Parameter Estimation for a Sinusoidal Signal with a Time-Varying Amplitude

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***Abstract*— This paper addresses the parameter estimation problem of a non-stationary sinusoidal signal with a time-varying amplitude, which is given by a known function of time multiplied by an unknown constant coefficient. A robust estimation algorithm is proposed for identifying the unknown frequency and the amplitude coefficient in real-time. The estimation algorithm is constructed based on the Volterra integral operator with suitably designed kernels and sliding mode adaptation laws. It is shown that the parameter estimation error converges to zero within an arbitrarily small finite time, and the robustness against bounded additive disturbances is certified by bounded-input-bounded-output arguments. The effectiveness of the estimation technique is evaluated and compared with other existing tools through numerical simulations.**

I. Introduction

Parameter estimation of sinusoidal signals has drawn continuously increasing attention in many engineering fields, such as communication, image processing, power electronics engineering, and control systems to mention a few [1], [2], [3], [4].

From the control perspective, online estimation is the crucial field of interest. A large amount of algorithms have been proposed using adaptive and nonlinear filtering techniques [5], [6], [7] and adaptive observers [8], [9], [10]. All of these methods formally guarantee the global stability of the estimator. Nevertheless, these methods are not widely used in real-world applications due to the lack of an extensive and comparable sample of practical case studies. Other, particularly those accommodated in power electronics applications, are using phase-locked-loop schemes [11], [12], [13], [14], where yet global convergence is usually not guaranteed. While the aforementioned algorithms provide asymptotic stability guarantees, there exist a few other methods that are capable to achieve finite-time convergence [15], [16]. This is a desirable feature in several application contexts, *e.g.* in micro-grid systems which is low-inertia and vulnerable to frequency fluctuations usually rely on fast estimation and detection to deliver prompt frequency control [17].

The aforementioned algorithms address in most cases only sinusoidal signals with stationary parameters. Such algorithms could be applied without modification to estimate piece-wise constant parameters, but are not suitable for more general non-stationary sinusoidal signals. From a practical point of view, it is worth discussing the parameter estimation of non-stationary signals, which are common in electric power systems and time-varying parameters can describe the qualitative behavior of the associated system. For example, oscillations with time-varying amplitude in power systems might be the precursor of instabilities due to equipment malfunctions or other faults. Some particular models of non-stationary signals have been studied in the literature. In [9], structured perturbations are modeled and incorporated in a sinusoidal estimation problem so as to represent high-order perturbations, such as the drift phenomena in real-life applications. A sinusoidal signal with an exponentially damped amplitude is investigated in [18], [19], [20]. In [21], an estimation approach is proposed for a sinusoidal signal with a linear time-varying amplitude that is characterized by a first-order time-polynomial. More recently, the work shown in [22] puts forward a class of frequency estimator for a sinusoidal signal with a time-varying amplitude, which is the product of a known time-varying function and an unknown time-invariant constant. Such formulation turns out to be useful for estimating the external wrenching force of robotic manipulators and for monitoring the angular speed of permanent magnet synchronous machines [23].

In this paper, we present a novel frequency estimation method for the non-stationary sinusoidal model proposed in [22]. With inspiration from the idea devised in [16], the proposed methodology employs Volterra operators with a typical class of non-asymptotic kernels functions, which can remove the dependency on the initial conditions. As a consequence, finite-time convergence of the estimation error can be achieved, which enables faster detection of the parameters compared to the convergence of the estimator presented in [22] without increasing the sensitivity to measurement noise. Furthermore, the proposed method can also estimate the unknown amplitude coefficient, which has not been addressed in previous works on this subject [23], [22].

The rest of this paper is organized as follows. Section II formulates the estimation problem for a sinusoidal signal with a non-stationary amplitude. Section III presents the frequency and amplitude coefficient estimation algorithm. Stability analysis and robustness analysis against measurement noise are carried out in Section IV. Section V gives some simulation examples. Finally, we conclude the paper in Section VI.

This work was supported in part by the Guangdong Basic and Applied Basic Research Foundation (2021A1515110262) and in part by the Shenzhen Fundamental Research Project (JCYJ20210324120400003).

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II. Problem Formulation and preliminaries

Consider a non-stationary sinusoidal signal

$$y(t) = \mu\alpha(t)\sin(\vartheta(t)), \tag{1}$$
$$\dot{\vartheta}(t) = \omega \tag{2}$$

where $\vartheta(0) = \phi_0 \in \mathbb{R}$ is the initial phase angle, $\mu\alpha(t) \in \mathbb{R}_{>0}$ is the time-varying amplitude of the signal, and $\omega \in \mathbb{R}_{>0}$ is the frequency. ω, μ, ϕ_0 are unknown constant parameters, while $\alpha(t)$ is a known function.

Assumption 1: $\alpha(t)$ is Lipschitz continuous and bounded of 2nd-order derivatives.

As discussed in [23], there are many applications that is consent to Assumption. 1. For instance, in the case of estimating the frequency of a carrier signal, in wireless communication, $\alpha(t)$ is a known modulating signal which is known and bounded.

The objective of this paper is to design an estimator that can identify ω and μ with a fast convergence rate from the signal measurement $y(t)$. Examples of such problem statement are also commonly seen in the frequency estimation of an amplitude modulated signal and angular velocity estimation for permanent magnet synchronous motors [24].

A key algbraic tool of the proposed estimation method is the Volterra integral operator and non-asymptotic kernel functions. For readers' convinience, basic concepts of the Volterra operator and the non-asymptotic algebra are briefly recalled hereafter. For deeper in sights, readers are advised to refer to [25] and [26] and the reference therein.

Given a function belongs to the Hilbert space of locally integrable function with domain $\mathbb{R}_{\geq 0}$ and range $\mathbb{R}$, i.e., $x(\cdot) \in \mathcal{L}^2_{loc}(\mathbb{R}_{\geq 0})$, its image by the *Volterra operator* $\mathcal{V}_K$ induced by a Hilbert-Schmidt ($\mathcal{HS}$) Kernel Function $K(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is denoted by $[\mathcal{V}_K x](\cdot)$ of the form

$$[\mathcal{V}_K x](t) \triangleq \int_0^t K(t,\tau)x(\tau)d\tau, \quad t \in \mathbb{R}_{\geq 0}\,.$$

To achieve the non-asymptotic convergence, we resorted to an Nth order Bivariate Feedthrough Non-asymptotic Kernel (BF-NK) designed in [26] which has the form of

$$K_h(t,\tau) = e^{-\beta_h(t-\tau)}\left(1 - e^{-\overline{\beta}\tau}\right)^N$$

which is parameterised by $\beta_h \in \mathbb{R}_{>0}$ and $\overline{\beta} \in \mathbb{R}_{>0}$. Such kernel function has an important feature that $K_h^{(j)}(t,0) = 0, \forall j \in \{0,1,\cdots,i-1\}, t \in \mathbb{R}_{\geq 0}$, where $K_h^{(i)}(t,\tau)$ denotes the ith derivative with respect to the second argument τ. As such, the Volterra image of the signal derivative $x^{(i)}(t), i \in \{1,\cdots,N\}$ can be expressed as

$$\begin{aligned}\left[\mathcal{V}_{K_h} x^{(i)}\right] &= \sum_{j=0}^{i-1}(-1)^{i-j-1}x^{(j)}(t)K_h^{(i-j-1)}(t,t) \\ &\quad + (-1)^i\left[\mathcal{V}_{K_h{}^{(i)}}x\right](t).\end{aligned} \tag{3}$$

In the light of the Leibniz Rule the Volterra image signal $\left[\mathcal{V}_{K_h^{(i)}}x\right](t)$, for any $i \in \{1,2,\cdots,N\}$ can be obtained as the output of a linear time-varying system as

$$\begin{cases} \xi(t) &= \left[\mathcal{V}_{K_h^{(i)}}x\right](t) \\ \dot{\xi}(t) &= K_h^{(i)}(t,t)x(t) + \displaystyle\int_0^t \left(\frac{\partial}{\partial t}K_h^{(i)}(t,\tau)\right)x(\tau)d\tau \\ &= -\beta_h\xi(t) + K_h^{(i)}(t,t)x(t) \end{cases} \tag{4}$$

with $\xi(0) = 0$. Being $K_h^{(i)}(t,t)$ bounded and ω strictly positive, it holds that the scalar dynamical system realization of the Volterra operators induced by the proposed kernels is BIBO stable with respect to $x(t)$.

III. Main algorithm

Starting from the noise-free signal $y(t)$, the time-derivative of $y(t)$ holds:

$$\dot{y}(t) = \mu\dot{\alpha}(t)\sin(\vartheta(t)) + \mu\alpha(t)\omega\cos(\vartheta(t)) \tag{5}$$
$$\begin{aligned}\ddot{y}(t) &= \mu\ddot{\alpha}(t)\sin(\vartheta(t)) + 2\mu\dot{\alpha}(t)\omega\cos(\vartheta(t)) \\ &\quad - \mu\alpha(t)\omega^2\sin(\vartheta(t))\end{aligned} \tag{6}$$

In view of (1) and (5), by cancelling the term associated with $\sin(\vartheta(t))$ and $\cos(\vartheta(t))$, the following relationship can be established:

$$\begin{aligned}&2\alpha(t)\dot{\alpha}(t)\dot{y}(t) - \alpha(t)^2\ddot{y}(t) \\ &\quad = 2\dot{\alpha}(t)^2y(t) - \alpha(t)\ddot{\alpha}(t)y(t) + \Omega\alpha(t)^2y(t),\end{aligned} \tag{7}$$

where $\Omega = \omega^2$ Consider $\gamma_1 = \alpha(t)^2y(t)$, $\gamma_2 = \alpha(t)\dot{\alpha}(t)y(t)$, which are composed of known signals. It holds that:

$$\begin{aligned}\alpha(t)^2\ddot{y}(t) &= \ddot{\gamma}_1 - 4\alpha(t)\dot{\alpha}(t)\dot{y}(t) - 2\dot{\alpha}(t)^2y(t) \\ &\quad - 2\alpha(t)\ddot{\alpha}(t)y(t)\end{aligned} \tag{8}$$
$$\alpha(t)\dot{\alpha}(t)\dot{y}(t) = \dot{\gamma}_2 - \dot{\alpha}(t)^2y(t) - \alpha(t)\ddot{\alpha}(t)y(t) \tag{9}$$

Substituting (8) and (9) in (7), $\alpha(t)\dot{\alpha}(t)\dot{y}(t)$ and $\alpha(t)^2\ddot{y}(t)$ can be replaced, yielding:

$$\begin{aligned}&6(\dot{\gamma}_2 - \dot{\alpha}(t)^2y(t) - \alpha(t)\ddot{\alpha}(t)y(t)) \\ &\quad - (\ddot{\gamma}_1 - 2\dot{\alpha}(t)^2y(t) - 2\alpha(t)\ddot{\alpha}(t)y(t)) \\ &\quad = 2\dot{\alpha}(t)^2y(t) - \alpha(t)\ddot{\alpha}(t)y(t) + \Omega\alpha(t)^2y(t)\end{aligned}$$

After rearrangements, it yields

$$6\dot{\gamma}_2 - 3\alpha(t)\ddot{\alpha}(t)y(t) - \ddot{\gamma}_1 - 6\dot{\alpha}(t)^2y(t) = \Omega\gamma_1(t) \tag{10}$$

Assuming that $K_h(\cdot,\cdot)$ are 2nd order non-asymptotic kernel functions i.e. $N = 2$. $\beta_h \in \mathbb{R}_{>0}, \forall h = \{1,2\}$ and $\bar{\beta} \in \mathbb{R}_{>0}$ are set by the designers subject to $\beta_1 \neq \beta_2$. Then it holds that:

$$\left[\mathcal{V}_{K_h}\gamma_1^{(2)}\right](t) = -\gamma_1(t)K_h^{(1)}(t,t) + \gamma_1^{(1)}(t)K_h(t,t) + \left[\mathcal{V}_{K_h^{(2)}}\gamma_1\right](t)$$
$$\left[\mathcal{V}_{K_h}\gamma_2^{(1)}\right](t) = \gamma_2(t)K_h(t,t) - \left[\mathcal{V}_{K_h^{(1)}}\gamma_2\right](t)$$

Applying K_1 and K_2 to both sides of (10), we get

$$\begin{aligned}&6\gamma_2(t)K_h(t,t) - 6\left[\mathcal{V}_{K_h^{(1)}}\gamma_2\right](t) - 3\left[\mathcal{V}_{K_h}\alpha\ddot{\alpha}y\right](t) \\ &\quad + \gamma_1(t)K_h^{(1)}(t,t) - \dot{\gamma}_1(t)K_h(t,t) - \left[\mathcal{V}_{K_h^{(2)}}\gamma_1\right](t) \\ &\quad - 6\left[\mathcal{V}_{K_h}\dot{\alpha}^2y\right](t) = \Omega\left[\mathcal{V}_{K_h}\gamma_1\right](t),\ h = 1,2\end{aligned} \tag{11}$$

Then, we can cancel the unavailable signal derivative $\dot{\gamma}_1(t)$ by manipulating the two equations of (11):

$$\begin{aligned}
&6\left[\mathcal{V}_{K_2^{(1)}}\,\gamma_2\right](t)K_1(t,t)-6\left[\mathcal{V}_{K_1^{(1)}}\,\gamma_2\right](t)K_2(t,t)\\
&+3\left[\mathcal{V}_{K_2}\,\alpha\ddot{\alpha}y\right](t)K_1(t,t)-3\left[\mathcal{V}_{K_1}\,\alpha\ddot{\alpha}y\right](t)K_2(t,t)\\
&+\gamma_1(t)K_1^{(1)}(t,t)K_2(t,t)-\gamma_1(t)K_2^{(1)}(t,t)K_1(t,t)\\
&+\left[\mathcal{V}_{K_2^{(2)}}\,\gamma_1\right](t)K_1(t,t)-\left[\mathcal{V}_{K_1^{(2)}}\,\gamma_1\right](t)K_2(t,t)\\
&+6\left[\mathcal{V}_{K_2}\,\dot{\alpha}^2y\right](t)K_1(t,t)-6\left[\mathcal{V}_{K_1}\,\dot{\alpha}^2y\right](t)K_2(t,t)\\
&=\Omega\left(\left[\mathcal{V}_{K_1}\,\gamma_1\right](t)K_2(t,t)-\left[\mathcal{V}_{K_2}\,\gamma_1\right](t)K_1(t,t)\right),
\end{aligned}\tag{12}$$

which can be rearranged in vector form as follows:

$$\mathbf{1}_{1\times 5}\zeta_1(t)\mathcal{K}(t,t)=\Omega\zeta_2(t)\mathcal{K}(t,t)\,,\tag{13}$$

with $\mathcal{K}(t,t)=[K_1(t,t)\;K_2(t,t)]^\top$

$$\zeta_1=\begin{bmatrix}
6\left[\mathcal{V}_{K_2^{(1)}}\,\gamma_2\right](t) & -6\left[\mathcal{V}_{K_1^{(1)}}\,\gamma_2\right](t)\\
3\left[\mathcal{V}_{K_2}\,\alpha\ddot{\alpha}y\right](t) & -3\left[\mathcal{V}_{K_1}\,\alpha\ddot{\alpha}y\right](t)\\
-\gamma_1K_2^{(1)}(t,t) & \gamma_1K_1^{(1)}(t,t)\\
\left[\mathcal{V}_{K_2^{(2)}}\,\gamma_1\right](t) & -\left[\mathcal{V}_{K_1^{(2)}}\,\gamma_1\right](t)\\
6\left[\mathcal{V}_{K_2}\,\dot{\alpha}^2y\right](t) & -6\left[\mathcal{V}_{K_1}\,\dot{\alpha}^2y\right](t)
\end{bmatrix}$$

and

$$\zeta_2=\left[\;-\left[\mathcal{V}_{K_2}\,\gamma_1 y\right](t)\quad\left[\mathcal{V}_{K_1}\,\gamma_1\right](t)\;\right].$$

Recalling (4), the Volterra images involved in ζ_1 and ζ_2 can be calculated the following LTV system.

$$\begin{cases}\dot{\xi}_h(t) &= G_h\xi_h(t)+E_h(t)u_\star(t),\\ \xi_h(0) &= 0,\;\; h=1,2\end{cases}\tag{14}$$

where

$$\begin{aligned}
\xi_h(t)=&\Big[[\mathcal{V}_{K_h^{(2)}}\gamma_1](t),[\mathcal{V}_{K_h}\gamma_1](t),[\mathcal{V}_{K_h}\dot{\alpha}^2y](t),\\
&\qquad[\mathcal{V}_{K_h}\alpha\ddot{\alpha}y](t),[\mathcal{V}_{K_h^{(1)}}\gamma_2](t)\Big]^\top,\\
G_h\;=&\operatorname{diag}\left(-\beta_h,-\beta_h,-\beta_h,-\beta_h,-\beta_h\right),\\
E_h(t)=&\operatorname{diag}(K_h^{(2)}(t,t),K_h(t,t),K_h(t,t),K_h(t,t),K_h^{(1)}(t,t)),\\
u_\star(t)=&\left[\gamma_1(t),\gamma_1(t),\dot{\alpha}(t)^2y(t),\alpha(t)\dot{\alpha}(t)y(t),\gamma_2(t)\right]^\top.
\end{aligned}$$

Notably, with such kernel functions tuned by β_1 and β_2 the following persistency of excitation condition has been embedded.

Lemma 3.1: Given the measurement $y(t)$ and the designed kernel functions $K_1(t,\tau)$ and $K_2(t,\tau)$, for any $\alpha(t)\not\equiv 0$, there exist ϵ_1 and $t_\epsilon\in\mathbb{R}_{\geq 0}$ such that

$$\frac{1}{t_\epsilon}\int_{t-t_\epsilon}^{t}|\zeta_2(\tau)\mathcal{K}(\tau,\tau)|d\tau\geq\epsilon_1,\quad t\geq t_\epsilon.\tag{15}$$

In order to avoid the zero-cross of $\zeta_2(t)\mathcal{K}(t,t)$ while solving for Ω by (13), we make use of filtering technique followed by a sliding mode adaptation law, *i.e.*

$$\begin{aligned}\dot{r}(t) &= \nu_1 r(t)+|\mathbf{1}_{1\times 5}\zeta_1(t)\mathcal{K}(t,t)|,\\ \dot{z}(t) &= \nu_1 z(t)+|\zeta_2(t)\mathcal{K}(t,t)|,\end{aligned}\tag{16}$$

where $z(0)=r(0)=0$ and $\nu_1\in\mathbb{R}_{<0}$ is a user-defined parameter that act as a forgetting factor.

Therefore, it can be concluded that the signal $z(t)$ is positive $\forall t>t_\epsilon$ in the sense that

$$\begin{aligned}z(t)&\geq\int_{t-t_\epsilon}^{t}e^{-\nu_1(t-\tau)}|\zeta_2(\tau)\mathcal{K}(\tau,\tau)|d\tau\\ &\geq e^{-\nu_1 t_\epsilon}\int_{t-t_\epsilon}^{t}|\zeta_2(\tau)\mathcal{K}(\tau,\tau)|d\tau\geq t_\epsilon\epsilon_1 e^{-\nu_1 t_\epsilon}.\end{aligned}$$

As a consequence, (13) becomes $r(t)=\Omega z(t)$, in which Ω can be estimated by a 1st-order sliding mode-based adaptation law

$$\dot{\hat{\Omega}}(t)=\begin{cases}z^{-1}(t)\Big[L_1 sign\left(R_1(t)\right)-\dot{r}(t)+\hat{\Omega}(t)\dot{z}(t)\Big],z\geq\delta_1,\\ 0,\quad\text{otherwise},\end{cases}\tag{17}$$

where $R_1(t)=r(t)-\hat{\Omega}(t)z(t)$ and $\delta_1\geq t_\epsilon\epsilon_1 e^{-\nu_1 t_\epsilon}$ is the singularity threshold with a positive small value. L_1 is the adaptive gain that can be chosen by the users.

Once $\hat{\Omega}$ is obtained, the frequency can be estimated by

$$\hat{\omega}(t)=\sqrt{\hat{\Omega}(t)}.\tag{18}$$

Next, we show how to estimate the coefficient μ. In view of (11), $\dot{\gamma}_1$ can be immediately obtained by utilizing $\hat{\Omega}$. From the fact that $\dot{\gamma}_1=2\alpha(t)\dot{\alpha}(t)y(t)+\alpha(t)^2\dot{y}(t)$, $\dot{y}(t)$ can be estimated. Thanks to

$$\dot{y}(t)\alpha(t)-y(t)\dot{\alpha}(t)=\mu\alpha(t)^2\omega\cos(\vartheta),$$

the following relationship can be established

$$\sqrt{(\dot{y}(t)\alpha(t)-y(t)\dot{\alpha}(t))^2+(\omega\alpha(t)y)^2}=\omega\mu\alpha(t)^2.\tag{19}$$

Similarly to (13), equation (19) can be used to identify μ resorting to the filtering and sliding mode technique. Let $R_2(t)=\rho_1(t)-\hat{\mu}(t)\rho_2(t)$ be a residual signal where $\rho_1(t)$ and $\rho_2(t)$ are generated by

$$\begin{aligned}\dot{\rho}_1(t)&=\nu_2\rho_1(t)+\sqrt{(\dot{y}(t)\alpha(t)-y(t)\dot{\alpha}(t))^2+(\hat{\omega}(t)\alpha(t)y)^2},\\ \dot{\rho}_2(t)&=\nu_2\rho_2(t)+\hat{\omega}(t)\alpha(t)^2,\end{aligned}\tag{20}$$

with $\rho_1(0)=\rho_2(0)=0$ and a user-defined gain $\nu_2\in\mathbb{R}_{<0}$. Then, the 1st-order sliding mode based adaptation law for μ can be constructed

$$\dot{\hat{\mu}}(t)=\begin{cases}\rho_2^{-1}(t)[L_2 sign\left(R_2(t)\right)-\dot{\rho}_1(t)+\hat{\mu}(t)\dot{\rho}_2(t)],\rho_2\geq\delta_2,\\ 0,\quad\text{otherwise},\end{cases}\tag{21}$$

where δ_2 is the small-valued singularity threshold and L_2 is a user-defined gain.

IV. Stability and robust analysis

Theorem 4.1: Given the sinusoidal signal $y(t)$, the estimated frequency $\hat{\Omega}(t)$ given by the adaptation law (17) converges to the true value Ω in finite time with any choice of L_1.

Proof: The candidate Lyapunov function is chosen as $V_\Omega(t)=|R_1(\hat{\Omega},t)|$, whose time-derivative is

$$\dot{V}_\Omega(t)=sign(R_1)\left[\dot{r}(t)-\dot{z}(t)\hat{\Omega}(t)-z(t)\dot{\hat{\Omega}}(t)\right].$$

Based on the sliding mode adaptation law in (17) when $z(t) \geq \delta_1$, it holds that

$$\dot{V}_\Omega(t) = -L_1, \quad \forall R_1 \neq 0,$$

which implies that the residual $R_1(t) \to 0$ in finite time with a constant rate of $-L_1$. Accordingly, one can conclude that the estimated $\hat{\Omega}(t)$ converge to its true value in a finite time. ■

In the same way of reasoning, the following theorem of the finite-time convergence of the adaptation law (21) can be concluded.

Theorem 4.2: Given the sinusoidal signal $y(t)$, the estimation of the amplitude coefficient $\hat{\mu}(t)$ given by the adaptation law (21) converges to the true value μ in finite time with any choice of L_2.

Next, the robustness of the algorithm will be analyzed. Assuming the measurement is affected by an additive disturbance, *i.e.*

$$y_d(t) = y(t) + d_y(t),$$

where the measurement noise is assumed to be bounded as $|d_y(t)| \leq \bar{d}_y$. By definition, the auxiliary signals become

$$\begin{aligned}\gamma_{1,d} &= \alpha(t)^2 \left(y(t) + d_y(t)\right) = \gamma_1(t) + \epsilon_{\gamma,1},\\ \gamma_{2,d} &= \alpha(t)\dot{\alpha}(t) \left(y(t) + d_y(t)\right) = \gamma_2(t) + \epsilon_{\gamma,2},\end{aligned}$$

where $\epsilon_{\gamma,1} \triangleq \alpha(t)^2 d_y(t)$ and $\epsilon_{\gamma,2} \triangleq \alpha(t)\dot{\alpha}(t) d_y(t)$. Thanks to the linearity of the Volterra operator, it holds that

$$\begin{aligned}\left[\mathcal{V}_{K_h} y_d\right] &= \left[\mathcal{V}_{K_h} y\right] + \left[\mathcal{V}_{K_h} d_y\right],\\ \left[\mathcal{V}_{K_h} \gamma_{1,d}\right] &= \left[\mathcal{V}_{K_h} \gamma_1\right] + \left[\mathcal{V}_{K_h} \epsilon_{\gamma,1}\right],\\ \left[\mathcal{V}_{K_h} \gamma_{2,d}\right] &= \left[\mathcal{V}_{K_h} \gamma_2\right] + \left[\mathcal{V}_{K_h} \epsilon_{\gamma,2}\right].\end{aligned}$$

As a consequence, the two auxiliary signals $\zeta_1(t)$ and $\zeta_2(t)$ are perturbed, leading to the following two error signals

$$\begin{aligned}\epsilon_{\zeta_1} &\triangleq \zeta_1(t) - \zeta_{1,d}(t)\\ &= \begin{bmatrix} 6\left[\mathcal{V}_{K_2^{(1)}}\, \epsilon_{\gamma_2}\right](t) & -6\left[\mathcal{V}_{K_1^{(1)}}\, \epsilon_{\gamma_2}\right](t)\\ 3\left[\mathcal{V}_{K_2}\, \alpha\ddot{\alpha} d_y\right](t) & -3\left[\mathcal{V}_{K_1}\, \alpha\ddot{\alpha} d_y\right](t)\\ -\epsilon_{\gamma_1} K_2^{(1)}(t,t) & \epsilon_{\gamma_1} K_1^{(1)}(t,t)\\ \left[\mathcal{V}_{K_2^{(2)}}\, \epsilon_{\gamma_1}\right](t) & -\left[\mathcal{V}_{K_1^{(2)}}\, \epsilon_{\gamma_1}\right](t)\\ 6\left[\mathcal{V}_{K_2}\, \dot{\alpha}^2 d_y\right](t) & -6\left[\mathcal{V}_{K_1}\, \dot{\alpha}^2 d_y\right](t) \end{bmatrix},\end{aligned}$$

and

$$\epsilon_{\zeta_2} \triangleq \zeta_2(t) - \zeta_{2,d}(t) = \left[\ -\left[\mathcal{V}_{K_2}\, \epsilon_{\gamma_1}\right](t) \quad \left[\mathcal{V}_{K_1}\, \epsilon_{\gamma_1}\right](t)\ \right],$$

where $\zeta_{1,d}(t)$ and $\zeta_{2,d}(t)$ are the noisy counterparts of $\zeta_1(t)$ and $\zeta_2(t)$ respectively.

In ϵ_{ζ_1}, being $\alpha(t)$, $K_h^{(1)}(t,t)$ and $d_y(t)$ bounded, for $h \in \{1,2\}$, it is straightforward to conclude

$$\begin{aligned}|\epsilon_{\gamma_1} K_h^{(1)}(t,t)| &\leq \sup_{0\leq\tau\leq t} \left|\alpha(\tau)^2 d_y(\tau) K_h^{(1)}(\tau,\tau)\right|\\ &\triangleq \bar{\epsilon}_{\zeta_1,3,h}, h \in \{1,2\}.\end{aligned}$$

The other elements of $\epsilon_{\zeta_1}(t)$ and $\epsilon_{\zeta_2}(t)$ are composed of Volterra images. Collecting corresponding Volterra images in a vector as $\epsilon_{\xi,h} = \Big[[\mathcal{V}_{K_h^{(2)}}\epsilon_{\gamma_1}](t), [\mathcal{V}_{K_h}\epsilon_{\gamma_1}], [\mathcal{V}_{K_h}\dot{\alpha}^2 d_y](t), [\mathcal{V}_{K_h}\alpha\ddot{\alpha} d_y](t), [\mathcal{V}_{K_h^{(1)}}\epsilon_{\gamma_2}](t)\Big]^\top$. Recalling (14), $\epsilon_{\xi,h}(t)$ is the output of the system

$$\begin{cases} \dot{\epsilon}_{\xi,h}(t) = G_h \epsilon_{\xi,h}(t) + E_h(t)\epsilon_{u_\star}(t),\\ \epsilon_{\xi,h}(0) = 0, \ \ h = 1,2, \end{cases} \tag{22}$$

with $\epsilon_{u_\star}(t) = \big[\epsilon_{\gamma_1}(t), \epsilon_{\gamma_1}(t),\ \dot{\alpha}(t)^2 d_y(t), \alpha(t)\dot{\alpha}(t) d_y(t), \epsilon_{\gamma_2}(t)\big]^\top$. Being G_h Hurwitz, and owing to the boundedness of α, $K_h(t,\tau)$ and their derivatives, $\epsilon_{\xi,h}$ is bounded elementwisely. Thus, it holds that

$$\mathbf{1}_{1\times 5}\epsilon_{\zeta_1}(t)\mathcal{K}(t,t) \leq 10\bar{\epsilon}_{\zeta_1} \sup_{0\leq\tau\leq t} \|\mathcal{K}(\tau,\tau)\|, \tag{23}$$

$$\epsilon_{\zeta_2}(t)\mathcal{K}(t,t) \leq 2\bar{\epsilon}_{\zeta_2} \sup_{0\leq\tau\leq t} \|\mathcal{K}(\tau,\tau)\|. \tag{24}$$

where $\bar{\epsilon}_{\zeta_1} \triangleq ||\epsilon_{\zeta_1}||$ and $\bar{\epsilon}_{\zeta_2} \triangleq ||\epsilon_{\zeta_2}||$.

Considering the P.E. condition in the noisy scenario, it is straightforward to see that there exists a constant $\epsilon_{1,d}$, such that

$$\frac{1}{t_\epsilon}\int_{t-t_\epsilon}^{t} |\mathcal{K}(\tau,\tau)\zeta_{2,d}(\tau)| d\tau \geq \epsilon_{1,d},$$

provided $y_d(t) \not\equiv 0$ and $\alpha(t) \not\equiv 0$, $\forall t \geq t_\epsilon$. Let $r_d(t)$ and $z_d(t)$ denote the counterpart signals of $r(t)$ and $z(t)$ in the noise environment. In view of (16) , it holds that

$$\begin{aligned}\dot{r}_d(t) &= \nu_1 r_d(t) + |\mathbf{1}_{1\times 5}\zeta_{1,d}(t)\mathcal{K}(t,t)|,\\ \dot{z}_d(t) &= \nu_1 z_d(t) + |\zeta_{2,d}(t)\mathcal{K}(t,t)|,\\ z_d(0) &= r_d(0) = 0,\end{aligned} \tag{25}$$

and the error signals with respect to these two signals satisfy

$$\begin{aligned}|\epsilon_r(t)| \triangleq |r(t) - r_d(t)| &\leq \int_0^t e^{-\nu_1\tau} |\mathbf{1}_{1\times 5}\bar{\epsilon}_{\zeta_1} \sup_{0\leq\tau\leq t} \mathcal{K}(\tau,\tau)| d\tau\\ &\leq \frac{10}{\nu_1}\bar{\epsilon}_{\zeta_1} \sup_{0\leq\tau\leq t} \|\mathcal{K}(\tau,\tau)\| \triangleq \bar{\epsilon}_r \end{aligned} \tag{26}$$

$$\begin{aligned}|\epsilon_z(t)| \triangleq |z(t) - z_d(t)| &\leq \int_0^t e^{-\nu_1\tau} |\bar{\epsilon}_{\zeta_2} \sup_{0\leq\tau\leq t} \mathcal{K}(\tau,\tau)| d\tau\\ &\leq \frac{2}{\nu_1}\bar{\epsilon}_{\zeta_2} \sup_{0\leq\tau\leq t} \|\mathcal{K}(\tau,\tau)\| \triangleq \bar{\epsilon}_z \end{aligned} \tag{27}$$

and

$$z_d(t) \geq \int_{t-t_\epsilon}^{t} e^{-\nu_1(t-\tau)} |\zeta_{2,d}(\tau)\mathcal{K}(\tau,\tau)| d\tau \geq t_\epsilon \epsilon_{1,d} e^{-\nu_1 t_\epsilon}.$$

Therefore, in the noisy scenario, the activation threshold of the sliding mode adaptation law (17) needs to be modified

$$\dot{\hat{\Omega}}(t) = \begin{cases} z_d^{-1}(t)\left[L_1 sign\left(R_{1,d}(t)\right) - \dot{r}_d(t) + \hat{\Omega}(t)\dot{z}_d(t)\right], \\ \qquad\qquad\qquad\qquad z_d(t) \geq \delta_{1,d},\\ 0, \quad \text{otherwise}, \end{cases} \tag{28}$$

where $R_{1,d}(t) = r_d(t) - \hat{\Omega}(t) z_d(t)$ and $\delta_{1,d} \geq t_\epsilon \epsilon_{1,d} e^{-\nu_1 t_\epsilon}$ is the activation threshold. The following Theorem characterizes the robustness feature of the proposed frequency estimation method (28).

Theorem 4.3: Given the amplitude-varying sinusoidal signal $y(t)$ and its noisy measurement $y_d(t)$, the estimates $\hat{\Omega}(t)$ given by the adaptation law (28) enters into a neighborhood of the true Ω in finite-time and the estimation error $\epsilon_\Omega(t) \triangleq \hat{\Omega}(t) - \Omega$ is bounded with respect to bounded measurement noise $d_y(t)$.

Proof: Consider a candidate Lyapunov function $V_d(t) = |R_{1,d}(\hat{\Omega}, t)|$. Following the proof carried out in Theorem 4.1, it is straightforward to show that the residual $R_{1,d}(\hat{\Omega})$ decays to 0 with a constant rate of L_1. The convergence time is $T_c = |R_{1,d}(0)/L_1|$. Retrieving the definition of $R_{1,d}$, we have that

$$\hat{\Omega}(t) = \frac{r_{d(t)}}{z_d(t)}, \quad \forall t > t_a + T_c.$$

where t_a denotes the activation time when $z_d(t)$ exceeds the threshold $\delta_{1,d}$. As such, with the proven boundedness of $z_d(t)$ and $r_d(t)$ (due to the boundedness of ϵ_r, ϵ_z), it turns out that the frequency estimates $\hat{\Omega}(t)$ is bounded $\forall t > 0$ and enters into the compact region

$$\hat{\Omega}(t) \in \left[\inf_{0\le\tau\le t}\left|\frac{r_d(\tau)}{z_d(\tau)}\right|, \sup_{0\le\tau\le t}\left|\frac{r_d(\tau)}{z_d(\tau)}\right|\right], \forall t \ge t_a + T_c, \tag{29}$$

that subsumes the true frequency square Ω with the estimation error $\epsilon_\Omega(t)$ depending on the bounded measurement noise $d_y(t)$. ■

As a result, the $\epsilon_\omega(t) \triangleq \omega - \hat{\omega}$ is bounded, denoting its upper bound by $\bar{\epsilon}_\omega$. The estimation of $\dot{\gamma}(t)$ is calculated via

$$\begin{aligned}\hat{\dot{\gamma}}_1(t) = K_h(t,t)^{-1}\Big[&6\gamma_{2,d}K_h(t,t) - 6\left[\mathcal{V}_{K_h^{(1)}}\,\gamma_{2,d}\right](t) \\ &-3\left[\mathcal{V}_{K_h}\,\alpha\ddot{\alpha}y_d\right](t) + \gamma_{1,d}K_h^{(1)}(t,t) - \left[\mathcal{V}_{K_h^{(2)}}\,\gamma_{1,d}\right](t) \\ &-6\left[\mathcal{V}_{K_h}\,\dot{\alpha}^2 y_d\right](t) - \hat{\Omega}(t)\left[\mathcal{V}_{K_h}\,\gamma_{1,d}\right](t)\Big], h \in \{1,2\}.\end{aligned}$$

The estimation error has the expression

$$\begin{aligned}\epsilon_{\dot{\gamma}_1}(t) = K_h(t,t)^{-1}\Big[&6\epsilon_{\gamma_2}K_h(t,t) - 6\left[\mathcal{V}_{K_h^{(1)}}\,\epsilon_{\gamma_2}\right](t) \\ &- 3\left[\mathcal{V}_{K_h}\,\alpha\ddot{\alpha}d_y\right](t) + \epsilon_{\gamma_1}K_h^{(1)}(t,t) - \left[\mathcal{V}_{K_h^{(2)}}\,\epsilon_{\gamma_1}\right](t) \\ &- 6\left[\mathcal{V}_{K_h}\,\dot{\alpha}^2 d_y\right](t) - \epsilon_\Omega(t)\left[\mathcal{V}_{K_h}\,\epsilon_{\gamma_1}\right](t)\Big].\end{aligned} \tag{30}$$

As $\dot{y}$ is estimated by

$$\hat{\dot{y}}(t) = \frac{\hat{\dot{\gamma}}_1(t) - 2\alpha(t)\dot{\alpha}(t)y_d(t)}{\alpha(t)^2},$$

Owing to the fact that $\mu\alpha \in \mathbb{R}_{>0}$, the corresponding estimation error takes on the following form

$$\epsilon_{\dot{y}}(t) = \hat{\dot{y}}(t) - \dot{y}(t) = \frac{\epsilon_{\dot{\gamma}_1}(t) - 2\alpha(t)\dot{\alpha}(t)d_y(t)}{\alpha(t)^2},$$

whose boundedness can be readily confirmed with respective to bounded noise $d_y(t)$ and $\alpha(t)$. By following the same process from (23) to (29), it is straightforward to show that $\forall t > t_a + T_c + |R_2(0)/L_2|$, $\hat{\mu}(t)$ goes into the compact region

$$\left[\inf_{0\le\tau\le t}\left|\frac{\rho_{1,d}(\tau)}{\rho_{2,d}(\tau)}\right|, \sup_{0\le\tau\le t}\left|\frac{\rho_{1,d}(\tau)}{\rho_{2,d}(\tau)}\right|\right]. \tag{31}$$

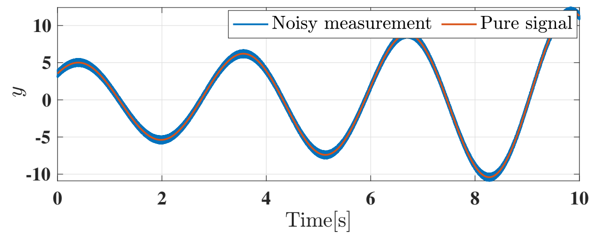

Fig. 1. True signal and noisy measurement in perturbed scenario.

where $\rho_{1,d}(t)$ and $\rho_{2,d}(t)$ are noisy counterparts of the $\rho_1(t)$ and $\rho_2(t)$ calculated by

$$\begin{aligned}\dot{\rho}_{1,d}(t) &= \nu_2\rho_{1,d}(t) \\ &\quad + \sqrt{(\hat{\dot{y}}(t)\alpha(t) - y_d(t)\dot{\alpha}(t))^2 + (\hat{\omega}(t)\alpha(t)y_d(t))^2}, \\ \dot{\rho}_{2,d}(t) &= \nu_2\rho_{2,d}(t) + \hat{\omega}(t)\alpha(t)^2,\end{aligned}$$

The region (31) is bounded due to the boundedness of all signals involved. This further implies that the estimation error $\epsilon_\mu(t) \triangleq \hat{\mu}(t) - \mu$ is bounded with respect to bounded measurement noise $d_y(t)$.

V. Numerical Example

The effectiveness of the proposed estimator for frequency and amplitude factor is examined by a numerical example. Consider a sinusoidal signal

$$y(t) = 5\alpha(t)\sin\left(2t + \frac{\pi}{3}\right) \tag{32}$$

with $\alpha(t) = 2 + \sin(0.2t - \pi/2)$. We also consider an uniformly distributed random noise within $[-0.5, 0.5]$ on the measurement, *i.e.*, $y_d(t) = y(t) + d_y(t)$. The signal $y(t)$ and the noisy measurement $y_d(t)$ are depicted as in Fig. 1. The estimation results of the proposed method have been compared with a recently designed frequency estimator [23]. It is worth noting such method provides frequency estimates only, as such, only estimated frequency are compared herein.

In the noise-free scenario, the estimation results of the two methods are shown as in Fig. 2, where it has been shown that both methods give accurate frequency estimates within a finite time. In the meantime, the proposed method is able to estimate μ with finite-time convergence.

In the noisy environment, the estimation results are shown as in Fig. 3. In such a scenario, the proposed method provides robust estimates for both ω and μ whereas the method in [23] is more sensitive to external noise, which is ubiquitous in practice.

VI. Conclusion

In this paper, the problem of parameter estimation has been addressed for sinusoidal signals subject to time-varying amplitude. Based on the Volterra integral operator and suitably designed non-asymptotic kernel functions, two adaptation laws are proposed to estimate the frequency and the amplitude coefficient that ensure the estimates converge to the true values in a finite time with constant converging rates. The robustness of the proposed estimation method has been proven under the effects of bounded measurement noise.

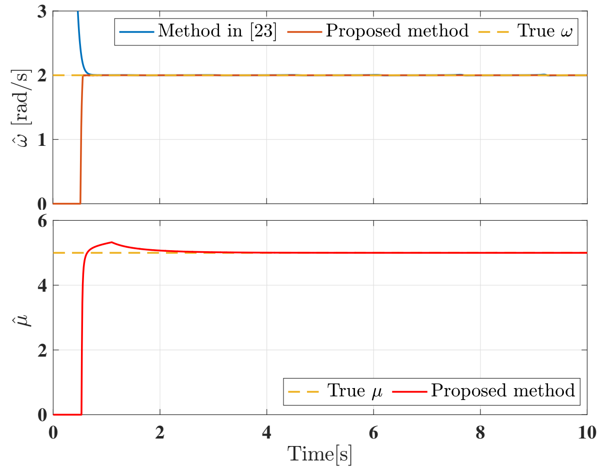

Fig. 2. Estimates of frequency ω and the amplitude coefficient μ in perturbation-free scenario.

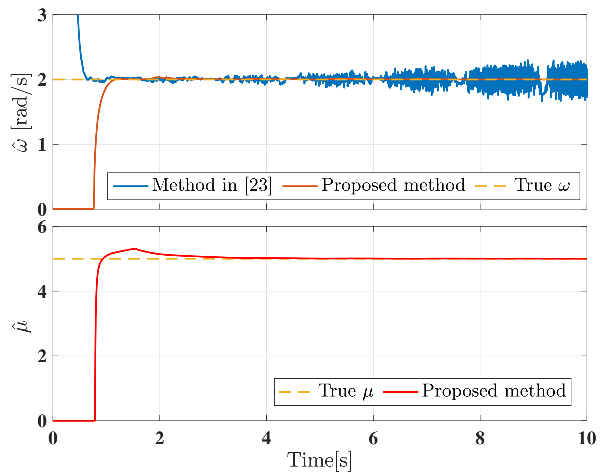

Fig. 3. Estimates of frequency ω and the amplitude coefficient μ in perturbed scenario.

Numerical simulations have been performed to examine the effectiveness of the proposed estimators with comparisons with a recently proposed frequency estimator.

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