ON THE PARAMETERISATION OF A CLASS OF DOUBLY-PERIODIC LATTICES OF EQUALLY-STRONG HOLES

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Summary

We construct an exact, explicit parameterisation of a class of doubly-periodic lattices of equally-strong holes in an infinite elastic plate that is in a state of plane stress. This parameterisation assumes no symmetries of the lattices' holes and allows for any finite number of holes per period cell. It is stated in terms of a conformal map from a circular domain. We construct this map in terms of the integrals of the first kind that are associated with a Schottky group that is generated from this circular domain. Key to our derivation of this parameterisation, is the observation that a doubly-periodic lattice of equally-strong holes is characterised by the property that the Schwarz functions of all of its holes' boundaries are identical up to additive constants. We also conjecture a condition that is necessary and sufficient for the existence of the class of lattices that are described by this parameterisation, although we are only able to verify this condition numerically here. We also present a selection of examples of such lattices, computed using this parameterisation.

1. Introduction

For the purpose of designing a perforated thin plate that is to be subjected to a given external loading, it is helpful to be able to determine *a priori* for which shapes and positions of the plate's holes the stress distribution that is induced in the plate by the loading is in some sense optimised. For instance, it is desirable to determine sets of holes along whose boundaries the induced stresses are uniformly distributed, as localised peaks of these boundary stresses are commonly a source of structural failure. Cherepanov (**1**) considered an inverse problem of this type for a perforated infinite, homogeneous, isotropic elastic plate. Assuming uniform loadings to be applied in the normal and tangential directions along the boundaries of the plate's holes, he sought sets of so-called 'equally-strong' holes. These possess the property that the induced tangential normal stresses - i.e., the 'hoop' stresses - take the same constant value along the boundaries of all of the holes.

Cherepanov (**1**) proposed an analytical method for constructing an exact, explicit parameterisation of sets of equally-strong holes. This parameterisation is of the domain exterior to such a set of holes as the image under a one-to-one conformal map of the domain exterior to a set of parallel slits. Cherepanov showed how, by first introducing the Kolosov-Muskhelishvili complex potentials, one can reformulate the governing boundary value problem as a coupled pair of homogeneous Schwarz problems, expressed in terms of his parameterising variable. Solutions of this pair provide one with the corresponding conformal map. Using this method, he constructed a parameterisation of a single hole and a set of two symmetric holes (for sets of a finite number of holes, the aforementioned loadings on the holes' boundaries are supplemented by a uniform loading at infinity), but also a doubly-periodic rectangular lattice of holes with a single, symmetric hole in each of its period cells. For the latter, he took the corresponding pre-image domain to be the exterior of a doubly-periodic rectangular array of parallel slits, and derived an exact expression for the corresponding conformal map in terms of elliptic functions associated with a period rectangle of this doubly-periodic pre-image slit domain. However, it is not straightforward to implement Cherepanov's method to construct parameterisations of more general sets of equally-strong holes, due to the difficulty of finding solutions to the attendant pair of Schwarz problems.

Recently, Antipov (**2**) developed an extension of Cherepanov's method that he used to construct a parameterisation of sets of any finite number of equally-strong holes (albeit restricted to sets whose pre-image domain is the exterior of a set of collinear slits). Antipov's approach involves rewriting the associated pair of Schwarz problems as a coupled pair of symmetric Riemann-Hilbert problems. He derived solutions to this pair and thus the corresponding conformal map in terms of hyperelliptic integrals. As for other parameterisations of doubly-periodic lattices of equally-strong holes, we first point out that sets of equally-strong holes in fact represent a limiting case of certain sets of equally-stressed elastic inclusions (embedded in an infinite elastic plate of another material). An exact, explicit parameterisation of a doubly-periodic rectangular lattice of such inclusions with a single, symmetric inclusion in each of its period cells, was constructed by Grabovsky & Kohn (**3**). (This lattice was first identified and constructed via a numerical approach by Vigdergauz (**4**).) Subsequently, Vigdergauz (**5**) constructed a parameterisation of a doubly-periodic rhombic lattice of such inclusions, again with a single, symmetric inclusion in each of its period cells. These parameterisations in (**3**) and (**5**) are also in terms of conformal maps from domains exterior to doubly-periodic arrays of parallel slits (a rectangular array in the case of (**3**), and a rhombic array in the case of (**5**)) and were constructed using Cherepanov's method (**1**). In both (**3**) and (**5**), the derived form for the corresponding conformal map is expressed in terms of elliptic functions associated with a period parallelogram of the corresponding doubly-periodic pre-image slit domain. We also mention that Vigdergauz (**6**) has devised a numerical algorithm for computing the boundaries of lattices of equally-strong holes whose periodic structures are of low rotational symmetry (this is based on single-layer potentials). However, to the best of our knowledge, no other exact parameterisations of doubly-periodic lattices of equally-strong holes (or the aforementioned elastic inclusions) have been constructed. Given the important role of lattice structures in solid mechanics, and the value (both theoretical and computational) of exact solutions, it is of interest to construct parameterisations of more general doubly-periodic lattices of equally-strong holes. That is precisely the aim of this paper.

In this paper, we construct an exact, explicit parameterisation of a class of doubly-periodic lattices of equally-strong holes. This parameterisation assumes no symmetries of the lattices' holes and allows for any finite number of holes per period cell. The method that we use to construct this parameterisation is different to that of Cherepanov (**1**). Rather, it is a natural generalisation of the method that was used in (**7**) to construct a parameterisation of sets of any finite number of equally-strong holes. In a broader sense, it is an extension of methods developed most notably by Crowdy over the course of the last twenty years or so, to construct solutions to a variety of problems in two-dimensional multiply connected domains. Further details of these, as well as many of their other applications and a description of the history of their development (including a list of further relevant references) can be found in Crowdy (**8**). The key components of these methods that we use in this paper are the Schwarz function, conformal mapping from a circular domain, Schottky groups and automorphic - or rather in our case, quasi-automorphic - functions. To elaborate further, here as in (**1**), we begin by introducing the Kolosov-Muskhelishvili complex potentials. However, rather than use the formulation of the problem as a coupled pair of Schwarz problems as proposed in (**1**), we instead proceed by identifying a characterisation of doubly-periodic lattices of equally-strong holes in terms of the Schwarz functions of the holes' boundaries. Like Cherepanov (**1**) (and as in (**7**)), in this paper we restrict ourselves to sets of equally-strong holes whose boundaries are smooth analytic curves. The Schwarz function, $S(z)$ say, of an analytic curve Γ, is the unique function that is analytic in a neighbourhood of Γ and has the property (e.g., see (**8**), Davis (**9**))

$$S(z) = \overline{z} \quad \text{for } z \in \Gamma. \tag{1.1}$$

For example, for a circle centred on the origin and of unit radius, $S(z)$ is simply $1/z$. In this paper, we show that a doubly-periodic lattice of equally-strong holes is characterised by the property that the Schwarz functions of all of its holes' boundaries are identical up to additive constants (see Proposition 2.1). The same property was identified for sets of a finite number of equally-strong holes in (**7**) (see Proposition 2.3 of the latter). We point out that one can identify the Schwarz function in Cherepanov (**1**, equation (2.5)), as well as in (**3**, equation (2.11)) and (**5**, equation (3.6)) - indeed, this is not surprising, as it is in fact related trivially to one of the Kolosov-Muskhelishvili complex potentials (see equation (2.14) of this paper). However, the Schwarz function is not identified explicitly in any of (**1**), (**3**), or (**5**) nor, to the best of our knowledge, in any of the other related literature.

Given this characterisation, we then seek to construct a parameterisation of a doubly-periodic lattice of equally-strong holes in terms of a conformal map from a circular domain. To do so, we introduce a certain Schottky group that is generated from this circular domain. Schottky groups have been used previously to construct conformal mapping parameterisations of numerous classes of singly-periodic lattices, a recent example of which is presented by Baddoo & Crowdy (**10**) (this describes singly-periodic domains with polygonal holes); other examples can be found in (**8**). However, to date, the use of Schottky groups for constructing parameterisations of doubly-periodic lattices appears to have been less common. One example is that which is presented in (**11**) (this describes doubly-periodic fluid regions that are formed by the injection of a viscous fluid into a Hele–Shaw cell through an array of point sources). However, the doubly-periodic lattices that are parameterised in (**11**) have a very symmetric structure: their period cells are squares, and each cell contains just a single, symmetric hole. Whilst it should be possible to adapt this parameterisation to parameterise square lattices of equally-strong holes with a single, symmetric hole in each period cell, it does not appear possible to extend it to construct parameterisations of more general doubly-periodic lattices of equally-strong holes. In this paper, to construct such a parameterisation, we use a similar type of pre-image domain to that which was used in (**11**) (although it is not restricted by the same symmetries - see section 3.1), but a different type of Schottky group. Indeed, the group that we shall use in this paper is of a different type to that which is used throughout (**8**). The Schottky group that is used in (**8**) (which is also the type of group that was used in (**7**) and also for the aforementioned parameterisations of singly-periodic lattices such as that in (**10**), as well as by Antipov (**12**) to construct a parameterisation of sets of any finite number of equally-stressed elastic inclusions) is one that is generated by a set of Möbius transformations, each of which 'pairs' one of the boundary circles of a corresponding circular domain (generally a pre-image domain, as it is in (**7**) and (**10**)) with its reflection in one of the other boundaries of this domain. (The group that was used in (**11**) is in fact a special subset of a group of this type - see also Vasconcelos *et al* (**13**).) The group that we use in this paper (see section 3.2) differs from that in (**8**) in that one of the Möbius transformations from which it is generated (which we label by $\theta_N(\zeta)$ - see equation (3.7)) pairs two of the boundaries of the pre-image circular domain with one another. (We also point out that Belokolos *et al* (**14**, Chapter 5) use other different types of Schottky groups, albeit in the context of constructing solutions of certain nonlinear integrable equations rather than conformal mapping parameterisations.) We construct an exact, explicit formula for our parameterising conformal map in terms of certain functions that are associated with this group, namely its integrals of the first kind. This formula is stated by equation (3.18) (equivalently, (3.19)). This map is quasi-automorphic with respect to our group - i.e., it is invariant under the action of this group, up to additive constants (e.g., see Ford (**15**)). However, this map must be univalent in the closure of our pre-image circular domain. We conjecture a condition that is necessary and sufficient for this to be the case, or equivalently, that is necessary and sufficient for the existence of the class of doubly-periodic lattices of equally-strong holes that are described by our parameterisation (see Conjecture 3.4). However, we are unable to prove this condition here, and instead are only able to verify it numerically for specific cases. Some examples of doubly-periodic lattices of equally-strong holes that we computed using our parameterisation are presented in section 4.

Finally, we point out that we do not claim that our parameterisation describes all doubly-periodic lattices of equally-strong holes. Furthermore, in this paper we restrict our attention almost solely to the parameterisation of such lattices, and leave an analysis of their physical properties for elsewhere.

2. Problem formulation

Consider an infinite, homogeneous, isotropic elastic plate that is perforated by a doubly-periodic lattice of holes. Suppose that the plate lies in a complex z-plane ($z = x + \mathrm{i}y$). Let Δ denote the (infinite) doubly-periodic domain that is occupied by the perforated plate, and $\partial\Delta$ denote the boundary of Δ (i.e., the union of the boundaries of all of the holes in Δ). Furthermore, let $\overline{\Delta}$ denote the closure of Δ (throughout this paper we use the 'overline' notation with respect to a domain to denote its closure). Next, let Λ denote the doubly-periodic lattice of holes in Δ ($\Lambda = \mathbb{C} \setminus \overline{\Delta}$). Label the two periods of Δ - equivalently, of Λ - by λ_1 and λ_2. We allow λ_1 and λ_2 to take any (non-zero) complex values such that $0 < \arg\{\lambda_2/\lambda_1\} < \pi$ (without loss of generality). Let $\mathcal{D}$ denote a single period cell of Δ. Suppose that $\mathcal{D}$ contains $N \geqslant 1$ holes (where N is finite), i.e., $\mathcal{D}$ is an $(N+1)$-connected domain. An example with $N = 3$ is sketched in Fig. 1. Label the holes in $\mathcal{D}$ by $\mathcal{D}_j$, $j = 0, 1, ..., N-1$. Furthermore, for $j = 0, 1, ..., N-1$, label the boundary of $\mathcal{D}_j$ by $\partial\mathcal{D}_j$. Let $\mathcal{L}$ denote the outer boundary of $\mathcal{D}$. Label the four sides of $\mathcal{L}$ by $\mathcal{L}_j$, $j = 1, ..., 4$, where these are arranged in this order in the clockwise direction around $\mathcal{D}$, and $\mathcal{L}_1$ maps onto $\mathcal{L}_3$ under $z \mapsto z + \lambda_2$, and $\mathcal{L}_2$ maps onto $\mathcal{L}_4$ under $z \mapsto z + \lambda_1$. Of course, to a large extent, the precise shape of $\mathcal{L}$ may be chosen arbitrarily (although obviously there are choices for it that are more natural than others when Λ is symmetrical). Furthermore, it is not necessary to specify the precise shape of $\mathcal{L}$ in order to determine a parameterisation of Λ as a whole. We shall not specify the precise shape of $\mathcal{L}$ *a priori*. Instead, this will be fixed automatically by the parameterisation that we are ultimately going to construct. In addition, let $\partial\mathcal{D}$ denote the complete boundary of $\mathcal{D}$, i.e., $\partial\mathcal{D} = \bigcup_{j=0}^{N-1} \partial\mathcal{D}_j \cup \mathcal{L}$, where $\mathcal{L} = \bigcup_{j=1}^{4} \mathcal{L}_j$.

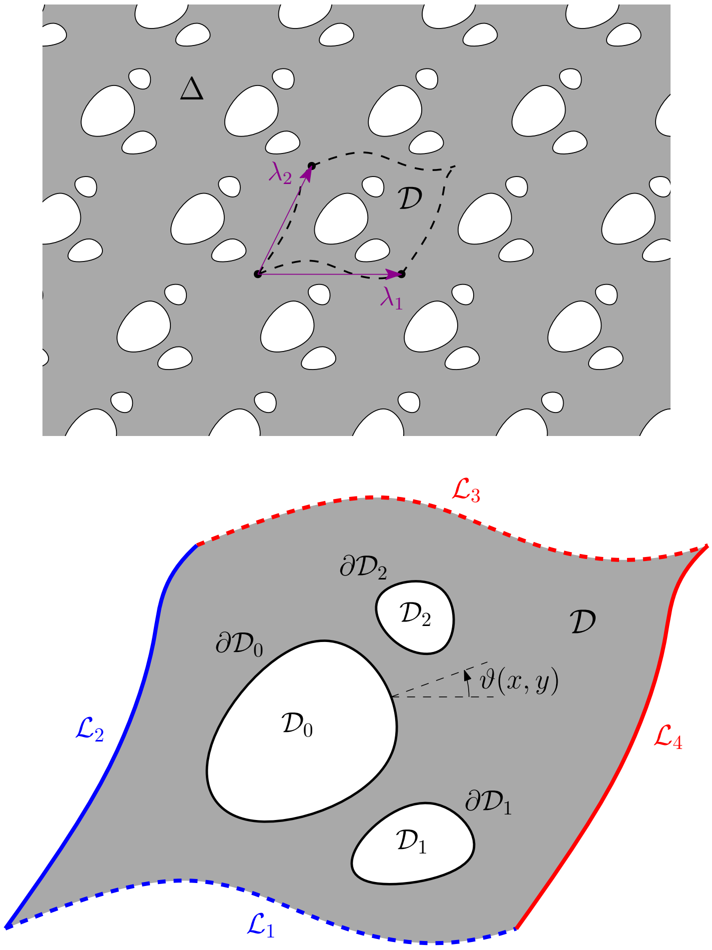


Fig. 1 A section of an infinite domain, Δ, that contains a doubly-periodic lattice of holes with periods λ_1 and λ_2. $\mathcal{D}$ is a single period cell of Δ.

Now suppose that our perforated plate is in a state of plane stress. We denote the components of the corresponding stress field as follows. Let $\sigma_x(x, y)$ and $\sigma_y(x, y)$ denote the normal stresses in the x- and y-directions, respectively, and $\tau_{xy}(x, y)$ denote the shearing stress at a point $(x, y) \in \overline{\Delta}$. Furthermore, let $\sigma_n(x, y)$ and $\tau_{nt}(x, y)$ denote the traction components in the normal and tangential directions, respectively, at a point $(x, y) \in \partial\Delta$, where we mean the normal direction pointing into Δ. In addition, let $\sigma_t(x, y)$ denote the traction component in the tangential normal direction - i.e., the hoop stress - at a point $(x, y) \in \partial\Delta$. These components are related by

$$\sigma_t(x, y) + \sigma_n(x, y) = \sigma_x(x, y) + \sigma_y(x, y), \tag{2.1a}$$

$$\sigma_t(x, y) - \sigma_n(x, y) + 2\mathrm{i}\tau_{nt}(x, y) = \mathrm{e}^{2\mathrm{i}\vartheta(x,y)}\left(\sigma_y(x, y) - \sigma_x(x, y) + 2\mathrm{i}\tau_{xy}(x, y)\right), \tag{2.1b}$$

for points $(x, y) \in \partial\Delta$, where $\vartheta(x, y)$ denotes the angle that the normal direction to $\partial\Delta$ (pointing into Δ) at a point $(x, y) \in \partial\Delta$ makes with the positive x-axis (see Fig. 1).

As in (**1**), suppose that uniform loadings are applied in the normal and tangential directions along $\partial\Delta$, i.e.,

$$\sigma_n(x, y) = p, \quad \tau_{nt}(x, y) = \tau \quad \text{for } (x, y) \in \partial\Delta, \tag{2.2}$$

for some constants p and τ. Then, the holes of the doubly-periodic lattice Λ are said to be *equally-strong* if (and only if) the induced hoop stress, $\sigma_t(x, y)$, takes the same constant value along all of their boundaries, i.e.,

$$\sigma_t(x, y) = \sigma \quad \text{for all } (x, y) \in \partial\Delta, \tag{2.3}$$

for some constant σ. In this paper, as in (**1**), we shall also assume the stress field to be bounded at all points in Δ, including at infinity. As stated in section 1, our aim is to construct a parameterisation of doubly-periodic lattices of equally-strong holes.

As in (**1**), we will make use of the Kolosov-Muskhelishvili complex potentials (e.g., see Muskhelishvili (**16**)), denoted here by $\Phi(z)$ and $\Psi(z)$. $\Phi(z)$ and $\Psi(z)$ are analytic for all $z \in \Delta$, except possibly at infinity. They are related to the components of the stress field by

$$\sigma_x(x, y) + \sigma_y(x, y) = 4\mathrm{Re}\{\Phi'(z)\}, \tag{2.4a}$$

$$\sigma_y(x, y) - \sigma_x(x, y) + 2\mathrm{i}\tau_{xy}(x, y) = 2\left(\overline{z}\Phi''(z) + \Psi'(z)\right) \tag{2.4b}$$

for all $z \in \overline{\Delta}$, where here and throughout this paper we use the notation $'$ with respect to a function to denote its derivative.

2.1 *Formulation in terms of the Schwarz function*

As stated in section 1, in this paper (as in (**1**)) we restrict ourselves to the consideration of lattices whose holes have boundaries that are smooth analytic curves (so they contain no corners or cusps). Then each of these boundaries possesses a Schwarz function. We shall now derive a characterisation of doubly-periodic lattices of equally-strong holes that identifies them in terms of properties of these Schwarz functions. For $j = 0, 1, ..., N-1$, let $\mathcal{S}_j(z)$ denote the Schwarz function of $\partial\mathcal{D}_j$.

PROPOSITION 2.1. Λ is a doubly-periodic lattice of equally-strong holes if and only if, for all $z \in \mathcal{D}$,

(i) $\mathcal{S}_0(z)$ is analytic and single-valued,

(ii)

$$\mathcal{S}_0(z+\lambda_j) = \mathcal{S}_0(z) + \overline{\mu_j} \quad \text{for } j = 1, 2, \tag{2.5}$$

for some complex constants μ_j, $j = 1, 2$,

(iii) and when $N > 1$,

$$\mathcal{S}_j(z) = \mathcal{S}_0(z) + \overline{\gamma_j} \quad \text{for } j = 1, ..., N-1, \tag{2.6}$$

for some complex constants γ_j, $j = 1, ..., N-1$.

Proof. First, suppose that Λ is a lattice of equally-strong holes. It follows from (2.1a), (2.2), (2.3) and (2.4a) that

$$4\mathrm{Re}\{\Phi'(z)\} = \sigma + p \tag{2.7}$$

for $z \in \partial\Delta$. But for all finite $z \in \Delta$, $\mathrm{Re}\{\Phi'(z)\}$ is harmonic since $\Phi(z)$ and hence $\Phi'(z)$ are analytic. Furthermore, since the stress field must be bounded for all $z \in \Delta$ including at infinity, it follows from (2.4a) that $\mathrm{Re}\{\Phi'(z)\}$ must also be bounded at infinity. One may then deduce that (2.7) must in fact hold for all $z \in \Delta$ (including at infinity). It then follows that $\Phi'(z) = (\sigma + p + \mathrm{i}c_1)/4$ for $z \in \Delta$, for some real constant c_1, and hence that

$$\Phi(z) = \frac{(\sigma + p + \mathrm{i}c_1)}{4} z + c_2 \quad \text{for } z \in \Delta, \tag{2.8}$$

for some complex constant c_2. The values of c_1 and c_2 do not affect the stress field (as follows from (2.4)) and are not needed hereafter, so we will not specify them.

Now, given (2.8), it follows from (2.1b), (2.2), (2.3) and (2.4b) that

$$\Psi'(z) = a\mathrm{e}^{-2\mathrm{i}\vartheta(x,y)} \quad \text{for } z \in \partial\Delta, \tag{2.9}$$

where a is the constant

$$a = \frac{\sigma - p}{2} + \mathrm{i}\tau. \tag{2.10}$$

But the unit tangent at a point z along $\partial\Delta$, pointing in the direction with the interior of Δ on the left, is given by $\mathrm{d}z/\mathrm{d}s$, where s denotes arc length measured along $\partial\Delta$ with the interior of Δ on the left. It follows that the unit normal pointing into Δ at this point is $\mathrm{i}\mathrm{d}z/\mathrm{d}s$. Hence

$$\mathrm{e}^{-2\mathrm{i}\vartheta(x,y)} = -\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^{-2} \quad \text{for } z \in \partial\Delta. \tag{2.11}$$

Now consider the boundary of just one of the holes in Δ, say $\partial\mathcal{D}_0$. From the general theory of the Schwarz function (e.g., see Davis (**9**)), we have

$$\mathcal{S}_0'(z) = \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^{-2} \quad \text{for } z \in \partial\mathcal{D}_0. \tag{2.12}$$

Hence, combining (2.11) and (2.12), it follows from (2.9) that

$$\Psi'(z) = -a\mathcal{S}_0'(z) \tag{2.13}$$

for $z \in \partial\mathcal{D}_0$. But $\Psi(z)$ is analytic for all finite $z \in \Delta$ and $\mathcal{S}_0(z)$ is analytic in a neighbourhood of $\partial\mathcal{D}_0$. Hence one may deduce by analytic continuation that (2.13) must in fact hold for all $z \in \Delta$, and hence that

$$\Psi(z) = -a\mathcal{S}_0(z) + c_0 \quad \text{for } z \in \Delta, \tag{2.14}$$

for some complex constant c_0. Hence $\mathcal{S}_0(z)$ must be analytic for all $z \in \Delta$, except possibly at infinity. (We also point out that, since the stress field must be bounded at infinity, it follows from (2.4b) and (2.8) that so must be $\Psi'(z)$ and hence also $\mathcal{S}_0'(z)$.) Furthermore, now let $u_x(x, y)$ and $u_y(x, y)$ denote the displacements in the x- and y-directions, respectively, associated with the stress field in Δ. Then (e.g., see (**16**, section 32)),

$$u_x(x, y) + \mathrm{i}u_y(x, y) = \frac{1}{2\mu}\left(\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)}\right), \tag{2.15}$$

where μ is the shear modulus and κ is defined in terms of Poisson's ratio ν by $\kappa = (3 - \nu)/(1 + \nu)$. These displacements are necessarily single-valued in Δ. Thus, given (2.8) and (2.14), it follows from (2.15) that $\mathcal{S}_0(z)$ must be single-valued in Δ.

Now, by the same reasoning as that which lead us to (2.14), one may deduce that the same relationship must hold between $\Psi(z)$ and the Schwarz function of the boundary of any other hole in Δ, up to a possibly different additive constant instead of c_0. Hence the Schwarz functions of the boundaries of all of the holes in Δ must be identical up to just additive constants. In particular, for $N > 1$, (2.6) must hold for $z \in \Delta$. But furthermore, for $j = 1, 2$, label the image of $\mathcal{D}$ under the translation $z \mapsto z + \lambda_j$ by $\mathcal{D}_{\lambda_j}$. Also, label the image of $\partial\mathcal{D}_0$ under this translation by $\partial\mathcal{D}_{0,\lambda_j}$, and the Schwarz function of $\partial\mathcal{D}_{0,\lambda_j}$ by $\mathcal{S}_{0,\lambda_j}(z)$. Then, for $j = 1, 2$, we must also have

$$\mathcal{S}_{0,\lambda_j}(z) = \mathcal{S}_0(z) + \overline{\lambda_j} - \overline{\mu_j}, \tag{2.16}$$

for $z \in \Delta$, for some complex constant μ_j. But for $z \in \partial\mathcal{D}_0$, we have $z + \lambda_j \in \partial\mathcal{D}_{0,\lambda_j}$, and thus it follows from (1.1) that

$$\mathcal{S}_{0,\lambda_j}(z + \lambda_j) = \overline{z + \lambda_j} = \mathcal{S}_0(z) + \overline{\lambda_j}. \tag{2.17}$$

But, by analytic continuation, one may deduce that (2.17) must in fact hold for all $z \in \Delta$. It then follows from (2.16) and (2.17) that (2.5) must also hold for $z \in \Delta$.

Thus, we have proved that if Λ is a lattice of equally-strong holes, then the properties (i)–(iii) of the statement of the proposition all hold (in fact, they all hold for all $z \in \Delta$, except that $\mathcal{S}_0(z)$ might not be bounded at infinity, as mentioned above).

One may prove the converse as follows. Assuming (i)–(iii) of the proposition statement, one may show (largely by simply reversing some of the arguments that were used in the first part of this proof) that $\Phi(z)$ and $\Psi(z)$ as given by (2.8) and (2.14), respectively, define complex potentials of a stress field that satisfies (2.2) and (2.3) (and is bounded at all points in Δ, including at infinity). □

We highlight the fact - demonstrated in the above proof of Proposition 2.1 - that the Schwarz functions of the boundaries of all of the holes of a doubly-periodic lattice of equally-strong holes are identical up to additive constants. One can show that these constants cannot all equal zero; for example, this follows from (2.16) and the fact - which we show in the final paragraph of section 6 - that $(\lambda_j - \mu_j) \neq 0$ for at least one of $j = 1$ and $j = 2$. We have not investigated the full ranges of the possible values of the constants μ_j, $j = 1, 2$, and γ_j, $j = 1, ..., N-1$, that appear in (2.5) and (2.6). Also, the only physical interpretation that we provide for these constants is that which is evident from the relationship (2.14) between the complex potential $\Psi(z)$ and the Schwarz function $\mathcal{S}_0(z)$. Also in section 6, using Proposition 2.1, we make some checks on the balances of forces and moments for a doubly-periodic lattice of equally-strong holes. There, we also derive formulae for some of these forces and moments.

3. Parameterisation by conformal mapping

Given Proposition 2.1, we are now going to construct a parameterisation of a class of doubly-periodic lattices of equally-strong holes in terms of a conformal mapping. (As stated in section 1, we do not claim that this parameterisation describes all such lattices.)

3.1 *The pre-image domain, D*

We shall first describe our pre-image domain, which we denote by D. An example with $N = 3$ is shown in Fig. 2. To begin, first let D' be an $(N+2)$-connected circular domain (by a circular domain, we mean a domain whose boundaries are all circles) that consists of the open unit disc $\mathbb{D}$ (i.e., centred on the origin, of unit radius) with $N+1$ (mutually disjoint) discs excised from its interior. We label the outer unit boundary circle of D' by C_0, $N-1$ of its inner boundary circles by C_j, $j = 1, ..., N-1$, and its two remaining inner boundary circles by C_N and C_{-N}. So the complete boundary of D' is $\partial D' = \bigcup_{j=0}^{N-1} C_j \cup C_{-N} \cup C_N$. For $j = 0, 1, ..., N, -N$, we label the centre and radius of C_j by δ_j and q_j, respectively (so in particular, $\delta_0 = 0$ and $q_0 = 1$). It is with no loss to the range of the class of lattices that are to be described by our parameterisation that we may fix δ_{-N} completely at some point on the real axis, and fix δ_N to be real (for example, we may always do so by the application of an automorphism of $\mathbb{D}$ (see (3.1) below)). For the time being, however, we shall only fix δ_{-N} and δ_N to both be real, and use the remaining degree of freedom that we have at our disposal to assume that the origin is contained strictly in the interior of D'. (We make the latter assumption simply for convenience; shortly, we will be considering the reflection of D' in C_0, and it is convenient to only have to discuss the case when this extends to infinity in all directions.) However, we point out that our parameterisation is still valid if one should wish to take one of the inner boundary circles of D' to contain the origin in its interior (or to pass through the origin) - indeed, we do so for some of the examples that we present in section 4. Furthermore, without loss of generality, we label the circles C_N and C_{-N} such that $\delta_{-N} < \delta_N$.

In addition to the above, we introduce a smooth, simple, open curve ℓ that has endpoints at $\zeta_{\pm N}$ that lie on $C_{\pm N}$, respectively, and which otherwise lies entirely in D'. Furthermore, we take $\zeta_N = \theta_N(\zeta_{-N})$, where $\theta_N(\zeta)$ is to be defined in the next section (see (3.7)). We shall not need to make any further specification of ℓ for the time being. We describe how our choice of ℓ affects our parameterisation in section 4. We also label by ℓ_- and ℓ_+ the sides of ℓ that are on our left and right, respectively, as we traverse it from ζ_{-N} to ζ_N. The introduction of ℓ reduces D' to an $(N+1)$-connected domain, the complete boundary of which is $\partial D = \bigcup_{j=0}^{N-1} C_j \cup L$ where $L = \ell_- \cup C_{-N} \cup \ell_+ \cup C_N$. It is this $(N+1)$-connected domain that we denote by D. Again, see Fig. 2 for an example.

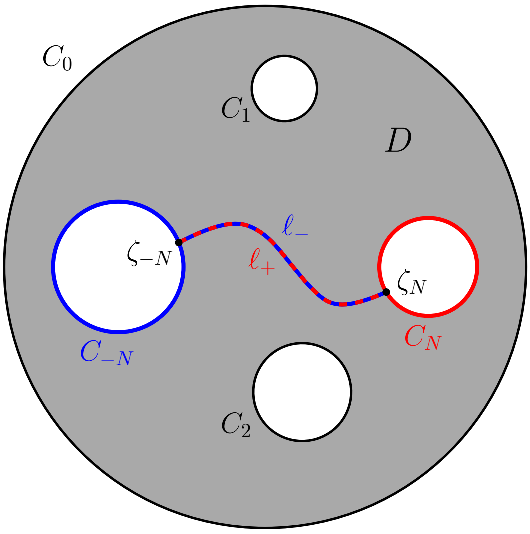


Fig. 2 An example of our pre-image domain D.

3.2 *An associated Schottky group,* Θ

We now introduce the following elementary transformations. First, note that a map $\psi(\zeta)$ is an automorphism of the unit disc $\mathbb{D}$ if and only if

$$\psi(\zeta) = \mathrm{e}^{-\mathrm{i}\alpha}\left(\frac{\beta-\zeta}{1-\overline{\beta}\zeta}\right), \tag{3.1}$$

for some $\alpha \in [0, 2\pi)$ and $\beta \in \mathbb{D}$. Next, for $j = 0, 1, ..., N, -N$, we define the reflection of a point ζ in the circle C_j to be $\varphi_j(\zeta)$ where

$$\varphi_j(\zeta) = \delta_j + \frac{q_j^2}{(\overline{\zeta} - \overline{\delta_j})} \quad \text{for } j = 0, 1, ..., N, -N, \tag{3.2}$$

so that

$$\varphi_j(\zeta) = \zeta \quad \text{for } \zeta \in C_j,\ j = 0, 1, ..., N, -N, \tag{3.3}$$

as one would expect. In particular, for $j = 0$, we have $\varphi_0(\zeta) = 1/\overline{\zeta}$. Note that

$$\varphi_j(\varphi_j(\zeta)) = \zeta \quad \text{for } j = 0, 1, ..., N, -N, \tag{3.4}$$

again as one would expect (as reflection in a circle should be a self-inverse transformation). We also introduce

$$\phi_j(\zeta) = \overline{\varphi_j(\zeta)} \quad \text{for } j = 0, 1, ..., N, -N. \tag{3.5}$$

Then, as follows from (3.3) and (3.5),

$$\phi_j(\zeta) = \overline{\zeta} \quad \text{for } \zeta \in C_j,\ j = 0, 1, ..., N, -N. \tag{3.6}$$

In particular, for $j = 0$, we have $\phi_0(\zeta) = 1/\zeta$.

Using the above transformations, we are now going to generate a special group of Möbius transformations, which we will denote by Θ. First, we seek the most general Möbius transformation that maps C_{-N} onto C_N and the exterior of the former onto the interior of the latter. We shall denote this transformation by $\theta_N(\zeta)$. One may think of $\theta_N(\zeta)$ as the composition of the transformation $\zeta \mapsto (\zeta - \delta_{-N})/q_{-N}$ followed by the automorphism $\psi(\zeta)$ that is given by (3.1), followed by the inversion $\zeta \mapsto 1/\zeta$ (i.e., $\phi_0(\zeta)$), followed finally by the map $\zeta \mapsto (q_N\zeta + \delta_N)$, and check that this gives

$$\theta_N(\zeta) = \delta_N + q_N \mathrm{e}^{\mathrm{i}\alpha} \left(\frac{\overline{\beta}\zeta - (\overline{\beta}\delta_{-N} + q_{-N})}{\zeta - (\delta_{-N} + \beta q_{-N})} \right). \tag{3.7}$$

Next, we define $\theta_{N+1}(\zeta)$ to be the image of ζ under the composition of reflection in C_0, followed by $\theta_N(\zeta)$, followed by reflection in C_0 again, i.e.,

$$\theta_{N+1}(\zeta) = \varphi_0\left(\theta_N\left(\varphi_0(\zeta)\right)\right). \tag{3.8}$$

Using (3.2) and (3.7), one may check that

$$\theta_{N+1}(\zeta) = \frac{(\delta_{-N} + \overline{\beta}q_{-N})\zeta - 1}{\left(\delta_N(\delta_{-N} + \overline{\beta}q_{-N}) + q_N\mathrm{e}^{-\mathrm{i}\alpha}(\beta\delta_{-N} + q_{-N})\right)\zeta - (\delta_N + \beta q_N \mathrm{e}^{-\mathrm{i}\alpha})}. \tag{3.9}$$

Finally, for $j = 1, ..., N-1$, we define $\theta_j(\zeta)$ to be the image of ζ under the composition of reflection in C_0 followed by reflection in C_j. One may check that

$$\theta_j(\zeta) = \varphi_j\left(\varphi_0\left(\zeta\right)\right) = \delta_j + \frac{q_j^2\zeta}{1 - \overline{\delta_j}\zeta} \quad \text{for } j = 1, ..., N-1. \tag{3.10}$$

Now, we label the reflection of D' in C_0 by D'_{-1}. D'_{-1} is the $(N+2)$-connected domain that is bounded by the circles C_0, $C_{-(N+1)}$, $C_{(N+1)}$ and C_{-j}, $j = 1, ..., N-1$, where $C_{-(N+1)}$ and C_{N+1} denote the reflections in C_0 of C_{-N} and C_N, respectively, while for $j = 1, ..., N-1$, C_{-j} denotes the reflection in C_0 of C_j. Furthermore, we introduce the $2(N+1)$-connected region, F, that is defined by

$$F = D' \cup D'_{-1} \cup \bigcup_{j=0}^{N+1} C_j. \tag{3.11}$$

So the boundary of F is

$$\partial F = \bigcup_{j=1}^{N+1} \left(C_{-j} \cup C_j\right), \tag{3.12}$$

but while C_j, $j = 0, 1, ..., N+1$, are contained in F, C_{-j}, $j = 1, ..., N+1$, are not. Note also that, since (by our assumption) the origin is contained in the interior of D', then D'_{-1} and hence F extend to infinity in all directions. See Fig. 3 for an example.

The maps $\theta_j(\zeta)$, $j = 1, ..., N+1$, are Möbius transformations. One may check that $\theta_j(\zeta)$ maps C_{-j} onto C_j and the exterior of C_{-j} onto the interior of C_j (as already stated for $j = N$). Then one may check that the set, Θ, that consists of all unique compositions of the maps $\theta_j(\zeta)$, $j = 1, ..., N+1$, and their inverses, is a (classical) *Schottky group* (e.g., see

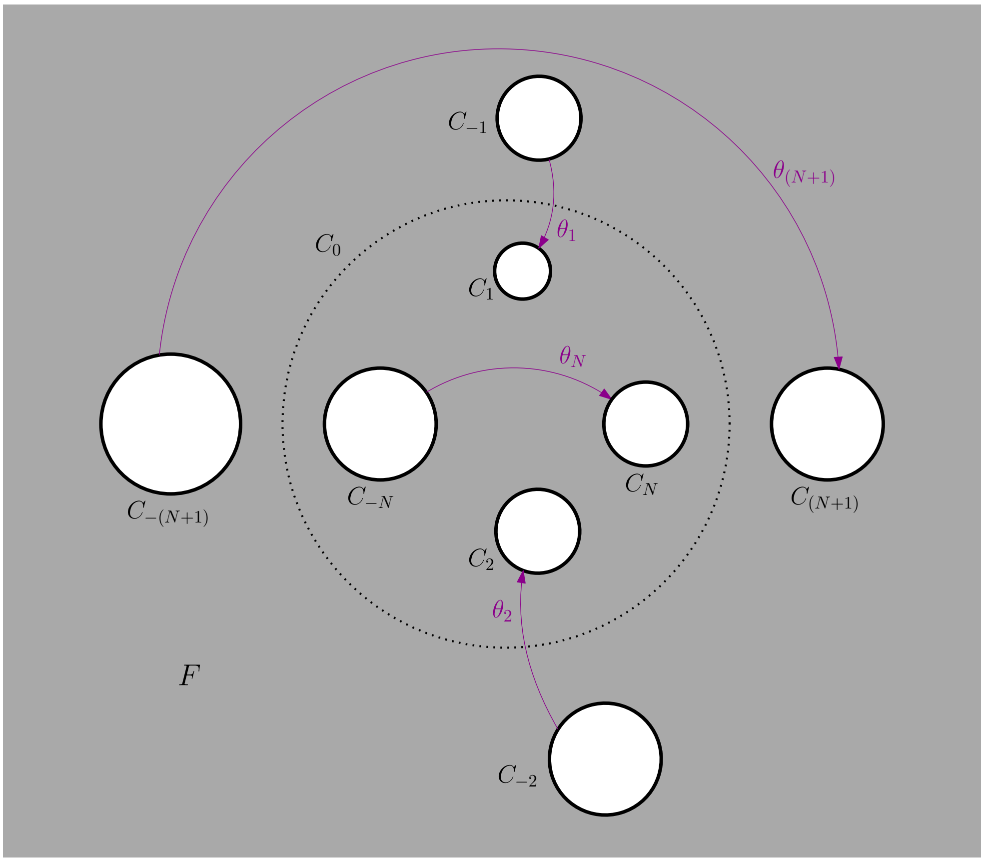


Fig. 3 Sketch of (a section of) the fundamental region, F, that is generated from a domain D of the type shown in Fig. 2 (with $N = 3$). F is bounded by the circles C_j, $j = \pm 1, \pm 2, \pm N, \pm(N+1)$ (C_0 is not a boundary of F) and extends to infinity in all directions. For $j = 1, 2, N, N+1$, C_{-j} is mapped onto C_j by the map $\theta_j(\zeta)$.

Crowdy (**8**), Belokolos *et al* (**14**), Baker (**17**)). We refer to the maps $\theta_j(\zeta)$, $j = 1, ..., N+1$, as the *generators* of Θ. Furthermore, the region F is a *fundamental region* of Θ; one can show that the images of F under all elements of Θ are mutually disjoint and cover the whole of the ζ-plane, the only exception being an infinite set of points known as the *limit points* of Θ (these are the points that can only be reached as the image of a point in F under a composition of infinitely many of the generators of Θ and their inverses).

In addition to the above, for $j = 1, ..., N+1$, we let b_j denote a simple, open curve that lies entirely in F and connects an arbitrary point ζ_{-j} on C_{-j} to the point $\zeta_j = \theta_j(\zeta_{-j})$ on C_j, and does not intersect b_k for $k = 1, ..., N+1, k \neq j$. Introducing these curves b_j, $j = 1, ..., N+1$, reduces F to a $(N+1)$-connected region. In particular, we take b_N to be our curve ℓ. Furthermore, we take b_{N+1} to be the reflection of ℓ in C_0, which we also label by ℓ_{-1}. As will become evident (see section 4), we shall not need to make any further specification of b_j, $j = 1, ..., N-1$.

Furthermore, for $j = 1, ..., N+1$, we use $\theta_{-j}(\zeta)$ and $\theta_j^{-1}(\zeta)$ to both denote the inverse of $\theta_j(\zeta)$. And for $n \in \mathbb{Z}$, we use $\theta_j^n(\zeta)$ to denote the 'n-th power' of $\theta_j(\zeta)$ in the obvious way, e.g., $\theta_j^{-2}(\zeta) = \theta_{-j}^2(\zeta) = \theta_{-j}(\theta_{-j}(\zeta))$, $\theta_N^0(\zeta) = \zeta$, etc. Also, we denote the image of a set of points under a map in the obvious way, e.g. $\theta_j(D)$ denotes the image of D under $\theta_j(\zeta)$.

Finally, we mention briefly that one could transform D by the application of a Möbius transformation that maps C_{-N} and C_N onto a pair of concentric circles, labelled $\hat{C}_{-N}$ and $\hat{C}_N$ respectively, say, centred on the origin and of radius 1 and $\hat{q}$ respectively, for some $0 < \hat{q} < 1$. Then a special case of the map (the analogue of $\theta_N(\zeta)$) that pairs these two circles, is of the form $\hat{\theta}(\zeta) = \hat{q}e^{\mathrm{i}\hat{\alpha}}\zeta$, for $\hat{\alpha} \in \mathbb{R}$. Then, considering in particular $N = 0$, it should be evident how (via a logarithmic transformation) one can make a connection between the functions to be used in this paper (to be introduced next) and elliptic functions (with periods 1 and $(\hat{\alpha} - \mathrm{i}\ln\hat{q})/(2\pi)$, where the latter is purely imaginary only if $\hat{\alpha} = 0$).

3.2.1 The integrals of the first kind

For any Schottky group, there exist what are known as its *integrals of the first kind* (e.g., see (**14**), (**17**)). For our group Θ, we denote these by $\upsilon_j(\zeta)$, $j = 1, ..., N+1$. They are defined by the following properties (i)–(iii):

(i) For $j = 1, ..., N+1$, $\upsilon_j(\zeta)$ is analytic for all $\zeta \in F$.

(ii)
$$\oint_{C_k} \mathrm{d}\upsilon_j(\zeta) = -\oint_{C_{-k}} \mathrm{d}\upsilon_j(\zeta) = \delta_{j,k} \quad \text{for } j, k = 1, ..., N+1, \tag{3.13}$$

where we integrate around $C_{\pm k}$ in the direction with the interior of F on the right (i.e., in the anticlockwise direction), and $\delta_{j,k}$ denotes the Kronecker delta function.

(iii) For all $\zeta \in F$,
$$\upsilon_j(\theta_k(\zeta)) - \upsilon_j(\zeta) = \tau_{j,k} \quad \text{for } j, k = 1, ..., N+1, \tag{3.14}$$

for some constants $\tau_{j,k}$, $j, k = 1, ..., N+1$.

It is evident from (3.13) that $\upsilon_j(\zeta)$, $j = 1, ..., N+1$, are multivalued. Thus we should really state (3.14) for defined branches of $\upsilon_j(\zeta)$, $j = 1, ..., N+1$, in order to make the constants $\tau_{j,k}$, $j, k = 1, ..., N+1$, definite. We shall take these branches to be ones with branch cuts along the curves b_j, $j = 1, ..., N+1$, and their images under all (non-identity) maps in Θ. Equivalently, one could interpret (3.14) as

$$\int_{b_k} \mathrm{d}\upsilon_j(\zeta) = \tau_{j,k} \quad \text{for } j, k = 1, ..., N+1, \tag{3.15}$$

where we integrate along b_k from ζ_{-k} to ζ_k. With such a set of cuts in place so that the constants $\tau_{j,k}$, $j, k = 1, ..., N+1$, are definite, it is known that they possess the following properties. Let $\boldsymbol{\tau}$ denote the square matrix of order $(N+1)$ with $\tau_{j,k}$ the entry in its j-th row and k-th column for $j, k = 1, ..., N+1$. Then $\boldsymbol{\tau}$ is symmetric and its imaginary part is positive definite, i.e.,

$$\tau_{j,k} = \tau_{k,j} \quad \text{for } j, k = 1, ..., N+1, \tag{3.16}$$

and

$$\sum_{j,k=1}^{N+1} x_j x_k \mathrm{Im}\{\tau_{j,k}\} > 0 \quad \text{for all } x_1, ..., x_{N+1} \in \mathbb{R}, \text{ not all zero.} \tag{3.17}$$

We add that properties (i)–(iii) above define $\upsilon_j(\zeta)$, $j = 1, ..., N+1$, for all ζ, not just for $\zeta \in F$. This is because (3.14) provides their continuations to all points that lie outside of F (recall that the images of F under all elements of Θ are mutually disjoint and cover the whole of the ζ-plane, the only exception being the limit points of Θ). One may deduce that $\upsilon_j(\zeta)$, $j = 1, ..., N+1$, are analytic for all ζ (including at infinity if F extends to infinity) except for essential singularities at the limit points of Θ. Additional properties of $\upsilon_j(\zeta)$, $j = 1, ..., N+1$, and $\tau_{j,k}$, $j, k = 1, ..., N+1$, that are specific to the particular Schottky group Θ that we have constructed in this paper, are stated by Lemma 7.1 in section 7.

3.3 *The conformal map, $z(\zeta)$, and our class of lattices*

Committing a slight abuse of notation, we now introduce the function $z(\zeta)$ which we define by

$$z(\zeta) = -\lambda_2 \upsilon_N(\zeta) + \mu_2 \upsilon_{N+1}(\zeta) + c, \tag{3.18}$$

where c is a (complex) constant. It follows from (7.1) that one may also write (3.18) as

$$z(\zeta) = -\lambda_2 \upsilon_N(\zeta) - \mu_2 \overline{\upsilon_N}\left(\zeta^{-1}\right) + \hat{c}, \tag{3.19}$$

where $\hat{c}$ is another constant.

Lemma 3.1. $z(\zeta)$ is analytic for all $\zeta \in F$ and possesses the properties

$$\oint_{C_N} \mathrm{d}z(\zeta) = -\oint_{C_{-N}} \mathrm{d}z(\zeta) = -\lambda_2, \tag{3.20a}$$

$$\oint_{C_{N+1}} \mathrm{d}z(\zeta) = -\oint_{C_{-(N+1)}} \mathrm{d}z(\zeta) = \mu_2, \tag{3.20b}$$

$$z(\theta_N(\zeta)) = z(\zeta) + \lambda_1, \tag{3.21a}$$

$$z(\theta_{N+1}(\zeta)) = z(\zeta) + \mu_1, \tag{3.21b}$$

with

$$\lambda_1 = -\lambda_2 \tau_{N,N} + \mu_2 \tau_{N+1,N}, \tag{3.22a}$$

$$\mu_1 = -\lambda_2 \tau_{N+1,N} - \mu_2 \overline{\tau_{N,N}}, \tag{3.22b}$$

where in (3.20) we integrate around $C_{\pm N}$ and $C_{\pm(N+1)}$ in the anticlockwise direction. Furthermore, when $N > 1$,

$$\oint_{C_j} \mathrm{d}z(\zeta) = -\oint_{C_{-j}} \mathrm{d}z(\zeta) = 0 \quad \text{for } j = 1, ..., N-1, \tag{3.23}$$

$$z\left(\theta_j(\zeta)\right) = z(\zeta) + \gamma_j \quad \text{for } j = 1, ..., N-1, \tag{3.24}$$

with

$$\gamma_j = -\lambda_2 \tau_{j,N} + \mu_2 \overline{\tau_{j,N}}. \tag{3.25}$$

Proof. It follows from property (i) of the integrals of the first kind listed above, that $z(\zeta)$ is analytic for all $\zeta \in F$. Next, it follows from (3.13) that $z(\zeta)$ also possesses the properties (3.20) and (3.23). Furthermore, it follows from (3.14) that

$$z(\theta_j(\zeta)) = z(\zeta) - \lambda_2 \tau_{N,j} + \mu_2 \tau_{N+1,j} \quad \text{for } j = 1, ..., N+1. \tag{3.26}$$

Then (3.21), (3.22), (3.24) and (3.25) follow from (3.26) given (3.16), (7.2a) and (7.2c). □

Evidently, given the properties (3.20), $z(\zeta)$ is multivalued. However, as one may deduce from our earlier remarks on the integrals of the first kind (see the paragraph that follows our statement of their properties (i)–(iii)), $z(\zeta)$ is single-valued when ζ is restricted to our pre-image domain D (ℓ is a branch cut of $z(\zeta)$, as is ℓ_{-1} and all images of ℓ and ℓ_{-1} under all (non-identity) maps in Θ). Furthermore, in view of (3.21) and (3.24), $z(\zeta)$ is evidently quasi-automorphic with respect to Θ.

We now take $\mathcal{D}$ to be the image of D under $z(\zeta)$. Furthermore, for $j = 0, 1, ..., N-1$, we take $\partial\mathcal{D}_j$ to be the image of C_j. We also take $\mathcal{L}_j$, $j = 1, ..., 4$, to be the images of ℓ_-, C_{-N}, ℓ_+ and C_N, respectively, so that $\mathcal{L}$ is the image of L (we have attempted to highlight these correspondences in Figs. 1 and 2). As we will now demonstrate, it is due to the properties (3.20a) and (3.21a) of $z(\zeta)$ that $\mathcal{D}$ is a period cell of the doubly-periodic domain exterior to a lattice of holes with periods λ_1 and λ_2, while it is a consequence of the additional properties (3.20b), (3.21b) and (3.24) that this lattice is in fact one of equally-strong holes. The only caveat to this is that we must also assume the additional conditions on $z(\zeta)$ that are stated in our next proposition.

Proposition 3.2. Suppose that $z(\zeta)$ is univalent for $\zeta \in D$ and that $z'(\zeta) \neq 0$ for all $\zeta \in L$. Then $\mathcal{D}$ is a period cell of a doubly-periodic domain with periods λ_1 and λ_2.

Proof. Since $z(\zeta)$ is analytic, univalent and also bounded for all $\zeta \in D$ (one may deduce the last of these properties from the properties of the integrals of the first kind - see section 3.2.1), it follows from the general theory of conformal mapping that $\mathcal{D}$ is a bounded (i.e., it does not extend to infinity) $(N+1)$-connected domain. One may show that $\mathcal{L}$ is the outer boundary of $\mathcal{D}$ as follows. Consider the integral $\oint_L \mathrm{d}\log z'(\zeta)$, where we integrate around L in the clockwise direction, i.e., with the interior of D on the left. Since $z'(\zeta) \neq 0$ for $\zeta \in L$ (and $z(\zeta)$ is bounded), it follows that this integral is well-defined. We shall evaluate it as follows. First, recall that, as stated in section 3.2, C_N is the image of C_{-N} under $\theta_N(\zeta)$, and that $\theta_N(\zeta)$ maps the exterior of C_{-N} onto the interior of C_N. It follows that $\theta_N(\zeta)$ traverses C_N in the clockwise direction as ζ traverses C_{-N} in the anticlockwise direction. Thus

$$\oint_{C_N} \mathrm{d}\log z'(\zeta) = -\oint_{C_{-N}} \mathrm{d}\log z'(\theta_N(\zeta)), \tag{3.27}$$

where we integrate around $C_{\pm N}$ in the clockwise direction. But it follows from (3.21a) that

$$z'(\theta_N(\zeta)) = \frac{z'(\zeta)}{\theta'_N(\zeta)}. \tag{3.28}$$

Furthermore, since $\theta_N(\zeta)$ maps the interior of C_{-N} onto the exterior of C_N, one may also deduce (from the general properties of Möbius maps) that the only singularity of $\theta_N(\zeta)$ is a simple pole that lies in the interior of C_{-N}. Hence the only singularity of $\theta'_N(\zeta)$ is a double pole that lies in the interior of C_{-N}. Furthermore, the only zero of $\theta'_N(\zeta)$ is a double zero at infinity. Then, it follows from the Argument Principle that

$$\frac{1}{2\pi\mathrm{i}}\oint_{C_{-N}} \mathrm{d}\log\theta'_N(\zeta) = 2. \tag{3.29}$$

It then follows from (3.27)–(3.29) that

$$\oint_{C_{-N}\cup C_N} \mathrm{d}\log z'(\zeta) = 4\pi\mathrm{i}. \tag{3.30}$$

Furthermore, it follows from (3.20a) that

$$\int_{\ell_-\cup\ell_+} \mathrm{d}\log z'(\zeta) = 0, \tag{3.31}$$

where we integrate along ℓ_- from ζ_{-N} to ζ_N, and along ℓ_+ in the opposite direction. Then, combining (3.30) and (3.31) gives

$$\oint_L \mathrm{d}\log z'(\zeta) = 4\pi\mathrm{i}. \tag{3.32}$$

But then, expanding the integral on the left-hand side of (3.32) gives

$$\oint_L (\mathrm{d}\arg\{\mathrm{d}z(\zeta)\} - \mathrm{d}\arg\{\mathrm{d}\zeta\}) = 4\pi. \tag{3.33}$$

And it is straightforward to check that

$$\oint_L \mathrm{d}\arg\{\mathrm{d}\zeta\} = -2\pi. \tag{3.34}$$

Hence, combining (3.33) and (3.34), one arrives at

$$\oint_L \mathrm{d}\arg\{\mathrm{d}z(\zeta)\} = 2\pi, \tag{3.35}$$

i.e., the total change in $\arg\{\mathrm{d}z(\zeta)\}$ after ζ has completed a circuit of L in the clockwise direction is 2π (rather than -2π). Hence one may deduce that $\mathcal{L}$ is the outer boundary of $\mathcal{D}$. (We also point out that, even though $z'(\zeta) \neq 0$ for $\zeta \in L$, both $\mathrm{d}z(\zeta)$ and $\mathrm{d}\zeta$ will vanish at $\zeta_{\pm N}$ if ℓ is tangential to $C_{\pm N}$, respectively, at these points. In such cases, $\arg\{\mathrm{d}z(\zeta)\}$ is obviously not continuous along L and so the integral in (3.35) is not well-defined. However, one may still deduce that the total change in $\arg\{\mathrm{d}z(\zeta)\}$ after ζ has completed a circuit of L in the clockwise direction is 2π. We omit further details here.)

Next recall that $\mathcal{L}_1$ and $\mathcal{L}_3$ are the images under $z(\zeta)$ of ℓ_- and ℓ_+, respectively. Then, since $z(\zeta)$ possesses the property (3.20a), it follows immediately that $\mathcal{L}_3$ is the translation of $\mathcal{L}_1$ by λ_2. Furthermore, $\mathcal{L}_2$ and $\mathcal{L}_4$ are the images under $z(\zeta)$ of C_{-N} and C_N, respectively. And C_N is the image of C_{-N} under $\theta_N(\zeta)$. Then, since $z(\zeta)$ also possesses the property (3.21a), it also follows that $\mathcal{L}_4$ is the translation of $\mathcal{L}_2$ by λ_1. One may then deduce from the general theory of tessellations that $\mathcal{D}$ is a period cell of a doubly-periodic domain with periods λ_1 and λ_2. □

We point out that it follows from (3.20a) that the image of D under the analytic continuation of $z(\zeta)$ around a circuit of C_N in the clockwise direction (or equivalently, around a circuit of C_{-N} in the anticlockwise direction) - i.e., the continuation of $z(\zeta)$ out of D across ℓ, from the side of ℓ_+ to the side of ℓ_- - is the translation of $\mathcal{D}$ by λ_2. And it follows from (3.21a) that the image of $\theta_N(D)$ under $z(\zeta)$ is the translation of $\mathcal{D}$ by λ_1. By extension, one may deduce how, by further continuation of $z(\zeta)$, one may map out (in a one-to-one manner) the whole of the doubly-periodic domain - say Δ - of which $\mathcal{D}$ is a single period cell. In particular, the pre-image of the whole of Δ consists of $\bigcup_{m\in\mathbb{Z}}\theta_N^m(\overline{D})$, plus all copies of this union (where, by a 'copy', we mean a region that is reached by continuation across $\theta_N^m(\ell)$, for some $m\in\mathbb{Z}$).

Evidently, we have yet to prove that $z(\zeta)$ is univalent for $\zeta\in D$ and that $z'(\zeta)\neq 0$ for all $\zeta\in L$. We shall address these conditions again after the statement of our next proposition, 3.3. We point out, however, that we do not need to assume these conditions in order to prove this next proposition. Before stating this result, we point out that, by construction, $\partial\mathcal{D}_j$, $j=0,1,...,N-1$, are analytic curves. Then, as in section 2.1, for $j=0,1,...,N-1$, we let $\mathcal{S}_j(z)$ denote the Schwarz function of $\partial\mathcal{D}_j$. Furthermore, we introduce the functions $S_j(\zeta)$, $j=0,1,...,N-1$, which we define by

$$S_j(\zeta)=\mathcal{S}_j(z(\zeta))\quad\text{for } j=0,1,...,N-1. \tag{3.36}$$

Proposition 3.3. For all $z\in\mathcal{D}$, $\mathcal{S}_0(z)$ possesses the properties (i)–(iii) of Proposition 2.1.

Proof. It follows from (3.36), (1.1) and the facts that $\partial\mathcal{D}_0$ is the image under $z(\zeta)$ of the unit circle C_0 and $\overline{\zeta}=1/\zeta$ for $\zeta\in C_0$, that

$$S_0(\zeta)=\overline{z}\left(\zeta^{-1}\right) \tag{3.37}$$

for $\zeta\in C_0$, where we define $\overline{z}(\zeta)\equiv\overline{z(\overline{\zeta})}$. But since $z(\zeta)$ is analytic for all $\zeta\in F$, it follows from the fact that the interior of F is mapped onto itself by reflection in C_0, as well as the Cauchy-Riemann relations, that $\overline{z}(1/\zeta)$ is also analytic for all $\zeta\in F$. Hence, by analytic continuation, one may deduce that (3.37) must in fact hold for all $\zeta\in F$. It follows that $S_0(\zeta)$ is analytic for all $\zeta\in D$, and hence that $\mathcal{S}_0(z)$ is analytic for all $z\in\mathcal{D}$. Next, since $z(\zeta)$ is single-valued in the region that is formed from F by the introduction of boundaries along ℓ and ℓ_{-1}, then so is $\overline{z}(1/\zeta)$ and hence also $S_0(\zeta)$. One may then deduce that $\mathcal{S}_0(z)$ is also single-valued for $z\in\mathcal{D}$.

Next note that, as ζ traverses $C_{\pm(N+1)}$ in the anticlockwise direction, so $1/\overline{\zeta}$ traverses $C_{\pm N}$, respectively, in the clockwise direction. Then it follows from (3.20b) and (3.37) that

$$\oint_{C_N}\mathrm{d}S_0(\zeta)=-\oint_{C_{-N}}\mathrm{d}S_0(\zeta)=\overline{\mu_2}. \tag{3.38}$$

But it follows from (3.20a) that as ζ completes a circuit of $C_{\pm N}$ in the clockwise direction, so $z(\zeta)$ is translated by $\pm\lambda_2$, respectively. Thus it follows from (3.38) that (2.5) holds for $j=2$. Furthermore, it follows from (3.21b) and (3.37), together with (3.8) and the fact that $\varphi_0(\zeta) = 1/\overline{\zeta}$, that

$$\begin{aligned} S_0(\theta_N(\zeta)) &= \overline{z\left(\varphi_0\left(\theta_N(\zeta)\right)\right)} \\ &= \overline{z\left(\theta_{N+1}\left(\varphi_0(\zeta)\right)\right)} \\ &= \overline{z\left(\varphi_0(\zeta)\right)} + \overline{\mu_1} \\ &= S_0(\zeta) + \overline{\mu_1}. \end{aligned} \tag{3.39}$$

It then follows from (3.21a) and (3.39) that (2.5) also holds for $j = 1$.

Finally, if $N > 1$, using arguments analogous to those that we used to deduce (3.37), one can show that

$$S_j(\zeta) = \overline{z}\left(\phi_j(\zeta)\right) \quad \text{for } j = 1, ..., N-1. \tag{3.40}$$

Then, it follows from (3.24) and (3.40), along with (3.5), the first identity in (3.10) and (3.37), that

$$S_j(\zeta) = \overline{z\left(\theta_j(\varphi_0(\zeta)\right)} = \overline{z}\left(\zeta^{-1}\right) + \overline{\gamma_j} = S_0(\zeta) + \overline{\gamma_j} \quad \text{for } j = 1, ..., N-1. \tag{3.41}$$

It then follows from (3.36) and (3.41) that (2.6) also holds. □

It follows from Propositions 2.1, 3.2 and 3.3, that our parameterisation describes a class of doubly-periodic lattices of equally-strong holes *provided* $z(\zeta)$ is univalent for $\zeta \in D$ and $z'(\zeta) \neq 0$ for all $\zeta \in L$ (as required by Proposition 3.2). We conjecture the following:

CONJECTURE 3.4. $z(\zeta)$ is univalent for $\zeta \in D$ and $z'(\zeta) \neq 0$ for all $\zeta \in L$ if and only if

$$\left|\frac{\mu_2}{\lambda_2}\right| \leqslant 1. \tag{3.42}$$

In other words, we conjecture that our parameterisation describes a class of doubly-periodic lattices of equally-strong holes if and only if (3.42) holds. Equivalently, we claim that (3.42) is a necessary and sufficient condition for the existence of the class of doubly-periodic lattices of equally-strong holes that are described by our parameterisation (recall that we do not claim that our parameterisation describes all such lattices). However, full details of a proof of Conjecture 3.4 remain to be resolved and are not presented in this paper. We have only verified this result numerically for specific cases, including for the examples that we present in section 4.

We highlight the fact that (3.42) depends only on the parameters λ_2 and μ_2, which are defined independently of any parameterisation. We also point out that (3.42) does not depend on λ_1 or μ_1. However, our parameterisation automatically fixes λ_1 and μ_1 by (3.22). In fact, for the class of lattices that are described by our parameterisation, one can show that (3.42) is equivalent to the condition $|\mu_1/\lambda_1| \leqslant 1$, as follows. By elementary algebra (making use of (7.2b)), one can show that with λ_1 and μ_1 as given by (3.22), we have

$$|\lambda_1|^2 - |\mu_1|^2 = \left(|\lambda_2|^2 - |\mu_2|^2\right)\left(|\tau_{N,N}|^2 - |\tau_{N+1,N}|^2\right). \tag{3.43}$$

But, using (3.17) (with $x_j = 0$ for $j = 1, ..., N-1$, $x_N = 1$ and $x_{N+1} = \pm 1$), along with (3.16), (7.2a) and (7.2b), one can show that

$$|\tau_{N+1,N}| < \mathrm{Im}\{\tau_{N,N}\}. \tag{3.44}$$

It then follows from (3.43) and (3.44) that in fact $|\mu_2/\lambda_2| < 1 \iff |\mu_1/\lambda_1| < 1$ and $|\mu_2/\lambda_2| = 1 \iff |\mu_1/\lambda_1| = 1$.

This completes the description of our parameterisation.

3.3.1 A limiting case

As a check on our parameterisation - including our conjectured condition (3.42) for existence - we shall now demonstrate that it retrieves as a limiting case the parameterisation that is presented in (**7**), which describes sets of a finite number of equally-strong holes (in an infinite plane). The holes in the period cell $\mathcal{D}$ should coincide with such a set in the limit as $|\lambda_1|$ and $|\lambda_2|$ tend simultaneously to infinity whilst the sizes and locations of the holes in $\mathcal{D}$ remain finite. It seems reasonable to expect that our parameterisation should describe this limit as q_{-N}, q_N and $|\delta_N - \delta_{-N}|$ all tend to 0, and thus in this case, reduce to the parameterisation of (**7**). We shall now sketch a demonstration of this.

Let us begin by considering the limit as just q_{-N} and $q_N \to 0$ (so for the time being we keep $|\delta_N - \delta_{-N}|$ to be of order 1). In this limit, $C_{\pm N}$ shrinks to the point at $\delta_{\pm N}$, respectively. Then, recalling (3.13), one may deduce that in this limit

$$\upsilon_N(\zeta) \sim \pm\frac{1}{2\pi\mathrm{i}}\log(\zeta - \delta_{\pm N}) + \mathcal{O}(1) \tag{3.45}$$

for ζ in some annular neighbourhood of $\delta_{\pm N}$, respectively. Let us now also let $|\delta_N - \delta_{-N}| \to 0$. More specifically, let us fix $\delta_{-N} = \zeta_\infty$ where $\zeta_\infty \neq 0$ (recall that we have the freedom to do this, as is mentioned in section 3.1), and let $\delta_N \to \zeta_\infty$. Letting $\varepsilon = \delta_N - \zeta_\infty$ (note that $\varepsilon > 0$ since $\delta_N > \delta_{-N} = \zeta_\infty$), it follows from (3.45) and the series expansion for $\log(1 - (\varepsilon/(\zeta - \zeta_\infty)))$ for $|\varepsilon/(\zeta - \zeta_\infty)| < 1$, that in the limit as q_{-N}, q_N, ε all $\to 0$,

$$\upsilon_N(\zeta) \sim \frac{-\varepsilon}{2\pi\mathrm{i}(\zeta - \zeta_\infty)} + \mathcal{O}(\varepsilon^2) \tag{3.46}$$

for ζ in some annular neighbourhood of ζ_∞. But in this combined limit (i.e., as $q_{-N}, q_N, \varepsilon \to 0$), one may also deduce the following. First, both C_{-N} and C_N shrink to the point at ζ_∞, while $C_{-(N+1)}$ and C_{N+1} shrink to the point at $1/\zeta_\infty$. Thus, both D' and D reduce to the N-connected circular domain that is bounded by the circles C_j, $j = 1, ..., N-1$. This is precisely the type of pre-image domain that is used for the parameterisation that is presented in (**7**). Meanwhile, F reduces to the $2(N-1)$-connected region that is bounded by the circles $C_{\pm j}$, $j = 1, ..., N-1$. Hence, the Schottky group Θ reduces to the group whose generators are just the $N-1$ maps θ_j, $j = 1, ..., N-1$, and their inverses. Then, recalling property (i) of the integrals of the first kind (see section 3.2.1), one may deduce that in this combined limit, $\upsilon_N(\zeta)$ must be analytic and single-valued in this reduced fundamental region except for the simple pole at ζ_∞ that is stated by (3.46). But furthermore, this limit of $\upsilon_N(\zeta)$ must still satisfy (3.14) for $k = 1, ..., N-1$ (for some limiting values of the constants $\tau_{N,k}$). Thus, in the limit as $q_{-N}, q_N, \varepsilon \to 0$, one may deduce that, up to an additive constant,

$$\upsilon_N(\zeta) \to \frac{\varepsilon}{2\pi\mathrm{i}}K(\zeta, \zeta_\infty), \tag{3.47}$$

where the function $K(\zeta, \zeta_\infty)$ is defined in section 3.2 of (**7**). By using similar arguments, one may also show that in this same limit, up to an additive constant,

$$\upsilon_{N+1}(\zeta) \to \frac{-\varepsilon}{2\pi \mathrm{i}\zeta_\infty^2} K(\zeta, \zeta_\infty^{-1}). \tag{3.48}$$

Hence, it follows that in this limit, (3.18) reduces to

$$z(\zeta) = \frac{-\lambda_2 \varepsilon}{2\pi \mathrm{i}} \left(K(\zeta, \zeta_\infty) + \left(\frac{\mu_2}{\lambda_2 \zeta_\infty^2} \right) K(\zeta, \zeta_\infty^{-1}) \right) + c', \tag{3.49}$$

for some constant c'. Then, taking $|\lambda_2| \sim \mathcal{O}(\varepsilon^{-1})$, so that $|\lambda_2| \to \infty$ while $|\lambda_2 \varepsilon| \sim \mathcal{O}(1)$, up to multiplication by a constant (of order 1), (3.49) is identical to the form of the conformal map for the parameterisation that is presented in (**7**) - see equation (3.24) of (**7**) - with

$$\frac{\mu_2}{\lambda_2} = \frac{\overline{b}}{\overline{a}}, \tag{3.50}$$

where the constant a is defined in (**7**) in the same way as we define it here (recall (2.9) and (2.10)), and the constant b is defined by (**7**, equation (2.3b)) (b relates to the corresponding loading at infinity). (Note that, as stated in the first paragraph of section 3 of (**7**), the parameter ζ_∞ that appears in (**7**, equation (3.24)) may be chosen freely, and in particular, may be chosen to be real and non-zero.) Hence, in this limiting case, our parameterisation (both the pre-image domain and the conformal map) for a doubly-periodic lattice of equally-strong holes, agrees with that of (**7**). Furthermore, the necessary and sufficient condition for the existence of finite sets of equally-strong holes that is stated in (**7**) is $|b/a| \leqslant 1$ (see Proposition 4.1 of (**7**)), and this, combined with (3.50), agrees with (3.42). In particular, it follows that in this limit with $N = 1$ - i.e., with just a single hole in each period cell of the lattice - the shape of this hole must tend to an ellipse, as this is the shape of a single equally-strong hole in an infinite plane (e.g., see Cherepanov (**1**)).

Finally, we mention that one could retrieve a parameterisation for a singly-periodic lattice of equally-strong holes by further consideration of the limit as just q_{-N} and $q_N \to 0$ (while $|\delta_N - \delta_{-N}|$ is kept at order 1), although we shall not do so in this paper. In this limit, D is the domain bounded by the circles C_j, $j = 0, 1, ..., N-1$, and the cut ℓ, which now joins the points δ_N and δ_{-N}. (Cherepanov (**1**) also constructed a parameterisation of a singly-periodic lattice of symmetric equally-strong holes.)

3.3.2 Some geometrical properties

We conclude this section by presenting some geometrical properties of the doubly-periodic lattices of equally-strong holes that are described by our parameterisation. To begin, we introduce the function $h(\zeta; \chi)$ which we define by

$$h(\zeta; \chi) = \upsilon_N(\zeta) + \mathrm{e}^{2\mathrm{i}\chi} \overline{\upsilon_N} \left(\zeta^{-1} \right) \quad \text{for } \chi \in \mathbb{R}. \tag{3.51}$$

LEMMA 3.5. For $j = 0, 1, ..., N-1$, $h(\zeta; \chi)$ maps C_j onto a straight slit (i.e., a section of a straight line) of finite length that is aligned at an angle χ to the direction of the positive real axis.

Proof. First, for $\zeta \in C_0$, since $\zeta = 1/\overline{\zeta}$,

$$\overline{h(\zeta;\chi)} = \overline{\upsilon_N}\left(\zeta^{-1}\right) + \mathrm{e}^{-2\mathrm{i}\chi}\upsilon_N(\zeta) = \mathrm{e}^{-2\mathrm{i}\chi}h(\zeta;\chi). \tag{3.52}$$

Hence $h(\zeta;\chi)$ maps C_0 onto a section of the straight line that is aligned at an angle χ to the positive real axis and passes through the origin. Similarly, for $j = 1, ..., N-1$, for $\zeta \in C_j$, recalling (3.3) and the first equality of (3.10), it follows that

$$\begin{aligned}\overline{h(\zeta;\chi)} &= \overline{\upsilon_N\left(\theta_j\left(\overline{\zeta}^{-1}\right)\right)} + \mathrm{e}^{-2\mathrm{i}\chi}\upsilon_N\left(\theta_{-j}(\zeta)\right)\\ &= \overline{\upsilon_N}\left(\zeta^{-1}\right) + \mathrm{e}^{-2\mathrm{i}\chi}\upsilon_N(\zeta) + \overline{\tau_{j,N}} - \mathrm{e}^{-2\mathrm{i}\chi}\tau_{j,N}\\ &= \mathrm{e}^{-2\mathrm{i}\chi}\left(h(\zeta;\chi) - \tau_{j,N}\right) + \overline{\tau_{j,N}}, \end{aligned} \tag{3.53}$$

where the second equality follows from (3.14) and (3.16). Hence for $j = 1, ..., N-1$, $h(\zeta;\chi)$ maps C_j onto a section of the straight line that is aligned at an angle χ to the positive real axis and passes through the point $\tau_{j,N}$. Finally, since $h(\zeta;\chi)$ is also bounded for all $\zeta \in \partial D$ (this follows from the properties of the integrals of the first kind - see section 3.2.1), then the slit images of C_j, $j = 0, 1, ..., N-1$, under $h(\zeta;\chi)$ are all of finite length. □

Proposition 3.6. If $|\mu_2/\lambda_2| = 1$, then for $j = 0, 1, ..., N-1$, $\partial\mathcal{D}_j$ is a straight slit of finite length that is aligned at an angle $\frac{1}{2}\arg\{\lambda_2\mu_2\} + \pi$ to the direction of the positive real axis.

Proof. With $|\mu_2/\lambda_2| = 1$, it follows from (3.19) and (3.51) that $z(\zeta) = -\lambda_2 h(\zeta;\hat{\chi}) + \hat{c}$, where $\hat{\chi} = \frac{1}{2}\arg\{\mu_2/\lambda_2\}$. The result then follows immediately from Lemma 3.5. □

Assuming our Conjecture 3.4, it follows from Proposition 3.6 that certain doubly-periodic arrays of parallel slits form a special class of lattices of equally-strong holes, albeit a degenerate one: it follows from (2.4b), (2.8), (2.9) and (2.11) that the stress fields corresponding to these lattices have singularities at the endpoints of the slits. Recall (see the first paragraph of section 2.1), that we have otherwise been restricting ourselves to lattices whose holes have boundaries that are smooth analytic curves. Some examples of doubly-periodic lattices of parallel slits that we computed using our parameterisation are shown in section 4. We point out that certain arrays of a finite number of parallel slits form a special class of sets of (a finite number) of equally-strong holes (e.g., see (**7**)).

For the rest of this section, we suppose that $|\mu_2/\lambda_2| \neq 1$. It follows from (3.19) and (3.51) (after some elementary algebra) that one may write

$$z(\zeta) = H(\zeta;\chi) + H\left(\zeta;\chi + \frac{\pi}{2}\right) + \hat{c} \tag{3.54}$$

for any $\chi \in \mathbb{R}$, where we define

$$H(\zeta;\chi) = t_1(\chi)h(\zeta;\chi) + \mathrm{i}t_2(\chi)h\left(\zeta;\chi + \frac{\pi}{2}\right) \tag{3.55}$$

with $t_1(\chi)$, $t_2(\chi) \in \mathbb{R}$ defined by

$$t_1(\chi) + \mathrm{i}t_2(\chi) \equiv T(\chi) = -\frac{1}{2}\left(\lambda_2 + \overline{\mu_2}\mathrm{e}^{2\mathrm{i}\chi}\right). \tag{3.56}$$

Note that with $|\mu_2/\lambda_2| \neq 1$, for all $\chi \in \mathbb{R}$ it follows from (3.56) that $T(\chi) \neq 0$, and hence from (3.55) that $H(\zeta;\chi)$ (considered as a function of ζ) is not identically zero.

LEMMA 3.7. For $j = 0, 1, ..., N-1$,

(i) $H(\zeta;\chi)$ maps C_j onto a straight slit of finite length that is aligned at an angle χ to the direction of the positive real axis;

(ii) each point on this slit is the image under $H(\zeta;\chi)$ of two distinct points on C_j, except for the two endpoints of this slit, each of which is the image of just a single point on C_j.

Proof. Property (i) follows immediately from Lemma 3.5. Now let $H'(\zeta;\chi)$ denote the derivative of $H(\zeta;\chi)$ with respect to ζ. For $j = 0, 1, ..., N-1$, $H'(\zeta;\chi)$ has a simple zero at each of the two points on C_j that maps to one of the two endpoints of the slit image of C_j under $H(\zeta;\chi)$. However, it follows from (3.51), (3.55), (3.56) and properties (i) and (iii) of the integrals of the first kind (see section 3.2.1) that $H(\zeta;\chi)$ is analytic for all $\zeta \in F$ (including at infinity) and also quasi-automorphic with respect to Θ. It then follows from Lemma 8.1 that $H'(\zeta;\chi)$ must have exactly $2(N+1)$ zeros in F, at least two of which must be at infinity. Thus, one may deduce that for $j = 0, 1, ..., N-1$, the aforementioned zeros of $H'(\zeta;\chi)$ on C_j (which correspond to the endpoints of the slit image of C_j under $H(\zeta;\chi)$) must be the only zeros of $H'(\zeta;\chi)$ on C_j. Hence follows property (ii). □

Now, following an approach similar to one taken by Garabedian and Schiffer in (**18**), we prove the following result.

PROPOSITION 3.8. If $|\mu_2/\lambda_2| \neq 1$, then for $j = 0, 1, ..., N-1$, $\partial\mathcal{D}_j$ is a convex curve.

Proof. It follows from (3.54) with $\chi = 0$, and (i) of Lemma 3.7, that for $j = 0, 1, ..., N-1$, and $\zeta \in C_j$,

$$\mathrm{Re}\{z(\zeta)\} = \mathrm{Re}\{H(\zeta;0)\} + \hat{a}, \tag{3.57}$$

where $\hat{a}$ is the constant real value of $H(\zeta;\pi/2) + \hat{c}$. It then follows from (3.57) and (ii) of Lemma 3.7, that for $\zeta \in C_j$, each value of $\mathrm{Re}\{z(\zeta)\}$ is attained exactly twice, except for its maximum and minimum values, which are each attained just once. Hence, any vertical line that intersects $\partial\mathcal{D}_j$, does so at exactly two points, except for lines through the two points where $\mathrm{Re}\{z(\zeta)\}$ attains its maximum and minimum values, respectively. But by the same reasoning, other than starting from (3.54) with any value of χ, one may deduce that any straight line (of *any* inclination) that intersects $\partial\mathcal{D}_j$, does so at exactly two points, except for tangent lines, which intersect $\partial\mathcal{D}_j$ at a single point. Hence one may deduce that $\partial\mathcal{D}_j$ is a convex curve. □

We remark that one may deduce from Proposition 3.8 that for $|\mu_2/\lambda_2| \neq 1$, $z(\zeta)$ is univalent for $\zeta \in C_j$, for $j = 0, 1, ..., N-1$. This observation may assist with a possible proof of Conjecture 3.4. We also add that Grabovsky & Kohn (**3**) point out that the boundaries of the inclusions in the doubly-periodic lattice that they parameterise are also convex curves. Furthermore, one can show that the boundaries of equally-strong holes of finite sets are also convex curves (e.g., by applying arguments similar to those used above to the parameterisation that is presented in (**7**)).

4. Examples

In this section we present some examples of doubly-periodic lattices of equally-strong holes that we computed using our parameterisation. Further details of how we performed these computations are provided in section 9.

We first point out that, in order to specify our parameterisation completely, we must specify values for the parameters δ_j and q_j, for $j = 1, ..., N, -N$, and α, β, c, λ_2 and μ_2 (the latter two appear explicitly in our form (3.18) for $z(\zeta)$). However, we must also fix which branch of $z(\zeta)$ that we wish to take, or rather, fix our branch cuts for $z(\zeta)$ (obviously, different branches of $z(\zeta)$ with the same cuts differ only by additive constants). As stated earlier (see the paragraph after the proof of Lemma 3.1), the branch cuts of $z(\zeta)$ consist of ℓ, ℓ_{-1} and the images of both of these curves under all (non-identity) maps in Θ. So these cuts are determined solely by our choice of ℓ (recall that ℓ_{-1} is the reflection of ℓ in C_0). However, one may deduce from (3.18) and the properties of the integrals of the first kind (see section 3.2.1) that the only effect that changing these branch cuts of $z(\zeta)$ has on any lattice that is described by our parameterisation, is to change the corresponding values of $\tau_{j,N}$, $j = 1, ..., N + 1$, and hence those of λ_1, μ_1 and γ_j, $j = 1, ..., N - 1$, as given by (3.22) and (3.25). It then follows that, having specified the parameters δ_j and q_j, for $j = 1, ..., N, -N$, and α, β, c, λ_2 and μ_2, in order to obtain the full range of lattices that are described by our parameterisation for these parameter values, we need only to be able to determine the full range of the possible corresponding values of $\tau_{j,N}$, $j = 1, ..., N + 1$. But it follows from (3.15), taking $k = N$ and recalling that we take b_N to be ℓ, that

$$\tau_{j,N} = \int_{\ell} \mathrm{d}\upsilon_j(\zeta) \quad \text{for } j = 1, ..., N + 1, \tag{4.1}$$

where we integrate along ℓ from its endpoint ζ_{-N} on C_{-N} to its endpoint $\zeta_N = \theta_N(\zeta_{-N})$ on C_N. We have yet to specify ℓ beyond what was said about it in the final paragraph of section 3.1. However, suppose that the values of δ_j and q_j, for $j = 1, ..., N, -N$, and α and β have been specified. Furthermore, suppose that $\hat{\ell}$ is a particular choice for ℓ in this case, and that $\hat{\tau}_{j,N}$ is the corresponding value of $\tau_{j,N}$, for $j = 1, ..., N + 1$. Then it follows from the properties of the integrals of the first kind, that for any other choice of ℓ we have $\tau_{j,N} = \hat{\tau}_{j,N}$ for $j = 1, ..., N - 1, N + 1$, while the set of all possible values of $\tau_{N,N}$ is simply $\{\hat{\tau}_{N,N} + m | m \in \mathbb{Z}\}$. (We omit further details here, except to mention that one must allow for the fact that ℓ may spiral around C_{-N} or C_N or both, and recall in particular the property (3.13).) Hence, having specified the parameters δ_j and q_j, for $j = 1, ..., N, -N$, and α, β, c, λ_2 and μ_2, in order to obtain the full range of the possible corresponding values of $\tau_{j,N}$, $j = 1, ..., N + 1$, we need only to compute these values for just a single choice of the curve ℓ, say $\hat{\ell}$ (as all other possible values of them are trivially related to those that correspond to $\hat{\ell}$, as we have just described). For brevity, we omit details of the specific choice that we made for ℓ in order to compute the examples that we present next. We also point out that since $z(\zeta)$ is constructed in terms of only $\upsilon_N(\zeta)$ and $\upsilon_{N+1}(\zeta)$ and none of the other integrals of the first kind, one may deduce that our parameterisation does not depend on our choice of the curves b_j, $j = 1, ..., N - 1$.

We also point out that for all of the examples presented here, we picked λ_2 and μ_2 to satisfy the condition (3.42).

An example of a 'square' lattice of equally-strong holes computed using our parameterisation with $N = 1$ - i.e., with one hole in each period cell - is shown in Fig. 4. In order to compute this example, we first fixed $\lambda_2 = \mathrm{i}$, $\mu_2 = 0$, $q_{\pm N} = 0.05$ and $\alpha = \beta = 0$. We also fixed $\delta_{-N} = -\delta_N$. We then solved the (now real) constraint (3.22a) numerically (using a Newton-type iteration) for the value of δ_N that gives $\lambda_1 = 1$, and found this to be $\delta_N = 0.582082$ (to 6 decimal places). This gives $\lambda_1 = 1.000000$. From (3.22b), these values also give $\mu_1 = 0.222786$. We also took $c = -0.209266$.

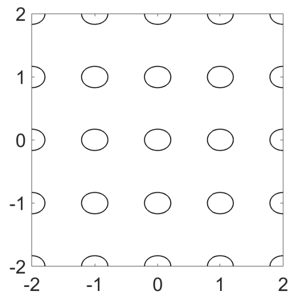
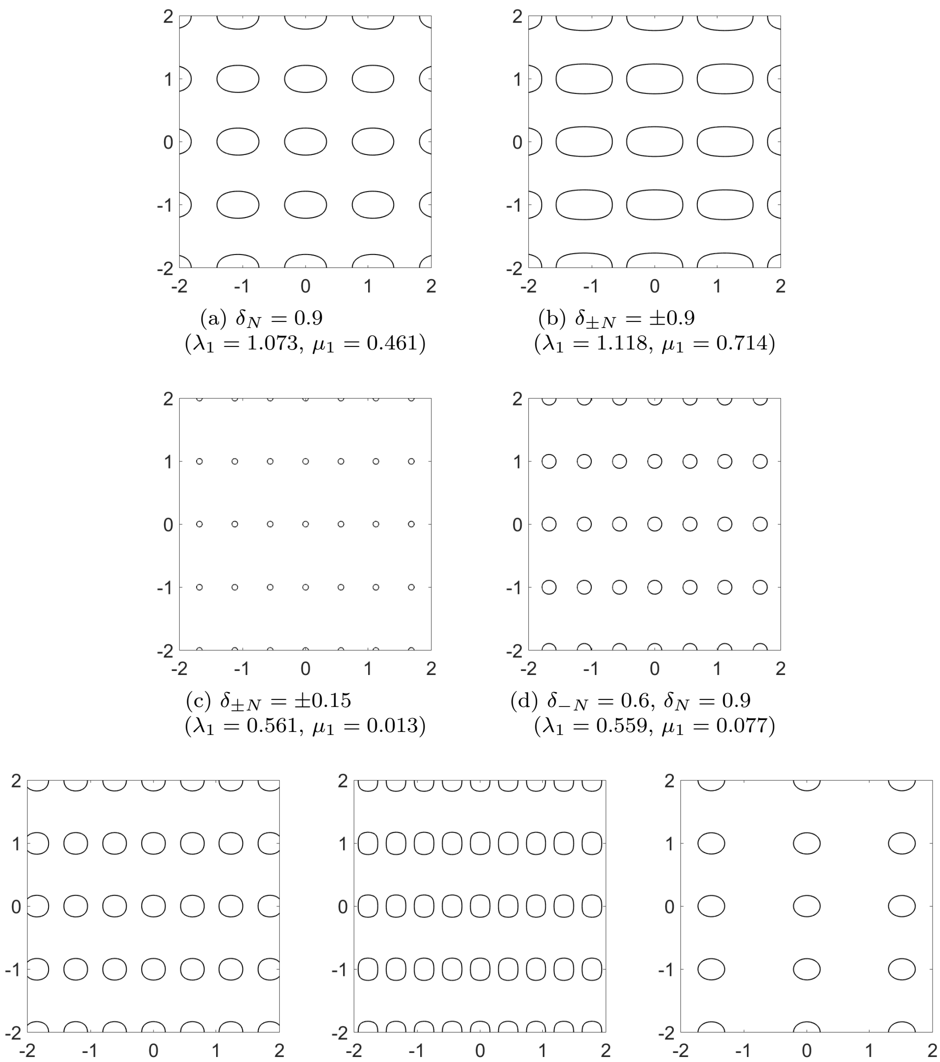


Fig. 4 An example of a 'square' lattice of equally-strong holes computed using our parameterisation with $N = 1$. For this, $\lambda_1 = 1.000000$, $\lambda_2 = \mathrm{i}$, $\mu_1 = 0.222786$, $\mu_2 = 0$.

As stated in section 3.1, it is with no loss of generality to the class of lattices that are described by our parameterisation that we may fix δ_{-N} completely and δ_N to be real. Furthermore, evidently our parameterisation allows us to choose λ_2, since it appears explicitly in (3.18). Also, obviously the constant c that appears in (3.18) corresponds simply to a translation, which one could think of as fixing the location of the centroid of one of the holes of the parameterised lattice, say $\mathcal{D}_0$. Then, of the eight real parameters that are represented by δ_N, $q_{\pm N}$, α, β and μ_2 (recall that β and μ_2 are complex), it seems natural to think of two as corresponding to λ_1 and of one as corresponding to the area of one of the holes, say $\mathcal{D}_0$ again. We make the following observations. Comparing Fig. 4 and Figs. 5a–5d, it appears that increasing $|\delta_N - \delta_{-N}|$ increases the area of $\mathcal{D}_0$ and the real part of λ_1. Furthermore, comparing Figs. 5c and 5d, it appears that moving δ_N and δ_{-N} closer to C_0 while keeping $|\delta_N - \delta_{-N}|$ fixed, also increases the area of $\mathcal{D}_0$. Comparing Figs. 4 and 5e–5g, it appears that increasing q_{-N} and q_N decreases the real part of λ_1 but has little effect on the area of $\mathcal{D}_0$. Considering also Figs. 6a and 6b, it appears that increasing α decreases the imaginary part of λ_1 but has little effect on the real part of λ_1 or on the area of $\mathcal{D}_0$. Also, from Figs. 6c and 6d, it appears that increasing $|\beta|$ increases the real part of λ_1, but that changing $\arg\{\beta\}$ has little effect. In addition, comparing Figs. 4 and 7, it appears that $|\mu_2/\lambda_2|$ and $\arg\{\mu_2/\lambda_2\}$ correspond to some sort of aspect ratio of the holes and some sort of common orientation of them about their centroids, respectively. The latter correspondence can be stated exactly when $|\mu_2/\lambda_2| = 1$, when the holes degenerate to slits, as stated by Proposition 3.6 (see Figs. 7d and 7e). One might think of these last two correspondences as accounting for the parameter μ_2. We point out that similar observations have been made for sets of a finite number of equally-strong holes (e.g., see (**7**), albeit with $\overline{b}/\overline{a}$ replacing μ_2/λ_2, as stated by (3.50)). Furthermore, for $j = 1, ..., N-1$, it seems natural to expect

δ_j and q_j to correspond to the location of the centroid of the hole $\mathcal{D}_j$ and also its area. Examples with $N > 1$ are shown in Figs. 8 and 9.



(a) $\delta_N = 0.9$ ($\lambda_1 = 1.073$, $\mu_1 = 0.461$)

(b) $\delta_{\pm N} = \pm 0.9$ ($\lambda_1 = 1.118$, $\mu_1 = 0.714$)

(c) $\delta_{\pm N} = \pm 0.15$ ($\lambda_1 = 0.561$, $\mu_1 = 0.013$)

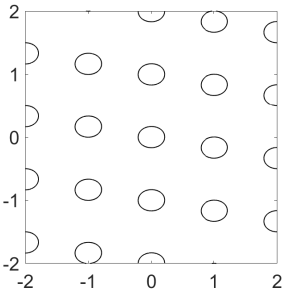
(d) $\delta_{-N} = 0.6$, $\delta_N = 0.9$ ($\lambda_1 = 0.559$, $\mu_1 = 0.077$)

(e) $q_N = 0.5$ ($\lambda_1 = 0.617$, $\mu_1 = 0.208$)

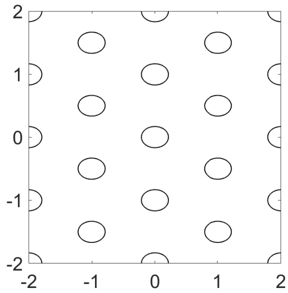
(f) $q_{\pm N} = 0.25$ ($\lambda_1 = 0.445$, $\mu_1 = 0.187$)

(g) $q_{\pm N} = 0.01$ ($\lambda_1 = 1.514$, $\mu_1 = 0.224$)

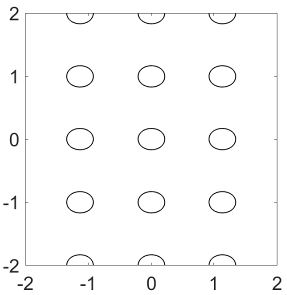
Fig. 5 Further examples of doubly-periodic lattices of equally-strong holes computed using our parameterisation with $N = 1$. For these, the corresponding values of $\delta_{\pm N}$, $q_{\pm N}$, α, β, λ_2 and μ_2 are the same as for the example shown in Fig. 4, except as stated. We also state the corresponding values of λ_1 and μ_1, as computed from (3.22) (we report these to 3 decimal places in order to save space; for the same reason, we omit the corresponding values of c).



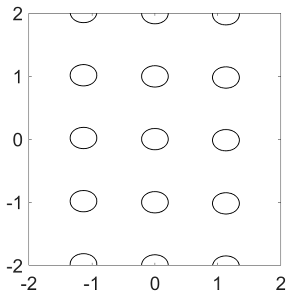
(a) $\alpha = \pi/3$
($\lambda_1 = 1.001 - 0.166\mathrm{i}$,
$\mu_1 = 0.224 + 0.000\mathrm{i}$)



(b) $\alpha = \pi$
($\lambda_1 = 1.004 - 0.500\mathrm{i}$,
$\mu_1 = 0.226 + 0.000\mathrm{i}$)



(c) $\beta = 0.75$
($\lambda_1 = 1.133$, $\mu_1 = 0.225$)



(d) $\beta = 0.75\mathrm{e}^{2\pi\mathrm{i}/3}$
($\lambda_1 = 1.133 - 0.018\mathrm{i}$,
$\mu_1 = 0.225 + 0.000\mathrm{i}$)

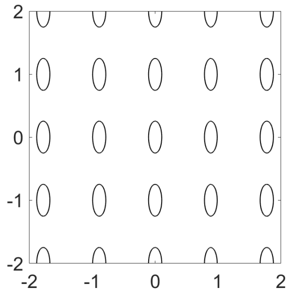
Fig. 6 Caption as for Fig. 5.

However, this still leaves three of the real parameters that are represented by δ_N, $q_{\pm N}$, α, β and μ_2 as unaccounted for. Some of these remaining parameters might correspond in some way to the value of μ_1. Or, it is possible that some of them might actually be redundant (so that it would be with no loss to the range of the class of lattices that the parameterisation describes that we may fix them too). In particular, some of these parameters might correspond only to the location and shape of the outer boundary $\mathcal{L}$ of the period cell $\mathcal{D}$ of the parameterised lattice and thus not have any effect on the lattice as a whole. However, we leave a more comprehensive analysis of the effects of all of these parameters and the range of the class of lattices that are described by the parameterisation for elsewhere.

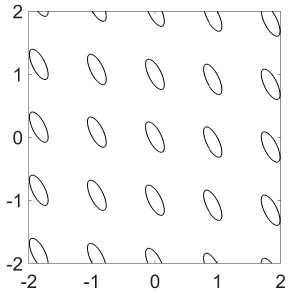
5. Closing remarks

To conclude, we suggest some other possible applications of the results that we have presented here, as well as some potential lines of further inquiry.

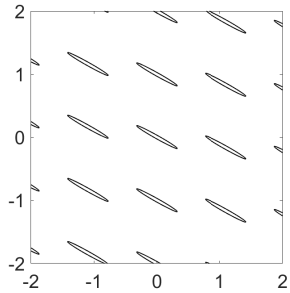
Evidently, it remains to prove that the condition (3.42) is necessary and sufficient for the existence of the class of doubly-periodic lattices of equally-strong holes that are described by our parameterisation. It may be possible to do this by following a similar approach to that which was taken in (**7**) to prove the corresponding existence condition for sets of a finite



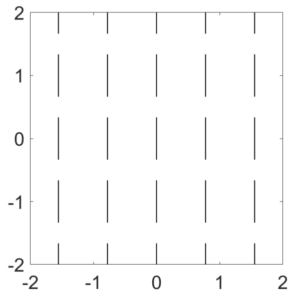
(a) $\mu_2/\lambda_2 = 0.5$
($\lambda_1 = 0.889$, $\mu_1 = -0.227$)



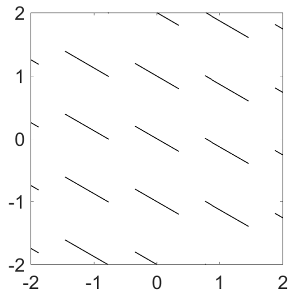
(b) $\mu_2/\lambda_2 = 0.5\mathrm{e}^{\pi\mathrm{i}/4}$
($\lambda_1 = 0.921 - 0.079\mathrm{i}$,
$\mu_1 = -0.131 - 0.354\mathrm{i}$)



(c) $\mu_2/\lambda_2 = 0.85\mathrm{e}^{2\pi\mathrm{i}/3}$
($\lambda_1 = 1.095 - 0.164\mathrm{i}$,
$\mu_1 = 0.648 - 0.736\mathrm{i}$)



(d) $\mu_2/\lambda_2 = 1$
($\lambda_1 = 0.777$, $\mu_1 = -0.777$)



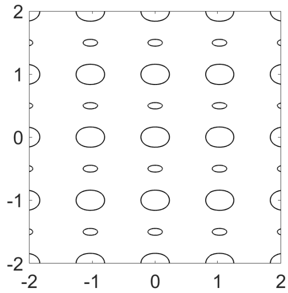
(e) $\mu_2/\lambda_2 = \mathrm{e}^{2\pi\mathrm{i}/3}$
($\lambda_1 = 1.111 - 0.193\mathrm{i}$,
$\mu_1 = 0.722 - 0.866\mathrm{i}$)

Fig. 7 Caption as for Fig. 5.

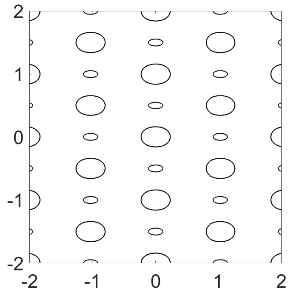
number of equally-strong holes (see Proposition 4.1 of (**7**)). It would also be of interest to determine whether or not our parameterisation in fact describes all doubly-periodic lattices of equally-strong holes.

One might envisage some of the doubly-periodic domains that are described by our parameterisation as defining geometries of interest for a variety of other problems. Perhaps most notable in this regard are the domains exterior to doubly-periodic arrays of parallel slits, and also those exterior to arrays of approximate ellipses and discs (e.g., see Figs. 4–7).

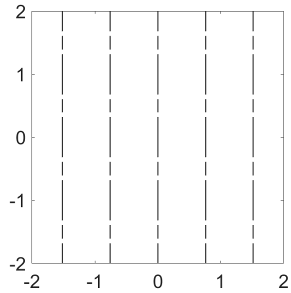
Furthermore, it may be that one could adapt our parameterisation to parameterise other interesting classes of doubly-periodic domains. For example, combined with the ideas of Baddoo & Crowdy (**10**), one might be able to construct a parameterisation of doubly-periodic domains with polygonal holes. Or, we point out that, given the property that is stated by Proposition 2.1, one may identify the domain exterior to a doubly-periodic lattice of equally-strong holes as being of a special class of planar domains known as quadrature domains, or rather what one might refer to as generalised quadrature domains. Domains exterior to sets of a finite number of equally-strong holes were identified as such in (**7**), and we refer the reader there for further details, and also to (**8**) and Gustafsson & Shapiro (**19**) for more information on quadrature domains in general. More commonly, a quadrature domain is characterised by the property that the Schwarz functions of the boundaries of all



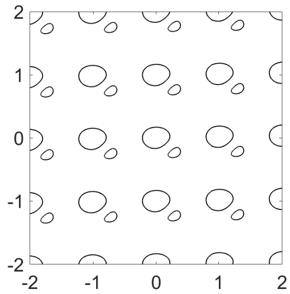
(a) ($\lambda_1 = 1.030$, $\mu_1 = 0.272$, $\gamma_1 = -0.500\mathrm{i}$)



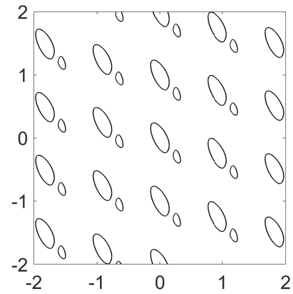
(b) $\alpha = \pi$ ($\lambda_1 = 1.033 - 0.500\mathrm{i}$, $\mu_1 = 0.275 + 0.000\mathrm{i}$, $\gamma_1 = -0.500\mathrm{i}$)



(c) $\mu_2/\lambda_2 = 1$ ($\lambda_1 = 0.758$, $\mu_1 = -0.758$, $\gamma_1 = 0$)



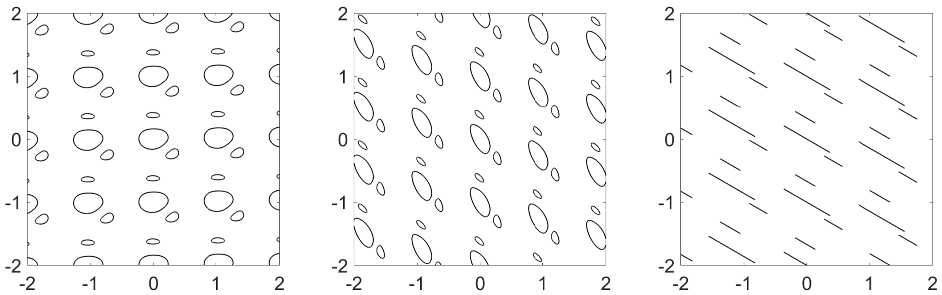
(d) ($\lambda_1 = 1.008 + 0.017\mathrm{i}$, $\mu_1 = 0.272 + 0.000\mathrm{i}$, $\gamma_1 = 0.170 - 0.194\mathrm{i}$)



(e) $\alpha = \pi/3$, $\mu_2/\lambda_2 = 0.5\mathrm{e}^{\pi\mathrm{i}/4}$ ($\lambda_1 = 0.911 - 0.244\mathrm{i}$, $\mu_1 = -0.033 - 0.409\mathrm{i}$, $\gamma_1 = 0.163 - 0.066\mathrm{i}$)

Fig. 8 Examples of lattices of equally-strong holes computed using our parameterisation with $N = 2$. For these, the corresponding values of $\delta_{\pm N}$, $q_{\pm N}$, α, β, λ_2 and μ_2 are the same as for the example shown in Fig. 4, except as stated. Furthermore, $\delta_1 = 0$, $q_1 = 0.15$ for those in Figs. 8a–8c, while $\delta_1 = 0.55 + 0.2\mathrm{i}$, $q_1 = 0.1$ for those in Figs. 8d and 8e. We also state the corresponding values of λ_1, μ_1 and γ_1, where the latter was computed from (3.25).

of its holes are in fact identical, and that this single Schwarz function has polar singularities in the domain (but is otherwise analytic in it). The domains that are parameterised in (**11**) in fact have this property (although this fact was not stated explicitly there) and may thus be identified as doubly-periodic quadrature domains. However, it might be possible to extend the parameterisation that we have presented in this paper to parameterise a more general class of doubly-periodic quadrature domains. Whilst parameterisations of general classes of finitely-connected quadrature domains are known (e.g., see (**8**)), parameterisations of doubly-periodic quadrature domains appear to be much less common - indeed, we are unaware of any other than that which is presented in (**11**) and that of the present paper. Such parameterisations would be of interest not only within the context of quadrature domain theory, but also for the numerous physical applications to which quadrature domains have been found to be relevant (e.g., see (**8**) and (**11**) for examples of these).



(a) ($\lambda_1 = 1.035 + 0.017\mathrm{i}$, $\mu_1 = 0.310 + 0.000\mathrm{i}$, $\gamma_1 = 0.166 - 0.204\mathrm{i}$, $\gamma_2 = -0.004 + 0.333\mathrm{i}$)

(b) $\alpha = \pi/3$, $\mu_2/\lambda_2 = 0.5\mathrm{e}^{\pi\mathrm{i}/4}$ ($\lambda_1 = 0.925 - 0.258\mathrm{i}$, $\mu_1 = -0.004 - 0.418\mathrm{i}$, $\gamma_1 = 0.154 - 0.074\mathrm{i}$, $\gamma_2 = 0.113 + 0.214\mathrm{i}$)

(c) $\mu_2/\lambda_2 = \mathrm{e}^{2\pi\mathrm{i}/3}$ ($\lambda_1 = 1.190 - 0.251\mathrm{i}$, $\mu_1 = 0.813 - 0.905\mathrm{i}$, $\gamma_1 = -0.094 - 0.162\mathrm{i}$, $\gamma_2 = 0.286 + 0.496\mathrm{i}$)

Fig. 9 Examples of lattices of equally-strong holes computed using our parameterisation with $N = 3$. For these, the corresponding values of $\delta_{\pm N}$, $q_{\pm N}$, α, β and μ_2 are the same as for the example shown in Fig. 4, except as stated. Furthermore, $\delta_1 = 0.55 + 0.2\mathrm{i}$, $q_1 = 0.1$, $\delta_2 = -0.25\mathrm{i}$ and $q_2 = 0.15$. We also state the corresponding values of λ_1, μ_1, γ_1 and γ_2.

6. Appendix A: Forces, moments and areas

Throughout this section, we take Λ to be a doubly-periodic lattice of equally-strong holes and make use of Proposition 2.1, including the forms (2.8) and (2.14) for corresponding complex potentials $\Phi(z)$ and $\Psi(z)$. We shall make the following checks on the balances of forces and moments. First, let $\mathcal{F}_x$ and $\mathcal{F}_y$ denote the resultant forces in the x- and y-directions, respectively, acting on an oriented arc Γ that is contained entirely in $\overline{\Delta}$, where these are the forces exerted by the right side of Γ on its left side. Then it is known (e.g., see Muskhelishvili (**16**, equation (33.1))) that

$$\mathcal{F}_x + \mathrm{i}\mathcal{F}_y = -\mathrm{i}\left[\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)}\right]_\Gamma, \tag{6.1}$$

where here and in the following, $[\ ]_\Gamma$ denotes the change of the bracketed quantity after z traverses Γ in its positive direction. It follows from (2.8) and the fact that $\Psi(z)$ is single-valued in Δ, that if Γ is a closed curve, then these resultant forces are zero. In particular, this includes if Γ is any one of $\partial\mathcal{D}_j$, $j = 0, 1, ..., N-1$, or $\mathcal{L}$.

Now, let $\mathcal{M}$ denote the resultant moment about the origin acting on Γ due to the forces $\mathcal{F}_x$ and $\mathcal{F}_y$. Then it is known (e.g., see Muskhelishvili (**16**, equation (33.3))) that

$$\mathcal{M} = \mathrm{Re}\left\{-\left[z\overline{z}\Phi'(z) + z\Psi(z)\right]_\Gamma + \int_\Gamma \Psi(z)\mathrm{d}z\right\}, \tag{6.2}$$

where we integrate along Γ in its positive direction. It follows again from (2.8) and the fact that $\Psi(z)$ is single-valued in Δ, that if Γ is a closed curve, then

$$\mathcal{M} = \mathrm{Re}\left\{\oint_\Gamma \Psi(z)\mathrm{d}z\right\}. \tag{6.3}$$

It then follows from (6.3), along with the fact that $\Psi(z)$ is analytic for all $z \in \mathcal{D}$ and

Cauchy's Theorem, that the resultant moment about the origin acting on $\mathcal{L}$ is equal but opposite to the sum of those acting on $\partial\mathcal{D}_j$, $j = 0, 1, ..., N-1$.

We shall now determine formulae for some particular forces and moments. For $j = 1, 2$, if one takes Γ to be $\mathcal{L}_j$, oriented from z_0 to $z_0 + \lambda_j$ - where here z_0 denotes the common endpoint of $\mathcal{L}_1$ and $\mathcal{L}_2$ - and lets $\mathcal{F}_{x,j}$ and $\mathcal{F}_{y,j}$ denote the forces $\mathcal{F}_x$ and $\mathcal{F}_y$, respectively, in this case, then it follows from (6.1) along with (2.5), (2.8), (2.10) and (2.14), that

$$\mathcal{F}_{x,j} + \mathrm{i}\mathcal{F}_{y,j} = -\mathrm{i}\frac{(\sigma + p)}{2}\lambda_j + \left(\tau + \mathrm{i}\frac{(\sigma - p)}{2}\right)\mu_j. \tag{6.4}$$

Alternatively, for $j = 0, 1, ..., N-1$, if one takes Γ to be $\partial\mathcal{D}_j$, oriented in the anticlockwise direction (i.e., with the interior of $\mathcal{D}$ on the right), and lets $\mathcal{M}_j$ denote the moment $\mathcal{M}$ in this case, then it follows from (6.3) along with (1.1), (2.6) and (2.14), that

$$\mathcal{M}_j = \mathrm{Re}\left\{-a\oint_{\partial\mathcal{D}_j}\overline{z}\mathrm{d}z\right\} = \tau\oint_{\partial\mathcal{D}_j}(x\mathrm{d}y - y\mathrm{d}x). \tag{6.5}$$

It is also evident from the first integral in (6.5) and the complex variable form of Stokes' Theorem, that for $j = 0, 1, ..., N-1$, $\mathcal{M}_j = 2\tau\mathcal{A}_j$, where $\mathcal{A}_j$ denotes the area of $\mathcal{D}_j$. Note also that, again by the complex variable form of Stokes' Theorem,

$$\begin{aligned}\sum_{j=0}^{N-1}\mathcal{A}_j &= \frac{1}{2\mathrm{i}}\sum_{j=0}^{N-1}\oint_{\partial\mathcal{D}_j}\overline{z}\mathrm{d}z \\ &= \frac{1}{2\mathrm{i}}\sum_{j=0}^{N-1}\oint_{\partial\mathcal{D}_j}\mathcal{S}_0(z)\mathrm{d}z \\ &= \frac{1}{2\mathrm{i}}\oint_{\mathcal{L}}\mathcal{S}_0(z)\mathrm{d}z \\ &= \frac{1}{2\mathrm{i}}\left(\lambda_2\overline{\mu_1} - \lambda_1\overline{\mu_2}\right),\end{aligned} \tag{6.6}$$

where the second equality follows from (1.1) and (2.6), the third equality follows from the fact that $\mathcal{S}_0(z)$ is analytic in $\mathcal{D}$ and Cauchy's Theorem, and the fourth equality follows from (2.5). Furthermore, the total area enclosed by $\mathcal{L}$ must equal that enclosed by a parallelogram with vertices at 0, λ_1, λ_2 and $\lambda_1 + \lambda_2$, which is $\mathrm{Im}\{\lambda_2\overline{\lambda_1}\}$. Then, it follows from (6.6) that

$$\mathcal{A}_{\mathcal{D}} = \mathrm{Im}\{\lambda_2\overline{\lambda_1}\} - \frac{1}{2\mathrm{i}}\left(\lambda_2\overline{\mu_1} - \lambda_1\overline{\mu_2}\right), \tag{6.7}$$

where $\mathcal{A}_{\mathcal{D}}$ denotes the area of $\mathcal{D}$. Evidently, since $\sum_{j=0}^{N-1}\mathcal{A}_j$ must be real, it follows from (6.6) that we must have $\mathrm{Re}\{\lambda_2\overline{\mu_1} - \lambda_1\overline{\mu_2}\} = 0$. Furthermore, since $\mathcal{A}_{\mathcal{D}}$ cannot equal zero, it follows from (6.7) that $\mu_j \neq \lambda_j$ for at least one of $j = 1$ and $j = 2$. One might derive other necessary conditions on the values of μ_j, $j = 1, 2$, using the above and the fact that we must also have $0 < \mathcal{A}_{\mathcal{D}} < \mathrm{Im}\{\lambda_2\overline{\lambda_1}\}$, although we shall not do so here.

7. Appendix B: Additional properties of our integrals of the first kind

LEMMA 7.1.

$$\overline{\upsilon_N}\left(\zeta^{-1}\right) = -\upsilon_{N+1}(\zeta) + a', \tag{7.1}$$

where we define $\overline{\upsilon_N}(\zeta) = \overline{\upsilon_N\left(\overline{\zeta}\right)}$, and a' is a constant. Furthermore,

$$\tau_{N+1,N+1} = -\overline{\tau_{N,N}}, \tag{7.2a}$$

$$\tau_{N+1,N} = -\overline{\tau_{N+1,N}}, \tag{7.2b}$$

$$\tau_{N+1,j} = \overline{\tau_{N,j}} \quad \text{for } j = 1, ..., N-1. \tag{7.2c}$$

Proof. First note that, since $\upsilon_N(\zeta)$ is analytic for all $\zeta \in F$, it follows from the fact that the interior of F is mapped onto itself by reflection in C_0, as well as the Cauchy-Riemann relations, that $\overline{\upsilon_N}(1/\zeta)$ is also analytic for all $\zeta \in F$.

Next note that, as ζ traverses $C_{\pm N}$ in the anticlockwise direction, so $1/\overline{\zeta}$ traverses $C_{\pm(N+1)}$, respectively, in the clockwise direction. Similarly, for $k = \pm 1, ..., \pm(N-1)$, as ζ traverses C_k in the anticlockwise direction, so $1/\overline{\zeta}$ traverses C_{-k} in the clockwise direction. It then follows from (3.13) that

$$\oint_{C_k} \mathrm{d}\overline{\upsilon_N}\left(\zeta^{-1}\right) = -\oint_{C_{-k}} \mathrm{d}\overline{\upsilon_N}\left(\zeta^{-1}\right) = -\delta_{N+1,k} \quad \text{for } k = 1, ..., N+1. \tag{7.3}$$

Furthermore, it follows from (3.14) that for $\zeta \in F$,

$$\overline{\upsilon_N}(1/\theta_{N+1}(\zeta)) = \overline{\upsilon_N\left(1/\overline{\theta_{N+1}(\zeta)}\right)} = \overline{\upsilon_N\left(\theta_N\left(\overline{\zeta}^{-1}\right)\right)} = \overline{\upsilon_N}\left(\zeta^{-1}\right) + \overline{\tau_{N,N}}, \tag{7.4}$$

where the second equality follows from (3.8) and the third from (3.14). Similarly,

$$\overline{\upsilon_N}(1/\theta_N(\zeta)) = \overline{\upsilon_N\left(1/\overline{\theta_N(\zeta)}\right)} = \overline{\upsilon_N\left(\theta_{N+1}\left(\overline{\zeta}^{-1}\right)\right)} = \overline{\upsilon_N}\left(\zeta^{-1}\right) + \overline{\tau_{N,N+1}}. \tag{7.5}$$

Also, recalling the first equality of (3.10), it follows that for $j = 1, ..., N-1$,

$$\overline{\upsilon_N}(1/\theta_j(\zeta)) = \overline{\upsilon_N\left(1/\overline{\theta_j(\zeta)}\right)} = \overline{\upsilon_N\left(\theta_{-j}\left(\overline{\zeta}^{-1}\right)\right)} = \overline{\upsilon_N}\left(\zeta^{-1}\right) - \overline{\tau_{N,j}}. \tag{7.6}$$

Thus (7.4)–(7.6) show that for $j = 1, ..., N+1$, $\overline{\upsilon_N}(1/\theta_j(\zeta)) - \overline{\upsilon_N}(1/\zeta)$ is a constant.

Hence, comparing the defining properties of $\upsilon_{N+1}(\zeta)$ (see section 3.2.1) with all of the properties of $\overline{\upsilon_N}(1/\zeta)$ that we have just derived, one may deduce (7.1). Furthermore, comparing (7.4)–(7.6) with (3.14) (and recalling (3.16)), one may deduce (7.2). □

8. Appendix C: A property of quasi-automorphic functions

LEMMA 8.1. Suppose that a function $f(\zeta)$ is analytic for all $\zeta \in F$ (including at infinity) and quasi-automorphic with respect to Θ. Then $f'(\zeta)$ has $2(N+1)$ zeros in F, at least two of which are at infinity.

Proof. Since $f(\zeta)$ is analytic for all $\zeta \in F$, then so is $f'(\zeta)$. Furthermore, in particular, $f'(\zeta) \sim \mathcal{O}(\zeta^{-M_\infty})$ as $|\zeta| \to \infty$, for some integer $M_\infty \geqslant 2$, i.e., $f'(\zeta)$ has a zero at infinity of order at least 2. Now, let M_I and M_B denote the numbers of zeros (counted according to their multiplicities) of $f'(\zeta)$ that are contained in the interior of F (but not at infinity) and on ∂F, respectively. Then, it follows from the Argument Principle that

$$M_I + \frac{M_B}{2} + M_\infty = \frac{1}{2\pi \mathrm{i}} \oint_{\partial F} \mathrm{d}\log f'(\zeta), \tag{8.1}$$

where the integral on the right-hand side of (8.1) is around ∂F in the direction with the interior of F on the left. If $M_B \neq 0$, then this integral is in fact a principal value integral.

Now recall that ∂F is given by (3.12). Also note that traversing ∂F with the interior of F on the left corresponds to traversing all of its boundary circles in the clockwise direction. Furthermore, recall that for $j = 1, ..., N+1$, C_j is the image of C_{-j} under $\theta_j(\zeta)$, and that $\theta_j(\zeta)$ maps the interior of C_{-j} onto the exterior of C_j. Then, it follows that $\theta_j(\zeta)$ traverses C_j in the clockwise direction as ζ traverses C_{-j} in the anticlockwise direction. But furthermore, by the same arguments as those applied to just $\theta_N(\zeta)$ in the proof of Proposition 3.2, one may deduce that the only singularity of $\theta'_j(\zeta)$ is a double pole in the interior of C_{-j}, and the only zero of $\theta'_j(\zeta)$ is a double zero at infinity. Also, since $f(\zeta)$ is quasi-automorphic with respect to Θ, one may deduce that

$$f'\left(\theta_j(\zeta)\right) = \frac{f'(\zeta)}{\theta'_j(\zeta)} \quad \text{for } j = 1, ..., N+1. \tag{8.2}$$

Given the above, one may deduce that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial F} \mathrm{d}\log f'(\zeta) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N+1} \oint_{C_{-j}} \mathrm{d}\log \theta'_j(\zeta) = 2(N+1), \tag{8.3}$$

where the last equality follows from another application of the Argument Principle.

Finally, it also follows from (8.2) and the aforementioned properties of $\theta_j(\zeta)$, that if $f'(\zeta)$ has a zero at some point $\zeta \in \partial F$, then it must also have a zero at $1/\overline{\zeta}$, which also lies on ∂F (at either $\theta_j(\zeta)$ or $\theta_{-j}(\zeta)$). However, recall that F contains C_j, $j = 1, ..., N+1$, but not C_{-j}, $j = 1, ..., N+1$. Hence one may deduce that M_B is twice the number of zeros of $f'(\zeta)$ that are contained in $F \cap \partial F$. Hence the result follows from (8.1) and (8.3). □

9. Appendix D: Computational details

We first describe how we computed the integrals of the first kind, $\upsilon_j(\zeta)$, $j = 1, ..., N+1$. For any Schottky group, there exists an explicit representation for the derivatives of its integrals of the first kind that is known to be valid under certain conditions (e.g., see (**14**), (**17**)). In our case, this representation may be stated as

$$\upsilon'_j(\zeta) = \frac{1}{2\pi \mathrm{i}} \sum_{\theta \in \Theta_j} \left(\frac{1}{\zeta - \theta(B_j)} - \frac{1}{\zeta - \theta(A_j)} \right) \quad \text{for } j = 1, ..., N+1. \tag{9.1}$$

Here, for $j = 1, ..., N+1$, Θ_j is the subset of Θ that consists of all maps whose composition in terms of the generators of Θ does not begin with θ_j or θ_{-j}. For example, Θ_N contains $\theta_N \theta_{N+1}$ but not $\theta_{N+1}\theta_N$. Note that the identity map is contained in Θ_j. A_j and B_j are the fixed points of $\theta_j(\zeta)$, i.e., the solutions of $\theta_j(\zeta) = \zeta$. One can show that one of these - that which we denote by A_j - is contained in the interior of C_{-j}, while the other - B_j - is contained in the interior of C_j. Also, A_j and B_j and their images under all $\theta \in \Theta_j$, are limit points of Θ. It is straightforward to derive expressions for A_j and B_j from (3.7), (3.9) and (3.10), or to just compute them. We omit any expressions for them here. Of course, one could easily integrate the sum on the right-hand side of (9.1) analytically, to obtain an explicit representation for $\upsilon_j(\zeta)$ itself, as a sum of logarithms. However, when computing

$\upsilon_j(\zeta)$, one must account for one's choice of branch cuts for it, and in practice it is easier to do so by integrating the representation (9.1) for $\upsilon_j'(\zeta)$ numerically, along a specified path.

The infinite sum in (9.1) is known to converge and provide a valid representation for $\upsilon_j'(\zeta)$ under the following conditions (e.g., see (**14**), (**17**)). It is valid for $N = 1$. For $N > 1$, it is valid provided that the circles that bound F are sufficiently small and well-separated. Furthermore, for $N > 1$, it is also valid if the centres of the circles that bound F lie on a common axis. We emphasise, however, that the integrals of the first kind exist for any Schottky group, and that our derivation of our parameterisation relies in no way on (9.1). Hence our parameterisation holds (for all N) regardless of the validity of (9.1).

We used (9.1) for the computations that we present in section 4, or rather, a finite truncation of (9.1). This truncation was the sum over all maps in Θ_j whose composition in terms of the generators of Θ consists of no more than five of these generators. For example, for $j = N$, we included $\theta_N^3\theta_{N+1}^2$ in our sum, but not $\theta_N^3\theta_{N+1}^3$. Of course, the number of terms that one includes in any such truncation must be sufficient for it to attain convergence to an acceptable degree of accuracy. We found this to be the case for this truncation that we used, for all of our computations.

For cases for which the representation (9.1) is not valid, or for which the number of terms that one must include in any finite truncation of it in order to attain sufficient accuracy is too large for it to be of practical use, it may be possible to develop alternative means of computing $\upsilon_j(\zeta)$, $j = 1, ..., N+1$. In particular, it may be possible to compute them by some adaptation of the novel numerical procedures that have been devised by Crowdy *et al* (**20**) - see also (**8**) - for the computation of the integrals of the first kind (and other functions) associated with the type of Schottky group that is used in (**8**) (as pointed out in section 1, the type of Schottky group that is used in (**8**) is slightly different to the group Θ that we use in this paper). This remains a matter for further investigation.

Finally, for $\zeta \in \overline{D}$, we computed $z(\zeta)$ by using (3.18) (rather than (3.19)). To do so, we computed $\upsilon_N(\zeta)$ and $\upsilon_{N+1}(\zeta)$ by numerically integrating our truncated forms of (9.1) for $j = N$ and $j = N+1$, respectively. We integrated these from a chosen base point - which we took simply to be 1 - to ζ, along a convenient path that lies entirely in $\overline{D}$. So, for example, in order to compute $z(\zeta)$ for $\zeta \in C_0$, we simply integrated along C_0. In addition, we computed values for $\tau_{j,N}$, $j = 1, ..., N+1$, by using (4.1) and numerically integrating our truncated form of (9.1) for $\upsilon_j'(\zeta)$ along our choice for the curve ℓ (we omit details of this choice). We then computed λ_1, μ_1 and γ_j, $j = 1, ..., N-1$, by using (3.22) and (3.25). We add that we carried out all of our computations in MATLAB.

References

1. G. P. Cherepanov, Inverse problems of the plane theory of elasticity, *J. Appl. Math. Mech.* **38** (1974) 915–931.

2. Y. A. Antipov, Slit maps in the study of equal-strength cavities in n-connected elastic planar domains, *SIAM J. Appl. Math.* **78** (2018) 320–342.

3. Y. Grabovsky and R. V. Kohn, Microstructures minimizing the energy of a two-phase elastic composite in two space dimensions. II: the Vigdergauz microstructure, *J. Mech. Phys. Solids* **43** (1995) 949–972.

4. S. B. Vigdergauz, Two-dimensional grained composites of extreme rigidity, *J. Appl. Mech.* **61** (1994) 390–394.

5. S. B. Vigdergauz, Rhombic lattice of equi-stress inclusions in an elastic plate, *Q. J. Mech. Appl. Math.* **49** (1996) 565–580.
6. S. B. Vigdergauz, Simply and doubly periodic arrangements of the equi-stress holes in a perforated elastic plane: the single-layer potential approach, *Math. Mech. Solids* **23** (2018) 805–819.
7. J. S. Marshall, On sets of multiple equally strong holes in an infinite elastic plate: parameterization and existence, *SIAM J. Appl. Math.* **79** (2019) 2288-2312.
8. D. G. Crowdy, *Solving problems in multiply connected domains* (SIAM, CBMS-NSF Regional Conference Series in Applied Mathematics 2020).
9. P. J. Davis, *The Schwarz function and its applications*, (Math. Assoc. of America, Washington 1974).
10. P. J. Baddoo and D. G. Crowdy, Periodic Schwarz–Christoffel mappings with multiple boundaries per period, *Proc. R. Soc. A* **475** (2019) 20190225.
11. J. S. Marshall, Exact solutions describing the injection-driven growth of a doubly-periodic fluid region in a Hele–Shaw cell, *IMA J. Appl. Math.* **81** (2016) 723–749.
12. Y. A. Antipov, Method of automorphic functions for an inverse problem of antiplane elasticity, *Q. J. Mech. Appl. Math.* **72** (2019) 213–234.
13. G. L. Vasconcelos, J. S. Marshall and D. G. Crowdy, Secondary Schottky–Klein prime functions associated with multiply connected planar domains, *Proc. R. Soc. A* **471** (2015) 20140688.
14. E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, *Algebro-geometric approach to nonlinear evolution equations*, (Springer-Verlag, Berlin 1994).
15. L. R. Ford, *Automorphic functions*, (Chelsea Pub. Co., New York 1972).
16. N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity: fundamental equations, plane theory of elasticity, torsion and bending*, (Noordhoff, Leiden 1975).
17. H. F. Baker, *Abelian functions: Abel's theorem and the allied theory of theta functions*, (Cambridge University Press, Cambridge 1995).
18. P. R. Garabedian and M. Schiffer, Identities in the theory of conformal mapping, *Trans. Amer. Math. Soc.* **65** (1949) 187–238.
19. B. Gustafsson and H. S. Shapiro, What is a quadrature domain?. In *Quadrature domains and their applications* (pp. 1–25), (Birkhäuser, Basel 2005).
20. D. G. Crowdy, E. H. Kropf, C. C. Green and M. M. S. Nasser, The Schottky–Klein prime function: a theoretical and computational tool for applications, *IMA J. Appl. Math.* **81** (2016) 589–628.