Multiple Laguerre polynomials: Combinatorial model and Stieltjes moment representation

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Abstract

I give a combinatorial interpretation of the multiple Laguerre polynomials of the first kind of type II, generalizing the digraph model found by Foata and Strehl for the ordinary Laguerre polynomials. I also give an explicit integral representation for these polynomials, which shows that they form a multidimensional Stieltjes moment sequence whenever $x \le 0$.

Key Words: Laguerre polynomial, multiple orthogonal polynomial, multiple Laguerre polynomial, Laguerre digraph, integral representation, Stieltjes moment sequence.

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1 Introduction

The monic Laguerre polynomials $\mathbf{L}_n^{(\alpha)}(x) = (-1)^n n!\, L_n^{(\alpha)}(x)$ can be defined as [1,13,21,28]

$$\mathbf{L}_n^{(\alpha)}(x) = (-1)^n\,(\alpha+1)^{\overline{n}}\;{}_1F_1\!\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\,\middle|\,x\right) \tag{1.1a}$$

$$= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\,(\alpha+1+k)^{\overline{n-k}}\,x^k \tag{1.1b}$$

where $r^{\overline{n}} \stackrel{\text{def}}{=} r(r+1)\cdots(r+n-1)$; note that they are polynomials (with integer coefficients) jointly in x and α. The monic Laguerre polynomials have the exponential generating function

$$\sum_{n=0}^{\infty}\mathbf{L}_n^{(\alpha)}(x)\,\frac{t^n}{n!} = (1+t)^{-(\alpha+1)}\,e^{xt/(1+t)}\,. \tag{1.2}$$

For $\alpha > -1$ they are orthogonal with respect to the measure $x^\alpha e^{-x}\,dx$ on $(0,\infty)$. Using Kummer's first transformation for the confluent hypergeometric function ${}_1F_1$ [13, eq. (1.4.11)], eq. (1.1a) can also be rewritten as

$$\mathbf{L}_n^{(\alpha)}(x) = (-1)^n\,(\alpha+1)^{\overline{n}}\,e^x\;{}_1F_1\!\left(\begin{matrix}\alpha+1+n\\ \alpha+1\end{matrix}\,\middle|\,-x\right). \tag{1.3}$$

Now fix an integer $r \ge 1$. The multiple Laguerre polynomials of the first kind of type II [13, section 23.4.1], denoted $\mathbf{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x)$ where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_r)$ and $\mathbf{n} = (n_1,\ldots,n_r)$, can be defined by a straightforward generalization of (1.3):

$$\mathbf{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x) = (-1)^{|\mathbf{n}|}\left(\prod_{i=1}^{r}(\alpha_i+1)^{\overline{n_i}}\right)e^x\;{}_rF_r\!\left(\begin{matrix}\alpha_1+1+n_1,\,\ldots,\,\alpha_r+1+n_r\\ \alpha_1+1,\,\ldots,\,\alpha_r+1\end{matrix}\,\middle|\,-x\right) \tag{1.4}$$

where $|\mathbf{n}| \stackrel{\text{def}}{=} n_1+\ldots+n_r$. It follows from known properties of the hypergeometric function ${}_rF_r$ that the right-hand side of (1.4) is an entire function of x that behaves asymptotically at infinity like $x^{|\mathbf{n}|}$; therefore it is a (monic) polynomial in x, of degree $|\mathbf{n}|$.[1] In fact, we have the explicit expression, which generalizes (1.1b):[2]

$$\mathbf{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}(-1)^{|\mathbf{n}|-|\mathbf{k}|}\left(\prod_{i=1}^{r}\binom{n_i}{k_i}\,(\alpha_i+1+k_1+\ldots+k_i)^{\overline{n_i-k_i}}\right)x^{|\mathbf{k}|}\,. \tag{1.5}$$

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[1]This reasoning goes back at least to Hille [12, p. 52]. The needed asymptotic expansion of ${}_rF_r$ can be found in [18, section 5.11.3] or [29].

[2]This formula follows from (1.4) by application of Karlsson's [16] identity for hypergeometric functions where the numerator and denominator parameters differ by integers, combined with ${}_0F_0(\,\overline{\underline{\quad}}\mid -x) = e^{-x}$ at the final stage. See also Srivastava [26] for a very simple proof of Karlsson's identity; and see [3,19] for some interesting generalizations.

When $\alpha_1, \ldots, \alpha_r > -1$ with $\alpha_i - \alpha_j \notin \mathbb{Z}$ for all pairs $i \neq j$, these polynomials are multiple orthogonal [13, Chapter 23] with respect to the collection of measures $x^{\alpha_i} e^{-x}\, dx$ on $(0,\infty)$ with $1 \le i \le r$. Finally, the multiple Laguerre polynomials have the multivariate exponential generating function [17]

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \mathbf{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x)\, \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!} \;=\; \biggl(\prod_{i=1}^{r} (1+t_i)^{-(\alpha_i+1)}\biggr) \exp\biggl[x\Bigl(1 - \prod_{i=1}^{r} \frac{1}{1+t_i}\Bigr)\biggr] \,. \tag{1.6}$$

Remark/Question. The multiple Laguerre polynomial $\mathbf{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x)$ is invariant under joint permutations of $\mathbf{n}$ and $\boldsymbol{\alpha}$: this is manifest in (1.4) and (1.6), but is far from obvious in the explicit formula (1.5). Is there some easy way of deriving this symmetry from (1.5)? And is there an alternate explicit formula in which this symmetry is manifest? ■

The purpose of the present paper is twofold: (a) to give a combinatorial interpretation of the multiple Laguerre polynomials (1.4)/(1.5), generalizing the digraph model found by Foata and Strehl [9] for the ordinary Laguerre polynomials; and (b) to give an explicit integral representation for these polynomials, showing that they form a multidimensional Stieltjes moment sequence whenever $x \le 0$.

2 Combinatorial model

Three decades ago, Foata and Strehl [9] introduced a beautiful combinatorial interpretation of the Laguerre polynomials. Let us define a ***Laguerre digraph*** to be a digraph in which each vertex has out-degree 0 or 1 and in-degree 0 or 1. It follows that each weakly connected component of a Laguerre digraph is either a directed path of some length $\ell \ge 0$ (where a path of length 0 is an isolated vertex) or else a directed cycle of some length $\ell \ge 1$ (where a cycle of length 1 is a loop). Let us write $\mathbf{LD}_n$ for the set of Laguerre digraphs on the vertex set $[n] \stackrel{\rm def}{=} \{1,\ldots,n\}$; and for a Laguerre digraph G, let us write $\mathrm{cyc}(G)$ [resp. $\mathrm{pa}(G)$] for the number of cycles (resp. paths) in G. Foata and Strehl [9] then showed that the monic unsigned Laguerre polynomials

$$\mathcal{L}_n^{(\alpha)}(x) \;\stackrel{\rm def}{=}\; n!\, L_n^{(\alpha)}(-x) \;=\; (-1)^n\, \mathbf{L}_n^{(\alpha)}(-x) \tag{2.1}$$

have the combinatorial representation

$$\mathcal{L}_n^{(\alpha)}(x) \;=\; \sum_{G \in \mathbf{LD}_n} x^{\mathrm{pa}(G)}\, (\alpha+1)^{\mathrm{cyc}(G)} \,. \tag{2.2}$$

Indeed, the proof of (2.2) is an easy argument using the exponential formula [27, chapter 5], or equivalently, the theory of species [2]: the number of directed paths on $n \ge 1$ vertices is $n!$, so with a weight x per path they have exponential generating function $xt/(1-t)$. The number of directed cycles on $n \ge 1$ vertices is $(n-1)!$, so with a weight $\alpha+1$ per cycle they have exponential generating function $-(\alpha+1)\log(1-t)$.

A Laguerre digraph is a disjoint union of paths and cycles, so by the exponential formula it has exponential generating function

$$\exp\biggl[\frac{xt}{1-t} \;-\; (\alpha+1)\log(1-t)\biggr] \;=\; (1-t)^{-(\alpha+1)}\, e^{xt/(1-t)} \;, \tag{2.3}$$

which coincides with (1.2) after $x \to -x$ and $t \to -t$. Foata and Strehl [9] also gave a direct combinatorial proof of (2.2) based on the definition (1.1); this requires a bit more work [9, Lemma 2.1].

Our first result is a combinatorial interpretation of the multiple Laguerre polynomials that extends the Foata–Strehl interpretation to $r > 1$. For $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we define a digraph $G_{\mathbf{n}} = (V_{\mathbf{n}}, \vec{E}_{\mathbf{n}})$ with vertex set

$$V_{\mathbf{n}} \;=\; \{(i,j)\colon\; 1 \le i \le r \text{ and } 1 \le j \le n_i\} \tag{2.4}$$

and edge set

$$\vec{E}_{\mathbf{n}} \;=\; \bigl\{\overrightarrow{(i,j)\,(i',j')}\colon\; i \le i'\bigr\} \;. \tag{2.5}$$

The vertex set is thus the disjoint union of "layers" $V_i \simeq [n_i]$ for $1 \le i \le r$; the edge set consists of all possible directed edges (including loops) within each layer V_i, together with all possible edges from a layer V_i to a layer $V_{i'}$ with $i' > i$. We then write $\mathbf{LD}_{\mathbf{n}}$ for the set of Laguerre digraphs that are spanning subdigraphs of $G_{\mathbf{n}}$, i.e. Laguerre digraphs of the form $(V_{\mathbf{n}}, A)$ with $A \subseteq \vec{E}_{\mathbf{n}}$. Note that in a Laguerre digraph $G \in \mathbf{LD}_{\mathbf{n}}$, every cycle must lie in a single layer V_i; we denote by $\mathrm{cyc}_i(G)$ the number of cycles in layer V_i. We then have:

Theorem 2.1. *The monic unsigned multiple Laguerre polynomials*

$$\mathcal{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x) \;\stackrel{\text{def}}{=}\; (-1)^{|\mathbf{n}|}\, \mathbf{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(-x) \tag{2.6}$$

have the combinatorial representation

$$\mathcal{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x) \;=\; \sum_{G \in \mathbf{LD}_{\mathbf{n}}} x^{\mathrm{pa}(G)} \prod_{i=1}^{r} (\alpha_i + 1)^{\mathrm{cyc}_i(G)} \;. \tag{2.7}$$

The proof of this result is a simple generalization of the argument just given for the Foata–Strehl formula (2.2):

Proof of Theorem 2.1. Denote the right-hand side of (2.7) by $\widehat{\mathcal{L}}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x)$, and consider its multivariate exponential generating function

$$F(t_1, \ldots, t_r) \;\stackrel{\text{def}}{=}\; \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \widehat{\mathcal{L}}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x)\, \frac{t_1^{n_1}}{n_1!} \,\cdots\, \frac{t_r^{n_r}}{n_r!} \;. \tag{2.8}$$

We again argue using the exponential formula. The multivariate exponential generating function for a single directed cycle in layer V_i is, as before, $-(\alpha_i + 1)\log(1 - t_i)$. Let us now look at paths. Every path P in the digraph $G_{\mathbf{n}}$ is of the following form: In each layer V_i choose a directed path P_i; the P_i are allowed to be empty, provided that they are not *all* empty. Let $i_1 < i_2 < \ldots < i_k$ be the indices with P_i nonempty, and construct the path P obtained from the union of the P_i by adjoining the edge linking the final vertex of P_{i_1} to the initial vertex of P_{i_2}, the edge linking the final vertex of P_{i_2} to the initial vertex of P_{i_3}, etc. With a weight x per path, the multivariate exponential generating function for a single such path is

$$x \left(\prod_{i=1}^{r} \frac{1}{1-t_i} \;-\; 1 \right) . \tag{2.9}$$

Therefore, by the exponential formula we have

$$F(t_1,\ldots,t_r) \;=\; \exp\!\left[-\sum_{i=1}^{r} (\alpha_i+1) \log(1-t_i) \;+\; x \left(\prod_{i=1}^{r} \frac{1}{1-t_i} \;-\; 1 \right) \right] , \tag{2.10}$$

which coincides with (1.6) after $x \to -x$ and $t_i \to -t_i$. $\square$

Remarks. 1. We leave it as an open problem to devise a direct combinatorial proof of (2.7) based on the explicit formula (1.5).

2. The combinatorial representation (2.7), unlike the explicit formula (1.5), manifestly exhibits the invariance of $\mathcal{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x)$ under joint permutations of $\mathbf{n}$ and $\boldsymbol{\alpha}$, since there is a weight-preserving bijection between the digraphs contributing to the right-hand side of (2.7) for the original and permuted cases. I thank an anonymous referee for pointing this out.

3. For the case $r = 2$, a slightly different combinatorial interpretation of the multiple Laguerre polynomials was found by Drake [6, Theorem 3.5.2]. But also this representation fails to manifestly exhibit the permutation symmetry. $\blacksquare$

3 Stieltjes moment representation

For the ordinary Laguerre polynomials ($r = 1$), a well-known integral representation [28, Theorem 5.4] asserts that

$$\mathcal{L}_n^{(\alpha)}(x) \;=\; n!\, L_n^{(\alpha)}(-x) \;=\; e^{-x} x^{-\alpha/2} \int\limits_0^\infty y^n \, e^{-y} \, y^{\alpha/2} \, I_\alpha(2\sqrt{xy}) \, dy \qquad \text{for } \alpha > -1 \,, \tag{3.1}$$

where I_α is a modified Bessel function of the first kind [30, p. 77]:

$$I_\alpha(z) \;=\; \sum_{k=0}^{\infty} \frac{(z/2)^{\alpha+2k}}{k!\, \Gamma(\alpha+k+1)} \tag{3.2a}$$

$$\;=\; \frac{1}{\Gamma(\alpha+1)} \, (z/2)^\alpha \; {}_0F_1\!\left(\begin{array}{c} - \\ \alpha+1 \end{array} \,\middle|\, z^2/4 \right) . \tag{3.2b}$$

Since I_α is nonnegative on $[0,\infty)$, it follows from (3.1) that the sequence $(\mathcal{L}_n^{(\alpha)}(x))_{n\ge 0}$ is a Stieltjes moment sequence whenever $\alpha \ge -1$ and $x \ge 0$: that is,

$$\mathcal{L}_n^{(\alpha)}(x) \;=\; \int\limits_0^\infty y^n \, d\mu_{\alpha,x}(y) \tag{3.3}$$

where

$$d\mu_{\alpha,x}(y) \;=\; \begin{cases} e^{-x}\, {}_0F_1\!\left(\begin{matrix} \text{---} \\ \alpha+1 \end{matrix} \,\middle|\, xy\right) \dfrac{1}{\Gamma(\alpha+1)}\, y^\alpha\, e^{-y}\, dy & \text{for } \alpha > -1 \\[4mm] x\, e^{-(x+y)}\, {}_0F_1\!\left(\begin{matrix} \text{---} \\ 2 \end{matrix} \,\middle|\, xy\right) dy & \text{for } \alpha = -1 \end{cases} \tag{3.4}$$

is a positive measure on $[0,\infty)$.[3]

We now give an integral representation for the multiple Laguerre polynomials that generalizes (3.1) to $r > 1$:

Theorem 3.1. *Let $\alpha_1,\ldots,\alpha_r \ge -1$ and $x \ge 0$. Then the multisequence $(\mathcal{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x))_{\mathbf{n}\in\mathbb{N}^r}$ of monic unsigned multiple Laguerre polynomials is a multidimensional Stieltjes moment sequence: that is, there exists a positive measure $\mu_{\boldsymbol{\alpha},x}$ on $[0,\infty)^r$ such that*

$$\mathcal{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x) \;=\; \int\limits_{[0,\infty)^r} \mathbf{y}^{\mathbf{n}} \, d\mu_{\boldsymbol{\alpha},x}(\mathbf{y}) \tag{3.5}$$

for all $\mathbf{n} \in \mathbb{N}^r$, where $\mathbf{y}^{\mathbf{n}} \stackrel{\text{def}}{=} \prod\limits_{i=1}^r y_i^{n_i}$. In fact, for $\alpha_1,\ldots,\alpha_r > -1$ we have the explicit formula

$$d\mu_{\boldsymbol{\alpha},x}(\mathbf{y}) \;=\; e^{-x}\, {}_0F_r\!\left(\begin{matrix} \text{---} \\ \alpha_1+1,\;\ldots,\;\alpha_r+1 \end{matrix} \,\middle|\, xy_1\cdots y_r\right) \prod_{i=1}^r \frac{1}{\Gamma(\alpha_i+1)}\, y_i^{\alpha_i}\, e^{-y_i}\, dy_i \,. \tag{3.6}$$

Proof. We begin from the exponential generating function (1.6) with $x \to -x$:

$$e^{-x}\left(\prod_{i=1}^r (1+t_i)^{-(\alpha_i+1)}\right) \exp\!\left[x \prod_{i=1}^r \frac{1}{1+t_i}\right] \;=\; e^{-x} \sum_{n=0}^\infty \frac{x^n}{n!} \prod_{i=1}^r (1+t_i)^{-(\alpha_i+1+n)} . \tag{3.7}$$

We now assume that $\alpha_1,\ldots,\alpha_r > -1$ and insert the integral representation

$$(1+t_i)^{-(\alpha_i+1+n)} \;=\; \frac{1}{\Gamma(\alpha_i+1+n)} \int\limits_0^\infty e^{-t_i y_i}\, y_i^{n+\alpha_i}\, e^{-y_i}\, dy_i \,. \tag{3.8}$$

[3] All this was observed a half-century ago by Karlin [14, p. 62] [15, pp. 440–441].

It follows that

$$e^{-x} \left(\prod_{i=1}^{r} (1+t_i)^{-(\alpha_i+1)} \right) \exp\!\left[x \prod_{i=1}^{r} \frac{1}{1+t_i} \right] \;=\; \int\limits_{[0,\infty)^r} e^{-\mathbf{t}\cdot\mathbf{y}} \, d\mu_{\boldsymbol{\alpha},x}(\mathbf{y}) \tag{3.9}$$

where

$$d\mu_{\boldsymbol{\alpha},x}(\mathbf{y}) \;=\; e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \prod_{i=1}^{r} \frac{y_i^{n+\alpha_i} \, e^{-y_i}}{\Gamma(\alpha_i+1+n)} \, dy_i \tag{3.10a}$$

$$\;=\; e^{-x} \, {}_0F_r\!\left(\begin{matrix} \text{—} \\ \alpha_1+1, \, \ldots, \, \alpha_r+1 \end{matrix} \middle| \, xy_1 \cdots y_r \right) \prod_{i=1}^{r} \frac{1}{\Gamma(\alpha_i+1)} \, y_i^{\alpha_i} \, e^{-y_i} \, dy_i \; . \tag{3.10b}$$

Extracting the coefficient of $\mathbf{t}^{\mathbf{n}}/\mathbf{n}!$, we conclude that

$$\mathcal{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x) \;=\; \int\limits_{[0,\infty)^r} \mathbf{y}^{\mathbf{n}} \, d\mu_{\boldsymbol{\alpha},x}(\mathbf{y}) \; . \tag{3.11}$$

This shows that $(\mathcal{L}_{\mathbf{n}}^{(\boldsymbol{\alpha})}(x))_{\mathbf{n}\in\mathbb{N}^r}$ is a multidimensional Stieltjes moment sequence whenever $\alpha_1, \ldots, \alpha_r > -1$; and it holds also for $\alpha_1, \ldots, \alpha_r \ge -1$ since the set of multidimensional Stieltjes moment sequences is closed under pointwise limits. $\square$

In particular, Theorem 3.1 implies:

Corollary 3.2. *Let $\alpha_1, \ldots, \alpha_r \ge -1$ and $x \ge 0$, and fix a multi-index $\mathbf{k} \in \mathbb{N}^r$. Then the sequence $(\mathcal{L}_{n\mathbf{k}}^{(\boldsymbol{\alpha})}(x))_{n\ge 0}$ is a Stieltjes moment sequence: that is, there exists a positive measure $\mu_{\boldsymbol{\alpha},x,\mathbf{k}}$ on $[0,\infty)$ such that*

$$\mathcal{L}_{n\mathbf{k}}^{(\boldsymbol{\alpha})}(x) \;=\; \int\limits_{[0,\infty)} y^n \, d\mu_{\boldsymbol{\alpha},x,\mathbf{k}}(y) \tag{3.12}$$

for all $n \ge 0$.

Corollary 3.2 can be restated in the language of total positivity. Recall that a finite or infinite matrix of real numbers is called *totally positive* (TP) if all its minors are nonnegative, and *totally positive of order r* (TP_r) if all its minors of size $\le r$ are nonnegative. Background information on totally positive matrices can be found in [8, 10, 15, 20]; they have application to many fields of pure and applied mathematics. In particular, it is known [11, Théorème 9] [20, section 4.6] that an infinite Hankel matrix $(a_{i+j})_{i,j\ge 0}$ of real numbers is totally positive if and only if the underlying sequence $(a_n)_{n\ge 0}$ is a Stieltjes moment sequence. So Corollary 3.2 asserts that, for every $\mathbf{k} \in \mathbb{N}^r$, every minor of the infinite Hankel matrix $(\mathcal{L}_{(i+j)\mathbf{k}}^{(\boldsymbol{\alpha})}(x))_{i,j\ge 0}$ is a polynomial in x and $\alpha_1, \ldots, \alpha_r$ that is nonnegative whenever $\alpha_1, \ldots, \alpha_r \ge -1$ and $x \ge 0$.

But much more appears to be true: namely, it seems that we have *coefficientwise* Hankel-total positivity [22–24] in the variables x and $\beta_i \stackrel{\text{def}}{=} \alpha_i + 1$:

Conjecture 3.3 (Coefficientwise Hankel-total positivity of the multiple Laguerre polynomials)**.** *For each multi-index* $\mathbf{k} \in \mathbb{N}^r$*, the sequence* $\big(\mathcal{L}_{n\mathbf{k}}^{(\boldsymbol{\beta}-\mathbf{1})}(x)\big)_{n\geq 0}$ *is coefficientwise Hankel-totally positive in the variables* x *and* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$*: that is, every minor of the infinite Hankel matrix* $\big(\mathcal{L}_{(i+j)\mathbf{k}}^{(\boldsymbol{\beta}-\mathbf{1})}(x)\big)_{i,j\geq 0}$ *is a polynomial in* x *and* $\boldsymbol{\beta}$ *with nonnegative coefficients.*

By symbolic computation using MATHEMATICA, I have verified this conjecture for the following cases:

- $r = 1$ and $\mathbf{k} = (1)$ up to the 11×11 Hankel matrix;
- $r = 2$ and $\mathbf{k} = (1, 1)$ up to the 9×9 Hankel matrix;
- $r = 2$ and $\mathbf{k} = (2, 1)$ up to the 8×8 Hankel matrix;
- $r = 2$ and $\mathbf{k} = (3, 1)$ up to the 8×8 Hankel matrix;
- $r = 2$ and $\mathbf{k} = (3, 2)$ up to the 8×8 Hankel matrix;
- $r = 3$ and $\mathbf{k} = (1, 1, 1)$ up to the 7×7 Hankel matrix;
- $r = 3$ and $\mathbf{k} = (2, 1, 1)$ up to the 6×6 Hankel matrix;
- $r = 3$ and $\mathbf{k} = (2, 2, 1)$ up to the 6×6 Hankel matrix;
- $r = 4$ and $\mathbf{k} = (1, 1, 1, 1)$ up to the 6×6 Hankel matrix;
- $r = 4$ and $\mathbf{k} = (2, 1, 1, 1)$ up to the 5×5 Hankel matrix;
- $r = 5$ and $\mathbf{k} = (1, 1, 1, 1, 1)$ up to the 4×4 Hankel matrix.

For the case of ordinary Laguerre polynomials ($r = 1$), this result was conjectured a few years ago by Sylvie Corteel and myself [4] and was proven very recently by Alex Dyachenko, Mathias Pétréolle and myself [7]. Our proof is based on constructing a quadridiagonal production matrix for the monic unsigned Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x)$ and then proving its total positivity; this construction is strongly motivated by the work of Coussement and Van Assche [5] on the multiple orthogonal polynomials associated to weights based on modified Bessel functions of the first kind [cf. (3.1)]. We have not yet succeeded in extending this proof to $r > 1$.

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