_{\mathfrak{s}_1}$ have multiplicity $|X| = [G : G_{\mathfrak{s}_1}]$ and all others have multiplicity $\mathrm{lcm}(|X|, \mathrm{denom}(d_{\mathfrak{s}_1}))$. Furthermore, the top cluster $\mathcal{R}$ is always in a trivial Galois orbit, so $M(\varepsilon) = 1$.

(iv) Blue vertices in $\underline{T}(\Sigma)$ are in one-to-one correspondence with orbits of non-übereven proper clusters in Σ. So, let $\mathfrak{s} \in \Sigma$ be a non-übereven cluster with Galois orbit X. Note that

$$\begin{aligned}\mathrm{denom}(d_{\mathfrak{s}}|X|) &= \frac{\mathrm{denom}(d_{\mathfrak{s}})}{\gcd(\mathrm{denom}(d_{\mathfrak{s}}), |X|)}, \\ &= \frac{\mathrm{lcm}(\mathrm{denom}(d_{\mathfrak{s}}), |X|)}{|X|}.\end{aligned}$$

We now consider the following cases separately:

- If $\mathfrak{s}$ is odd and has no orphan proper children then the edge from v_X to $v_{P(X)}$ (or the open edge ε if $\mathfrak{s} = \mathcal{R}$) is blue, of multiplicity $|X|$, and all other incident edges have multiplicity $\mathrm{lcm}(|X|, \mathrm{denom}(d_{\mathfrak{s}})) \geq |X|$. Since $\mathfrak{s}$ is odd we have from Definition 2.1.14 that $|\tilde{\mathfrak{s}}| = 2g_{\mathrm{ss}}(\mathfrak{s})+1$. By [Bis19, Theorem 1.3 (iii)], the length of orbits of non-orphan children of $\mathfrak{s}$ under $G_{\mathfrak{s}}$ is $\mathrm{denom}(d_{\mathfrak{s}}|X|)$. Under the assumption that $\mathfrak{s}$ has no proper orphaned children, all proper odd children must be in orbits of size $\mathrm{denom}(d_{\mathfrak{s}}|X|)$ under $G_{\mathfrak{s}}$, that is $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid \#\{\text{odd proper children of } \mathfrak{s}\}$. Since $\mathfrak{s}$ has no orphan proper children, $\mathfrak{s}$ will have either one orphan singleton, or no orphans at all. These two possibilities correspond to $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid (|\mathfrak{s}_{\mathrm{sing}}|-1)$ or $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid |\mathfrak{s}_{\mathrm{sing}}|$ respectively. By definition, $|\tilde{\mathfrak{s}}| = \#\{\text{odd proper children of } \mathfrak{s}\} + |\mathfrak{s}_{\mathrm{sing}}| = 2g_{\mathrm{ss}}(\mathfrak{s}) + 1$, so $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid 2g_{\mathrm{ss}}(\mathfrak{s}) + 1$ or $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid 2g_{\mathrm{ss}}(\mathfrak{s})$.
- If $\mathfrak{s}$ is even and has no orphan proper children then a similar result follows, the only difference is that $|\tilde{\mathfrak{s}}| = 2g_{\mathrm{ss}}(\mathfrak{s}) + 2$. The outcome is that $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid 2g_{\mathrm{ss}}(\mathfrak{s}) + 2$ or $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid 2g_{\mathrm{ss}}(\mathfrak{s}) + 1$, if $\mathfrak{s}$ has no orphans, or if $\mathfrak{s}$ has an orphaned singleton respectively.
- If $\mathfrak{s}$ is odd and has an orphaned proper child, then all singletons of $\mathfrak{s}$ must be non-orphans. That is, all singletons and all but one proper child are in orbits of size $\mathrm{denom}(d_{\mathfrak{s}}|X|)$. So $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid |\mathfrak{s}_{\mathrm{sing}}|$. If the orphan is even then we have that $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid \#\{\text{ odd proper children of } \mathfrak{s}\}$, otherwise the orphan is odd and $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid \#\{\text{ odd proper children of } \mathfrak{s}\}-1$. Therefore $\mathrm{denom}(d_{\mathfrak{s}}|X|) \mid 2g_{\mathrm{ss}}(\mathfrak{s}) + 1$ if the orphan is even (i.e. there is one blue and one yellow edge of multiplicity $|X|$ incident to v_X), or

denom($d_{\mathfrak{s}}|X|$) | $2g_{\mathrm{ss}}(\mathfrak{s})$ if the orphan is odd (i.e. there are two blue edges of multiplicity $|X|$ incident to v_X).

- Similarly, is $\mathfrak{s}$ is even and has a proper orphan, one can show that denom($d_{\mathfrak{s}}|X|$) | $2g_{\mathrm{ss}}(\mathfrak{s})+2$ if the orphan is even (i.e. there are two yellow edges of multiplicity $|X|$ incident to v_X), or denom($d_{\mathfrak{s}}|X|$) | $2g_{\mathrm{ss}}(\mathfrak{s})+1$ if the orphan is odd (i.e. there is one blue and one yellow edge of multiplicity $|X|$ incident to v_X).

(v) Let X be an orbit of non-übereven proper clusters such that every $\mathfrak{s} \in X$ has genus 0, that is $v_X \in V(\underline{T}(\Sigma))$ is blue and $g(v_X) = 0$. Then, since $|\tilde{\mathfrak{s}}| \in \{2g_{\mathrm{ss}}(\mathfrak{s})+1, 2g_{\mathrm{ss}}(\mathfrak{s})+2\}$, every $\mathfrak{s} \in X$ has either one or two odd children. As $\mathfrak{s}$ is a proper cluster, $\mathfrak{s}$ must have at least 2 children. If $\mathfrak{s}$ is even then the edge from $v_{P(X)}$ to v_X (or the open edge if $X = \{\mathcal{R}\}$ is yellow and we are done. Otherwise $\mathfrak{s}$ is odd and $\mathfrak{s}$ has exactly one odd child. So, $\mathfrak{s}$ must have at least one even child also (so as to have at least 2 children). Even children are always proper and give rise to yellow edges. So, every genus 0, blue vertex in $\underline{T}(\Sigma)$ has at least one yellow edge.

(vi) Let X be an orbit of proper clusters in Σ, and let $\mathfrak{s} \in X$. Then by Definition 2.1.14 we know that $g(v_X) = g_{\mathrm{ss}}(\mathfrak{s})$ is such that $|\tilde{\mathfrak{s}}| \in \{2g_{\mathrm{ss}}(\mathfrak{s})+1, 2g_{\mathrm{ss}}(\mathfrak{s})+2\}$. If $\mathfrak{s}$ is even then $|\tilde{\mathfrak{s}}| = 2g_{\mathrm{ss}}(\mathfrak{s})+2$ and the edge from v_X to $v_{P(X)}$ (or the open edge if $\mathfrak{s} = \mathcal{R}$) is yellow, so

$$2g_{\mathrm{ss}}(\mathfrak{s})+2 = |\tilde{\mathfrak{s}}| \geq \#\{\text{ odd proper children of } \mathfrak{s}\} = \sum_{\substack{e,\text{ blue edge} \\ \text{incident to } v}} \frac{M(e)}{M(v)}.$$

If instead $\mathfrak{s}$ is odd then $|\tilde{\mathfrak{s}}| = 2g_{\mathrm{ss}}(\mathfrak{s})+1$ and the edge from v_X to $v_{P(X)}$ (or the open edge if $\mathfrak{s} = \mathcal{R}$) is blue, so

$$2g_{\mathrm{ss}}(\mathfrak{s})+1 = |\tilde{\mathfrak{s}}| \geq \#\{\text{odd proper children of } \mathfrak{s}\} = -1 + \sum_{\substack{e,\text{ blue edge} \\ \text{incident to } v}} \frac{M(e)}{M(v)},$$

where the '-1' accounts for the edge from v_X to $v_{P(X)}$ being blue but not corresponding to an odd child of $\mathfrak{s}$.

Therefore, $\underline{T}(\Sigma)$ is indeed an open quotient BY tree. It follows immediately from the above work that $\underline{T}(\Sigma)$ is in fact a metric open quotient BY tree in the sense of Definition 4.1.13. □

This concludes the proof of Theorem 5.1.2 and allows us to easily translate work on open quotient BY trees to work on cluster pictures, and vice versa.

The goal is to use the equivalence relation we constructed on open quotient BY trees to study reduction types of tame hyperelliptic curves. It is worth noting that, now we have this one-to-one correspondence, an equivalence relation on open quotient BY trees will induce an equivalence relation on cluster pictures of polynomial type. It is only now that the equivalence relation and canonical representative of classes of open quotient BY trees becomes practically useful for us.

5.2 The Bruhat-Tits Tree

In the next few sections we build towards proving two things:

- Given cluster pictures of polynomial type with sets of roots $\mathcal{R}$ and $\mathcal{R}'$ and associated open quotient BY trees T and T' respectively, if T and T' are equivalent then there is a Möbius transformation taking $\mathcal{R}$ to $\mathcal{R}'$.

- Conversely, given any Möbius transformation ψ, and any cluster picture Σ of polynomial type with roots $\mathcal{R}$, let

$$\mathcal{R}' = \begin{cases} \{\psi(r) \mid r \in \mathcal{R}\} \setminus \{\infty\} & \text{if } \mathcal{R} \text{ is even,} \\ \{\psi(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\} & \text{if } \mathcal{R} \text{ is odd,} \end{cases}$$

  then the cluster picture of polynomial type, Σ' with roots $\mathcal{R}'$, is such that the open quotient BY trees $\underline{T}(\Sigma)$ and $\underline{T}(\Sigma')$ are equivalent.

This will get us part of the way to being able to classify reduction types of hyperelliptic curves. However, note that just because there is a Möbius transformation between $\mathcal{R}$ and $\mathcal{R}'$ this does not mean that C and C' are necessarily isomorphic. It is important to check how Möbius transformations affect the leading coefficients. This is dealt with in Section 6.3.

Roughly, our approach will be to embed open quotient BY trees and their cores as subgraphs of the Bruhat-Tits tree. We will then study the effect that applying Möbius transformations has on the Bruhat-Tits tree. We will prove in Section 5.3 that the data needed to construct the canonical representative is unchanged by Möbius transformations, that is cores are preserved, up to isomorphism, and marked points are integer distances apart. Thus, applying Möbius transformations results in equivalent open quotient BY trees. In practice, for an open quotient BY tree T, we actually embed what looks like $q^{-1}(T)$, but where edge lengths remain unchanged, into the Bruhat-Tits tree. Let us start by defining the Bruhat-Tits tree.

5.2.1 Discs and the Bruhat-Tits Tree

There are several different descriptions of the Bruhat-Tits tree in literature such as in [Bra08, §3] and [Cas14]. Here we are interested in the definition where vertices are taken to be discs and the edge set is given by maximal inclusions.

In [DDMM18], discs were defined to have elements in $\bar{K}$. For our purposes it is enough to instead take elements in a finite extension L/K. So, we define a *disc* of L/K to be a subset

$$D_{z,d} := \{x \in L \mid v_K(x-z) \geq d\},$$

with $z \in L$ and $d \in \mathbb{Q}$. Here d is an invariant of the disc, the *depth*, denoted d_D. If a disc D has depth d_D and $z \in D$ is any element of the disc, then $D = D_{z,d_D}$. For that reason we call any $z \in D$ a *centre* of D. In practice, we will take $L = K(\mathcal{R})$ where $\mathcal{R}$ is a set of roots of a square free polynomial $f \in K[x]$ whose splitting field is tamely ramified.

Definition 5.2.1. A disc is *integral* if it has centre in K and integer depth.

Definition 5.2.2. If $\mathfrak{s} \subseteq \mathcal{R}$ is a proper cluster in $(\mathcal{R}, \Sigma, d)$ then we call the unique smallest disc of $K(\mathcal{R})/K$ cutting out $\mathfrak{s}$ the *defining disc of* $\mathfrak{s}$, denoted $D(\mathfrak{s})$. It is useful to note that, by definition, for any proper cluster $\mathfrak{s}$, the disc $D(\mathfrak{s})$ has depth $d_\mathfrak{s}$ and any root $r \in \mathfrak{s}$ can be taken to be a centre.

Definition 5.2.3. Let $\mathcal{R}$ be the set of roots of a square free polynomial $f(x)$ defined over K with tamely ramified splitting field. We define the Bruhat-Tits tree of $K(\mathcal{R})$ to be the graph whose vertices are discs

$$D = D_{z,d} := \{x \in K(\mathcal{R}) \mid v_K(x-z) \geq d\}$$

with $z \in K(\mathcal{R})$ and $d \in \frac{1}{b}\mathbb{Z}$, where $b = [K(\mathcal{R}) : K]$, and whose edges are given by maximal inclusions. We denote this by $\mathscr{T}_{K(\mathcal{R})}$. We can give $\mathscr{T}_{K(\mathcal{R})}$ a metric by taking the length of an edge between a disc D and a maximal sub-disc $D' \subset D$ to be $d_{D'} - d_D$. In particular this means that every edge of $\mathscr{T}_{K(\mathcal{R})}$ has length $\frac{1}{b}$.

Example 5.2.4. In simple cases it is much easier to visualise the Bruhat-Tits tree. For instance if we look over $\mathbb{Q}_p$ then $\mathscr{T}_{\mathbb{Q}_p}$ is a $p+1$-regular tree. For example, the maximal proper subdiscs of $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p = D_{0,0}$ are precisely the disjoint discs $D_{0,1}, D_{1,1}, \ldots, D_{p-1,1}$. There is also a unique maximal disc in which $\mathbb{Z}_p$ is properly contained, namely $D_{0,-1}$. These observations hold for arbitrary discs $D_{z,d}$, that is there are exactly $p+1$ edges incident to each vertex. It is unfortunately not so straightforward when the residue field is algebraically closed and in general, for our purposes, the Bruhat-Tits tree is not a $p+1$-regular tree. However, this example should help illustrate the general idea.

Remark 5.2.5. The boundary of the Bruhat-Tits tree describes $\mathbb{P}^1(K(\mathcal{R})) = K(\mathcal{R}) \cup \{\infty\}$. One can see this by noting that any infinite descending chain of discs converges to a unique number whose terms in its p-adic expansion are determined by these discs. All strictly increasing sequences of discs differ by finitely many vertices, and are therefore said to be equivalent. Under this equivalence relation, this equivalence class of strictly increasing chains of discs corresponds to a single point, which we call the *point at infinity.*

5.2.2 Visualising Open Quotient BY Trees and Their Cores Using the Bruhat-Tits Tree

Here we describe how we can make a simple construction of a subtree of the Bruhat-Tits tree which allows us to visualise open quotient BY trees and their cores as objects arising directly from $\mathscr{T}_{K(\mathcal{R})}$. First let us discuss the construction for open quotient BY trees, before moving on to look at their cores.

Construction 5.2.6. Let $f \in K[x]$ be a square free polynomial with tamely ramified splitting field with set of roots $\mathcal{R}$ in $\bar{K}$, such that $d_{\mathcal{R}} \geq 0$. Using $\mathcal{R}$ we can construct a tree as a subtree of $\mathscr{T} = \mathscr{T}_{K(\mathcal{R})}$ as follows:

- For every pair of roots $r, r' \in \mathcal{R}$ link r and r' by the unique path between them in $\mathscr{T}$
- link the point at infinity (as defined in Remark 5.2.5) to each root $r \in \mathcal{R}$
- take the vertex set to be all vertices of $\mathscr{T}$ of degree ≥ 3 on these paths
- link vertices by an edge if they are linked by a path in $\mathscr{T}$ and are adjacent (i.e. no other vertex of our tree that is under construction lies on the path between them in $\mathscr{T}$) and let this edge length be equal to the length of the path between them in $\mathscr{T}$.
- additionally adjoin one open edge at the closest vertex to infinity. Note that by construction the closest point, on the union of the vertices and edges defined above, to infinity is unique since $\mathscr{T}$ is a tree, and will be a vertex of our subtree.

- label the point arising from the disc $D_{z_{\mathcal{R}},0}$ by m. This will be a point on the open edge, or the unique vertex incident to the open edge, since $d_{\mathcal{R}} \geq 0$, so $r \in D_{z_{\mathcal{R}},0}$ for all $r \in \mathcal{R}$.

Label the tree constructed from $\mathscr{T}$ in this way $\mathscr{T}(f/K)$. We define a colouring, blue or yellow, on the vertices and edges as follows. Starting at the leaves and working towards the unique vertex $v_{\mathcal{R}}$ incident to the open edge, we can colour the tree in the following way. Colour every leaf blue, and colour the unique edge incident to it blue if there are an odd number of roots in $\mathcal{R}$ contained in the disc and yellow otherwise. Continuing in this way up every branch to $v_{\mathcal{R}}$ we colour a vertex v once every edge and vertex below it has been coloured. We colour v blue if there is a root contained in this disc that is not contained in any of the vertices below it or if there is an edge incident to v that has already been coloured blue, otherwise we colour v yellow. We then colour the unique edge incident to the v that lies on the path between v and $v_{\mathcal{R}}$ (or the open edge if $v = v_{\mathcal{R}}$) blue if there are an odd number of roots contained in it, and yellow otherwise. The genus of a vertex v is defined to be $g(v) = 0$ if v is yellow, and

$$g(v) = \frac{\#\{r \in v \mid D(r \wedge r') \supseteq v \text{ for all } r' \in \mathcal{R}\} + \deg_{\mathscr{T}(f/K)_b}(v) - 2}{2},$$

where $D(r \wedge r')$ is the smallest disc containing both r and r', and $\mathscr{T}(f/K)_b$ is the blue part of $\mathscr{T}(f/K)$. That is, we can calculate the genus of v when v is blue from the number of roots contained in the disc v, and the number of blue edges incident to v.

Example 5.2.7. Consider the polynomial $f(x) = (x-\sqrt{7}+7)(x-\sqrt{7}-7)(x+\sqrt{7}+7)(x+\sqrt{7}-7)((x-1)^3-7^7)(x-8)(x-2)(x-3)$ over $K = \mathbb{Q}_7^{\mathrm{ur}}$. This has set of roots

$$\mathcal{R} = \{\sqrt{7}+7, \sqrt{7}-7, -\sqrt{7}+7, -\sqrt{7}-7, 7^{\frac{7}{3}}+1, \zeta_3 7^{\frac{7}{3}}+1, \zeta_3^2 7^{\frac{7}{3}}+1, 8, 2, 3\},$$

where ζ_3 is a third root of unity. Following Construction 5.2.6 we can construct the subtree $\mathscr{T}(f/k)$ of $\mathscr{T}_{K(\mathcal{R})}$ as pictured in Figure 5.4, where we have also shown the paths to the roots with dashed lines. We can see that the vertices have been taken to be the vertices of $\mathscr{T}$ of degree ≥ 3 on the paths between the roots in $\mathcal{R}$ and the point at infinity. We now continue to follow Construction 5.2.6 to give a colouring to this tree, and genera to its vertices. For example $D_{1,\frac{7}{3}}$ contains 3 roots of $\mathcal{R}$ and is a leaf thus we colour it blue and its unique incident edge blue. Similarly, $D_{\sqrt{7},1}$ and $D_{-\sqrt{7},1}$ are leaves so we

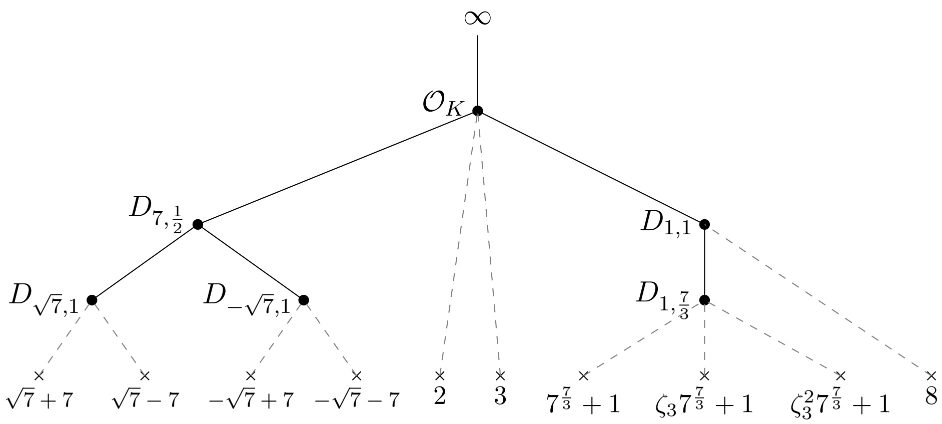


Figure 5.4: A subtree $\mathscr{T}(f/k)$ of $\mathscr{T}$ constructed by considering the paths between the roots in $\mathcal{R}$, and ∞.

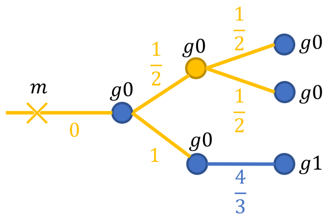


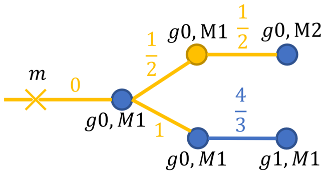
Figure 5.5: $\mathscr{T}(f/K)$ with colouring and genera.

colour them blue and the edges incident to them yellow. We then can colour $D_{7,\frac{1}{2}}$ and the edge between it and $\mathcal{O}_K$. Since $D_{7,\frac{1}{2}}$ does not have any edges incident to it already coloured blue, and no root is a child of it, we colour $D_{7,\frac{1}{2}}$, and the edge to $\mathcal{O}_K$ yellow. We can also assign genera to the vertices. For example $g\left(D_{7,\frac{1}{2}}\right) = 0$ since it is coloured yellow, and $g\left(D_{1,\frac{7}{3}}\right) = \frac{3+1-2}{2} = 1$. Continuing to follow Construction 5.2.6 in this way we give a colouring to all of $\mathscr{T}(f/K)$, and genera to all its vertices, as shown in Figure 5.5.

Before the following construction it is useful to note that, by Lemma B.1 in [DDMM18], if a disc D is fixed by G_K then D has a rational centre, that is there exists some $z \in K$ such that z is a centre of D.

Construction 5.2.8. Let $\mathscr{T}(f/K)$ be as in Construction 5.2.6. Note that the Galois orbits of elements of $\mathcal{R}$ induces orbits on the vertices of $\mathscr{T}_{K(\mathcal{R})}$. So we have a Galois action on the entire Bruhat-Tits tree. Galois preserves $\mathscr{T}(f/K)$, hence acting on it by automorphisms that preserves the colouring and genera. Write $q(\mathscr{T}(f/K))$ for the quotient of $\mathscr{T}(f/K)$, with colouring and genera.

Example 5.2.9. Let f, K, and $\mathscr{T}(f/K)$ be as in Example 5.2.7. Then the quotient of $\mathscr{T}(f/K)$ is

$q(\mathscr{T}(f/K)) =$ 

Note that this is in fact isomorphic to $\underline{T}(\Sigma)$, where Σ is the cluster picture associated to the set of roots of f. This leads us onto the following proposition.

Proposition 5.2.10. *Let $(\mathcal{R}, \Sigma, d)$ be a cluster picture of polynomial type arising from $f \in K[x]$ (square free with tamely ramified splitting field). Then the open quotient BY tree $\underline{T}(\Sigma)$ is isomorphic to the tree $q(\mathscr{T}(f/K))$. This isomorphism preserves colouring, multiplicities of edges and vertices, the genera of vertices, and distances.*

Proof. Denote by S the subtree of $\mathscr{T}$ created by the unique embedded paths between r and r' for all pairs of roots $r, r' \in \mathcal{R}$, as well as the path between ∞ and some $r \in \mathcal{R}$. First, let us show that the vertices of $q(\mathscr{T}(f/K))$ are in one-to-one correspondence with the vertices of T. By definition, it is clear that for a root $r \in \mathcal{R}$, the only discs containing r are those that lie on the infinite path between ∞ and r. Furthermore, again by definition, as we move towards r the discs are smaller. So, for any two roots $r, r' \in \mathcal{R}$ if we let D be the disc on the path between r and r' which is closest to infinity, then D is the smallest disc containing both r and r'. So, certainly a vertex D in S of degree ≥ 3 will correspond to a proper cluster in Σ, that is there exists some proper cluster $\mathfrak{s} \in \Sigma$ such that $D = D(\mathfrak{s})$. It remains to show that every proper cluster in Σ can be seen as a degree ≥ 3 vertex in S in this way.

Let $\mathfrak{s}$ be a proper cluster in Σ. By definition of being proper, any such $\mathfrak{s}$ has at least two children. So, let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be children of $\mathfrak{s}$, where $|\mathfrak{s}_i| \geq 1$. As we discussed in Remark 5.2.5, two roots r_1 and r_2 both lie in a disc if and only if the corresponding terms in their expansions are the same. Roots in $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have distinct next terms, so the path from $D(\mathfrak{s})$ to any root $r_1 \in \mathfrak{s}_1$ must lie on a completely separate branch from the path from $D(\mathfrak{s})$ to any root $r_2 \in \mathfrak{s}_2$. The path to infinity in S from $D(\mathfrak{s})$ consists of discs which contain $D(\mathfrak{s})$, so $D(\mathfrak{s})$ has degree ≥ 3 in S. So the proper clusters of Σ are in one to one correspondence with the vertices of our tree $\mathscr{T}(f/K)$. The action on the vertices of S is precisely the action of G_K on their corresponding clusters. After taking the quotient by the action of G_K we get $q(\mathscr{T}(f/K))$. Namely we have shown that the vertices of $q(\mathscr{T}(f/K))$ are in one-to-one correspondence with orbits of such clusters. So, vertices in T (as defined in Definition D.6 in [DDMM18]) are in one-to-one correspondence with G_K orbits of such degree ≥ 3 discs in S, i.e. vertices of $q(\mathscr{T}(f/K))$.

To see that the edge set is the same and there is an isomorphism is now not hard. By construction, vertices (i.e. discs) in $\mathscr{T}$ are joined by an edge if one is maximally contained in the other. So two discs $D(\mathfrak{s})$ and $D(\mathfrak{s}')$, where $\mathfrak{s}$ and $\mathfrak{s}'$ are proper clusters, are connected by an edge if and only if, either $\mathfrak{s}' < \mathfrak{s}$, $\mathfrak{s} < \mathfrak{s}'$. The lengths of edges equal $\delta(\mathfrak{s}', \mathfrak{s})$ by construction of $\mathscr{T}$. The marked point of $\mathscr{T}(f/K)$ is, by construction, distance $d_{\mathcal{R}}$ along the open edge of $\mathscr{T}(f/K)$. Taking the quotient by the action of G_K as one would expect (constructed explicitly in Construction 5.2.15) yields the result we are looking for.

Finally, now that we have this relation between the vertices of $\mathscr{T}(f/K)$ and specific proper clusters of Σ, it is straightforward to see that $q(\mathscr{T}(f/K))$ has the same colouring and genera as described by Construction 5.1.5. In particular, it is easy to see that, in $\mathscr{T}(f/K)$, a vertex is coloured yellow if it corresponds to an übereven cluster, and blue otherwise. If v' and v are adjacent vertices of $\mathscr{T}(f/K)$ with v' a disc that is contained in the disc v, then they arise from clusters $\mathfrak{s}'$ and $\mathfrak{s}$ respectively with $\mathfrak{s}' < \mathfrak{s}$, and the edge between v and v' is coloured yellow if $\mathfrak{s}'$ is even, and blue otherwise. Finally the open edge of $\mathscr{T}(f/K)$ is coloured yellow if $\mathcal{R}$ is even, and blue otherwise. Similarly, it is easy to then see that the genus of a vertex $v \in V(\mathscr{T}(f/K))$ is the semistable genus of its corresponding cluster. This is because blue edges correspond to proper odd children, with the exception of an additional blue edge incident to v if v corresponds to an odd cluster, and it is clear that the number of singletons is the same. So, after taking the quotient we get that $q(\mathscr{T}(f/K))$ and T are isomorphic and the colouring and genera are preserved. $\square$

Remark 5.2.11. Note that Construction 5.2.8 is similar to the quotient of the open BY tree in Construction 4.2.2. For semistable curves we can think of open quotient BY trees and open BY trees interchangeably as the only difference between these two trees is that the open quotient BY tree has an additional marked point along the open edge, whose distance along the open edge gives us $d_{\mathcal{R}}$. So, for a hyperelliptic curve $C : y^2 = f(x)$ over K with L/K such that C is semistable over L, $\mathscr{T}(f/L)$ can be thought of as the open BY tree associated to $\Sigma(C/L)$. We constructed a quotient of $\mathscr{T}(f/L)$ in Construction 4.2.2. We can also note that $\mathscr{T}(f/L)$ is isomorphic to $\mathscr{T}(f/K)$ but all lengths of edges have been scaled by $[L : K]$ since $\mathscr{T}(f/L)$ takes its lengths from the normalised valuation over L, v_L, whereas $\mathscr{T}(f/K)$ takes its lengths from the normalised valuation over K, v_K. From noting this it is clear that the quotient we construct in Construction 5.2.8 gives a tree which is isomorphic to the open quotient BY tree of $\Sigma(C/K)$.

We can prove a very similar result for the cores of open quotient BY trees. Let us first make some similar constructions.

Construction 5.2.12. Let $f \in K[x]$ be a square free polynomial with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$, such that $d_{\mathcal{R}} \geq 0$. Using $\mathcal{R}$ we can construct a tree from $\mathscr{T} = \mathscr{T}_{K(\mathcal{R})}$ as follows:

- For every pair of roots $r, r' \in \mathcal{R}$ link r and r' by the unique path between them in $\mathscr{T}$;
- If $\mathcal{R}$ is odd link the point at infinity (as defined in Remark 5.2.5) to each root $r \in \mathcal{R}$ by the unique path between them in $\mathscr{T}$;
- Take the vertex set to be all vertices of $\mathscr{T}$ of degree ≥ 3 on these paths;
- Link vertices by an edge if they are linked by a path in $\mathscr{T}$ and no other vertex lies on this path. Let this edge length be equal to the length of the path between them in $\mathscr{T}$.

Label the tree constructed from $\mathscr{T}$ in this way $\tilde{\mathscr{T}}(f/K)$. Furthermore, note that $\tilde{\mathscr{T}}(f/K)$ is a subtree of $\mathscr{T}(f/K)$, so the same colouring, blue or yellow, can be given to all vertices and edges, and the same genera can be given to every vertex.

Remark 5.2.13. Note that an alternative way of viewing this colouring on $\tilde{\mathscr{T}}(f/K)$ is to note that an edge will be coloured blue if there are an odd number of elements in $\mathcal{R}^+$ on either side of the edge, and yellow if there are an even number on either side of the edge, where

$$\mathcal{R}^+ = \begin{cases} \mathcal{R} & \text{if } \mathcal{R} \text{ is even,} \\ \mathcal{R} \cup \{\infty\} & \text{if } \mathcal{R} \text{ is odd.} \end{cases}$$

Note that $\mathcal{R}^+$ will always be even, so we only need check the number of elements of $\mathcal{R}^+$ lying on one side as the number on the other side will have the same parity. Furthermore, after colouring all edges in this manner, if all edges surrounding a vertex are coloured yellow and the vertex is not the closest vertex to any element of $\mathcal{R}^+$ then the vertex gets coloured yellow. Otherwise a vertex is coloured blue.

The genera can then be read off the colouring of the edges, vertices, and the number of roots for which any given vertex is the closest vertex.

In fact, it is helpful to note that all of the above is completely determined by the number of elements of $\mathcal{R}^+$ each vertex is the closest vertex to.

Example 5.2.14. Consider the polynomial $f(x) = (x-1)(x^3-7^5)(x-7-7^2)(x-7+7^2)$ over $K = \mathbb{Q}_7^{\mathrm{ur}}$. Let ζ_3 be a third root of unity. Following Construction 5.2.6 we can construct the subtree $\tilde{\mathscr{T}}(f/k)$ of $\mathscr{T}_{K(\mathcal{R})}$ as pictured in Figure 5.4, where we have also shown the paths to the roots with dashed lines and the disc $\mathcal{O}_K$. Note that in this example $D(\mathcal{R}) = \mathcal{O}_K$, however this

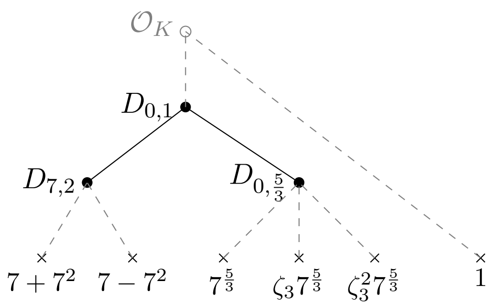


Figure 5.6: The subtree $\tilde{\mathscr{T}}(f/k)$ of $\mathscr{T}$ constructed by considering the paths between the roots in $\mathcal{R}^+$.

only has degree 2 on the union of all the paths between elements of $\mathcal{R}^+$ so is not a vertex of $\tilde{\mathscr{T}}(f/K)$. We can then give a colouring to all of $\tilde{\mathscr{T}}(f/K)$, and genera to all its vertices, as shown in Figure 5.7.

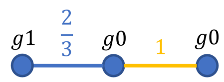


Figure 5.7: $\tilde{\mathscr{T}}(f/K)$ with colouring and genera.

In order to view the core of an open quotient BY tree as arising from the Bruhat-Tits tree we construct a quotient in a very similar way to Construction 5.2.8. The only slight complication is that one minor adjustment is sometimes needed to deal with the vertex set in the case where $\Sigma(C/K)$ is a union of two clusters that are swapped by Galois. This is due to a modification in the vertex set of cores of open quotient BY trees in comparison to cores of open BY trees, as discussed in Remark 4.3.8, to ensure that cores of open quotient BY trees always have a vertex of multiplicity 1.

Construction 5.2.15 ($q(\tilde{\mathscr{T}}(f/K))$)**.** Let $(\mathcal{R}, \Sigma, d)$ be a cluster picture of polynomial type arising from $f \in K[x]$ (square free with tamely ramified splitting field), and let $\tilde{\mathscr{T}}(f/K)$ be as defined in Construction 5.2.12. As in Construction 5.2.8, we have an action of $\mathrm{Gal}(K(\mathcal{R})/K)$ on the vertices of $\mathscr{T}_{K(\mathcal{R})}$. Note that this action is the same as the action on clusters, i.e. if $\mathfrak{s}, \mathfrak{s}' \in \Sigma$ are in the same Galois orbit, then so are the vertices $D(\mathfrak{s})$ and $D(\mathfrak{s}')$ of $\mathscr{T}_{K(\mathcal{R})}$. We can use this action to define a quotient of $\tilde{\mathscr{T}}(f/K)$, in the way one would expect, which we will denote $q(\tilde{\mathscr{T}}(f/K))$.

In particular, we define the vertex and edge sets of $q(\tilde{\mathscr{T}}(f/K))$ as follows:

- For every Galois orbit of vertices X in $V(\tilde{\mathscr{T}}(f/K))$ we define one vertex v_X of $q(\tilde{\mathscr{T}}(f/K))$, and write $q(v) = v_X$ for all $v \in X$.

- If X and X' are two distinct Galois orbits of vertices of $\tilde{\mathscr{T}}(f/K)$ and there exist vertices $v, v' \in V(\tilde{\mathscr{T}}(f/K))$ in X and X' respectively, such that $[v, v'] \in E(\tilde{\mathscr{T}}(f/K))$, then we define an edge $[v_X, v_{X'}]$ of $q(\tilde{\mathscr{T}}(f/K))$, and write $q([v, v']) = [v_X, v_{X'}]$ for all $[v, v'] \in E(\tilde{\mathscr{T}}(f/K))$ with $v \in X$ and $v' \in X'$.

- If there exist two vertices $v, v' \in V(\tilde{\mathscr{T}}(f/K))$ with v and v' in the same Galois orbit X, and $e = [v, v'] \in E(\tilde{\mathscr{T}}(f/K))$, then we must have that $X = \{v, v'\}$ is their Galois orbit. Label the midpoint of e, $m(e)$. Add one additional vertex to $V(q(\tilde{\mathscr{T}}(f/K)))$ and label it $v_{m(e)}$. Then add an edge to $q(\tilde{\mathscr{T}}(f/K))$ between v_X and $v_{m(e)}$. Write $q(m(e)) = v_{m(e)}$.

All other points on $\tilde{\mathscr{T}}(f/K)$ are mapped by q as one would expect.

Define a multiplicity function $M : V(q(\tilde{\mathscr{T}}(f/K))) \sqcup E(q(\tilde{\mathscr{T}}(f/K))) \to \mathbb{Z}_{>0}$ to be the number of vertices or edges in the preimage of q, except in the case when v and v' are in an orbit and there is an edge e between them. In this case we define $M([m(e), v_X]) = 2$ and $M(v_{m(e)}) = 1$. That $M(v_X) = 2$ in this exceptional case follows from above.

We define genera of vertices $v \in V(q(\tilde{\mathscr{T}}(f/K)))$ to be equal to the genera of vertices in the preimage, i.e. if $v' \in V(\tilde{\mathscr{T}}(f/K)))$ is such that $q(v') = v$ then we define $g(v) = g(v')$. Note that since all vertices in the preimage of v are in a Galois orbit, they must have equal genera, so this is well defined. In the exceptional case that $v \in V(q(\tilde{\mathscr{T}}(f/K)))$ is such that $v = v_{m(e)}$ for some $e \in E(\tilde{\mathscr{T}}(f/K))$, we define $g(v) = 0$.

Finally, we can colour edges and vertices in $q(\tilde{\mathscr{T}}(f/K))$ according to the colouring of their preimage, where if $v \in V(q(\tilde{\mathscr{T}}(f/K)))$ is such that $v = v_{m(e)}$ for some $e \in E(\tilde{\mathscr{T}}(f/K))$ we colour v the same colour as e. Again, this is well defined.

Construction 5.2.16. We can define a metric on $q(\tilde{\mathscr{T}}(f/K))$ as follows. If P, P' are points on $q(\tilde{\mathscr{T}}(f/K))$ then we define $d(P, P') = \min\{d(Q, Q') \mid Q \in q^{-1}(P),\ Q' \in q^{-1}(P')\}$. Recall that an edge $e \in E(q(\tilde{\mathscr{T}}(f/K)))$ is either of the form $e = [q(v), q(v')]$ where $e' = [v, v'] \in E(\tilde{\mathscr{T}}(f/K))$ (in which case we write

$e = q(e'))$, or of the form $e = [q(v), q(m(e'))]$ where $m(e')$ is the mid point of an edge $e' = [v, v'] \in E(\tilde{\mathscr{T}}(f/K))$ where v and v' are in the same Galois orbit (in which case we write $e = q([v, m(e')]) = q([v', m(e')]))$. Note that this gives us that the length $l(e)$ of an edge $e \in E(q(\tilde{\mathscr{T}}(f/K)))$ is $l(e) = l(e')$ if $e = q(e')$, or $l(e) = \frac{l(e')}{2}$ if $e = q([v, m(e')]) = q([v', m(e')])$, where $e' = [v, v']$, and v and v' are in the same Galois orbit.

Example 5.2.17. Consider the polynomial $f(x) = x^6 - 21x^4 - 1911x^2 - 23667$ over $\mathbb{Q}_7^{\mathrm{ur}}$, with roots $\zeta_6^i 7^{\frac{5}{6}} + 7^{\frac{1}{2}}$ and $\zeta_6^j 7^{\frac{5}{6}} - 7^{\frac{1}{2}}$ for $i = 0, 2, 4$, $j = 1, 3, 5$ and ζ_3 a third root of unity. We can see how $\tilde{\mathscr{T}}(f/K)$ is constructed in Figure 5.8. Note that $D(\mathcal{R}) = D_{0,\frac{1}{2}}$ only has degree 2 on the union of all the paths

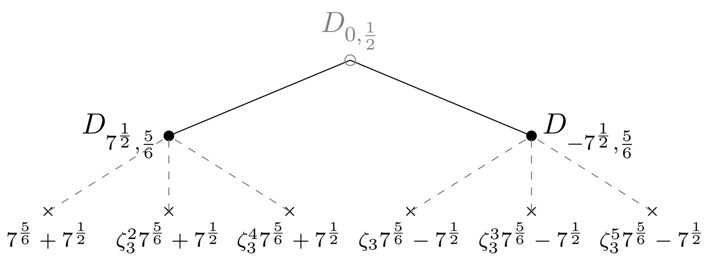


Figure 5.8: The subtree $\tilde{\mathscr{T}}(f/k)$ of $\mathscr{T}$ constructed by considering the paths between the roots in $\mathcal{R}^+$.

between elements of $\mathcal{R}^+ = \mathcal{R}$ so is not a vertex of $\tilde{\mathscr{T}}(f/K)$. We can then colour $\tilde{\mathscr{T}}(f/K)$, and assign genera to its vertices, as shown in Figure 5.9. Taking the

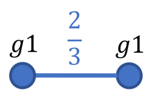


Figure 5.9: $\tilde{\mathscr{T}}(f/K)$ with colouring and genera.

quotient of $\tilde{\mathscr{T}}(f/K)$ as described in Construction 5.2.15 to obtain $q(\tilde{\mathscr{T}}(f/K))$, and giving $q(\tilde{\mathscr{T}}(f/K))$ the metric described in Construction 5.2.16 gives the tree shown in Figure 5.10.

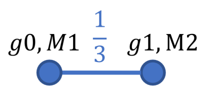


Figure 5.10: $q(\tilde{\mathscr{T}}(f/K))$ with colouring and genera.

Theorem 5.2.18. *Let $f \in K[k]$ be a square free polynomial with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$, with $d_{\mathcal{R}} \geq 0$. Let $\tilde{T}$ be the core of the of the open quotient BY tree $T = \underline{T}(\Sigma)$. Then $q(\tilde{\mathscr{T}}(f/K))$, as defined in Construction 5.2.15, is isomorphic to $\tilde{T}$. Distances, genera, multiplicities and colouring are preserved under this isomorphism.*

Proof. The approach here is very similar to that of Proposition 5.2.10. Denote by S the subtree of $\mathscr{T}$ created by the unique embedded paths between r and r' for all pairs of roots $r, r' \in \mathcal{R}$, as well as the path between ∞ and some $r \in \mathcal{R}$ if $\mathcal{R}$ is odd. First, let us show that the vertices of $q(\tilde{\mathscr{T}}(f/K))$ are in one-to-one correspondence with the vertices of $\tilde{T}$. By definition, it is clear that for a root $r \in \mathcal{R}$, the only discs containing r are those that lie on the infinite path between ∞ and r. Furthermore, again by definition, as we move towards r the discs are smaller. So, for any two roots $r, r' \in \mathcal{R}$ if we let D be the disc on the path between r and r' which is closest to infinity, then D is the smallest disc containing both r and r'. So, certainly a vertex D in S of degree ≥ 3 will correspond to a proper cluster in Σ, that is there exists some proper cluster $\mathfrak{s} \in \Sigma$ such that $D = D(\mathfrak{s})$. However, not every proper cluster in Σ can be seen as a degree ≥ 3 vertex in S in this way. We need to check which proper clusters these degree ≥ 3 vertices in S correspond to.

Suppose that $\mathcal{R}$ is even. Then if a vertex $D(\mathfrak{s})$ has degree ≥ 3 in S then there are at least two roots lying in the cluster $\mathfrak{s}$ (since the degree indicates that $\mathfrak{s}$ has $\deg(v) - 1$ proper children). Note that if $\mathcal{R}$ has a child $\mathfrak{s}$ of size $2g+1$, then there exists a root $r \in \mathcal{R}$ such that $\mathcal{R} = \mathfrak{s} \sqcup \{r\}$. So, for any root $r' \in \mathfrak{s}$ the path from $D(\mathcal{R})$ to r' passes through $D(\mathfrak{s})$, and has no common discs other than $D(\mathcal{R})$ with the path from $D(\mathcal{R})$ to r. That is, $D(\mathcal{R})$ has degree 2 in S. Similarly, if $\mathcal{R}$ is a union of two proper clusters then $D(\mathcal{R})$ has degree 2 in S since there exist children $\mathfrak{s}, \mathfrak{s}' < \mathcal{R}$ with all roots lying in either $\mathfrak{s}$ or $\mathfrak{s}'$. It remains to show that for all other proper clusters $\mathfrak{s} \in \Sigma$, $D(\mathfrak{s})$ has degree 3 in S. By definition of being proper, any such $\mathfrak{s}$ has at least two children. So, let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be children of $\mathfrak{s}$, where $|\mathfrak{s}_i| \geq 1$. As we discussed in Remark 5.2.5, two roots r_1 and r_2 both lie in a disc if and only if the corresponding terms in their expansions are the same. Roots in $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have distinct next terms, so the path from $D(\mathfrak{s})$ to any root $r_1 \in \mathfrak{s}_1$ must lie on a completely separate branch from the path from $D(\mathfrak{s})$ to any root $r_2 \in \mathfrak{s}_2$. Note that, since either $\mathfrak{s} \neq \mathcal{R}$ or if $\mathfrak{s} = \mathcal{R}$ then $\mathfrak{s}$ is not a union of two clusters, there exists a root $r \in \mathcal{R} \setminus \mathfrak{s}_1 \cup \mathfrak{s}_2$. Again, r will have a different expansion to roots in $\mathfrak{s}_1 \cup \mathfrak{s}_2$, regardless of whether $r \in \mathfrak{s}$ or $r \notin \mathfrak{s}$. So, $D(\mathfrak{s})$ has degree ≥ 3 in S, and $D(\mathfrak{s})$ is a vertex of $\tilde{\mathscr{T}}(f/K)$. The G_K-action on the vertices of S is precisely the action on their corresponding clusters. After taking the quotient by the action of G_K we get $q(\tilde{\mathscr{T}}(f/K))$. Namely we have shown that the vertices of $q(\tilde{\mathscr{T}}(f/K))$ are in one-to-one correspondence with orbits of such clusters. So, vertices in $\tilde{T}$ (as defined in Definition D.6 in [DDMM18]) are in one-to-one correspondence with G_K orbits of such degree ≥ 3 discs in S, i.e. vertices of $q(\tilde{\mathscr{T}}(f/K))$.

Suppose instead that $\mathcal{R}$ is odd. Note that for any root $r \in \mathcal{R}$ the path between r and ∞ passes through $D(\mathcal{R})$. Furthermore $\mathcal{R}$ has at least 2 children, so as above there are at least two distinct branches from $D(\mathcal{R})$ to roots in $\mathcal{R}$. So, $D(\mathcal{R})$ always has at least degree 3 with one edge incident to it on the path to infinity and at least two edges incident on paths to roots. All other proper clusters in Σ correspond to vertices of S of degree ≥ 3 just as in the proof of the $\mathcal{R}$ even case. So, by the same justification as in the $\mathcal{R}$ even case, when $\mathcal{R}$ is odd vertices of $q(\tilde{\mathscr{T}}(f/K))$ are in one to one correspondence with G_K orbits of such proper clusters in Σ, which, by Definition D.6 in [DDMM18]), are in one to one correspondence with vertices of $\tilde{T}$.

To see that the edge set is the same and there is an isomorphism is now not hard. By construction, vertices (i.e. discs) in $\mathscr{T}$ are joined by an edge if one is maximally contained in the other. So two discs $D(\mathfrak{s})$ and $D(\mathfrak{s}')$, where $\mathfrak{s}$ and $\mathfrak{s}'$ are proper clusters (where if either $\mathfrak{s}$ or $\mathfrak{s}'$ equals $\mathcal{R}$ then it is not a union of two clusters), are connected by an edge if and only if either $\mathfrak{s}' < \mathfrak{s}$, or $\mathfrak{s} < \mathfrak{s}'$, or $\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$ with $\mathcal{R}$ even. The lengths of edges equal $\delta(\mathfrak{s}', \mathfrak{s})$ by construction of $\mathscr{T}$. Taking the quotient by the action of G_K as one would expect (constructed explicitly in Construction 5.2.15) yields the result we are looking for and gives us an isomorphism.

Finally, we can use Proposition 5.2.10 to give us that the colouring and genera are preserved by this isomorphism. $\square$

Example 5.2.19. Let f/K be as in Example 5.2.17. The open quotient BY tree T associated to f is as follows:

$T =$ 

Indeed, $q(\tilde{\mathscr{T}}(f/K))$, shown in Figure 5.10, is isomorphic to the core of T.

Notation 5.2.20. Following Theorem 5.2.18, it now makes sense to use the notation $q^{-1}(\tilde{T}) = \tilde{\mathscr{T}}(f/K)$, after noting that $q^{-1}(q(\tilde{\mathscr{T}}(f/K))) \cong \tilde{\mathscr{T}}(f/K)$, where q^{-1} is given explicitly in Construction 4.3.12. We will also introduce similar notation for T, namely we will write $q^{-1}(T) = \mathscr{T}(f/K)$, taking $q^{-1}(T)$ as given explicitly in Construction 4.2.2. Note that, as mentioned in Remark 4.2.4, when referring to $q^{-1}(T)$ and $q^{-1}(\tilde{T})$ we are now referring to these as metric quotient BY trees. In particular, for an edge $e \in E(T)$ we define the length of each $q^{-1}(e)_i$ to be

$$l(q^{-1}(e)_i) = l(e)$$

Similarly in $\tilde{T}$, except for an exceptional edge between $q^{-1}(v')_1$ and $q^{-1}(v')_2$, which we give length $2l(e)$, where e is the edge between v' and the exceptional vertex in $\tilde{T}$ which gives rise to this edge.

Later, in Section 6.1 we will see a different metric on $q^{-1}(T)$ and $q^{-1}(\tilde{T})$.

Remark 5.2.21. Note that we are only able to view T itself on $\mathscr{T}_{K(\mathcal{R})}$ if every vertex of T has multiplicity 1, that is $T^1 = T$. Note that this is the case if and only if Σ has no non-trivial Galois orbits of proper clusters. Likewise, we can only view $\tilde{T}$ itself on $\mathscr{T}_{K(\mathcal{R})}$ if every vertex of $\tilde{T}$ has multiplicity 1, that is $\tilde{T}^1 = \tilde{T}$. Again, since only multiplicity 1 vertices and edges are ever removed when passing from T to the core $\tilde{T}$, this is the case if and only if Σ has no non-trivial Galois orbits of proper clusters.

Proposition 5.2.22. *Let $f \in K[x]$ be a square free polynomial with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$. Let Σ be the cluster picture and $T = \underline{T}(\Sigma)$ the associated open quotient BY tree. Label the marked point of T by m. Let B be as in Construction 4.5.1. Then the discs of the form $D_{z,d}$ with $z \in K$ and $d \in \mathbb{Z}$ on $\mathscr{T}_{K(\mathcal{R})}$ correspond to points m' on B which are integer distance from m.*

Proof. By Theorem 5.2.10, $q(\mathscr{T}(f/K))$ is isomorphic to T. So, we can think of $q(\mathscr{T}(f/K))^1$ as being T^1, thus lying on B. When we "undo the quotient" we get $\mathscr{T}(f/K)$, lying on the Bruhat-Tit tree $\mathscr{T}_{K(\mathcal{R})}$. As $q(\mathscr{T}(f/K))^1$ is the multiplicity 1 component of $q(\mathscr{T}(f/K))$ we have an isomorphism between $q(\mathscr{T}(f/K))^1$ and $\mathscr{T}(f/K)^1$. Similarly, since B has only multiplicity 1 components we can picture B as lying on part of the Bruhat-Tits tree which is Galois invariant. By construction of $\mathscr{T}(f/K)$, the marked point arises from a shift of $D_{z_{\mathcal{R}},0}$ by $z_{\mathcal{R}} \in K$, i.e. $D_{z_{\mathcal{R}},0}$. Thus any point m', an integer distance from m on B, corresponds to a point on $\mathscr{T}_{K(\mathcal{R})}$ which is fixed by G_K and integer distance from $D_{z_{\mathcal{R}},0}$. Certainly any disc $D_{z,d}$ with $z \in K$ and $d \in \mathbb{Z}$ will be integer distance from $D_{z_{\mathcal{R}},0}$ in $\mathscr{T}$. Conversely if a disc is integer distance from $D_{z_{\mathcal{R}},0}$ then it must have an integer depth, and by [DDMM18, Lemma B.1], any disc which is fixed by G_K has a centre in K. So, any point on B which is an integer distance from $D_{z_{\mathcal{R}},0}$ when viewed on the Bruhat-Tits tree must be an integral disc. Therefore, the discs $D_{z,d}$ with $z \in K$ and $d \in \mathbb{Z}$ correspond to such points m'. □

These result allow us to work with the Bruhat-Tits tree in place of quotient BY trees and points which are integer distance from the marked point. As such, we will work explicitly with discs in the following section.

5.3 Invariance under Möbius Transformations

Our aim is to classify reduction types of tame hyperelliptic curves. To do this we want to show that the canonical representative of the equivalence class of open quotient BY trees is model invariant. That is, choosing a different model for a hyperelliptic curve does not change the canonical representative.

To do this we need to study the effect of Möbius transformations on open quotient BY trees. In this section we will verify that applying a Möbius transformation results in an equivalent open quotient BY tree. In Section 5.4, we will show that given any cluster picture Σ of polynomial type arising from $f \in K[x]$, and any $T' \sim \underline{T}(\Sigma)$, there exists an $f' \in K[x]$ isomorphic to f (obtained by applying a Möbius transformation) with open quotient BY tree isomorphic to T'. Recall that we gave a complete description of the equivalence class of open quotient BY trees in 4.6.4. We are now able to study open quotient BY trees via the Bruhat-Tits tree. So, let us now investigate how Möbius transformations act on the discs that form the vertices of $\mathscr{T}_{K(\mathcal{R})}$.

Let ϕ be a Möbius transformation defined over K, that is

$$\phi(z) = \frac{az+b}{cz+d}, \quad \text{with } a, b, c, d \in K,$$

where $ad - bc \neq 0$. Then ϕ can be expressed as a composition of simple transformations, namely shifts, scalings, or inversions. These simple transformations are themselves Möbius transformations, in particular they are as follows:

- Scaling: $\phi(z) = az$,
- Shift: $\phi(z) = z + b$,
- Inversion: $\phi(z) = \frac{1}{z}$.

As such, it is enough for us to discuss what effect applying these simple Möbius transformations has on the Bruhat-Tits tree $\mathscr{T}_K(\mathcal{R})$, and as a result on cores and canonical representatives of open quotient BY trees. We will discuss each of these simple transformations separately.

5.3.1 Scaling

Here we consider the effect of Möbius transformations of the form

$$\phi(z) = az \quad 0 \neq a \in K.$$

Let $f \in K[x]$ be square free with tamely ramified splitting field, set of roots $\mathcal{R}$, cluster picture Σ, and open quotient BY tree $T = \underline{T}(\Sigma(f/K))$ with marked point m. Firstly, it is important to note that scaling all roots in $\mathcal{R}$ by an integer n, that is applying $\phi(z) = \pi^n z$ to $\mathcal{R}$, simply increases the depth of all proper clusters in Σ by n. So, as briefly mentioned at the start of Section 4.1, without loss of generality, we can assume that $d_{\mathcal{R}} \geq 0$. In particular, for a hyperelliptic curve C/K this translates into us assuming that we have chosen a model for C with $d_{\mathcal{R}} \geq 0$.

Lemma 5.3.1. *Consider a Möbius transformation $\phi(z) = az$, defined over K, with $0 \neq a \in K$. Let $f \in K[x]$ be square free with tamely ramified splitting field, and set of roots $\mathcal{R}$. Let $f' \in K[x]$ have set of roots $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\}$. Then $\tilde{\mathscr{T}}(f/K) \cong \tilde{\mathscr{T}}(f'/K)$ and the genera and colouring are preserved by this isomorphism.*

Proof. Every non-zero element of K can be written as $\pi^n u$ where $n \in \mathbb{Z}$ and u is a unit. It is clear that scaling by a unit simply rotates discs around, preserving adjacency. So, assume that $a = \pi^n$ for some $n \in \mathbb{Z}$. It is clear that ϕ is an isomorphism of $\mathscr{T}_{K(\mathcal{R})}$ and for any $\alpha, \beta \in \mathbb{P}^1(K(\mathcal{R}))$ the line between α and β in $\mathscr{T}_{K(\mathcal{R})}$ is mapped entirely to the line between $\pi^n\alpha$ and $\pi^n\beta$. Since $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\}$, regardless of the parity of $\mathcal{R}$, we include the path to ∞ in construction of $\tilde{\mathscr{T}}(f'/K)$ from $\mathscr{T}_{K(\mathcal{R})}$ if and only if we did so for $\tilde{\mathscr{T}}(f/K)$. The point of $\tilde{\mathscr{T}}(f/K)$ which is closest to ∞ is mapped to the point of $\tilde{\mathscr{T}}(f'/K)$ which is closest to infinity. Distances are clearly preserved so $\tilde{\mathscr{T}}(f/K) \cong \tilde{\mathscr{T}}(f'/K)$.

Furthermore, a disc forming a vertex of $\tilde{\mathscr{T}}(f/K)$ which is closest to an element $\alpha \in \mathbb{P}^1(K(\mathcal{R}))$ is mapped to the vertex of $\tilde{\mathscr{T}}(f'/K)$ which is closest to $\phi(\alpha)$. So, by Remark 5.2.13, the colouring and genera and preserved. □

5.3.2 Shift

Now let us consider the effect of Möbius transformations of the form

$$\phi(z) = z + b, \quad 0 \neq b \in K.$$

This case can be proved similarly to the case of scaling. That is, a shift is an isomorphism on $\mathscr{T}_{K(\mathcal{R})}$, and preserves distances and Galois orbits of roots.

Lemma 5.3.2. *Consider the Möbius transformation $\phi(z) = z+b$, defined over K, with $0 \neq b \in K$. Let $f \in K[x]$ be square free with tamely ramified splitting field, and set of roots $\mathcal{R}$. Let $f' \in K[x]$ have set of roots $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\}$.*

Then $\tilde{\mathscr{T}}(f/K) \cong \tilde{\mathscr{T}}(f'/K)$ and the genera and colouring are preserved by this isomorphism.

Proof. Let $f' \in K[x]$ have set of roots $\mathcal{R}'$ and cluster picture Σ', with associated open quotient BY tree T'. The proof that $\tilde{\mathscr{T}}(f/K)$ and $\tilde{\mathscr{T}}(f'/K)$ are isomorphic, and the colouring and genera are preserves follows similarly to when we applied a scaling above in Lemma 5.3.1. □

5.3.3 Inversion

Note that if we apply $z \mapsto \frac{1}{z}$ to a disc D which contains 0, its image is not a disc since elements of D closer to 0 get mapped closer to ∞. We are therefore unable to apply this Möbius transformation directly to the vertices of the Bruhat-Tits tree $\mathscr{T}_{K(\mathcal{R})}$ as we did for scalings and shifts. It would however, be useful if we were able to do something like this. So, we make a slight modification to how we think of vertices of $\mathscr{T}_{K(\mathcal{R})}$. This will enable us to apply $z \mapsto \frac{1}{z}$ directly to vertices. It is for this reason that dealing with inversion will take significantly more work than was required for scalings and shifts. If a disc D contains 0 then we can take 0 as a centre and write

$$D = \{x \in K(\mathcal{R}) \mid v_K(x) \geq d_D\}.$$

Applying $z \mapsto \frac{1}{z}$ gives the set $\{x \in K(\mathcal{R}) \mid v_K(x) \leq -d_D\}$, which is not a disc. If instead we associate an annulus A to D defined by

$$A = A_{0,d_D} = \{x \in K(\mathcal{R}) \mid v_K(x) = d_D\},$$

then we can redefine $\mathscr{T}$ to be the tree with these annuli in the place of any disc with 0 as a centre. We say that A_{0,d_D} is an annulus of radius d_D centred at 0. It is then easy to see that any such annulus A is mapped to another annulus

$$A^{-1} = A_{0,-d_D} = \{y \in K(\mathcal{R}) \mid v_K(y) = -d_D\}.$$

Note that the discs that we have replaced with annuli are precisely the vertices between 0 and ∞ in $\mathscr{T}_{K(\mathcal{R})}$. So, annuli are mapped to annuli and it remains to check that all remaining discs get mapped to discs under $z \mapsto \frac{1}{z}$. Let D be a disc with centre $\alpha \in K(\mathcal{R})$, radius d_D, and suppose that $0 \notin D$. We must have $d_D > v_K(\alpha)$, else 0 would lie in D. So,

$$D = \{x \in K(\mathcal{R}) \mid v_K(x - \alpha) \geq d_D\},$$

and any element $x \in D$ can be written as $x = \alpha + p^r u$ where $u \in \mathcal{O}_K$ and $r \geq d_D$. If we invert an element x of D we get

$$\begin{aligned}\frac{1}{x} &= \frac{1}{\alpha + p^r u},\\ &= \frac{1}{\alpha}\frac{1}{1+p^r u\alpha^{-1}},\\ &= \alpha^{-1}(1+p^r u\alpha^{-1})^{-1},\\ &= \alpha^{-1}(1-p^r u\alpha^{-1}+\dots),\\ &= \alpha^{-1} - p^r u\alpha^{-2}+\dots,\end{aligned}$$

where we have replaced $(1+p^r u\alpha^{-1})^{-1}$ by its binomial expansion. So,

$$v\left(\frac{1}{x}-\frac{1}{\alpha}\right) = v(-p^r u\alpha^{-2}) = r - 2v(\alpha) + v(u) \geq d_D - 2v(\alpha).$$

The converse is also true, and proved similarly. We can conclude from this that $x \in D$ if and only if $\frac{1}{x} \in D^{-1}$, where D^{-1} is the disc

$$D^{-1} = \{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq d_D - 2v(\alpha)\}.$$

We will talk about both ways of viewing the vertices of $\mathscr{T}_{K(\mathcal{R})}$ interchangeably, but will often refer to this alternative description using annuli as the *modified Bruhat-Tits tree*. With this new way of viewing of the Bruhat-Tits tree we can safely apply $z \mapsto \frac{1}{z}$ to vertices of $\mathscr{T}_{K(\mathcal{R})}$.

Let $f \in K[x]$ be square free with tamely ramified splitting field, set of roots $\mathcal{R}$, cluster picture Σ, and open quotient BY tree $T = \underline{T}(\Sigma(f/K))$ with marked point m. Recall that, by Construction 5.2.12, the vertices of $\tilde{\mathscr{T}}(f/K)$ are precisely the meeting points of triples of distinct elements of $\mathcal{R}$ if $\mathcal{R}$ is even, or of $\mathcal{R} \cup \{\infty\}$ if $\mathcal{R}$ is odd. In order to prove that the Möbius map $\phi : z \mapsto \frac{1}{z}$ preserves the core $\tilde{\mathscr{T}}(f/K)$ we will prove the following:

- Adjacent vertices of $\tilde{\mathscr{T}}(f/K)$ are mapped to adjacent vertices by ϕ. Moreover distances between vertices are preserved.
- For $r \in \mathcal{R}$ any disc sufficiently close to r is mapped to a disc close to $\frac{1}{r}$.
- If r and r' are two distinct roots in $\mathcal{R}$ then the unique path between them in $\mathscr{T}_{K(\mathcal{R})}$ is mapped to the unique path between $\frac{1}{r}$ and $\frac{1}{r'}$.
- Consequently, for any three roots α, β, γ the unique triple intersection point between them in $\mathscr{T}_{K(\mathcal{R})}$ is mapped to the unique triple intersection

point of $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$.

- Finally, we will conclude that the Möbius map $z \mapsto \frac{1}{z}$ is an isomorphism on $\tilde{\mathscr{T}}(f/K)$ which preserves genera, colouring and Galois orbits.

Lemma 5.3.3. *Let v_1 and v_2 be two vertices of the modified Bruhat-Tits tree described above. That is, they are either a disc of the form $\{x \in K(\mathcal{R}) \mid v(x-\alpha) \geq n\}$, where $\alpha \neq 0$ and $n > v(\alpha)$, or an annulus of the form $\{x \in K(\mathcal{R}) \mid v(x) = n\}$. Then v_1 and v_2 are adjacent in $\mathscr{T}_{K(\mathcal{R})}$ if and only if their images under $\phi : z \mapsto \frac{1}{z}$ are also adjacent.*

Proof. Suppose that v_1 and v_2 are both discs, D_1 and D_2 respectively. Since they are adjacent, one must be maximally contained in the other, so without loss of generality we can assume that D_2 is maximally contained in D_1. Let α be a centre of D_2, then we can also choose α to be a centre of D_1 since $\alpha \in D_2 \subset D_1$. Since D_1 and D_2 are adjacent, we must have that $d_{D_1} = d_{D_2} - \frac{1}{b}$, where $b = [K(\mathcal{R}) : K]$. So, if we write $d_{D_1} = n$, we have

$$D_1 = \{x \in \bar{K} \mid v(x-\alpha) \geq n\},$$
$$D_2 = \left\{x \in \bar{K} \mid v(x-\alpha) \geq n + \frac{1}{b}\right\},$$

and since $0 \notin D_1, D_2$ we have that $n > v(\alpha)$. Write

$$D_1^{-1} = \left\{y \in K(\mathcal{R}) \mid \frac{1}{y} \in D_1\right\}, \text{ and } D_2^{-1} = \left\{y \in K(\mathcal{R}) \mid \frac{1}{y} \in D_2\right\}.$$

We want to show that D_1^{-1} and D_2^{-1} are both discs and one is maximally contained in the other. We have already shown above that

$$D_1^{-1} = \left\{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq d_{D_1} - 2v(\alpha)\right\},$$
$$D_2^{-1} = \left\{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq d_{D_2} - 2v(\alpha)\right\}.$$

So, both D_1^{-1} and D_2^{-1} have $\frac{1}{\alpha}$ as a centre, and $d_{D_1^{-1}} = d_{D_2^{-1}} - \frac{1}{b}$, which by construction of $\mathscr{T}_{K(\mathcal{R})}$ means that D_2^{-1} is maximally contained in D_1^{-1}, in particular D_1^{-1} and D_2^{-1} are adjacent.

Suppose instead that v_1 and v_2 are both annuli, A_1 and A_2 respectively. Since A_1 and A_2 are adjacent we can assume without loss of generality that

A_2 is maximally contained in A_1 and

$$A_1 = \{x \in K(\mathcal{R}) \mid v(x) = n\},$$
$$A_2 = \left\{x \in K(\mathcal{R}) \mid v(x) = n + \frac{1}{b}\right\},$$

for some $n \in \mathbb{Z}$. Write $A_i^{-1} = \left\{y \in K(\mathcal{R}) \mid \frac{1}{y} \in A_i\right\}$. Then, by above, we have

$$A_1^{-1} = \{y \in K(\mathcal{R}) \mid v\,(y) = -n\}\,,$$
$$A_2^{-1} = \left\{y \in K(\mathcal{R}) \mid v\,(y) = -n - \frac{1}{b}\right\}.$$

If we consider the discs corresponding to A_1^{-1} and A_2^{-1}, we see that the disc $\{y \in K(\mathcal{R}) \mid v(y) \geq -n\}$ is maximally contained in the disc $\left\{y \in K(\mathcal{R}) \mid v(y) \geq -n - \frac{1}{b}\right\}$. So we have that A_1^{-1} and A_2^{-1} are adjacent.

Finally we need to consider when one of v_1 and v_2 is a disc and the other is an annulus. Without loss of generality we can assume that v_1 is an annulus A_1 and v_2 is a disc D_2. Write $A_1 = \{x \in K(\mathcal{R}) \mid v(x) = n\}$. Then A_1 corresponds to a disc D_1 with centre 0 which we can write as $D_1 = \{x \in K(\mathcal{R}) \mid v(x) \geq n\}$. Note that, since v_2 is a disc D_2, we must have that $0 \notin D_2$. Under our assumption that v_1 and v_2 are adjacent, we must have that either D_1 is maximally contained in D_2, or D_2 is maximally contained in D_1. However $0 \in D_1$ and $0 \notin D_2$, so we must have that D_2 is maximally contained in D_1. Pick $\alpha \in K(\mathcal{R})$ to be a centre of D_2, that is $D_2 = \{x \in K(\mathcal{R}) \mid v(x - \alpha) \geq n + \frac{1}{b}\}$. Then α could also be chosen as a centre of D_1 giving $D_1 = \{x \in K(\mathcal{R}) \mid v(x - \alpha) \geq n\}$. Since $0 \in D_1$ we must have that $v(\alpha) = n$. Defining A_1^{-1} and D_2^{-1} as before we get

$$A_1^{-1} = \{y \in K(\mathcal{R}) \mid v\,(y) = -n\}\,,$$
$$D_2^{-1} = \left\{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq n + \frac{1}{b} - 2v(\alpha) = -n + \frac{1}{b}\right\}.$$

Write $D_1^{-1} = \left\{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq -n\right\}$, the disc corresponding to the annulus A_1^{-1}. Then D_2^{-1} is maximally contained in D_1^{-1}, since their depths differ by $\frac{1}{b}$, and the centre $\frac{1}{\alpha}$ of D_2^{-1} has $v\left(\frac{1}{\alpha}\right) = -n$ so lies in D_1^{-1} and can therefore also be chosen to be a centre of D_1^{-1}. So A_1^{-1} and D_2^{-1} are adjacent.

Note that, since ϕ is self inverse, the converse is also immediately true. So v_1 and v_2 are adjacent in $\mathscr{T}_{K(\mathcal{R})}$ if and only if their images under $\phi : z \mapsto \frac{1}{z}$ are also adjacent. $\square$

Lemma 5.3.4. *Let v_1 and v_2 be two vertices in $\mathscr{T}_{K(\mathcal{R})}$ and let v_1^{-1} and v_2^{-1} be their images under $\phi : z \mapsto \frac{1}{z}$. Then $d(v_1, v_2) = d(v_1^{-1}, v_2^{-1})$ in $\mathscr{T}_{K(\mathcal{R})}$.*

Proof. Let $v_1 = w_1, w_2, \ldots, w_n, w_{n+1} = v_2$ be the unique path of vertices of $\mathscr{T}_{K(\mathcal{R})}$ between v_1 and v_2. Then w_i and w_{i+1} are adjacent for all $1 \leq i \leq n$ and distance $\frac{1}{b}$ from each other. So, the distance between v_1 and v_2 is $\frac{n}{b}$. Let w_i^{-1} be the image of w_i under ϕ. By Lemma 5.3.3, for $1 \leq i, j \leq n$, w_i^{-1} and w_j^{-1} are adjacent if and only if w_i and w_j are. That is, w_i^{-1} and w_j^{-1} are adjacent if and only if $i = j \pm 1$. So we obtain a path $v_1^{-1} = w_1^{-1}, w_2^{-1}, \ldots, w_n^{-1}, w_{n+1}^{-1} = v_2^{-1}$, which does not contain any backtrackings, between v_1^{-1} and v_2^{-1}. In particular, the distance between v_1^{-1} and v_2^{-1} is also $\frac{n}{b}$. □

For the next lemma we will use the following notation to talk about discs or annuli being closer to elements of $\mathbb{P}^1(K(\mathcal{R}))$ than others.

Notation 5.3.5. Take $\alpha \in K(\mathcal{R})$ and let D_1 and D_2 be two discs containing α then we say that D_2 is closer to α than D_1 if $d_{D_2} > d_{D_1}$. If D_1 and D_2 contain 0, and so the corresponding vertices in the modified Bruhat-Tits tree are annuli, A_1 and A_2 respectively, then we say that A_2 is closer to α than A_1. In this case, we also say that A_1 is closer to ∞ than A_2.

Lemma 5.3.6. *Let $\alpha \in \mathbb{P}^1(K(\mathcal{R})) = K(\mathcal{R}) \cup \{\infty\}$. If $\alpha \neq 0, \infty$ then let D_1 and D_2 be two discs containing α with D_2 closer to α than D_1, and $d_{D_1} > v(\alpha)$. Write D_1^{-1} and D_2^{-1} for the images of D_1 and D_2 under the Möbius transformation $\phi : z \mapsto \frac{1}{z}$ as in the proof of Lemma 5.3.3. Then both D_1^{-1} and D_2^{-1} are discs containing $\frac{1}{\alpha}$, and in particular D_1^{-1} and D_2^{-1} are vertices of the modified Bruhat-Tits tree. Moreover, D_2^{-1} is closer to $\frac{1}{\alpha}$ than D_1^{-1} is.*

If $\alpha \in \{0, \infty\}$ then let D_1 and D_2 be two discs centred at 0 with A_1 and A_2 their corresponding annuli and vertices of the modified Bruhat-Tits tree. Suppose that A_2 is closer to α than A_1 and let A_1^{-1} and A_2^{-1} be the images of A_1 and A_2 under ϕ respectively. Then both A_1^{-1} and A_2^{-1} are annuli centred at 0 and A_2^{-1} is closer to $\frac{1}{\alpha}$ than A_1^{-1} is, where $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof. Let $\alpha \in K(\mathcal{R}) \cup \{\infty\} = \mathbb{P}^1(K(\mathcal{R}))$ be any point on the boundary of the Bruhat-Tits tree. If $\alpha \neq 0, \infty$ then subsequent discs containing α are $\cdots \supset D_{\alpha, n-\frac{1}{b}} \supset D_{\alpha,n} \supset D_{\alpha, n+\frac{1}{b}} \supset \ldots$, where $b = [K(\mathcal{R}) : K]$, and

$$D_{\alpha,n} = \{x \in \bar{K} \mid v(x - \alpha) \geq n\}.$$

If $n > v(\alpha)$ then $0 \notin D_{\alpha,n}$, so $D_{\alpha,n}$ is a vertex of the modified Bruhat-Tits tree. Write $D_{\alpha,n}^{-1}$ for the image of $D_{\alpha,n}$ under ϕ. As above, we get that

$$D_{\alpha,n}^{-1} = \{y \in \bar{K} \mid v\left(y - \frac{1}{\alpha}\right) \geq n - 2v(\alpha)\}.$$

Note that, since $n > v(\alpha)$, $n - 2v(\alpha) > -n$, we have that $0 \notin D_{\alpha,n}^{-1}$. So, $D_{\alpha,n}^{-1}$ is also a vertex of the modified Bruhat-Tits tree. It is clear that $D_{\alpha,n}^{-1}$ contains $\frac{1}{\alpha}$, and that as $n \to \infty$, $D_{\alpha,n}$ gets closer to α whilst $D_{\alpha,n}^{-1}$ gets closer to $\frac{1}{\alpha}$.

If $\alpha \in \{0, \infty\}$ then we must instead consider annuli arising from discs $D_{0,n}$, namely the vertices along the path from ∞ to 0 are the annuli $\ldots, A_{0,n-\frac{1}{b}}, A_{0,n}, A_{0,n+\frac{1}{b}}, \ldots$ for $n \in \mathbb{Z}$, where $A_{0,n} = \{x \in \bar{K} \mid v(x) = n\}$, and the larger n is the closer $A_{0,n}$ is to 0 and the closer $A_{0,n}^{-1}$ is to ∞. $\square$

Lemma 5.3.7. *Let $\alpha, \beta \in \mathbb{P}^1(K(\mathcal{R}))$ be distinct. Then the unique embedded path between α and β in $\mathscr{T}_{K(\mathcal{R})}$ is mapped to the unique embedded path between $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ in $\mathscr{T}_{K(\mathcal{R})}$ under the Möbius transformation $\phi : z \mapsto \frac{1}{z}$, where $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.*

Proof. First, we will find two vertices v_1 and v_2 that lie on the path between α and β with v_1 closer to α than v_2 is, and v_2 closer to β than v_1 is, where their images v_1^{-1} and v_2^{-1} under ϕ lie on the path between $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ with v_1^{-1} closer to $\frac{1}{\alpha}$ than v_2^{-1} is, and v_2^{-1} closer to $\frac{1}{\beta}$ than v_1^{-1} is. Furthermore v_1 and v_2 will be such that every vertex between v_1 and α maps to a vertex between v_1^{-1} and $\frac{1}{\alpha}$, and every vertex between v_2 and β maps to a vertex between v_2^{-1} and $\frac{1}{\beta}$.

If $\alpha, \beta \notin \{0, \infty\}$ then we define the wedge of α and β just as we did in cluster pictures, namely $D(\{\alpha\} \wedge \{\beta\})$ is the smallest disc containing both α and β. Write $v_{\alpha\wedge\beta}$ for the vertex of $\mathscr{T}_{K(\mathcal{R})}$ corresponding to $D(\{\alpha\} \wedge \{\beta\})$. So, $v_{\alpha\wedge\beta}$ is either a disc or an annulus. Vertices lying between $v_{\alpha\wedge\beta}$ and α or β are either discs themselves, or annuli corresponding to discs, that contain α or β respectively.

Let us now concentrate on α, as the same argument can then be applied to β. By construction, $d_{D(\{\alpha\}\wedge\{\beta\})} = v(\alpha - \beta) \geq \min\{v(\alpha), v(\beta)\}$. So, a vertex lying strictly between $v_{\alpha\wedge\beta}$ and α is either a disc of the form $D_{\alpha,n}$ where $n > v(\alpha - \beta)$ if $0 \notin D_{\alpha,n}$, or an annulus corresponding to such a disc if $0 \in D_{\alpha,n}$. Taking $n > v(\alpha)$ gives that $0 \notin D_{\alpha,n}$, so $D_{\alpha,n}$ is a vertex of the modified Bruhat-Tits tree. So let $n > \max\{v(\alpha), v(\alpha - \beta)\}$. By Lemma 5.3.6, these discs map to discs containing $\frac{1}{\alpha}$ under ϕ. Moreover, Lemma 5.3.6 also gives that the closer $D_{\alpha,n}$ is to α, the closer its image under ϕ is to $\frac{1}{\alpha}$. Recall that the image of $D_{\alpha,n}$ is

$$D_{\alpha,n}^{-1} = \{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq n - 2v(\alpha).$$

Since $v\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) = v(\alpha - \beta) - v(\alpha) - v(\beta)$, we can write

$$D\left(\left\{\frac{1}{\alpha}\right\} \wedge \left\{\frac{1}{\beta}\right\}\right) = \{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq v(\alpha - \beta) - v(\alpha) - v(\beta)\}.$$

Discs which are strictly contained in $D(\{\frac{1}{\alpha}\} \wedge \{\frac{1}{\beta}\})$ and which contain $\frac{1}{\alpha}$ are of the form

$$\{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{\alpha}\right) \geq m\},$$

where $m > v(\alpha - \beta) - v(\alpha) - v(\beta)$. Recall that we have assumed that $n > \max\{v(\alpha), v(\alpha - \beta)\}$. Now note that:

- If $v(\alpha) > v(\beta)$: then $v(\alpha - \beta) = v(\beta)$, so $v(\alpha - \beta) + v(\alpha) - v(\beta) = v(\alpha)$. Therefore, since $n > v(\alpha) = v(\alpha - \beta) + v(\alpha) - v(\beta)$, we get that

$$n - 2v(\alpha) > v(\alpha - \beta) - v(\alpha) - v(\beta) = v\left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

- If $v(\beta) > v(\alpha)$: then $v(\alpha - \beta) = v(\alpha)$, so $v(\alpha - \beta) + v(\alpha) - v(\beta) = 2v(\alpha) - v(\beta) < v(\alpha)$. Since $n > v(\alpha)$, we get that

$$n - 2v(\alpha) > v(\alpha - \beta) - v(\alpha) - v(\beta) = v\left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

- If $v(\alpha) = v(\beta)$: then $v(\alpha - \beta) + v(\alpha) - v(\beta) = v(\alpha - \beta)$. So, since $n > v(\alpha - \beta)$, we get that

$$n - 2v(\alpha) > v(\alpha - \beta) - v(\alpha) - v(\beta) = v\left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

So, in all cases, our assumption that $n > \max\{v(\alpha), v(\alpha - \beta)\}$ gives that $n - 2v(\alpha) > v\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$. In particular, $D_{\alpha,n}^{-1}$ lies between $D(\{\frac{1}{\alpha}\} \wedge \{\frac{1}{\beta}\})$ and $\frac{1}{\alpha}$, and the greater the value of n, the closer $D_{\alpha,n}$ is to α and the closer $D_{\alpha,n}^{-1}$ is to $\frac{1}{\alpha}$. As already mentioned, the same argument can be applied to β. So, when $\alpha, \beta \in K(\mathcal{R}) \setminus \{0\}$, taking $v_1 = D_{\alpha,n}$ and $v_2 = D_{\beta,m}$ for any fixed $n > \max\{v(\alpha), v(\alpha - \beta)\}$ and $m > \max\{v(\beta), v(\alpha - \beta)\}$ has the desired properties.

It remains to find such vertices v_1 and v_2 when either $\alpha = 0$ and $\beta \in K(\mathcal{R}) \setminus \{0\}$, $\alpha = \infty$ and $\beta \in K(\mathcal{R}) \setminus \{0\}$, or $\alpha = 0$ and $\beta = \infty$.

If $\alpha = 0$ and $\beta \in K(\mathcal{R}) \setminus \{0\}$ then $v(\alpha - \beta) = v(\beta)$, so $D(\{\alpha\} \wedge \{\beta\})$ has depth $v(\beta)$ and can be written

$$D(\{\alpha\} \wedge \{\beta\}) = \{x \in K(\mathcal{R}) \mid v(x) \geq v(\beta)\}.$$

The corresponding vertex on the modified Bruhat-Tits tree is the annulus

$$A = \{x \in K(\mathcal{R}) \mid v(x) = v(\beta)\}.$$

Under ϕ, the image of A is

$$A^{-1} = \{x \in K(\mathcal{R}) \mid v(x) = -v(\beta)\},$$

which corresponds to a disc, say D^{-1}, where $D^{-1} = \{x \in K(\mathcal{R}) \mid v(x) \geq -v(\beta)\}$. Clearly $\frac{1}{\beta}$ lies in D^{-1}, so A^{-1} is on the path between $\frac{1}{\alpha} = \infty$ and $\frac{1}{\beta}$. Likewise for any annulus $A_{0,n}$ with $n > v(\beta)$. So, we can take $v_1 = A$. Note that discs containing $\frac{1}{\beta}$ are in one-to-one correspondence with vertices on the path between $\frac{1}{\beta}$ and ∞. Any disc of the form $D_{\beta,n}$, where $n > v(\beta)$, is a vertex and lies on the path between β and 0. As above, discs of this form get mapped to discs containing $\frac{1}{\beta}$ with $D_{\beta,n}^{-1}$ getting closer to $\frac{1}{\beta}$ the larger n is. So, take $v_2 = D_{\beta,n}$ for some fixed $n > v(\beta)$.

If $\alpha = \infty$ and $\beta \in K(\mathcal{R}) \setminus \{0\}$ then similarly to above we can take $v_1 = A$ and $v_2 = D$ where

$$A = \{x \in K(\mathcal{R}) \mid v(x) = v(\beta)\},$$
$$D = \{x \in K(\mathcal{R}) \mid v(x - \beta) = n\},$$

for some fixed $n > v(\beta)$. Note that this works since any annulus closer to ∞ than A is will map to an annulus closer to 0 than A^{-1} is, and any disc closer to β than D is will map to a disc (that does not contain 0) closer to $\frac{1}{\beta}$ than D^{-1} is under ϕ.

Finally, if $\alpha = 0$ and $\beta = \infty$ then the path between α and β has vertices which are precisely the set of annuli centred at 0. Take $v_1 = A_1$ to be any such annulus and $v_2 = A_2$ to be the unique annulus adjacent to v_1 which is closer to ∞ than A_1 is. By Lemma 5.3.6 A_1^{-1} and A_2^{-1} are both annuli centred at 0, and lie on the path between 0 and ∞ with A_1^{-1} closer to 0 than A_2^{-1} is to 0.

In all cases v_1 and v_2 lie on the path between α and β and are such that

v_1^{-1} and v_2^{-1} lie on the path between $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, and vertices which are closer to α (resp. β) than v_1 (resp. v_2) is, get mapped to vertices which are closer to $\frac{1}{\alpha}$ (resp. $\frac{1}{\beta}$) than v_1^{-1} (resp. v_2^{-1}) is. Now by Lemma 5.3.3 adjacency is preserved so we get the path from v_1 to α (resp. v_2 to β) maps to the path from v_1^{-1} to $\frac{1}{\alpha}$ (resp. v_2^{-1} to $\frac{1}{\beta}$) and the path between v_1 and v_2 maps to the path between v_1^{-1} and v_2^{-1}. Moreover, this gives us that ϕ is an isomorphism between the path from α to β and the path from $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, this is because if the image had any backtrackings this would contradict the path selected from α to β being the shortest path. □

Corollary 5.3.8. *Let $f \in K[x]$ be square free with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$. Take $r_1, r_2 \in \mathcal{R}$ to be distinct roots of $f(x)$. Then the unique path between r_1 and r_2 in $\mathscr{T}_{K(\mathcal{R})}$ gets mapped to the unique path between $\frac{1}{r_1}$ and $\frac{1}{r_2}$ under the Möbius map $\phi : z \mapsto \frac{1}{z}$, where $\frac{1}{r_i} = \infty$ if $r_i = 0$. Furthermore, for any root $r \in \mathcal{R}$ the unique path between r and ∞ gets mapped to the unique path between $\frac{1}{r}$ and 0.*

Proof. Follows as a direct consequence of Lemma 5.3.7 □

Lemma 5.3.9. *Let $f \in K[x]$ be square free with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$. Take r, s, t to be distinct, with $r, s, t \in \mathcal{R}$ if $\mathcal{R}$ is even, and $r, s, t \in \mathcal{R} \cup \{\infty\}$ if $\mathcal{R}$ is odd. Write c for the unique point that lies on all three the paths between r and s, r and t, and s and t. Then c is mapped to the unique point, c^{-1}, that lies on all three of the embedded paths between $\frac{1}{r}$ and $\frac{1}{s}$, $\frac{1}{r}$ and $\frac{1}{t}$, and $\frac{1}{s}$ and $\frac{1}{t}$ under $\phi : z \mapsto \frac{1}{z}$, where $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.*

Proof. Recall that c is either a disc or an annulus and its image under ϕ is as described earlier in this section and denoted by c^{-1}. By Lemma 5.3.7, since c lies on all three of the paths between r and s, r and t, and s and t, c^{-1} lies on all three of the paths between $\frac{1}{r}$ and $\frac{1}{s}$, $\frac{1}{r}$ and $\frac{1}{t}$, and $\frac{1}{s}$ and $\frac{1}{t}$. That is, c^{-1} is the unique intersection point of all three paths. □

We now have enough to prove that $\tilde{\mathscr{T}}(f/K)$ is invariant under the Möbius transformation $z \mapsto \frac{1}{z}$.

Lemma 5.3.10. *Consider the Möbius transformation $\phi(z) = \frac{1}{z}$, defined over K. Let $f \in K[x]$ be square free with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$. Let $f' \in K[x]$ have set of roots $\mathcal{R}'$, where $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even, and $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd. Then $\tilde{\mathscr{T}}(f/K) \cong \tilde{\mathscr{T}}(f'/K)$ and the genera and colouring are preserved by this isomorphism.*

Proof. Recall that the vertices of $\tilde{\mathscr{T}}(f/K)$ are precisely the triple intersection points of three distinct elements of $\mathcal{R}$ if $\mathcal{R}$ is even, and of elements of $\mathcal{R} \cup \{\infty\}$ if $\mathcal{R}$ is odd. Similarly, vertices of $\tilde{\mathscr{T}}(f'/K)$ are the vertices of $\mathscr{T}_{K(\mathcal{R})}$ which are triple intersection points of three distinct elements of $\mathcal{R}'$ if $\mathcal{R}'$ is even and of $\mathcal{R}' \cup \{\infty\}$ if $\mathcal{R}'$ is odd.

Suppose that $\mathcal{R}$ is even. Then $\mathcal{R}' = \{\frac{1}{r} \mid r \in \mathcal{R}\} \backslash \{\infty\}$. So $\mathcal{R}'$ is even if and only if $0 \notin \mathcal{R}$, in which case $\infty \notin \{\frac{1}{r} \mid r \in \mathcal{R}\}$ so we can write $\mathcal{R}' = \{\frac{1}{r} \mid r \in \mathcal{R}\}$. If $\mathcal{R}'$ is odd then $\infty \in \{\frac{1}{r} \mid r \in \mathcal{R}\}$ so we have that $\mathcal{R}' \cup \{\infty\} = \{\frac{1}{r} \mid r \in \mathcal{R}\}$. By Lemma 5.3.9, the intersection point in $\mathscr{T}_{K(\mathcal{R})}$ of every triple of elements in $\mathcal{R}$ gets mapped to the intersection point of the corresponding triple of elements in $\{\frac{1}{r} \mid r \in \mathcal{R}\}$. Therefore, we have an isomorphism between the vertices of $\tilde{\mathscr{T}}(f/K)$ and $\tilde{\mathscr{T}}(f'/K)$. Since adjacency is preserved by Lemma 5.3.3, and distance is preserved by Lemma 5.3.4, we have an isomorphism between $\tilde{\mathscr{T}}(f/K)$ and $\tilde{\mathscr{T}}(f'/K)$ that preserves distances. It remains to show that the genera of vertices and colouring of the edges and vertices are preserved. However, by Lemma 5.3.7, a vertex of $\tilde{\mathscr{T}}(f/K)$ which is closest to an element $\alpha \in \mathbb{P}^1(K(\mathcal{R}))$ is mapped to the vertex of $\tilde{\mathscr{T}}(f'/K)$ which is closest to $\phi(\alpha)$. So, it follows by Remark 5.2.13, that the colouring and genera are preserved. □

5.3.4 Cores and Canonical Representatives are Model Invariant

All three of our discussions about simple Möbius transformations lead us to the following result.

Proposition 5.3.11. *Let ϕ be any Möbius transformation, defined over K. Let $f \in K[x]$ be square free with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$. Let $f' \in K[x]$ have set of roots $\mathcal{R}'$, where $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\} \backslash \{\infty\}$ if $\mathcal{R}$ is even, and $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \backslash \{\infty\}$ if $\mathcal{R}$ is odd. Then $\tilde{\mathscr{T}}(f/K) \cong \tilde{\mathscr{T}}(f'/K)$ and the genera and colouring are preserved by this isomorphism.*

Proof. Any Möbius transformation can be broken down into simple Möbius transformations, namely scalings, shifts, and inversions. The result follows from Lemmas 5.3.1, 5.3.2, and 5.3.10. □

Recall that, if $d_{\mathcal{R}}, d_{\mathcal{R}'} \geq 0$, Construction 5.2.15 and Theorem 5.2.18 give us a description of how to obtain $\tilde{T}$ and $\tilde{T}'$ from $\mathscr{T}_{K(\mathcal{R})}$. In particular we take the quotients of $\tilde{\mathscr{T}}(f/K)$ and $\tilde{\mathscr{T}}(f'/K)$ by their induced Galois actions.

Lemma 5.3.12. *If r and r' are in the same Galois orbit then $\phi(r)$ and $\phi(r')$ are in the same orbit as each other also, for any Möbius transformation ϕ.*

Proof. Let $\phi(z) = \frac{az+b}{cz+d}$ be a Möbius transformation. Write $G_K = \langle \sigma \rangle$. Suppose that $r' = \sigma(r)$. Since $a, b, c, d \in K$ it is clear that $\phi(\sigma(r)) = \phi(r') = \frac{ar'+b}{cr'+d} = \frac{a\sigma(r)+b}{c\sigma(r)+d} = \sigma\left(\frac{ar+b}{cr+d}\right) = \sigma(\phi(r))$. □

So, as a direct consequence of this and Proposition 5.3.11 we get the following theorem.

Theorem 5.3.13. *Let $f \in K[x]$ be square free with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$ such that $d_{\mathcal{R}} \geq 0$. Let T be the open quotient BY tree associated to f. Take ϕ to be any Möbius transformation, defined over K, such that $d_{\mathcal{R}'} \geq 0$, where $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even, and $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd. Let $f' \in K[x]$ have set of roots $\mathcal{R}'$ and let T' be the open quotient BY tree associated to f'. Then $\tilde{T} \cong \tilde{T}'$ and the genera and colouring are preserved by this isomorphism.*

Proof. By Lemma 5.3.12, applying ϕ does not change the Galois orbits. So, the quotient described in Construction 5.2.15 gives that $q(\tilde{\mathscr{T}}(f/K))$ and $q(\tilde{\mathscr{T}}(f'/K))$ are isomorphic trees and their colouring, genera, multiplicities and distances are all preserved by this isomorphism. By Theorem 5.2.18, since $d_{\mathcal{R}}, d_{\mathcal{R}'} \geq 0$, $q(\tilde{\mathscr{T}}(f/K)) \cong \tilde{T}$ and $q(\tilde{\mathscr{T}}(f'/K)) \cong \tilde{T}'$. □

Recall that two open quotient BY trees T and T' are equivalent if and only if their canonical representatives are isomorphic. That is their cores $\tilde{T}$ and $\tilde{T}'$ are isomorphic, and on the extended trees as described in Construction 4.5.1 their marked points are integer distance from each other (although not every tree satisfying this will be equivalent to T and T'). Recall also that we discussed in Proposition 5.2.22 how these marked points will always correspond to integral discs $D_{z,d}$ with $z \in K$ and $d \in \mathbb{Z}$. So, it is important to show that under any Möbius transformation vertices of the modified Bruhat-Tits tree corresponding to discs of the form $D_{\alpha,d}$ with $\alpha \in K$ and $d \in \mathbb{Z}$ get mapped to vertices corresponding to discs of the same form.

Proposition 5.3.14. *Let D be an integral disc, that is, a disc of the form $D_{\alpha,d}$, with $\alpha \in K$ and $d \in \mathbb{Z}$. Then D (or its corresponding annulus) is mapped to an integral disc (or its corresponding annulus) under any Möbius transformation ϕ defined over K.*

Proof. Suppose that $\phi(z) = \frac{1}{z}$. We noted at the start of the discussion about inversion that a disc D, $0 \notin D$ is mapped to

$$D^{-1} = \{y \in K(\mathcal{R}) \mid v\left(y - \frac{1}{z_D}\right) \geq d_D - 2v(z_D)\}.$$

Likewise, if $0 \in D$ then the corresponding annulus A is mapped to

$$A^{-1} = \{y \in K(\mathcal{R}) \mid v(y) = -d_D\}.$$

It is therefore easy to see that any integral disc (or its corresponding annulus) is mapped to an integral disc (or its corresponding annulus) under $z \mapsto \frac{1}{z}$. We can similarly apply shifts and scalings to discs and it is easy to show that they also send integral discs to integral discs. $\square$

It is important to note that whilst the vertex that the marked point arises from on the Bruhat-Tits tree might be mapped to a different vertex under a Möbius transformation this does not mean that the marked point has moved in the sense of Theorem 4.6.4. This is because the marked point needs to have moved relative to the core of the open quotient BY tree and when we visualise this on the Bruhat-Tits tree, the whole core may also be moved by the Möbius map, for instance under the map $z \mapsto z + \beta$. Likewise, just because the vertex of the Bruhat-Tits tree which corresponds to the marked point is fixed by a Möbius map, this does not mean that the marked point has not moved in the sense of Theorem 4.6.4. For instance under the map $z \mapsto \frac{1}{z}$ if $D_{z_{\mathcal{R}},0} = \mathcal{O}_K$. As such we always need to be careful what we mean when we say the marked point has moved. The default assumption is that we are speaking in the sense of Theorem 4.6.4.

Finally, we are able to conclude and prove that the canonical representative is invariant under Möbius transformations.

Theorem 5.3.15. *Let $f \in K[x]$ be square free with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$ such that $d_{\mathcal{R}} \geq 0$. Write T for the open quotient BY tree associated to f. Take a Möbius transformation $\phi(z)$, defined over K, such that $d_{\mathcal{R}'} \geq 0$, where $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even, and $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd. Let $f' \in K[x]$ have set of roots $\mathcal{R}'$ and let T' be the open quotient BY tree of f'. Then T and T' are equivalent.*

Proof. By Lemma 5.3.13 $\tilde{T} \cong \tilde{T}'$ and their colouring, genera, multiplicities and distances are all preserved by this isomorphism. Write m and m' for the marked points of T and T' respectively. By Proposition 5.3.14 m and m' are integer distance apart. It follows that if there is a unique closest integral disc D to the centre c of $\tilde{T}$ then there is a unique closest integral disc D' to the centre c' of $\tilde{T}'$. Similarly, if there are two closest discs to c, then there are such discs for c'. We will deal with the former situation however, as the case when there are two closest integral discs follows similarly. It remains to prove that the open edge is added to $\tilde{T}$ in the same way as it is to $\tilde{T'}$. If D lies on $\tilde{T}^1$ (when viewed on $\mathscr{T}$) then we are done much like in the proof of Theorem 4.6.8. So, suppose this is not the case. Recall that we can visualise B and B', the extended trees of T and T' respectively on $\mathscr{T}$. Note that shifts and scalings clearly have no effect on whether green edges are added or not, since they do not change the colouring of the open edge, nor move its positioning relative to $\tilde{T}$. It is not hard to show that when D does not lie on $\tilde{T}$, whether or not D is coloured green on the extended tree B is entirely determined by the number of elements of $\mathcal{R}^+$ each vertex of $\tilde{T}$ is closest to, and the denominator of the distance of each vertex to the marked point. Both of these are unchanged by Mobius transformation (since lines are mapped to lines and adjacency is preserved - proved in the lemmas earlier in this section). Therefore, D lies on a green part of B if and only if D' lies on a green part of B'.

This is the only information we need to calculate their canonical representative, so this shows that their canonical representatives are isomorphic. Namely T and T' are equivalent. □

We translate this into the setting of hyperelliptic curves in Section 6.1.

5.4 Möbius Maps Between Equivalent Open Quotient BY Trees

Here, to complete our classification, we study how one can find a Möbius map between any two equivalent open quotient BY trees. This is formally stated as follows.

Theorem 5.4.1. *Let $f \in K[x]$ be square free with tamely ramified splitting field and set of roots $\mathcal{R}$ in $\bar{K}$ such that $d_{\mathcal{R}} \geq 0$. Write Σ for the cluster picture, and T for the open quotient BY tree associated to f. Let T' be an open quotient BY tree equivalent to T. Then there exists a Möbius transformation ψ over K such that for*

$$\mathcal{R}' = \begin{cases} \{\psi^{-1}(r) \mid r \in \mathcal{R}\} \setminus \{\infty\} & \text{if } \mathcal{R} \text{ is even,} \\ \{\psi^{-1}(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\} & \text{if } \mathcal{R} \text{ is odd,} \end{cases}$$

the associated cluster picture $\Sigma' = (\Sigma', \mathcal{R}', d')$ has $d'_{\mathcal{R}'} \geq 0$ and $\underline{T}(\Sigma') \cong T'$.

The rough method for proving this result is as follows. Visualise $q^{-1}(T)$ on the Bruhat-Tits tree $\mathscr{T}_{K(\mathcal{R})}$ via our usual method described in Section 5.2.

Since T and T' are equivalent open quotient BY trees they have isomorphic cores. So, we can visualise the preimage $q^{-1}(\tilde{T}')$ of the core of T' as lying in $q^{-1}(T)$, and think of it as a subtree of $\mathscr{T}_{K(\mathcal{R})}$. By definition $\tilde{T}'$ is a subtree of T' so we can use this to visualise $q^{-1}(T')$ and thus the marked point m' of T' (recall m' has multiplicity 1 so there is precisely one point in the preimage $q^{-1}(m')$ of m', so we think of this as also being labelled m') as lying on $\mathscr{T}_{K(\mathcal{R})}$ overlapping $q^{-1}(T)$ at the preimage of their cores.

Example 5.4.2. Consider the polynomial $f(x) = (x-1)(x^3-7^5)(x-7-7^2)(x-7+7^2)$ over $K = \mathbb{Q}_7^{\mathrm{ur}}$. Let T be the open quotient BY tree associated to f/K. In Example 5.2.14 we found $\tilde{\mathscr{T}}(f/k)$, as pictured in Figure 5.4. We can similarly view $q^{-1}(T)$ on $\mathscr{T}$ by constructing $\mathscr{T}(f/K)$. Note that in this case $T^1 = T$, so in fact $T \cong q^{-1}(T) \cong \mathscr{T}(f/K)$. This is pictured in Figure 5.11 below, where ζ_3 is a third root of unity. Let T' be the following open

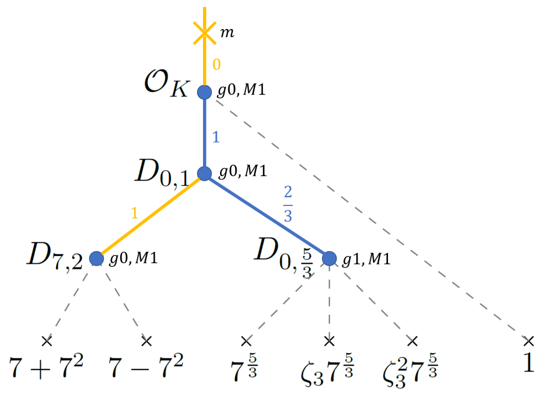
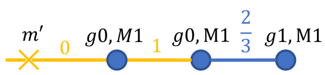


Figure 5.11: $T \cong q^{-1} \cong \mathscr{T}(f/k)$ visualised on the Bruhat-Tits tree.

quotient BY tree:

$T' =$ 

Similarly, $q^{-1}(T') \cong T'$ so we can visualise T' as lying on the Bruhat-Tits tree by overlapping T' and T along their core, as shown in Figure 5.12 below.

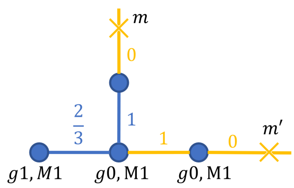


Figure 5.12: $T \cup T'$.

We will show that m' corresponds to a vertex of $\mathscr{T}_{K(\mathcal{R})}$ which is an integral disc. Denote this disc by $D_{m'} = D_{\alpha,n}$ for some $\alpha \in K$ and $n \in \mathbb{Z}$. We can map

$D_{m'}$ to the ring of integers $\mathcal{O}_{K(\mathcal{R})}$ by applying a shift and scaling. Depending on the case, we make an assumption about whether or not a root lies at 0, possibly by applying a further shift. We can then apply an inversion so that, roughly speaking, all vertices (i.e. discs corresponding to vertices of $q^{-1}(T)$ after the shifts and scalings above have been applied) get mapped to discs that lie inside the image of the ring of integers, or get mapped to annuli corresponding to such discs. That is, the images of all other vertices get mapped to vertices hanging below the image of the ring of integers.

Example 5.4.3. Continuing with Example 5.4.2, we can consider m' as corresponding to the disc $D_{7,2}$, a vertex of T. This is shown in Figure 5.13.

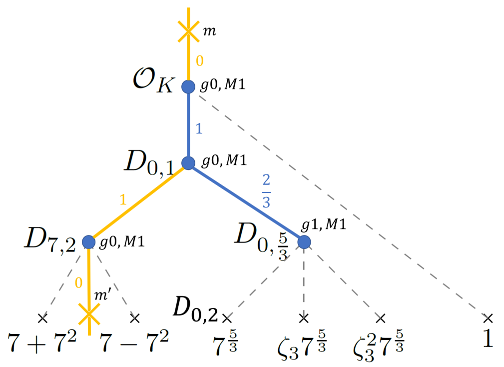


Figure 5.13: $T \cup T'$ on the Bruhat-Tits tree.

To map $D_{m'} = D_{7,2}$ to $\mathcal{O}_K$ we first shift by 7 so it becomes centred at 0, and then scale by 7^2. That is we can apply the transformation $z \mapsto \frac{z-7}{7^2}$ to the set of roots $\mathcal{R}$ of f. This maps $\mathcal{R}$ to the set $\{1, -1, \frac{1}{7^{\frac{1}{3}}} - \frac{1}{7}, \zeta_3 \frac{1}{7^{\frac{1}{3}}} - \frac{1}{7}, \zeta_3^2 \frac{1}{7^{\frac{1}{3}}} - \frac{1}{7}, \frac{1}{7^2} - \frac{1}{7}\}$. We can apply $z \mapsto \frac{1}{z}$, and denote the resulting set by $\mathcal{R}'$. When viewed on $\mathscr{T}$ the open quotient BY tree afforded by Construction 5.2.6 is indeed isomorphic to T'. This is shown pictorially in Figure 5.14.

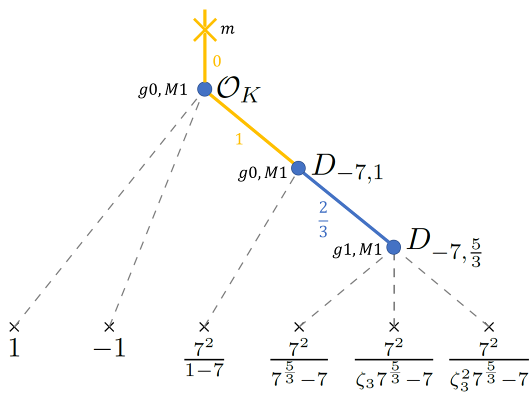


Figure 5.14: Open subtree of $\mathscr{T}$ arising from $\mathcal{R}'$, isomorphic to T'.

The process will differ slightly depending on the coloring of the open edges of both T and T'. Note that at different points in this proof we will have to be careful about which description of the Bruhat-Tits tree we use, be that either the standard definition or the modified definition to include annuli. This is not something to worry about as we are able to use them interchangeably but it is a subtlety that we need to be aware of in order to complete the proof. In particular, whenever we are performing a shift or scaling we use the standard definition, but when we perform an inversion we need to make use of the modified definition.

Proof of Theorem 5.4.1. Let m and m' be the marked points of T and T' respectively. By definition of T and T' being equivalent, their cores $\tilde{T}$ and $\tilde{T}'$ are isomorphic. Note that, since $d_{\mathcal{R}}, d_{\mathcal{R}'} \geq 0$, by Theorems 5.2.10 and 5.2.18, $q(\mathscr{T}(f/K)) \cong T$, $q(\mathscr{T}(f'/K)) \cong T'$, $q(\tilde{\mathscr{T}}(f/K)) \cong \tilde{T}$, and $q(\tilde{\mathscr{T}}(f'/K)) \cong \tilde{T}'$. So, we are instead able to use notation such as $q^{-1}(T)$ in the place of $\mathscr{T}(f/K)$, as defined in Notation 5.2.20. Recall, by Remark 5.2.21, that if Σ_C has a non-trivial Galois orbit of proper clusters then we cannot visualise T and $\tilde{T}$ themselves as lying on the Bruhat-Tits tree, but instead $q^{-1}(T)$ and $q^{-1}(\tilde{T})$. Since $\tilde{T}$ and $\tilde{T}'$ are isomorphic it is clear from Construction 5.2.15 that $q^{-1}(\tilde{T})$ and $q^{-1}(\tilde{T}')$ are also. Note that $q^{-1}(T')$ can also then be visualised as lying on the Bruhat-Tits tree, by overlapping $q^{-1}(T)$ at $q^{-1}(\tilde{T})$. It is important to note that in this visualisation the open edge of $q^{-1}(T)$ will be thought of as going off to the point at infinity of $\mathscr{T}_{K(\mathcal{R})}$, whereas it is likely that the open edge of $q^{-1}(T')$ will instead need to be thought of as going off to some element of K on the boundary of the Bruhat-Tits tree $\mathscr{T}_{K(\mathcal{R})}$. The image of any multiplicity 1 point under q^{-1} has size 1, so we are able to think of T^1 as lying on $q^{-1}(T)$ and therefore as lying on $\mathscr{T}_{K(\mathcal{R})}$. In particular, as stated in Proposition 5.2.22, m' corresponds to a vertex of $\mathscr{T}_{K(\mathcal{R})}$ which is a disc with centre in K and integer depth (or equivalently an annulus corresponding to a such disc if we are using the modified Bruhat-Tits tree). For now let us take the standard description of the Bruhat-Tits tree where all vertices are discs, and let us denote the disc which m' corresponds to as $D_{m'} = D_{\alpha,n}$ where α is some element of K and $n \in \mathbb{Z}$. Applying the shift $\phi_1 : z \mapsto z - \alpha$ to $K(\mathcal{R})$, maps $D_{m'}$ to the disc $D_{0,n}$. Further applying the scaling $\phi_2 : z \mapsto \pi^{-n}z$ maps $D_{0,n}$ to the ring of integers of $K(\mathcal{R})$, namely $D_{0,0}$. We proved above in Proposition 5.3.11 that Möbius transformations are isomorphisms on $q^{-1}(\tilde{T})$. So, under $\phi_2 \circ \phi_1$ when we restrict to $q^{-1}(\tilde{T})$ this is an isomorphism, and $D_{m'}$ is mapped to the ring of integers of $K(\mathcal{R})$. So we can assume without loss of generality that $\alpha = 0$

and $d_{D_{m'}} = 0$.

We now subdivide into three different cases depending on what must be removed from T' to obtain $\tilde{T}' = \tilde{T}$. Recall, by Proposition 4.3.9, that to obtain the core we either:

(a) delete just the open edge,

(b) delete the open edge and view the vertex it was attached to as a point on an edge,

(c) or delete the open edge along with a genus 0 blue vertex and a closed blue edge.

So, in order to know which Möbius transformations to apply, we must consider what case T' is in.

(a1) Suppose first that $\tilde{T}'$ is obtained from T' by simply deleting the open edge and that T' has a yellow open edge. That is, we are in case (A1) or (E1) of Theorem 4.6.4. After shifting further by some $y \in \mathcal{O}_K$ we can assume that all elements of $D_{m'} \cap \mathcal{R}$ lie in $\mathcal{O}_{K(\mathcal{R})}$ and are units, that is every $r \in \mathcal{R}$ such that $r \in D_{m'}$ has $v(r) = 0$. All other roots $r \notin D_{m'}$ will have valuation $d_{D_{m'} \wedge \{r\}} < 0$. We now make a shift and consider the modified Bruhat-Tits tree instead of the standard definition. As such we now need to consider the annulus corresponding to $D_{m'}$ which we will denote $A_{m'} = A_{0,0}$. As discussed earlier, the Möbius transformation $\phi_3 : z \mapsto \frac{1}{z}$ is an isomorphism on $q^{-1}(\tilde{T})$. Furthermore, the image of $A_{m'}$ under ϕ_3 is itself. Let us denote $\mathcal{R}' = \{\psi(r) \mid r \in \mathcal{R}\}$, where $\psi = \phi_3 \circ \phi_2 \circ \phi_1$. Let $f' \in K[x]$ have roots $\mathcal{R}'$. All roots $r' \in \mathcal{R}'$ have images with non-negative valuation, that is they lie in $D_{m'}$, so we are now able to apply Construction 5.2.6. Linking up all our roots as in Construction 5.2.6 we certainly obtain a tree which is isomorphic to $q^{-1}(T')$, since linking all the vertices without the additional edge certainly gives us $q^{-1}(\tilde{T})$ as premiages of cores under q (and therefore cores themselves) are invariant under Möbius transformation, and the marked point arises from $\mathcal{O}_{K(\mathcal{R})}$, namely where $D_{m'}$ was mapped to by the inversion. Distances are unchanged under Möbius maps, so the marked point is certainly the correct distance along the open edge. The colouring is also as required since the colouring of the core remains unchanged by the Möbius transformations, and the colouring of the open edge comes from the size of $\mathcal{R}'$, which is even so coloured yellow. By Lemma 5.3.12 the Galois orbits of roots remain unchanged. So, after taking the quotient, we have that f' has open quotient BY tree isomorphic to T' as required.

(a2) Suppose now that $\tilde{T}'$ is obtained from T' by simply deleting the open edge but in this case the open edge is coloured blue. In this situation, when considered in B (the full extended tree of T as defined in Construction 4.5.1), we have the following cases:

(i) Theorem 4.6.4 case (A2): m' is a blue vertex of $\tilde{T} = \tilde{T}'$ with

$$2g(m') + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } m'}} M(e),$$

and if $g(m') = 0$ then m' has at least one yellow incident edge in $\tilde{T}$.

(ii) Theorem 4.6.4 case (C2): m' lies on a green part of B and the open edge of T' is attached at the vertex $v_c \in V(\tilde{T})$ which is closest to m' in B. Note that v_c is coloured blue. We also have that $\mathrm{denom}(d(v,m)) \nmid s(v,T)$, or $g(v_0) = 0$ and there is exactly one closed edge incident to v_0, which is blue of multiplicity 1 and v_c is the unique vertex adjacent to v_0 in T, or $v_c = v_0$ and the open edge of T is blue.

(iii) Theorem 4.6.4 case (E2): m' does not lie on $\tilde{T}$, and in B m' is not coloured green. The open edge of T' is attached at the vertex $v_c \in V(\tilde{T})$ which is closest to m' in B and v_c is blue with

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e),$$

and if $g(v_c) = 0$ then v_c has at least one yellow incident edge in $\tilde{T}$. Furthermore, if $\#\{\text{incident blue edges to } v_c \text{ in } T\} \neq \#\{\text{incident blue edges to } v_c \text{ in } T'\}$ then $d(m', v_c) \in \mathbb{Z}$.

We will consider each of these cases separately.

(i) Recall that $s(m',T) = 2g(m') + 2 - \sum_{\substack{e \in E(T) \text{ blue,} \\ \text{incident to } m'}} M(e)$. In case (i) we have

$$2g(m') + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } m'}} M(e).$$

Recall also that we need to also consider what edges must be removed from T to obtain $\tilde{T}$. So, if $\#\{e \in E(\tilde{T}) \text{ blue incident to } m'\} =$

$\#\{e \in E(T)$ blue incident to $m'\}$ we have that $s(m', T) > 0$. By construction, $d(m, m') \in \mathbb{Z}$ which means all singletons in $D_{m'}$ are fixed by G_K, that is there exists at least one root $s \in \mathcal{R}$ with $s \in D_{m'}$ in $q^{-1}(T) \subseteq \mathscr{T}_{K(\mathcal{R})}$ which is fixed by G_K, i.e. such that $s \in \mathcal{O}_K$, and $\{s\}$ is a child of the cluster corresponding to $D_{m'}$ in Σ. As before, we are able to assume that $D_{m'} = D_{0,0}$. All $r \in D_{m'}$ have $v(r) \geq 0$, so, $v(s) \geq 0$. If $v(s) > 0$ we can apply a shift by some element of K, such as $z \mapsto z - 1$, so we can assume without loss of generality that $v(s) = 0$, that is s is a unit. Shifting by s we can assume that $s = 0$. Now since s is a child of the cluster corresponding to $D_{m'}$ we have that $v(s - r) = 0$ for all other roots $r \in D_{m'}$, in particular, $v(r) = 0$. So, all other roots in $D_{m'}$ are units. Finally we apply $\phi_3 : z \mapsto \frac{1}{z}$ and take $\mathcal{R}' = \{\phi_3^{-1}(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even and $\mathcal{R}' = \{\phi_3^{-1}(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd. In either situation under our assumption that $s = 0$ is a root we get that $\mathcal{R}'$ is odd. So, if f' has roots $\mathcal{R}'$ then its associated open quotient BY tree is isomorphic to T'.

If instead

$$\#\{e \in E(\tilde{T}_b) \text{ incident to } m'\} \neq \#\{e \in E(T_b) \text{ incident to } m'\},$$

then m' must have had a blue edge deleted from it when passing from T to $\tilde{T}$. In particular either T has a blue open edge attached to $\tilde{T}$ at m', or to obtain $\tilde{T}$ from T we delete "yellow open edge $\to$ blue genus 0 multiplicity 1 vertex $\to$ blue closed edge" from m'. In the first instance a simple scaling of $z \mapsto \pi^n z$ with $n = -d_{\mathcal{R}}$ gives the desired result. In the second instance, there exists some $s \in \mathcal{R}$ with $s \notin D_{m'}$, all other $r \in \mathcal{R}$ lie in $D_{m'}$. Certainly s is fixed by G_K, and therefore lies in K. We can assume, after possibly scaling, that $D(\mathcal{R}) = D_{0,0}$, so $s \in \mathcal{O}_K$. Shifting by s we can assume that s is in fact 0. The shift $z \mapsto z - s$ does not change that $D(\mathcal{R}) = D_{0,0}$, so we have that, for all $r \in D_{m'}$, $v(r) = v(r - s) = d_{\mathcal{R}} = 0$. So we can assume without loss of generality that $s = 0$ and all elements of $\mathcal{R} \setminus \{s\}$ are units. We now apply $\phi_3 : z \mapsto \frac{1}{z}$ and, since $\mathcal{R}$ is even, take $\mathcal{R}' = \{\phi_3^{-1}(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even. In particular this gives that $\mathcal{R}' = \{\frac{1}{r} \mid r \neq 0 \in \mathcal{R}\}$ and $\mathcal{R}'$ is odd. Note that since $v(r) = 0$ for all $r \in \mathcal{R} \setminus \{s\}$ we have that $v\left(\frac{1}{r} - \frac{1}{r'}\right) = v(r - r')$. So, similarly to before, if f' has roots $\mathcal{R}'$ then its associated open quotient BY tree isomorphic to T'.

(ii) Recall that in this case m' is on a green part of B and the open edge of T' is attached at the closest vertex v_c to m' in B. As above, $\mathrm{denom}(d(v_c, m)) \nmid s(v_c, T)$, or $v_c = v_0$ and the open edge of T is blue, or v_c is a unique vertex incident to v_0 with $g(v_0) = 0$ and there is exactly one incident closed edge to v_0 in T which is blue.

First let us suppose that $\mathrm{denom}(d(v_c, m)) \nmid s(v_c, T)$, and we are not also in either of the other two situations. That is (because $M(v_c) = 1$) there exists some root $s \in \mathcal{R}$ that is fixed by G_K and is a child of the cluster associated to the disc v_c, that is $s \in K$. Note that since $\mathrm{denom}(d(v_c, m)) \nmid s(v_c)$ we must have $d(v_c, m) \notin \mathbb{Z}$ and all other singletons of v_c are in non-trivial G_K orbits. m' must not contain any roots that lie in proper children of v_c (else there would be a vertex v' corresponding to a proper child of v_c with m' lying on the path between v' and v_c i.e. m' would lie on the core so could not lie on a green part of B). Since m' is fixed by G_K we can choose $D_{m'}$ to contain $s \in K$. Note that, $D_{m'}$ cannot contain a second root of $\mathcal{R}$ since $D_{m'} \subset D_v \subseteq \mathcal{R}$ so if $s \neq r \in D_{m'}$ either r lies in a proper child of D_{v_c} (which we have already mentioned above cannot happen) or $\{r\} \wedge \{s\} = D_{v_c}$, both of which give contradictions. In particular, under the transformations we made above that saw us move $D_{m'}$ to $D_{0,0}$ (in the more general setting, before we entered this specific case) we also move s to 0. So, we can assume without loss of generality that $D_{m'} = D_{0,0}$ and that $s = 0$ is a root in $\mathcal{R}$. Now for all roots $r \in \mathcal{R} \setminus \{0\}$, the cluster $\{r\} \wedge \{0\}$ contains the cluster $D_{m'}$ and therefore has depth $d_{\{r\}\wedge\{0\}} < 0$. Furthermore, $v(r) = v(r-s) = d_{\{r\}\wedge\{0\}}$, so $v(r) < 0$ for all $r \in \mathcal{R}\setminus\{0\}$. Finally we can apply $\phi_3 : z \mapsto \frac{1}{z}$ and find that all roots $r \in \mathcal{R}\setminus\{0\}$ are mapped to elements of $D_{0,0}$, which is the image of $D_{m'}$. Take $\mathcal{R}' = \{\phi_3^{-1}(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even and $\mathcal{R}' = \{\phi_3^{-1}(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd. In either situation, under our assumption that $s = 0$ is a root, we get that $\mathcal{R}'$ is odd. So, if f' has roots $\mathcal{R}'$ then its associated open quotient BY tree isomorphic to T'.

Suppose instead that $v_c = v_0$ and the open edge of T is blue. Note that it is also possible that $\mathrm{denom}(d(v_c, m)) \nmid s(v_c, T)$, in which case there will be two green edges attached to v_c in blue. If we lie on the green edge arising from the fact that $\mathrm{denom}(d(v_c, m)) \nmid s(v_c, T)$, then v_c will lie on the path between m and m'. This situation is handled in the paragraph above. Otherwise, a simple scaling gives the desired result.

Suppose instead that $g(v_0) = 0$, there is exactly one closed edge incident to v_0 in T, which is blue and has multiplicity 1, and v_c is the unique vertex adjacent to v_0 in T. Note that, by Proposition 4.3.9, v_0 and the unique closed blue edge incident to v_0 gets removed (along with the open edge of T) when passing from T, furthermore the open edge of T must be yellow. Note that again, it is possible that $\mathrm{denom}(d(v_c, m)) \nmid s(v_c, T)$ and v_c lies on the path between m and m'. The first paragraph deals with this situation. So let's assume that v_c is not on the path between m and m'. Since $g(v_0) = 0$, we have that $s(v_0, T) = 1$ and $d(m, v_0) \in \mathbb{Z}$. Let us call the one singleton of $D(v_0)$, s. Note that s is fixed by G_K, therefore lies in K. The only children of $\mathfrak{s}(v_0)$ are the one proper odd child, and $\{s\}$. All roots in $\mathcal{R} \setminus \{s\}$ lie in the disc corresponding to $v_0' = v_c$. Note that $D(\mathfrak{s}(v_0))$ has integer radius and the centre can be chosen to be s. So, after a shift and scaling we can assume that $D(\mathfrak{s}(v_0)) = D_{0,0}$ and $s = 0$. Since $\{s\} < \mathfrak{s}(v_0)$ we have that for all $r \in \mathcal{R} \setminus \{s\}$ $\{r\} \wedge \{s\} = \mathfrak{s}(v_0)$. In particular, we have that $v(r) = v(r - s) = d_{\{r\}\wedge\{s\}} = 0$. After inversion we get $\mathcal{R}' = \{\frac{1}{r} \mid r \in \mathcal{R}\} \setminus \{\infty\}$. Note that $D(\mathfrak{s}(v_0)) = D_{0,0}$ is fixed by this transformation, and the image of $D(\mathfrak{s}(v_c))$ is contained in $D_{0,0}$. However, since $s = 0$ is sent to infinity, the image of $D(\mathfrak{s}(v_c))$ is now the smallest cluster containing all roots in $\mathcal{R}'$, which is odd. So far, our transformations result in an open quotient BY tree with core $\tilde{T}$, and blue open edge attached to $\tilde{T}$ as v_c, with marked point distance $d(v_0, v_c)$ along the open edge. Scaling further by $d(m', v_0)$ gives the desired result. Note that this gives the total scaling to be $\frac{\pi^{d(m',v_0)}}{\pi^{d(m,v_0)}} = \pi^{\pm d(m,m')}$.

(iii) The final situation that can happen in this case is that m' does not lie on $\tilde{T}$, and in B, m' is not coloured green. As stated above, the open edge of T' is attached at the vertex $v_c \in V(\tilde{T})$ which is closest to m' in B. So, $v_0' = v_c$ and have that v_0' is blue with

$$2g(v_0') + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_0'}} M(e).$$

Furthermore, if $g(v_0') = 0$ then v_0' has at least one yellow incident edge in $\tilde{T}$, and if $\#\{\text{incident blue edges to } v_0' \text{ in } T\} \neq \#\{\text{incident blue edges to } v_0' \text{ in } T'\}$ then $d(m', v_0') \in \mathbb{Z}$.

First, suppose that

$$\#\{e \in E(T_b) \text{ incident to } v_0'\} = \#\{e \in E(T_b') \text{ incident to } v_0'\}.$$

That is there exists a blue edge incident to v_0' in T that gets deleted when passing to the core. We know v_0' lies on the core, so there are two possibilities. Either $v_0 = v_0'$ and the open edge of T is coloured blue, or to obtain T from $\tilde{T}$ we attach "open edge $\to$ genus 0 multiplicity 1 blue vertex $\to$ closed blue edge" to v_0'. In the first instance a simple scaling will give the desired result. In the second instance, much as in one situation of the previous case (ii), there is precisely one root of $\mathcal{R}$, say s which does not lie in the disc associated to v_0'. Furthermore, since m' is not green, m' must lie on an open black edge attached to v_0', thus $d(m, v_0') \in \mathbb{Z}$. Without loss of generality we can assume that $s = 0$ and the disc associated to v_0 is $D_{0,0}$. All other roots $r \in \mathcal{R} \setminus \{0\}$ will then have $v(r) = 0$, and an inversion (followed perhaps by a scaling) gives the desired result as in previous cases.

Suppose instead that

$$\#\{e \in E(T_b) \text{ incident to } v_0'\} \neq \#\{e \in E(T_b) \text{ incident to } v_0'\},$$

so we also have that $d(m', v_0') \in \mathbb{Z}$. Similarly to in case (a2)(ii), we can assume that $D_{m'} = D_{0,0}$ and, after perhaps shifting further by some $y \in \mathcal{O}_K$, we can also assume that some $s \in \mathcal{R}$ is such that $s = 0 \in D_{m'}$. Every $r \in D_{m'} \setminus \{0\}$ must be a unit and all other roots $r \notin D_{m'}$ have $v(r) < 0$. Applying $z \mapsto \frac{1}{z}$ and taking $\mathcal{R}' = \{\frac{1}{r} \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even, and $\mathcal{R}' = \{\frac{1}{r} \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd gives the desired result.

(b) Next, let us suppose that $\tilde{T}'$ is obtained from T' by removing the open edge and viewing v_0' as a point on an edge. Note that in this situation the open edge of T' is always coloured yellow. We have the following cases:

(i) Theorem 4.6.4 case (B1): m' lies on an edge e of $\tilde{T}$ (NB: it may be a vertex of B but not of $\tilde{T}$). Then T' is obtained from $\tilde{T}$ by creating a genus 0, multiplicity 1 vertex at m', the same colour as the edge e, and adding an open yellow edge at m'.

(ii) Theorem 4.6.4 case (D1): m' does not lie on $\tilde{T}$, m' is not green, and the closest point P on $\tilde{T}$ to m' is not a vertex of $\tilde{T}$. Then T' is obtained from $\tilde{T}$ creating a genus 0, multiplicity 1 vertex at P on $\tilde{T}$ to m', coloured the same as the edge P lies on in $\tilde{T}$ and attaching an open yellow edge here.

Note that, by construction of B, in case (ii) the point P must be an integer distance from m'. In particular, (ii) can be obtained from (i) by a simple scaling $z \mapsto \pi^{d(P,m')}z$. For this reason we will not address (ii) here as it will follow from our proof of (i).

(i) Suppose that m' lies on an edge e of $\tilde{T}$ and that T' is obtained from $\tilde{T}$ by creating a genus 0, multiplicity 1 vertex at m', coloured the same as e, and adding an open yellow edge at m'. Certainly $D_{m'}$ can be chosen so that at least one root $s \in \mathcal{R}$ lies in $D_{m'}$. As in some of the previous cases, we can assume that $D_{m'} = D_{0,0}$ and all roots in $D_{m'}$ are units. All roots outside of $D_{m'}$ must have negative valuation, since if $r \notin D_{m'}$ then

$$v(r) = v(r-0) = d_{\{r\}\wedge\{0\}} < d_{D_{m'}} = 0.$$

Taking $v \mapsto \frac{1}{z}$ gives the desired result.

(c) Finally, let us suppose that $\tilde{T}'$ is obtained from T' by removing "open edge $\to$ genus 0 mult 1 blue vertex $\to$ closed blue edge". In this situation, when considered in B (the full extended tree of T as defined in Construction 4.5.1), we have the following cases:

(i) Theorem 4.6.4 case (C1): m' does not lie on $\tilde{T}$ and m' lies on a green part of B. In this case we can create a blue genus 0, multiplicity 1 vertex v_0' which is attached to the vertex of $\tilde{T}$ which is closest to m' in B via a blue multiplicity 1 closed edge, and add a yellow open edge at v_0'.

(ii) Theorem 4.6.4 case (E3): m' does not lie on $\tilde{T}$ and is not coloured green in B, and the closest point on $\tilde{T}$ to m' is a vertex $v_c \in V(\tilde{T})$

where v_c is blue,

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e),$$

and if $g(v_c) = 0$ then m' has at least one incident yellow edge in $\tilde{T}$. Finally, we require that if $\#\{\text{incident blue edges to } v_c \text{ in } T\} \neq \#\{\text{incident blue edges to } v_c \text{ in } T'\}$ then $d(m', v_c) \in \mathbb{Z}$. In this case we can create a blue genus 0, multiplicity 1 vertex v_0' which is attached to v_c via a blue multiplicity 1 closed edge, and add a yellow open edge at v_0', insuring that $d(m', v_0') \in \mathbb{Z}$.

Again we will consider both of these cases separately.

(i) Suppose that m' does not lie on $\tilde{T}$, m' is green, and T' is created from T by adding to $\tilde{T}$ in the following way. Let v_c denote that vertex in $V(\tilde{T})$ which is closest to m' in B. Create a new genus 0, multiplicity 1 vertex v_0' such that v_0' lies between v_c and m' in B, and v_0' is an integer distance from m'. To obtain T', attach v_0' to $\tilde{T}$ at v_c via a closed blue edge and add an open yellow edge at v_0' with the distances between v_0', v_c, and m' as they are in B. Note that, by Construction 4.5.1 at least one of the following hold:

- $v_c = v_0$ and the open edge of T is blue,
- v_0 (the unique vertex incident to the open edge in T) is blue of genus 0 and there is exactly one closed edge incident to v_0 in T (the edge to v_c), and this edge is blue and has multiplicity 1,
- or v_c is such that $\mathrm{denom}(d(v_c, m)) \nmid s(v_c, T)$.

Let us first suppose that $\mathrm{denom}(d(v_c, m)) \nmid s(v_c, T)$ and $D_{m'} \subset D_{s(v_c)}$. That is, as in case (ii) in the previous situation, because $M(v_c) = 1$ there exists some root $s \in \mathcal{R}$ that is fixed by G_K and is a child of the cluster associated to the disc in $\mathscr{T}_{K(\mathcal{R})}$ corresponding to v_c. That is, $s \in K$. Again, we have that $d(v_c, m) \notin \mathbb{Z}$ and all other singletons of v_c are in non-trivial Galois orbits. Note that $D_{m'}$ must not contain any roots that lie in proper children of v and $D_{m'}$ must contain exactly one root of v_c, else this would contradict v_c being the closest vertex on $\tilde{T}$ to m'. Under the usual transformations we can assume without loss of generality that $D_{m'} = D_{0,0}$, and $s = 0$. All other roots $r \in \mathcal{R} \setminus \{s\}$ have $v(r) = v(r - s) = d_{\{r\} \wedge \{s\}} < d_{D_{m'}} = 0$. Let $d = d(m', v_0')$ and note that $d \in \mathbb{Z}_{\geq 0}$. Note that $d(m', v_0')$, i.e.

the distance to m' along the open edge of T' can be achieved by a simple scaling at the end. As such, we will assume for simplicity that $d(m', v_0') = 0$ and correct this at the end by a simple scaling if this is not the case. We now make a shift by a unit, say 1, $z \mapsto z+1$ so that $s = 1$ is a unit, and all other roots have had their valuations unchanged by this shift. Note that under this shift $D_{0,0}$ is fixed, that is $D_{m'}$ remains in the same place. Inversion, $z \mapsto \frac{1}{z}$, and taking taking $\mathcal{R}' = \{\frac{1}{r} \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even and $\mathcal{R}' = \{\frac{1}{r} \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd, now gives us the desired result. To see this note that $D_{m'}$ is mapped to itself, i.e. $D_{0,0}$, all roots are mapped to roots in $D_{0,0}$ (since $v(\frac{1}{r}) = -v(r)$) and finally $1 \mapsto 1$, whilst all other roots r get mapped to roots with positive valuation, meaning $v(1 - \frac{1}{r}) = 0$, in particular $\{1\} \wedge \{\frac{1}{r}\} = D_{0,0}$, so this is a vertex of the resulting open quotient BY tree. That the correct colouring is given is clear since $\mathcal{R}'$ is even regardless of the parity of $\mathcal{R}$.

Now let us suppose that $D_{m'} \supseteq D_{v_c}$ and v_0 is blue, of genus 0, multiplicity 1, has exactly one closed incident edge in T, and this edge is blue and has multiplicity 1. Note that v_0 will get removed when passing from T to $\tilde{T}$ and that the unique vertex incident to v_0 in T will be v_c. In particular this means that to get from $\tilde{T}$ to T we must have to add "open edge $\to$ genus 0 multiplicity 1 blue vertex $\to$ closed blue edge" to v_c. If $d(v_0, v_c) = d(v_0', v_c)$ then we can apply a simple scaling to get the desired result. So, let us suppose that $d(v_0, v_c) \neq d(v_0', v_c)$. Let us also suppose for now that $d(v_0, v_c) < d(v_0', v_c)$. By construction, the disc corresponding to v_0 contains all roots $r \in \mathcal{R}$. So, given m' is on the open edge of T, $r \in D_{m'}$ for all $r \in \mathcal{R}$. All but one root in $\mathcal{R}$ lies in the disc corresponding to v_c. Let us denote the unique element of $\mathcal{R}$ that does lie in $D(\mathfrak{s}(v_c))$ by s. After a shift and a scaling we can assume that $D(\mathfrak{s}(v_0)) = D_{0,0}$, and $s = 0$. Since $\{s\} < D(\mathfrak{s}(v_0))$, we have that $v(r) = v(r - s) = d_{\{s\} \wedge \{r\}} = 0$. Scaling by $d(v_0, v_0')$, that is applying $z \mapsto \pi^{-d(v_0, v_0')} z$, scales the roots in $\mathcal{R} \setminus \{s\}$ but leaves $s = 0$ fixed. Note that every $r \in \mathcal{R} \setminus \{s\}$ is mapped to something with valuation $-d(v_0, v_0')$ We can now shift by a unit, say 1, $z \mapsto z + 1$ so that all $r \in \mathcal{R}$ are still mapped to elements with valuation $-d(v_0, v_0')$, and s is mapped to a unit. Inversion, followed by a scaling $z \mapsto \pi^{d(v_0', m')}$ so that m' is the correct distance along the open edge, now gives the desired result. If instead $d(v_0, v_c) > d(v_0', v_c)$ then, we instead note that taking a different marked point lying at v_0' satisfies the required conditions for (c)(ii), with $\#\{\text{incident blue edges to } v_c \text{ in } T\} \neq \#\{\text{incident blue edges to } v_c \text{ in } T'\}$. So, we refer to this part of the proof instead. Taking v_0' to be the marked point in the proof of (c)(ii) (where it is labeled m'), and then performing a simple scaling of magnitude $d(v_0, m')$ at the end gives the desired result.

Suppose instead that $D_{m'} \supseteq D_{v_c}$ and $v_0 = v_c$ and the open edge of T is blue. Then we can scale and then shift all roots $r \in \mathcal{R}$ so that $v(r) = 0$ and $d_{\mathcal{R}} = d(v_c, v_0')$. Inverting and then scaling if necessary gives the desired result.

(ii) Suppose now that m' does not lie on $\tilde{T}$, is not coloured green in B, and the closest vertex of $\tilde{T}$ to m' is v_c where v_c is blue,

$$2g(v_c) + 2 > \sum_{\substack{e \in E(\tilde{T}) \text{ blue,} \\ \text{incident to } v_c}} M(e),$$

and if $g(v_c) = 0$ then m' has at least one incident yellow edge in $\tilde{T}$. Finally we have that, if $\#\{\text{incident blue edges to } v_c \text{ in } T\} \neq \#\{\text{incident blue edges to } v_c \text{ in } T'\}$ then $d(m', v_c) \in \mathbb{Z}$. To obtain T' we create a blue genus 0, multiplicity 1 vertex v_0' which is attached to v_c via a blue multiplicity 1 closed edge, and add a yellow open edge at v_0', so that $d(v_0', m') \in \mathbb{Z}$.

First let us suppose that

$$\#\{e \in E(T_b) \text{ incident to } v_c\} = \#\{e \in E(T_b') \text{ incident to } v_c\}.$$

So, there are two possibilities, either $v_c = v_0$ and the open edge of T is coloured blue, or to obtain T from $\tilde{T}$ add "open edge $\to$ genus 0 multiplicity 1 blue vertex $\to$ closed blue edge" to v_c. In the first instance, as above we can assume that all roots are units (perhaps after a shift) and that $D_{m'} = D_{0,0}$ before inverting which gives us the desired result. To see this just note that, since $D_{m'} = D_{0,0}$ and all roots in $\mathcal{R}$ are units, inversion has no effect on the distances between these roots but does introduce an additional root at 0. In the second instance, m' lies on a black open edge which is attached at v_c. There is precisely one root, say $s \in \mathcal{R}$ that is not in the disc associated to v_c. Taking m'' to be a disc on the open edge of T with $d(m,m') = d(m,m'')$ and adding "open edge $\to$ genus 0 multiplicity 1 blue vertex $\to$ closed blue edge" to v_c with m'' as the marked point results in exactly the same picture T'. However, in this situation m'' is green and we lie in a case previously discussed so we can apply the same result.

Suppose instead that

$$\#\{e \in E(T_b) \text{ incident to } v_c\} \neq \#\{e \in E(T_b') \text{ incident to } v_c\}.$$

In particular, this means that no blue edge is removed from v_c when passing from T to $\tilde{T}$, that is:

$$\#\{e \in E(T_b) \text{ incident to } v_c\} \neq \#\{e \in E(\tilde{T}_b) \text{ incident to } v_c\}.$$

So, in T, $s(v_c, T) > 0$ and there exists at least one singleton in the cluster corresponding to v_c. Furthermore, under our requirement that in this case $d(m', v_c) \in \mathbb{Z}$, all children of $\mathfrak{s}(v_c)$ are fixed by G_K. So, in T, v_c contains a root, say $s \in \mathcal{R}$ which lies in no other clusters and is fixed by Galois, i.e. $s \in K$. Note that in this situation, either $v_c = v_0$, in which case the open edge ε of T must be coloured yellow, or $v_c \neq v_0$, in which case $D_{m'} \subset D(\mathfrak{s}(v_c))$. In the first instance, note that, since $d(v_c, m) \in \mathbb{Z}$, v_c has an open black edge attached to it in B. We can assume that m' lies on this open black edge rather than on the open edge of T. The reason we can assume this is because we are only looking for f' with BY tree isomorphic to T' and the combinatorial construction of T' does not mind where we picked our m' to be as long as the distance from v_c is correct and m' satisfies any conditions required to be able to create T' from $\tilde{T}$. So, since every edge and vertex in B has multiplicity 1 (or no multiplicity assigned i.e. multiplicity 1), we can consider B as lying on the Bruhat-Tits tree in such a way that m' contains s and $D_{m'} \subset D(\mathfrak{s}(v_c))$. So regardless of whether $v_c = v_0$ or not, we can assume that $D_{m'} \subset D(\mathfrak{s}(v_c))$ and $s \in D_{m'}$. Since $D_{m'} \subsetneq D(\{s\} \wedge \{r\})$ for all $r \in \mathcal{R} \setminus \{s\}$ we must have that all other roots $r \neq s$ do not lie in $D_{m'}$. As in many other cases, we can assume that $D_{m'} = D_{0,0}$ and $s = 0$, therefore $v(r) < 0$ for all $r \neq s$. Shifting by a unit we can assume that s is instead a unit. Inversion then gives the desired result.

$\square$

Chapter 6

Curves and Open Quotient BY Trees

6.1 Results for Hyperelliptic Curves

For simplicity, the preceding sections stated results in more generality than we require. Given we are concerned with classification of hyperelliptic curves, we now want to pull results from earlier sections, stating them for hyperelliptic curves rather than general open quotient BY trees or cluster pictures.

Definition 6.1.1. Let $C : y^2 = f(x)$ be a hyperelliptic curve over K with tame reduction. Then f is a square free polynomial in $K[x]$ with tamely ramified splitting field and the cluster picture $\Sigma_{C/K}$ is of polynomial type. As such, we define the *open quotient BY tree associated to C over K* to be $\underline{\mathrm{T}}(\Sigma_{C/K})$.

Open quotient BY trees have been carefully constructed with this in mind. In particular they have been constructed to be quotients of open BY trees, in the sense of Definition 2.2.3, hence their name. This is something we are now able to study, after constructing a sensible metric on $q^{-1}(T)$.

Construction 6.1.2. Here we make the comments in Remark 4.2.4 more formal. If T is the metric open quotient BY tree associated to a hyperelliptic curve C/K and L/K is such that C/L is semistable, then we can define a metric associated to L/K on $q^{-1}(T)$. In particular, for an edge $e \in E(T)$ we define the length of each $q^{-1}(e)_i$ to be

$$l(q^{-1}(e)_i) = \begin{cases} [L:K]l(e) & \text{if } e \text{ is blue,} \\ 2[L:K]l(e) & \text{if } e \text{ is yellow.} \end{cases}$$

Writing $T' = q^{-1}(T)$, this gives that for any edge $e' \in E(T')$,

$$l(q(e')) = \begin{cases} \frac{l(e')}{[L:K]} & \text{if } e' \text{ is blue,} \\ \frac{l(e')}{2[L:K]} & \text{if } e' \text{ is yellow.} \end{cases}$$

Proposition 6.1.3. *Let C/K be a hyperelliptic curve with tame reduction, and let L/K be a field extension such that C is semistable over L. Let T be the open quotient BY tree associated to C/K, then $q^{-1}(T)$ (with the metric defined in Construction 6.1.2) is isomorphic to the metric open BY tree associated to C/L.*

Proof. We already proved in Theorem 4.2.6 that $q^{-1}(T)$ is an open BY tree. It is not hard to see from the construction that every proper cluster in $\Sigma_{C/K}$ contributes one vertex to $q^{-1}(T)$, which is coloured yellow if the cluster is übereven and blue otherwise. Proper clusters in $\Sigma_{C/K}$ are in one-to-one correspondence with proper clusters in $\Sigma_{C/L}$. Likewise, it is not hard to see that the edge set, colouring of edges, and genera of vertices are as required. So certainly the non-metric version of this statement is true. To see that the metrics are the same it is enough to note that when we take a field extension, the effect on the depths is that they are multiplied by the degree of the extension. Taking into account the convention for yellow edges in BY trees to have length $2\delta_{\mathfrak{s}}$, rather than $\delta_{\mathfrak{s}}$, gives us that the metrics are indeed the same. □

Remark 6.1.4. If T' is a metric closed quotient BY tree, then we can define the lengths of edges just as in Construction 6.1.2, except for an exceptional edge between $q^{-1}(v')_1$ and $q^{-1}(v')_2$, which we give length

$$l([q^{-1}(v')_1, q^{-1}(v')_2]) = \begin{cases} 2[L:K]l(e) & \text{if } e \text{ is blue,} \\ 4[L:K]l(e) & \text{if } e \text{ is yellow,} \end{cases}$$

where e is the edge between v' and the exceptional vertex in T' which gives rise to this edge.

We can use our equivalence relation on open quotient BY trees to study hyperelliptic curves. In particular, the work in Sections 5.3 and 5.4 gives us the following two results. Firstly, for a hyperelliptic curve C/K one can find an isomorphic hyperelliptic curve with any given open quotient BY tree equivalent to that of C. Stated more formally:

Theorem 6.1.5. *Let $C : y^2 = f(x)$ be a hyperelliptic curve with cluster picture Σ, $d_{\mathcal{R}} \geq 0$, and open quotient BY tree T. Let T' be an open quotient BY tree equivalent to T, then there is a K-isomorphic curve C'/K with cluster picture Σ', and $d_{\mathcal{R}'} \geq 0$ such that $\underline{T}(\Sigma') \cong T'$.*

This follows as a direct result of Theorem 5.4.1 along with the following remark.

Remark 6.1.6. Suppose that $C : y^2 = f(x)$ is a hyperelliptic curve. Just as stated in the proof of Proposition 14.6 in [DDMM18], for a Möbius transformation $\psi(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in K$, a change of variables of the form

$$x = \frac{ax' + b}{cx' + d}, \quad y = \frac{y'}{(cx' + d)^{g+1}}$$

gives a model for C/K of the form $y'^2 = f'(x')$. The set of roots of $f'(x')$ is precisely $\mathcal{R}' = \{\psi^{-1}(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even and $\mathcal{R}' = \{\psi^{-1}(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd.

So, we can take C'/K to be the model afforded by making the appropriate change of coordinates for the Möbius transformations in the proof of Theorem 5.4.1. Writing Σ' for the cluster picture of C'/K, gives $\underline{T}(\Sigma') \cong T'$ as required.

Conversely, we can find a Möbius map between any two equivalent open quotient BY trees. This is formally stated below in Theorem 6.1.5. It is worth noting that this a generalisation of [DDMM18, Corollary 14.7].

Theorem 6.1.7. *Let C be a hyperelliptic curve over K, with set of roots $\mathcal{R}$ such that $d_{\mathcal{R}} \geq 0$, and open quotient BY tree T. Take a Möbius transformation $\phi(z)$ such that $d_{\mathcal{R}'} \geq 0$, where $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R}\} \setminus \{\infty\}$ if $\mathcal{R}$ is even, and $\mathcal{R}' = \{\phi(r) \mid r \in \mathcal{R} \cup \{\infty\}\} \setminus \{\infty\}$ if $\mathcal{R}$ is odd. Let C'/K be a hyperelliptic curve with set of roots $\mathcal{R}'$ and let T' be the open quotient BY tree of C'/K. Then T and T' are equivalent.*

This follows directly from Theorem 5.3.15 by writing $C : y^2 = f(x)$ and $C' : y^2 = g(x)$, and applying Theorem 5.3.15 to f and g. It is worth noting that if we wish to work with curves with negative top cluster depths, then a simple scaling (at either end of the process) will allow us to work with a model with non-negative top cluster instead and thus apply Theorem 5.3.15. What this shows us is, the canonical representative of an open quotient BY trees associated to a hyperelliptic curve is model invariant. That is, choosing a different model for a hyperelliptic curve does not change the canonical representative.

6.2 Genus

It would be useful, in order to produce a usable classification, to have a formula for the genus of a quotient BY tree. In particular, if T is an open quotient BY tree, we would like the genus of T to equal the genus of a hyperelliptic curve with cluster picture $\underline{\Sigma}(T)$. This will give a way of listing all open quotient

BY trees corresponding to hyperelliptic curves of a given genus. To do this we return to the semistable situation, and examine what is known there.

Definition 6.2.1 ([DDMM17, Definition 3.23])**.** For a closed or open BY tree T, the genus of T is

$$g(T) = \#\{\text{connected components of } T_b\} - 1 + \sum_{v \in V(T)} g(v).$$

There are several useful propositions in [DDMM17] that tell us how the genus of a BY tree relates to that of a hyperelliptic curve (with semistable reduction). In particular, by [DDMM17, Proposition 4.17], for an open BY tree T, $g(T) = g(\underline{\Sigma}(T))$. Conversely, by [DDMM17, Proposition 4.19], if Σ is a cluster picture then $g(\Sigma) = g(\underline{T}(\Sigma))$. Note that if Σ is a cluster picture of polynomial type, and is the cluster picture of a hyperelliptic curve, say C, then $g(\Sigma) = g(C)$. Other useful results include [DDMM17, Proposition 5.7] which gives that if T is an open BY tree with core $\tilde{T}$, then $g(T) = g(\tilde{T})$. We can use quotients of BY trees to easily obtain similar results for open quotient BY trees and hyperelliptic curves with tame reduction. We define the genus of quotient BY tree as follows.

Definition 6.2.2. Let T be a (closed or open) quotient BY tree and let $B_1, \ldots, B_n$ be the connected components of T_b. Then the genus of T is

$$g(T) = \left(\sum_{i=1}^{n} \min_{w \in V(B_i)} \{M(w)\} \right) - 1 + \sum_{v \in V(T)} g(v) M(v).$$

It is worth noting the following:

Proposition 6.2.3. *For an open quotient BY tree T, $g(T) = g(\tilde{T})$.*

Proof. By Proposition 4.3.9, the only vertices removed when passing to the core of an open quotient BY tree are of genus 0, so the last part of the genus formula is clearly unaltered. It remains to show that the first term in the formula remains unchanged when passing from T to $\tilde{T}$.

Suppose first that v_0 is yellow. Then the open edge ϵ must be yellow and, by Proposition 4.3.9, we know that no blue vertex or edge is deleted when passing from T to $\tilde{T}$. So we are done.

Suppose instead that v_0 is blue. Deleting the open edge, regardless of the colouring, does not have any effect on the genus formula. Let B_1 be the connected component of T_b which contains v_0. Recall that $M(v_0) = 1$. By

Proposition 4.3.9, we know that v_0 gets deleted (along with a closed blue multiplicity 1 edge) if $g(v_0) = 0$ and it has only one incident closed edge, which is blue of multiplicity 1. If instead $g(v_0) = 0$ and v_0 has exactly two closed incident edges, both coloured the same as v_0 and of multiplicity 1, then v_0 is viewed as a point on an edge. In each of these cases v_0 has an incident closed blue edge of multiplicity 1, that is there exists some blue, multiplicity 1 vertex $v' \in V(T)$ that lies in B_1. Neither of these two situations leaves B_1 disconnected after removing v_0 (be that deleting v_0 along with a closed edge or viewing v_0 as a point on an edge). All other connected components of T_b remain untouched. In particular, the connected components of $\tilde{T}_b$ are B_1' and $B_2, \dots, B_n$, where $B_1' \subseteq B_1$ and $\min_{w \in V(B_1')}\{M(w)\} = \min_{w \in V(B_1)}\{M(w)\} = 1$. □

Corollary 6.2.4. *Suppose $T \sim T'$ are two equivalent open quotient BY trees, then $g(T) = g(T')$.*

Proof. Since T and T' are equivalent, $\tilde{T} \cong \tilde{T}'$. The result follows. □

The remainder of this section is dedicated to proving that this definition does what we want, in particular proving the following two propositions.

Proposition 6.2.5. *Let T be an open quotient BY tree with open edge ε, and associated cluster picture $\underline{\Sigma}(T) = (R, \Sigma)$. Then*

$$|\mathcal{R}| = \begin{cases} 2g(T) + 2 & \text{if } \varepsilon \text{ is yellow,} \\ 2g(T) + 1 & \text{if } \varepsilon \text{ is blue.} \end{cases}$$

In particular, $g(T) = g(\Sigma) = g(C)$, where C is a hyperelliptic curve with cluster picture Σ.

Proposition 6.2.6. *If Σ is a cluster picture, with associated open quotient BY tree $\underline{T}(\Sigma)$, then $g(\Sigma) = g(\underline{T}(\Sigma))$.*

The idea behind the proofs of these propositions is simple; for a hyperelliptic curve C with cluster picture Σ, the genus is unaffected by taking a field extension. Extending the field to obtain a semistable hyperelliptic curve C' enables us to see that $g(C) = g(T')$ where T' is the open BY tree associated to C'. The genus of T' is phrased in terms of the connected components of $(T')_b$ and the genus of vertices of T'. These can be rephrased in terms of T using the quotient map from T' to T. This allows us to prove that the formula defined above indeed gives $g(T) = g(\Sigma) = g(C)$. For the converse, we use that if T is an open quotient BY tree then we can construct a BY tree T' such that T is the "quotient" of T'.

Lemma 6.2.7. *Let T be an open quotient BY tree, and $T' = q^{-1}(T)$ as defined in Construction 4.2.2. Let B be a connected component of T_b then there exist $\min_{v \in V(B)} M(v)$ connected components in $(T')_b$ which map to B under q.*

Proof. Let $u \in V(B)$ be the vertex closest to the open edge of T. Note that u is necessarily unique in this way since T is a tree. By definition of T being an open quotient BY tree, multiplicities decrease as we head towards the open edge. Thus, $\min_{v \in V(B)} M(v) = M(u)$. By construction, u gives $M(u)$ vertices in T', and every vertex $v \preceq u$ gives $\frac{M(v)}{M(u)}$ vertices $\preceq q^{-1}(u)_i$ for each $1 \leq i \leq M(u)$. Again, by construction, there are no edges in T' between $q^{-1}(u)_i$ and $q^{-1}(u)_j$, therefore, there are precisely $\min_{v \in V(B)} M(v)$ connected components in $(T')_b$ which map to B under q. $\square$

Proof of Proposition 6.2.5. Let T be an open quotient BY tree with open edge ε and associated cluster picture $\underline{\Sigma}(T) = (R, \Sigma)$. By Theorem 5.1.19 we know that $\underline{\Sigma}(T)$ is of polynomial type, that is there exists some hyperelliptic curve C/K with tame reduction such that $\Sigma_{C/K} \cong \underline{\Sigma}(T)$. Let L/K be a field extension such that C/L is semistable. Note that the genus of C is not changed by extending the field. Then, by Proposition 6.1.3, $T' = q^{-1}(T)$ is the open BY tree associated to $\Sigma_{C/L} = (R', \Sigma')$. So, $\underline{\Sigma}(T') \cong \Sigma_{C/L}$ and, by [DDMM17, Proposition 4.17], we have that

$$|R'| = \begin{cases} 2g(T') + 2 & \text{if } \varepsilon \text{ is yellow,} \\ 2g(T') + 1 & \text{if } \varepsilon \text{ is blue.} \end{cases}$$

In particular, $g(T') = g(\Sigma_{C/L}) = g(C) = g(\Sigma_{C/K})$. By definition

$$g(T) = \left(\sum_{i=1}^{n} \min_{w \in V(B_i)} \{M(w)\} \right) - 1 + \sum_{v \in V(T)} g(v) M(v),$$

and by Lemma 6.2.7 and Construction 4.2.2 we can conclude that

$$\begin{aligned} g(T) &= \left(\sum_{i=1}^{n} \min_{w \in V(B_i)} \{M(w)\} \right) - 1 + \sum_{v \in V(T)} g(v) M(v), \\ &= \#\{\text{connected components of } T'_b\} - 1 + \sum_{v' \in V(T')} g(v'), \\ &= g(T'). \end{aligned}$$

Since the number of roots is unaltered by extending the field we have that $|R'| = |R|$. □

Proof of Proposition 6.2.6. This follows by a similar argument to the proof of Proposition 6.2.5, and evoking Proposition 4.19 from [DDMM17]. □

6.3 Leading coefficient

It remains to discuss how the valuation of the leading coefficient can change under Möbius transformation. This will allow us to give a relation between the leading coefficients of two hyperelliptic curves with equivalent open quotient BY trees, which ensures that they have the same reduction type. We will prove the following theorem:

Theorem 6.3.1. *Let $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ be hyperelliptic curves of genus g over K, with open quotient BY trees $T = \underline{T}(\Sigma_{C/K})$ and $T' = \underline{T}(\Sigma_{C'/K})$. Suppose that the sets of roots $\mathcal{R}$ and $\mathcal{R}'$ of f and f' respectively are such that $d_{\mathcal{R}}, d_{\mathcal{R}'} \geq 0$. Write c_f and $c_{f'}$ for the leading coefficients of f and f'. Then the dual graphs of the special fibres of the minimal SNC models of C and C' are isomorphic if T and T' are equivalent and:*

- *when g is even: if*

$$v\left(\mathrm{disc}\left(\frac{f}{c_f}\right)\right) - v\left(\mathrm{disc}\left(\frac{f'}{c_{f'}}\right)\right) \equiv 2(g+1)(2g+1)d(m,m') \mod 4(2g+1)$$

  then $v(c_f) \equiv v(c_{f'}) \mod 2$, else $v(c_f) \not\equiv v(c_{f'}) \mod 2$

- *when g is odd: then*

$$\frac{v\left(\mathrm{disc}\left(\frac{1}{c_f}f\right)\right) - v\left(\mathrm{disc}\left(\frac{1}{c_{f'}}f'\right)\right)}{2(2g+1)} \equiv v(c_f) - v(c_{f'}) \mod 2$$

If either $d_{\mathcal{R}}, d_{\mathcal{R}'} < 0$ then note that a simple scaling gives us a change of model, and will allow us to transform the cluster picture into something with non-negative top cluster depth. We will discuss how such a transformation affects the leading coefficient later. Piecing this together with the theorem above will allow us to handle changes in leading coefficients regardless of what the value of the top cluster depths are.

For any given hyperelliptic curve C/K, our main use of this theorem is to allow us to select an appropriate leading coefficient to go along with our canonical representative of $\underline{T}(\Sigma_{C/K})$, ensuring the reduction type of our canonical representative and this leading coefficient is the same as that of C/K. Before we prove Theorem 6.3.1, let us look at a relation between the discriminants of isomorphic hyperelliptic curves due to [Liu96].

6.3.1 Leading Coefficient and Discriminants

We define the discriminant of a hyperelliptic curve as in [DDMM18] and [Liu96].

Definition 6.3.2. Let $C : y^2 = f(x)$ be a hyperelliptic curve of genus g over K. The discriminant Δ_C of C is

$$\Delta_C = 16^g c_f^{4g+2} \mathrm{disc}\left(\frac{1}{c_f} f(x)\right).$$

Let $y^2 = f(x)$ and $y'^2 = f'(x')$ be two equations for a hyperelliptic curve C/K with discriminants Δ and Δ' respectively. By [Liu96, §2], we have that there exist substitutions

$$x = \frac{ax' + b}{cx' + d}, \quad y = \frac{ey'}{(cx' + d)^{g+1}},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K), \quad e \in K^*.$$

Furthermore, we obtain the following relation between their discriminants:

$$\Delta' = \Delta e^{-4(2g+1)} (ad - bc)^{2(g+1)(2g+1)}. \tag{6.1}$$

Remark 6.3.3. In [Liu96] there is an additional term, $H(x') \in K[x]$, in the y-coordinate substitution, but we omit this here since we take all our hyperelliptic curves to be of the form $y^2 = f(x)$.

This relation between Δ and Δ' provides us with an easy way to check whether two hyperelliptic curves are isomorphic. Note that isomorphic curves will always have the same reduction type, however the converse is not true and just because two curves have the same reduction type does not mean that they are isomorphic. Reduction types are a more crude classification than isomorphism classes of curves. This is important to note as it is a subtlety that means our theorems need to be worded carefully. The following example illustrates this.

Example 6.3.4. Consider the two hyperelliptic curves $C : y^2 = f(x)$ and $C' : y^2 = pf(x)$ over $\mathbb{Q}_p^{\mathrm{ur}}$, where $f(x) = x(x^2 - p^3)(x - 1)((x - 1)^2 - p^5)$.

Both C and C' have Namikawa-Ueno type III-III*-1. However, C and C' are not isomorphic. This can be easily seen by checking how their discriminants compare. By [Liu96, §2] we know that if C and C' are isomorphic then their respective discriminants Δ and Δ' will differ by a 20^{th} and 30^{th} power. That is, there exist some $a, b, c, d \in K$ and $e \in K^*$ with $ad - bc \neq 0$ such that $\Delta' = \Delta e^{20}(ad-bc)^{30}$. However, if we compare their discriminants, we have $\Delta = 16^2\mathrm{disc}(f(x))$ and $\Delta' = 16^2 p^{10}\mathrm{disc}(\frac{pf(x)}{p}) = p^{10}\Delta$. That is, we do not satisfy the discriminant condition that isomorphic curves must satisfy. So C and C' are not isomorphic even though the dual graphs of the special fibres of their minimal regular models are isomorphic.

To prove Theorem 6.3.1, we will make use of the effect that applying Möbius transformations has on the discriminant of a curve. So, before we proceed to the proof let us first prove a short lemma.

Lemma 6.3.5. *Let C be a hyperelliptic curve and ϕ be a Möbius transformation with $\phi(z) = \frac{az+b}{cz+d}$. Suppose that C' is the curve obtained from C by a change of coordinates*

$$x = \frac{ax'+b}{cx'+d}, \quad y = \frac{y'}{(cx'+d)^{g+1}}.$$

Then we have the following relationships between the valuations of the discriminants of C and C' when ϕ is a scaling, shift or inversion:

(i) If $\phi(z) = az$ then $v(\Delta_{C'}) \equiv v(\Delta_C) - v(a)2(g+1)(2g+1) \mod 4(2g+1)$.

(ii) If $\phi(z) = z + b$ then $v(\Delta_{C'}) \equiv v(\Delta_C) \mod 4(2g+1)$.

(iii) If $\phi(z) = \frac{1}{z}$ then $v(\Delta_{C'}) \equiv v(\Delta_C) \mod 4(2g+1)$.

Proof. This follows directly from relation (6.1). □

It is also helpful to note the following from [DDMM18, Theorem 16.2]:

Lemma 6.3.6. *Let $C : y^2 = f(x)$ be a hyperelliptic curve of genus g over K, and let Σ be the associated cluster picture. Then*

$$v(\Delta_C) = v(c_f)(4g+2) + \sum_{\mathfrak{s} \text{ proper}} d_{\mathfrak{s}} \left(|\mathfrak{s}|^2 - \sum_{\mathfrak{s}' < \mathfrak{s}} |\mathfrak{s}'|^2 \right).$$

Remark 6.3.7. Note that [DDMM18, Theorem 16.2] also then gives us that

$$v\left(\mathrm{disc}\left(\frac{1}{c_f} f\right)\right) = \sum_{\mathfrak{s} \text{ proper}} d_{\mathfrak{s}} \left(|\mathfrak{s}|^2 - \sum_{\mathfrak{s}' < \mathfrak{s}} |\mathfrak{s}'|^2 \right),$$

that is, if two hyperelliptic curves $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ have isomorphic cluster pictures then, $v\left(\mathrm{disc}\left(\frac{1}{c_f} f\right)\right) = v\left(\mathrm{disc}\left(\frac{1}{c_{f'}} f'\right)\right)$.

It is clear from this remark that two hyperelliptic curves $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$, with associated open quotient BY trees T and T' respectively, will have $v\left(\mathrm{disc}\left(\frac{1}{c_f} f\right)\right) = v\left(\mathrm{disc}\left(\frac{1}{c_{f'}} f'\right)\right)$ if T and T' are isomorphic. It is however useful to formally rephrase Lemma 6.3.6 in terms of open quotient BY trees. Recall in Construction 5.1.9 we defined a partial order on the vertices of T by setting $v' \preceq v$ if v lies on the embedded path from m to v', where m is the marked point of T.

Definition 6.3.8. For a vertex v of an open quotient BY tree, we define $|v|$ to be the size of each of the clusters $\mathfrak{s}_{v,1}, \mathfrak{s}_{v,2}, \dots \mathfrak{s}_{v,M(v)}$ as defined in Construction 5.1.9. That is,

$$|v| = \sum_{v' \preceq v} s(v', T) \frac{M(v')}{M(v)}.$$

Lemma 6.3.9. *Let $C : y^2 = f(x)$ be a hyperelliptic curve of genus g over K, and let T be the associated open quotient BY tree. Then*

$$\begin{aligned} v(\Delta_C) &= v(c_f)(4g+2) + \sum_{v \in V(T)} M(v) d(v,m) \left(|v|^2 - \sum_{v' < v} |v'|^2 \tfrac{M(v')}{M(v)} - s(v,T) \right), \\ &= v(c_f)(4g+2) + \sum_{v \in V(T)} M(v) \delta_v |v| \left(|v| - 1 \right), \end{aligned}$$

where $\delta_v = length(e_v)$, the length of the edge incident to v lying on the embedded path between v and m, and $v' < v$ if v' is adjacent to v with $v' \preceq v$. If $v = v_0$, the unique vertex incident to the open edge, then we take $\delta_{v_0} = d(v_0, m)$.

Proof. Every vertex v contributes $M(v)$ clusters, $\mathfrak{s}_{v,1}, \dots \mathfrak{s}_{v,M(v)}$, each of size $|v|$ and depth $d(v,m)$, to the associated cluster picture. A child v' of v contributes $\frac{M(v')}{M(v)}$ proper children of $\mathfrak{s}_{v,i}$ for each $1 \leq i \leq M(v)$. The only children of a cluster $\mathfrak{s}_{v,i}$ that do not arise in this way are singletons. Each $\mathfrak{s}_{v,i}$ has precisely $s(v,T)$ singletons. Putting all of this information into Lemma 6.3.6 we get the first formula. To see that the second formula holds we note by [BBB+20, Theorem 15.1] that

$$v(\Delta_C) = v(c_f)(4g+2) + \sum_{\mathfrak{s} \text{ proper}} \delta_{\mathfrak{s}} |\mathfrak{s}| \left(|\mathfrak{s}| - 1 \right),$$

where if $\mathfrak{s} = \mathcal{R}$ we take $\delta_{\mathcal{R}} = d_{\mathcal{R}}$. ☐

We now proceed to prove Theorem 6.3.1.

Proof of Theorem 6.3.1. Since $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ have equivalent open quotient BY trees T and T', we know by Theorem 6.1.5 that there is some substitution of the form $x = \frac{ax'+b}{cx'+d}$, $y = \frac{y'}{(cx'+d)^{g+1}}$ that when put in to $C : y^2 = f(x)$ yields a hyperelliptic curve $C'' : y^2 = f'(x)$ with open quotient BY tree isomorphic to T'. Certainly the special fibres of the minimal SNC models of C and C'' are isomorphic, since C and C'' are isomorphic. Since C' and C'' have isomorphic open quotient BY trees they must have isomorphic cluster pictures. By Theorem 9.2.3 ([FN20, Theorem 7.12]), the cluster picture and the valuation of the leading coefficient of the defining polynomial completely determine the structure of the special fibre of the minimal SNC model. Furthermore, since we can always make a change of variables $y = ey'$ which is an isomorphism and will send $v(c_f) \mapsto v(c_f) - 2v(e)$, the parity of the valuation of leading coefficient is actually all that influences the special fibre. As such, if $v(c_{f'}) \equiv v(c_{f''}) \mod 2$ then we must have that C' has special fibre isomorphic to that of C'' and therefore to that of C. It remains to work out what effect the Möbius transformations we describe in the proof of Theorem 6.1.5 have on the valuation of the leading coefficient. This will give us conditions for when $v(c_{f''}) \equiv v(c_f) \mod 2$ and when $v(c_{f''}) \not\equiv v(c_f) \mod 2$, thus allowing us to take $v(c_{f'}) \equiv v(c_{f''}) \mod 2$ and ensuring that C and C' have isomorphic special fibres.

Since we can always perform such a substitution $y = ey'$, we can assume that $v(c_f), v(c_{f''}) \in \{0, 1\}$. Since C and C'' are isomorphic, we have the following relation between their discriminants, Δ and Δ'', due to [Liu96]:

$$\Delta'' = \Delta e^{-4(2g+1)}(ad - bc)^{2(g+1)(2g+1)}.$$

Taking valuations we get that

$$v(\Delta'') = v(\Delta) - 4(2g+1)v(e) + 2(g+1)(2g+1)v(ad - bc).$$

Note that by Theorem 6.1.5 to go from C to C'' we make a series of scalings, shifts and inversions. The total scaling is always $z \mapsto \alpha z$, where $v(\alpha) = n$ and $n \equiv \pm d(m, m') \equiv d(m, m') \mod 2\mathbb{Z}$ (if we denote the open quotient BY tree of C'' by T'' with marked point m'' then $T'' \cong T'$, so $d(m, m') = d(m, m'')$). By Lemma 6.3.5, shifts and inversions have no effect on the valuation of the discriminant mod $4(2g+1)$. As such, we obtain the following relation between Δ and Δ'':

$$v(\Delta'') \equiv v(\Delta) + d(m, m')2(g+1)(2g+1) \mod 4(2g+1).$$

In fact, since $2d(m,m')2(g+1)(2g+1) \equiv 0 \mod 4(2g+1)$, we have

$$|v(\Delta'') - v(\Delta)| \equiv d(m,m')2(g+1)(2g+1) \mod 4(2g+1).$$

Suppose that

$$v\Big(\mathrm{disc}\left(\tfrac{f'}{c_{f'}}\right)\Big) - v\Big(\mathrm{disc}\left(\tfrac{f}{c_f}\right)\Big) \equiv d(m,m')2(g+1)(2g+1) \mod 4(2g+1).$$

By Remark 6.3.7, $v\left(\mathrm{disc}\left(\frac{1}{c_{f'}}f'\right)\right) = v\left(\mathrm{disc}\left(\frac{1}{c_{f''}}f''\right)\right)$, therefore

$$v(\Delta'') - v(\Delta) = 2(2g+1)(v(c_{f''}) - v(c_f)) + v\Big(\mathrm{disc}\left(\tfrac{f'}{c_{f'}}\right)\Big) - v\Big(\mathrm{disc}\left(\tfrac{f}{c_f}\right)\Big).$$

So, $|v(\Delta'') - v(\Delta)| \equiv d(m,m')2(g+1)(2g+1) \mod 4(2g+1)$ if and only if

$$2(2g+1)(v(c_{f''}) - v(c_f)) \equiv 0 \mod 4(2g+1),$$

which is the case if and only if $v(c_{f''}) \equiv v(c_f) \mod 2$. So, in this case, if $v(c_{f'}) \equiv v(c_f) \mod 2$ this certainly results in C and C' having isomorphic special fibres. Similarly, if instead

$$v\Big(\mathrm{disc}\left(\tfrac{f'}{c_{f'}}\right)\Big) - v\Big(\mathrm{disc}\left(\tfrac{f}{c_f}\right)\Big) \not\equiv d(m,m')2(g+1)(2g+1) \mod 4(2g+1),$$

then if $v(c_{f'}) \not\equiv v(c_f) \mod 2$, C and C' have isomorphic special fibres.

Note that when g is odd this simplifies slightly since then

$$d(m,m')2(g+1)(2g+1) \equiv 0 \mod 4(2g+1),$$

regardless of the parity of $d(m,m')$. So, in fact the distance between m and m' does not play a role in ensuring which parity of $v(c_{f'})$ will certainly give isomorphic special fibres. We can simply check whether

$$v\Big(\mathrm{disc}\Big(\tfrac{1}{c_{f'}}f'\Big)\Big) - v\Big(\mathrm{disc}\Big(\tfrac{1}{c_f}f\Big)\Big) \equiv 0 \mod 4(2g+1),$$

and if it is we know that if $v(c_{f'}) \equiv v(c_f) \mod 2$ the special fibres will be isomorphic, otherwise if $v(c_{f'}) \not\equiv v(c_f) \mod 2$ the special fibres will be isomorphic. $\square$

6.4 Minimal Discriminant

Recall that the definition of the centre of the core of an open quotient BY tree T is the minimising vertex or midpoint of an edge of the weight function ϕ as defined in Definition 4.4.1.

In the semistable situation, by [BBB+20, §18], the open BY tree created by adding an open yellow edge at the centre, c, of $\tilde{T}$ has minimal discriminant, that is a minimal Weierstrass model of C has this associated open BY tree. Note that, technically, in [BBB+20, §18] they work with closed, rather than open, BY trees. However, by [BBB+20, Remark 18.9] we can instead consider the open BY tree obtained from $\tilde{T}$ by adding an open yellow edge at its centre. In this section we discuss whether we are able to draw a similar conclusion for our more general setting. Unfortunately, as one may recall from Construction 4.5.5, it is not always possible to attach an open edge at the centre c of $\tilde{T}$ if C is not semistable. Instead, we choose a marked point m' to be as close to the centre as possible, which is integer distance from the marked point m of T. We remarked earlier, that in the case when $\mathrm{denom}(d(m,c)) = 2$, there is not a unique choice of point closest to c integer distance from m, as there will be two choices for m' each distance $\frac{1}{2}$ from c. The centre minimises the discriminant in the semistable case, so it would be nice if one of the two points either side of the centre (not necessarily the closest point to the centre) would minimise the discriminant in the non-semistable case. We will discuss and prove what we can of this in this section.

Conjecture 6.4.1. *let $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ be two semistable hyperelliptic curves over K, with equivalent open quotient BY trees T and T' respectively, such that their core is obtained from each of them by removing their open edges, and possibly viewing v_0 or v_0' (the unique vertices incident to the open edges of T and T' respectively) as points on an edge. Let c be the centre of their core $\tilde{T}$. If v_0 lies on the embedded path between v_0' and c, then $v\left(\mathrm{disc}\left(\frac{1}{c_f} f(x)\right)\right) \leq v\left(\mathrm{disc}\left(\frac{1}{c_{f'}} f'(x)\right)\right)$.*

We do not prove all of this here but we are able to prove the following. Suppose that the centre c of $\tilde{T}$ is a vertex, and c is integer distance from the marked point m. Recall that $q^{-1}(c)$ is the centre of $\widetilde{q^{-1}(T)}$ and note that taking a ramified extension only multiplies all depths in the cluster picture by the degree of the extension, so all this does is scale the discriminants so won't change where the minimum is attained. If the canonical representative T^* of the equivalence class of T (i.e. the open quotient BY tree obtained by adding an open yellow edge at c and taking c to be the marked point) did not give the minimal discriminant then this would contradict that the canonical representative of $q^{-1}(T)$ (i.e. the open BY tree obtained by adding an open yellow edge at $q^{-1}(c)$) has minimal discriminant.

In all other situations we conjecture that the minimal discriminant is obtained by attaching an open yellow edge at one of the two choices of marked point either side of the centre. Suppose that this conjecture holds. It would be convenient if we were able to further specify that the minimal discriminant would always arise from the choice of marked point which is closest to the centre (if such a unique choice existed). Unfortunately it is more messy than this. Even if Conjecture 6.4.1 were true, in order to find which of these two trees results in the minimum discriminant the easiest thing to do is simply calculate both. Of course, which tree gives the minimal discriminant will also depend on the original choice of model and leading coefficient. Let's finish by illustrating this with some examples which demonstrate that lots of different things may happen.

Example 6.4.2. Let $C : y^2 = f(x) = ((x-p)^3-p)((x+p)^3-p)(x^2-p^2)\ /\mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 5$, then the cluster picture associated to C has an orbit X of three twins with depth $d_X = 1$, an additional twin $\mathfrak{t}$ with depth $d_{\mathfrak{t}} = 1$ and top cluster $\mathcal{R}$ with depth $d_{\mathcal{R}} = \frac{1}{3}$. So, C has open quotient BY tree T as follows:

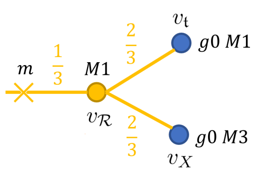


Figure 6.1: Open quotient BY tree T associated to C

The core of T is obtained by simply removing the open edge. We can calculate that $\phi(v_{\mathcal{R}}) = 2$ and $\phi(v_{\mathfrak{t}}) = 6$, so $\min_{v\in V(\tilde{T}^1)} \phi(v) = \phi(v_{\mathcal{R}})$ giving that $v_{\mathcal{R}}$ is the centre of $\tilde{T}$. Note that $d(m, v_{\mathcal{R}}) = \frac{1}{3}$, so the two closest possible marked points for an equivalent open quotient BY tree are distance $\frac{1}{3}$ and $\frac{2}{3}$. Taking a marked point distance $\frac{1}{3}$ from the centre results in T. The other option results in the following equivalent open quotient BY tree T':

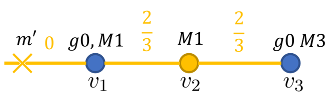


Figure 6.2: Open quotient BY tree T'

Let $C' : y^2 = f'(x)$ be a hyperelliptic curve, isomorphic to C with open quotient BY tree T' as given by Theorem 6.1.5. We can calculate the discriminants of C and C' from their leading coefficients and open quotient BY trees. First

note that $|v_{\mathcal{R}}| = |v_1| = 8$, $|v_X| = |v_{\mathfrak{t}}| = |v_3| = 2$, and $|v_2| = 6$, so

$$\begin{aligned}
v\left(\operatorname{disc}\left(\frac{1}{c_f}f\right)\right) &= \sum_{v\in V(T)} M(v)\delta_v|v|(|v|-1),\\
&= \frac{1}{3}\cdot 8\cdot 7 + 3\cdot\frac{2}{3}\cdot 2 + \frac{2}{3}\cdot 2,\\
&= 24,\\
v\left(\operatorname{disc}\left(\frac{1}{c_{f'}}f'\right)\right) &= \sum_{v\in V(T')} M(v)\delta_v|v|(|v|-1),\\
&= 0 + 3\cdot\frac{2}{3}\cdot 2 + \frac{2}{3}\cdot 6\cdot 5,\\
&= 24.
\end{aligned}$$

By Theorem 6.3.1, we can assume that $v(c_{f'}) = 0$. So, $v(\Delta_C) = v(\Delta_{C'}) = 24$. Assuming Conjecture 6.4.1 is true, this gives $v(\Delta_C^{\min}) = 24$.

Example 6.4.3. Let $C : y^2 = f(x) = x(x^5 - p)\ /\mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 7$, then the cluster picture associated to C has just one proper cluster $\mathcal{R}$ with depth $d_{\mathcal{R}} = \frac{1}{5}$. So, C has open quotient BY tree T as follows:

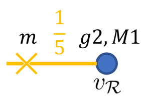


Figure 6.3: Open quotient BY tree T associated to C

The core of T is obtained by simply removing the open edge. There is only one vertex of $\tilde{T}$, namely $v_{\mathcal{R}}$, so it follows that this is the centre. Note that $d(m, v_{\mathcal{R}}) = \frac{1}{5}$, so the two closest possible marked points for an equivalent open quotient BY tree are distance $\frac{1}{5}$ and $\frac{4}{5}$. Taking a marked point distance $\frac{1}{5}$ from the centre results in T. The other option results in the following equivalent open quotient BY tree T':

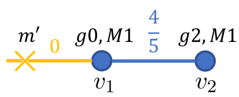


Figure 6.4: Open quotient BY tree T'

Let $C' : y^2 = f'(x)$ be a hyperelliptic curve, isomorphic to C with open quotient BY tree T' as given by Theorem 6.1.5. We can calculate the discriminants of C and C' from their leading coefficients and open quotient BY trees. First

note that $|v_{\mathcal{R}}| = |v_1| = 6$, and $|v_2| = 5$, so

$$\begin{aligned} v\left(\operatorname{disc}\left(\frac{1}{c_f}f\right)\right) &= \sum_{v\in V(T)} M(v)\delta_v|v|(|v|-1), \\ &= 6, \\ v\left(\operatorname{disc}\left(\frac{1}{c_{f'}}f'\right)\right) &= \sum_{v\in V(T')} M(v)\delta_v|v|(|v|-1), \\ &= 16. \end{aligned}$$

Note that by Theorem 6.3.1, since

$$v\left(\operatorname{disc}\left(\frac{f}{c_f}\right)\right) - v\left(\operatorname{disc}\left(\frac{f'}{c_{f'}}\right)\right) = -10 \equiv 30 = 2(g+1)(2g+1)d(m,m') \mod 20,$$

we can assume that $v(c_{f'}) = 0$. So, we get $v(\Delta_C) = 6$ and $v(\Delta_{C'}) = 16$. Assuming Conjecture 6.4.1 is true, this gives $v(\Delta_C^{\min}) = 6$.

Example 6.4.4. Let $C : y^2 = f(x) = x(x^5 - p^2)$ $/\mathbb{Q}_p^{\text{ur}}$ for $p \geq 7$, then the cluster picture associated to C has just one proper cluster $\mathcal{R}$ with depth $d_{\mathcal{R}} = \frac{2}{5}$. So, C has open quotient BY tree T as follows:

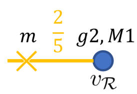


Figure 6.5: Open quotient BY tree T associated to C

The core of T is obtained by simply removing the open edge. There is only one vertex of $\tilde{T}$, namely $v_{\mathcal{R}}$, so it follows that this is the centre. Note that $d(m, v_{\mathcal{R}}) = \frac{2}{5}$, so the two closest possible marked points for an equivalent open quotient BY tree are distance $\frac{2}{5}$ and $\frac{3}{5}$. Taking a marked point distance $\frac{2}{5}$ from the centre results in T. The other option results in the following equivalent open quotient BY tree T':

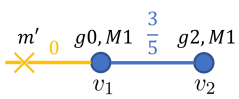


Figure 6.6: Open quotient BY tree T'

Let $C' : y^2 = f'(x)$ be a hyperelliptic curve, isomorphic to C with open quotient BY tree T' as given by Theorem 6.1.5. We can calculate the discriminants of C and C' from their leading coefficients and open quotient BY trees. First

note that $|v_{\mathcal{R}}| = |v_1| = 6$, and $|v_2| = 5$, so

$$\begin{aligned} v\left(\operatorname{disc}\left(\frac{1}{c_f}f\right)\right) &= \sum_{v\in V(T)} M(v)\delta_v|v|(|v|-1), \\ &= 12, \\ v\left(\operatorname{disc}\left(\frac{1}{c_{f'}}f'\right)\right) &= \sum_{v\in V(T')} M(v)\delta_v|v|(|v|-1), \\ &= 12. \end{aligned}$$

Note that by Theorem 6.3.1, since

$$v\left(\operatorname{disc}\left(\frac{1}{c_f}f\right)\right) - v\left(\operatorname{disc}\left(\frac{1}{c_{f'}}f'\right)\right) = 0 \not\equiv 30 = 2(g+1)(2g+1)d(m,m') \mod 20,$$

we can assume that $v(c_{f'}) = 1$. So, we get $v(\Delta_C) = 12$, and $v(\Delta_{C'}) = (4g+2) + 12 = 22$. Assuming Conjecture 6.4.1 is true, this gives $v(\Delta_C^{\min}) = 12$. Note that here, unlike if we had done the same in the previous example, if we instead took $C : y^2 = pf(x)$ then we would have $v(c_{f'}) = 0$ which would give us $v(\Delta_C) = 22$ and $v(\Delta_{C'}) = 12$. So $v(\Delta_C^{\min}) = 12$ again, however in this instance the minimal discriminant is attained by the open quotient BY tree with marked point being the second closest integral disc to the centre rather than the closest.

Example 6.4.5. Let $C : y^2 = f(x)/\mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 7$ be a genus 5 hyperelliptic curve such that f has leading coefficient c_f and roots $\pm p^{3/2} + \zeta_3^i p^{1/3}$, $\zeta_5^j p^{6/5}$, 0, for $0 \leq i \leq 2$, $0 \leq j \leq 4$, ζ_3 a third root of unity, and ζ_5 a fifth root of unity. The cluster picture associated to C has an orbit X of three twins $\mathfrak{s}_1, \mathfrak{s}_2$, and $\mathfrak{s}_3$ with depth $d_X = \frac{3}{2}$, a cluster $\mathfrak{s}_4$ of size 6 with depth $d_{\mathfrak{s}} = \frac{6}{5}$ and top cluster $\mathcal{R}$ with depth $d_{\mathcal{R}} = \frac{1}{3}$. So, C has open quotient BY tree T as follows:

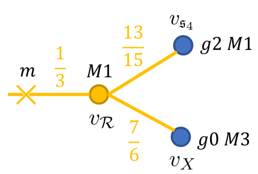


Figure 6.7: Open quotient BY tree T associated to C

The core of T is obtained by simply removing the open edge. We can calculate that $\phi(v_{\mathcal{R}}) = 6$ and $\phi(v_{\mathfrak{s}_4}) = 6$, so $\min_{v\in V(\tilde{T}^1)} \phi(v)$ is attained at both $v_{\mathcal{R}}$ and $v_{\mathfrak{s}_4}$ giving that the midpoint of the edge between them is the centre c of $\tilde{T}$. Note that $d(m,c) = \frac{23}{30}$, so the two closest possible marked points for an equivalent open quotient BY tree are distance $\frac{23}{30}$ and $\frac{7}{30}$. Taking a marked point distance $\frac{23}{30}$ from the centre results in T. The other option results in the following equivalent open quotient BY tree T':

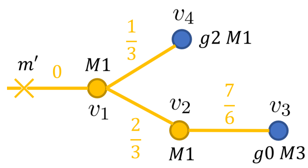


Figure 6.8: Open quotient BY tree T'

Let $C' : y^2 = f'(x)$ be a hyperelliptic curve, isomorphic to C with open quotient BY tree T' as given by Theorem 6.1.5. As in previous examples, we can calculate the discriminants of C and C' from their leading coefficients and open quotient BY trees. First note that

$$\begin{aligned} v\left(\mathrm{disc}\left(\frac{1}{c_f} f\right)\right) &= \sum_{v \in V(T)} M(v)\delta_v |v|(|v|-1), \\ &= 77, \\ v\left(\mathrm{disc}\left(\frac{1}{c_{f'}} f'\right)\right) &= \sum_{v \in V(T')} M(v)\delta_v |v|(|v|-1), \\ &= 37. \end{aligned}$$

Note that by Theorem 6.3.1, we can assume that $v(c_{f'}) \equiv v(c_f) \mod 2$. So, regardless of the valuation of c_f, T' will always give a smaller discriminant than T. In particular, assuming Conjecture 6.4.1, we get $v(\Delta_C^{\min}) = 37$ if $v(c_f) \equiv 0 \mod 2$, and $v(\Delta_C^{\min}) = 4g + 2 + 37 = 55$ if $v(c_f) \equiv 1 \mod 2$.

Chapter 7

Background - Models of Curves

We now turn our attention to studying models of hyperelliptic curves. Recall that our goal is to show that the structure of the minimal SNC model of a tame hyperelliptic curve is completely determined by the cluster picture and leading coefficient of the defining polynomial. We begin with a brief introduction to models, and some important background work.

7.1 Models

Let C be a hyperelliptic curve over K. If you want to study C/K then the idea is to do this by reducing, as much as possible, any questions you might have to ones over k. In this way one can often reduce questions to a finite computation. The key method for moving from K to k is given by the theory of models.

Formally, a *model* of C/K is a scheme $\mathscr{X}/\mathcal{O}_K$, of finite type, flat and proper over $\mathcal{O}_K$, equipped with an isomorphism

$$\mathscr{X} \times_{\mathcal{O}_K} K \to C.$$

We refer to $\mathscr{X} \times_{\mathcal{O}_K} K$ as the *generic fibre* of C and define its *special fibre* to be the k-scheme $\mathscr{X}_k = \mathscr{X} \times_{\mathcal{O}_K} k$.

Roughly, finite type means there are finitely many equations and variables. Flatness ensures that the resulting reduction retains information about C such as being connected, having dimension 1 and having arithmetic genus equal to the genus of C. Properness ensures projectivity of the reduction and the existence of a reduction map on points.

Less formally, we can think of a model in the following way. $\mathrm{Spec}(\mathcal{O}_K)$ consists of just two points: the maximal ideal $\mathfrak{m}$, and the ideal (0). So, a scheme over $\mathcal{O}_K$ consists of a fibre above each of these two points, the generic fibre over (0) and the special fibre over $\mathfrak{m}$. This is a model of C if the generic fibre "looks like" C, and the special fibre is a way of studying the curve over k. Pictorially, this looks something like Figure 7.1

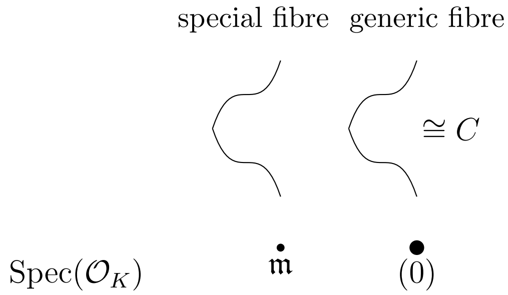


Figure 7.1: Model of C/K

Example 7.1.1. Take $E : y^2 = x^3 + p$ over $\mathbb{Q}_p$. The simplest model of E is to take the generic fibre to be $E/\mathbb{Q}_p$ itself, and simply reduce E modulo p to obtain the special fibre. This is called a *Weierstrass model* and for E is pictured in Figure 7.2.

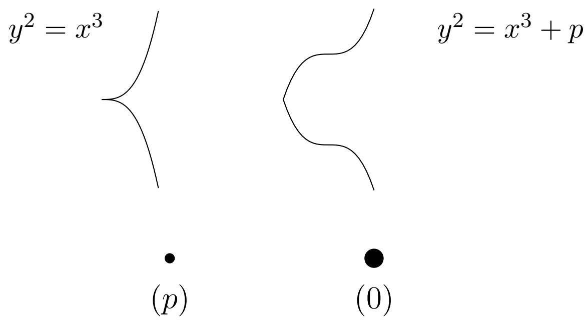
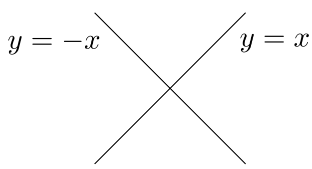


Figure 7.2: Weierstrass model of $E/\mathbb{Q}_p$

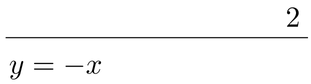
It won't always be the case that a special fibre can be described by just one irreducible equation. There may be several equations each describing a component which intersect the other components.

Example 7.1.2. For instance, if we consider the curve $C/\mathbb{Q}_5 : y^2 = 5x^3 + x^2 + 15$, the reduction mod 5 is $\bar{C}/\mathbb{F}_5 : y^2 = x^2$. This is equivalent to $(y-x)(y+x) = 0$. So, the reduction is the union of two lines $y = x$ and $y = -x$.



It is also possible to get a repeated factor when we reduce. Although this does not add to the solutions, it is important to distinguish between when $\mathscr{X}_k$ is defined by $f(x, y) = 0$ and when it is defined by $f(x, y)^2 = 0$.

Example 7.1.3. Let $C/\mathbb{Q}_5$ be defined by $y^2 + 2xy = 5x^3 - x^2 + 15$. The reduction mod 5 is $\bar{C}/\mathbb{F}_5 : y^2 + 2xy = -x^2$, equivalently $(y + x)^2 = 0$. So we can see that the line y=-x appears in the defining equation of the special fibre with multiplicity 2.



These examples demonstrate how models are a way of visualising a curve over K and k. As such, the defining equation needs coefficients in $\mathcal{O}_K$, else we are unable to "reduce mod p". So, essentially we can think of a model as being a choice of equation with coefficients in $\mathcal{O}_K$ (or equivalently a choice of substitution which yields such an equation). Therefore, there are many possible models of any given curve. We need a way of specifying which of these models is the "best" to look at. There are several different ways one could do this, but here we choose to specify a "best" model as being regular. Roughly this means that we ask that the tangent space has the "correct" dimension. More formally this is defined as follows.

Definition 7.1.4. A scheme $\mathscr{X}$ is *regular* at $x \in \mathscr{X}$ if

$$\dim \mathcal{O}_{\mathscr{X},x} = \dim_k \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2},$$

where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{\mathscr{X},x}$ at $x \in X$. Otherwise, $\mathscr{X}$ is *singular* at x. We say a model $\mathscr{X}$ is *regular* if the underlying scheme $\mathscr{X}$ is.

Note that, since we are only concerned with models of curves, in all of our cases $\mathscr{X}$ can be thought of as a surface and is 2 dimensional, so $\dim \mathcal{O}_{\mathscr{X},x} = 2$.

Example 7.1.5. Consider $C : y^2 = x^3 + 5^n$, $n \geq 1$ over $\mathbb{Q}_5$. Take $\mathscr{X}$ to be the model with generic fibre C and special fibre $\overline{C}/\mathbb{F}_p$. The only potentially singular point on the special fibre is the point corresponding to the maximal ideal $\mathfrak{m} = (x, y, 5)$. To check if $\mathscr{X}$ is regular it suffices to check regularity at this point. If $n = 1$ then $5 = y^2 - x^3$, so $5 \in \mathfrak{m}^2 = (x, y, 5)^2$. Therefore

$$\dim_{\mathbb{F}_5} \frac{\mathfrak{m}}{\mathfrak{m}^2} = \dim_{\mathbb{F}_5} \frac{(x, y, 5)}{(x, y, 5)^2} = \dim_{\mathbb{F}_5} \langle x, y \rangle = 2,$$

and $\mathscr{X}$ is regular. If $n > 1$, $5 \notin \mathfrak{m}^2$, so $\dim \frac{\mathfrak{m}}{\mathfrak{m}^2} \geq 3$ and $\mathscr{X}$ is singular at $\mathfrak{m}$.

We say a regular model $\mathscr{X}$ for C is *minimal* if for every other regular model $\mathscr{Y}$ for C, the map between their generic fibres corresponding to the identity on C extends to a morphism $\mathscr{Y} \to \mathscr{X}$. In other words, a regular model $\mathscr{X}$ is minimal if $\mathscr{X}$ cannot be obtained by blowing up a point on another regular model. Every hyperelliptic curve has a (unique minimal) regular model. This is a good reason to choose to ask that our models be regular.

Another advantage of regular models is that there is an intersection pairing on the irreducible components of the special fibre. Let $\mathscr{X}$ be a regular model. If $E_1, \ldots, E_r$ are the irreducible components of $\mathscr{X}$ with multiplicities m_i then:

- $\sum_{i=1}^{r} m_i E_i \cdot E_j = 0$ for all $1 \leq j \leq r$,
- $E_i \cdot E_j \geq 0$ for $i \neq j$, and $E_i \cdot E_i = E_i^2 < 0$ (the latter is the self intersection number of E_i).

It is possible to blow down a component in a regular model if and only if it is isomorphic to $\mathbb{P}^1_k$ with self intersection -1. (Castelnuovo's Criterion, [Liu02, Theorem 9.3.8])

Strict normal crossing (SNC) models are models which are regular as schemes and whose special fibre $\mathscr{X}_k$ is an SNC divisor - that is, a curve over k whose worst singularities are normal crossings i.e. the singularities "locally" look like the union of two axes (possibly with multiplicity). Note that we do not insist that the irreducible components themselves are smooth. For a given curve, there is a unique SNC model $\mathscr{X}^{\min}$ which is minimal in the sense that any map of SNC models $\mathscr{X}^{\min} \to \mathscr{X}$ is an isomorphism ([Liu02, Proposition 9.3.36]).

Another class of models that are of particular interest to us are semistable models. These are SNC models which have a reduced special fibre. Curves which have a semistable model are said to have *semistable reduction*. The minimal SNC models of such curves can be calculated explicitly from the cluster picture, this is done in [DDMM18].

We now collate some facts about models from [Lor90], [CES03], and [DDMM18] for the convenience of the reader. Similar techniques concerning quotients of models are also used in [Hal10].

7.1.1 Chains of Rational Curves

Chains of rational curves are central to our descriptions of SNC models. The following definition explains what we mean by a chain of rational curves and defines the three main types of chains we are interested in: *tails*, *linking chains* and *crossed tails*.

Definition 7.1.6. Let $\mathscr{X}$ be a SNC model of a hyperelliptic curve defined over K. Suppose that $E_1, \ldots, E_\lambda$ are smooth irreducible rational components of $\mathscr{X}_k$. A divisor $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ is a *chain of rational curves* if

(i) $(E_i \cdot E_{i+1}) = 1$ for all $1 \leq i < \lambda$ and $(E_i \cdot E_j) = 0$ for $j \neq i \pm 1$,

(ii) $(E_1 \cdot \overline{\mathscr{X}_k \setminus \mathcal{C}}) = 1$,

(iii) $(E_i \cdot \overline{\mathscr{X}_k \setminus \mathcal{C}}) = 0$ for $i \neq 1, \lambda$,

where $(E \cdot F)$ is the usual intersection pairing defined on regular models. If $(E_\lambda \cdot \overline{\mathscr{X}_k \setminus \mathcal{C}}) = 0$ then $\mathcal{C}$ is a *tail.* If $(E_\lambda \cdot \overline{\mathscr{X}_k \setminus \mathcal{C}}) = 1$ then $\mathcal{C}$ is a *linking chain.*

We say a chain of rational curves $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ is a *loop* if $\mathcal{C}$ is a linking chain such that E_1 and E_λ both intersect the same component of $\overline{\mathscr{X}_k \setminus \mathcal{C}}$.

Furthermore, if $(E_\lambda \cdot \overline{\mathscr{X}_k \setminus \mathcal{C}}) = 2$ then $\mathcal{C}$ is a *crossed tail* if E_λ intersects two rational components of $\overline{\mathscr{X}_k \setminus \mathcal{C}}$, say $E_{\lambda+1}^{+}$ and $E_{\lambda+1}^{-}$, such that $(E_{\lambda+1}^{\pm} \cdot E_\lambda) = 1$ and $(E_{\lambda+1}^{\pm} \cdot \overline{\mathscr{X}_k \setminus E_\lambda}) = 0$. We call the components $E_{\lambda+1}^{\pm}$ the *crosses* of $\mathcal{C}$.

Illustrations of the definitions of tails, linking chains, loops, and crossed tails are shown in Figures 1.3 and 1.4 in Section 1.

Blowing down a component E results in a regular model if and only if it is rational and has self intersection -1 (Castelnuovo's Criterion, [Liu02, Theorem 9.3.8]). However, blowing down a general rational component of $\mathscr{X}_k$ of self intersection -1 will not necessarily produce an SNC model. For example the resulting model obtained by blowing down the component of multiplicity 3 in the minimal SNC model of an elliptic curve of Kodaira type IV is no longer an SNC model. This is shown in Figure 7.3 below.

(a) Special fibre of the minimal SNC model.

(b) Special fibre of the minimal regular model.

Figure 7.3: Elliptic curve of Kodaira Type IV.

After blowing down a component of a chain of rational curves of self-intersection -1, the special fibre is still an SNC divisor. Therefore, we will be interested in blowing down all such components. If a chain of rational curves cannot be blown down any further, we call it *minimal.*

Definition 7.1.7. A chain of rational curves $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ is *minimal* if $(E_i \cdot E_i) \leq -2$ for every i.

7.1.2 Quotients of Models

This section collates several results from [Lor90] and [CES03] concerning taking quotient of models. Let C be a hyperelliptic curve over K and let L/K be a tame field extension of degree e such that $C_L = C \times_K L$ is semistable over L. Note that the cluster picture of C_L/L is the same as the cluster picture of C/K, except all the depths have been multiplied by e. Since k is algebraically closed, the extension L/K is totally tamely ramified, hence L/K is Galois with $\mathrm{Gal}(L/K)$ cyclic.

Let $\mathscr{Y}$ be the minimal semistable model of $C_L/\mathcal{O}_L$, so $\mathscr{Y}_k$ is a reduced, SNC divisor of $\mathscr{Y}$. Any $\sigma \in \mathrm{Gal}(L/K)$ induces a unique automorphism of $\mathscr{Y}$ of the same degree which makes the following diagram commute [Lor90, p. 136]:

$$\begin{tikzcd} \mathscr{Y} \arrow[r, "\sigma"] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathcal{O}_L \arrow[r, "\sigma"] & \mathcal{O}_L \end{tikzcd}$$

Although a slight abuse of notation, we will also refer to this automorphism on $\mathscr{Y}$ as σ, and define $G = \langle \sigma : \mathscr{Y} \to \mathscr{Y} \rangle$ where σ generates $\mathrm{Gal}(L/K)$. The model $\mathscr{Y}$, as well as the automorphism induced on the special fibre, will be given more explicitly in Section 7.1.3.

Since $\mathscr{Y}$ is projective, the quotient $\mathscr{X} = \mathscr{Y}/G$ given by $q : \mathscr{Y} \to \mathscr{X}$ can be constructed by glueing together the rings of invariants of G-invariant affine open sets of $\mathscr{Y}$. The resulting scheme $\mathscr{X}/\mathcal{O}_K$ is a model of C/K. Furthermore, since $\mathscr{X}$ is a normal scheme, its singularities are closed points lying on the special fibre $\mathscr{X}_k$. The following proposition, from [Lor90, p. 137], gives the multiplicities of the components of $\mathscr{X}_k$.

Proposition 7.1.8. *Let $Y \subseteq \mathscr{Y}_k$ be an irreducible component of $\mathscr{Y}_k$. Then $Z = q(Y)$ is a component of $\mathscr{X}_k$ of multiplicity $e/|Stab(Y)|$, where $Stab(Y)$ is the pointwise stabiliser of Y.*

Blowing up a singularity on $\mathscr{X}_k$ results in a chain of rational curves, as in Definition 7.1.6. It is well known (e.g. [Lor90, Fact V], [Lip78]) that blowing up the singularities on $\mathscr{X}_k$ and blowing down all rational components in chains with self intersection -1 results in a minimal SNC model.

The singularities on $\mathscr{X}_k$ are *tame cyclic quotient singularities*, and there is a precise description of the chain of rational curves that arises after resolving them. We will prove in Proposition 8.1.11 that singularities $z \in \mathscr{X}_k$ which lie on precisely one irreducible component of $\mathscr{X}_k$ are tame cyclic quotient singularities. The definition is as follows:

Definition 7.1.9. Let S be a scheme over $\mathcal{O}_K$ and let $s \in S$ be a closed point. The point s is a *tame cyclic quotient singularity* if there exists

- a positive integer $m > 1$ which is invertible in k,
- a unit $r \in (\mathbb{Z}/m\mathbb{Z})^\times$,
- integers $m_1 > 0$ and $m_2 \geq 0$ satisfying $m_1 \equiv -rm_2 \mod m$

such that $\widehat{\mathcal{O}_{S,s}}$ is isomorphic to the subalgebra of μ_m-invariants in $k[\![t_1, t_2]\!]/(t_1^{m_1}t_2^{m_2} - \pi_K)$ under the action $t_1 \mapsto \zeta_m t_1, t_2 \mapsto \zeta_m^r t_2$. We call the pair (m, r) the *tame cyclic quotient invariants* of s.

The following theorem, [CES03, Theorem 2.4.1], tells us how to resolve tame cyclic quotient singularities.

Theorem 7.1.10. *Let S be a flat, proper, normal curve over $\mathcal{O}_K$ with smooth generic fibre. Suppose $s \in S_k$ is a tame cyclic quotient singularity with tame cyclic quotient invariants (m, r), as in Definition 7.1.9 above.*

Consider the Hirzebruch-Jung continued fraction expansion of $\frac{m}{r}$ given by

$$\frac{m}{r} = b_\lambda - \frac{1}{b_{\lambda-1} - \frac{1}{\cdots - \frac{1}{b_1}}},$$

where $b_i \geq 2$ for all $1 \leq i \leq \lambda$. Then the minimal regular resolution of s is a chain of rational curves $\bigcup_{i=1}^{\lambda} E_i$ such that E_i has self intersection $-b_i$.

Remark 7.1.11. Note that in [CES03] the E_i are labeled in the opposite order. Instead we use the same labeling of the components in our chain as in both [Dok18] and [Lor90].

7.1.3 Semistable Models

A critical step in the proof of the main theorem in Chapter 9 will be extending the field so that C has semistable reduction. The following theorem, a criterion for C to have semistable reduction in terms of the cluster picture of C, allows us to do just that. First we need the following definition:

Definition 7.1.12. For a proper cluster $\mathfrak{s} \in \Sigma_C$ define $\nu_{\mathfrak{s}} = v_K(c_f) + \sum_{r \in \mathcal{R}} d_{\mathfrak{s} \wedge r}$.

Theorem 7.1.13 (The Semistability Criterion)**.** *Let $C : y^2 = f(x)$ be a hyperelliptic curve, and let $\mathcal{R}$ be the set of roots of $f(x)$ in $\overline{K}$. Then C has semistable reduction over L if and only if* *(i) the extension $L(\mathcal{R})/L$ has ramification degree at most* 2*,*

(ii) every proper cluster of $\Sigma_{C/L}$ is G_L invariant,

(iii) every principal cluster $\mathfrak{s} \in \Sigma_{C/L}$ has $d_\mathfrak{s} \in \mathbb{Z}$ and $\nu_\mathfrak{s} \in 2\mathbb{Z}$.

Once the field K has been extended so that C has semistable reduction over L, there is a very explicit description of the special fibre of $\mathscr{Y}$ in terms of the cluster picture of C/L in [DDMM18, Theorem 8.5]. For this we need some definitions. Write $e = [L : K]$. To simplify some invariants, we assume that all clusters $\mathfrak{s} \in \Sigma_{C/K}$ have $e\delta_\mathfrak{s} > \frac{1}{2}$, since a cluster $\mathfrak{s}$ with $e\delta_\mathfrak{s} = \frac{1}{2}$ introduces singular irreducible components. This will be sufficient for our purposes since these invariants are used to describe the explicit automorphism on $\mathscr{Y}_k$ and we can always extend our field so that the minimal semistable model has no singular components, i.e. all components are smooth. Note that the valuation on $\overline{K}$ is normalised with respect to K, such that the valuation of a uniformiser π_L of L is $v_K(\pi_L) = \frac{1}{e}$.

Definition 7.1.14. For $\sigma \in \mathrm{G}_K$ let

$$\chi(\sigma) = \frac{\sigma(\pi_L)}{\pi_L} \mod \mathfrak{m}.$$

For a proper cluster $\mathfrak{s} \in \Sigma_C$ define

$$\lambda_\mathfrak{s} = \frac{\nu_\mathfrak{s}}{2} - d_\mathfrak{s} \sum_{\mathfrak{s}' < \mathfrak{s}} \left\lfloor \frac{|\mathfrak{s}'|}{2} \right\rfloor.$$

Define $\theta_\mathfrak{s} = \sqrt{c_f \prod_{r \notin \mathfrak{s}} (z_\mathfrak{s} - r)}$. If $\mathfrak{s}$ is even or a cotwin,we define $\epsilon_\mathfrak{s} : \mathrm{G}_K \to \{\pm 1\}$ by

$$\epsilon_\mathfrak{s}(\sigma) \equiv \frac{\sigma(\theta_{\mathfrak{s}^*})}{\theta_{(\sigma\mathfrak{s})^*}} \mod \mathfrak{m}.$$

For all other clusters $\mathfrak{s}$ set $\epsilon_\mathfrak{s}(\sigma) = 0$. We write $\epsilon_\mathfrak{s}$ without reference to any $\sigma \in \mathrm{Gal}(L/K)$ for $\epsilon_\mathfrak{s}(\sigma)$, $\sigma \in \mathrm{Gal}(L/K)$ a generator. [DDMM18, Definition 8.2]

Remark 7.1.15. By [DDMM18, Theorem 8.7], the quantity $\epsilon_\mathfrak{s}(\sigma) = -1$ if and only if σ swaps the two points at infinity of $\Gamma_{\mathfrak{s},L}$. When $k = \overline{k}$, $\epsilon_\mathfrak{s} = (-1)^{\nu_{\mathfrak{s}^*} - |\mathfrak{s}^*| d_{\mathfrak{s}^*}}$ since

$$\nu_{\mathfrak{s}^*} = v_K\left(c_f \prod_{r \notin \mathfrak{s}^*} (z_{\mathfrak{s}^*} - r)\right) + \sum_{r \in \mathfrak{s}^*} d_{\mathfrak{s}^*}.$$

Remark 7.1.16. Our definition of $\lambda_\mathfrak{s}$ differs slightly from that in [DDMM18]. In [DDMM18] $\lambda_\mathfrak{s}$ is defined to be $\frac{\nu_\mathfrak{s}}{2} - d_\mathfrak{s} \sum_{\mathfrak{s}' < \mathfrak{s}, \delta_{\mathfrak{s}'} > \frac{1}{2}} \lfloor \frac{|\mathfrak{s}'|}{2} \rfloor$, and a second quantity

$\tilde{\lambda}_{\mathfrak{s}}$ is defined to be $\frac{\nu_{\mathfrak{s}}}{2} - d_{\mathfrak{s}} \sum_{\mathfrak{s}' < \mathfrak{s}} \lfloor \frac{|\mathfrak{s}'|}{2} \rfloor$. This is to account for singular components of the special fibre. Given our assumption that every cluster in $\Sigma_{C/L}$ has relative depth $> \frac{1}{2}$, when we calculate these for C/L we find that $\lambda_{\mathfrak{s}} = \tilde{\lambda}_{\mathfrak{s}}$, so for simplicity we do not write the tilde.

Definition 7.1.17. Let $\mathfrak{s} \in \Sigma_{C/K}$ be a principal cluster. Define $c_{\mathfrak{s}} \in k^{\times}$ by

$$c_{\mathfrak{s}} = \frac{c_f}{\pi_L^{v_K(c_f)}} \prod_{r \notin \mathfrak{s}} \frac{z_{\mathfrak{s}} - r}{\pi_L^{v_K(z_{\mathfrak{s}} - r)}} \mod \mathfrak{m}.$$

These definitions are key for the description of the minimal SNC model of C. In the interest of brevity, we will not restate [DDMM18, Theorem 8.5] here, which is a simplification of Theorem 1.3.6 to the case of semistable reduction, and also gives the action of $\mathrm{Gal}(\overline{K}/K)$ on the minimal SNC model. However, we recommend that the reader familiarise themselves with this theorem as it is helpful for understanding the case when C does not have semistable reduction. The main idea is that principal non-übereven (resp. übereven) clusters each have one (resp. two) components associated to them, and components of parents and odd (resp. even) children are linked by one (resp. two) chain(s) of rational curves. The Galois action on components is induced by the Galois action on clusters, and the two components (resp. two linking chains) of an übereven cluster (resp. even child) $\mathfrak{s}$ are swapped precisely when $\epsilon_{\mathfrak{s}} = -1$.

7.2 Models of Curves via Newton polytopes

In this section we describe a method from [Dok18] for calculating a SNC model of a curve C/K which is Δ_v-regular. The notion of Δ_v-regularity, given in [Dok18, Definition 3.9], applies to more general smooth projective curves. Here we restrict to the case where C has a nested cluster picture (as defined in Definition 2.1.15), and note that this condition implies Δ_v-regularity. The results are applied in Section 8.2.

7.2.1 Newton polytopes

Here we briefly collate the key definitions regarding Newton polytopes necessary for this thesis. We begin with the definition of a Newton polytope.

Definition 7.2.1. Let $G(x, y) = y^2 - f(x) = \sum a_{ij} x^i y^j$ be the defining equation of a hyperelliptic curve C over K. The *Newton polytopes* of C over K and

$\mathcal{O}_K$ respectively are:

$$\Delta(C) = \text{convex hull } \{(i,j) \mid a_{ij} \neq 0\} \subseteq \mathbb{R}^2,$$
$$\Delta_v(C) = \text{lower convex hull } \{(i,j,v_K(a_{ij})) \mid a_{ij} \neq 0\} \subseteq \mathbb{R}^2 \times \mathbb{R}.$$

Above every point $P \in \Delta(C)$ there is exactly one point $(P, v_K(P)) \in \Delta_v(C)$. This defines a piecewise affine function $v_{\Delta(C)} : \Delta(C) \to \mathbb{R}$. When there is no risk of confusion, we may sometimes write $\Delta = \Delta(C)$, and $\Delta_v = \Delta_v(C)$ and the pair (Δ, v_Δ) determines Δ_v. [Dok18, § 3]

Definition 7.2.2. Under the homeomorphic projection $\Delta_v \to \Delta$, the images of the 1 and 2 dimensional open faces of Δ_v are called *v-edges*, and *v-faces* respectively. Note that a v-edge (often denoted L) is homeomorphic to an open interval, and a v-face (often denoted F) is homeomorphic to an open disc (see [Dok18, Definition 3.1]).

Notation 7.2.3. For a v-edge L and a v-face F we write

$$L(\mathbb{Z}) = L \cap \mathbb{Z}^2, \quad F(\mathbb{Z}) = F \cap \mathbb{Z}^2, \quad \Delta(\mathbb{Z}) = (\Delta^{\circ}) \cap \mathbb{Z}^2,$$

and $\overline{L}(\mathbb{Z})$, $\overline{F}(\mathbb{Z})$, $\overline{\Delta}(\mathbb{Z})$ to include points on the boundary. We use subscripts to restrict to the set of points P with $v_K(P)$ in a given set, for instance $F(\mathbb{Z})_{\mathbb{Z}} = \{P \in F(\mathbb{Z}) \mid v_\Delta(P) \in \mathbb{Z}\}$.

Definition 7.2.4. The *denominator* δ_λ, for every v-face or v-edge λ is defined to be the common denominator of $v_\Delta(P)$ for $P \in \overline{\lambda}(\mathbb{Z})$. For two alternate, but equivalent, definitions see [Dok18, Notation 3.2].

Remark 7.2.5. We shall see that the denominator of a v-face or v-edge λ, in some sense, tells us the multiplicity of the component or chain of the SNC model arising from λ. Roughly, for a v-face F, δ_F is the multiplicity of the component Γ_F, and for a v-edge L, δ_L is the minimum multiplicity appearing in the chain of rational curves arising from L.

We distinguish between v-edges which lie on precisely one or two v-faces of the Newton polytope, the former giving rise to tails and the latter to linking chains.

Definition 7.2.6. A v-edge L is *inner* if it is on the boundary of two v-faces. Otherwise, if L is only on the boundary of one v-face, L is *outer*.

7.2.2 Calculating a Model

Before we begin, we give a few constants related to v-faces and v-edges which will be necessary for our description.

Definition 7.2.7. Let L be a v-edge on the boundary of a v-face F. Write

$$L^* = L^*_{(F)} = \text{ the unique affine function } \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z} \text{ with } L^*|_{\bar{L}} = 0, \text{ and } L^*|_F \geq 0.$$

Definition 7.2.8. Let L be a v-edge. If L is inner it bounds two v-faces, say F_1 and F_2. If L is outer it bounds one v-face, say F_1. Choose $P_0, P_1 \in \mathbb{Z}^2$ with $L^*_{(F_1)}(P_0) = 0$, and $L^*_{(F_1)}(P_1) = 1$. The *slopes* $[s_1^L, s_2^L]$ at L are

$$s_1^L = \delta_L(v_1(P_1) - v_1(P_0)), \quad \text{and} \quad s_2^L = \begin{cases} \delta_L(v_2(P_1) - v_2(P_0)) & \text{for } L \text{ inner}, \\ \lfloor s_1^L - 1 \rfloor & \text{for } L \text{ outer}, \end{cases}$$

where v_i is the unique affine function $\mathbb{Z}^2 \to \mathbb{Q}$ that agrees with v_Δ on F_i.

Theorem 7.2.9. *Suppose $C : y^2 = f(x)$ is a nested hyperelliptic curve over K. Then there exists a regular model $C_\Delta/\mathcal{O}_K$ of C/K with strict normal crossings. Its special fibre is as follows:*

(i) Every v-face F of Δ gives a complete smooth curve Γ_F/k of multiplicity δ_F and genus $|F(\mathbb{Z})_\mathbb{Z}|$.

(ii) For every v-edge L with slopes $[s_1^L, s_2^L]$ pick $\frac{m_i}{d_i} \in \mathbb{Q}$ such that

$$s_1^L = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \cdots > \frac{m_\lambda}{d_\lambda} > \frac{m_{\lambda+1}}{d_{\lambda+1}} = s_2^L, \;\; \textit{with} \; \begin{vmatrix} m_i & m_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1. \tag{7.1}$$

Then L gives $|\overline{L}(\mathbb{Z})_\mathbb{Z}| - 1$ chains of rational curves of length λ from Γ_{F_1} to Γ_{F_2} (if L is outer these chains are tails of Γ_{F_1}) with multiplicities $\delta_L d_1, \ldots, \delta_L d_\lambda$. [Dok18, Theorem 3.13]

Remark 7.2.10. In Theorem 7.2.9 (ii), $\lambda = 0$ indicates that Γ_{F_1} and Γ_{F_2} intersect $|\overline{L}(\mathbb{Z})_\mathbb{Z}| - 1$ times in the inner case, and that L contributes no tails in the outer case.

Remark 7.2.11. An explicit equation for Γ_F is given in [Dok18, Definition 3.7], where it is denoted by $\overline{X}_F$. However this is more information than necessary for our situation so we do not give this description here. A description of a similar object X_L is also given in [Dok18, Definition 3.7], and in Theorem 3.13 of [Dok18] the number of rational chains that a v-edge L

gives rise to is described in terms of $|X_L(\overline{k})|$. However, it is straightforward to show that in our case, $|X_L(k)| = |L(\mathbb{Z})_{\mathbb{Z}}| - 1$, noting that in our situation k is algebraically closed so $\bar{k} = k$, so we omit this description also.

Remark 7.2.12. To see that the sequences in Theorem 7.2.9 exist, take all numbers in $[s_2^L, s_1^L] \cap \mathbb{Q}$ of denominator $\leq \max\{\mathrm{denom}(s_1^L), \mathrm{denom}(s_2^L)\}$ in decreasing order. This is essentially a Farey series, so satisfies the determinant condition in (7.1). One can then repeatedly remove, in any order, terms of the form

$$\cdots > \frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d} > \cdots \mapsto \cdots > \frac{a}{b} > \frac{c}{d} > \ldots,$$

where $(a+c)$ and $(b+d)$ are coprime, until no longer possible. This corresponds to blowing down $\mathbb{P}^1$s of self intersection -1 (see Remark 3.16 in [Dok18]). The resulting minimal sequence is unique (else this would contradict uniqueness of minimal SNC model), and still satisfies the determinant condition. If $(s_2^L, s_1^L) \cap \mathbb{Z} = \{N, \ldots, N+a\} \neq \emptyset$ this minimal sequence has the form

$$s_1^L = \frac{m_0}{d_0} > \cdots > \frac{m_h}{d_h} > N + a > \cdots > N > \frac{m_l}{d_l} > \cdots > \frac{m_{\lambda+1}}{d_{\lambda+1}} = s_2^L, \quad (7.2)$$

with $d_0, \ldots, d_h$ strictly decreasing and $d_l, \ldots, d_{\lambda+1}$ strictly increasing. If $(s_2^L, s_1^L) \cap \mathbb{Z} = \emptyset$ this minimal sequence has the form

$$s_1^L = \frac{m_0}{d_0} > \cdots > \frac{m_l}{d_l} > \frac{m_{l+1}}{d_{l+1}} > \cdots > \frac{m_{\lambda+1}}{d_{\lambda+1}} = s_2^L, \quad (7.3)$$

with $d_0, \ldots, d_l$ strictly decreasing, $d_{l+1}, \ldots, d_{\lambda+1}$ strictly increasing, and $d_i > 1$ for all $1 \leq i \leq \lambda$. Notice that shifting either s_1^L or s_2^L by an integer does not change the denominators d_i, that appear in this sequence. If $s_2 > 0$ (else shift by an integer), the numbers $N > \frac{m_l}{d_l} > \cdots > \frac{m_{\lambda+1}}{d_{\lambda+1}}$ are the approximants of the Hirzebruch-Jung continued fraction expansion of s_2^L, similarly for $\frac{m_0}{d_0} > \cdots > \frac{m_h}{d_h} > N + a$ consider the expansion of $1 - s_1^L$. [Dok18, Remark 3.15]

7.2.3 Sloped Chains

The following definition allows us to talk about different parts of chains of rational curves arising from v-edges in the Newton polytope of C.

Definition 7.2.13. Let $t_1, t_2 \in \mathbb{Q}$ and $\mu \in \mathbb{N}$. Pick m_i, d_i as in Theorem 7.2.9; that is, such that

$$\mu t_1 = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \cdots > \frac{m_\lambda}{d_\lambda} > \frac{m_{\lambda+1}}{d_{\lambda+1}} = \mu t_2, \text{ and } \begin{vmatrix} m_i & m_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

with $d_0 \geq \cdots \geq d_l$ and $d_l \leq \cdots \leq d_{\lambda+1}$, for some $0 \leq l \leq \lambda + 1$.

Let $A = \{i \mid 1 \leq i \leq \lambda \text{ and } d_i = 1\}$. If A is non-empty, let a_0 be the minimal element of A and let a_1 be the maximal element of A. Suppose $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ is a chain of rational curves where E_i has multiplicity μd_i. Then $\mathcal{C}$ is a *sloped chain of rational curves* with parameters (t_2, t_1, μ) and we split $\mathcal{C}$ into three sections. If $A \neq \emptyset$ we define the following:

(i) $E_1 \cup \cdots \cup E_{a_0-1}$, the *downhill section*,

(ii) $E_{a_0} \cup \cdots \cup E_{a_1}$, the *level section*,

(iii) $E_{a_1+1} \cup \cdots \cup E_{\lambda}$, the *uphill section*.

If instead $A = \emptyset$ we define:

(i) $E_1 \cup \cdots \cup E_l$, the *downhill section*,

(ii) $E_{l+1} \cup \cdots \cup E_{\lambda}$, the *uphill section*,

and there is no *level section*.

We define the length of each section to be the number of E_i contained in it, and each section is allowed to have length 0. For instance, the level section has length 0 if and only if $A = \emptyset$, and the downhill section has length 0 if and only if $1 \in A$.

Remark 7.2.14. A tail is a sloped chain with level section of length 1 and no uphill section. Therefore any tail can be given by just two parameters, namely t_1 and μ (since $t_2 = \frac{1}{\mu}\lfloor \mu t_1 - 1 \rfloor$). We will often refer to a tail as a tail with parameters (t_1, μ). It follows from Remark 7.2.12 that a tail with parameters (t_1, μ) has the same multiplicities as the tail obtained by resolving a tame cyclic quotient singularity with tame cyclic quotient invariants $\frac{m}{r} = 1 - \mu t_1$.

Remark 7.2.15. All of our chains of rational curves, be they tails, linking chains or crossed tails, are sloped chains. For example, a linking chain in a semistable model will consist of only a level section, since all components have multiplicity 1. Both tails and crossed tails in a minimal SNC model will have no uphill section.

Chapter 8

Base Case and Linking Chains

8.1 Tame Potentially Good Reduction

In this section we calculate the minimal SNC model of a hyperelliptic curve C/K with genus $g = g(C) \geq 1$ which has tame potentially good reduction. That is, there exists a field extension L/K of degree e such that e and p are coprime, and C has a smooth model over $\mathcal{O}_L$. In order to calculate this model, we assume that L is the minimal such extension.

The minimal SNC model of a hyperelliptic curve has a rather straightforward description: it consists of a central component with some tails (in the sense of Definition 7.1.6) whose multiplicities can be explicitly described using the results of Section 7.1.2. Since C has tame potentially good reduction, by [DDMM18, Theorem 1.8(3)] we can assume (possibly after a Möbius transform) that the cluster picture of C over K consists of a single proper cluster $\mathfrak{s}$. We will discuss this in more detail shortly, but first we note the following: The size and depth of the unique proper cluster $\mathfrak{s}$, as well as the valuation of the leading coefficient c_f will be sufficient to calculate the (dual graph with multiplicity of the) minimal SNC model of C over K.

Theorem 8.1.1. *Let C be hyperelliptic curve over K with tame potentially good reduction. Then the special fibre $\mathscr{X}_k$ of the minimal SNC model $\mathscr{X}$ of C/K consists of a component $\Gamma = \Gamma_{\mathfrak{s},K}$, the central component, of multiplicity e. Furthermore, if $e > 1$ the following tails intersect the central component Γ:*

Name	*Number*	*Condition*
T_∞	1	$\mathfrak{s}$ *odd*
$T_\infty^\pm$	2	$\mathfrak{s}$ *even and* $v_K(c_f)$ *even*
T_∞	1	$\mathfrak{s}$ *even*, $e > 2$ *and* $v_K(c_f)$ *odd*
$T_{y_\mathfrak{s}=0}$	$\lfloor \frac{\lvert\mathfrak{s}_{\mathrm{sing}}\rvert}{b_\mathfrak{s}} \rfloor$	$e = 2b_\mathfrak{s}$
$T_{x_\mathfrak{s}=0}$	1	$b_\mathfrak{s} \mid \lvert\mathfrak{s}\rvert$, $\lambda_\mathfrak{s} \notin \mathbb{Z}$ *and* $e > 2$
$T_{x_\mathfrak{s}=0}^\pm$	2	$b_\mathfrak{s} \mid \lvert\mathfrak{s}\rvert$ *and* $\lambda_\mathfrak{s} \in \mathbb{Z}$
$T_{(0,0)}$	1	$b_\mathfrak{s} \nmid \lvert\mathfrak{s}\rvert$

Remark 8.1.2. The genus of the central component can be calculated using Riemann Hurwitz, and we prove an explicit formula for it in Proposition 8.1.24.

We now expand slightly on our assumption above that $\Sigma_{C/K}$ has a unique proper cluster. Since C has tame potentially good reduction, by [DDMM18, Theorem 1.8(3)] we know that $\Sigma_{C/K}$ has no proper clusters of size $< 2g+1$. We can assume that $\Sigma_{C/K}$ consists of a single proper cluster $\mathfrak{s}$ since, if $\Sigma_{C/K}$ has clusters of size $2g+1$ and of size $2g+2$ then, by applying an appropriate choice of Möbius transformation we can obtain a cluster picture with just one proper cluster. After an appropriate shift of the affine line we can assume further that $\mathfrak{s}$ is centered around 0 and that C is given by one of the following two equations:

$$y^2 = c_f \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi^{d_{\mathfrak{s}}}), \quad \text{or} \quad y^2 = c_f x \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi^{d_{\mathfrak{s}}}),$$

if $b_{\mathfrak{s}} \mid |\mathfrak{s}|$ or $b_{\mathfrak{s}} \nmid |\mathfrak{s}|$ respectively, where the $u_r \in K$ are units.

We will proceed in the manner of Section 7.1.2. Let $\mathscr{Y}$ be the smooth Weierstrass model of C over L. This is in general obtained by a substitution $x_L = \pi^{-d_{\mathfrak{s}}} x$, $y_L = \pi^{-\lambda_{\mathfrak{s}}} y$ and will be given by the equation

$$y_L^2 = c_{f,L} \prod_{0 \neq r \in \mathcal{R}} (x_L - u_r), \quad \text{or} \quad y_L^2 = c_{f,L} x_L \prod_{0 \neq r \in \mathcal{R}} (x_L - u_r),$$

if $b \mid |\mathfrak{s}|$ or $b \nmid |\mathfrak{s}|$ respectively, and where $c_{f,L} = \frac{c_f}{\pi_K^{v_K(c_f)}}$.

Let $q : \mathscr{Y} \to \mathscr{Z}$ be the quotient map induced by the action of $\mathrm{Gal}(L/K)$. We will explicitly describe the singular points of $\mathscr{Z}$, show that they are tame cyclic quotient singularities in the sense of Definition 7.1.9, and give their tame cyclic quotient invariants in Proposition 8.1.11. Theorem 7.1.10 then tells us the self intersection numbers of the rational curves in the tails obtained by resolving the tame cyclic quotient singularities. After using intersection theory, this allows us to describe the special fibre of the minimal SNC model $\mathscr{X}$ of C/K in full.

8.1.1 The Automorphism and its Orbits

To describe the singularities on $\mathscr{Z}_k$, we must first explicitly describe the Galois automorphism on the unique component $\Gamma_{\mathfrak{s},L} = \Gamma \subseteq \mathscr{Y}_k$ of the special fibre of the smooth Weierstrass model of C over L. The following fact from [Lor90, Fact IV p. 139] describes the singularities of $\mathscr{Z}_k$ in terms of the quotient $q : \mathscr{Y} \to \mathscr{Z}$.

Proposition 8.1.3. *Let $z_1, \ldots, z_d$ be the ramification points of the morphism $q : \Gamma \to \mathscr{Z}_k$. Then $\{z_1, \ldots, z_d\}$ is precisely the set of singular points of $\mathscr{Z}_k$.*

Furthermore, the ramification points of q correspond to points whose preimage is an orbit of size strictly less than e.

Definition 8.1.4. Let X be an orbit of points of $\mathscr{Y}_k$. If $|X| < e$, we say that X is a *small orbit.*

So, describing the singular points of $\mathscr{Z}_k$ is equivalent to describing the small orbits of $\mathrm{Gal}(L/K)$. In order to list these orbits, we simplify some cluster invariants from 7.1.12 and 7.1.14.

Lemma 8.1.5. *Let C/K be a hyperelliptic curve with tame potentially good reduction and unique proper cluster $\mathfrak{s}$. Then:*

$$\nu_{\mathfrak{s}} = |\mathfrak{s}|d_{\mathfrak{s}} + v_K(c_f), \quad \lambda_{\mathfrak{s}} = \frac{\nu_{\mathfrak{s}}}{2} = \frac{|\mathfrak{s}|d_{\mathfrak{s}} + v_K(c_f)}{2}, \quad \epsilon_{\mathfrak{s}} = (-1)^{v_K(c_f)},$$

and any $\sigma \in Gal(\overline{K}/K)$ induces on the special fibre

$$\sigma|_{\Gamma} : (x_{\mathfrak{s}}, y_{\mathfrak{s}}) \longmapsto (\chi(\sigma)^{ed_{\mathfrak{s}}} x_{\mathfrak{s}}, \chi(\sigma)^{e\lambda_{\mathfrak{s}}} y_{\mathfrak{s}}),$$

where $x_{\mathfrak{s}}, y_{\mathfrak{s}}$ are coordinates on the special fibre.

Proof. Definitions 7.1.12 and 7.1.14 and [DDMM18, Theorem 8.5]. $\square$

Since $\chi(\sigma)^{ed_{\mathfrak{s}}}$ and $\chi(\sigma)^{e\lambda_{\mathfrak{s}}}$ are non-zero and k is algebraically closed, the only points which can lie in orbits of size strictly less than e are points at infinity, or points where $x_{\mathfrak{s}} = 0$ or $y_{\mathfrak{s}} = 0$. This gives four cases which we will take care to distinguish between, as it will make it easier to describe the minimal SNC model for a general cluster picture. With this in mind we make the following definitions:

Definition 8.1.6. We split the small orbits that can occur into the following types.

- ∞-*orbits*: orbits on the point(s) at infinity of Γ,
- $(y_{\mathfrak{s}} = 0)$-*orbits*: orbits on images in Γ of non-zero roots (i.e. the $\overline{u_r}$),
- $(x_{\mathfrak{s}} = 0)$-*orbits*: orbits on the points $(0, \pm\sqrt{\overline{c_{f,L}}}) \in \Gamma$,
- $(0,0)$-*orbits*: the orbit on the point $(0, 0) \in \Gamma$.

The following lemmas describe in which situations we see these small orbits. We will assume $e > 1$ since no small orbits occur when $e = 1$.

Lemma 8.1.7. *If* $\deg(f)$ *is odd then there is a single* ∞*-orbit consisting of a single point. If* $\deg(f)$ *is even and* $v_K(c_f) \in 2\mathbb{Z}$ *then there are two* ∞*-orbits each of size 1. If* $\deg(f)$ *is even,* $v_K(c_f) \notin 2\mathbb{Z}$ *and* $e > 2$ *then there is a single* ∞*-orbit of size 2.*

Proof. Let $u_\mathfrak{s} = 1/x_\mathfrak{s}, v_\mathfrak{s} = y_\mathfrak{s}/x_\mathfrak{s}^{g+1}$ denote the coordinates at infinity of Γ. The curve Γ has a single point at infinity $(u_\mathfrak{s}, v_\mathfrak{s}) = (0,0)$ if $\deg(f)$ is odd, and two points at infinity $(u_\mathfrak{s}, v_\mathfrak{s}) = (0, \pm\sqrt{c_{f,L}})$ if $\deg(f)$ is even. In the latter case, Lemma 8.1.5 gives the action at infinity $\sigma : (0, \sqrt{c_{f,L}}) \mapsto (0, \chi(\sigma)^{e(\lambda_\mathfrak{s}-(g+1)d_\mathfrak{s})}\sqrt{c_{f,L}})$. Therefore, when $\deg(f)$ is even, the points at infinity are swapped if and only if $\chi(\sigma)^{e\lambda_\mathfrak{s}} = -1$ for some $\sigma \in \mathrm{Gal}(L/K)$. This is the case if and only if $v_K(c_f)$ is odd. In this case, the orbit at infinity has size 2 and is only a small orbit if $e > 2$. $\square$

Lemma 8.1.8. *If* $f(0) = 0$ *then there is a single* $(0,0)$*-orbit consisting of a single point. Otherwise* $f(0) \neq 0$*, and if* $\lambda_\mathfrak{s} \in \mathbb{Z}$ *then there are two* $(x_\mathfrak{s} = 0)$*-orbits of size 1, else* $\lambda_\mathfrak{s} \notin \mathbb{Z}$ *and, if* $e > 2$ *then, there is a single* $(x_\mathfrak{s} = 0)$*-orbit of size 2.*

Proof. If $f(0) = 0$ then $\{(0,0)\} \in \Gamma$ is the unique $(0,0)$-orbit. If $f(0) \neq 0$ then $(0, \pm\sqrt{c_{f,L}}) \in \Gamma$, and these points are swapped by some element of the Galois group (see Lemma 8.1.5) if and only if $\lambda_\mathfrak{s} \notin \mathbb{Z}$. If $\lambda_\mathfrak{s} \notin \mathbb{Z}$ then the orbit has size 2 hence it is only a small orbit if $e > 2$. $\square$

Lemma 8.1.9. *Either* $e = b_\mathfrak{s}$ *or* $e = 2b_\mathfrak{s}$*, where* $b_\mathfrak{s}$ *is the denominator of* $d_\mathfrak{s}$*. In particular* $e = 2b_\mathfrak{s}$ *if and only if* $b_\mathfrak{s}\nu_\mathfrak{s} \notin 2\mathbb{Z}$*.*

Proof. By Theorem 7.1.13, e is the minimal integer such that $ed_\mathfrak{s} \in \mathbb{Z}$ and $e\nu_\mathfrak{s} \in 2\mathbb{Z}$. Since $ed_\mathfrak{s} \in \mathbb{Z}$, we can deduce that $b_\mathfrak{s} \mid e$. Since $2b_\mathfrak{s}\nu_\mathfrak{s} \in 2\mathbb{Z}$, $e = b_\mathfrak{s}$ or $e = 2b_\mathfrak{s}$. We can check that the other conditions of Theorem 7.1.13 are satisfied over a field extension of degree e. $\square$

Lemma 8.1.10. *If* $e > b_\mathfrak{s}$ *then there are* $\frac{|\mathfrak{s}|}{b_\mathfrak{s}}$ $(y_\mathfrak{s} = 0)$*-orbits if* $b_\mathfrak{s} \mid |\mathfrak{s}|$*, or* $\frac{|\mathfrak{s}|-1}{b_\mathfrak{s}}$ $(y_\mathfrak{s} = 0)$*-orbits if* $b_\mathfrak{s} \nmid |\mathfrak{s}|$*.*

Proof. The non-zero points with $y_\mathfrak{s} = 0$ are of the form $(\zeta_{b_\mathfrak{s}}^i, 0)$ for $\zeta_{b_\mathfrak{s}}$ a primitive $b_\mathfrak{s}^{\text{th}}$ root of unity. The $(y_\mathfrak{s} = 0)$-orbits have size $b_\mathfrak{s}$ so if $e = b_\mathfrak{s}$ then the $(y_\mathfrak{s} = 0)$-orbits are not small orbits. $\square$

These lemmas allow us to fully describe how many singularities $\mathscr{X}_k$ has. The following proposition tells us that they are tame cyclic quotient singularities in the sense of Definition 7.1.9. Theorem 7.1.10 then allows us to resolve these singularities.

Proposition 8.1.11. *Let $z \in \mathscr{X}_k$ be a singularity which is the image of a Galois orbit $Y \subseteq \mathscr{Y}_k$. Then z is a tame cyclic quotient singularity. In addition, with notation as in Definition 7.1.9, $\frac{m}{r} = \frac{e}{r}$ where $1 \leq r < e$ and $r \mod e$ is given in the following table:*

Orbit Type	$r \mod e$	*Condition*
∞	$e\lambda_{\mathfrak{s}} - e(g(C)+1)d_{\mathfrak{s}}$	$\mathfrak{s}$ *odd*
∞	$-ed_{\mathfrak{s}}\lvert Y\rvert$	$\mathfrak{s}$ *even*
$y_{\mathfrak{s}} = 0$	$e\lambda_{\mathfrak{s}}\lvert Y\rvert$	*None*
$x_{\mathfrak{s}} = 0$	$ed_{\mathfrak{s}}\lvert Y\rvert$	*None*
$(0,0)$	$e\lambda_{\mathfrak{s}}$	*None*

Proof. Recall that for z to be a tame cyclic quotient singularity, there must exist $m > 1$ invertible in k, a unit $r \in (\mathbb{Z}/m\mathbb{Z})^\times$ and integers $m_1 > 0$ and $m_2 \geq 0$ such that $m_1 \equiv -rm_2 \mod m$, and such that $\mathcal{O}_{\mathscr{X},z}$ is equal to the subalgebra of μ_m-invariants in $k[\![t_1,t_2]\!]/(t_1^{m_1}t_2^{m_2} - \pi_K)$ under the action $t_1 \mapsto \zeta_m t_1, t_2 \mapsto \zeta_m^r t_2$. We will show that $m = \frac{e}{|Y|} = |\mathrm{Stab}(Y)|$, $m_1 = e$, $m_2 = 0$ and will explicitly calculate r.

Let $Y \subseteq \mathscr{Y}_k$ be a small orbit and let $Q \in Y$. Then $\mathcal{O}_{\mathscr{X},z}$ is the subalgebra of μ_m-invariants of $\mathcal{O}_{\mathscr{Y},Q}$ under the action of $\mathrm{Stab}(Y)$, where $m = |\mathrm{Stab(Y)}|$. This follows from the definition of $\mathscr{X}$ as the quotient of $\mathscr{Y}$ under the action of $\mathrm{Gal}(L/K)$, which for a generator $\sigma \in \mathrm{Gal}(L/K)$ sends

$$\sigma : \pi_L \longmapsto \chi(\sigma)\pi_L, \quad \sigma : x_{\mathfrak{s}} \longmapsto \chi(\sigma)^{ed_{\mathfrak{s}}}x_{\mathfrak{s}}, \quad \sigma : y_{\mathfrak{s}} \longmapsto \chi(\sigma)^{e\lambda_{\mathfrak{s}}}y_{\mathfrak{s}}.$$

To prove that z is a tame cyclic quotient singularity we must calculate $\mathcal{O}_{\mathscr{Y},Q}$.

First, suppose Y is a $(y_{\mathfrak{s}} = 0)$ or a $(0,0)$-orbit, and write $Q = (x_Q, 0)$. Then $\mathcal{O}_{\mathscr{Y},Q}$ is generated by $\pi_L, x_{\mathfrak{s}} - x_Q$ and $y_{\mathfrak{s}}$. However, since $x_{\mathfrak{s}} - x_Q = uy^2$ for a unit $u \in \mathcal{O}_{\mathscr{Y},Q}$, $\mathcal{O}_{\mathscr{Y},Q}$ is generated by π_L and $y_{\mathfrak{s}}$. Therefore, $\mathcal{O}_{\mathscr{Y},Q} \cong k[\![\pi_L, y_{\mathfrak{s}}]\!]/(\pi_L^e - \pi_K)$, and $\mathcal{O}_{\mathscr{X},z}$ is the subalgebra of μ_m-invariants of this under the action $\pi_L \mapsto \zeta_m\pi_L, y_{\mathfrak{s}} \mapsto \zeta_m^{e\lambda_{\mathfrak{s}}}y_{\mathfrak{s}}$ where $\zeta_m = \chi(\sigma)^{|Y|}$ generates $\mathrm{Stab}(Y)$ (as $\mathrm{Gal}(L/K)$ is cyclic). Let r be such that $0 < r < m$ and $r \equiv e\lambda_{\mathfrak{s}}|Y| \mod m$. Then to prove z is a tame cyclic quotient singularity all that is left to show is that r is a unit in $(\mathbb{Z}/m\mathbb{Z})^\times$ and that $e \equiv 0 \mod m$. The second is clear, and for the first note that since ζ_m^r also generates $\mathrm{Stab}(Y)$, it must be a primitive m^{th} root of unity hence r must be a unit.

If Y is an $(x_{\mathfrak{s}} = 0)$-orbit, then $Q = (0, \pm\sqrt{c_{f,L}})$. By a similar argument to above, $\mathcal{O}_{\mathscr{Y},Q} \cong k[\![\pi_L, x_{\mathfrak{s}}]\!]/(\pi_L^e - \pi_K)$ and $\mathcal{O}_{\mathscr{Z},z}$ is the subalgebra of μ_m invariants under the action $\pi_L \mapsto \zeta_m \pi_L$, $x_{\mathfrak{s}} \mapsto \zeta_m^r x_{\mathfrak{s}}$, where $m = \frac{e}{|Y|}$ and r is such that $0 < r < m$ and $r \equiv ed_{\mathfrak{s}}|Y| \mod m$.

If Y is an ∞ orbit, then we can calculate m, r, m_1 and m_2 explicitly by going to the chart at infinity. □

Corollary 8.1.12. *If Y is a $(y_{\mathfrak{s}} = 0)$-orbit which gives rise to a tame cyclic quotient singularity $z \in \mathscr{Z}_k$, then the tame cyclic quotient invariants (m, r) of z are such that $\frac{m}{r} = 2$.*

Proof. The orbit Y is a $(y_{\mathfrak{s}} = 0)$-orbit hence has size $b_{\mathfrak{s}}$. Lemma 8.1.9 tells us that, $|Y| < e$ if and only if $e = 2b_{\mathfrak{s}}$. In this case $e\lambda_{\mathfrak{s}}|Y| = 2b_{\mathfrak{s}} \cdot \frac{\nu_{\mathfrak{s}}}{2} \cdot b_{\mathfrak{s}} = b_{\mathfrak{s}}^2 \nu_{\mathfrak{s}}$. Since $b_{\mathfrak{s}} = \frac{e}{2}$ and $b_{\mathfrak{s}}\nu_{\mathfrak{s}}$ is an odd integer, this gives $e\lambda_{\mathfrak{s}}|Y| \equiv \frac{e}{2} \mod e$, hence $\frac{m}{r} = 2$. □

8.1.2 Tails

Resolving singularities as in Section 8.1.1 results in tails. These are chains of rational curves intersecting the central component once and intersecting the rest of the special fibre nowhere else. It is useful to distinguish between tails based on the type of orbit they arise from.

Definition 8.1.13. Define the following tails based on the type of singularity of $\mathscr{Z}_k$ they arise from:

- ∞-*tail*: arising from the blow up of a singularity of $\mathscr{Z}_k$ which arose from an ∞-orbit,
- $(y_{\mathfrak{s}} = 0)$-*tail*: arising from the blow up of a singularity of $\mathscr{Z}_k$ which arose from an orbit of non-zero roots,
- $(x_{\mathfrak{s}} = 0)$-*tail*: arising from the blow up of a singularity of $\mathscr{Z}_k$ which arose from an orbit on the points $(0, \pm\sqrt{c_{f,L}})$,
- $(0, 0)$-*tail*: arising from the blow up of a singularity of $\mathscr{Z}_k$ which arose from the point $(0, 0)$.

Remark 8.1.14. The tails defined in Definition 8.1.13 are the only tails that can possibly occur in $\mathscr{X}_k$. This is because any tail must arise from a singularity of $\mathscr{Z}_k$ which lies on just one component, namely a singularity which arises from one of the small orbits discussed in Section 8.1.1.

Proof of Theorem 8.1.1. The central component Γ is the image of the unique component of $\mathscr{Y}_k$ under q. Since blowing up points on Γ does not affect its multiplicity, this has multiplicity e, by Proposition 7.1.8. The description of the tails follows from Lemmas 8.1.7, 8.1.8, and 8.1.10, since the tails are in a bijective correspondence with the orbits of points of $\mathscr{Y}_k$ of size strictly less than e. We must check that Γ really appears in the minimal SNC model. Suppose Γ is exceptional. Then $g(\Gamma) = 0$ and Riemann-Hurwitz says

$$\sum_{z \in \mathscr{X}_k} \left(\frac{e}{|q^{-1}(z)|} - 1 \right) \geq e.$$

Therefore there must be at least three ramification points, so Γ intersects at least three tails. $\square$

Remark 8.1.15. The method for calculating the multiplicities of the rational curves in these tails is described in Theorem 7.1.10 using the tame cyclic quotient invariants given in Proposition 8.1.11.

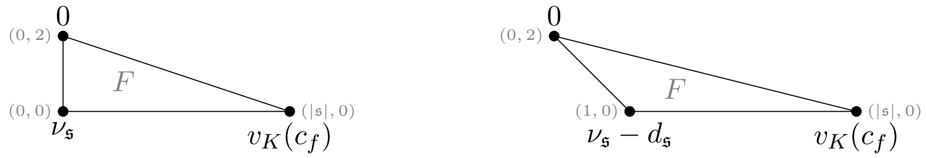
Remark 8.1.16. The central component Γ is the only component of $\mathscr{X}_k$ which may have non-zero genus. Its genus, $g(\Gamma)$, can be calculated via the Riemann-Hurwitz formula. An even more explicit calculation of $g(\Gamma)$ in terms of the Newton polytope is given in Proposition 8.1.24.

8.1.3 Relation to Newton polytopes

Up to this point, this section has described the minimal SNC model of a hyperelliptic curve C/K with tame potentially good reduction using the methods from Section 7.1.2. However, such a hyperelliptic curve has a nested cluster picture so we can also calculate the minimal SNC model using Newton polytopes and the techniques described in Section 7.2. By the uniqueness of the minimal SNC model, these two methods will give the same result: for the reader's sanity, in this section we will show that this is indeed the case. Recall that without loss of generality we can assume that C/K with tame potentially good reduction is given by one of the following two equations:

$$y^2 = c_f \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi_K^{d_{\mathfrak{s}}}), \qquad \text{if } b_{\mathfrak{s}} \mid |\mathfrak{s}|,$$
$$y^2 = c_f x \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi_K^{d_{\mathfrak{s}}}), \qquad \text{if } b_{\mathfrak{s}} \nmid |\mathfrak{s}|.$$

The Newton polytope of C is shown in Figure 8.1a if $b_{\mathfrak{s}} \mid |\mathfrak{s}|$, and in Figure 8.1b if $b_{\mathfrak{s}} \nmid |\mathfrak{s}|$. In each case there is exactly one v-face of $\Delta_v(C)$, which we shall label F. Therefore, by Theorem 7.2.9, the minimal SNC model consists of a central component $\Gamma_{\mathfrak{s}} = \Gamma_F$, and possibly tails arising from the three outer v-edges of F.



(a) If $0 \notin \mathcal{R}$.

(b) If $0 \in \mathcal{R}$.

Figure 8.1: $\Delta_v(C)$ of a hyperelliptic curve C with tame potential good reduction.

Lemma 8.1.17. *The multiplicity of $\Gamma_{\mathfrak{s}} = \Gamma_F$ is δ_F; that is $\delta_F = e$.*

Proof. We will first show that $e \mid \delta_F$, and then that $\delta_F \mid e$. Note that, in both Newton polytopes in Figure 8.1, the valuation map is given by the affine function $v_\Delta(x, y) = \nu_{\mathfrak{s}} - d_{\mathfrak{s}}x - \frac{\nu_{\mathfrak{s}}}{2}y$. Since e is such that $ed_{\mathfrak{s}} \in \mathbb{Z}$ and $e\nu_{\mathfrak{s}} \in 2\mathbb{Z}$, we have $ev_\Delta(x, y) = e\nu_{\mathfrak{s}} - ed_{\mathfrak{s}}x - e\frac{\nu_{\mathfrak{s}}}{2}y \in \mathbb{Z}$. As δ_F is the common denominator of all $v_\Delta(x, y)$ for $x, y \in \Delta$, this gives that $\delta_F \mid e$. Note that $\delta_F\left(v_\Delta(n-1, 0) - v_\Delta(n, 0)\right) = \delta_F d_{\mathfrak{s}} \in \mathbb{Z}$, and $\delta_F\left(v_\Delta(1, 0) - v_\Delta(1, 1)\right) = \delta_F\frac{\nu_{\mathfrak{s}}}{2} \in \mathbb{Z}$. By minimality of e, this implies $e \mid \delta_F$. $\square$

Lemma 8.1.18. *The ∞-tails arise from the outer v-edge of $\Delta_v(C)$ between $(0, 2)$ and $(|\mathfrak{s}|, 0)$.*

Proof. We will first check that this v-edge gives the correct number of ∞-tails, and then calculate the slope to check that the multiplicities of the components are the same.

Let us call this v-edge L. By Theorem 7.2.9 then L contributes $|\bar{L}(\mathbb{Z})_{\mathbb{Z}}| - 1$ tails to the SNC model. Since the points $(0, 2), (|\mathfrak{s}|, 0) \in \bar{L}(\mathbb{Z})_{\mathbb{Z}}$, it contributes two tails if and only if $P = (\frac{|\mathfrak{s}|}{2}, 1) \in \bar{L}(\mathbb{Z})_{\mathbb{Z}}$. If $\mathfrak{s}$ is odd then $P \notin \bar{L} \cap \mathbb{Z}^2$, hence L contributes one tail. If $\mathfrak{s}$ is even then $v_\Delta(P) = \frac{v_K(c_f)}{2}$, hence $P \in \bar{L}(\mathbb{Z})_{\mathbb{Z}}$ if and only if $v_K(c_f) \in 2\mathbb{Z}$. Therefore L contributes one tail if $\mathfrak{s}$ is even and $v_K(c_f)$ is odd, and two tails if $\mathfrak{s}$ and $v_K(c_f)$ are even. This agrees with Theorem 8.1.1.

A quick calculation tells us that $\delta_L = 2$ if and only if $\mathfrak{s}$ is even and $v_K(c_f) \notin 2\mathbb{Z}$, and that $\delta_L = 1$ otherwise. Therefore, $\delta_L = |Y|$, where Y is the orbit at infinity. The unique surjective affine function which is zero on L and non-negative on F is $L_F^*(x, y) = 2|\mathfrak{s}| - 2x - |\mathfrak{s}|y$ if $\mathfrak{s}$ is odd, and $L_F^*(x, y) = |\mathfrak{s}| - x - \frac{1}{2}|\mathfrak{s}|y$ if $\mathfrak{s}$ is even. Therefore, $s_1^L = (g+1)d_{\mathfrak{s}} - \lambda_{\mathfrak{s}}$ if $\mathfrak{s}$ is odd, and $s_1^L = -d_{\mathfrak{s}}|Y|$ if $\mathfrak{s}$ is even. Since the multiplicities of the components of a tail are the Hirzebruch-Jung approximants of the slopes, we are done after comparing the slopes to the table in Proposition 8.1.3.

If $e = 2$ (when $\mathfrak{s}$ is even and $v_K(c_f)$ is odd) has $s_L^1 \in \mathbb{Z}$, so the associated tail is empty, which agrees with the table in Theorem 8.1.1. □

Lemma 8.1.19. *In both cases, when $0 \in \mathcal{R}$ and when $0 \notin \mathcal{R}$, the $(y_{\mathfrak{s}} = 0)$-tails arise from the outer v-edge of $\Delta_v(C)$ on the x-axis. Also, if $b_{\mathfrak{s}} \mid |\mathfrak{s}|$ then the $(x_{\mathfrak{s}} = 0)$-tails arise from the v-edge between $(0,0)$ and $(0,2)$. Else the $(0,0)$-tail arises from the v-edge between $(1,0)$ and $(0,2)$.*

Proof. This follows after a similar calculation to Lemma 8.1.18. Denote this v-edge by L. We will check that $|\overline{L}(\mathbb{Z})_{\mathbb{Z}}| = \lfloor |\mathfrak{s}_{\text{sing}}|/b_{\mathfrak{s}} \rfloor$ and that $s_1^L = -\lambda_{\mathfrak{s}} b_{\mathfrak{s}}$. Then, by comparing to the tables in Proposition 8.1.3 and Theorem 8.1.1, we will done.

First let us calculate $|\overline{L}(\mathbb{Z})_{\mathbb{Z}}|$. The valuation on the x-axis is given by $v_{\Delta}(x, 0) = \nu_{\mathfrak{s}} - d_{\mathfrak{s}} x$. Since $\nu_{\mathfrak{s}} \in \mathbb{Z}$, we have that $v_{\Delta}(x, 0) \in \mathbb{Z}$ if and only if $b_{\mathfrak{s}} \mid x$. From this we see that $|L(\mathbb{Z})_{\mathbb{Z}}| = \lfloor |\mathfrak{s}_{\text{sing}}|/b_{\mathfrak{s}} \rfloor$.

Now, $\delta_L = b_{\mathfrak{s}}$, the size of any $(y_{\mathfrak{s}} = 0)$-orbit, and the unique surjective affine function which is zero on L and non-negative on F is $L^*_{(F)}(x, y) = y$. Therefore,

$$s_1^L = \delta_L \left(v_{\Delta}(1, 0) - v_{\Delta}(0, 0) \right) = -b_{\mathfrak{s}} \lambda_{\mathfrak{s}}.$$

Observe that this gives rise to a non-empty tail if and only if $s_1^L \notin \mathbb{Z}$, which occurs if and only if $e = 2b_{\mathfrak{s}}$. □

8.1.4 Curves Associated to Principal Clusters

To conclude this section, we drop the requirement for C/K to have tame potentially good reduction. We will describe a hyperelliptic curve with potentially good reduction which we associate to a principal cluster $\mathfrak{s} \in \Sigma_C$ with $g_{\text{ss}}(\mathfrak{s}) > 0$. This new curve, which we will denote by $C_{\tilde{\mathfrak{s}}}$, will be invaluable in describing the components of the minimal SNC model of C/K which are associated to $\mathfrak{s} \in \Sigma_C$. For $\mathfrak{s} \in \Sigma_{C/K}$ with $g_{\text{ss}}(\mathfrak{s}) > 0$, the cluster picture $\Sigma_{\tilde{\mathfrak{s}}}$ of $C_{\tilde{\mathfrak{s}}}/K$ will be such that the singletons in $\Sigma_{\tilde{\mathfrak{s}}}$ correspond to odd children of $\mathfrak{s}$ and the even children of $\mathfrak{s}$ are in effect discarded. The leading coefficient of $C_{\tilde{\mathfrak{s}}}/K$ is chosen so that everything behaves well, and allows us to make the comparisons we wish between the minimal SNC model of C/K and the minimal SNC model of $C_{\tilde{\mathfrak{s}}}/K$. This can be formally described as follows:

Definition 8.1.20. Let C/K be a hyperelliptic curve, not necessarily with tame potentially good reduction. Let $\mathfrak{s} \in \Sigma_{C/K}$ be a principal cluster with

$g_{\mathrm{ss}}(\mathfrak{s}) > 0$ such that $\mathfrak{s}$ is fixed by G_K. Suppose furthermore that $\sigma(z_{\mathfrak{s}'}) = z_{\sigma(\mathfrak{s}')}$ for any $\sigma \in \mathrm{G}_K$, $\mathfrak{s}' \in \Sigma_{C/K}$. We define another hyperelliptic curve $C_{\tilde{\mathfrak{s}}}/K$ by

$$C_{\tilde{\mathfrak{s}}} : y^2 = c_{f_{\mathfrak{s}}} \prod_{\mathfrak{o} \in \tilde{\mathfrak{s}}} (x - z_{\mathfrak{o}}), \text{ where } c_{f_{\mathfrak{s}}} = c_f \prod_{r \notin \mathfrak{s}} (z_{\mathfrak{s}} - r).$$

Write $\Sigma_{\tilde{\mathfrak{s}}/K} = \Sigma_{\tilde{\mathfrak{s}}} = \Sigma(C_{\tilde{\mathfrak{s}}}/K)$ for the cluster picture of $C_{\tilde{\mathfrak{s}}}/K$, and $\mathscr{X}_{\tilde{\mathfrak{s}}}$ for the minimal SNC model of $C_{\tilde{\mathfrak{s}}}/K$. The special fibre of the minimal SNC model of $C_{\tilde{\mathfrak{s}}}$ is denoted $\mathscr{X}_{\tilde{\mathfrak{s}},k}$, and the central component is denoted $\Gamma_{\tilde{\mathfrak{s}}}$. We also write $\mathcal{R}_{\tilde{\mathfrak{s}}}$ for the set of all roots of $c_{f_{\mathfrak{s}}} \prod_{\mathfrak{o} \in \tilde{\mathfrak{s}}} (x - z_{\mathfrak{o}})$, and define $d_{\tilde{\mathfrak{s}}} = d_{\mathcal{R}_{\tilde{\mathfrak{s}}}}$, $\nu_{\tilde{\mathfrak{s}}} = \nu_{\mathcal{R}_{\tilde{\mathfrak{s}}}}$, and $\lambda_{\tilde{\mathfrak{s}}} = \lambda_{\mathcal{R}_{\tilde{\mathfrak{s}}}}$.

Remark 8.1.21. Let $\mathscr{Y}$ be the minimal semistable model of C over $\mathcal{O}_L$, for some L/K such that C/L is semistable. Let $\mathfrak{s}$ be a principal cluster with $g_{\mathrm{ss}}(\mathfrak{s}) > 0$. If we reduce $C_{\tilde{\mathfrak{s}}}$ mod $\mathfrak{m}$, we obtain $\Gamma_{\mathfrak{s},L}$, the component of $\mathscr{Y}_k$ corresponding to $\mathfrak{s}$ (see [DDMM18, Theorem 8.5] for the equation of $\Gamma_{\mathfrak{s},L}$). In addition, $c_{f_{\mathfrak{s}}}$ has been carefully chosen so that $d_{\mathfrak{s}} = d_{\tilde{\mathfrak{s}}}$, $\nu_{\mathfrak{s}} = \nu_{\tilde{\mathfrak{s}}}$ and $\lambda_{\mathfrak{s}} = \lambda_{\tilde{\mathfrak{s}}}$. In particular, the automorphisms induced by Galois on $\Gamma_{\mathfrak{s},L}$ and $\Gamma_{\tilde{\mathfrak{s}},L}$ are the same.

Definition 8.1.22. For a principal, Galois-invariant cluster $\mathfrak{s}$, define $e_{\mathfrak{s}}$ to be the minimum integer such that $e_{\mathfrak{s}} d_{\mathfrak{s}} \in \mathbb{Z}$ and $e_{\mathfrak{s}} \nu_{\mathfrak{s}} \in 2\mathbb{Z}$. Furthermore, if $g_{\mathrm{ss}}(\mathfrak{s}) > 0$ define $g(\mathfrak{s})$ to be the *genus of* $\Gamma_{\tilde{\mathfrak{s}}}$ and if $g_{\mathrm{ss}}(\mathfrak{s}) = 0$ define $g(\mathfrak{s}) = 0$. We call $g(\mathfrak{s})$ the *genus of* $\mathfrak{s}$.

Remark 8.1.23. By the Semistability Criterion [DDMM18, Theorem 1.8], if $\mathfrak{s}$ is not übereven then $e_{\mathfrak{s}}$ is the minimum integer such that $C_{\tilde{\mathfrak{s}}}$ has semistable reduction over a field extension L/K of degree $e_{\mathfrak{s}}$. In particular, the central component $\Gamma_{\tilde{\mathfrak{s}}}$ of $\mathscr{X}_{\tilde{\mathfrak{s}},k}$ has multiplicity $e_{\mathfrak{s}}$ and genus $g(\mathfrak{s})$. If $e_{\mathfrak{s}} = 1$ then $g_{\mathrm{ss}}(\mathfrak{s}) = g(\mathfrak{s})$, but the converse is not necessarily true.

Proposition 8.1.24. *If $g_{ss}(\mathfrak{s}) > 0$, the genus $g(\mathfrak{s})$ is given by*

$$g(\mathfrak{s}) = \begin{cases} \lfloor \frac{g_{ss}(\mathfrak{s})}{b_{\mathfrak{s}}} \rfloor & \lambda_{\mathfrak{s}} \in \mathbb{Z}, \\ \lfloor \frac{g_{ss}(\mathfrak{s})}{b_{\mathfrak{s}}} + \frac{1}{2} \rfloor & \lambda_{\mathfrak{s}} \notin \mathbb{Z}, b_{\mathfrak{s}} \text{ even}, \\ 0 & \lambda_{\mathfrak{s}} \notin \mathbb{Z}, b_{\mathfrak{s}} \text{ odd}. \end{cases}$$

Proof. By Theorem 7.2.9, we know $g(\mathfrak{s})$ is given by $|F(\mathbb{Z})_{\mathbb{Z}}|$. This is the number of interior points with integer valuation of the unique face F of the Newton polytope of $C_{\tilde{\mathfrak{s}}}$. By examining Figure 8.1, we see that all interior points are of the form $(x, 1)$ with $1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s})$. For such points, $v_\Delta(x, 1) = \lambda_\mathfrak{s} - d_\mathfrak{s}x$. Therefore,

$$g(\mathfrak{s}) = \left|\{x : 1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s}), \lambda_\mathfrak{s} - xd_\mathfrak{s} \in \mathbb{Z}\}\right|.$$

When $\lambda_\mathfrak{s} \in \mathbb{Z}$ this is therefore equal to

$$|\{x : 1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s}), b_\mathfrak{s} \mid x\}| = \left\lfloor \frac{g_{\mathrm{ss}}(\mathfrak{s})}{b_\mathfrak{s}} \right\rfloor.$$

When $\lambda_\mathfrak{s} \notin \mathbb{Z}$, this is equal to

$$\left|\left\{x : 1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s}), xd_\mathfrak{s} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}\right\}\right|.$$

When $\lambda_\mathfrak{s} \notin \mathbb{Z}$ and $b_\mathfrak{s}$ is odd this set is always empty, and when $\lambda_\mathfrak{s} \notin \mathbb{Z}$ and $b_\mathfrak{s}$ is even it has size $\left\lfloor \frac{g_{\mathrm{ss}}(\mathfrak{s})}{b_\mathfrak{s}} + \frac{1}{2} \right\rfloor$. □

Lemma 8.1.25. *Let C be a hyperelliptic curve and let $\mathfrak{s} \in \Sigma_{C/K}$ be a principal cluster which is fixed by Galois. Let L be an extension such that C is semistable over L, and let σ generate Gal(L/K). Then $\sigma|_{\Gamma_{\mathfrak{s},L}} : \Gamma_{\mathfrak{s},L} \to \Gamma_{\mathfrak{s},L}$ has degree $e_\mathfrak{s}$.*

Proof. The map $\sigma|_{\Gamma_{\mathfrak{s},L}}$ is given by $(x_\mathfrak{s}, y_\mathfrak{s}) \mapsto (\chi(\sigma)^{e_\mathfrak{s}d_\mathfrak{s}}x_\mathfrak{s}, \chi(\sigma)^{e_\mathfrak{s}\lambda_\mathfrak{s}}y_\mathfrak{s})$. The result follows as $e_\mathfrak{s}$, by definition, is the minimal integer such that $e_\mathfrak{s}d_\mathfrak{s}, e_\mathfrak{s}\lambda_\mathfrak{s} \in \mathbb{Z}$. □

8.2 Calculating Linking Chains

The minimal SNC model of a general hyperelliptic curve C/K can roughly be described as follows. Each principal cluster of Σ_C has one or two central components, and some tails associated to it. These central components are linked by chains of rational curves. Section 8.1 will allow us to describe these central components and tails, while this section will be used to describe these linking chains. This includes describing any loops. We will also see the simplest example of the general philosophy that the components of the special fibre of the minimal SNC model of C/K associated to a principal cluster $\mathfrak{s}$ "look like" the special fibre of the minimal SNC model of $C_{\tilde{\mathfrak{s}}}/K$.

Throughout the rest of this section we will take C/K to be a hyperelliptic curve such that $\Sigma_{C/K}$ consists of exactly two proper clusters: a proper cluster $\mathfrak{s}$ and a unique proper child $\mathfrak{s}' < \mathfrak{s}$. This is pictured in Figure 8.2. Note that

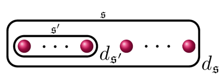


Figure 8.2: Cluster picture with parent $\mathfrak{s}$ and one unique proper child $\mathfrak{s}'$ with no proper children of its own.

$d_{\mathfrak{s}'} > d_{\mathfrak{s}}$ and $|\mathfrak{s}| > |\mathfrak{s}'|$. If C is such that $\mathfrak{s}$ is even and $|\mathfrak{s}| = |\mathfrak{s}'| + 1$ then C/K has potentially good reduction, this case is covered in Section 8.1. To avoid this case we will assume that if $\mathfrak{s}$ is even then $|\mathfrak{s}| \geq |\mathfrak{s}'| + 2$. Since hyperelliptic curves of this type are nested we can directly apply the methods from [Dok18]. Before we apply Theorem 7.2.9, we need to understand the Newton polytope of C/K.

8.2.1 The Newton polytope

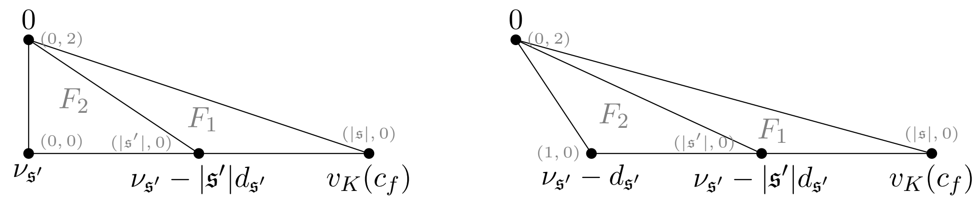
Without loss of generality, we can assume that the defining equation of C/K will be either

$$y^2 = c_f \prod_{r \in \mathcal{R} \setminus \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}}}\right) \prod_{r \in \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}'}}\right), \tag{8.1}$$

or

$$y^2 = c_f x \prod_{r \in \mathcal{R} \setminus \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}}}\right) \prod_{0 \neq r \in \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}'}}\right), \tag{8.2}$$

where the u_r are units. If C has defining equation (8.1), then $\nu_{\mathfrak{s}'} = v_K(c_f) + (|\mathfrak{s}| - |\mathfrak{s}'|)d_{\mathfrak{s}} + |\mathfrak{s}'|d_{\mathfrak{s}'}$, and the Newton polytope $\Delta_v(C)$ of C will be as shown in Figure 8.3a. If instead C has defining equation (8.2), the Newton polytope will be as shown in Figure 8.3b.

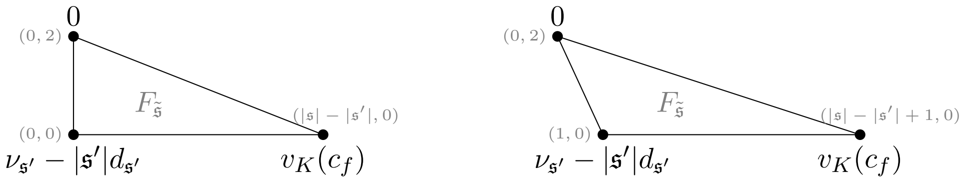


(a) if C has defining equation (8.1) **(b)** if C has defining equation (8.2)

Figure 8.3: Newton polytope $\Delta_v(C)$ of C.

Lemma 8.2.1. *Let C have Newton polytope as in Figure 8.3a. Then there is an isomorphism $\psi : \overline{F_1} \to \Delta_v(C_{\tilde{\mathfrak{s}}})$, from the closure of the v-face marked F_1 to the Newton polytope of $C_{\tilde{\mathfrak{s}}}$ (whose only v-face we label $F_{\tilde{\mathfrak{s}}}$), shown in Figure 8.4. In particular ψ preserves valuations and $\delta_{F_1} = \delta_{F_{\tilde{\mathfrak{s}}}}$. In this sense we say that F_1* corresponds *to the cluster $\mathfrak{s}$. Similarly the v-face F_2 in Figure 8.3a corresponds to $\mathfrak{s}'$.*

Proof of Lemma 8.2.1. Let us compare the v-face F_1 in Figure 8.3a to the Newton polytope, $\Delta_v(C_{\tilde{\mathfrak{s}}})$, of $C_{\tilde{\mathfrak{s}}}$. This is given in Figure 8.4a if $\mathfrak{s}'$ is even, and given in Figure 8.4b if $\mathfrak{s}'$ is odd.



(a) if $\mathfrak{s}'$ is even **(b)** if $\mathfrak{s}'$ is odd

Figure 8.4: Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}}})$, where C is given by (8.1) or (8.2).

If $\mathfrak{s}'$ is even we can define

$$\psi : \overline{F_1} \to \Delta_v(C_{\tilde{\mathfrak{s}}}) : (x, y) \mapsto \left(x - \frac{|\mathfrak{s}'|}{2}(2 - y), y \right).$$

It is easy to show that this is an isomorphism, and that the valuations are preserved. Similarly if $\mathfrak{s}'$ is odd we can define

$$\psi : \overline{F_1} \to \Delta_v(C_{\tilde{\mathfrak{s}}}) : (x, y) \mapsto \left(x - \frac{(|\mathfrak{s}'| + 1)}{2}(2 - y), y \right),$$

which is also an isomorphism that preserves the valuations. In particular, in both cases we have $\delta_{F_1} = \delta_{F_{\tilde{\mathfrak{s}}}}$, and if v_1 is the unique affine function agreeing with $v_{\Delta(C)}$ on F_1, then $v_1(x, y) = v_{\Delta_{\tilde{\mathfrak{s}}}}(\psi(x, y))$, where $v_{\Delta_{\tilde{\mathfrak{s}}}} = v_{\Delta(C_{\tilde{\mathfrak{s}}})}$.

Similarly, we can see that the v-face F_2 in Figure 8.3a corresponds to $\mathfrak{s}'$ by considering the Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}}'})$ of $C_{\tilde{\mathfrak{s}}'}$. This is shown in Figure 8.5.

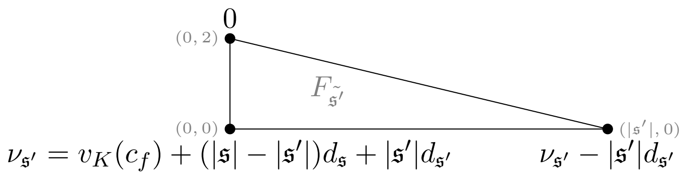


Figure 8.5: Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}}'})$ of $C_{\tilde{\mathfrak{s}}'}$, where C is given by (8.1).

We see that the map

$$\overline{F_2} \to \Delta_v(C_{\tilde{\mathfrak{s}}'}) : (x, y) \mapsto (x, y)$$

is an isomorphism that preserves the valuations, that is $v_2(x, y) = v_{\Delta(C_{\tilde{\mathfrak{s}}'})}(x, y)$, and $\delta_{F_2} = \delta_{F_{\tilde{\mathfrak{s}}'}}$, where v_2 is the unique affine function agreeing with $v_{\Delta(C)}$ on F_2. $\square$

We can make a similar comparison of the v-faces of the Newton polytope in Figure 8.3b.

Lemma 8.2.2. *Let C have Newton polytope as in Figure 8.3b. Then the v-face marked F_1 in Figure 8.3b corresponds to the cluster $\mathfrak{s}$. That is there is a valuation preserving isomorphism between $\overline{F_1}$ and $\Delta_v(C_{\tilde{\mathfrak{s}}})$, and $\delta_{F_1} = \delta_{F_{\tilde{\mathfrak{s}}}}$, where $F_{\tilde{\mathfrak{s}}}$ is the unique v-face of $\Delta_v(C_{\tilde{\mathfrak{s}}})$. Similarly the v-face marked F_2 on the Newton polytope in Figure 8.3b corresponds to the cluster $\mathfrak{s}'$.*

Proof. Again, we can see that the v-face marked F_1 on the Newton polytope in Figure 8.3b corresponds to the cluster $\mathfrak{s}$ by looking at the Newton polytope of $C_{\tilde{\mathfrak{s}}}$. This is shown in Figure 8.4a if $\mathfrak{s}'$ is even, and in Figure 8.4b if $\mathfrak{s}'$ is odd. Take ψ to be exactly as in the proof of Lemma 8.2.1 in both the $\mathfrak{s}'$ even and $\mathfrak{s}'$ odd cases. This gives us an isomorphism between $\overline{F_1}$ and $\Delta_v(C_{\tilde{\mathfrak{s}'}})$ which preserves the valuations. We can also see that $\delta_{F_1} = \delta_{F_{\tilde{\mathfrak{s}}}}$.

Similarly we can see that the v-face marked F_2 on the Newton polytope in Figure 8.3b corresponds to the cluster $\mathfrak{s}'$ by looking at the Newton polytope of $C_{\tilde{\mathfrak{s}'}}$. This is shown in Figure 8.6.

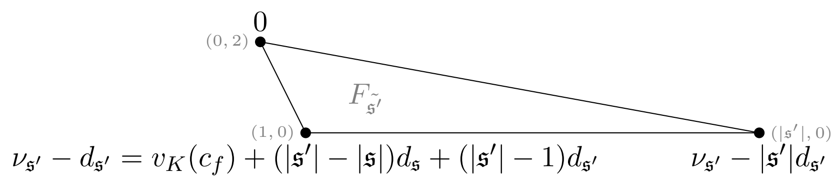


Figure 8.6: Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}'}})$ of $C_{\tilde{\mathfrak{s}'}}$, where C is given by (8.2).

The affine map $\overline{F_2} \to \Delta_v(C_{\tilde{\mathfrak{s}'}}) : (x, y) \mapsto (x, y)$ is an isomorphism which preserves the valuations, and we can see that $\delta_{F_2} = \delta_{F_{\tilde{\mathfrak{s}'}}}$. □

8.2.2 Structure of the SNC Model

The following theorem describes the structure of the special fibre of the minimal SNC model for hyperelliptic curves whose cluster picture looks like Figure 8.2.

Theorem 8.2.3. *Let C/K be a hyperelliptic curve with cluster picture as in Figure 8.2. If $\mathfrak{s}$ is principal, then the special fibre of the minimal SNC model has a component $\Gamma_{\mathfrak{s},K}$ arising from $\mathfrak{s}$ with multiplicity $e_{\mathfrak{s}}$ and genus $g(\mathfrak{s})$. If $\mathfrak{s}'$ is principal then there is a component $\Gamma_{\mathfrak{s}',K}$ arising from $\mathfrak{s}'$ of multiplicity $e_{\mathfrak{s}'}$ and genus $g(\mathfrak{s}')$. These are linked by sloped chain(s) of rational curves with parameters $(t_1 - \delta, t_1, \mu)$, which are described in the following table:*

Name	*From*	*To*	t_1	δ	μ	*Conditions*
$L_{\mathfrak{s},\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}}$	$\Gamma_{\mathfrak{s}'}$	$-\lambda_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'}/2$	1	$\mathfrak{s}$ *principal,* $\mathfrak{s}'$ *odd, principal*
$L^+_{\mathfrak{s},\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}}$	$\Gamma_{\mathfrak{s}'}$	$-d_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'}$	1	$\mathfrak{s}$ *and* $\mathfrak{s}'$ *principal,* $\mathfrak{s}'$ *even,* $\epsilon_{\mathfrak{s}'} = 1$
$L^-_{\mathfrak{s},\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}}$	$\Gamma_{\mathfrak{s}'}$	$-d_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'}$	1	$\mathfrak{s}$ *and* $\mathfrak{s}'$ *principal,* $\mathfrak{s}'$ *even,* $\epsilon_{\mathfrak{s}'} = 1$
$L_{\mathfrak{s},\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}}$	$\Gamma_{\mathfrak{s}'}$	$-d_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'}$	2	$\mathfrak{s}$ *and* $\mathfrak{s}'$ *principal,* $\mathfrak{s}'$ *even,* $\epsilon_{\mathfrak{s}'} = -1$
$L_{\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}}$	$\Gamma_{\mathfrak{s}}$	$-d_{\mathfrak{s}}$	$2\delta_{\mathfrak{s}'}$	1	$\mathfrak{s}$ *principal,* $\mathfrak{s}'$ *twin,* $\epsilon_{\mathfrak{s}'} = 1$
$T_{\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}}$	-	$-d_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'} + \frac{1}{2}$	2	$\mathfrak{s}$ *principal,* $\mathfrak{s}'$ *twin,* $\epsilon_{\mathfrak{s}'} = -1$
$L_{\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}'}$	$-d_{\mathfrak{s}}$	$2\delta_{\mathfrak{s}'}$	1	$\mathfrak{s}$ *cotwin,* $v_K(c_f) \in 2\mathbb{Z}$
$T_{\mathfrak{s}'}$	$\Gamma_{\mathfrak{s}'}$	-	$-d_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'} + \frac{1}{2}$	2	$\mathfrak{s}$ *cotwin,* $v_K(c_f) \notin 2\mathbb{Z}$

The chains where the "To" column has been left empty are crossed tails with crosses of multiplicity 1*. If* $\mathfrak{s}$ *is principal and* $e_{\mathfrak{s}} > 1$ *then* $\Gamma_{\mathfrak{s}}$ *has the following tails with parameters* (t_1, μ)*:*

Name	*Number*	t_1	μ	*Condition*
T_∞	1	$(g(\mathfrak{s})+1)d_{\mathfrak{s}} - \lambda_{\mathfrak{s}}$	1	$\mathfrak{s}$ *odd*
$T^\pm_\infty$	2	$-d_{\mathfrak{s}}$	1	$\mathfrak{s}$ *even and* $\epsilon_{\mathfrak{s}} = 1$
T_∞	1	$-d_{\mathfrak{s}}$	2	$\mathfrak{s}$ *even,* $\epsilon_{\mathfrak{s}} = -1$ *and* $e_{\mathfrak{s}} > 2$
$T_{y_{\mathfrak{s}}=0}$	$\lvert\mathfrak{s}_{\text{sing}}\rvert/b_{\mathfrak{s}}$	$-\lambda_{\mathfrak{s}}$	$b_{\mathfrak{s}}$	$e_{\mathfrak{s}} = 2b_{\mathfrak{s}}$

If $\mathfrak{s}'$ *is principal and* $e_{\mathfrak{s}'} > 1$ *then* $\Gamma_{\mathfrak{s}'}$ *has the following tails with parameters* (t_1, μ)*:*

Name	*Number*	t_1	μ	*Condition*
$T_{y_{\mathfrak{s}}=0}$	$\lfloor\lvert\mathfrak{s}'_{\text{sing}}\rvert/b_{\mathfrak{s}'}\rfloor$	$-\lambda_{\mathfrak{s}'}$	$b_{\mathfrak{s}'}$	$e_{\mathfrak{s}'} = 2b_{\mathfrak{s}'}$
$T_{x_{\mathfrak{s}}=0}$	1	$-d_{\mathfrak{s}'}$	2	$b_{\mathfrak{s}'} \mid \lvert\mathfrak{s}'\rvert$, $\lambda_{\mathfrak{s}'} \notin \mathbb{Z}$ *and* $e_{\mathfrak{s}'} > 2$
$T^\pm_{x_{\mathfrak{s}}=0}$	2	$-d_{\mathfrak{s}'}$	1	$b_{\mathfrak{s}'} \mid \lvert\mathfrak{s}'\rvert$, $\lambda_{\mathfrak{s}'} \in \mathbb{Z}$
$T_{(0,0)}$	1	$-\lambda_{\mathfrak{s}'}$	1	$b_{\mathfrak{s}'} \nmid \lvert\mathfrak{s}'\rvert$

Remark 8.2.4. For this particular type of hyperelliptic curve, $\mathfrak{s}$ will be principal unless it is a cotwin (i.e. if $|\mathfrak{s}'| = 2g(C)$), and $\mathfrak{s}'$ will be principal unless it is a twin. Since we have assumed that $g \geq 2$, these cases cannot coincide. Note that neither $\mathfrak{s}$ nor $\mathfrak{s}'$ can be übereven in this case.

Remark 8.2.5. Suppose $\mathfrak{s}$ is principal. In $\mathscr{X}_k$ we can see most of the components of $\mathscr{X}_{\tilde{\mathfrak{s}},k}$. The central component $\Gamma_{\mathfrak{s}}$ will have the same multiplicity and genus as $\Gamma_{\tilde{\mathfrak{s}}}$, and will have almost the same tails. The only difference being that one or two of the tails (the $(0,0)$-tail in the case $\mathfrak{s}'$ is odd and the $(x_{\mathfrak{s}} = 0)$-tail(s) otherwise) will instead form either part of a linking chain between $\Gamma_{\mathfrak{s}}$ and $\Gamma_{\mathfrak{s}'}$ (in the case $\mathfrak{s}'$ principal); or a loop or a crossed tail associated to $\mathfrak{s}'$ (in the case where $\mathfrak{s}'$ is a twin). We will say that the downhill section of the linking chain *corresponds* to this tail. If the linking chain, loop or crossed tail in $\mathscr{X}_k$ has a non-trivial level section, then all the components of the tails in $\mathscr{X}_{\mathfrak{s},k}$ appear in the linking chain(s) in $\mathscr{X}_k$. If the level section has length zero then some of the lower multiplicity components do not appear - we expand on this in Section 8.2.3.

Similarly, if $\mathfrak{s}'$ is principal, we see most of the components of $\mathscr{X}_{\tilde{\mathfrak{s}}',k}$ in $\mathscr{X}_k$. In this case, $\Gamma_{\mathfrak{s}'}$ has the same tails as $\Gamma_{\tilde{\mathfrak{s}}'}$ except that the infinity tail(s) of the latter are absorbed into the linking chain(s) $L_{\mathfrak{s},\mathfrak{s}'}$ (or the loop or crossed tail arising from $\mathfrak{s}$ if it is a cotwin). In this case, we say that the uphill section of the linking chain *corresponds* to the infinity tail in $\mathscr{X}_{\tilde{\mathfrak{s}}',k}$. We shall see that this is a phenomenon which generalises to the main theorems in Section 9.

Remark 8.2.6. The length of the level section of a linking chain, loop or crossed tail $\mathcal{C} \subseteq \mathscr{X}_k$ (that is, the number of $\mathbb{P}^1$s with multiplicity μ) is equal to $|(\mu(t_1 - \delta), \mu t_1) \cap \mathbb{Z}|$. Let $\mathscr{Y}$ be the minimal regular model of C over L, $q : \mathscr{Y} \to \mathscr{Z}$ be the quotient by $\mathrm{Gal}(L/K)$ and $\phi : \mathscr{X} \to \mathscr{Z}$ the resolution of singularities. Then any irreducible component E in the level section of $\mathcal{C}$ is *not* an exceptional divisor - that is to say, it is the image of μ components of $\mathscr{Y}_k$ which are permuted by $\mathrm{Gal}(L/K)$. This can be seen by looking at the explicit automorphisms on the components of $\mathscr{Y}$ given in [DDMM18, Theorem 6.2].

Example 8.2.7. Consider the hyperelliptic curve $C : y^2 = (x^2 - p)(x^3 - p^5)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$, for $p \geq 5$. The special fibre of the minimal SNC model of C/K can be seen in Figure 8.7. The central components $\Gamma_{\mathfrak{s}}$ and $\Gamma_{\mathfrak{s}'}$ are labeled and shown in bold.

(a) Σ_C

(b) Special fibre

Figure 8.7: Cluster picture and special fibre of the minimal SNC model of C.

If we consider the curves $C_{\tilde{\mathfrak{s}}}$ and $C_{\tilde{\mathfrak{s}}'}$ and the special fibres of their minimal SNC models we find that they are as pictured in Figure 8.8 below. We can see

(a) where $C_{\tilde{\mathfrak{s}}} : y^2 = x(x^2 - p)$.

(b) where $C_{\tilde{\mathfrak{s}}'} : y^2 = p(x^3 - p^5)$.

Figure 8.8: The special fibres of the minimal SNC models of $C_{\tilde{\mathfrak{s}}}$ and $C_{\tilde{\mathfrak{s}}'}$

that all the components in both Figures 8.8a and 8.8b also appear in the special fibre of the minimal SNC model of C. They are glued together along one of their multiplicity one components which forms the linking chain in Figure 8.7. This provides a visualisation of what we mean when we say the tails of $\Gamma_{\mathfrak{s}}$ correspond to those of $\Gamma_{\tilde{\mathfrak{s}}}$, the tails of $\Gamma_{\mathfrak{s}'}$ correspond to those of $\Gamma_{\tilde{\mathfrak{s}}'}$, and that some of these tails form part of the linking chains of the special fibre of the minimal SNC model of C.

Before proving Theorem 8.2.3, we will first prove the following lemmas.

Lemma 8.2.8. *If $\mathfrak{s}$ is principal then the special fibre has an irreducible component $\Gamma_{\mathfrak{s}} = \Gamma_{F_1}$ of multiplicity $e_{\mathfrak{s}}$ and genus $g(\mathfrak{s})$. If $\mathfrak{s}'$ is principal then there is a component $\Gamma_{\mathfrak{s}'} = \Gamma_{F_2}$ of multiplicity $e_{\mathfrak{s}'}$ and genus $g(\mathfrak{s}')$.*

Proof. Follows from Lemmas 8.2.1 and 8.2.2. □

Remark 8.2.9. Lemma 8.2.8 further proves that $\delta_{F_1} = e_{\mathfrak{s}}$ and $\delta_{F_2} = e_{\mathfrak{s}'}$ since, by Theorem 7.2.9, Γ_{F_i} has multiplicity δ_{F_i}.

Lemma 8.2.10. *If $\mathfrak{s}$ is principal and $e_{\mathfrak{s}} > 1$, the following tails of $\Gamma_{\mathfrak{s}}$ arise from outer v-edges of the v-face F_1 in Figure 8.3, with conditions as in Theorem 8.2.3:*

(i) ∞-tail(s) arising from the v-edge connecting $(0,2)$ and $(|\mathfrak{s}|,0)$,

(ii) $(y_{\mathfrak{s}} = 0)$-tail(s) arising from the v-edge connecting $(|\mathfrak{s}'|,0)$ and $(|\mathfrak{s}|,0)$.

Proof. This is a consequence of our discussion above, relating F_1 to the Newton polytope of $C_{\tilde{\mathfrak{s}}}$. The conditions in Theorem 8.2.3 for the tails to occur follow since $\epsilon_{\mathfrak{s}} = (-1)^{v_K(c_f)}$. □

Lemma 8.2.11. *If $\mathfrak{s}'$ is principal and $e_{\mathfrak{s}'} > 1$, the following tails of $\Gamma_{\mathfrak{s}'}$ arise from outer v-edges of the v-face F_2 in Figure 8.3, with conditions as in Theorem 8.2.3:*

(i) if $b_{\mathfrak{s}'} \mid |\mathfrak{s}'|$, $(x_{\mathfrak{s}} = 0)$-tail(s) arise from the v-edge connecting $(0,0)$ and $(0,2)$,

(ii) if $b_{\mathfrak{s}'} \nmid |\mathfrak{s}'|$, a $(0,0)$-tail arises from the v-edge connecting $(1,0)$ and $(0,2)$,

(iii) in both cases, $(y_{\mathfrak{s}} = 0)$-tail(s) arise from the v-edge intersecting the x-axis.

Proof. This is a consequence of our discussion above, relating F_2 to the Newton polytope of $C_{\tilde{\mathfrak{s}}'}$. The conditions in Theorem 8.2.3 for these tails to occur follow since $\epsilon_{\mathfrak{s}'} = (-1)^{\nu_{\mathfrak{s}'} - |\mathfrak{s}'| d_{\mathfrak{s}'}}$. □

In order to find the lengths of the level sections of the linking chains, we must calculate the slopes of the unique inner v-edge L, adjacent to both v-faces F_1 and F_2 in Figure 8.3.

Lemma 8.2.12. *If $\mathfrak{s}'$ is odd $s_1^L = -\lambda_{\mathfrak{s}}$, and $s_2^L = -\lambda_{\mathfrak{s}} - \frac{\delta_{\mathfrak{s}'}}{2}$. Else $s_1^L = -\delta_L d_{\mathfrak{s}}$, and $s_2^L = -\delta_L d_{\mathfrak{s}'}$.*

Proof. Suppose s' is odd. Then the only points in $\overline{L}(\mathbb{Z})$ are the endpoints $(0,2)$ and $(|\mathfrak{s}'|,0)$, so $\delta_L = 1$. The unique function $L^*_{F_1} : \mathbb{Z}^2 \to \mathbb{Z}$ such that $L^*_{F_1}\big|_L = 0$ and $L^*_{F_1}\big|_{F_1} \geq 0$ is given by

$$L^*_{F_1}(x,y) = 2x + |\mathfrak{s}'|y - 2|\mathfrak{s}'|.$$

To calculate s_1^L and s_2^L we need P_0 and P_1 such that $L^*_{F_1}(P_0) = 0$ and $L^*_{F_1}(P_1) = 1$. We will take $P_0 = (|\mathfrak{s}'|,0)$ and $P_1 = \left(\frac{|\mathfrak{s}'|+1}{2}, 1\right)$. The unique affine function which agrees with v_Δ on F_1 is defined by $v_1(x,y) = \nu_{\mathfrak{s}} - d_{\mathfrak{s}}x - \frac{\nu_{\mathfrak{s}}}{2}y$. Therefore,

$$\begin{aligned} s_1^L &= \delta_L(v_1(P_1) - v_1(P_0)), \\ &= \nu_{\mathfrak{s}} - d_{\mathfrak{s}}\frac{|\mathfrak{s}'|+1}{2} - \frac{\nu_{\mathfrak{s}}}{2} - \nu_{\mathfrak{s}} + d_{\mathfrak{s}}|\mathfrak{s}'|, \\ &= -\left(\frac{\nu_{\mathfrak{s}}}{2} - d_{\mathfrak{s}}\frac{|\mathfrak{s}'|-1}{2}\right), \\ &= -\lambda_{\mathfrak{s}}. \end{aligned}$$

The calculations for s_2^L and $\mathfrak{s}'$ even are similar. □

Proof of Theorem 8.2.3. Recall that $e_{\mathfrak{s}}$ is the minimum integer such that $e_{\mathfrak{s}}d_{\mathfrak{s}} \in \mathbb{Z}$, and $e_{\mathfrak{s}}\nu_{\mathfrak{s}} \in 2\mathbb{Z}$. If $e_{\mathfrak{s}} = 1$ then $d_{\mathfrak{s}}, \lambda_{\mathfrak{s}} \in \mathbb{Z}$, hence the slopes of the outer v-edges of F_1 are integers and $\Gamma_{\mathfrak{s}}$ has no tails. If $e_{\mathfrak{s}} > 1$ then Lemma 8.2.10 describes the tails of $\Gamma_{\mathfrak{s}}$. Similarly if $e_{\mathfrak{s}'} = 1$ then $\Gamma_{\mathfrak{s}'}$ has no tails and if $e_{\mathfrak{s}'} > 1$ then Lemma 8.2.11 describes the tails of $\Gamma_{\mathfrak{s}'}$. The statement on the parameters of the tails and the linking chain follows from Remark 7.2.12 and the calculation of the slopes in Lemma 8.2.12. The multiplicity of the level section is δ_L where L is the inner v-edge between F_1 and F_2.

The two cases left to worry about are when $\mathfrak{s}'$ is a twin or when $\mathfrak{s}$ is a cotwin. We will only argue the case where $\mathfrak{s}'$ is a twin, as the case where $\mathfrak{s}$ is a cotwin is proved similarly. Recall from Remark 7.1.15 that $\epsilon_{\mathfrak{s}'} = (-1)^{\nu_{\mathfrak{s}'} - |\mathfrak{s}'|d_{\mathfrak{s}'}}$. So, $\epsilon_{\mathfrak{s}'} = 1$ if and only if $v_\Delta((|\mathfrak{s}'|,0)) = \nu_{\mathfrak{s}'} - |\mathfrak{s}'|d_{\mathfrak{s}'} \in 2\mathbb{Z}$.

Suppose that $\epsilon_{\mathfrak{s}'} = 1$. Since $v_\Delta(0,2) = 0 \in 2\mathbb{Z}$ and $|\mathfrak{s}'| = 2$ we have that $(\frac{|\mathfrak{s}'|}{2}, 1) = (1,1) \in \mathbb{Z}^2$, and $v_\Delta(1,1) \in \mathbb{Z}$. So, $|\overline{L}(\mathbb{Z})_{\mathbb{Z}}| = 3$ and by Theorem 7.2.9 there are two linking chains from $\Gamma_{\mathfrak{s}}$ to the component Γ_{F_2} arising from the

v-face F_2 of $\Delta_v(C)$ in Figure 8.3. The component Γ_{F_2} is exceptional by [Dok18, Proposition 5.2] and the linking chains between $\Gamma_{\mathfrak{s}}$ and Γ_{F_2} are minimal. After blowing down Γ_{F_2}, this results in a loop from $\Gamma_{\mathfrak{s}}$ to itself.

Suppose instead that, $\epsilon_{\mathfrak{s}'} = -1$. Then there is a single chain of rational curves from $\Gamma_{\mathfrak{s}}$ to Γ_{F_2}, and Γ_{F_2} has two other rational curves intersecting it transversely (which arise from the v-edge connecting $(0,0)$ and $(0,2)$). Therefore, Γ_{F_2} is not exceptional and must appear in the minimal SNC model. This means, if we consider Γ_{F_2} as a component of the level section, that this chain of rational curves is a crossed tail. $\square$

8.2.3 Small Distances

Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be the principal clusters such that there is a linking chain $\mathcal{C} \subseteq \mathscr{X}_k$ from $\Gamma_{\mathfrak{s}_1}$ to $\Gamma_{\mathfrak{s}_2}$. If $\mathcal{C}$ has level section of length greater than 0, it is straightforward to compare the multiplicities of $\mathcal{C}$ to those of the corresponding tails (see Remark 8.2.5). All of the multiplicities of the corresponding tails appear in the uphill and downhill sections of $\mathcal{C}$. However, if the level section is empty and the downhill section of $\mathcal{C}$ corresponds to a tail, say $\mathcal{T}_1$, then not all of the multiplicities of $\mathcal{T}_1 \subseteq \mathscr{X}_{\tilde{\mathfrak{s}}_1,k}$ appear in the downhill section of $\mathcal{C}$. The situation is similar if the uphill section corresponds to a tail, say $\mathcal{T}_2 \subseteq \mathscr{X}_{\tilde{\mathfrak{s}}_2,k}$. We shall show that in this case, $\mathcal{T}_1$ and $\mathcal{T}_2$ "meet" at a component of second least common multiplicity. In other words, if we consider a chain of rational curves $\mathcal{C}'$ such that $\mathcal{C}'$ has level section of length 1, and whose downhill and uphill sections correspond to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, then we "cut out" a section of $\mathcal{C}'$ to obtain $\mathcal{C}$.

Example 8.2.13. Consider the hyperelliptic curves given by $y^2 = (x^4-p)(x^5-p^{2+10n})$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 7$ and $n \in \mathbb{Z}_{\geq 0}$, with cluster pictures shown in Figure 8.9. The level section of the linking chain between $\Gamma_{\mathcal{R}}$ and $\Gamma_{\mathfrak{s}}$ has length

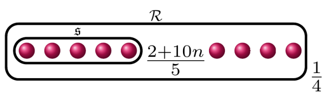
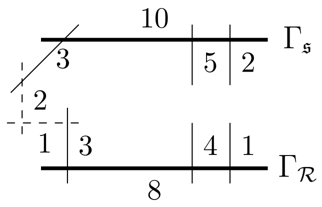
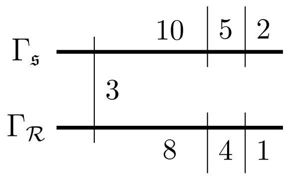


Figure 8.9: Cluster picture Σ_C of $C : y^2 = (x^4-p)(x^5-p^{2+10n})$.

n. Figure 8.10 shows the special fibres of the minimal SNC models for both when $n = 1$, and the small distance case (when $n = 0$). Here we can see that when $n = 1$ the uphill and downhill sections of the linking chain have a common multiplicity greater than 1, namely 3, and that to obtain the $n = 0$ case we remove the dashed section of the linking chain and glue back along the multiplicity 3 components.



(a) $n = 1$ case.



(b) $n = 0$ case.

Figure 8.10: Example of "cutting out" a section of linking chain to obtain the small distance case.

Let us solidify this with a precise statement.

Theorem 8.2.14. *Let $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ be a sloped chain of rational curves with parameters (t_2, t_1, μ), as in Definition 7.2.13. Suppose that $\mathcal{C}$ has level section length 0 and $[\mu t_2, \mu t_1] \subset (0, 1)$. Suppose E_i has multiplicity μ_i; the downhill section comprises E_i for $1 \leq i \leq l$, for some $l \in \mathbb{Z}$ with $1 \leq l \leq \lambda$; and all remaining components form the uphill section. Write $\mu_0 = \mathrm{denom}(\mu t_1)$ and $\mu_{\lambda+1} = \mathrm{denom}(\mu t_2)$. Let $\mathcal{T}_j = \bigcup_{i=1}^{\lambda_j} F_i^{(j)}$ for $j = 1, 2$ be tails (with $\mathcal{T}_j$ possibly empty, in which case $\lambda_j = 0$), where $\mathcal{T}_1$ has parameters (t_1, μ) and $\mathcal{T}_2$ has parameters $(\frac{1}{\mu} - t_2, \mu)$. Let $F_i^{(j)}$ have multiplicity $\mu_i^{(j)}$ (and write $\mu_0^{(j)} = \mathrm{denom}(\mu t_j)$), and let $l_j < \max(1, \lambda_j)$ be maximal such that $\mu_{l_1}^{(1)} = \mu_{l_2}^{(2)}$. Then $l = l_1 = \lambda - l_2$, $\mu_i = \mu_i^{(1)}$ for $0 \leq i \leq l_1$ and $\mu_{\lambda+1-i} = \mu_i^{(2)}$ for $0 \leq i \leq l_2$.*

Remark 8.2.15. Let $\mathcal{C}$ be as in Theorem 8.2.14. Since the level section of $\mathcal{C}$ is empty, it must be the case that $(\mu t_2, \mu t_1) \cap \mathbb{Z} = \emptyset$. Therefore, after shifting μt_2 and μt_1 by an integer if necessary, we may insist that $[\mu t_2, \mu t_1] \subseteq [0, 1]$. If $\mu t_2 \in \mathbb{Z}$ (hence $\mathcal{T}_2$ is empty) then it is immediate from Remark 7.2.12 that $\lambda = \lambda_1 - 1$ and $\mu_i = \mu_i^{(1)}$ for $1 \leq i \leq \lambda$, since the multiplicities come from the same sequence of fractions. A similar conclusion applies if $\mu t_1 \in \mathbb{Z}$. So we are able to assume without loss of generality that $\mu t_2, \mu t_1 \notin \mathbb{Z}$, hence our assumption in Theorem 8.2.14 that $[\mu t_2, \mu t_1] \subset (0, 1)$.

Roughly, Theorem 8.2.14, states that when there is no level section, rather than seeing all of the multiplicities of the tails which the uphill and downhill sections correspond to, the two tails "meet" at the component of minimal shared multiplicity greater than μ. Before we prove this theorem, let us prove a couple of lemmas.

Lemma 8.2.16. *Let $q_1, q_2 \in \mathbb{Q}$ with $[q_1, q_2] \cap \mathbb{Z} = \emptyset$. Then there is a unique fraction with minimal denominator in the set $[q_1, q_2] \cap \mathbb{Q}$, when written with coprime numerator and denominator.*

Proof. Suppose not, and suppose $r_1, r_2 \in [q_1, q_2] \cap \mathbb{Q}$ with $r_1 < r_2$ can be written $r_i = \frac{m_i}{d}$ with m_i, d coprime and d the minimal denominator of elements in the set $[q_1, q_2] \cap \mathbb{Q}$. We will show that there exists a rational number r lying between r_1 and r_2 of denominator $< d$.

Write $r_i = \frac{m_i(d-1)}{d(d-1)}$, and consider the set $S = [m_1(d-1), m_2(d-1)] \cap \mathbb{Z}$. Since $m_2 > m_1$ and $m_1, m_2 \in \mathbb{Z}$, $|S| \geq d$ and there must exist a multiple of d in S. That is, there exists $m \in \mathbb{Z}$ such that $md \in S$. Since m_i and d are coprime, we have $m_1 < md < m_2$. Therefore,

$$\frac{m_1(d-1)}{d(d-1)} < \frac{md}{d(d-1)} < \frac{m_2(d-1)}{d(d-1)} \Longrightarrow r_1 < \frac{m}{d-1} < r_2,$$

which contradicts the minimality of d. $\square$

Lemma 8.2.17. *With notation as in Theorem 8.2.14, there exists some $l_j < \lambda_j$, for $j = 1, 2$, such that $\mu^{(1)}_{l_1} = \mu^{(2)}_{l_2}$.*

Proof. Write $s_i = \mu t_i$. Recall that we assumed that, $[s_2, s_1] \subset (0, 1)$, so $[s_2, s_1] \cap \mathbb{Z} = \emptyset$. Let $\frac{m}{d}$ be the unique fraction of minimal denominator in $[s_2, s_1]$, which exists by Lemma 8.2.16. Then if

$$s_1 = \mu t_1 = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \cdots > \frac{m_\lambda}{d_\lambda} > \frac{m_{\lambda+1}}{d_{\lambda+1}} = \mu t_2 = s_2,$$

is the reduced sequence giving rise to the linking chain $\mathcal{C}$, as in Remark 7.2.12, where $(m_i, d_i) = 1$, $d_0 > \cdots > d_l$ and $d_l < \cdots < d_{\lambda+1}$ for some $1 \leq l \leq \lambda$, we must have that $d_l = d$.

Consider the following two reduced sequences:

$$\mu t_1 = \frac{m^{(1)}_0}{d^{(1)}_0} > \frac{m^{(1)}_1}{d^{(1)}_1} > \cdots > \frac{m^{(1)}_{\lambda_1}}{d^{(1)}_{\lambda_1}} > \frac{m^{(1)}_{\lambda_1+1}}{d^{(1)}_{\lambda_1+1}} = -1,$$
$$1 - \mu t_2 = \frac{m^{(2)}_0}{d^{(2)}_0} > \frac{m^{(2)}_1}{d^{(2)}_1} > \cdots > \frac{m^{(2)}_{\lambda_2}}{d^{(2)}_{\lambda_2}} > \frac{m^{(2)}_{\lambda_2+1}}{d^{(2)}_{\lambda_2+1}} = -1.$$

These give rise to the multiplicities $\mu^{(j)}_i = \mu \cdot d^{(j)}_i$ for $1 \leq i \leq \lambda_j$, $j = 1, 2$ of the tails $\mathcal{T}_j$. We will show that there exist $0 \leq l_1 < \lambda_1 + 1$ and $0 \leq l_2 < \lambda_2 + 1$ with $d^{(1)}_{l_1} = d = d^{(2)}_{l_2}$.

We will first prove that $d^{(1)}_{l_1} = d$ for some $l_1 \in \mathbb{Z}$. Since $[s_2, s_1] \subset (0, 1)$, we have that $s_2 > \lfloor s_1 \rfloor = 0$. So, some fraction of denominator d, say $\frac{m}{d}$, appears in the full sequence of fractions in $[\lfloor s_1 \rfloor, s_1] \cap \mathbb{Q}$ of denominator less than or equal to $\max\{d_0, d_{\lambda+1}\}$. To obtain a reduced sequence, we remove all terms of the form

$$\cdots > \frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d} > \cdots \mapsto \cdots > \frac{a}{b} > \frac{c}{d} > \ldots,$$

as in Remark 7.2.12. We can only remove $\frac{m}{d}$ if there exists some $q \in \mathbb{Q}$ with $\mathrm{denom}(q) < d$ and $s_1 > q > \frac{m}{d}$. No such q can exists since d is the minimal denominator of any element of $[s_2, s_1] \cap \mathbb{Q}$. Therefore, $\frac{m}{d}$ cannot be removed in the reduction process and so must appear in the reduced sequence. Therefore there exists $0 \leq l_1 < \lambda_1 + 1$ such that $d^{(1)}_{l_1} = d$. Proving that there exits $0 \leq l_2 < \lambda_2 + 1$ such that $d^{(1)}_{l_1} = d = d^{(2)}_{l_2}$ is done similarly. □

We can now prove Theorem 8.2.14.

Proof of Theorem 8.2.14. The fractions $\frac{m_0}{d_0}, \frac{m_1}{d_1}, \ldots, \frac{m_l}{d_l}$ in the reduced sequence depend only on the elements of $[s_1, \frac{m_l}{d_l}]$ of denominator less than or equal to $\max(d_0, d_{\lambda+1})$, as do the fractions $\frac{m^{(1)}_0}{d^{(1)}_0}, \ldots, \frac{m^{(1)}_{l_1}}{d^{(1)}_{l_1}} = \frac{m_l}{d_l}$. This proves that $d^{(1)}_i = d_i$ hence $\mu_i = \mu^{(1)}_i$ for $0 \leq i \leq l_1$. Similarly $d^{(2)}_i = d_{\lambda+1-i}$ hence $\mu_{\lambda+1-i} = \mu^{(2)}_i$ for $0 \leq i \leq l_2$. It remains to show maximality of l_1 and l_2.

Suppose there is some r_1, r_2 such that $\lambda_i > r_i > l_i$ and $\mu^{(1)}_{r_1} = \mu^{(2)}_{r_2} < \mu^{(1)}_{l_1}$. In addition to this, $d^{(1)}_{r_1} = d^{(2)}_{r_2} < d$ (recall $\frac{m}{d}$ is the unique fraction with least denominator in $[s_2, s_1] \cap \mathbb{Q}$). Therefore $q_2 = 1 - \frac{m^{(2)}_{r_2}}{d^{(2)}_{r_2}} \in (s_1, 1]$ and $q_1 = \frac{m^{(1)}_{r_1}}{d^{(1)}_{r_1}} \in [0, s_2)$. Let q' be the unique rational with least denominator d' in $[q_1, q_2]$. By uniqueness, $d' < d^{(1)}_{r_1} < d$. Therefore, $q' \in (s_1, q_2)$ or (q_1, s_2). Suppose for now that $q' \in (s_1, q_2)$, and consider again the reduced sequence

$$1 - \mu t_2 = \frac{m^{(2)}_0}{d^{(2)}_0} > \frac{m^{(2)}_1}{d^{(2)}_1} > \cdots > \frac{m^{(2)}_{\lambda_2}}{d^{(2)}_{\lambda_2}} > \frac{m^{(2)}_{\lambda_2+1}}{d^{(2)}_{\lambda_2+1}} = -1.$$

However $1 - q_2$ cannot appear in this reduced sequence since a fraction with smaller denominator, $1 - q'$, appears to the left of it in the non-reduced sequence. So, at some step in the reduction process $1 - q_2$ would have been removed. Therefore, $q' \notin (s_1, q_2)$. Similarly, one can show that $q' \notin (q_1, s_2)$. This is a contradiction. So no such r_1 and r_2 exist. □

Chapter 9

Hyperelliptic Curves with Tame Reduction

The previous chapter looked at the minimal SNC models of specific cases of hyperelliptic curves. In this chapter, we state our main theorems in full generality. Theorem 9.2.3 gives the structure of the special fibre of the minimal SNC model, and Theorems 9.3.1 and 9.3.2 give more explicit descriptions of multiplicities and genera of components appearing in the special fibre.

9.1 Orbits of Clusters

First we need to extend some definitions. The definitions in Section 2.1 come from [DDMM18], where everything is semistable, so do not deal with orbits of clusters. Here we make some new definitions which extend the preexisting ones to orbits. First let us recall Definitions 5.1.3 and 5.1.4.

Definition 9.1.1 (Definition 5.1.3)**.** Let X be a Galois orbit of clusters. Then X is *übereven* if for all $\mathfrak{s} \in X$, $\mathfrak{s}$ is übereven. Define an orbit X to be *odd*, *even*, and *principal* similarly.

Definition 9.1.2 (Definition 5.1.4)**.** An orbit X' is a *child* of X, written $X' < X$, if for every $\mathfrak{s}' \in X'$ there exists some $\mathfrak{s} \in X$ such that $\mathfrak{s}' < \mathfrak{s}$. Define $\delta_{X'} = \delta_{\mathfrak{s}'}$ for some $\mathfrak{s}' \in X'$.

Definition 9.1.3. Let X be an orbit of clusters. Define K_X/K to be the field extension of K of degree $|X|$. By Lemma 2.1.16, K_X/K is the minimal field extension over which for any $\mathfrak{s} \in X$, $\sigma \in \mathrm{Gal}(\overline{K}/K_X)$ we have $\sigma(\mathfrak{s}) = \mathfrak{s}$.

Definition 9.1.4. For X be Galois orbit of clusters, with some $\mathfrak{s} \in X$, define

$$d_X = \frac{a_X}{b_X} = d_{\mathfrak{s}}, \quad \nu_X = \nu_{\mathfrak{s}}, \quad \lambda_X = \lambda_{\mathfrak{s}}, \quad g_{\mathrm{ss}}(X) = g_{\mathrm{ss}}(\mathfrak{s}), \quad \text{and} \quad \epsilon_X = \epsilon_{\mathfrak{s}}^{|X|}.$$

There are well defined, i.e they do not depend on the choice of $\mathfrak{s} \in X$.

Definition 9.1.5. Let X be a principal orbit of clusters with $g_{\mathrm{ss}}(X) > 0$ and fix some $\mathfrak{s} \in X$. Define $C_{\widetilde{X}}$ to be the curve $C_{\tilde{\mathfrak{s}}}$ over K_X. Denote the minimal SNC model of $C_{\widetilde{X}}/K_X$ by $\mathscr{X}_{\widetilde{X}}/\mathcal{O}_{K_X}$, and the central component by $\Gamma_{\widetilde{X}}/k$.

Remark 9.1.6. The curve $C_{\widetilde{X}}$ depends on a choice of $\mathfrak{s} \in X$, but the combinatorial description of the special fibre of the minimal SNC model will not. Since this is what we need $C_{\widetilde{X}}$ for, we do not need to worry about this.

Definition 9.1.7. Let X be a principal orbit of clusters. Define $e_X \in \mathbb{Z}_{>0}$ to be minimal such that $e_X|X|d_{\mathfrak{s}} \in \mathbb{Z}$ and $e_X|X|\nu_{\mathfrak{s}} \in 2\mathbb{Z}$ for all $\mathfrak{s} \in X$. Define $g(X) = g(\mathfrak{s})$ for $\mathfrak{s} \in X$ over K_X, where $g(\mathfrak{s})$ is as defined in Definition 8.1.22.

Remark 9.1.8. Analogously to Lemma 8.1.25, the curve $C_{\widetilde{X}}/K_X$ is semistable over an extension L/K_X of degree e_X and the quotient map $\Gamma_{\mathfrak{s},L} \to \Gamma_{\mathfrak{s},K_X}$ has degree e_X for $\mathfrak{s} \in X$.

9.2 Special Fibre of the Minimal SNC Model

We state here the first of our main theorems. Roughly this tells us that the cluster picture, the leading coefficient of f, and the action of G_K on the cluster picture is enough to calculate the structure of the minimal SNC model, along with the multiplicities and genera of the components.

Theorem 9.2.1. *Let K be a complete discretely valued field with algebraically closed residue field of characteristic $p > 2$. Let $C : y^2 = f(x)$ be a hyperelliptic curve over K with tame reduction and cluster picture $\Sigma_{C/K}$. Then the dual graph, with genus and multiplicity, of the special fibre of the minimal SNC model of C/K is completely determined by $\Sigma_{C/K}$ (with depths), the valuation of the leading coefficient $v_K(c_f)$ of f, and the action of G_K.*

Remark 9.2.2. If K does not have algebraically closed residue field, then the Frobenius action on the dual graph is determined by this data, as well as the values of $\epsilon_X(\mathrm{Frob})$ for each orbits of clusters X. We will not discuss this further here, but one can refer to [FN20].

The proof of this will follow from the theorems proved in the rest of this section, and we make this more precise later. First we split Theorem 9.2.1 into several smaller theorems. The first tells us which components appear in the special fibre of the minimal SNC model. Roughly, there is a central component for every orbit of principal non-übereven clusters, one or two central components for every orbit of principal übereven clusters, and a chain of rational curves associated to each orbit of twins. These central components are linked by chains of rational curves, and certain central components will also have tails intersecting them. The following theorem gives us the structure of the special fibre but is missing important details such as multiplicities, genera and lengths of these chains. These remaining details will be discussed in Theorem 9.3.2.

Theorem 9.2.3 (Structure of SNC model)**.** *Let K be a complete discretely valued field with algebraically closed residue field of characteristic $p > 2$. Let C/K be a hyperelliptic curve with tame reduction. Then the special fibre of its minimal SNC model is structured as follows. Every principal Galois orbit of clusters X contributes one central component Γ_X, unless X is übereven with $\epsilon_X = 1$, in which case X contributes two central components Γ_X^+ and Γ_X^-.*

These central components are linked by chains of rational curves, or are intersected transversely by a crossed tail in the following ways (where, for any orbit Y, we write $\Gamma_Y^+ = \Gamma_Y^- = \Gamma_Y$ if Y is not übereven):

Name	*From*	*To*	*Condition*
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	$X' < X$ *both principal,* X' *odd*
$L^+_{X,X'}$	Γ^+_X	$\Gamma^+_{X'}$	$X' < X$ *both principal,* X' *even with* $\epsilon_{X'} = 1$
$L^-_{X,X'}$	Γ^-_X	$\Gamma^-_{X'}$	$X' < X$ *both principal,* X' *even with* $\epsilon_{X'} = 1$
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	$X' < X$ *both principal,* X' *even with* $\epsilon_{X'} = -1$
$L_{X'}$	Γ^-_X	Γ^+_X	X *principal,* $X' < X$ *orbit of twins,* $\epsilon_{X'} = 1$
$T_{X'}$	Γ_X	-	X *principal,* $X' < X$ *orbit of twins,* $\epsilon_{X'} = -1$

Note that any chain where the "To" column has been left blank is a crossed tail. If $\mathcal{R}$ is not principal then we also get the following chains of rational curves:

Name	*From*	*To*	*Condition*
$L_{\mathfrak{s}}$	$\Gamma^-_{\mathfrak{s}'}$	$\Gamma^+_{\mathfrak{s}'}$	$\mathfrak{s} < \mathcal{R}$, $\mathfrak{s}$ *a cotwin,* $\mathfrak{s}' < \mathfrak{s}$ *child of size* $2g$, $\epsilon_{\mathfrak{s}'} = 1$
$T_{\mathfrak{s}}$	$\Gamma_{\mathfrak{s}'}$	-	$\mathfrak{s} < \mathcal{R}$, $\mathfrak{s}$ *a cotwin,* $\mathfrak{s}' < \mathfrak{s}$ *child of size* $2g$, $\epsilon_{\mathfrak{s}'} = -1$
$L_{\mathcal{R}}$	$\Gamma^-_{\mathfrak{s}}$	$\Gamma^+_{\mathfrak{s}}$	$\mathcal{R}$ *a cotwin,* $\mathfrak{s} < \mathcal{R}$ *principal of size* $2g$, $\epsilon_{\mathfrak{s}} = 1$
$T_{\mathcal{R}}$	$\Gamma_{\mathfrak{s}}$	-	$\mathcal{R}$ *a cotwin,* $\mathfrak{s} < \mathcal{R}$ *principal of size* $2g$, $\epsilon_{\mathfrak{s}} = -1$
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma_{\mathfrak{s}_1}$	$\Gamma_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, *with* $\mathfrak{s}_i$ *both principal, odd and stable*
T_X	Γ_X	-	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{s_1, s_2\}$ *principal, odd orbit*
$L^+_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma^+_{\mathfrak{s}_1}$	$\Gamma^+_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *stable, principal and even,* $\epsilon_{\mathfrak{s}_i} = 1$
$L^-_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma^-_{\mathfrak{s}_1}$	$\Gamma^-_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *stable, principal and even,* $\epsilon_{\mathfrak{s}_i} = 1$
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma_{\mathfrak{s}_1}$	$\Gamma_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *stable, principal and even,* $\epsilon_{\mathfrak{s}_i} = -1$
T^+_X	Γ^+_X	-	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{s_1, s_2\}$ *principal and even,* $\epsilon_{\mathfrak{s}_i} = 1$
T^-_X	Γ^-_X	-	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{s_1, s_2\}$ *principal and even,* $\epsilon_{\mathfrak{s}_i} = 1$
T_X	Γ_X	-	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{s_1, s_2\}$ *principal and even,* $\epsilon_{\mathfrak{s}_i} = -1$
$L_{\mathfrak{t}}$	$\Gamma^-_{\mathfrak{s}}$	$\Gamma^+_{\mathfrak{s}}$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal and even,* $\mathfrak{t}$ *a twin,* $\epsilon_{\mathfrak{t}} = 1$
$T_{\mathfrak{t}}$	$\Gamma_{\mathfrak{s}}$	-	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal and even,* $\mathfrak{t}$ *a twin,* $\epsilon_{\mathfrak{t}} = -1$

Finally, a central component Γ_X is intersected transversally by some tails if and only if $e_X > 1$. These are explicitly described in Theorem 9.3.2.

Remark 9.2.4. At no point do we give explicit equations for the central components $\Gamma_X^{\pm}$. However, these can be calculated using the method laid out in this thesis. In particular, one can take the explicit equations given in [DDMM18, Theorem 8.5] for the components $\Gamma_{\mathfrak{s},L}^{\pm}$ in the semistable model of C/L and the Galois action on these components, and apply [DD18, Theorem 1.1].

Before we prove this, let us prove a couple of results. Recall that L is a field over which C has semistable reduction and that $\Gamma_{\mathfrak{s},L}$ is the component associated to a cluster $\mathfrak{s}$ in the special fibre of the minimal semistable model $\mathscr{Y}$ of C over L.

Lemma 9.2.5. *Let $\mathfrak{s}$ be a principal cluster with $g_{ss}(\mathfrak{s}) = 0$.*

(i) If $\mathfrak{s} = \mathcal{R}$ and $\mathfrak{s}$ is not übereven (resp. übereven) then $\Gamma_{\mathfrak{s},L}$ (resp. each of $\Gamma_{\mathfrak{s},L}^{+}$ and $\Gamma_{\mathfrak{s},L}^{-}$) intersects at least two other components.

(ii) If $\mathfrak{s} \neq \mathcal{R}$ and $\mathfrak{s}$ is not übereven (resp. übereven) then $\Gamma_{\mathfrak{s},L}$ (resp. each of $\Gamma_{\mathfrak{s},L}^{+}$ and $\Gamma_{\mathfrak{s},L}^{-}$) intersects at least three other components.

Proof. (i) Let $\mathfrak{s} = \mathcal{R}$ and suppose $\mathfrak{s}$ is not übereven. Since $g_{\mathrm{ss}}(\mathfrak{s}) = 0$, $\mathfrak{s}$ can have at most two odd children and in particular at most two singletons. Since, $g(C) \geq 2$, we have $|\mathfrak{s}| \geq 5$. If $|\mathfrak{s}|$ is odd then $\mathfrak{s}$ must have an even child $\mathfrak{s}'$ and, by [DDMM18, Theorem 8.5], $\Gamma_{\mathfrak{s},L}$ is intersected by the two linking chains to $\Gamma_{\mathfrak{s}',L}$. Since $\mathfrak{s}$ is principal, $\mathfrak{s}$ cannot be the union of two odd clusters. So, if $|\mathfrak{s}|$ is even then $\mathfrak{s}$ has an even child and we are done by [DDMM18, Theorem 8.5].

If $\mathfrak{s} = \mathcal{R}$ is übereven then every child of $\mathfrak{s}$ is even. In particular, there are at least two even children $\mathfrak{s}_1$ and $\mathfrak{s}_2$. So, each of $\Gamma_{\mathfrak{s},L}^{\pm}$ intersects $L_{\mathfrak{s}_1}^{\pm}$ and $L_{\mathfrak{s}_2}^{\pm}$ (the linking chains to the children).

(ii) Let $\mathfrak{s} \neq \mathcal{R}$ and suppose $\mathfrak{s}$ is not übereven. Since $\mathfrak{s}$ is principal, we know $|\mathfrak{s}| \geq 3$. Therefore, $\mathfrak{s}$ must have at least one proper child $\mathfrak{s}'$. Suppose that $P(\mathfrak{s})$ is principal. If $\mathfrak{s}' < \mathfrak{s}$ is even then $\Gamma_{\mathfrak{s},L}$ intersects the linking chain to $\Gamma_{P(\mathfrak{s}),L}$ and the two linking chains to $\Gamma_{\mathfrak{s}',L}$. Otherwise $\mathfrak{s}$ must be the union of two odd clusters, hence $\mathfrak{s}$ is even. In this case there are two linking chains to $\Gamma_{P(\mathfrak{s}),L}$ and one to $\Gamma_{\mathfrak{s}',L}$. A similar argument works if $\mathfrak{s}$ is übereven. If $P(\mathfrak{s}) = \mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}_2$ is not principal, the argument is similar, but linking chains to $\Gamma_{P(\mathfrak{s}),L}$ are replaced by linking chains to $\Gamma_{\mathfrak{s}_2,L}$. □

Proposition 9.2.6. *Let $\mathscr{Y}$ be the semistable model of C/L and $\mathscr{Z}$ the image under the quotient map. Let $\mathscr{X}$ be the SNC model obtained by resolving the singularities of $\mathscr{Z}$ such that all rational chains are minimal. Let X be a principal orbit of clusters. Let $\Gamma_{X,K} \in \mathscr{X}_k$ be the image of $\Gamma_{\mathfrak{s},L}$ for some $\mathfrak{s} \in X$*

under the quotient by Gal(L/K)*. Then if* $g(\Gamma_{X,K}) = 0$ *and* $(\Gamma_{X,K} \cdot \Gamma_{X,K}) = -1$*,* $\Gamma_{X,K}$ *intersects at least three other components of the special fibre (i.e. blowing down* $\Gamma_{X,K}$ *would not result in an SNC model).*

Proof. [FN20, Proposition 7.15]. □

We are now able to prove our structure theorem (Theorem 9.2.3).

Proof of Theorem 9.2.3. First let us find which central components appear. Over L, by [DDMM18, Theorem 8.5], we know there is a component for every principal, non-übereven cluster, and we know the action of $\mathrm{Gal}(L/K)$ on these central components is the same as the action on the clusters. After taking the quotient by $\mathrm{Gal}(L/K)$, we get a component for every orbit of principal, non übereven clusters. Similarly over L, by [DDMM18, Theorem 8.5], we know there are two components for every übereven cluster $\mathfrak{s}$. These are swapped by Galois if and only if $\epsilon_{\mathfrak{s}} = -1$. Taking the quotient gives us two components for an übereven orbit X if $\epsilon_X = 1$ and a single component if $\epsilon_X = -1$. We call these components the central components. Showing which linking chains appear is done similarly, using information given in [DDMM18, Theorem 8.5].

To ensure these central components do in fact appear in the minimal SNC model, we must check that they cannot be blown down. Any central component $\Gamma_{X,K} \in \mathscr{X}_k$ is the image of $\Gamma_{\mathfrak{s},L} \in \mathscr{Y}_k$ for some $\mathfrak{s} \in X$. A central component $\Gamma_{X,K}$ can only be blown down if $g(\Gamma_{X,K}) = 0$, and $(\Gamma_{X,K} \cdot \Gamma_{X,K}) = -1$. However, by Proposition 9.2.6, any central component $\Gamma_{X,K}$ with $g(\Gamma_{X,K}) = 0$ and $(\Gamma_{X,K} \cdot \Gamma_{X,K}) = -1$ intersects at least three other components of the special fibre. Therefore, if $\Gamma_{X,K}$ were to be blown down, $\mathscr{X}_k$ would no longer be an SNC divisor. So $\Gamma_{X,K}$ appears in the special fibre of the minimal SNC model. □

Remark 9.2.7. A linking chain can have length 0, and indicates an intersection between central components (when $X' < X$ both principal) or a singular central component (when X is principal and $X' < X$ is an orbit of twins).

9.3 Explicit Description of the Special Fibre

Theorem 9.2.3 describes the structure of the special fibre, but says nothing about the multiplicity or genera of the components. The following theorems fill in these details. The first focuses on the central components, and the second describes the chains of rational curves present in the special fibre.

Theorem 9.3.1 (Central Components)**.** *Let* K *and* C/K *be as in Theorem 9.2.3. Let* X *be an orbit of clusters in* $\Sigma_{C/K}$*. Then* $\Gamma_X^{\pm}$ *has multiplicity* $|X|e_X$ *and genus* $g(X)$*. Note that if* X *is übereven then* Γ_X *has genus* 0*.*

Proof. Let X be a principal, orbit, and choose some $\mathfrak{s} \in X$. Recall that K_X is the minimal field extension of K such that the clusters of X are fixed by $\mathrm{Gal}(\overline{K}/K_X)$, and L is the minimal field extension of K such that C is semistable over L. The image $\Gamma_{\mathfrak{s},K_X}$ of $\Gamma_{\mathfrak{s},L}$ after taking the quotient by $\mathrm{Gal}(L/K_X)$ has multiplicity e_X, since the action on $\Gamma_{\mathfrak{s},L}$ has multiplicity e_X (by Lemma 8.1.25). There are $|X|$ such components, which are permuted by $\mathrm{Gal}(K_X/K)$ in the minimal SNC model of C/K_X. So, Γ_X has multiplicity $|X|e_X$ by [Lor90, Fact IV].

To find the genus of the central components, note that (since genus cannot increase by taking the quotient) if $g(\Gamma_{\mathfrak{s},L}) = 0$ then $g(\Gamma_{X,K}) = 0$. So let us assume that $g(\Gamma_{\mathfrak{s},L}) > 0$. In this case, as mentioned in Remark 8.1.21, $\Gamma_{\mathfrak{s},L}$ is isomorphic to the special fibre of the smooth model of $C_{\widetilde{\mathfrak{s}}}$ over L. Furthermore, the action on $\Gamma_{\mathfrak{s},L}$ is the same as the action on $\Gamma_{\widetilde{\mathfrak{s}},L}$. Hence, the genus of $\Gamma_{\mathfrak{s},K_X}$ is $g(X)$, and also the genus of $\Gamma_{X,K}$. $\square$

Theorem 9.3.2 (Description of Chains)**.** *Let K and C/K be as in Theorem 9.2.3. Let X be a principal orbit of clusters. Choose some $\mathfrak{s} \in X$ of depth $d_{\mathfrak{s}}$ with denominator $b_{\mathfrak{s}}$. If $e_X > 1$, then the central component(s) associated to X are intersected transversely by the following sloped tails with parameters (t_1, μ) (writing $\Gamma_X = \Gamma_X^+ = \Gamma_X^-$ if X is not übereven):*

Name	*From*	*Number*	t_1	μ	*Condition*
T_∞	Γ_X	1	$(g+1)d_{\mathcal{R}} - \lambda_{\mathcal{R}}$	1	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *odd*
$T_\infty^\pm$	$\Gamma_X^\pm$	2	$-d_{\mathcal{R}}$	1	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *even*, $\epsilon_{\mathcal{R}} = 1$
T_∞	Γ_X	1	$-d_{\mathcal{R}}$	2	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *even*, $e_{\mathcal{R}} > 2$, $\epsilon_{\mathcal{R}} = -1$
$T_{y_{\mathfrak{s}}=0}$	Γ_X	$\lfloor \frac{\lvert\mathfrak{s}_{\mathrm{sing}}\rvert \lvert X\rvert}{b_X} \rfloor$	$-\lambda_X$	b_X	$\lvert\mathfrak{s}_{\mathrm{sing}}\rvert \geq 2$, *and* $e_X > b_X/\lvert X\rvert$
$T_{x_{\mathfrak{s}}=0}$	Γ_X	1	$-d_X$	$2\lvert X\rvert$	*X has no stable child*, $\lambda_X \notin \mathbb{Z}$, $e_X > 2$, *and either* $g_{\mathrm{ss}}(X) > 0$ *or X is übereven*
$T_{x_{\mathfrak{s}}=0}^\pm$	$\Gamma_X^\pm$	2	$-d_X$	$\lvert X\rvert$	*X has no stable child*, $\lambda_X \in \mathbb{Z}$, *and either* $g_{\mathrm{ss}}(X) > 0$ *or X is übereven*
$T_{(0,0)}$	Γ_X	1	$-\lambda_X$	$\lvert X\rvert$	*[X has an orphan singleton], or [$g_{\mathrm{ss}}(X) = 0$, X is not übereven and X has no proper stable odd child]*

Furthermore, regardless of whether $e_X > 1$ or not, for $X' < X$ an orbit of clusters, the central components are intersected by the following sloped chains of rational curves with parameters $(t_1 - \delta, t_1, \mu)$:

Name	t_1	δ	μ	*Condition*
$L_{X,X'}$	$-\lambda_X$	$\delta_{X'}/2$	$\lvert X'\rvert$	X', X *principal*, X' *odd*
$L^+_{X,X'}$	$-d_X$	$\delta_{X'}$	$\lvert X'\rvert$	X', X *principal*, X' *even*, $\epsilon_{X'} = 1$
$L^-_{X,X'}$	$-d_X$	$\delta_{X'}$	$\lvert X'\rvert$	X, X *principal*, X' *even*, $\epsilon_{X'} = 1$
$L_{X,X'}$	$-d_X$	$\delta_{X'}$	$2\lvert X'\rvert$	X', X *principal*, X' *even*, $\epsilon_{X'} = -1$
$L_{X'}$	$-d_X$	$2\delta_{X'}$	$\lvert X'\rvert$	X *principal*, X' *orbit of twins*, $\epsilon_{X'} = 1$
$T_{X'}$	$-d_X$	$\delta_{X'} + \frac{1}{\mu}$	$2\lvert X'\rvert$	X *principal*, X' *orbit of twins*, $\epsilon_{X'} = -1$

If $\mathcal{R}$ *is not principal we get additional sloped chains with parameters* $(t_1 - \delta, t_1, \mu)$ *as follows:*

Name	t_1	δ	μ	*Condition*
$L_{\mathfrak{s}}$	$-\delta_{\mathfrak{s}}$	$2\delta_{\mathfrak{s}'}$	1	$\mathfrak{s} < \mathcal{R}$ *cotwin*, $\mathfrak{s}' < \mathfrak{s}$ *child of size* $2g$, $v(c_f) \in 2\mathbb{Z}$
$T_{\mathfrak{s}}$	$-\delta_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'} + \frac{1}{\mu}$	2	$\mathfrak{s} < \mathcal{R}$ *cotwin*, $\mathfrak{s}' < \mathfrak{s}$ *child of size* $2g$, $v(c_f) \notin 2\mathbb{Z}$
$L_{\mathcal{R}}$	$-d_{\mathcal{R}}$	$2\delta_{\mathfrak{s}}$	1	$\mathcal{R}$ *a cotwin*, $\mathfrak{s} < \mathcal{R}$ *child of size* $2g$, $v_K(c_f) \in 2\mathbb{Z}$
$T_{\mathcal{R}}$	$-d_{\mathcal{R}}$	$\delta_{\mathfrak{s}} + \frac{1}{\mu}$	2	$\mathcal{R}$ *a cotwin*, $\mathfrak{s} < \mathcal{R}$ *child of size* $2g$, $v_K(c_f) \notin 2\mathbb{Z}$
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$(g(\mathfrak{s}_1)+1)d_{\mathfrak{s}_1} - \lambda_{\mathfrak{s}_1}$	$\frac{1}{2}\delta(\mathfrak{s}_1,\mathfrak{s}_2)$	1	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *principal, odd,*
T_X	$(g(\mathfrak{s}_1)+1)d_{\mathfrak{s}_1} - \lambda_{\mathfrak{s}_1}$	$\frac{1}{2}\delta(\mathfrak{s}_1,\mathfrak{s}_2)$	2	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{\mathfrak{s}_1,\mathfrak{s}_2\}$ *principal, odd orbit*
$L^+_{\mathfrak{s}_1,\mathfrak{s}_2}$	$d_{\mathfrak{s}_1}$	$\delta(\mathfrak{s}_1,\mathfrak{s}_2)$	1	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *principal, even,* $\epsilon_{\mathfrak{s}_i} = 1$
$L^-_{\mathfrak{s}_1,\mathfrak{s}_2}$	$d_{\mathfrak{s}_1}$	$\delta(\mathfrak{s}_1,\mathfrak{s}_2)$	1	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *principal, even,* $\epsilon_{\mathfrak{s}_i} = 1$
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$d_{\mathfrak{s}_1}$	$\delta(\mathfrak{s}_1,\mathfrak{s}_2)$	2	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *principal, even,* $\epsilon_{\mathfrak{s}_i} = -1$
L^+_X	$d_{\mathfrak{s}_1}$	$\delta(\mathfrak{s}_1,\mathfrak{s}_2)$	2	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{\mathfrak{s}_1,\mathfrak{s}_2\}$ *principal, even orbit, and* $\epsilon_{\mathfrak{s}_i} = 1$
L^-_X	$d_{\mathfrak{s}_1}$	$\delta(\mathfrak{s}_1,\mathfrak{s}_2)$	2	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{\mathfrak{s}_1,\mathfrak{s}_2\}$ *principal, even orbit, and* $\epsilon_{\mathfrak{s}_i} = 1$
T_X	$d_{\mathfrak{s}_1}$	$\delta(\mathfrak{s}_1,\mathfrak{s}_2) + \frac{1}{\mu}$	4	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{\mathfrak{s}_1,\mathfrak{s}_2\}$ *principal, even orbit, and* $\epsilon_{\mathfrak{s}_i} = -1$
$L_{\mathfrak{t}}$	$d_{\mathfrak{s}}$	$2\delta(\mathfrak{s},\mathfrak{t})$	1	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal even,* $\mathfrak{t}$ *twin,* $\epsilon_{\mathfrak{t}} = 1$
$T_{\mathfrak{t}}$	$d_{\mathfrak{s}}$	$\delta(\mathfrak{s},\mathfrak{t}) + \frac{1}{\mu}$	2	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal even,* $\mathfrak{t}$ *twin,* $\epsilon_{\mathfrak{t}} = -1$

Finally, the crosses of any crossed tail have multiplicity $\frac{\mu}{2}$.

Proof. Postponed to Section 9.4. □

Remark 9.3.3. If there is any confusion over which central components linking chains or tails intersect, the reader is urged to refer back to the tables in Theorem 9.2.3. This information has been omitted from the tables in Theorem 9.3.2 due to spatial concerns.

Remark 9.3.4. Let X be a principal orbit of clusters in $\Sigma_{C/K}$. As in Remark 8.2.5, we compare the rational chains intersecting a central component $\Gamma_X \in \mathscr{X}_k$ to the tails in the special fibre of the minimal SNC model $\mathscr{X}_{\widetilde{X}}$. The central component $\Gamma_X \in \mathscr{X}_k$ will have the same genus as the central component $\Gamma_{\widetilde{X}} \in \mathscr{X}_{\widetilde{X},k}$ and the multiplicity is multiplied by $|X|$. It will have the same tails (with all multiplicities multiplied by $|X|$) except these tails will make up part of the linking chains intersecting Γ_X in the following cases:

- If $X \neq \mathcal{R}$ and $P(X)$ is principal, an ∞-tail in $\mathscr{X}_{\widetilde{X},k}$ will form the uphill section of one of the linking chains $L^{\pm}_{P(X),X}$,
- If $X < \mathcal{R}$ and $\mathcal{R}$ is not principal, then any ∞-tail in $\mathscr{X}_{\widetilde{X},k}$ will form the uphill section of a chain: the linking chain between $\Gamma_{\mathfrak{s}_1}$ and $\Gamma_{\mathfrak{s}_2}$ if $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ and $X = \{\mathfrak{s}_1\}$; the crossed tail if $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ and $X = \{\mathfrak{s}_1, \mathfrak{s}_2\}$; and the loop or crossed tail arising from $\mathcal{R}$ if $\mathcal{R}$ is a cotwin,
- If $X' < X \neq \mathcal{R}$ and X is not principal (that is, X' must have size $2g$, and X has size $2g+1$), then any ∞-tail in $\mathscr{X}_{\widetilde{X'},k}$ will form the uphill section of a loop or crossed tail arising from X,
- a $(y_{\mathfrak{s}} = 0)$-tail will form the downhill section of a linking chain $L_{X,X'}$ if there exists some $X' < X$, a non-trivial orbit of odd, principal children,
- a $(x_{\mathfrak{s}} = 0)$-tail will form the downhill section of a linking chain $L^{\pm}_{X,X'}$ if there exists some $\{\mathfrak{s}'\} = X' < X$, a stable even child,
- a $(0,0)$-tail will form the downhill section of a linking chain $L_{X,X'}$ if there exists some $\{\mathfrak{s}'\} = X' < X$, a unique stable odd child,

where again, all multiplicities are multiplied by $|X|$.

9.4 Proof of Theorem 9.3.2

To prove Theorem 9.3.2, we will proceed by induction on two things: the number of proper clusters in $\Sigma_{C/K}$, and the degree $e = [L : K]$ of the minimal extension L/K such that C/L is semistable. The base cases for these are when $\Sigma_{C/K}$ consists of a single proper cluster (which is covered in Section 8.1, in particular Theorem 8.1.1 and Proposition 8.1.11), and when C has semistable reduction over K i.e. $e = 1$ (which is covered in Section 7.1.3). For our inductive hypothesis, suppose that for any hyperelliptic curve where the number of proper clusters in its cluster picture is strictly less than that of C/K, or the degree of an extension needed such that it is semistable is strictly less than that of C, we can completely determine the special fibre of its minimal SNC model.

9.4.1 Principal Top Cluster

We start by assuming that the top cluster $\mathcal{R}$ is principal, and that it has a Galois invariant proper child $\mathfrak{s}$. We will calculate the tails of $\Gamma^{\pm}_{\mathcal{R},K}$ and, if $\mathfrak{s}$ is principal, $\Gamma^{\pm}_{\mathfrak{s},K}$. We will also calculate the linking chain(s) (or the chain arising from $\mathfrak{s}$ if $\mathfrak{s}$ is a twin) between them. This will be done by comparing the linking chain(s) to those in the special fibre of the minimal SNC model of another hyperelliptic curve over K, which we will call C^{new}. We will write $C^{\text{new}} : y^2 = f^{\text{new}}(x)$, and denote the set of roots of f^{new} over $\overline{K}$ by $\mathcal{R}^{\text{new}}$. The curve C^{new}/K is chosen so that $\Sigma_{C^{\text{new}}/K}$ has a unique proper cluster $\mathfrak{s}^{\text{new}} \neq \mathcal{R}^{\text{new}}$, enabling us to apply the results of Section 8.2. We will then use induction to deduce the components of the model arising from the subclusters of $\mathfrak{s}$. Finally, we will remove the assumption that $\mathfrak{s}$ is Galois invariant.

Lemma 9.4.1. *Let $\mathcal{R}$ be principal and suppose that $e_{\mathcal{R}} > 1$. The tails of the central component(s) associated to $\mathcal{R}$ are as described in Theorem 9.3.2.*

Proof. First suppose that $\mathcal{R}$ is not übereven. Let $\mathscr{Y}$ be the semistable model of C/L and consider $\Gamma_{\mathcal{R},L} \subseteq \mathscr{Y}$. The stabiliser of $\mathcal{R}$ has order $e_{\mathcal{R}}$. Under the quotient map, a Galois orbit T of points of $\Gamma_{\mathcal{R},L}$ gives rise to a singularity on $\Gamma_{\mathcal{R},K}$ lying on precisely one component of $\mathscr{X}_K$ if and only if $|T| < e_{\mathcal{R}}$ and the points of T lie on $\Gamma_{\mathcal{R},L}$ and no other components of $\mathscr{Y}_k$.

Suppose that $g(\Gamma_{\mathcal{R},L}) = 0$. There are only two orbits with size less than $e_{\mathcal{R}}$, which after an appropriate shift we can assume are at $x_{\mathcal{R}} = 0$ and $x_{\mathcal{R}} = \infty$. The point at ∞ certainly lies on no other component of $\mathscr{Y}_k$ by [DDMM18, Propositions 5.5,5.20], so $\Gamma_{\mathcal{R},K}$ will always have ∞-tails. By [DDMM18, Proposition 5.20], the point $x_{\mathcal{R}} = 0$ lies on no other component of $\mathscr{Y}_k$ if and only if $\mathcal{R}$ has no stable proper odd child. This is because if $\mathfrak{s} < \mathcal{R}$ is a stable odd child then $L_{\mathcal{R},\mathfrak{s}}$ intersects $\Gamma_{\mathcal{R},L}$ at $x_{\mathcal{R}} = 0$, however no other linking chain to a child will ever intersect $\Gamma_{\mathcal{R},L}$ at $x_{\mathcal{R}} = 0$. Therefore $\Gamma_{\mathcal{R},K}$ will have a $(0,0)$-tail if and only if it has no stable proper odd child. The description of the tails follows.

Suppose instead that $g(\Gamma_{\mathcal{R},L}) > 0$. The orbits of points on $\Gamma_{\mathcal{R},L}$ of size less than $e_{\mathcal{R}}$ are the same as the small orbits of points on $\Gamma_{\widetilde{\mathcal{R}},L}$, which are described in Lemmas 8.1.7 - 8.1.10. To complete the description, we must calculate when these small orbits are intersection points with other components. We do this using the explicit description of the components of $\mathscr{Y}_k$ given in [DDMM18, Proposition 5.20] and how they glue in [DDMM18, Proposition 5.5]. From this, we can deduce that the points at ∞ never lie on a component other than $\Gamma_{\mathcal{R},L}$, $(y_{\mathfrak{s}} = 0)$-orbits are intersection points if and only if $\mathfrak{s}$ has a non-trivial orbit of proper odd children, $(x_{\mathfrak{s}} = 0)$-orbits are intersection points if and only if $\mathfrak{s}$ has a stable even child, and the $(0,0)$-orbit is an intersection point if and only if $\mathcal{R}$ has a proper stable odd child.

Now suppose $\mathcal{R}$ is übereven. Then each $\Gamma^{\pm}_{\mathcal{R},L}$ has two orbits of size less than $e_{\mathcal{R}}$, $\{x_{\mathcal{R}} = 0\}$ and $\{x_{\mathcal{R}} = \infty\}$. The points at ∞ do not lie on any other components of $\mathscr{Y}_k$. The points at 0 lie on no other component of $\mathscr{Y}_k$ if and only if $\mathcal{R}$ has no stable child. So, $\Gamma^{\pm}_{\mathcal{R},K}$ has a $(x_{\mathfrak{s}} = 0)$-tail if and only if $\mathcal{R}$ does not have a stable child. The description of the tails follows. □

Lemma 9.4.2. *Let $\mathfrak{s} < \mathcal{R}$ be a principal, Galois invariant cluster with $e_{\mathfrak{s}} > 1$. Then the tails intersecting the central component(s) associated to $\mathfrak{s}$ are as described in Theorem 9.3.2.*

Proof. The proof is similar to that of the previous lemma, noting that all of the orbits at infinity are the intersection points of $\Gamma^{\pm}_{\mathfrak{s},L}$ and the linking chain between $\Gamma^{\pm}_{\mathcal{R},L}$ and $\Gamma^{\pm}_{\mathfrak{s},L}$. □

Following is a technical lemma allowing us to compare the chain(s) appearing between $\Gamma_{\mathcal{R},K}$ and $\Gamma_{\mathfrak{s},K}$ to those of a simpler curve C^{new}.

Lemma 9.4.3. *Let $\mathfrak{s}_1, \mathfrak{s}_2$ be two Galois invariant principal clusters (resp. a principal cluster and a twin) such that either $\mathfrak{s}_2 < \mathfrak{s}_1$, or $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal. Then any linking chain between $\Gamma^{\pm}_{\mathfrak{s}_1,K}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,K}$ (resp. the chain of rational curves arising from $\mathfrak{s}_2$ intersecting $\Gamma^{\pm}_{\mathfrak{s}_1,K}$) is determined entirely by $\lambda_{\mathfrak{s}_i}$ mod $\mathbb{Z}$, the parity of $|\mathfrak{s}_2|$, $d_{\mathfrak{s}_i}$, and when $\mathcal{R}$ is not principal $d_{\mathcal{R}}$.*

Proof. Assume that both $\mathfrak{s}_i$ are principal, Galois invariant clusters. From Section 7.1.2, we know that a linking chain between $\Gamma^{\pm}_{\mathfrak{s}_1,K}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,K}$ is completely determined by the length and number of linking chains between $\Gamma^{\pm}_{\mathfrak{s}_1,L}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,L}$, the order of the action of $\mathrm{Gal}(L/K)$ on any individual component of a linking chain between $\Gamma^{\pm}_{\mathfrak{s}_1,L}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,L}$, and the nature of the singularities at the intersection points of components after taking the quotient. By [DDMM18, Theorem 8.5], there is one linking chain, say $\mathcal{C}$, between $\Gamma^{\pm}_{\mathfrak{s}_1,L}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,L}$ if $\mathfrak{s}_2$

is odd and two linking chains, say $\mathcal{C}^+$ and $\mathcal{C}^-$, if $\mathfrak{s}_2$ is even. We will write $\mathcal{C} = \mathcal{C}^+ = \mathcal{C}^-$ if $\mathfrak{s}_2$ is odd. Furthermore, by [DDMM18, Theorem 8.5], the length of $\mathcal{C}^\pm$ is determined by $\delta(\mathfrak{s}_1, \mathfrak{s}_2)$, which is given in terms of $d_{\mathfrak{s}_1}$ and $d_{\mathfrak{s}_2}$ (and $d_{\mathcal{R}}$ in the case where $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal).

Let P be an intersection point of components $E_1, E_2 \in \{\Gamma_{\mathfrak{s}_1,L}, \Gamma_{\mathfrak{s}_2,L}, \mathcal{C}^\pm\}$, and σ_{E_i} the induced G_K action on E_i for a generator $\sigma \in \mathrm{Gal}(L/K)$. Suppose $\sigma_{E_1}^a$, and $\sigma_{E_2}^b$, generate the stabilisers of P in E_1 and E_2 respectively. Then $q(P)$ is a tame cyclic quotient singularity with parameters

$$n = \gcd(o(\sigma_{E_1}^a), o(\sigma_{E_2}^b)), \quad m_1 = o(\sigma_{E_1}^a)/n, \quad m_2 = o(\sigma_{E_2}^b)/n,$$

$$\text{and } r = \begin{cases} \frac{d_{E_1}^{-a} d_{E_2}^b}{n^2} & \mathfrak{s}_2 \text{ even}, \\ \frac{\lambda_{E_1}^{-a} \lambda_{E_2}^b}{n^2} & \mathfrak{s}_2 \text{ odd}, \end{cases}$$

where for $\tau \in \mathrm{Gal}(L/K)$, $o(\tau)$ is the order of τ. In other words, the tame cyclic quotient singularity is determined entirely by the automorphisms on the E_i and the parity of $\mathfrak{s}_2$. Therefore, since the automorphisms on E_i are determined entirely by the invariants in the statement of the theorem (by [DDMM18, Theorem 6.2]), we are done. The case where $\mathfrak{s}_2$ is a twin follows similarly. $\square$

For the following lemma we first need some notation. Recall that a child of $\mathfrak{s} \in \Sigma_{C/K}$ is *stable* if has the same stabiliser as $\mathfrak{s}$. Let $\widehat{\mathfrak{s}}^{\mathrm{f}}$ denote the set of stable children of $\mathfrak{s}$, and $\widehat{\mathfrak{s}}^{\mathrm{nf}}$ denote the set of unstable children of $\mathfrak{s}$. Note that here the superscripts 'f' and 'nf' stand for 'fixed' and 'not fixed' respectively.

Lemma 9.4.4. *Let C/K be a hyperelliptic curve with $\mathcal{R}$ principal, and let $\mathfrak{s} < \mathcal{R}$ be a Galois invariant proper child. We can construct a hyperelliptic curve, C^{new}, such that the cluster picture $\Sigma_{C^{\mathrm{new}}}$ of C^{new} consists of two proper clusters $\mathfrak{s}^{\mathrm{new}} < \mathcal{R}^{\mathrm{new}}$, where $|\mathfrak{s}| \equiv |\mathfrak{s}^{\mathrm{new}}| \mod 2, d_{\mathcal{R}} = d_{\mathcal{R}^{\mathrm{new}}}, d_{\mathfrak{s}} = d_{\mathfrak{s}^{\mathrm{new}}}$ and $\lambda_{\mathcal{R}} - \lambda_{\mathcal{R}^{\mathrm{new}}}, \lambda_{\mathfrak{s}} - \lambda_{\mathfrak{s}^{\mathrm{new}}} \in \mathbb{Z}$.*

Proof. Let C^{new} be the hyperelliptic curve over K defined by $C^{\mathrm{new}} : y^2 = c_f f_{\mathcal{R}} f_{\mathfrak{s}}$, where

$$f_{\mathcal{R}} = \begin{cases} \displaystyle\prod_{\mathfrak{s} \neq \mathfrak{o} \in \widetilde{\mathcal{R}}} (x - z_{\mathfrak{o}}) & |(\widetilde{\mathcal{R}} \cup \mathfrak{s}) \setminus \mathfrak{s}| \geq 2, \\ \pi_K^{|\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}}} \displaystyle\prod_{\mathfrak{s} \neq \mathfrak{s}' < \mathcal{R}} (x - z_{\mathfrak{s}'}) & \text{otherwise}, \end{cases}$$

$$f_{\mathfrak{s}} = \begin{cases} \prod_{\mathfrak{o} \in \widetilde{\mathcal{R}}} (x - z_{\mathfrak{o}}) & |\widetilde{\mathfrak{s}}| \geq 2, \\ \prod_{\mathfrak{o} \in \widetilde{\mathfrak{s}}^{\mathrm{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}}^{\mathrm{nf}}} (x - z_{\mathfrak{s}'}) & |\widetilde{\mathfrak{s}}| \leq 1 \text{ and } |\widehat{\mathfrak{s}}^{\mathrm{nf}}| \text{ even}, \\ \prod_{\mathfrak{o} \in \widetilde{\mathfrak{s}}^{\mathrm{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}}^{\mathrm{nf}}} (x - z_{\mathfrak{s}'})(x + z_{\mathfrak{s}'}) & |\widetilde{\mathfrak{s}}| \leq 1 \text{ and } |\widehat{\mathfrak{s}}^{\mathrm{nf}}| \text{ odd}. \end{cases}$$

It is clear that $\Sigma_{C_{\mathrm{new}}/K}$ consists of proper two clusters which we will call $\mathcal{R}^{\mathrm{new}}$ and $\mathfrak{s}^{\mathrm{new}}$, where $\mathcal{R}_{\mathrm{new}}$ consists of the roots of $f_{\mathcal{R}} \cdot f_{\mathfrak{s}}$, and $\mathfrak{s}^{\mathrm{new}}$ consists of the roots of $f_{\mathfrak{s}}$. It follows that $\mathfrak{s}^{\mathrm{new}} < \mathcal{R}^{\mathrm{new}}$. It remains to check how the cluster invariants of $\mathcal{R}^{\mathrm{new}}$ and $\mathfrak{s}^{\mathrm{new}}$ compare to those of $\mathcal{R}$ and $\mathfrak{s}$. Since any root in a cluster can be taken as its center, it is immediate that $d_{\mathcal{R}} = d_{\mathcal{R}^{\mathrm{new}}}$ and $d_{\mathfrak{s}} = d_{\mathfrak{s}^{\mathrm{new}}}$. By comparing $\deg(f_{\mathfrak{s}})$ to $|\mathfrak{s}|$ we see that $|\mathfrak{s}| \equiv |\mathfrak{s}^{\mathrm{new}}| \mod 2$.

It remains to check that $\lambda_{\mathcal{R}} - \lambda_{\mathcal{R}^{\mathrm{new}}}, \lambda_{\mathfrak{s}} - \lambda_{\mathfrak{s}^{\mathrm{new}}} \in \mathbb{Z}$. Let us begin with the first. By construction, $\mathfrak{s}^{\mathrm{new}}$ is odd if and only if $\mathfrak{s}$ is. Therefore, if $|(\widetilde{\mathcal{R}} \cup \mathfrak{s}) \setminus \mathfrak{s}| \geq 2$ it follows that $\lambda_{\mathcal{R}^{\mathrm{new}}} = \lambda_{\mathcal{R}}$. Else,

$$2(\lambda_{\mathcal{R}^{\mathrm{new}}} - \lambda_{\mathcal{R}}) = v_K(c_f) + |\widehat{\mathcal{R}}| d_{\mathcal{R}} + |\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}} - v_K(c_f) - |\widetilde{\mathcal{R}}| d_{\mathcal{R}} = 2|\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}}.$$

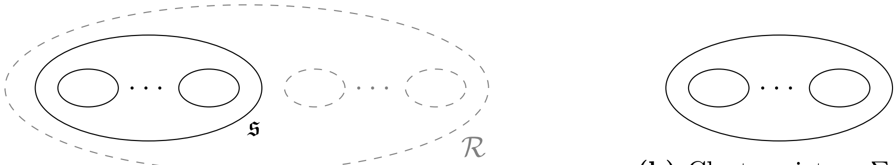
If $d_{\mathcal{R}} \in \mathbb{Z}$, then clearly $\lambda_{\mathcal{R}^{\mathrm{new}}} - \lambda_{\mathcal{R}} \in \mathbb{Z}$. Otherwise, $d_{\mathcal{R}} \notin \mathbb{Z}$. By Lemma 2.1.16, the children of $\mathcal{R}$ must lie in orbits of size $b_{\mathcal{R}} > 1$. Therefore, all even children are in orbits of size $b_{\mathcal{R}}$, since $\mathfrak{s} < \mathcal{R}$ is fixed so all other children have orbit sizes $b_{\mathcal{R}}$. Hence, $|\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}} \in \mathbb{Z}$, and so $\lambda_{\mathcal{R}^{\mathrm{new}}} - \lambda_{\mathcal{R}} \in \mathbb{Z}$. It can be checked similarly that $\lambda_{\mathfrak{s}^{\mathrm{new}}} - \lambda_{\mathfrak{s}} \in \mathbb{Z}$. □

By the above lemmas and Theorem 8.2.3, we have proved the statements in Theorem 9.2.3 about the linking chain(s) between $\Gamma^{\pm}_{\mathfrak{s},K}$ and $\Gamma^{\pm}_{\mathcal{R},K}$ where $\mathfrak{s} < \mathcal{R}$ is a Galois invariant proper child.

We now turn our focus to the components of $\mathscr{X}_k$ which arise from $\mathfrak{s}$ and its subclusters. In order to do this, we construct another new hyperelliptic curve, which we shall call C', given by

$$C' : y^2 = c'_f \prod_{r \in \mathfrak{s}} (x - r), \text{ where } c'_f = c_f \prod_{r \notin \mathfrak{s}} (z_{\mathfrak{s}} - r). \tag{9.1}$$

Note that C' is also semistable over L, and let $\mathscr{Y}'$ be the semistable model of C' over L. Comparing the cluster pictures of C' and C, we see that the cluster picture $\Sigma_{C'}$ appears within the cluster picture Σ_C of C. This is illustrated in Figure 9.1. In particular, $\mathfrak{s}$ and all of its subclusters in Σ_C are drawn in solid black in Figure 9.1a. These are exactly the clusters that make up $\Sigma_{C'}$, also shown in solid black.

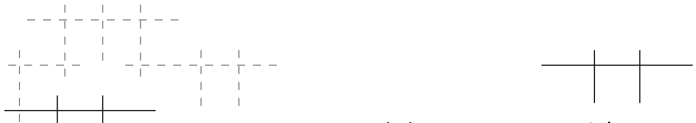


(a) Cluster picture Σ_C.

(b) Cluster picture $\Sigma_{C'}$.

Figure 9.1: Comparison of the cluster pictures of C and C'

The leading coefficient of C' has been chosen so that the corresponding clusters in Σ_C and $\Sigma_{C'}$ have the same size modulo 2, and the same cluster invariants modulo $\mathbb{Z}$ (as in the theorem statement above). Therefore, there is a closed immersion $\mathscr{Y}'_k \to \mathscr{Y}_k$ which commutes with the action of G_K. The existence of this immersion is illustrated in Figure 9.2. It is possible to see this explicitly by calculating the equations of the components of $\mathscr{Y}'$ and using the Galois action on these components given in [DDMM18, Theorem 8.5]. Therefore, this immersion also commutes with the quotient by $\mathrm{Gal}(L/K)$.



(a) Special fibre $\mathscr{Y}_k$, of the minimal SNC model of C/L.

(b) Special fibre $\mathscr{Y}'_k$, of the minimal SNC model of C'/L.

Figure 9.2: Comparison of the special fibres of the minimal SNC models of C and C'

After taking this quotient by $\mathrm{Gal}(L/K)$, and performing any appropriate blow ups and blow downs, we obtain a closed immersion $\overline{\mathscr{X}'_k \setminus T_\infty} \to \mathscr{X}_k$, where $\mathscr{X}'$ is the minimal SNC model of C'/K and T_∞ is the set of infinity tails of $\mathscr{X}'_k$. We remove the infinity tails since in the small distance case (see Section 8.2.3) the whole tails do not appear in $\mathscr{X}_k$. By our inductive hypothesis (since the number of proper clusters in $\Sigma_{C'}$ is strictly less than that in Σ_C), we can calculate $\mathscr{X}'_k$. This gives us a full description of the components of $\mathscr{X}_k$ which arise from the subclusters of $\mathfrak{s}$.

Finally let us remove the assumption that $\mathfrak{s}$ is G_K invariant. Let $X < \mathcal{R}$ be a non-trivial orbit of children. Extend K by degree $|X|$ to the field K_X, the minimal extension such that each cluster in X is fixed by $\mathrm{Gal}(\overline{K}/K_X)$. By our inductive hypothesis (since C/K_X needs an extension of degree strictly less than C/K does in order to have semistable reduction), we can calculate the minimal SNC model of C over K_X, which we denote $\mathscr{X}_X$. Since each cluster of X is fixed by $\mathrm{Gal}(L/K_X)$, there is a divisor $D_\mathfrak{s}$ corresponding to every cluster

$\mathfrak{s} \in X$ and all of the subclusters of $\mathfrak{s}$. Let $D_X = \bigcup_{\mathfrak{s}\in X} D_{\mathfrak{s}}$ be the union of these divisors. Since $\mathrm{Gal}(K_X/K)$ simply permutes these divisors, after taking the quotient by $\mathrm{Gal}(K_X/K)$, the image of D_X consists of precisely the same components as $D_{\mathfrak{s}}$ for some $\mathfrak{s} \in X$, but with all the multiplicities multiplied by $|X|$. See Figure 9.3 for an illustration. This concludes the proof when $\mathcal{R}$ is principal.

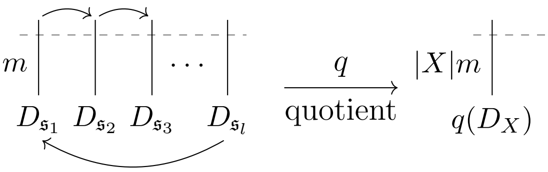


Figure 9.3: Divisors $D_{\mathfrak{s}_i}$, where $X = \{\mathfrak{s}_1 \ldots, \mathfrak{s}_l\}$, are permuted by $\mathrm{Gal}(K_X/K)$. After taking the quotient the image of $D_X = \bigcup_{i=1}^{l} D_{\mathfrak{s}_i}$ consists of the components of $D_{\mathfrak{s}_i}$ but where a component of multiplicity m in $D_{\mathfrak{s}_i}$ now has multiplicity $|X|m$.

9.4.2 Not Principal Top Cluster

Now suppose that $\mathcal{R}$ is not principal. If $\mathcal{R}$ is a cotwin, then the contribution to the special fibre of the minimal SNC model from $\mathcal{R}$ can be deduced using Remark 8.2.5 and Lemmas 9.4.3 and 9.4.4. The contribution of $\mathfrak{s} < \mathcal{R}$, the child of size $2g$, can be calculated by induction using a curve C' as in (9.1) above.

If $\mathcal{R}$ is not principal and not a cotwin then $\mathcal{R}$ is even and the union of two children. In this case, we will write $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$. Here the s_i are either fixed or swapped by G_K. We will deal with the case when the s_i are swapped at the end of this section, so for now suppose that both s_i are fixed by G_K. Let us also suppose for now that both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are proper clusters. We will deal with the case when one of $\mathfrak{s}_i$ has size 1 shortly. The first of these lemmas shows that there is a Möbius transform taking a certain class of curves with $\mathcal{R}$ not principal to the curves we studied in Section 8.2.

Lemma 9.4.5. *Let C/K be a hyperelliptic curve with cluster picture $\Sigma_{C/K}$, and set of roots $\mathcal{R}$.*

(i) Let $\mathfrak{s} \in \Sigma_{C/K}$ be a cluster with centre $z_{\mathfrak{s}}$. Write every root $r \in \mathfrak{s}$ as $r = z_{\mathfrak{s}} + r_h$, where $v_K(r_h) \geq d_{\mathfrak{s}}$. Then there exists at most one $r \in \mathfrak{s}$ such that $v_K(r_h) > d_{\mathfrak{s}}$.

(ii) If $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $d_{\mathcal{R}} \geq 0$, where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both fixed by $Gal(L/K)$, have no proper children, and $z_{\mathfrak{s}_1} = 0$. Then the Möbius transform ψ :

$r \mapsto \frac{1}{r}$ takes C to a new curve C_M which has cluster picture $\Sigma_M = \{\mathcal{R}_M = \mathfrak{s}_{1,M}, \mathfrak{s}_{2,M}\}$, with $\mathfrak{s}_{1,M} = \{\frac{1}{r} : 0 \neq r \in \mathfrak{s}_1\}$, $\mathfrak{s}_{2,M} = \{\frac{1}{r} : r \in \mathfrak{s}_2\}$, $d_{\mathfrak{s}_{1,M}} = -d_{\mathfrak{s}_1}$ and $d_{\mathfrak{s}_{2,M}} = d_{\mathfrak{s}_2} - 2d_{\mathcal{R}}$.

Proof. (i) Suppose there are two roots r and r' such that $v_K(r_h), v_K(r'_h) > d_{\mathfrak{s}}$. Then $d_{\mathfrak{s}} = v_K(r - r') = v_K(r_h - r'_h) \geq \min(v_K(r_h), v_K(r'_h)) > d_{\mathfrak{s}}$.

(ii) Since $z_{\mathfrak{s}_1} = 0$, we have that $v_K(r) = d_{\mathfrak{s}_1}$ for any $0 \neq r \in \mathfrak{s}_1$. Note also that, $v_K(z_{\mathfrak{s}_2}) = d_{\mathcal{R}}$, hence $v_K(r) = d_{\mathcal{R}}$ for any $r \in \mathfrak{s}_2$. The statement then follows from the fact that $v_K\left(\frac{1}{x} - \frac{1}{y}\right) = v_K(x - y) - v_K(x) - v_K(y)$. □

Remark 9.4.6. Note that $\delta_{\mathfrak{s}_{1,M}} = \delta_{\mathfrak{s}_1} + \delta_{\mathfrak{s}_2}$, $\lambda_{\mathfrak{s}_{1,M}} = \lambda_{\mathfrak{s}_1} - (g(\mathfrak{s}) + 1)d_{\mathfrak{s}}$ and $\lambda_{\mathfrak{s}_2} - \lambda_{\mathfrak{s}_{2,M}} = (|s_1| - |s_2|)d_{\mathcal{R}} \in 2\mathbb{Z}$.

The next lemma is analogous to Lemma 9.4.4, it gives us the existence of some new curve, which we will again call C^{new}, to which we can apply Lemma 9.4.5. This will allow us to calculate the linking chain(s) between $\Gamma^{\pm}_{\mathfrak{s}_1}$ and $\Gamma^{\pm}_{\mathfrak{s}_2}$, by using Lemma 9.4.3.

Lemma 9.4.7. *Let $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $\mathfrak{s}_i$ both fixed by Galois. Then there exists a hyperelliptic curve $C^{\mathrm{new}} : y^2 = f^{\mathrm{new}}(x)$ whose set of roots of f^{new} we denote by $\mathcal{R}^{\mathrm{new}}$, such that $\mathcal{R}^{\mathrm{new}} = \mathfrak{s}_1^{\mathrm{new}} \sqcup \mathfrak{s}_2^{\mathrm{new}}$, where $\mathfrak{s}_i^{\mathrm{new}}$ has no proper children, $|\mathfrak{s}_i| - |\mathfrak{s}_i^{\mathrm{new}}| \in 2\mathbb{Z}$, $d_{\mathfrak{s}_i} = d_{\mathfrak{s}_i^{\mathrm{new}}}$ and $\lambda_{\mathfrak{s}_i} - \lambda_{\mathfrak{s}_i^{\mathrm{new}}} \in \mathbb{Z}$ for $i = 1, 2$.*

Proof. For $i = 1, 2$ define

$$f_{\mathfrak{s}_i} = \begin{cases} \displaystyle\prod_{\mathfrak{o} \in \widetilde{\mathfrak{s}_i}} (x - z_{\mathfrak{o}}) & g(\Gamma_{\mathfrak{s}_i, L}) > 0, \\ \displaystyle\prod_{\mathfrak{o} \in \widehat{\mathfrak{s}_i}^{\mathrm{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}_i}^{\mathrm{nf}}} (x - z_{\mathfrak{s}'}) & g(\Gamma_{\mathfrak{s}_i, L}) = 0 \text{ and } |\widehat{\mathfrak{s}_i}^{\mathrm{nf}}| \text{ even}, \\ \displaystyle\prod_{\mathfrak{o} \in \widehat{\mathfrak{s}_i}^{\mathrm{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}_i}^{\mathrm{nf}}} (x - z_{\mathfrak{s}'})(x + z_{\mathfrak{s}'}) & g(\Gamma_{\mathfrak{s}_i, L}) = 0 \text{ and } |\widehat{\mathfrak{s}_i}^{\mathrm{nf}}| \text{ odd}. \end{cases}$$

Let $f^{\mathrm{new}} = c_f f_{\mathfrak{s}_1} f_{\mathfrak{s}_2}$, so $C^{\mathrm{new}} : y^2 = c_f f_{\mathfrak{s}_1} f_{\mathfrak{s}_2}$. Proving this satisfies the conditions in the statement of this lemma is similar to the proof of Lemma 9.4.4. □

So, if $\mathcal{R}$ is not principal and is a union of two proper clusters s_i which are fixed by G_K then, by Lemma 9.4.7, Lemma 9.4.3, and Lemma 9.4.5, we know now the linking chain(s) between $\Gamma^{\pm}_{\mathfrak{s}_1}$ and $\Gamma^{\pm}_{\mathfrak{s}_2}$. We can calculate the components associated to $\mathfrak{s}_i$ and its subclusters by induction, constructing a curve as in (9.1). Therefore this gives us the full special fibre of the minimal SNC model of C/K when $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal and the $\mathfrak{s}_i$ are fixed by Galois.

Suppose now that $\mathcal{R}$ is even with $\mathcal{R} = \mathfrak{s} \sqcup \{r\}$, that is $\mathfrak{s} < \mathcal{R}$ is a cotwin. Then, by Theorem 5.4.1 and 6.3.1, there exists an isomorphic curve C' whose cluster picture is isomorphic to $\mathfrak{s}$ and whose leading coefficient is the same as C. The result then follows from the case when the top cluster is a cotwin.

It remains to consider the case when $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal and the $\mathfrak{s}_i$ are swapped by Galois. This is solved by extending the field K to K_X, an extension of degree two. Here, C/K_X has a non principal top cluster $\mathcal{R}' = \mathfrak{s}_1' \sqcup \mathfrak{s}_2'$, where the s_i' are both proper clusters, and are fixed by $\mathrm{Gal}(\overline{K}/K_X)$. So we can apply the above lemmas to find the special fibre of the minimal SNC model of C/K_X. Taking the quotient by $\mathrm{Gal}(K_X/K)$, which we know how to do by Section 7.1.2, gives the special fibre of the minimal SNC model of C/K. This completes the cases when $\mathcal{R}$ is not principal.

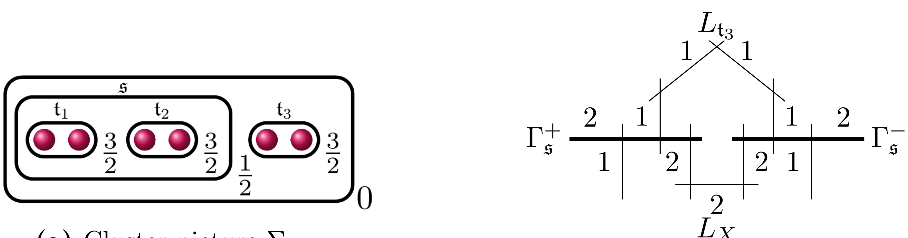
Proof of Theorem 9.2.1. Combining the results proved in the rest of this section proves this. □

Recall that in Section 1 we assumed that $\mathcal{R}$ was principal, and gave some examples. We conclude with a couple of additional examples of when $\mathcal{R}$ is not principal. Let $K = \mathbb{Q}_p^{\mathrm{ur}}$.

Example 9.4.8. Consider the hyperelliptic curve

$$C : y^2 = \left((x^2 - p)^2 + p^4\right)\left((x-1)^2 - p^3\right)$$

over K. Note that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are swapped by G_K and denote their orbit by X. This is a hyperelliptic curve of Namikawa-Ueno type II_{2-4} as in [NU73, p. 183]. Note that $\mathfrak{s}$ is übereven and $\epsilon_{\mathfrak{s}} = 1$, hence $\mathfrak{s}$ gives rise to two components; X is an orbit of twins with $\epsilon_X = 1$, so gives rise to a linking chain, and $\mathcal{R}$ is a cotwin (Definition 2.1.8) so gives rise to a linking chain. Also $e_{\mathfrak{s}} = 2$ so $\Gamma_{\mathfrak{s}}^{\pm}$ are both intersected by tails.



(a) Cluster picture $\Sigma_{C/K}$.

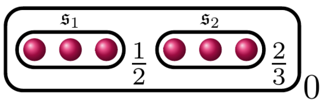
(b) Special fibre of the minimal SNC model of C/K.

Figure 9.4: $C : y^2 = ((x^2-p)^2+p^4)((x-1)^2-p^3)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

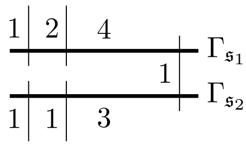
Example 9.4.9. Let $p \geq 5$ and C/K be the hyperelliptic curve given by

$$C : y^2 = x(x^2 - p)\left((x-1)^3 - p^2\right).$$

This curve has Namikawa-Ueno type IV − III−0 as in [NU73, p. 167]. Observe that $\mathcal{R}$ is not principal so gives rise to a linking chain between $\Gamma_{\mathfrak{s}_1}$ and $\Gamma_{\mathfrak{s}_2}$. Note that the special fibre here is the same as in Example 8.2.7, and there is in fact a Möbius transform between the two curves.



(a) Cluster picture $\Sigma_{C/K}$.



(b) Special fibre of the minimal SNC model of C/K.

Figure 9.5: $C : y^2 = x(x^2 - p)((x-1)^3 - p^2)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

Appendix A

Appendix

A.1 Naming Convention

For our classifications, we introduce a naming convention for open quotient BY trees. A naming convention for open BY trees is proposed in [DDMM17, §8.1]. We extend this here.

Notation A.1.1. Let T be an open quotient BY tree with open edge ε, and v_0 the unique vertex incident to the open edge. For the edges we use:

$\cdot$	blue edge
$:$	yellow edge
$\cdot_d, :_d$	edge of length d

For the vertices we use:

U	yellow vertex
$0, 1, 2, \ldots$	blue vertex of multiplicity 1 and genus $0, 1, 2, \ldots$
$0^M, 1^M, 2^M, \ldots$	blue vertex of multiplicity $M > 1$ and genus $0, 1, 2, \ldots$

As a topological space, T decomposes into the disjoint union

$$T = \{\varepsilon\} \cup \{v_0\} \cup t_1 \cup \cdots \cup t_n \cup T_1 \cup \cdots \cup T_N,$$

where the t_i are open trees consisting of an open yellow edge, say of length d_i, and a genus 0 blue vertex of any multiplicity say m_i, and the T_j are the remaining connected components of $T \setminus \{\varepsilon, v_0\}$. To define a naming convention or "Type" T we inductively define

$$\mathrm{Type}(T) = \mathrm{Type}(\varepsilon)\mathrm{Type}(v_0)_{[d_1]^{m_1},\ldots,[d_n]^{m_n}}\mathrm{Type}(T_1)\ldots\mathrm{Type}(T_N),$$

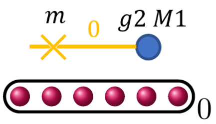
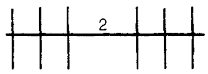
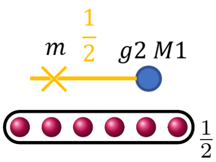
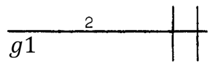
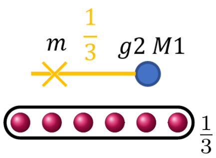
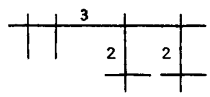
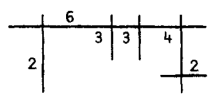
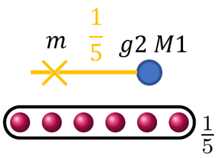
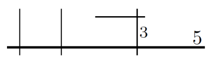
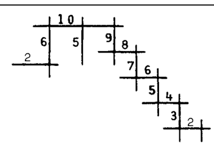
where $\mathrm{Type}(\varepsilon)$ and $\mathrm{Type}(v_0)$ is the notation for the edge ε and the vertex v_0 as above, and when $m_i = 1$, $[d_i]^{m_i}$ is simplified to d_i. To avoid any possible ambiguity, when $N > 0$ and T is not the full tree we are interested in, we bracket everything after the notation for the open edge and write

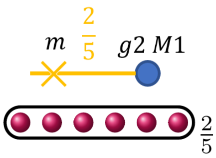
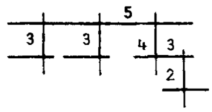
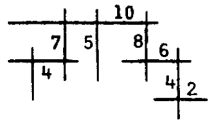
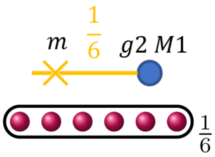
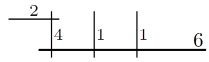
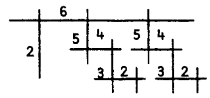
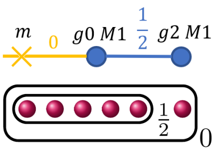
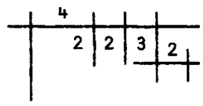
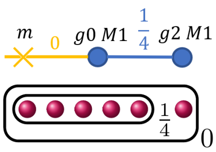
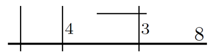
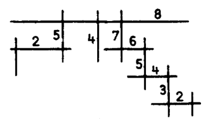
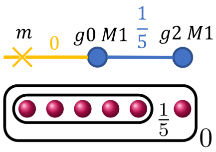
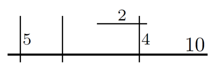
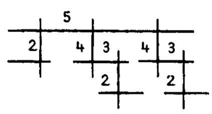
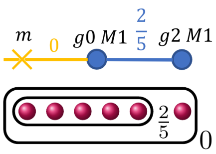
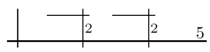
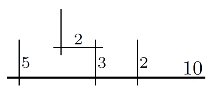
$$\mathrm{Type}(T) = \mathrm{Type}(\varepsilon)(\mathrm{Type}(v_0)_{[d_1]^{m_1},\ldots,[d_n]^{m_n}}\mathrm{Type}(T_1)\ldots\mathrm{Type}(T_N)).$$

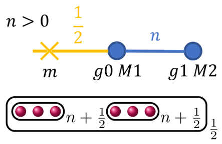
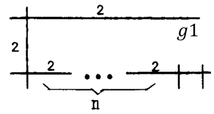
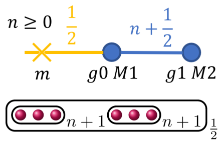
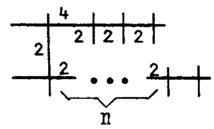
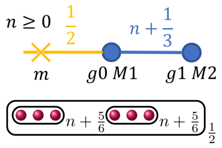
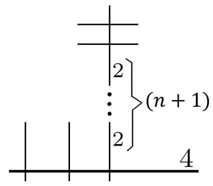
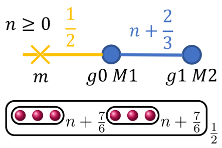
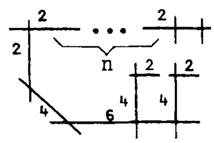
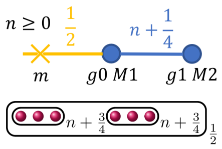
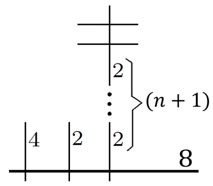
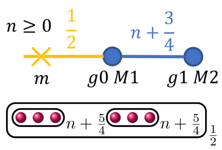
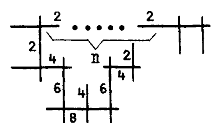
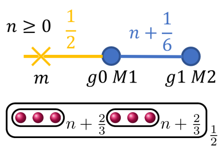
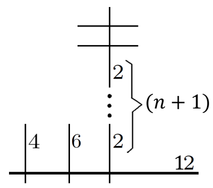
In the non-metric case, the subscripts $[d_i]^{m_i}$ are simply placeholders to record the number of genus 0 leaves, rather than lengths.

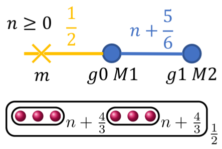
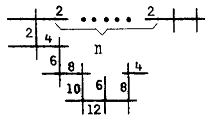
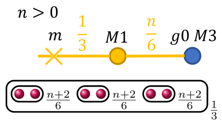
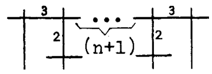
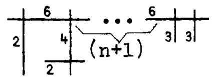
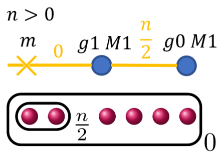
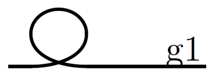
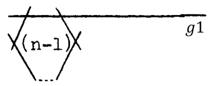
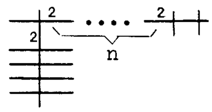
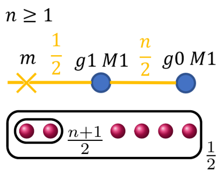
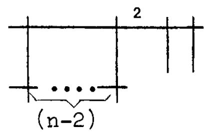
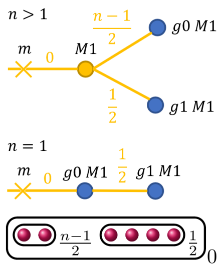
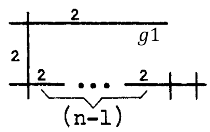
A.2 Genus 2 Classification

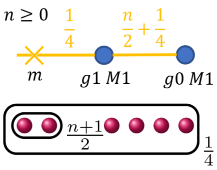
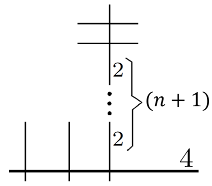
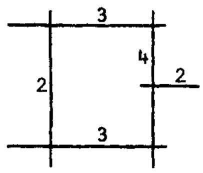
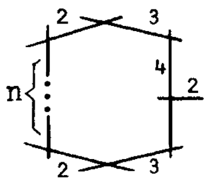
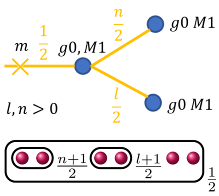
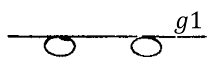
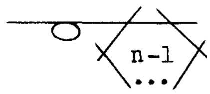
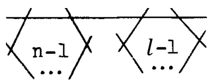
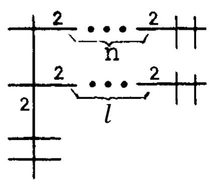
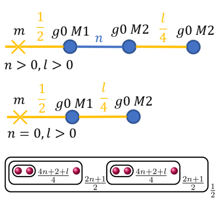
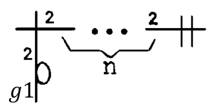
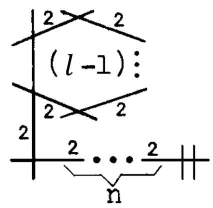
The following table presents the canonical representatives for the equivalence classes of metric open quotient BY trees of genus 2, alongside their associated cluster picture, Namikawa-Ueno type, special fibre of the minimal SNC model, and type name as introduced above.

T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_0 2$	I_{0-0-0} if $v(c_f) \in 2\mathbb{Z}$	
	I^*_{0-0-0} if $v(c_f) \notin 2\mathbb{Z}$	
 $:_{\frac{1}{2}} 2$	II	
 $:_{\frac{1}{3}} 2$	III if $v(c_f) \in 2\mathbb{Z}$	
	IV if $v(c_f) \notin 2\mathbb{Z}$	
 $:_{\frac{1}{5}} 2$	IX-2 if $v(c_f) \in 2\mathbb{Z}$	
	VII-4 if $v(c_f) \notin 2\mathbb{Z}$	

T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_{\frac{2}{3}} 2$	IX-4 if $v(c_f) \in 2\mathbb{Z}$	
	VII-2 if $v(c_f) \notin 2\mathbb{Z}$	
 $:_{\frac{1}{6}} 2$	V if $v(c_f) \in 2\mathbb{Z}$	
	V* if $v(c_f) \notin 2\mathbb{Z}$	
 $:_0 0 \cdot_{\frac{1}{2}} 2$	VI	
 $:_0 0 \cdot_{\frac{1}{4}} 2$	VII if $v(c_f) \in 2\mathbb{Z}$	
	VII* if $v(c_f) \notin 2\mathbb{Z}$	
 $:_0 0 \cdot_{\frac{1}{5}} 2$	VIII-1 if $v(c_f) \in 2\mathbb{Z}$	
	IX-3 if $v(c_f) \notin 2\mathbb{Z}$	
 $:_0 0 \cdot_{\frac{2}{5}} 2$	IX-1 if $v(c_f) \in 2\mathbb{Z}$	
	VIII-3 if $v(c_f) \notin 2\mathbb{Z}$	

T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_{\frac{1}{2}}\ 0 \cdot_{n}\ 1^2$	$2\mathrm{I}_0$-n	
 $:_{\frac{1}{2}}\ 0 \cdot_{n+\frac{1}{2}}\ 1^2$	$2\mathrm{I}_0^*$-n	
 $:_{\frac{1}{2}}\ 0 \cdot_{n+\frac{1}{3}}\ 1^2$	$2\mathrm{IV}$-n	
 $:_{\frac{1}{2}}\ 0 \cdot_{n+\frac{2}{3}}\ 1^2$	$2\mathrm{IV}^*$-n	
 $:_{\frac{1}{2}}\ 0 \cdot_{n+\frac{1}{4}}\ 1^2$	$2\mathrm{III}$-n	
 $:_{\frac{1}{2}}\ 0 \cdot_{n+\frac{3}{4}}\ 1^2$	$2\mathrm{III}^*$-n	
 $:_{\frac{1}{2}}\ 0 \cdot_{n+\frac{1}{6}}\ 1^2$	$2\mathrm{II}$-n	

T^*, $\Sigma(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_{\frac{1}{2}}\ 0 \cdot_{n+\frac{5}{6}} 1^2$	2II*-n	
 $:_{\frac{1}{3}}\ U_{[\frac{n}{d}]^3}$	III_n if $v(c_f) \in 2\mathbb{Z}$	
	III_n^* if $v(c_f) \notin 2\mathbb{Z}$	
 $:_0\ 1_{\frac{n}{2}}$	I_{1-0-0} if $n = 1$, $v(c_f) \in 2\mathbb{Z}$	
	I_{n-0-0} if $n > 1$, $v(c_f) \in 2\mathbb{Z}$	
	I_{n-0-0}^* if $v(c_f) \notin 2\mathbb{Z}$	
 $:_{\frac{1}{2}}\ 1_{\frac{n}{2}}$	II_{n-0} if $v(c_f) \in 2\mathbb{Z}$	
 If $n > 1$: $:_0\ U_{\frac{n-1}{2}}\ :_{\frac{1}{2}}\ 1$ If $n = 1$: $:_0\ 0_{\frac{1}{2}}$	II_{n-0}^* if $v(c_f) \notin 2\mathbb{Z}$	

T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_{\frac{1}{4}}\ 1_{\frac{n}{2}+\frac{1}{4}}$	III-II$_n^*$ if $n \geq 0$, $v(c_f) \in 2\mathbb{Z}$	
	III*-II$_0$ if $n = 0$, $v(c_f) \notin 2\mathbb{Z}$	
	III*-II$_n$ if $n > 0$, $v(c_f) \notin 2\mathbb{Z}$	
 $:_{\frac{1}{2}}\ 0_{\frac{n}{2},\frac{l}{2}}$	I_{1-1-0} if $l = n = 1$, $v(c_f) \in 2\mathbb{Z}$	
	I_{1-n-0} if $n > l = 1$, $v(c_f) \in 2\mathbb{Z}$	
	I_{l-n-0} if $l, n > 1$, $v(c_f) \in 2\mathbb{Z}$	
	I^*_{l-n-0} if $v(c_f) \notin 2\mathbb{Z}$	
 If $n > 0$: $:_{\frac{1}{2}}\ 0 \cdot_n 0_{[\frac{l}{4}]^2}$ If $n = 0$: $:_{\frac{1}{2}}\ 0_{[\frac{l}{4}]^2}$	$2I_1$-n if $l = 1$	
	$2I_l$-n if $l > 1$	

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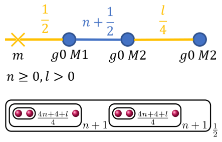
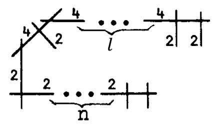
<tr><td>

$:_{\frac{1}{2}} 0 \cdot_{n+\frac{1}{2}} 0_{[\frac{l}{4}]^2}$</td><td>$2I_l^*$-$n$</td><td>
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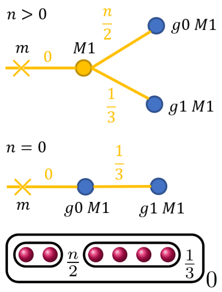
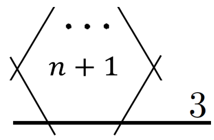
<tr><td rowspan="2">

If $n > 0$:
$:_0 U_{\frac{n}{2}} :_{\frac{1}{3}} 1$
If $n = 0$:
$:_0 0 :_{\frac{1}{3}} 1$</td><td>IV-II$_n$ if $v(c_f) \in 2\mathbb{Z}$</td><td>
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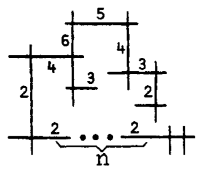
<tr><td>II*-II$_n^*$ if $v(c_f) \notin 2\mathbb{Z}$</td><td>
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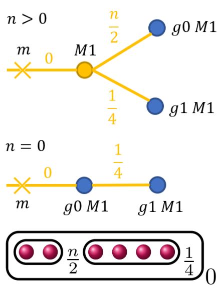
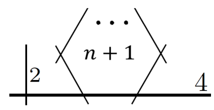
<tr><td rowspan="2">

If $n > 0$:
$:_0 U_{\frac{n}{2}} :_{\frac{1}{4}} 1$
If $n = 0$:
$:_0 0 :_{\frac{1}{4}} 1$</td><td>III-II$_n$ if $v(c_f) \in 2\mathbb{Z}$</td><td>
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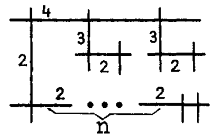
<tr><td>III*-II$_n^*$ if $v(c_f) \notin 2\mathbb{Z}$</td><td>
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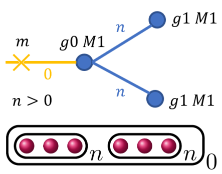
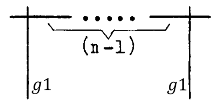
<tr><td rowspan="2">

$:_0 0 \cdot_n (1) \cdot_n (1)$</td><td>I$_0$-I$_0$-$n$ if $n \equiv v(c_f) \mod 2$</td><td>
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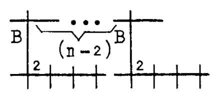
<tr><td>I$_0^*$-I$_0^*$-$(n-1)$ if $n \not\equiv v(c_f) \mod 2$</td><td>
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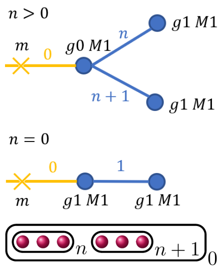
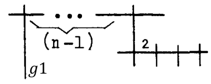
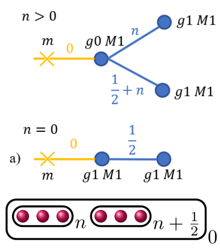
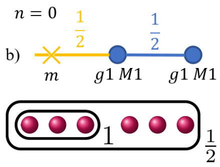
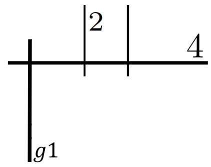
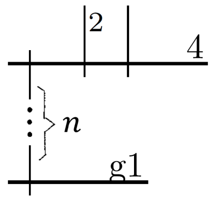
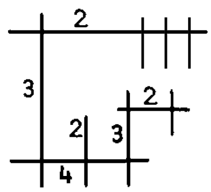
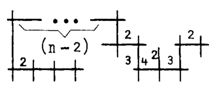
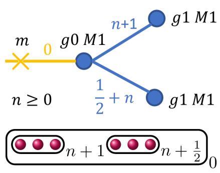
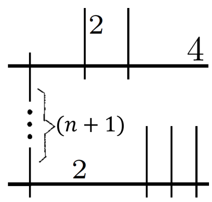
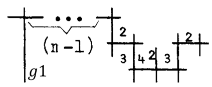
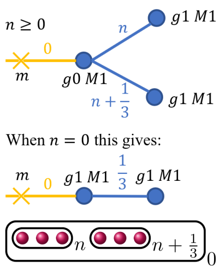
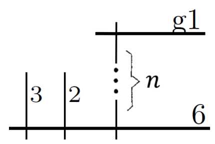
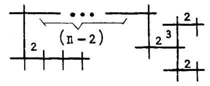
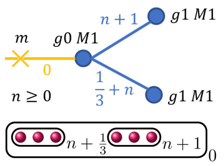
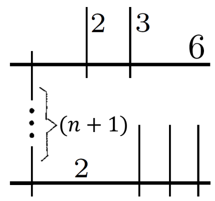
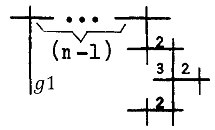
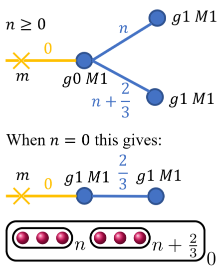
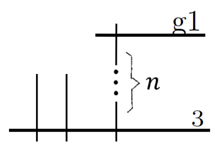
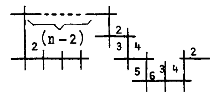
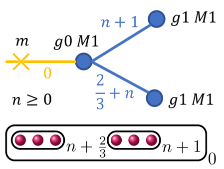
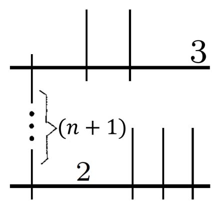
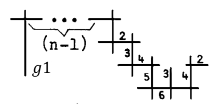
T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 If $n > 0$: $:_0\ 0 \cdot_n (1) \cdot_{n+1} (1)$ If $n = 0$: $:_0\ 1 \cdot_1 1$	I_0-I_0^*-n	
 If $n > 0$: $:_0\ 0 \cdot_n (1) \cdot_{n+\frac{1}{2}} (1)$ If $n = 0$: $:_0\ 1 \cdot_{\frac{1}{2}} 1$ When $n = 0$ there is a second canonical representative:  $:_{\frac{1}{2}}\ 1 \cdot_{\frac{1}{2}} 1$	I_0-III-0 if $n = 0$ and $v(c_f) \notin 2\mathbb{Z}$ for case a), or and $v(c_f) \in 2\mathbb{Z}$ for case b)	
	I_0-III-n if $n > 0$ and $v(c_f) \in 2\mathbb{Z}$	
	I_0^*-III*-α if $n = 0$ and $v(c_f) \in 2\mathbb{Z}$ for case a), or and $v(c_f) \notin 2\mathbb{Z}$ for case b)	
	I_0^*-III*-$(n-1)$ if $n > 0$, $v(c_f) \notin 2\mathbb{Z}$	
 $:_0\ 0 \cdot_{n+1} (1) \cdot_{n+\frac{1}{2}} (1)$	I_0-III*-n if $v(c_f) \in 2\mathbb{Z}$	
	I_0^*-III-n if $v(c_f) \notin 2\mathbb{Z}$	

T^*, $\Sigma(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
	I_0-II-n if $n \equiv v(c_f) \mod 2$	
If $n > 0$: $:_0\ 0 \cdot_n (1) \cdot_{n+\frac{1}{3}} (1)$ If $n = 0$: $:_0\ 1 \cdot_{\frac{1}{3}} 1$	I_0^*-IV*-$(n-1)$ if $n \not\equiv v(c_f) \mod 2$	
 $:_0\ 0 \cdot_{n+1} (1) \cdot_{n+\frac{1}{3}} (1)$	I_0^*-II-n if $n \equiv v(c_f) \mod 2$	
	I_0-IV*-n if $n \not\equiv v(c_f) \mod 2$	
	I_0-IV-n if $n \equiv v(c_f) \mod 2$	
If $n > 0$: $:_0\ 0 \cdot_n (1) \cdot_{n+\frac{2}{3}} (1)$ If $n = 0$: $:_0\ 1 \cdot_{\frac{2}{3}} 1$	I_0^*-II*-$(n-1)$ if $n \not\equiv v(c_f) \mod 2$	
 $:_0\ 0 \cdot_{n+1} (1) \cdot_{n+\frac{2}{3}} (1)$	I_0-II*-n if $n \equiv v(c_f) \mod 2$	
	I_0^*-IV-n if $n \not\equiv v(c_f) \mod 2$	

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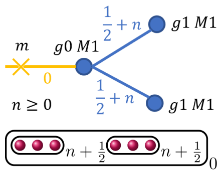
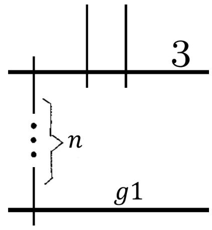
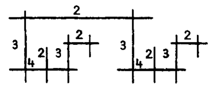
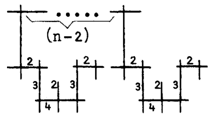
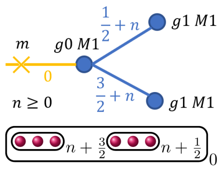
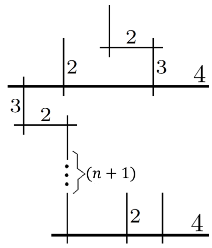
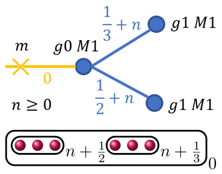
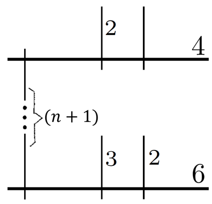
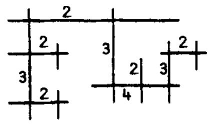
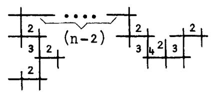
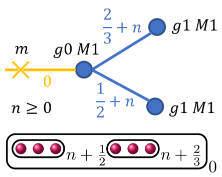
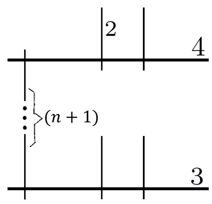
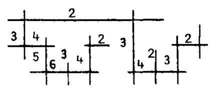
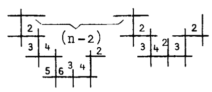
<tr><td rowspan="3">
$:_0 0 \cdot_{n+\frac{1}{2}} (1) \cdot_{n+\frac{1}{2}} (1)$</td><td>III-III-$n$ if $n \equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>III*-III*-α if $n = 0$ and $v(c_f) \notin 2\mathbb{Z}$</td><td></td></tr>
<tr><td>III*-III*-$(n-1)$ if $n > 0$, $n \not\equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>
$:_0 0 \cdot_{n+\frac{1}{2}} (1) \cdot_{n+\frac{3}{2}} (1)$</td><td>III-III*-$n$</td><td></td></tr>
<tr><td rowspan="3">
$:_0 0 \cdot_{n+\frac{1}{3}} (1) \cdot_{n+\frac{1}{2}} (1)$</td><td>II-III-$n$ if $n \equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>III*-IV*-α if $n = 0$ and $v(c_f) \notin 2\mathbb{Z}$</td><td></td></tr>
<tr><td>III*-IV*-$(n-1)$ if $n > 0$, $n \not\equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td rowspan="3">
$:_0 0 \cdot_{n+\frac{2}{3}} (1) \cdot_{n+\frac{1}{2}} (1)$</td><td>IV-III-$n$ if $n \equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>II*-III*-α if $n = 0$ and $v(c_f) \notin 2\mathbb{Z}$</td><td></td></tr>
<tr><td>II*-III*-$(n-1)$ if $n > 0$, $n \not\equiv v(c_f) \mod 2$</td><td></td></tr>
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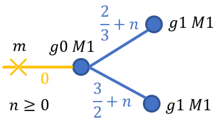
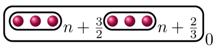
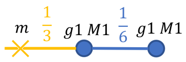
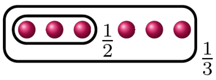
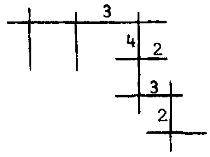
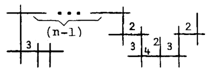
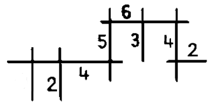
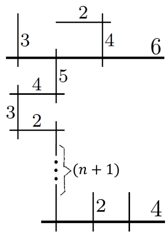
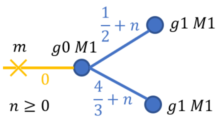
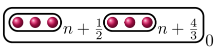
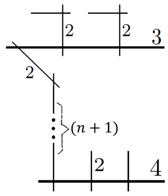
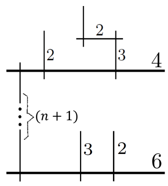
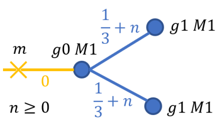
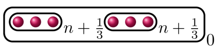
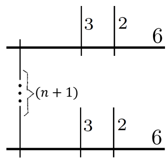
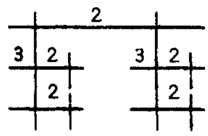
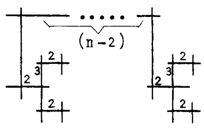
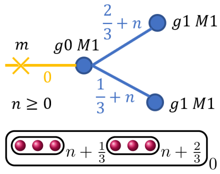
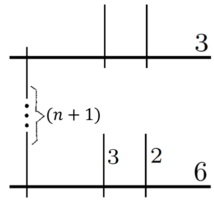
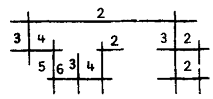
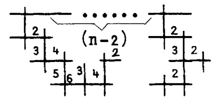
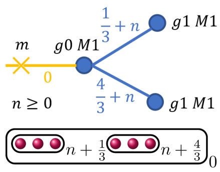
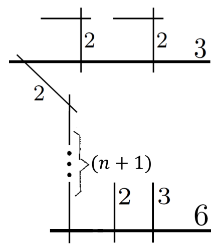
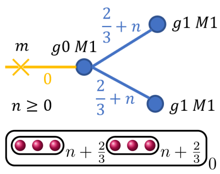
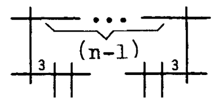
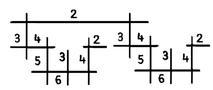
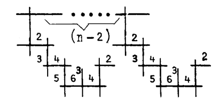
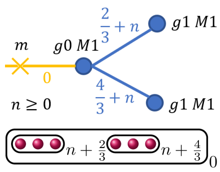
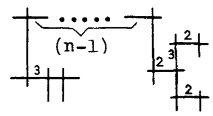
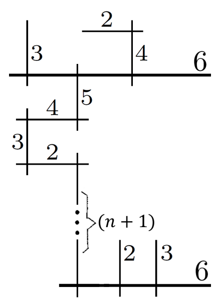
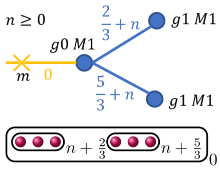
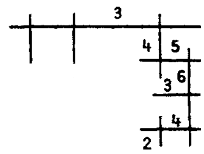
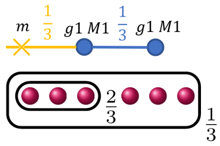
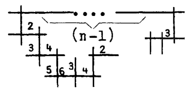
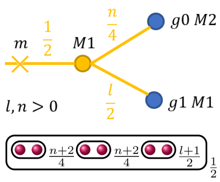
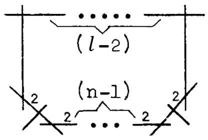
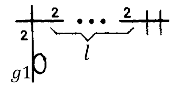
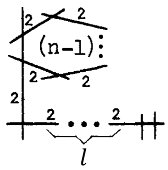
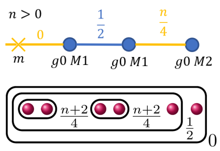
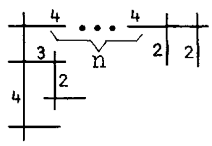
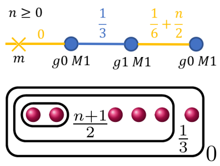
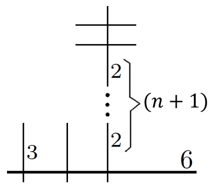
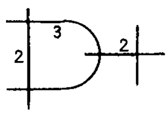
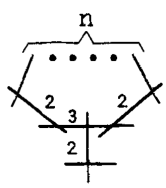
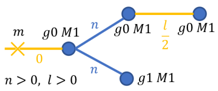
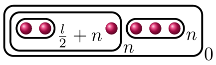
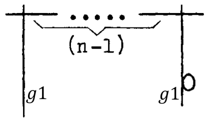
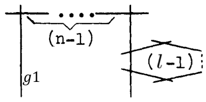
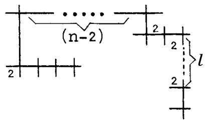
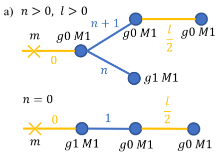
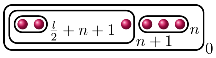
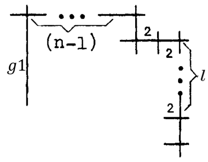
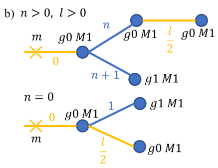
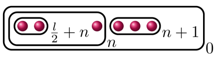
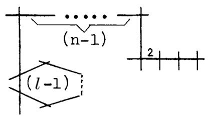
T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_0 0 \cdot_{n+\frac{2}{3}} (1) \cdot_{n+\frac{3}{2}} (1)$ Along with the following small distance case (n = −1): $:_{\frac{1}{3}} 1 \cdot_{\frac{1}{6}} 1$	IV-III*-α if $n = -1$ and $v(c_f) \in 2\mathbb{Z}$	
	IV-III*-n if $n \geq 0$, $n \equiv v(c_f) \mod 2$	
	II*-III-α if $n = -1$ and $v(c_f) \notin 2\mathbb{Z}$	
	II*-III-$(n-1)$ if $n \geq 0$, $n \not\equiv v(c_f) \mod 2$	
 $:_0 0 \cdot_{n+\frac{1}{2}} (1) \cdot_{n+\frac{4}{3}} (1)$	IV*-III-n if $n \equiv v(c_f) \mod 2$	
	II-III*-n if $n \not\equiv v(c_f) \mod 2$	
 $:_0 0 \cdot_{n+\frac{1}{3}} (1) \cdot_{n+\frac{1}{3}} (1)$	II-II-n if $n \equiv v(c_f) \mod 2$	
	IV*-IV*-α if $n = 0$ and $v(c_f) \notin 2\mathbb{Z}$	
	IV*-IV*-$(n-1)$ if $n > 0$, $n \not\equiv v(c_f) \mod 2$	

T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_0 0 \cdot_{n+\frac{2}{3}} (1) \cdot_{n+\frac{1}{3}} (1)$	II-IV-n if $n \equiv v(c_f) \mod 2$	
	II*-IV*-α if $n = 0$ and $v(c_f) \notin 2\mathbb{Z}$	
	II*-IV*-$(n-1)$ if $n > 0$, $n \not\equiv v(c_f) \mod 2$	
 $:_0 0 \cdot_{n+\frac{1}{3}} (1) \cdot_{n+\frac{4}{3}} (1)$	II-IV*-n	
 $:_0 0 \cdot_{n+\frac{2}{3}} (1) \cdot_{n+\frac{2}{3}} (1)$	IV-IV-n if $n \equiv v(c_f) \mod 2$	
	II*-II*-α if $n = 0$ and $v(c_f) \notin 2\mathbb{Z}$	
	II*-II*-$(n-1)$ if $n > 0$, $n \not\equiv v(c_f) \mod 2$	
 $:_0 0 \cdot_{n+\frac{2}{3}} (1) \cdot_{n+\frac{4}{3}} (1)$	IV-IV*-n if $n \equiv v(c_f) \mod 2$	
	II-II*-n if $n \not\equiv v(c_f) \mod 2$	

T^*, $\Sigma(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 $:_0\ 0 \cdot_{n+\frac{2}{3}} (1) \cdot_{n+\frac{5}{3}} (1)$	II*-IV-α if $n = -1$	
Along with the following small distance case (n = −1):  $:_{\frac{1}{3}}\ 1 \cdot_{\frac{1}{3}} 1$	II*-IV-n if $n \geq 0$	
 $:_{\frac{1}{2}}\ U_{\frac{l}{2},[\frac{n}{4}]^2}$	II_{l-n} if $v(c_f) \in 2\mathbb{Z}$	
	II^*_{l-1} if $n = 1$, $v(c_f) \notin 2\mathbb{Z}$	
	II^*_{l-n} if $n > 1$, $v(c_f) \notin 2\mathbb{Z}$	
 $:_0\ 0 \cdot_{\frac{1}{2}} 0_{[\frac{n}{4}]^2}$	III_n	
 $:_0\ 0 \cdot_{\frac{1}{3}} 1_{\frac{n}{2}+\frac{1}{6}}$	II-II^*_n if $n \in 2\mathbb{Z}$	
	IV*-II_0 if $n \notin 2\mathbb{Z}$	
	IV*-II_n if $n \notin 2\mathbb{Z}$	

T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
  $:_0 0 \cdot_n (1) \cdot_n (1_{\frac{l}{2}})$	I_1-I_0-n if $l = 1$ and $n \equiv v(c_f) \mod 2$	
	I_l-I_0-n if $l > 1$ and $n \equiv v(c_f) \mod 2$	
	I_0^*-I_l^*-$(n-1)$ if $n \not\equiv v(c_f) \mod 2$	
  If $n > 0$: $:_0 0 \cdot_n (1) \cdot_{n+1} (0_{\frac{l}{2}})$ If $n = 0$: $:_0 1 \cdot_1 0_{\frac{l}{2}}$	I_0-I_l^*-n if $n \equiv v(c_f) \mod 2$ in a), or if $n \not\equiv v(c_f) \mod 2$ in b)	
  If $n > 0$: $:_0 0 \cdot_{n+1} (1) \cdot_n (0_{\frac{l}{2}})$ If $n = 0$: $:_0 0_{\frac{l}{2}} \cdot_1 1$	I_l-I_0^*-n if $n \not\equiv v(c_f) \mod 2$ in a), or if $n \equiv v(c_f) \mod 2$ in b)	

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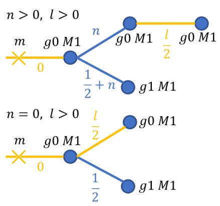
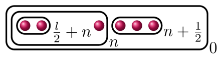
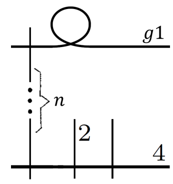
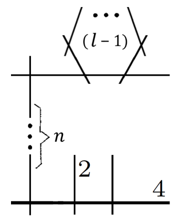
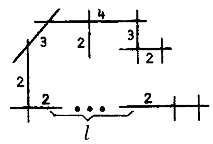
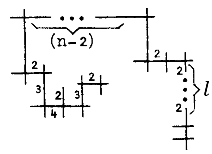
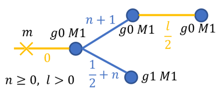
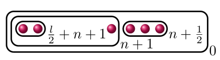
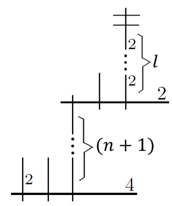
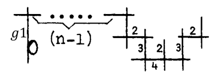
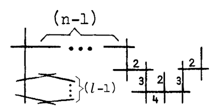
<tr><td rowspan="4">

If $n > 0$:
$:_0\ 0 \cdot_{n+\frac{1}{2}} (1) \cdot_n (0_{\frac{l}{2}})$
If $n = 0$:
$:_0\ 0_{\frac{l}{2}} \cdot_{\frac{1}{2}} 1$</td><td>III-$\mathrm{I}_l$-1 if $n = 1$, $1 \equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>III-I_l-n if $n > 1$, $n \equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>III*-I_l^*-α if $n = 1$, $0 \equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>III*-I_l^*-n if $n > 1$, $0 \not\equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td rowspan="3">

$:_0\ 0 \cdot_{n+\frac{1}{2}} (1) \cdot_{n+1} (0_{\frac{l}{2}})$</td><td>III-$\mathrm{I}_l^*$-$n$ if $n \equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>III*-I_1-n if $l = 1$, $n \not\equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>III*-I_l-n if $l > 1$, $n \not\equiv v(c_f) \mod 2$</td><td></td></tr>
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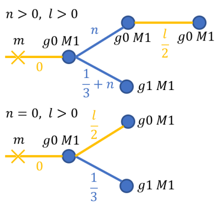
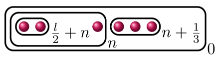
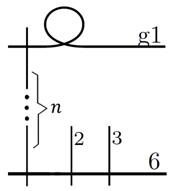
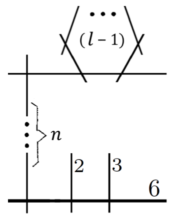
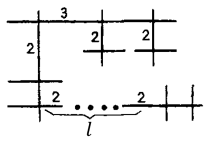
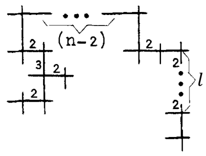
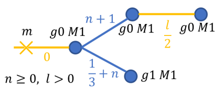
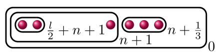
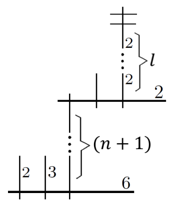
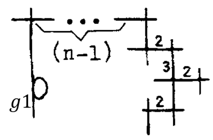
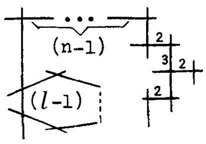
T^*, $\underline{\Sigma}(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
  If $n>0$: $:_0\ 0 \cdot_{n+\frac{1}{3}} (1) \cdot_n (0_{\frac{l}{2}})$ If $n=0$: $:_0\ 0_{\frac{l}{2}} \cdot_{\frac{1}{3}} 1$	II-I$_1$-n if $l=1$, $n \equiv v(c_f) \mod 2$	
	II-I$_l$-n if $l>1$, $n \equiv v(c_f) \mod 2$	
	IV*-I$_l^*$-α if $n=0$, $0 \not\equiv v(c_f) \mod 2$	
	IV*-I$_l^*$-$(n-1)$ if $n>0$, $0 \not\equiv v(c_f) \mod 2$	
  $:_0\ 0 \cdot_{n+\frac{1}{3}} (1) \cdot_{n+1} (0_{\frac{l}{2}})$	II-I$_l^*$-n if $n \equiv v(c_f) \mod 2$	
	IV*-I$_1$-n if $l=1$, $n \not\equiv v(c_f) \mod 2$	
	IV*-I$_l$-n if $l>1$, $n \not\equiv v(c_f) \mod 2$	

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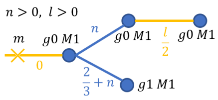
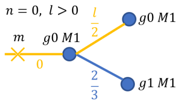
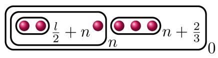
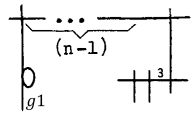
<tr><td rowspan="4">




If $n > 0$:
$:_0 0 \cdot_{n+\frac{2}{3}} (1) \cdot_n (0_{\frac{l}{2}})$
If $n = 0$:
$:_0 0_{\frac{l}{2}} \cdot_{\frac{2}{3}} 1$</td><td>IV-I$_1$-$n$ if $l = 1$, $n \equiv v(c_f) \mod 2$</td><td>
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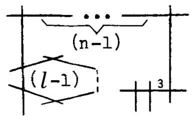
<tr><td>IV-I$_l$-n if $l > 1$, $n \equiv v(c_f) \mod 2$</td><td>
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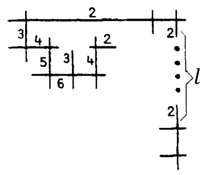
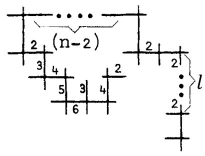
<tr><td>II*-I$_l^*$-α if $n = 0$, $0 \not\equiv v(c_f) \mod 2$</td><td></td></tr>
<tr><td>II*-I$_l^*$-$(n-1)$ if $n > 0$, $0 \not\equiv v(c_f) \mod 2$</td><td>
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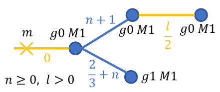
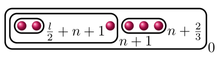
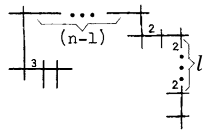
<tr><td rowspan="3">


$:_0 0 \cdot_{n+\frac{2}{3}} (1) \cdot_{n+1} (0_{\frac{l}{2}})$</td><td>IV-I$_l^*$-$n$ if $n \equiv v(c_f) \mod 2$</td><td>
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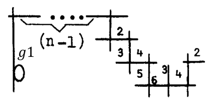
<tr><td>II*-I$_1$-n if $l = 1$, $n \not\equiv v(c_f) \mod 2$</td><td>
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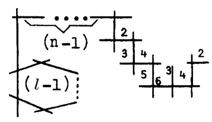
<tr><td>II*-I$_l$-n if $l > 1$, $n \not\equiv v(c_f) \mod 2$</td><td>
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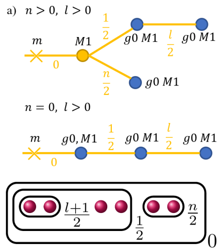
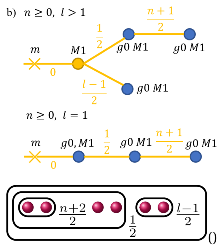
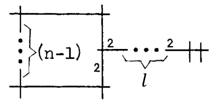
<tr><td>

If $n > 0$:
$:_0 U_{\frac{n}{2}} :_{\frac{1}{2}} 0_{\frac{l}{2}}$
If $n = 0$:
$:_0 0 :_{\frac{1}{2}} 0_{\frac{l}{2}}$


If $l > 0$:
$:_0 U_{\frac{l-}{2}} :_{\frac{1}{2}} 0_{\frac{n+1}{2}}$
If $l = 0$:
$:_0 0 :_{\frac{1}{2}} 0_{\frac{n+1}{2}}$</td>
<td>$\mathrm{II}_{n+1,l}$</td>
<td>
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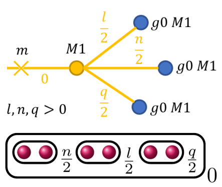
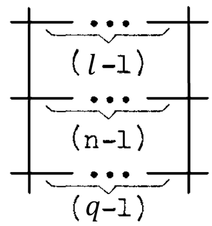
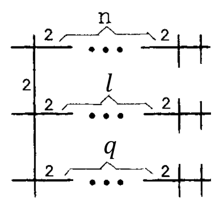
<tr><td rowspan="2">

$:_0 U_{\frac{l}{2},\frac{n}{2},\frac{q}{2}}$</td>
<td>I_l-I_n-I_q if $0 \equiv v(c_f) \mod 2$</td>
<td>
</td></tr>
<tr><td>I_l^*-I_n^*-I_q^* if $1 \equiv v(c_f) \mod 2$</td>
<td>
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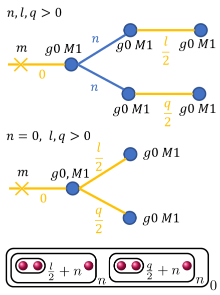
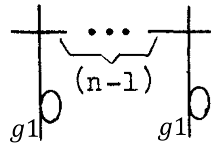
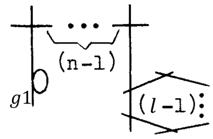
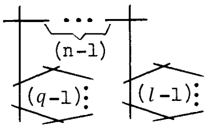
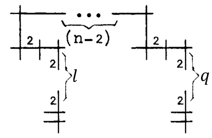
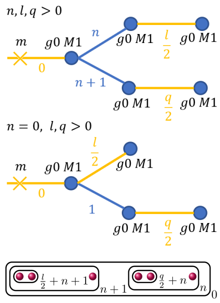
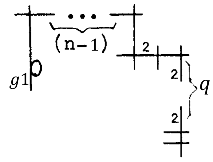
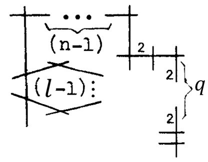
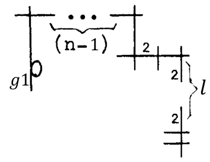
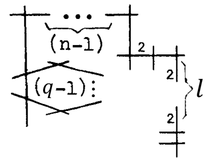
T^*, $\Sigma(T^*)$	Namikawa-Ueno type	$\mathscr{X}_k$
 If $n>0$: $:_0 0 \cdot_n (0_{\frac{l}{2}}) \cdot_n (0_{\frac{q}{2}})$ If $n=0$: $:_0 0 :_{\frac{l}{2},\frac{q}{2}}$	I_1-I_1-n if $l=q=1$, $n \equiv v(c_f) \mod 2$	
	I_1-I_l-n if $l>q=1$, $n \equiv v(c_f) \mod 2$	
	I_q-I_l-n if $l,q>1$, $n \equiv v(c_f) \mod 2$	
	I_l^*-I_q^*-n if $n \not\equiv v(c_f) \mod 2$	
 If $n>0$: $:_0 0 \cdot_n (0_{\frac{l}{2}}) \cdot_{n+1} (0_{\frac{q}{2}})$ If $n=0$: $:_0 0 :_{\frac{l}{2},\frac{q}{2}}$	I_1-I_q^*-n if $l=1$, $n \equiv v(c_f) \mod 2$	
	I_l-I_q^*-n if $l>1$, $n \equiv v(c_f) \mod 2$	
	I_q-I_l^*-n if $q=1$, $n \equiv v(c_f) \mod 2$	
	I_q-I_l^*-n if $q>1$, $n \not\equiv v(c_f) \mod 2$	

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