6 Students' Justification Strategies on the Correctness and Equivalence of Computer-Based Algebraic Expressions

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Introduction

As many authors (e.g., Arcavi et al., 2017; Mason, 1996; Radford, 2011) have argued arithmetic is the precursor of and prerequisite for algebra and even though algebra is considered "generalised arithmetic", arithmetic and algebra have different foci. Arithmetic encourages students to find numerical answers, whereas algebra encourages students to identify and express mathematical structures. While algebra is a system that supports structural sense and expresses generalisation, its teaching often prioritises the acquisition of its transformation rules, rather than structural thinking and generalisation itself. There have, of course, been numerous attempts to address this challenge, and more generally, to come to grips with the difficulties that students face in their transition from arithmetic to algebraic thinking. For example, activities based around generalising patterns of various descriptions have been widely considered as a potentially powerful way to help students learn how to "see the structure" and generalise and are used as a common route for introducing algebra (e.g., Küchemann, 2010; Mason, 1996; Radford, 2010; Radford et al., 2007). However, the use of these activities needs care, as they can easily encourage trial-and-error techniques and a focus on the term-to-term (or in other words additive) rule, which do not necessarily lead to mathematically valid generalisation strategies that promote the acquisition of structural sense for the learner (Dörfler, 2008; Hart, 1981; Küchemann, 2010; Küchemann & Hoyles, 2009; Radford, 2011). The challenge, therefore, is to introduce patterns and generalisation in ways that promote algebraic thinking, that is, to identify and express structural commonalities and relationships (Geraniou et al., 2009; Noss et al., 2009).

Our aim is to investigate the impact of collaborative computer-based tasks involving figural patterns to students' justification strategies for discussing the equivalence and correctness of algebraic expressions and subsequently students' development of algebraic generalisation and structural sense in Algebra. In the analysis of our empirical data involving such tasks, we will be discussing the interconnectedness of figural

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and numerical patterns for the development of algebraic structural sense and generalisation.

Different authors have defined mathematical generalisation in various ways: "as a mathematical rule about relationships or properties" (Ellis, 2007, p. 196); as the process of extending one's scope of reasoning beyond the case or cases considered (Harel & Tall, 1991); or as communicating at a level "where the focus is no longer on the cases or situations themselves, but rather on the patterns, procedures, structures, and the relationships across and among them" (Kaput, 1999, p. 137). More recently, Radford (2010) argues, "generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S" (Radford, 2010, p. 42). Ellis (2011) argues similarly, but adds a reference to the sociomathematical contexts in which students are engaged.

In this chapter, we refer to a restricted class of generalisation: namely, the process of noticing the structure of a figural pattern, identifying what is repeated, and expressing the rule that corresponds to this structure symbolically. Our interest also encompasses the role of justification of generality in a carefully designed collaborative social context, as a step towards the adoption of algebraic ways of thinking.

While it is relatively straightforward for students to illustrate some elements of generality using numbers and gestures, expressing generality in words or in algebraic form has proved more problematic (Arcavi, Drijvers, & Stacey, 2017; Filloy and Rojano, 1989; Noss, Healy, & Hoyles, 1997; Noss et al., 2009; Radford, 2010, 2011;). Frequently, students view the algebraic expression as disconnected from the structure of the problem, to be added as an optional and seemingly arbitrary endpoint.

In this chapter, we present a research study, which aimed at bridging the gap between identifying and expressing pattern, by encouraging students to identify the structure of the pattern *through its construction* and providing them with the necessary tools to express generality. Thus, the underlying theory that guided our activity design is constructionism (see Harel & Papert, 1991) and the claim is that this experience will support the expression of generality by focusing on the structural sense in Algebra, as well as shape how the generality is expressed. We used the microworld, eXpresser, a toolkit for working on the construction of tiling patterns (more details given later), and activity sequences that ended in a reflective and a collaborative phase aiming to provide students with a rationale and opportunity to justify their pattern constructions to each other. We focus on this final phase of the activity sequences. We present data from several studies in three English schools, illustrating how 11- to 14-year old students who had engaged with eXpresser were able to reflect upon their own and their peers' solution strategies and employ a range of strategies to justify the correctness and – where appropriate – the equivalence of their computer-based algebraic rules. Our research questions were: What is the impact of computer-based collaborative tasks involving figural pattern generalisation on students' justification strategies for the equivalence and correctness of algebraic expressions? How do such tasks support the development of algebraic generalisation and structural sense in Algebra?

Theoretical Background

Perceiving and Expressing the Structure of Figural Patterns

The pattern-generalising process involves various steps that students typically follow in their efforts to reach meaningful generalisations. Dretske (1990) argues that the initial act of coming to see a pattern is of two types: sensory, which refers to individuals perceiving an object as a mere object-in-itself; and cognitive, when perception moves beyond sensory perception by recognising a fact or a property related to the object in question. We will argue that Dretske's distinction between these two modes might not be so clear-cut, depending crucially on the tools at hand, and the contexts in which they are used. A key issue, however, as Rivera (2010) points out, is seeing or recognizing a fact or a property in relation to an object, and doing so cognitively or theoretically rather than practically. In fact, the way in which the structure of a pattern is conceived will depend critically on the unit of repeat perceived, and – this is the crucial point – the tools available for repeating.

If and when students recognise what is repeated, they are often capable of expressing a general rule through the use of words like "always" or "every", but struggle to use letters and symbols (see Warren & Cooper, 2008). Additionally, Radford's (2009) discussion of "objectification" shows how students' inexperience with mathematical language prevents them from "translating" the structure of a pattern to an algebraic expression after noticing what is repeated). To address this, Rivera (2010) advocates an "*abductive-inductive action on objects*" that involves "employing different ways of counting and structuring discrete objects or parts in a pattern in an algebraically useful manner" (ibid., p. 4). The idea is that such an approach allows students to perceive the structure of a pattern that could lead to a meaningful, but also mathematically viable, pattern generalisation, which could form the basis for deriving a formula.

After this initial step of recognising the structure of the pattern, students are usually encouraged (e.g., in mathematics lessons by a teacher) to adopt a suitable symbol system to reason about and to express generalisations (Arcavi et al., 2017; Kieran, 1989), which is once again a far from trivial process. This action has been characterised as "*symbolic action*" by Rivera (2010) and involves translating the visually perceived structure of a pattern into the form of an expressed generalisation. This transition from arithmetic to algebra, or "didactical cut" as described by Filloy and Rojano (1989) involves a substantive shift from operating with numbers and operating with the unknown (see also Filloy et al., 2010). To effect this transition, Filloy, Rojano, and Puig (2008) argue that a teacher's intervention is crucial.

The pedagogical strategy used by Filloy, Rojano, and Puig (2008) involved two concrete models: the *balance scales* model, in which the equation was represented as a balance between two weights in two pans, and the *geometric model*, in which the letters and algebraic expressions were represented as lengths and areas of rectangles. The lesson we take from this work is that while a modelling approach can support the development of algebraic thinking, it can also become a hindrance in the absence of carefully designed activities to support pedagogically the transition to algebraic expression.

Algebraic expression, of course, does not have to be verbal or written. Noss and Hoyles (1996), for example, use the idea of *situated abstraction* to describe how students can express abstractions *within* a "concrete" symbolic medium, such as a computer program; Radford (2010) similarly invokes the notion of *semiotic means of objectification* to capture the means used by students to express a general rule, such as gestures, signs, etc. Designing semiotic systems is therefore a worthwhile challenge in the attempt to foster the expression of generality. However, the utility of such systems is sensitive to their structure: there are, for example, several studies (e.g., Lee, 1996; Stacey, 1989) that highlight students' tendency to focus on recursive rather than functional relationships, which can present barriers towards generalising "any" case. Furthermore, generalisation tasks that are presented as a sequence of consecutive terms often lead students to seek empirical generalisation rather than a structural one (Bills & Rowland, 1999). More recent research documents the different strategies students employ when constructing the algebraic rules that underpin patterns, which allow them to support the correctness of their general rules. For example, Rivera and Becker (2008) differentiate between constructive and deconstructive generalisation, depending on whether or not students perceive the figural pattern as having overlapping components, and Chua and Hoyles (2011) refer to *reconstructive* generalisation, where components of the pattern are rearranged to reveal the pattern structure.

Collaboration and Reflection

Alongside the affordances of novel representational systems for describing generality, algebraic thinking can be further supported by opportunities for reflection, to assist students in expressing structural relationships and distinguishing variants and invariants (see for example, Ellis, 2007). Encouraging students to reflect on their actions can promote their justification skills and support the development of their algebraic thinking.

Unsurprisingly, reflection can be strengthened by suitably designed collaboration tasks. As students explain their ideas and solutions to their peers, they are encouraged to resolve conflicts and develop a deeper understanding than those who do not (Cohen & Lotan, 1995; Leonard, 2001; Lou et al., 1996). Wood (1988) argues that discussion and interaction are critical to avoid “misconceptions”: “A trouble shared, in mathematical discourse, may become a problem solved” (p. 210). When collaborating in small groups, students are more likely to ask questions, reflect on their own work and attempt to make sense of each other’s work (Linchevski & Kutscher, 1998). In the context of generalisation, research suggests that working in small groups is advantageous for students’ deeper understanding of generalisation, equivalence of rules and algebraic thinking (Ellis, 2011).

Leonard (2001) argues for the value of forming heterogeneous groups, especially for lower-attainment sixth grade mathematics students who were grouped with higher attainment students and achieved more although other disagree (see for example, Carter & Jones, 1993, as cited in Fuchs et al., 1998). Criteria such as these for grouping students are dependent upon the mathematical tasks they are asked to tackle and the learning objectives assigned by the teacher. Furthermore, students who work in small groups seem to learn more when the outcome depends upon all of the group members’ efforts (Lou et al., 1996). A cooperative learning strategy encourages students to share arguments and consider different approaches that could even be shared with the rest of the class. Hopefully such an approach minimises the issue of one student overpowering the group. The question is how to *design* group work so it is most likely to lead to optimal results for learning (Abdu, Olsher & Yerushalmy, 2019; Healy et al., 1995).

Argumentation and Justification

Even though students are capable of generalising a pattern or a rule, few are able to explain why the rule is valid and justify their actions (Coe and Ruthven, 1994; Ellis, 2007; Healy and Hoyles, 2000; Küchemann and Hoyles, 2009). Many tend to rely on empirical examples to justify the truth of statements: it would hardly be surprising if a student who generalises based solely on specific cases were to use one or more examples as a form of justification.

Research suggests that a student who generalises by attending to the structure of a pattern and relating each algebraic expression to its corresponding part of the pattern-model-construction, has a better chance of justifying the generality of their expression and possibly producing a general argument to justify the equivalence of rules (Küchemann, 2010). Helping students develop their own algebraically powerful generalisations will likely aid in their abilities to provide symbolically-expressed justifications or in other words some form of proof (Ellis, 2007). Thus setting challenges for students to reflect, recognise, and justify general rules and actions to themselves as well as to others, is a strong candidate for a strategy to enhance students' generalisation capabilities. Moreover, it seems that focusing on justification activities may not only enhance students' expression of their existing generalisations, but also aid in the development of subsequent, more powerful generalisations (Ellis, 2007).

Mercer (1995; 2000, as cited in Swan, 2006) has shown that attention should also be directed at promoting exploratory talk (critical and constructive exchanges) among the participants of the group rather than disputational (disagreement and individualised decision-making) and cumulative talk (build positively on others' input). Working collaboratively (rather than competitively), students tend to be more committed to overcoming conflict in their efforts to master a task and coordinate different pints of view into new ones (Laborde, 1994). Furthermore, promoting a collaborative "knowledge building" culture as envisaged by Scardamalia and Bereiter (2006) during collaboration, can advance students' knowledge.

Collaborative Interaction in Relation to Computers

There is a long research tradition in collaborative learning with computers in mathematics education. Early work from Teasly and Roschelle (1993) and Healy et al. (1995) highlighted the importance of expressing ideas in words and establishing a common group goal. Similarly, a key research finding is that a characteristic of computational artefacts is that students' focus of attention can gradually change from being computer-oriented to being focused on the mathematical aspects of the task at hand (e.g., Kieran, 2001; Lavy & Leron, 2004).

Among other relevant findings, Bereiter and Scardamalia (2003) and Moss and Beatty (2006) discuss how students make progress not only in improving their own knowledge but also in developing "collective knowledge" by contributing to their peers' comments. The WebLabs project took this a step further through its "Webreports" system (Mor et al., 2005) that facilitated distance collaboration of students who constructed models. This study demonstrated the positive effects of sharing, commenting, making changes and allowing students to reflect on each other's artefacts both synchronously (face-to-face) and asynchronously (Mor et al., 2006). The importance of students' engaging with, or talking about, the product of their work and the opportunities for building on each other's ideas, learning to participate in a community of practice and benefit from the reflection that occurs from the interaction with others has also been widely identified (Vahey et al., 2000; Vahey et al., 2007).

Ellis's (2011) research on the role of collaboration in the development and refinement of students' mathematical generalisations highlighted how generalisation can be viewed as "a dynamic, socially situated process that can evolve through collaborative acts" (p. 308) where, in the classroom situation outcomes are influenced by the interconnected actions of students, teachers, problems, representations, and artefacts. In her research, students were stimulated publicly to generalise, share, build, and encourage justifications or clarifications in their attempts to understand a new mathematical domain. Moreover, her research revealed that an initial generalisation evolves through extensive interactions, phases of reflection, and takes many different forms until the final, stabilised version of generalisation, which cannot be claimed to have been developed in isolation. Rather, every student in a group is responsible for the final generalisation. This is described as collective generalising (ibid., 2011).

The eXpresser Microworld and Activities

The key idea of the eXpresser microworld is that students first *identify* the structure of a pattern of squares presented dynamically, next *construct* the pattern, and finally *express* a general rule for the number of tiles in a general pattern. Thus, there is a tight coupling between building the pattern, and being able to describe how it is built – between the "algebra" and the objects the algebra aims at expressing. The quotes round the word algebra signify, as we will see, that the language of expression is algebra, but not as we know it.

For our purposes here, we see students' work in eXpresser to solve a generalisation activity as involving two phases, the construction and the collaboration phases. In the construction phase, students go through the following actions: (1) visual perception of the model presented, (2) inductive action on the model, to realise what stays the same and what is repeated, (3) constructionist action, building the model (using one or more patterns), (4) expressive action, expressing the constructed model in the form of a general symbolic rule that colours the model. Then in the collaboration phase, students continue their interactions with: (5) reflective action with students writing arguments for or against their models and particularly their derived rules (these arguments are written at the end of the construction phase and used during subsequent collaboration, (6) justification action, viewing other students' rules and validating/justifying correctness and equivalence, and (7) collaborative reflective action, involving students in groups to reach an agreed statement regarding the equivalence and correctness (or not) of their shared rules.

A typical activity in eXpresser will ask the student to reproduce what is presented as a dynamic model of a tiling pattern shown in a window that appears on the side of the activity screen. In eXpresser, an initial figure is presented dynamically in order to draw students' attention to the general problem, rather than the static and inevitably therefore specific problem that could otherwise be posed on paper. Figure 6.1a shows the "Train-Track" model: it is animated randomly from left to right[1] with the value of the model number changing accordingly.[2] Students are asked to construct the Train-Track model in eXpresser using patterns and combinations of patterns of their own choosing (examples are shown in Figure 6.1), depending on their perceptions of the Train-Track's structure. Students are encouraged to colour the patterns with different colours to represent the way they visualise the structure of the model. This happens by providing an expression that "tells the computer" the number of tiles in each coloured pattern. They then seek to derive a general rule for the total number of tiles needed for *any* Model Number. The generality of the rule can be tested by "animating" the figure to see if the model matches the activity model, is drawn correctly, and remains coloured for any model number the computer chooses randomly.

Since the activity model is presented dynamically, as an animated pattern, students are given the opportunity to perceive the model visually and construct it in any way they see it. They can therefore identify the structural relationships in the model in an abstract way that would materialise as they carry on constructing the model and make meaningful generalisations. Based on their visual perception of the activity-model, students are expected to derive a rule that expresses their method of construction. Students identify a common unit in the activity-model, build it, and repeat it to form a pattern. A number of patterns can make their model. In an effort to discourage students from thinking from term-to-term and therefore additively, eXpresser is designed to help students to express the relationship between the common unit and the number of its repetitions. This common unit is referred to as a "building block": the idea of using

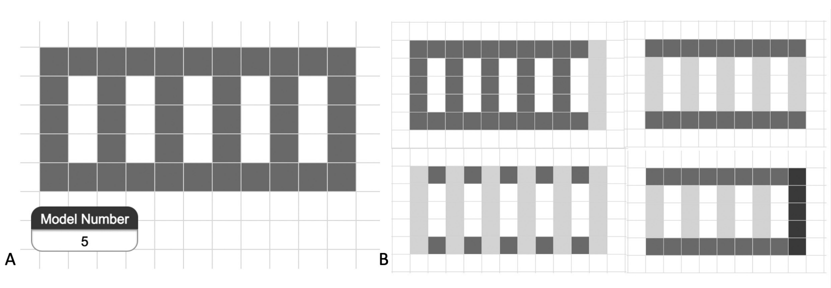

Figure 6.1 (a) The Train-Track activity and (b) six different students' perceptions of it.

different building blocks for different patterns comes from the prevailing view in recent studies involving figural pattern generalisation activities that students see a pattern in different ways depending on how they conceptualize such units (see the ZDM issue on pattern generalisation for references to most studies; Rivera & Becker, 2008). This process comprises the second step in the generalisation activity diagram. This common unit is the coefficient (constant) of students' variable in the general rule and as it will be shown later in this chapter, students employed the notion of the constant to justify the correctness of their rules.

The eXpresser capitalises on visual dynamic representations and feedback[3] and, in addition, on the simultaneous representation of a specific and a generalised model, called "My Model" and "General Model" (see Figure 6.2 presenting the model of a year 8 student, Alicia). The model is built by combining patterns and there is a close alignment of the symbolic expression, the Model Rule and the structure of the model. In the General Model, a value of the variable[4] ("'Model Number" in this example) is chosen automatically at random (it is "6" in Figure 6.2. It will generally be different from that in the specific model ("3" in Figure 6.2).

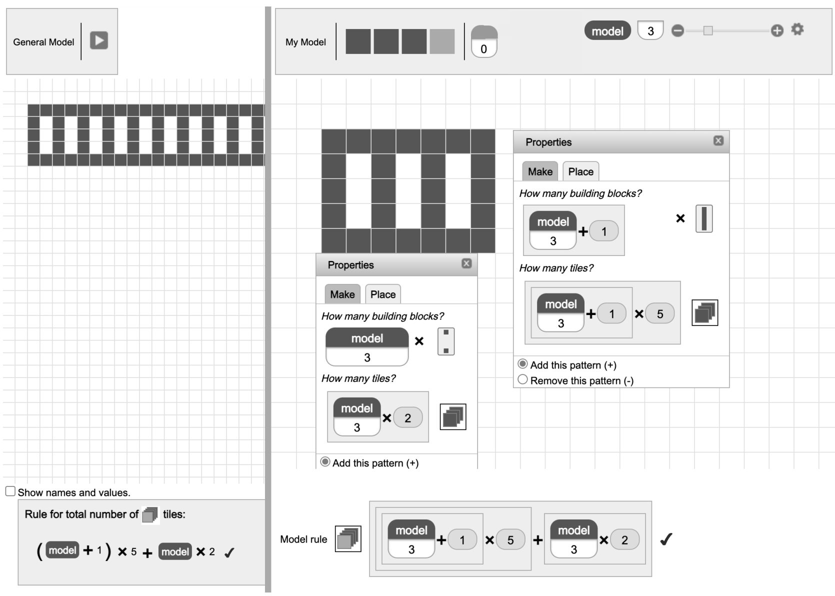

Figure 6.2 Alicia's Train-Track model showing her constructed model on the right, where she chose her variable "model" to be 3. On the left hand-side, there is the General Model window, where her constructed model is demonstrated for the value of 7 for the 'model' variable. Also, below Alicia's Model, one can see her derived general rule given in the eXpresser language. On the General Model Window, Alicia's Model rule is presented in an algebraic form.

So the General Model indicates to students whether their constructions are structurally correct for the different values of the variable(s) assigned to the various properties.

The system is designed to encourage students to express the structure of a pattern using the eXpresser representational form (which is the "Model Rule" in Figure 6.2). Students construct a model rule for the total number of tiles, and validation of its correctness is made evident by the system through colouring: patterns are *only* coloured if the rule for the number of tiles required is correct. Motivation for generality is thus provided by the main goal of the activity, that is, to produce a model that will animate correctly by colouring the exact number of tiles required, in combination with a pedagogical strategy that challenges students to construct models that are impervious to changes in the values of the variables.

Designing for Collaboration in eXpresser

Students' notions of generality, even when constructed in an environment designed to scaffold mathematical generality, might not match what it is required for a meaningful generalisation. We therefore designed individual reflective activities, in which students were required to think of and write down their arguments for their rules' correctness. They were prompted to think carefully how best to explain and justify this in written form, to encourage them to come up with good arguments and express them explicitly. This activity would also prepare them for the second part of the collaborative activity where they discuss the equivalence of different rules.

As illustrated in the final stage of the MiGen Activity Diagram (Figure 6.1), students were grouped in pairs by their teacher with the advice of the Grouping Tool (for a description see later) based on their different, yet equivalent, derived rules. During their collaboration, processes of ascertaining and persuading each other of their rules' correctness and equivalence were revealed. In fact, there was a process of ascertaining when a student was going through the reflection phase, which was followed by a process of persuading when they were paired during the collaboration phase. Also, collaboration encouraged the development of reasoning skills since students were encouraged to explain why their generalisations make sense.

Having reviewed the literature, we based our approach on the assumption that collaborating *about* and *with* something concrete (in this case, virtual) is more likely to lead to effective learning than without. The activities were designed to prompt students to revisit their derived rules and through discussion with their peers reflect on their transition from simple to more complex expressions and then to algebraically accepted ones.

We conjecture that focusing on figural pattern activities with tuned tools and carefully structuring and providing a context for students' discussions on the correctness and equivalence of models and related algebraic expressions could prove to be a powerful approach to fostering algebraic thinking. In the context of this carefully designed exploratory learning environment eXpresser, students are given the means to relate the symbolic representation to the relevant parts of the pattern, give meaning to symbols, and form justifications in an algebraic manner.

Interactions in eXpresser

To give the reader some understanding of students' interactions with the eXpresser microworld, we will briefly outline Alicia's interactions with the Train-Track activity before we move on to consider the research methodology employed mainly for the collaborative activities.

Alicia was presented with the animated Train-Track model and was asked to construct it and find the general rule. She placed five red (dark grey in Figure 6.2) tiles to form a column. She wanted to repeat this column a number of times towards the right direction, but always leaving a gap in between each two consecutive columns. She therefore characterised this column in eXpresser as a building block. She decided to repeat this building block three times to make her blue pattern. She then made a building block of two green (light grey in Figure 6.2) tiles placed vertically, but with a gap of three tiles in between, and repeated it three times. She noticed that the end of her model did not match the activity model as there was a column missing. She then derived a relationship between her two patterns that for **every number of green building blocks, she always needs an extra red building block**. Using the eXpresser's language she was able to express this relationship. For any number of green building blocks, for example [model:3], she needs [model:3] + 1 red building blocks. To colour each of her patterns, she had to multiply the number of repeats for the building block with the number of tiles in one building block. So, for the green pattern, she built the expression [model:3] × 2 and for the red pattern, {[model:3]+1} × 5. Since the activity was to find the rule that gives the number of tiles in the model for any model number, she added the two expressions that coloured her two patterns and got her general rule: {{[model:3]+1} × 5} + {[model:3] × 2}.

We now return to the study itself, and focus on the methodological approach regarding the reflective and collaborative aspects of our study.

Methodology

The purpose of our research study was to collect data that illustrate how students who had engaged with eXpresser were able to reflect upon their own and their peers' solution strategies and employ a range of strategies

Table 6.1 Sample

Schools in London	*Year group*	*Number of students in class*	*Number of students interviewed during collaboration*
A	Year?	20	6
B	Year?	24	6
B	Year 8	22	6
C	Year 8	16	16
C	Year 9	28	14
TOTAL			**48**

to justify the correctness and – where appropriate – the equivalence of their computer-based algebraic rules. Such data would allow us to discuss the impact of computer-based collaborative tasks involving figural pattern generalisation on students' justification strategies for the equivalence and correctness of algebraic expressions. Subsequently, we would make inferences regarding the degree to which such tasks would support the development of algebraic generalisation and structural sense in algebra.

The data presented in this chapter are derived from 48 mixed ability year 7, 8, and 9 students (aged 11–14 years) from three different schools in England. Table 6.1 presents the number of students interviewed from each school.

The following sections describe the sequence of activities students undertook, how the Grouping Tool, another tool of the MiGen system, paired students, and the data analysis process that was followed to reach our results.

Students' Activities in eXpresser

Students were familiarised with the eXpresser in two lessons through a number of introductory activities and practice activities, asking students to construct figural models. Afterwards, they were given the Train-Track activity and were asked to build the model on their own. The activity text comes with suggested goals as follows: "Construct the Train-Track model. Use more than one pattern to make the model. Use different colours for each pattern to show to other people how you made your model. Find a rule for the number of tiles for any Model Number. Use pattern(s) to construct the model; make sure "My Model" is always coloured; check that the "General Model" animates without messing up; make sure that the "General Model" is coloured always". Students constructed the model in many different ways, some of which are presented in Figure 6.1.

After constructing their model, students were asked to reflect on it (reflective action) by answering the following questions: *(1) Use your* *model to find the number of tiles for Model Numbers: 6, 12, 1, and 100. (2) Is your rule correct or not? In the next activity, you will discuss with another student. Make some notes here to explain why your rule is correct or not to prepare for this group activity.* Based on having different student models and rules in any pair, students were paired to discuss the correctness and equivalence of their rules as mentioned earlier.

Students' Collaboration

Student pairs were formed by exploiting information provided by the Grouping Tool,[5] which was designed to support the teacher in deciding upon the best possible pairs of students in terms of potentially worthwhile discussions on the equivalence of eXpresser rules. The grouping tool retrieves all students' models from the database and analyses them on the basis of three criteria: (1) the similarity of the building blocks, (2) the values of the right and downward displacement of the building blocks, and (3) the similarity of expressions that relate properties of the models (e.g., that a building block is repeated twice as another). Based on its analysis, it suggests to teachers' possible groupings based on the dissimilarity of students' models. The final decision for the best pairings of students of course lies with the teacher, so the tool is designed to allow the teacher to change the suggested pairs, as they deem appropriate.

Having been assigned to their pairs the students were asked the following two questions: *(1) Convince each other that your rules are correct, (2) Can you explain why the rules look different but are equivalent? Discuss and write down your explanations.* Students were asked to write their final arguments on paper to share with the rest of the class, to stimulate discussion orchestrated by the teacher.

Data Collection and Analysis

For each school study, based on the grouping tool's and the teacher's suggestions, a number of pairs of students were chosen to be interviewed outside the classroom during their collaboration (in some cases we interviewed all students, whereas in other cases we interviewed at least half a class). The rest of the students worked in pairs in the classroom. This decision served our research purposes as it allowed us to better insight into their ways of thinking in a quiet space (outside a noisy classroom) and where their discussions could be recorded.[6] In the interviews, the two students' models together with their rules were opened in eXpresser on the same machine so the pair could explore and interact with the models if they wished. A snapshot of each student's final model and rule in eXpresser was also presented colour-printed on paper in front of the students. Students' interactions were video-recorded and their verbal discussions recorded, transcribed, and analysed. The students were encouraged to write some arguments to prepare for their discussions, as mentioned earlier, and this written work provided more data as to how they expressed their arguments in written as well as verbal form.

When analysing the 24 transcripts, our focus was on the students' strategies in supporting their arguments to their peer. By the time the study was undertaken, students in years 8 and 9 had been introduced to generalisation activities in their normal classes whereas this was not the case for the younger year 7 students. But, as we understood from the teachers, no student had participated in any discussions on the topic of equivalence of algebraic expressions. Our interest, therefore, extended to seeing how students would reflect upon their general rules prior to the collaborative activity as well as during the discussions with their peers and assess their understanding of a general rule.

Throughout their interactions with eXpresser, support was provided through the intelligent support of the system, the teacher and the researchers. During collaboration, the researcher's role was to initiate the discussions by introducing the activity and intervened only to clarify issues that were raised by the students by asking them to provide more details for their explanations. The researcher encouraged students to ask their fellow student questions if they were unsure of the other's rule or their arguments. They reminded occasionally students of their collaborative activity and prompted them to clarify their arguments or ideas to their peer (especially if they seem confused and didn't take the initiative of asking if they were unsure).

The students' discussions, orchestrated by a researcher as described above, were audio-recorded, transcribed, and analysed qualitatively following the *open coding, axial coding*, and *selective coding* analytical processes as described by Strauss and Corbin (1998). To facilitate this process, the Transana[7] software was used, which allowed us to annotate all the transcripts and create codes that later formed categories and themes. During the open coding process, the transcripts were conceptualised line-by-line to create codes that were constantly compared, renamed, modified, or even merged into new concepts so as to reach a saturated list of codes. After constant comparison of the data and several modifications of the derived concepts, and an intent to sharpen the emerging theory, a number of categories that described the justification strategies students were using to argue about their rules' correctness and equivalence were generated. Once a first set of categories had been established, the raw data were revisited to assess their validity and evaluate whether all the justification strategies used by students were adequately captured. When this was not the case, a new category was incorporated and validated against the data. This iterative cycle was repeated a number of times until the set of justification strategies was finalised.

After the categories and subcategories were finalised and we reached what Straus and Corbin refer to as "theoretical saturation" (Strauss & Corbin, 1998), students' different justification processes were grouped under two themes: (a) Justification for Correctness and (b) Justification for Equivalence. It was at this point that we integrated all major categories and formed a coherent theoretical scheme (*coding scheme*) to present our conceptualisation of students' approach towards justifying their generalisations. In what follows we outline the classification of strategies based on the coding scheme in its final state and show how it relates to the theoretical framework we outlined at the outset.

Justification Strategies

Justification for Correctness

During the first part of the collaborative activity, students in their pairs were asked to convince each other that their rules are correct. In their efforts to give reasons for their rules being correct and therefore convince their pair, all students referred to the arguments they had written during the reflective action from the construction phase as preparation for their discussions and used them to start off their discussions.

During their collaborations, students used a number of different strategies as identified in the data. Note that 18 out of 48 students used more than one strategy to justify the rules' correctness. Some students used their models to match each part of the rule to the corresponding coloured pattern of their model. Such an approach was characterised as *constructive justification* (number of responses: *26*). For example, a Year 7 student, Janet, said for her model (see Figure 6.3):

> Janet: that's the number of red. But there's 3 tiles in each of the blue building blocks, and the one plus 6 is because the 6 is the number of these and there's always one more here, there's one more of blue building blocks than the red because of the green one on the end and then the 2 green ones.

Other students used their models to explain the coefficients in their rule based on the number of tiles in each repeated building block. Such an approach was characterised as *structural justification* (number of responses: *29*). For example, a Year 7 student, Patricia, said:

> *Patricia:* we have the blue bits, which is 2 of them, each are blue, and then we have the red block. And the other 2 tiles. So that makes 4 blue […] then add it together with red. Add these two together.
>
> [That's the same with Emily's because she is talking about the C].

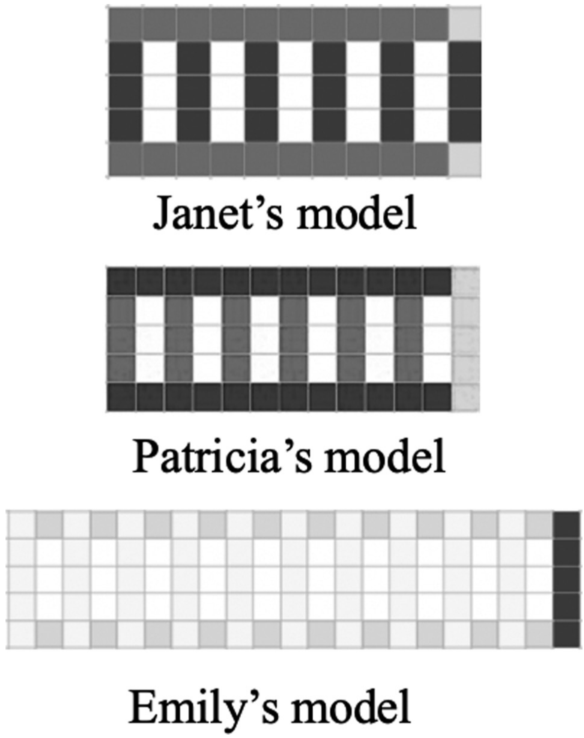

Figure 6.3 Three example models.

In both Patricia and Janet's strategy, we see traces of the construction each had pursued, and in both cases, the justification clearly took for granted that the audience (i.e., the other student) had participated in a similar activity. Indeed, in these cases, it is not difficult to share the situatedness of each abstraction, in which the constructive aspects of the spoken generalisation are expressed in the medium of the activity (e.g., "add it together with red"). These students directly related either the different patterns in their models to the different terms in their rules, or the number of tiles in their constructed building blocks to the coefficients in their rules.

As we might expect, students often used examples as a "proof" for their conjectures. Such an approach was characterised as *empirical justification* (number of responses: *10*) and was chosen by a few students as a strategy that allowed them to test their rules with the help of the eXpresser's feedback, that is, if their rule was correct, choosing a different value for the independent variable would maintain the model's colouring. For example, a Year 7 student, Emily, said:

Emily: We answered correct when we typed in the model number.

Emily tried a number of different values for the model number and each time her model remained coloured and did not “mess-up”.[8] The idea of the computer “not being wrong” seemed to play a crucial factor in her judgement as in addition to her tendency to focus on a few cases to generalise for *any* case, she cleverly relied on eXpresser’s immediate feedback. During her collaboration with her pair, Patricia, Emily was challenged to use a different strategy later in their discussions that of constructive justification, as is evident from the results presented in Table 6.2, described below. Similarly, Susan and Tod, the only Year 8 students who used the empirical justification, only used this strategy initially, and later relied on a different one as their collaboration in their pairs progressed. In fact, all the Year 9 students used the empirical justification strategy as an additional one to the ones they used initially (which were mainly the constructive or the structural ones), to support further their arguments with the support from the eXpresser’s immediate feedback.

On one occasion, it was interesting to hear the following dialogue of two students:

NANCY: I understand the rule so I don’t see a reason why it shouldn’t be correct.
JANET: Yeah. I understand yours.
RESEARCHER: So you don’t see any reason why it’s not correct either?
JANET: No.

Mainly Nancy, but also Janet influenced by her partner, failed to see the need to justify their rules. Their correctness was so obvious in their minds and so “understandable” that justification seemed unnecessary. In these cases, the relationship between the generic and general seems to be self-evident, thanks to the relationship between construction and expression: the situated abstraction goes something like this – “if you build it like that, and you say what you’ve built, it must be right”. Such an approach was characterised as *authoritarian justification* (number of responses: *3*). It is worth mentioning though that after their initial reaction to use the authoritarian strategy, as their discussions continued, they used other strategies too. The same outcome holds for Emily and Maria, as will be presented below in the summative results in Table 6.2.

Summative Results

In Table 6.2, we summarise the results by presenting the different strategies students used in their pairs. Each column that corresponds to a strategy is split into two cells to distinguish through the use of a tick whether the student named first or the one named second in the pair or both students used that strategy. As mentioned earlier, some students used more than one strategy. For example, Emily and Patricia were two

Table 6.2 Justification Strategies for Rules Correctness Used by Each Pair of Students from Each School. A Tick Means That the Correspondent Student Has Used That Strategy. For Example, Emily Used Both Constructive and Empirical Justification, Whereas, Patricia, Only Structural Justification

	STUDENTS	*Constructive*		*Structural*		*Empirical*		*Authoritarian*	
School A (Yr 7)	Emily + Patricia	3			3	3			
	Alex + Anne	3	3						
	Janet + Nancy	3		3				3	3
School B (Yr 7)	Alan + Simon			3	3				
	Fiona + Jackie	3	3	3	3				
	Susan + Dorothy	3	3	3	3				
School B (Yr 8)	Neil + Rex	3			3				
	Randy + Susan	3					3		
	Tod + Ally		3			3			
School C (Yr 8)	Colin + Lara			3	3				
	Maria + Mike	3	3	3	3				
	Penny + Leo			3	3				
	Alicia + Greg		3		3				
	Abigail + Mark	3	3						
	Scot + Louise	3		3	3				
	Amy + Nick			3	3				
	Eleanor + Trevor			3	3				
School C (Yr 9)	Andy + Penny	3	3			3	3		
	Bill + Dave	3		3		3	3		
	Carey + Teddy	3			3	3			
	Nick + Scot			3	3	3			
	Colin + Maria	3		3					3
	Eleanor + Mark	3	3		3				
	Greg + Leo	3	3		3	3			
		16	10	13	16	7	3	1	2
	TOTAL	26		29		10		3	

Year 7 students from school A. Emily used the constructive and the empirical strategies, whereas Patricia used the structural one only.

The last row in Table 6.2 reveals the total number of students who used each strategy. Out of the 68 times all strategies were used, the constructive strategy was used 26 times and the structural 29. Thus a justification strategy relevant to the structure of the constructed model was used to support the rules' correctness 55 times. In total, about four-fifths of students' strategies relied on the structure of the model.

Given the size of the sample, we do not of course claim causality. However, we do conjecture that there could be some link to the main Train-Track activity students had worked on prior to their collaborative activity. It seems that the system's design to encourage students to focus on the structure prevailed against their tendency to focus on recursive rather than functional relationships, a strongly prevalent tendency revealed in other studies (e.g., Lee, 1996; Stacey, 1989). Even though we noticed that a few students (2 Year 7 and 1 Year 8) followed the empirical strategy initially, we can see how their further interaction with eXpresser and their constructed models during their collaboration shaped their thinking in a direction that takes into account the structure of the pattern.

Justification for Equivalence

During the second part of the collaborative activity, students in their same pairs compared each other's rules and discussed their equivalence. They were asked: *Can you explain together why your rules look different but are equivalent? Discuss and write your explanations.*

After extensive analysis, coding and recoding, the data ended up grouped into three main categories: *structural, symbolic,* and *empirical*, described below along with their subcategories. Note that 22 out of the 48 students used more than one strategy to justify the rules' equivalence.

Structural Justification for Equivalence

Justifications in this category all focused on the structural aspect of the pattern by, for example, comparing the building blocks used in the different patterns and making arguments as to their equivalence with little if any reference to the symbolic rule. We distinguished three subcategories illustrated below with data from the study.

1. Reconstructive Justification (number of responses: *30*). In this subcategory, different building blocks are compared or reconfigured as illustrated by the case of Janet and Nancy (see Figure 6.4).

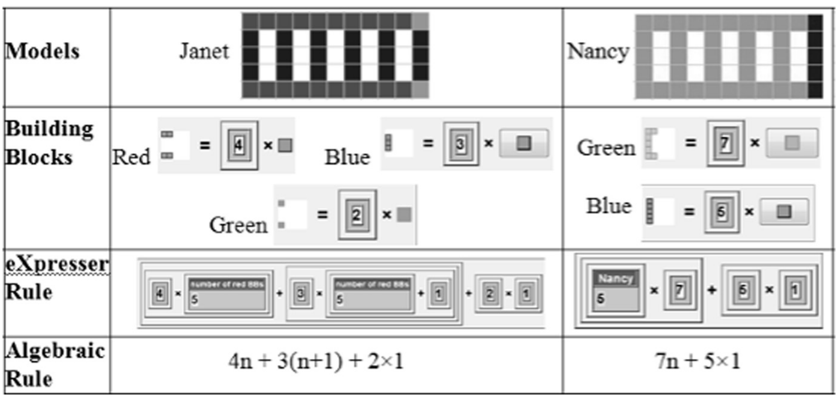

Figure 6.4 Janet and Nancy's model, building blocks and general rules.

Nancy compared her building block with that of Janet's:

NANCY: Yeah it's one red building block plus one blue building block so that would actually kind of make the…

JANET: Yeah, it would make the same shape…

NANCY: Because one red building block added to one blue building block…

JANET: And that's the same as one of my green building blocks.

Students complemented each other's arguments and concluded that their building blocks were the same. They either explicitly related the models to their rules or linked the number of tiles in each block to the coefficients in the algebraic expressions. Rather, they simply compared the building blocks underlying the patterns used, basing their verbal interactions on their shared experience of construction and (algebraic) expression.

2. Experimental Justification (number of responses: *8*). In this subcategory, students chose a specific case and compared their two models and rules for this case, as illustrated by Alex and Anne.

ALEX: I kind of got a C, but coloured them in different ways so I mean the 5 is only added at the end…

ANNE: Then there are just 7 tiles in one model.

ALEX: Yes, but your first model has 12 tiles and your second model has 7 tiles. For 5 red blocks I have 5 blue extra tiles, but you have 12 blue extra tiles.

Anne was able to read Alex's rule and recognised the configuration of tiles that formed a similar building block to hers. Yet it was

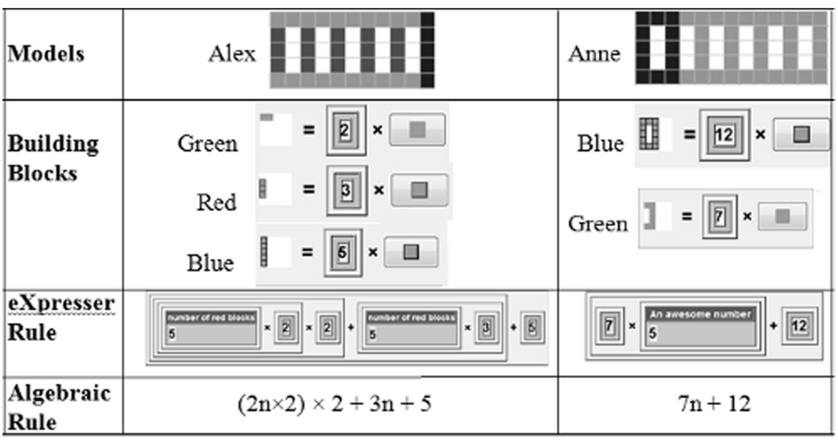

Figure 6.5 Alex's and Anne's model, building blocks, and general rules.

evident that both students considered each building block as a separate model. At first, Alex chose to change the number of red blocks in her model to 5 to match Anne's model, but then realised that it was just not possible to match: the two models, in fact, had different constant terms. Alex decided to compare the two models for the *same* model number and then justified the non-equivalence of the two rules.

This strategy was used by six other students, who all chose to select a value for their model number and use it in both rules (their own and their pair's one), but then compared their models structurally focusing mainly on their length, but also on their building blocks. Such a strategy is mathematically valid for justifying non-equivalence (as demonstrated by Alex and Anne's example above), since finding one case for which two rules are not equivalent is enough to disprove equivalence. However, experimenting with one case is not enough to generalise the equivalence for any case.

3. Justification by Contradiction (number of responses: 7). Students used the same model number and calculated the number of tiles used, and noticed that they obtain different answers, as illustrated by Amy and Nick (Figure 6.6). They had to go back to comparing their models structurally.

Nick noticed that for the same value of the independent variable, their models could *never* be the same. His justification was based on a contradiction expressed both numerically and structurally.

There were five more students that used justification by contradiction and they all resorted to compare the structure of their models.

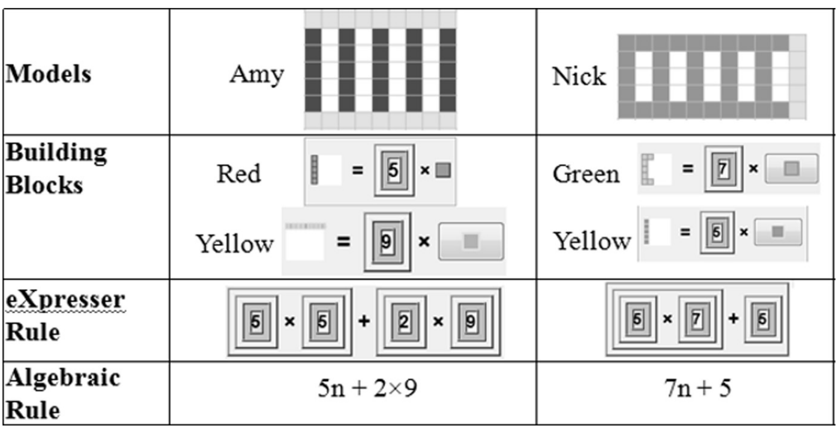

Figure 6.6 Amy's and Nick's model, building blocks and general rules.

Symbolic Justification (Number of Responses: 21)

This category comprised student justifications focused on their eXresser rules and justified their equivalence by adding the constants and variables in each rule and comparing them as illustrated by Leo and Penny's case (Figure 6.7).

When paired, Leo realised that his rule was incorrect, but was able to derive a correct general rule that he wrote on paper as [5] × 9 – [5] × 2 + 5. This is what they both compared with Penny's rule.

LEO: I had 5 times 9 because I had 9 things but I have to take away 2 of my red building block, so I have to take away 10 tiles because I need to have 5 sevens. I had that many on the end of each one [pointing at his model]. That is why I have to take away 2 and then plus 5 because I need an extra line at the end. The 9 minus 2 is equal to plus 7 and the 5 is the same and then the 5 is the same so they're the same rule but written differently.

PENNY: Mine is 5 times 1 plus 8 times 7. These 8 times 7 because we've got 8 of the 7 blocks and so 8 times 9 minus times 2 is 8 times 7.

They concluded that Leo's second rule on paper was equivalent to Penny's rule. In this example, the value of reflecting on one's own rule, triggered by collaboration is revealed.

There are 19 more students that followed this strategy and as shown in Table 6.3, the majority were Year 9 students. This might be due to their greater experience compared to Year 7 and 8 students with algebraic language. Two Year 9 students, Eleanor and Carey, resorted initially to the symbolic justification (see Table 6.3) and then used some type of a structural justification to support further their arguments and visually justify their rules' equivalence or non-equivalence. The

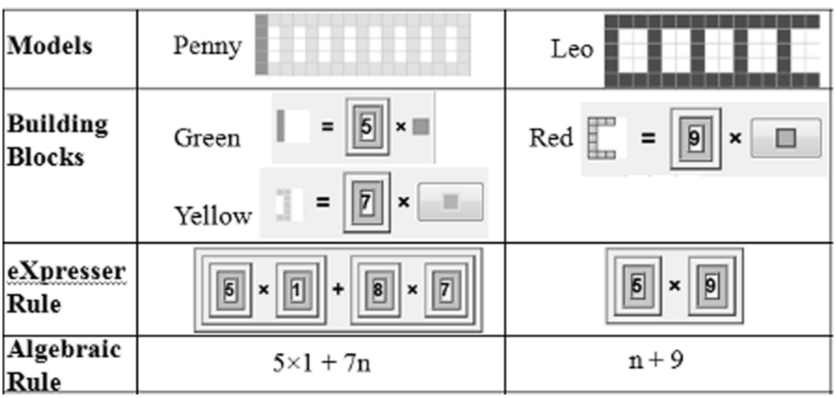

Figure 6.7 Penny's and Leo's model, building blocks, and general rules.

Table 6.3 Justification Strategies for Rules Equivalence Used by Each Pair of Students from Each School. Similarly to Table 6.2, a Tick Means that the Correspondent Student Has Used That Strategy. For Example, Emily Used Only the Symbolic Justification Strategies, Whereas Her Partner, Patricia, Used Only the Reconstructive Justification Strategy

	STUDENTS	*Rec.*		*Exper.*		*Contr.*		*Sym.*		*Match*		*Eval.*	
School A (Yr 7)	Emily + Patricia		3					3					
	Alex + Anne	3	3	3	3	3	3			3		3	
	Janet + Nancy	3	3					3					
School B (Yr 7)	Alan + Simon			3	3					3	3		
	Fiona + Jackie	3	3					3	3				
	Susan + Dorothy	3	3					3	3				
School B (Yr 8)	Neil + Rex		3	3									
	Randy + Susan	3	3										
	Tod + Ally	3	3										
School C (Yr 8)	Colin + Lara	3	3										
	Maria + Mike							3	3				
	Penny + Leo		3					3	3				
	Alicia + Greg	3	3										
	Abigail + Mark	3	3	3	3								
	Scot + Louise	3	3			3							
	Amy + Nick				3	3	3		3			3	
	Eleanor + Trevor	3	3					3					
School C (Yr 9)	Andy + Penny							3	3	3			
	Bill + Dave	3						3					
	Carey + Teddy	3						3				3	3
	Nick + Scot	3	3								3		
	Colin + Maria							3	3				
	Eleanor + Mark		3			3	3	3					
	Greg + Leo							3		3			3
		13	17	4	4	4	3	13	8	3	2	3	2
	TOTAL	30		8		7		21		5		5	

rest of the students started off structurally and then moved on to symbolic justification.

Empirical Justification for Equivalence

Some students focused solely on the numerical aspect of the rules, avoiding any reference to the structure of their model constructions. Two subcategories were distinguished:

1. Matching-Terms Justification (number of responses: 5). In this category students pick a constant or a variable and compare with the

equivalent term in the other students' rules. Here is Alex at an early stage of her collaboration with Anne:

ALEX: They both have 7 in them plus something to make the end of the pattern.

She picked a constant in her rule and identified it in Anne's rule too (see Figure 6.10). She noticed the similarities in the algebraic expressions, but also the difference in the added constant term (5 in Alex's rule, but 12 in Anne's rule). This initial reaction to the collaborative activity reveals their tendency for an exploration of the rules, a rather important problem-solving skill, and does allow them to "read" both their rules and identify commonalities and differences.

Only two Year 9 students, Andy and Scot, as presented in Table 6.3, used this strategy after they used a symbolic and a structural justification strategy respectively. It seems that this helped them to simplify further the rules and prove that their rules were equivalent and identified their simplest form as 7n+5. Similarly, two Year 7 students, Alan and Simon, as also presented in Table 6.3, used the matching-terms strategy to support further their arguments after using the experimental justification strategy. Both these strategies revealed those students' tendency to compare models first and then rules for specific cases preventing them from focusing on the general case and their rules' equivalence for any value for the model number. Alex, who was the other Year 7 student, used this strategy too. She seemed eager to use many different arguments to support her view on her pair's rule not being equivalent to hers and she was the only student to use five different strategies.

2. Evaluating-Terms Justification (number of responses: 5). In this category, students compared the number of tiles for different model numbers. Later in their discussion, Alex chose a value for the independent variable and compared the answers for the two rules:

ALEX: Model number 1 is blue blocks and it's got 12 tiles in total. The backwards C is model number 2. So, we have 12 plus 7…19 tiles.

ANNE: No, model number 2 is 2 backward Cs plus the blue block. So, 2 times 7 plus 12…26 tiles.

Anne's answer included seven more tiles because of the blue block she had added to her model. The students were confused at this point as to what the model was and what the model number was. This was the last strategy Alex used to support her argument on non-equivalence of their rules.

All three Year 9 students who used this strategy, except for Teddy, which was his only strategy, used it as an additional justification strategy, but focused mostly on their original one, which was the symbolic strategy. A similar story holds for the only Year 8 student, Amy, who used the evaluating-terms justification strategy.

Summative Results

In Table 6.3, we summarise the results by presenting the different strategies students used in their pairs to justify their rules' equivalence or non-equivalence. The same structure as for Table 6.2 is used. This means that each column that corresponds to a strategy is split into two cells to distinguish through the use of a "tick" whether the student named first or the one named second in the pair or both students used that strategy.

The last row in Table 6.3 reveals the total number of students who used each strategy. Out of the 76 times all strategies were used, a type of the structural justification strategy was used 45 times with the reconstructive strategy dominating (30 times) and the symbolic one 21 times: the empirical strategy was the least used (10 times). About three-fifths of the time, students' strategies relied on the structure of the model and two-fifths on the rules alone through manipulation of the terms. Compared to the first part of the collaborative activity on discussing correctness, students seem to rely mostly on the way they constructed their models. Some of the students, the majority of whom were in Year 9, focused on manipulating their rules algebraically to transform them into their simplest form. These students revealed their confidence in using the eXpresser language but crucially related it to formal algebraic language identifying correctly and manipulating successfully constants and variables. Despite the obvious limitations of this quantitative overview, it suggests that a focus on structure remained the dominant choice for students. This, in combination with the general tendency in the literature and our previous anecdotal observations and evidence that seemed to suggest a prevalence of empirical justifications, supports the design decisions of the eXpresser and how it can act as a context for the collaboration activities.

Discussion

The data in the previous section support the value of reflective and collaborative activities. Similarly to many researchers we mentioned earlier, such as Ellis (2011), Wood (1988) and Lou et al. (1996), students engaged in acts of argumentation to support their own solution strategies, but also recognise and understand those of their peers. Such actions enhanced their justification skills and their algebraic thinking. For example, there were many cases, both during individual and initial collaborative work, that manifested students' particular misconceptions such as their tendency to focus on the additive principle as has been reported before (e.g., Hart, 1981; Lee, 1996; Stacey, 1989) or what the general rule is and what "n" represents. In our case, the crucial point is that the collaborative activity helped students to share their misunderstandings and support each other to overcome them *because they had an object to share.*

Besides challenging misconceptions about generality, putting students in a collaborative setup to discuss their modelling approaches helped them reflect on the components of their models in relation to the corresponding rules. From judging the justification strategies students used, it seemed that most students made sense of the "unknown" as they were given a rationale for its use (c.f. Filloy et al., 2008). Also, students were able to manipulate the rules and avoid mistakes, such as $Ax + B = (A + B)x$, as they could easily identify the components in their models' rules.

In terms of forming heterogeneous groups, as suggested by Leonard (2001), our first criterion was for students to have derived *different* rules for the argumentation activity to be meaningful. The second criterion, the students' characters and which pairs would collaborate sensibly and constructively, relied a lot upon the teacher who advised us based on the suggested pairs by the Grouping Tool. Students were prompted by the activity questions and also the researchers and the teacher reminded them on a regular basis to share arguments and consider all their different approaches. This established a collaborative learning approach as emphasized by Lou et al. (1996) and promoted a collaborative "knowledge building" culture as described by Scardamalia and Bereiter (2006), which led students to use more than one justification strategy, resolve contradictions, and therefore challenged and broadened their algebraic ways of thinking.

Revisiting Dretcke's (1990) distinction between sensory and cognitive mode as two types for the act of coming to visually perceive a pattern, it can be argued that the distinction between the two relies crucially on the tools for expression the student is given as well as the context in which they are used. In the case of MiGen, it can be argued that when students interacted individually with the eXpresser, they could have seen patterns and their components as mere objects (sensory) and they could have derived expressions to "link" them procedurally. However, when they became involved in the collaborative activity, they were encouraged to reflect upon these objects, recognise their properties, and argue about their expressions-rules (cognitive). Even though eXpresser is designed to support students' expressions of generalisations, we argue that it is students' engagement in acts of justification through the collaborative activity that supported their understanding and "forced" them to reflect upon their visual perception of the figural patterns.

Since students are usually presented with the method of generalising based on noticing a commonality among the terms in a pattern sequence, one outstanding challenge posed by some researchers is to understand how students come to generalise what they notice to the *whole pattern* (e.g., Radford, 2010). In eXpresser, this turns out to be relatively straightforward, as with eXpresser's functionality of building a pattern using a building block, the arbitrary repetition of this building block when the model is, animated, shows how the model is "generalised". We found that structural and reconstructive justifications were the main strategies used, a result that seems to align with students' intuitive explanation of their construction method. As also claimed by Ellis (2007), Küchemann (2010), and Arcavi et al. (2017), attending to the structure of a pattern has a better chance of justifying students' expressions generally and the findings from our studies seem to support this claim as the students successfully reached a good understanding of generality as demonstrated by their justification strategies.

The choice of a "unit of repetition" or (in the eXpresser's language) a "building block" proved to be a crucial step towards a correct generalisation, as suggested by Rivera (2010). Supported by the intelligent support of the MiGen system, students could find the minimum number of tiles that could be grouped into a building block and then repeated to form a pattern. Those who didn't were challenged by their pair during collaboration (see, for example, the case of Anne, who chose 12 tiles for her blue building block instead of five, and was challenged by Alex), an outcome that emphasises the value of collaboration towards forming correct generalisations.

Students' investment in building their own models supported them in deriving generalisations by directing their focus towards relationships between quantities. The algebraic discourse of the eXpresser – the grammar of objects and relationships between them – gave students a means to express generalisation without the formal machinery of algebra. We argue that students were supported in expressing a general rule by the eXpresser's language that gave meaning to the 'unknown', allowing them to name it as it made sense to them and use this as an intermediate step to formal algebraic language. Therefore, we supported the transition from natural language, as "always" and "every", which is more intuitive for students (e.g., Warren & Cooper, 2008), to formal algebraic language. This action, which we refer to as "expressive action" in Figure 6.1 and which could be linked to the "symbolic action" described by Rivera (2010), was supported further by the collaborative activity when students were encouraged to reflect upon their expressions and endorse them.

In more detail, during the collaboration phase, students were encouraged to revise their rules and identify the constants and the variables. The metaphor of the unlocked number was particularly salient here. Perhaps surprisingly all the students realised that in this case this was also the model number, "the number that can change" or in other words the *variable*.

An interesting observation is that some students who used algebraic symbolic justification responded in ways that could allow us to assume that the transition to the use of letters to represent the unknown seemed easier for them. For example, Penny, during collaboration:

RESEARCHER: How many tiles in model number 6?
PENNY: So it will be 5 times 1 plus x times 7.
RESEARCHER: And what's x?
PENNY: x is 6.

Although it is too early to claim that this is the case and despite lack of specific data on this question directly, we can surmise based on the above and similar interchanges that students mostly seemed to understand what the "n" stands for and equally important, convinced of a rationale for having a general rule. Out of all year groups, most students (nine in total) who used a symbolic justification strategy were Year 9 students (six were Year 7 and six were Year 8), which could mean that they are more experienced with algebraic notation or they have gained more expertise in the use of eXpresser.

The difference between algebraic thinking and algebraic symbolism is evident in these data. Students were able to express generality verbally and in written form. If we consider the trajectory, there is a change from the first time they interacted with the eXpresser and their latest interactions in terms of their expressions of generality. By the end, most students were able to write down the rules using the eXpresser language (not the boxes, but e.g., 5 × Model Number + 3, or 5 × Unlocked Number + 3).

What is encouraging is that most students' activities after the engagement described here tended to continue to focus on the structure of the patterns in order to articulate the general rule. In their efforts to justify their general rules, students revisited their generalising actions, built on them, and constructed ones that were more powerful and meaningful. They succeeded in reaching rich justifications for the correctness and equivalence of their derived algebraic expressions for the linear Train-Track pattern.

The findings point to the students' preference for referring to the structure of their models to justify correctness of their rules, since most students (55 out of 68 times all strategies were used or 81%) used either the constructive or the structural justification strategies. There was a similar preference when students justified the equivalence of their rules, since most students (45 out of 76 times all strategies were used or 59%) used a type of the structural justification strategy. The second most common strategy (21 out of 76 times all strategies were used or 39%) in terms of equivalence was symbolic justification. This result supports the usefulness of the eXpresser for students' introduction to algebra and the collaborative activity for a possible introduction to proof (as a next step from justification).

In summary, we can claim that students' engagement in acts of justifying through collaboration seemed to support their generalisation skills in a number of ways that is: recognise the importance of seeing the structure, find the constants and the variables in their model and rule, express relationships using an independent variable to link patterns within their models and see the rationale and recognise the power of structural sense and algebraic generalisation. But to achieve this they had engaged in a carefully designed sequence of activities in the context of the eXpresser, which we suggest played a key role in their learning outcomes and assisted the integration of transition from arithmetic to algebra.

Acknowledgements

Early stages of the analysis of the data discussed in this chapter were presented at the 35th Conference of the International Group for the Psychology of Mathematics Education, 10–15 July 2011, in Ankara, Turkey. The reader is referred to Geraniou, E., Mavrikis, M., Hoyles, C. and Noss, R. (2011). "Students" justification strategies on equivalence of quasi-algebraic expressions', *PME35: International Group for the Psychology of Mathematics Education,* Vol. 2, pp.393–400. 10th–15th of July 2011, Ankara, Turkey. We would like also to thank Professors Celia Hoyles and Richard Noss for their leadership in the MiGen project as well as the rest of the MiGen team for their support and ideas in the work presented here.

Notes

1 For this and other design priorities and rationales, the reader is referred to Noss et al. (2012). Mavrikis et al. (2013b) provides a more detailed discussion on how the design of eXpresser supports the development of algebraic ways of thinking.
2 To help the reader, the number of white gaps inside the Train-Track model could be mapped to the value of the Model Number.
3 Apart from immediate feedback from the visualisations and representations of the microworld (e.g., lack of colouring of a pattern) the MiGen system incorporates intelligent components that analyse students' activities and provide explicit feedback on their actions. This involves nudges to draw students' attention to inconsistencies in their model compared to the activity model, for example, the lack of colouring or structural generality of the pattern and other prompts to help them reflect explicitly on their actions, especially when they request additional help. For more details and examples, see Noss et al. (2012) and Mavrikis et al. (2013a).
4 All numbers in eXpresser are *constants* by default, referred to as "locked" numbers. When the user "unlocks a number", it is possible to change its value; it becomes a *variable*.
5 For more information on the Grouping Tool, see Noss et al. (2012) and Gutierrez-Santos et al. (2017).

6 The collaborative activity, though, is designed to be carried out in the classroom where the teacher is expected to run a classroom discussion at the end of the lesson.
7 http://www.transana.org/
8 When interacting with students, a pedagogical strategy, referred to as "messing-up" (Healy et al., 1994) was used. This strategy challenges students to construct models that are impervious to changing values of the various properties of their construction.

References

Abdu, R., Olsher, S., & Yerushalmy, M. Towards automated grouping: Unraveling mathematics teachers' considerations. In Barzel, B., Berbinik, R., Göbel, L., Pohl, M., Ruchniewicz, H., Schacht, F., & Thurm, D. (Eds.) *Proceedings of the 14th International Conference on Technology in Mathematics Teaching-ICTMT 14* (Essen, Germany, July 2019). [S.147] https://duepublico2.uni-due.de/receive/duepublico_mods_00048820

Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The learning and teaching of algebra: Ideas, Insights and Activities*. New York: Routledge.

Bereiter, C., & Scardamalia, M. (2003). Learning to work creatively with knowledge. In E. D. Corte, L. Verschaffel, N. Entwistle, & J. V. Merriënboer (Eds.), *Powerful learning environments: Unravelling basic components and dimensions* (pp. 73–78). Oxford: Elsevier Science.

Bills, L., & Rowland, T. (1999). Examples, generalisation and proof. In L. Brown (ed.), *Making meanings in mathematics* (pp. 103–11). York: QED.

Carter, G., & Jones, G. M. (1993). *The relationship between ability-paired interactions and the development of fifth graders' concepts of balance*. Paper presented at the Conference of the National Association of Research in Science Teaching: Atlanta. (ERIC Document Reproduction Service No. ED 361 175).

Chua, B. L., & Hoyles, C. (2011). Secondary school students' perception of best help generalising strategies. *Paper presented at the Seventh Congress of the European Society for Research in Mathematics Education*. Rzeszów, Poland.

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal*, *20*(1), 41–53.

Cohen, E. G., & Lotan, R. A. (1995). Producing equal-status interaction in the heterogeneous classroom. *American Educational Research Journal*, *32*(1) resulta), 99–120.

Dörfler, W. (2008). En route from patterns to algebra: Comments and reflections. *ZDM – The International Journal on Mathematics Education*, *40*(1), 143–160.

Dretske, F. (1990). Seeing, believing, and knowing. In D. Osherson, S. M. Kosslyn, & J. Hollerback (Eds.), *Visual cognition and action: An invitation to cognitive science* (pp. 129–148). Cambridge, Massachusetts: MIT Press.

Ellis, A. B. (2007). Connections between generalising and justifying: students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, *38*(3), 194–229.

Ellis, A. B. (2011). Generalising-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, *42*(4), 308–341.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 12–25.

Filloy, E., Rojano, T., & Puig, L. (2008). *Educational algebra; a theoretical and empirical approach*. Springer: New York.

Filloy, E., Rojano, T., & Solares, A. (2010). Problems dealing with unknown quantities and two different levels of representing unknowns. *Journal for Research in Mathematics Education*, *41*(1), 52–80.

Fuchs, L., Fuchs, D., Karns, K., Hamlett, C., Katzaroff, C., & Dutka, S. (1998). Comparisons among individual and cooperative performance assessments and other measures of mathematics competence. *The Elementary School Journal*, *99*(1), 23–52.

Geraniou, E., Mavrikis, M., Kahn, K., Hoyles, C., & Noss, R. (2009). *Developing a Microworld to Support Mathematical Generalisation. Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, 49–56). Thessaloniki, Greece.

Gutierrez-Santos, S., Mavrikis, M., Geraniou, E., & Poulovassilis, A. (2017). Similarity-based grouping to support teachers on collaborative activities in an exploratory mathematical microworld. *IEEE Transactions on Emerging Topics in Computing*, *5*(1), 56–68.

Harel, I., & Papert, S. (1991). *Constructionism*. Norwood, N.J.: Ablex Publishing Corporation.

Harel, G., & Tall, D. (1991). The general, the abstract and the generic in advanced mathematics. *For the Learning of Mathematics*, *11*(1), 38–42.

Hart, K. (1981) Ratio and proportion. In *Children's understanding of mathematics 11–16*, edited by Kathleen Hart, 88–101. London: John Murray.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428.

Healy, L., Pozzi, S., & Hoyles, C. (1995). Making sense of groups, computers and mathematics. *Cognition and Instruction*, *13*(4), 505–523.

Healy, L., Hoelzl, R., Hoyles, C., & Noss, R. (1994). Messing up. *Micromath*, *10*(1), 14–17.

Kaput, J. (1999). Teaching and learning a new algebra. In T. Romberg & E. Fennema (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Hillsdale, NJ: Lawrence Erlbaum.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner, & C. Kieran (Eds.), *Research Issues in the learning and teaching of algebra*. (pp.33–56). VA: LEA.

Kieran, C. (2001). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. *Educational Studies in Mathematics*, *46*(13), 187–228.

Küchemann, D., & Hoyles, C. (2009). From empirical to structural reasoning in mathematics: Tracking changes over time. In D. Stylianou, M. Blanton & E. Knuth (Eds.), *Teaching and learning proof across the grades K-16 perspective* (pp. 171–191). London: LEA.

Küchemann, D. (2010). Using patterns generically to see structure. *Pedagogies: An International Journal*, *5*(3), 233–250.

Laborde, C. (1994). Working in small groups: A learning situation? In R. Beiler (Ed.), *Didactics of mathematics as a scientific discipline* (pp. 147–157). Dordrecht: Kluwer.

Lavy, I., & Leron, U. (2004). The emergence of mathematical collaboration in an interactive computer environment. *International Journal of Computers for Mathematical Learning*, *9*(1), 1–23.

Lee, L. (1996). An initiation to algebraic culture through generalization activities. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching* (pp. 87–106). Dordrecht, The Netherlands: Kluwer Academic.

Leonard, J. (2001). How group composition influenced the achievement of sixth-grade mathematics students. *Mathematical Thinking and Learning*, *3*(2&3), 175–200.

Linchevski, L., & Kutscher, B. (1998). Tell me with whom you're learning, and I'll tell you how much you've learned: Mixed-ability versus same-ability grouping in mathematics. *Journal for Research in Mathematics Education*, *29*(5), 533–554.

Lou, Y., Abrami, P., Spence, J., Poulsen, C., Chambers, B., & D'Apollonia, S. (1996). Within-class grouping: A meta-analysis. *Review of Educational Research*, *66*(4), 423–458.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra – perspectives for research and teaching* (pp. 65–86). The Netherlands: Kluwer Academic Publishers.

Mavrikis, M., Gutierrez-Santos, S., Geraniou, E., & Noss, R. (2013). Design requirements, student perception indicators and validation metrics for intelligent exploratory learning environments. *Personal and Ubiquitous Computing*, *17*(8), 1605–1620. doi: http://dx.doi.org/10.1007/s00779-012-0524-3.

Mavrikis, M., Noss, R., Geraniou, E., & Hoyles, C. (2013b). Sowing the seeds of algebraic generalisation: Designing epistemic affordances for an intelligent microworld. In R. Noss & A. DiSessa (Eds.), *Special issue on knowledge transformation, design and technology, journal of computer assisted learning*, **29**(1), 68–85. doi: 10.1111/j.1365-2729.2011.00469.x

Moss, J., & Beatty, R. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *Computer-Supported Collaborative Learning*, *1*(4), 441–465.

Mercer, N. (1995). *The guided construction of knowledge*. Clevendon: Multilingual Matters.

Mercer, N. (2000). *Words and Minds: how we use language to think together*. London: Routledge.

Mor, Y., Tholander, J., & Holmberg, J. (2005). Designing for cross-cultural web-based knowledge building. In Timothy Koschmann, Daniel D. Suthers, & Tak-Wai Chan (eds.), "*The 10th Computer Supported Collaborative Learning (CSCL) Conference (2005)*" (pp. 450–459). Lawrence Erlbaum Associates, Taipei, Taiwan.

Mor, Y., Noss, R., Hoyles, C., Kahn, K., & Simpson, G. (2006). Designing to see and share structure in number sequences. *International Journal for Technology in Mathematics Education*, *13*(2), 65–78.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics*, *33*(2), 203–233.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht: Kluwer Academic Publishers.

Noss, R., Hoyles, C., Mavrikis, M., Geraniou, E., Gutierrez-Santos, S., & Pearce, D. (2009). Broadening the sense of `dynamic': A microworld to support students' mathematical generalisation. *ZDM*, *41*(4), 493–503.

Noss, R., Poulovassilis, A., Geraniou, E., Gutierrez-Santos, S., Hoyles, C., Kahn, K., Magoulas, G., & Mavrikis, M. (2012). The design of a system to support exploratory learning of algebraic generalisation. *Computers and Education*, *59*(1), 63–81.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, *70*(2), 111–126.

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA-Pensamiento Númerico Avanzado*, *4*(2), 37–62.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 303–322). Berlin: Springer-Verlag.

Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the general. The multi-semiotic dimension of students' algebraic activity. *Journal for Research in Mathematics Education*, *28*(5), 507–530.

Rivera, F., & Becker, J. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear and figural patterns. *ZDM*, *40*(1), 65–82.

Rivera, F. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics*, *73*(3), 297–328.

Scardamalia, M., & Bereiter, C. (2006). Knowledge building: Theory, pedagogy, and technology. In K. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (pp. 97–118). New York: Cambridge University Press.

Stacey, K. (1989). Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics*, *20*(2), 147–164.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: techniques and procedures for developing grounded theory*. London: Sage.

Swan, M. (2006). *Collaborative learning in mathematics: A challenge to our beliefs and practices*. London: National Institute for Advanced and Continuing Education (NIACE); National Research and Development Centre for Adult Literacy and Numeracy (NRDC).

Teasley, S. D., & Roschelle, J. (1993). Constructing a joint problem space: The computer as a tool for sharing knowledge. In S. P. Lajoie & S. J. Derry (Eds.), *Computers as cognitive tools* (pp. 229–258). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Vahey, P., Enyedy, N., & Gifford, B. (2000). Learning probability through the use of a collaborative, inquiry-based simulation environment. *Journal of Interactive Learning Research*, *11*(1), 51–84. Charlottesville, VA: AACE.

Vahey, P., Tatar, D., & Roschelle, J. (2007). Using handheld technology to move between private and public interactions in the classroom. In M. van 't Hooft & K. Swan (Eds.), *Ubiquitous computing in education: Invisible technology, visible impact* (pp. 187–210). Mahway, NJ: Lawrence Erlbaum Associates.

Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics*, *67*(2), 171–185.

Wood, D. (1988). *How children think and learn*. Oxford and Cambridge, MA: Blackwell.

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