BRILL

IMPLEMENTATION AND REPLICATION STUDIES IN MATHEMATICS EDUCATION 2 (2022) 21–44

IRME

brill.com/irme

Operationalising Vergnaud's Notion of Scheme in Task Design in Online Learning Environments to Support the Implementation of Formative Assessment

Morten Elkjær | ORCID: 0000-0001-6035-882X
Danish School of Education (DPU), Aarhus University, Tuborgvej 164, 2400 Copenhagen, Denmark
me@edu.au.dk

Jeremy Hodgen | ORCID: 0000-0002-9196-4088
Department of Curriculum, Pedagogy and Assessment, IOE, UCL's Faculty of Education and Society, University College London, 20 Bedford Way, London WC1H 0AL, UK
Corresponding author, e-mail: jeremy.hodgen@ucl.ac.uk

Abstract

This paper presents an implementation process model for designing and implementing tasks that provide formative feedback in the online learning environment of mathematics classrooms. Specifically, the model operationalises components of Vergnaud's notion of scheme. The implementation process model features a task sequence guided by controlled variation and a 'dual scheme idea'. Using such a sequence of tasks, this work illustrates how Vergnaud's notion of scheme can be used to aid teachers in hypothesising about their learners' understanding of problems involving linear equations, ultimately providing improved feedback for teachers and improved opportunities for student learning in online environments. In Denmark, the online environment matematikfessor.dk is used by approximately 80% of Danish K-9 students.

The impact sheet to this article can be accessed at 10.6084/m9.figshare.19493846.

Keywords

implementation process model – diagnostic tasks – task design – online learning environment – linear equations

 | DOI:10.1163/26670127-BJA10002

1 Introduction

This paper presents the process of developing a framework for implementing formative feedback for teachers engaging in online learning environments. Online learning environments can play a significant role in the implementation of research-based knowledge in the classroom, as online learning environments are capable of providing feedback directly to end users, teachers and students (Dyssegaard et al., 2017). In addition, these environments can produce substantial assessment data with relatively little effort from teachers. However, little research has investigated how online learning environments can best provide feedback to enable teachers to provide effective feedback to their students. There is an extensive research base on how diagnostic tasks performed and why the tasks are believed to be sensible choices for exploring certain areas of difficulty in learning mathematics (Rhine et al., 2018). However, few papers have systematically addressed the considerations or design principles underlying the construction of diagnostic tasks to provide relevant feedback to teachers.

1.1 *The Promise of Online Learning Environments for Mathematics Education*

Online learning environments allow teachers, students themselves or 'intelligent systems' to tailor the content in the environment to the learning needs of students, and these environments can dynamically assign tasks based on students' previous responses (Steenbergen-Hu & Cooper, 2013). Moreover, these environments can 'mark' students' responses almost immediately, thus providing automatic feedback to both students (Cavalcanti et al., 2021) and teachers regarding students' individual and group performance.

However, the process of providing high-quality feedback is not straightforward. Typically, the user is allowed only certain input types: multiple-choice items or numbers. As a result, most of the tasks in online learning environments are closed. Inferring students' mathematical understanding from such tasks can be difficult because a correct answer may be the result of incorrect, or only partially correct, reasoning. This becomes a significant constraint in task design, particularly when working with algebraic expressions, as it is not possible to prompt students for an algebraic expression without presenting them with a multiple-choice option. One possible solution is for the task designer to create distractors that can be set as viable multiple-choice options. Preferably, such distractors would be chosen based on scientific or experimental findings that qualify the option as a good distractor if it indicates a particular difficulty or well-known misunderstanding. However, even good distractors have limitations since it is usually not possible to probe student thinking to infer a student's reasoning behind a particular response.

As online learning environments become more ubiquitous, it is crucial to develop an explicit understanding of the process of designing tasks for implementation in online environments to enable and support teachers' application of the 'big idea' in formative assessment to better their teaching practices (Black & Wiliam, 2009).

1.2 *Matematikfessor.dk: The Context for This Paper*

Matematikfessor.dk, the environment discussed in this paper, has been running in Denmark for over 10 years, and approximately 80% of Danish schools are subscribers to their services. In Denmark, there are approximately 700,000 students in primary school and lower secondary school combined. On an average day, Danish students provide answers to around 1.5 million tasks on *matematikfessor.dk*. This means that in a Danish school year, on average, 250 million tasks are provided with answers on *matematikfessor.dk*. Online learning environments, such as *matematikfessor*, therefore have access to a large amount of data and can potentially provide valuable feedback to teachers regarding the difficulties that students encounter in learning mathematics. *Matematikfessor.dk* provides students with feedback and suggestions on how a task could have been completed if a wrong answer was given. However, like many other online learning environments, student responses are limited to multiple-choice or numeric inputs.

2 Formative Assessment and Related Issues

There is a great deal of evidence that formative assessment can have a positive impact on learning (Black & Wiliam, 1998). Indeed, formative assessment is one of the most widely adopted teaching and feedback provision strategies worldwide. However, attempts to promote formative assessment have often resulted in teachers facing substantial difficulties implementing these ideas (Smith & Gorard, 2005). To help address this implementation problem, Black and Wiliam (2009) proposed five key strategies to support teachers in enacting the 'big idea' of formative assessment: 'evidence about student learning is used to adjust instruction to better meet student needs — in other words, that teaching is adaptive to the student's learning needs' (Wiliam & Thompson, 2007: p. 15). The relationships between these strategies and different aspects of formative assessment are illustrated in Figure 1.

	Where is the learner going	Where is the learner right now	How to get there
Teacher	**1(a)**: Clarifying learning intentions and criteria for success	**2**: Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding	**3**: Providing feedback that moves students forward
Students as peers	**1(b)**: Understanding and sharing learning intentions and criteria for success	**4** Activating students as instructional resources for one another	
Students as self-regulated learners	**1(c)**: Understanding learning intentions and criteria for success	**5** Activating students as the owners of their own learning	

FIGURE 1 Aspects of formative assessment
ADAPTED FROM BLACK & WILIAM, 2009: P. 8

In this paper, we mainly focus on Strategies 2 and 3, which spotlight feedback for teachers with the purpose of improving future learning situations for students. These strategies engineer items that elicit evidence of student understanding *for* teachers and thus support teachers in providing feedback that helps students advance forward.

In a critical review of Black and Wiliam's (1998, 2009) approach to formative assessment, Bennett (2011) identified six issues with the implementation of formative assessment in the classroom, of which four are relevant to our analysis: the *definitional* issue, the *domain dependency* issue, the *measurement* issue and the *professional development* issue.

Bennett's definitional issue refers to two contrasting views of formative assessment as an instrument or a process. The instrumental view, common among test publishers and many online learning environments, posits that formative assessment is a test that produces a score or set of scores that has 'diagnostic value'. In contrast, the process view, the focus of this paper, sees formative assessment as producing insights into student understanding that can 'actually [be] used to adapt the teaching to meet student needs' (Black & Wiliam, 1998: p. 140).

In highlighting the domain dependency issue, Bennett (2011) argued that eliciting evidence and providing feedback requires an understanding of mathematical learning and development that goes far beyond the generic. To do this, teachers need a 'reasonably deep cognitive-domain understanding … includ[ing] the processes, strategies and knowledge important for proficiency in a domain, the habits of mind that characterise the community of practice in that domain, and the features of tasks that engage those elements' (Bennett, 2011: p. 15).

In focusing on the measurement issue, Bennett argued that assessment is not simply a process of observing students' responses and noting errors or difficulties. Rather, it is an inferential process that requires teachers to have substantial knowledge and expertise that enables them to make productive 'formative hypotheses' and then to act on these. Bennett (2011) argued that this may involve engaging with the student to probe why the student gave a particular answer. Additionally, the teacher could provide more tasks that attempt to determine a pattern in the answers consistent with the hypothesis.

> It is worth noting that the generation and testing of hypotheses about student understanding is made stronger to the extent that the teacher has a well developed, cognitive-domain model. Such a model can help direct an iterative cycle, in which the teacher observes behaviour, formulates hypotheses about the causes of incorrect responding, probes further, and revises the initial hypotheses. In addition, if the underlying model is theoretically sound, it can help the teacher discount student responding that may be no more than potentially misleading noise (e.g., slips that have no deep formative meaning).
>
> BENNETT, 2011: p. 17

The final issue that Bennett raised was the professional development needed for teachers to develop the 'substantial knowledge [required] to implement formative assessment effectively in classrooms' (Bennett, 2011, p. 20). In this paper, our focus is the design of formative tasks within online learning environments that operationalise the components of Vergnaud's (2009) notion of the scheme to enable teachers to better interpret and respond to learners' errors while overcoming some of the issues mentioned by Bennett (2011). In a later section, we return to the notion of scheme and the functionality of its components, as well as how these might serve as guidance in designing formulations of tasks that address the mentioned issues.

3 Research Question

This research was conducted to respond to the following question: How can the notion of scheme guide the design of diagnostic tasks for implementation in online learning environments, specifically regarding known difficulties with the concept of linear equations and the equals sign, to enable teachers to better interpret or hypothesise about learners' difficulties?

The following sections propose a framework for establishing principles to answer the research question. The notion of scheme in the research question refers to the work of Gérard Vergnaud, who established the theory of conceptual fields, including the notion of scheme as an important concept (e.g., Vergnaud, 2009). We expand on the role of this theory in a later section.

This framework is specifically intended to aid in designing diagnostic tasks that enable teachers to hypothesise about their learners' difficulties. Of specific interest in this paper are the difficulties related to the properties of the equals sign in situations involving linear equations. We further elaborate on these properties and difficulties related to students' comprehension of the properties of the equals sign in the following section.

4 Task Design Principles

This section first introduces some key principles that go into the task design and implementation process to address the posed research question, and we further elaborate on the notion of scheme and its role in the task design. The overarching design principle idea stems from the work by Ahl and Helenius (2018) who presented a situation where a student was asked to calculate the average speed. The student invoked a scheme seemingly capable of handling average speed to some extent but ended up invoking and working with a scheme that incorrectly interpreted average speed using an alternative (and insufficient) scheme involving a different average, namely arithmetic mean. Although the student approached the task with knowledge and procedures (a scheme) connected to working with speed, the student ended up applying a scheme that handled the inappropriate arithmetic mean. Because the student was not able to arrive at a satisfactory solution to the task, he ended up invoking another scheme due to the word 'average' (in Swedish, as in English, the word 'average' is used both when discussing average speed and arithmetic mean). We believe this observation made by Ahl and Helenius is a vital finding and opens up a discussion about how to teach students about situations such as this one.

Inspired by Ahl and Helenius (2018), we formulated a sequence of tasks based on an initial task that invokes two potential paths to a solution, one *expected* and one *preferred*. Unlike the situation considered by Ahl and Helenius (2018), we aimed to formulate situations (a sequence of tasks) in which the two paths to the solution are both viable and should result in a correct answer. To create this task sequence, we drew on descriptions of hypothetical (learner) responses guided by variation theory (Watson & Mason, 2006; Marton, 2015). Applying variation theory (Marton, 2015) when designing tasks for this context makes it possible to meaningfully design a sequence of tasks that may be seen as a whole (Watson & Mason, 2006).

When designing such a sequence of tasks, Watson and Mason (2006) proposed the *controlled variation* of tasks and presented the following elements as key factors:

- Analys[e] of concepts in the conventional canon that one hopes learners will encounter.
- Identif[y] of regularities in conventional examples of [...] concept[s] [...] that might help learners (re)construct generalities associated with the concept. [...]
- Identif[y] of variation(s) that would exemplify these generalities;
- Decide which dimensions to vary and how to vary them;
- Construct exercises that offer micro-modelling opportunities, by presenting controlled variation, so that learners might observe regularities and differences, develop expectations, make comparisons, have surprises, test, adapt and confirm their conjectures within the exercise;
- Provide sequences of micro-modelling opportunities, based on sequences of hypothetical responses to variation, that nurture shifts between focusing on changes, relationships, properties, and relationships between properties. (Watson & Mason, 2006: pp. 108–109)

With this task sequence, we created a series of situations through which a learner might experience the schematic shift from the expected path to the preferred path to the solution. The goal of variation theory in our context is to construct the sequence of tasks so that the expected path to the solution increases in difficulty while the preferred path to the solution remains equally efficient. In our case, the intended shift is an extension of the knowledge of the properties of the equals sign. We established variations in situations through a sequence of tasks where students would get to experience the substitution property of equivalence — that if $a = b$, then a can be substituted for b, and vice versa, in any, to the situation relevant equation. We set up the task sequence so that the expected path to the solution would revolve around a scheme guided by equation-solving strategies. However, the intention was that the students may end up observing the substitution property. Before proceeding with the design principles, we discuss some of the difficulties related to the equals sign.

We created this specific sequence of tasks for the contexts of linear equations and the equals sign and drew on research on learners' errors and difficulties with these concepts (Kieran, 1981; Jones et al., 2012; Rhine et al., 2018). Several programmes have attempted to document specific difficulties in learning mathematics. These projects have contributed to the understanding of many misconceptions and other obstacles related to learning mathematics (Rhine et al., 2018). We adopted the view of Jankvist and Niss (2015) who identified genuine difficulties in learning mathematics as 'those seemingly unsurmountable obstacles and impediments — stumbling blocks — which some students encounter in their attempt to learn the subject' (Jankvist & Niss, 2015: p. 260). One kind of stumbling block that many students experience at the beginning of lower secondary school or when attempting to learn to solve more 'abstract' (Vlassis, 2002) linear equations is the role and interpretation of the equals sign (Kieran, 1981). Jones et al. (2012) argued that to achieve a better understanding of the role of the equals sign, one must learn to substitute one representation for a different representation that is equal to the original one. Many strategies for solving equations exist and are all sensibly tied to different tasks and/or situations. However, at some point, even linear equations can become abstract or complicated to the extent where only one strategy is truly viable. Many of these strategies, such as 'guess and check' or 'working backwards', do not necessarily require a deep understanding of the role or interpretation of the equals sign (Linsell, 2009). However, to apply more advanced equation-solving strategies, students must become more flexible in their understanding of the equals sign (Matthews et al., 2012; Kieran, 1981; Rhine et al., 2018).

We utilised controlled variation in the construction of each task in the sequence. By doing this, we attempted to address the importance of learning about the properties of the equals sign mentioned in the previous section. This made it possible to develop a meaningful sequence of tasks that could be evaluated as a whole. When discussing this sequence of nearly similar tasks, Bokhove (2014) suggested that *the element of crisis* is an important factor. Such a crisis occurs when the student completing a range of tasks — or parts of the task range — encounters a task that is impossible or nearly impossible to solve. This element of crisis resembles a cognitive conflict (e.g., Tall, 1977) or an inadequate conceptual field (Vergnaud, 2009). Bokhove and Drijvers (2012) used the element of crisis in their variations when designing a sequence of nearly similar tasks to show that students attempting to solve a crisis-provoking task using strategies appropriate for pre-crisis tasks may result in incompleteness because the earlier strategy is inadequate. In a later section, we return to what we refer to as the 'dual scheme idea', which works implicitly with the element of crisis (Bokhove, 2014).

5 The Notion of Scheme and Its Role in Activity

Vergnaud's (2009) work demonstrates how the scheme as a concept works as an organiser of action or activity when faced with a situation or a class of situations:

> [Schemes] describe ordinary ways of doing, for situations already mastered, and give hints on how to tackle new situations. Schemes are adaptable resources: they assimilate new situations by accommodating to them. Therefore, the definition of schemes must contain ready-made rules, tricks and procedures that have been shaped by already mastered situations.
>
> VERGNAUD, 2009: p. 88

Such a situation or class of situations could be equated to working with algebraic expressions or engaging in solving linear equations. If we accept that schemes are organisers of an individual's activity, we can create assumptions about students' schemes by observing their actions in desired situations. Ahl and Helenius (2018) claimed that this is why schemes are both didactically and analytically more interesting than the idea of conceptual understanding.

Vergnaud (2009) defined a scheme as having four aspects:

> The *intentional aspect* involves a goal or several goals that can be developed in subgoals and anticipations. The *generative aspect* involves rules to generate activity, namely the sequences of actions, information gathering, and controls. The *epistemic aspect* involves operational invariants, namely concepts-in-action and theorems-in-action. Their main function is to pick up and select the relevant information and infer from it goals and rules. The *computational aspect* involves possibilities of inference. They are essential to understand that thinking is made up of an intense activity of computation, even in apparently simple situations; even more in new situations. We need to generate goals, subgoals and rules, also properties and relationships that are not observable.
>
> The main points I needed to stress in this definition are the generative property of schemes, and the fact that they contain conceptual components, without which they would be unable to adapt activity to the variety of cases a subject usually meets.
>
> VERGNAUD, 2009: p. 88, our emphasis on the aspects

Essential to schemes from Vergnaud's perspective are the operational invariants (the epistemic aspect of schemes) consisting of *concepts-in-action* and *theorems-in-action*. A concept-in-action is 'an object, a predicate, or a category that is held to be relevant' (Vergnaud, 1988: p. 168). In every mathematical action, we choose certain objects, predicates or categories that are believed to be relevant to the current situation or setting. A theorem-in-action is a proposition held to be true. When we engage in a mathematical situation, we believe certain 'theorems' to be true or false regarding the objects relevant to the situation. Vergnaud stated that there is a dialectic connection between theorems and concepts, and this emerges from the fact that more advanced mathematical concepts originate from theorems, and vice versa. Nonetheless, it is important to distinguish the cognitive functions of the operational invariants in this precise manner. Concepts-in-action are individually available concepts in a, for the enactor, relevant representation to the situation. We emphasise that Vergnaud's (1988) interpretation of representation is similar to what others call a conception, a concept image or an invoked concept image (Tall & Vinner, 1981). Concepts-in-action bear no value in terms of logical truth, just relevance to the situation. Theorems-in-action are by nature true or false. These entities are sentences (or propositions) that provide the concepts with the possibility of inferences taking place. The *rules of action* are not to be confused with theorems-in-action. The function of the rules of action (the generative aspect of the scheme) is to be appropriate and efficient, but they rely implicitly on theorems-in-action (Vergnaud, 1997). Vergnaud (2009) emphasised that schemes are efficient organisers of activity by nature, and should they also become effective, the scheme can be considered an algorithm. He further clarified that schemes do not have all the characteristics of algorithms. The effectiveness of algorithms allows them to find a solution to a task using a finite number of steps (if a solution is possible).

6 Implementation of the 'Dual Scheme Idea' in a Sequence of Tasks

This section presents the task design and the reasoning behind it. The sequence of tasks was generated using the principles of controlled variation (Watson & Mason, 2006) based on the 'dual scheme idea' with expected and preferred paths to a solution. We also present formulations of a task from the sequence guided by the four components of the scheme (Vergnaud, 2009). An added discussion of the considerations regarding implementation in an online environment precedes each task formulation.

As mentioned, we set up a sequence of tasks (situations), of identical form, where two different schemes might be in play at the same time. The first line in each task can be interpreted as an equation that needs to be solved for the unknown value x. The second line in each task presents an expression for evaluation based on the knowledge acquired in the first line. However, the solution to a task from the sequence does not indicate what path (scheme) the students might use to obtain the solution. The alternate formulations guided by the components of the scheme should reflect which scheme got the upper hand, providing teachers with feedback that can help them generate 'formative hypotheses' regarding their students' schemes. In some cases, we proposed several formulations of the task in an attempt to address different aspects of the components of the scheme. We argue that this 'dual scheme idea', together with the principle of controlled variation, can help task designers better focus the intention and purpose of tasks in online learning environments. These principles lead to a clearer distinction between the expected and preferred paths to a solution and thereby a more explicit articulation of the learning objective. We focused the variations on the value of the unknown. This choice stemmed from the desire to create a sequence of increasingly more 'difficult' tasks based on the expected path to the solution where the preferred path remains at the original level of difficulty. We remind the reader of the role of the epistemic aspect of the scheme. Focused on the operational invariants, this aspect is such an essential part of the scheme that it and its elements (concepts-in-action and theorems-in-action) inform or are present in the other three components. For example, one would simply not be able to establish goals without having at least partial access to a concept relevant to the situation — in other words, a concept-in-action. The focus of the formulations guided by the components of the scheme is on evaluating whether students invoke schemes capable of handling substitutions based on equality, or rather, a scheme suitable for solving equations to solve the task and thus provide teachers with information on students' schemes.

1. What number should go into the empty space?
   $3x = 9$
   $3x + 4 =$ ___
2. What number should go into the empty space?
   $4x = 10$
   $4x + 3 =$ ___
3. What number should go into the empty space?
   $3x = 2$
   $3x + 4 =$ ___
4. What number should go into the empty space?
   $3x = 11$
   $3x + 4 =$ ___
5. What number should go into the empty space?
   $3x = -6$
   $3x + 4 =$ ___

Applying a scheme for handling and substituting equal terms will, in most cases, be the 'easier' path to the goal of solving the above tasks. The term $3x$ can be treated as an object. An expert equation solver would choose the most efficient scheme to solve the task: 'I know x has a specific value and I could easily calculate it, but I do not need to for this task because the object $3x$ is present in both equations'. However, many students believe that such tasks need to be solved using the method that they have been taught, and then they encounter difficulties implementing it. The sequence of tasks is designed to target and isolate these difficulties. The expected path to the solution among students involves solving the first equations for x and then substituting the value of x into the second equation, allowing them to do calculations to fill in the empty space.

In Task 1, a lower secondary school student might look at it and just 'know' the value of x (Linsell, 2009) or simply ignore the letter (Küchemann, 1981). In our experience, many students in lower secondary school become able to read an equation, such as the top equation in Task 1, and imagine it as a command that sorts out the value of a spot in a multiplication table. Traditionally, students around the world would be familiar with tasks such as $4 + 3 =$ ___, where the goal is simply to fill in the empty space, and they would do so by adding the numbers on the left side of the equals sign. Similarly, students would have experienced tasks such as $3 \cdot$ ___ $= 12$, having worked with all four basic operations. The difference in this task sequence, and a difference one should be aware of, is the missing multiplication symbol between the coefficient and the unknown. From previous work, we know that expressions such as $3x$ are easily misinterpreted by students (Rhine et al., 2018).

In Task 2, a lower secondary school student might still be able to guess/know the value of x. However, if the student is not comfortable or familiar with the idea that non-integer numbers could be solutions to an equation, such as the ones presented in this context, the student would have to alter their strategy based on this crisis or cognitive conflict. The coefficient to x is not a true divisor of the number to the right of the equals sign. This does not necessarily mean that a student in lower secondary school does not know or is able to guess the value of the unknown.

In Task 3, we go a step further. To determine the value of x, the student should be comfortable with numbers between 0 and 1. If the student is also uncomfortable with fractions, they would end up with an infinite decimal number. We expect that most students will experience difficulties knowing or guessing the value of the unknown at this stage.

Task 4 is in many ways like Task 3. However, in this task, the students solving the equations deal with an improper fraction or an infinite decimal number.

Task 5 changes the situation a little bit. As a final task in the sequence, we let the unknown be a negative number. Vlassis (2002) demonstrated that linear equations become significantly more difficult when they involve negatives. In this task, we chose *not* to have the coefficient take a negative value because this might make the task more difficult than having the multiplication result in a negative number and because we wanted to simply vary the value of the unknown.

The common idea among all these tasks is that if students can take the preferred path by substituting the right side of the top expression for the term with the unknown in the bottom expression (in the first tasks, substitute $3x$ for 9), our assumption is that most secondary school students can then straightforwardly solve the task sequence. As mentioned, the tasks were designed so that a 'correct' answer might obscure some difficulties related to linear equations and substitution and therefore are items that generate answers that may be misinterpreted by teachers. If students are more inclined to determine the value of x, because that is what is 'expected' when faced with linear equations, then they might face, as the difficulty of solving the equations increases through the sequence due to the way the numbers were chosen. Each task becomes gradually more difficult if a student attempts to calculate the value of the unknown number x. We realised that the students could reach a fraction as the solution in every task. To solve the task (finding the number that should go in the empty space), students would first have to do the division and end up with increasingly more complex fractions. The same number that just served as the divisor should then be multiplied onto these fractions, resulting in the same number that served as a dividend. However, if students pursue this path and do not preserve the fraction, multiplication could lead to tricky situations dealing with infinite decimal numbers. In addition, a student able to solve the sequence of equations for x (accepting fractions as solutions) should eventually realise that, when inserting the value of x in the second line, they will arrive at the same number that was just on the right side of the first equation and begin to make inferences about substitutional properties.

In *matematikfessor.dk*, tasks must meet certain criteria regarding structure and user input types. The input types are restricted to inputting either a number or multiple-choice selection. Each task must have a unique 'right answer' and must be presented in such a way as to make immediate feedback possible. The tasks presented in this paper are suited for formative and educational purposes rather than training exercises, a categorisation that pertains to many other short-in-formulation equation-solving tasks.

Before we present the formulations guided by the components of the scheme, we emphasise our methodological intention for future teachers working with the task sequence and the formulations based on it. With the overall aim being the implementation of new possibilities for formative feedback for teachers using online learning environments, the intention of this framework is not to leave the teachers without any instructions associated with the task sequence. When working with a task sequence designed using hypothetical responses to variations that nurture shifts between focuses, leading to potential new learning (adaptations in schemes), teachers are provided with a clear common learning goal. Therefore, working with task sequences, such as the above, can lead to possibilities for classroom discussion of the differences in paths to solutions the students might have taken. Alternatively, the task formulations guided by the components can help teachers hypothesise about learners' schemes and formulate a basis for classroom discussion. In our opinion, the latter provides teachers with an opportunity to work with schemes and their components as a more practical tool for teaching problem-solving or engaging in mathematical situations in general.

6.1 *Using the Components of the Scheme as a Guide for Asking Formative Questions*

With the sequence of tasks established, we now present alternate formulations guided by the components of the scheme. In the following sections, we go through the four components (intentional, generative, epistemic and computational) to make alternative formulations of the tasks from the sequence that enable teachers to hypothesise about their students' schemes and difficulties.

6.2 *Setting 'Goals and Anticipations'* (*the Intentional Part of the Scheme*)

Under this category, we present formulations where setting a goal for or anticipation of the task forms the solution. Before choosing a strategy to arrive at a solution, one must set a goal for what solves the task and what is expected to arrive at a solution. In this intentional aspect of the scheme, one also establishes what the task (the situation) anticipates and what one anticipates from the situation. In many cases, one could expect that, when confronted with a task containing a linear equation, finding the unknown value would be crucial. We propose that when focusing on the goals and anticipation part of the scheme, the task could be formulated as follows (we demonstrate this by using the values from initial Task 4):

> Is it necessary to know the value of x to fill in the empty space?
>
> $3x = 11$
>
> $3x + 4 = \_\_\_$

Alternative wording could be as follows:

> Is it *beneficial* to know the value of x to fill in the empty space?
> Would the task be much easier if you knew the value of the unknown x?

Goals and rules are set and established based on the concepts-in-action and the theorems-in-action (the epistemic aspect of the scheme). Whether a student takes note of the equality between $3x$ and 11 and the fact that the term $3x$ is present in both equations as relevant information (concepts-in-action) could indicate whether they are capable of substituting the term $3x$ for 11 in the two equations. Should students not choose the substitutional link between the two equations to be relevant, we expect that they would argue that they would like to know the value of x to fill in the empty space. One might argue that the goal of the task is blurred by the new formulations, since there is an empty line 'begging' for a number to be put on it, but the task is answered by a simple yes or no. However, for the purpose of providing feedback to teachers, following our intention to focus on learning according to Strategies 2 and 3 (Black & Wiliam, 2009), teachers might learn about their students' schemes with this formulation as opposed to just receiving a correct or incorrect answer from students filling in the empty space.

The students might expect that if the unknown value were provided, the value would 'make sense' to the situation and therefore be a small natural number because the numbers present are such. Even if a student were to choose to answer yes to one of the formulations, the student would most likely not expect to be provided an improper fraction as the value of the unknown.

6.3 *Applying 'Rules of Action'* (*the Generative Part of the Scheme*)

Working with this component of the scheme, we attempted to uncover the rule, or strategy, that students would apply to solve the tasks considering the two schemes. As mentioned in the above category, operational invariants help set the goal or apply rules. After the student has established a goal and/or anticipations, the student can choose appropriate rules to generate action. When forming a rule or a strategy to solve a task, theorems-in-action might be more in focus. Therefore, we formulated the task not in terms of concept relevance but rather in theorems-in-action that lead to rules of action. To confirm that the first equation presented in each of the initial tasks (e.g., $4x = 10$) is in fact important in filling in the empty space, we propose a formulation of the tasks that hints at what 'path to the goal' a student would rather choose: an equation-solving strategy or a substitution of equal terms strategy.

How is $3x = 11$ important in filling in the empty space?

$$3x \quad = 11$$
$$3x + 4 = \_\_\_$$

1) Because $3x$ and 11 can be substituted in both lines.
2) Because it lets me calculate the value of x.
3) Another reason.
4) It is not important.

In this formulation, the teacher will get a slightly different view of what path a student wants to choose. A further formulation of the task focused on rules to generate action could appear as follows:

How would you attempt to find the number that goes in the empty space?

1) I would find the value of x.
2) I would substitute $3x$ for 11.
3) I would do something else.
4) I don't know.

If the scheme(s) upon which the student is drawing are only able to apply the rule of action to determine the value of the unknown, the teacher can be provided with valuable information. In this way, a teacher gets a different perspective on essentially the same task but in a different formulation and with a different focus or aim. The rules of action might differ when students engage with the formulations of the different tasks from the sequence due to the increased difficulty when applying an equations-solving scheme.

6.4 *Handling Information with 'Operational Invariants'* (*the Epistemic Aspect of the Scheme*)

With this category, we enter the more complex part of the scheme. Concepts-in-action are concepts relevant to the student engaging in the task, while theorems-in-action are propositions held to be true in a given situation. Therefore, we were careful not to change the situation when we created formulations that attempt to engage with either theorems or concepts related to the task sequence. We remind the reader that in the epistemic part of the scheme, these operational invariants are in play as an underlying aspect of the other parts of the scheme. However, we find it fit to ask questions regarding the relevance of objects or questions regarding the truth value of statements in an attempt to uncover the structure of students' schemes. We begin by addressing the theorem that applies when solving Task 4 to observe whether students might agree to it.

Is it true that the number that goes on the empty line is 4 more than 11?

Another way to address this matter could be via a multiple-choice question:

Can '$3x$' simply be thought of as another way to write '11'?
1) Yes, because that's what the equals sign means.
2) No, because it says that 3 multiplied by some number makes 11.
3) I think both 1 and 2 sounds correct.
4) I do not agree with either 1 or 2.

Or,

Is it okay to substitute $3x$ with 11 in the two lines/equations in the task?

When working with concepts-in-action, we attempted to address the relevance of the objects present in the task. It can be very tricky to ask questions about the relevance of a concept in a given situation. We remind the reader that, in this context, the aim is to assess what students consider relevant objects/concepts.

Is it okay not to care what the value of the unknown is when filling in the empty line?

If the scheme(s) upon which the student is drawing are only able to determine what number that go into the empty space by performing calculations with the value of the unknown number, we expect students not to instinctively agree with such statement.

6.5 *Generating Space for 'Possible Inferences'* (*the Computational Aspect of the Scheme*)

In this last example, we attempt to address the computational part of the scheme with another formulation of the initial task. Specifically, this formulation aims to determine whether a student makes inferences about the element $3x$ when they compare it to a similar-looking task, but where the scheme for solving equations should be rendered useless because $3x$ has been replaced by a blue box.

Does the same number go into the empty space in both tasks?

$3x \quad = 11$

$3x + 4 = \_\_\_$

☒ = 11
☒ + 4 = ___

If a student does not see the similarities between the two tasks and uses a scheme to solve equations by working with the leftmost task and substituting the blue box and the number 11 in the rightmost task, the student might become suspicious. One might expect the student to wonder why this is the case or why the tasks perform differently, yet so similarly. A different formulation could therefore be as follows:

> Is it surprising that the same number goes into the empty space in both tasks?

The reason for wording the task this way is to establish a cognitive conflict (an element of crisis) in students who are surprised that 15 is the correct number for both empty spaces. This formulation attempts to create a link to a scheme we consider to be like the scheme capable of substituting mathematical equal terms by introducing the task with the blue box.

Another idea is to flip the situation to observe whether the student is willing to infer.

> Is it equally difficult to fill in the empty space in both tasks?
>
> $3x = 11$
> $3x + 4 =$ ___
> $3k + 4 = 15$
> $3k =$ ___

The task on the right-hand side resembles a more ordinary equation-solving situation since the second line presents fewer terms than the first.

7 Discussion

This paper explored the considerations and reasoning behind an implementation process model that can guide diagnostic task design for an online learning environment using an illustrative exemplar set of tasks. This exercise operationalised Vergnaud's idea of the scheme to enable or improve teachers' opportunities to interpret or hypothesise about learners' difficulties within the constraints of the online learning environment.

This section presents a more generalised overview of the proposed framework as an *implementation process model* (Nilsen, 2015). The motivation for establishing such model was the desire to explore the potential of asking formative questions guided by components suitable for online environments. Another part of the motivation stemmed from Bennett's (2011) critical review of formative assessment, which identified six major issues with managing or implementing formative assessment in schools. We set out to address three of the major issues, namely the measurement issue, the domain dependency issue and the definitional issue. In the following, we present our condensed implementation model before going into a discussion of how this model addresses the three mentioned issues with formative assessment.

7.1 *Implementation Process Model*

The first step in our model is to establish the task sequence based on an important property or difficulty in mathematics. In this paper, we looked at the property of the equals sign. The second part of the implementation process model involves transforming the task sequence into questions guided by the components of the scheme. This adds a different layer of variation.

To create a sequence of tasks for use in online learning environments that can improve feedback for teachers and students, we propose the following steps as part of the implementation process model:

1. *Identify regularities* in examples of a concept that might *help learners* (*re*)*construct the generalities* associated with that concept. This identification could be guided by research findings on the difficulties students experience in the different subject areas of mathematics education.
2. Establish the *dual scheme idea* with the preferred and expected paths to the solution. This step is extremely important as a guide when designing the variations within the task sequence and within the formulations guided by the components of the scheme.
3. Design a *sequence of tasks* using controlled variation.

The second part of the implementation process model becomes the framework for transforming the task sequence into questions related to the four components of the scheme. This step in the model enables improved feedback to teachers working with online learning environments.

4. Formulate questions that are related to the individual *components of the scheme*, with an emphasis on involving the *dual scheme idea*, as

the questions strengthen the opportunities for the intended learning objective to be successful. Furthermore, these alternate formulations enhances the possibilities for teachers to hypothesise about their learners' schemes.

The implementation process model is illustrated in Figure 2. The steps on the left represent the four steps in the model. The difference in feedback is illustrated on the right.

We realise that the 'dual scheme idea' does not necessarily encapsulate all possible objectives of learning. However, from a teaching perspective, this helps control the situations and enables a clear path for instruction when discussing how the two different schemes handle the tasks, especially in relation to the four major formative assessment issues of measurement, domain dependency, definitional and professional development (Bennett, 2011). We believe that we are now in the business of providing teachers that use online learning environments with useful classroom materials that integrate pedagogical, domain and measurement knowledge regarding formative assessment.

Regarding the definitional issue, Bennett (2011) argued that seeing formative assessment as a process or a test/instrument is an oversimplification. If formative assessment were to be thought of as an instrument, Bennett argued that a carefully developed, research-based instrument is still unlikely to be effective in instruction if the process surrounding its use is flawed. When working with online learning environments that operationalise the components of the scheme as in our model, we get a step closer to creating an instrument with a functioning process supporting it.

In our implementation process model is a designed framework that accommodates for the measurement issue. This issue might also be the most tangible of the three issues discussed here. The task design, considerations and ideas

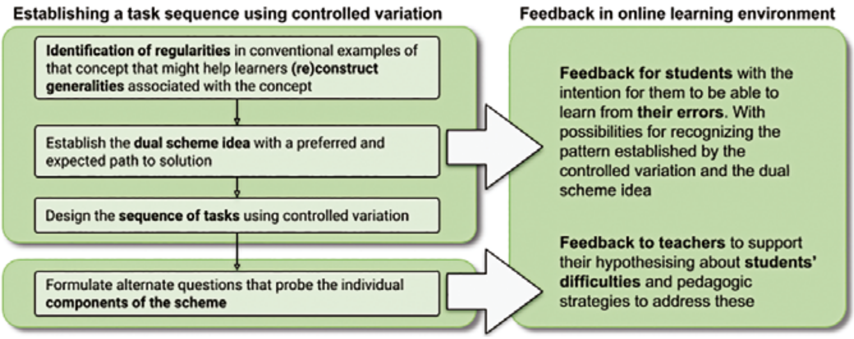

FIGURE 2 Diagram of the implementation process model

behind the implementation process model have been discussed. The next step is to gather feedback from teachers and study teachers' use of this model.

Finally, the use of the components of the scheme enabled us to address the domain dependency issue. Asking questions using the components can provide a deep cognitive-domain understanding that has the potential to reveal general principles, strategies, techniques and knowledge important for proficiency in mathematics.

7.2 *The What, How and Why*

In this paper, we considered just one sequence of tasks to demonstrate how the four components of Vergnaud's idea of the scheme can be operationalised to support teachers in formatively assessing students' understanding within the constraints of an online learning environment. The intentional aspect relates to *what* needs to be done to solve the task or *what* is expected of the student as they attempt to reach the anticipated goal. The generative aspect relates to *how* expectations are met or *how* progress is to be made in the situation. Finally, the computational aspect relates to *why* the desired goal is achieved or *why* new connections to other schemes or concepts make sense or might be established. We have intentionally left out the epistemic aspect of the scheme in this section since the epistemic is such an essential part of the scheme and will be present when working with the other components. We argue that this resembles how clarifying questions asked by teachers should always take form.

The *what*, *how* and *why* might potentially contribute to the shared or agreed upon theory of change, understood as such that all stakeholders involved with the chain of implementation, so to speak, agree on the framework (Jankvist et al., 2021). This understanding can thereby strengthen the implementability of the tasks, as the idea of respecting the *what*, *how* and *why* among teachers using this framework with online learning environments will be shared (Jankvist et al., 2021).

8 Concluding Remarks

The next steps for the implementation process model include implementing the model in the design of further tasks in online learning environments, such as *matematikfessor*. It is also important to examine whether, and how, the resulting tasks actually enable teachers to hypothesise about their learners' schemes. We used Vergnaud's components of the scheme to demonstrate how an array of exemplar items can be designed to distinguish between the different schemes that students use to tackle a task.

We believe that Vergnaud's components provide a productive approach to describing students' math-related actions to teachers. We also believe that online environments, such as *matematikfessor*, could help implement this for many teachers. However, our paper highlights an urgent need for work on task design for online environments such as *matematikfessor*. One affordance of working with online environments is that teachers, as well as students, can get easy access to feedback. However, we do not claim that the task of creating a task sequence, such as the one presented in this paper, is necessarily easy. We do, however, believe that this could also be considered an advantage of online learning environments. With professional task designers and a framework for generating tasks with added formative feedback, task sequences might benefit teachers in planning future lessons.

Further steps include the design of additional task sequences in different areas of mathematics education. These sequences of tasks, alongside formulations using the components of the scheme, should then be implemented in an online learning environment, such as *matematikfessor*. Then, a study examining teachers' experiences with the formulations of the sequences using the components of the scheme should take place. Getting feedback on not only the interpretational potential and effect but also the way the teachers go about operationalising the feedback they receive from using the formulations is important. Knowledge sharing and feedback from teachers using the tasks will pave the way for the sensible implementation of a range of sequences of tasks.

Impact Sheet

The impact sheet to this article can be accessed at 10.6084/m9.figshare.19493846.

References

Ahl, L. M. & Helenius, O. (2018). Why is calculating the average speed difficult? In E. Bergqvist, M. Österholm, C. Granberg & L. Sumper (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 3–10). PME.

Bennett, R. E. (2011). Formative assessment: A critical review. *Assessment in Education: Principles, Policy & Practice*, *18*(1), 5–25. https://doi.org/10.1080/0969594X.2010.513678.

Black, P. & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: Principles, Policy & Practice*, *5*(1), 7–74. https://doi.org/10.1080/0969595980050102.

Black, P. & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability*, *21*(1), 5. https://doi.org/10.1007/s11092-008-9068-5.

Bokhove, C. (2014). Using crises, feedback, and fading for online task design. *PNA*, *8*(4), 127–138. https://doi.org/10.30827/pna.v8i4.6111.

Bokhove, C. & Drijvers, P. (2012). Effects of feedback in an online algebra intervention. *Technology, Knowledge and Learning*, *17*(1–2), 43–59. https://doi.org/10.1007/s10758-012-9191-8.

Cavalcanti, A. P., Barbosa, A., Carvalho, R., Freitas, F., Tsai, Y.-S., Gašević, D. & Mello, R. F. (2021). Automatic feedback in online learning environments: A systematic literature review. *Computers and Education: Artificial Intelligence*, *2*, 100027. https://doi.org/10.1016/j.caeai.2021.100027.

Dyssegaard, C. B., Egelund, N. & Sommersel, H. B. (2017). *A systematic review of what enables or hinders the use of research-based knowledge in primary and lower secondary school*. Danish Clearinghouse for Educational Research; Department of Education; Aarhus University, Aarhus.

Jankvist, U. T., Gregersen, R. M. & Lauridsen, S. D. (2021). Illustrating the need for a 'Theory of Change' in implementation processes. *ZDM — Mathematics Education*, *53*(5), 1047–1057. https://doi.org/10.1007/s11858-021-01238-1.

Jankvist, U. T. & Niss, M. (2015). A framework for designing a research-based "maths counsellor" teacher programme. *Educational Studies in Mathematics*, *90*(3), 259–284. https://doi.org/10.1007/s10649-015-9629-8.

Jones, I., Inglis, M., Gilmore, C. & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of Experimental Child Psychology*, *113*(1), 166–176. https://doi.org/10.1016/j.jecp.2012.05.003.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317–326. https://doi.org/10.1007/BF00311062.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11–16* (pp. 102–119). Murray, London.

Linsell, C. (2009). A hierarchy of strategies for solving linear equations. In R. Hunter, B. Bicknell & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 331–338). MERGA.

Marton, F. (2015). *Necessary conditions of learning*. Routledge, Abingdon.

Matthews, P., Rittle-Johnson, B., McEldoon, K. & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43*(3), 316–350. https://doi.org/10.5951/jresematheduc.43.3.0316.

Nilsen, P. (2015). Making sense of implementation theories, models and frameworks. *Implementation Science*, *10*, 53. https://doi.org/10.1186/s13012-015-0242-0.

Rhine, S., Harrington, R. & Starr, C. (2018). *How students think when doing algebra*. Information Age Publishing, Charlotte, NC.

Smith, E. & Gorard, S. (2005). 'They don't give us our marks': The role of formative feedback in student progress. *Assessment in Education: Principles, Policy & Practice*, *12*(1), 21–38. https://doi.org/10.1080/0969594042000333896.

Steenbergen-Hu, S. & Cooper, H. (2013). A meta-analysis of the effectiveness of intelligent tutoring systems on K–12 students' mathematical learning. *Journal of Educational Psychology*, *105*(4), 970–987. https://doi.org/10.1037/a0032447.

Tall, D. (1977). Cognitive conflict and the learning of mathematics. In *Proceedings of the First Conference of The International Group for the Psychology of Mathematics Education*. PME. https://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1977a-cog-confl-pme.pdf.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. https://doi.org/10.1007/BF00305619.

Vergnaud, G. (1988). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior*, *17*(2), 167–181. https://doi.org/10.1016/S0364-0213(99)80057-3.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective*. Psychology Press, Hove.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, *52*(2), 83–94. https://doi.org/10.1159/000202727.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, *49*(3), 341–359. https://doi.org/10.1023/A:1020229023965.

Watson, A. & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*, *8*(2), 91–111. https://doi.org/10.1207/s15327833mtl0802_1.

Wiliam, D. & Thompson, M. (2007). Integrating assessment with learning: What will it take to make it work? In C. A. Dwyer (Ed.), *The future of assessment: Shaping teaching and learning*, 1st edn. (pp. 53–82). Routledge, Abingdon.