Monads, applicatives and the semantics of perspective: a solution to the problems of logical omniscience and granularity

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If the intension signifies the meaning of a word, and the extension is the number of actual objects of which the meaning can be truly predicated, then both extension and intension are relative to our knowledge, and naturally fluctuate with altering experience. For instance, "mammal" is a term whose meaning has changed and will change. We can fix no limit to the possible information the word may convey, for we do not know how many attributes in the end may be found to be implied in the quality of giving suck. And the number of objects we denominate "mammal" is of course not stationary. Such considerations may seem too obvious to be ignored, but their neglect has given rise to a serious mistake.

F.H BRADLEY

Lo suo tacere e 'l trasmutar sembiante puoser silenzio al mio cupido ingengno che già nuove questioni avea davante

DANTE ALIGHIERI

Declaration

I, Luke Burke, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Chapters 4-7 of this thesis extend and are in part based on, work originally contained in:

- (Burke 2019a) [Monads for hyperintensionality? A situation semantics for hyperintensional side effects. In Proceedings of the Sixth Workshop on Natural Language and Computer Science (pp. 34-43), the 13th International Conference on Computational Semantics (IWCS 2019).]

- (Burke 2019b) [P-HYPE: A monadic situation semantics for hyperintensional side effects. In Proceedings of Sinn und Bedeutung (Vol. 23, No. 1, pp. 201-218).]

- (Burke 2021) [Metalinguistic Focus in P-HYPE Semantics. In Logic in High Definition (pp. 203-240). Springer.]

- (Mendler et al. 2021) [Mendler, M., Scheele, S., & Burke, L. (2021, September). The Došen Square Under Construction: A Tale of Four Modalities. In International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (pp. 446-465). Springer.]

Abstract

This thesis elaborates a framework, P-HYPE, in which to capture fine-grained, 'hyperintensional' distinctions in meaning and in which to model the compositional sub-sentential semantics of a variety of natural language constructions.

In **part I** of the thesis, two problems which natural language semantic theory has faced from its inception are discussed: the problem of granularity and the problem of logical omniscience. The former problem regards issues that arise if we treat all sentences which express mathematical truths as having the same meaning, and issues that arise if we treat seemingly synonymous predicates such as *doctor* and *physician* as semantically equivalent. The latter problem concerns certain closure conditions on knowledge and belief, which entail, contrary to fact, that agents are omniscient in their logical powers.

In **part II** of the thesis, it is argued that the fine-grained semantic values which we require in order to solve the problems of granularity and logical omniscience can be understood as linguistic side-effects (Shan 2007), where the notion of a 'side effect' is taken from the literature on functional programming (Wadler 1995). In this way, we build upon a recent tradition in natural language semantic theory, which argues that various semantic phenomena can be understood as linguistic side-effects (Shan 2007, Charlow 2014).

We then develop P-HYPE, which is a combination of the logic HYPE (Leitgeb 2019) with the perspective relative semantic theory of (Asudeh and Giorgolo 2016). In P-HYPE, perspective relativity is used to capture the subjective understanding of words and expressions of individual language users. Via perspective relativity, we

can model a variety of phenomena, such as the behaviour of co-intensional predicates, focus, metalinguistic focus and anaphora. In addition, we show that P-HYPE can be used to solve the problem of logical omniscience, without introducing so-called 'impossible worlds' into the semantic theory, and in a way which is fully compositional at the subsentential level, unlike certain accounts of logical omniscience (Berto and Jago 2019, Solaki et al. 2019).

Impact statement

This thesis examines the problem of logical omniscience and granularity from the standpoint of natural language semantics. It clarifies the import of these principles for natural language semantics and formulates a logical framework, P-HYPE, in which these problems can be solved. P-HYPE combines ideas from the literature on perspective relativity with the logic HYPE. Our central contribution is the observation that perspective relativity can contribute to a solution to the problems of granularity and logical omniscience, so that the semantics of perspective relativity plays a role outside those phenomena to which it has traditionally been confined (e.g, predicates of taste and aesthetic predicates). For this reason, the thesis will be of interest to philosophers and linguists interested in the semantics of perspective. A second contribution we make is to analyse fine-grained meanings as 'side-effects' as this notion is understood in functional programming and, to this end, P-HYPE incorporates various mathematical structures employed in computer science for modelling side effects, such as monads, applicatives and continuations. Our research will therefore also be of interest to computer scientists who are interested in the applications of these mathematical structures.

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Part I: Granularity and logical omniscience from a natural language perspective

Chapter 0

Introduction

0.1 The problems of granularity and logical omniscience informally described

This thesis provides novel solutions to two foundational problems which arise in model-theoretic natural language semantic theories–the *problem of granularity* and the problem of *logical omniscience*.[1] The problem of granularity, as we will understand it, consists of two sub-problems. The first is the problem that results in natural language semantic theory from treating sentences which express mathematical truths as semantically equivalent and from treating sentences which express the negations of mathematical truths as semantically equivalent. The second problem arises from the assumption that seemingly synonymous predicates, such as *oculist* and *ophthalmologist* are semantically equivalent. The problem of logical omniscience concerns instead the question of how to model agents' reasoning capabili-

[1]Our thesis will concentrate on how these problems arise in model-theoretic natural language semantic theories, and we will not be considering proof-theoretic approaches to natural language semantics (though see (Francez 2015) for a recent monograph on the proof-theoretic approach).

ties without imputing to them unlimited logical powers.

Why are we treating what appear to be two separate problems in one thesis? Firstly, both problems have something in common: they both raise questions about exactly how fine-grained meanings are, and both have been perceived to pose a challenge of a foundational character to traditional natural language semantic theories (Partee 1982, Fox and Lappin 2008). Secondly, the problems are often treated together, often under the rubric of 'hyperintensionality' (Berto and Jago 2019, Jago 2014a, Pollard 2015), so that there is sometimes an expectation that they be treated together.

We call the class of model-theoretic natural language semantic theories in which we will examine these problems *state-based (intensional) semantic theories*, which employ the notion of a state familiar from the Kripke semantics of modal logic (Kripke 1959). In the simplest state-based semantic theories, states are primitive entities (colourfully called 'possible worlds') relative to which the truth of various formulas is evaluated (Cresswell 1973, Stalnaker 1984), but in more complex state-based semantic theories states may be pairs of possible worlds and assignment functions (Beaver 2001), or tuples of parameters relative to which the truth of a formula is evaluated (Montague 1974).

State-based semantic theories are typically animated by a key idea: that the semantic interpretation of a sentence is a set of states, or equivalently, the characteristic function associated with a set of states (Montague 1974, Dowty et al. 1981, Cann 1993, Carpenter 1997, Heim & von Fintel 2011, Kratzer 2012). State-based semantic theories often extend this basic animating idea, treating sentences as functions from various parameters (besides states) to Boolean truth values, or as functions from states to a set of 3 or more truth values.

The problem of granularity, as we understand it in this thesis,[2] is a problem

[2](Barwise 1997, Pollard 2015, Berto 2010) use the expression *granularity problem* to refer to

concerning how finely to individuate sentence meanings in state-based semantic theories. In particular, state-based semantic theories have been charged with lacking the fine-grained distinctions necessary for individuating the meanings of sentences expressing mathematical truths or of sentences expressing mathematical falsities, and as being insufficiently fine-grained for capturing the meanings of sentences expressing belief or knowledge or other so-called propositional attitudes (Hintikka 1975, Barwise and Perry 1983, Richard 1990, Jago 2015).[3] If this charge is correct, it raises the question of how a model-theoretic semantic theory which utilises states ought to treat the meanings of mathematical sentences and how such a semantic theory ought to treat the meanings of sentences expressing belief or knowledge or other so-called propositional attitudes.

The problem of logical omniscience—at least in the context of natural language semantics—concerns certain closure principles on the semantics of the verbs *know* and *believe* (and other propositional attitude verbs) which are often adopted in state-semantic theories. These closure principles appear to be undesirably strong, since they entail that agents are logically omniscient. For example, if we treat belief

the same problem. The expression *hyperintensional* and its cognates are sometimes used to refer to semantic theories which avoid the problem of granularity. Our disatissfaction with this term is explained in 1.4

[3]Certain sentences involving clause embedding verbs report the mental state or an act of communication, such as

(i) a. Peter believes that it is raining.
  b. Kim wants to go to sleep.
  c. John promised that he would return.

Believe and *want* express mental attitudes, whereas *promise* is a verb of communication. Both classes of verbs are called *propositional attitude verbs* and sentences whose main verbs are propositional attitude verbs are called (*propositional*) *attitude reports*.

via a normal doxastic modal operator as in (Hintikka 1962), our belief is closed under logical consequence, so that if ψ is a logical consequence of φ, then whoever believes φ, believes ψ. This conflicts with speakers' intuitions about the entailments between natural language sentences, thereby generating a rupture between certain intuitions about entailment in natural language and model-theoretic representations of natural language entailment.

0.2 Our solution outlined

In this thesis we argue that the problem of granularity can be resolved if we stay within the confines of state-based semantic theories but move to a richer notion of the meaning of sentences; rather than expressing functions from states to truth values, they will express something like[4] functions from perspective indices to states to truth values. We will treat sentences which express different mathematical truths as different functions from perspectives indices to states to truth values, in order to capture the fact that, relative to the perspective of certain individuals, certain mathematical truths fail to hold at certain states and certain mathematical falsities hold at them. We will avoid the charge of relativism— that our account implies that mathematical truths themselves vary in truth relative to people's perspectives– since we will distinguish the notion of a sentence being true *simpliciter* and the notion of a sentence being true with respect to a perspective index. Truth simpliciter is truth with respect to a special, designated perspective index– the enlightened perspective index.[5] Relative to the enlightened perspective index, sentences expressing mathe-

[4]Sentences will express more complicated functions in our theory, but this is simplification which should suffice for present purposes.

[5]The eventual shape of our account will be more complex than this, but this simplification doesn't matter for present purposes.

matical truths will be true, and those expressing mathematical falsities will be false. The notion of truth relative to a perspective is employed to capture the idiolectical and private aspects of natural language meanings, as opposed to any notion of objective truth. The perspective-relativity of sentences is in many cases not apparent, because we interpret the sentences relative to the enlightened perspective, so that sentences which traditionally have been held to denote the same characteristic function of a set of states, have the same semantic value in our framework with respect to the enlightened perspective index. When a propositional attitude verb like *believe* combines with a *that*-clause, the expressions in the *that*-clause may be evaluated from the perspective of an agent who is salient in the context, so that we evaluate the truth of the sentence with respect to their perspective index.

Just as sentences express functions from perspective indices and states to truth values, so predicates will be treated as state and perspective-relative, so that an n-ary predicate of individuals will express a function from perspective indices and states to a curried n-ary function from individuals to truth values. Predicates such as *oculist* and *ophthalmologist*—predicates which are often treated as having the same semantic value in many semantic theories—will express different functions from perspective indices, states and individuals to truth values. In this way, we can account for the fact that, relative to a given person's understanding of language, the predicates *oculist* and *ophthalmologist* might differ in meaning. Our semantics of perspective will develop that of (Asudeh and Giorgolo 2016), who argue that proper names are perspective-relative. We will build upon their account by exploring the consequences of making predicates and other expressions perspective-relative.

We implement our theory by using monads and by combining the logic HYPE Leitgeb (2019) into partial *Ty2* Lepage et al. (1992). For our purposes, monads are a way of compositionally incorporating different types of semantic value into one semantic theory, without necessitating revisions of compositional rules as more types of semantic value are added to the mix. But in addition, they provide a nice conceptual payoff, in that they enable us to conceptualise intensionality and 'hyperintensionality'[6] (Cresswell 1975) as side-effects, as this term is understood in the literature on functional programming (see part II for further discussion). Our reason for employing the logic HYPE is that we wish to vindicate Leitgeb (2019)'s claim that the logic HYPE is appropriate for modelling the problem of granularity (or rather, what Leitgeb (2019), following (Cresswell 1975) calls the *hyperintensional contexts*). In particular, we wish to see whether HYPE can form the basis of a subsentential semantics of natural language adequate for dealing with the so-called 'hyperintensional contexts' and this thesis can be seen as proof that HYPE can.

Regarding the problem of logical omniscience, we will argue that the problem can be resolved by taking into account, in a more fine-grained way, the rules of inference which are available to agents in context and the way that agents are able to use these in order to arrive at knowledge of an inconsistency. Our solution will make use of some ideas currently explored in the literature on dynamic epistemic logic Skipper and Bjerring (2018), Solaki et al. (2019), in order to avoid closure under HYPE logical consequence; however, unlike Skipper and Bjerring (2018), Solaki et al. (2019) we will not employ an explicitly dynamic framework. Furthermore, we will combine our solution with the semantics of perspective relativity which we elaborate in response to the problem of granularity, tagging the inference rules which are available in a context to those which are available relative to certain perspective indices.

Our account of the problem of logical omniscience will have two conceptual advantages over the accounts of Skipper and Bjerring (2018), Solaki et al. (2019).

[6]See 1.4 for discussion of so-called 'hyperintensionality'.

Firstly, we are able to simulate the role of certain states (often called ‘impossible worlds’) utilised in these theories (Skipper and Bjerring (2018), Solaki et al. (2019)) to which the valuation of a model assigns formulas in a non-recursive fashion, without employing the controversial ontology of impossible worlds. Secondly, Skipper and Bjerring (2018), Solaki et al. (2019) do not give a compositional subsentential semantics of natural language which addresses the problem of logical omniscience, whereas our treatment of the problem of logical omnscience will issue in a compositional subsentential semantics of natural language.

0.3 Roadmap of the thesis

The reader is encouraged to consult the introduction of each chapter for a fuller description of its contents.
The thesis consists of two parts: part I and part II, and an optional appendix. Part I consists of chapters **1** and **2**. Chapters **3**- **7** comprise part II.

Chapter 1 provides a detailed overview of the problems of logical omniscience and granularity, illustrating the various linguistic phenomena in which the problems are manifest.

Chapter 2 has two parts. The first half discusses two approaches to the problem of granularity: one based on extending the semantics of *de re* noun phrases to predicates, and the other based on the celebrated technique of diagonalisation introduced in (Stalnaker 1978). We argue that both approaches have serious pitfalls and point towards the desirability of a perspective-relative semantic theory of the kind we pursue. The second half discusses two solutions to the problem of logical omniscience, and points out various problems with them which motivate the development of an alternative approach.

Chapter 3 motivates the idea that fine-grained semantic values be treated as so-called linguistic side-effects (Shan 2007) and discusses how monads and applicatives—which will be used in P-HYPE—have been used to capture these. This opens up the possibility that many fine-grained semantic theories can be captured by using monads and applicatives. We then introduce HYPE and the perspective-relative semantic theory of (Asudeh and Giorgolo 2016), on which P-HYPE is based. Certain limitations with HYPE and with (Asudeh and Giorgolo 2016)'s theory are pointed out, which motivate the development of P-HYPE in the subsequent chapter.

Chapter 4 introduces P-HYPE, a semantic theory combining (Leitgeb 2019)'s logic, HYPE and (Asudeh and Giorgolo 2016)'s theory of perspective-relative interpretation. P-HYPE generalises the notion of perspective in (Asudeh and Giorgolo 2016) by applying it to predicates, so that predicates can differ in interpretation with respect to perspective indices. Unlike (Asudeh and Giorgolo 2016), we incorporate an enlightened perspective index, whose role is to ensure that certain obvious truths and entailments are preserved in our account. We show that P-HYPE can capture some aspects of (Asudeh and Giorgolo 2016)'s semantic theory which they do not formalise, and explain and discuss various lexical entries in P-HYPE.

Chapter 5 shows how P-HYPE resolves the problem of granularity and deals with various objections to our proposed resolution. We present compositional derivations of various sentences in our framework.

Chapter 6 extends the P-HYPE framework to cover the interaction of perspective and focus, so that we can capture predicates which bear intonational focus. The combination of perspective relativity and focus allows us to deal with certain cases of intonational focus which are tricky to capture. We then discuss how our semantics deals with metalinguistic focus on predicates and argue that the the semantics of metalinguistic focus in (Li 2017) does not work in all cases. Instead, we propose two alternative solutions to this problem, without deciding between them.

Finally, in **Chapter 7**, we present our solution to the problem of logical omniscience. Our solution is similar in spirit to certain 'dynamic' theories in the literature Skipper and Bjerring (2018), Solaki et al. (2019), except, unlike these, we provide a sub-sentential compositional semantics and we incorporate the notion of perspective into our account.

In the optional appendix (sections **A- B**), we elaborate the technical background necessary for understanding the thesis, introducing partial *Ty2* and our approach to compositional semantics. We discuss the formal details of (Berto and Jago 2019)'s account of the problem of logical omniscience in section **C**. Finally, section **E** sketches a treatment of intensional anaphora in P-HYPE using higher-order continuations (Barker and Shan 2014) which illustrates the potential of our perspective-relative semantic theory to be applied to other issues in semantic theory.

Chapter 1

The problem of granularity and the problem of logical omniscience

1.1 Introduction

In this chapter, we first examine the problem of granularity and then we examine the problem of logical omniscience. In 1.2 we explain how the problem of granularity arises if two principles constrain semantic theorising: *Mathematical Uniformity* and *Predicate Uniformity*. In 1.2.1 we explore the problematic predictions of *Mathematical Uniformity* and in 1.2.2 we explore the problematic predictions of *Predicate Uniformity*.

In 1.3 we explain how the problem of logical omniscience is first and foremost a problem in epistemic and doxastic logic, but it spills over into natural language semantic theories which treat propositional attitude verbs such as *believe* and *know* as modal operators which obey certain closure principles. The closure principles make some bad predictions about knowledge and belief and in 1.3 we discuss these bad predictions. Finally, in 1.4 we discuss the relationship between the problem of granularity and so-called ‘hyperintensionality’.

The reader may find section A- B of the appendix useful in understanding this chapter.

1.2 The problem of granularity

Consider the following principles (where $\mathfrak{M}$ is the model used in some state-based natural language semantic theory):

- *Mathematical uniformity*: There are natural language sentences which express mathematical truths and all natural language sentences which express mathematical truths are rendered[1] as formulas which have as their interpretation in $\mathfrak{M}$ the constant function from any state to 1 (i.e, the formulas which render sentences which express mathematical truths have as their interpretation the characteristic function of the set of all states).

- *Predicate uniformity*: Intuitively synonymous predicates such as *oculist* and *ophthalmologist* have the same semantic value in $\mathfrak{M}$.

These principles are somewhat vague. What counts as a sentence which expresses a mathematical truth? For example, does (1) express a mathematical truth?

(1) If there are two loaves of bread in room A and two loaves of bread in room B, then room A and B contain the same number of loaves of bread.

In one sense, (1) does express a mathematical truth, because it has some mathematical content (it refers to numbers and a notion of equality between them). On

[1] See A.1 for the notion of rendering.

the other hand, it seems to stretch the notion of a mathematical truth somewhat to allow mathematical truths to encompass facts about loaves, their background arithmetical content notwithstanding. Nonetheless, there are (relatively)[2] clear cases of sentences which express mathematical truths, such as (2a) and (2b):

(2) a. The prime numbers are infinite.
    b. Every Lie algebra is a vector space.

We will stick in our discussion of *Mathematical Uniformity* to clear-cut cases such as these. As for *Predicate Uniformity*, we will consider pairs of predicates which have been considered intuitively synonymous in the literature. The typical examples are pairs such as *oculist/ophthalmologist*, *bachelor/unmarried man*, *xerox/photocopy*.

Mathematical Uniformity and *Predicate Uniformity* are widely adopted by semantic theorists and philosophers who employ what we call *classical state-based semantic theories*, which are state-based semantic theories built on classical logic (Montague 1974, Dowty et al. 1981, Cann 1993, Carpenter 1997, Heim & von Fintel 2011, Kratzer 2012).

From now on, we will say that a semantic theory *induces* the problem of granularity if and only if it satisfies at least one of the following conditions:

- It adopts *Mathematical Uniformity*

[2]Leaving aside the fact that some mathematicians may reject as untrue certain classical mathematical theorems. For example, a hardcore constructivist who considers classical logic to be incorrect in the context of reasoning about infinities might reject as mathematical truths certain mathematical theorems which depend essentially in their proof on the axiom of choice or the law of the excluded middle. Thus it is controversial what mathematical truths to include for the purposes of *Mathematical Uniformity*.

- It adopts *Predicate Uniformity*.

We will call semantic theories which do not induce the problem of granularity *fine-grained semantic theories*. The reason we refer to the *problem* of granularity will become apparent in the next two sections in which we discuss the principles of *Mathematical uniformity* and *Predicate uniformity*.

The reader should note that we have excluded certain topics from the definition of the problem of granularity which are usually included under it (Barwise 1997), such as *Frege's puzzle* and the semantics of counterpossibles. Our rationale for not addressing these issues is that we thereby avoid being embroiled in the larger questions these topics give rise to, such as the nature of fictional discourse, in the case of *Frege's puzzle*, and the semantics of conditionals, in the case of counterpossibles. Discussion of these topics would lengthen the thesis considerably and they deserve separate treatment in their own right. For this reason we will remain neutral about what the best treatment of these topics is.

1.2.1 Sentences expressing mathematical truths

Mathematical uniformity is counterintuitive: it entails that all sentences expressing mathematical truths have the same semantic value; to wit, the constant function from all states to 1. Given semantic values are the model-theoretic objects used to capture natural language meanings, this predicts, counterintuitively, that sentences such as (3a) and (3b), which state mathematical truths, have the same meaning (see Cresswell (1975), Bigelow (1978), Cresswell and Stechow (1982), Cresswell (1985), Richard (1990), Jago (2014a), Égré (2020) for discussion):

(3) a. The prime numbers are infinite.

b. Every Lie algebra is a vector space.

The other side of the coin is that mathematical falsities, such as negations of (3a) and (3b) are then predicted (again counterintuitively) to have the same semantic value, since they are false in all states. Consequently, if someone believes the negation of a sentence which expresses a mathematical truth, then they are on a par with someone who believes an arbitrary contradiction.[3]

There have been valiant defenders of this treatment of mathematical sentences, in particular (Stalnaker 1984). However, in the main, the defences offered have largely been at the service of buttressing a metaphysical theory about mathematics (Kripke 1980, Fine 1981, Stalnaker 1984, Lewis 1986, Perry 1998, Jago 2014a, Williamson 2016)—such as buttressing a platonistic view of mathematical discourse—as opposed to offering empirical reasons from linguistic theory for treating mathematical sentences differently from others. In fact, as far as we are aware, no purely empirical arguments relating to the syntax and semantics of these sentences have been adduced for upholding *Mathematical Uniformity* in natural language semantic theory, which is surprising in the light of semantic theorists' adherence to it.[4] (Stalnaker 1984)'s position on this will be discussed in 2.2.2, but for now, from a linguistic point of view, such metaphysical arguments clash with our naive judgement about sentences such as (3a) and (3b), which differ in meaning in

[3]We will focus on the problems induced by treating sentences which express mathematical truths as denoting the constant function from all states to 1. The problem induced by treating sentences which express mathematical falsehoods as denoting the constant function from all states to 0 is the same as the problem induced by treating sentences which express mathematical truths as denoting the constant function from all states to 1; namely, the negations of sentences expressing mathematical truths are not always substitutable *salva veritate*.

[4]One might suppose that intuitions that sentences expressing mathematical truths must be the case, objectively speaking, as one piece of evidence in favour of *Mathematical Uniformity*. But this is weak evidence, given the gap between how reality is constituted and how we capture and represent it in thought.

much the same way that (4a) and (4b) do:

(4) a. Mary is sleeping.
    b. John was shocked.

Given the intuitively clear difference in meaning between different mathematical sentences, the onus is on the heroic defender of *Mathematical Uniformity* to justify her position against what seems to be the default assumption: that sentences expressing mathematical truths can differ in meaning. In the remainder of this section we will discuss further problems induced by *Mathematical Uniformity* which add to the case against this principle, at least as a principle to be adopted in natural language semantic theory as opposed to a metaphysical principle regarding mathematical truth.

The intuitive difference in meaning between sentences expressing mathematical truths persists when sentences expressing mathematical truths are embedded under *believe* (as well as other propositional attitude verbs), in an entirely parallel way to which differences in meaning between sentences expressing non-mathematical truths persist when the sentences are embedded under *believe*:

(5) a. Pat believes that Mary is sleeping.
    b. Pat believes that John was shocked.
    c. Pat believes that the prime numbers are infinite.
    d. Pat believes that every Lie algebra is a vector space.

Unfortunately, *Mathematical Uniformity* stands in the way of capturing this parallel between sentences expressing mathematical truths embedded under *believe* and non-mathematical sentences embedded under *believe*.

Further reasons for doubting *Mathematical Uniformity* emerge once we consider disjunctions and conjunctions of sentences expressing mathematical truths. *Prima facie*, disjunctions and coordinations of sentences expressing mathematical truths are not semantically equivalent to their disjuncts (or their conjuncts), whereas *Mathematical Uniformity* predicts they should be:

(6) a. The prime numbers are infinite or every Lie algebra is a vector space.
  b. The prime numbers are infinite and every Lie algebra is a vector space.

A defender of *Mathematical Uniformity* might argue that (3a), (6a) and (6b) have the same meaning but differ in their non-literal meaning.[5] For example, someone who utters (6a) communicates that they consider as live options only one of the disjuncts (as with an exclusivity implicature) or that they are unsure which disjunct is true (as with an ignorance implicature) and an utterance of (3a) does not give rise to such implicatures.

This proposed solution merely generates another problem, for it is unclear how implicatures are to be computed with respect to disjunctions consisting of sentences which express logical or mathematical truths, if the assumption of *Mathematical uniformity* is in place. If we calculate the proposed implicature by considering relevant alternatives, à la Grice (see (Schlenker 2016)), an utterance of (6a) will convey that the speaker is unsure of the stronger relevant alternative (6b) and thus that the speaker does not know that this stronger alternative is true. But if we represent knowledge via a simple universal modal operator over possible states (so that an agent knows p if p is true in all her epistemically accessible states), given *Mathematical uniformity*, it cannot be the case that the speaker is unsure of the truth of (6b); rather, the speaker must know (6b), since, *ex hypothesi*, it is true in all states.

[5](Stalnaker 1984) argues that sentences expressing mathematical truths differ in their non-literal meaning. We will examine his theory in 2.2.2.

So the conclusion that the speaker is unsure of the stronger relevant alternative (6b) is not possible to make, let alone any further conclusion that might be drawn from an utterance of (6a) (for example, the conclusion that the speaker knows that (6b) is false, or that the speaker does not know which of the conjuncts is true.) A broadly Gricean explanation of the contrast between uttering (6a) and (6b) therefore requires a special story in these sorts of cases. Consequently, absent a special story explaining how implicatures arise in these cases, implicature generation should be blocked. This shows that *Mathematical Uniformity* stands in the way of calculating the implicatures of sentences such as (6a).[6]

Finally, further problems with *Mathematical Uniformity* emerge if we cast our linguistic nets further, and consider questions and anaphora. For if (7b) and (7c)

[6] *Mathematical Uniformity* also predicts that an utterance of (6a) should be infelicitous, given Hurford's constraint (Chierchia et al. 2009). According to Hurford's constraint, a disjunction *A or B* is infelicitous if one of the disjuncts entails the other. For example, the disjunction *Either Mary is in France or she is in Paris* is infelicitous in a context where Paris is assumed by the speaker to be in France and *either John is here or someone is here* is infelicitous in a context in which John is assumed to be an animal and not an inanimate object. By contrast, although, given *Mathematical uniformity*, every state in which it is true that prime numbers are infinite is a state in which it is true that Lie algebras are vector spaces, and *vice versa*, (6b) is not infelicitous, contra Hurford's constraint. Thus, given (3a), (6a) and (6b) all have the same truth value at all states, an utterance of (6a) or (6b) should be redundant and given Hurford's constraint, and utterance of (6a) should be infelicitous. This conflicts with the strong intuition that (6a) and (6b) have perfectly non-redundant and felicitous readings. Moreover, if (3a), (6a) and (6b) all have the same truth value at all states, then an utterance of (6a) should be just as strong as an utterance of (3a) or (6b). This conflicts with the intuition that an assertion of a disjunctive coordination such as (6a) is weaker than an assertion of one of its disjuncts such as (3a), and that an assertion of a conjunctive coordination (6b) is stronger than both an assertion of one of its conjuncts such as (3a) and an assertion of a disjunction such as (6a). It is unclear how an appeal to the implicatures of (3a), (6a) and (6b) can cast any light on their intuitive difference in meaning.

had the same semantic value, we could answer by uttering (7c), which clashes with our intuition that (7c) is not an appropriate answer to the question:[7]

(7) a. Are there infinitely many primes?
b. There are infinitely many primes.
c. Bertrand's postulate is true.

Finally, if (7b) and (7c) had the same semantic value, the anaphoric pronoun *it* in (8a) could substitute for "Bertrand's postulate is true", in which case (8a) should have a reading (it does not) on which it has the same meaning as (8b):

(8) a. Sarah believes that there are infinitely many Mersenne primes and John believes it, too.
b. Sarah believes that there are infinitely many Mersenne primes and John believes that Bertrand's postulate is true, too.

In 2.2.2 we will discuss the most prominent way of responding to some of the problems we have adumbrated in this section. But *prima facie*, it seems that, in the light of the examples discussed in this section, whatever the metaphysical status of *Mathematical Uniformity* may be, it is of rather dubious status as a principle which governs the interpretation of mathematical sentences in natural language.

1.2.2 Predicates

We now consider the second half of the problem of granularity, which consists in the predictions that arise from assigning certain predicates the same interpreta-

[7]Furthermore, as Daniel Rothschild (p.c) has pointed out, on the partition semantics of questions (Groenendijk and Stokhof 1984) it is not clear that (7a) would be felicitous, since, if *Mathematical Uniformity* holds, the only answer it yields is the trivial partition of all states.

tion in all admissible models. The predicates in question are predicates such as *oculist* and *ophthalmologist*, which appear to be synonymous. In many situations, we can substitute such predicates for one another *salva veritate*, so that simple inferences such as *John is a bachelor, therefore John is an unmarried man* are truth preserving. In order to ensure that such inferences come out as truth preserving in all admissible partial *Ty2* models (see the appendix, section A.5 for the notion of a partial *Ty2* model), we could (see (Zimmermann 1999) for discussion of non-logical axioms in natural language semantics) impose the condition in (1.1) (where $bachelor, unmarried.man \in Con_{e \to s \to t}$ and where $\mathcal{M}$ is a class of admissible models):

$$\forall \mathfrak{M} \in \mathcal{M}\, \forall g \in G\, \forall d \in D_e\, \forall s \in D_s \\ ([\![bachelor]\!]^{\mathfrak{M},g}\, d\, s = [\![unmarried.man]\!]^{\mathfrak{M},g}\, d\, s) \tag{1.1}$$

In this way, $bachelor$ and $unmarried.man$ would be guaranteed to be co-intensional, where two predicates are *co-intensional* or *intensionally equivalent* if they have the same extension at every state. A weaker axiom would restrict the quantification over states in (1.1) to a proper subset of the set of all states. Then inferences such as *John is a bachelor, therefore John is an unmarried man.* would be truth-preserving only with respect to this particular subset of states.[8]

[8] An alternative to imposing axioms of this kind which we will not discuss is to employ the strategy of lexical decomposition (Engelberg 2011). For example, instead of imposing axioms on the interpretation of the constant $bachelor$, we define a constant $bachelor := \lambda i, x.unmarried(x)(i) \wedge man(x)(i) : s \to e \to t$, assuming we have the constants $unmarried$ and man in the language. In this case, $bachelor$ and *unmarried man* would come out as equivalent by definition. Montague formulated such non-logical axioms in the object language, but we could formulate them instead in a metalanguage, and in many cases this would be more appropriate. For example, if object language quantifiers only range over variables of type e in the model, but we wanted a logical axiom which

(Mates 1950) observed that seemingly synonymous predicates such as *bachelor* and *unmarried man* are not always replaceable with one another *salva veritate*. He observed that, for natural language predicates D and D' which seem intuitively to be synonymous, (9a) and (9b) may differ in truth-value:

(9) a. Nobody doubts that whoever believes that John is a D believes that John is a D.
b. Nobody doubts that whoever believes that John is a D believes that John is a D'.

(Mates 1950)'s observation also holds of the following pairs:

(10) a. Jason believes that Harvard is a fine university.[9]
b. Jason believes that Hahvahd is a fine university.

(11) a. Mary believes that John is an oculist.
b. Mary believes that John is an ophthalmologist.

(12) a. Max believes that Mary xeroxed War and Peace
b. Max believes that Mary photocopied War and Peace.

Each of these pairs can sometimes bear a different interpretation, in circumstances in which we take into account someone's beliefs about the denotation of the two predicates. For example, Mary might believe that not every oculist is an ophthalmologist, or Max might believe that *xerox* and *photocopy* refer to two distinct activities. For this reason, it seems that co-intensional predicates embedded under

quantified over predicate variables of some type, then we might formulate an appropriate non-logical axiom in the metalanguage. We could alternatively adopt a first-order schema instead.

[9]Here Harvard represents Jason's pronunciation of the string *Harvard* and Hahvahd is some other pronunciation of the string, salient in the context of utterance.

attitude verbs (as well as predicates not so embedded) seem to differ in their contribution to the meaning of the sentences they are contained within.

We can also explicitly deny that one of two seemingly synonymous predicates applies whilst maintaining that another synonymous predicate applies:

(13) John thinks that he is a bachelor, not an unmarried man.

(14) Peter believes that John is an oculist, but he doesn't believe that John is an ophthalmologist.

If the predicates $bachelor$ and $unmarried.man$ have the same semantic value in all admissible models and are used to render the words *bachelor* and *unmarried man* in (13) (respectively), then (13) should be contradictory. However, (13) has a true reading in a scenario in which John associates a different interpretation with *bachelor* than with *unmarried man* and we are talking about these predicates from John's perspective. From an utterance of (14) we would usually infer that Peter thinks that being an oculist is not the same thing as being an ophthalmologist. That is, we take the ascriber's (Peter) point of view on what the ascribee (John) thinks, and the ascribee in this instance is taken to distinguish the meaning of two expressions which have the same meaning for the ascriber. Observations of this kind have been widely discussed in the literature on attitude reports ((Mates 1950), (Partee 1973), (Kripke 1979), (Larson and Ludlow 1993), (Fiengo and May 1996)), and the upshot is that, sometimes, in order for an utterance to be felicitous we have to take into account the way the grammatical subject of the sentence thinks about the matter, or we have to take into account the fact that they made an utterance, or have made utterances, in a particular language.[10] (13) and (14) presuppose that either John or the utterer

[10](Yagisawa 1984) thinks the following sort of cases casts into doubt (Mates 1950)'s observation:

of the sentence is an English speaker (or at least has uttered certain sounds which resemble the sounds of certain English words) and, in many cases, that either John or the utterer is confused about the meaning of *oculist* and *ophthalmologist*.[11]

Similar observations also apply to mathematical predicates. In the next chapter, we will focus on a particular case in which mathematical predicates which are co-intensional (given appropriate mathematical axioms have been laid down) but differ in meaning in certain circumstances. Consider the class of *verbs of inference* (Partee 1973), such as *deduce, figure out, prove, establish, show, discover* and *infer*. Here we use *prove/conclude/establish* in our examples, but similar examples

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(i) a. Nobody doubts that whoever believes that 14 is a number believes that 14 is a number.

b. Nobody doubts that whoever believes that 14 is a number believes that fourteen is a number.

(Yagisawa 1984) claims that (ia) and (ib) cannot differ in truth value, and so these examples are counterexamples to (Mates 1950)'s observation. However, consider a child who has only learnt the roman numerals and thinks that $fourteen$ is a word with a completely different meaning. It is coherent to suppose that a child might, when asked if fourteen is a number, ask whether the person asking the question means '14' or 'fourteen', and give a different answer depending on whether she thinks the person asking the question is using the expression '14' or the expression 'fourteen'. In these cases, (ii) might have a true reading:

(ii) The child believes that 14 is a number, but she doesn't believe that fourteen is a number.

For this child, 'fourteen' and '14' are two entirely different words (albeit pronounced in the same way). We can say that the child has a different perspective on the interpretation of the two words. But the fact that she has a perspective on the interpretation of these words doesn't entail that we should model her belief by introducing quotation into the semantics. Arguably the puzzle also has to do with the different perspective that language users have on the interpretation of expressions. In the next chapter we will have a look at a notion of perspective which can capture this.

[11](Rieber 1992) points out that there are certain cases in which John might think that oculist and ophthalmologist have a different meaning but understand these expressions perfectly well.

could be constructed with the other verbs of inference. In (Cresswell 1985: p.82) a set is defined to be 'finite' iff it cannot be put into a one-one correspondence with a proper subset of itself, and a set is 'inductive' iff it can be put into a one-one correspondence with a proper initial segment of the natural numbers. Readers may be more familiar with the terms 'Dedekind finite' for the former term and 'finite' for the latter (Cameron 2012). The inductive sets and the finite sets are provably co-extensional in ZFC.[12] So if *Mathematical Uniformity* holds, (15) and (16) have the same intension (here 'the prime numbers' denotes the set of prime numbers), as do (17) and (18):

(15) Kim proved that the prime numbers are infinite.

(16) Kim proved that the prime numbers are not inductive.

(17) The prime numbers are infinite.

(18) The prime numbers are not inductive.

Consider the verb pairs *assume/conclude, suppose/establish, consider/deduce*, which are pairs of verbs of inference (Partee 1973). The second of each of these pairs we call 'result verbs' since they express result of a process described by the first verb. But now consider the contrast between (19b) and (19c) in the scenario (19a), where we prefix '??' to example sentences to indicate that they are infelicitous:

(19) a. *Scenario*: Harold is told the primes are not inductive and via deduction arrives at the conclusion that the primes are not finite, a conclusion which he may have rejected before.

[12] Lavinia Picollo urged that we write 'provably co-extensional' here, as opposed to 'provably equivalent', since this latter phrase would 'assume a standard notion of equivalence between sets', whereas this notion might be controversial.

b. Harold was given that the primes are not inductive and concluded/ established that the primes are not finite.

c. ?? Harold was given that the primes are not finite and concluded/ established that the primes are not finite.

d. ?? Harold was given that the primes are not inductive and concluded/ established that either first order logic is complete or it is not complete.

(19b) is felicitous in the scenario described but (19c) is not, on account of its redundancy, nor is (19d).[13] If 'the primes are not inductive' and 'the primes are not finite' are both given the same semantic value this difference between (19b) and (19c) is not easily predicted.[14] The contrast between (19b) and (19d) is even starker: (19d) in no way seems to describe the process that Harold has gone through in making his discovery, even if the truth of the statement that either first order logic is complete or it is not complete is secured in all states of every admissible model by appropriate non-logical axioms.

Now consider a similar example, (20a), in the same scenario. (20a) is perfectly felicitous in the scenario described. But if we have imposed non-logical axioms making *finite* and *inductive* equivalent, (20a) would be treated as necessarily false, like the contradictory (20b) or the mathematical falsity described in (20c):

[13]There is also a reading of (19c) which is felicitous, which depends on the sentence being interpreted from the standpoint of the utterer, who is assumed to know the relevant mathematical equivalence.

[14]The difference in felicity might also have something to do with the fact that uttering (19c) would be misleading on the part of an utterer who knew the mathematical equivalence of the predicates 'is not prime' and 'is not inductive', in a context where it was known by the utterer that the addressee was ignorant of the equivalence.

(20) a. Harold proved the prime numbers are not finite, not that the prime numbers are not inductive.

b. ?? Harold proved the prime numbers are not finite, not that the prime numbers are not finite.

c. ?? John proved that there are finitely many prime numbers.

Again, this contrast suggests that, *prima facie*, the predicates *is finite* and *is inductive* make distinct semantic contributions to these sentences.

The linguistic judgements we have made about the cases discussed above are easily explained if we suppose that the scenario (19a) is one in which Harold discovers something which is genuinely new to him as a result of his calculations. Harold's cognitive state at the end of this process of deduction is genuinely different from how it was before he began the proof, and this is reflected in the judgements he makes about what he has discovered. In fact, we may imagine that, had Harold already known the conclusion, he may have decided to do something else. This difference in how he would have behaved had he known the conclusion before setting out to prove it is arguably best explained by supposing that he came to know something he did not know previously, and thus that what he came to know is distinct from what he knew when he began the proof (see (Heck 2014) for arguments to this end).

1.3 The problem of logical omniscience

The problem of logical omniscience concerns the modelling of knowledge and belief, and it was Hintikka (see (Hintikka 1962, Fagin et al. 2003) and references therein) who was one of the first to discuss the problem in the context of formal epistemic and doxastic logics. In the approach to epistemic and doxastic logic pioneered by (Hintikka 1962), modal operators are used to represent the beliefs and knowledge of agents (see the appendix, section A.7). Such modal operators are indexed by a non-empty set I of agents, where, for each agent $i \in I$, the epistemical modal operator $\mathcal{K}_i$ and the doxastic modal operator $\mathcal{B}_i$ are intended to represent (respectively) the knowledge and the beliefs of the agent i. Syntactically, such epistemic and doxastic modal operators combine with formulas φ, ψ to form formulas $\mathcal{K}_i\varphi$, $\mathcal{B}_i\psi$ in order to represent that the agent believes or knows the given formula. Typically, in the model-theoretic semantics of epistemic and doxastic logics, for every agent, there is an associated *accessibility relation* which holds between states. A *doxastic accessibility relation* for a given agent captures which states are compatible with that agent's beliefs at a given state, and an *epistemic accessibility relation* for a given agent captures which states are compatible with that agent's knowledge at a given state. We then semantically interpret formulas such as $\mathcal{B}_i\varphi$ as the set of states S such that all the states doxastically accessible from any member of S are states in which φ is true.

It is common to discuss the problem of logical omniscience from the vantage point of a simple propositional modal logic. For example, Fagin et al. (2003) isolate various forms of logical omniscience in the context of (classical) propositional modal logic, the strongest being full logical omniscience, which Fagin et al. (2003) define as follows:

> "An agent is *fully logically omniscient* with respect to a class $\mathcal{M}$ of structures if, whenever he knows all of the formulas in a set Ψ, and Ψ logically implies the formula ϕ with respect to $\mathcal{M}$, then the agent also knows ϕ."

Fagin et al. (2003) define the notion of logical implication, in turn, as follows:

"...we say that Ψ *logically implies* ϕ *with respect to a class* $\mathcal{M}$ *of structures* if, for all $M \in \mathcal{M}$ and all states s in M, whenever $M, s \models \psi$ for every $\psi \in \Psi$, then $M, s \models \phi$."

Let us transpose these notions to the richer, higher-order setting of partial *Ty2*, making use of ideas presented in the appendix, in section A.6. Let $\mathcal{M} \subseteq \mathbf{M}$ be a class of admissible partial *Ty2* models. Using the notation of (Popkorn 1994), in partial *Ty2* we can formulate full logical omniscience as (1.2), which is equivalent to (1.3) (see (8.12) in the appendix):

$$\mathcal{M}\ ;\ \boldsymbol{\Psi} \models_i \boldsymbol{\phi} \quad \Rightarrow \quad \mathcal{M}\ ;\ \Box\boldsymbol{\Psi} \models_i \Box\boldsymbol{\phi} \tag{1.2}$$

$$\mathcal{M}\ ;\ ;\ \models_i \boldsymbol{\Psi} \rightarrow \boldsymbol{\phi} \quad \Rightarrow \mathcal{M}\ \ ;\ ;\ \ \models_i \Box\boldsymbol{\Psi} \rightarrow \Box\boldsymbol{\phi} \tag{1.3}$$

(1.3) clearly holds in *Ty2*, as a simple inductive argument shows.

Full logical omniscience implies the following family of closure principles—called *forms of logical omniscience*:[15]

Necessitation

$$\mathcal{M},\ \boldsymbol{\phi}\ ;\ \models_i \Box\boldsymbol{\phi}$$

Closure under logical consequence

[15] Note that, given the definitions in the appendix, $\mathcal{M}\ \ \boldsymbol{\phi}\ ;\ \models_i \Box\boldsymbol{\phi}$ states:

$\mathcal{M}\ \ \boldsymbol{\phi}\ ;\ \models_i \Box\boldsymbol{\phi}$ if and only if $\forall \mathfrak{M} \in \mathcal{M}\, \forall g (\mathfrak{M}, g \models \forall s \boldsymbol{\phi}\, s \Rightarrow \mathfrak{M}, g \models \forall s\, (\Box\, \boldsymbol{\phi})\, s$

if and only if $\forall \mathfrak{M} \in \mathcal{M}\, \forall g (\mathfrak{M}, g \models \forall s \boldsymbol{\phi}\, s \Rightarrow$
$\mathfrak{M}, g \models \forall s\, (\lambda p_{s \rightarrow t}, w_s . \forall w' (R\, w\, w' \rightarrow p\, w')\, \boldsymbol{\phi})\, s$

if and only if $\forall \mathfrak{M} \in \mathcal{M}\, \forall g (\mathfrak{M}, g \models \forall s \boldsymbol{\phi}\, s \Rightarrow$
$\mathfrak{M}, g \models \forall s\, \forall w' (R\, s\, w' \rightarrow \boldsymbol{\phi}\, w')$

$$\mathcal{M}\ ;\ ;\ \models_i \phi \rightarrow \psi \quad \Rightarrow \quad \mathcal{M}\ ;\ ;\ \models_i \Box\phi \rightarrow \Box\psi$$

Closure under logical equivalence

$$\mathcal{M}\ ;\ ;\ \models_i \phi \leftrightarrow \psi \quad \Rightarrow \quad \mathcal{M}\ ;\ ;\ \models_i \Box\phi \leftrightarrow \Box\psi$$

Closure under material implication:

$$\mathcal{M}\ ;\ ;\ \models_i (\Box\phi \wedge \Box(\phi \rightarrow \psi)) \rightarrow \Box\psi$$

Distribution of $\Box$ *with respect to* $\wedge$

$$\mathcal{M}\ ;\ ;\ \models_i \Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$$

Agglomeration

$$\mathcal{M}\ ;\ ;\ \models_i \Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$$

Disjunctive weakening on the right

$$\mathcal{M}\ ;\ ;\ \models_i \Box\phi \rightarrow \Box(\phi \vee \psi)$$

Disjunctive weakening on the left

$$\mathcal{M}\ ;\ ;\ \models_i \Box\phi \rightarrow \Box(\psi \vee \phi)$$

What exactly is the worry with the forms of logical omniscience above? Fundamentally, the worry here is that agents may often not put ‘two and two together’ and so their beliefs may fail to be closed under a given form of logical omniscience, even if they are perfectly informed about relevant facts and otherwise reason perfectly logically. To suppose otherwise would appear to belie agents’ evident cognitive limitations, attributing to them omniscient powers of deduction. And whilst it may be useful to ‘idealise away’ the limitations of agents for certain theoretical purposes (see (Holliday 2015) for discussion), if we want our semantic theories to provide a descriptively adequate account of belief and knowledge, then such limitations must eventually be addressed. From now on, we understand *the problem of logical omniscience* to consist in the implausibly strong conditions that the above forms of logical omniscience impose on the concepts of knowledge and belief. A semantic theory *induces* or *gives rise to* the problem of logical omniscience insofar as it entails the correctness of at least one of the forms of logical omniscience above.

Now we will consider the relation between the problem of logical omniscience and natural language semantics. But first, note that, in this thesis, our discussion will generally concentrate on *believe*, *know* (and later, *prove*), but the issues we discuss arise for other clause-embedding verbs, such as *hope* and *dream* (Égré 2020), insofar as they are closed under certain forms of logical omniscience. We will use *embedded sentence* to refer to a clause which is the syntactic argument of an embedding verb, and we will say that the sentence is *embedded under* the (embedding) verb. An *unembedded sentence* is a clause which is not the argument of an embedding verb.

The problem of logical omniscience spills over into those model-theoretic natural language semantic theories in which the meanings of sentences are taken to be sets of states and in which doxastic or epistemic modal operators closed under some of the forms of logical omniscience above are used to represent the meanings of sentences such as *Mary knows the axioms of Peano arithmetic*. For these assumptions will issue in concrete predictions about the entailment atterns of natural language sentences, and their meanings. For example, *Necessitation* predicts that *Mary believes that S* is always true whenever *S* expresses a logical truth, despite the infinitely many logical truths and their complexity, and closure under logical equivalence implies that whenever *Mary believes that S*, *Mary believes that S'*, if *S* and *S'* are logically equivalent sentences, even though it may be beyond Mary's capabilities to actually demonstrate or come to realise that *S* and *S'* are logically equivalent.[16] If belief is closed under logical consequence, whenever *Mary believes the axioms of Peano arithmetic* is true at a state s, and the axioms of Peano arithmetic have been adjoined to the logic used to represent Mary's beliefs, then, it is predicted that it will be true to say that Mary believes all the consequences of the Peano axioms. Many theorists Cresswell (1975), Bigelow (1978), Cresswell and Stechow (1982), Cresswell (1985), Richard (1990), Jago (2014a), Égré (2020) have deemed it quite implausible to maintain that *Mary believes the axioms of Peano arithmetic* entails that Mary believes all the far flung logical consequences of these axioms; consequently, in this case they dispute the descriptive adequacy of the closure of belief under logical consequence, on the grounds that it contradicts an intuitive notion of semantic entailment standing between sentences in natural language which semantic theories ought to capture. Since it is one of the purposes of natural language semantic theory to capture entailments which stand between natural

[16]Here we gloss over an important issue; namely, that we cannot always take for granted that there is sufficient agreement about which sentences of natural language are logically equivalent. This is because, in many cases, the path from a natural language sentence to an appropriate analysis of its logical form and compositional semantics is an object of intense theoretical scrutiny, and not widely agreed upon. The linguistic literature is full of examples in which sentences which seem naively to have a certain structure or certain truth conditions have been argued to have a more complicated structure or semantics on analysis. To give just one example, (Kratzer 2012) claims that:

> "The history of the conditional is the story of a syntactic mistake. There is no two-place *if* . . . then connective in the logical forms for natural languages. *If*-clauses are devices for restricting the domains of various operators. Whenever there is no explicit operator, we have to posit one."

For the same reason we cannot take for granted whether two sentences are logically equivalent, we cannot always take for granted whether a sentence expresses a tautology. It may be the case that sentences which appear to have the form of tautologies, in fact have a more complex structure or a more complex semantics.

language sentences and to rule out non-entailments, this is a problem for natural language semantics.

Consider the consequences of combining *Mathematical Uniformity* with *Closure under logical equivalence*. If the verb *dream* is rendered as universal quantifier over states in the range of some suitable accessibility relation, whenever *Mary dreamt that two plus two is four* is true (or some other sentence in which *dream* embeds a clause expressing a mathematical truth), then it will be true to say that *Mary dreamt that S*, for a sentence *S* expressing some arbitrary mathematical truth, and so (21) will come out true:

(21) Mary dreamt that two plus two is four, so Mary dreamt that sixty-eight times fourteen is nine hundred and fifty two.

In addition, the following implications would come out correct in a circumstance in which Mary dreams that sixty-eight times fourteen is nine hundred and fifty two:

(22) a. Mary dreamt that sixty-eight times fourteen is nine hundred and fifty two, so Mary believes that sixty-eight times fourteen is nine hundred and fifty two.

b. Mary dreamt that two plus two is four, so Mary believes that two plus two is four.

(22) contradicts the intuitive datum that Mary may in fact not believe what she dreams, when she wakes up, whereas (22a) seems to bear no relation whatsoever to the circumstances described in Mary's dream, and the conclusions that she may draw from it.

1.4 How do the problems of granularity and logical omniscience relate to the issue of so-called ‘hyperintensionality’?

Often the problems of granularity and logical omniscience are both subsumed under the notion of *hyperintensionality* Jago (2014a). Following (Barwise 1997, Berto 2010), we have chosen to use the term *granularity* where others would use *hyperintensionality*, because we think the term *hyperintensional* and its cognates are either incredibly misleading or tendentious.

In the literature, a *hyperintensional semantic theory* is often understood as a semantic theory which can distinguish the meanings of sentences which are logically equivalent, particularly within the scope of certain operators, called the *hyperintensional operators*. For example, (Cresswell 1975) and Jago (2014a) define an operator O as *hyperintensional* if OA and OB may differ in truth value with respect to a state in a model, even when A and B are logically equivalent and have the same truth value at all states and models. For this reason, Cresswell argues that ‘x believes that’ (and presumably ‘$x\ Vs$ that’, for other clause-embedding verbs V) creates a *hyperintensional context*, understood as a context in whose scope logically equivalent expressions are not substitutable *salva veritate*.

Used in this way, the word *hyperintensional* is redundant. For hyperintensional operators, as Cresswell and Jago understand them, are simply operators which are not closed under logical equivalence, and closure under logical equivalence is a form of logical omniscience which we discussed above. The question of how to deal with hyperintensional operators is therefore simply the question of how to deal with closure under logical equivalence. (Berto and Nolan 2021) claim that a hyperintensional semantic theory is best understood more broadly to denote a semantic theory in which we can distinguish the meanings of sentences which are necessarily equivalent, where the relevant notion of necessity might, besides logical necessity, include mathematical and metaphysical necessity. However, in a state-based semantic framework, the default assumption would surely be that such types of necessity be understood as restricted forms of logical equivalence (i.e, logical equivalence with respect to a subset of states in certain models), and we are unaware of any proof that this cannot be done.[17] If this is the case, then the question of how to deal with hyperintensional operators is therefore simply the question of how to deal with closure under different forms of (restricted) logical equivalence.

Secondly, we will avoid the term 'hyperintensional' because it is commonly used with a presupposition in mind: the presupposition that a hyperintensional semantic theory is not simply an intensional semantic theory, but is based on a special kind of logic that is not merely an intensional classical logic (i.e, a classical modal logic) of the kind used in the classical state-based semantic theories.[18] For example,

[17]It doesn't help(Berto and Nolan 2021) do not clarify what they mean by mathematical and metaphysical necessity in technical terms.

[18]The most rigorous definition of *hyperintensional logic* we know of is provided by (Odintsov and Wansing 2019). This is based on the concept of self-extensionality from abstract algebraic logic (Font 2016) and perhaps captures what some philosophers have in mind by the concept of hyperintensionality. Given a consequence relation $\vdash$ of a logic L, and where $\psi \dashv\vdash \varphi := \psi \vdash \varphi$ and $\varphi \vdash \psi$, then L is said to be *self-extensional* if for all formulas φ, ψ, θ and propositional variables p from the language of L:

$$\psi \dashv\vdash \varphi \text{ if and only if } \theta[\phi/p] \dashv\vdash \theta[\psi/p]$$

Here $\theta[\varphi/p]$ and $\theta[\psi/p]$ result, respectively, by substituting φ for p and ψ for p in θ. (Odintsov and Wansing 2019) define a logic L as *hyperintensional* if and only if it is not self-extensional. They then define a hyperintensional n-place connective in terms of the concept of congruentiality. An n-place connective $\#$ is *congruential* if and only if the following rule is admissible; otherwise $\#$ is recently truthmaker semantics (Fine 2017) has been touted as a 'paradigm springing the bounds of standard modal logic' (van Benthem 2019).

'Hyperintensional' used in this way implies something that requires demonstration; namely, that a special non-classical logic is necessary or desirable for an account of the so-called hyperintensional contexts. This is a rather strong assumption which requires argument, and should not be smuggled in by terminological fiat. The strength of the assumption can be seen by reflecting on the fact that classical logic is extremely powerful, and we can embed many different logics into classical logic. Often semantic theories of logics are declared 'hyperintensional' without considering whether they can be translated into classical logic. But often such logics can be translated into classical logic. Consider truthmaker semantics. (van Benthem 2019) has proven that there is a translation from truthmaker semantics into temporal S4, and S4 can itself be translated into classical logic via the standard translation (Blackburn et al. 2001). Additionally, consider HYPE (Leitgeb 2019), which we will use in this thesis. (Odintsov and Wansing 2019) has proven that we can translate HYPE into a classical bimodal logic, which can in turn be translated into classical logic via the standard translation (Blackburn et al. 2001). Consequently, in order to avoid the implication that we need some special non-classical logic to model the hyperintensional contexts which is not merely a classical intensional logic, we avoid the term 'hyperintensional' and its cognates.

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hyperintensional:

$$\varphi_i \dashv\vdash \psi_i \text{ for } i = 1, ..., n, \text{ then } \#(\varphi_1, ..., \varphi_n) \dashv\vdash \#(\psi_1, ..., \psi_n)$$

1.5 Conclusion

The problem of granularity is generated by adherence to the principles of *Mathematical Uniformity* and *Predicate Uniformity*. The problems with these principles that we have adumbrated in this chapter constitute a *prima facie* case against them, at least from the standpoint of natural language semantics. The principles of *Mathematical Uniformity* and *Predicate Uniformity* seem to rest on metaphysical assumptions, rather than on any empirical arguments from linguistic theory. The principle of *Mathematical Uniformity*, gives rise to counter-intuitive predictions in propositional attitude contexts, as well as in the semantics of questions and anaphora. It also creates problems in accounting for the implicatures that sentences have. *Predicate Uniformity* cannot distinguish the semantic contribution of certain predicates in propositional attitude contexts and stands in the way of capturing the way that an agent's limited information interact with the semantics of propositional attitude reports.

The problem of logical omniscience is generated by certain closure principles on modal operators representing knowledge or belief, which make predictions about agents' knowledge or beliefs which are hard to square with their limited cognitive capacities. If such modal operators are used to capture the meanings of natural language expressions such as *know* and *believe*, the problem of logical omniscience makes predictions about what sentences are true in natural language. Here, again, the demands of the closure principles seem to conflict with the belief and knowledge attributions we are willing to make about agents based on the limitations of their informational perspective.

Sometimes the problem of granularity and logical omniscience are discussed under the banner of *hyperintensionality*. The word *hyperintensional* and its cognates are often used tendentiously, and obfuscate the debate. We therefore propose they be abandoned in natural language semantics, though they may be useful as a technical term to isolate categories of logics in abstract algebraic logic (Odintsov and Wansing 2019).

In the next chapter we will see how some theorists have dealt with the problems of granularity and logical omniscience.

Chapter 2

On some solutions to the problems of granularity and logical omniscience

2.1 Introduction

The chapter will discuss some solutions to the problem of granularity (2.2), and some solutions to the problem of logical omniscience (2.3), with a view to motivating an alternative approach based on a semantics of perspective, which we pursue in part II of the thesis. We have had to be selective in the theories we examine, given the vast literature on both problems.[1] We examine two kinds of solution to the problem of granularity: *de re* approaches (Cresswell and Stechow 1982, Tancredi and Sharvit 2020) and an approach based on diagonalisation (Stalnaker 1978). *De re* approaches are examined in 2.2.1 and 2.2.1.2 and the approach based on diagonalisation is examined in 2.2.2. With reference to both the *de re* approaches and the diagonalisation approach, we point towards the need to incorporate a notion of

[1]See (Stalnaker 1984, Cresswell 1985, Richard 1990, Crimmins 1992, Bäuerle and Cresswell 2003, Jago 2006, Fox and Lappin 2008, Jago 2014a) and references therein.

perspective in order to solve the problems we identify.

In the second half of the chapter we examine two solutions to the problem of logical omniscience: that of (Solaki et al. 2019)–examined in 2.3.1—and that of (Berto and Jago 2019)—examined in 2.3.2. Both solutions aim to capture the fact that agents do not usually believe obvious contradictions whilst rejecting the forms of logical omniscience discussed in 1.3. We argue that certain aspects of (Solaki et al. 2019), such as their incorporation of resources into their model, are artificial from the standpoint of natural language semantics, and we argue that (Berto and Jago 2019)'s theory does not allow for the possibility of felicitous attributions of obvious contradictions.

2.2 Some solutions to the problem of granularity adopted in the literature

2.2.1 *De re* approaches

Consider (Quine 1956)'s story of Ralph and Ortcutt. On one occasion, Ralph sees a man wearing a brown hat who he thinks is behaving like a spy. On another occasion, Ralph sees a man on the beach who he judges not to exhibit spy-like behaviour. Unbeknownst to Ralph, on both occasions he has seen Ortcutt, his friend. With reference to the first occasion, we can truly utter (1a), whilst with reference to the second occasion we can truly utter (1b):

(1) a. Ralph believes that Ortcutt is a spy.
    b. Ralph believes that Ortcutt is not a spy.

In the readings of both (1a) and (1b), *Ortcutt* is interpreted *de re*. According to the *de dicto* reading of (1a), in all of Ralph's doxastic alternatives, Ortcutt is a spy, whilst according to the *de dicto* reading of (1b), in all of Ralph's doxastic alternatives, Ortcutt is not a spy. But the *de re* readings of (1a) and (1b) can be true even if Ralph has no beliefs about whether the individual he sees in both cases is Ortcutt.

Certain approaches to the semantics of *de re* noun phrases have been extended so as to deal with the problem of granularity and the problem of logical omniscience. We call these *de re* approaches. The *locus classicus* of this kind of approach is (Cresswell and Stechow 1982) (see also (Égré 2014) in which a similar theory is developed). It would certainly be attractive if a solution to the problem of granularity fell neatly out of a semantics of the *de re*. But we will see in this section that *de re* approaches have important limitations in dealing with the problem of granularity; limitations which motivate the introduction of our own solution to the problem of granularity in terms of perspective-relativity. We will only have the space to examine (Cresswell and Stechow 1982)'s approach (and a refinement of it given by (Tancredi and Sharvit 2020)), though we believe that the issues we here identify also cause problems for the *de re* approach pursued in (Égré 2014), and that the problems we identify would also cause problems for theoretically possible *de re* approaches which have not been developed by anyone in the literature and which are based on the semantic theories of *de re* noun phrase interpretation given in any of the following citations: (Percus and Sauerland 2003, Aloni 2005, Égré 2014, Holliday and Perry 2014).

2.2.1.1 (Cresswell and Stechow 1982)

Though Cresswell and Stechow (1982) do not present the semantics of any particular formal language in their paper, it is clear that Cresswell and Stechow (1982) have in mind a type-theoretic language such as *Ty2*, whose models (see appendix, section A.3) are based on a collection of domains stratified by type. The driving idea of Cresswell and Stechow (1982) is that, on occasion, sentences may have as their model-theoretic interpretation a so-called *structured proposition*: a non-empty tuple whose members are the semantic values of parts of the sentence. They concede that sentences expressing mathematical truths often denote the same model-theoretic object–namely, the function from all states of a model to 1–but argue that such sentences can sometimes also denote different structured propositions.

In this section we will refer to a fixed model $\mathfrak{M}$ and assignment g of the kind that Cresswell and Stechow (1982) have in mind in the examples we discuss. According to Cresswell and Stechow (1982)'s semantics, belief is a relation between an agent and a structured proposition. Two structured propositions are equivalent in $\mathfrak{M}$ under g if and only if they are identical tuples in $\mathfrak{M}$ under g. Importantly, (Cresswell and Stechow 1982) make the standard assumption that seemingly synonymous predicates such as *oculist* and *ophthalmologist* have the same semantic value (being functions in $D_{e\to s\to t}$), and that mathematical predicates such as *finite* and *inductive* have the same semantic value, so that the tuples created by substituting these predicates with one another will be identical.

In the simplest case, a structured proposition consists of an n-ary property of individuals. Here an *n-ary property of individuals* is understood either as an *extensional property*– a function from n individuals to truth values–or as an *intensional property*, understood as a function from n individuals to a state to a truth value. For example, *Two plus two equals four* might express the structured proposition in (2.1), which consists of the (extensional) property of being identical to two plus two as its first member and the number four as its second member:

$$\langle\ [\![equals_{e\to e\to t}(plus_{e\to e}\, two_e\, two_e)]\!]^{\mathfrak{M},g},\ [\![four_e]\!]^{\mathfrak{M},g}\ \rangle \tag{2.1}$$

In more complex cases, a structured proposition might consist of an n-ary property had by properties of individuals and a property of individuals, as in (2.2), which consists of a generalised quantifier and an argument to it:

$$\langle [\![\lambda Q_{e\to s\to t}, s_s. \forall x_e\, (man_{e\to s\to t}\, x\, s \to Q\, x\, s)]\!]^{\mathfrak{M},g}, [\![sleep_{e\to s\to t}]\!]^{\mathfrak{M},g} \rangle \tag{2.2}$$

Cresswell and Stechow (1982) assume that a single sentence can, depending on the context, be semantically represented via different tuples, which we call its *structurings* (following (Tancredi and Sharvit 2020)). For example, a given utterance of *Two plus two equals four* might have any of the following structurings as its semantic value:

(2) a. $\langle\ [\![equals_{e\to e\to t}(plus_{e\to e\to e}\, two_e\, two)\, four_e]\!]^{\mathfrak{M},g}\ \rangle$
    b. $\langle\ [\![\lambda y_e, x_e.\, equals(plus\, y\, x)\, four]\!]^{\mathfrak{M},g},\ [\![two]\!]^{\mathfrak{M},g},\ [\![two]\!]^{\mathfrak{M},g}\ \rangle$
    c. $\langle\ [\![\lambda f_{e\to e\to t}, g_{e\to e\to e}, z_e, u_e, v_e.\, g(f(z,u),v)]\!]^{\mathfrak{M},g},$
    $[\![plus]\!]^{\mathfrak{M},g},\ [\![equals]\!]^{\mathfrak{M},g},\ [\![two]\!]^{\mathfrak{M},g},\ [\![two]\!]^{\mathfrak{M},g},\ [\![four]\!]^{\mathfrak{M},g} \rangle$

An important question that arises is what the semantics of *believe* should be like, in order that it may combine with structured propositions so as to capture, *inter alia*, the semantics of *de re* belief reports. (Cresswell and Stechow 1982)'s proposal *à propos* is (following (Kaplan 1968)) to define the semantic value of *believe* in terms of so-called *suitable relations* between individuals and entities holding at a state, or between individuals and properties holding at a state. The notion of a suitable relation is tricky to spell out in full, and we have nothing illuminating to say about it in the following (though see (Lewis 1979)). Briefly put, the idea is that a relation is suitable if it puts the believer in 'cognitive contact' with an individual, or a property, and such relations are thought to be necessary in order that we make true *de re* belief attributions. For example, in (Quine 1956)'s well-known story about Ortcutt, we have two suitable relations ξ_1 and ξ_2, which, crucially unbeknownst to Ralph, put him into cognitive contact with Ortcutt: ξ_1 is the relation 'x sees y in a brown hat in the context c_1'—relative to which (3a) is true *de re* — and ξ_2 is the relation 'x sees y on the beach in a different context c_2'—relative to which (3b) is true *de re*:

(3) a. Ralph believes that Ortcutt is a spy.
b. Ralph believes that Ortcutt is not a spy.

According to (Cresswell and Stechow 1982), a sentence of the form 'A believes that *S*' is true in world w under a structuring $\langle \omega, b_1, ..., b_n \rangle$ of *S* consisting of an n-ary property ω and n arguments if and only if A *ascribes* the property ω to $b_1, .., b_n$ in w, where A *ascribes* the property ω *to* $b_1, .., b_n$ *in* w if and only if there are suitable relations $\xi_0, ..., \xi_n$ such that:

(a) $\forall y (w \in \xi_0 \,(a,\, y) \Leftrightarrow y = \omega)$

(b) For $1 \leqslant i \leqslant n$, $\forall y (w \in \xi_i \,(a,\, y) \Leftrightarrow y = b_i))$

(c) a self-ascribes ϕ in w, where ϕ is that property such that for any world w' and any individual c, $w' \in \phi(c)$ iff $\exists \omega, x_1, ..., x_n (\forall \omega', x'_1, ..., x'_n (w' \in \xi_0 \,(c,\, \omega') \Leftrightarrow \omega' = \omega$ & for $1 \leqslant i \leqslant n$, $(w' \in \xi_i \,(c,\, x'_i) \Leftrightarrow x'_i = x_i)$ & $w' \in \omega'(x'_1, ..., x'_n)))$

(Cresswell and Stechow 1982) suggest that either *self-ascription* is taken as a primitive notion, or defined in terms of a relation DOX of doxastic accessibility between world-individual pairs, such that $(w, a) DOX (w', a')$ if and only if a's beliefs in w don't rule out the possibility that she is a' in the world w'. In this latter case, a self-ascribes ϕ in w iff $\{(w', a') \mid (w, a) DOX (w', a')\} \subseteq \phi$.

Importantly, regardless of whether self-ascription is primitive or not, the relations ξ_i, for $0 \leqslant i \leqslant n$, are closed under substitution of semantic equivalents, in the sense that, if $\xi_i(a,p)$ holds and p and q are the same objects in a model (i.e, $p, q \in D_\alpha$ for some $\alpha \in TYPE$ and $p = q$), then $\xi_i(a,q)$ holds. In conjunction with their assumption that seemingly synonymous predicates such as *oculist* and *ophthalmologist* have the same semantic value (being functions in $D_{e \to s \to t}$), and that mathematical predicates such as *finite* and *inductive* have the same semantic value, this can create a number of problems. We will now examine these problems and suggest that *de re* solutions to the problem of granularity have important limitations. Then we will discuss another related problem raised by (Tancredi and Sharvit 2020), who attempt to solve the problem by refining the semantics of (Cresswell and Stechow 1982). (Tancredi and Sharvit 2020)'s approach has a number of shortcomings, which will motivate the perspective relative approach to predicates that we take in Part II of this thesis.

(Cresswell and Stechow 1982)'s semantics has problems accounting for the felicitousness of utterances which deny tautologies. For example, suppose that John does not accept the validity of classical logic for reasoning about infinite sets, and so does not believe the Goldbach conjecture–the conjecture that every even number greater than two can be written as the sum of two primes– is either true or false. Knowing this, we might, for example, truly assert (4):

(4) John doesn't believe that either every even number greater than two can be written as the sum of two primes or it is not the case that every even number greater than two can be written as the sum of two primes.

(Cresswell and Stechow 1982) might propose that (4) denotes the structured proposition (2.3), using the predicate $Goldbach : s \to t$:

$$\langle [\![\lambda s_s.(Goldbach\, s) \vee \neg(Goldbachs)]\!]^{\mathfrak{M},g} \rangle \tag{2.3}$$

$$\langle [\![\lambda p_{s\to t}, s_s.(p\, s) \vee \neg(ps)]\!]^{\mathfrak{M},g}, [\![Goldbach]\!]^{\mathfrak{M},g}\rangle \tag{2.4}$$

(2.3) and (2.4) are the same structured proposition, respectively, as (2.5) and (2.6), given the classical logical equivalence of $A \vee \neg A$ and $\neg(A \wedge \neg A)$:

$$\langle [\![\lambda s_s.\neg(Goldbach\, s \wedge \neg(Goldbachs))]\!]^{\mathfrak{M},g}\rangle \tag{2.5}$$

$$\langle [\![\lambda p_{s\to t}, \lambda s_s.\neg(p\, s \wedge \neg(ps))]\!]^{\mathfrak{M},g}, [\![Goldbach]\!]^{\mathfrak{M},g}\rangle \tag{2.6}$$

This therefore predicts that, whenever (4) is true, (5) is:

(5) John doesn't believe that it is not the case that both every even number greater than two can be written as the sum of two primes and it is not the case that every even number greater than two can be written as the sum of two primes.

(5) commits John to not believing an instance of the law of non-contradiction ($\neg(A \wedge \neg A)$); however, John may well endorse the law of non-contradiction, consistently with his intuitionistic beliefs about number theory. There does not seem to be a sense in which (5) is true in the situation described.

Of course, (Cresswell and Stechow 1982) can propose that, given the syntactic differences between (4) and (5), an utterance of (5) would usually suggest a different structured proposition from (4), such as:

$$\langle [\![\lambda f_{t\to t}, p_{s\to t}, s_s.f(p\, s \wedge f(ps))]\!]^{\mathfrak{M},g}, [\![\lambda p_t.\neg\, p]\!]^{\mathfrak{M},g}, [\![Goldbach]\!]^{\mathfrak{M},g}\rangle \tag{2.7}$$

But something is missing in the reduction of every difference in meaning to a difference in structure: the element missing is the perspective that agents take to the words they are using. In other words, structure doesn't make the only difference. This is brought through in the next examples we discuss, and is the intuition based on which we develop the perspective relative semantic theory in part II of this thesis.

Suppose that Peter is confused about the meaning of *oculist*, and opinionated about whether John is an oculist or not (i.e, he has a definite opinion about the matter). In fact, suppose that Peter thinks that John is an opthalmologist, but that John is not an oculist. In such a situation, either of the sentences (6) or (7) may be truly uttered:

(6) Peter believes that John is an ophthalmologist, but he doesn't believe that John is an oculist.

(7) Peter believes that John is an ophthalmologist and he believes that John is not an oculist.

Note that, since (Cresswell and Stechow 1982) assign the same semantic value to *oculist* and *ophthalmologist*, (6) is felicitous and possibly true if and only if (8a) and (8b) are; likewise, (7) is felicitous and possibly true, if and only if (8c) and (8d) are:

(8) a. Peter believes that John is an ophthalmologist, but he doesn't believe that John is an ophthalmologist.
   b. Peter believes that John is an oculist, but he doesn't believe that John is an oculist.
   c. Peter believes that John is an ophthalmologist, but he believes that John is not an ophthalmologist.
   d. Peter believes that John is an oculist, but he believes that John is not an oculist.

One issue here is that, (6) is more felicitous than (8a) and (8b), whereas (Cresswell and Stechow 1982) have offered no explanation for this difference in felicity. In fact, for some of our informants, (8a) and (8b) are never felicitous. The same can be said of (7) and (8c) and (8d): (7) is more felicitous than (8c) and (8d). This could be a pragmatic effect: it might be clearer to use rather than (8a) or (8b) to express the relevant reading, so that (8) is preferable on pragmatic grounds, and the same might be said of (7) and (8c) or (8d). It seems feasible in principle that a pragmatic explanation along these lines could be elaborated by (Cresswell and Stechow 1982) ((Asudeh and Giorgolo 2016) also discuss a similar pragmatic explanation of similar data involving co-referring proper names).

There is a deeper problem, however. Consider the question of whether (6) and (7) can possibly be true in the semantics of (Cresswell and Stechow 1982). The answer is: it depends on the structuring of the clauses embedded under *believe*. Take (6) first. If the first *that*-clause in (6) denotes (2.8) and the second *that*-clause in (6) denotes (2.9), then, given *oculist* and *ophthalmologist* have the same semantic value, (6) will be a contradiction in (Cresswell and Stechow 1982)'s semantics:

$$\langle [\![ophthalmologist_{e\rightarrow s\rightarrow t}]\!]^{\mathfrak{M},g}, [\![john_e]\!]^{\mathfrak{M},g} \rangle \tag{2.8}$$

$$\langle [\![oculist_{e\rightarrow s\rightarrow t}]\!]^{\mathfrak{M},g}, [\![john_e]\!]^{\mathfrak{M},g} \rangle \tag{2.9}$$

On the other hand, regarding (7), if the first *that*-clause in (7) denotes (2.10) and the second *that*-clause in (7) denotes (2.11), then (7) will not be contradictory in (Cresswell and Stechow 1982)'s semantics, simply because the the second *that*-clause states a *de re* belief about a negative property, unlike the first *that*-clause, and so an agent can consistently stand in distinct suitable relations to both properties:

$$\langle [\![ophthalmologist_{e\rightarrow s\rightarrow t}]\!]^{\mathfrak{M},g}, [\![john_e]\!]^{\mathfrak{M},g} \rangle \tag{2.10}$$

$$\langle [\![\lambda x_e, s_s.\neg(\textit{oculist}_{e\to s\to t}\, x\, s)]\!]^{\mathfrak{M},g}, [\![\textit{john}_e]\!]^{\mathfrak{M},g}\rangle \tag{2.11}$$

This asymmetry between (6)—which expresses a contradiction on the structuring proposed—and (7)—which does not express a contradiction on the structuring proposed—does not seem warranted. For neither (6) nor (7) is intuitively more infelicitous than the other, and both sentences can be conjointly true. The asymmetry between (6) and (7) is avoided and (6) is rendered consistent, if the first conjunct of (6) denotes the structured proposition (2.8), but the second conjunct denotes the structured proposition (2.12):

$$\langle [\![\lambda P_{e\to s\to t}, x_e, s_s.(P\, x\, s)]\!]^{\mathfrak{M},g}, [\![\textit{oculist}_{e\to s\to t}]\!]^{\mathfrak{M},g}, [\![\textit{john}_e]\!]^{\mathfrak{M},g}\rangle \tag{2.12}$$

It would, however, contradict the spirit of their theory for (Cresswell and Stechow 1982) to propose that, in (6), the isomorphically structured *that*-clauses *that John is an ophthalmologist* and *that John is an oculist* denote completely different structured propositions. For, after all, one of the ideas motivating the theory of structured propositions is that sentences that express mathematical truths differ in meaning solely because of their structure, despite the *that*-clauses in (6) not exhibiting any relevant structural differences. Of course, (Cresswell and Stechow 1982) could claim that *that John is an ophthalmologist* and *that John is an oculist*, though apparently structurally isomorphic on the surface, are actually structured in the more complicated way that they suggest, on a deeper analysis, so that there apparent surface structure is not indicative of their real structure. It is not clear what evidence they could adduce in favour of this claim, however; it would seem rather to be a move they could make to avoid problems with their position, as opposed to being based on some important structural difference between the two *that*-clauses.

For this reason, we conclude that (Cresswell and Stechow 1982)'s treatment of (6) is problematic, on the grounds that it gives rise to puzzling asymmetries in the treatment of sentences like (6) and (7). Such asymmetries are a bi-product of the attempt to reduce all differences in the meaning of sentences to their structure, and of the desire to hold onto *Predicate Uniformity* for predicates such as *oculist* and *ophthalmologist*. In the semantic theory we elaborate in part II of this thesis, *oculist* and *ophthalmologist* will express two distinct perspective relative properties, and we will avoid the problems which (Cresswell and Stechow 1982)'s account engenders. We will now consider some examples recently discussed by (Tancredi and Sharvit 2020), which also suggest that the notion of perspective is important in the attribution of propositional attitudes.

2.2.1.2 (Tancredi and Sharvit 2020)

(Tancredi and Sharvit 2020) consider the case of John, a monolingual English speaker who has a non-standard understanding of the natural language predicate *prime*, according to which a natural number is prime if and only if it is equal to $x^3 - 1$, for some natural number x. This person, who we shall call 'John', may assert that 26 is a prime number, knowing that it has exactly four natural number factors. In such cases, when speaking about John, we– who are not confused about the meaning of *prime*–might truthfully utter (33b), (33c) and their conjunction (33d):

(9) a. John believes that 26 is a prime number
  b. John doesn't believe that 26 has exactly two natural number factors.
  c. John believes that 26 is a prime number, but he doesn't believe that 26 has exactly two natural number factors.

(Tancredi and Sharvit 2020) label the mode of attribution under which these sentences are true *de translato*, and argue that it is distinct though compatible with *de* *dicto* and *de re* interpretation.

Consider now how the (Cresswell and Stechow 1982) might account for the felicitousness and truth of (33b), in the context described. The most obvious possibility is that we model (33b) via the structured proposition $\langle prime, 26\rangle$, in terms of the attribution of the property $prime \in D_{e\to s\to t}$, to the number $26 \in D_e$. Then John ascribes $prime$ to 26 at state w iff there is a suitable relation ξ_0 which stands uniquely between John and $prime$ at w and a suitable relation ξ_1 which stands uniquely between John and 26 at w, and John self-ascribes the property of being an individual a' in a world w' in which the unique property in $D_{e\to s\to t}$ which John stands in ξ_0 to in w' is true of the unique entity in D_e which John stands in ξ_1 to in w'. For example, ξ_1 might be the relation of 'being the number which is represented for John by the numeral "26"' and ξ_0 might be the relation of 'being the property which is represented for John by the natural language predicate 'is prime'.' Here the property represented for John by 'is prime' is not $prime \in D_{e\to s\to t}$, but is rather the property of being a natural number y such that there is a natural number x for which $y = x^3 - 1$. For this reason, ξ_0 in the situation described cannot be the relation of 'being the property which is represented for John by the natural language predicate 'is prime''. Suppose instead that suitable relations ξ_0, ξ_1 can be found, which relate John respectively to $prime \in D_{e\to s\to t}$ and $26 \in D_e$, and that John ascribes $prime$ to 26. Since $prime \in D_{e\to s\to t}$ and $exactly.two.factors \in D_{e\to s\to t}$ have the same semantic value according to (Cresswell and Stechow 1982), it will then follow that John ascribes $exactly.two.factors$ to 26. But, in the situation described, John believes that 26 does not have exactly two natural number factors. Thus, we should be able to felicitously and truly utter (10):

(10) John believes that 26 does not have exactly two natural number factors and he believes that 26 has exactly two natural number factors.

Clearly, it is not felicitous to utter (10) in the situation described. To account for the truth of (33b), we need a way of cleaving apart the semantic value of the natural languages predicates *is prime* and *has exactly two natural number factors*.

(Tancredi and Sharvit 2020) offer an analysis of sentences (33b)-(33d) which involves the perspective that John takes towards the predicates of his language. This analysis has some shortcomings and which will provide the opportunity to show why a different theory, along the lines we give in part II of this thesis, is superior.

(Tancredi and Sharvit 2020) argue that the true utterance of sentence (33b) involves a modification of the I-language (in the sense of (Chomsky 1986)) of the speaker, so that she interprets the natural language predicate *prime* as John does. They argue that this is a general phenomenon which can occur in utterances of propositional attitude constructions. In the case of (33b), the speaker modifies her I-language, reinterpreting the natural language predicate *prime* so that it denotes the property of being a natural number which is equal to the cube of a some natural number minus 1. Their initial proposal is that linguistic expressions are interpreted with respect to a language, and that a linguistic expression of the form x_L indicates that the expression x is syntactically indexed with the language L. For all expression x and languages L, L', they then require that 2.13 holds:

$$[\![x_{L'}]\!]^L = [\![x]\!]^{L'} \tag{2.13}$$

It is clear from (Tancredi and Sharvit 2020)'s article that they understand $[\![\cdot]\!]^L$ to be a function whose domain consist of certain formal representations of English or Japanese strings which can be subscripted with variables over languages. We will call the entities in the domain of $[\![\cdot]\!]^L$ *formalised strings*. But they do not specify either:

(i) Whether the function $[\![\cdot]\!]^L$ applied to something in its domain returns a model-theoretic object, or a lambda term.

(ii) How to evaluate the right hand side of the equality in 2.13

Regarding (i), we will suppose $[\![\cdot]\!]^L$ applies to formalised strings and returns a lambda term. This fits with how they use the notation. For example, they later write the following formula:

$$justified_a(believe(john, \langle [\![prime]\!]^{T_j(L),j}, 26\rangle)) = \\ justified_a(believe(john, \langle \lambda x.\exists n(x = n^3 - 1), 26\rangle)) \tag{2.14}$$

where it seems that the equality $[\![prime]\!]^{T_j(L),j} = \lambda x.\exists n(x = n^3 - 1)$ is used.

Regarding (ii), we will suppose that the right hand side of the equality in 2.13 is evaluated as in 2.15, where the language L is a function from lambda terms to lambda terms (as we are assuming $[\![x]\!]^L$):

$$\text{For any language } L : [\![x]\!]^L = L(x) \tag{2.15}$$

2.15 is consistent with the fact that (Tancredi and Sharvit 2020: p.322) use variables over languages, such as L, as functions which apply to the formal representatives of linguistic expressions and return lambda expressions.

(Tancredi and Sharvit 2020) first propose that (33b) be given the following analysis, where the notion of ascription–they claim– is the same as that of (Cresswell and Stechow 1982):

(11) $[\![$*John believes that 26 is prime*$_{L_j}]\!]^{@,L} = 1$ under the structuring $\langle [\![prime]\!]^{L_j}, [\![26]\!]^L\rangle$ iff John ascribes $[\![prime]\!]^{L_j}$ to $[\![26]\!]^L$

It is clear then, that here they presuppose that the notion of ascription applies to lambda terms, in the sense that, in an attitude report the subject of the attitude report ascribes lambda terms to other lambda terms. In fact, they often seem to talk as though it is lambda terms which are ascribed to other lambda terms. For example, with reference to the n-tuple $\langle \lambda x, w.x$ is prime in $w, 26\rangle$, whose first component is abbreviated $prime'$, they write that "Under this analysis [*the analysis of* (Cresswell and Stechow 1982)], the sentence is predicted to be true iff John ascribes the property $prime'$ to the number 26." Later on they write that the sentence (12) (discussed below) "comes out true as desired because John ascribes the property $\lambda n.\exists x(n = x^3 - 1)$ to the number 26".

Based on their understanding of ascription (which we will soon argue is confused), they reason as follows: since $[\![prime]\!]^{L_j}$ and $[\![$*has exactly two factors*$]\!]^{L_j}$ may return distinct lambda terms, it is possible for John to ascribe $[\![prime]\!]^{L_j}$ to $[\![26]\!]^{L}$ without ascribing
$[\![$*has exactly two factors*$]\!]^{L_j}$ to $[\![26]\!]^{L}$. A problem then arises, if the function $[\![\cdot]\!]^{L}$ is a function which is only defined for those expressions which have a meaning in the language L. Plausibly, L_j, John's language, is not defined for *sosuu*, the Japanese word for *prime*, since John was by our assumption a monolingual English speaker. But, if this is the case, we will predict, falsely, that the Japanese translation of (33b) in (12) is false or perhaps undefined, since $[\![$*sosuu*$\cdot]\!]^{L}$ is undefined, and so $[\![$*sosuu*$\cdot]\!]^{L}$ cannot be ascribed to $[\![26]\!]^{L}$:

(12) John-wa 26-ga sosuu de-aru to omotteiru

For these reasons, (Tancredi and Sharvit 2020) propose that we make use of a transformation function T_j, which applies to a language L to generate a language $T_j(L)$. They define this notion as follows:

For all expressions z where there is no evidence to indicate otherwise, (2.16)

$L(z) = T_j(L)(z)$.

$L(prime) = \lambda n.has.exactly.two.factors\, n$.

$T_j(L)(prime) = \lambda n.\exists x(n = x^3 - 1)$.

They then require that when a certain word z' in a language L'– for example, *prime*– is the translation of a another word z in a language L (for example, *sosuu*), then the language L applied to z must return the same lambda term as L' applied to z', and that for any transformation T, $T(L')$ assigns to z' the same value that $T(L)$ assigns to z.

Armed with transformation functions, they replace $[\![\cdot]\!]^L$ with $[\![\cdot]\!]^{L,p}$, which parameterises the formal representatives of linguistic expressions to both a language and a perspective parameter,[2] and formal representatives of linguistic expressions can be of the form x_T, indicating that the expression x is syntactically indexed with the transformation function T. 2.13 is replaced with 2.17:

$$[\![x_T]\!]^{L,p} = [\![x]\!]^{T_p(L),p} \tag{2.17}$$

$T_p(L)$ is supposed to indicate the language which is just like L except that is assigns to some expressions a value which reflects p's presumed I-language. When evaluating full sentences, they stipulate that p is identified with the agent of the context, but importantly for their purposes, when evaluating the argument of an attitude predicate, p is stipulated to denote the agent who is the grammatical subject

[2]As far as we understand, (Tancredi and Sharvit 2020) parameterise $[\![\cdot]\!]$ to both a language and a perspective parameter, and not just to a language, on the basis that this is independently needed to account for perspective relativity.

of the attitude verb.

With this set up in place, (33b) is then analysed as follows:

(13) $[\![$*John-wa 26-ga sosuu*$_T$ *de-aru to omotteiru*$]\!]^{@,L,a} = 1$ under the structuring $\langle\ [\![$*sosuu*$_T$ *de aru*$]\!]^{L,j},\ [\![$ *26* $]\!]^{L,j}\ \rangle$ iff John ascribes $[\![$ *sosuu* $]\!]^{T_j(L),j}$ to $[\![$ *26* $]\!]^{L,j}$

The proposed analysis, however, rests on a misunderstanding of (Cresswell and Stechow 1982)'s notion of ascription, and once this misunderstanding has been clarified, a simpler extension of (Cresswell and Stechow 1982)'s analysis which can account for the same data is available, which avoids having to employ transformation function and having to parameterise expressions to a language. (Cresswell and Stechow 1982) clearly use the notion of ascription to refer to a relation standing between the ascriber, an n-place property (for $n \geqslant 0$) in some domain of the model and various objects in D_e to which the property is ascribed.[3] That is, the notion of ascription is semantic: an ascriber ascribes the model-theoretic interpretation of a lambda term (a property), to the model theoretic interpretation of one or more arguments of this lambda term. This is not how (Tancredi and Sharvit 2020) understand ascription, as we have emphasised above. But once we understand ascription in the way that (Cresswell and Stechow 1982) understand it, the problem regarding Japanese translations which (Tancredi and Sharvit 2020) raise, can be be avoided via a simpler analysis. In particular, there is no need for language parameters or

[3](Cresswell and Stechow 1982):

> "In possible-worlds semantics (where a proposition is a set of possible worlds) a (one-place) property is a function ω to which associates with any individual a in its domain a set $\omega(a)$ of possible worlds. If to is the property of being a spy then to(a) (the proposition that a is a spy) is the set of worlds in which a is a spy."

transformation functions.

First, note that, in the method of indirect interpretation pursued in this thesis, both the English predicate *prime* and its Japanese translation *sosuu*, can be rendered as the same expression of partial *Ty2*, which is then interpreted in a model. If we use direct interpretation, we can map *prime* and its Japanese translation *sosuu* to the same model-theoretic object. What is needed in the cases at hand, is for the English predicate *prime* and its Japanese translation *sosuu* to be assigned the same model-theoretic interpretation in the case of (33b) and its Japanese translation (12). From John's perspective, 26 is prime, whereas from the utterer of (33b) and its Japanese translation (12), 26 is not prime. The simplest way to account for this observation is to make natural language predicates functions which take a perspective index as argument. For example, suppose that *prime* and its Japanese translation *sosuu* are rendered as a constant $P \in Con_{e \to p \to s \to t}$, where p is the type of perspective indices. Then we could ensure that $[\![P\, 26\, i\, @]\!]^{\mathfrak{M},h} = 1$, where $[\![\cdot]\!]^{\mathfrak{M},h}$ is the interpretation function of a partial *Ty2* model, $26 \in Con_e$, $i \in Con_p$ and is the perspective index associated with John and $@ \in D_s$ and denotes the actual world. If $[\![P\, 26\, i\, w']\!]^{\mathfrak{M},h} = 1$ holds in all John's doxastic alternatives w', then we would capture John's belief that 26 is a prime number. Similarly, we could map the English language predicate *has exactly two factors* to a lambda constant $E \in Con_{e \to p \to s \to t}$ and ensure that $[\![E\, 26\, i]\!]^{\mathfrak{M},g} \neq [\![P\, 26\, i]\!]^{\mathfrak{M},g}$, because John doesn't treat *prime* and *has exactly two factors* as synonymous. As such, it could be the case that, in all of John's doxastic alternatives w', $[\![E\, 26\, i\, w']\!]^{\mathfrak{M},h} = 0$ and $[\![P\, 26\, i\, w']\!]^{\mathfrak{M},h} = 1$.

An analysis on this kind does without the transformation function and language parameters essential to (Tancredi and Sharvit 2020)'s account, nor is it based on the notion of attribution taken from (Cresswell and Stechow 1982). In fact, if, as (Tancredi and Sharvit 2020) argue, *de translato* is independent of the the *de dicto/re* distinction, then we do not see the need for analysing it using the framework of

(Cresswell and Stechow 1982). Nevertheless, it would be simple to amend the analysis of attribution in (Cresswell and Stechow 1982), by relativising it to a perspective parameter, and in so doing we can account for *de translato* interpretation. Let $\omega \in (D_e)^n \rightarrow D_{s\rightarrow p\rightarrow t}$ be a function which takes n individuals in D_e, a state $s \in D_s$, a perspective index $i \in D_p$ and returns a truth value, and let $b_1, ..., b_n \in D_e$. The suitable relations ξ_0 in (Cresswell and Stechow 1982)'s definition of attribution will continue to relate agents to either individuals or to properties in $D_{e\rightarrow s\rightarrow t}$ as per their definition.

The revised analysis of attribution becomes, a *ascribes* $\omega \in (D_e)^n \rightarrow D_{s\rightarrow p\rightarrow t}$ *to* $b_1, .., b_n$ (for some $n \geqslant 0$) in w *relative to perspective index* $j \in D_p$ if and only if there are suitable relations $\xi_0, ..., \xi_n$ such that:

(a) $\forall y \in (D_e)^n \rightarrow D_{s\rightarrow p\rightarrow t}\Big(w \in \xi_0\,(a,\, y) \Leftrightarrow \forall z_1, ..., z_n \in D_e\,\big(\,y\,z_1, ..., z_n\,w\,j\, = \omega\,x_1, ...x_n\,w\,j\,\big)\Big)$

(b) For $1 \leqslant i \leqslant n$, $\forall y(w \in \xi_i\,(a,\, y) \Leftrightarrow y = b_i)$

(c) a self-ascribes ϕ in w, where ϕ is that property such that for any world w' and any individual c, $w' \in \phi(c)$ iff $\exists \omega, x_1, ..., x_n(\forall y' \in (D_e)^n \rightarrow D_s \rightarrow D_p \rightarrow D_t, x'_1, ..., x'_n \in D_e(w' \in \xi_0\,(c,\, y') \Leftrightarrow \forall z_1, ..., z_n \in D_e$ $(y'\,z_1, ..., z_n\,w'\,j\, = \omega\,x_1, ...x_n\,w'\,j$ & for $1 \leqslant i \leqslant n$, $(w' \in \xi_i\,(c,\, x'_i) \Leftrightarrow x'_i = x_i)$ & $w' \in y'x'_1, ..., x'_n))$

The revised definition of ascription will produce different results for $\langle [\![prime_{e\rightarrow p\rightarrow s\rightarrow t}]\!]^{\mathfrak{M},g},\ [\![26_e]\!]^{\mathfrak{M},g}\rangle$ and $\langle [\![exactly.two.factors_{e\rightarrow p\rightarrow s\rightarrow t}]\!]^{\mathfrak{M},g},\ [\![26_e]\!]^{\mathfrak{M},g}\rangle$, at least relative to the perspective index j (John's perspective index), given the set of individuals satisfying $[\![prime_{e\rightarrow p\rightarrow s\rightarrow t}]\!]^{\mathfrak{M},g}$ and $[\![exactly.two.factors_{e\rightarrow p\rightarrow s\rightarrow t}]\!]^{\mathfrak{M},g}$ are different with respect to the perspective index j. And since

$[\![prime_{e\to p\to s\to t}]\!]^{\mathfrak{M},g}$ and $[\![exactly.four.factors_{e\to p\to s\to t}]\!]^{\mathfrak{M},g}$ denote the same set of individuals with respect to j, $\langle[\![prime_{e\to p\to s\to t}]\!]^{\mathfrak{M},g},\ [\![26_e]\!]^{\mathfrak{M},g}\rangle$ and $\langle[\![exactly.two.factors_{e\to p\to s\to t}]\!]^{\mathfrak{M},g},\ [\![26_e]\!]^{\mathfrak{M},g}\rangle$ will also denote different structured propositions.

This motivates the account of predicates that we will give in the second half of this thesis, according to which a predicate such as the English predicate *prime*, is not a constant in $Con_{e\to s\to t}$, but a constant $Con_{e\to p\to s\to t}$, where p is the type of perspective indices. With this typing, we can say that, from John's perspective 26 is prime, whereas from the perspective of the utterer of (33b), 26 is not a prime number, if the utterer knows that to be prime is to have exactly two natural number factors. This accounts for the fact that (33b) is arguably both true and false in the situation (Tancredi and Sharvit 2020) describe, depending on whether we are considering things from John's perspective or from some other perspective. Moreover, the fact that (14) is infelicitous, will be accounted for in our framework by a special perspective– the enlightened perspective:

(14) ?? John knows that 26 is prime.

The idea behind the enlightened perspective is that it simply preserves the ordinary intension of any expression, as if the expression were not itself perspective sensitive. In order for (14) to be true, on our account, the complement *that*-clause must be true from the enlightened perspective, and since this impossible given the fact that the ordinary intension of '26 is prime' is false in all states (since the number 26 is not prime), (14) cannot be true. Our theory thus accounts for the data which (Tancredi and Sharvit 2020) discuss in a simpler way. Moreover, it can be used to give a solution to the problems of granularity and logical omniscience.

2.2.2 Stalnaker and Diagonalisation

(Stalnaker 1978) has exploited a technique known as diagonalisation in order to give a solution to what we are calling the problem of granularity, as well as a solution to Frege's puzzle (Salmon 1986). He has also advocated a 'fragments of belief' approach to the problem of logical omniscience (Stalnaker 1984).

According to Stalnaker, a sentence, when uttered in a given context, expresses a function from states to truth values, which he calls a *proposition*. Whether an utterance is true depends on two sorts of facts: the facts which fix what is said by an utterance, and the 'worldly' facts which determine whether what is said by the utterance is true. Stalnaker argues that sentences can be associated with *propositional concepts* to capture this double dependence of truth on the facts, where a propositional concept is a function from states to propositions. Formally, $[\![A : s \to s \to t]\!]^{\mathfrak{M},g}$ is the propositional concept expressed by the term A in a partial *Ty2* model under assignment g. The first state argument s of a propositional concept fixes what is said by an utterance of a given sentence in a given state, and the second state argument s' fixes whether what is said by that utterance in s is true at s'.

According to Stalnaker, under the right pragmatic conditions, mathematical sentences are associated with different *diagonal propositions*, where the diagonal proposition associated with a term $A \in Con_{s\to s\to t}$ in a partial *Ty2* model is as defined in (2.18), and the diagonal operator $\dagger : (s \to s \to t) \to s \to t$ is as defined in (2.19):

$$[\![\ \lambda s_s.A\, s\, s\ ]\!]^{\mathfrak{M},g} \tag{2.18}$$

$$\dagger_{(s\to s\to t)\to s\to t} =_{df} \lambda A_{s\to s\to t},\ s_s.\ A\, s\, s \tag{2.19}$$

The diagonal proposition expressed by an utterance of a natural language sentence is an proposition (in Stalnaker's sense) whose truth value at the situation s is the truth value the utterance of S would have if it were uttered in the context s, where s may be a context in which the sentence has a completely different meaning from that which it ordinarily has. The diagonal proposition associated with different sentences which express mathematical truths will be false in those states at which the sentence expresses something false.

According to Stalnaker, when we doubt whether a sentence expresses a mathematical truth, we are unsure of whether the state we are in is one in which the diagonal proposition expressed by the given sentence is true in that state. We are thus unsure of whether we are in a state s such that the sentence *given its meaning in* s expresses a truth in s. For Stalnaker, therefore, we can only believe that a sentence which actually expresses a mathematical truth is false if we do not know what it means. Similarly, if Mary believes that John is an oculist and not an ophthalmologist she does not know that we are in a state in which the sentence 'John is an oculist and not an ophthalmologist' expresses a truth. An account of the meanings of sentences expressing mathematical truths along these lines therefore makes disputes about whether a given mathematical sentence is true, metalinguistic disputes: they concern whether we inhabit a state in which the given sentence expresses a sentence which is true.

There is, however, an internal tension in Stalnaker's account. Consider a sentence such as (15):

(15) The prime numbers are infinite or the prime numbers are not infinite.

Stalnaker assumes that for a sentence to be felicitous, it must not be trivially true, in the sense of being true in all states, or trivially false, in the sense of being false in all states. This is embedded in a particular theory of how to model contexts which for the purpose of our discussion we can abstract from. Since (15) is true in all states—being an instance of the law of the excluded middle—it should not be felicitous. Given utterances of (15) can in fact be felicitous, Stalnaker has three clear options: he can either try to argue that his 'fragments of belief' approach can account for the felicitous utterance (we will explain this approach below), or that diagonalisation can, or that (15) is felicitous because it gives rise to certain non-trivial implicatures. All these alternatives face a number of problems.

Regarding the third option, Stalnaker could propose that the reason that (15) is felicitous is because it tends to communicate a stronger proposition than that expressed by its literal meaning. For example, often an utterance of a disjunction gives rise to an exclusivity implicature, to the effect that not both of the disjuncts are true. When someone utters "S or S'" this gives rise either to a primary implicature that they do not believe that S and S' are both true, or a stronger secondary implicature that they believe that it is not the case that S and S' are both true, so that we know that in all their doxastic alternatives, either S is true or S' is true, but not both. Utterances of disjunctions can also give rise to ignorance implicatures, to the effect that the speaker does not know which disjunct is true. On the suggestion under examination it is only in virtue of such implicatures that an utterance of (15) can be non-redundant.

Regarding exclusivity implicatures, this proposal doesn't fare well. The primary implicature that there are doxastic alternatives of the agent at which it is false that A and $\neg A$, is trivial, for A and $\neg A$ cannot both be true at any state. This also makes the secondary implicature, that A and $\neg A$ cannot both be true at any doxastic alternative, trivially true. The upshot is that an exclusivity implicature will fail to provide any information to the hearer that she doesn't already have. For similar reasons, the ignorance implicature associated with an utterance of (15) will be trivial. We therefore conclude that an utterance of (15) cannot be felicitous because it gives rise to certain non-trivial implicatures.

Consider next the possibility that the fragments of belief approach will be able to capture the fact that an utterance of (15) can be felicitous. According to the fragments of belief approach, which Stalnaker does not formally develop, agents' beliefs are fragmented across multiple states of mind. Because of this, agents do not always pool the information in their fragments of mind together and deduce the consequences of what they believe. Consequently, someone can believe S relative to one fragment and they can believe S' relative to another fragment without believing S and S' relative to any fragment.

The fragments of belief approach has been modelled formally by (Fagin and Halpern 1987). They employ a classical modal language (with n belief modalities for $n \geqslant 1$ agents), and define a model for n agents as a structure

$$\mathfrak{M} = (W, V, L_1, ..., L_n) \tag{2.20}$$

consisting of a set W of states, a valuation function V sending propositional letters and states to 1 or 0 and, for each $i \leqslant n$, L_i is a function $W \to 2^{2^W}$ associated with the agent i, sending states to non-empty sets of sets of states. When $L_i\, w = \{X_1, ..., X_k\}$, each member of $L_i\, w$ is a *state of mind* or *fragment* indexed by some $j \leqslant k$, and each X_j is the set of states i considers possible in the state of mind j. The satisfaction of a formula at a state in a model $\mathfrak{M}$ is standard for the logical connectives, but a formula of the form $\Box_i\, \varphi$ is true at a state w in a model $\mathfrak{M}$ iff φ is true throughout one of i's states of mind at w:

$$\mathfrak{M}, w \models \Box_i \varphi \text{ iff for some } X \in L_i\, w \text{ and for all } w' \in X, \mathfrak{M}, w' \models \varphi$$

Since the logical connectives behave classically at every state of a model and each state of mind is built on a classical set of states, $\varphi \vee \neg\phi$ will be true in all states of any model. Consequently, an agent will automatically be treated as believing any instance of the law of the excluded middle, and Stalnaker cannot account for the felicity of (15). In addition, whenever $\varphi \rightarrow \psi$ is valid on this semantics and i believes φ, it follows that i believes ψ, and so this semantics cannot avoid the problem of logical omniscience. As it stands, the fragment of belief approach is therefore inadequate to deal with the problem of granularity and the problem of logical omniscience.

Stalnaker could alternatively propose that diagonalisation will associate a non-trivially true proposition with (15). But diagonalisation won't make a difference, unless *or* and *not* as they occur in (15), can be given a different interpretation at some states; otherwise, (15) will continue to express an instance of the excluded middle. Since neither diagonalisation nor the fragments of belief are sufficient to capture the fact that (15) can be felicitously uttered, if Stalnaker is to capture this, he cannot render *or* and *not* as the classical logical constants $\vee$ and $\neg$, respectively. This leaves two choices: either Stalnaker is forced to accept that the words *or* and *not* denote non-logical constants, or he abandons the assumption that logical constants have the same interpretation at all states, a possibility he has elsewhere (Stalnaker 1996) rejected. Besides his having already argued against it, this latter option would be problematic for a number of reasons. If Stalnaker were to allow the interpretation of the logical constants to vary, this would seem to contradict the spirit of his argument that proper names like *Hesperus* and *Phosphorus* have the same semantic interpretation. This argument was based on the idea that proper names are so-called *rigid designators* (Kripke 1980)— they denote the same object in every state in which that object exists— and this is based (in part) on the intuition that it is impossible to imagine a coherent situation in which Hesperus and Phosphorus are non-identical. But the intuition that it is impossible for the logical constants to change their meaning depending on the state is surely just as strong, or if not stronger, than the intuition that is impossible to imagine a coherent situation in which Hesperus and Phosphorus are non-identical. For example, it would surely be stranger to admit states at which "$A \wedge B$" can be true at state s even though both A and B are false than to admit states at which the proper names *Hesperus* and *Phosphorus* have a different interpretation.

For this reason, if Stalnaker allowed the interpretation of the logical constants to vary, it would be be a short step to allowing the interpretation of predicates such as *oculist* and *ophthalmologist* to be distinct at certain states and the interpretation of proper names such as *Hesperus* and *Phosphorus* to be distinct at certain states. And if, as the problem of logical omniscience seems to suggest, that for almost any syntactically distinct sentences *S* and *S'*, it is possible that there is an agent who believes one and not the other due to not putting 'two and two' together, then Stalnaker will have to allow the behaviour of logical constants to vary arbitrarily at certain non-standard states to capture this. But if Stalnaker did allow for such non-standard states, this would be tantamount to letting in so-called impossible worlds (Berto 2010) (which he rejects in (Stalnaker 1996)) by the backdoor, or at least to allowing worlds which were not constrained by the classical definition of truth at a world.

In the light of Stalnaker's other commitments, it is more tenable for Stalnaker to accept that the words *or* and *not* denote non-logical constants, distinct from $\vee$ and $\neg$, whose interpretation differs at certain states from the interpretation of the logical constants. But in this case, it is not clear what the semantic interpretation of such non-logical constants would be. Moreover, allowing the words *or* and *not* to denote non-logical constants would render Stalnaker's 'fragments of belief' approach to logical omniscience superfluous. For example, if *and* is a non-logical constant, we can allow for states at which an agent believes S and at which they believe S', without believing S and S', without employing the 'fragments of belief' strategy, and the same would hold, *mutis mutandis* for any of the other kinds of phenomena which Stalnaker hoped to capture using the 'fragments of belief' strategy.

We conclude that there are serious empirical difficulties with Stalnaker's approach to the problem of granularity. But there are also a number of conceptual problems at the heart of his account. We said above that sentences can be associated with diagonal propositions under the right pragmatic conditions, rather than their expressing diagonal propositions as part of their literal meaning. Stalnaker is forced into this position, since he endorses *Mathematical Uniformity* (repeated below), whilst agreeing with the intuition that utterances of sentences expressing mathematical truths can sometimes have different meanings.

- *Mathematical uniformity*: There are natural language sentences which express mathematical truths and all natural language sentences which express mathematical truths are rendered as formulas which have as their interpretation the constant function from any state to 1 (i.e, the formulas which render sentences which express mathematical truths have as their interpretation the characteristic function of the set of all states).

His solution is to weaken *Mathematical uniformity*, to something like the following principle:

- *Weakened mathematical uniformity*: There are natural language sentences which express mathematical truths and all natural language sentences which express mathematical truths are rendered as formulas which have as their interpretation the constant function from any state to 1. However, sometimes such sentences are used to convey something other than their literal meaning, when certain pragmatic conditions obtain, and in this case they denote functions from some proper subset of the set of states to 1.

It appears that Stalnaker thinks diagonalisation is forced given the assumption that mathematical truths are metaphysically necessary and given the assumption that it is metaphysically impossible for someone to be an oculist but not an ophthalmologist, or to be an ophthalmologist but not an oculist. But the move here from facts about the purported metaphysical constitution of reality to a proposed semantic treatment of sentences expressing mathematical truths or containing predicates such as *oculist* and *ophthalmologist* is open to challenge.

First of all, Stalnaker's argument is based on a distinction between mathematical and empirical sentences which is not argued for on linguistic grounds, regardless of whether the distinction is respectable metaphysically speaking. According to Stalnaker's proposal, disbelief in mathematical logical truths is semantic ignorance, whereas disbelief in ordinary empirical matters is not. But the distinction between sentences of an empirical nature on the one hand, and sentences of a mathematical nature on the other seems to be an artifact of the assumption of *Mathematical Uniformity* and is not clearly motivated from a purely linguistic standpoint.

Secondly, if the metaphysical facts are supposed to determine our semantic treatment of mathematical sentences, what we require is a reason to think that the way that reality is constituted, metaphysically speaking, ought in the case at issue to be hardwired into a semantic account of the functioning of the natural language. This question is especially important to answer given the significant differences between the way that reality is, and our conceptions of reality in language; indeed, philosophers are often anxious to avoid making metaphysical conclusions based on the way we talk about reality (see (Button 2020) for a recent article evincing such anxiety), thereby recognising the gap between reality itself and our conceptions of it.

In fact, it has been argued by a number of natural language semanticists since the 1980s (Bach 1986), (Pelletier 2011), (Moltmann 2019) that we ought to be very careful not to import assumptions about the metaphysics of reality into our semantic theories, lest we distort the semantic phenomena we are trying to capture. Consider the mass/count distinction. According to (Pelletier 2011) different languages make different syntactically and semantically motivated distinctions between 'things' and 'stuff' (see also (Chierchia 2021) for further considerations). But this doesn't entail any such semantic distinction is mirrored in physical reality; nor do the physical facts about the universe force a particular conception of the mass/count distinction. For another kind of example, consider expressions which seem to express properties, such as *is wise*. *is wise* can function as either a predicate (16a), or a subject of a sentence (16b) (see (Chierchia 1984) for discussion):

(16) a. John is wise.
 b. Being wise is wise.

But this grammatical fact about the distribution of certain expressions in a certain language does not entail that properties ought to be treated a certain way in a metaphysical theory of properties, unless we have reason to believe that language tracks the metaphysical facts so that the fact that *is wise* functions in this way reveals something fundamental about properties that metaphysics should take heed of. Finally, consider the fact that, in a standard state-based semantic theory, disjunctive predicates such as *is red or round* are mapped to functions from individuals to states to truth values, and such functions are often called properties. The notion of property as a function from individuals to states to truth values may be good enough for the purposes of modelling some natural language phenomena. But whether there can be genuine disjunctive properties–such as the property of being red or round–is a matter of controversy in metaphysics (see (Audi 2013) and references therein), because metaphysicians have different explanatory goals and aims from semanticists.

The upshot of this is that, if Stalnaker's adoption of diagonalisation is motivated in part by his desire to retain *Mathematical Uniformity* or a weakened form of it, the motivation is based on the problematic assumption that the question of the nature of mathematical truth, metaphysically speaking, must necessarily impinge on the way that sentences of natural language expressing mathematical truths are modelled. The diagonalisation strategy also begs the question with regard to predicates such as *oculist* and *ophthalmologist*, since the fact that an individual is an oculist if and only if they are an ophthalmologist does not mean that some individuals do not draw a semantically relevant distinction between the predicates, albeit based on their incomplete knowledge. In the perspective relative semantic theory we present in part II of this thesis, we will be able to capture the fact that, for most people, the predicates *oculist* and *ophthalmologist* have the same meaning, whilst allowing these predicates to vary in their semantic value relative to certain perspective indices.

2.3 Some solutions to the problem of logical omniscience adopted in the literature

The upshot of our discussion in 1.2 seems to be that, given our inability to put 'two and two' together on occasions, it is possible that, for almost any two orthographically distinct sentences *S, S'*, there is some agent A and some potential context (however far-fetched), such that 'A believes *S*' and 'A believes *S'*' can differ in truth value, as (Berto and Jago 2019) describe:

> "Take any putative closure principle P, from $A_1, A_2, ...$ to (distinct) C. If all worlds are closed under P, and we analyse belief in terms of accessible worlds, then any agent who believes that $A_1, A_2, ...$ will thereby be modelled

as believing that C, too. But it is at least possible for some agent to believe $A_1, A_2, ...$ but not believe that C. So not all worlds are closed under P. The argument is quite general. If some possible agent can believe the premises of some logical principle but not the conclusion, then there must be an epistemically accessible world which represents those premises but not the conclusion."

It would be easy to simply block all forms of logical omniscience described in 1.3, by having a set of non-normal states at which the truth of formulas was simply stipulated without regard to their syntactic structure, and then allowing such states to be doxastically accessible. This is the route taken by (Rantala 1982), who blocks the logical rule of the replacement of logical equivalents (see (Muskens 1991) for analysis of this option).

But according to the approaches to the problem of logical omniscience we will now examine, an adequate solution to the problem of logical omniscience must do more than simply blocking the closure principles on knowledge and belief. It must also capture two important regularities about human agents. The first regularity is that, in many cases, humans can and do regularly make trivial inferences, so that human reasoning is not entirely undisciplined and anarchic. The second regularity is that, more often than not, human agents don't usually believe outright, blatant contradictions of the form *S and not S*. In fact, often, agents reason about subtly impossible situations—situations which, as ((Hintikka 1975:ṗ.476–8)) remarked, 'look possible but which contain hidden contradictions'—and we reject suppositions that lead fairly to blatant impossibilities (as (Lewis 2004) points out). For example, we might reason about whether a certain non-obviously true mathematical hypothesis is true, but the assumption that it is true might in fact be inconsistent. The distinction between blatant inconsistencies and more subtle ones is relevant to the practice of belief attribution itself. For perhaps one of the reasons why we do not usually attribute obvious logical oversights to people—at least if we can find an alternative rationalising explanation of their behaviour—is that *ceteris paribus* we find it more plausible that someone believe something which is inconsistent, though subtly so, than that they believe something which is obviously inconsistent.

2.3.1 Dual-process plausibility models (Solaki et al. 2019)

In this section we will present (Solaki et al. 2019)'s solution to the problem of logical omniscience. We have decided to give a fairly detailed presentation of their account, since our resolution of the problem of logical omniscience in 7.1 is based in part on ideas form their account (in particular on their definition of R_k-accessibility).

(Solaki et al. 2019) employ dynamic epistemic logic (Baltag and Renne 2016) in order to solve the problem of logical omniscience. In dynamic epistemic logic, an action such as inferring a conclusion is represented via a model transformation which takes an initial model and adjusts it to represent the result of a given action. (Solaki et al. 2019) employ so-called *plausibility models*. In the literature on dynamic epistemic logic, a plausibility model is a model in which there is a relation $\geqslant$ between states such that $u \geqslant w$ reads "w is considered no more plausible than u", or "w is either more plausible than u or equally plausible". $\geqslant$ is a reflexive, transitive, locally connected, and converse well-founded, relation on the set of states. (Solaki et al. 2019) employ so-called *pointed plausibility models* (models will be defined below), which are pairs $(\mathfrak{M}, w)$, consisting of $\mathfrak{M}$, a so-called *dual-process plausibility model* and w a designated state in it.

We will now present the logic presented by (Solaki et al. 2019), starting with the language they use:[4]

Definition 1 (Language) Given a set Φ of propositional atoms and a set of inference rules R available to the agent, the language $\mathcal{L}$ is inductively defined from:

$$\phi \quad ::= \quad A \quad \neg\phi \quad \phi \wedge \phi \quad \Box\phi \quad B\phi \quad \langle R_k \rangle$$

where

- $A \in \Phi$
- $\Box A$ expresses that "the agent defeasibly knows that A"
- $B\,A$ expresses that "the agent believes that A"
- $\langle R_k \rangle A$, for each rule $R_k \in R$, expresses that "after some application of inference rule R_k, A is true".

The concept of a *dual-process plausibility model* is defined as follows:

Definition 2 (Dual-process plausibility model) Given a set of inference rules R, the set of inference rules available to the agent, and a finite set Res of resources (for example, memory, time etc.), and where $r := |Res|$, i.e., the number of resources, a dual-process plausibility model is a tuple $\mathfrak{M} = (W^P, W^I, ord, V, C, cp)$ where:

- W^P, W^I are countable non-empty sets of possible and impossible states respectively, where satisfaction at impossible states differs from satisfaction at possible states, as we shall soon see.
- $ord : W \to \Omega$ is a function from $W := (W^P \cup W^I)$ to the class of ordinals Ω, which assigns an ordinal to each state.

[4] For simplicity, we have chosen not to include their operator $[\Psi \Uparrow]$ in the language, so that our definition of the language is slightly different.

- $V : W \to \mathscr{P}(\mathcal{L})$ is a function assigning assigning a set of sentences in $\mathcal{L}$ to each world in W, where the propositional atoms are taken from a fixed set Φ.

- $C : R \to \mathbb{N}^r$ is a function assigning a cognitive cost for each resource to every inference rule $R_k \in R$.

- cp denotes the agent's cognitive capacity, i.e., $cp \in \mathbb{N}^r$, intuitively standing for what the agent is able to afford with regard to each resource.

The intended function of $ord : W \to \Omega$ is to assign ordinals to states such that the smaller the ordinal, the more plausible the state is. From dual-process models we can define the plausibility $\geqslant$ as follows: $u \geqslant w$ if and only if $ord(u) \geqslant ord(w)$. As in (Rantala 1982), the truth-definition allows $w \in W^I$ to verify formulas non-recursively, without reference to their syntactic structure. However, states in $w \in W^I$ are not entirely unconstrained, since (Solaki et al. 2019) impose a condition of *minimal consistency* on $w \in W^I$, according to which $\{\phi, \neg\phi\} \nsubseteq V(w)$, for all $w \in W^I$.

(Solaki et al. 2019) define a *pointed plausibility model* (or *pointed model* for short) to be a tuple $(\mathfrak{M}, w)$ consisting of a dual-process plausibility model $\mathfrak{M}$ and a designated world w in it. A key component of their logic is the notion of R_k-accessibility, which relates pointed plausibility models and is used in the semantic definition of the $\langle R_k \rangle$ operator. Roughly, when a pointed plausibility model $(\mathfrak{M}', w)$ is R_k-accessible from $(\mathfrak{M}, w)$, the states at least as plausible as w (i.e, the members of the set $P_{\geqslant}(w) := \{u \in W \mid w \geqslant u\}$) are replaced with states that can be reached by an application of the rule R_k to the the members of $P_{\geqslant}(w) := \{u \in W \mid w \geqslant u\}$. In addition, the ordering ord is modified, and cp us reduced, to reflect the cost of a rule application. Given their condition of minimal consistency, if a rule application results in a state that verifies a contradiction, then this state cannot be a member of the set of states arrived at via the model transformation, and it is thrown away.

2.3.1.1 The notion of R_k-accessibility

The notion of R_k-accessibility is defined as follows.

Definition 3 (Propositional truths) Let $\mathfrak{M}$ be a dual-process plausibility model, $w \in W$ a world of the model and $\mathcal{L}_P$ a language as defined in **Definition** (1). The set of *propositional truths* of $w \in W^P$ is $V^*(w) = \{\phi \in \mathcal{L}_P \mid \mathfrak{M}, w \models \phi\}$ and the set of *propositional truths* of $w \in W^I, V^*(w) = \{\phi \in \mathcal{L}_P \mid \mathfrak{M}, w \models \phi\}$

A given instance of the inference rule R_k consists of a set $pr(R_k)$ of and a conclusion, denoted by $con(R_k)$. The condition of *Succession* allows us to relate states which meet the following condition:

For every, if $w \in W$:

1. $pr(R_k) \subseteq V^*(w)$

2. $\neg con(R_k) \notin V^*(w)$

3. $con(R_k) \neq \neg\phi$, for all $\phi \in V^*(w)$

then there is $w \in W$ such that $V^*(u) = V^*(w) \cup \{con(R_k)\}$

$V^*(w) \vdash_{R_k} V^*(u)$ expresses that *u expands w* via *Succession* with respect to some instance of R_k. When there is no instance of R_k for which $pr(R_k) \subseteq V^*(w)$, w has itself only as an R_k expansion, in order to reflect the fact that applying R_k does not expand w. When either 2. or 3. is violated but $pr(R_k) \subseteq V^*(w)$ for some instance of R_k, there is no R_k expansion via *Succession* with regard to this instance, in order to reflect that inconsistency in the form of a formula ϕ and its negation would result in applying R_k (thereby violating the *minimal consistence* condition aforementioned). If we apply the rules $R_1, ..., R_n$ successively, we can generalise $\vdash_{R_k}$ to $\vdash_{R_1,...,R_n}$.

Definition 4 (Rule-specific radius) For inference rule $R_k \in R$, the R_k-radius of a world $w \in W$ is $w^{R_k} = \{u \mid V^*(w) \vdash_{R_k} V^*(u)\}$.

w^{R_k} collects R_k-expansions of w. Whenever $w \in W^P$, the deductive closure of possible worlds ensures that $w^{R_k} = \{w\}$ whilst the R_k-radius of an impossible worlds can contain different R_k-expansions. R_k-expansions are monotonic, so that applications of rules cannot disturb inferences already made.

Definition 5 (Choice function) Let $\mathcal{C} : \mathscr{P}(\mathscr{P}(W)) \to \mathscr{P}(\mathscr{P}(W))$ be a choice function. $\mathcal{C}$ takes a set $\mathcal{W} = \{W_1, ..., W_n\}$ of sets of worlds and returns the set $\mathcal{C}(\mathcal{W})$ of sets of worlds which results from all the possible combinations in which we can select exactly one element from each non-empty $W_i \in \mathcal{W}$. A *choice* of $\mathcal{W}$ is a member of $\mathcal{C}(\mathcal{W})$.

$\mathcal{C}$ is used to define the transformation on a plausibility model induced by the application of an instance of a certain inference rule R_k. $\mathcal{C}$ applies to the radii of worlds, and the choices that it yields correspond to different ways of applying a certain rule of inference. If a $v \in P_{\geqslant}(w) := \{u \in W \mid w \geqslant u\}$ cannot survive the application of a rule R_k then it must be an impossible world, and the fact it doesn't survive a rule application means an agent can rule it out as doxastically or epistemically possible. Worlds which don't survive rule application aside, the ordering of plausibility between worlds is preserved.

Let us spell out the transformation technically. Let $\mathfrak{M} = (W^P, W^I, ord, V, C, cp)$ be a plausibility model. The transformation induced by the application of an instance of a certain inference rule R_k is in three steps:

Step 1 Let $(\mathfrak{M}, w)$ be a pointed model and R_k be an inference rule. Then $P^{R_k}(w) := c$ where $c \in \mathcal{C}(\{v^{R_k} \mid v \in P_{\geqslant}(w)\})$. c is a choice of R_k-expansions of worlds considered at least as plausible as w.

Step 2 If $u \in P_{\geqslant}(w)$ but $u \notin P^{R_k}(w)$ then u doesn't survive application of R_k, so that it must be excluded as an epistemic or doxastic possibility from the new R_k accessible pointed model. The R_k accessible pointed model $(\mathfrak{M}', w)$ has as its set of worlds $W^{R_k} = W \backslash \{u \in P_{\geqslant}(w) \mid u \notin P^{R_k}(w)\}$

Step 3 Following application of R_k, ord^{R_k} is defined as follows. Let $u \in W^{R_k}$:

(1) If $u \notin P_{\geqslant}(w) \cup P^{R_k}(w)$, then $ord^{R_k}(u) = ord(u)$, i.e., worlds that were less plausible than w and are not in the choice are assigned the same ranking as before.

(2) Consider a $u \in P^{R_k}(w)$. If $u \in P^{R_k}(w)$, then for the particular choice c, $u \in v^{R_k}$ for one or more $v \in P_{\geqslant}(w)$. The set of such v's is denoted T and $ord^{R_k}(u) = ord(z)$ for $z \in min(T)$, so that u takes the position of the most plausible of the worlds from which it originated.

Step 4 For $u, v \in W^{R_k} : u \geqslant_{R_k} v$ if and only if $ord^{R_k}(u) \leqslant ord^{R_k}(v)$.

Step 5 V is restricted to the worlds in W^{R_k} and $cp^{R_k} := cp - C(R_k)$.

An example illustrating R_k-accessibility is given in the appendix, section D.

2.3.1.2 Truth at a state

With the definition of R_k-accessibility in place, we can state the truth definition for the language presented in **Definition 1**:

Definition 6 (Semantics) For any dual-process plausibility model $\mathfrak{M}$, state s and formulas ϕ, ψ, $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \dashv \phi$, which are read respectively as 'formula ϕ is true at w in $\mathfrak{M}$' and 'formula ϕ is false at w in $\mathfrak{M}$', are defined as follows:

For $w \in W^P$:

- $\mathfrak{M}, w \models A$ if and only if $A \in V(w)$, where $p \in \Phi$
- $\mathfrak{M}, w \models \neg\phi$ if and only if $\mathfrak{M}, w \not\models \phi$
- $\mathfrak{M}, w \models \phi \wedge \psi$ if and only if $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\phi$ if and only if $\mathfrak{M}, w' \models \phi$, for all w' such that $w \geqslant w'$
- $\mathfrak{M}, w \models B\phi$ if and only if $\mathfrak{M}, w' \models \phi$ for all $w' \in min(W)$
- $\mathfrak{M}, w \models \langle R_k \rangle\phi$ if and only if $\mathfrak{M}', w \models \phi$ for some $(\mathfrak{M}', w)$ which is R_k-accessible from $(\mathfrak{M}, w)$.
- $\mathfrak{M}, w \dashv \phi$ if and only if $\mathfrak{M}, w \not\models \phi$

For $w \in W^I$:

- $\mathfrak{M}, w \models \phi$ if and only if $\phi \in V(w)$
- $\mathfrak{M}, w \dashv \phi$ if and only if $\neg\phi \in V(w)$

A sentence is *valid* in a model if and only if it is true at every $w \in W^P$, so that validity is assessed only with respect to possible states.

We don't have much to say by way of criticism of (Solaki et al. 2019)'s approach, since the approach we develop to the problem of logical omniscience will be closely based on their account. In particular, our account will be based on the idea that worlds are related by rules of inference, though without employing the model transformation of R_k-accessibility. Nevertheless, in our theory the R_k accessibility relation used to define $\langle R_k \rangle$ will be internalised within a static semantic theory, unlike their dynamic one.

There is one aspect of (Solaki et al. 2019)'s account which we think is problematic from the standpoint of natural language semantics. (Solaki et al. 2019) try to capture the role resources play in how agents reason, by assigning numerical values to the resources that agents have, and calculating how these resources change via reasoning. Though this may be useful for certain modelling purposes—for example, if we want a way to analyse the inferences that people will make in a given task—we think that from the standpoint of natural language semantics it is useful to abstract from such details, which don't obviously have to do with the meanings of any words or constructions in language. It would, indeed, be quite bold to claim that the semantic value or meaning of natural language sentences implicitly refers to some quantity of resource that agents have, and we are sceptical that assigning precise numerical values determining what resources people have has much relevance to how we interpret and understand sentences. For this reason, we think that a more general approach along the lines we elaborate in part II of this thesis, which abstracts from the question of how to represent resources numerically, is superior for the purposes of natural language semantics.

Regarding the dynamic modality $\langle R_k \rangle$, it is also unclear how such a modality could be relevant to natural language semantic theory. The modality is supposed to capture how agents are able to extend their beliefs by making n-steps of reasoning via a given rule of inference. But it doesn't capture the semantics of any particular word or construction in contrast to belief or knowledge operators. The dynamic modality has an attractive property, however. It illustrates how agents can construct and extend their beliefs by applying rules of inference in a particularly perspicuous way, by relating states in a step-by-step construction. This motivates the search for a semantic theory which can internalise this idea of incremental extension, step-by-step, into the semantic values of sentences themselves, so that whether an agent believes a sentence will depend on whether they have undergone a step-by-step construction of a certain kind. This is the approach we will pursue in part II of the thesis.

2.3.2 (Berto and Jago 2019)

In this section we will present (Berto and Jago 2019)'s theory informally, since our criticism of their theory will not require that the reader understands the formal details of their account. Nevertheless, in the appendix, section C, we have included a summary of their formal account, lest the reader be interested to know the details.

(Berto and Jago 2019) call the "problem of rational knowledge" [19, p. 1152] the incompatibility of the following two platitudes:

(i) Rational agents seemingly know the trivial consequences of what they know.

(ii) Rational agents do not know all logical consequences of what they know.

Just like (Bjerring and Skipper 2019) and (Solaki et al. 2019), (Berto and Jago 2019) conclude that agents are either logically omniscient, or agents fail to know or believe some trivial logical consequence of what they believe and thus are prone to what (Berto and Jago 2019) call *epistemic oversights*. Nevertheless, according to (Berto and Jago 2019), attributing an epistemic oversight to someone is infelicitous. As proof of this contention they point out that it would sound odd to assert (17a) or (17b), thereby seeming to assert that John cannot infer *the streets are wet* from *if it is raining then the streets are wet* and John cannot infer *Paris is the capital of France and that Paris is in France* from the conjuncts of this sentence:

(17) a. John knows that if it is raining then the streets are wet and he knows that it is raining, but he doesn't know that the streets are wet.

b. John knows that Paris is the capital of France and he knows that London is the capital of England, but he doesn't knows both that Paris is the capital of France and that London is the capital of England.

In order to understand their explanation of the infelicity of utterances of (17a) or (17b), we need to consider their general picture of belief and knowledge, according to which "It's indeterminate what an agent believes (or knows), given what else she believes (or knows)." The contention that what an agent knows or believes is indeterminate, give what else she believes or knows is defended by an analogy with the Sorites paradox familiar from discussions of vagueness (Sorensen 2018). (Jago 2014b: p.1155) asks us to imagine a deduction from premisses that an agent knows to a conclusion that the agent does not know. The deduction consists of a chain of steps each of which is a trivial logical consequence of the other (their definition of the notion of trivial logical consequence, which appears vague, is examined in the appendix, section C). Given the assumption that if a moderately rational agent knows one of the steps then she knows every trivial logical consequence of it, then the agent would know the conclusion of the deduction, *contra* our assumption that the agent does not know. So it must be the case that one step of the deduction makes a difference somehow, yet, according to (Jago 2014b: p.1155), we cannot pinpoint which particular deductive step makes the difference. Thus "states of belief and knowledge are themselves vague, because it's indeterminate which logical consequences of her beliefs an agent believes." Moreover, not only states of belief and knowledge are vague, but the notion of trivial logical consequence itself is vague, so that whether something is a trivial logical consequence of another thing can fail to be determinate.

According to (Berto and Jago 2019), because of the fact that it's indeterminate which trivial logical consequences of her beliefs an agent knows, whilst it may be the case that an agent knows that ϕ and does not know that ψ, where ψ is a trivial consequence of ϕ, it cannot be determinately true that the agent does not know ψ; in other words, the following principle holds:

(TRIV) If it's determinate that an agent knows that ϕ, and the inference from ϕ to ψ is trivial, then she cannot determinately fail to know that ψ.
(Berto and Jago 2019: p.230)

(TRIV) in turn is supposed to be justified by a general norm governing assertion, that we can only assert things which are determinately true:

> "Suppose that one may assert only what is determinately true...one can assert that agent x believes that A, but not B, only if it is determinate both that x believes that A and that she does not believe that B. But, if the inference from 'A' to 'B' is trivial, this is precisely the situation ruled out by *TRIV*. So, when 'A' trivially entails 'B', we can never assert that an agent believes that A but not that B."

Here (Berto and Jago 2019) is relying on one feature of vague predicates, known as *unassertability at the borderline*. When it is not determinate whether a property holds of an object– i.e, when that object is a borderline case of the property–then an assertion that the object has the property is open to criticism, on the grounds that we should only assert what is determinately true. Attributing an epistemic oversight to someone thus violates this norm of assertion.

(Jago 2014a) claims that it is always infelicitous to attribute an obvious logical oversight to an agent, and gives a theory of what the distinction between obvious and subtle contradictions amounts to (see 2.3.2 for discussion and C for the technical details of Jago's theory). If his argument is correct, then semantic theorists need to pay attention to the distinction between obvious and subtle contradictions in order to account for the (in)felicitousness of certain knowledge and belief reports. His argument that it is infelicitous to attribute obvious logical oversights is based on the perceived infelicity of examples such as (18a) or (18b), which attribute to John an inability to make a simple deduction via modus ponens:

(18) a. John knows that if it is raining then the streets are wet and he knows that it is raining, but he doesn't know that the streets are wet.

b. John knows that Paris is the capital of France and he knows that London is the capital of England, but he doesn't knows both that Paris is the capital of France and that London is the capital of England.

According to (Jago 2014a), both (18a) or (18b) are infelicitous, because it is infelicitous to attribute an obvious logical oversight to someone. This may be correct in most cases, but in this section we argue that it is not always correct.

Consider someone who has certain theoretical commitments to a particular logic, such as a constructive mathematician who eschews classical logic. In order to represent the beliefs of a constructive mathematician of this kind, we might represent a particular belief of hers by saying that she doesn't believe *S* or *not S*, where neither the sentence *S* nor its negation has been proven. This might be the case, crucially, even if we ourselves accept think *S* or *not S* is trivially true. Likewise, in order to represent the beliefs of a dialetheist (Priest et al. 2018), who believes that there are true contradictions, we might say that she believes *S* and not *S*, regardless of whether we think that it is possible for there to be true contradictions. For example, with reference to the paraconsistent set theory (Priest 2011) which Graham Priest advocates, the following sentence seems perfectly felicitous and true:

(19) Graham thinks that the Russell set both is and isn't a member of itself.

One way of understanding this, is that, in both cases, we make an utterance from the perspective of a different agent, in order to capture how she views a given situation, and in order to signal her theoretical commitments or beliefs. In such cases, our assertions provide 'meta-linguistic' information about her beliefs and how she is using certain logical words.[5] When we assert that the constructive mathematician doesn't believe *S* or *not S* we are making a metalinguistic statement, to the effect that, a certain individual uses the words *or* and *not* in a certain manner, and that it is relevant to take account of this in the conversation. This idea is one of the motivations for elaborating the perspective relative semantic theory we present in part II.

Returning to the sentences (18a) and (18b), if it is clear in the context that John has some pretty strange logical beliefs—including the belief that modus ponens sometimes fails[6]— and we wish to signal this to our interlocuter, then we may indeed express this by uttering (18a) or (18b), and these sentences might be a correct way of describing John's beliefs, *from his perspective*. This is consistent with it being the case that, from our perspective, we can reserve a sense in which John does know the conclusion of the simple modus ponens inferences in discussion.

[5]This seems similar to certain metalinguistic uses of adjectives which (Barker 2002) discusses. For example, suppose Feynman is at a party in which everyone knows his height (this example is from (Barker 2002)). Someone at the party asks another person what counts as tall in her country and she replies "Feynman is tall". In such a case, the utterance does not communicate the height of Feynman, which everyone knows *ex hypothesi*; rather, it communicates something metalinguistic: what counts as tall in the relevant country under discussion.

[6](McGee 1985) argues that modus ponens is not valid of English *If...then* constructions.

For this reason, (Jago 2014a)'s assumption that it is always infelicitous to attribute what an obvious logical inconsistency to someone cannot be correct. This will be reflected in our solution to the problem of logical omniscience part II.

The problem seems to be that (Berto and Jago 2019) have no scope in their account for the idea that what counts as an obvious logical consequence of a belief, is a highly context-bound and relative question, and one, we would argue—thus motivating the approach taken in part II of this thesis— that depends heavily on what perspective we take to interpreting someone's utterance. For these reasons, we think it is worth developing an alternative account to the problem of logical omniscience, which doesn't rule out *tout court* the possibility that someone can felicitously deny a seemingly tautologous statement or something which seems to follow obviously from her beliefs. And an account, more importantly for the argument of this thesis, that takes the notion of a perspective seriously.

2.4 Conclusion

(Cresswell and Stechow 1982)'s *de re* approach to the problem of granularity gives rise to puzzling asymmetries in its treatment of certain sentences, asymmetries which can be removed at the cost of permitting questionable structural analyses of certain sentences. (Tancredi and Sharvit 2020) articulate a refined version of (Cresswell and Stechow 1982)'s approach which allows intuitively synonymous predicates to sometimes diverge in meaning and which introduces perspectives, allowing them to capture the perspective-relativity of certain attitude reports. However, their account is unnecessarily baroque and it is based on a serious misunderstanding of the notion of attribution as understood in (Cresswell 1982).

As for the method of diagonalisation, (Stalnaker 1978)'s resolution of the problem of granularity cannot account for the felicitousness of certain tautologies, such as instances of the excluded middle, and his ‘fragments of belief’ approach to logical omniscience cannot account for them either. He could account for the felicitousness of what appear to be instances of the excluded middle simply by treating the English words *or* and *not* as non-logical constants, but then this raises the question of what their semantic interpretation ought to be. The alternative option of allowing the logical constants $\vee$ and $\neg$ to vary in interpretation at certain states is inconsistent with the arguments that (Stalnaker 1996) propounds, and undercuts the argument from rigidity which underlies his account of diagonalisation.

The accounts of logical omniscience we examined are both limited, from the standpoint of natural language semantics, in that they do not relate their proposals to a systematic procedure for interpreting the constituents of sentences. (Solaki et al. 2019)’s account includes a special dynamic modality, but it is unclear whether such a modality is lexicalised by any natural language expression. Their inclusion of resources tied to specific numerical quantities also lacks clear motivation from the standpoint of natural language semantics. (Berto and Jago 2019)’s account of logical omniscience assumes that it is always infelicitous to deny that someone believes an obvious logical consequence of their beliefs. But we discussed cases in which this prediction is incorrect.

We have seen in this chapter that in many cases the truth value of certain sentences which appear to be logically equivalent comes apart when we take account of certain aspects of an agent’s perspective on the world in attributing beliefs to her; in particular, this happens either when we take into account an agent’s cognitive or logical limitations, or when we take into account their theoretical beliefs about the functioning of words such as *and*, *or* and *not*. The accounts of the problems of granularity and logical omniscience we examined do not pay enough attention to the role of agents’ perspective on the world. For this reason, we urge that perspective be integrated into any solution of the problems of granularity and logical omniscience. This integration will be now be pursued.

Part II

Part II: P-HYPE

Chapter 3

HYPE and perspective relativity: fine-grained semantic values as side-effects

3.1 Introduction

In this chapter, we introduce the notion of a linguistic side effect, and propose that fine-grained semantic values be treated as linguistic side effects. We then introduce monads and applicatives, which have been used for capturing side effects in functional programming languages. The logic HYPE (Leitgeb 2019), and the perspective relative semantic theory of (Asudeh and Giorgolo 2016) are introduced, and their theory of perspective relativity is considered.

This chapter will have the following structure. In 3.2, we argue that fine-grained semantic values are linguistic side effects (in the sense of (Shan 2007)). In 3.3 we propose that applicatives and monads offer a compelling lens with which to capture fine-grained semantic values as 'side-effects' of semantic computations. In 3.4 we introduce HYPE, and discuss some of its limitations; namely, HYPE as it stands does not avoid the problem of logical omniscience and HYPE does not incorporate perspective relativity, which previous chapters have found to be an important aspect of propositional attitude semantics. In 3.5 we introduce (Asudeh and Giorgolo 2016)'s perspective-relative semantic theory, and then, finally, in 3.6 we discuss a limitation of AG's account and propose a way forward.

3.2 Fine-grained semantic values as side effects

The *Principle of Extensional Compositionality* (discussed in the appendix, section A.1) states that the extension of a complex expression is determined by the extensions of its parts and the way they are combined. If the extension of a sentence is its truth value and suppose that attitude verbs are functions of type $t \to e \to t$, we run into the problems discussed in the appendix, section A.1. Following (Zimmermann 2012), we say that the assignment of truth values to sentences together with the Principle of Extensional Compositionality results in a *compositionality problem* (Zimmermann 2012):[1] we may know the extension of the sentence *John*

[1] In (Zimmermann 2012), Zimmermann describes three kinds of problems in determining how the extension of a complex expression (construction) is determined by its parts. There he assumes that the extension of a sentence is a truth value. A *type 1* problem occurs when we know the extension of the complex expression but we cannot find an appropriate extension for one of the parts which, together with the extension of the other parts of the complex expression and certain modes of semantic composition, can generate the extension of the complex expression. In the case of propositional attitude constructions such as *John says that Mary sings.*, the extension of the embedded clause is a truth value (under Zimmermann's assumptions) as is the extension of *John says that Mary sings*. But if the extension of *say* simply takes the extension of *Mary sings.* as an argument, we falsely predict that we may substitute any sentence with the same truth value under *say salva veritate*. In Zimmermann's typology of problems, propositional attitude constructions thus present a Type 1 problem, to be solved by finding a suitable extension for a given clause embedding verb. We may know the extension of *John says that Mary sings.* (*viz.* whether it is true), but we need to find a verb denotation which is appropriate for generating this.

says that Mary sings. (i.e, we may know the sentence's truth value), but–without intensions–we are at a loss as to how we could compositionally generate this extension. Consequently, propositional attitude reports appear to be non-compositional relative to the *Principle of Extensional Compositionality* and an assignment of truth values as the extensions of sentences. Often a weaker principle of compositionality is adopted in response, which we call the **Fregean Principle of Compositionality**:

> **Fregean Principle of Compositionality** (Zimmermann 1991)
> The extension of a complex expression is determined by the extensions or intensions of its parts and the way in which they are combined.

In effect, we have restored a notion of compositionality in the face of certain substitution failures, by introducing a dimension of meaning which goes beyond the mere reference or extension of a sentence (*viz.* a truth value).

This illustrates a pattern. Time and time again in semantic theory we find that a naive compositional assignment of semantic values fails to capture some dimension of a given phenomenon. We face a compositionality problem of how to compositionally generate a certain semantic value. Then, to arrive at a solution to the compositionality problem, special semantic values are introduced (in the above case, intensions). For example, (Charlow 2014) introduces non-deterministic and state-changing operators to capture the anaphoric potential of indefinite noun phrases. (Charlow 2014) treats indefinites as non-deterministic analogs of proper names, and defines state-changing operations which allow for the addition of discourse referents in a discourse. Besides anaphora and intensionality, we could mention also focus, variable binding and presupposition (Karttunen and Peters 1979), (Rooth 1985), (Romero and Novel 2013)– all phenomena which have been treated by enriching a basic intensional semantics by introducing new kinds of semantic value.

Compositionality problems also occur in programming languages. Consider the following piece of code from the programming language Python (the example is from (Shan 2007)):

(1) a. $f(2) \times f(3)$

b. $f(3) \times f(2)$

Given the commutativity of $\times$, these programs intuitively compute the same function, which consists of computing the values of two functions and then multiplying the result. However, there are contexts in which we might distinguish (1a) and (1b). Consider a function which prints its argument and then returns it:

(2) def $f(x)$

print $x \times 10$

return x

In this case $f(2) \times f(3)$ prints out "20 30" and then returns 6, whereas $f(3) \times f(2)$ prints out "30 20" and then returns 6. Commands like "print" induce non-compositionality, as input commands like "read" and control commands like "goto". In functional programming, the characteristic effects of commands such as 'print' are often called 'side effects'. This terminology is appropriate if we view a program first and foremost mathematically, as a function and the characteristic effects generated by problematic seemingly non-compositional elements of a programme as secondary effects.

(Shan 2005) was one of the first to elaborate on the parallels between programming languages and natural languages in this respect:

"The commonality here between natural and programming language seman-

> tics is that we have some pretheoretic notion of what phrases mean—for example, the expression $f(2)$ "means" whatever you get when you feed the function f the number 2, and the noun phrase *the morning star* "means" Venus—yet two phrases that supposedly mean the same thing turn out to be distinguished by a troublemaking context involving verbs like *think* or commands like "print". This kind of situation–where, in short, equals cannot be substituted for equals–is what I take *referential opacity* to mean (as opposed to *referential transparency*, which is when equals can be substituted for equals). A vaguer way to define referential opacity is that it's when meaning "depends on context". Worse than referential opacity, sometimes we don't have any pretheoretic notion of meaning. For example, what does *the king of France* mean, and what does "goto" mean, anyway?"

Shan had the intuition that we can model certain phenomena in language as 'side effects' of computing the main value of an expression and that we could use monads to capture such linguistic phenomena, just like monads capture side effects in functional programming languages (Moggi 1988, Wadler 1995, Hyland et al. 2002). (Shan 2005) defines a natural language phenomenon as a *linguistic side effect* if "it involves either referential opacity or the lack of any pretheoretic notion of meaning to even challenge compositionality with", and he includes amongst so-called linguistic side effects intensionality, binding, quantification, interrogatives, focus and presuppositions. Since (Shan 2005), monads have become a popular method of modelling linguistic side effects (see (Giorgolo and Unger 2009, Van Eijck and Unger 2010, Asudeh and Giorgolo 2012, Unger 2011, Charlow 2014, Barker and Shan 2014, Bumford 2015, Charlow 2020)).

We will define monads in the next section. But for now, we can say that, intuitively, their role in natural language semantic theories is to provide a compositional regime for integrating new types of semantic values into a semantic theory with minimal disruption to compositional derivations induced by the need to inflate the stock of compositional rules devised to combine the new types of semantic value. A popular way to treat side effects *sans* monads in both linguistics and computer science is to type-lift (Partee 1986, Hendriks 2020). For example, we lift the denotation of names from things of type e to functions from predicates of type $e \rightarrow t$ to truth values, or–in programming language– we might deal with “print” by lifting denotations from numbers to number-string pairs. But type-lifting has a cost: in order to compositionally combine intensions we need a new rule of semantic composition for combining intensions together, such as intensional function application (Heim and Kratzer 1998), in addition to the basic rule of extensional function application. Similar observations could be made of focus-sensitive semantic values, and of the novel rules of semantic composition which have been introduced to combine them together. Monads map operations and values in a given type-space–for example, the type space provided by the simply typed λ-calculus with basic types for individuals and Booleans–with operations and values in an enriched type-space (which might include such exotic semantic values as intensions, focus-sensitive and perspective relative semantic values[2]) whilst preserving ordinary extensional function application as the main compositional principle. Moreover, they do this without generalising to the worst case (Shan 2002), where, by ‘generalising to the worst case’ we mean the strategy of assigning all expressions of some syntactic category

[2]By focus-sensitive and perspective-relative semantic values, we mean semantic values which incorporate the truth-conditional effects of focus and of perspective. For discussion of focus sensitive semantic values see (Rooth 1992). For discussion of perspective relative semantic values, see (Lasersohn 2017). The focus relative semantic values in (Rooth 1992) have been incorporated into the monadic fold by (Shan 2002), as we later shall see in chapter 6. And perspective relative semantic values can be captured via the reader monad, as we will shortly see.

a complicated semantic type, whenever one expression of that syntactic category requires that complicated type to deal with some phenomena.[3] Shan's broader picture is that, aided with monads, richer semantic theories can be constructed which solve the compositionality problems caused by a given linguistic phenomena by incorporating richer types of semantic value and providing new ways of combining semantic values. In addition, there is a nice conceptual boon to employing monads: monads enable us to reveal underlying technical commonalities between diverse phenomena such as intensionality, focus and quantificational scope by abstracting away from the surface differences between them. This is *a priori* preferable from having to have a number of separate mechanisms with which to model these phenomena.

It is natural to ask whether the linguistic phenomena that exemplify the problems of logical omniscience and granularity can be thought of as linguistic side effects on Shan's definition, in the same way that he understands intensionality as a linguistic side-effect. Arguably, they can. The linguistic phenomena which exemplify the problems of logical omniscience and granularity involve failures of compositionality in the following sense: given an assignment of bog standard intensions to the parts of the sentence and certain basic rules of semantic composition, we cannot calculate an appropriate semantic value for a whole sentence, just as in

[3] Montague required that every expression of the same syntactic category must be assigned the same semantic type, and so whenever a certain expression of some syntactic category requires a complex semantic type, all expressions of that syntactic category must be given that type. For example, although an extensional transitive verb *find* expresses a relation between individuals of type $e \to e \to t$, since an intensional transitive verb like *seek* requires a more complex type assignment, the orthodox Montagovian strategy requires that we give *find* a more complicated type. This is known as 'generalising to the worst case' and it has been criticised on various grounds. In particular, it has been argued that allowing the semantic types of expressions to be shifted via type shifters better captures certain empirical data, and also allows us to adopt simple lexical entries.

the case of intensionality with respect to the naïf understanding of compositionality and of the semantic value of sentences as truth values. That there are problems with producing a compositional semantics that deals with the linguistic phenomena that exemplify the problems of logical omniscience and granularity is discussed in the literature (King 2007). Furthermore, the linguistic phenomena that exemplify the problems of logical omniscience and granularity involve referential opacity, as Shan understands that term in the above quotation, since it is not clear what distinction to make between the semantic contribution of logically equivalent statements, nor how their meanings relate to their truth conditions. And the linguistic phenomena that exemplify the problems of logical omniscience and granularity involve sentences in which it is desirable to block the 'replacement of equals for equals', as we discussed in the previous chapter, where we pointed out that fine-grained phenomena have motivated theorists such as (Rantala 1982) to abandon the rule of the replacement of logical equivalents. We therefore propose that the linguistic phenomena that exemplify the problems of logical omniscience and granularity be added to the list of linguistic side-effects, and that they should be studied from this vantage point. Once we add them to the list of linguistic side-effects, we can then see whether fine-grained semantic values can be modelled using monads and whether different fine-grained semantic theories can be seen as giving different answers to the question of what kinds of side effect are necessary to model the phenomenon.

Discussions of the problem of granularity often include many different kinds of phenomena: the question of the treatment of propositional attitudes, the treatment of co-referring names, the role of 'metalinguistic' or syntactic information in propositional attitude reports and the treatment of resource-bounded reasoning. Different fine-grained semantic theories have explored these topics in different ways with different methods. Monads, we suggest, may be a natural way of integrating the side effects used to study these phenomena in one language. Indeed, in (Charlow

2014) monads are used as a tool to integrate intensionality, focus semantics, dynamic semantics and continuations, and the integration of multiple 'side-effects' via monads in functional programming is a well-established topic in theoretical computer science (Hyland et al. 2006). We therefore expect that many of the traditional fine-grained semantic theories can be formulated in monadic terms. In this and the proceeding chapters, we will show how monads and the closely related concept of an applicative, can be used to capture fine-grained semantic values.

3.3 Monads and applicatives: an introduction

We will now present monads and applicatives. First let us consider monads. Monads are mathematical structures from Category theory (Mac Lane 2013), and are an everyday part of the functional programming toolkit. Here we discuss monads as they are usually presented in the functional programming literature (Wadler 1995), but for the relationship between monads as used in functional programming and the equivalent definition usually employed in Category theory, see (nLab authors 2020). Given a set TYPE of types, we can define a *reader monad* as a triple $(\Diamond, \eta, \bigstar)$. $\Diamond$ is a so-called *type-constructor*, which takes a type as argument and returns another type. In Haskell terms $\Diamond$ is of type $\star \rightarrow \star$, the type of unary type constructors, where $\star$ is a kind.

$\Diamond$ behaves as a special modal operator in Lax logic (Fairtlough and Mendler 1997). In terms of Category theory, $\Diamond$ is an endofunctor; that is, a functor that maps a category to itself (nLab authors 2020), in our case the category of types, whose objects are types and whose morphisms are functions between types. A functor maps objects of the category to objects of another (possibly identical) category and maps morphisms in one category to morphisms in another. In the case of a reader monad, $\Diamond$ maps any type τ to $\rho \rightarrow \tau$ (where ρ is some fixed type upon which the reader monad is built). For all types α, β, $\Diamond$ maps a function $f : \alpha \to \beta$ to a function $f' : \Diamond\alpha \to \Diamond\beta$ such that $f'\,x = \lambda i.f\,(x\,i)$. The second component of a reader monad $(\Diamond, \eta, \bigstar)$ is $\eta : \tau \to \Diamond\tau$, and η combines with lambda terms to construct lambda terms; it takes a non-monadic value $x : \tau$ and trivially upgrades it to a monadic value by forming a constant function from things of type ρ to x. It is called the *unit* of the monad:

Definition 7 : $\eta\,x =_{def} \lambda i.x : \rho \to \alpha$

Finally, $\bigstar$ (called *bind*) is a polymorphic binary infix operator defined as in 8:

(3) $\bigstar : \Diamond\tau \to (\tau \to \Diamond\delta) \to \Diamond\delta$

Definition 8 : $a\bigstar f =_{def} \lambda i.f\,(a\,i)\,i$ where $a : \Diamond\tau, f : \tau \to \Diamond\delta$

$\bigstar$ applies a monadic function f to a monadic argument a, by threading a variable $i : \rho$ through a and f.

Monads satisfy the following coherence laws:

Definition 9 (The monad laws)

Left Id $$(\eta\,x) \bigstar f = f\,x \tag{3.1}$$

Right Id $$x \bigstar \eta = x \tag{3.2}$$

Associativity $$(f\bigstar g) \bigstar h = f \bigstar \lambda x.\,(g\,x \bigstar h) \tag{3.3}$$

Left ID requires that wrapping a term in η protects it and enables it to be supplied directly to the function that it combines with via $\bigstar$. *Right ID* requires enables us to ignore the function η in the configuration above, so that the result is just the first argument of $\bigstar$. Associativity ensures that order of evaluation matters in lambda terms in which there are a number of occurrences of $\bigstar$, and not the hierarchy of the groupings within the lambda term.[4]

In any monad $(M, \eta, \bigstar)$ we can define a form of monadic functional application $\mathtt{A}\ m\ n$ between a function $m : M(\alpha \to \beta)$ and an argument of type $n : M\alpha$, for all types α, β using $\bigstar$ and η as below:

$$\begin{aligned} &\mathtt{A}\ m\ n := \\ &m_{M(\alpha\to\beta)}\ \bigstar\ \lambda x_{\alpha\to\beta}.\ n_{M\alpha}\ \bigstar\ \lambda y_{\alpha}.\ \eta\,(\,x\,y\,) : M\beta \end{aligned} \tag{3.4}$$

Alternatively, we can use the methods of an applicative functor to allow us to combine a function $m : M(\alpha \to \beta)$ and an argument of type $n : M\alpha$. In the next chapter we will make use of applicative functors. An applicative functor consists of a functor $\odot$, an injection $\eta : \alpha \to \odot\,\alpha$ and an infix, associative function $\bullet : \odot(\alpha \to \beta) \to \odot\alpha \to \odot\beta$, which, for all $\alpha, \beta, \delta \in TYPE$, obey the following laws (here $id : \alpha \to \alpha$ is the (polymorphic) identity function and $\mathbf{;} : (\beta \to \delta) \to (\alpha \to \beta) \to (\alpha \to \delta)$ is function composition):

Identity:	$\eta(id) \bullet a_{\Diamond\alpha} = a_{\Diamond\alpha}$
Composition:	$(\eta\ \mathbf{;}) \bullet\ a_{\Diamond(\beta\to\delta)} \bullet b_{\Diamond(\alpha\to\beta)} \bullet c_{\Diamond\alpha} = a_{\Diamond(\beta\to\delta)} \bullet (b_{\Diamond(\alpha\to\beta)} \bullet c_{\Diamond\alpha})$
Homomorphism:	$\eta(f_{\alpha\to\beta}\ x_{\alpha}) = \eta(f) \bullet \eta(x)$
Interchange:	$a_{\Diamond(\alpha\to\beta)} \bullet (\eta\ b_{\alpha}) = \eta(\lambda f_{\alpha\to\beta}.f\ b_{\alpha}) \bullet a_{\Diamond(\alpha\to\beta)}$

We use the same symbol η for the injection of a monad, as for the injection of an applicative. A monad is an applicative functor equipped with a 'join' $\mu : \odot\odot\,\alpha \to$

[4] We will not prove the monad laws are satisfied for the monads we present, since, in all cases the proof that the structures we present satisfy the monad laws is trivial. Likewise, we will not prove the applicative laws (below) are satisfied for the monads we present, since, in all cases the proof that the structures we present satisfy the applicative laws is trivial.

$\odot\alpha$, which enables us to derive from every lambda term of type $\odot\odot A$ a lambda term of type $\odot A$. In an applicative we have neither the 'join' $\mu : \odot\odot\alpha \rightarrow \odot\alpha$ nor $\bigstar$. The join is interderivable with $\bigstar$, given $fmap : (\alpha \rightarrow \beta) \rightarrow \odot\alpha \rightarrow \odot\beta$, a functor which belongs to both monads and applicatives:

(4) $\quad m \bigstar f = \mu(fmap\, f\, m)$

We will use the following two reader applicatives in the next chapter, though discussion of their use will have to wait until then:

<u>The κ applicative</u> :

$\blacklozenge\, \alpha = (s \rightarrow p) \rightarrow \alpha$

$\eta_{sp}(x_\alpha) =_{df} \lambda\delta.\ x : (s \rightarrow p) \rightarrow \alpha$

$f \bullet_{sp} a =_{df} \lambda\delta.\ f\,\delta\,(a\,\delta)$ where $a : \blacklozenge\alpha, f : \blacklozenge(\alpha \rightarrow \beta)$

<u>The reader applicative on s</u> :

$\Diamond\alpha = s \rightarrow \alpha$

$\eta_s(x_\alpha) =_{df} \lambda i.x : s \rightarrow \alpha$

$f \bullet_s a =_{df} \lambda i.f\, i\,(a\, i)$ where $a : \Diamond\alpha, f : \Diamond(\alpha \rightarrow \beta)$

Nota bene: we will often write the type $s \rightarrow p$ as $\Diamond p$ so that, for example, instead of η_{sp} or $\bullet_{sp}$, we may write $\eta_{\Diamond p}$ or $\bullet_{\Diamond p}$.

In a reader monad, we reduce a lambda term of type $\odot\odot\alpha$ to a lambda term of type $\odot\alpha$ as follows:

(5) $\quad \mu_{\odot\odot\alpha\rightarrow\odot\alpha}\ (\lambda i, j.\ f\, i\, j)_{\odot\odot\alpha} = (\lambda i.\ f\, i\, i)_{\odot\alpha}$

Where $\odot\alpha := s \rightarrow \alpha$, readers may recall that the operation performed by μ in this case is in fact the operation performed by the celebrated diagonalisation operator † from (Stalnaker 1978). If we needed the diagonalisation operator in our semantic theory, a reader monad would be very useful. But since we do not need it, we will not use a reader monad.

Before moving on to discuss HYPE, we will illustrate how we can use $\bullet_s$ and $\bullet_{sp}$ to lift lexical entries, using labelled binary trees (see the appendix, section B). The types $\blacklozenge\Diamond e$ and $\blacklozenge\Diamond t$ will be very important in the next chapter (see 4.3), but unfortunately this reason for this cannot be explained here. Instead, we simply show the mechanics of various operations which will prove useful in the next chapter. Suppose we have a predicate $is.ten.years.old : e \to t$ and we want to combine this predicate with a constant $sm : \blacklozenge\Diamond e$ rendering some proper name. We can do this by lifting $is.ten.years.old : e \to t$:

(6)

$$(\eta_{sp}\,((\eta_s\, is.ten.years.old)^{\bullet_s}))^{\bullet_{sp}}\ sm : \blacklozenge\Diamond t$$

$$sm : \blacklozenge\Diamond e \qquad (\eta_{sp}\,((\eta_s\, is.ten.years.old)^{\bullet_s}))^{\bullet_{sp}} : \blacklozenge\Diamond e \to \blacklozenge\Diamond t$$

$$\big|\ \bullet_{sp}$$

$$\eta_{sp}\,((\eta_s\, is.ten.years.old)^{\bullet_s}) : \blacklozenge(\Diamond e \to \Diamond t)$$

$$\big|\ \eta_{sp}$$

$$(\eta_s\, is.ten.years.old)^{\bullet_s} : \Diamond e \to \Diamond t$$

$$\big|\ \bullet_s$$

$$\eta_s\, is.ten.years.old : \Diamond(e \to t)$$

$$\big|\ \eta_s$$

$$is.ten.years.old : e \to t$$

In this way, we can combine a possibly perspective relative proper name–sm– with a predicate, $\eta_{sp}\,(\eta_s\, is.ten.years.old)$ which is not perspective relative. Likewise, we can use similar methods to combine a possibly perspective relative predicate, such as $is.amusing : \blacklozenge\lozenge(e \rightarrow t)$ with a proper name $mary : e$ which is not perspective relative:

(7)

$$is.amusing \bullet (\eta_{sp}(\eta_s mary)) : \blacklozenge\lozenge t$$

- $\eta_{sp}\,(\eta_s\, mary) : \blacklozenge\lozenge e$
  - $\eta_s\, mary : \lozenge e$
    - $mary : e$
- $is.amusing^{\bullet} : \blacklozenge\lozenge e \rightarrow \blacklozenge\lozenge t$
  - $\bullet_{sp}$: $is.amusing : \blacklozenge\lozenge(e \rightarrow t)$

For us, the methods of the applicative will be particularly useful for logical connectives such as and and *or* and *not*, where this latter is classical negation. For, as the following tree shows, using the methods of the applicatives we have introduced we can lift classical negation so that it can combine with lambda terms of type $\blacklozenge\lozenge t$:

(8)

$$(\eta_{sp}((\eta_s(\lambda p_t.\neg p))^{\bullet_s}))^{\bullet_{sp}}\ (\eta_{sp}\,\eta_s\, is.ten.years.old \bullet sm) : \blacklozenge\lozenge t$$

- $\blacklozenge\lozenge t \rightarrow \blacklozenge\lozenge t$
  - $\bullet_{sp}$: $\blacklozenge(\lozenge t \rightarrow \lozenge t)$
    - η_{sp}: $\lozenge t \rightarrow \lozenge t$
      - $\bullet_s$: $\lozenge(t \rightarrow t)$
        - η_s: $\lambda p_t.\neg p : t \rightarrow t$
- $\eta_{sp}\,((\eta_s\, is.ten.years.old)^{\bullet_s}))^{\bullet_{sp}}\ sm : \blacklozenge\lozenge t$

For logical connectives which take more than one argument, such as and and or, it is useful to use a special combinator, $\mathbf{zip}^n_{\mathtt{M}}$, where $\mathtt{M}$ stands for an arbitrary applicative:[5]

$$\mathbf{zip}^n_{\mathtt{M}} : (\alpha_1 \rightarrow \alpha_2 \cdot \cdot \cdot \rightarrow \cdot \cdot \cdot \alpha_n) \rightarrow (\mathtt{M}\alpha_1 \rightarrow \mathtt{M}\alpha_2 \cdot \cdot \cdot \rightarrow \cdot \cdot \cdot \mathtt{M}\alpha_n)$$

We define $\mathbf{zip}^n_{\mathtt{M}}$ as follows:

$$\lambda f_{\alpha_1 \rightarrow \alpha_2 \cdot \cdot \cdot \rightarrow \cdot \cdot \cdot \alpha_n}, a_1 : \mathtt{M}\alpha_1, a_2 : \mathtt{M}\alpha_2, ..., a_{n-1} : \mathtt{M}\alpha_{n-1}. \, ((...(((\eta_{\mathtt{M}}\, f)^{\bullet_{\mathtt{M}}} a_1)^{\bullet_{\mathtt{M}}}\, a_2)...)^{\bullet_{\mathtt{M}}}\, a_{n-1})$$

Where f is instantiated to $\lambda q, p.p \wedge q$, we have:

$$\mathbf{zip}^3_{\mathtt{M}}(\lambda q, p.p \wedge q) = \lambda b_{\mathtt{M}t}, a_{\mathtt{M}t}. \, ((\eta_{\mathtt{M}}(\lambda q, p.p \wedge q))^{\bullet_{\mathtt{M}}} a)^{\bullet_{\mathtt{M}}}\, b$$

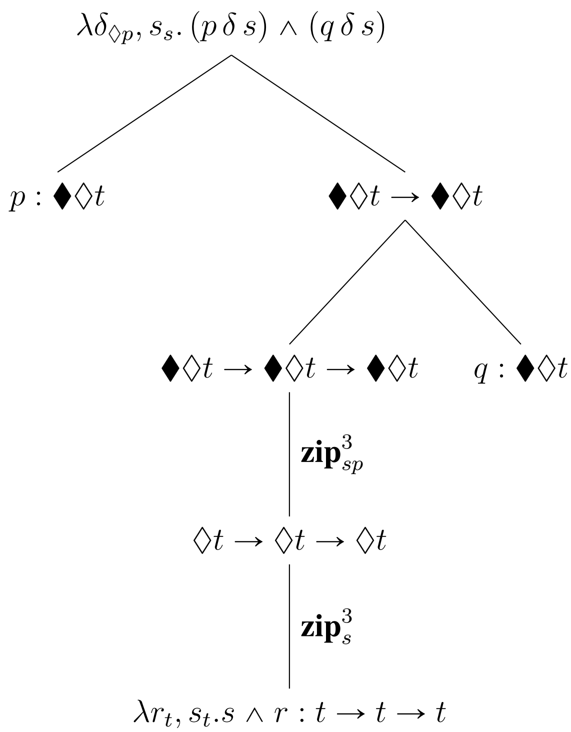
If $\mathtt{M}$ is the reader applicative, then

$$\mathbf{zip}^3_{\mathtt{M}}(\lambda q, p.p \wedge q) = \lambda b_{\mathtt{M}t}, a_{\mathtt{M}t}, s. \;\; (a\, s) \wedge (b\, s)$$

Using $\mathbf{zip}^3_s$ and $\mathbf{zip}^3_{sp}$ we can lift and as follows:

[5] Haskellers will recognise this function as none other than $zipWith$.

(9)



If we have a function $f : \mathtt{M}(\alpha \to \beta)$ and want a function $g : \mathtt{M}\alpha \to \mathtt{M}\beta$ we use the combinator $\mathbf{zipalt}^n_{\mathtt{M}}$, whose type is (3.5) and whose definition is in (3.6):

$$\mathbf{zipalt}^n_{\mathtt{M}} : \mathtt{M}(\alpha_1 \to \alpha_2 \cdot \cdot \cdot \to \cdot \cdot \cdot \alpha_n) \to (\mathtt{M}\alpha_1 \to \mathtt{M}\alpha_2 \cdot \cdot \cdot \to \cdot \cdot \cdot \mathtt{M}\alpha_n) \tag{3.5}$$

$$\lambda f, a_1 : \mathtt{M}\alpha_1, a_2 : \mathtt{M}\alpha_2, ..., a_{n-1} : \mathtt{M}\alpha_{n-1}.\,((...((f \bullet_{\mathtt{M}} a_1) \bullet_{\mathtt{M}} a_2)...) \bullet_{\mathtt{M}} a_{n-1}) \tag{3.6}$$

3.4 Introducing HYPE

We now present HYPE, and then show how to encode HYPE into partial *Ty2* with monads. HYPE is a logic which has been used to give an analysis of the semantic paradoxes. From a logical point of view, HYPE bears a number of relationships to classical and intuitionistic logic, as well as to first-degree entailment (Belnap and

Dunn 1975). The name *HYPE* was chosen by (Leitgeb 2019) to reflect the possibility that the logic HYPE be useful in modelling the so-called hyperintensional contexts.

Before looking at the logic HYPE, we need to make an important caveat. We will not be claiming in this thesis that HYPE is superior to other logics in helping to resolve the problem of granularity or logical omniscience. Indeed, we will not be claiming that classical logic is insufficient to resolve the problems of granularity and logical omniscience either. As discussed before, this would be a rather strong claim, given the fact that there are full and faithful translations of many logics into classical logic. Our use of HYPE is rather based on the conviction that a so-called hyperintensional logic worth its salt which has been claimed by (Leitgeb 2019) as a possible framework in which to study hyperintensional operators would ideally have something to say about the semantics of these operators in natural language, and in particular, how their semantics might relate to a compositional, subsentential semantic theory of natural language. Our use of HYPE can therefore be seen as an attempt to show that HYPE can be used as a basis for a compositional semantics of natural language, and so fulfill its promise to provide a framework in which to study hyperintensional operators. If it transpires that there are, after all, good arguments for using HYPE in the compositional semantics of natural language over other logical approaches, then hopefully P-HYPE will be of interest. Our proposals as to how to solve the problems of granularity and omniscience are based on a semantics of perspective and an approach to resource bound reasoning (presented in (7.1)), and these proposals are detachable from our use of HYPE.

Certainly, HYPE has many features which may make it attractive conceptually to some semanticists and philosophers. Firstly, the semantics of HYPE is based on situations, as opposed to possible worlds. Situations have been argued to be more plausible in many applications (Kratzer 2012), as they may be partial, and situations may be combined together cumulatively to form more complex situations. The semantics of HYPE allows states to be gappy (i.e, to verify neither φ nor $\neg\varphi$) and glutty (i.e, to verify both φ and $\neg\varphi$) and situations can be combined together with the use of a fusion operator. Secondly, HYPE may be attractive because of certain properties of negation in the logic. Negation in HYPE is treated as a negative modal operator. It is weaker than classical logic since it only admits a form of contraposition which applies to theorems and not to arbitrary premisses (see rule (3.29) of the axioms of first-order HYPE below). This may be useful in the semantics of natural language conditional constructions. Whilst HYPE negation only allows contraposition in limited cases, HYPE negation validates many classical laws (for, example, the De Morgan laws and double negation elimination).

With this caveat explained we can now consider HYPE. HYPE (Leitgeb 2019) employs states/situations relative to which the truth of formulas is determined.[6] One nice feature of HYPE is that it behaves entirely classically at a subset of states (see (Leitgeb 2019) for the details); as such, linguistic analyses couched in classical logics can be transferred to HYPE. HYPE, as we will presently see, makes use of a special incompatibility $\perp$ relation between states and a fusion operator $\circ$ which combines states together. They are also used in the satisfaction clauses for negation and the conditional, somewhat like (Veltman 1987: pp. 202–7) and truthmaker semantics.

The language of first-order HYPE $\mathcal{L}_{HYPE}$ consists of

- A non-empty countable set VAR of individual variables $x, x', x''...$
- A non-empty countable set $PRED$ of predicates of finite arity $P_1, P_2, ...$

[6]We will use the words 'states' and 'situations' interchangeably and we steer clear of metaphysical issues relating to states in this thesis. Possible worlds are therefore states of a certain kind, those which are neither gappy nor glutty.

- Logical symbols: $\neg, \wedge, \vee, \supset, \forall, \exists$

- Auxiliary symbols: (,)

The set $FORM_{HYPE}$ of well-formed formulas of first-order HYPE is defined as in first-order logic. $A \supset B \wedge B \supset A$ is abbreviated as $A \equiv B$. 'A', 'B', 'C' and 'D' will stand for arbitrary HYPE formulas and capital greek letters like Φ stand for sets of formulas. $\neg$ has greater binding strength than the other binary connectives and $\wedge$, $\vee$ have greater binding strength than $\supset$ or $\equiv$. Variable assignments g and their modified variants $g[d/x]$ behave as in Classical Predicate logic.

The logic of first-order HYPE is given by the universal closures of instances of the following axioms together with Modus Ponens (3.28) and admissible contraposition ((3.29)) as the rules of inference:

$$\vdash \top \tag{3.7}$$

$$\vdash A \supset A \tag{3.8}$$

$$\vdash A \supset (B \supset A) \tag{3.9}$$

$$\vdash A \supset (B \supset C) \supset ((A \supset B) \supset (A \supset C)) \tag{3.10}$$

$$\vdash A \wedge B \supset A. \tag{3.11}$$

$$\vdash A \wedge B \supset B. \tag{3.12}$$

$$\vdash A \supset A \vee B. \tag{3.13}$$

$$\vdash B \supset A \vee B. \tag{3.14}$$

$$\vdash A \supset (B \supset A \wedge B). \tag{3.15}$$

$$\vdash (A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C)) \tag{3.16}$$

$$\vdash A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C). \tag{3.17}$$

$$\vdash A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C). \tag{3.18}$$

$$\vdash \neg\neg A \equiv A. \tag{3.19}$$

$$\vdash \neg(A \wedge B) \equiv \neg A \vee \neg B. \tag{3.20}$$

$$\vdash \neg(A \vee B) \equiv \neg A \wedge \neg B. \tag{3.21}$$

$$\forall x(A \supset B) \supset (\forall x A \supset \forall x B) \tag{3.22}$$

$$\forall x A \supset A[t/x] \tag{3.23}$$

$$\forall x(A \supset B) \supset (A \supset \forall y B[y/x]) \tag{3.24}$$

(where x does not occur free in A, and $y = x$ or y does not occur free in B)

$$A[t/x] \supset \exists x A \tag{3.25}$$

$$\forall x(A \supset B) \supset (\exists y A[y/x] \supset B) \tag{3.26}$$

(where x does not occur free in B, and $y = x$ or y does not occur free in A)

$$\neg \forall x A \equiv \exists x \neg A \tag{3.27}$$

$$\frac{A \supset B \qquad A}{A}\ \text{Modus Ponens} \tag{3.28}$$

$$\frac{\vdash A \supset B}{\vdash \neg B \supset \neg A}\ \text{Admissible contraposition} \tag{3.29}$$

A first-order HYPE model is a structure $\mathfrak{M} = (S_H, D, V, \circ_H, \perp_H)$, such that:

- $S_H \neq \varnothing$ is a set of states.[7]
- $D \neq \varnothing$ is the domain of individuals.
- $V : S_H \to \mathscr{P}(SoA)$ is a valuation function, where SoA is defined as follows:

Definition 10 The set of states of affairs, SoA, relative to a given domain D and vocabulary, is the set of all tuples of the form

[7] We use S_H to denote the set of HYPE states because we later need to use the symbol S in the semantics of P-HYPE.

$$(P,\, d_1, ..., d_n) \text{ and } (\overline{P},\, d_1, ..., d_n)$$

where P is an n-ary predicate ($n \geqslant 0$) such that $\overline{P}$ is its negation, $P = \overline{\overline{P}}$ and each $d_i \in D\,(1 \leqslant i \leqslant n)$.

- $\circ_H$ and $\perp_H$ are the fusion and incompatibility operators, respectively, such that:

- $\circ_H : S_H \times S_H \to S_H$ is a partial commutative, associative binary function (called *fusion*), such that:

  - Either $s \circ_H s'$ is undefined, or $s \circ_H s'$ is defined (and hence in S) in which case $V(s \circ_H s') \supseteq V(s) \cup V(s')$.
  - $s \circ_H s$ is defined, and $s \circ_H s = s$.

- $\perp_H$ is a binary symmetric relation on S_H (the incompatibility relation), such that:

  - If there is a $(P, d_1, ..., d_n) \in SoA$ such that either $(P, d_1, ..., d_n) \in V(s)$ and $(\overline{P}, d_1, ..., d_n) \in V(s')$ or $(\overline{P}, d_1, ..., d_n) \in V(s)$ and $(P, d_1, ..., d_n) \in V(s')$, then $s \perp_H s'$.
  - If $s \perp_H s'$ and both $s \circ_H s''$ and $s' \circ_H s'''$ are defined, then $s \circ_H s'' \perp_H s' \circ_H s'''$.

$\circ_H$ gives rise to a partial order $\leqslant_H$, such that, for all $s, s' \in S_H$, $s \leqslant_H s'$ iff $s \circ_H s'$ is defined and $s \circ_H s' = s'$. Importantly, truth is monotonic under fusion extension: for all models $\mathfrak{M}$ and states s, if $\mathfrak{M}, s \models \phi$ and $s \circ_H s'$ is defined, then $\mathfrak{M}, s \circ_H s' \models \phi$.

In HYPE, it is assumed that for every s in S_H there is a unique $s^* \in S_H$ (called the *star image of* s), such that:

1. $V(s^*) = \{(\overline{P}, d_1, ..., d_n)$
   $\mid (P, d_1, ..., d_n) \notin V(s)\} \cup \{(P, d_1, ..., d_n) \mid (\overline{P}, d_1, ..., d_n) \notin V(s)\}$

2. $s^{**} = s$.

3. s and s^* are not incompatible with each other: $s \not\perp_H s^*$.

4. s^* is the "largest" state having the previous compatibility property with respect to s: if $s \not\perp_H s'$, then $s' \circ_H s^*$ is defined and $s' \circ_H s^* = s^*$.

If $(P, d_1, ..., d_n) \in V(s)$ and $(\overline{P}, d_1, ..., d_n) \in V(s)$, for some $(P, d_1, ..., d_n)$ in the SoA, then s is called a *glutty* state. If neither $(P, d_1, ..., d_n) \in V(s)$ nor $(\overline{P}, d_1, ..., d_n) \in V(s)$, for some $(P, d_1, ..., d_n)$ in the SoA, then s is called a *gappy state*. We will call s a *non-classical state* if it is either gappy or glutty. A state which is neither gappy nor glutty is called a *classical state*. Where S_c and S_n denote, respectively, the set of classical and non-classical HYPE states, S_H can be defined as $S_H = S_c \uplus S_n$.

Satisfaction of a formula A is defined relative to a model, state and a variable assignment (written: $\mathfrak{M}, s, g \models A$), and the clauses for the logical symbols are as usual, except for $\neg$ and $\supset$, which have a distinctly modal flavour:

$$\mathfrak{M}, s, g \models \neg A \text{ iff } \forall s' \ (\mathfrak{M}, s', g \models A \Rightarrow s \perp_H s') \tag{3.30}$$

$$\mathfrak{M}, s, g \models A \supset B \text{ iff } \forall s' \,(\mathfrak{M}, s', g \models A \text{ and } s \circ_H s' \text{ is defined} \tag{3.31}$$
$$\Rightarrow \mathfrak{M}, s \circ_H s', g \models B)$$

The notions of logical consequence and validity are defined as follows:

Definition 11

Logical consequence:

For any $\Phi, B \subseteq FORM_{HYPE}$: $\Phi \models B$ iff for all HYPE-models $\mathfrak{M}$, for all s in S_H, for all variable assignments σ, if $s, \sigma \models A$, for all $A \in \Phi$, then $s, \sigma \models B$.

'$\Phi \models B$' is read 'B is a *logical consequence* of Φ'.

Validity:

$\models B$ iff for all HYPE-models $\mathfrak{M}$, for all $s \in S_H$, for all variable assignments σ : $s, \sigma \models B$.

'$\models B$' is read 'B is *valid*'.

(Leitgeb 2019) proves the following monotonicity lemma (**Lemma 9**):

(*Monotonicity Lemma*) (3.32)

For all HYPE models $\mathfrak{M}$ and states $s \in S_H$: if $\mathfrak{M}, s \models A$ and $s \circ_H s'$ is defined, then $\mathfrak{M}, s \ \circ_H \ s' \models A$.

Leitgeb remarks that (3.32), the model conditions on $\circ_H$ and the fact that, for all $s, s' \in S_H$, $s \leqslant_H s'$ if and only if $s \circ_H s'$ is defined and $s \circ_H s' = s'$, implies that the clause for $\supset$ can be written as in (3.33). We can then curry (3.33), arriving at (3.34):

$$\mathfrak{M}, s, g \models A \supset B \text{ iff } \forall s' \, (s \leqslant_H s' \wedge \mathfrak{M}, s', g \models A \Rightarrow \mathfrak{M}, s', g \models B) \tag{3.33}$$

$$\mathfrak{M}, s, g \models A \supset B \text{ iff } \forall s' \, (s \leqslant_H s' \Rightarrow (\mathfrak{M}, s', g \models A \Rightarrow \mathfrak{M}, s', g \models B)) \tag{3.34}$$

Rewriting the clause for $\supset$ in this way brings out its underlying similarity to the intuitionistic conditional in the Kripke semantics for Intuitionistic logic. Notice, also, that since $\Rightarrow$ is simply the material conditional in the metalanguage, if there were a symbol $\rightarrow$ in HYPE expressing the material conditional, we could rewrite (3.33) and (3.34), as (3.35), or as in (3.36), respectively:

$$\mathfrak{M}, s, g \models A \supset B \text{ iff } \forall s' \, (s \leqslant_H s' \wedge \mathfrak{M}, s', g \models A \rightarrow B) \tag{3.35}$$

$$\mathfrak{M}, s, g \models A \supset B \text{ iff } \forall s' \, (s \leqslant_H s' \Rightarrow (\mathfrak{M}, s', g \models A \rightarrow B)) \tag{3.36}$$

The reader should note that the partial *Ty2* formulas we will later use in our merger of HYPE and partial *Ty2* all have the form

$$\forall s'(s \leqslant_H s' \wedge A\, s' \rightarrow B\, s) \tag{3.37}$$

We will translate HYPE into *Ty2* in the sense that we will take the definition of the HYPE conditional in the metalanguage and the definition of HYPE negation in the metalanguage and transplant them into the partial *Ty2* setting. However, we are not claiming to be giving a translation or embedding of HYPE into partial *Ty2* in the logician's technical sense. Since a result of (Odintsov and Wansing 2019) shows that HYPE can be embedded into the classical bimodal logic $S4 \times KD!B$ (the fusion of two normal modal logics), we could easily translate HYPE into classical (non-partial) *Ty2* by making use of the standard translation of modal operators into classical logic. But since we are employing partial *Ty2*, the translation of (Odintsov and Wansing 2019) cannot be guaranteed to work in the context of partial *Ty2*.

3.4.1 Two limitations of HYPE in resolving the problems of granularity and logical omniscience

Before discussing the notion of a perspective in the theory of (Asudeh and Giorgolo 2016) and P-HYPE, which combines HYPE with a semantics of perspective, we would like to draw our reader's attention to two limitations of HYPE as it stands as a means of solving the problems of granularity and logical omniscience.

3.4.1.1 HYPE and logical omniscience

HYPE of itself does not offer an obvious way of dealing with the problem of logical omniscience, because the forms of logical omniscience we discussed in 1.3 are valid in HYPE. For example, in HYPE certain formulas are valid in the sense of being true at every state of every HYPE model and logically equivalent formulas are substitutable with one another *salva veritate* (This is proven by (Leitgeb 2019) as **Observation 13**). Nor does HYPE avoid closure under logical omniscience, since whenever $A \supset B$ is true at a HYPE state, and A is true at a HYPE state, then B will be true, by *Modus Ponens*. Likewise, the other forms of logical omniscience we discussed in 1.3 would all hold in HYPE if we introduced a normal doxastic modality operator interpreted as a universal quantifier over doxastically accessible states. We will deal with the apparent limitations of HYPE in resolving the problem of logical omniscience in (7.1), in which we present our solution to the problem. But for now, we ask our reader to ignore the issue until later.

3.4.1.2 HYPE and the notion of a perspective

HYPE as it stands has nothing to say about the perspectives that language users have on expressions of their language. Given that in this chapter and previous chapters we have argued for the importance of perspective in an account of the propositional attitudes, it would be desirable to combine HYPE with perspective relativity.
We will now discuss the perspective relative semantic theory of (Asudeh and Giorgolo 2016), since P-HYPE results from combining their approach to perspective with HYPE.

3.5 AG's semantic theory

Here we discuss (Asudeh and Giorgolo 2016)'s account of perspective relative interpretation with a view to introducing our own perspective relative semantic theory. Since (Asudeh and Giorgolo 2016) introduce their theory in order, *inter alia*, to resolve *Frege's puzzle*, and they include examples involving fictional names, we will discuss their examples. Nevertheless, in line with our remarks in 1.2, we wish to remain neutral as to whether (Asudeh and Giorgolo 2016)'s treatment of proper names is better than alternatives and whether (Asudeh and Giorgolo 2016)'s treatment of fictional names is fully adequate. The fictional names in examples discussed in this section could easily be substituted with non-fictional names without changing our exposition of their theory.

The general picture of AG's theory is that proper names can be interpreted from various perspectives; for example, a proper name can be interpreted from the perspective of the grammatical subject of a sentence in which the proper name occurs, or from the perspective of the utterer of the sentence in which the proper name occurs. They argue that when proper names are interpreted from a given perspective they denote (private) mental representations, which they call *perspectives*. To each agent there corresponds a private mental lexicon—called her *Perspective* (capital 'P') or *mental model* —-consisting of the set of perspectives that that agent associates with terms of her language.

The picture just outlined is motivated by sentences such as (10b), uttered in the scenario (10a):

(10) a. *Scenario:* Mary Jane does not know Peter Parker is Spiderman and she loves the man she calls 'Peter Parker'. A speaker who knows

or is ‘enlightened’ (Zimmermann 2005)[8] about Peter Parker’s secret identity utters (10b).

b. Mary Jane loves Spiderman.

c. Mary Jane loves Peter Parker, but she doesn’t love Spiderman.

(11) Mary Jane punched Spiderman.

According to AG, there is a sense in which sentences like (10b) are true from the perspective of an enlightened utterer (i.e, an utterer who knows that Spiderman is none other than Peter Parker), but false from Mary Jane’s perspective and there is a sense in which (10c) is true from Mary Jane’s perspective but false from the perspective of an enlightened utterer. AG contrast (10b) and (10c) with (11), which they claim does not exhibit this ambiguity depending on whether we consider the perspective of Mary Jane or an enlightened perspective. They conclude that certain verbs, such as *love*, are perspective sensitive, whereas others, such as *punch* are not. In their account, AG adopt as base types e, t and p, where p is the type of perspective indices, and proper names can have type $p \to e$. Consequently, the interpretation of certain names is made perspective relative, so that Mary Jane can associate a distinct denotation with the names ‘Spiderman’ and ‘Peter Parker’. Perspective sensitive transitive verbs, such as *love*, are sensitive to the perspective index supplied to their objects, and are assigned type $(p \to e) \to e \to t$. Perspective insensitive transitive verbs, such as *punch*, are assigned type $e \to e \to t$.

AG think their theory can be used to solve *Frege’s puzzle* (discussed in (Salmon 1986)) if names embedded under *believe* are assigned a perspective relative inter-

[8]The distinction between enlightened and unenlightened speakers relates to (Forbes 1999)’s use of these terms. Our distinction was taken from (Zimmermann 2005) and (Asudeh and Giorgolo 2016), who explicitly reference (Forbes 1999). So as far as we know our distinction is the same as in (Forbes 1999).

pretation. To this end, they propose that (12) has a reading which is true if John associates a different perspective with ‘Hesperus’ from the perspective he associates with ‘Phosphorus’, and it has a reading which is false; namely, the reading on which ‘Hesperus’ and ‘Phosphorus’ are interpreted relative to the perspective of an enlightened person who utters (12):

(12) John thinks that Hesperus isn’t Phosphorus.

AG suggest that perspective relativity is sometimes constrained depending on the verbs used in a given sentence. Consider a cases like (13a) and (13b) :

(13) a. Mary Jane saw the morning star.
b. Mary saw the morning star, but she didn’t see the evening star.

According to AG,[9] (13a) can only receive an interpretation with respect to the perspective index of the individual who utters it and that this is because the verb *saw* has different lexical properties from the verb *love*. If the utterer of (13a) is enlightened as to the identity of the morning and evening star, ‘the morning star’ can only receive one interpretation in this case: it denotes the planet Venus.[10] The difference between *saw* and *love* is brought out by cases like (13b), which are contradictory if we are aware that the morning star is the evening star and we are aware that the

[9]AG do not give this example, but we assume that this is what they would say in this case, since they make similar remarks about the verb *punch*: that it does not induce a perspective relative interpretation.

[10]They do not discuss the case in which (13a) or (13b) are uttered by a non-enlightened speaker, but if we consider non-enlightened speakers we would have to say that either (13b) could receive a true reading from the perspective of an un-enlightened utterer, or we would have to ensure its falsity in some other way. Below we introduce an enlightened perspective, relative to which (13b) would be false.

utterer of (13b) is aware of this, whilst similar sentences like (10c) are not contradictory, even when we are aware that Spiderman is Peter Parker and we are aware that the utterer of (10c) is aware of this. The difference between verbs like *saw* and *love* will be ignored from now on, but AG propose a different typing for the verbs, and the lexical entries they give for the verbs, together with the way they render certain sentences involving the verbs, ensure that the difference between *saw* and *love* can be captured.

Let us consider the details of their account. Let '$\Diamond$' be the endofunctor of the reader monad defined on the type p, the type of perspective indices. To every agent in a discourse there corresponds a perspective index, which we can think of as a sort of tag. According to AG, certain names are perspective relative, in the sense that they vary in their denotation relative to a perspective index, and these names are type $\Diamond e$. Names which are perspective relative are parasitic on certain names of type e which we call *default names*. Default names do not vary in perspective and have default status (see (Zimmermann 2005) for more discussion), in the following sense: if someone becomes *enlightened*, learning that two proper names refer to the same individual, then they will, by and large[11], tend to use one and not the other proper name. To give one example, an utterance of 'I've never been to Constantinople' by an utterer who knows that Constantinople is Istanbul, generally sounds infelicitous or semantically anomalous, which is consistent with the assumption that an enlightened individual tends to prefer to use the name 'Istanbul' in favour of 'Constantinople'. Likewise, someone who learns that Peter Parker and Spiderman are the same individual, would tend to prefer 'Spiderman' to 'Peter Parker', unless

[11](Zimmermann 2005) discusses some of the exceptions to this claim. Zimmermann gives examples in which we engage in speech-acts with a counterfactual flavour, acts in which we utter a name like 'Spiderman' as if we were an unenlightened person who did not know that Spiderman is Peter Parker.

the individual is trying to convey information about another individual who is not enlightened about the identity of Peter Parker and Spiderman.

Consider the lexicon (in Table 3.1 below) which represents part of the mental model of an enlightened speaker, whose perspective index is $\sigma : p$. σ in Table 3.1 is used as a subscript, and used in this way it is not a lambda term (nor is it the type of a lambda term), but simply a way to index constants and variables to the private mental lexicon of the individual who bears perspective index σ. When σ is used as a lambda term, we will write $\sigma : p$ to disambiguate:[12]

[12] We presume this is the function of subscripting such constants, though AG are not clear about this. Alternatively, the subscript could indicate an argument of type p, which is fed to the constants mj, pp, sm, etcetera.

WORD	DENOTATION	TYPE
Mary Jane	mj_σ	e
Peter Parker	pp_σ	e
believe	$\lambda c.\lambda s.\, B\,(c\,(\kappa_{e\to p}\,s))\,s$	$\Diamond t \to e \to t$
love	$\lambda c.\lambda s.\, love\,(c\,(\kappa_{e\to p}\,s))\,s$	$\Diamond e \to e \to t$
Spider-Man	$\lambda i. \begin{cases} sm_\sigma\, i & \text{if } i = \kappa\, mj_\sigma \\ pp_\sigma & \text{if } i = (\sigma : p) \end{cases}$	$\Diamond e$

Table 3.1: Lexicon of σ, the enlightened speaker

The default names in Table 3.1 are mj_σ and pp_σ. Notice the $\kappa_{e\to p}$ operator in the denotation of *believe* and *love*. Given a model $\mathfrak{M}$ and assignment g, $[\![\kappa_{e\to p}]\!]^{\mathfrak{M},g} = \boldsymbol{\kappa}$, where $\boldsymbol{\kappa} : D_{e\to p}$ is a function which takes an individual and returns a perspective index correlated with that individual. It has the effect that, if a lambda term is fed to the rendering of *believe* or *love* in Table 3.1, then it will be interpreted with respect to the perspective index of the grammatical subject of a sentence whose main verb is *believe* or *love*.

AG stipulate that the speaker's perspective is the one fed to an expression of the form $a_{\Diamond\alpha}\bigstar f_{\alpha\rightarrow\Diamond\beta}$. It is unclear how they aim to enforce this stipulation, but the stipulation is not arbitrary: the stipulation allows us to give a a particular logical form to sentences which are interpreted from the perspective of their utterers, and this could be motivated by appealing to the empirical claim (Lasersohn 2017) that, in the default case of perspective-relative interpretation, sentences or expressions which are perspective relative are interpreted relative to the perspective of the utterer of them. Consequently, terms that scope above $\bigstar$, as in (14) are interpreted relative to the default perspective of the utterer, so that the lambda term rendering *Spiderman* in (14) is interpreted relative to perspective index of the speaker, who they assume to be enlightened. By contrast, terms which are arguments of $love$ are caught by $\kappa\ s$, as in (15) and so are interpreted with respect to the perspective index of second argument of $love$ (i.e, the argument representing the lover):

(14) $$\left(\lambda i.\begin{cases} sm_\sigma\, i & \text{if } i = \kappa\, mj_\sigma : p \\ pp_\sigma & \text{if } i = (\sigma : p) \end{cases} \bigstar \lambda z.\eta\,(love_{e\rightarrow e\rightarrow t}\; z\; mj_\sigma)\right)\;\sigma$$

(15) $$(\lambda c_{\Diamond e}.\lambda s_e.\, love_{e\rightarrow e\rightarrow t}\,(c\,(\kappa\, s))\; s)\left(\lambda i.\begin{cases} sm_\sigma\, i & \text{if } i = \kappa\, mj_\sigma \\ pp_\sigma & \text{if } i = (\sigma : p) \end{cases} (\kappa\, mj_\sigma)\right)\;mj_\sigma$$

(14) β reduces to (16), sm is interpreted with respect to σ, and (15) β reduces to (17), in which sm is caught by $\kappa\ mj$:

(16) $$love\,(\,sm_\sigma\,(\kappa\,\sigma)\,)\;\; mj_\sigma$$

(17) $$love_{e\rightarrow e\rightarrow t}\,(sm_\sigma\,(\kappa\, mj_\sigma))\;\; mj_\sigma$$

3.6 A limitation of AG's semantic theory

Consider a false utterance of *Mary Jane loves Spiderman* by an enlightened utterer, in which *Spiderman* seems to be interpreted from Mary Jane's perspective. AG argue that, in such an utterance, *Spiderman* is not, in fact, utterered from Mary Jane's perspective; rather, the name *Spiderman* is interpreted from the utterer's perspective of what Mary Jane's perspective on the name is. This makes sense if we consider that, for AG, a name denotes some kind of private mental representation, to which the utterer can only have indirect access to.

Unfortunately, this 'solipsistic' aspect of their account is not reflected in their renderings of natural language sentences. Consider (17). In (17), *Spiderman* is interpreted with respect to Mary Jane's perspective index, $\kappa\, mj$. But nothing in the formula (17) requires that $(sm\,(\kappa\, mj))$ captures the utterer's perspective of what Mary Jane's perspective on the name is.

One possibility would be to replace the $\kappa : e \rightarrow p$ function with a function $\kappa^1 : e \rightarrow e \rightarrow p$, where, given a model $\mathfrak{M}$ and assignment g, $[\![\kappa^1\ u\ mj]\!]^{\mathfrak{M},g}$ is the perspective index which u associates with Mary Jane; that is, what u considers Mary Jane's perspective index to be. If we keep both κ and κ^1, we could then require that, for a given model $\mathfrak{M}$ and assignment g, (3.38) holds:

$$[\![\kappa^1\ u\ mj]\!]^{\mathfrak{M},g} \neq [\![\kappa\ mj]\!]^{\mathfrak{M},g} \tag{3.38}$$

(3.38) would entail that u can never access Mary Jane's perspective index, so that u cannot ever access the interpretations she attaches to expressions. Alternatively, if we drop κ, then we could require that, for a given model $\mathfrak{M}$ and assignment g, (3.39) holds:

$$[\![\kappa^1\ u\ mj]\!]^{\mathfrak{M},g} \neq [\![\kappa^1\ mj\ mj]\!]^{\mathfrak{M},g} \tag{3.39}$$

(3.39) would entail that u can never access what Mary Jane considers her own perspective index to be.

An account along either of these lines would somehow have to feed the argument u to κ in the course of compositional derivations and it is not clear from the structure of sentences such as *Mary Jane loves Spiderman* how reference to u would be incorporated. Plus, if we admit the κ^1 function, why not admit a $\kappa^2 : e \rightarrow e \rightarrow e \rightarrow p$ function where $[\![\kappa^2\ u\ v\ mj]\!]^{\mathfrak{M},g}$, is the perspective index which u considers v to associate with Mary Jane, or a κ^n function, taking n type e arguments, etcetera? There seems no reason, philosophically at least, not to generalise in this way. The consequence, however, would be that we would need to think how all such type e arguments could be compositionally fed to κ^n.

We think that incorporating states into AG's semantic theory can help solve the problem in a more elegant and compositionally satisfying manner. Introduce a (state-sensitive) function $\pi_{p\rightarrow s\rightarrow s\rightarrow t}$. For some perspective index i and state s, $\pi_{p\rightarrow s\rightarrow s\rightarrow t}\ i\ s$ returns the characteristic function of a set of states, where we call this set of states, the *p-set* (*perspective set*) *of the agent* (the agent with the perspective index i) *at the state* s. The perspective set of an agent is supposed to capture the perspective or mental model of that agent. It is a set of states which, for all that agent knows, might be the state she inhabits.

Further, suppose that κ is now a function of type $e \rightarrow s \rightarrow p$, so that it takes both a state and an individual and returns a perspective index. Then, if we want to capture that Mary Jane's perspective index itself is not accessible to the utterer, but only a representation of Mary Jane's perspective index in the p-set of the utterer, then we can bind the type s argument in κ, as in (3.40):

$$\forall s'(\pi_{p\to s\to s\to t}\ (\kappa_{e\to s\to p}\ u\ s)\ s\ s' \qquad (3.40)$$
$$\to love_{e\to e\to s\to t}\ (sm_{\Diamond e}\ (\kappa\, mj\ s'))\ mj_e\ s')$$

If we can ensure that, for any s', $(sm\ \ (\kappa\, mj\ s'))$ is private to the utterer's p-set, then we will have ensured that the binding of s' implements AG's idea that *Spiderman* is interpreted from the utterer's perspective of what Mary Jane's perspective on the name is. One way of thinking of $\kappa : e \to s \to p$ is that it is like a counterpart function (Lewis 1968), except for perspective indices. According to counterpart theory, every object inhabits exactly one state, and objects have counterparts at other states. Rather than having a counterpart function which applies to a perspective relative inhabits of $D_{p\to s\to e}$ (or to plain individuals in D_e) and returns a counterpart of it, we can generate different counterpart perspective indices at different states. This captures the idea that, what perspective one has depends on the state one is in.

In the next chapter (4.2), we will present P-HYPE. P-HYPE combines HYPE with the perspective relative semantic theory of AG and builds on the semantic theory of AG by allowing predicates to be perspective relative in addition to names. In 4.4 we aim to construct a class of models which captures the idea that certain expressions occurring in a sentence only really ever receive a denotation in the mental model of the utterer of that sentence, without entering into the question of whether, ultimately, this is defensible on linguistic grounds.

3.7 Conclusion

Linguists have spoken of 'linguistic side effects' and enumerated various types of linguistic side effect. But missing from the list have been the sorts of fine-grained semantic values needed to solve the problems of granularity and logical omniscience. It is illuminating to view these, too, as side effects, just as focus and intensionality are according to (Shan 2007). Just like typical examples of linguistic side effects, fine-grained semantic phenomena in natural language have been held to constitute problems for the compositional treatment of natural language, and to upset notions of equality based on the substitution of extensions for extensions.

One question is therefore whether we can understand perspective-relative interpretation as a kind of linguistic side-effect. The semantic theory of (Asudeh and Giorgolo 2016) shows that we can. The problem is, their theory does not incorporate intensionality, and (Asudeh and Giorgolo 2016) do not formally capture some key aspects of their view of perspective-relative interpretation, according to which the denotations of natural language expressions are private to each individual. The question therefore remains of how to combine intensionality with perspective relativity and how to capture their view of perspective-relative interpretation. That is the subject of the next chapter.

Chapter 4

P-HYPE: models, frames and lexical entries

4.1 Introduction

In this chapter, we introduce a compositional fine-grained semantic theory, P-HYPE. P-HYPE is based on the logic HYPE (Leitgeb 2019) and on the semantics of perspective in (Asudeh and Giorgolo 2016). The notion of a perspective is not itself part of the logic HYPE (hence the name 'P' in P-HYPE), and is taken and developed from the work of (Asudeh and Giorgolo 2016). But whereas (Asudeh and Giorgolo 2016) chiefly use this notion as a means of interpreting proper names as entities in $D_{p \to e}$, where p is the type of perspectives, following suggestions in their paper, we also treat predicates as perspective relative entities.

P-HYPE utilises applicatives (structures from category theory), and we later augment it with monads (structures from category theory). So far as we know, P-HYPE is the first semantic theory which uses monads as a tool of compositional semantics in order to provide a fine-grained semantics for predicates. The fact that

P-HYPE provides a compositional analysis of sentences at the subsentential level is already a point in its favour in comparison to some alternative fine-grained semantic theories in the philosophical literature (Jago 2014a), (Fine 2017), (Yablo 2014). These theories do not present a systematic method of pursuing a compositional analysis of sentences at the subsentential level, or suggest what syntax-semantics interface they would employ; ultimately, however, it is incumbent on theorists who aim to resolve the problem of granularity in natural language to pinpoint how their formal theories fit into a systematic and formal account of the syntax and semantics of natural language.

In 4.2 we introduce frames and models, which are based on partial *Ty2* frames and models. In 4.3 we introduce the main ideas behind the lexical entries in P-HYPE, and present lexical entries for *love*, *prove*, *believe* and various predicates and proper names. In 4.4 we isolate a class of P-HYPE models which capture some ideas behind AG's semantic theory which they do not themselves formalise.

4.2 P-HYPE: frames and models

We are now ready to introduce P-HYPE. P-HYPE models are partial *Ty2* models (which are described in A.3).

Definition 12 (Types) Let $e, t, s_c, s_n, p, time$ be six distinct, primitive objects and let $TYPE$ be the smallest set such that:

- $e, t, s_c, s_n, p, time \in TYPE$
- $\alpha, \beta \in TYPE$ implies $\alpha \rightarrow \beta \in TYPE$
- $\alpha, \beta \in TYPE$ implies $\alpha \times \beta \in TYPE$

- $\alpha, \beta \in TYPE$ implies $\alpha + \beta \in TYPE$

Note that, in addition to the type of functions from one type to another, we also admit pair types $\alpha \times \beta$ and the disjoint union type $\alpha + \beta$. Type e, t, s_c, s_n, p and $time$ are the type of individuals, truth values, classical HYPE states, non-classical HYPE states, perspective indices and times respectively. The type s will be used extensively to abbreviate the type $(s_c + s_n) \times time$.

We now introduce the language of P-HYPE, which includes the symbols $\circ_{s \to s \to s}$, $\leqslant_{s \to s \to t}$, $\perp_{s \to s \to t}$, which (as we clarify later) are related to, but not to be confused with, the HYPE symbols $\circ_H, \leqslant_H, \perp_H$.

Definition 13 (Language of P-HYPE) The language of P-HYPE is the language of partial *Ty2*, save for the addition of the following distinguished non-logical constants:

- For every $\alpha, \beta \in TYPE$ a pair-forming operator $(\cdot) \in CON_{\alpha \to \beta \to \alpha \times \beta}$.
- For every $\alpha_1, ..., \alpha_n \in TYPE$, each $n \in \mathbb{N}$ and every i such that $1 \leqslant i \leqslant n$, a projection function $\pi_i^n \in CON_{(\alpha_1 \times \alpha_2 \times ... \times \alpha_n) \to \alpha_i}$ which selects the i-th element of an n-ary pair. The superscript n will often be admitted, which should not cause confusion.
- For every $\alpha, \beta \in TYPE$, a function $inl \in CON_{\alpha \to \alpha + \beta}$
- For every $\alpha, \beta \in TYPE$, a function $inr \in CON_{\alpha \to \beta + \alpha}$
- For every $\alpha, \beta \in TYPE$, a function $cases \in CON_{(\alpha+\beta) \to (\alpha \to \gamma) \to (\beta \to \gamma) \to \gamma}$, allows us to eliminate a term of disjoint union type.
- $classical \in CON_{s \to t}$ will select classical HYPE states.

- $DOX \in CON_{e \to s \to s \to t}$ is a doxastic modal operator.
- $PROV \in CON_{e \to s \to s \to t}$, is a provability modal operator.
- E_p is a special designated constant which will denote what we call the *enlightened perspective index*.
- u_e is a constant denoting the utterer.
- $\circ_{s \to s \to s}$.
- $\leqslant_{s \to s \to t}$
- $\pi_{p \to s \to s \to t}$ is a function which can be thought of as a kind of accessibility relation labelled by a perspective index. When π is defined on $\kappa\ x\ s$ for some $x : e$ and $s : s$ we call the set of states associated with $\kappa\ x\ s$ at s the *perspective set* or *p*-set of x at s. The perspective set is supposed to represent the *perspective* or *mental model* of an agent, in the sense discussed in (3.5).
- $\kappa_{e \to s \to p}$
- $\perp_{s \to s \to t}$
- $C_{\mathbf{p}} : s \to p \to t$
  $C_{\mathbf{p}} \in CON_{s \to p \to t}$ is a function from states to the characteristic function of a set of perspective indices. The idea behind this function is that, it returns true when applied to a state and a perspective index, if that perspective index is contextually available, or salient. This function will allow us to constrain which perspective indices can be utilised in a given context.

$\circ_{s\to s\to s}, \leqslant_{s\to s\to t}, \perp_{s\to s\to t}$ are used as infix operators and we write (A, B) instead of $(\cdot)\ A\ B$.

The constant u_e is probably inadequate for some purposes, given that there may be many different utterers in different states and even relative to the same state. In this thesis we will generally ignore this issue as a convenient simplification.

Note that, in the language of P-HYPE, we can define HYPE negation and the HYPE conditional directly, via universal quantification over states:

$$\neg_H := \lambda p_{s\to t}, s_s.\ \forall s'(p\, s' \to s \perp s') \tag{4.1}$$

$$\supset := \lambda p_{s\to t}, q_{s\to t}, s_s.\ \forall s'(s \leqslant s' \wedge p\, s' \to q\, s') \tag{4.2}$$

To form a P-HYPE frame we need

- A non-empty set $S = S_c \uplus S_n$ of classical (HYPE) states and a (possibly empty) set of non-classical (HYPE) states S_n.
- The domain D of individuals of a HYPE model.
- A non-empty set P of perspective indices.
- A non-empty set T of times, which are linearly ordered by the usual temporal precedence relation $\leqslant_T$.

Definition 14 An *admissible P-HYPE* frame $\mathcal{F}$ is a partial general *Ty2* frame meeting the following two conditions:

1. $\mathcal{F}$ has the following base domains:

    - <u>*Base domains*</u>

$D_e = D \cup \{\#\}$ (domain of individuals)

$D_{time} = T \cup \{\#\}$ (domain of times)

$D_{s_c} = S_c \cup \{\#\}$ (domain of classical HYPE states)

$D_{s_n} = S_n \cup \{\#\}$ (domain of non-classical HYPE states)

$D_s = ((D_{s_c} \uplus D_{s_n}) \times D_{time}) \cup \{\#\}$ (domain of P-HYPE states)

$D_p = P \cup \{\#\}$ (domain of perspective indices)

$D_t = \{0, 1, \#\}$ (domain of truth-values)

- Besides the function domain $D_{\alpha \to \beta}$, for any $\alpha, \beta \in TYPE$ (as defined in the appendix, section A.5, **Definition 26**), the following 'complex' domains:

- *Complex domains*

  $D_{\alpha \times \beta} = \{(a, b) \mid a \in D_\alpha, b \in D_\beta$ and, if $a' \in D_\alpha$ and $b' \in D_\beta$ then $(a, b) \leqslant_{PD} (a', b')$ iff $a \leqslant_{PD} a'$ and $b \leqslant_{PD} b'\}$

  $D_{\alpha + \beta} = \{(x, i) \mid$ either $x \in D_\alpha$ and $i = 0,$ or $x \in D_\beta$ and $i = 1,$ and, if $x' \in D_\alpha$ and $i' \in \{0, 1\}$ then $(x, i) \leqslant_{PD} (x', i')$ iff $x \leqslant_{PD} x'$ and $i \leqslant_{PD} i'\}$

2. $\mathcal{F}$ is the union of a partial general *Ty2* frame with the following distinguished functions, which are stipulated to return $\#$ whenever they are applied to arguments whose subformulas contain $\#$:

   - For every $i \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $1 \leqslant i \leqslant n$ and for every $\alpha_1, ..., \alpha_n \in TYPE$, a function $proj_i^n \in D_{(\alpha_1 \times \alpha_2, ..., \times \alpha_n) \to \alpha_i}$, the i-th projection, which extracts the i-th element from an n-tuple. We often omit the superscript n.

- $[classical] \in (S_c \cup S_n) \to D_t$ where $[classical]\ s = 1$ iff $s \in S_c$.
- $\blacksquare_{DOX} \in D_e \to (S_c \cup S_n) \to (S_c \cup S_n) \to D_t$ is a doxastic modality.
- $\blacksquare_{PROV} \in D_e \to (S_c \cup S_n) \to (S_c \cup S_n) \to D_t$ is a provability modality.
- $\circ_H \in (S_c \cup S_n) \to (S_c \cup S_n) \to (S_c \cup S_n)$, is the partial binary function of fusion defined on the set of states of a first-order HYPE model.
- $\leqslant_H \in (S_c \cup S_n) \to (S_c \cup S_n) \to D_t$, is the partial order between states of a first-order HYPE model.
- $[\kappa] \in D_e \to (S_c \cup S_n) \to D_p$, if defined, is a function which generates the perspective index of an individual at a given state.
- $[\pi] \in D_p \to (S_c \cup S_n) \to (S_c \cup S_n) \to D_t$
- $[E] \in P$ is a distinguished perspective index called the *enlightened perspective (index)*, such that:

  (i) for all $s, s' \in (S_c \cup S_n)$: $[\pi]\,[E]\,s\,s' = 1$ iff $s = s'$ and $s \in S_c$.

  (ii) $[E]$ is not in the range of $[\kappa]$ at any $s \in (S_c \cup S_n)$ (i.e $\neg\exists x \in D_e, \exists s \in (S_c \cup S_n)\ ([\kappa]\,x\,s = [E])$.

  (iii) For all $s, s', s'' \in (S_c \cup S_n)$, $x \in D_e$ and $\Delta \in \{\blacksquare_{DOX}, \blacksquare_{PROV}\}$: If $([\pi]\,[E]\,s\,s' = 1$ and $\Delta\,x\,s'\,s'' = 1$, then $s, s', s'' \in S_c$.

- $\perp_H \in (S_c \cup S_n) \to (S_c \cup S_n) \to D_t$ is the incompatibility relation between the states of a first-order HYPE model.

- $[C_{\mathbf{p}}] \in (S_c \cup S_n) \to D_p \to D_t$ is a function associating the characteristic function of a set of perspective indices with a state; intuitively, the set of perspective indices which are relevant at that state. $\forall s \in (S_c \cup S_n) : ([C_{\mathbf{p}}]\, s\, [E] = 1$ and $C_{\mathbf{p}}\, s\, ([\kappa]\, u\, s) = 1)$.

With the notion of an admissible P-HYPE frame, we can define models:

Definition 15 An admissible *P-HYPE* model is a partial general *Ty2* model $\mathfrak{M} = (\mathcal{F}, I)$ based on an admissible P-HYPE frame $\mathcal{F}$ such that:

- For every $\alpha, \beta \in TYPE$, $A \in D_\alpha$ and $B \in D_\beta$: If $(\cdot) \in CON_{\alpha \to \beta \to (\alpha \times \beta)}$, then $[\![(\cdot)]\!]^{\mathfrak{M},g}\, A\, B = (A,\ B) \in D_{\alpha \times \beta}$.

- For every $\alpha_1, ..., \alpha_n \in TYPE$, every $n \in \mathbb{N}$ and every i such that $1 \leqslant i \leqslant n$: if $\pi_i^n \in CON_{(\alpha_1 \times \alpha_2 \times, ..., \times \alpha_n) \to \alpha_i}$, then $[\![\pi_i^n]\!]^{\mathfrak{M},g} = proj_i^n$

- For every $\alpha, \beta \in TYPE$ and $A \in D_\alpha$: if $inl \in CON_{\alpha \to \alpha + \beta}$ and $inr \in CON_{\alpha \to \beta + \alpha}$, then $[\![inl]\!]^{\mathfrak{M},g}\ A = (A,\ 0) \in D_{\alpha + \beta}$ and $[\![inr]\!]^{\mathfrak{M},g}\ A = (A,\ 1) \in D_{\beta + \alpha}$.

- For every $\alpha, \beta, \gamma \in TYPE$, $p \in D_{\alpha + \beta}$, $f \in D_{\alpha \to \gamma}$ and $g \in D_{\beta \to \gamma}$: if $cases \in CON_{(\alpha + \beta) \to (\alpha \to \gamma) \to (\beta \to \gamma) \to \gamma}$, and $p = (a, 0)$, then $[\![cases]\!]^{\mathfrak{M},g}\ p\ f\ g = f\ a$; otherwise $p = (b, 1)$ and $[\![cases]\!]^{\mathfrak{M},g}\ p\ f\ g = g\ b$.

- For any $s \in D_s : [\![classical]\!]^{\mathfrak{M},g}\ s = [classical]\ (proj_1\ (proj_1\ s))$

- For any $x \in D_e$ and $t, t' \in D_s$: $[\![DOX]\!]^{\mathfrak{M},g}\ x\ t\ t' = \blacksquare_{DOX}\ x\ (proj_1\ (proj_1\ t))\ (proj_1\ (proj_1\ t'))$

- For any $x \in D_e$ and $t, t' \in D_s$:
$[\![PROV]\!]^{\mathfrak{M},g}\ x\ t\ t' = \blacksquare_{PROV}\ x\ (proj_1\ (proj_1\ t))\ (proj_1\ (proj_1\ t'))$

- For any $((s,i),t),\ ((s',j),t') \in D_s$:

  (a) If (i) $i = j = 0$, (ii) $s \circ_H s' = \#$ or (iii) $\#$ is a subformula of either $((s,i),t)$ or $((s',j),t')$, then $[\![\circ_{s\to s\to s}]\!]^{\mathfrak{M},g}((s,i),t)\ ((s',j),t') = \#$

  (b) If $s \circ_H s' \in S_c$, then
  $[\![\circ_{s\to s\to s}]\!]^{\mathfrak{M},g}((s,i),t)\ ((s',j),t') = ((s \circ_H s', 0), max\{t,t'\})$

  (c) If $s \circ_H s' \in S_n$ then
  $[\![\circ_{s\to s\to s}]\!]^{\mathfrak{M},g}((s,i),t)\ ((s',j),t') = ((s \circ_H s', 1), max\{t,t'\})$

- For any $((s,i),t),\ ((s',j),t') \neq \# \in D_s$:

  (a) if $[\![\circ_{s\to s\to s}]\!]^{\mathfrak{M},g}\ ((s,i),t)\ ((s',j),t') = \#$, or $\#$ is a subformula of either $((s,i),t)$ or $((s',j),t')$, then
  $[\![\leqslant_{s\to s\to t}]\!]^{\mathfrak{M},g}((s,i),t)\ ((s',j),t') = \#$

  (b) Otherwise, $[\![\leqslant_{s\to s\to t}]\!]^{\mathfrak{M},g}((s,i),t)\ ((s',j),t') = 1$ iff
  $[\![\circ_{s\to s\to s}]\!]^{\mathfrak{M},g}\ ((s,i),t)\ ((s',j),t') = ((s',j),t')$

- For any $i \in D_p, s, s' \in D_s$:
$[\![\pi_{p\to s\to s\to t}]\!]^{\mathfrak{M},g}\ i\ s\ s' = [\pi]\ i\ (proj_1\ (proj_1\ s))\ (proj_1\ (proj_1\ s'))$

- For any $x \in D_e, s \in D_s$: $[\![\kappa_{e\to s\to p}]\!]^{\mathfrak{M},g} x\ s = [\kappa]\ x\ (proj_1\ (proj_1\ s))$

- $[\![E]\!]^{\mathfrak{M},g} = [E]$

- For any $s, s' \in D_s$:
$[\![\bot_{s\to s\to t}]\!]^{\mathfrak{M},g}\ s\ s' = \ \bot_H\ (proj_1\ (proj_1\ s))\ (proj_1\ (proj_1\ s'))$

- For all $s \in D_s, i \in D_p$:
  $[\![C_{\mathbf{p}} : s \to p \to t]\!]^{\mathfrak{M},g} s\ i = \ [C_{\mathbf{p}}]\ (proj_1\ (proj_1\, s))\ \ i$

Note that, by the conditions we placed on the interpretation of π and E above (specifically, conditions (i) and (iii)), the truth of (4.3) (where $\chi \in \{DOX, PROV\}$), requires that for every classical state $s^1 \in D_{s_c}$, if $s^1 = s$ and $s^1 \leqslant s^1$, then in all of the doxastically reachable states s'' of x via s^1, $A\ s''$ is true. As a consequence, (4.4) and (4.3) are logically equivalent:

$$\lambda s.\ \forall s'((s \leqslant s' \wedge\ \pi\, E\, s\, s') \to \forall s''(\chi_{esst}\, x\ s'\, s'' \to A_{pst}\, E\, s'')) \tag{4.3}$$

$$\lambda s. \forall s''(\chi_{esst}\, x\, s\, s'' \to A_{pst}\, E\, s'') \tag{4.4}$$

Thus, conditions (i) and (iii) governing the interpretation of E and its interaction with π and $\Delta \in \{\blacksquare_{DOX}, \blacksquare_{PROV}\}$, ensure that, if we supply the enlightened perspective index to certain functions, we arrive at the interpretation of *believe* and *prove* on which they are simple Hintikkian universal quantifiers over accessible classical states. The enlightened perspective index thus preserves the ordinary intensional interpretation of sentences involving *believe* and *prove*.

4.3 Key P-HYPE lexical entries

We will now discuss in four subsections some key aspects of the lexical entries which will be used in subsequent chapters in compositional derivations. These are

- The lexical entries of *Love*, *believe* and *prove*
- Proper names in P-HYPE
- Predicates in P-HYPE

- Assumptions about plurality

Here we will concentrate on technical issues, and in the next chapter we will explain how the lexical entries will ultimately be used to solve the problem of granularity.

4.3.1 *Love*, *believe* and *prove*

In AG's account, perspective-relative proper names have type $p \rightarrow e$ and perspective-relative predicates such as *believe* have type $(p \rightarrow t) \rightarrow e \rightarrow t$. For technical reasons we will describe in this section, we have decided to adopt our more complex typing for proper names and predicates: proper names will have type $\blacklozenge\Diamond e$ and n-ary natural language predicates of individuals will have type $\blacklozenge\Diamond\alpha_1 \rightarrow ... \rightarrow \blacklozenge\Diamond\alpha_n \rightarrow \blacklozenge\Diamond t$, where each $\alpha_i = e$, for each i such that $1 \leqslant i \leqslant n$. Consequently, sentences will be of type $\blacklozenge\Diamond t$.

We explained previously that, our use of P-HYPE will be based on partial *Ty2* formulas of the form (4.5), which by definition of $\supset$ would be semantically equivalent to formulas of the form (4.6) if we were using (non-partial) *Ty2*:

$$\forall s'(s \leqslant s' \wedge A\ s' \rightarrow B\ s') \quad (4.5)$$

$$A \supset B \quad (4.6)$$

By ensuring that sentences are rendered as formulas of the form (4.5), we ensure that our merger of HYPE with partial *Ty2* stays close to HYPE, as formulas of the form (4.5) simply translate the truth conditions of $\supset$ into the object language of partial *Ty2*.

Given we want formulas to have the form in (4.5) and to incorporate perspective indices and states, an initial proposal would be to render *love*, *believe* and *prove* as follows, where $C_{\mathbf{p}}$ is used to filter out perspective indices which are not contextually available:

$$\lambda y_{p\to s\to e}, x_e, i_p, i'_p, s'. \begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\, i\, s'\, s'' & \text{if } C_{\mathbf{p}}\, s\, i = \top \\ \to love\,(y\, i'\, s'')\, x\, i'\, s'') & \text{and } C_{\mathbf{p}}\, s\, i' = \top \\ \# & \text{otherwise} \end{cases} \tag{4.7}$$

$$\lambda q_{p\to s\to t}, x_e, i_p, s'_s. \begin{cases} \forall s''[\,[s' \leqslant s'' \wedge \pi\ i\ s'\ s''] & \text{if } C_{\mathbf{p}}\, s\ i = \top \\ \to \forall s'''[DOX\ x\ s''\ s''' \to & \\ q\,(\kappa\, x\, s''')\, s'''\,]\,] & \\ & \\ \# & \text{otherwise} \end{cases} \tag{4.8}$$

$$\lambda q_{p\to s\to t}, x_e, i_p, s'_s. \begin{cases} \forall s''[\,[s' \leqslant s'' \wedge \pi\ i\ s'\ s''] & \text{if } C_{\mathbf{p}}\, s\ i = \top \\ \to \forall s'''[PROV\ x\ s''\ s''' \to & \\ q\,(\kappa\, x\, s''')\, s'''\,]\,] & \\ & \\ \# & \text{otherwise} \end{cases} \tag{4.9}$$

In AG's account, the object of *love* is interpreted with respect to what the utterer considers to be the perspective of the grammatical subject. π in our case serves to delimit the states considered to those that are someone's p-set, so that the complement of *love* (and the complement clauses of *believe* and *prove*) will be interpreted with respect to states in the p-set of that individual, even if the complement of the sentence also takes as argument the perspective index associated with the grammatical subject. Thus, if i in (4.7) (or (4.8) or (4.9)) is substituted with the perspective index of the utterer of a sentence, then π will generate the characteristic function of a set of states in the π set of the utterer, and this will form the domain in which we interpret the complement of *love* (or the complement clauses of *believe* or *prove*). If, to the outermost π argument of a formula we only ever supply the perspective index of the utterer, we will capture the intuition of AG that the complement of a verb–whether it be clausal or nominal– is always interpreted relative to what the utterer thinks is the perspective of another person, and never from the perspective of that other person.

In order for the rendering of a sentence to combine with (4.8) or (4.9) it must be of type $p \to s \to t$, and in order for the rendering of a proper name to combine with (4.7) it must be of type $p \to s \to e$. With these lexical entries we might then want to derive formulas such as (4.10), (4.11) and (4.12), in which the state argument of each $\kappa\ mj$ is fed the same argument as that fed to π:

$$\begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\,(\kappa\,u\,s')\,s'\,s'' & \text{if } C_{\mathbf{p}}\ s\ (\kappa\,u\,s') = \top \\ \to love\,(l_{p\to s\to e}\,(\kappa\,mj\,s'')\,s'')\,x_e\ (\kappa\,mj\,s'')\ s'') & \text{and } C_{\mathbf{p}}\ s\ (\kappa\,mj\,s'') = \top \\ \# & \text{otherwise} \end{cases} \tag{4.10}$$

$$\begin{cases} \forall s''[\,[s' \leqslant s'' \wedge \pi\ (\kappa\,u\,s')\ s'\,s''] \to & \text{if } C_{\mathbf{p}}\,s\ (\kappa\,u\,s') = \top \\ \forall s'''[DOX\ x\ s''\,s'''\ \to & \\ q\,(\kappa\,mj\ s''')\,s'''\,]\,] & \\ & \\ \# & \text{otherwise} \end{cases} \tag{4.11}$$

$$\begin{cases} \forall s''[\,[s' \leqslant s'' \wedge \pi\,(\kappa\, u\, s')\; s'\, s''] \rightarrow & \text{if } C_{\mathbf{p}}\, s\; (\kappa\, u\, s') = \top \\ \forall s'''[PROV\, x\;\, s''\, s''' \;\rightarrow & \\ q\,(\kappa\, mj\; s''')\, s'''\,]\,] & \\ & \\ \# & \text{otherwise} \end{cases} \tag{4.12}$$

As discussed in 3.6, a reason for wanting the state argument of each $\kappa\ mj$ to be bound by the same variable fed to π is that this variable is a state in the p-set of the utterer, and we want to interpret the clausal argument of a sentence relative to the perspective index which the utterer thinks Mary Jane has in the states accessible via DOX or $PROV$.

Unfortunately, (4.7), (4.8) and (4.9) will not allow us to generate formulas such as (4.10), (4.11) and (4.12). Take (4.7) and suppose we want to derive (4.10) (repeated below as (4.13)) from (4.7), by applying (4.7) to (4.14), (4.15) and (4.16):

$$\begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\,(\kappa\, u\, s')\, s'\, s'' & \text{if } C_{\mathbf{p}}\; s\; (\kappa\, u\, s') = \top \\ \rightarrow love\,(l_{p\rightarrow s\rightarrow e}\,(\kappa\, mj\; s'')\, s'')\, x_e\;(\kappa\, mj\; s'')\;\, s'') & \text{and } C_{\mathbf{p}}\; s\; (\kappa\, mj\; s'') = \top \\ \# & \text{otherwise} \end{cases} \tag{4.13}$$

$$l_{p\rightarrow s\rightarrow e} \tag{4.14}$$

$$mj_e \tag{4.15}$$

$$(\kappa\, mj\; s'')_p \tag{4.16}$$

This won't work: substituting $(\kappa\, m\, s'')$ for i or i' in (4.7), results in the s'' in $(\kappa\, m\, s'')$ becoming accidentally bound by the quantifier $\forall s''$, so that this is not a licit substitution.

For this reason, we adopt a more complicated typing, and this more complicated typing will allow us to make use of both the ⧫ and the ◊ modalities we introduced in 3.3. Suppose instead the word *love* is rendered as (4.17) and instead of (4.14) we use (4.18), (4.20) and (4.20):

$$\lambda y_{\blacklozenge\Diamond e}, x_e, \delta'_{\Diamond p}, \delta_{\Diamond p}, s'_s. \begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\,(\delta\, s')\, s'\, s'' & \text{if } C_{\mathbf{p}}\, s\ (\delta\, s') = \top \\ \to love\,(y\,\delta'\,s'')\,x\ (\delta'\,s'')\,s'') & \text{and } C_{\mathbf{p}}\, s\ (\delta'\,s'') = \top \\ \# & \text{otherwise} \end{cases} \tag{4.17}$$

$$\lambda\delta_{s\to p}, s_s.\ \ l_{p\to s\to e}(\delta\, s)\, s \tag{4.18}$$

$$mj_e \tag{4.19}$$

$$(\kappa\, mj)_{s\to p} \tag{4.20}$$

Relative to this more complex typing, we avoid the problem of accidental variable capture altogether, and, as remarked above, the relevant typing forces proper names to be of type $\blacklozenge\Diamond e$ instead of type $p \to s \to e$ and it forces sentences to be of type $\blacklozenge\Diamond t$ instead of $p \to s \to t$. This systematic replacement of $p \to s \to \alpha$ with $(s \to p) \to s \to \alpha$, for $\alpha \in \{e, t\}$ will be important in the following sections and in the rest of the thesis. The final lexical entries for *love*, *believe* and *prove* will be the following:

$$\lambda y_{\blacklozenge\Diamond e}, x_e, \delta'_{s\Diamond p}, \delta_{\Diamond p}, s'_s. \begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\,(\delta\, s')\, s'\, s'' & \text{if } C_{\mathbf{p}}\, s\ (\delta\, s') = \top \\ \to love\,(y\,\delta'\,s'')\,x\ (\delta'\,s'')\,s'') & \text{and } C_{\mathbf{p}}\, s\ (\delta'\,s'') = \top \\ \# & \text{otherwise} \end{cases} \tag{4.21}$$

$$\lambda p_{\blacklozenge\Diamond t}, x_e, \delta_{\Diamond p}, s_s. \begin{cases} \forall s'[\,[s \leqslant s' \wedge \pi(\delta\, s)\, s\, s'] \rightarrow & \text{if } C_{\mathbf{p}}\, s\ (\delta\, s) = \top \\ \forall s''[DOX\ x\ s'\, s'' \rightarrow & \\ p\,(\eta_s(\kappa\, x\, s''))\, s''\,]\,] & \\ & \\ \# & \text{otherwise} \end{cases} \tag{4.22}$$

$$\lambda p_{\blacklozenge\Diamond t}, x_e, \delta_{\Diamond p}, s_s. \begin{cases} \forall s'[\,[s \leqslant s' \wedge \pi(\delta\, s)\, s\, s'] \rightarrow & \text{if } C_{\mathbf{p}}\, s\ (\delta\, s) = \top \\ \forall s''[PROV\ x\ s'\, s'' \rightarrow & \\ p\,(\eta_s(\kappa\, x\, s''))\, s''\,]\,] & \\ & \\ \# & \text{otherwise} \end{cases} \tag{4.23}$$

An alternative solution to the problem of accidental variable capture just described would be to render *love* as (4.24), so that it takes an extra argument z of type e, and to keep (4.14), (4.15) and (4.16):

$$\lambda y_{p\rightarrow s\rightarrow e}, x_e, z_e, s'_s. \begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\,(\kappa\, z\ \ s')\, s'\, s'' & \text{if } C_{\mathbf{p}}\, s\ (\kappa\, z\ \ s') = \top \\ \rightarrow love\,(y\,(\kappa\, x\, s'')\ s'')\, x\ (\kappa\, x\ \ s'')\, s'') & \text{and } C_{\mathbf{p}}\, s\ (\kappa\, x\, s'') = \top \\ \# & \text{otherwise} \end{cases} \tag{4.24}$$

This has the advantage of being a simpler typing. But suppose we wanted to generate the formula (4.27):

$$\begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\, E_p\, s'\, s'' & \text{if } C_{\mathbf{p}}\, s\ E_p = \top \\ \rightarrow love\,(l\, E_p\ s'')\, mj_e\ E_p\, s'') & \text{and } C_{\mathbf{p}}\, s\ E_p = \top \\ \# & \text{otherwise} \end{cases} \tag{4.25}$$

(4.27) cannot be arrived at using (4.24), unless we introduce a term $@_e$, for which $\kappa \,@\, s = E$. Perhaps we could stipulate that the term $\#$ is such that $\kappa \,\#\, s = E$. But if we do this, we would have to abandon the constraint in P-HYPE models described in 4.2 that $[E]$ is not in the range of $[\kappa]$ at any $s \in D_s$ (i.e $\neg\exists x \in D_e, \exists s \in D_s$ $([\kappa]\, x\, s = [E]$), which was imposed to capture the intuition that no one is completely enlightened in all matters. It would also be unclear what the philosophical import of attributing the enlightened perspective to the undefined object $\#$ would be; for example, just what does it mean to say that the undefined object has a perspective? For this reason we will not pursue the simpler typing suggested.

Another possible solution is to redefine the type s, as follows:

$$s := \Diamond p \times ((s_c + s_n) \times time) \tag{4.26}$$

The *love* could be rendered as (4.27):

$$\lambda y_{p \to s \to e}, x_e, s'_s. \begin{cases} \forall s''_s(s' \leqslant s'' \wedge \pi\ (\ (\ \pi_1\, s'\ )\ s')\ s'\ s'' & \text{if} \\ \wedge(\ (\pi_1\ s'')\ s'') = (\kappa\, x\, s'') & C_{\mathbf{p}}\, s\ (\ (\ \pi_1\ s')\ s') \\ \to love\,(y\,(\ (\pi_1\ s'')\ s'')\ s'')\,x\ (\ (\pi_1\ s'')\ s'')\ s'') & = \top \\ & \text{and} \\ & C_{\mathbf{p}}\, s\ ((\pi_1\, s'')\, s'') \\ & = \top \\ \# & \text{otherwise} \end{cases} \tag{4.27}$$

The only reason we have to favour our approach (on which *love* is rendered as (4.21)) to an approach which renders *love* as (4.27) is that we have found in practice our approach to be less notationally cumbersome. In fact, the alternative approach which we have just described seems to be a mere notational variant of our own approach.

4.3.2 Proper names in P-HYPE

In this and the next chapter, certain proper names, such as *Peter Parker* and *Spiderman* are treated as potentially perspective relative and are of type $\blacklozenge\Diamond e$, whereas others, such as *Harold*, *Mary Jane*, *John* and *Mary* are not, and are of type e:

$$\textbf{sm} := \lambda\delta_{\Diamond p}, s_s. \begin{cases} sm_{p\to s\to e}(\delta\, s)\, s & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) = \top \\ \\ \# & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) \neq \top \\ \\ pp_{p\to s\to e}\ E\ s & \text{if } \delta\, s = E \end{cases} \tag{4.28}$$

$$\textbf{pp} := \lambda\delta_{\Diamond p}, s_s. \begin{cases} pp_{p\to s\to e}(\delta\, s)\, s & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) = \top \\ \\ \# & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) \neq \top \\ \\ pp_{p\to s\to e}\ E\ s & \text{if } \delta\, s = E \end{cases} \tag{4.29}$$

(4.30)

$$\textbf{h} := harold_e \tag{4.31}$$

$$\textbf{mj} := mary.jane_e \tag{4.32}$$

$$john_e \tag{4.33}$$

$$mary_e \tag{4.34}$$

Like the lexical entries for *believe* and *prove*, the lexical entries (4.28) and (4.29) involve a case analysis. In the first case, we interpret the inner constant $pp_{p\to s\to e}$ or $sm_{p\to s\to e}$ with respect to a perspective index $\delta\, s$ which is not the enlightened perspective but is contextually available. In the second case, the term is undefined, because the perspective index $\delta\, s$ is not contextually available. In the third case, we interpret the inner constant $pp_{p\to s\to e}$ or $sm_{p\to s\to e}$ with respect to the enlightened perspective index.

(4.28) and (4.29) β-reduce to an inner constant sm or pp or type $p \to s \to e$. This has the result that, when (4.28), (4.36), (4.37) and (4.38) are fed to (4.21) (here repeated as (4.35)), via β-reduction we arrive at (4.39):

$$\lambda y_{\blacklozenge\Diamond e}, x_e, \delta'_{s\Diamond p}, \delta_{\Diamond p}, s'_s . \begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\,(\delta\, s')\, s'\, s'' & \text{if } C_{\mathbf{p}}\, s\, (\delta\, s') = \top \\ \to love\,(y\,\delta'\, s'')\, x\, (\delta'\, s'')\, s'') & \text{and } C_{\mathbf{p}}\, s\, (\delta'\, s'') = \top \\ \# & \text{otherwise} \end{cases} \tag{4.35}$$

$$(\kappa\, \mathbf{mj})_{\Diamond p} \tag{4.36}$$

$$(\kappa\, u)_{\Diamond p} \tag{4.37}$$

$$s^1 \tag{4.38}$$

$$\begin{cases} \forall s''(s^1 \leqslant s'' \wedge \pi\,(\kappa\; u\; s^1)\, s^1\, s'' & \text{if } C_{\mathbf{p}}\, s\, (\kappa\; u\; s^1) = \top \\ \to love\,(\; sm\; (\kappa\; \mathbf{mj}\; s''\;)\; s'')\, \mathbf{mj}\; (\kappa\, \mathbf{mj}\, s'')\, s'') & \text{and } C_{\mathbf{p}}\, s\, (\kappa\, \mathbf{mj}\, s'') = \top \\ \# & \text{otherwise} \end{cases} \tag{4.39}$$

Since proper names of type $\blacklozenge\Diamond p$ normalise to terms of type $p \to s \to e$, they are similar to proper names in AG's account, except that we make the inner constant on which the rendering of a proper name is built be perspective and state relative. We think this fits in better with an intuitive picture of perspectives, according to which the perspective an agent has on a proper name varies depending on which state they find themselves in.

4.3.3 Predicates in P-HYPE

Since sentences will have type $\blacklozenge\Diamond t$ in our account and proper names have type $\blacklozenge\Diamond e$, n-ary predicates over entities of type $\blacklozenge\Diamond e$ will have to be functions from n arguments of type $\blacklozenge\Diamond e$ into functions of type $\blacklozenge\Diamond t$.

In this and the next chapter, we render the predicates *oculist*, *ophthalmologist*, *finite*, *inductive* and *man*, as follows (in 4.3.4 we will introduce an operator to pluralise these predicates):

$$\textbf{oculist} \ := \tag{4.40}$$
$$\lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} oculist_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ oculist_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{ophthalmologist} \ := \tag{4.41}$$
$$\lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} ophthal_{epst}\,(x\,\delta\ s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ ophthal_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{inductive} \ := \tag{4.42}$$
$$\lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} inductive_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ inductive\,(x\ (\eta_s E)\ s)\ E\ s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{finite} \ := \tag{4.43}$$

$$\lambda x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s. \begin{cases} finite_{epst}\,(x\,\delta\ s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ finite\,(x\ (\eta_s E)\ s)\ E\ s & \text{if } \delta\,s = E \end{cases}$$

man := (4.44)

$$\lambda x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s. \begin{cases} man_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ man_{epst}\,(x\ (\eta_s E)\ s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

In the lexical entries for intransitive predicates above, there are three cases to consider. In the first case, a term of type $\lozenge p$ is substituted for $\delta_{\lozenge p}$ which doesn't form E when it combines with s; rather, it returns a non-enlightened perspective index which is sent to 1 by $C_{\mathbf{p}}\,s$. In the second case, a term of type $\lozenge p$ is substituted for $\delta_{\lozenge p}$ which, combined with s, forms a perspective which $C_{\mathbf{p}}\,s$ does not send to 1, and is therefore irrelevant in the context, and the result is undefined. In the third case, a term of type $\lozenge p$ is substituted for $\delta_{\lozenge p}$ which combines with s to form E. The first case captures the meaning of an intransitive predicate from the utterer's perspective, or any other perspective relevant in the context. The third case captures the interpretation of a intransitive predicate from the enlightened perspective.

The lambda terms **oculist**, **ophthalmologist**, **inductive**, **finite** and **man** are built on the perspective relative predicates $oculist_{epst}$, $ophthal_{epst}$, $inductive_{epst}$, $finite_{epst}$ (respectively), which can differ in semantic value with respect to certain perspective indices. When we supply a perspectival proper name— such as that in (4.28)— as well as (4.36), and (4.38) to 5.1, via β-reduction we arrive at (4.45):

$$\begin{cases} oculist_{epst}\,(sm\,(\kappa\,\mathbf{mj}\,s^1)\,)\,(\kappa\,\mathbf{mj}\,s^1)\,s & \text{if } (\kappa\,\mathbf{mj}\,s^1) \neq E \\ & \text{and } C_{\mathbf{p}}\ s^1\ (\kappa\,\mathbf{mj}\,s^1) = \top \\ \# & \text{if } (\kappa\,\mathbf{mj}\,s^1) \neq E \\ & \text{and } C_{\mathbf{p}}\ s\ (\kappa\,\mathbf{mj}\,s^1) \neq \top \\ oculist_{epst}\,(sm\ E\ s)\ E\,s & \text{if } (\kappa\,\mathbf{mj}\,s^1) = E \end{cases} \tag{4.45}$$

4.3.4 Assumptions about plurals

In 5.5 we will try to capture the semantics of certain plural noun phrases such as the following:

(1) a. Harold proved that the primes are infinite.
    b. Harold proved that the primes are not inductive.
    c. (No,) Harold proved that the primes are infinite, not that the primes are not inductive.

The sentences (1a) - (1c) include the plural noun phrase *the primes* (elliptical for *the prime numbers*. In (1a) - (1c), the predicate *are finite* is interpreted collectively. For example, (1a) bears the interpretation that the set of prime numbers is infinite, not that each individual prime number is infinite.

We will now briefly outline our assumptions about plurality so that by 5.5 we can present our compositional derivations. A fuller treatment of plural interpretation would have to enter into more details than we find necessary here, and so our assumptions about plurality are quite basic.Following (Landman 2008), we will model plurality by using complete atomic Boolean algebras. We will now define these in terms by means of the concept of a complemented, distributive lattice.

Definition 16 A *lattice* $(L, \sqsubseteq)$ is a partially-ordered set in which, for any two $x, y \in L$, the least upper bound and greatest lower bound of x and y exist. We will denote these $\sqcap$ and $\sqcup$, respectively, and we will call $\sqcup$ the *sum*-operator.

$\sqcup$ and $\sqcap$ are associative, commutative and idempotent:

1. *Associativity:* For all $a, b, c \in B$ and $\circ \in \{\sqcup, \sqcap\}$: $(a \circ b) \circ c = a \circ (b \circ c)$
2. *Commutativity*: For all $a, b \in B$ and $\circ \in \{\sqcup, \sqcap\}$: $a \circ b = b \circ a$
3. *Idempotence*: For all $a \in B$ and $\circ \in \{\sqcup, \sqcap\}$: $a \circ a = a$

A lattice L is said to be *bounded* if there are elements $0, 1 \in L$ such that:

$$0 \leqslant x, \text{ for all } x \in L \tag{4.46}$$

$$x \leqslant 1, \text{ for all } x \in L \tag{4.47}$$

Definition 17 A *complemented lattice* is a bounded lattice in which, for every element $a \in L$ there exists a $b \in L$, which is such that:

$$a \vee b = 1 \tag{4.48}$$

$$a \wedge b = 0 \tag{4.49}$$

We will use $-a$ to indicate the complement of a.

Definition 18 A lattice is *distributive* if, for any $x, y, z \in L$:

$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (s \sqcup z) \tag{4.50}$$

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (s \sqcap z) \tag{4.51}$$

In fact, we only need to assume one of (4.50) and (4.51) to prove the other (Landman 1991: p.262-5).

Given the above notions, we define a *Boolean algebra* $(B, \sqsubseteq)$ as a complemented distributive lattice. A Boolean algebra is said to be *complete* if, given any arbitrary subset $X \subseteq B$, $\sqcup X$ and $\sqcap X$ exist. In a Boolean algebra $(B, \sqsubseteq)$, there may exist a set $ATOM$ of atoms, defined as follows:

$$ATOM = \{c \in B \mid c \neq 0 \text{ and there is no } d \text{ such that } d \in B \backslash \{0, c\} \text{ and } d \sqsubseteq c\}$$

An *atomic Boolean algebra* is a Boolean algebra in which every element is the sum of some atoms, so that the following condition of atomicity holds:

- *Atomicity*: For all $b \in B \backslash \{0\}$: $ATOM\, b \neq \varnothing$

Put together, a complete atomic Boolean algebra is a Boolean algebra which is both complete and atomic.

From now on, the domain of individuals D_e will be a complete atomic Boolean algebra $(B, \sqsubseteq)$, where $ATOM = D$, where D is as in previous sections. The reader may notice that complete Boolean algebras contain a bottom element, which is not commonly employed in semantic treatments of plurality. It is more common to employ free i-join semilattices in the lattice-theoretic semantics of plurality. But the free i-join semilattices are exactly the complete atomic Boolean algebras with the bottom element removed and the operations restricted to (generalized) sum (see (Landman 1991: p.262-5)), so readers who object to the bottom element can simply remove it and restrict the operations of the algebra accordingly (though for a series of arguments that complete Boolean algebras are preferable to free-i join semilattices in modelling plurality, see (Landman 2008)).

One of the central ideas articulated in (Link 1983) and took on by researchers since (Landman 2008) (Champollion 2017), is that singular individuals like *John* denote atoms in D_e and plural noun phrases, such as *John and Mary*, denote the sum of the atomic individuals John and Mary. Since D_e consists of both plural and atomic individuals, we now allow predicates to have plural individuals in their extension. This allows for a collective interpretation of predicates—which arises when a predicate has only plural individuals in its extension. We do not need to capture distributive interpretations of predicates to model our examples, and so do not adopt an approach to distributive predication here.

Since (Link 1983), singular count nouns such as *boy* are typically rendered as constants of type $e \to t$ and are semantically interpreted as functions from atomic individuals to truth values (i.e, they are subsets of $ATOM$). We turn the rendering of a singular count noun like *boy* into its plural counterpart *boys* by a *pluralisation* operator $*$ which closes singular count nouns under sum. Where $g \in Con_{e\to t}$, and $* \in Con_{(e\to t)\to(e\to t)}$, $[\![*g]\!]^{\mathfrak{M},g}$ is defined as follows:

(2) For $g \in Con_{e\to t}$, $[\![*g]\!]^{\mathfrak{M},g}$ is the smallest function $h \in D_{et}$, such that, for all $x \in D_e$, if $[\![g]\!]^{\mathfrak{M},g}x = 1$ then $[\![h]\!]^{\mathfrak{M},g}x = 1$ and $\forall S'(S' \subseteq \{x \mid [\![g]\!]^{\mathfrak{M},g}x = 1\} \to h(\sqcup S') = 1)$

$*g$ closes the individuals in the domain of the function g under sum, sending singular individuals to 1 if g sends them to 1, and sending plural individual to 1 if all the parts of the plural individual are sent by g to 1. Consider the following partial function $[\![boy]\!]^{\mathfrak{M},g} \in D_{et}$:

$$[\![boy]\!]^{\mathfrak{M},g} = \{(a,1),(b,1),(c,1),(d,0)\}$$

By the definition in (3), we then have:

$[\![*boy]\!]^{\mathfrak{M},g} = \{(0,1),(a,1),(b,1),(c,1),(d,0),(a \sqcup b,1),(a \sqcup c,1),$
$(a \sqcup d,0),(c \sqcup d,0),(b \sqcup c,1),(b \sqcup d,0),(a \sqcup b \sqcup c,1),(a \sqcup b \sqcup c \sqcup d,0)\}$

We can now generalise $*$ to n-ary functions for $n > 1$. Where $g \in Con_{\alpha_1 \to \alpha_2,...,\alpha_n \to t}$, and $* \in Con_{(\alpha_1 \to \alpha_2,...,\alpha_n \to t) \to (\alpha_1 \to \alpha_2,...,\alpha_n \to t)}$, $[\![*g]\!]^{\mathfrak{M},g}$ is defined as follows:

(3) For $g \in Con_{\alpha_1 \to \alpha_2,...,\alpha_n \to t}$, $[\![*g]\!]^{\mathfrak{M},g}$ is the smallest function $h \in D_{\alpha_1} \to D_{\alpha_2},...,D_{\alpha_n} \to D_t$, such that, for all $z^1 \in D_{\alpha_1}, z^2 \in D_{\alpha_2},...,z^n \in D_{\alpha_n}$: if $[\![g]\!]^{\mathfrak{M},g} z^1\, z^2 ... z^n = 1$ then $[\![h]\!]^{\mathfrak{M},g} z^1\, z^2 ... z^n = 1$ and $\forall A_1, A_2, ..., A_n (A_1 \times A_2 \times, ..., \times A_n \subseteq \{(x_1,...,x_n) \mid [\![g]\!]^{\mathfrak{M},g} x^1\, x^2,...,x^n = 1\}$
$\Rightarrow h(\sqcup A_1)\,(\sqcup A_2),\cdots,(\sqcup A_n) = 1)$.

Note that $*$ requires that the domains $D_{\alpha_1}, D_{\alpha_2},...,D_{\alpha_n}$ on which the function g is defined are complete Boolean algebras, since $*$ takes the sum of an arbitrary subset of each domain D_{α_i}, for $1 \leqslant i \leqslant n$. For this reason, the definition (3) must be restricted accordingly (by fixing the types $\alpha_1,...,\alpha_n$ to certain types) if one of the domains is not a complete Boolean algebra. We note this technical complication only to put it aside, though in a fuller treatment we would need to consider what structure is imposed on our domains.

In our setting, unary predicates are of type $\blacklozenge\lozenge e \to \blacklozenge\lozenge t$, which raises the question of how we intend to use $*$ to pluralise them. Note, however, that the predicates of type $\blacklozenge\lozenge e \to \blacklozenge\lozenge t$ in our examples are all built on predicates of type $e \to p \to s \to t$. For example, consider **oculist**:

$$\lambda x, \delta, s. \begin{cases} oculist_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ oculist_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

oculist : $\blacklozenge\lozenge e \rightarrow \blacklozenge\lozenge t$ is built on the predicate $oculist_{epst}$, which applies to individuals which have been fed a perspective index and a state. To pluralise **oculist** we therefore need to pluralise its internal argument $oculist_{epst}$, since only this internal argument will appear when we normalise lambda terms in which **oculist** features and it is the individuals which saturate the argument of type e in $oculist_{epst}$ that we need to sum over. Since $oculist_{epst}$ is a predicate of individuals and not of functions of type $\blacklozenge\lozenge e$, the pluralisation operator introduced above will fit the bill perfectly. To combine a predicate like *oculists* or *ophthalmologists* with a plural subject, such as *John and Mary*, *oculists* and *ophthalmologists* will therefore be rendered as the following lambda terms (respectively), in which $*$ is applied to $oculist_{epst}$ and $ophthalmologist_{epst}$:

$$* \textbf{oculist} = \tag{4.52}$$
$$\lambda x, \delta, s. \begin{cases} *oculist_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ *oculist_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

$$* \textbf{ophthalmologist} = \tag{4.53}$$
$$\lambda x, \delta, s. \begin{cases} *ophthal_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ *ophthal_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

How will we form the plural individuals which can be arguments to ***oculist**? Since the parameter x in ***oculist** is fed an argument of type $\lozenge p$ and an argument of type s, any plural individual substituted for x must be of type $\blacklozenge\lozenge e$. We could directly sum functions of type $\blacklozenge\lozenge e$, but this wouldn't ensure that such plural individuals were bound by the same abstractors of type $\Diamond p$ and type s. What we want is for plural individuals that substitute x in *****oculist** to be reduced to an expression of type e in which the parameters of type $\Diamond p$ and type s have all been substituted with the same arguments, namely the arguments δ and s in *****oculist**. To ensure this, we proceed as follows. In the object language we have a sum operator $\oplus : e \rightarrow e \rightarrow e$, which is such that $[\![\oplus]\!]^{\mathfrak{M},g} = \sqcup \in D_{(ee)e}$, the sum operator of our complete atomic Boolean algebra. We then lift $\oplus$ in the following way, so that it combines with terms of type $\blacklozenge\Diamond e$:

$$\mathbf{zip}^3_{sp}(\eta_{sp}(\mathbf{zip}^3_s(\eta_s\oplus))) = \lambda v_{\blacklozenge\Diamond e}, w_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s.\ (v\,\delta\,s) \oplus (w\,\delta\,s)$$

Crucially, this lifting ensures that v and w are fed the same arguments of type $\Diamond p$ and type s, just as we desired. Given (3), $[\![*oculist_{epst}]\!]^{\mathfrak{M},g}$ will then be the smallest function $h \in D_{epst}$, such that, for all $z^1 \in D_e, z^2 \in D_p, z^3 \in D_s$: if $[\![oculist_{epst}]\!]^{\mathfrak{M},g} z^1\, z^2\, z^3 = 1$ then $[\![h]\!]^{\mathfrak{M},g} z^1\, z^2\, z^3 = 1$ and $\forall S'(S' \subseteq \{x \in D_{\alpha_1} \mid [\![oculist_{epst}]\!]^{\mathfrak{M},g} x\, z^2\, z^n = 1\} \rightarrow h(\sqcup S')\, z^2\, z^3 = 1)$. We will abbreviate such functions by writing $[\![*oculist_{epst}]\!]^{\mathfrak{M},g} = h_{*oculist}$ and $[\![*oculist_{epst}]\!]^{\mathfrak{M},g} = h_{*ophthal}$.

Now we have introduced the pluralisation operator $*$, we can consider how to form plural definite noun phrases, such as *The boys*. (Landman 2008), following earlier ideas of (Sharvy 1980), proposes that the English definite article *the* be rendered as an operator $the : (e \rightarrow t) \rightarrow e$ which takes a function $f : e \rightarrow t$ and returns the supremum of the set of objects which are each mapped to 1 by f, if the supremum of that set is itself mapped to 1 by f:

$$[\![the_{(e\rightarrow t)\rightarrow e}\ \ f_{e\rightarrow t}]\!]^{\mathfrak{M},g} =$$

$$\begin{cases} \sqcup\{x \in D_e \mid [\![f]\!]^{\mathfrak{M},g}\, x = 1\} & \text{If } [\![f]\!]^{\mathfrak{M},g}(\sqcup\{x \mid [\![f]\!]^{\mathfrak{M},g}\, x = 1\}) = 1 \\ \# & \text{otherwise} \end{cases}$$

In this way, *The boys* denotes the maximal plural object in the extension of *boys*, if it exists. Since for us, unary predicates are of type $\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond t$, we will semantically interpret *the* as follows:

$$[\![\sigma_{(\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t)\rightarrow\blacklozenge\Diamond e}]\!]^{\mathfrak{M},g} = \begin{cases} \text{That function } G_\sigma \in D_{(\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t)\rightarrow\blacklozenge\Diamond e} & \text{if } \sqcup\{x\,\delta\,s \mid \\ \text{which is such that, for} & x \in D_{\blacklozenge\Diamond e} \text{ and} \\ f \in D_{\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t}, \delta \in D_{\Diamond p} \text{ and} & f\,x\,\delta\,s = 1\} \\ s \in D_s : G_\sigma\,f\,\delta\,s = & \in \{x\,\delta\,s \mid x \in D_{\blacklozenge\Diamond e} \\ \sqcup\{x\,\delta\,s \in D_e \mid x \in D_{\blacklozenge\Diamond e} \text{ and} & \text{and} \\ f\,x\,\delta\,s = 1\} & f\,x\,\delta\,s = 1\} \\ \\ \# & \text{otherwise} \end{cases}$$

We will abbreviate the function $[\![\sigma_{(\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t)\rightarrow\blacklozenge\Diamond e}]\!]^{\mathfrak{M},g}$ by writing G_σ.

Given the previous rendering of *ophthalmologist*, we can then render *The oculists are ophthalomologists*, as follows:

$$*\textbf{ophthalmologist}(\sigma_{(\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t)\rightarrow\blacklozenge\Diamond e}\ *\textbf{oculist}) : \blacklozenge\Diamond t \tag{4.54}$$

If we then feed this lambda term $\kappa\,h$ and s^1 and $\kappa\,h\,s^1 \in C_{\mathbf{p}}\,s$, then (4.54) β-reduces to (4.55), which β-reduces to :

$$\Bigg(\lambda\delta_{\Diamond p}, s_s.*ophthal_{epst} \tag{4.55}$$

$$\left(\sigma_{(\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t)\rightarrow\blacklozenge\Diamond e}\left(\lambda x, \delta', s'.*oculist_{epst}\,(x\,\delta'\,s')\,(\delta'\,s')\,s'\right)\delta\,s\right)(\delta\,s)\,s\Bigg)\,(\kappa\,h)\ \ s^1$$

$$=_\beta$$

$$* \, ophthal_{epst} \tag{4.56}$$

$$\Big(\sigma_{(\blacklozenge\lozenge e\to\blacklozenge\lozenge t)\to\blacklozenge\lozenge e}\Big(\lambda x,\delta',s'.\, *\, oculist_{epst}\,(x\,\delta'\,s')\,(\delta'\,s')\;s'\Big)\,(\kappa\,h)\,s^1\Big)\,(\kappa\,h\,s^1)\;\; s^1$$

where (4.56) when evaluated with $[\![\ ]\!]^{\mathfrak{M},g}$, reduces to:

$$h_{*ophthal}\Big(\sqcup\{x\,(\kappa\,h)\,s\in D_e \mid h_{*oculist}\,(x\,(\kappa\,h)s)\,(\kappa\,h)\,s=1\}\Big)\,(\kappa\,h\,s)\;s=1$$

We now have the basic semantics of plurality in place needed for our examples.

4.4 Solipsistic P-HYPE models

In this section we outline the requirements on P-HYPE models necessary for implementing AG's account. According to AG's account, each individual has her own mental model and the denotation of predicates and names is wholly solipstic, in the sense that each name and predicate has a different interpretation for each individual. Models of this kind will eventually be presented in (19), and are called *Solipsistic P-HYPE models*. Again, we would like to emphasise that one of the aims in this thesis has been to capture formally the kinds of models that AG require for their account, leaving the question of whether the solipsistic properties of such models is ultimately tenable for the semantics of natural language.

In order to specify constraints on solipsistic models and on the interpretation of lambda terms in such models, we will need a function ST, which returns the subterms of a lambda term. $ST(A)$, where A is an arbitrary term of type $\alpha \in TYPE$ is defined recursively, as follows:

If $A \in Var_\alpha$ for $\alpha \in TYPE$, then $ST(A) = A$

If $A \in Con_\alpha$ for $\alpha \in TYPE$, then $ST(A) = A$

If $A \in TERM_\alpha$ is of the form $(\bullet\psi)$ and

$\bullet \in \{\eta_s, \eta_{\Diamond p}, \neg, \lambda x., \forall x, \exists x, \Box, \Diamond, PROV, DOX\}$, then $ST(A) = \{\bullet\psi\} \cup ST(\psi)$

If $A \in TERM_\alpha$ is of the form $\rho \circ \eta$ and $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \bullet_s, \bullet_{\Diamond p}, =\}$,

then $ST(A) = \{\rho \circ \eta\} \cup ST(\rho) \cup ST(\eta)$

If $A \in TERM_\alpha$ is of the form $(B_{\beta\rightarrow\alpha}\, C_\beta)$ for some $\beta \in TYPE$ then

$ST(A) = \{B\, C\} \cup ST(B) \cup ST(C)$

As a first step towards building solipsistic P-HYPE models, we will consider those admissible P-HYPE models which are such that certain conditions, now to be outlined, hold.

Firstly, we require that (4.57)- (4.58) hold:

$$\forall x, y \in D_e, \forall s, s' \in D_s\, (\llbracket\kappa\rrbracket^{\mathfrak{M},g}\, x\, s = \llbracket\kappa\rrbracket^{\mathfrak{M},g}\, y\, s' \rightarrow x = y) \tag{4.57}$$

$$\forall i \in D_p(i = E \vee \exists x \in D_e\, \exists s \in D_s\, (\llbracket\kappa\rrbracket^{\mathfrak{M},g}\, x\, s = i)) \tag{4.58}$$

(4.57) states that κ is an injective function, so that each member of D_e for which κ is defined is sent to a different perspective index. (4.58) states that the only things in the set D_p of perspective indices are E or generated from κ applied to some individual at some state. (4.57) assures that κ sends different individuals to distinct perspective indices, each individual has her own private perspective index. (4.58) instead is motivated by the need to keep P-HYPE models simple, and to clarify the intended role of κ as a generator of all the perspective indices, save E.

In particular, in the next chapter, we will render propositional attitude reports of the form *John Xs that the earth is round*, where Xs is a tensed verb, in the following way (where u_e is the utterer of the sentence and $\kappa\, u_e\, s$ is the perspective index corresponding to u and χ_{esst} is the rendering of the verb X):

(4) $\forall s'[\,[s \leqslant s' \wedge \pi\,(\kappa\, u_e\, s)\, s\, s'] \rightarrow \forall s''[\chi_{esst}\, x\, s'\, s^2 \;\rightarrow A\, s''\,]\,]$

Since $\pi\,(\kappa\, u_e\, s)\, s\, s'$ if and only if s' is in the perspective set of the utterer u at s, and we want this perspective set to be private, so as to formalise AG's idea of perspectives, we must first impose the condition (4.59), according to which the perspective set of each individual in D_e for which π is defined is distinct:

$$\forall d_1, d_2 \in CON_e \cup VAR_e\, \forall s,\ s' \in CON_s \cup VAR_s \,\Big(\llbracket\, d_1\,\rrbracket^{\mathfrak{M},g} \neq \llbracket\, d_2 \rrbracket^{\mathfrak{M},g} \qquad (4.59)$$
$$\rightarrow \{s'' \in D_s \mid \llbracket\, \pi(\kappa\, d_1\, s)\, s'\, s''\,\rrbracket^{\mathfrak{M},g} = 1\} \cap \{s'' \in D_s \mid \llbracket \pi(\kappa\, d_2\, s)\, s'\, s''\,\rrbracket^{\mathfrak{M},g} = 1\} = \varnothing\,\Big)$$

But (4.59) by itself does not capture their idea that names and predicates are wholly subjective. Consider

(5) John believes that Mary Jane loves Spiderman.

According to AG's account, there is an interpretation of this sentence on which *Spiderman* is interpreted relative to what the utterer thinks is Mary Jane's perspective. In AG's account the name *Spiderman* is fed the perspective index associated with Mary Jane, but they do not capture the informal idea motivating their account, according to which the name denotes not what Mary Jane thinks the name denotes, but what the utterer thinks Mary Jane thinks the name denotes. To ensure that names which occur in A in (4) denote what the utterer thinks a possible different individual (Mary Jane, for example) thinks the name denotes, we require that there is a non-empty set $\mathcal{N} \subseteq CON_{p\rightarrow s\rightarrow e}$ of perspective-relative names of type $p \rightarrow s \rightarrow e$. We want the members of $\mathcal{N}$ to be constants with no non-trivial internal syntactic structure, and so, for every $n \in \mathcal{N}$ we impose the condition (4.60) (see below). Note, this rules out trivial liftings of constants or variables of type e, such as $\lambda i_p, \eta_s.john_e$ from being members of $\mathcal{N}$. Members of $\mathcal{N}$ must also meet the following 'injectivity' conditions (4.61) and (4.62) hold:

$$\forall n_{pse} \in \mathcal{N}(ST[n] = \{n\}) \tag{4.60}$$

$$\forall n_{pse} \in \mathcal{N}\ \forall s,\ s' \in D_s\, \forall i,\ i' \in D_p([\![n]\!]^{\mathfrak{M},g}\, i\, s = [\![n]\!]^{\mathfrak{M},g}\, i'\, s' \rightarrow i = i') \tag{4.61}$$

$$\forall n_{pse} \in \mathcal{N}\ \forall s,\ s', s'', s''' \in D_s, \forall i,\ i' \in D_p(i \neq i' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i\, s\, s' \wedge \tag{4.62}$$
$$[\![\pi]\!]^{\mathfrak{M},g}\, i'\, s''\, s''' \rightarrow [\![n]\!]^{\mathfrak{M},g}\, i\, s' \neq [\![n]\!]^{\mathfrak{M},g}\, i\, s''' \wedge [\![n]\!]^{\mathfrak{M},g}\, i'\, s''' \neq [\![n]\!]^{\mathfrak{M},g}\, i'\, s')$$

From now on, terms in $TERM_{pse}$ will be called *perspectival names*, and perspectival names in $\mathcal{N}$ (i.e, those perspectival names satisfying the conditions (4.60), (4.61) and (4.62)) will be called *injective perspectival names*.

(4.61) requires that the interpretation of an injective perspectival name is never shared across perspective indices, whilst (4.62) requires that the interpretation of an injective name is never shared across states accessible via π, even if those name have been supplied the same perspective index. (4.62) is motivated by our desire to implement AG's account, according to which sentences such as *John believes that Mary Jane loves Spiderman* will be rendered as formulas of the form (4), in which *Spiderman* is a sub-term of A which denotes not what Mary Jane thinks the name denotes, but what the utterer thinks Mary Jane thinks the name denotes. Consequently, even though *Spiderman* may be supplied the perspective index associated with Mary Jane, it will not bear the interpretation which Mary Jane herself attaches to the name.

Next, we require that there is a non-empty set $\mathcal{C} \subseteq CON_{epst}$ of predicates of type $e \rightarrow p \rightarrow s \rightarrow t$ (we will use C_{epst} as a variable over members of $\mathcal{C}$) and a non-empty set of predicates $\mathcal{D} \subseteq CON_{eepst}$ of type $e \to e \to p \to s \to t$ (we will use D_{eepst} as a variable over members of $\mathcal{D}$). We want the members of $\mathcal{C}$ and $\mathcal{D}$ to be constants with no non-trivial internal syntactic structure, and so, for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$ we impose the condition (4.63) (see below). Note, this rules out trivial liftings of constants or variables of type t, such as $\lambda x_e, i_p, \eta_s.\top$ from being members of $\mathcal{C}$ and lambda terms such as $\lambda x_e, y_e, i_p, \eta_s.\top$ from being members of $\mathcal{D}$. In addition, predicates in $\mathcal{C}$ and $\mathcal{D}$ must satisfy the 'injectivity' requirements (4.64)-(4.67):

$$\forall C_{epst} \in \mathcal{C}, D_{eepst} \in \mathcal{D}(ST[C] = C \wedge ST[D] = D) \tag{4.63}$$

$$\forall C_{epst} \in \mathcal{C}\ \forall s, s' \in D_s, \forall i_1, i_2 \in D_p : \tag{4.64}$$
$$\{ e \mid [\![C_{epst}]\!]^{\mathfrak{M},g}\, e\, i_1\, s = 1\} = \{e \mid [\![C_{epst}]\!]^{\mathfrak{M},g}\, e\, i_2\, s' = 1\} \to i_1 = i_2$$

$$\forall D_{eepst} \in \mathcal{D}\ \forall s, s' \in D_s, \forall i_3, i_4 \in D_p : \tag{4.65}$$
$$\{ (e_1, e_2) \mid [\![D_{eepst}]\!]^{\mathfrak{M},g}\, e_1\, e_2\, i_3\, s = 1\} = \{(e_3, e_4) \mid [\![D_{eepst}]\!]^{\mathfrak{M},g}\, e_3\, e_3\, i_4\, s' = 1\}$$
$$\to i_3 = i_4$$

$$\forall C_{epst} \in \mathcal{C}\ \forall s,\ s', s'', s''' \in D_s\, \forall i,\ i' \in D_p\big(i \neq i' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i\, s\, s' \tag{4.66}$$
$$\wedge [\![\pi]\!]^{\mathfrak{M},g}\, i'\, s''\, s''' \to \{ e \mid [\![C_{epst}]\!]^{\mathfrak{M},g}\, e\, i\, s' = 1\} \neq \{e \mid [\![C_{epst}]\!]^{\mathfrak{M},g}\, e\, i\, s''' = 1\}$$
$$\wedge \{ e \mid [\![C_{epst}]\!]^{\mathfrak{M},g}\, e\, i'\, s' = 1\} \neq \{e \mid [\![C_{epst}]\!]^{\mathfrak{M},g}\, e\, i'\, s''' = 1\}\big)$$

$$\forall D_{eepst} \in \mathcal{D}\ \forall s,\ s', s'', s''' \in D_s\, \forall i,\ i' \in D_p \tag{4.67}$$
$$(i \neq i' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i\, s\, s' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i'\, s''\, s'''$$
$$\to \{ (e_1, e_2) \mid [\![D_{eepst}]\!]^{\mathfrak{M},g}\, e_1\, e_2\, i\, s' = 1\} \neq \{(e_3, e_4) \mid [\![D_{eepst}]\!]^{\mathfrak{M},g}\, e_3\, e_4\, i\, s''' = 1\}$$

$$\wedge \{ (e_1, e_2) \mid [\![D_{eepst}]\!]^{\mathfrak{M},g}\, e_1\, e_2\, i'\, s' = 1\} \neq \{(e_3, e_4) \mid [\![D_{eepst}]\!]^{\mathfrak{M},g}\, e_3\, e_4\, i'\, s''' = 1\}$$

From now on, terms in $TERM_{epst}$ and terms in $TERM_{eepst}$ will be called *perspectival predicates*, and perspectival predicates in $\mathcal{C}$ or $\mathcal{D}$ (i.e, those satisfying conditions (4.63)-(4.67)) will be called *injective perspectival predicates*.

In the next chapter it will be useful to talk about injective perspectival predicates which have the same interpretation with respect to the enlightened perspective index and injective perspectival names which have the same interpretation with respect to the enlightened perspective index. Thus, we will require that there be:

- A non-empty set of injective perspectival names $\mathcal{E}_n \subseteq \mathcal{N}$, where, for each $n \in \mathcal{E}_n$, there is an $n' \in \mathcal{E}_n$ such that n and n' satisfy the condition (4.68) below.

- A non-empty set of injective perspectival predicates $\mathcal{E}_{1-place} \subseteq \mathcal{C}$, where, for each $C \in \mathcal{E}_{1-place}$, there is an $C' \in \mathcal{E}_{1-place}$ such that C and C' satisfy the condition (4.69) below.

- A non-empty set of injective perspectival predicates $\mathcal{E}_{2-place} \subseteq \mathcal{D}$, where, for each $D \in \mathcal{E}_{2-place}$, there is an $D' \in \mathcal{E}_{2-place}$ such that D and D' satisfy the condition (4.71) below.

$$\forall s \in D_s([\![n_{pse}]\!]^{\mathfrak{M},g}\, E\, s = [\![n'_{pse}]\!]^{\mathfrak{M},g}\, E\, s) \tag{4.68}$$

$$\forall x \in D_e\, \forall s \in D_s \tag{4.69}$$

$$([\![C_{epst}]\!]^{\mathfrak{M},g}\, x\, E\, s = 1 \text{ if and only if } [\![C'_{epst}]\!]^{\mathfrak{M},g}\, x\, E\, s = 1) \tag{4.70}$$

$$\forall x, y \in D_e\, \forall s \in D_s \tag{4.71}$$

$$([\![D_{eepst}]\!]^{\mathfrak{M},g}\, x\, y\, E\, s = 1 \text{ if and only if } [\![D'_{eepst}]\!]^{\mathfrak{M},g}\, x\, y\, E\, s = 1) \tag{4.72}$$

We call names satisfying (4.68) *E-synonmyous names* and predicates satisfying (4.69) or (4.71), *E-synonmyous predicates*.

4.4.1 The injective perspectival predicates/names and E-synonymous names/predicates in our account

From now till the end of the thesis, we will treat the following predicates as injective perspectival predicates:

$$oculist_{epst} \tag{4.73}$$

$$ophthal_{epst} \tag{4.74}$$

$$finite_{epst} \tag{4.75}$$

$$inductive_{epst} \tag{4.76}$$

$$prime.numbers_{epst} \tag{4.77}$$

$$is^1_{eepst} \tag{4.78}$$

We will treat $oculist_{epst}$ and $ophthal_{epst}$ as E-synonymous and $inductive_{epst}$ and $finite_{epst}$ as E-synonymous.

The injective perspectival names we will be considering in the next chapter are:

$$sm_{pse} \tag{4.79}$$

$$pp_{pse} \tag{4.80}$$

$$(\sigma_{(\blacklozenge\Diamond e\blacklozenge\Diamond t)\blacklozenge\Diamond e}\ C_{\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t}\ \ \delta_{\Diamond p}\ s_s \tag{4.81}$$

where $C_{\blacklozenge\Diamond e\rightarrow\blacklozenge\Diamond t} \in \{$**primes**, $*$**oculist**, $*$**ophthalmologist**, **man**$\}$

The predicate **primes** will be introduced in 5.5. sm_{pse} and pp_{pse} are E-synonymous and ***oculist** and ***ophthalmologist** will be built on the E-synonymous predicates

$oculist_{epst}$ and $ophthal_{epst}$.

4.4.2 Special top level terms

In the admissible P-HYPE models we will discuss in the next chapter, there are certain constraints governing perspective indices and state constants that occur in a certain class of terms called the *special top-level terms* which we now define.

In particular, we will consider certain normalised lambda terms of type e, or which are injective perspectival names in CON_{pse} which have been applied to terms of type p and type s to generate terms of type e. Normalised terms $x : e$ which are such that $ST[x] = \{x\}$ (i.e, which have only themselves as sub-terms), or normalised terms which contain as their subterms variables or constants of type p and type s, or variables or constants of type $s \to p$ and type s, are called *top-level type e terms*. The set of top-level type e terms will be written TOP_e. We call a top-level term $x \in TOP_e$ a *special top-level type e term* if either (4.82) and (4.83) both jointly hold, or (4.84) holds:

$$\forall i'_p \left(i' \in ST\left[x\right] \Rightarrow i' = E\right) \wedge \forall \delta_{s\to p} \left(\delta \in ST\left[x\right] \Rightarrow \delta = \eta_s E\right) \tag{4.82}$$

$$\forall s_s (s \in ST\left[x\right] \Rightarrow proj_1\ proj_1\ [\![s]\!]^{\mathfrak{M},g} \in D_{s_c} \tag{4.83}$$

$$\forall i''_p (i'' \in ST\left[x\right] \Rightarrow i'' \neq E) \tag{4.84}$$

T_e will be the set of special top-level type e terms. For any $x \in T_e$, either both all variables or constants of type p are identical to E and all variables or constants of type s are semantically interpreted as elements built up from a classical state, or there are no variables or constants of type p which are identical to E in x. In the next chapter, we will adopt the following condition on top-level terms x:

$$\forall x(\, x \in TOP_e \text{ and } x \notin T_e \Rightarrow [\![\, x \,]\!]^{\mathfrak{M},g} = \#) \wedge \exists y(\, y \in T_e \wedge [\![\, y \,]\!]^{\mathfrak{M},g} \in D_e \backslash \# \,) \tag{4.85}$$

In admissible P-HYPE models in which (4.85) is true, some special top-level terms of type e are interpreted as members of $D_e\backslash\{\#\}$, but all non-special top-level terms of type e denote $\#$.

We have defined the notion of a special top-level type e term. We now define the notion of special top-level type t term. Let us make the following abbreviations. Where $s^0, s', s^2 \in CON_s \cup VAR_s$, $i^0 \in CON_p \cup VAR_p$ and $x^0 \in CON_e \cup VAR_e$:

$$\mathbf{S}[s^0\ s'\ i^0] =_{df} \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i^0\, s^0\, s'] \rightarrow$$

$$\mathbf{P}[x^0\ s'\ s^2] =_{df} \forall s^2[\Delta_{esst}\, x^0\, s'\, s^2\ \rightarrow$$

In the next chapter, we will only consider normalised lambda terms of type t of the form (4.86) or (4.87), where A_t and B_t are terms of type t containing certain injective perspectival predicates $C_{epst} \in \mathcal{C}$ and $D_{eepst} \in \mathcal{D}$ of the form in (i) or (ii) below:

$$\mathbf{S}\,[s^0\ s'\ i^0]\,\mathbf{P}[x^0\ s'\ s'']\,A_t \tag{4.86}$$

$$\forall s'(s^0 \leqslant s^1 \wedge \pi\, i^0\, s^0\, s' \rightarrow B_t) \tag{4.87}$$

(i) A_t is of the form:

$$C_{epst}(x\, i^1\, s'')\, i^2\, s'' \tag{4.88}$$

$$D_{eepst}\,(x\, i^1\, s'')\,(x'\, i^2\, s'')\, i^3\, s'' \tag{4.89}$$

$$\exists x_{\blacklozenge\lozenge e}(C_{epst}(x\, \delta_{s\rightarrow p}\, s'')\, i^2\, s'') \tag{4.90}$$

$$\exists x_{\blacklozenge\Diamond e}(D_{eepst}\,(x\,\delta_{s\to p}\,s'')\,(x'\,i^2\,s'')\,i^3\,s'') \tag{4.91}$$

$$\exists x'_{\blacklozenge\Diamond e}(D_{eepst}\,(x\,i^1\,s'')\,(x'\;\delta\;s'')\,i^3\,s'') \tag{4.92}$$

or A_t is built by applying $\mathbf{S}\,[s^0\;\;s'\;\;i^0]\,\mathbf{P}[x^0\;\;s'\;\;s'']$ a finite number of times to (4.88), (4.89), (4.90) or (4.91).

(ii) B_t is of the form:

$$C_{epst}(x\,i^1\,s')\,i^2\,s' \tag{4.93}$$

$$D_{eepst}\,(x\,i^1\,s')\,(x'\,i^2\,s')\,i^3\,s' \tag{4.94}$$

$$\exists x_{\blacklozenge\Diamond e}\;(C_{epst}(x\,\delta_{s\to p}\,s')\,i^2\,s') \tag{4.95}$$

$$\exists x_{\blacklozenge\Diamond e}(D_{eepst}\,(x\,\delta_{s\to p}\,s')\,(x'\,i^2\,s')\,i^3\,s') \tag{4.96}$$

$$\exists x'_{\blacklozenge\Diamond e}(D_{eepst}\,(x\,i^1\,s')\,(x'\;\delta\;s')\,i^3\,s') \tag{4.97}$$

We call terms of the form (4.86) or (4.87) where A_t and B_t satisfy (i) and (ii) above (respectively), *top-level type t* terms. The set of top-level type t terms will be writen TOP_t. Let u_e be a constant such that $[\![\,u\,]\!]^{\mathfrak{M},g}$ denotes the utterer of a sentence. We call a term in TOP_t of the form (4.86) where A_t is as above a *special top-level type t term* if either (4.98) and (4.99) both hold, or (4.100) holds:

$$\forall i'_p\,(i' \in ST\;[\;\;\mathbf{S}\,[s^0\;\;s'\;\;i^0]\,\mathbf{P}[x^0\;\;s'\;\;s'']A_t\;\;] \tag{4.98}$$
$$\Rightarrow i' = E) \wedge \forall\delta_{s\to p}\,(\delta \in ST\;[\;\;\mathbf{S}\,[s^0\;\;s'\;\;i^0]\,\mathbf{P}[x^0\;\;s'\;\;s'']A_t\;\;] \Rightarrow \delta = \eta_s E)$$

$$\forall s_s(s \in ST\;[\mathbf{S}\,[s^0\;\;s'\;\;i^0]\,\mathbf{P}[x^0\;\;s'\;\;s'']\,A_t] \tag{4.99}$$
$$\Rightarrow s \in CON_{s_c} \cup VAR_{s_c})$$

$$\forall i''_p, \delta_{s\to p}(i'', \delta \in ST\ [\mathbf{S}\,[s^0\ s'\ i^0]\,\mathbf{P}[x^0\ s'\ s'']\,A_t] \Rightarrow i'' \neq E \wedge \delta \neq (\eta_s E) \wedge i^0 = \kappa\, u\, s^0) \tag{4.100}$$

We call a term in TOP_t of the form (4.87) where B_t is as above a *special top-level type t term* if either (4.101) and (4.102) both hold, or (4.103) holds (again u_e is a constant such that $[\![\, u\,]\!]^{\mathfrak{M},g}$ denotes the utterer of a sentence):

$$\forall i'_p\,(i' \in ST[\forall s'(s \leqslant s' \wedge \pi\,(\delta^0\, s)\, s\, s' \to B_t] \Rightarrow i' = E) \wedge \forall \delta_{s\to p}\,(\delta \in ST\ [\forall s'(s \leqslant s' \wedge \pi\,(\delta^0\, s)\, s\, s' \to B_t] \Rightarrow \delta = \eta_s E) \tag{4.101}$$

$$\forall s_s(s \in ST[\forall s'(s \leqslant s' \wedge \pi\,(\delta^0\, s)\, s\, s' \to B_t] \Rightarrow s \in CON_{s_c} \cup VAR_{s_c}) \tag{4.102}$$

$$\forall i''_p, \delta_{s\to p}(i'', \delta \in ST\ [\forall s'(\pi\, i^0\, s^0\, s' \to B_t] \Rightarrow i'' \neq E \wedge \delta \neq (\eta_s\, E) \wedge i^0 = \kappa\, u\, s^0) \tag{4.103}$$

T_t is the set of special top-level type t terms and the *special top-level terms* are the terms contained in $T_e \cup T_t$ (i.e, the terms which are either special top-level type e terms or special top-level type t terms). In the next chapter, we will only be interested in models in which special top-level type t terms may be true in a model, but in which non-special top-level type $x \in TOP_t$ terms are undefined. To this end, we will impose the condition (4.104):

$$\forall x(x \in TOP_t \text{ and } x \notin T_t \Rightarrow [\![\, x\,]\!]^{\mathfrak{M},g} = \#) \wedge \exists y(y \in T_t \wedge [\![\, y\,]\!]^{\mathfrak{M},g} \in D_t \backslash \#) \tag{4.104}$$

In admissible P-HYPE models in which (4.104) is true, some special top-level terms of type t are interpreted as members of $D_t \backslash \{\#\}$, but all non-special top-level terms of type e denote $\#$.

By confining models to those in which only special top-level type t terms may be true in a model, and non-special top-level type t terms are undefined, we eliminate certain 'mixed' readings of top-level terms which would otherwise arise which one expression is interpreted with respect to the enlightened perspective and another is interpreted relative to another perspective index. Furthermore, if a special top-level type t term of the form (4.86) is true in a model, then the first π occurring in it can only be saturated with $\kappa\, u$ or with $\eta_s E$. This is meant to capture, in part, AG's intuition that an utterance is only really ever from the perspective of the utterer, except that we also allow an utterance to be given from the enlightened perspective.

4.4.3 *Prove* and veridicality

Verbs such as *regret* and *resent* are called *factive verbs* (see (Djarv 2019) and references therein for analysis and discussion of factivity) since they give rise to the presupposition that their complements are true. For example, both an un-negated sentence of the form *Pat regretted he ate the Ramen* and its negation *Pat didn't regret he ate the Ramen* presuppose that Pat ate the Ramen. In addition, a sentence whose main verb is factive typically implies that the person uttering takes the complement to be true, so that the perspective of the utterer of a sentence containing a factive verb is relevant to its interpretation. Exactly how to cash out the presuppositions and inferential patterns of factive verbs is a delicate matter, and the reader is referred to (Djarv 2019) for further discussion.

In this section we are concerned only to secure certain inferential patterns of the verb *prove*. The verb *prove* is not *factive*, since *A didn't prove that S* doesn't entail *S* is true. Consider by way of example (6c) and (6d). The truth of (6c) doesn't require the truth of (6d); rather, in the jargon. (Egré 2008, Uegaki et al. 2015), the verb *prove* is *veridical*, since *A proved that S* entails *S* is true, hence an utterance of (6a) presupposes the truth of (6b):

(6) a. John proved that the prime numbers are infinite.
  b. The prime numbers are infinite.
  c. John didn't prove that the prime numbers are infinite.
  d. The prime numbers are not infinite.

We now wish to ensure that, if, relative to the enlightened perspective index, it is true that someone proved that *S* then the rendering of *S* is true is true with respect to the enlightened perspective index and that if they have proven that the sentence which expresses the negation of *S* is true, then the rendering of the sentence which expresses the negation of *S* is true is true with respect to the enlightened perspective index.[1] We now lay down some conditions in order to ensure this. Similar conditions could be imposed to capture the entailment patterns of factive verbs such as *regret*, but the issues such verbs give rise to are orthogonal to the thesis, and so we will not give an analysis of factive verbs.

Let C_{epst} be in $\mathcal{C}$, the set of injective perspectival predicates of type $e \to p \to s \to t$ and D_{epst} be in $\mathcal{D}$, the set of injective perspectival predicates of type $e \to e \to p \to s \to t$. Then, for any $s^0 \in TERM_s$ and $d \in TERM_e$, and for all

[1]We said above that sentences whose main verbs are factive verb with *that*-clause complements (such as *regret*) typically imply that the person uttering them takes the complement to be true. For this reason, we could also lay down further conditions that require that if Pat utters that John has proven that *S* is true, and *S* is true either a negated or un-negated sentence, then the rendering of *S* is true with respect to Pat's perspective index. We won't lay down these conditions but they should be simple to impose.

$i, i^1, i^2, i^3 \in TERM_p$, relative to an admissible model $\mathfrak{M}$ and assignment g, we require that (4.125)-(4.128) hold:

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow C_{epst}(x\, i^1\, s'')\, i^2\, s''] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket C_{epst}(x\, E\, s^0)\, E\, s^0 \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.105}$$

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow D_{eepst}\,(x\, i^1\, s'')\,(x'\, i^2\, s'')\, i^3\, s''] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket\, D_{eepst}\,(x\, E\, s^0)\,(x'\, E\, s^0)\, E\, s^0 \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.106}$$

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow \forall s'''(C_{epst}(x\, i^1\, s''')\, i^2\, s''' \rightarrow s'' \bot s'''] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket\, \forall s'(C_{epst}(x\, E\, s')\, E\, s' \rightarrow s' \bot s^0) \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.107}$$

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow \forall s'''(D_{eepst}\,(x\, i^1\, s''')\,(x'\, i^2\, s''')\, i^3\, s''' \rightarrow s'' \bot s''')] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket\, \forall s'(D_{eepst}\,(x\, E\, s')\,(x'\, E\, s')\, E\, s' \rightarrow s' \bot s^0\,) \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.108}$$

With these conditions to hand, sentences of the form *A proved that S* or *A proved that not S*, which are semantically interpreted as functions $g \in D_{\blacklozenge\lozenge t}$, are true with respect to a function $f \in D_{s\rightarrow p}$ and a state $s \in D_s$ if and only if both (i) $g\, f\, s = 1$ and (ii) g is true with respect to the enlightened perspective at s (i.e, $g\, h\, s = 1$, where $h = \llbracket \eta\, E \rrbracket^{\mathfrak{M},g}$). Since these sentences must be true with respect to the enlightened perspective, they must be true, and not merely true relative to a perspective index, given the enlightened perspective index captures the classical truth conditions of a sentence.

We can now define the notion of a solipsistic P-HYPE model:

Definition 19 An admissible P-HYPE model $\mathfrak{M}$ is called a *Solipsistic P-HYPE* model if it is such that:

- κ and π satisfy the conditions (4.57), (4.58) and (4.59), repeated here as (4.109), (4.110) and (4.111):

$$\forall x, y \in D_e, \forall s, s' \in D_s \left([\![\kappa]\!]^{\mathfrak{M},g}\, x\, s = [\![\kappa]\!]^{\mathfrak{M},g}\, y\, s' \to x = y \right) \tag{4.109}$$

$$\forall i \in D_p (i = E \vee \exists x \in D_e \exists s \in D_s \left([\![\kappa]\!]^{\mathfrak{M},g}\, x\, s = i \right)) \tag{4.110}$$

$$\forall d_1, d_2 \in CON_e \cup VAR_e \forall s,\ s' \in CON_s \cup VAR_s \tag{4.111}$$
$$\big([\![\, d_1 \,]\!]^{\mathfrak{M},g} \neq [\![\, d_2]\!]^{\mathfrak{M},g} \to$$
$$\{s'' \in D_s \mid [\![\, \pi(\kappa\, d_1\, s)\, s'\, s'' \,]\!]^{\mathfrak{M},g} = 1\} \ \cap \{s'' \in D_s \mid [\![\pi(\kappa\, d_2\, s)\, s'\, s'' \,]\!]^{\mathfrak{M},g} = 1\}$$
$$= \varnothing \,\big)$$

- There is a non-empty set $\mathcal{N} \subseteq CON_{pse}$ of injective perspectival names, which are terms satisfying (4.112), (4.61) and (4.62), here repeated as (4.112), (4.113) and (4.114) (respectively):

$$\forall n_{pse} \in \mathcal{N} (ST[n] = \{n\}) \tag{4.112}$$

$$\forall n_{pse} \in \mathcal{N}\ \forall s,\ s' \in D_s\, \forall i,\ i' \in D_p \tag{4.113}$$
$$([\![n]\!]^{\mathfrak{M},g}\, i\, s = [\![n]\!]^{\mathfrak{M},g}\, i'\, s' \ \to i = i')$$

$$\forall n_{pse} \in \mathcal{N}\ \forall s,\ s', s'', s''' \in D_s\, \forall i,\ i' \in D_p \tag{4.114}$$
$$(i \neq i' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i\, s\, s' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i'\, s''\, s'''$$
$$\to [\![n]\!]^{\mathfrak{M},g}\, i\, s' \neq [\![n]\!]^{\mathfrak{M},g}\, i\, s''' \wedge [\![n]\!]^{\mathfrak{M},g}\, i'\, s''' \neq [\![n]\!]^{\mathfrak{M},g}\, i'\, s')$$

- There is a non-empty set $\mathcal{C} \subseteq CON_{epst}$ and a non-empty set $\mathcal{D} \subseteq CON_{eepst}$ of injective perspectival predicates, which are terms satisfying the conditions (4.63), (4.64), (4.65), (4.66), and (4.67), here repeated as (4.115), (4.117), (4.118), and (4.119), respectively:

$$\forall C_{epst} \in \mathcal{C}\ \forall s, s' \in D_s, \forall i_1, i_2 \in D_p : \tag{4.115}$$

$$\forall C_{epst} \in \mathcal{C}\ \forall s, s' \in D_s, \forall i_1, i_2 \in D_p : \tag{4.116}$$

$$\{\, e \mid [\![\, C_{epst} \,]\!]^{\mathfrak{M},g}\, e\, i_1\, s\ = 1\} = \{e \mid\ [\![\, C_{epst} \,]\!]^{\mathfrak{M},g}\, e\, i_2\, s' = 1\} \rightarrow i_1\ =\ i_2$$

$$\forall D_{eepst} \in \mathcal{D}\ \forall s, s' \in D_s, \forall i_3, i_4 \in D_p : \tag{4.117}$$

$$\{\, (e_1, e_2) \mid [\![\, D_{eepst} \,]\!]^{\mathfrak{M},g}\, e_1\, e_2\, i_3\, s\ = 1\} = \{(e_3, e_4) \mid\ [\![\, D_{eepst} \,]\!]^{\mathfrak{M},g}\, e_3\, e_3\, i_4\, s' = 1\}$$

$$\rightarrow i_3\ =\ i_4$$

$$\forall C_{epst} \in \mathcal{C}\ \forall s,\ s', s'', s''' \in D_s\, \forall i,\ i' \in D_p\big(i \neq i' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i\, s\, s' \tag{4.118}$$

$$\wedge\, [\![\pi]\!]^{\mathfrak{M},g}\, i'\, s''\, s''' \rightarrow \{\, e \mid [\![\, C_{epst} \,]\!]^{\mathfrak{M},g}\, e\, i\, s'\ = 1\} \neq \{e \mid\ [\![\, C_{epst} \,]\!]^{\mathfrak{M},g}\, e\, i\, s''' = 1\}$$

$$\wedge\, \{\, e \mid [\![\, C_{epst} \,]\!]^{\mathfrak{M},g}\, e\, i'\, s'\ = 1\} \neq \{e \mid\ [\![\, C_{epst} \,]\!]^{\mathfrak{M},g}\, e\, i'\, s''' = 1\}\big)$$

$$\forall D_{eepst} \in \mathcal{D}\ \forall s,\ s', s'', s''' \in D_s\, \forall i,\ i' \in D_p \tag{4.119}$$

$$(i \neq i' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i\, s\, s' \wedge [\![\pi]\!]^{\mathfrak{M},g}\, i'\, s''\, s'''$$

$$\rightarrow \{\, (e_1, e_2) \mid [\![\, D_{eepst} \,]\!]^{\mathfrak{M},g}\, e_1\, e_2\, i\, s'\ = 1\} \neq$$

$$\{(e_3, e_4) \mid\ [\![\, D_{eepst} \,]\!]^{\mathfrak{M},g}\, e_3\, e_4\, i\, s''' = 1\} \wedge \{\, (e_1, e_2) \mid [\![\, D_{eepst} \,]\!]^{\mathfrak{M},g}\, e_1\, e_2\, i'\, s'\ = 1\}$$

$$\neq \{(e_3, e_4) \mid \; [\![D_{eepst}]\!]^{\mathfrak{M},g} \, e_3 \, e_4 \, i' \, s''' = 1\}$$

- There is a non-empty set $\mathcal{E}_n \subseteq \mathcal{N}$ of E-synonymous names, a non-empty set $\mathcal{E}_{1-place} \subseteq \mathcal{C}$ of E-synonymous predicates and a non-empty set $\mathcal{E}_{2-place} \subseteq \mathcal{D}$, of E-synonymous predicates. E-synonymous names satisfy (4.68) (repeated below as (4.120)) and E-synonymous predicates satisfy (4.121) or (4.122):

$$\forall s \in D_s([\![n_{pse}]\!]^{\mathfrak{M},g} \, E \, s = [\![n'_{pse}]\!]^{\mathfrak{M},g} \, E \, s) \tag{4.120}$$

$$\forall x \in D_e \, \forall s \in S([\![C_{epst}]\!]^{\mathfrak{M},g} \, x \, E \, s = 1 \text{ if and only if } [\![C'_{epst}]\!]^{\mathfrak{M},g} \, x \, E \, s = 1) \tag{4.121}$$

$$\forall x, y \in D_e \, \forall s \in S([\![D_{eepst}]\!]^{\mathfrak{M},g} \, x \, y \, E \, s = 1 \text{ if and only if } [\![D'_{eepst}]\!]^{\mathfrak{M},g} \, x \, y \, E \, s = 1) \tag{4.122}$$

- Every member of the set T_e of special top-level terms of type e is such that such that the condition (4.85) (repeated below as (4.123)) holds and every member of the set T_t of special top-level terms of type t is such that the condition (4.104) (repeated below as (4.124)) holds:

$$\forall x(x \in TOP_e \text{ and } x \notin T_e \Rightarrow [\![\, x \,]\!]^{\mathfrak{M},g} = \#) \wedge \exists y(y \in T_e \wedge [\![\, y \,]\!]^{\mathfrak{M},g} \in D_e \backslash \#) \tag{4.123}$$

$$\forall x(x \in TOP_t \text{ and } x \notin T_t \Rightarrow [\![\, x \,]\!]^{\mathfrak{M},g} = \#) \wedge \exists y(y \in T_t \wedge [\![\, y \,]\!]^{\mathfrak{M},g} \in D_t \backslash \#) \tag{4.124}$$

- Where $C_{epst} \in \mathcal{C}$ and $D_{epst} \in \mathcal{D}$, for any $s^0 \in TERM_s$ and $d \in TERM_e$, and for all $i, i^1, i^2, i^3 \in TERM_p$, (4.125)-(4.128) hold:

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow C_{epst}(x\, i^1\, s'')\, i^2\, s''] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket C_{epst}(x\, E\, s^0)\, E\, s^0 \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.125}$$

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow D_{eepst}\,(x\, i^1\, s'')\,(x'\, i^2\, s'')\, i^3\, s''] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket\, D_{eepst}\,(x\, E\, s^0)\,(x'\, E\, s^0)\, E\, s^0 \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.126}$$

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow \forall s'''(C_{epst}(x\, i^1\, s''')\, i^2\, s''' \rightarrow s'' \bot s'''] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket\, \forall s'(C_{epst}(x\, E\, s')\, E\, s' \rightarrow s' \bot s^0) \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.127}$$

$$\llbracket \forall s'[\,[s^0 \leqslant s' \wedge \pi\, i\, s^0\, s'] \rightarrow \forall s''[PROV_{esst}\, d\, s'\, s'' \rightarrow \forall s'''(D_{eepst}\,(x\, i^1\, s''')\,(x'\, i^2\, s''')\, i^3\, s''' \rightarrow s'' \bot s''')] \,\rrbracket^{\mathfrak{M},g} = 1 \Rightarrow \llbracket\, \forall s'(D_{eepst}\,(x\, E\, s')\,(x'\, E\, s')\, E\, s' \rightarrow s' \bot s^0\,) \,\rrbracket^{\mathfrak{M},g} = 1 \tag{4.128}$$

4.5 Conclusion

P-HYPE provides one way of capturing the solipsistic semantic theory of (Asudeh and Giorgolo 2016). P-HYPE builds upon their account of perspective relative interpretation by extending it to predicates. But it turns out that once we try to flesh out their semantic theory, we are led to a more complex typing for proper names, predicates and sentences than they propose, and we are led to a more complicated semantic theory than the one they proposed. In solipsistic P-HYPE models, κ is an injective function, and perspective indices are generated only via κ, with the sole exception of E. In addition, no state is ever in the perspective set of two distinct individuals. In solipsistic P-HYPE models, the interpretation of injective perspectival names is never the same with respect to two or more distinct perspective indices and states, and such names do not have the same interpretation in states accessible via π, even when they are fed the same perspective index. Consequently, even though *Spiderman* may be supplied the perspective index associated with Mary Jane, it will not bear the interpretation which Mary Jane herself attaches to the name. Likewise, in solipsistic P-HYPE models, the interpretation of injective perspectival predicates is never the same with respect to two or more distinct perspective indices and states, and such predicates do not have the same interpretation in states accessible via π, even when they are fed the same perspective index. Nonetheless, certain injective perspectival names have the same interpretation with respect to the enlightened perspective, and certain injective perspectival predicates have the same interpretation with respect to the enlightened perspective. In solipsistic P-HYPE models, the enlightened perspective index projects through formulas, in the sense that, if a formula is defined and contains the enlightened perspective index, then any other perspective index in the formula is the enlightened perspective index, and the formula contains state variables which range only over classical states. In addition, $PROV$ is required to be such that its complement sentence is true with respect to the enlightened perspective at the state of evaluation.

In the next chapter we will see how P-HYPE solves the problems of granularity. We will assume that the models we are using are solipsistic P-HYPE models. But ultimately, solipsistic P-HYPE models may be untenable, for empirical or philosophical reasons. If they are, then we P-HYPE models may still prove adequate for our purposes. The idea of a solipsistic P-HYPE model was arrived at, not as part of a claim that natural language is best understood on the model that (Asudeh and Giorgolo 2016) propose, but rather as a way of trying to formalise their proposal.

Chapter 5

P-HYPE and the problem of granularity: predicates

5.1 Introduction

This chapter shows how P-HYPE can solve the problem of granularity, using perspective relativity to distinguish mathematical and other seemingly synonymous predicates. It then (in 5.3) discusses various objections to the proposed solution: that the solution leaves no space for objectivity (5.3.1); that it leads inexorably to problems in capturing the possibility of coherent disagreement (5.3.2); that a simpler alternative that treats predicates as intensional functions of type $s \to e \to t$ is superior (5.3.3); and that the proposed solution engenders problems regarding the explanation of communication (5.3.4). In the second half of the chapter we discuss how unembedded sentences are modelled in P-HYPE (in 5.4) and give compositional derivations to illustrate, and then we discuss how embedded sentences are treated 5.5, again with compositional derivations to illustrate. In 5.6 we compare P-HYPE to the semantic theories of (Cresswell and Stechow 1982), (Tancredi and

Sharvit 2020) and (Stalnaker 1978), which we examined in chapter 2.

5.2 How does P-HYPE solve the problem of granularity?

The basic strategy for dealing with the problem of granularity in P-HYPE is this: in P-HYPE, the semantic values of sentences are functions in $D_{\blacklozenge\lozenge t}$, which include a type p argument. Since the semantic values of sentences include a type p argument, we avoid the problematic predictions of *Mathematical Uniformity* because sentences expressing mathematical truths can be false at certain perspective indices, and sentences expressing mathematical falsities can be true at certain perspective indices. Likewise, we avoid the problematic predictions of *Predicate Uniformity* because seemingly synonymous predicates such as *oculist* and *ophthalmologist* are rendered as the lambda terms 5.1 and 5.2 (abbreviated as **oculist** and **ophthalmologist**, respectively). **oculist** and **ophthalmologist** are built on the perspective relative predicates $oculist_{epst}$ and $ophthal_{epst}$ (respectively), which can differ in semantic value with respect to certain perspective indices, so that we can distinguish the semantic contribution of the natural language predicates *oculist* and *ophthalmologist* in propositional attitude reports:

$$\textbf{oculist} := \tag{5.1}$$

$$\lambda x, \delta, s. \begin{cases} oculist_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ oculist_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{ophthalmologist} := \tag{5.2}$$

$$\lambda x, \delta, s. \begin{cases} ophthal_{epst}\,(x\,\delta\;s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\;s\;(\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\;s\;(\delta\,s) \neq \top \\ ophthal_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

Likewise, we avoid the problematic predictions of *Predicate Uniformity* in the case of mathematical predicates, because seemingly synonymous mathematical predicates such as *inductive* and *finite* are rendered as the lambda terms 5.3 and 5.4 (abbreviated as **inductive** and **finite**, respectively). **inductive** and **finite** are built on the perspective relative predicates $inductive_{epst}$ and $finite_{epst}$ (respectively), which can differ in semantic value with respect to certain perspective indices, so that we can distinguish the semantic contribution of the natural language predicates **inductive** and **finite** in propositional attitude reports:

$$\textbf{inductive} \;:= \tag{5.3}$$
$$\lambda x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s. \begin{cases} inductive\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\;s\;(\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\;s\;(\delta\,s) \neq \top \\ inductive\,(x\;(\eta_s E)\;s)\;E\;s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{finite} \;:= \tag{5.4}$$
$$\lambda x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s. \begin{cases} finite\,(x\,\delta\;s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\;s\;(\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\;s\;(\delta\,s) \neq \top \\ finite\,(x\;(\eta_s E)\;s)\;E\;s & \text{if } \delta\,s = E \end{cases}$$

For example, on our analysis, (1a) might be true whilst (1b) is false, despite the fact that the predicates *inductive* and *infinite* are equivalent in ZFC on the definition of *inductive* and *finite* discussed in (Cresswell 1985: p.82) according to which a set is defined to be ‘finite’ iff it cannot be put into a one-one correspondence with a proper subset of itself, and a set is ‘inductive’ iff it can be put into a one-one correspondence with a proper initial segment of the natural numbers:

(1) a. Harold proved that the prime numbers are infinite.
    b. Harold proved that the prime numbers are not inductive.

(1a) and (1b) may differ in truth value since the predicates *inductive* and *finite* express functions from perspective indices to functions from HYPE states to the characteristic function of a set of entities.

Au fond, the idea here is that the truth value of (1a) varies depending on whether the utterer or another salient agent is in an informational state which distinguishes *infinite* and *not inductive*, and this informational state plausibly varies depending on the context. This coheres with the following attractive picture of information, from (Barwise 1997). As we learn more about the world, certain possibilities become less plausible, because we increase the information at our disposal and so are able to eliminate them. *Prima facie*, mathematical inquiry is such that, as we increase the available mathematical information at our disposable, things we considered mathematically possible before, are no longer considered to be so. In this respect, mathematical inquiry is similar to other forms of inquiry, in which certain possibilities become less plausible and are eliminated as we increase the information at our disposal.[1] As (Barwise 1997) points out, attention to the limited informational perspective of agents in various situations allows us to contemplate a situation in which an agent might consider possible some mathematical or physical impossibility (such as a round square). The picture of information just outlined fits nicely with a view like ours, on which the question of whether two natural language predicates have the same meaning can vary depending on the perspective induced

[1] There are also important differences between mathematical inquiry and other forms of inquiry. For example, when we prove something, there is a sense in which the conclusion of the proof is already contained in the premisses of the proof.

by what information speakers have at their disposal.

On most approaches to perspective relativity in natural language semantics, the contrast between 'objective' and 'subjective' predicates is encoded in the typing of predicates. Numerical predicates such as *is 4 metres tall* and mathematical predicates such as *infinite* and *not inductive* which appear to express objective properties, are given types in which p does not occur. In contrast to most approaches to perspective-relativity, however, we will treat all predicates of individuals as being of type $e_0 \rightarrow .... \rightarrow e_n \rightarrow p \rightarrow s \rightarrow t$, or some more complex type of which p is a part. Instead, the intuition that the predicate *is 4 metres tall* does not apply subjectively to certain objects is captured by the fact that relative to the enlightened perspective index, *is 4 metres tall* has its usual interpretation, so that the entities which are really 4 metres tall are those which are 4 metres tall according to the enlightened perspective. If *is 4 metres tall* is given a type in which p occurs, we can account for the possibility that there are certain agents who believe that an object is 4 metres tall but who do not believe that the object is 13.12336 feet tall, even though the set of objects that are 4 metres tall is the same set as the set of objects which are 13.12336 feet tall.

In addition, we claim that for utterances of unembedded sentences, the default perspective index in many contexts is the enlightened perspective index. If this were not the case, then clear cases of objectively true sentences such as *The prime numbers are infinite* would regularly come out false, relative to the perspective index of certain agents. It is only in certain contexts that people's perspectives on the interpretation of their words becomes important, and amongst these are contexts in which propositional attitude reports are made, but we sometimes distinguish the meaning of sentences outside of propositional attitude contexts. For example, even though the rendering of *The prime numbers are infinite* may be true with respect to the enlightened perspective index if and only if the rendering of *The prime numbers* *are not inductive* is true with respect to the enlightened perspective index, and the set of states in which these sentences are true are the same with respect to the enlightened perspective index, these sentences may denote different sets of states with respect to a non-enlightened perspective index. The fact that the default perspective index in many contexts is the enlightened perspective index is analogous to the fact that the default state in which we evaluate our assertions in many contexts is the actual state which we inhabit, or to the fact that the default context relative to which we fix the denotation of certain words is the context of utterance in the actual state we inhabit.

Perspective indices, as we understand them, are somewhat like the speaker parameter sometimes included in the context of utterance (Kaplan 1989) in order to model indexical pronouns; however, their role is not to pick out any particular agent, but rather to pick out a domain over which expressions will be interpreted, a domain which is supposed to represent the perspective of a given agent on the universe that they inhabit. When we supply a perspective index to a perspective relative expression, it picks out something in the domain associated with that perspective, and this domain may not overlap with the domain of objects in the state we inhabit. Accordingly, predicates such as *inductive* and *finite*, relative to a perspective index other than the enlightened perspective index, will not denote, respectively, the class of inductive sets or finite sets proper; rather, they will denote some private mental proxy of the class of inductive sets and the class of finite sets. Likewise, *the prime numbers*, taken as a function from a perspective index to a function from states to entities, will not denote the prime numbers proper relative to a perspective index i and state s, but some private mental proxy of the prime numbers at s associated with the perspective index i, and this private mental proxy is intended to capture part of the perspective on the state of the individual associated with the perspective index i.[2] Nevertheless, we incorporate a special designated perspective index called the *enlightened perspective* relative to which expressions receive their ordinary intensional interpretation, and relative to which (1a) and (1b) have the same truth value. The role of the enlightened perspective in our account is to avoid some bad predictions our theory would otherwise make. For example, intuitively, (2a) entails (2b):

(2) a. John believes that Susan arrived at 5 o'clock

b. John believes that Susan arrived.

Normally, this entailment would be captured by having the set of states (or the set of events in an event semantics) in which a person arrives at a certain location before a certain time being a subset of the set of states (or events) in which that person arrived at that location. However, in our semantic theory, John could associate a distinct semantic value with the predicates *arrived at 5 o'clock* and the predicate *arrived*. This would happen, for example, if John had certain unorthodox beliefs about the word *arrive*, whence the inference from (2a) to (2b) would not be truth-

[2] We presume that, in order to have a perspective on the prime numbers, Harold has to grasp, in some way or another, what the prime numbers are, or be aware of them, but he will not necessarily have come across the English predicate *prime number*, but that nevertheless the semantic interpretation he attaches to *prime number* is his own private interpretation. If perspectives are constrained in this way, we could then require that a belief report uttered relative to the perspective index of the subject of the belief report is only defined if the subject is aware of the concepts involved in the belief report in some way. We are not sure exactly how to cash out this proposal, but it would be interesting in future work to explore how the notion of perspective could be combined with the notion of awareness as explored in awareness logics logics (Sillari 2006), in which each agent is assigned a set of formulas that comprise that person's 'awareness set', intuitively, the set containing formulas that the agent is aware of.

preserving! Relative to the enlightened perspective index, the inference from (2a) to (2b) goes through, however, which is a good result, since, intuitively, (2b) follows from (2a) in many contexts of utterance. Thus any logical consequence that follows when we use a simple Hintikkian belief operator can be captured in our framework by employing the enlightened perspective index.

One question which naturally arises is which perspective indices can we interpret sentences relative to. In the literature on perspective relativity (see (Lasersohn 2017) and references therein, (Pearson 2013), (Sauerland and Schenner 2009), (Asudeh and Giorgolo 2016)) the perspective of the grammatical subject of a sentence containing an attitude verb and the perspective of the utterer of that sentence are held to play a particularly important role in the semantics of perspectival language.[3] Accordingly, in P-HYPE the perspective index associated with the grammatical subject and the perspective index associated with the utterer of a sentence play an important role (in addition to the enlightened perspective index). In addition to these particular perspective indices, P-HYPE will also allow expressions to be interpreted relative to perspective indices of other agents who may be present or salient in the context of utterance.[4] The role of $C_{\mathbf{p}}$ is precisely to filter out the

[3]Consider an utterance of (these examples are from (Pearson 2020)):

(i) John thinks that the vase is to the left of the couch.

(i) can be used to indicate that the case is to the left of the couch from the position of the speaker, or from the position of John.

[4]The literature on perspective relativity discusses cases in which we take the perspective of another individual salient in the context. Consider (i):

(i) John thinks the cat food is tasty, so we don't have to worry about the cat.

In certain situations, (i) has a reading on which we interpret *is tasty* relative to the perspective of the perspectives which are not available in the context, though we have nothing illuminating to say in this thesis about how we decide what perspective indices are available in a context and we leave this aspect of our account for future work.

We employ a more radical form of perspective relativity than that usually encountered in the literature, in two ways. Firstly, on our account, predicates which are usually thought to express paradigmatically objective properties, such as *finite* and *inductive*, will be perspective relative, and their perspective relativity will be manifest in both sentences embedded under clause-taking verbs and in unembedded sentences like (3) and (4), unless we interpret the sentences relative to the enlightened perspective index, on which they are equivalent:

(3) The prime numbers are infinite.

(4) The prime numbers are not inductive.

This contrasts from the standard use of perspective relativity in the semantics literature, in which such seemingly 'objective' predicates are not treated as perpective relative (see (Lasersohn 2017) and references therein).

The second way in which our perspective relative semantic theory may be deemed more radical than others is, that, in principle, we leave open the possibility that perspectives may be associated with non-sentient objects, such as mathematical theories (like ZFC), though we will avoid these in this thesis, and we don't have anything to say in general about when perspectives associated with non-human objects arise and how they are constrained. The advantage of associating a perspective with a mathematical theory is that it allows us to bring out the theory-relativity of certain mathematical utterances. For example, whilst the finite and the inductive sets are identical in ZFC, they are not identical in some constructive forms of set theory.

cat.

Taking this into account, suppose we are in a context in which we are discussing a constructive form of set theory which rejects the equivalence of the predicates *finite* and *inductive* is under discussion and we are considering the availability of constructive proofs of certain theorems. In such a context, it might be true to assert (1a)—knowing that Harold has produced a constructive proof of the primes being infinite– and false to assert (1b), since in the constructive set theory being discussed the equivalence of *finite* and *inductive* cannot be relied upon. We won't discuss this particular case in what follows, but this particular reading of (1a) and (1b) could be modelled in our theory by having the predicates *finite* and *inductive* interpreted with respect to a perspective index associated with the relevant constructive form of set theory.

5.3 Philosophical and linguistic concerns with our solution of the problem of granularity

In this section we will discuss with three issues with our proposed resolution of the problem of granularity. The first issue concerns how we allow for a notion of objective truth in P-HYPE. The second issue concerns how we allow for the possibility that people disagree in P-HYPE. The third issue concerns whether a treatment of seemingly synonymous natural language predicates such as *oculist* and *ophthalmologist* as denoting distinct functions in $D_{s \to e \to t}$ would be possible, instead of supposing that such predicates are perspective relative.

5.3.1 Objective truth

According to one objection, our account would make the truth of sentences such as *The prime numbers are infinite* entirely subjective, which clashes with the intuition that sentences expressing mathematical truths are paradigm cases of sentences which are objectively true. Our response is to draw a contrast between the objective truth-conditions of a sentence, understood as a the actual conditions in which a sentence is true, from what we might call its *perspectival* or *subjective* truth conditions.

For many unembedded sentences, the objective truth conditions of a sentence are those arrived at via the enlightened perspective index. For example, the objective truth conditions of *The prime numbers are infinite* in our account will be generated if we supply the enlightened perspective index to the renderings of the predicates in the sentence *The prime numbers are infinite*. The objective truth conditions will be a function from classical states to truth values, and in all classical states, relative to the enlightened perspective, the prime numbers are infinite. The subjective truth conditions of *The prime numbers are infinite* will be arrived at if we interpret the sentence relative to perspective indices which are not the enlightened perspective index. Our account, contrary to the objection, does not imply that it is a wholly subjective matter whether *The prime numbers are infinite* is true, since the truth of the sentence *The prime numbers are infinite* is arrived at by interpreting it relative to the enlightened perspective. More generally, in order to determine whether certain unembedded sentences are objectively true or false and respect the facts outside our semantic theory, we need to consider the verdict of the enlightened perspective index. Consequently, whilst the rendering of the sentence *The prime numbers are not infinite* is true relative to certain perspective indices, it is objectively false, and so false relative to the enlightened perspective index, at least in those P-HYPE models in which the enlightened perspective index does not implement a strict-finitist view of mathematics! So in the case of sentences like *The prime numbers are infinite* we can distinguish *truth* (lower case) *relative to a perspective index* from *Truth* (capital 't') in some intended P-HYPE model (presumably not the model in which strict-finitism is true). The same remarks could be made of a non-mathematical sentence, such as (5):

(5) The morning star is not the evening star.

(5) is false *objectively speaking* and outside of the semantic theory, because it is false with respect to the enlightened perspective index, though (5) may be true relative to a non-enlightened perspective.

Sometimes the objective truth of a sentence outside the semantic theory is not arrived at by interpreting the sentence relative to the enlightened perspective index. Consider (6b) in the context (6a):

(6) a. *Scenario:* Mary Jane does not know Peter Parker is Spiderman and she loves the man she calls 'Peter Parker'. A speaker who knows or is 'enlightened' (Zimmermann 2005) about Peter Parker's secret identity utters (10b)

b. Mary Jane loves Spiderman.

The claim of AG– or if not, the claim we are making– about sentences such as (6b) is that they can be true or false, in the objective sense, from a given perspective index. In particular:

1. There is a reading of (6b) in scenario (6a) on which it is false, *objectively speaking* and outside of the semantic theory, and that reading is provided by interpreting *Spiderman* relative to Mary Jane's perspective index.

2. There is a reading of (6b) in scenario (6a) on which it is true, *objectively speaking* and outside of the semantic theory, and that reading is provided by interpreting *Spiderman* relative to the enlightened perspective index.

If an unembedded sentence is true relative to the enlightened perspective, then that sentence is objectively true. But if an unembedded sentence is objectively true on a given reading, then it is not necessarily true with respect to the enlightened perspective index. For example, the enlightened perspective index only gives us the truth of one reading of (6b).

We therefore conclude that some unembedded sentences such as *The prime numbers are infinite* and *The morning star is not the evening star* are only objectively true on one reading, and whether they are objectively true is revealed by supplying the enlightened perspective index. Instead, other unembedded sentences such as (6b) are objectively true on two or more readings. There is a class of verbs, such as *love*, whose objective truth does vary according to perspective. But there is another class of verbs, such as *is* when used as an equative in unembedded sentences not containing predicates of personal state (like *fun* or *tasty*), whose objective truth does not vary according to perspective.

So far we have distinguished objective and subjective truth conditions for unembedded sentences. But what about sentences containing a clause-taking verb such as *believe* or *know*? In this case, the objective truth of a sentence of the form *A X that p*, where X is a clause-taking transitive verb, may only require that its complement sentence be true with respect to a perspective index which is salient in the context (except for verbs like *prove*, which we discuss in the next section). Suppose that in all Pierre's doxastically accessible states the prime numbers are not infinite, when the predicates *prime numbers* and *infinite* are interpreted relative to Pierre's perspective index. Then in certain contexts of use in which Pierre's perspective index is the most salient, the objective truth conditions of (7) will be arrived at by interpreting (7) relative to Pierre's perspective index:

(7) Pierre believes that the prime numbers are not infinite.

The corollary of our account is that our semantic theory is not only in the business of always capturing the objective truth of certain sentences, but in capturing their perspective relative truth. Our semantic theory in this way captures a dimension of the meaning of a sentence which goes beyond the objective truth conditions of a sentence, if we understand the objective truth conditions of a sentence as the actual conditions in which a sentence is objectively true. Consequently, we will sharply distinguish 'objective truth'–understood as the truth of a sentence outside a particular semantic theory– from truth relative to a state or a perspective, internally to a semantic theory like P-HYPE. Truth relative to a state or a perspective is internally defined within a semantic theory, and is supposed to capture certain ordinary judgements of truth of falsity. Objective truth is not defined internally to a semantic theory, and is a notion which is supposed to capture what is actually true. Such a distinction is less radical than it might seem. The model of the common ground which (Stalnaker 2002) elaborates is supposed to capture truth for the purposes of a conversation, as the output of a co-operative endeavour, based on pragmatic norms of reasoning about the common ground. On the Stalnakerian picture of the common ground, a sentence might be true relative to the information at the disposal of discourse participants, but false, objectively speaking, and the falsity of the sentence might not be secured by the particular model of the common ground at issue. This will happen, for example, if it is common ground that some sentence is true, when, as a matter of fact, that sentence is false.

5.3.2 Disagreement

Consider utterances of the following sentences, which contain the factive verb *prove*:

(8) a. Harold proved that the primes are infinite.

b. Harold proved that the primes are not inductive.

c. (No,) Harold proved that the primes are infinite, not that the primes are not inductive.

One possibly controversial aspect of our account is that we are treating utterances of sentences such as (8a)-(8c) similarly to utterances of sentences in which indexical pronouns occur, such as *I am hungry*, which differ in truth value depending on who utters them. In particular, according to our theory, utterances of (8a) vary in truth value depending on whether their utterers know whether *infinite* and *not inductive* pick out the same set of objects.

The theory might be retained to inexorably lead to a problem regarding disagreement. One type of disagreement occurs when one person utters a sentence and another person utters its negation. Intuitively, this seems to occur in (9a) and (9b). Consider the following dialogue between A and B:

(9) a. A: Harold proved that the primes are infinite.

b. B: (No), Harold didn't prove that the primes are infinite(, he proved that the primes are not inductive).

Our problem is that our formal renderings of *Harold proved that the primes are infinite* as uttered by A and *Harold didn't prove that the primes are infinite* as uttered by B, will not contradict each other, since the former can be true relative to A's perspective index compatibly with *Harold didn't prove that the primes are infinite* being true with respect to B's perspective index. But then it seems our account will have trouble accounting for the datum that *Harold proved that the primes are infinite* and *Harold didn't prove that the primes are infinite* contradict one another, regardless of who utterers them.

Our response to this problem gives us the occasion to clarify a crucial contrast between cases of disagreement such as (9a) and (9b) and traditional cases disagreement involving perspective relativity such as (10a) and (10b):

(10) a. John: This rollercoaster is fun!
b. Mary: This rollercoaster isn't fun!

Intuitively, (10b) contradicts (10a). One characteristic of perspective relative predicates such as *fun* or *tasty* is that they seem to give rise to *faultless disagreement*. Roughly, a faultless disagreement is a disagreement in which none of the parties are making a factual error and we may regard each of them as correct, despite the fact that they seem to contradict one another. In the case of (10a) and (10b), there seems to be a disagreement in which no one is at error. We claim that (9a) and (9b), however, do not give rise to faultless disagreement. They don't give rise to it, since it is not the case that both parties are correct. The correctness of the claim that *Harold proved that the primes are infinite* is given by the interpretation of this sentence relative to the enlightened perspective. So the kind of perspective-relativity which we employ in our semantic theory is importantly different from the kind of perspective relativity identified in the literature hertofore on perspective relativity.

Instead, our analysis of the disagreement in (9a) and (9b) is that it is metalinguistic: the two speakers are disagreeing about what the enlightened perspective on these sentences is. One of them claims that their perspective on the interpretation of the sentence *Harold proved that the primes are infinite* is to count as enlightened in the context of utterance, and the other claims that her perspective on this sentence is to count as enlightened in the context. In cases of disagreement involving factive predicates such as *prove*, an assertion carries the metalinguistic commitment that what is asserted is true relative to the enlightened perspective, and a denial of that assertion carries the metalinguistic commitment that what is denied is false relative to the enlightened perspective.

(Lasersohn 2017: p.38-42) criticises a similar account of disagreement for the following reason:

> "As we have already seen, two people may disagree even if they are geographically separated and unaware of each other's existence: If John, in Urbana, Illinois, asserts that licorice is tasty, and Mary, a stranger he has never met, in Santa Cruz, California, asserts that licorice is not tasty, their assertions contradict each other, even though they are not engaged in conversation with each other and do not presume each other to share a conversational context. Their disagreement cannot be over the nature of "the context." Likewise, John might be a monolingual speaker of English, and Mary a monolingual speaker of German. If John says *Licorice is tasty* and Mary says *Lakritze ist nicht lecker*, they are disagreeing—contradicting each other, in fact—even though they are not in conflict over the meanings of any English or German words."

Lasersohn's criticism, applied to our analysis of disagreement, would be that our analysis of disagreement assumes that the two parties who contradict one another are actually involved in a verbal dispute, perhaps located in the same place. Lasersohn's intuition is that the fact that (10b) contradicts (10a), and which underpins the idea that they disagree, does not depend on there being a dispute between John and Mary in a particular location. Our theory accounts for this intuition nicely. For even if John and Mary are separated physically and do not know of each other's existence, they can still think their answer to the question "What is the enlightened perspective on this sentence?" is true, and that other answers are false. It is in this sense that we claim that they disagree. An analysis of an utterance of (9a) and an utterance of (9b) which captures the fact that they disagree is possible, and does not require that there exists a verbal dispute between the utterer's of these sentences.

We give a similar analysis of the disagreement between A and B in the dialogue below:

(11) a. A: Mary believes that John is an oculist.
b. B: (No), Mary doesn't believe that John is an oculist(, she believes that he is an ophthalmologist).

In this case, whether *Mary believes that John is an oculist* is objectively true in a context in which Mary's perspective index is the most salient, is determined by considering Mary's perspective index, and the disagreement in this case is over what Mary's perspective is. Again, it is not the case that both parties A and B are correct, and so faultless disagreement does not arise. In cases of disagreement involving non-factive predicates such as *believe*, an assertion carries the metalinguistic commitment that what is asserted is true relative to perspective index of the grammatical subject, and a denial of that assertion carries the metalinguistic commitment that what is denied is false relative to the perspective index of the grammatical subject.

5.3.3 Seemingly synonymous predicates as bog-standard intensional predicates

One might wonder what is inadequate with the treatment of the natural language predicates *oculist* and *ophthalmologist* as constants $oculist' : s \rightarrow e \rightarrow t$ and $ophthalmologist' : s \rightarrow e \rightarrow t$ which denote different functions in $D_{s \rightarrow e \rightarrow t}$, where type s is defined as $(s_c + s_n) \times time$, as in 4.2.[5] We think it is better to attach a

[5]In 4.3.1 we also discussed an alternative treatment of type s as $\Diamond p \times ((s_c + s_n) \times time)$. Our criticisms in this section do not apply to that alternative treatment.

perspective relative semantic value to the natural language predicates *oculist* and *ophthalmologist*, for the following reasons. Firstly, making the denotation of *infinite* and *not inductive* distinct functions in $D_{s \to e \to t}$ does not allow these predicates to differ in meaning to certain agents. This doesn't fit well with the view of possibilities (inspired by (Barwise 1997)) we outlined above, according to which the meanings that we attach to words, and the possibilities we consider live, vary depending on the situations in which we find ourselves, and in one and the same state two agents can bear different cognitive relations to the same natural language predicate. A semantics of predicates which takes into account the cognitive relations that agents bear to natural language predicates is therefore philosophically superior to one which adopts a one sized fits all approach on which natural language predicates have the same meaning for all agents. By contrast, in our account, whether two predicates are taken to have the same meaning to an agent will depend on the information that agent has at their disposal and their mental attitude towards the predicates – their perspective. It is this aspect of the semantics of predicates that the notion of perspective is supposed to capture, just as AG are trying to capture in the case of proper names.

Treating $oculist' : s \to e \to t$ and $ophthalmologist' : s \to e \to t$ as two different functions in $D_{s \to e \to t}$ is also inadequate for two further reasons. Firstly, if the semantic value of $oculist' : s \to e \to t$ and $ophthalmologist' : s \to e \to t$ were distinct, then this would predict that (12b) is just as felicitous as (12a), since the semantic value of $ophthalmologist' : s \to e \to t$ and $baker : s \to e \to t$ will presumably be treated as distinct:

(12) a. Mary believes that John is an ophthalmologist, but she doesn't believe that he's a baker.

b. Mary believes that John is an ophthalmologist, but she doesn't believe he's an oculist.

This is a bad prediction, because, in most contexts, sentences such as (12b) are not felicitous, whereas sentences such as (12a) are. In the perspective relative semantic theory which we adopt, the difference between (12b) and (12a) is that, relative to most perspective indices, the natural language predicates *oculist* and *ophthalmologist* have the same interpretation, whereas relative to most perspective indices, the natural language predicates *ophthalmologist* and *baker* have a distinct interpretation, and this is why utterances of (12b) are usually not felicitous, whereas utterances of (12a) are. Treating $oculist' : s \rightarrow e \rightarrow t$ and $ophthalmologist' : s \rightarrow e \rightarrow t$ as denoting different functions in $D_{s \rightarrow e \rightarrow t}$ is also inadequate because it fails to get at what grounds the contradictory from the non-contradictory reading of (12b). In particular, whether (12b) is contradictory depends, intuitively, on what we are assuming about the lexical knowledge and particularities of the utterer of (12b), or of people salient in the context; in sum, it depends on what perspective people salient in the context take on the interpretation of the predicate *oculist*. This is the key idea behind our account.

5.3.4 Communication

It might be objected that the semantic theory of AG cannot explain how communication occurs. The semantic theory of AG is consistent with the dominant idiolectical approach to semantic theory in linguistics (Chomsky 1986, Chomsky et al. 2000, Adger and Trousdale 2007, Lohndal and Narita 2009, Pietroski 2017) according to which the semantic values of linguistic expressions do not represent objects 'out there' in the real world, but pick out private mental representations of some kind. The idiolectical approach is often counterposed to a picture on which a semantic theory must always attach semantic values to expressions which represent objects which are external to the mind of individual speakers (Dummett 1976) on the basis that communication can only proceed if semantic values are shared between different speakers.

We won't enter into the details of objections of this kind, and limit ourselves here to three remarks. First, the objection seems to beg the question; namely, that it is the job of semanticists to construct a model of communication. We rather take the view that the explanation of communication is something which cognitive scientists or biologists have to explain, and it is not clear how semanticists actually have the right skill sets to address this question. Of course, some philosophers have outlined models of communication, such as (Stalnaker 1999). But, it seems to us that, until these models of communication are backed up with evidence from cognitive and biological science, they are merely speculative models of communication.

Second, we should point out that the (speculative) model of communication on which communication involves shared contents or semantic values that agents attach to expressions is not the only model of communication, and there are in fact philosophical models of communication which have been offered for centered world theories of content, which also make contents idiolectical (Weber 2013). As far as we know, the viability of such models of communication is still an open question, and we would require evidence from the cognitive and biological sciences to decide between idiolectical and non-idiolectical models of communication.

Third, if we distinguish the propositional content of sentences from the semantic values of sentences (please see the appendix, section A.2, for the distinction between the propositional content of sentences and their semantic values), there might be conceptual space for adopting an idiolectical approach to semantic values in natural language but a non-idiolectical account of propositional content which assigns shareable contents to linguistic expressions. To the extent that such an approach is tenable, the semantic theory of AG is not necessarily incompatible with a theory in which contents are shareable.

For these reasons, we are not convinced of the argument that the semantic theory of AG will necessarily have problems in explaining how communication occurs.

5.4 Compositional derivations of some sentences not embedded under a clause-taking verb

In this section we consider sentences which are not embedded under clause-taking verbs. In the previous chapter we introduced π_{psst}, a function assigning the characteristic function of a set of states to a perspective index at a state. Every unembedded sentence which has a perspective relative interpretation will be rendered as a universally quantified formula, either of the form (5.5), or of the form (5.6)

$$\forall s'(s \leqslant s' \land \pi\,(\kappa\, u\, s)\, s\, s' \to A_t) \tag{5.5}$$

$$\forall s'(s \leqslant s' \land \pi\, E\, s\, s' \to A_t) \tag{5.6}$$

in which π is either parameterised to the utterer's perspective index (as in (5.5)) or to the enlightened perspective index (as in (5.6)). If we treat unembedded sentences in this way we ensure that our renderings of unembedded sentence will have the form of the HYPE conditional (in line with the arguments made in 4.3). By the conditions on E and π discussed in the previous chapter, formulas of the form (5.5) are semantically equivalent to formulas of the form (5.8):

$$\forall s'(s \leqslant s' \land \pi\, E\, s\, s' \to A_t) \tag{5.7}$$

$$A_t \tag{5.8}$$

When π is parameterised to the utterer's p-set, this represents the fact that an utterance of an unembedded sentence is a speech act ordinarily made from the standpoint of the utterer. Consider the following lexical entries (we have abbreviated some lambda terms using constants in boldface):

λ-term

$$\mathbf{a} := \lambda Q_{\blacklozenge\Diamond e \to \blacklozenge\Diamond t}, P_{\blacklozenge\Diamond e \to \blacklozenge\Diamond t}, \delta_{\Diamond p}, s_s. \begin{cases} \exists x_{\blacklozenge\Diamond e}(Q\,x\,\delta\,s \wedge P\,x\,\delta\,s) & \text{if } \delta\,s \neq E \\ & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \\ & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ \exists x_{\blacklozenge\Diamond e}(Q\,x\,(\eta_s E)\,s & \\ \wedge\, P\,x\,(\eta_s E)\,s) & \text{if } \delta\,s = E \end{cases}$$

a oculist $:=$

$$\lambda P_{\blacklozenge\Diamond e \to \blacklozenge\Diamond t}, \delta_{\Diamond p}, s_s. \begin{cases} \exists x_{\blacklozenge\Diamond e}(oculist_{epst}(x\,\delta\,s) & \text{if } \delta\,s \neq E \\ (\delta s)\,s \wedge P\,x\,\delta\,s & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \\ & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ \exists x_{\blacklozenge\Diamond e}(oculist_{epst}(x\,(\eta_s E)\,s)\,E\,s & \\ \wedge P\,x\,(\eta_s E)\,s & \text{if } \delta\,s = E \end{cases}$$

a ophthalmologist $:=$

λ-term

$$\lambda P_{\blacklozenge\lozenge e\to\blacklozenge\lozenge t}, \delta_{\lozenge p}, s_s. \begin{cases} \exists x(ophthal_{epst}(x\,\delta\,s) & \text{if } \delta\,s \neq E \\ (\delta s)\,s \wedge P\,x\,\delta\,s & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ & \\ \# & \text{if } \delta\,s \neq E \\ & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ & \\ \exists x(ophthal_{epst}(x\,(\eta_s E)\,s) & \\ E\,s \wedge P\,x\,(\eta_s E)\,s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{believe} \quad := \quad \lambda p_{\blacklozenge\lozenge t}, x_e, \delta_{\lozenge p}, s_s. \begin{cases} \forall s'[\,[s \leqslant s' \wedge \pi(\delta\,s)\,s\,s'] \to & \text{if } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \forall s''[DOX\ x\ s'\,s'' \to & \\ p\,(\eta_s(\kappa\,x\,s''))\,s''\,]\,] & \\ & \\ \# & \text{if } \delta\,s \notin C_{\mathbf{p}} \end{cases}$$

$$and \quad := \quad \lambda q_t, p_t.p \wedge q$$

$$\textbf{is}^1 \quad := \quad \lambda x_{\blacklozenge\lozenge e}, y_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s.\, is^1(x\,\delta\ s)(y\,\delta\,s)\,(\delta s)\,s$$

$$\textbf{BE} \quad := \quad \lambda \mathcal{P}_{(\blacklozenge\lozenge e\to\blacklozenge\lozenge t)\to\blacklozenge\lozenge t}, y_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s.\mathcal{P}(\lambda z_{\blacklozenge\lozenge e}, \delta'_{\lozenge p}, s'_s.is^1\,(z\,\delta'\,s')\,(y\,\delta\,s')\,(\delta'\,s')\,s')\,\delta\,s$$

$$\mathbf{G}_{(\alpha_n\to\beta)\to(\alpha_1\to\alpha_2\to,\ldots,\to\alpha_{n-1}\to\alpha_n)\to\alpha_1\to\alpha_2,\ldots,\alpha_{n-1}\to\beta} :=$$
$$\lambda f_{\alpha_n\to\beta}, g_{\alpha_1\to\alpha_2\to,\ldots,\to\alpha_{n-1}\to\alpha_n}, x^1_{\alpha_1}, x^2_{\alpha_1}, \ldots, x^{n-1}_{\alpha_{n-1}}.f(g\,x^1\,x^2_{\alpha_1}\ldots x^{n-1})$$

λ**-term**
lift[1] := $\lambda p_{\blacklozenge\lozenge t}, \delta_{\lozenge p}, s_s.\, \forall s'(s \leqslant s' \wedge \pi\, E\, s\, s' \to p\, \delta\, s')$
lift[2] := $\lambda p_{\blacklozenge\lozenge t}, \delta_{\lozenge p}, s_s.\, \forall s'(s \leqslant s' \wedge \pi\, (\kappa\, u\, s)\, s' \to p\, \delta\, s')$
$\neg_H := \lambda p_{s\to t}, s_s.\, \forall s'(p\, s' \to s \perp s')$
not[1,H] := $\lambda P_{\blacklozenge\lozenge e\to\blacklozenge\lozenge t}, x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s.\neg_H(P\, x\, \delta)\, s$
$\mathbf{not}^{2,H}_{\blacklozenge\lozenge t\ \blacklozenge\lozenge t} = \lambda p_{\blacklozenge\lozenge t}, \delta_{\lozenge p}, s_s.\neg_H(p\, \delta)\, s$
$john_e$
$mary_e$

Table 5.1: Simplified lexical entries

$john$ and $mary$ are constants of type e, and is just classical conjunction, and **BE** is a perspective-relative version of the following lambda term, which (Montague 1973) adopts for both predicate and equative versions of the copula:

$$\lambda \mathcal{P}_{(e\to t)\to t},\, y_e.\, \mathcal{P}\,(\ \lambda z_e.\, z = y\ ) \tag{5.9}$$

$\mathbf{G}_{(\alpha_n\to\beta)\to(\alpha_1\to\alpha_2\to,\ldots,\to\alpha_{n-1}\to\alpha_n)\to\alpha_1\to\alpha_2,\ldots,\alpha_{n-1}\to\beta}$ is functional composition, for $\alpha_i \in TYPE$ (for $1 \leqslant i \leqslant n$) and $\beta \in TYPE$.

It is useful for allowing negation to apply to both predicates and sentences. For example, $\mathbf{not}^{1,H}$ and $\mathbf{not}^{2,H}$ can be defined as (5.10) and (5.10), respectively:

$$\mathbf{G} \neg_H h_{\blacklozenge\lozenge e \to \blacklozenge\lozenge t} \tag{5.10}$$
$$=_{df\,\mathbf{G}} [\lambda f_{\lozenge t \to \lozenge t}, g_{\blacklozenge\lozenge e \to \blacklozenge\lozenge t}, x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}. f(g\, x\, \delta)] \neg_H h_{\blacklozenge\lozenge e \to \blacklozenge\lozenge t}$$
$$=_\beta \lambda x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}. \neg_H (h\, x\, \delta)$$
$$=_{\beta, df \neg_H} \lambda x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s.\forall s'(h\, x\, \delta'\, s' \to s \bot s')$$

$$\mathbf{G} \neg_H p_{\blacklozenge\lozenge t} \tag{5.11}$$
$$=_{df\,\mathbf{G}} [\lambda f_{\lozenge t \to \lozenge t}, g_{\blacklozenge\lozenge t}, \delta_{\lozenge p}. f(g\, \delta)] \neg_H p_{\blacklozenge\lozenge t}$$
$$=_\beta \lambda \delta. \neg_H (h\, \delta)$$
$$=_{\beta, df \neg_H} \lambda \delta, s.\forall s'(h\, \delta\, s' \to s \bot s')$$

The term **lift**^{1} takes a lambda term $p_{\blacklozenge\lozenge t}$, and returns a lambda term of type $\blacklozenge\lozenge t$ in which p is interpreted with respect to a state s' in the π-set associated with the enlightened perspective index. **lift**^{2} takes a lambda term $p_{\blacklozenge\lozenge t}$, and returns a lambda term of type $\blacklozenge\lozenge t$ in which p is interpreted with respect to a state s' in the π-set associated with the utterer's perspective index. **neg**^{1} takes predicates of type $\blacklozenge\lozenge e \to \blacklozenge\lozenge t$ and negates them. **a oculist** and **a ophthalmologist** are quantifiers which take predicates of type $\blacklozenge\lozenge e \to \blacklozenge\lozenge t$ and return objects of type $\blacklozenge\lozenge t$.

With the lexical entries above in hand, we are ready to consider our treatment of predicates. Consider the following sentences:

(13) a. John is an oculist, not an ophthalmologist.
b. Mary believes that John is an oculist, but she doesn't believe that John is an ophthalmologist.

A detailed compositional derivation of (13a) would have to give an analysis of ellipsis, and as this would involve irrelevant complications, we forego that here. Instead, for simplicity, we will treat (13a) as elliptical for its more stilted cousin (14):

(14) John is an oculist and John is not an ophthalmologist.

The compositional derivation of (5.12) (the rendering of (14)) is given by the lambda term in (5.13):

$$\forall s'(s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s' \rightarrow oculist_{epst}\, john\,(\kappa\, mary\, s')\, s' \wedge \\ \forall s''(\ (\ ophthal_{epst}\ john\ (\kappa\, mary\, s'')\ s''\ ) \rightarrow s' \bot s''\ )\ ) \tag{5.12}$$

$$[\,\mathbf{lift}^2_{\blacklozenge\lozenge t}\left((\mathbf{zip}^3_{sp}(\mathbf{zip}^3_s\, and))_{\blacklozenge\lozenge t\rightarrow\blacklozenge\lozenge t\rightarrow\blacklozenge\lozenge t}\right. \\ (\mathbf{not}^{1,H}(\mathbf{BE}\,(\mathbf{a\ ophthalmologist}))\,(\eta_{sp}(\eta_s john))_{\blacklozenge\lozenge e}\ )_{\blacklozenge\lozenge t} \\ \left.(\mathbf{BE}\,(\mathbf{a\ oculist})\,(\eta_{sp}(\eta_s john))_{\blacklozenge\lozenge e}\ )_{\blacklozenge\lozenge t}\right)\,]\,(\kappa\, mary)\ \ s \tag{5.13}$$

Via β-reduction, the lambda term in (5.13) reduces to the formula in (5.14):

$$\forall s'(s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s' \rightarrow \\ \exists x\,(\ oculist_{epst}(x\,(\kappa\, mary)\, s')(\kappa\, mary\, s')\, s' \wedge \\ is^1\,(x\,(\kappa\, mary)\, s')\, john\,(\kappa\, mary\, s')\, s'\ ) \\ \wedge\ \forall s''\,[\exists y\,(\ ophthal(y\,(\kappa\, mary)\, s'')(\kappa\, mary\, s'')\, s'' \wedge \\ is^1\,(y\,(\kappa\, mary)\, s'')\, john\ \ (\kappa\, mary\, s'')\ s''\ )\ \rightarrow s' \bot s''] \tag{5.14}$$

Suppose we adopt the following axiom, where $\chi \in \{oculist_{epst}, ophthal_{epst}\}$:

$$\forall \delta_{\lozenge p} \forall s_s[\exists y.(\chi_{epst}\,(y\,(\delta\, s)\, s)\,(\delta\, s)\, s \wedge is^1(y\,\delta\ s)(a\,\delta\, s)\,(\delta s)\, s] \\ \Leftrightarrow \chi_{epst}\,(a\,(\delta\, s)\, s)\,(\delta\, s)\, s) \tag{5.15}$$

(5.14) and (5.15) imply (15) (= 5.12):

(15) $$\forall s'(s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s' \rightarrow oculist_{epst}\, john\,(\kappa\, mary\, s')\, s' \wedge \forall s''(\ (\ ophthal_{epst}\ john\ (\kappa\, mary\, s'')\ s''\ ) \rightarrow s' \bot s''\ )\ )$$

(15) expresses that, relative to the perspective index of Mary at state s as conceived by the utterer u, and relative to the state s, John is an oculist and not an ophthalmologist. (13b) is derivable similarly from the lexical entries above.

Besides (14) having the reading (15), we can also derive readings such as (16), in which *oculist* is interpreted relative to the utterer's perspective index in one case, and relative to John's perspective index in the other case (all from within the utterer's π-set):

(16) $$\forall s'(s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s' \rightarrow oculist_{epst}\, john\,(\kappa\, u\, s')\, s' \wedge \neg oculist_{epst}\, john\,(\kappa\, j\, s')\, s')$$

This reading would arise if the utterer and her audience were aware that John had a strange understanding of the predicate *oculist*– perhaps John considers an oculist to be a specialised eye doctor of some kind–and the utterer is trying to convey that John is not some kind of specialised eye-doctor, but a standard one.

One unusual feature of our account of unembedded sentences is that we will treat the logical form of an unembedded sentence differently from the logical form of this sentence when it is embedded under a clause-taking verb. Our account thus rejects a principle similar to that which (Holliday and Perry 2014) have termed the **Complement = Operand Hypothesis**:[6]

[6]With regard to what (Holliday and Perry 2014) call the *problem of cognitive fix*– essentially the problem induced by the *Frege's puzzle* and the fact that proper names cannot always be substituted for one another *salva veritate*— they conclude that:

> **Complement = Operand Hypothesis**: in epistemic logic, as in alethic modal logic, the formula to which the modal operator is applied—the operand—is the formalization of the sentence embedded in the 'that'-clause—the complement sentence—in the natural language belief/knowledge ascription. (Holliday and Perry 2014)

To give a concrete example, when *John is an oculist, not an ophthalmologist* is embedded under a clause-taking verb such as *believe*, the sentence *Peter believes that John is an oculist, not an ophthalmologist* will not be rendered as a universally quantified formula of the form (5.5), or of the form (5.6) (as we will see in the next section). Thus the logical form of a sentence like *John is an oculist, not an ophthalmologist* will change depending on whether it is an embedded sentence or an unembedded sentence. One justification for this might draw on speech act theory. To utter an unembedded sentence such as *John is an oculist, not an ophthalmologist* is to perform a particular speech act, whereas, to utter *John is an oculist, not an ophthalmologist* as part of the utterance of the larger sentence *Peter believes that*

> "...the solution to the problem of the cognitive fix is not to treat names, worlds, or individuals differently in epistemic logic than in alethic logic...*the solution requires giving up the idea that translating belief ascriptions into modal logic follows the simple pattern of translating necessity claims, what we called the Complement = Operand Hypothesis.* We argued for an alternative approach to formalizing belief reports, based on making explicit the *unarticulated constituents* of such reports. Taking these unarticulated constituents to be the *roles* that the objects of belief play in the cognitive life of the believer, we carried out the formalizations in a version of Fitting's First-Order Intensional Logic. We applied the idea of agent-relative roles to the Hintikka-Kripke Problem for alethic-epistemic logic, to quantification into epistemic contexts, and to multi-agent belief ascriptions."

One way of looking at κ, $\mathbf{lift}^1$ and $\mathbf{lift}^2$ in our account, is that they are unarticulated constituents of the kind that (Holliday and Perry 2014) describe in the above quotation.

John is an oculist, not an ophthalmologist is not to perform a speech act;[7] rather the utterance of the whole sentence *Peter believes that John is an oculist, not an ophthalmologist* is the performance of a speech act. If this is correct, then the speech act role of the embedded sentence *John is an oculist, not an ophthalmologist* differs depending on whether it is embedded under a clause-taking verb. The different logical forms we give to sentences depending on whether they are embedded under clause-taking verbs or not might then be justified by reference to speech act theory, as semanticisations of two different speech acts. Since nothing we have said so far has implied that logical forms are never assigned pragmatically, this is compatible with a traditional picture of speech acts, on which the force of speech acts is generated by pragmatic norms.

5.5 Compositional derivations of some examples with *prove*

In this section, we will concentrate on giving a detailed treatment of a type of mathematical example exemplified in sentences (17a) - (17c), in which two mathematical predicates which express the same properties of sets in ZFC set theory, *finite* and *inductive*, seem to have a different semantic value from one another when embedded under the verb *prove*:

(17) a. Harold proved that the primes are infinite.
b. Harold proved that the primes are not inductive.
c. (No,) Harold proved that the primes are infinite, not that the primes are not inductive.

[7] Whilst speech act embedding occurs, it is generally quite a limited phenomenon (Krifka 2014).

Nothing hangs on our choice of the natural language predicates *inductive* and *finite* and the reader can substitute her favourite example of two mathematical predicates which pick out the same objects but which are intuitively distinct in meaning, if she finds the example of *inductive* and *finite* problematic.

From now on, we will employ the following terminology:

- When all the perspective index arguments in a sentence denotation are saturated with the perspective index associated with the utterer u_e at some state, we call this the *extreme u* interpretation.

- We call the *extreme* E interpretation of a sentence, the interpretation which arises when all the arguments requiring a perspective index are saturated with E.

We will now present some lexical entries, and show how they can model sentences involving attitude verbs, such as *prove* and *believe*. Our lexical entries will omit various details which a semantic theory might take into account.

λ-term

the := $\sigma_{(\blacklozenge\lozenge e \to \blacklozenge\lozenge t)\to\blacklozenge\lozenge e}$

$$\textbf{primes} := \lambda x_{\blacklozenge\lozenge e}, \delta_{\lozenge p}, s_s. \begin{cases} *prime.numbers\,(x\,\delta\,s) & \text{if } \delta\,s \neq E \\ (\delta\,s)\,s & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \\ & \text{and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ *prime.numbers\,(x\ (\eta_s E)\ s) & \text{if } \delta\,s = E \\ E\ s & \end{cases}$$

λ-term

$$\textbf{finite} := \lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} finite\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ finite\,(x\ (\eta_s E)\ s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{inductive} := \lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} inductive\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ inductive\,(x\ (\eta_s E)\ s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{man} := \lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} man\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ man\,(x\ (\eta_s E)\ s)\,E\,s & \text{if } \delta\,s = E \end{cases}$$

$$\textbf{love} := \lambda y_{\blacklozenge\Diamond e}, x_e, \delta'_{s\Diamond p}, \delta_{\Diamond p}, s'_s. \begin{cases} \forall s''(s' \leqslant s'' \wedge \pi\,(\delta\,s')\,s'\,s'' & \text{if } C_{\mathbf{p}}\,s\ (\delta\,s') = \top \\ \rightarrow love\,(y\,\delta'\,s'')\,x\ (\delta'\,s'')\,s'') & \text{and } C_{\mathbf{p}}\,s\ (\delta'\,s'') = \top \\ \# & \text{otherwise} \end{cases}$$

$$\textbf{prove} := \lambda p_{\blacklozenge\Diamond t}, x_e, \delta_{\Diamond p}, s_s. \begin{cases} \forall s'[\,[s \leqslant s' \wedge \pi(\delta\,s)\,s\,s'] \rightarrow & \text{if } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \forall s''[PROV\ x\ s'\ s'' \rightarrow & \\ p\,(\eta_s(\kappa\,x\,s''))\,s''\,]\,] & \\ \# & \text{otherwise} \end{cases}$$

$$\textbf{is}^2/\textbf{are}^2 := \lambda P_{\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond t}.P$$

λ**-term**
pp $:= \lambda\delta_{\Diamond p}, s_s. \begin{cases} pp(\delta\, s)\, s & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) = \top \\ \# & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) \neq \top \\ pp\ E\ s & \text{if } \delta\, s = E \end{cases}$
sm $:= \lambda\delta_{\Diamond p}, s_s. \begin{cases} sm(\delta\, s)\, s & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) = \top \\ \# & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) \neq \top \\ pp\ E\ s & \text{if } \delta\, s = E \end{cases}$
h := $harold_e$
mj := $mary.jane_e$

Table 5.2: Simplified lexical entries for embedded sentences

We are now ready to discuss the various interpretations of (17a) and (17b) which our theory generates. The extreme E interpretation of (17a) and (17b) are (18a) and (18b):

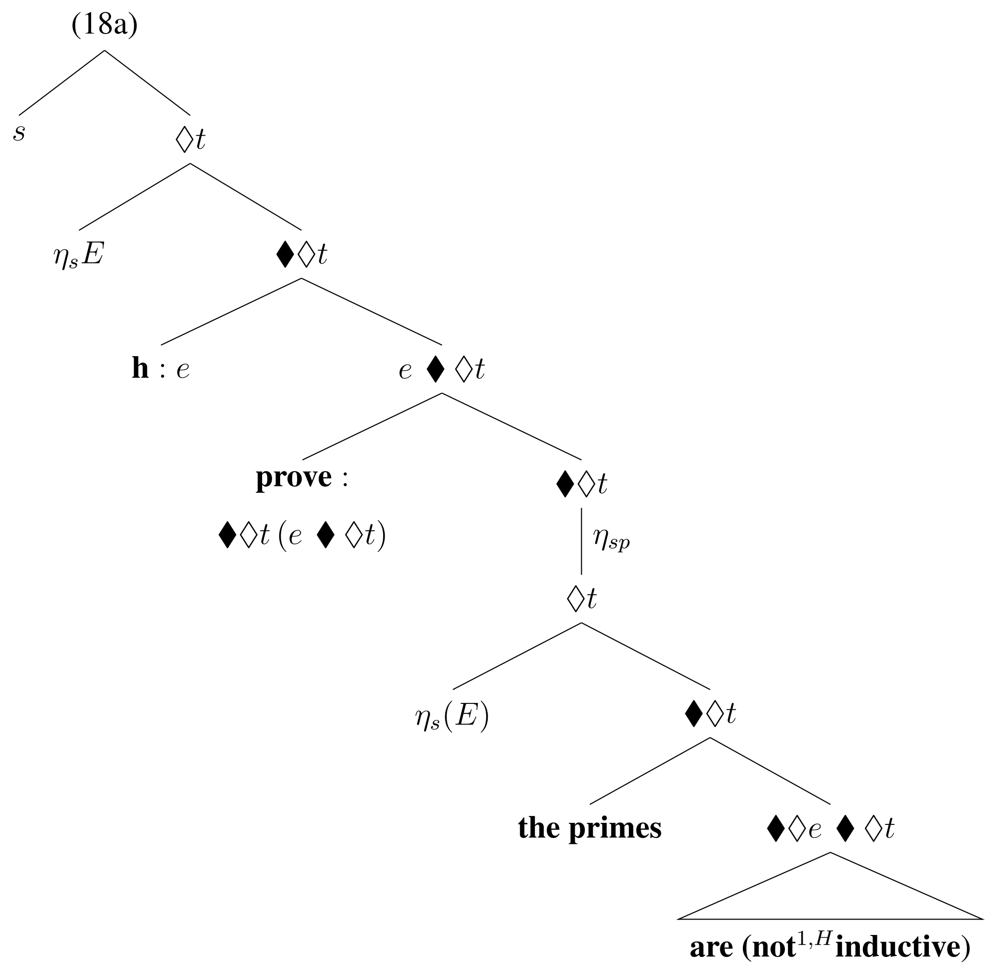
(18) a. $\lambda s.\forall s'\,[\, s \leqslant s' \wedge \pi\, E\, s\, s'] \rightarrow \forall s''[PROV\, h\, s'\, s'' \rightarrow$
$\forall s'''(finite\,((\sigma\ \textbf{primes})\ (\eta_s\, E)\ s''')\ E\ s''' \rightarrow s''\bot s'''\ )\,]$

b. $\lambda s.\forall s'\,[\, s \leqslant s' \wedge \pi\, E\, s\, s'] \rightarrow \forall s''[PROV\, h\, s'\, s'' \rightarrow$
$\forall s'''(inductive\,((\sigma\ \textbf{primes})\ (\eta_s\, E)\ s''')\ E\ s''' \rightarrow s''\bot s'''\ )\,]$

(18a) and (18b) are equivalent, since *not inductive* and *not finite* have the same semantic value relative to the enlightened perspective index. (18a) via the tree in

(19) below (see the appendix, section B, for the conventions we follow regarding trees):

(19)



Let us now consider the extreme u interpretations of (17a) and (17b). In P-HYPE, in the scenario described in (20a), (20b) would be false, and (20c) would be true:

(20) a. *Scenario*: Someone discovers a proof of Harold's that the primes are infinite. They does not know that *not inductive* and *infinite* pick out the same set of objects and believe that the primes are inductive. They

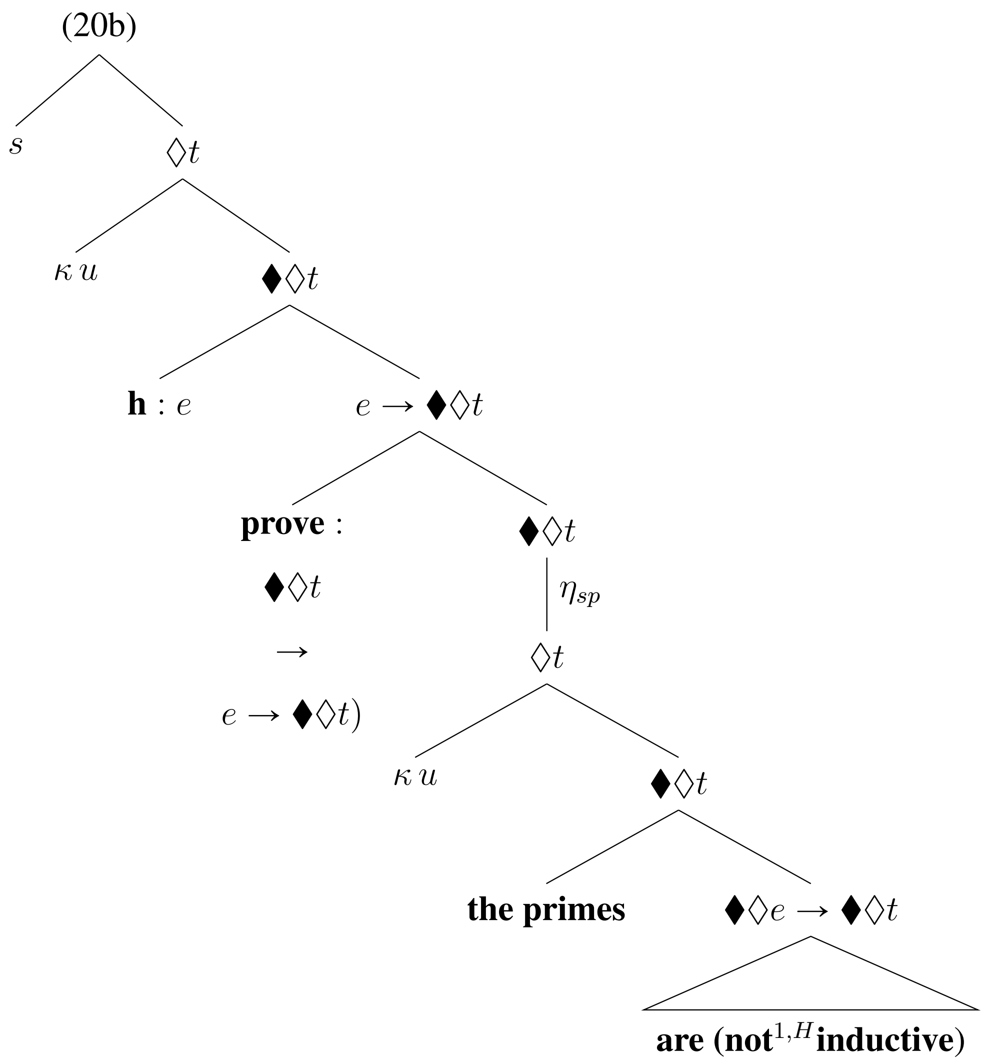
utter *Harold proved that the primes are infinite.* (= (17a))

b. $\lambda s.\forall s' \left[s \leqslant s' \wedge \pi \; (\kappa \, u \, s) \; s \; s' \right] \rightarrow \forall s''[PROV \; h \; s' \, s'' \rightarrow$
$\forall s'''(\; (inductive \; ((\sigma \; \textbf{primes}) \; (\kappa \; u) \; s''') \; (\kappa \; u \; s''') \; s''' \;) \rightarrow s'' \perp s''' \;]$

c. $\lambda s.\forall s' \left[s \leqslant s' \wedge \pi \; (\kappa \, u \, s) \; s \; s' \right] \rightarrow \forall s''[PROV \; h \; s' \, s'' \rightarrow$
$\forall s'''(\; (finite \; ((\sigma \; \textbf{primes}) \; (\kappa \; u) \; s''') \; (\kappa \; u \; s''') \; s''' \;) \rightarrow s'' \perp s''' \;]$

(20b) is derived via the following tree, and (20c) would be derived in the same way with **finite** in place of **inductive**:

(21)



So far we have described the extreme E and the extreme u interpretations of sentences. But there are interpretations of sentences in which π is fed u, but other expressions are fed a perspective index associated with the grammatical subject of the sentence.

For example, (17c) can also be felicitously uttered by someone who is trying to take Harold's perspective, in the scenario described in (22a) and in that scenario (22b) is true, whilst (22c) is false, because the utterer assumes that Harold associates distinct denotations with *inductive* and *finite*:

(22) a. *Scenario*: Someone knows that *not inductive* and *infinite* pick out the same objects but thinks that both Harold and the other's present in the conversation are unaware of this and draw a similar distinction in meaning between *not inductive* and *infinite*. Since Harold thinks that he has proved that the prime numbers are infinite, they utter to someone else, *(No,) Harold proved that the primes are infinite, not that the primes are not inductive* (= (17c)).

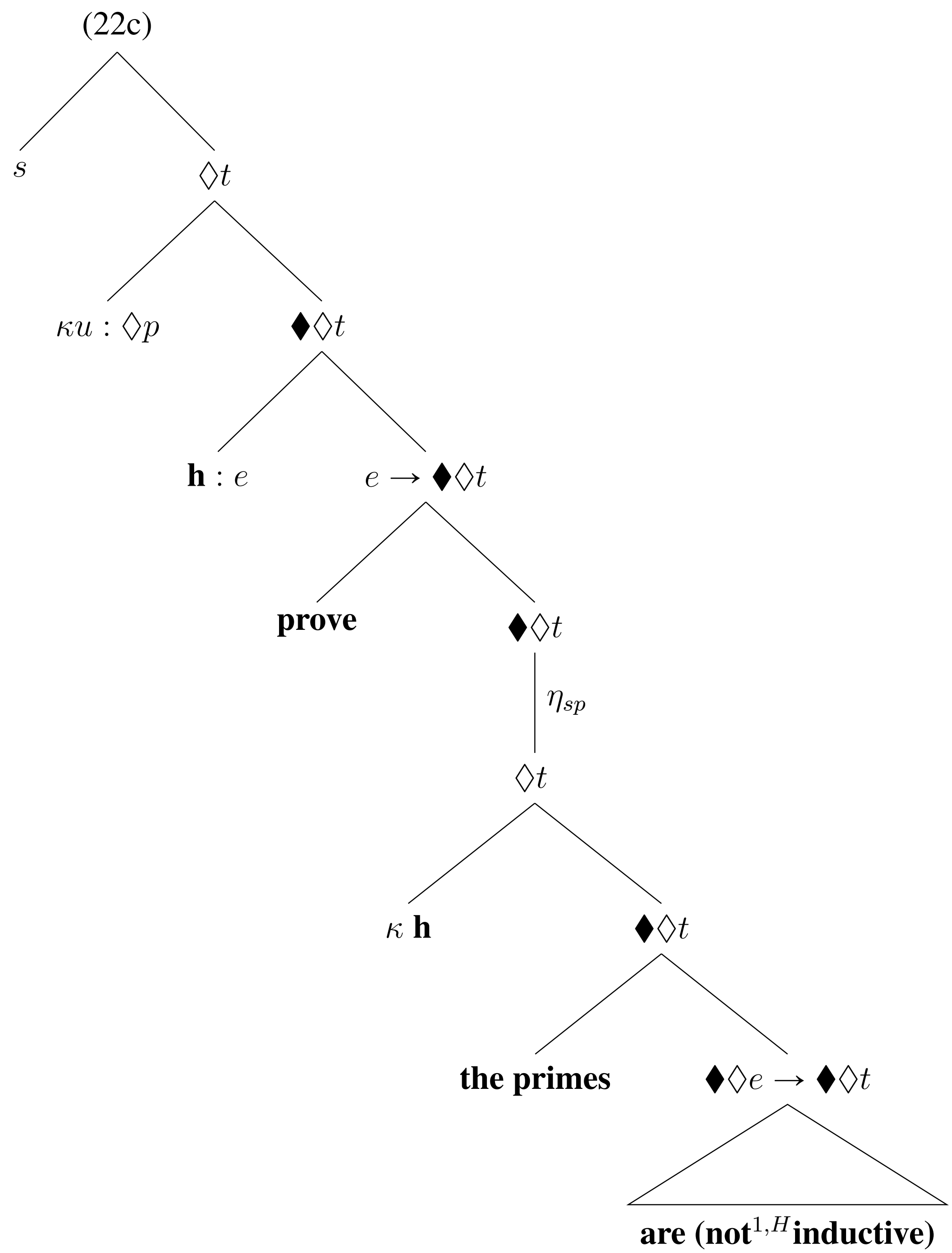
b. $\lambda s.\forall s'\,[\,[\,s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s'] \rightarrow \forall s''[PROV\; h\, s'\, s'' \rightarrow$
$\forall s'''(finite\;((\sigma\;\textbf{primes})\;(\kappa\;\textbf{h})\;s''')\;(\kappa\,\textbf{h}\,s''')\,s''' \rightarrow s'' \perp s''')\,]\,]$

c. $\lambda s.\forall s'\,[\,s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s'] \rightarrow \forall s''[PROV\;\textbf{h}\, s'\, s'' \rightarrow$
$\forall s'''(inductive\;((\sigma\;\textbf{primes})\;(\kappa\;\textbf{h})\;s''')\;(\kappa\,\textbf{h}\,s''')\,s''' \rightarrow s'' \perp s''')\,]$

If the utterer utters (17a) in the scenario (22a), she temporarily tries to take the perspective of Harold on the interpretation of *infinite* and on the interpretation of *the prime numbers*.

The derivation of (22c) is produced by the following tree:

(23)



Finally, we also derive the following 'mixed readings' for (17c), in which the subject of the embedded clause and the predicate occurring in it are interpreted with respect to different perspective indices:

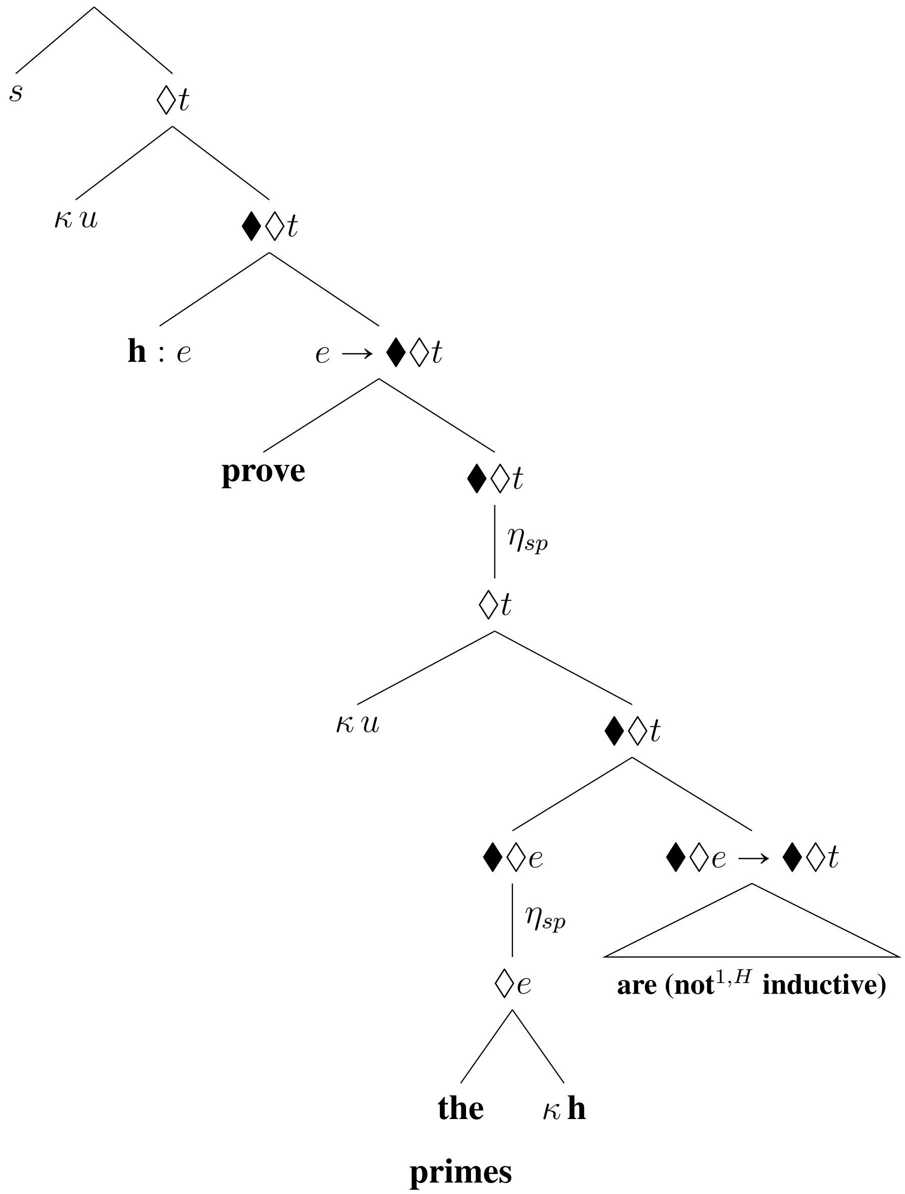
(24) $\lambda s.\forall s' [\, s \leqslant s' \wedge \pi\ (\kappa\, u\, s)\ s\ s'] \rightarrow \forall s''[PROV\, \mathbf{h}\ s'\, s'' \rightarrow$
$\forall s'''(inductive\ ((\sigma\ \textbf{primes})\ (\kappa\ \mathbf{h})\ s''')\ (\kappa\ u\ s''')\ s''' \rightarrow s'' \perp s''')\ ]$

(25) $\lambda s.\forall s' [\, s \leqslant s' \wedge \pi\ (\kappa\, u\, s)\ s\ s'] \rightarrow \forall s''[PROV\, \mathbf{h}\ s'\, s'' \rightarrow$
$\forall s'''(inductive\ ((\sigma\ \textbf{primes})\ (\kappa\ u)\ s''')\ (\kappa\ \mathbf{h}\ s''')\ s''' \rightarrow s'' \perp s'''\ )\ ]$

(24) arises when *inductive* is fed the utterer's perspective index whilst *the primes* is fed Harold's perspective index:

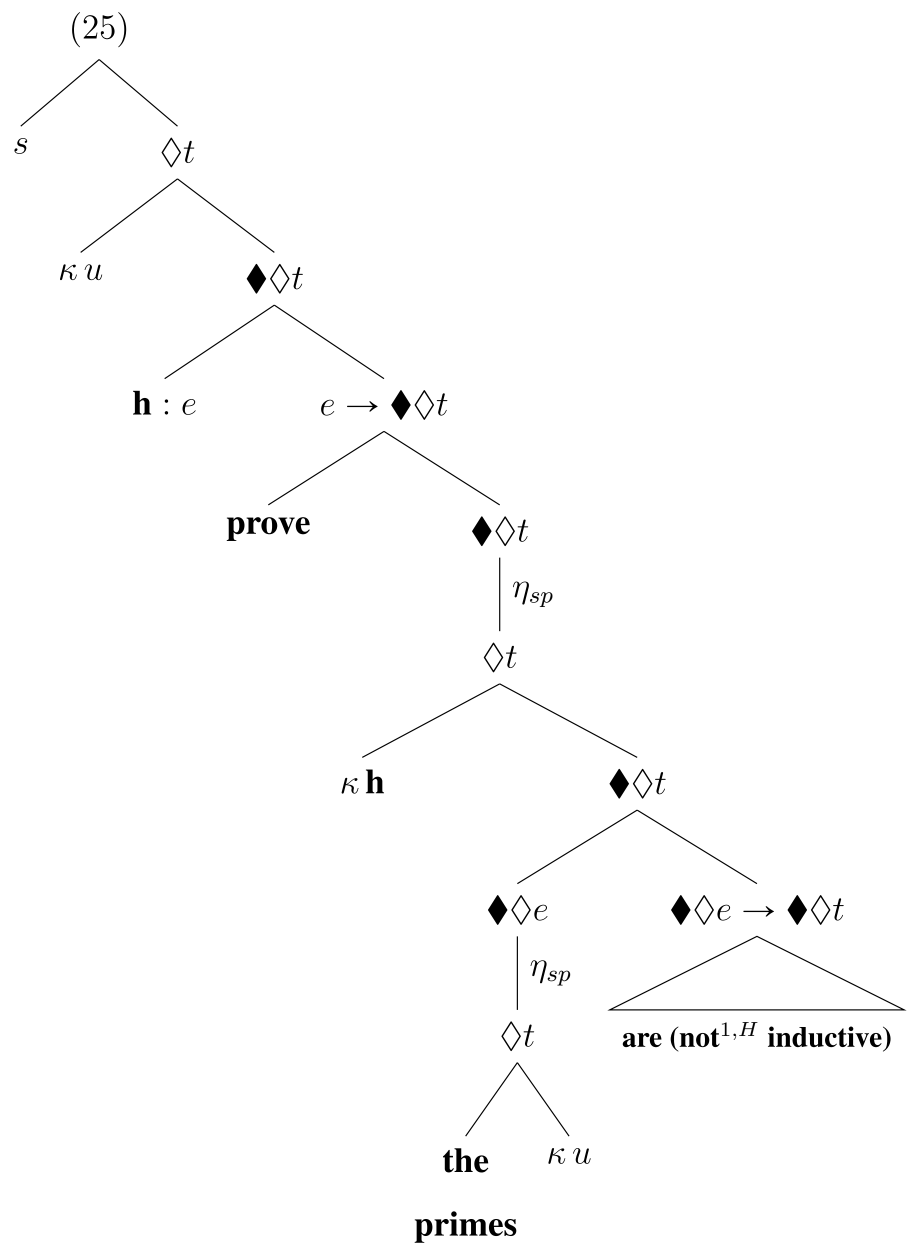
(26)

(24)



The second, (25), arises when *inductive* is fed the Harold's perspective index whilst *the primes* is fed the utterer's perspective index:

(27)



These mixed readings may be marginal, or inexistent. In any case, if they do not occur they can be ruled out as follows. Take any top-level term of the form $[\ \mathbf{S}\,[s']\,\mathbf{P}[s'']A_t\ ]$ as described in the previous chapter. We call a top-level term of the form $[\ \mathbf{S}\,[s']\,\mathbf{P}[s'']A_t\ ]$ *perspectivally uniform* whenever the following holds:

$$\forall x_t(x \in ST[A_t] \wedge PROV, DOX, \pi \notin ST[x] \wedge \quad (5.16)$$
$$\forall i_p, j_p(i, j \in ST[x] \Rightarrow i = j) \wedge \forall \delta_{sp}\, \delta'_{sp}(\delta, \delta' \in ST[x] \Rightarrow \delta = \delta'))$$

We could then impose the condition that, in any admissible P-HYPE model, only perspectivally uniform terms are defined. This may be a rather 'brute force' way of ruling out the relevant mixed readings, and in future work we want to investigate whether ideas in the literature on *de re* noun phrase interpretation (Keshet 2008, Elliott 2020, Percus 2020) can help in constraining perspective relativity.

We close this section with an observation. Notice that our semantic theory predicts that sentences like (28) should be possible, when *inductive* is fed the perspective index of an enlightened utterer in the first part of the sentence, and the perspective index of Harold in the second half of the sentence:

(28) Harold proved that the prime numbers are inductive, but he didn't prove the prime numbers are inductive.

(Asudeh and Giorgolo 2016) discuss similar sentences and we also discussed such sentences in 2.2.1.1. To reiterate what we wrote 2.2.1.1, like (Asudeh and Giorgolo 2016) in connection with their similar examples, we think that (28) can be felicitous in exceptional circumstances, but that it is usually pragmatically dispreferred. The pragmatically preferred way of expressing the intended meaning of (28) is (29):

(29) Harold proved that the prime numbers are infinite, but he didn't prove the prime numbers are inductive.

5.6 Comparisons

In 2.2 we criticised some alternative solutions of the problem of granularity. An in-depth comparison of these theories with our own will not be attempted here. Instead, we wish briefly show how P-HYPE deals with the problems we identified with the accounts we examined in 2.2.

5.6.1 P-HYPE compared to (Cresswell and Stechow 1982)

We criticised the theory of (Cresswell 1982) for predicting there is an asymmetry between (30) and (31) in terms of felicity, whereas both sentences are just as felicitous as the other:

(30) Peter believes that John is an ophthalmologist, but he doesn't believe that John is an oculist.

(31) Peter believes that John is an ophthalmologist and he believes that John is not an oculist.

In P-HYPE, no such asymmetry is apparent, so that we avoid the problematic prediction of (Cresswell 1982). Another related criticism of their approach is that it predicts spurious ambiguities. For example, a sentence such as *John is an oculist* can have the structurings (5.17) or (5.18), which are different structured propositions:

$$\langle [\![\lambda Q_{e\to s\to t}, s_s . Q\, john_e\, s]\!]^{\mathfrak{M},g}, [\![oculist_{e\to s\to t}]\!]^{\mathfrak{M},g} \rangle \tag{5.17}$$

$$\langle [\![oculist_{e\to s\to t}]\!]^{\mathfrak{M},g}, [\![john_e]\!]^{\mathfrak{M},g} \rangle \tag{5.18}$$

Since (5.17) and (5.18) are different structured propositions, this allows for a consistent reading of (32b) and (32c) in the scenario (32a):

(32) a. *Scenario*: Peter is not confused about the meaning of *oculist* and *ophthalmologist* and thinks they denote the same property. Peter thinks John is an oculist. Peter utters (32b):

b. John is an oculist and John is not an oculist.

c. Peter believes that John is an oculist, but he doesn't believe that John is an oculist.

There is, however, no coherent sense in which the sentences (32b) and (32c) are true in the scenario (32a). In P-HYPE (32b) and (32c) can only be true if we interpret *oculist* relative to distinct perspective indices in each of the sentences. We thus avoid the awkward prediction of (Cresswell 1982)'s account.

5.6.2 P-HYPE compared to (Tancredi and Sharvit 2020)

In 2.2.1.2 we discussed the theory of (Tancredi and Sharvit 2020), who argue that (33b), (33c) and their conjunction (33d) are true in the scenario described in (33a):

(33) a. *Scenario*: John, a monolingual English speaker who has a non-standard understanding of the natural language predicate *prime*, according to which a natural number is prime if and only if it is equal to $x^3 - 1$, for some natural number x. This person, who we shall call 'John', may assert that 26 is a prime number, knowing that it has exactly four natural number factors.

b. John believes that 26 is a prime number

c. John doesn't believe that 26 has exactly two natural number factors.

d. John believes that 26 is a prime number, but he doesn't believe that

26 has exactly two natural number factors.

We argued in 2.2.1.2 that a simpler analysis of sentences like (33b), (33c) and (33d) than (Tancredi and Sharvit 2020) provide is available if predicates are treated as perspective relative. In P-HYPE (33b)-(33d) might be jointly true, since the predicates *is a prime number* and *has exactly two natural number factors* can be treated as different perspective-relative predicates. Furthermore, we can capture (Tancredi and Sharvit 2020)'s intuition that (34) is neither true nor false in the scenario (33a) described if we require that sentences in which the verb *know* occurs require, in order for them to be defined, that the complement sentence *know* combines with is true with respect to the enlightened perspective:

(34) ?? John knows that 26 is prime

In some respects, our proposal fares better than (Tancredi and Sharvit 2020)'s. For example, in P-HYPE we can capture the possibility that (35a) is true:

(35) a. John believes that 978 and 7 are both prime numbers.

In P-HYPE (35a) might be true when *prime numbers* is given a single semantic value, relative to John's perspective. But in (Tancredi and Sharvit 2020) $[\![prime\ numbers]\!]^{T_j(L),j}$ must be mapped to a single lambda term l which applies to 978 and 7 and returns true. But in the situation described it may be that John considers a number as prime if and only if it is greater than 1 and has no positive divisors other than 1 and itself, so that $[\![prime\,numbers]\!]^{T_j(L),j} = \lambda n.n > 1 \wedge \forall x(divisorx\ n \rightarrow (x = 1 \vee x = n))$. However, since 978 is not a prime number, this will not work. So it seems that, unlike P-HYPE, (Tancredi and Sharvit 2020)'s account predicts (35a) to be false. In order to avoid this prediction, they will have to complicate their (already complicated) account even more. By comparison, in P-HYPE (35a) may be true simply because, 978 and 7 are sent to 1 by the perspective relative predicate *prime number*, when it is fed John's perspective index. So the P-HYPE account is likely to be preferable on simplicity grounds.

5.6.3 P-HYPE compared to (Stalnaker 1978)

In 2.2.2, we saw that Stalnaker couldn't account for felicitous assertions of certain tautologies, such as (36):

(36) The prime numbers are infinite or the prime numbers are not infinite.

The problem was that, in Stalnaker's framework, (36), and any sentence which expresses a classical logical truth, is true in all states, whereas, according to his account of information, a sentence is only felicitous if it false at at least one state. In P-HYPE, sentence (36), if it is rendered as a formula of the form $\varphi \vee \neg\varphi$ may fail to hold at a state, because states are gappy, so that we avoid the problem we described for Stalnaker's theory. But Stalnaker seems to rule out the possible of partial states (see (Stalnaker 1984)).
Nevertheless, all HYPE validities will hold at every HYPE state of all HYPE models, so that we will not be able to distinguish their semantic contribution. We will see how we can do this in 7.1.

5.7 Conclusion

If we are correct, both 'objective' and 'subjective' predicates can have a perspective-relative, 'subjective' interpretation in at least some cases. In many cases the subjectivity of predicates is simply hidden from view, since we interpret them with respect to the enlightened perspective. If we take this view, we can use P-HYPE to solve the problem of granularity without sacrificing mathematical objectivity at the altar of subjectivity, and the worries with the view proposed in this chapter can all be met. Moreover, P-HYPE can avoid many of the problems we identified with the semantic theories which we examined in 2.

In P-HYPE, unembedded and embedded sentences have a different logical form in P-HYPE, which we defended on speech-act grounds. Unembedded sentences which contain a predicate of taste such as *fun* or *tasty*, or which contain a verb such as *love*, whose complement can be sensitive to the perspective of its grammatical subject, can be objectively true relative to different perspective indices, but unembedded sentences which do not contain such predicates are objectively true only with respect to the enlightened perspective index. In propositional attitude contexts, for verbs like *believe* the objective truth of a sentence varies depending on what perspective indices are available in the context, but for verbs such as *prove*, the complement sentence which is its argument must be true with respect to the enlightened perspective index.

Chapter 6

P-HYPE and the problem of granularity: predicates and focus

6.1 Introduction

This short chapter will be devoted to how focus interacts with the semantics of predicates in P-HYPE. In 6.2 we show how perspective relativity, in combination with focus semantics, can capture certain semantic properties of focus in cases in which the principle of *Predicate Uniformity* causes problems. In 6.3 we show how the semantics of metalinguistic focus negation in (Li 2017) and perspective relativity interact, describe cases of metalinguistic focus which cannot be captured by (Li 2017), and propose a solution to the problem there described.

6.2 Focus on predicates in P-HYPE

In this section we discuss how to integrate focus into our analysis of predicates. Our analysis here sets the groundwork for the next section, in which we examine some potential problems with the analysis of predicates we have adopted.

Expressions can sometimes bear contrastive focus, as in the following examples ($[\text{ e }]_F$ indicates that the linguistic expression 'e' is focused):

(1) Jason believes that $[\textsf{Harvard}]_F$ is a fine university, but he doesn't believe that $[\textsf{Hahvahd}]_F$ is.

(2) Mary believes that John is an $[\text{oculist}]_F$, but she doesn't believe that he is an $[\text{ophthalmologist}]_F$

(3) Max believes that Mary xeroxed War and Peace, but he doesn't believe that Mary $[\text{photocopied}]_F$ War and Peace.

(4) John didn't $[\text{photocopy}]_F$ the letter, he $[\text{xeroxed}]_F$ it.

(5) John $[\text{believes}]_F$ that Mary will come to the party, but he doesn't $[\text{think}]_F$ she will.

There are two principal sorts of cases in which these examples involving contrastive focus could be uttered. Take (5). This could be uttered by someone who thinks that John has some odd linguistic beliefs about the verbs *believe* and *think* which she doesn't share. Or it could be uttered by someone who herself has some odd linguistic beliefs about the verbs *believe* and *think*, regardless of whether John shares these beliefs or not. The perspective relative semantic theory presented in the previous section would treat these predicates as perspective relative and could derive both these readings. However, in order to capture the semantics of examples such as these, we need to integrate focus into our semantic theory.

In the focus semantics of (Rooth 1992), focus is modelled via alternatives, in the following sense. Consider an utterance of 'John likes $[\text{tea}]_F$ with milk'. Intuitively, this suggests that there are alternative substances besides tea which are salient in the context, but which John does not like to have with milk. In (Rooth 1992), a linguistic expression of type e which bears focus is semantically interpreted as a pair of an element in D_e, and a set of more than one entities in D_e. For example, the focused word 'tea' could be semantically interpreted as a pair of a mass noun entity tea in D_e and a set of alternative mass nouns $alt\,tea \subseteq D_e$ of the same type as tea (and of which tea is a subset), where $alt\ tea$ might be, for example $\{\ y \in D_e \mid y = \mathit{coffee}$ or $y = orange.juice$ or $y = tea\ \}$. A linguistic expression which does not bear focus, is semantically interpreted as the pair of an entity in D_e and the singleton set containing that entity. A sentence with an expression bearing focus, such as 'John likes $[\text{tea}]_F$ with milk' would be semantically interpreted as a pair whose first member is the ordinary semantic value of the sentence 'John likes tea with milk'— for example, a function in $D_{\Diamond t}$—and whose second member is a set of alternative functions in $D_{\Diamond t}$. This set of alternative functions in $D_{\Diamond t}$ might be the ordinary semantic values expressed by the sentences 'John likes coffee with milk', 'John likes cereal with milk', etc. Since words are now rendered as pairs, we require a special mode of composition to combine them. In Roothian focus semantics $A : (\sigma \to \tau) \times ((\sigma \to \tau) \to t)$ and $B :\ \sigma \times (\sigma \to t)$ are combined via pointwise function application (**PFA**). According to *pointwise function application*:

$$\textbf{PFA}\ A\,B = \Big((\pi_1 A)\,(\pi_1 B),\ \{ g\,a\ \mid g \in \pi_2 A \text{ and } a \in \pi_2 B \} \Big)$$

Suppose that we render the word *John* and the expressions *likes* $[\text{tea}]_F$ *with milk* as below, where $alt : \alpha \to \alpha \to t$ forms a set of alternatives:

$(\ john_e,\ \{\ john\ \}\ )$

$\big(\ \lambda x_e.\,(with_{e \to t \to t}\, milk_e)\,(like_{e \to e \to t}\, tea\, x),$

$$\{ \lambda x_e.(with_{e\to t\to t}\, milk_e)\,(like_{e\to e\to t}\, a_e\, x) \} \mid a \in alt\ tea \} \big)$$

By PFA, these combine to form:

$$\big(\ (with_{e\to t\to t}\, milk_e)\,(like_{e\to e\to t}\, tea\, john)\,,\ \{ (with\, milk)\,(like\, a_e\, john) \mid a \in alt\ tea\} \big) \tag{6.1}$$

Roothian focus semantics can be implemented via the 'pointed power-set' monad (Shan 2002), (Charlow 2014), (Li 2017). The monad is defined as follows (π_i are projection functions returning the ith argument of the expression they apply to, and $F : TYPE \to TYPE$ is the monadic type constructor of the monad $(F, \eta_F, \bigstar_F)$:

(6) $\eta_F x = (\ x,\ \{ x \}\ )$

(7) $a \bigstar_F\ f = (\ \pi_1(f\,(\pi_1\, a)),\ \ \bigcup_{x' \in \pi_2\, a} \pi_2(f\, x')\ )$

η_F forms a pair consisting of the argument x that η_F is applied to, and the singleton containing it. Lambda terms of type $F\alpha$ are pairs whose first members is of type α and whose second members are either sets of alternatives of type α, or singleton sets containing the first member of the pair (which arise when η_F is applied to some term). For example, we might have a pair $\big(john,\ alt\, john\big)$ whose first member is the constant $john$ and whose second member are various other individuals salient in the context of utterance. $a \bigstar_F f$ forms a pair consisting of the first member of the function f applied to the first member of a, and the set of things produced by applying f to the members of $\pi_2 a$.

In fact, for the examples in this section, we only need to use the F-applicative, which is defined as follows:

The F applicative :

$F\alpha = \alpha \times (\alpha \rightarrow t)$

$\eta_F x =_{def} \big(x,\ \{x\}\big)$

$f \bullet_F a =_{def} \big(\ (\pi_1 f)\,(\pi_1 a),\ \ \{\ gx' \mid g \in \pi_2 f,\ x' \in \pi_2 a\ \}\ \big)$

where $a : F\alpha, f : F(\alpha \rightarrow \beta)$

P-HYPE can avoid some of the difficulties which are encountered in accounting for (1), (2), (4) and (5) in a semantic theory in which *Predicate Uniformity* is assumed to hold. If we integrated the focus semantics of (Rooth 1992) with our semantics, then, roughly, (4) would be semantically interpreted as a pair containing the ordinary semantic value of *John didn't photocopy the letter, John xeroxed the letter*— in P-HYPE, a function in $D_{\blacklozenge\Diamond t}$—together with a set of alternatives of the same type, for each sentence of the form *John didn't photocopy the letter, John* x *the letter*, where x is a tensed transitive verb. Formally, a word such as *photocopy* is treated as a pair of a lambda term $\lambda y, x.photocopy\,y\,x$ and a set of alternative lambda terms of the same type, such as $\lambda y, x.sent\,y\,x$, etc. A lambda term in the first co-ordinate of this pair may be replaced with any lambda term with the same semantic value, by substitution of equals. If we make the standard assumption of *Predicate Uniformity*, then seemingly synonymous pairs like *photocopy* and *xerox* will have the same semantic value, but then their alternatives will be the same, because, for all $\alpha \in TYPE$,

$$\forall z, z' \in D_\alpha\big(z = z' \rightarrow [\![\,alt]\!]^{\mathfrak{M},g}\,z = [\![\,alt]\!]^{\mathfrak{M},g}\,z'\big)$$

(In (Rooth 1992) the set of alternatives is defined on the denotation of the lambda term in the first co-ordinate, which is by hypothesis the same in both cases). Consequently, if the predicates *photocopy* and *xerox* have the same semantic value, then (4) will express a contradiction. By contrast, in P-HYPE we can assign the predicates *photocopy* and *xerox* a different, perspective relative semantic value, and so we won't run into this problem.

We will now present the compositional derivation of (8) and of (9):

(8) Harold $[\text{believes}]_F$ that the prime numbers are not finite but he doesn't $[\text{think}]_F$ that the prime numbers are not finite.

(9) Harold loves [harvard$]_F$, but he doesn't love [hahvahd$]_F$.

In the previous section, we rendered the verb *believe* as **believe** (see 5.5). We now render *think* as the following lambda term, which we abbreviate using **think**:

$$\lambda p_{\blacklozenge\lozenge t}, x_e, \delta_{\lozenge p}, s_s. \begin{cases} \forall s'[\,[s \leqslant s' \wedge \pi(\delta\, s)\, s\, s'] \rightarrow & \text{if } C_{\mathbf{p}}\ s\ (\delta\, s) = \top \\ \forall s''[THINK\, x\ s'\, s'' \rightarrow & \\ p\,(\eta_s(\kappa\, x\, s''))\, s''\,]\,] & \\ & \\ \# & \text{if } C_{\mathbf{p}}\ s\ (\delta\, s) \neq \top \end{cases}$$

In order to ensure that **believe** and **think** are interpreted the same with respect to the enlightened perspective index, we require that (6.2) is true of all admissible P-HYPE models:

$$\forall p \in D_{\blacklozenge\lozenge t}, x \in D_e, s \in D_s \tag{6.2}$$
$$\big(\ [\![\ \textbf{believe}\ ]\!]^{\mathfrak{M},g}\, p\, x\, [\![(\eta_s E)]\!]^{\mathfrak{M},g}\ s = [\![\ \textbf{think}\ ]\!]^{\mathfrak{M},g}\, p\, x\, [\![(\eta_s E)]\!]^{\mathfrak{M},g}\ s\ \big)$$

$[\text{believes}]_F$ could then be rendered as in (10), and $[\text{think}]_F$ could be rendered as (11), where, for $x : \alpha$, $alt\, x : \alpha \rightarrow t$, is a set of lambda terms of the same type of x:

(10) $(\mathbf{believe},\ alt\ \mathbf{believe}) : F(\blacklozenge\lozenge t \to e \to \blacklozenge\lozenge t)$

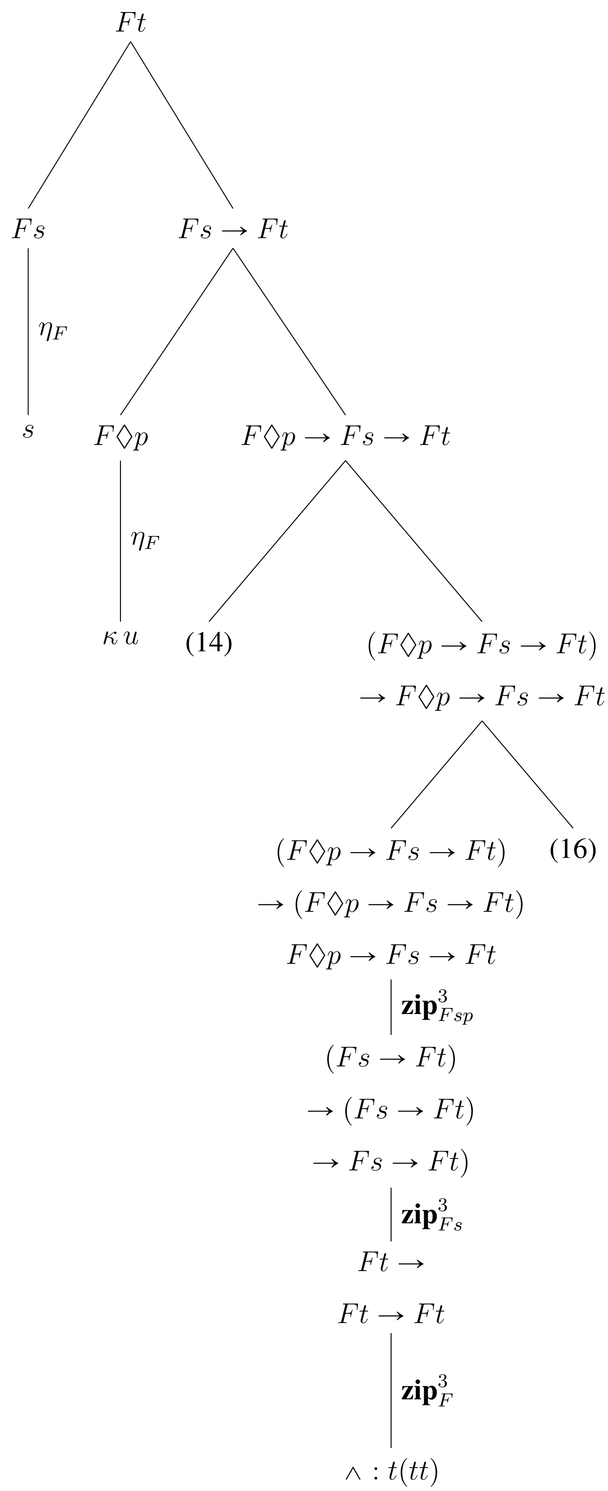
(11) $(\mathbf{think},\ alt\,\mathbf{think}) : F(\blacklozenge\lozenge t \to e \to \blacklozenge\lozenge t)$

Supposing that (8) is uttered from the perspective of Harold at the state s, we render (8) (repeated here as (12)) as (6.3), whose compositional derivation is given in (13):

(12) Harold [believes]$_F$ that the prime numbers are not finite but he doesn't [think]$_F$ that the prime numbers are not finite.

$$\begin{aligned}
&\Big(\forall s' \,[\,[\, s \leqslant s' \wedge \pi\,(\kappa\ u\ s)\, s\, s'] \to \forall s''[DOX\,\mathbf{h}\, s'\, s'' \qquad (6.3)\\
&\to \forall s'''\,(finite\,(\sigma\ \mathbf{primes}\,(\kappa\ \mathbf{h})\ s'''\ )\ (\kappa\,\mathbf{h}\, s''')\, s'''\ \to s'' \perp s''')\,]\,]\\
&\wedge \forall s^0\ (\ \forall s'\,[\,[\ s^0 \leqslant s' \wedge \pi\,(\kappa\ u\ s^0)\, s^0\, s'\ ] \to \forall s''\ [\ THINK\,\mathbf{h}\, s'\, s'' \to\\
&\forall s^1(finite\,(\sigma\ \mathbf{primes}\ (\kappa\ \mathbf{h})\ s^1\ )\ (\kappa\,\mathbf{h}\, s^1\ )\ s^1\ \to\ s'' \perp s^1\,)\,]\ \to\ s \perp s^0\ )\ ],\\
&\Big\{\big(f\ (\eta_{\lozenge p}\ (\ (\ \mathbf{are}\,(\ \mathbf{not}^{1,H}\ \mathbf{finite}\ )\ )\ (\sigma\ \mathbf{primes})\ (\ \kappa\ \mathbf{h}\ )\ )\ )\ \ \mathbf{h}\ \ (\kappa\, u)\ s\ \big) \wedge\\
&\mathbf{not}^{2,H}\ \big(\ g\ \ (\eta_{\lozenge p}\ (\ \mathbf{are}\,(\ \mathbf{not}^{1,H}\ \mathbf{finite}\ )\ )\ (\sigma\ \mathbf{primes})\ (\ \kappa\ \mathbf{h}\ )\ )\ \mathbf{h}\ \big)\ (\kappa\, u)\ \ s\\
&\mid\ f \in alt\,\mathbf{believe}, g \in alt\,\mathbf{think}\Big\}\Big)\ :\ Ft
\end{aligned}$$

(13)


Ft
Fs
$Fs \to Ft$
η_F
s
$F\Diamond p$
$F\Diamond p \to Fs \to Ft$
η_F
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(14)
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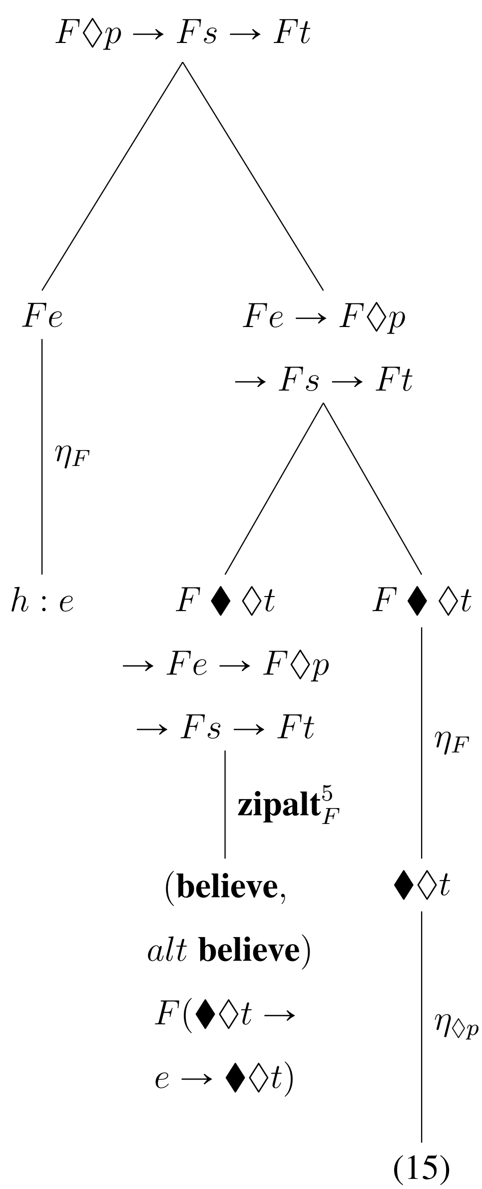
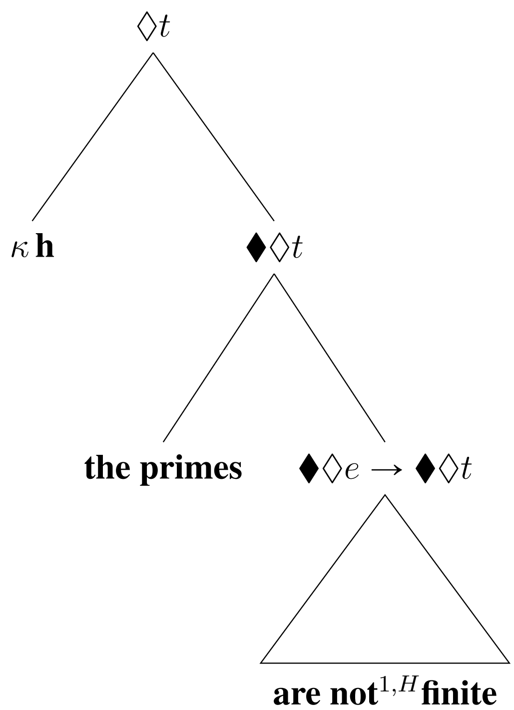
$(F\Diamond p \to Fs \to Ft)$
$\to (F\Diamond p \to Fs \to Ft)$
$F\Diamond p \to Fs \to Ft$
(16)
$\mathbf{zip}^3_{Fsp}$
$(Fs \to Ft)$
$\to (Fs \to Ft)$
$\to Fs \to Ft)$
$\mathbf{zip}^3_{Fs}$
$Ft \to$
$Ft \to Ft$
$\mathbf{zip}^3_F$
$\wedge : t(tt)$

(14), (15) and (16) in the tree (13) refer to the trees given below:

(14)

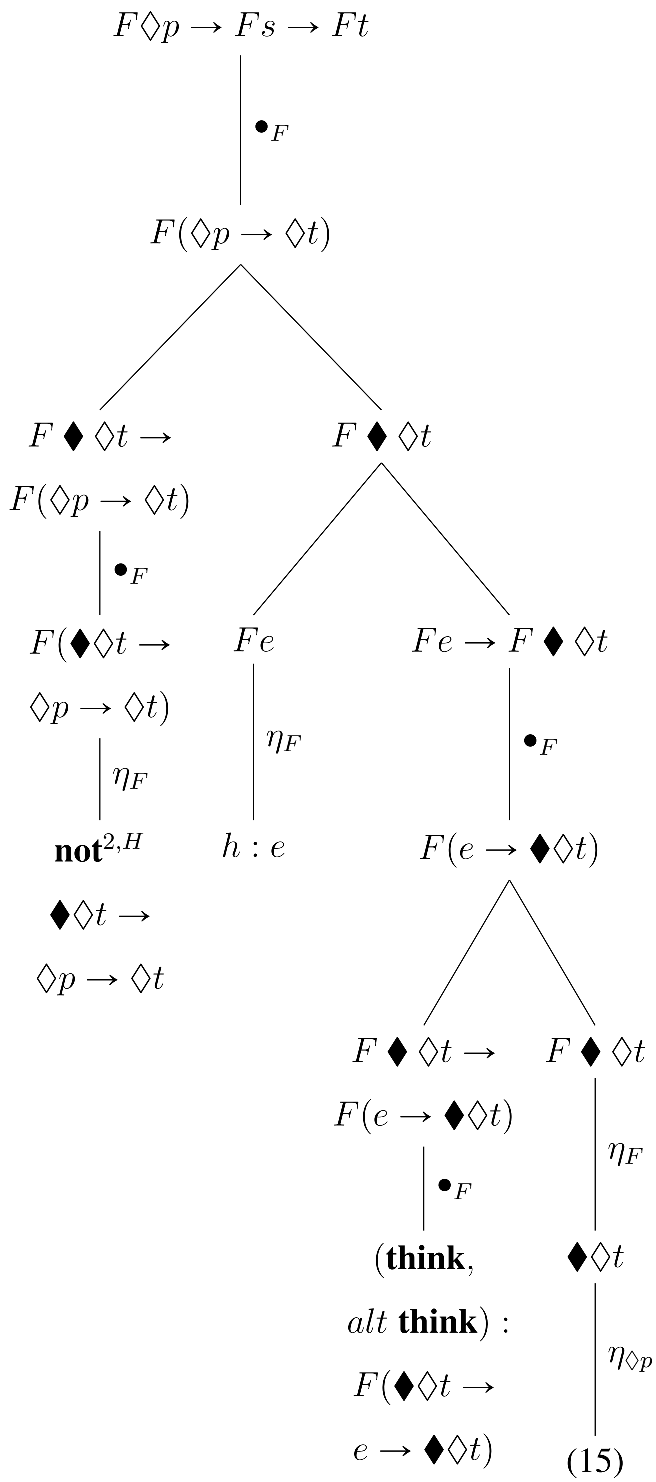


(15)



$\Diamond t$
κ **h**
$\blacklozenge\Diamond t$
the primes
$\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond t$
are not^{1,H}**finite**

(16)


F◊p → Fs → Ft
•F
F(◊p → ◊t)
F ♦ ◊t →
F(◊p → ◊t)
•F
F(♦◊t →
◊p → ◊t)
ηF
not2,H
♦◊t →
◊p → ◊t
F ♦ ◊t
Fe
ηF
h : e
Fe → F ♦ ◊t
•F
F(e → ♦◊t)
F ♦ ◊t →
F(e → ♦◊t)
•F
(think,
alt think) :
F(♦◊t →
e → ♦◊t)
F ♦ ◊t
ηF
♦◊t
η◊p
(15)

(6.3) consists of a pair whose first member is the lambda term which renders the unfocused sentence *Harold believes that the prime numbers are not finite but he doesn't think that the prime numbers are not finite*, and whose second member is a set of alternative lambda terms which are renderings of sentences of the form *Harold* x *that the prime numbers are not finite but he doesn't* y *that the prime numbers are not finite*, where x and y are variables standing for transitive verbs. Since **think** and **believe** have a different model-theoretic interpretation, their focus alternatives can be different and so an utterance of (12) will not be contradictory.

The case of (1), unlike the other cases, involves someone who thinks that the pronunciations Hahvahd and Harvard are pronunciations of different words. From this person's perspective, there *are* two words (see (Asudeh and Giorgolo 2016) for discussion). We could therefore render Hahvahd as (17) and Harvard as (18), by two distinct lambda terms, which we abbreviate respectively by **harvard** and **hahvahd**):

(17) $$\lambda\delta, s. \begin{cases} harvard_{pse}\,(\delta\, s)\, s & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) = \top \\ \# & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) \neq \top \\ harvard_{pse}\ E\ s & \text{if } \delta\, s = E \end{cases}$$

(18) $$\lambda\delta, s. \begin{cases} hahvahd_{pse}\,(\delta\, s)\, s & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) = \top \\ \# & \text{if } \delta\, s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\, s) \neq \top \\ harvard_{pse}\ E\ s & \text{if } \delta\, s = E \end{cases}$$

In order to ensure that **harvard** and **hahvahd** are interpreted the same with respect to the enlightened perspective index, we require that (6.4) is true of all admissible P-HYPE models:

$$\forall x \in D_{\blacklozenge\lozenge e}, s \in D_s \quad (6.4)$$
$$\left([\![\ \textbf{harvard} \ ]\!]^{\mathfrak{M},g}\, x\, [\![(\eta_s E)]\!]^{\mathfrak{M},g}\, s = [\![\ \textbf{hahvahd} \ ]\!]^{\mathfrak{M},g}\, x\, [\![(\eta_s E)]\!]^{\mathfrak{M},g}\, s \right)$$

Then relative to the enlightened perspective index, the phonological unit harvard and the phonological unit hahvahd have the same interpretation, but relative to Harold's perspective index, these phonological units may denote different things. In particular, for Harold, the focus alternatives of $harvard$ and $hahvahd$ may be different, given $[\![\textbf{harvard}]\!]^{\mathfrak{M},g} \neq [\![\textbf{hahvahd}]\!]^{\mathfrak{M},g}$. Consequently, Harold might have conventional beliefs about Harvard—familiar to him via the string Harvard—but think the string Hahvahd denotes a financial institution, so that the focus alternatives of Hahvahd are various financial institutions. We will render $[\text{Hahvahd}]_F$ and $[\text{Harvard}]_F$ by (19) and (20), respectively, where the alternatives of **harvard**, might be **harvard** itself, alongside $\textbf{MIT}_{\blacklozenge\lozenge e}$ and $\textbf{Princeton}_{\blacklozenge\lozenge e}$, and the alternatives of **hahvahd**, might be **hahvahd** itself, alongside $\textbf{Santander}_{\blacklozenge\lozenge t}$ and $\textbf{HSBC}_{\blacklozenge\lozenge t}$:

(19) $(\ \textbf{hahvahd},\ \ alt\,\textbf{hahvahd}\ ) : F \blacklozenge \lozenge e$

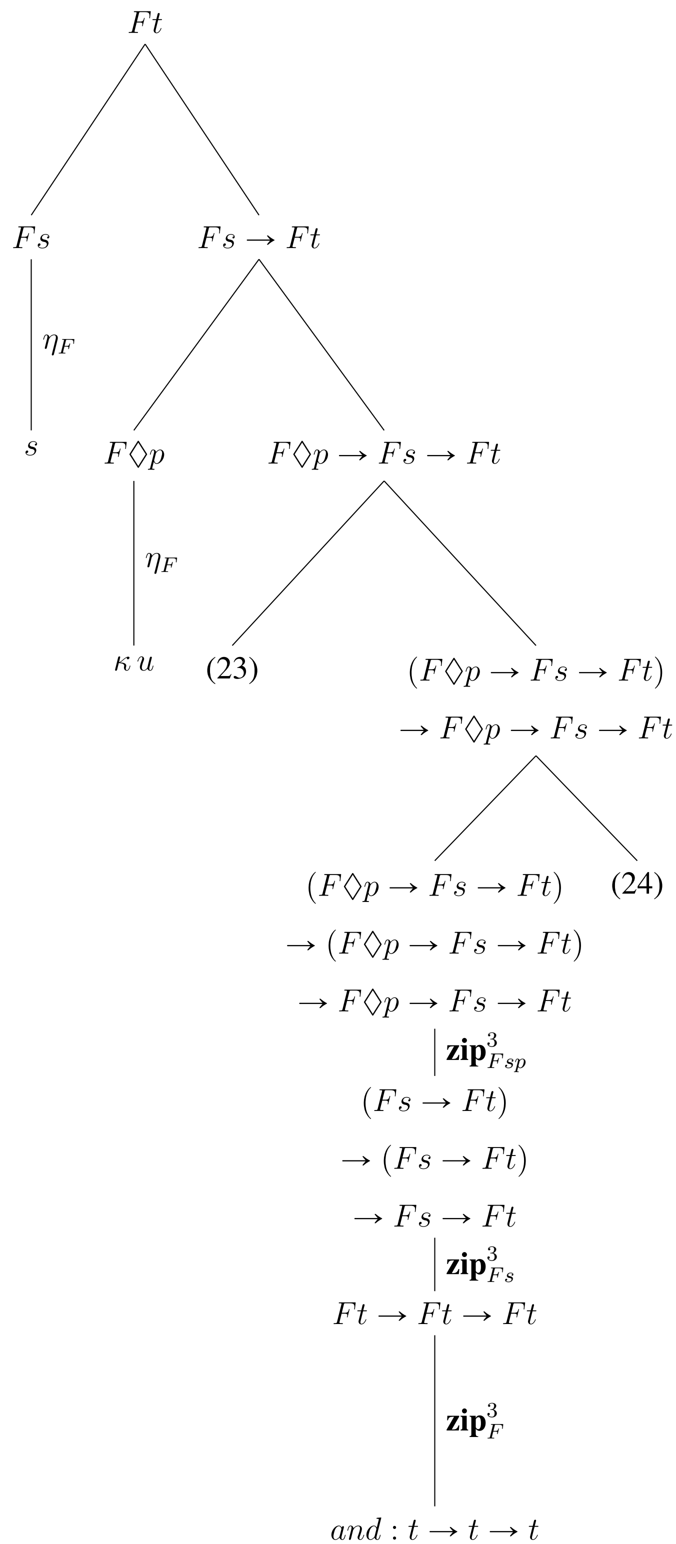
(20) $(\ \textbf{harvard},\ \ alt\,\textbf{harvard}\ ) : F \blacklozenge \lozenge e$

Supposing that (9) (repeated here as (21)) is uttered from the perspective of Harold at the state s, we render (21) as (6.5), whose compositional derivation is given in (22), where (23) and (24) refer to the trees given below:

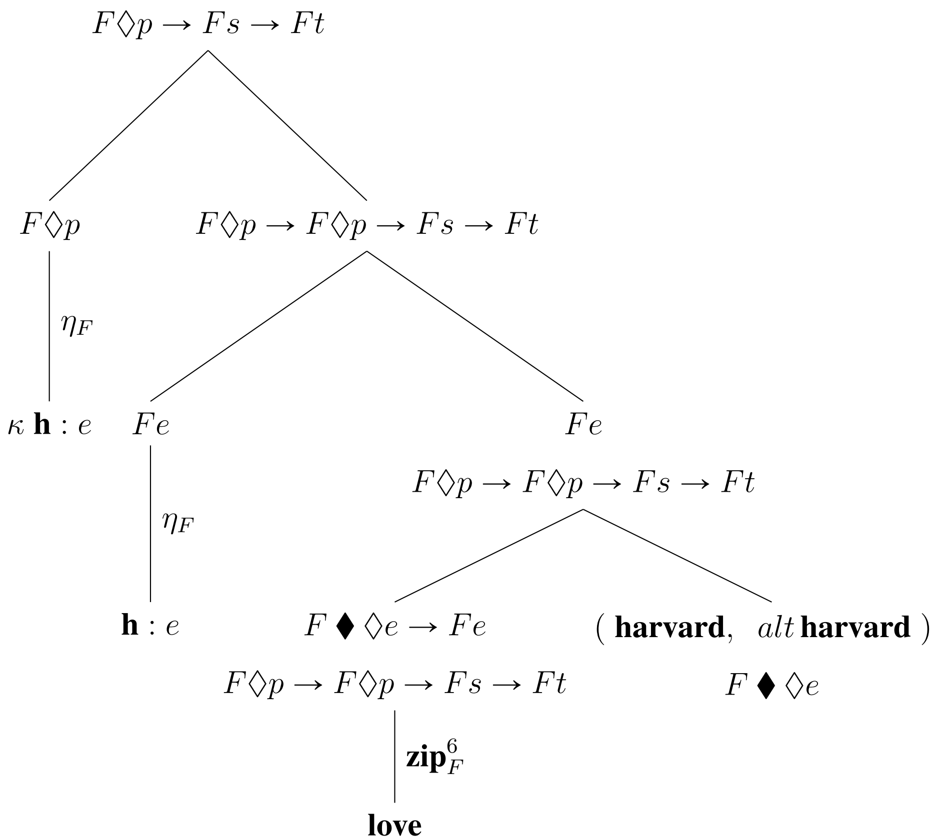
(21) Harold loves $[\text{ harvard}]_F$, but he doesn't love [hahvahd]

$$
\begin{cases}
\Bigg((\ \forall s'[s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s' & \text{if } C_{\mathbf{p}}\, s\ (\kappa\, u\, s) = \top \\
\rightarrow love\,(harvard\ (\kappa\ \mathbf{h}\, s')\ s')\,\mathbf{h}\ (\kappa\ \mathbf{h}\ s')\ s'] & \text{and } C_{\mathbf{p}}\, s\ (\kappa\ \mathbf{h}\, s') = \top \\
\wedge \forall s^3\ (\ \forall s'\ [\ s^3 \leqslant s' \wedge \pi\,(\kappa\, u\, s^3)\, s^3\, s' \rightarrow & \\
love\,(hahvahd\ (\kappa\ \mathbf{h}\ s')\ s')\,\mathbf{h}\ (\kappa\,\mathbf{h}\, s')\ s'\ ] & \\
\rightarrow\ \ s \perp s^3), & \\
\Big\{ (\mathbf{love}\ x\ \mathbf{h}\ (\kappa\,\mathbf{h})\ (\kappa\, u)\ s) & \\
\wedge(\ \mathbf{not}^{2,H}\ (\mathbf{love}\ y\ \mathbf{h}\ (\kappa\,\mathbf{h})\ )\ (\kappa\, u)\ s) & \\
\mid x \in alt\,\mathbf{harvard}, y \in alt\,\mathbf{hahvahd} \Big\} \Bigg) & \\
\# & \text{otherwise}
\end{cases}
\tag{6.5}
$$

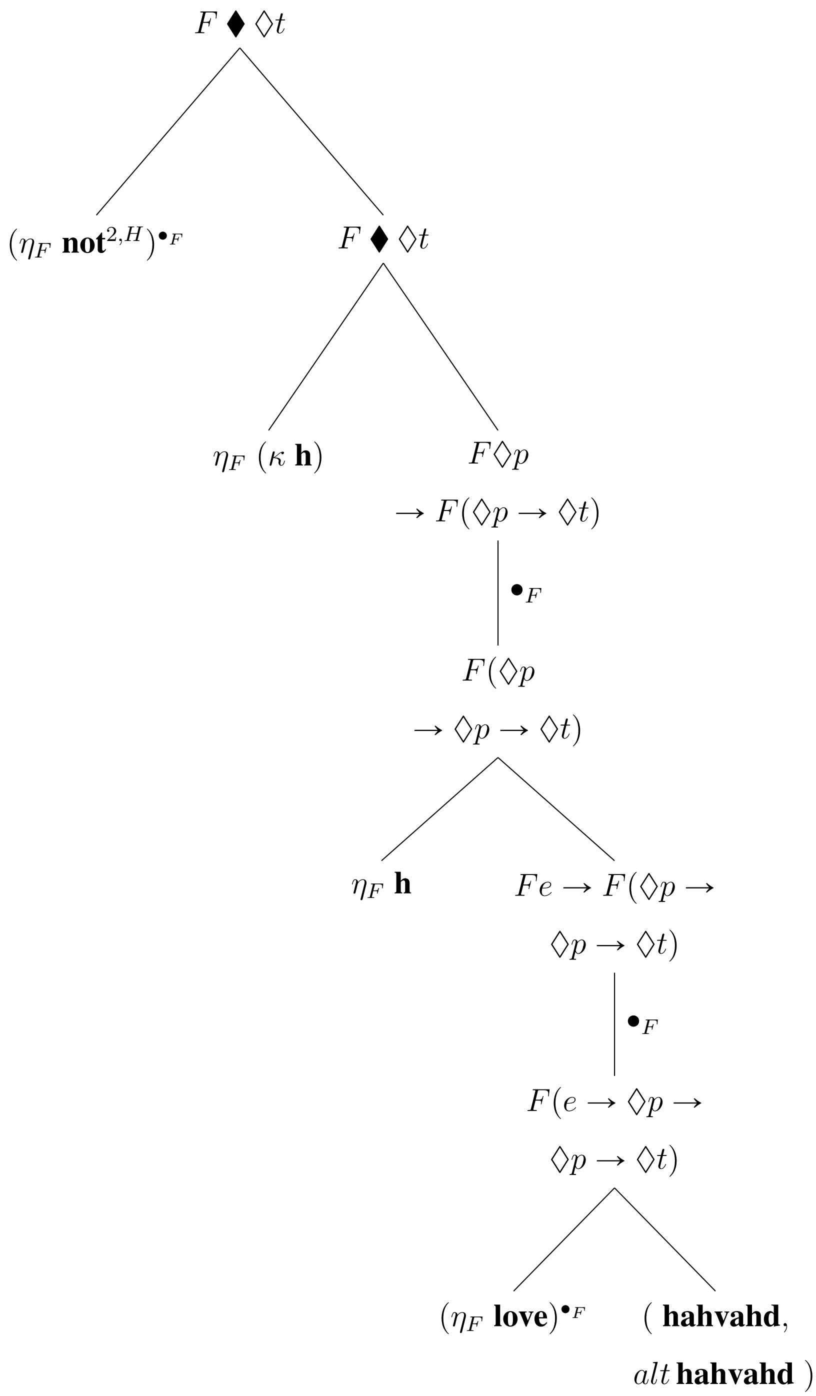
(22)


Ft
Fs
Fs → Ft
ηF
s
F◊p
F◊p → Fs → Ft
ηF
κ u
(23)
(F◊p → Fs → Ft)
→ F◊p → Fs → Ft
(F◊p → Fs → Ft)
→ (F◊p → Fs → Ft)
→ F◊p → Fs → Ft
(24)
zip3Fsp
(Fs → Ft)
→ (Fs → Ft)
→ Fs → Ft
zip3Fs
Ft → Ft → Ft
zip3F
and : t → t → t

(23)


$F\Diamond p \to Fs \to Ft$
$F\Diamond p$
$F\Diamond p \to F\Diamond p \to Fs \to Ft$
η_F
κ **h** : e
Fe
Fe
$F\Diamond p \to F\Diamond p \to Fs \to Ft$
η_F
h : e
$F \blacklozenge \Diamond e \to Fe$
$F\Diamond p \to F\Diamond p \to Fs \to Ft$
(**harvard**, alt **harvard**) :
$F \blacklozenge \Diamond e$
$\mathbf{zip}_F^6$
love

(24)


$F \blacklozenge \Diamond t$
$(\eta_F$ **not**$^{2,H})^{\bullet_F}$
$F \blacklozenge \Diamond t$
$\eta_F\ (\kappa$ **h**$)$
$F\Diamond p \rightarrow F(\Diamond p \rightarrow \Diamond t)$
$\bullet_F$
$F(\Diamond p \rightarrow \Diamond p \rightarrow \Diamond t)$
η_F **h**
$Fe \rightarrow F(\Diamond p \rightarrow \Diamond p \rightarrow \Diamond t)$
$\bullet_F$
$F(e \rightarrow \Diamond p \rightarrow \Diamond p \rightarrow \Diamond t)$
$(\eta_F$ **love**$)^{\bullet_F}$
(**hahvahd**, alt **hahvahd**)

6.3 Metalinguistic focus on predicates in P-HYPE

Having seen the basic strategy that we would pursue in order to integrate focus with our perspective relative semantic theory, we now turn to consider the meaning of certain utterances which seem to involve special difficulties for our semantic theory. These utterances involve contrastive focus on seemingly synonymous predicates such as *inductive* and *not finite*. In the case we will discuss, the semantic contribution of *inductive* and *not finite* seems to differ though they they have the same interpretation with respect to each perspective index which is relevant in the context of utterance.

Consider an utterance of (25b) followed by an utterance of (25c) in the scenario (25a):

(25) a. *Scenario*: Harold and Bill are discussing a proof in ZFC which Harold has produced, in which the conclusion is that the set of prime numbers does not belong to the class of finite sets. Bill's proof explicitly involves the ZFC definition of finite sets, but the proof nowhere involves the explicit ZFC definition of an inductive set. Harold and Bill both know that each other knows that, in ZFC, the prime numbers are finite if and only if they are inductive. In addition, Harold and Bill both know that they assign the same interpretation to both predicates as each other, and that they assign the same interpretation to *inductive* as to *finite*. Harold and Bill both know that they interpret 'the prime numbers' in the same way, as denoting the set of prime numbers. No one else's opinion (other than Harold and Bill's opinions) about the predicates is salient to Harold and Bill in the context. Harold utters (25b) to Bill. In response, Bill utters (25c) to Harold:[1]

[1](25c) could be felicitously uttered by Bill in a different scenario in which Bill is considering

b. I proved that the prime numbers are not inductive.

c. You proved the prime numbers are not $[\text{finite}]_F$, not that the prime numbers are not $[\text{inductive}]_F$.

Let us suppose that Harold and Bill both interpret *inductive* and *finite* with respect to the enlightened perspective, and that Harold and Bill both interpret *the prime numbers* with respect to the enlightened perspective. In scenario (25a), Bill's utterance of (25c) should be infelicitous, since, by hypothesis, *inductive* and *finite* have the same interpretation. Consequently, Bill's utterance of (25c) should have the same interpretation as an assertion of "You proved the prime numbers are not inductive, not that the prime numbers are not inductive" normally has in the scenario (25a). But (25c) is felicitous in the scenario (25a). We must therefore somehow account for the felicity of Bill's utterance of (25c).

The felicitous utterance of (25c) in the scenario described is arguably (Chris Barker, p.c) a case of metalinguistic focus (Li 2017), in which Bill is rejecting the appropriateness of using 'not inductive' as opposed to 'not finite', perhaps because the conclusion of the proof explicitly states that the set of prime numbers does not belong to the class of finite sets. There might be many reasons motivating Bill's rejection. He might for instance think that using 'inductive' is unhelpful or potentially distracting in the context in which the proof is being presented, if the proof is to be presented to a class of set theory novices. The correction is then a way of signalling to Harold that his word choice might be potentially confusing. Alternatively, Bill might simply be being obtuse: there is no logical explanation for his word preference (or he perhaps merely pretends to have a preference), and he

some constructive form of set theory on which the predicates *inductive* and *finite* are not equivalent, or in a scenario in which there is someone salient in the context– either by being physically present or by having been discussed– who distinguishes the predicates 'finite' and 'inductive'.

simply wishes to be a nuisance.

In a typical case of metalinguistic focus negation, focus and negation are used to object to the words or expressions someone has used to express their thought, rather than to negate the thought expressed itself. As we saw above, in Roothian focus semantics, the alternatives of a focused word are generated from the denotation of the word. So if $geese$ is a constant of type $e \to t$ rendering the English word 'geese', then the alternatives of $[\text{geese}]_F$ will be other entities in $D_{e\to t}$. But in natural language, focus alternatives aren't always determined by the denotation of a word that a word is conventionally assigned; in particular, sometimes the focus alternatives of a word can be other words or linguistic expressions.

This comes through in the dialogue (26):

(26) a. A: Look! Some gooses are flying.

b. B: No. Some $[\text{geese}]_F$ are flying.

In this dialogue, B is not objecting to the truth of what A says–she might, for instance, concur that some geese are flying–but to the grammatical appropriateness of A's use of 'gooses', since 'gooses' is not the correct plural form of 'goose' in standard English. B's correction thus has an 'expressive' dimension (Potts 2007b): it conveys non-at-issue information about the corrector's attitude towards the expressions the speaker uses in her utterance and about what alternative expressions could be used to express exactly the same thing. In the case of (26), a certain string is rejected as ungrammatical, but in other cases of metalinguistic focus negation the words used by a speaker might be rejected for other reasons. We propose that we analyse (25c) as rejecting Bill's use of the grammatical predicate 'not inductive', in favour of 'not finite'.

This form of metalinguistic focus negation cannot be captured simply by using

Roothian focus semantics, as (Li 2017) points out. For Rooth posits a so-called *focus-licensing operator* $\sim$, which licenses focus only if a certain contextual restriction is satisfied. Let $\Gamma : \alpha \times (\alpha \rightarrow t)$ be the rendering of a complex linguistic expression in which a focused constituent occurs, and $C : \alpha \rightarrow t$ be the relevant contextual restriction, which amounts to a set of contextual salient alternatives to Γ. These contextually salient alternatives are taken from antecedent sentences. The precise mechanism for determining this will not be discussed here, but often focus alternatives are taken from an immediately preceding sentence, as in the dialogue (27), where the focus alternatives of $[\text{James}]_F$ include the individual John, who has been introduced in a previous sentence:

(27) a. A: John married Mary?
  b. B: No, $[\text{James}]_F$ married Mary.

We interpret $\Gamma \sim C$ as follows:

$$[\![\Gamma \sim C]\!]^{\mathfrak{M},g} = \begin{cases} [\![\Gamma]\!]^{\mathfrak{M},g} & \text{if } [\![C]\!]^{\mathfrak{M},g} \subseteq [\![\pi_2\Gamma]\!]^{\mathfrak{M},g} \wedge [\![\pi_1\Gamma]\!]^{\mathfrak{M},g} \in [\![C]\!]^{\mathfrak{M},g} \\ & \wedge \exists y(y \in [\![C]\!]^{\mathfrak{M},g} \wedge y \neq [\![\pi_1\Gamma]\!]^{\mathfrak{M},g}) \\ \\ \# & \text{otherwise} \end{cases} \tag{6.6}$$

This requires the contextually salient alternatives in $[\![C]\!]^{\mathfrak{M},g}$ be amongst the alternatives in $[\![\pi_2\Gamma]\!]^{\mathfrak{M},g}$ and for the ordinary denotation of the complex linguistic expression Γ to be in $[\![C]\!]^{\mathfrak{M},g}$; in addition, it requires $[\![C]\!]^{\mathfrak{M},g}$ to contain an alternative distinct from $[\![\pi_1\Gamma]\!]^{\mathfrak{M},g}$. If B assigns the same interpretation to $gooses$ and $geese$, then the first co-ordinate of the rendering of ‘Some gooses are flying’ will have the same interpretation as the first co-ordinate of the rendering of ‘Some geese are flying'. As a result, assuming the discourse is complete as it stands, B's assertion will not meet the requirement of having a contextually salient antecedent which differs from it in its interpretation, and so B's assertion will not focus-licensed. It seems that in dialogues like (26), B is pointing A's attention to salient linguistic alternatives.

We will now introduce a proposed semantics for metalinguistic focus negation (Li 2017). We argue that this account of metalinguistic focus negation is not the full story on the phenomenon, since it cannot capture the felicitous utterance of (25c) in the scenario (25a). Then we will discuss our own proposed solution to the problems (Li 2017)'s account runts into and show how our solution can capture the felicitous utterance of (25c) in the scenario (25a).

The core intuition of (Li 2017)'s account is that focus is sometimes assigned to linguistic expressions, thus giving rise to focus alternatives which are linguistic, unlike in Rooth's account. According to the analysis of (Li 2017), in the dialogue (26), the salient alternative to the linguistic expression *geese* in the context is the linguistic expression *gooses* which is used in the first sentence of (26). If we assign a semantics to these two linguistic expressions which takes into account their phonological differences, then they will differ from one another, and thus *geese* will be focused licensed.

In order to talk about the interpretation of linguistic expressions in contexts of utterance, (Li 2017) adds a domain of contexts of utterance D_c and a domain of possible phonological strings D_u to his semantic theory. (Li 2017) follows (Potts 2007a), who proposes a two-dimensional semantics of linguistic expressions on which they are rendered as pairs of lambda terms interpreted relative to a given context of utterance, and of formulas expressing the information that a certain linguistic expression has been used in a context of utterance in order to express whatever is expressed by the semantic value of the first member of the pair (we will see an example soon). (Li 2017) follows (Kaplan 1989)'s treatment of utterance contexts, in which utterance contexts are tuples containing a state, a speaker, a time, a location and possibly other features of the non-linguistic context in which the utterance takes place. In (Kaplan 1989), the role of such utterance contexts is, *inter alia*, to fix information about the denotation of context-sensitive expressions (personal pronouns, etc). (Kaplan 1989) distinguishes the *character* of an expression–a function from contexts of utterance to intensions–from what he calls their *content* in a context c, which is the intension obtained by applying the character of an expression to the context c.

(Li 2017) uses utterance contexts to provide the metalinguistic information that a particular string has been used in a given context of utterance, and that this string has a certain intension in that context. The members of D_u are closed under concatenation, so that, where $a^\frown b$ is the concatenation of strings a and b, $a, b \in D_u$ implies $a^\frown b \in D_u$. Let $(\!|\cdot|\!) : u \rightarrow c \rightarrow \alpha$, be a function from phonological strings to utterance contexts c to entities of type α, for $\alpha \in TYPE$. Intuitively, $(\!|\cdot|\!) : u \rightarrow c \rightarrow \alpha$ takes a string and a context of utterance and returns a lambda term whose semantic value is the semantic value of the string in the given context. If someone uses 'gooses' to talk about geese in a given context c in the actual state w_0, then $(\!|goose s|\!)\, c\, w_0 = (\!|geese|\!)\, c\, w_0$. Let $\ulcorner \cdot \urcorner : u \rightarrow (\alpha \times t)$, for arbitrary $\alpha \in TYPE$ apply to a string u, producing a pair $\big((\!|u|\!)\, c,\ exp\, c\, u\, ((\!|u|\!)\, c)\big)$ of $(\!|u|\!)\, c$, where c is the context of utterance, and a formula $exp_{c \rightarrow u \rightarrow (s \rightarrow \alpha) \rightarrow t}\, c\, u\, ((\!|u|\!)\, c)$, where $exp_{c \rightarrow u \rightarrow (s \rightarrow \alpha) \rightarrow t}\, c\, u\, ((\!|u|\!)\, c)$ says that the string u was used to express $(\!|u|\!)\, c$ in c:

$$\ulcorner u \urcorner = \begin{cases} \left((\!|u|\!)\, c,\ exp\, c\, u\, ((\!|u|\!)\, c) \right) & \text{if } u \text{ is a meaning bearing element in } c \\ \# & \text{otherwise} \end{cases}$$

Consider (28b) and (28a):

(28) a. $\left((\!|gooses_u|\!)\, c,\ \ exp\ c\ gooses\ ((\!|gooses|\!)\ c) \right)$

b. $\left((\!|geese_u|\!)\, c,\ \ exp\ c\ geese\ ((\!|geese|\!)\ c) \right)$

(28b) and (28a) do not necessarily have the semantic value in a particular context c, simply because substituting two expressions of type u for one another in the second co-ordinate of each of these pairs is not guaranteed to preserve truth, as $gooses_u$ and $geeses_u$ are not the same strings. But recall the dialogue (26), repeated here as (29):

(29) a. A: Look! Some gooses are flying.

b. B: No. Some $[\text{geese}]_F$ are flying.

In this dialogue, we want to distinguish the meaning of *Some gooses are flying* and *Some*$[\text{geese}]_F$ *are flying*, but in the context of utterance in which the dialogue (29) occurs, both (28b) and (28a) are identical, since *goose* and *gooses* are both used to express the property of being a goose, and so both co-ordinates of (28b) and (28a) have the same semantic value, so that (6.7) holds:

$$[\![*geese_{e\to s\to t}]\!]^{\mathfrak{M},g} = [\![*gooses_{e\to s\to t}]\!]^{\mathfrak{M},g} \tag{6.7}$$

In order to distinguish the semantic contribution of *Some gooses are flying* and

Some$[\text{geese}]_F$ *are flying*, we must therefore require that A's utterance is one context of utterance c (relative to which the second co-ordinate of (28b) is false) and B is another, c' (relative to which the second co-ordinate of (28a) is false), so that (28b) and (28a) are semantically distinct. All in all, therefore we are assuming that (6.8), (6.9) and (6.10) hold, where c is the context of A's utterance in the dialogue and c' is the context of B's reply:

$$[\![(\!| gooses |\!)\ c]\!]^{\mathfrak{M},g} = [\![(\!| geese |\!)\ c]\!]^{\mathfrak{M},g} = [\![(\!| gooses |\!)\ c']\!]^{\mathfrak{M},g} = [\![(\!| geese |\!)\ c']\!]^{\mathfrak{M},g} \quad (6.8)$$
$$= [\![*geese_{e \to s \to t}]\!]^{\mathfrak{M},g} = [\![*gooses_{e \to s \to t}]\!]^{\mathfrak{M},g}$$

$$exp\ c\ gooses\ ((\!| gooses |\!)\ c) = 1 \neq exp\ c\ geese\ ((\!| gooses |\!)\ c) \quad (6.9)$$

$$exp\ c'\ geese\ ((\!| gooses |\!)\ c') = 1 \neq exp\ c'\ gooses\ ((\!| gooses |\!)\ c') \quad (6.10)$$

One consequence of having to distinguish contexts in this way is that we may even have to differentiate contexts of utterance mid-sentence in order to account for examples such as (30):

(30) Some gooses— $[\text{geese}]_F$, rather—are flying.

(Li 2017) is not clear about the need to distinguish contexts so finely, and this may be a controversial requirement. For simplicity, we will ignore this problem in the following and assume contexts cut very finely. But, unless we are to abandon (6.7), we could avoid having to cut contexts so finely by making exp a function of type $c \to u \to (s \to \alpha) \to \Diamond t$ (instead of a function of type $c \to u \to (s \to \alpha) \to t$), so that $exp\ c\ gooses\ ((\!| gooses |\!)\ c)$ expresses the set of states relative to which the string *gooses* was used in the context c to express $(\!| gooses |\!)\ c$.

In order to talk about the phonological focus alternatives of a given string, we need to integrate focus with $\ulcorner \cdot \urcorner$. Using the monad $(F, \eta_F, \bigstar_F)$ and the monad

$(U, \eta_U, \bigstar_U)$ (described below), (Li 2017) shows that we can capture the metalinguistic character of B's response in the dialogue (26) above:

<u>The U monad</u> :

$U\alpha = \alpha \times t \quad$ for $\alpha \in TYPE$

$\eta_U\, x = (\ x, \top\ )$

$a \bigstar_U f = \Big(\ \pi_1(f\,(\pi_1 a)),\ \ \pi_2\, a \wedge \pi_2\,(f\,(\pi_1\, a))\ \Big)$ where $a : U\alpha$ and $f : \alpha \to U\beta$

η_U forms a pair consisting of the argument x and $\top$. Impure monadic types are pairs which express that a certain string was used to express a certain intension in a given context of utterance, as in the examples described above. B's utterance of 'Some $[\text{geese}]_F$ are flying' when evaluated at the state s is rendered as (31):

(31) $$\Big(\ (\ \exists x(\llparenthesis geese \rrparenthesis\, c\, x\, s \wedge fly\, x\, s),\ \ exp\, c\, geese\, (\llparenthesis geese \rrparenthesis\, c)\ ),$$
$$\Big\{ \Big(\ \exists x(\llparenthesis u' \rrparenthesis\, c\, x\, s \wedge fly\, x\, s),\ \ exp\, c\, u'(\llparenthesis u' \rrparenthesis\, c)\Big) \mid u' \in alt\, geese : u \Big\} \Big)$$

The first member of (31) is a pair whose first member is the ordinary intension of the sentence 'Some geese are flying', and whose second member indicates that the string 'geese' was used in c to express the relevant property. The second member of (31) consists of a set of meaning expression pairs containing the metalinguistic alternatives u' that are relevant in the context and could be used in the context to express the ordinary semantic value of 'geese'. These might, for example, include 'gooses'. For the compositional derivation of (31), see (Li 2017). (Li 2017) proposes adopting the following focus licensing operator $\sim: F\alpha \to \alpha$:

$$[\![\ (\ x,\ X\ ) \sim C\ ]\!]^{\mathfrak{M},g} = \begin{cases} [\![\ x\ ]\!]^{\mathfrak{M},g} & \text{if } [\![C]\!]^{\mathfrak{M},g} \subseteq [\![X]\!]^{\mathfrak{M},g} \wedge [\![x]\!]^{\mathfrak{M},g} \in [\![C]\!]^{\mathfrak{M},g} \\ & \wedge \exists y (y \in [\![C]\!]^{\mathfrak{M},g} \wedge y \neq [\![x]\!]^{\mathfrak{M},g}) \\ \# & \text{otherwise} \end{cases}$$

In the context described C contains an alternative distinct from x (namely *gooses*), and so focus is licensed, and so $\sim$ maps (31) to (34), a lambda term of type Ut:

(32) $$\left(\exists x ((\!|geese|\!)\, c\, x\, s \wedge fly\, x\, s),\quad exp\, c\, geese\, ((\!|geese|\!)\, c) \right)$$

The problem with (Li 2017)'s account of metalinguistic focus negation is that it would render infelicitous the dialogue (33), which is a variant on (26):

(33) a. A: Look! Some gooses are flying.

b. B: No. It's not true that some $[\text{gooses}]_F$ are flying, but it's true that some $[\text{geese}]_F$ are flying.

B's response in (33) is felicitous. Consider how (Li 2017)'s semantic theory might be used to represent this. In the original dialogue (26) we assumed that A's utterance (*Look! Some gooses are flying*) and B's utterance (*No. Some* $[geese]_F$ *are flying*) took place in two separate contexts c and c', so as to distinguish the semantic contribution of A and B's utterances.

We might therefore propose the following rendering of (33):

(34) $$\Big((\neg \exists x ((\!|gooses|\!)\, c\, x\, s \wedge fly\, x\, s) \wedge \exists x ((\!|geese|\!)\, c'\, x \wedge fly\, x),$$
$$exp\, c\, gooses\, ((\!|gooses|\!)\, c) \wedge exp\, c'\, geese\, ((\!|geese|\!)\, c')\,),$$
$$\Big\{ \Big(\neg \exists x ((\!|u|\!)\, c\, x\, s \wedge fly\, x\, s) \wedge \exists x ((\!|u'|\!)\, c'\, x\, s \wedge fly\, x\, s) \mid u \in alt\, (gooses : u), u' \in alt\, (geese : u) \Big\} \Big)$$

This, however, will not work, on the assumption that (6.8) holds. For if (6.8) holds, the component formula of ((34)), namely (6.11), will express a contradiction, since, by (6.8), $[\![(\!| gooses |\!)\, c]\!]^{\mathfrak{M},g} = [\![(\!| geese |\!)\, c']\!]^{\mathfrak{M},g}$:

$$\neg\exists x((\!| gooses |\!)\, c\, x\, s \wedge fly\, x\, s) \wedge \exists x((\!| geese |\!)\, c'\, x\, s \wedge fly\, x\, s) \tag{6.11}$$

The only way out of this dilemma is for (Li 2017) to argue that $[\![(\!| gooses |\!)\, c]\!]^{\mathfrak{M},g} \neq [\![(\!| geese |\!)\, c']\!]^{\mathfrak{M},g}$. However, it is not clear exactly what entity the semantic interpretation function $[\![\cdot]\!]^{\mathfrak{M},g}$ should map $(\!| gooses |\!)\, c$ to, consistent with (Li 2017)'s assumption that $[\![(\!| gooses |\!)\, c]\!]^{\mathfrak{M},g}, [\![(\!| geese |\!)\, c']\!]^{\mathfrak{M},g} \in D_{e\to s\to t}$. Matters are somewhat improved if we suppose instead that the words *gooses* and *geese* are rendered, respectively, as (6.12) and (6.13), exactly as we have rendered predicates such as *oculist*:

$$\lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} *gooses_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ *gooses_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases} \tag{6.12}$$

$$\lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} *geese_{epst}\,(x\,\delta\,s)\,(\delta\,s)\,s & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) = \top \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\ s\ (\delta\,s) \neq \top \\ *geese_{epst}\,(x\,(\eta_s E)\,s)\,E\,s & \text{if } \delta\,s = E \end{cases} \tag{6.13}$$

With the above renderings, we could allow that the interpretation of *gooses* and *geese* differs with respect to B's perspective index, so that $[\![(\!| gooses |\!)\, c]\!]^{\mathfrak{M},g} \neq [\![(\!| geese |\!)\, c']\!]^{\mathfrak{M},g}$. Consequently, (6.11) would no longer express a contradiction.

However, this will not solve all the potential problems. For B's utterance is intuitively felicitous in some circumstances, even if she attaches the same interpretation to *gooses* and *geese*, and so even if we assume for all $s \in D_s$, $\{x \in D_{\blacklozenge\Diamond e} \mid [\![\textbf{gooses}]\!]^{\mathfrak{M},g}\ x\,[\![(\kappa\, B)]\!]^{\mathfrak{M},g}\, s = 1\} = \{y \in D_{\blacklozenge\Diamond e} \mid [\![\textbf{geese}]\!]^{\mathfrak{M},g}\, x\,[\![(\kappa\, B)]\!]^{\mathfrak{M},g}\, s = 1\}$. In other words, our intuition is that B's utterance (*No. It's not true that some* $[gooses]_F$ *are flying, but it's true that some* $[geese]_F$ *are flying.*) in dialogue (33) is similar to Harold's reply (*You proved the prime numbers are not* $[finite]_F$, *not that the prime numbers are not* $[inductive]_F$) in dialogue (25c), in that, in both cases, the reply is felicitous even if the two focused predicates are given the same interpretation relative to a perspective index.

To see that they are in the same boat, consider how (Li 2017)'s semantics might treat Harold's felicitous reply (*You proved the prime numbers are not* $[finite]_F$, *not that the prime numbers are not* $[inductive]_F$) in dialogue (25c). The sentence (35a) intuitively has the same meaning as (35b), and we will discuss how (Li 2017) might treat (35b) on the assumption that this is the case:

(35) a. You proved the prime numbers are not [finite $]_F$, not that the prime numbers are not [inductive $]_F$.

b. You proved the prime numbers are not [finite $]_F$, but you didn't prove that the prime numbers are not [inductive $]_F$.

We assumed in the scenario (25a) in which the dialogue (25c) takes place, that 'finite', 'inductive' and 'the prime numbers' are interpreted from the enlightened perspective on which 'finite' and 'inductive' are E-synonymous (i.e (6.14) holds):

$$\forall x \in D_{\blacklozenge\Diamond e}, s \in D_s \quad ([\![\textbf{ finite }]\!]^{\mathfrak{M},g}\ x\ (\eta_s\ E)\ s = [\![\textbf{ inductive }]\!]^{\mathfrak{M},g}\ x\ (\eta_s\ E)\ s) \tag{6.14}$$

In addition, we assume that (6.15) holds; since, as with *gooses* and *geese*, it is not clear how *inductive* and *finite* in B's utterance (35b) can differ in semantic value in the scenario described:

$$\forall c, c' \in D_c([\![\, (\!| \, finite \, |\!) \,]\!]^{\mathfrak{M},g} \; c \; = [\![\, (\!| \, inductive \, |\!) \,]\!]^{\mathfrak{M},g} \; c') \tag{6.15}$$

Relative to these assumptions the unfocused sentences *You proved the prime numbers are not finite but you didn't prove that the prime numbers are not inductive* can be rendered as 6.16 ($addressee : c \to e$, denotes the individual picked out by 'you' in a given context):

$$\begin{aligned}
&\big(\forall s' \; [\; [\; s \leqslant s' \land \pi \; E \; s \; s'] \to \forall s'' \; [\; PROV \; (addressee \; c) \; s' \; s'' \\
&\to \forall s''' \; (\; (\!| finite |\!) \; c \; (\sigma \; \textbf{primes} \; (\eta_s \; E) \; s''' \;) \; E \; s''' \; \to s'' \perp s''') \;] \;] \\
&\land \forall s^0 \\
&(\; \forall s' \; [\; [\; s^0 \leqslant s' \land \pi \; E \; s^0 \; s' \;] \to \forall s'' \; [\; PROV \; (addressee \; c) \; s' \; s'' \\
&\to \forall s^1 \; (\; (\!| inductive |\!) \; c \; \; (\sigma \; \textbf{primes} \; (\eta_s \; E) \; s^1 \;) \; E \; s^1 \; \to \; s'' \perp s^1) \;] \;] \\
&\to \; s \perp s^0 \;)
\end{aligned} \tag{6.16}$$

But given the assumptions (6.14) and (6.15), (6.16) is equivalent to (6.17), which is a contradiction:

$$\begin{aligned}
&\big(\forall s' \; [\; [\; s \leqslant s' \land \pi \; E \; s \; s'] \to \forall s'' \; [\; PROV \; (addressee \; c) \; s' \; s'' \\
&\to \forall s''' \; (\; (\!| finite |\!) \; c \; (\sigma \; \textbf{primes} \; (\eta_s \; E) \; s''' \;) \; E \; s''' \; \to s'' \perp s''') \;] \;] \\
&\land \forall s^0 \\
&(\; \forall s' \; [\; [\; s^0 \leqslant s' \land \pi \; E \; s^0 \; s' \;] \to \forall s'' \; [\; PROV \; (addressee \; c) \; s' \; s'' \\
&\to \forall s^1 \; (\; (\!| finite |\!) \; c \; \; (\sigma \; \textbf{primes} \; (\eta_s \; E) \; s^1 \;) \; E \; s^1 \; \to \; s'' \perp s^1) \;] \;] \\
&\to \; s \perp s^0 \;)
\end{aligned} \tag{6.17}$$

For this reason, unless we reject either (6.14) or (6.15), (Li 2017)'s semantics of metalinguistic focus negation cannot capture Harold's felicitous reply (*You proved the prime numbers are not* $[$*finite*$]_F$, *not that the prime numbers are not* $[$*inductive*$]_F$) in dialogue (25c).

We now propose two possible solutions to the problem. One solution to the problem is to propose that Harold's felicitous reply (*You proved the prime numbers are not* $[$*finite*$]_F$, *not that the prime numbers are not* $[$*inductive*$]_F$) in dialogue (25c) is felicitous because we widen the domain of relevant perspectives picked out by $C_{\mathbf{p}}\, s$ in **finite** and **inductive**, to incorporate some perspective index on which the two predicates differ in denotation. Here we suppose that something similar to the phenomenon of domain widening (Kadmon and Landman 1993) is occurring. In normal cases the semantics of a quantifier is restricted to a contextually salient group of individuals. For example, an utterance of *A/some cat is in my garden* would usually be interpreted as restricting the domain of the quantifier *some* to a proper subset of the set of all cats in the universe: for example the cats who live in a certain quarter of London. According to (Kadmon and Landman 1993), the quantifier *any* has a semantics like *some*, except that utterances of sentences containing *any* invite the hearer to widen the domain of the contextually relevant group of individuals over which the existential quantifier ranges, so as to incorporate cases which may otherwise be deemed marginal or irrelevant. This proposal has been used to explain the contrast in felicity between the felicitous sentences (36a), (36b) and the infelicitous sentence (36c):

(36) a. There's some cat in my garden.
b. There isn't any cat in my garden.
c. *There's any cat in my garden.

The explanation is as follows. Suppose the domain of the existential quantifier is set of cats which are local to the utterer's house, so that (36a) expresses that one of those cats is in the utterer's garden. On the assumption that *any* induces domain widening, then in uttering (36c) we widen the domain of cats under consideration. But widening the domain will not lead to an informationally stronger statement, since, if some local cat is in the utterer's garden, this will remain true no matter how the domain of cats is extended. The purpose of domain widening is therefore not clear in the case of utterances of (36c) and so, the explanation goes, utterances of (36c) are likely to be infelicitous.

By contrast, if we suppose that *any* widens the relevant domain of cats in an utterance of (36b), we end up with a larger domain of cats which incorporates the previous domain, and (36c) then denies that any cats amongst a larger group of cats are in the utterer's garden, which is an informationally stronger statement than saying that none of the local cats are in the utterer's garden, and something which the utterer might have reason to utter.

Bracketing whether this argument applies perfectly to the case of *any*, we propose that, in some cases, in order to render a statement felicitous, we broaden the domain of perspective indices under consideration. One such case might be Harold's felicitous reply (*You proved the prime numbers are not* $[$*finite*$]_F$, *not that the prime numbers are not* $[$*inductive*$]_F$) in dialogue (25c). In such a case, no one else is present in the situation. But Harold could use his utterance to signal the existence of another perspective, according to which the predicates *finite* and *inductive* differ in interpretation. This might be a rational thing to do, if, for example Harold and Bill have a friend in common and expanding the domain of relevant perspective indices to include the perspective index of this friend would make his utterance true. It might be the case, for example, that Harold, Bill and their friend had been discussing the predicates *finite* and *inductive* on another occasion. Harold's assertion than signifies that he has come to realise that they know someone in common who is such as to distinguish the meaning of *finite* and *inductive*. This is one possible explanation of the felicity of Harold's utterance in the dialogue (25c).

Another possible explanation is that no domain widening is going on in the dialogue (25c). Instead, the verb *prove* has a metalinguistic meaning which differs from its customary meaning. According to this alternative proposal, the verb *prove* can sometimes have a meaning which is similar to the meaning of the transitive predicate *provide a proof whose conclusion is the sentence*, on which this predicate relates an agent to a sentence:

(37) a. You provided a proof whose conclusion is the sentence *the primes are not finite*, but you didn't provide a proof whose conclusion is the sentence *the primes are not inductive*.

Semantically this could be captured by allowing it to be the case that, in some contexts of utterance $[\![(\!| prove_u |\!)\ c]\!]^{\mathfrak{M},g} = [\![prove']\!]^{\mathfrak{M},g}$, where $prove' : u \rightarrow e \rightarrow \Diamond t$ is defined as follows:

$$prove' =_{df} \tag{6.18}$$
$$\lambda u_u, x_e, s_s.\exists y_e(produced_{e \rightarrow e \rightarrow s \rightarrow t}\ y\ x\ s \wedge proof_{e \rightarrow t}\ y \wedge conclusion_{e \rightarrow e \rightarrow t}\ u\ y)$$

We propose that (35b), repeated here as (38a), be rendered as the lambda term (6.19):

(38) a. You proved the prime numbers are not $[$ finite $]_F$, but you didn't prove that the prime numbers are not $[$ inductive $]_F$.

$$\Big(prove'\ \text{the prime numbers are not finite}\ \ (addressee\ c)\ s\ \wedge \tag{6.19}$$

$\forall s'(\ prove'\ \textsf{the prime numbers are not inductive}\ \ (addressee\ c)\ s' \rightarrow s' \bot s\ ),$

$\Big\{\ prove'\ \ (\ \textsf{the prime numbers are not}\ x_u^{'}\ )\ \ (addressee\ c)\ s\ \wedge$

$\forall s'(\ prove'\ \ (\ \textsf{the prime numbers are not}\ y_u^{'}\ )\ \ (addressee\ c)\ s' \rightarrow s' \bot s\ )$

$\mid x_u^{'} \in alt\ finite, y_u^{'} \in alt\ inductive \Big\}\Big)$

The lambda term (6.19) is a pair of type Ft. The first component of (6.19) is a formula which expresses that the string the prime numbers are not finite is the conclusion of a proof which the addressee of the context proved in a state s, whereas the string the prime numbers are not inductive is not the conclusion of a proof which the addressee of the context proved in a state s. The second component of (6.19) is a set of alternative formulas expressing that strings of the form the prime numbers are not x', for alternatives x' to the string *finite* are the conclusions of proofs which the addressee of the context proved in a state s, whereas strings of the form the prime numbers are not y', for alternatives y' to the string *inductive* are not the conclusion of a proof which the addressee of the context proved in a state s.

We will now explain how the lambda term (6.19) can be compositionally generated. To produce the lambda term in (6.19), we need to generate a lambda term of type u which $prove' : u \rightarrow e \rightarrow t$ can combine with. To do this, we will use the applicative defined in (6.20), where ε is the empty string and is such that, for all $x \in D_u : \varepsilon^\frown x = x^\frown \varepsilon = x$, and where $utter : e \rightarrow u \rightarrow t$ is a constant which applies to an agent, a string and returns 1 iff the agent uttered the string:

The G applicative :

$G\,\alpha = (\alpha \times u) \times (u \rightarrow t)$

$\eta_G\, x =_{df} ((x,\ \varepsilon), \{\varepsilon\})$

$$f_{G(\alpha\to\beta)} \bullet_G a_{G\,\alpha} =_{df}$$

$$\begin{cases} \left(\left((\pi_1(\pi_1 f))\,(\pi_1(\pi_1 a)),\ \ (\pi_2(\pi_1 f))^\frown(\pi_2(\pi_1 a)) \right),\ \left\{ f'^\frown a' \mid f' \in \pi_2 f, a' \in \pi_2 a \right\} \right) & \text{if } utter\ u_e\ (\pi_2(\pi_1 f))^\frown(\pi_2(\pi_1 a)) \\ \left(\left((\pi_1(\pi_1 f))\,(\pi_1(\pi_1 a)),\ \ (\pi_2(\pi_1 a))^\frown(\pi_2(\pi_1 f)) \right),\ \left\{ a'^\frown f' \mid f' \in \pi_2 f, a' \in \pi_2 a \right\} \right) & \text{if } utter\ u_e\ (\pi_2(\pi_1 f))^\frown(\pi_2(\pi_1 a)) \\ \# & \text{otherwise} \end{cases}$$

With the G applicative to hand, we propose that, in some contexts c, the following renderings hold:

$$(\!|the_u|\!)\ c = \tag{6.20}$$

$$\left(\left(\textbf{the}_{(\blacklozenge\lozenge e\to\blacklozenge\lozenge t)\to\blacklozenge\lozenge e}\ ,\ the_u \right),\ \{\ the_u\ \} \right) : G((\blacklozenge\lozenge e \to \blacklozenge\lozenge t) \to \blacklozenge\lozenge e)$$

$$(\!|(prime\ numbers)_u|\!)\ c = \tag{6.21}$$

$$\Bigg(\Bigg(\textbf{primes}_{\blacklozenge\lozenge e \to \blacklozenge\lozenge t} \ , \ (prime\ numbers)_u \Bigg),$$

$$\{ \ (prime\ numbers)_u \ \} \Bigg) : G(\blacklozenge\lozenge e \to \blacklozenge\lozenge t)$$

$$(\!|are_u|\!)\ c = \tag{6.22}$$

$$\Bigg(\Bigg(\textbf{are}_{\substack{(\blacklozenge\lozenge e \to \blacklozenge\lozenge t) \\ \to \blacklozenge\lozenge e \to \blacklozenge\lozenge t}} \ , \ are_u \Bigg), \ \{ \ are_u \ \} \Bigg) : G((\blacklozenge\lozenge e \to \blacklozenge\lozenge t) \to \blacklozenge\lozenge e \to \blacklozenge\lozenge t)$$

$$(\!|not_u|\!)\ c = \tag{6.23}$$

$$\Bigg(\Bigg(\textbf{not}^{1,H}_{\substack{(\blacklozenge\lozenge e \to \blacklozenge\lozenge t) \\ \to \blacklozenge\lozenge e \to \blacklozenge\lozenge t}} \ , \ not_u \Bigg), \ \{ \ not_u \ \} \Bigg) : G((\blacklozenge\lozenge e \to \blacklozenge\lozenge t) \to \blacklozenge\lozenge e \to \blacklozenge\lozenge t)$$

$$(\!|finite_u|\!)\ c = \tag{6.24}$$

$$\Bigg(\Bigg(\textbf{finite}_{\blacklozenge\lozenge e \to \blacklozenge\lozenge t} \ , \ finite_u \Bigg) \ , \ alt\ finite_u \Bigg) : G(\blacklozenge\lozenge e \to \blacklozenge\lozenge t)$$

$$(\!|inductive_u|\!)\ c = \tag{6.25}$$

$$\Bigg(\Bigg(\textbf{inductive}_{\blacklozenge\lozenge e \to \blacklozenge\lozenge t} \ , \ inductive_u \Bigg) \ , \ alt\ inductive_u \Bigg) : G(\blacklozenge\lozenge e \to \blacklozenge\lozenge t)$$

Given the renderings above, we can render *The prime numbers are not* $[$*finite*$]_F$ and *The prime numbers are not* $[$*inductive*$]_F$, respectively, as the type $G\blacklozenge\lozenge t$ lambda terms (6.26) and (6.27), respectively:

$$\Bigg(\Bigg(\textbf{are} \ (\ \textbf{not}^{1,H}\textbf{finite} \ ) \ (\ \textbf{the primes} \ ), \tag{6.26}$$

$$the^\frown prime^\frown numbers^\frown are^\frown not^\frown finite \ \Bigg),$$

$$\left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2 \; (\; (\!|finite_u|\!) \; c \;) \right\}\Bigg)$$

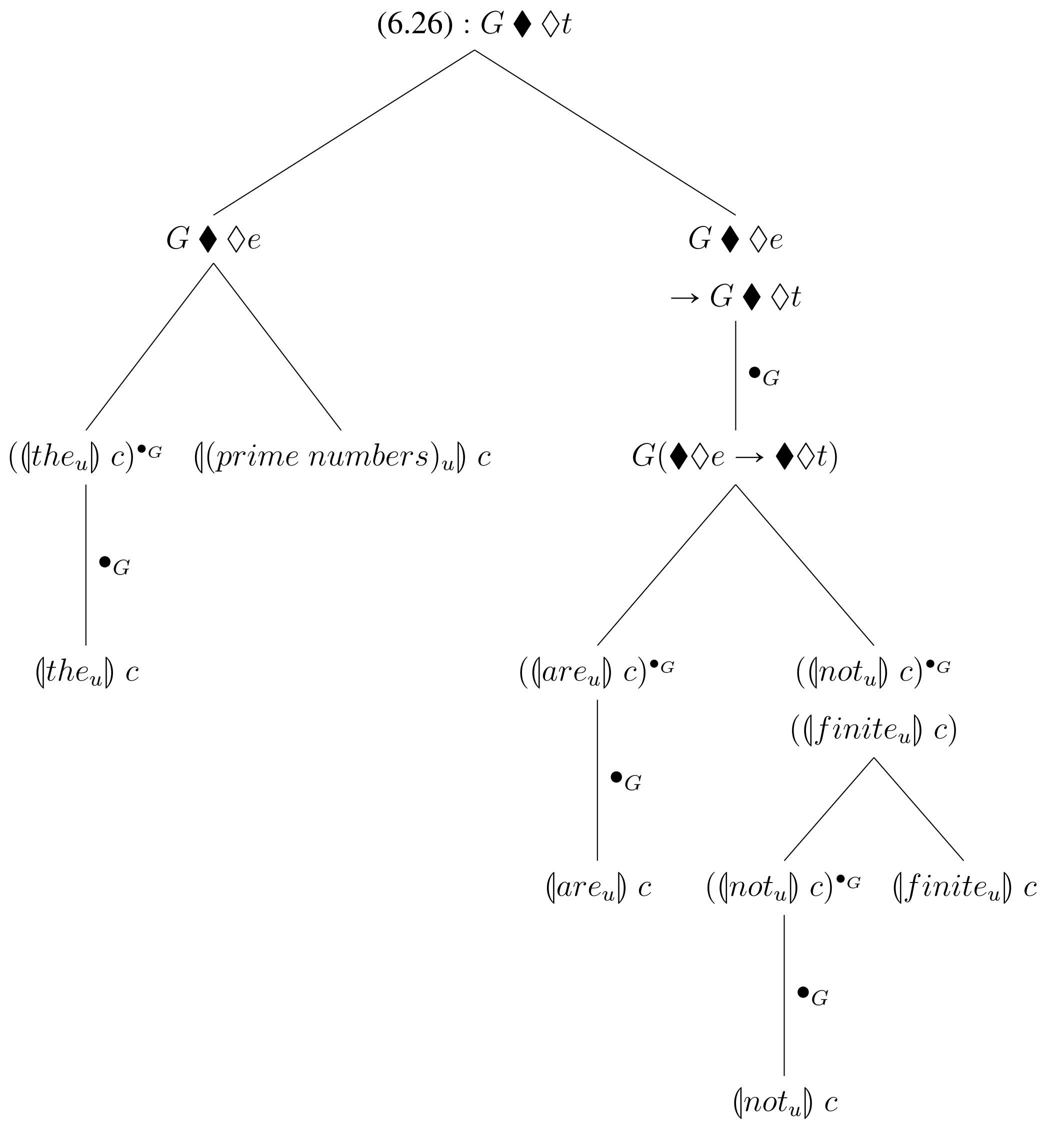
$$\Bigg(\Bigg(\textbf{are} \; (\; \textbf{not}^{1,H}\textbf{finite} \;) \; (\; \textbf{the primes} \;), \tag{6.27}$$

$$the^\frown prime^\frown numbers^\frown are^\frown not^\frown finite \Bigg),$$

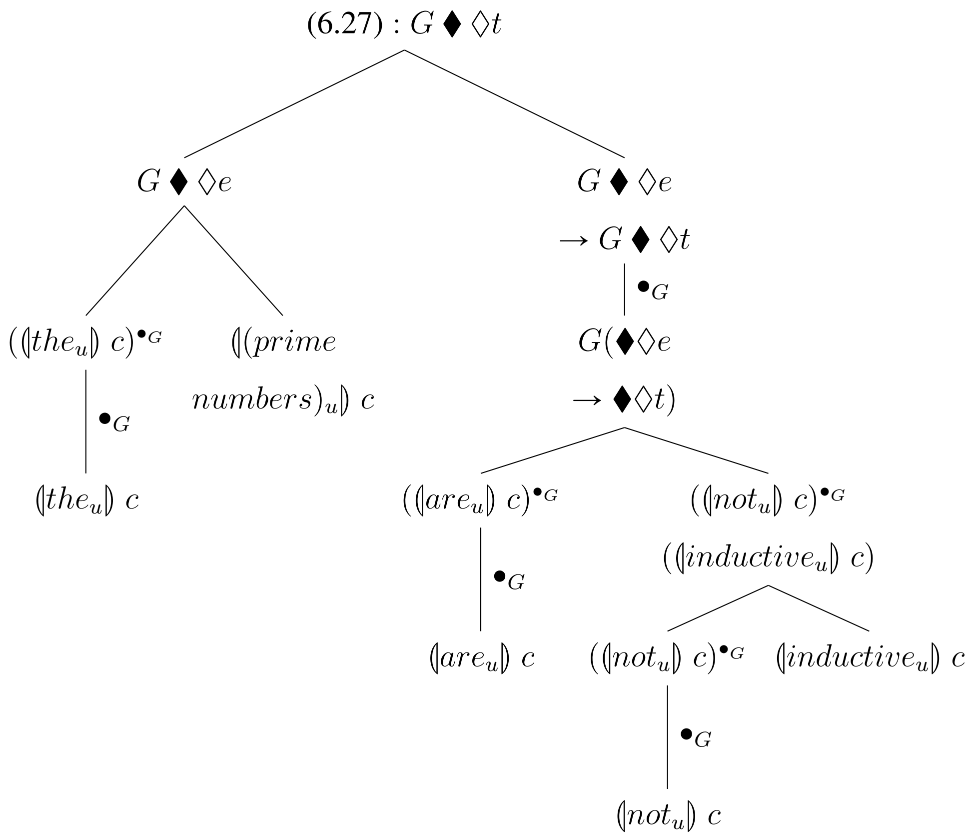
$$\left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2 \; (\; (\!|inductive_u|\!) \; c \;) \right\}\Bigg)$$

The lambda terms (6.26) and (6.27) can be compositionally derived via the (39) and (40), respectively:

(39)


(6.26) : G ♦ ◊t
G ♦ ◊e
G ♦ ◊e
→ G ♦ ◊t
•G
((|the_u|) c)^{•G}
(|(prime numbers)_u|) c
G(♦◊e → ♦◊t)
•G
(|the_u|) c
((|are_u|) c)^{•G}
((|not_u|) c)^{•G}
((|finite_u|) c)
•G
(|are_u|) c
((|not_u|) c)^{•G}
(|finite_u|) c
•G
(|not_u|) c

(40)



In order to generate the type Ft lambda term in (6.19), we need to extract from a lambda term of type Fu from the lambda term (6.26) and a lambda term of type Fu from the lambda term (6.27), which are then of the right type to combine with $(\eta_F prove')^{\bullet_F} : Fu \rightarrow F(e \rightarrow \Diamond t)$. We can do this using the combinator (6.28), and the result of applying this combinator to (6.26) and (6.27) are, respectively (6.29) and (6.30)—both lambda terms of type Fu:

$$\textbf{select}_{G\blacklozenge\Diamond t\rightarrow Fu} =_{df} \lambda g_{G\blacklozenge\Diamond t} \,.\, (\ \pi_2(\pi_1 g),\ \pi_2\ g) \tag{6.28}$$

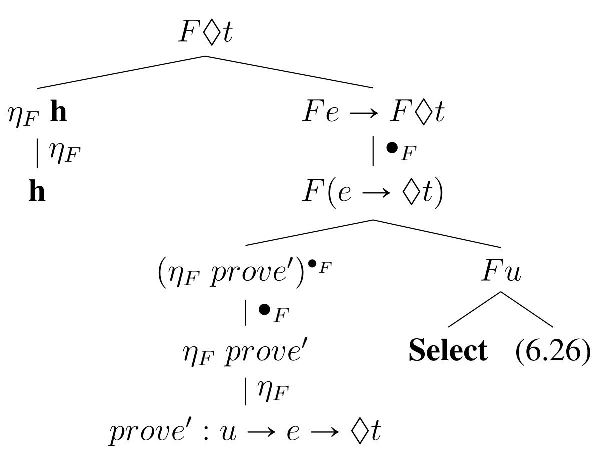
$$\textbf{select}\left(\left(\textbf{are}\ (\ \textbf{not}^{1,H}\textbf{finite}\ )\ (\ \textbf{the primes}\ ), \right.\right. \tag{6.29}$$
$$\left. the^\frown prime^\frown numbers^\frown are^\frown not^\frown finite \right),$$

$$\left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2 \ (\ (\!| finite_u |\!) \ c \ ) \right\} \Bigg)$$
$$= \Bigg(the^\frown prime^\frown numbers^\frown are^\frown not^\frown finite,$$
$$\left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2 \ (\ (\!| finite_u |\!) \ c \ ) \right\} \Bigg) : Fu$$

$$\textbf{select} \Bigg(\Bigg(\textbf{are} \ (\ \textbf{not}^{1,H}\textbf{inductive} \ ) \ (\ \textbf{the primes} \ ), \tag{6.30}$$
$$the^\frown prime^\frown numbers^\frown are^\frown not^\frown inductive \Bigg),$$
$$\left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2 \ (\ (\!| inductive_u |\!) \ c \ ) \right\} \Bigg)$$
$$= \Bigg(the^\frown prime^\frown numbers^\frown are^\frown not^\frown inductive,$$
$$\left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2 \ (\ (\!| inductive_u |\!) \ c \ ) \right\} \Bigg) : Fu$$

(6.26) can then each be fed to $prove'$, once it has been lifted via η_F and $\bullet_F$, as can **h**:

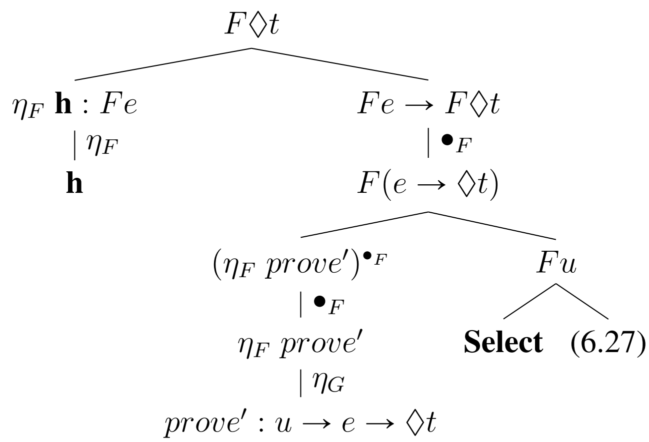
(41)



The root of the tree (41) is the lambda term:

$$\left(prove'\ the^\frown prime^\frown numbers^\frown are^\frown not^\frown finite\ \mathbf{h}, \right. \tag{6.31}$$
$$\left. \left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2\ (\ \llparenthesis finite_u \rrparenthesis\ c\ ) \right\} \right) : F\Diamond t$$

Similarly, we can feed (6.27) to $prove'$, once it has been lifted via η_F and $\bullet_F$, as can **h**:

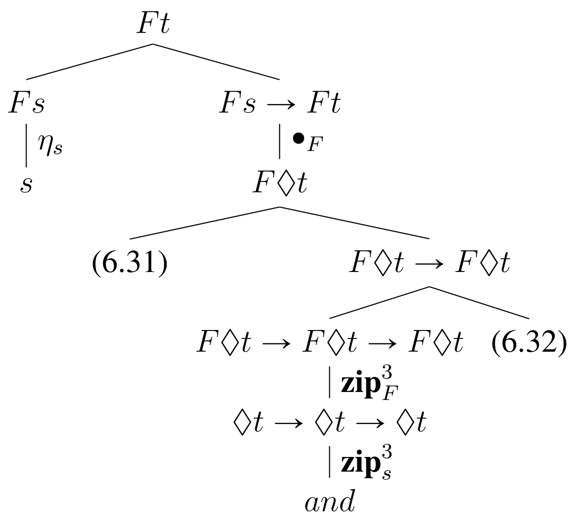


(42)

The root of the tree (42) is the lambda term:

$$\left(prove'\ the^\frown prime^\frown numbers^\frown are^\frown not^\frown inductive\ \mathbf{h}, \right. \tag{6.32}$$
$$\left. \left\{ the^\frown prime^\frown numbers^\frown are^\frown not^\frown x' \mid x' \in \pi_2\ (\ \llparenthesis inductive_u \rrparenthesis\ c\ ) \right\} \right)$$

We then need to combine the trees (41) and (42), by lifting and so that it can combine with two lambda terms of type $F\Diamond t$. Lifting the result of this operation and feeding a lambda term of type Fs, results in the following tree, whose root is type Ft:



(43)

The root of (43) is the lambda term (6.19), which we wanted to compositionally generate.

6.4 Conclusion

Both focus and metalinguistic focus negation have previously argued to be examples of linguistic side effects (Shan 2002), (Charlow 2014), (Li 2017). Our discussion of focus and metalinguistic focus negation allows us to show how we can deal with the accusation that P-HYPE is insufficiently fine-grained to deal with certain phenomena. Combining focus and perspective relativity enables us to capture certain cases of focus which are not captured easily under the assumption of *Predicate Uniformity*. Some puzzling uses of predicates in the scope of *prove* involve metalinguistic focus negation, but the semantics of metalinguistic focus negation in (Li 2017) cannot capture them. To capture these cases we can either invoke domain widening, a phenomenon familiar from the literature on quantifiers (Kadmon and Landman 1993) or we can adopt a different semantics for *prove* in the relevant cases.

Chapter 7

P-HYPE and the problem of logical omniscience

7.1 Introduction

In this chapter we provide a solution to the problem of logical omniscience in the form of a compositional natural language semantic theory. In 7.2 we will describe one approach that could be taken to logical omniscience in P-HYPE and give some reasons why we will not pursue it. On this approach, *and*, *or* and negation are treated as perspective-relative non-logical constants. We will express doubts that the approach can capture the distinction between subtle and obvious contradictions. In 7.3 we describe two approaches to logical omniscience: the single perspective approach (7.3.1) and the double perspective approach (7.3.2). Both approaches are based on the accessibility relation of a dynamic modality from the modal logics of (Solaki et al. 2019).

7.2 Perspective relativity, all the way down: *and*, *or* and *not* as non-logical, perspective relative constants

Recall the following family of closure principles, called *forms of logical omniscience*, which were discussed in chapter 2:

Necessitation

$\mathcal{M}\ \phi\ ;\ \models_i \Box\phi$

Closure under logical consequence

$\mathcal{M}\ ;\ ;\ \models_i \phi \rightarrow \psi \quad \Rightarrow \quad \mathcal{M}\ ;\ ;\ \models_i \Box\phi \rightarrow \Box\psi$

Closure under logical equivalence

$\mathcal{M}\ ;\ ;\ \models_i \phi \leftrightarrow \psi \quad \Rightarrow \quad \mathcal{M}\ ;\ ;\ \models_i \Box\phi \leftrightarrow \Box\psi$

Closure under material implication:

$\mathcal{M}\ ;\ ;\ \models_i (\Box\phi \wedge \Box(\phi \rightarrow \psi)) \rightarrow \Box\psi$

Distribution of $\Box$ *with respect to* $\wedge$

$\mathcal{M}\ ;\ ;\ \models_i \Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$

Agglomeration

$\mathcal{M}\ ;\ ;\ \models_i \Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$

Disjunctive weakening on the right

$\mathcal{M}\ ;\ ;\ \models_i \Box\phi \rightarrow \Box(\phi \vee \psi)$

Disjunctive weakening on the left

$$\mathcal{M}\ ;\ ;\ \models_i \Box\boldsymbol{\phi} \rightarrow \Box(\boldsymbol{\psi} \vee \boldsymbol{\phi})$$

In the first approach to the problem of logical omniscience under consideration, the above closure rules may be true of $\Box$, $\wedge$, $\vee$, but these logical operators are deemed inadequate renderings of the words *believe*, *and* and *or*; in addition, classical and HYPE negation are deemed inadequate renderings of negation. Suppose that sentences denote perspective and state relative functions in $D_{p\rightarrow s\rightarrow t}$. Then, according to the first approach under examination, *believe*, *and* and *or*, and sentential negation, are rendered as non-logical, perspective relative and state relative constants of the following type:

$$believe : (p \rightarrow s \rightarrow t) \rightarrow e \rightarrow (p \rightarrow s \rightarrow t)$$
$$and : t \rightarrow t \rightarrow (p \rightarrow s \rightarrow t)$$
$$or : t \rightarrow t \rightarrow (p \rightarrow s \rightarrow t)$$
$$not : t \rightarrow (p \rightarrow s \rightarrow t)$$

Adopting non-logical constants of this kind is compatible with allowing them to behave classically at a certain subset of states relative to a subset of perspective indices. Moreover, an approach along these lines would not require that we admit states at which the definition of truth at a state is non-recursive, as in (Rantala 1982). But given these non-logical constants may vary in denotation with respect to both perspective index and state, they are not forced to obey closure principles such as the family of closure principles listed above. For example, we could allow for a partial *Ty2* model $\mathfrak{M}$ and assignment g for which, for certain $p, q \in D_{p\rightarrow s\rightarrow t}$, $k \in D_p$, $s \in D_s$ and $x \in D_e$, (7.1), (7.2) and (7.3) hold:

$$[\![believe]\!]^{\mathfrak{M},g}\, p\, x\, k\, s = 1 \tag{7.1}$$

$$\text{For all } k' \in D_p \text{ and } s' \in D_s : p\, k'\, s' = 1 \text{ implies } q\, k'\, s' = 1 \tag{7.2}$$

$$[\![believe]\!]^{\mathfrak{M},g} q\, x\, k\, s = 0 \tag{7.3}$$

In this way, we can block all the analogues of the logical omniscience principles above, except *closure under logical equivalence*. *Closure under logical equivalence* will still hold, since whenever two terms $a, b \in Con_{p\to s\to t} \cup Var_{p\to s\to t}$ are logically equivalent, $[\![a]\!]^{\mathfrak{M},g} = [\![b]\!]^{\mathfrak{M},g}$, and so by substitution of equivalents, we can substitute a for b and *vice versa*. However, the counter-intuitive consequences of *closure under logical equivalence* might be avoided, if no renderings of two distinct natural language sentences are treated as having the same semantic value in a model.

We think that the proposed approach would not lead to an illuminating account of the problem of logical omniscience, for the following reasons. It would not be clear how the approach would enable us to give an analysis of the distinction between obvious contradictions and subtle contradictions, which we have discussed in previous chapters. This distinction is arguably important in natural language semantic theory, since asserting that someone believes something obviously contradictory is usually more infelicitous than asserting that someone believes some subtle contradiction (for example, by denying a particular mathematical truth, or a complex tautology). Perhaps a proponent of this approach could make the following suggestion. If sentences are semantically interpreted as elements of $D_{p\to s\to t}$, it could be proposed that a subtle contradiction is something that is false only relative to a few perspective indices and states, whereas an obvious contradiction is false relative to most or all perspectives and states. The problem with this suggestion is that it omits any mention of the logical rules (*modus ponens*, *conjunction introducton*, etc) used to infer sentences from other sentences, which plausibly feature in an explanation of why a certain contradiction is subtly as opposed to obviously contradictory. In sum, the logical and syntactic complexity of a sentence may obscure the fact that it follows logically from another sentence, and its contradictoriness will only become apparent once we have broken it up into component expressions, using certain logical rules. The number and complexity of such rules might therefore pose an impediment to realising the obvious logical equivalence of two sentences, or to realising that a given sentence follows logically from another. An approach which tries to distinguish obvious and subtle contradictions solely based on perspective is unable to capture this intuition, or at least is not clearly grounded in this intuition.

7.3 Resource-bounded reasoning in P-HYPE

In this section we will provide two different ways of extending some of the ideas in recent work on resource-bounded reasoning presented in (Bjerring and Skipper 2019) and (Solaki et al. 2019) to a compositional semantics of natural language. The first we call the *single perspective approach* and the second we call the *double perspective approach*. Unlike the approaches of (Bjerring and Skipper 2019) and (Solaki et al. 2019) our account will not involve dynamic model operators which induce model transformations (i.e, operations which take models and transform them into other models). In addition, our approach will provide a semantics at the subsentential level, unlike (Bjerring and Skipper 2019) and (Solaki et al. 2019) who do not discuss how their proposals could be integrated with a compositional natural language semantic theory. Indeed (Skipper and Bjerring 2018) consider it an open question how the logic they discuss will be extended to a compositional semantics of natural language.

Key to both approaches is the idea, developed in (Jago 2014a, Bjerring and Skipper 2019, Solaki et al. 2019), that rules of inference relate states in which the premisses of an inference are true to states in which their premisses plus their conclusions are true. The dynamic modalities in (Bjerring and Skipper 2019, Solaki et al. 2019) allow us to express that, after applying certain inference rules a certain number of times, an agent can believe some consequence of her beliefs. Since they also adopt states at which formulas are evaluated non-recursively, as in (Rantala 1982), they allow agents to not believe any logical consequence of her beliefs. However, they think that, via their special modalities, they can capture the intuition that a given agent is moderately rational, in the sense that she can come to believe certain logical consequences of her beliefs after applying certain inference rules a limited number of times, in proportion to the resources that she has.

We propose a modification of this type of account, along the following lines. Firstly, from the point of view of natural language semantics, whilst an operator expressing that an agent can come to believe something after reasoning a number of steps may be useful in modelling how agents reason, no natural language, as far we know, lexicalises such a modality, and the modality itself doesn't tell us how we ought to treat the semantics of sentences in natural language words, such as *believe*. (Bjerring and Skipper 2019) and (Solaki et al. 2019) allow the semantics of *believe* to be completely unrestrained, since at certain states, we can simply stipulate how the valuation function behaves. Consequently, the rich structure of their account of inference rules on the reasoning of agents in terms of inference rules which relate states to states is not part of the semantic account they give of *believe*, and since they have not offered a subsentential semantics, they have not explored the question of whether their dynamic modality could itself be used in giving a compositional semantics of natural language. We think that this opens up a space for an alternative way of viewing the rich structure of their account of inference rules on the reasoning of agents in terms of inference rules which relate states to states. Rather than seeing this structure as constraining how agents are able to reason, we can see it as determining the semantic value of sentences, so that the semantic value of certain sentences itself includes references to particular chains of reasoning which an agent can perform, chains which are themselves related to the relations between states which inference rules are supposed to give rise to in (Bjerring and Skipper 2019) and (Solaki et al. 2019). In this way, our account sees the semantics of certain sentences as involving constructions on premisses involving inference rules which are available to agents in context. If agents are able to carry through certain constructions leading to a formula ϕ, then those agents believe ϕ. We can therefore see the account of the problem of logical omniscience which we will give as internalising aspects of the dynamic modality in (Bjerring and Skipper 2019) and (Solaki et al. 2019) to the semantics of sentences themselves. Unlike (Bjerring and Skipper 2019) and (Solaki et al. 2019), our account will not include reference to the specific inference rules that agents use in reaching a certain deductive consequence of their beliefs in the object language; rather, our account will existentially quantify over such inference rules which have been used in deducing certain consequences in the metalanguage. By doing this, we avoid the commitment that the semantic value of a given sentence refers explicitly to rules of inference used by agents; rather, the semantic value of the sentence will only assert that there has been some inference rule or other used in deducing a certain consequence. Instead of using dynamic modal operators, we will use the concept of a *chain*, which is a list of characteristic functions of sets of names of terms of type t (i.e, formulas) which are related to one another via inference rules, in a way we will later specify. In our account, inference rules will stand between names of formulas, and we will introduce a type f of names of formulas. Whether the name of a formula is reachable via an inference rule after applying an inference rule for a certain agent and given her limited resources will be determined by whether the name of that formula is sent to 1 by the characteristic function which is the final element of a chain which is available to the agent in the context, where such a chain of formulas is called an *available chain* This should become clear after we introduce some details.

The machinery we are soon to introduce here does not disturb the general perspective relative semantic theory we have elaborated in previous chapters, for only in special cases will we need to invoke the available chains of various discourse participants. In particular, for derivations involving the enlightened perspective, the availability of certain chains will not figure in the sentence's denotation, on the basis that considerations of resource bound reasoning and logical omniscience are irrelevant to the enlightened perspective, which is omniscient. Our general picture will be this. On some occasions, the words used in a sentence are relevant to its semantic value, and other cases they are not. For example (see (Yalcin 2016) for a similar example), if you swerve to avoid hitting an animal on the motorway, intuitively it may be true to say that, implicitly, you believe that the animal you just missed was larger than a golfball, even though you would have not put it this way because golfballs were far from your mind at the time. In other cases the truth of belief attributions is sensitive to the linguistic material used to express the belief, as with the examples of metalinguistic focus negation discussed in the previous chapter. No doubt there is lots to say about exactly when the words used in a sentence are relevant to determining its semantic value, but we abstract from the complexity of these considerations here, and complexities of this kind will, in any case, be relevant to all natural language semantic theories (though see (Cresswell and Stechow 1982) for some suggestions about how to differentiate cases in which the linguistic material used is important to individuating the semantic value of sentences and cases in which it is not).

7.3.1 The single perspective approach

Both the single perspective approach and the double perspective approach will be based on a collection of rules of inference. In the double perspective approach this collection of rules will be parameterised to contexts states and two perspective indices, as we will discuss later, but in the single perspective approach, we parameterise sets of inference rules to contexts and states and a single perspective index. Given a model $\mathfrak{M}$, where $\mathcal{R}^{i,c,s}$ denotes the set of rules of inference available to the perspective index $i \in D_p$ in the context $c \in D_c$ at state $s \in D_s$, we define the following global set of inference rules (7.29):

$$\bigcup_{\substack{i \in D_p \\ c \in D_c \\ s \in D_s}} \mathcal{R}^{i,c,s} \tag{7.4}$$

As we shall understand it, rules of inference operate on names of arbitrary formulas, not on formulas, which are terms of type t in partial *Ty2*, nor do they operate on names of formulas; rather, particular *instances* of rules of inference act on names of formulas. As an example of a rule of inference, consider modus ponens (MP), where "ϕ" and "ψ" stand for names of arbitrary formulas, where names of formulas are indicated by quotation marks:

$$\frac{\text{“}\phi\text{”} \qquad \text{“}\phi \rightarrow \psi\text{”}}{\text{“}\psi\text{”}}\ \text{MP}$$

We can represent MP as a pair whose first member is the set containing the names of the arbirary formulas ϕ and ψ, and whose second member is the name of the arbitrary formula ψ:

$$(\{\text{“}\phi\text{”}, \text{“}\phi \rightarrow \psi\text{”}\}, \text{“}\psi\text{”})$$

Particular instances of MP act not on names of arbitrary formulas, but names of substitution instances of arbitrary formulas. Given names of formulas “A_f” and “$(A \rightarrow B)$” (which are substitution instances of “ϕ” and “$\phi \rightarrow \psi$”, respectively), we can represent a particular instance of MP, as follows:

$$(\{\text{“}A\text{”}, \text{“}A \rightarrow B\text{”}\}, \text{“}B\text{”})$$

We write **R** in bold, for inference rules, and R (not bold) for a particular instance of **R**. R^{σ} is a substitution instance of the inference rule R, where σ is a substitution.

We propose that substitutions be chosen from a set $SUB^{p,c,s}$ of substitutions, which varies depending on perspective index, context and state, so that SUB, the set of all substitutions in a model, is the following union:

$$SUB = \bigcup_{\substack{i \in D_p \\ c \in D_c \\ s \in D_s}} SUB^{c,i,s} \tag{7.5}$$

Defining substitutions in this way will be useful, since it will allow us to model agents who apply certain rules of inference generally, but not to certain names of formulas.

Given inference rules can be thought of in terms of their premisses and conclusions, we have in addition:

$pr\ R^{\sigma}$ is the set of names of the formulas which are premisses of some particular instance of an inference rule **R**

$con\ R^{\sigma}$ is the name of the formula which is the conclusion of the same instance of the inference rule **R**

We will now introduce a f of names of formulas, a type n of natural numbers and a type $List\,(f \to t)$ of lists of functions from formulas to truth values. We will use terms of type $(f \to t)$ to denote the characteristic functions of sets of names of formulas Our revised type formation rules will now be:

The set $TYPE$ of P-HYPE types is the smallest set of strings for which (i)-(iii) holds:

(i) $e, t, s, p, u, c, f, n \in TYPE$

(ii) If $\alpha \in TYPE$ and $\beta \in TYPE$, then $\alpha \to \beta \in TYPE$

(iii) If $\alpha \in TYPE$ and $\beta \in TYPE$, then $\alpha \times \beta \in TYPE$

(iv) $List\,(f \to t) \in TYPE$

We add the following rules of term formation:

- $p \in TERM_t$, then "p" $\in TERM_f$
- If $f_1, ..., f_n \in TERM_{f \to t}$, then $[f_1, ..., f_n] \in TERM_{List(f \to t)}$. When $n = 0$, we have $[\ ] \in TERM_{List(f \to t)}$, the empty list.

The domains for types e, t, s, p, u and c are the same as before, but we add $D_n = \mathbb{N}$ and $D_f = TERM_t$. For any $f_{f \to t} \in Con_{f \to t}$, we write $\mathbf{f}^{\mathfrak{M},g}$ instead of $[\![f]\!]^{\mathfrak{M},g}$. We then set that:

- If $p \in TERM_f$, then $[\![\ p\ ]\!]^{\mathfrak{M},g} = p$

- If $g \in CON_{List\,(f\to t)}$ is a non-empty list with n elements of the form $[f_1, ..., f_n]$, then $[\![\ g\ ]\!]^{\mathfrak{M},g} =$ the function $h \in D_{n\to(f\to t)}$ whose domain is $\{1, ..., n\}$ and which is such that for all i such that $1 \leqslant i \leqslant n$, $h(i) \neq \#$ and $h(i) = \mathbf{f}_i^{\mathfrak{M},g} \in D_{f\to t}$. If $g = [\,] \in CON_{List\,(f\to t)}$ is the empty list, then $[\![\ [\ \ ]\ ]\!]^{\mathfrak{M},g} = \#$.

A key aspect of our account will be the role of the non-logical constant $\sqsubseteq$: $(f \to t) \to (f \to t) \to c \to p \to \Diamond t \in Con_{(f\to t)\to(f\to t)\to c\to p\to\Diamond t}$. We write $f \sqsubseteq_{c,i,s} g$, instead of $\sqsubseteq f\,g\,c\,i\,s$. $f \sqsubseteq_{c,i,s} g$ is read as $\mathbf{g}^{\mathfrak{M},g}$ *is a minimal consistent deductive extension of* $\mathbf{f}^{\mathfrak{M},g}$ *according to perspective index i at state s in context c*. The semantics of $\sqsubseteq$ is based on the condition of *Succession* imposed by (Solaki et al. 2019) in their definition of the notion of R_k accessibility, as discussed in (2.3.1.1)-(D).We set that $[\![\sqsubseteq]\!]^{\mathfrak{M},g} = \sqsubseteq^{\mathfrak{M},g}$, where $\sqsubseteq^{\mathfrak{M},g}$ is defined as in (7.6), and expresses the characteristic function of a reflexive relation between sets of names of formulas $\mathbf{f}^{\mathfrak{M},g}, \mathbf{g}^{\mathfrak{M},g} \in D_{f\to t}$ relative to a certain perspective index, state and context:

For all $\mathbf{f}^{\mathfrak{M},g}, \mathbf{g}^{\mathfrak{M},g} \in D_{f\to t}, s \in D_s, c \in D_c$ and $i \in D_p$: $\mathbf{f}^{\mathfrak{M},g} \sqsubseteq^{\mathfrak{M},g}_{c,i,s} \mathbf{g}^{\mathfrak{M},g}$ iff (7.6)

$$\forall s' \Big[\ s \leqslant s' \Rightarrow \exists \mathbf{R} \in \mathcal{R}^{c,i,s}\ \exists \sigma \in SUB^{c,i,s} \Big[$$

$$\Big(\ \forall \mathbf{a}^{\mathfrak{M},g} \in D_f\ (\mathbf{a}^{\mathfrak{M},g} \in pr\ R^{\sigma} \Rightarrow \mathbf{f}^{\mathfrak{M},g}\ \mathbf{a}^{\mathfrak{M},g} = 1\ \Big)\ \&$$

$$\forall \mathbf{b}^{\mathfrak{M},g} \forall \mathbf{c}^{\mathfrak{M},g} \Big[$$

$$\Big(\mathbf{g}^{\mathfrak{M},g}\ \mathbf{b}^{\mathfrak{M},g} = 1 \Rightarrow \Big(\ \Big(\mathbf{f}^{\mathfrak{M},g}\ \mathbf{b}^{\mathfrak{M},g} = 1 \text{ or } \mathbf{b} = con\ R^{\sigma}\Big)\ \&$$

$$\exists d \in TERM_t\ \Big(\ \mathbf{b}^{\mathfrak{M},g} = [\![\text{“}d\text{”}_f]\!]^{\mathfrak{M},g}\ \&\ \mathbf{f}^{\mathfrak{M},g}\ [\![\text{“}\neg_H d\text{”}_f]\!]^{\mathfrak{M},g} \neq 1\ \Big)\ \Big)\ \&$$

$$\exists e \in TERM_t\ \Big($$

$$\Big(\mathbf{c}^{\mathfrak{M},g} = [\![\text{“}e\text{”}_f]\!]^{\mathfrak{M},g}\ \Big)\&\ \Big(\mathbf{f}^{\mathfrak{M},g}\ \mathbf{c}^{\mathfrak{M},g} = 1\ \Rightarrow \mathbf{f}^{\mathfrak{M},g}\ [\![\text{“}\neg_H e\text{”}_f]\!]^{\mathfrak{M},g} \neq 1\ \Big)\ \Big)\ \Big]$$

$\& \; \mathbf{g}^{\mathfrak{M},g} = \mathbf{f}^{\mathfrak{M},g} \cup \{con \; R^{\sigma}\} \Big] \Big]$

We assume that $\sqsubseteq^{\mathfrak{M},g}_{c,i,s}$ is undefined when $i = E$, the enlightened perspective index. This condition is imposed to ensure that the enlightened interpretation of sentences does not enter into the determination of how far a particular agent's beliefs or knowledge extend deductively; rather, as before, the enlightened interpretation of a sentence is simply its ordinary intensional interpretation as a set of classical states.

The condition (7.6) that, relative to perspective index i, state s and context c, there is some rule of inference $\mathbf{R} \in \mathcal{R}^{c,i,s}$, and some contextually available substitution $\sigma \in SUB^{c,i,s}$, such that $\mathbf{g}^{\mathfrak{M},g}$ contains the conclusion of a particular instance of an inference rule whose premisses are in $\mathbf{f}^{\mathfrak{M},g}$. In detail, (7.6) states that $\mathbf{f}^{\mathfrak{M},g} \sqsubseteq^{\mathfrak{M},g}_{c,i,s} \mathbf{g}^{\mathfrak{M},g}$ iff there is some rule of inference $\mathbf{R} \in \mathcal{R}^{c,i,s}$, and some substitution σ, such that $\mathbf{g}^{\mathfrak{M},g}$ contains the conclusion of a particular instance of an inference rule whose premisses are in $\mathbf{f}^{\mathfrak{M},g}$ such that for every name of a formula $\mathbf{f}^{\mathfrak{M},g}\mathbf{a}^{\mathfrak{M},g} = 1$ in the premiss set $pr \; R^{\sigma}$, and for every name of a formula $\mathbf{b}^{\mathfrak{M},g}$ such that $\mathbf{g}^{\mathfrak{M},g}\mathbf{b}^{\mathfrak{M},g} = 1$, either $\mathbf{f}^{\mathfrak{M},g}\mathbf{b}^{\mathfrak{M},g} = 1$ or $\mathbf{b}^{\mathfrak{M},g}$ is the conclusion of R^{σ} and $\mathbf{f}^{\mathfrak{M},g}\mathbf{b}^{\mathfrak{M},g} = 0$ and $\mathbf{b}^{\mathfrak{M},g}$ is not the negation of any name of a formula $\mathbf{c}^{\mathfrak{M},g}$, such that $\mathbf{f}^{\mathfrak{M},g}\mathbf{c}^{\mathfrak{M},g} = 1$. At least for classical negation, (7.6) guarantees the condition of *minimal rationality* (Cherniak 1986), whereby an agent is modelled as being able to eliminate some, but not all, inconsistencies in her beliefs. For if $\mathbf{f}^{\mathfrak{M},g} \sqsubseteq^{\mathfrak{M},g}_{c,i,s} \mathbf{g}^{\mathfrak{M},g}$, then neither $\mathbf{f}^{\mathfrak{M},g}$ nor $\mathbf{g}^{\mathfrak{M},g}$ can be such that $\mathbf{f}^{\mathfrak{M},g}$"$\phi$"$^{\mathfrak{M},g} = 1$ and $\mathbf{f}^{\mathfrak{M},g}$"$\neg\phi$"$^{\mathfrak{M},g} = 1$ (and the same for $\mathbf{g}^{\mathfrak{M},g}$), though it is possible that $\mathbf{f}^{\mathfrak{M},g}$"$\neg_H\phi$"$^{\mathfrak{M},g} = 1$ and $\mathbf{f}^{\mathfrak{M},g}$"$\phi$"$^{\mathfrak{M},g} = 1$ (and likewise for $\mathbf{g}^{\mathfrak{M},g}$), where $\neg_H$ is HYPE negation. Note, (7.6) doesn't rule out classically inconsistent names of formulas which are not of the form "ϕ"$^{\mathfrak{M},g}$ and "$\neg\phi$"$^{\mathfrak{M},g}$ from being sent to 1 by $\mathbf{f}^{\mathfrak{M},g}$ and $\mathbf{g}^{\mathfrak{M},g}$.

Given we allow terms of type $List(f \to t)$, we can now form lists of the characteristic functions of sets of names of formulas, such that each member of the list (except the final element) stands in the relation $\sqsubseteq$ to the member which immediately follows it. The semantic interpretation of such a list in a given context c and state s represents the application of different inference rules in $\mathcal{R}^{c,i,s}$ to the first member of the list, and is called a *chain*, and represent the inferences that a certain agent has performed, with the last member of the list containing the names of formulas which that agent believes. We want a lambda term which forms chains; moreover, they should be chains which actually reflect processes of reasoning that agents have undergone. For this reason, we shall shortly introduce a lambda term $available.chain_{List(f \to t) \to c \to \blacklozenge \Diamond List(f \to t)}$ which returns a list if it is a chain and if it is available, in the sense of reflecting a process of reasoning which an agent has performed. We will now add some details to this picture, by providing certain non-logical constants which will enable us to define $available.chain_{List(f \to t) \to c \to \blacklozenge \Diamond List(f \to t)}$:

- $nat : n \to t \in Con_{n \to t}$, a constant for representing the characteristic function of the set of natural numbers.

- $< : n \to n \to t \in Con_{n \to n \to t}$, a constant for representing the relation of inequality on the natural numbers

- $head : List(f \to t) \to f \in Con_{List(f \to t) \to f}$, a function which retrieves the first member of a list of objects of type $f \to t$, and which is such that, for all $\mathfrak{M}$ and assignments g, (7.7) holds:

$$[\![head]\!]^{\mathfrak{M},g} = \text{the function } G \in D_{List[f \to t] \to f}, \text{ such that,} \tag{7.7}$$

for any sequence $h \in D_{n \to (f \to t)}, G\, h = \#$ if $h = \#$ and $G\, h = h(1),$ otherwise.

We stipulate that, for all $n \in \mathbb{N}$, if $n = 0$, then $head^n = head$, otherwise $head^n\, A = head^{n-1}\,(head\, A)$.

- $tail : List(f \to t) \to List(f \to t) \in Con_{List(f \to t) \to f}$, which retrieves the tail of a list (i.e, the list containing all the elements of the input list, save the first member). We stipulate that, for all $n \in \mathbb{N}$, if $n = 0$, then $tail^n = tail$, otherwise $tail^n\, A = tail^{n-1}\,(head\, A)$, and, for all $\mathfrak{M}$ and assignments g, (7.8) holds:

$$[\![\, tail]\!]^{\mathfrak{M},g} = \text{the function } H \in D_{List[f \to t] \to f},\ \text{such that,} \tag{7.8}$$

for any sequence $h \in D_{n \to (f \to t)}$ whose domain is $\{1, ..., n\}$ for some $n \in \mathbb{N}^0$, $H\ h = \#$ if $h = \#$ and $G\, h = h'$ where h' is the function which is exactly like h except that it is not defined on 1.

- $member.of : (f \to t) \to List(f \to t) \to t$ applies to a function of type $f \to t$ and a list of functions of type $f \to t$ and returns true if the function is a member of the list, and which is such that, for all $\mathfrak{M}$ and assignments g, (7.9) holds:

$$[\![\, member.of]\!]^{\mathfrak{M},g} = \text{the function } H \in D_{(f \to t) \to List(f \to t) \to t}, \tag{7.9}$$

such that, for any $\mathbf{f}^{\mathfrak{M},g} \in D_{f \to t}$ and $h \in D_{n \to (f \to t)}, H\ \mathbf{f}^{\mathfrak{M},g}\ h = \#$ if $h = \#$ and $H\, \mathbf{f}^{\mathfrak{M},g}\ h = 1$ if $h(i) = \mathbf{f}^{\mathfrak{M},g},$ for some $i \leqslant length\ h.$

- $length : List(f \to t) \to n \in Con_{List(f \to t) \to n}$, a function which returns the length of a list and which is such that, for all $\mathfrak{M}$ and assignments g, (7.9) holds:

$$[\![\ length]\!]^{\mathfrak{M},g} = \text{the function } H \in D_{List(f\to t)\to n}, \text{ such that,} \tag{7.10}$$
$$\text{for any } h \in D_{n\to(f\to t)}, H\ h = \# \text{ if } h = \# \text{ and}$$
$$H\ h = n \text{ if for all i, } 1 \leqslant i \leqslant n, h(i) \neq \# \text{ and for all } j > n, h(j) = \#.$$

- $last : List(f \to t) \to f \to t \in Con_{List(f\to t)\to f\to t}$ is a function which, applied to a list of the characteristic functions of sets of names of formulas, and to the name of a formula, returns 1 if the last element of the list returns 1 when applied to the name of the formula. $last : List(f \to t) \to f \to t \in Con_{List(f\to t)\to f\to t}$ is such that, for all $\mathfrak{M}$ and assignments g, (7.11) holds:

$$[\![\ last]\!]^{\mathfrak{M},g} = \text{the function } H \in D_{List(f\to t)\to f\to t}, \text{ such that,} \tag{7.11}$$
$$\text{for any } h \in D_{n\to(f\to t)} \text{ and any } \mathbf{f}^{\mathfrak{M},g} \in D_{f\to t}, \text{ if } h = \# \text{ then } H\ h\ \mathbf{f}^{\mathfrak{M},g} = \#.$$
$$\text{Otherwise } H\ h\ \mathbf{f}^{\mathfrak{M},g} = 1 \text{ if and only if } \mathbf{f}^{\mathfrak{M},g} = h(n) \text{ for some } n \neq 0 \in \mathbb{N},$$
$$\text{and } h \text{ is not defined on any } m > n.$$

- $A_{List(f\to t)\to p\to c\to s\to t} \in Con_{List(f\to t)\to p\to c\to s\to t}$ is a function which returns 1 when supplied a list of functions, a perspective index i, a state s and a context c if and only if the list of functions is available from the perspective index i in the context c at c. From A we can construct the lambda term:

$$\textbf{Available} := \tag{7.12}$$
$$\lambda f_{List(f\to t)}, \delta_{\Diamond p}, c_c, s_s.\ A_{List(f\to t)\to p\to c\to s\to t}\ f\ (\delta\ s)\ c\ s$$

Using the non-logical constants above, we can define the lambda term $available.chain_{List(f\to t)\to c\to\blacklozenge\Diamond List(f\to t)}$, which selects chains which are available to a perspective index, relative to a given context and state:

- $available.chain_{List(f\to t)\to c\to\blacklozenge\Diamond(List(f\to t))} :=$

$$\lambda f_{List(f\to t)}, c_c, \delta_{\Diamond p}, s_s. \begin{cases} f & \text{if } \textbf{Available}\, f\,(\delta\, s)\, c\, s \wedge \exists a_{f\to t}(member.of\; a\; f) \wedge \\ & \forall a_{f\to t}(member.of_{(f\to t)\to List(f\to t)\to t}\; a\; f \\ & \to \exists x_f(a\, x = 1)) \wedge \Big(length\, f \geqslant 2 \to \\ & head\, f \sqsubseteq_{c,(\delta\, s),s} head\,(tail\, f) \\ & \wedge \forall i(nat\, i \;\wedge\; i \geqslant 2 \;\wedge i < (length\, f) - 1) \\ & \to head(tail^{i-1}\, f) \sqsubseteq_{c,(\delta\, s),s} head(tail^{i} f)\Big) \\ \# & \text{otherwise} \end{cases}$$

This function takes a list of the characteristic functions associated with sets of names of formulas and returns the input list, provided it is an available chain, according to a given perspective index at a certain state and context.

We are now ready to state how the single perspective approach resolves the problem of logical omniscience. In the semantic theory discussed in part II of this thesis, sentences have denoted functions of type $\blacklozenge\Diamond t$. In the single perspective approach we feed such terms to the following function:

$$\textbf{lift}_{\textbf{non-final}} := \tag{7.13}$$
$$\lambda p_{\blacklozenge\Diamond t}, c_c, \delta, s_s.\; \forall s'(s \leqslant s' \to \exists f_{List(f\to t)}(last\;(available.chain\, f\, c\, \delta\, s')\; \text{``}\; p\, \delta\, s'\; \text{''}))$$

This function takes a term $p_{\blacklozenge\Diamond t}$, a context c, a function δ and a state s and returns 1 if in all states s' accessible via $\leqslant$ there is a chain which is available in c at s' according to the perspective index $\delta\, s'$ and whose last element returns 1 when applied to "$p\, \delta\, s'$" the name of the formula $p\, \delta\, s'$. Since the elements of this list are the characteristic functions of sets of names of formulas, they are not closed under any logical rules, so that, for example, it could be the case that an element of a chain sends "$p\,\delta\,s'$ " to 1 and "$q\,\delta\,s'$ " to 1 but that it does not send "$p\,\delta\,s' \;\; \wedge\, q\,\delta\,s'$ " to 1, or "$p\,\delta\,s' \;\; \vee\, q\,\delta\,s'$ " to 1. In this way, the elements of chains can mimic the non-normal or 'impossible' states in (Rantala 1982, Jago 2014a, Bjerring and Skipper 2019, Solaki et al. 2019) which are not closed under the recursive rules of satisfaction at a state. $\mathbf{lift}_{\textbf{non-final}}$, if applied to any terms of type t, would imply that the truth conditions of sentences depended generally on the reasoning capabilities of agents. A more modest proposal would restrict $\mathbf{lift}_{\textbf{non-final}}$, so that it only applies to names of formulas of the form "$\mathbf{Believe}_{\blacklozenge\lozenge t\to e\to\blacklozenge\lozenge t}\ p_{\blacklozenge\lozenge t}\ x_e\ \delta_{\lozenge p}\ s_s$". This more modest proposal provides one way of avoiding the problem of logical omniscience, since we can restrict which names of formulas are sent to 1 by the final element of a chain arbitrarily.

Let us give an example. Consider an utterance of (1):

(1) Harold believes that *S* and Harold believes that *S'*, but Harold doesn't believe that *S* and *S'*.

We have seen in previous chapters that, in extreme cases, an agents beliefs may not be closed under conjunction, and in such cases, (1) may be felicitous. Suppose that *that S* and *that S'* are rendered as two lambda terms $p_{\blacklozenge\lozenge t}$ and $q_{\blacklozenge\lozenge t}$. Then, given the lexical entries presented in the previous chapter, (1) can be rendered as (7.14), which β reduces to (7.15):

$$\left(\mathbf{zip}^3_{sp}\ (\ \mathbf{zip}^3_{s}\, and\ )\right)_{\blacklozenge\lozenge t\to\blacklozenge\lozenge t\to\blacklozenge\lozenge t} \tag{7.14}$$

$$\left(\left(\mathbf{zip}^3_{sp}\ (\ \mathbf{zip}^3_{s}\, and\ )\right)_{\blacklozenge\lozenge t\to\blacklozenge\lozenge t\to\blacklozenge\lozenge t}(\mathbf{believe}\, p\, \mathbf{h})_{\blacklozenge\lozenge t}\ (\mathbf{believe}\, q\, \mathbf{h})_{\blacklozenge\lozenge t}\right)$$

$$\left[\ \mathbf{not}^{2,H}_{\blacklozenge\lozenge t\to\blacklozenge\lozenge t}\left(\mathbf{believe}\left(\left(\mathbf{zip}^3_{sp}\ (\ \mathbf{zip}^3_{s}\, and\ )\right)_{\blacklozenge\lozenge t\to\blacklozenge\lozenge t\to\blacklozenge\lozenge t}\ p\ q\right)\mathbf{h}\right)\right]$$

$$\lambda\delta, s.\ \forall s' \,[\,[\, s \leqslant s' \wedge \pi\,(\delta\, s)\, s'] \rightarrow \forall s''[DOX\,\mathbf{h}\, s'\, s'' \rightarrow \tag{7.15}$$
$$p(\eta_s(\kappa\,\mathbf{h}\, s''))\ )\ s''\,]\,]\wedge$$
$$\forall s' \,[\,[\, s \leqslant s' \wedge \pi\,(\delta\, s)\, s'] \rightarrow \forall s''[DOX\,\mathbf{h}\, s'\, s'' \rightarrow$$
$$q(\eta_s(\kappa\,\mathbf{h}\, s''))\ )\ s''\,]\,]\wedge$$
$$\forall s^1\ [\ \forall s' \,[\,[\, s^1 \leqslant s' \wedge \pi\,(\delta\, s^1)\, s'] \rightarrow \forall s''[DOX\,\mathbf{h}\, s'\, s'' \rightarrow$$
$$p(\eta_s(\kappa\,\mathbf{h}\, s''))\ s'' \wedge q(\kappa\,\mathbf{h}\, s'')\ s''\,]\,] \rightarrow\ s \perp s^1\ ]$$

(7.15) cannot be true with respect to any perspective index and state, however, since, for any function δ of type $\Diamond p$ and state s, if p is true with respect to δ and s and q is, then $p \wedge q$ is. However, abbreviating the lambda term in (7.15) as $r_{\blacklozenge\Diamond t}$, it is perfectly possible for the formula (7.16) to be satisfiable:

$$\forall s' \Bigg(s^0 \leqslant s' \rightarrow \exists f_{List(f\rightarrow t)} \tag{7.16}$$
$$\Bigg(last\ (available.chain\, f\, (\kappa\,\mathbf{h}\, s')\, c\, s')\ \text{“}r_{\blacklozenge\Diamond t}\,(\kappa\, u)\, s^0\text{”} \Bigg)\Bigg)$$

This would happen if the particular substitution instance of the rule depicted below is unavailable to Harold, for whatever reason, so that we cannot go from a set of names of formulas containing the names of its premisses, to the set of names of formulas containing the name of its conclusion:

$$\frac{\begin{array}{c}\text{“}\forall s'\,[\,s^0 \leqslant s' \wedge \pi\,(\delta\, s)\, s'] \rightarrow \\ \forall s''[DOX\,\mathbf{h}\, s'\, s'' \rightarrow p(\eta_s(\kappa\,\mathbf{h}\, s''))\ s''))\,]\text{”}\end{array} \qquad \begin{array}{c}\text{“}\forall s'\,[\,s \leqslant s' \wedge \pi\,(\delta\, s)\, s'] \rightarrow \\ \forall s''[DOX\,\mathbf{h}\, s'\, s'' \rightarrow q(\eta_s(\kappa\,\mathbf{h}\, s''))\ s''))\,]\text{”}\end{array}}{\begin{array}{c}\text{“}\forall s'\,[\,s \leqslant s' \wedge \pi\,(\delta\, s)\, s'] \rightarrow \forall s''[DOX\,\mathbf{h}\, s'\, s'' \rightarrow \\ p(\eta_s(\kappa\,\mathbf{h}\, s''))\ s'' \wedge q(\kappa\,\mathbf{h}\, s'')\ s''\,]\text{”}\end{array}}\ Bel\ \wedge$$

At other contexts and states, Harold may be perfectly able or willing to apply the rule to the relevant names of formulas. Notice, this is compatible with the agent being able to apply conjunction introduction to other names of formulas. For example, the available chain quantified over in (7.16) might include two adjacent elements a_i, a_{i+1} such that conjunction introduction was used to reach a_{i+1} from a_i. Notice, in addition, that (7.16) also makes (2), the Italian translation of (1), true, given that the rendering of the Italian sentence into a formula of *partial Ty2* is the same as the rendering of the English sentence:

(2) Harold crede che *S* e Harold crede che *S*', ma (Harold) non crede che *S* e *S*'.

In some cases it will be reasonable to assume that two sentences of different languages that are translations of one another be given the same renderings, given they are structurally similar. This is the case with the (1) and (2). In such cases we avoid the notorious 'Translation argument' discussed in the literature, for the same reason that the Translation argument doesn't arise when considering an ordinary intensional semantic theory on which sentences are taken to denote the characteristic functions of sets of states: there is nothing incompatible with supposing that two sentences of two distinct languages can be given the same renderings, so that they are assigned the same semantic value in a model.

But what about languages in which the translation of (1) needs to be rendered in a more elaborate way, such that the syntactic structure of the translation of (1) is markedly different from its English counterpart? In such cases, we must require that there is a contextually available substitution in $SUB^{s,c}$ which permits as a substitution instance the name of a formula which renders the relevant sentence.

In some extreme cases, however, the language that a sentence is uttered may itself be a component of its meaning. (Partee 1973) points out that verbs such as *giggle* and *holler* seem to refer to some overt utterance or communication, and to combine with near quotations, i.e., quotations excluding personal pronouns, of the embedded sentence:

(3) The criminal in court hollered that them uns wouldn't put him in prison.

For such cases, $available.chain$ will not do, as we do not want to allow substitutions which treat translations of sentences as appropriate, given the fact that in such sentences the words used are essential to the meaning conveyed. This won't work, however, if the rendering of, for example, an Italian and an English sentence are the same, for then if the rendering of the English sentence is an appropriate substitution instance for a given rule, then the rendering of the Italian sentence will be.

This problem can be avoided using the techniques discussed in the previous chapter; in particular, we can use the U monad there defined, and here repeated:

The U monad :

$U\alpha = \alpha \times t$ for $\alpha \in TYPE$

$\eta_U\, x = (\ x, \top\ )$

$a \bigstar_U f = \Big(\ \pi_1(f\,(\pi_1 a)),\ \ \pi_2\, a \wedge \pi_2\,(f\,(\pi_1\, a))\ \Big)$ where $a : U\alpha$ and $f : \alpha \rightarrow U\beta$

In fact, in order to model (3), we only need the U applicative, as defined below:

The U applicative :

$U\alpha = \alpha \times t$

$\eta_U x =_{def} (x,\ \top)$

$f \bullet_U a =_{def} \big(\ \pi_1 f\, \pi_1 a,\ \ \pi_2 f \wedge \pi_2 a\ \big)$ where $a : U\alpha, f : U(\alpha \rightarrow \beta)$

Supposing that in an utterance of (3), the expression being quoted is *them uns*, then we can assign this expression (7.17), whose second co-ordinate is constrained by (7.18):

PEOPLE $:=$ (7.17)

$$\Big((those\,people)_{\blacklozenge\lozenge e},$$

$$exp\ \ c\ \ \textsf{them uns}\ \ ((\!|\textsf{them uns}|\!)\ c)\Big) : U \blacklozenge \lozenge e$$

$$((\!|\textsf{them uns}|\!)\ c) = (those\,people)_{\blacklozenge\lozenge e} \tag{7.18}$$

In order to combine **PEOPLE** with the other elements of the sentence, we need to lift the constants representing other parts of the sentence, as below:

$$(\mathbf{zip}^2_U hollered)_{U\blacklozenge\lozenge e \to U\blacklozenge\lozenge t} \tag{7.19}$$

$$(\mathbf{zip}^2_U(\mathbf{zip}^2_{\lozenge p}(\mathbf{zip}^2_s\,\neg)))_{U(\blacklozenge\lozenge t \to U\blacklozenge\lozenge t)} \tag{7.20}$$

$$\mathbf{zip}^4_U put_{U\blacklozenge\lozenge e \to U\blacklozenge\lozenge e \to U\blacklozenge\lozenge e \to U\blacklozenge\lozenge t} \tag{7.21}$$

$$(\eta_U in\,prison)_{U\blacklozenge\lozenge e} \tag{7.22}$$

$$(\eta_U him)_{U\blacklozenge\lozenge e} \tag{7.23}$$

$$(\eta_U the\,criminal)_{U\blacklozenge\lozenge e} \tag{7.24}$$

$$(\eta_U(\kappa\ u)) \tag{7.25}$$

$$(\eta_U s^0_s) \tag{7.26}$$

Relative to these renderings (7.27) β-reduces to (7.28):

$$(\mathbf{zip}^2_U\ hollered)_{U\blacklozenge\lozenge e \to U\blacklozenge\lozenge t}\Big((\mathbf{zip}^2_U(\mathbf{zip}^2_{\lozenge p}(\mathbf{zip}^2_s\,\neg)))_{U(\blacklozenge\lozenge t \to U\blacklozenge\lozenge t)} \tag{7.27}$$

$$\Big(\ (\mathbf{zip}^4_U\ put\ )_{U\blacklozenge\lozenge e \to U\blacklozenge\lozenge e \to U\blacklozenge\lozenge e \to U\blacklozenge\lozenge t}\ (\eta_U in\,prison)_{U\blacklozenge\lozenge e}\ (\eta_U him)_{U\blacklozenge\lozenge e}$$

$$\mathbf{PEOPLE}_{U\blacklozenge\lozenge e}\ \Big)\ \Big)(\eta_U the\,criminal)_{U\blacklozenge\lozenge e}\ (\eta_U(\kappa\ u))\ (\eta_U s^0_s)$$

$$\Big(hollered\,(put(in\,prison)\,him\,(those\,people)) \tag{7.28}$$

$$(the\,criminal)\ (\kappa\ u)\ s^0_s),\quad exp\ c\ \textsf{them uns}\ ((\!|(\textsf{them uns})_u|\!)\ c)\Big)$$

(7.28) is a pair consisting of the evaluation of a lambda term of type $\blacklozenge\lozenge t$ at $\kappa\ u$ and s^0_s, whilst the second member of the pair consists formula expressing that in the context c the string $\textsf{them uns}$ was used to express $(\!|\textsf{them uns}|\!)\ c$.

So far we have proposed a way of resolving the problem of logical omniscience which relies on the concept of an available chain. It might be thought, however, that the proposal contradicts, or is in tension with, the account of propositional attitudes given by (Asudeh and Giorgolo 2016), according to which utterances of propositional attitude sentences are not made from the perspective of the grammatical subject of those sentences, but from the perspective of the utterer, who is trying to capture the perspective of the grammatical subject of sentences. In this thesis we have aimed to develop (Asudeh and Giorgolo 2016)'s approach, and in the previous chapter we implemented their approach to the propositional attitudes by requiring that the outermost occurrence of π in a formula always take the perspective index associated with the utterer of the sentence. But here we have defined the predicate $available.chain$ only in terms of one perspective index. We could therefore feed the perspective index of the grammatical subject of the sentence to $available.chain$ instead of the perspective index of the utterer, and then the propositional attitude report would be made from the perspective of the grammatical subject, contradicting (Asudeh and Giorgolo 2016)'s proposal. In the next section, we therefore propose a more complicated semantics, on which both $available$ and chains themselves, are parameterised to two perspective indices. On this more complicated account, we preserve the intuition of (Asudeh and Giorgolo 2016) according to which utterances of propositional attitude sentences are not made from the perspective of the grammatical subject of those sentences, but from the perspective of the utterer, who is trying to capture the perspective of the grammatical subject of sentences.

7.3.2 The double perspective approach

In the double perspective approach, the set of rules in a model and the set of substitutions is defined relative to two perspective indices. The picture is a natural development of (Asudeh and Giorgolo 2016), in the following sense. According to (Asudeh and Giorgolo 2016),in an utterance of a sentence such as *John believes that the prime numbers are inductive*, the utterer cannot access the private perspective of John, but only the utterer's representation of this perspective. The rules which the subject of a sentence is able to use cannot also be known by the utterer of the sentence, and even if the utterer did know them, she couldn't in principle know exactly how the rule of inference a person is using represented in their mental lexicon. The set of rules can therefore be understood as an abstraction based on what someone thinks about how the rules a person is following are mentally represented. This leads to the following proposal. Given a model $\mathfrak{M}$, $\mathcal{R}^{c,i,(\kappa\, u\, s),s}$ denotes the set of rules of inference available in the context $c \in D_c$ at state $s \in D_s$ to the perspective index $i \in D_p$ according to the utterer's perspective index $(\kappa\, u\, s)$. The set of rules in a model is then defined as follows:

$$\bigcup_{\substack{i \in D_p \\ c \in D_c \\ s \in D_s}} \mathcal{R}^{c,i,(\kappa\, u\, s),s} \tag{7.29}$$

We use $SUB^{c,i,(\kappa\, u\, s),s}$ to denote the set of substitutions available in context c for the perspective index i according to the perspective index the utterer's perspective index $(\kappa\, u\, s)$ at state s. From this we can define the set of all substitutions in a model as follows:

$$SUB = \bigcup_{\substack{i \in D_p \\ c \in D_c \\ s \in D_s}} SUB^{c,i,(\kappa\, u\, s),s} \tag{7.30}$$

In the double perspective approach $\sqsubseteq$ takes an extra perspective argument, so that it is of type $(f \to t) \to (f \to t) \to c \to p \to p \to \Diamond t$. This will (shortly) be fixed to the utterer's perspective index, thereby capturing the idea that the utterer can never really grasp what chains other people have in their mental lexicons, but at best can only surmise. So $\mathbf{f}^{\mathfrak{M},g} \sqsubseteq_{c,i,j,s} \mathbf{g}^{\mathfrak{M},g}$ will hold if $\mathbf{g}^{\mathfrak{M},g}$ is a minimally consistent extension of $\mathbf{f}^{\mathfrak{M},g}$, in the perspective index i at context c in state s, *according to the perspective index* j. This function, which corresponds to a reflexive relation between $\mathbf{f}^{\mathfrak{M},g}, \mathbf{g}^{\mathfrak{M},g} \in D_{f\to t}$ at a state s, context c and at perspective indices i, j, is defined as follows:

$$\text{For all } \mathbf{f}^{\mathfrak{M},g}, \mathbf{g}^{\mathfrak{M},g} \in D_{f\to t}, s \in D_s, c \in D_c \text{ and } i \in D_p : \mathbf{f}^{\mathfrak{M},g} \sqsubseteq^{\mathfrak{M},g}_{c,i,j,s} \mathbf{g}^{\mathfrak{M},g} \text{ iff} \tag{7.31}$$

$$\forall s' \Bigg[s \leqslant s' \Rightarrow \exists \mathbf{R} \in \mathcal{R}^{c,i,j,s}\, \exists \sigma \in SUB^{c,i,j,s} \Bigg[$$
$$\Bigg(\forall \mathbf{a}^{\mathfrak{M},g} \in D_f\ (\mathbf{a}^{\mathfrak{M},g} \in pr\, R^{\sigma} \Rightarrow \mathbf{f}^{\mathfrak{M},g}\ \mathbf{a}^{\mathfrak{M},g} = 1 \Bigg) \ \&$$
$$\forall \mathbf{b}^{\mathfrak{M},g} \forall \mathbf{c}^{\mathfrak{M},g} \Bigg[$$
$$\Bigg(\mathbf{g}^{\mathfrak{M},g}\ \mathbf{b}^{\mathfrak{M},g} = 1 \Rightarrow \Bigg(\Bigg(\mathbf{f}^{\mathfrak{M},g}\ \mathbf{b}^{\mathfrak{M},g} = 1 \text{ or } \mathbf{b} = con\, R^{\sigma} \Bigg) \ \&$$
$$\exists d \in TERM_t \Bigg(\mathbf{b}^{\mathfrak{M},g} = [\![\text{“}d\text{”}_f]\!]^{\mathfrak{M},g} \ \&\ \mathbf{f}^{\mathfrak{M},g}\ [\![\text{“}\neg_H d\text{”}_f]\!]^{\mathfrak{M},g} \neq 1 \Bigg) \Bigg) \ \&$$
$$\exists e \in TERM_t \Bigg($$
$$\Bigg(\mathbf{c}^{\mathfrak{M},g} = [\![\text{“}e\text{”}_f]\!]^{\mathfrak{M},g} \Bigg) \& \Bigg(\mathbf{f}^{\mathfrak{M},g}\ \mathbf{c}^{\mathfrak{M},g} = 1 \Rightarrow \mathbf{f}^{\mathfrak{M},g}\ [\![\text{“}\neg_H e\text{”}_f]\!]^{\mathfrak{M},g} \neq 1 \Bigg) \Bigg) \Bigg]$$
$$\&\ \mathbf{g}^{\mathfrak{M},g} = \mathbf{f}^{\mathfrak{M},g} \cup \{con\ R^{\sigma}\} \Bigg] \Bigg]$$

We assume that $\sqsubseteq^{\mathfrak{M},g}_{c,i,j,s}$ is undefined when $i = E$ or $j = E$. This condition is imposed for the same reason as before, to ensure that the enlightened interpretation of sentences does not enter into the determination of how far a particular agent's beliefs or knowledge extend deductively; rather, the enlightened interpretation of a sentence is simply its ordinary intensional interpretation as a set of classical states.

In the double perspective approach, A is redefined, as follows, so that it takes an extra perspective index argument:

- $A_{List(f \to t) \to p \to p \to c \to s \to t} \in Con_{List(f \to t) \to p \to p \to c \to s \to t}$ is a function which returns 1 when supplied a list of functions, a perspective index i, a perspective index j, a state s and a context c if and only if the list of functions is available from the perspective index i according to perspective index j in the context c at c. From A we can construct the lambda term, which fixes the availability of a chain to what the utterer thinks is available via a certain perspective index:

$$\textbf{Available} := \tag{7.32}$$
$$\lambda f_{List(f \to t)}, \delta_{\Diamond p}, c_c, s_s.\ A_{List(f \to t) \to p \to p \to c \to s \to t}\ f\ \ (\delta\ s)\ \ (\kappa\ u\ s)\ c\,s$$

$available.chain_{List(f \to t) \to c \to \blacklozenge \Diamond List(f \to t)}$, is defined as in (7.33):

$$available.chain_{List(f \to t) \to c \to \blacklozenge \Diamond (List(f \to t))} := \tag{7.33}$$

$$\lambda f_{List(f\to t)}, c_c, \delta_{\Diamond p}, s_s. \begin{cases} f & \text{if } \textbf{Available}\, f\,(\delta'\, s)\,(\kappa\ u\ s)\,c\,s \\ & \wedge \exists a_{f\to t}(member.of\ a\ f) \wedge \forall a_{f\to t} \\ & (member.of_{(f\to t)\to List(f\to t)\to t}\ a\ f \\ & \to \exists x_f(a\,x = 1)) \wedge \Big(length\, f \geqslant 2 \\ & \to head\, f \sqsubseteq_{c,(\delta'\, s),(\kappa\ u\ s),s} head\,(tail\, f) \\ & \wedge \forall i(nat\, i\ \wedge\ i \geqslant 2\ \wedge i < (length\, f) - 1 \\ & \to head(tail^{i-1}\, f) \sqsubseteq_{c,(\delta'\, s),(\kappa\ u\ s),s} \\ & head(tail^{i} f)\Big) \\ \# & \text{otherwise} \end{cases} \tag{7.34}$$

Notice that both **Availability**–via the definition of this predicate in (7.32)–and $available.chain_{List(f\to t)\to c\to\blacklozenge\Diamond(List(f\to t))}$ as defined in (7.33), are relativised to the utterer's perspective.

In the semantic theory discussed in part II of this thesis, sentences have denoted functions of type $\blacklozenge\Diamond t$. In the double perspective approach we feed such terms to the following function, but this time $available.chain$ has the definition given in (7.33):

lift$_{\textbf{non-final}}$ $:=$ (7.35)

$$\lambda p_{\blacklozenge\Diamond t}, c_c, \delta, s_s.\ \forall s'(s \leqslant s' \to \exists f_{List(f\to t)}(last\ (available.chain\ f\,\delta\,c\,s')\text{ ``}\ p\,\delta\,s'\text{ ''}))$$

This function takes a term $p_{\blacklozenge\Diamond t}$, a context c, a function δ and a state s and returns 1 if in all states s' accessible via $\leqslant$, according to the perspective index of the utterer $(\kappa\ u\ s')$ at s', the perspective index i has an available chain in c at s' whose last element returns 1 when applied to "$p\,\delta\,s'$" the name of the formula $p\,\delta\,s'$.

Suppose we want to use **lift** to derive (7.36), where the quoted formula in (7.36) is (7.37):

$$\forall s' \Big(s \leqslant s' \rightarrow \exists f_{List(f \rightarrow t)} \Big(last\ (available.chain\, f\, (\kappa\, \mathbf{h})\, c\, s') \tag{7.36}$$

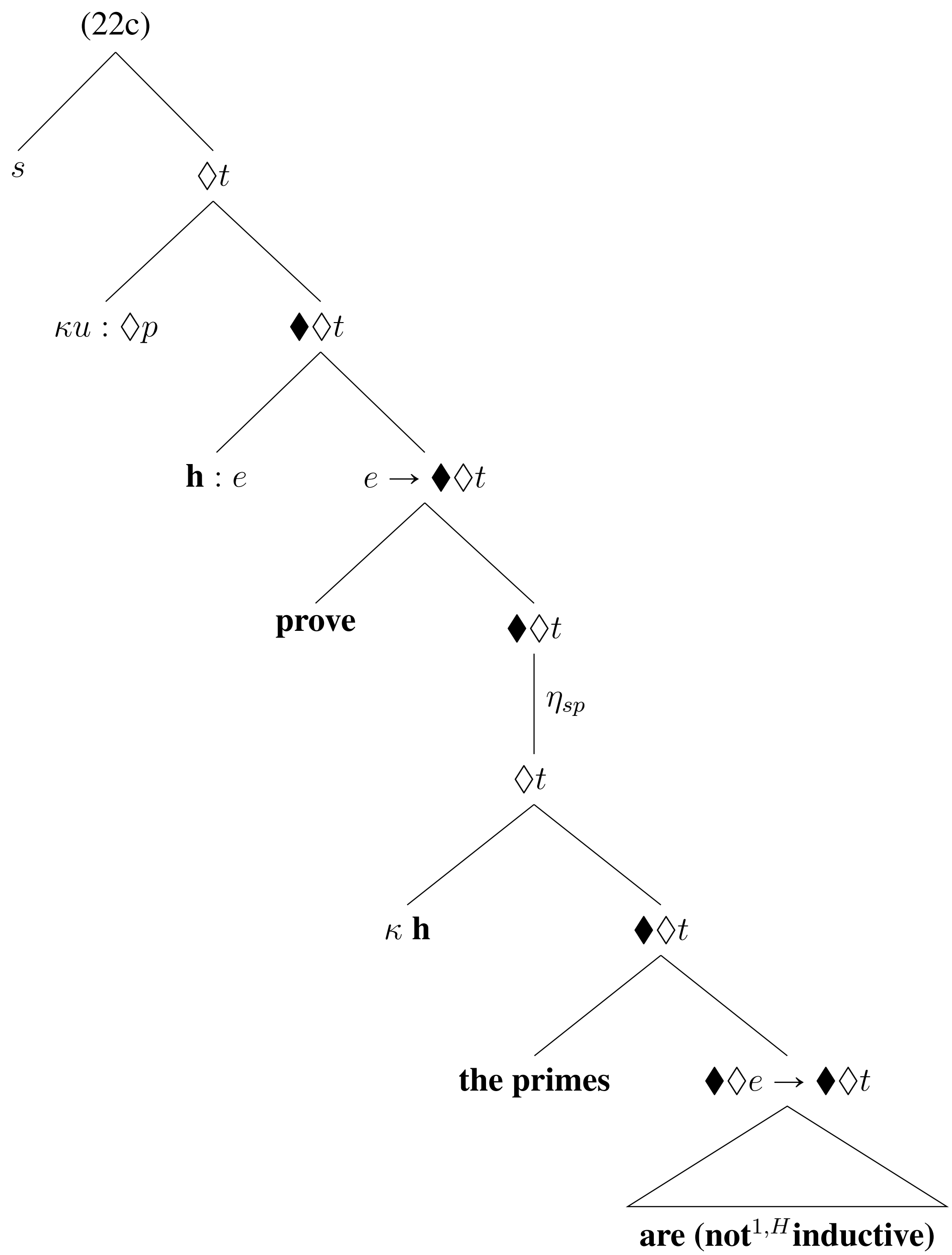
$$\text{“}\ \forall s'\ [\ [\ s \leqslant s' \wedge \pi\, (\kappa\, u\, s)\, s\, s'\ ] \rightarrow \forall s''\ [\ PROV\, h\, s'\, s'' \rightarrow$$

$$\forall s'''\ (\ inductive\ (\ (\ \sigma\ \textbf{primes}\ )\ (\ \kappa\ \mathbf{h}\ )\ s''''\ )\ (\ \kappa\, \mathbf{h}\, s'''\ )\ s'''$$

$$\rightarrow s'' \perp s'''\ )\ ]\ ]\ \text{”}\ \Big) \Big)$$

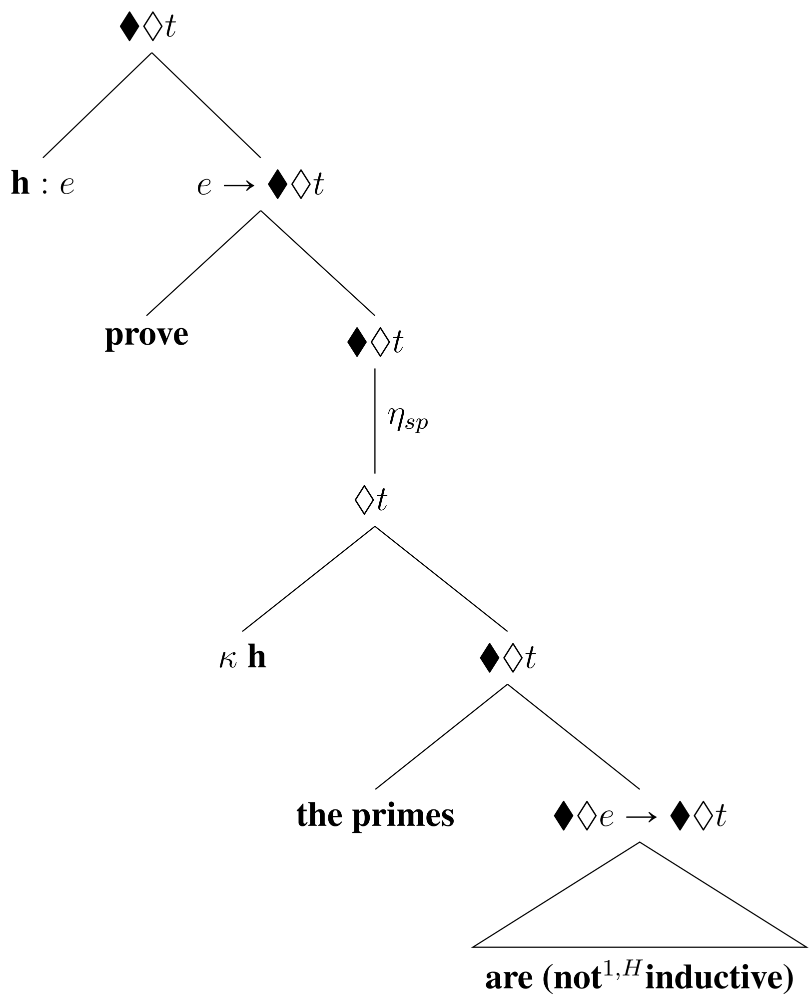
$$\forall s'\ [\ [\ s \leqslant s' \wedge \pi\, (\kappa\, u\, s)\, s\, s'\ ] \rightarrow \forall s''\ [\ PROV\, h\, s'\, s'' \rightarrow \tag{7.37}$$

$$\forall s'''\ (\ inductive\ \ (\ (\ \sigma\ \textbf{primes}\ )\ (\ \kappa\ \mathbf{h}\ )\ s'''\ )\ (\ \kappa\, \mathbf{h}\, s'''\ )\ s''' \rightarrow s'' \perp s'''\ )\ ]\ ]$$



We might try feeding the root of the tree (5) to **lift**, given the fact that its root is type $\blacklozenge\lozenge t$ and given it β-reduces to (7.38), which is a lambda term which, if supplied with $\kappa\, u$ and s, will produce the quoted lambda formula in (7.36):

(5)

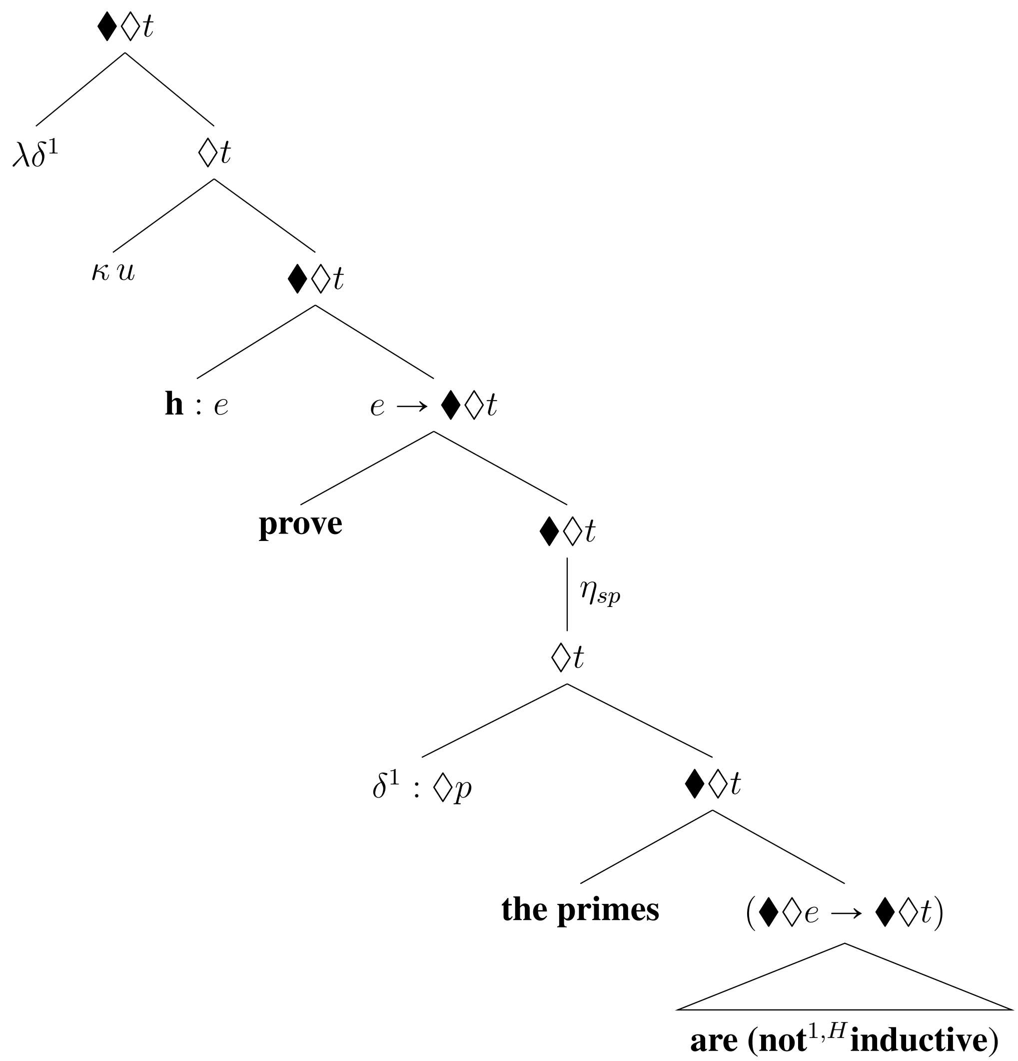


$$\lambda\delta, s.\ \forall s'\ [\ [\ s \leqslant s' \wedge \pi\,(\delta\, s)\, s\, s'\ ] \to \forall s''\ [\ PROV\, h\, s'\, s'' \to \forall s'''\ (\ inductive\ \ (\ (\ \sigma\ \textbf{primes}\ )\ (\ \kappa\ \mathbf{h}\ )\ s'''\ )\ (\ \kappa\, \mathbf{h}\, s'''\ )\ s''' \to s'' \perp s'''\ )\ ]\ ] \tag{7.38}$$

Unfortunately, this won't work, as then the predicate $available.chain$ will also be fed $\kappa\, u$, whereas, in order to arrive at (7.36) $available.chain$ must be fed $\kappa\, \mathbf{h}$.

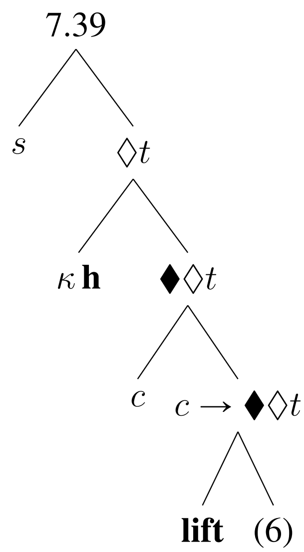
In order to arrive at (7.36), we need to feed the root of the following tree to **lift**:

(6)



This fixes the perspective index supplied to π to the utterer's perspective index, whilst binding the perspective index supplied to the rendering of the complement sentence to a variable δ which is abstracted over higher in the tree. If we then feed the lambda term at the root of the tree in (6) to **lift**, we derive the following tree (7) whose root node β-reduces to 7.39, which is the same formula as (7.36):

(7)



$$\forall s'\Bigg(s \leqslant s' \rightarrow \exists f_{List(f\rightarrow t)} \Bigg(last\ (available.chain\, f\, (\kappa\, \mathbf{h})\, c\, s') \tag{7.39}$$
$$\text{“}\ \forall s'\ [\ [\ s \leqslant s' \wedge \pi\,(\kappa\, u\, s)\, s\, s'\ ] \rightarrow \forall s''\ [\ PROV\, h\, s'\, s'' \rightarrow$$
$$\forall s'''\ (\ inductive\ (\ (\ \sigma\ \mathbf{primes}\ )\ (\ \kappa\ \mathbf{h}\ )\ s''''\ )\ (\ \kappa\, \mathbf{h}\, s'''\ )\ s'''$$
$$\rightarrow s'' \perp s'''\ )\ ]\ ]\ \text{”}\ \Bigg)\Bigg)$$

The upshot is that, in order to derive (7.36) (i.e, (7.39)), we need to feed the lambda term at the root of (6) to **lift**. This also allows us to avoid some possibly unattested readings. For the tree in (6) ensures that the argument of type $\Diamond p$ in (6) (the δ variable) is bound also to the argument of type $\Diamond p$ that is abstracted over in the definition of **lift**. If this were not the case, the $available.chain$ predicate could be saturated by an argument of type $\Diamond p$ which is completely different from the argument of type $\Diamond p$ which is supplied to the lambda term at the root of (6). Consequently, we could produce lambda terms such as the following, in which the $available.chain$ predicate is saturated with a type $\Diamond p$ function relating to Mary Jane's perspective, so that it will be true that **Available** $f\ (\kappa$ **mj** $s)\ (\kappa\ u\ s)\ c\ s$ (i.e, that the chain is a available to Mary Jane's perspective index according to the utterer):

$$\forall s' \Big(s \leqslant s' \rightarrow \exists f_{List(f\rightarrow t)} \Big(last \ (\ available.chain \ f \ (\ \kappa \ \mathbf{mj} \ ) \ c \ s' \ ) \qquad (7.40)$$

$$\text{“} \ \forall s' \ [\ [\ s \leqslant s' \wedge \pi \ (\ \kappa \, u \, s \ ) \, s \, s' \ ] \rightarrow \forall s'' \ [\ PROV \, h \, s' \, s'' \rightarrow$$

$$\forall s''' \ (\ inductive \ (\ (\ \sigma \ \mathbf{primes} \ ) \ (\ \kappa \ \mathbf{h} \ ) \ s'''' \ ) \ (\ \kappa \, \mathbf{h} \, s''' \ ) \, s'''$$

$$\rightarrow s'' \perp s''' \ ) \ ] \ ] \text{”} \ \Big) \Big)$$

We doubt that readings such as (7.40) arise, however, or if they do then they are unusual.

7.4 Obvious and subtle contradictions and the moderately rational reasoner

We must now consider the place of the ‘moderately rational reasoner’ who is usually disposed to make trivial inferences, in our account. At various points in the thesis we stressed that it is often infelicitous to attribute an ‘obvious’ contradiction to someone. (Bjerring and Skipper 2019) and (Solaki et al. 2019) provided their own solution to the problem of logical omniscience with the aim of capturing a moderately rational reasoner, who doesn’t believe obvious contradictions. Rather than using a dynamic modality to capture this (as in (Bjerring and Skipper 2019) and (Solaki et al. 2019)), we claim that to capture the capacities of a moderately rational reasoner in our approach, we need to restrict models to those in which, in most available chains, whenever ϕ and ψ are available formulas $\phi \wedge \psi$ are available formulas, and so on and so forth for other trivial inferences, where whether an inference is trivial may vary from context to context. This would reflect the fact that, for most moderately rational agents, it is generally the case that, whenever they believe ϕ and they believe ψ, they believe $\phi \wedge \psi$. This leaves open the possibility that, in certain cases, moderately rational reasoners will not draw simple inferences, as such as the inference from ϕ and ψ to $\phi \wedge \psi$. In such cases ϕ and ψ are available formulas but $\phi \wedge \psi$ is not an available formula. Since we allow for the possibility of available chains the last member of which send formulas ϕ and ψ to 1 but send $\phi \wedge \psi$ to 0, we can allow for felicitous readings of sentences such as *John believes S and he believes S', but he doesn't believe S' and S'*, in the case when John has an unorthodox theory about *and*, unlike (Berto and Jago 2019)'s theory.

7.5 Conclusion

P-HYPE can solve the problem of logical omniscience. Limits on reasoning capabilities are modelled as limitations incurred by only being able to apply certain inference rules, and the stock of inference rules an agent can make use of depends on her perspective. Though most reasoners have a perspective which makes them believe obvious logical consequences of their view most of the time, there are agents for whom this is not the case for systematic reasons owing to their perspective on 'logical' words such as *and*, *or* and *not*.

Taking account of perspective is therefore of importance not only in solving the problem of granularity, but in solving the problem of logical omniscience. But the connection between perspective and the problem of logical omniscience has not, to our knowledge, ever been clearly drawn. The P-HYPE solution to logical omniscience allows us to block any form of logical omniscience, without introducing impossible worlds. In addition, it is fully compositional, and so avoids the objections of those who have argued against impossible worlds on the grounds that they are incompatible with a compositional semantics (Williamson 2020). In future work we hope to refine the P-HYPE solution to logical omniscience, and compare its predictions to other solutions in greater detail.

Chapter 8

Conclusions and further questions

Let us take stock, and reflect on the significance of what we have shown in the thesis. We found that various accounts of the problems of granularity and logical omniscience fail to adequately take into account the perspective of agents. For this reason, we then elaborated a perspective-relative semantic theory which builds upon the perspective relative semantics of (Asudeh and Giorgolo 2016) and on the logic HYPE (Leitgeb 2019), and renders some of (Asudeh and Giorgolo 2016)'s assumptions about the privacy of perspectives concrete, by offering a class of models—the solipsistic P-HYPE models— which capture some of their ideas.

The significance of P-HYPE is that it demonstrates that the semantics of perspective can be used to solve the problems of granularity and logical omniscience. As far as we are aware, no one has claimed this before. The problems we identified with other solutions derive, in part, from not taking perspective seriously in capturing the meaning of mathematical sentences and the semantics of certain predicates. In addition, combining the semantics of perspective relativity with the semantics of focus and metalinguistic focus enables us to avoid certain problems that accounts of these phenomena have if they assume *Predicate Uniformity*.

The P-HYPE solution of the problem of logical omniscience is fully-compositional at the subsentential level, unlike the accounts of (Solaki et al. 2019) and (Berto and Jago 2019), who have not shown how their accounts relate to a compositional natural language semantics. The P-HYPE solution avoids 'impossible worlds' whilst allowing us to make fine-grained distinctions between formulas which represent classical tautologies.

There are five questions which we hope to address in future work. Firstly, we would like to consider how our perspective-relative semantic theory relates to the predictions made by (Charlow 2014), who argues that the greater scopal freedom of indefinite quantifiers with respect to other types of quantifier can be understood in terms of the monadic side effects that indefinites induce, which enable them to scope out of scope islands. We have not defended a particular semantic account of quantifiers in this thesis (though see the appendix, section E for the sketch of a semantics of indefinite quantifiers and anaphora), but it does seem that certain indefinites are plausibly analysed as having a semantics involving perspective, and such indefinites may scope above non-finite clauses, which are traditionally held to be scope islands. Consider again (Tancredi and Sharvit 2020)'s example (discussed in the previous chapter) of a monolingual English speaker named John who has a non-standard understanding of the natural language predicate *prime*, according to which a natural number is prime if and only if it is equal to $x^3 - 1$, for some natural number x. John may assert that 26 is a prime number, knowing that it has exactly four natural number factors. Suppose that John has asserted that 26 does not have exactly two natural number factors. In such a scenario, when speaking about John, we– who are not confused about the meaning of *prime*–might truthfully utter (1a), on the reading in which *a prime number* scopes above negation, and we might truthfully utter (1b), on the reading in which *a prime number* scopes out of a relative clause (thereby seeming to escape a complex DP island):

(1) a. John thinks it's not the case that a prime number has exactly two natural number factors.

b. Everyone heard John's claim that a prime number does not have exactly two natural number factors.

Crucially, in such cases, *prime number* is interpreted in the way that John is presumed to understand this word, and so the exceptional scope of the indefinite noun phrase *a prime number* and a perspective relative interpretation together seem to scope above a scope island. In other words, the perspective relative interpretation of *prime number* projects beyond the confines of syntactic islands. For this reason, we predict that our semantic theory is consistent with (Charlow 2014)'s central claim, that indefinites induce side effects which scope out of syntactic islands. However, if we treat universal quantifiers as perspective-relative, we also predict that these induce side effects too, unlike in (Charlow 2014)'s theories. In future work we therefore hope to discuss perspective relative quantifiers in more detail.

Secondly, our thesis raises questions about the status of HYPE in natural language semantics. Are modal negations (Berto and Restall 2019, Dosen 1986, Odintsov and Wansing 2019, Mendler et al. 2021) useful in natural language semantics? To our knowledge no one has explored this question in any depth.

Thirdly, we would like to explore how we can adequately constrain perspective relativity. We have made predicates perspective relative in our account, but we have offered few suggestions about how the context narrows down on a particular perspective index which is suitable. Here the literature on constraints on the interpretation of *de re* noun phrases might provide some way of thinking about constraints on perspective relativity, as well as the proposals made in the literature on perspective relative interpretation, for example in (Lasersohn 2017).

Fourthly, we would like to explore the connections between our semantics of perspective relativity and the semantics of the *de re* and the *de se*? (Asudeh and Giorgolo 2016) analyse *de re* proper names as perspective relative, but do not explore in detail the question of the relationship of their semantics to the semantics of *de re* noun phrases. Nor are we clear whether our semantics might be extended to model *de se* pronouns, though see (Pearson 2015, Lasersohn 2017) for discussion relating the *de se* and perspective relativity.

Finally, we would like to explore the connections between our semantic theory and an alternative semantic theory on which the natural language predicates we have discussed are treated as quotational, and attitude reports are treated via mixed quotation (see (Maier 2020) for discussion of mixed quotation). As far as we are aware, no account of this kind has been pursued with respect to predicates such as *oculist* and *ophthalmologist*.

In conclusion, we hope to have motivated the desirability of taking perspective seriously in the semantics of natural language outside of the traditional contexts in which it has usually been advocated.

Appendix

A Technical preliminaries

A.1 Compositionality and intensionality in natural language: a snapshot

In model-theoretic natural language semantics, sentences and their parts are assigned model-theoretic interpretations called *semantic values*, which are supposed to represent their meanings, or aspects thereof.[1] The assignment of semantic values is done either directly, via a systematic translation from natural language strings into model theoretic objects, or indirectly, in which case sentences and their parts are first *rendered* (as (Moschovakis 2006) terms the process) into expressions of an (intermediary) logic which are then interpreted in a model.[2]

A central concern of semanticists has been to provide models of how the seman-

[1]See (Zimmermann 2006, 2011) for further discussion of this basic idea and its theoretical ramifications.

[2]The rendering relation stands between natural language expressions and formal expressions of an intermediary logic. We write $a \rightsquigarrow b$ when the linguistic expression a is rendered as b. In several of the papers in (Montague 1974), Montague rigorously defines a translation procedure between natural language strings and expressions of the logic IL. But in modern work the translation is not always fully spelled out.

tic values of parts of sentences combine to form the semantic values of whole sentences, based on their structure (Montague 1974), (Gamut 1990) and (Heim & von Fintel 2011), and semanticists have also been concerned with how the semantic values of sentences combine to form the semantic values of larger discourses (Heim 1982) (Kamp 1981).[3] A guiding principle in this respect (though not uncontroversial)[4] has been the principle of compositionality; informally put, the principle states that the meaning of a complex expression is a function of its parts and how they are syntactically combined. In Montague's approach, the syntax of a natural language is an algebra SYN, consisting of expressions and syntactic operations and the semantics of a natural language is an algebra SEM, consisting of semantic values and operations on them. A meaning assignment $\rho : SYN \rightarrow SEM$ is then said to be *compositional* if ρ is a *homomorphism*, in the sense that, for every *n*-ary syntactic operation F and arguments $X_1, ..., X_n$, there is a semantic operation $G : SEM \rightarrow SEM$, such that:

[3]See (Zimmermann 2006) for reflection on the nature of semantic theories, and for a discussion of different ways of understanding the role of discourses versus sentences in a semantic theory. It is often claimed that a semantic theory must be compositional. But just how to formulate in rigorous terms the required notion of compositionality is controversial (see (Jönsson 2008, Nefdt 2020) for discussion) and requiring that a semantic theory be compositional by definitional fiat would entail that there are no non-compositional semantic theories. This might (depending on the definition of compositionality) misfire, by entailing that some obvious semantic theories–such as some versions of DRT (which have been claimed to be non-compositional in their original presentations (Groenendijk and Stokhof 1990))–from being semantic theories.

[4]For example (Zadrozny 1994), (Kazmi and Pelletier 1998). (Nefdt 2020) writes:

> "Unfortunately, there is still no consensus on the correct definition of compositionality in natural language. Furthermore, it is not clear if it is exclusively a methodological principle (Dever 1999) or can be empirically tested (Dowty 2007)."

$$\rho(F(X_1, ..., X_n)) = G(\rho(X_1), ..., \rho(X_n))$$

There are different ways of developing the underlying algebraic picture which Montague toyed with (Janssen and Partee 1997), (Hodges 2001), (Kracht 2006), (Hodges 2012). But any semantic theorist faces the question of how we get from a natural language expression to its interpretation in a model of a given intermediary logic. There are two particularly well-known methodological approaches to this question: the method of *indirect interpretation* and the method of *direct interpretation*. The direct and indirect interpretations can be schematised as below:

Indirect interpretation: *natural language* $\rightsquigarrow$ *intermediary logic* $\overset{[\![\,]\!]}{\Longrightarrow}$ *model*

Direct interpretation: *natural language* $\overset{[\![\,]\!]}{\Longrightarrow}$ *model*

Indirect interpretation (adopted in (Montague 1973)'s *PTQ* and (Kratzer 2005)) associates a natural language expression with its interpretation in two steps: step one renders strings of English into formulas of a sufficiently expressive intermediary logic–which for Montague was intensional logic (IL)– and step two assigns formulas of the intermediary logic an interpretation in a model. Whenever both step one and step two are homomorphisms, by function composition we derive a homomorphism from the syntax to the semantics, rendering the intermediary language in principle dispensable, as Montague desired.[5] According to (Partee and Hendriks 1997: p.24), Montague "viewed the use of an intermediate language as motivated by [. . .] the expectation (which has been amply realized in practice) that a

[5]This depends on an elementary set-theoretic fact, that the function composition of two homomorphisms is a homomorphism.

sufficiently well designed language such as his Intensional Logic with a known semantics could provide a convenient tool for giving the semantics of various fragments of various natural languages". In contrast, in direct interpretation (pursued in (Montague 1970), (Keenan and Faltz 1985)) one constructs a direct mapping from syntactical structures to their model theoretic interpretation, without employing an intermediary logic.

Direct interpretation can be rather notationally heavy (see (Janssen 2017) for discussion) and so indirect interpretations are generally more popular. Typically, terms of the typed lambda calculus are used to render natural language expressions, and these terms are the leaf nodes of binary trees which represent either the surface syntactic structure of sentences, or, in minimalist syntactic theories (Adger 2003)) their syntactic structure at a certain level of syntactic representation, such as logical form (Fox 1995). A limited set of rules of semantic composition, including function application (Heim and Kratzer 1998), are then used to combine lambda terms with each other in a way which mirrors the way meanings combine together.[6] These lambda terms are then assigned an interpretation or semantic value in a model, and such semantic values are supposed to capture the meanings of natural language expressions.[7]

[6]There is probably no agreed upon rigorous definition of what a natural language semantic theory is, but, in recent years, a number of theorists (Hodges 2012, Jönsson 2008, Pagin 2003) have pursued an algebraic approach to the definition of natural language semantic theories. On this approach a natural language semantic theory is a structure $\langle \mathscr{G}, [\![\,]\!], \Psi \rangle$ consisting of a term algebra $\mathscr{G}$ of syntactic structures, an interpretation function $[\![\,]\!]$ assigning semantic values to members of $\mathscr{G}$ and a set of rules Ψ that combine semantic values (see (Jönsson 2008) for further discussion and (Hodges 2012) for an elaborate mathematical development of the underlying algebraic picture and a mathematical definition of the syntactic notion of constituent structure.)

[7]For more on the role of model theory in natural language semantics, see (Zimmermann 2011) and references therein.

Ty2 is a type-logic widely used by semanticists (Gamut 1990, Groenendijk and Stokhof 1982, Muskens 1995) as an intermediary logic in order to give an indirect interpretation of fragments of natural language. It was introduced in (Gallin 1975) as a streamlined version of Montague's logic IL.[8] For $m \in \mathbb{N}$, let *Tym* denote an m-sorted type-logic, such that it has $m + 1$ base types including type t, the type of boolean truth values. Gallin's *Ty2* (Gallin 1975) has basic types s and types e, (in addition to type t), and incorporate function types $\alpha \to \beta$, where α and β are $Ty2$ types (the rules for type formation will be discussed in the next section), and so *Ty2* is a 2-sorted type-logic. The types s and types e are respectively the types of indices[9] and the type of possible entities. (Gallin 1975) defines the intension and extension of a $Ty2$ term in a model and relative to an assignment. Roughly, the *intension* of a $Ty2$ term (in a model and relative to an assignment) is a function from any index to its extension at that index in the model and relative to the assignment. For example, the extension of a term of type $s \to t$ at index w is the truth value of the term at w, the extension of a term of type $s \to e$ at w is an individual, and the extension of a term of type $s \to e \to e \to t$ at w is binary function from individuals to truth values. We will be lax and talk about the intension/extension of a linguistic expression of a given grammatical category (for example, we will talk about the intension of the sentence *John sleeps*, the intension of a predicate *sigh*, the intension

[8]In fact, the logics are closely related, since there exists an embedding of *IL* into $Ty2$ ((Gallin 1975)), and an embedding of $Ty2$ into IL (Zimmermann 1989). There are two reasons why we present *Ty2* instead of *IL*. Firstly, much recent work (Keshet 2008), based on (Cresswell 1990), has argued for the need to have variables denoting indices in the object-language, as opposed to the metalanguage, and Ty2 allows us to do this by admitting a type of indices. Secondly, *Ty2* has certain desirable meta-logical properties which *IL* lacks; for example, in *IL* the Church-Rosser property, Beta conversion, Leibniz' law and Universal instantiation all fail (Muskens 1995), whereas in $Ty2$ they do not.

[9]Throughout this thesis, the words *index*, *state* and *situation* will be used interchangeably.

of a name *Mary* or a noun phrase *every student*), where the intension/extension is the interpretation of the rendering of that linguistic expression in a particular model and relative to a certain assignment. In many applications in philosophy and linguistics, indices can be taken as characterisations of situations which are compatible in some sense with the current stage of a conversation or inquiry at a time (Stalnaker 1984), (Stalnaker 1998). Alternatively indices could be thought of as so-called *possible worlds*–which specify alternative ways that the universe could be, epistemically, metaphysically or logically speaking (Lewis 1986)– or they can be thought of as world-time pairs, or tuples of parameters (Montague 1974) (for example, an index could be a tuple containing a world, a time, a location and a speaker).[10]

In the early development of intensional semantics, theorists (for example, (Cresswell 1973)) found it was necessary to proliferate parameters in the index to account for more context-dependent phenomena. For example, (Lewis 1970) uses tuples of a time, a place, an agent, an audience, a list of indicated objects, a segment of previous discourse, and an assignment function, and the list of required parameters may have to be even more elaborate in certain cases (See Cresswell (1973)). In this respect, theorists were following the advice of Dana Scott (Scott 1970) who wrote that:

> "For more general situations one must not think of the [point of reference] as anything as simple as instants of time or even possible worlds. In general we will have
>
> $i = (w, t, p, a, ...)$

[10]See Menzel (2017) and see (Lewis 1986, Stalnaker 2003) for philosophical discussion of the nature of indices.

where the index i has many coordinates: for example, w is a world, t is a time, $p = (x, y, z)$ is a (3-dimensional) position in the world, a is an agent, etc. All these coordinates can be varied, possibly independently, and thus affect the truth-values of statements which have indirect references to these coordinates." (Scott 1970: p.151)

(Kaplan 1989) subsequently adduced various arguments for what we call context-index frameworks, which repackage elements of the index into two separate components, the *context of utterance* and the *index* (though the success of some of these arguments has been questioned. See (Schlenker 2003), (Anand and Nevins 2004), (Rabern 2012b) and (Santorio 2019)). Formally, a context of utterance is a tuple consisting of at least the parameters present in the index, including an assignment function. We could implement the context-index framework by having a type of contexts of utterance.

Due to their complexity, index frameworks and context-index frameworks allow us to define various notions of semantic value. For example, in (Montague 1970), Montague makes a threefold distinction between the *denotation*, the *sense*, and the *meaning* of an expression. The denotation of a sentence is a truth value, the sense (or intension) of a sentence is a function from indices– which are pairs of a world and a moment of time– to truth values, and the meaning of a sentence is a function from contexts of use–which are tuples of contextually relevant parameters– to senses.[11]

There are numerous motivations (see (Cresswell 1990), (Heim & von Fintel 2011) and references therein) for employing indices and intensions in a seman-

[11](Kaplan 1989) adopts a similar distinction between indices of reference and contexts of use, but for different reasons than Montague (see (Rabern 2012b)).

tic theory and we shall mention just one type of argument, popularised by (Frege 1892). Consider the following *Principle of Extensional Compositionality*:

> *Principle of Extensional Compositionality* :
>
> The extension of a compound expression is a function of the extensions of its immediate parts and the way they are composed.

Take any two true sentences, such as (2a) and (2b):

(2) a. London is in England.
    b. Bristol is in England.

Given that the extension of a sentence is its truth value, (2a) and (2b) have the same extension. We can calculate the extensions of these sentences in line with the *Principle of Extensional Compositionality* if we assume the extension of *is in England* is $\{x \mid x \text{ is in England}\}$ and the extensions of *London* and *Bristol* are the cities London and Bristol, respectively.

But consider (3a) and (3b):

(3) a. John believes London is in England.
    b. John believes Bristol is in England.

Assuming these sentences are structurally isomorphic, they should have the same extension. So, if, for example (3a) has the extension 1, then (3b) should. However, this is not so: (3a) and (3b) may plausibly differ in extension, since John might believe London is in England but not that Bristol is in England. The principle of extensional compositionality cannot therefore hold, and it must fail in a context involving a propositional attitude verb, like *believe*. Environments in which extensional compositionality fails are called *intensional contexts*. In intensional contexts, the extension of a compound expression is a function of the intension of its immediate parts and the way they are composed. We can say that (2a) and (2b) have the same extension, but differ in intension.

A.2 On propositions and contents

Various philosophers and linguists typically make claims to the effect that sentences, or utterances of them, have (propositional) *content* (see (Stalnaker 1984, Kaplan 1989, Stanley 1997, King 2007, Lasersohn 2016, Rabern 2017, Stojnić 2017) for discussion).

In this thesis, we will (in the main) avoid talk of propositions and of contents altogether, and we will refer to the *semantic value* of a sentence, which we take to be some model-theoretic object, such as a function from states to truth values, or a function from other parameters to states to truth values. Our reasons for avoiding the notion of a proposition and the notion of content will now be described.

The first worry is that it is not clear that there is a single entity which fulfill the roles that some philosophers have wished propositions and contents to fulfill (see (Lewis 1980, Rabern 2012a) for discussion). (Stalnaker 1984) claims that both mental states and linguistic expressions have content and that sentences have propositional content in the sense that they express propositions, understood. Propositions play a cluster of theoretical roles; for example, they are supposed to be the objects of thought, provide the content of mental attitudes, and to provide the so-called assertoric content of sentences (Evans 1979, Dummett 1981, Rabern 2012a), or the information they convey. (Lewis 1980, 1986, Rabern 2012a) argue that there is no reason why there should be a single entity which plays all of these theoretical roles and to our knowledge there is no knockdown argument for the necessity of treating such diverse theoretical roles via a single notion, namely (propositional) content.

More fundamentally, there is little agreement about what contents are. One popular proposal (Stalnaker 1984), is that propositional contents are the characteristic functions of sets of states. But the claim that the objects of mental states are functions from states to truth values would surely require evidence which bears on the nature of mental states themselves, and there is none, to our knowledge. It seems to us that, whatever contents are, the notion of content is a notion which, if anything, falls within the province of neuroscience and cognitive science, and is best left to those fields to study. That said, sometimes semanticists and philosophers use the term *proposition* simply to refer to the semantic value of a sentence and not to endorse the cluster of theoretical roles that some philosophers have attributed to propositions. But, in our mind, the heavy-loaded philosophical use of these terms in the philosophy literature has tainted the terms *proposition* and *(propositional) content*, and so, to avoid confusion, we will avoid them altogether.

The reason for doubting that functions from states to truth values are contents also applies to other semantic values which semantic theorists may assign to sentences. As we have explained above, the semantic values of sentences in a semantic theory are often quite elaborate critters indeed. In some semantic theories, the entity relative to which a given formula is true might be, for example, a sextuple, consisting of a point, a time, a location, an agent, a variable assignment and a standard of precision. The claim that the objects of thought or contents of mental attitudes are functions from such sextuples to truth values would presumably require a rather elaborate defence (see (Rabern 2012a) for philosophical reflection on this point); moreover, if what we have argued is correct, it would require evidence from neuroscience or cognitive science. It thus seems that, whatever contents are (and supposing they actually exist), they are probably not semantic values. If the notion of content is best studied by cognitive science, then, to quote (Montague 1973), “this is an issue on which it would be unethical for me as a logician (or linguist or grammarian or semanticist, for that matter) to take a stand.”

A.3 Partial *Ty2*

(Lapierre 1992) introduces a partial semantics for the logic IL (Gallin 1975) and proves that if IL is sound and complete for its standard semantics, then it is sound and complete for the partial semantics (Gallin 1975) introduces. IL differs from *Ty2* in that it does not allow terms of type s, nor does it allow functions whose range is an entity of type s. Nevertheless (as mentioned in ft. 8), there exists an embedding of *IL* into $Ty2$ ((Gallin 1975)), and an embedding of $Ty2$ into IL (Zimmermann 1989). Now Ty2 is particular theory within the simply-typed lambda calculus whose base types include at least e, s (besides the type t of truth values). By inspecting the proof of (Lapierre 1992), it seems clear that the partial semantics which (Lapierre 1992) provides for IL can be used to give a sound and complete semantics of the simply-typed lambda calculus, simply by omitting the parts of the proof used to deal with the peculiarities of the IL type system. We therefore refer to *partial Ty2* as the system obtained by using the standard deductive system of *Ty2* (from (Gallin 1975)) presented in A.4 with the partial semantics provided by (Lapierre 1992), presented in A.5.

We now introduce partial *Ty2*. Partial *Ty2* types are formed by the following recursive rule (α, β are variables standing for arbitrary types):

Definition 20

The set $TYPE$ of *Ty2* types is the smallest set of strings for which (i)-(ii) holds:

(i) $e \in TYPE, t \in TYPE$ and $s \in TYPE$

(ii) If $\alpha \in TYPE$ and $\beta \in TYPE$, then $\alpha \to \beta \in TYPE$

The types formed via (i) are called **base types**. The types formed via (ii) are called **function types**. We understand functions set-theoretically, as sets of ordered pairs. We use '→' to form function types, overloading the symbol '→' (which is used for implication), but this should not cause confusion.

We now introduce a set of constants and variables:

Definition 21 For each $\tau \in TYPE$, Con_τ and Var_τ are each denumerable sets of *constants* and *variables* (respectively) of type τ, such that:

- $Con = \bigcup_\tau Con_\tau$
- $Var = \bigcup_\tau Var_\tau$

For all $A \in Con_\alpha$ and $A \in Var_\alpha$, $A : \alpha$ reads 'A is of type α'. We also use A_α to indicate this typing information, especially when trying to save space or to improve readability.

We sometimes write A instead of $A : \alpha$ or A_α, when the typing of A is clear.

The set $TERM$ shall now be defined. Terms of type t will be called *formulas*.

Definition 22 For all types $\tau \in TYPE$, the set $TERM_\tau$, of *terms of type* τ is the smallest set such that:

1. $c \in TERM_\tau$ and $x \in TERM_\tau$, for all constants $c : \tau$ and variables $x : \tau$

2. $\top, \bot$ and $\#$ are formulas.

3. If φ and ψ are formulas, then $\neg\varphi$ and $\varphi \wedge \psi$ are formulas.

4. If φ is a formula and x is a variable of any type, then $\forall x \varphi$ is a formula.

5. If $A : \alpha \to \beta$ and $B : \alpha$, then $(A\,B) : \beta$.

6. If $A : \beta$ and $x : \alpha$, then $(\lambda x.\, A) : \alpha \to \beta \in TERM_{\alpha\to\beta}$.

7. If $A : \tau$ and $B : \tau$, then $A = B$ is a formula.

We can then set:

$$TERM = \bigcup_{\tau \in TYPE} TERM_{\tau}$$

For any type α, if A is a term of type α, A is called a *term*.

In fact, following (Lapierre 1992), $\top, \bot, \neg, \wedge, \vee, \to, \forall, \exists$ are abbreviations of the following lambda terms:

Definition 23

$$\top := \lambda x_t.x = \lambda x_t.x$$

$$\bot := \lambda x_t.x = \lambda x_t.\top$$

$$\neg A_t := A = F$$

$$\wedge_{t(tt)} := \lambda x_t\, y_t.(\lambda z_{t(tt)}(\,(z\,x)\,y)\,) = \lambda z_{t(tt)}\,(z\,\top)\,\top)$$

$$\vee_{t(tt)} := \lambda x_t\,, y_t.\neg(\neg x \wedge \neg y)$$

$$\rightarrow_{t(tt)} := \lambda x_t\,, y_t.(\neg x \vee y)$$

$$\forall x_\alpha\, A_t := \lambda x_\alpha.\, A = \lambda x.\top$$

$$\exists x_\alpha\, A_t := \neg\forall x_\alpha\, \neg A$$

Γ, Δ and other capitalised greek letters will be used for sets of formulas, and φ, ψ and other lower-case greek letters will be used to stand for arbitrary formulas. '$\bigwedge_{\gamma\in\Gamma}\gamma$' is occasionally used to denote the conjunction of a finite set Γ of formulas. We will use lower-case greek letters in bold (e.g, $\boldsymbol{\varphi}, \boldsymbol{\phi}, \boldsymbol{\gamma}$) for *Ty2* terms of type $s \rightarrow t$ and capital greek letters in bold for sets of *Ty2* terms of type $s \rightarrow t$. Where $\boldsymbol{\Gamma}$ is a set of *Ty2* terms of type $s \rightarrow t$ and $s \in Var_s \cup Con_s$, we use $\boldsymbol{\Gamma}\, s_s$ to denote $\bigwedge_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \boldsymbol{\gamma}\, s$, so that we can form formulas such as $\boldsymbol{\Gamma}\, s \rightarrow \boldsymbol{\varphi}\, s$. We omit outer brackets when this is clearer, and we sometimes impose brackets on an *ad hoc* basis to improve readability. When parentheses are omitted, association is to the left, so that $A\, B\, C\, D = (((AB)\, C)\, D)$.

We will often indicate the type of a variable or constant at the beginning of a term, and then omit it when the variable or constant occurs later on in the term. We will also use $\lambda x_1, ..., x_n.\varphi$ to abbreviate $\lambda x.\lambda x_2....\lambda x_n.\varphi$.

For example, instead of writing

$$\lambda x_e\, \lambda z_e\, \lambda y_{s\rightarrow e\rightarrow e}.\, \forall s_s (y_{s\rightarrow e\rightarrow e}\, s_s\, z_e = x_e) \tag{8.1}$$

we will write

$$\lambda x_e, z_e, y_{s\rightarrow e\rightarrow e}.\, \forall s_s (y\, s\, z = x) \tag{8.2}$$

or, if the typing of (8.1) is clear from the context, we will write

$$\lambda x, z, y. \forall s(y\, s\, z = x) \tag{8.3}$$

Definition 24 Where A is a term of any type, the *free variables of* A, $FV(A)$, are defined as follows:

- If $A \in Con_\alpha$ for $\alpha \in TYPE$, then $FV(A) = \varnothing$
- If $A \in Var_\alpha$ for $\alpha \in TYPE$, then $FV(A) = \{A\}$
- If A is of the form $(\neg\psi)$, then $FV(A) = FV(\psi)$
- If A is of the form $\rho \circ \eta$ and $\rho \in \{\wedge, \vee, \rightarrow\}$, then $FV(A) = FV(\psi) \cup FV(\eta)$
- If A is of the form $(\forall x_\alpha.\psi)$ and $\alpha \in TYPE$, then $FV(A) = FV(\psi)\backslash\{x\}$
- If A is of the form $(B_{\alpha\rightarrow\beta}\, C_\alpha)$ and $\alpha, \beta \in TYPE$, then $FV(A) = FV(B) \cup FV(C)$
- If A is of the form $(\lambda x.B : \alpha \rightarrow \beta)$ and $\alpha, \beta \in TYPE$, then $FV(A) = FV(B)\backslash\{x\}$
- If A is of the form $B = C$, then $FV(A) = FV(B) \cup FV(C)$

Definition 25 We use $B[A/x]$ to denote the operation of *substitution*, which substitutes all free occurrences of the variable x in B by the term A, where A and x have the same type. $B[A/x]$ is defined as follows:

- $x[A/x] = A$

- $x'[A/x] = x'$, if $x' \neq x$

- $(\Delta\varphi)[A/x] = \Delta(\varphi[A/x])$, for $\Delta \in \{\neg, \Box, \Diamond\}$

- $(\varphi\,\rho\,\psi)[A/x] = (\varphi[A/x]\,\rho\,\psi[A/x])$, for $\rho \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

- $(\lambda y.B : \alpha \rightarrow \beta)[A/x] = \lambda y.B$ if $x = y$

- $(\lambda y.B : \alpha \rightarrow \beta)[A/x] = \lambda y.(B[A/x])$ if $x \neq y$ and $y \notin FV(A) \cup FV(B)$

- $(\lambda y.B : \alpha \rightarrow \beta)[A/x] = \lambda z.((B[z/y])[A/x])$ if $x \neq y$ and $y \in FV(A)$ and $z \notin FV(A)$

- $(B = C)[A/x] = (B[A/x] = C[A/x])$

- For $Q \in \{\forall, \exists\}$: $(Qy\varphi)[A/x] = Qy(\varphi)$ if $x = y$

- For $Q \in \{\forall, \exists\}$: $(Qy\varphi)[A/x] = Qy(\varphi[A/x])$ if $y \notin FV(A)$.

- For $Q \in \{\forall, \exists\}$: $(Qy\varphi)[A/x] = Qz((\varphi[z/y])[A/x])$, if $y \in FV(A)$ and where $z \notin FV(A) \cup FV(\varphi)$

Note that this definition entails that substituting a term for a variable cannot result in accidental variable capture, so that the notion of a term being *free for* a variable x is not needed.

A.4 Proof theory of partial *Ty2*

The theory of *Ty2*

Axioms of Ty2

$A1.\ \ g_{t\to t}\top \wedge g_{t\to t}\bot = \forall x_t(g\,x)$

$A2.\ \ x_\alpha = y_\alpha \to f_{\alpha\to t}\,x = f_{\alpha\to t}\,y$

$A3.\ \ \forall x_\alpha(f_{\alpha\to\beta}x = g_{\alpha\to\beta}\,x) = (f = g)$

$AS4.\ \ (\lambda x_\alpha.A_\beta(x))\,B_\alpha = A_\beta\,(B_\alpha),$ where $A_\beta\,(B_\alpha)$ is obtained from $A_\beta\,(x_\alpha)$ by substituting all free occurrences of x by B_α in $A_\beta\,(x_\alpha)$

Rule of Inference

(R) : From $A_\alpha = A'_\alpha$ and formula B infer the formula B',

where B' is obtained by substituting one occurrence of A not immediately preceded by λ with A'

We will call $AS4$, β-conversion.

A *proof* is a sequence of formulas such that each formulas is either an axiom or obtainable from an earlier formula by rule (R). A formula A is *provable* or a *theorem* if it is the last line of a proof.

(Gallin 1975) remarks that (unlike IL), the following schemata are provable in *Ty2*:

$$\forall x_\alpha A(x) \to A(B_\alpha),\ \text{where the term } B \text{ is substituted with } x \text{ in the formula } A(x) \tag{8.4}$$

$$(B_\alpha = C_\alpha) \to (A_\beta(B) = A_\beta(C)),\ \text{where } B \text{ and } C \text{ are substituted with } x \text{ in } A_\beta(x_\alpha) \tag{8.5}$$

A.5 Model theory of partial *Ty2*

We will shortly define the notion of a partial *Ty2* frame. Partial frames can be thought of as extensions of total frames, where a total frame is defined as follows.

Let D and S be two disjoint, non-empty sets. A *total frame* is a family of domains $\{D_\alpha \mid \alpha \in TYPE\}$, where:

- $D_e = D$
- $D_s = S$
- $D_t = \{0, 1\}$
- $D_{\alpha \to \beta} = D_\beta^{D_\alpha}$

Partial frames will be constructed from domains which include an undefined object, $\#$.

Definition 26 Let D and S be two disjoint, non-empty sets. A *standard partial Ty2 frame* $\mathcal{F}$ based on D and S is a family of domains $\{PD_\alpha \mid \alpha \in TYPE\}$, such that:

(i) $PD_e = D \cup \{\#\}$, where, for all $x, y \in PD_e, x \leqslant_{PD} y$ if and only if $x = \#$ or $x = y$

(ii) $PD_t = \{0, 1, \#\}$, where, for all $x, y \in PD_t, x \leqslant_{PD} y$ if and only if $x = \#$ or $x = y$

(iii) $PD_{\alpha \to \beta} \subseteq \{f \mid PD_\alpha \to PD_\beta\}$, where the following condition (*MON*) is satisfied: for all $f, g \in PD_{\alpha \to \beta}$,
$f \leqslant_{PD} g$ if and only if for all $x \in PD_\alpha$, $f\, x \leqslant_{PD} g\, x$

A partial $Ty2$ frame is *standard* when $PD_{\alpha \to \beta} = \{f \mid f : PD_\alpha \to PD_\beta\}$
A partial $Ty2$ frame is *general* when $PD_{\alpha \to \beta} \neq \{f \mid f : PD_\alpha \to PD_\beta\}$
F denotes the class of all partial *Ty2* frames.

$\mathbf{F}_s$ denotes the class of all partial standard *Ty2* frames.

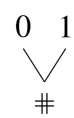
$\mathbf{F}_g$ denotes the class of all partial general *Ty2* frames.

(Lepage et al. 1992) proves that for all $\alpha \in TYPE$, PD_α is a meet semilattice, where $x_\alpha \wedge y_\alpha$, the infimum of x and y is defined as follows:

1. If $\alpha = e$ or $\alpha = t$, then $x \wedge y = x$, if $x = y$; otherwise $x \wedge y = \#$.

2. If $\alpha = \beta \rightarrow \gamma$, then $f \wedge g = h$, where $h : PD_{\beta\rightarrow\gamma}$ is the function such that, for every $x \in \beta$, $h(x) = f(x) \wedge g(x)$.

For example, the meet semilattice associated with PD_T can be represented as follows:



A notion of *strong difference* is then defined as follows:

Definition 27 For every $\alpha \in TYPE$, and any x_α, y_α, $x \neq^* y$ (‘*x strongly differs from y*’) is defined as follows:

1. If $\alpha = e$ or $\alpha = t$, then $x \neq^* y$ iff $x \neq \#$ and $y \neq \#$ and $x \neq y$.

2. If $\alpha = \beta \rightarrow \gamma$, then $f \neq^* g$ iff there is a $x \in \beta$ such that $f(x) =^* g(x)$

We use $\neg x \neq^* y$ to express that x and y are not strongly different.

Proposition 1

For every α, and every $x, x', y, y' \in PD_\alpha$: if $x \neq^* y$ and $x \leqslant x'$ and $y \leqslant y'$, then $x' \neq^* y'$.

Proof: See (Lapierre 1992), **Proposition 6**.

Proposition 2

For any $\alpha \in TYPE$, we can collect together those sets of objects in PD_α which are not strongly different into a set C_α:

$$C_\alpha = \{X \subseteq PD_\alpha \mid X \neq \emptyset \& \forall x, y \in PD_\alpha (x, y \in X \Rightarrow \neg x \neq^* y)\} \tag{8.6}$$

For every $X \in C_\alpha$, $\bigvee X$ exists in PD_α.

Proof: See (Lapierre 1992), **Proposition 7**.

For each partial domain PD_α, (Lapierre 1992) defines the notion of being a total object in PD_α, as follows:

Definition 28 For any $\alpha \in TYPE$, the set $PT_\alpha \subseteq PD_\alpha$ is defined as follows:

1. If $\alpha = e$ or $\alpha = t$, then $PT_\alpha = \{x \in PD_\alpha \mid x \neq \#\}$
2. If $\alpha = \beta \rightarrow \gamma$, then $PT_\alpha = \{f \in PD_\alpha \mid \forall x \in PT_\beta \, (f(x) \in PT_\gamma)\}$

An equivalence relation between objects in PT_α, for any $\alpha \in TYPE$, can then be defined:

Definition 29 For any $x_\alpha, y_\alpha \in PT_\alpha$, $x < > y$ is defined as follows:

1. If $\alpha = e$ or $\alpha = t$, then $x < > y$ iff $x = y$.
2. If $\alpha = \beta \rightarrow \gamma$, then $f < > y$ iff, for every $x \in PT_\beta$, $f(x) < > g(x)$.

When $x <> y$ we say that x and y are *equivalent total objects*. When x and y are total objects which are not equivalent we write $\neg x <> y$.

Proposition 3

For every $\alpha \in TYPE$, for every $x, z \in PT_\alpha$: if $x \leqslant z$, then $z \in PT_\alpha$ and $x <> z$.

Proof: See (Lapierre 1992), **Proposition 11**.

Proposition 4

For every $\alpha \in TYPE$, for every $x, y \in PT_\alpha$: if $\neg x <> y$, then $\neg x \neq^* y$.

Proof: See (Lapierre 1992), **Proposition 12**.

Definition 30 For every $\alpha \in TYPE$, and $x \in PT_\alpha$, $< x >= \{z \in PT_\alpha \mid x <> z\}$ and $\Pi_\alpha = \{< x >| x \in PT_\alpha\}$.

Proposition 5

Let $x \in PD_\alpha$. If $x \in PT_\alpha$, then $\bigvee < x >$ exists and is contained in $< x >$ and for every $z \in PT_\alpha$ such that $x \neq^* z$, $\neg x <> z$. If $x \notin PT_\alpha$, then there exists a $y \in PT_\alpha$ such that $x \leqslant y$.

Proof: See (Lapierre 1992), **Proposition 14**.

Proposition 6

For every $\beta \rightarrow \gamma \in TYPE$, every $f \in PT_{\beta\rightarrow\gamma}$ and all $x, y \in PT_\beta$: if $x <> y$, then $f(x) <> f(y)$.

Proof: See (Lapierre 1992), **Proposition 15**.

Proposition 7

Let $f \in PT_{\beta\rightarrow\gamma}$, $x \in PT_\beta$ and $y \in PT_\gamma$ be such that $\bigvee < f > (\bigvee < x >) \in < y >$; then $\bigvee < f > (\bigvee < x >) \in \bigvee < y >$.

Proof: See (Lapierre 1992), **Proposition 16**.

Definition 31 The functions

$$\Phi : \bigcup_{\alpha \in TYPE} M_\alpha \to \bigcup_{\alpha \in TYPE} \Pi_\alpha$$

$$\Theta : \bigcup_{\alpha \in TYPE} \Pi_\alpha \to \bigcup_{\alpha \in TYPE} M_\alpha$$

are defined as follows:

(i) If $\alpha = e$ or $\alpha = t$, then $\Phi(x) = \{x\}$ and $\Theta\{x\} = x$.

(ii) If $\alpha = \beta \to \gamma$, then:

$$\Phi(f) = \{f' \in PT_\alpha \mid \forall x \in M_\beta \, (\forall y \in \Phi(x)(f'(y) \in \Phi(f(x))))\}$$

$\Theta(F) = h \in M_{\beta \to \gamma},$ where h is the function such that, for all $x \in M_\beta$,

$h(x) = \Theta(\{z \in PT_\gamma \mid \forall g \in F(\forall y \in \Phi(x) \; (z <> g(y)))\})$

Proposition 8

Φ is bijective and $\Phi^{-1} = \Theta$.

Proof:

See (Lapierre 1992), **Proposition 18**.

Proposition 8 establishes an isomorphism between M_α and Π_α for all $\alpha \in TYPE$ and **Proposition 5** establishes a correspondence between any total object $x \in M_\alpha$ and the object $\bigvee \Phi(x) \in PT_\alpha$, the unique partial total object which maximally approximates x. This allows us to identify each $x \in M_\alpha$ with its maximal approximation, and consider any $y \in PD_\alpha$ for which $y \leqslant \bigvee \Phi(x)$ as an approximation of x.

Let us notate the unique partial total object which maximally approximates a given x of a given type $\alpha \in TYPE$ AS "$ma(x)$".

Proposition 8:

Given any $f \in PD_{\beta\to\gamma}$ and $x \in PD_{\beta}$, $(ma(f))(ma(x)) = ma(f(x))$. ***Proof***: See (Lapierre 1992), **Proposition 21**.

From now on we will use D_{α} instead of PD_{α} and in our applications in the coming chapters we will be utilising general frames. (Henkin 1950) works with general frames in order to ensure that the entailment relation of the logic he discusses is recursively axiomatisable. General frames also make sense from a linguistic point of view, because, presumably not all functions will capture the meaning of some natural language expression. For example, it is hard to think of a natural language expression which has the type $s \to (s \to e \to t) \to (t \to s) \to e \to t$, though in a standard frame we would have a non-empty domain of functions $D_{s\to(s\to e\to t)\to(t\to s)\to e\to t}$.

In this thesis, we will be interested in the subclass of general partial *Ty2* frames in which there is a set D of individuals and a set S of indices, and amongst the ground domains are the following sets:

$D_e = D \cup \{\#\}$	(domain of individuals)
$D_s = S \cup \{\#\}$	(domain of indices.)
$D_t = \{0, 1, \#\}$	(domain of truth-values)

In line with mathematical practice, we sometimes use '2' to refer to $\{0, 1\}$, where 0 and 1 are intended to denote, respectively, truth and falsity and $0 = \varnothing$ and $1 = \{\varnothing\}$ (as in the Von Neumann coding).

We can now define the notion of a partial *Ty2 model*:

Definition 32 • A partial *Ty2 model* $\mathfrak{M} = (\mathcal{F}, I)$ *based on* $\mathcal{F}$ is a tuple, where:

- $\mathcal{F}$ is a partial *Ty2* frame $\mathcal{F}$
- I is an *Interpretation function*, (i.e a function such that, for every constant c_α, for $\alpha \in TYPE$, $I(c_\alpha) \in D_\alpha$).

A model based on a standard frame $\mathcal{F} \in \mathbf{F}_s$ is called a *partial standard Ty2 model*.

A model based on a general frame $\mathcal{F} \in \mathbf{F}_g$ is called a *partial general Ty2 model*.

• We use $VAL_{\mathcal{F}}$ for the class of partial *Ty2* interpretation functions on the frame $A : s \to t$ and VAL for the class of all partial *Ty2* interpretation functions.

• We use $\mathbf{M}$ for the class of all partial *Ty2* models.

• We use $\mathbf{M}_s$ for the class of all partial standard *Ty2* models.

• We use $\mathbf{M}_g$ for the class of all partial general *Ty2* models.

• We use $\mathbf{M}^s_{\mathcal{F}}$ for the class of all partial standard *Ty2* models based on the frame $\mathcal{F}$.

- We use $\mathbf{M}^g_{\mathcal{F}}$ for the class of all partial general *Ty2* models based on the frame $\mathcal{F}$.

We now define the value $[\![A]\!]^{\mathfrak{M},g}$ of a term in a *Ty2* model $\mathfrak{M}$ under an assignment g (sometimes written $[\![A]\!]$, when this presents no confusion), where G is the set of assignments. An assignment $g \in G$ sends variables of a given type α to D_α. We use $g[d/x]$ in order to indicate the assignment which is just like g but which maps x to d.

Definition 33 : Tarski truth definition

1. $[\![c]\!]^{\mathfrak{M},g} = I(c)$, where $c \in Con_\alpha$ and $\alpha \in TYPE$.
2. $[\![x]\!]^{\mathfrak{M},g} = g(x)$, where $x \in Var_\alpha$ and $\alpha \in TYPE$.
3. $[\![\top]\!]^{\mathfrak{M},g} = 1$
4. $[\![\bot]\!]^{\mathfrak{M},g} = 0$
5. $[\![\#]\!]^{\mathfrak{M},g} = \#$
6. $[\![A\,B]\!]^{\mathfrak{M},g} = [\![A]\!]^{\mathfrak{M},g}([\![B]\!]^{\mathfrak{M},g})$
7. $[\![\lambda\, u_\alpha.B_\beta]\!]^{\mathfrak{M},g} = h : D_\alpha \to D_\beta$, where for any $d \in D_\alpha$, $h(d) = [\![B]\!]^{\mathfrak{M},g[d/u]}$
8. $$[\![A_\alpha = B_\alpha]\!]^{\mathfrak{M},g} = \begin{cases} 1 & \text{if } [\![A]\!]^{\mathfrak{M},g}, [\![B]\!]^{\mathfrak{M},g} \in PT_\alpha \\ & \text{and } [\![A]\!]^{\mathfrak{M},g} <> [\![B]\!]^{\mathfrak{M},g} \\ 0 & \text{if } [\![A]\!]^{\mathfrak{M},g} \neq^* [\![B]\!]^{\mathfrak{M},g} \\ \# & \text{otherwise} \end{cases}$$

We say that a set of formulas Γ is *true in* $\mathfrak{M}$ *relative to* g if and only if $[\![\varphi]\!]^{\mathfrak{M},g} =$

1 for every $\varphi \in \Gamma$, and write (8.7):

$$\mathfrak{M}, g \models \Gamma \tag{8.7}$$

When $[\![\varphi]\!]^{\mathfrak{M},g} = 0$ for at least one $\varphi \in \Gamma$, we write (8.8) and we say that Γ is *not true in* $\mathfrak{M}$ relative to g:

$$\mathfrak{M}, g \not\models \Gamma \tag{8.8}$$

When $[\![\varphi]\!]^{\mathfrak{M},g} = 1$ or $[\![\varphi]\!]^{\mathfrak{M},g} = \#$ for every $\varphi \in \Gamma$, we write (8.9) and we say that Γ is *true or undefined in* $\mathfrak{M}$ *relative to* g:

$$\mathfrak{M}, g \models^{p} \Gamma \tag{8.9}$$

When $\Gamma = \{\varphi\}$ and $\rhd \in \{\models, \not\models, \models^{p}\}$, in order to avoid clutter we write $\mathfrak{M}, g \rhd \varphi$ instead of $\mathfrak{M}, g \rhd \{\varphi\}$ and we say, respectively, that φ *is true in* (or *not true* or *true or undefined*) in $\mathfrak{M}$ *relative to* g.

(Lapierre 1992) remarks that the clauses in the truth definition for lambda abstraction, equality, function application and the interpretation of constants and variables, together with the lambda terms used to represent $\neg$, $\wedge$, $\forall$, $\vee$, $\rightarrow$ and $\exists$ given in **Definition 23**, ensure these operators have the following interpretation, which is the interpretation these operators have in Strong Kleene Logic (Bergmann 2008):

$$[\![\neg A_t]\!]^{\mathfrak{M},g} = \begin{cases} 1 & \text{if } [\![A]\!]^{\mathfrak{M},g} = 0 \\ 0 & \text{if } [\![A]\!]^{\mathfrak{M},g} = 1 \\ \# & \text{otherwise} \end{cases}$$

$$[\![A_t \wedge B_t]\!]^{\mathfrak{M},g} = \begin{cases} 1 & \text{if } [\![A]\!]^{\mathfrak{M},g} = [\![B]\!]^{\mathfrak{M},g} = 1 \\ 0 & \text{if } [\![A]\!]^{\mathfrak{M},g} = 0 \text{ or } [\![B]\!]^{\mathfrak{M},g} = 0 \\ \# & \text{otherwise} \end{cases}$$

$$[\![\forall x_\alpha A_t]\!]^{\mathfrak{M},g} = \begin{cases} 1 & \text{if for every } z \neq \# \in D_\alpha, [\![A]\!]^{\mathfrak{M},g[z/x]} = 1 \\ 0 & \text{if for some } z \neq \# \in D_\alpha, [\![A]\!]^{\mathfrak{M},g[z/x]} = 0 \\ \# & \text{otherwise} \end{cases}$$

$$[\![A_t \vee B_t]\!]^{\mathfrak{M},g} = \begin{cases} 1 & \text{if } [\![A]\!]^{\mathfrak{M},g} = 1 \text{ or } [\![B]\!]^{\mathfrak{M},g} = 1 \\ 0 & \text{if } [\![A]\!]^{\mathfrak{M},g} = 0 \text{ and } [\![B]\!]^{\mathfrak{M},g} = 0 \\ \# & \text{otherwise} \end{cases}$$

$$[\![A_t \rightarrow B_t]\!]^{\mathfrak{M},g} = \begin{cases} 1 & \text{if } [\![A]\!]^{\mathfrak{M},g} = 0 \text{ or } [\![B]\!]^{\mathfrak{M},g} = 1 \\ 0 & \text{if } [\![A]\!]^{\mathfrak{M},g} = 1 \text{ and } [\![B]\!]^{\mathfrak{M},g} = 0 \\ \# & \text{otherwise} \end{cases}$$

$$[\![\exists x_\alpha A_t]\!]^{\mathfrak{M},g} = \begin{cases} 1 & \text{if for some } z \neq \# \in D_\alpha, [\![A]\!]^{\mathfrak{M},g[z/x]} = 1 \\ 0 & \text{if for all } z \neq \# \in D_\alpha, [\![A]\!]^{\mathfrak{M},g[z/x]} = 0 \\ \# & \text{otherwise} \end{cases}$$

Nota bene: Throughout the thesis we will often speak of a class of *admissible* (partial *Ty2*) *models*. By this, we mean a model in which certain axioms hold which govern the interpretation of the non-logical vocabulary in order that certain lexical inferences come out valid, such as the inference from 'John is a vegetarian' to 'John does not eat meat'. We therefore use 'admissible' as a placeholder for a fully specified partial *Ty2* model with the appropriate non-logical axioms in place, whatever they are (to our knowledge, no-one has supplied a complete list!). In any case, what counts as admissible will vary depending on the application.

A.6 Validity and logical consequence

We now turn to the question of how to define validity and logical consequence. On the classical conception of logical consequence, a set of formulas Δ is a logical consequence of Γ if and only if every model in which all members of Γ are true is a model in which at least one member of Δ is true, and Γ is valid if every member of Γ is a logical consequence of the empty set. (Lapierre 1992) points out that in Strong Kleene Logic no formulas are valid in this sense. Instead (Lapierre 1992) utilises the notion of *weak consequence* from (van Benthem 1986), according to which a set of formulas Δ is a logical consequence of Γ if and only if every model in which Γ is true is a model in which some member of Δ is not false (i.e, in which some member of Δ is either true or undefined). Weak consequence induces a logic in which the class of valid formulas is the same as that of classical logic.

Different notions of validity of a set Γ of formulas, depending on whether we are quantifying over assignments, interpretation functions or classes of frames:

Total validity relations	Abbreviation
$\forall g' \in G\, \forall\varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I),g'} = 1)$	$(\mathcal{F}, I) \models \Gamma$
$\forall I' \in VAL\, \forall\varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g} = 1)$	$\mathcal{F}, g \models \Gamma$
$\forall I' \in VAL\, \forall g' \in G \forall\varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1)$	$\mathcal{F} \models \Gamma$
$\forall\mathfrak{F} \in \mathbf{F}_s\, \forall I' \in VAL\, \forall g' \in G \forall\varphi \in \Gamma(\, [\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1)$	$\mathbf{M}_s \models \Gamma$
$\forall\mathfrak{F} \in \mathbf{F}_g\, \forall I' \in VAL\, \forall g' \in G \forall\varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1)$	$\mathbf{M}_g \models \Gamma$
$\forall\mathfrak{F} \in \mathfrak{F}\, \forall I' \in VAL\, \forall g' \in G \forall\varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1)$	$\mathfrak{F} \models \Gamma$
$\forall\mathfrak{F} \in \mathbf{F}\, \forall I' \in VAL\, \forall g' \in G \forall\varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1)$	$\mathbf{M} \models \Gamma$

Table A.1: Total validity relations

When $(\mathcal{F}, I) \models \varphi \leftrightarrow \psi$ we say that φ and ψ are *logically equivalent in the model* $(\mathcal{F}, I)$, or φ and ψ are *model equivalent*. When $\mathcal{F} \models \varphi \leftrightarrow \psi$ we say that φ and ψ are *logically equivalent in the frame* $\mathcal{F}$, or φ and ψ are *frame equivalent*. When $\mathbf{M} \models \varphi \leftrightarrow \psi$, we say that φ and ψ are *logically equivalent*. φ and ψ are then logically equivalent for classes of models (or frames) if they are logically equivalent for each model in the class (or each frame in the class).

We can also define another hierarchy of validity relations (the *partial* validity relations) in terms of $\models^p$:

Partial validity relations	Abbreviation
$\forall g' \in G\, \forall \varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I),g'} = 1 \vee [\![\varphi]\!]^{(\mathcal{F},I),g'} = \#)$	$(\mathcal{F}, I) \models^p \Gamma$
$\forall I' \in VAL\, \forall \varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g} = 1 \vee [\![\varphi]\!]^{(\mathcal{F},I),g'} = \#)$	$\mathcal{F}, g \models^p \Gamma$
$\forall I' \in VAL\, \forall g' \in G \forall \varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1 \vee [\![\varphi]\!]^{(\mathcal{F},I),g'} = \#)$	$\mathcal{F} \models^p \Gamma$
$\forall \mathfrak{M} \in \mathbf{F}_s\, \forall I' \in VAL\, \forall g' \in G \forall \varphi \in \Gamma(\, [\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1 \vee [\![\varphi]\!]^{(\mathcal{F},I),g'} = \#)$	$\mathbf{M}_s \models^p \Gamma$
$\forall \mathfrak{M} \in \mathbf{F}_g\, \forall I' \in VAL\, \forall g' \in G \forall \varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1 \vee [\![\varphi]\!]^{(\mathcal{F},I),g'} = \#)$	$\mathbf{M}_g \models^p \Gamma$
$\forall \mathfrak{M} \in \mathfrak{F}\, \forall I' \in VAL\, \forall g' \in G \forall \varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1 \vee [\![\varphi]\!]^{(\mathcal{F},I),g'} = \#)$	$\mathfrak{F} \models^p \Gamma$
$\forall \mathfrak{M} \in \mathbf{F}\, \forall I' \in VAL\, \forall g' \in G \forall \varphi \in \Gamma([\![\varphi]\!]^{(\mathcal{F},I'),g'} = 1 \vee [\![\varphi]\!]^{(\mathcal{F},I),g'} = \#)$	$\mathbf{M} \models^p \Gamma$

Table A.2: Partial validity relations

'Δ is a logical consequence of Γ' can be defined in one of the following ways:

Forms of logical consequence	Abbreviation
$\forall g \in G((\forall \varphi \in \Gamma[\![\varphi]\!]^{(\mathcal{F},I),g'}) \Rightarrow \exists \psi \in \Delta([\![\psi]\!]^{(\mathcal{F},I),g'} = 1 \vee [\![\psi]\!]^{(\mathcal{F},I),g'} = \#))$	$\forall g' ((\mathcal{F}, I), g' \models \Gamma \Rightarrow \exists \psi \in \Delta \; (\mathcal{F}, I), g' \models^p \psi)$
$\forall \mathfrak{M} \in \mathbf{M} \, \forall g \in G\big(\; (\forall \varphi \in \Gamma[\![\varphi]\!]^{\mathfrak{M},g'}) \Rightarrow \exists \psi \in \Delta\big([\![\psi]\!]^{\mathfrak{M},g'} = 1 \vee [\![\psi]\!]^{\mathfrak{M},g'} = \#)\big)$	$\forall \mathfrak{M} \in \mathbf{M}_s \, \forall g' \; (\mathfrak{M}, g' \models \Gamma \Rightarrow \exists \psi \in \Delta \, \mathfrak{M}, g' \models^p \psi)$
$\forall I \in VAL\big((\forall g' \in G \forall \varphi \in \Gamma[\![\varphi]\!]^{(\mathcal{F},I),g'}) \Rightarrow \exists \psi \in \Delta \forall g' \in G \; ([\![\psi]\!]^{(\mathcal{F},I'),g'} = 1 \vee [\![\psi]\!]^{(\mathcal{F},I'),g'} = \#)\big)$	$\forall I'(\, (\mathcal{F}, I') \models \Gamma \Rightarrow \exists \psi \in \Delta \, (\mathcal{F}, I') \models^p \psi)$
$\forall \mathcal{F} \in \mathfrak{F}\big(\forall I \in VAL \forall g' \in G \, \forall \varphi \in \Gamma[\![\phi]\!]^{(\mathcal{F},I'),g'} \Rightarrow \exists \psi \in \Delta \forall I \in VAL \forall g' \in G \; ([\![\psi]\!]^{(\mathcal{F},I'),g'}) = 1 \vee [\![\psi]\!]^{(\mathcal{F},I'),g'} = \#\big)$	$\forall \mathcal{F} \in \mathfrak{F}(\mathcal{F} \models \Gamma \Rightarrow \exists \psi \in \Delta(\mathcal{F} \models^p \psi))$
$\big(\forall \mathcal{F} \in \mathfrak{F} \forall I \in VAL \forall g' \in G \, \forall \varphi \in \Gamma[\![\phi]\!]^{(\mathcal{F},I'),g'}\big) \Rightarrow \big(\exists \psi \in \Delta \forall \mathcal{F} \in \mathfrak{F} \forall I \in VAL \forall g' \in G \; ([\![\psi]\!]^{(\mathcal{F},I'),g'}) = 1 \vee [\![\psi]\!]^{(\mathcal{F},I'),g'} = \#)\big)$	$(\forall \mathcal{F} \in \mathfrak{F} \; (\mathcal{F} \models \Gamma \,)) \Rightarrow \exists \psi \in \Delta(\forall \mathcal{F} \in \mathfrak{F} \; (\mathcal{F} \models^p \psi \,))$

Table A.3: Forms of logical consequence

Note that, in partial *Ty2*, indices are *consistent* in the sense that it's not the case that there is a model $\mathfrak{M}$, an index w, valuation $[\![\;]\!]^{\mathfrak{M},g}$ and term $A : s \to t$ such that both $(w, 1) \in [\![A]\!]^{\mathfrak{M},g}$ and $(w, 1) \notin [\![A]\!]^{\mathfrak{M},g}$. For if this were the case, $A : s \to t$ would be a function which mapped w to both 1 and did not map w to 1. In partial *Ty2* indices are not however required to be *complete* in the sense that for all terms $A : s \to t$, valuations $[\![\ ]\!]^{\mathfrak{M},g}$ and all indices $w \in D_s$, $(w, 1) \in [\![A]\!]^{\mathfrak{M},g}$ or $(w, 1) \notin [\![A]\!]^{\mathfrak{M},g}$. For we may have that $(w, \#) \in [\![A]\!]^{\mathfrak{M},g}$ and $(w, \#) \notin [\![A]\!]^{\mathfrak{M},g}$.

As we have seen above, in partial $Ty2$ the Tarski truth definition is defined for a formula φ in a model $\mathfrak{M}$ relative to variable assignment g (written: $\mathfrak{M}, g \models \varphi$), without indices being involved. In IL (Gallin 1975) the Tarski truth definition is defined for a formula φ in a model $\mathfrak{M}$ at index w and relative to variable assignment g and this is written $\mathfrak{M}, g, w \models \varphi$. Certain sets $\mathbf{\Gamma} \subseteq TERM_{s\to t}$ might be true at all indices, or true in all models based on a class of frames, or true in all frames. Axioms true on all frames of a certain kind are called *frame axioms*. The classical tautologies will be frame axioms for any frame. Other axioms can be imposed to be true in all models on a class of frames, and these are called *model axioms*. For example, we might require all models on a class of frames verify a certain propositional variable P_t at all states. Finally, some axioms can be imposed on certain indices in a particular model, and these are called *state axioms*. In the first line of the table below, we define '$\mathfrak{M}, g, w \models_i \mathbf{\Gamma}$' in *Ty2*, so that we have a convenient way of expressing that $\mathbf{\Gamma} \subseteq Con_{s\to t}$ is true at the index $w \neq \# \in D_s$ in a *Ty2* model $\mathfrak{M}$ with respect to an assignment g. The rest of the table includes other relevant notions of validity which we can define in terms of the notion on the first line of the table (where $\mathcal{M}$ is some class of *Ty2* models):

Truth at an index and related notions	Notation
$\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\,w = 1\,)$	$\mathfrak{M}, g, w \models_i \boldsymbol{\Gamma}$
$\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\,w = 1\,)$	$\mathfrak{M} \models_i \boldsymbol{\Gamma}$
$\forall I \in VAL\,\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = 1\,)$	$\mathbf{M}_{\mathcal{F}} \models_i \boldsymbol{\Gamma}$
$\forall \mathcal{F} \in \mathfrak{F}\,\forall I \in VAL\,\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = 1\,)$	$\mathfrak{F} \models_i \boldsymbol{\Gamma}$
$\forall \mathfrak{M} \in \mathbf{M}\,\forall I \in VAL\,\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = 1\,)$	$\mathbf{M} \models_i \boldsymbol{\Gamma}$

Table A.4: Truth at an index and related notions

Again, corresponding to $\models^p$, where $w \neq \# \in D_s$, we can define '$\mathfrak{M}, g, w \models_i^p \boldsymbol{\Gamma}$' in *Ty2* and various related notions of partial truth:

Partial truth at an index and related notions	Notation
$\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\,w = 1$ $\vee [\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\,w = \#)$	$\mathfrak{M}, g, w \models_i^p \boldsymbol{\Gamma}$
$\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\,w = 1$ $\vee [\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\,w = \#)$	$\mathfrak{M} \models_i^p \boldsymbol{\Gamma}$
$\forall I \in VAL\,\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = 1$ $\vee [\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = \#)$	$\mathbf{M}_{\mathcal{F}} \models_i^p \boldsymbol{\Gamma}$
$\forall \mathcal{F} \in \mathfrak{F}\,\forall I \in VAL\,\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = 1$ $\vee [\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = \#)$	$\mathfrak{F} \models_i^p \boldsymbol{\Gamma}$
$\forall \mathfrak{M} \in \mathbf{M}\,\forall I \in VAL\,\forall w \in D_s\,\forall g \in G\,\forall \boldsymbol{\varphi} \in \boldsymbol{\Gamma}\,(\,[\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = 1$ $\vee [\![\boldsymbol{\varphi}]\!]^{(\mathcal{F},I),g}\,w = \#)$	$\mathbf{M} \models_i^p \boldsymbol{\Gamma}$

Table A.5: Partial truth at an index and related notions

As above, when $\Gamma = \{\boldsymbol{\varphi}\}$ and $\rhd \in \{\models_i, \not\models_i, \models_i^p\}$, in order to avoid clutter we write $\mathfrak{M}, g, w \;\rhd\; \boldsymbol{\varphi}$ instead of $\mathfrak{M}, g, w \;\rhd\; \{\boldsymbol{\varphi}\}$ and we say that $\boldsymbol{\varphi}$ *is true in* (or *not true* or *false*) at index w in $\mathfrak{M}$ *relative to* g.

We will use $\wedge_i, \vee_i, \neg_i, \rightarrow_i$, etcetera) on the right-hand side of $\models_i$ as abbreviations of the following lambda terms (here $R : s \to s \to t$ is some accessibility relation between indices), which (modulo partiality) are the familiar intensional liftings of the connectives and the modal operator $\Box$ discussed in the literature (Heim and Kratzer 1998), (Heim & von Fintel 2011), (Shan 2002):

Symbol	Lambda term
$\wedge_i$	$\lambda p_{s\to t}, q_{s\to t}, w_s.\, p_s\, w \wedge q\, w$
$\vee_i$	$\lambda p_{s\to t}, q_{s\to t}, w_s.\, p\, w \vee q\, w$
$\neg_i$	$\lambda p_{s\to t}, w_s.\, \neg(p\, w)$
$\rightarrow_i$	$\lambda p_{s\to t}, q_{s\to t}, w_s.\, p\, w \to q\, w$
$\Box_i$	$\lambda p_{s\to t}, w_s.\forall w'(R\, w\, w' \to p\, w')$

Table A.6: Intensional liftings of logical connectives and operators

For any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in Con_{s\to t}$, any *Ty2* model $\mathfrak{M} = (\mathcal{F}, I)$, assignment $g \in G$ and $s \in CON_s$ such that $[\![s]\!]^{\mathfrak{M},g} = w$, where w is a fixed index $w \neq \# \in D_s \in \mathcal{F}$ let $\models_i$ be defined as follows:

$\mathfrak{M}, g, w \models_i \boldsymbol{\varphi} \wedge_i \boldsymbol{\psi}$	if and only if	$\mathfrak{M}, g \models \boldsymbol{\varphi}\, s \wedge \boldsymbol{\psi}\, s$
$\mathfrak{M}, g, w \models_i \boldsymbol{\varphi} \vee_i \boldsymbol{\psi}$	if and only if	$\mathfrak{M}, g \models \boldsymbol{\varphi}\, s \vee \boldsymbol{\psi}\, s$
$\mathfrak{M}, g, w \models_i \neg_i \boldsymbol{\psi}$	if and only if	$\mathfrak{M}, g \models \neg\, (\boldsymbol{\psi}\;\, s)$
$\mathfrak{M}, g, w \models_i \boldsymbol{\varphi} \rightarrow_i \boldsymbol{\psi}$	if and only if	$\mathfrak{M}, g \models \boldsymbol{\varphi}\, s \rightarrow \boldsymbol{\psi}\, s$
$\mathfrak{M}, g, w \models_i \Box_i \boldsymbol{\psi}$	if and only if	$\mathfrak{M}, g \models \Box_i\, \psi\, s$

Where $\mathbf{\Psi} \subseteq TERM_{s \to t}$ and $\boldsymbol{\varphi} \in TERM_{s \to t}$, when using $\models_i$, we will sometimes write $\mathfrak{M}, g, w \models_i \mathbf{\Psi} \to_i \boldsymbol{\varphi}$. In such cases, this is shorthand for $\mathfrak{M}, g, w \models_i \left(\bigwedge^i_{\boldsymbol{\psi} \in \mathbf{\Psi}} \psi \right) \to_i \boldsymbol{\varphi}$, where $\bigwedge^i_{\boldsymbol{\psi} \in \mathbf{\Psi}} \boldsymbol{\psi}$ indicates the application of $\wedge_i$ to each member of $\mathbf{\Psi}$. Similarly, we will sometimes write $\mathfrak{M}, g, w \models_i \Box_i \mathbf{\Psi}$ as shorthand for $\mathfrak{M}, g, w \models_i \Box_i \boldsymbol{\psi}$, for all $\boldsymbol{\psi} \in \mathbf{\Psi}$. We will usually omit the subscript on the logical symbols $\wedge_i, \vee_i, \neg_i, \to_i$ and on the modal operator $\Box_i$, since this should present no confusion. At some points we will use $\Box$ to refer to a doxastic or epistemic modal operator, with the context serving to clarify which kind of modal operator we have in mind, and with a generic agent assumed. We might also sometimes assume that we are dealing with a language in which we only have a doxastic modal operator (and no epistemic modal operator), or *vice versa*. Whenever we do this, we do not intend to be taking sides on what properties the accessibility relation has (so that what we discuss using $\Box$ will be true in the basic modal logic K). At some points we will use $\Diamond$ to refer to the dual of a doxastic or epistemic modal operator (i.e, $\Diamond := \neg \Box \neg$, with the context serving to clarify which kind of modal operator we have in mind, and with a generic agent assumed.

A.7 Doxastic and epistemic modal operators

For the purposes of introducing properties of accessibility relations we will temporarily use $\mathcal{B}_i$ as doxastic modal operator for an agent i and $\mathcal{K}_i$ as an epistemic modal operator for an agent i. On the classic account of (Hintikka 1962), such operators are treated as universal quantifiers over the indices that are compatible with that agent's beliefs (or their knowledge). The basic idea is that a sentence of the form 'a believes that S' ('a knows that S is true') is true iff the rendering of S is true in all her doxastically (or epistemically) accessible indices. Such operators can be defined directly in the object language of *Ty2*, via an accessibility relation, $R_{\mathcal{K}_i}$, for knowledge, and an accessibility relation, $R_{\mathcal{B}_i}$, for belief. We can require these relations satisfy certain properties (see the table below). For example, an accessibility relation for capturing what knowledge sentences are true of an agent might be reflexive, symmetric and transitive and an accessibility relation for capturing what belief sentences are true of an agent might be serial, transitive, and euclidean. A frame axiom (as we will see in more detail below) is a formula true in a class of *Ty2* models. Such properties of accessibility relations determine frame axioms (summarised in the table below), which require sentences involving *know* to have certain properties (we have used $(\cdot)^{\mathsf{D}}$ as an operator which applies to a modality to produce its dual). These are all properties which, at some time or other, have been mooted as sensible principles for knowledge, albeit disputed (see (Williamson 2002) and references therein).

There are a number of apparent shortcomings with the simple Hintikkian analysis which we we abstract from in this thesis but which are discussed in the literature. For example, the verb *believe*, unlike *know*, can be modified by degree modifiers like *partially/ sort of* or *fully/completely*, just like gradable adjectives such as *deaf*, and a more sophisticated analysis of *believe* would have something to say about how the degree semantics for gradable adjectives (Morzycki 2020) and the Hintikkian analysis combine with one another (see (Koev 2019) for a discussion of this). Furthermore, a more sophisticated semantics of belief would have something to say about whether utterances of sentences of the form *A believes S* require *A* to be certain that *S*, or only that A presumes S is likely (for discussion, see (Hawthorne et al. 2016)). Finally, the doxastically accessible indices, as defined above, are the indices which are objectively compatible with her state, and this doesn't take into account a subject's perception of her location in time or space, leading some theorists (Lewis 1979) to argue that the Hintikkian analysis fails to account for sentences involving *de se* pronouns. Nevertheless, since the simple Hintikkian account is clear, familiar and widely adopted, and since the problem of logical omniscience was first raised with respect to it, in this thesis we will concentrate on how the problem of logical omniscience arises in semantic theories which treat propositional attitudes as Hintikka did.

PROPERTY	FRAME AXIOM
KNOW	
Reflexivity:	$\mathcal{K}_i\varphi \rightarrow \varphi$
$\forall w(wR_{\mathcal{K}_i}w)$	
Symmetry:	$\varphi \rightarrow \mathcal{K}_i\,\mathcal{K}_i^{\mathsf{D}}\varphi$
$\forall w, w'(wR_{\mathcal{K}_i}w' \rightarrow w'R_{\mathcal{K}_i}w)$	
Transitivity:	$\mathcal{K}_i\varphi \rightarrow \mathcal{K}_i\,\mathcal{K}_i\varphi$
$\forall w, w', w''(wR_{\mathcal{K}_i}w' \wedge w'R_{\mathcal{K}_i}w'' \rightarrow wR_{\mathcal{K}_i}w'')$	
BELIEVE	
Serial:	$\mathcal{B}_i\varphi \rightarrow \mathcal{B}_i^{\mathsf{D}}\varphi$
$\forall w.\exists v.R_{\mathcal{B}_i}(w, w')$	
Transitivity:	$\mathcal{B}_i\varphi \rightarrow \mathcal{B}_i\,\mathcal{B}\varphi$
$\forall w, w', w''(wR_{\mathcal{B}_i}w' \wedge w'R_{\mathcal{B}_i}w'' \rightarrow wR_{\mathcal{B}_i}w'')$	
Euclidean:	$\mathcal{B}_i^{\mathsf{D}}\varphi \rightarrow \mathcal{B}_i\mathcal{B}_i^{\mathsf{D}}\varphi$
$\forall w, w', w''.(R_{\mathcal{B}_i}(w, w') \wedge R_{\mathcal{B}_i}(w, w'') \rightarrow R_{\mathcal{B}_i}(w', w''))$	

We will now consider whether a set $\mathbf{\Lambda} \subseteq TERM_{s\rightarrow t}$ is a logical consequence of a set $\mathbf{\Delta} \subseteq TERM_{s\rightarrow t}$ relative to the consequence relation $\models_i$ introduced above.

We adopt the approach of Popkorn (1994), who axiomatises the different assumptions corresponding to frame-level, model-level and state-level assumptions into the satisfaction relation itself. Since Popkorn (1994) is working in propositional modal logic, his definition is slightly different than the one we will adopt for the higher-order framework of *Ty2*. Where $\mathcal{M} \subseteq \mathbf{M}$ and $\mathbf{\Theta}, \mathbf{\Gamma}, \mathbf{\Delta}, \mathbf{\Lambda}$ are sets of *Ty2* terms of type $s \to t$, we write:

$;\ ;\ ;$	$\models_i$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathbf{M}(\mathfrak{M} \models \forall s\, \mathbf{\Lambda}\, s)$
$\mathcal{M}\ ;\ ;$	$\models_i$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\ \ (\mathfrak{M} \models \forall s\, \mathbf{\Lambda}\, s)$
$\mathcal{M}\ \ \mathbf{\Gamma}\ ;$	$\models_i$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\, \forall g (\mathfrak{M}, g \models \forall s \mathbf{\Gamma}\, s \Rightarrow \mathfrak{M}, g \models \forall s\, \mathbf{\Lambda}\, s)$
$\mathcal{M}\ ;\ \mathbf{\Delta}$	$\models_i$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\, \forall g\, \forall s (\mathfrak{M},\ g \models \mathbf{\Delta}\, s \Rightarrow \mathfrak{M},\ g \models \mathbf{\Lambda}\, s)$
$\mathcal{M}\, \mathbf{\Gamma}\, \mathbf{\Delta}$	$\models_i$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\, \forall g \in G\, ((\mathfrak{M}, g \models \forall s\, \mathbf{\Gamma} s)$
				$\Rightarrow \forall s \in D_s\, (\mathfrak{M}, g \models \mathbf{\Delta}\, s \Rightarrow \mathfrak{M}, g \models \mathbf{\Lambda}\, s))$

Note that when $\mathcal{M}\ ;\ \mathbf{\Delta} \models_i\ \mathbf{\Lambda}$ then (8.10) holds:

$$\begin{aligned} &\forall \mathfrak{M} \in \mathcal{M}\, \forall g\, \forall s \big(\{s \in S \mid [\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\, s = 1 \text{ for all } \boldsymbol{\varphi} \in \mathbf{\Delta}\} \\ &\subseteq \{s \in S \mid [\![\boldsymbol{\psi}]\!]^{\mathfrak{M},g}\, s = 1 \text{ for all } \boldsymbol{\psi} \in \mathbf{\Lambda}\}\big) \end{aligned} \tag{8.10}$$

When $\mathcal{M}\ ;\ \mathbf{\Delta} \models_i\ \mathbf{\Lambda}$ and $\mathcal{M}\ ;\ \mathbf{\Lambda} \models_i\ \mathbf{\Delta}$ we say that $\mathbf{\Lambda}$ and $\mathbf{\Delta}$ are *i-logically equivalent* in the class $\mathcal{M}$, and (8.11) holds:

$$\begin{aligned} &\forall \mathfrak{M} \in \mathcal{M}\, \forall g\, \forall s \big(\{s \in S \mid [\![\boldsymbol{\psi}]\!]^{\mathfrak{M},g}\, s = 1 \text{ for all } \boldsymbol{\varphi} \in \mathbf{\Delta}\} \\ &= \{s \in S \mid [\![\boldsymbol{\varphi}]\!]^{\mathfrak{M},g}\, s = 1 \text{ for all } \boldsymbol{\psi} \in \mathbf{\Lambda}\}\big) \end{aligned} \tag{8.11}$$

The hierarchy above can also be replicated with $\models_i^p$:

; ; ;	$\models_i^p$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathbf{M}\ \ (\mathfrak{M} \models^p \forall s\, \mathbf{\Lambda}\, s)$
$\mathcal{M}$; ;	$\models_i^p$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\ \ (\mathfrak{M} \models^p \forall s\, \mathbf{\Lambda}\, s)$
$\mathcal{M}\ \mathbf{\Gamma}$;	$\models_i^p$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\, \forall g (\mathfrak{M}, g \models \forall s \mathbf{\Gamma}\, s \Rightarrow \mathfrak{M}, g \models^p \forall s\, \mathbf{\Lambda}\, s)$
$\mathcal{M}$; $\mathbf{\Delta}$	$\models_i^p$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\, \forall g\, \forall s (\mathfrak{M},\ g \models \mathbf{\Delta}\, s \Rightarrow \mathfrak{M},\ g \models^p \mathbf{\Lambda}\, s)$
$\mathcal{M}\ \mathbf{\Gamma}\ \mathbf{\Delta}$	$\models_i^p$	$\mathbf{\Lambda}$	if and only if	$\forall \mathfrak{M} \in \mathcal{M}\, \forall g \in G\, ((\mathfrak{M}, g \models \forall s\, \mathbf{\Gamma} s)$ $\Rightarrow \forall s \in D_s\, (\mathfrak{M}, g \models \mathbf{\Delta}\, s \Rightarrow \mathfrak{M}, g \models^p \mathbf{\Lambda}\, s))$

If any of $\mathbf{\Theta}, \mathbf{\Gamma}, \mathbf{\Delta}, \mathbf{\Lambda}$ are singleton sets containing some unique formula φ and occurring either to the right or to the left of $\models_i$, then we write φ and not $\{\varphi\}$.

Clearly, $\top$ and $\bot \rightarrow \bot$ will be valid in the class of all *Ty2* models, so that

$$\forall \mathfrak{M} \in \mathbf{M}\, (\mathfrak{M}\ \models\ \top)$$

But the fact that all *Ty2* models validate $\top$ doesn't give us any distinctive information about particular sub-classes of the class of all *Ty2* models and it may not be possible in some cases to determine the class of axioms which are valid in a given class of *Ty2* models. When this is possible, however, and we are able to determine a class of non-trivial axioms which are valid in a given class $\mathcal{M}$ of models and at every index of those models, where $\mathbf{\Xi}$ is a non-trivial set of *Ty2* terms of type $s \rightarrow t$, we will have:

$$\forall \mathfrak{M} \in \mathcal{M}\, (\mathfrak{M}\ \models \forall s\, \mathbf{\Xi}\, s)$$

When such non-trivial model axioms are valid in a class of models $\mathcal{M}$, we have that $\mathcal{M}\ \mathbf{\Gamma}\ \mathbf{\Delta} \models_i\ \mathbf{\Lambda}$ holds if and only if

$$\forall \mathfrak{M} \in \mathcal{M}(\mathfrak{M} \models \forall s\, \mathbf{\Xi}\, s) \Rightarrow$$
$$\forall g \in G\, ((\mathfrak{M}, g \models \forall s\, \mathbf{\Gamma} s) \Rightarrow \forall s \in D_s\, (\mathfrak{M}, g \models \mathbf{\Delta}\, s \Rightarrow \mathfrak{M}, g \models \mathbf{\Lambda}\, s)))$$

Popkorn (1994)'s notation has been found useful in the literature Mendler and De Paiva (2005), Mendler and Scheele (2008), Mendler, Scheele, and Burke (2021), though it is not universally deployed.[12] The advantage of Popkorn (1994)'s notation is that it allows us, in principle, to factor into our definition of logical consequence various frame, model and state axioms so as to be precise about different notions of validity which may be at stake. For example, (8.12) holds, since (8.13) holds:

$$\mathbf{M}\, ;\, \mathbf{\Psi}\, \models_i \boldsymbol{\varphi} \;\Rightarrow\; \mathbf{M}\, ;\, ;\, \models_i \mathbf{\Psi} \rightarrow \boldsymbol{\varphi} \tag{8.12}$$

$$\forall \mathfrak{M}\, \forall g\, \forall s(\mathfrak{M},\, g \models \mathbf{\Psi}\, s \Rightarrow \mathfrak{M},\, g \models \boldsymbol{\varphi}\, s) \Rightarrow \forall \mathfrak{M}\, \forall g\, \forall s(\mathfrak{M},\, g \models \mathbf{\Psi}\, s \rightarrow \boldsymbol{\varphi}\, s) \tag{8.13}$$

However, (8.14) does not hold, because (8.15) fails:

$$\mathbf{M}\, ;\, \mathbf{\Psi}\, ;\, \models_i \boldsymbol{\varphi} \;\not\Rightarrow\; \mathbf{M}\, ;\, ;\, \models_i \mathbf{\Psi} \rightarrow \boldsymbol{\varphi} \tag{8.14}$$

$$\forall \mathfrak{M}\, \forall g(\mathfrak{M}, g \models \forall s \mathbf{\Psi}\, s \Rightarrow \mathfrak{M}, g \models \forall s\, \boldsymbol{\varphi}\, s) \Rightarrow \forall \mathfrak{M}\, \forall g\, \forall s(\mathfrak{M},\, g \models \mathbf{\Psi}\, s \rightarrow \boldsymbol{\varphi}\, s) \tag{8.15}$$

(8.15) fails for the following reason. Suppose we set $\mathbf{\Psi}, \boldsymbol{\varphi}$, respectively to:

[12] One further possible advantage of Popkorn (1994)'s notation for natural language semantics, which we will not explore here is that it could be used to represent axioms which hold of contexts of utterance (Kaplan 1989), or axioms which hold of a restricted class of worlds which are designated in some model. In this way, (Kaplan 1989)'s distinction between validity in a context of utterance and validity in a model, to give one just example, could be perspicuously represented.

$$\boldsymbol{\Psi} = \{\boldsymbol{A} : s \rightarrow t\} \qquad\qquad \boldsymbol{\varphi} = \Box\boldsymbol{A} \tag{8.16}$$

Clearly, for any model $\mathfrak{M}$ and assignment g, we have (8.17), so that (8.18) holds:

$$\forall\mathfrak{M}\,\forall g(\mathfrak{M}, g \models \forall s\, \boldsymbol{A}\, s \Rightarrow \forall s\,(\mathfrak{M}, g \models\ \forall s'(R\, s\, s' \rightarrow \boldsymbol{A}\, s'))) \tag{8.17}$$

$$\mathbf{M}\,;\ \ \boldsymbol{A} : s \rightarrow t\ ;\ \ \models_i \Box\boldsymbol{A} \tag{8.18}$$

But there might be a model $\mathfrak{M}$, an assignment g and a state w such that $\mathfrak{M}, g, w \not\models_i$ $\boldsymbol{A} \rightarrow \Box\boldsymbol{A}$, for $\boldsymbol{A}$ might be locally true at w, and not true at any other state w' in the model. Thus (8.19) fails and so (8.14) fails, too:

$$\mathbf{M}\ ;\ ;\ \models_i \boldsymbol{A}\ \rightarrow \Box\boldsymbol{A} \tag{8.19}$$

Similarly (8.20) cannot hold, since by (8.12) we would then have (8.19):

$$\mathbf{M}\ ;\ \boldsymbol{A} : s \rightarrow t\ \ \models_i \Box\boldsymbol{A} \tag{8.20}$$

B Labelled binary trees

In this thesis we will present our compositional derivations via labelled binary trees (see (4) below). Formally, the nodes of these trees consist of triples. For example, a linguistic expression such as *John* could be a triple $John\ :\ (j'\ :\ e)\ :\ DP$, consisting of a string, a lambda term j' and a syntactic category DP. We will not be presenting a formal syntax of our examples, but in principal, the syntax of most of our examples could be captured by a simple categorial grammar (essentially that presented in (Moortgat 2014) but excluding product types), in which syntactic categories are built up recursively via the following type language (or perhaps a simple enrichment of this language which takes into account syntactic agreement, along the lines of (Steedman 2019)):

$$A,\, B ::= p \mid A\backslash B \mid B/A$$

Here lower case p ranges over atomic syntactic categories, and A, B over arbitrary syntactic categories. The atomic syntactic categories could include DP and S (representing the basic syntactic categories Determiner Phrase and Sentence) and possibly other categories. The slashes \ and / are interpreted as follows: $A\backslash B$ combines with A to its left to form a B and A/B combines with B to its right to form an A. Given this kind of setup, ***John sleeps*** could be given the following representation:

(4)

$$John\,sleeps : (sleep' : e \rightarrow t)(j' : e) : S$$

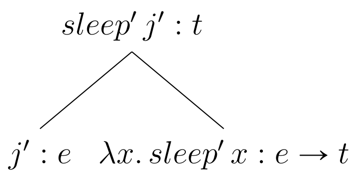
$$John : (j' : e) : DP \quad sleeps : (sleep' : e \rightarrow t) : DP\backslash S$$

Since we are not presenting a formal syntax of our examples, we will ignore the syntactic component of our trees, and we will omit the first member (e.g, the string $John$) and the third member (e.g, the syntactic category DP) of linguistic expressions in our binary trees.

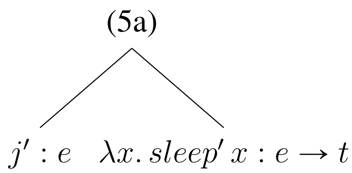
We will label the nodes of our trees with lambda terms or with the types of lambda terms and nodes of trees will often be labelled with example numbers, as in the tree (5c):

(5) a. $sleep'\, j'$

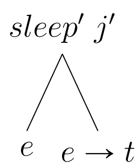
b.



c.

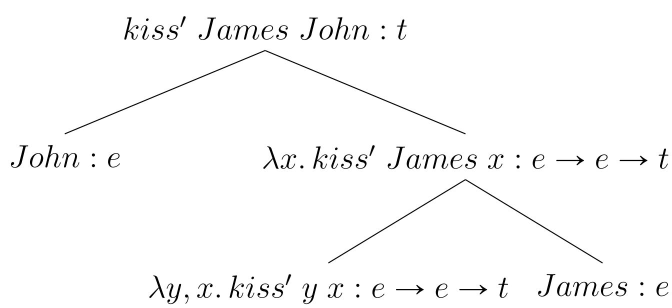


d.

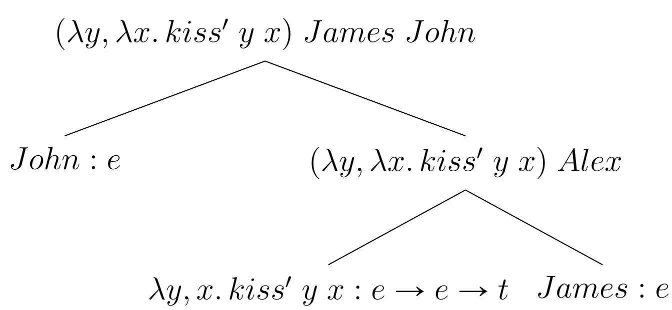


Compare the trees (6a) and (6a):

(6) a.



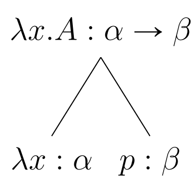
b.



In (6a) each node is labelled with a lambda term, but crucially β reduction has been performed wherever possible. Instead, in (6b), nodes are labelled with lambda terms and no instance of β reduction has been performed, though the root is labelled with a lambda term which can be reduced to normal form. We will tend to present trees in the first style applying β-reduction as we move to the root of a tree.

We follow the usual convention in linguistics of abbreviating parts of trees by using triangles, in order to avoid presenting certain details of the derivations that take up unnecessary space. We follow the general convention of *type-driven* composition (Klein and Sag 1985), (Heim and Kratzer 1998) that, in the case of a binary branching node α consisting of daughters β, γ, either β is a function which applies to γ, or γ is a function which applies to β. For this reason, we often omit the lambda term to the left of the colon and label nodes with types only, since we can infer the lambda term given the general convention of type-driven interpretation. One exception to type-drive composition is that we allow trees such as the tree in (7):

(7)



In such trees, one of the daughters is labelled with $\lambda x : \alpha$ and the other is labelled as $A : \beta$, where α is the type of the variable x and λx cannot combine with A via function application, satisfying the convention of type-driven composition.

C The formal details of (Berto and Jago 2019)

Formally, (Berto and Jago 2019)'s theory consists of three main components: a definition of *epistemically possible state*; a definition of *trivial logical consequence*; and a formal logic including doxastic and epistemic modal operators. These components will now be discussed in turn and our discussion will be limited to an exposition of the theory without criticism (our main criticisms of the theory having been given in 2.3.2).

C.1 Epistemically possible states

To implement their theory, (Berto and Jago 2019) take proof rules such as disjunction introduction to be relations between states. Roughly, proof rules relate a state in which the premisses of the rule are true to a state in which the conclusion of the rule is true, in addition to its premisses. Their aim is to use relations between worlds to give a definition of the notion of an epistemically possible state. (Berto and Jago 2019) define the notion of an epistemically possible state via the proof rules of the following sequent calculus, which consists of id plus the strong inference rules from (Buss 1998), modified so that formulas appearing in the lower sequent of a rule appear in all upper sequents:

$$\frac{\Gamma,\ \neg A \vdash\ A,\Delta}{\Gamma, \neg A\ \vdash\ \Delta}\ \neg L$$

$$\frac{\Gamma,\ A \vdash\ \neg A,\Delta}{\Gamma\ \vdash\ \neg A,\ \Delta}\ \neg R$$

$$\frac{\Gamma\ \ A\vee B, A \vdash\ \Delta \qquad \Gamma, A\vee B, B \vdash\ \Delta}{\Gamma, A\vee B \vdash \Delta}\ \vee L$$

$$\frac{\Gamma,\ \vdash\ A,B,A\vee B,\Delta}{\Gamma,\ \vdash\ A\vee B,\Delta}\ \vee R$$

$$\frac{\Gamma\ \ A\wedge B,A,B \vdash\ \Delta}{\Gamma, A\wedge B \vdash\ \Delta}\ \wedge L$$

$$\frac{\Gamma\ \vdash\ A,A\wedge B,\ \Delta \qquad \Gamma\ \vdash\ B,A\wedge B,\ \Delta}{\Gamma \vdash A\wedge B,\Delta}\ \wedge R$$

$$\frac{\Gamma,A\to B \vdash\ A,\Delta \qquad \Gamma,A\to B,B \vdash \Delta}{\Gamma, A\to B \vdash \Delta}\ \to L$$

$$\frac{\Gamma,A \vdash\ B,A\to B,\Delta}{\Gamma \vdash\ A\to B,\Delta}\ \to R$$

$$\frac{}{\Gamma,A\ \ \vdash\ A,\Delta}\ id$$

Given a state w, the set of formulas verified by w is $|w^+|$ and the set of formulas falsified by w is $|w^-|$. Then a sequent, such as the following

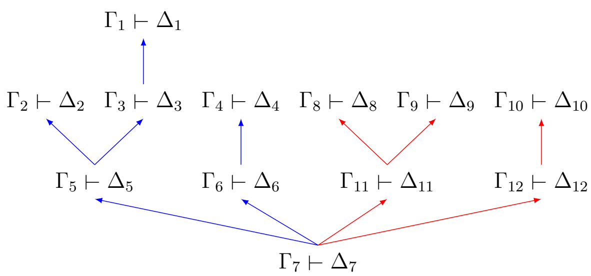
$$\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_2 \vdash \Delta_2}$$

can be interpreted as a link between two states w_2 and w_1, such that $|w_1^+| = \Gamma_1 \cup \Gamma_2$, $|w_1^-| = \Delta_1 \cup \Delta_2$ and $|w_2^+| = \Gamma_1 \cup \Gamma_2$ and $|w_2^-| = \Delta_2$. In the case of proof rules with two upper sequents there will therefore be a link between the state corresponding to the lower sequent and two states. Using this technique, we can code up proofs in the sequent calculus.

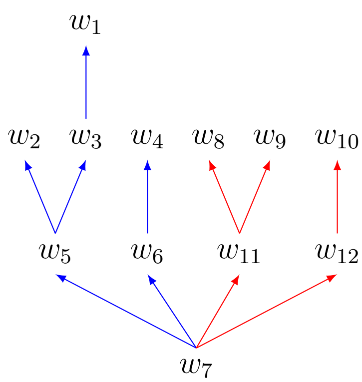
The graph corresponding to sequent calculus proof is a tree, so that, replacing each node of the tree with its corresponding state, we have a tree consisting of nodes labelled by states (a *state diagram*). By way of example, consider the trees in (8):

the first is a representation of two proofs of $\Gamma_7 \vdash \Delta_7$, where nodes collected by edges of the same colour belong to the same proof; the second is the corresponding state diagram of this proof:

(8)



(9)



Subtrees of state diagrams are called *state proofs* when they meet the following conditions:

$$|w^+| \cap |w_1^-| \neq \varnothing, \text{ when } w \text{ is a leaf node in the state diagram} \tag{8.21}$$

Every non-leaf node of the state diagram has at most two edges leading away from it (8.22)

If the edges (w_1, w_3) and (w_2, w_3) are contained in the state diagram, then the proof system contains a rule instance of the following kind: (8.23)

$$\frac{|w_1^+| \vdash |w_1^-| \qquad |w_2^+| \vdash |w_2^-|}{|w_3^+| \vdash |w_3^-|}$$

(Berto and Jago 2019: p.201) then propose the following informal definition of an epistemically possible state:

> (EP) State w is epistemically possible just in case w isn't the root of any small state proof.

Since 'small' is a vague predicate, the notion of epistemically possible state is vague. The inherent vagueness of this notion is a property which (Berto and Jago 2019) endorse. The idea behind (EP) is that if a w is the root of a large state proof then unwinding the contradiction in it requires many applications of proof rules and so the state verifies a subtle contradiction. By contrast, if w is the root of a small state proof then unwinding the contradiction in it requires few applications of proof rules, and so w verifies an obvious contradiction.

Formally, let $|G|$ (the *length* of G) be the number of non-leaf vertices the state proof G contains and $G_w = \{ H \mid H \text{ is a state proof whose root is } w \}$. We define $\#w$ as follows:

$$\#w = \begin{cases} min\{ |G| \quad | \ G \in G_w\} & \text{if}\, G_w \neq \varnothing \\ \\ \omega & \text{otherwise} \end{cases}$$

Intuitively, $\#w$ returns the length of the shortest state proof in G whose root is w. (Berto and Jago 2019) then partially orders states via the length of the state proofs associate with them. Let $w \leqslant u$ iff $\#w \leqslant \#u$. This is an ordering on states depending on how easy it is to demonstrate that they contain contradictions. Intuitively, the states assigned smaller numbers by $\#$ are more obviously contradictory and states assigned larger numbers by $\#$ contain contradictions which are harder to uncover (or which cannot be uncovered, if there is no proof associated with a given state), with the latter class of states constituting the epistemically possible states.

C.2 Trivial logical consequence

Setting $||\phi||^+$ to the set of states which verify ϕ and $||\phi||^-$ to the set of states which falsify ϕ, (Berto and Jago 2019) define the 'content' of a deduction as $\Gamma \vdash \psi$ as the set of epistemically possible states in $||\Gamma||^+ \cap ||\psi||^-$, where $||\Gamma||^+ = \{w \mid w \in ||\phi||^+ \text{ for all } \phi \in \Gamma\}$ and $||\Gamma||^- = \{w \mid w \in ||\phi||^- \text{ for some } \phi \in \Gamma\}$. In this way, $||\{\phi, \phi \rightarrow \psi\}||^+ \cap ||\psi||^-$ will come out as empty, since the state proof associated with modus ponens is small, and so the root of the world proof is not an epistemically possible state. Likewise, $||\phi \vee \neg\psi||^-$ will come out as empty, since the state proof associated with $\phi \vee \neg\phi$ is small. (Berto and Jago 2019) then define a deduction $\Gamma \vdash \Delta$ as trivial if $||\Gamma||^+ \cap ||\Delta||^- \neq \emptyset$. To this end, they define a relation of *trivial consequence* for a class of models and the following propositional language:

Definition 34 (Language) Given a set P of propositional atoms the language $\mathcal{L}$ is inductively defined from:

$$\phi \quad ::= \quad p \quad \neg\phi \quad \phi \wedge \phi \quad \Delta\phi \quad B\phi \quad K\phi$$

where

- $p \in P$
- Δ reads "it is determinately the case that" and its dual $\neg\Delta\neg$ reads "it is indeterminate whether"
- K reads "the agent knows that"

$\phi \circ \phi$ for $\circ \in \{\vee, \rightarrow\}$ are defined in the usual way from $\neg$ and $\wedge$.

According to the notion of trivial consequence they define, 'a trivial consequence holds when the truth of the premisses guarantee falsity avoidance for the conclusion across appropriate models.' The models relative to which they define the notion of trivial consequence are tuples of the form $\mathfrak{M} = (W, V^{+}, V^{-}, \#)$, such that:

- W is a set of states
- $V^{+} : W \rightarrow 2^{\mathscr{L}}$ and $V^{-} : W \rightarrow 2^{\mathscr{L}}$ are functions assigning sentences to states.
- $\# : W \rightarrow N \cup \{\omega\}$, assigns an ordinal number to $w \in W$.

$min\{\#w \mid w \in W\}$, is called the *rank of* $\mathfrak{M}$ and as above, is intended as a measure of how trivially impossible a world in $\mathfrak{M}$ is. For example, when $\mathfrak{M}$ has rank 0, for instance, then, for some $w \in W, V^{+}w \cap V^{-}w \neq \emptyset$, and so $\mathfrak{M}$ contains a trivially impossible world. But when $\mathfrak{M}$ has a large rank, then it contains no trivially impossible world. Consider the set of sequent rules above defined over the language $\mathscr{L}$. A proof is associated with $w \in W$ iff $V^{+}w \vdash V^{-}w$ is at the root of the tree and the size of a proof is as above (the number of its non-leaf nodes). Consider the class of models **M** such that $\#w$ is the size of the smallest proof associated with w, if there is one, and ω otherwise (so that $\#$ receives the interpretation discussed above). The members of **M** are intended to give different sharpenings of the set of epistemically possible states.

The rank of pointed models $(\mathfrak{M}, w)$, where $\mathfrak{M} \in \mathbf{M}, w \in W$ is the rank of $\mathfrak{M}$. A notion of consequence can then be defined for pointed models $(\mathfrak{M}, w)$, where $\mathfrak{M} \in \mathbf{M}, w \in W$. $\mathfrak{M}, w \Vdash A$ iff $A \in V^{+}w$ and $\mathfrak{M}, w \dashv\!\mid A$ iff $A \in V^{-}w$ in $\mathfrak{M}$.

Then for set of formulas Γ, $\mathfrak{M}$, $\mathfrak{M} \Vdash \Gamma$ iff $\mathfrak{M} \Vdash A$ for all $A \in \Gamma$ and $\mathfrak{M} \dashV \Gamma$ iff $\mathfrak{M} \dashV \Gamma$ for at least one $A \in \Gamma$. From these notions we can define a relation, $triv_n$, for all $n \in \mathbb{N}$. $triv_n$ provides a non-transitive logical entailment relation that, for each $n \in \mathbb{N}$, captures different sharpenings of the notion of a trivial consequence of a set of premisses:

Definition 35 (Trivial consequence) For all $n \in \mathbb{N}$, $A \in \mathscr{L}$ and $\Gamma \subseteq \mathscr{L}$, $triv_n(\Gamma, A)$ (in prose: A is a trivial consequence of Γ), if and only if, for all $\mathfrak{M} \in \mathbf{M}$ of rank $r > n$, $\mathfrak{M} \Vdash \Gamma$ only if $\mathfrak{M} \not\dashV A$.

(Jago 2014a) proves that $triv_n$ has the following properties, for $n \in \mathbb{N}$:

(a) If $triv_n(\Gamma, A)$ then $triv_{n+1}(\Gamma, A)$.

(b) If $triv_n(\Gamma, A)$ and $\Gamma \subseteq \Delta$ then $triv_n(\Delta, A)$.

(c) $triv_n(\Gamma, A)$ only if Γ classically entails A.

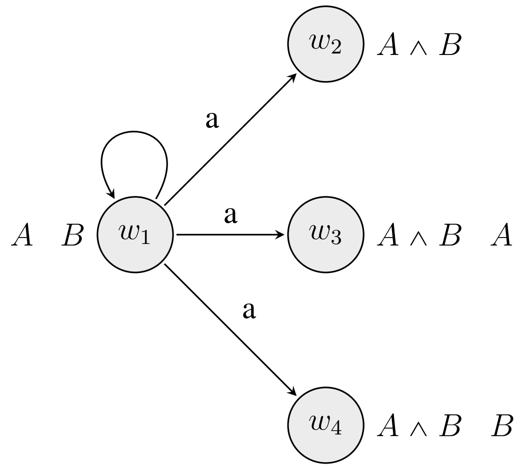
(d) $triv_n$ is reflexive

(e) $triv_o(\Gamma, A)$ if and only if $A \in \Gamma$

(f) For $n \geqslant 1,$ is non-transitive and does not satisfy cut in the sense that, if $triv_n(\Gamma, A)$ and $triv_n(\Gamma \cup \{A\}, B)$ then $triv_n(\Gamma, B)$

C.3 Formal logic

Since (Berto and Jago 2019) allow incomplete states to be epistemically possible, they allow for states in which $A \wedge B$ is verified but in which its conjuncts are neither verified nor falsified. This allows for the kind of situation represented in the diagram below:



Here the label a connects two states in the relation of being an epistemic alternative for a certain agent, and the agent is represented as knowing $A \wedge B$ but not knowing either A nor B.

Nevertheless, according to (Berto and Jago 2019)'s theory of epistemic oversights described above, if an agent knows $A \wedge B$ then it cannot be determinate that she fails to know A and it cannot be determinate that she fails to know B, since these are trivial consequences of $A \wedge B$. As a consequence, (Berto and Jago 2019) argue that whether a state is an epistemic or doxastic alternative of another is itself vague, so that epistemic and doxastic accessibility relations are themselves vague. However, they require that it be determinate that some incomplete state is epistemically accessible in order to guarantee that an agent's belief state is not logically closed. Indeterminacy in the accessibility relation is understood in terms of there being a range of different accessibility relations for each agent, each of which is a 'sharpening' of the accessibility relation for that agent which forms part of the model. If a formula is true in all the agent's accessible worlds given the accessibility relation which forms part of the model, then it is true (simpliciter) that that agent knows the formula, whereas if a formula is true in all an agent's accessible worlds given all sharpenings of the accessibility relation in the model, then it is determinately true that that agent knows the formula. We will now define the models which (Berto and Jago 2019) use, and then the notion of an alternative sharpening of an accessibility relation can be defined with more precision:

Definition 36 (Epistemic model) An epistemic model for k agents is a tuple $\mathfrak{M} = (W^P, W^I, V^+, V^-, \#, f_1, ..., f_k)$, where W^P and W^I are sets of worlds (thought of as sets of possible and impossible worlds, respectively), and $V^+, V^-, \#$ are as above. $W = W^P \cup W^I$ and $f_1, ..., f_k$ are functions from $W \to 2^W$. The rank of $\mathfrak{M}$ is $\min\{\#w \mid w \in W\}$.

Given some $f_i \in \mathfrak{M}$ and some $A \in \mathscr{L}$, we define f_i^A:

Definition 37 (A-variant of f_i)

$$f_i^A w = \begin{cases} (f_i w \cap \{w \mid A \in V^+ w\}) \cup (f_i w \cap W^P) & \text{if } f_i w \subseteq \{w \mid A \notin V^- w\} \\ f_i w & \text{otherwise} \end{cases}$$

Let $f_i^{\mathscr{L}} = \{f_i\} \cup \{f_i^A \mid A \in \mathscr{L}\}$

Definition 38 (Alternative sequences)

Given an epistemic model $\mathfrak{M}$, and where $(a_1, ..., a_n)$ represents a sequence consisting of $a_1, ..., a_n$,

$$\alpha_{\mathfrak{M}} = \{(g_1, ..., g_k) \mid g_i \in f_i^{\mathscr{L}}, i \leqslant k\}$$

is called the set of *alternative sequences*.

We use the notation $\vec{g}$ to denote a member of $\alpha_{\mathfrak{M}}$, and for any i, $\vec{g}^i = f_i$, when $\vec{g} = (f_i, ..., f_k)$.

Definition 39 Let $\mathfrak{M}$ be an epistemic model, $w \in W$ and $\vec{g} \in \alpha_{\mathfrak{M}}$. Then $\mathfrak{M}, w \Vdash_{\vec{g}} \phi$ and $\mathfrak{M}, w \dashv\!\mid_{\vec{g}} \phi$, are defined as follows:

For $w \in W^P$:

$$\mathfrak{M}, w \Vdash_{\vec{g}} p \text{ iff } p \in V^+w$$

$$\mathfrak{M}, w \Vdash_{\vec{g}} \neg\phi \text{ iff } \mathfrak{M}, w \nVdash_{\vec{g}} \phi$$

$$\mathfrak{M}, w \Vdash_{\vec{g}} \phi \wedge \psi \text{ iff } \mathfrak{M}, w \Vdash_{\vec{g}} \phi \text{ and } \mathfrak{M}, w \Vdash_{\vec{g}} \psi$$

$$\mathfrak{M}, w \Vdash_{\vec{g}} \phi \vee \phi \text{ iff } \mathfrak{M}, w \Vdash_{\vec{g}} \phi \text{ or } \mathfrak{M}, w \Vdash_{\vec{g}} \psi$$

$$\mathfrak{M}, w \Vdash_{\vec{g}} \phi \rightarrow \psi \text{ iff } \mathfrak{M}, w \nVdash_{\vec{g}} \phi \text{ or } \mathfrak{M}, w \Vdash_{\vec{g}} \psi$$

$$\mathfrak{M}, w \Vdash_{\vec{g}} K_i\phi \text{ iff } \mathfrak{M}, u \Vdash_{\vec{g}} \phi \text{ for all } u \in \vec{g}^i w$$

$$\mathfrak{M}, w \Vdash_{\vec{g}} \triangle\phi \text{ iff } \mathfrak{M}, w \Vdash_{\vec{h}} \phi \text{ for all } h \in \alpha_{\mathfrak{M}}$$

$$\mathfrak{M}, w \dashv\!\mid_{\vec{g}} \phi \text{ iff } \mathfrak{M}, w \nVdash_{\vec{g}} \phi$$

For $w \in W^I$:

$$\mathfrak{M}, w \Vdash_{\vec{g}} \phi \text{ iff } \phi \in V^+w$$

$$\mathfrak{M}, w \dashv\!\mid_{\vec{g}} \phi \text{ iff } \phi \in V^-w$$

Definition 40 (n-entailment) Given an epistemic model $\mathfrak{M} = (W^P, W^I, V^+, V^-, \#, f_1, ..., f_k)$ and $w \in W^P$, we can form a pointed model $\mathfrak{M}' = (\mathfrak{M}, w)$. Then:

$$\mathfrak{M}' \Vdash A \text{ iff } \mathfrak{M}, w \Vdash_{(f_1,...,f_n)} A$$

$\mathfrak{M}' \Vdash \Gamma$ iff $\mathfrak{M}' \Vdash A$ for every $A \in \Gamma$. For any $n \in \mathbb{N}$, logical n-entailment is defined as follows: $\Gamma \models_n A$ iff, for every pointed model $\mathfrak{M}$ of rank $\geqslant n$, $\mathfrak{M} \Vdash \Gamma$ only if $\mathfrak{M} \Vdash A$.

(Jago 2014b) establishes a number of theorems. First, he proves the following theorem, showing that $\models_n$ preserves classical entailment for any $n \in \mathbb{Z}^+ \cup \{\omega\}$:

Theorem 1 For all $n \in \mathbb{Z}^+ \cup \{\omega\}$: if Γ classically entails A, then $\Gamma \models_n A$

Proof: See (Jago 2014a)

We earlier discussed the principle (TRIV). (Jago 2014b) demonstrates the following theorem, which shows that, for any n, epistemic models satisfy (TRIV):

Theorem 2 For all $n \in \mathbb{Z}^+ \cup \{\omega\}$, if $triv_n(\Gamma, A)$, then $\{\triangle K_i\phi \mid \phi \in \Gamma\} \models_n \neg\triangle\neg K_i\psi\}$

Proof: See (Jago 2014a)

Corollary 1 For all $n \in \mathbb{Z}^+ \cup \{\omega\}$, if $triv_n(\Gamma, A)$, then $\{\triangle K_i\phi \mid \phi \in \Gamma\} \cup \{\neg K_i\psi\} \models_n \nabla K_i\psi\}$

Proof: See (Jago 2014a)

Corollary 2 For all $n \in \mathbb{Z}^+ \cup \{\omega\}$, if $n \geqslant 3$, then $\models_n \neg\triangle\neg k_i(A \vee \neg A)$ and $\neg K_i(A \vee \neg A) \models_n \nabla K_i(A \vee \neg A)$

Proof: See (Jago 2014a)

This theorem thus ensures that, even though it may be true that someone suffers from an epistemic blindspot, it can never be determinately true that someone has an epistemic blindspot.

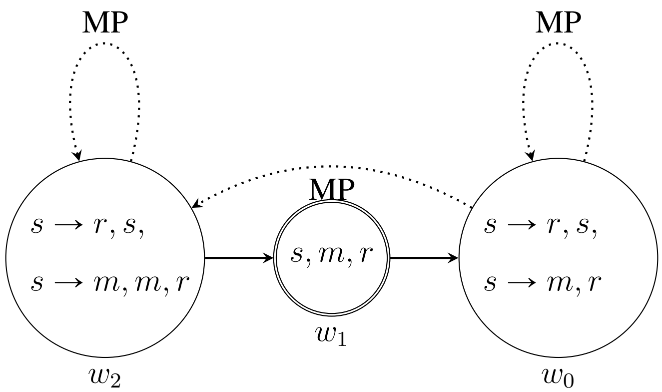


Figure D.1: Initial model

D An example of R_k-accessibility

Example 1 (taken from (Solaki et al. 2019)).

Consider the model represented in figure (D.1), where:

- a dashed arrow from w to u means that u is an MP-expansion of w
- a thick arrow from w to u means that u is more plausible than w.
- the thicker node w_1 indicates that $w_1 \in W^P$

The model represents the worlds which an agent is entertaining. Here s stands for "the odds of survival one month after surgery are 90%", m for "mortality within one month of surgery is 10%" and r for "the surgery is safe". $W^P = \{w_1\}$ and $W^I = \{w_2, w_0\}$. Note only the propositional atoms are written down for w_1, since truth at a state can be calculated recursively for $w \in W^P$, whereas all the propositional formulas true at for $w \in W^I$ are written down in the diagram. The ordering on worlds is such that $ord(w_2) = 2$, $ord(w_1) = 1$, $ord(w_0) = 0$. The set of resources

R in the model involved are time and memory and the cost of an application of Modus Ponens with respect to these resources is $C(MP) = (3, 2)$. The agent's capacity is $cp = (15, 9)$. Note that m doesn't hold in w_0, the most plausible world, even though $s \to r$ and $s \to m$ hold in all worlds. This represents that the agent modelled has not made the inference to m from $s \to m$ and s, though she has made the inference to r from $s \to r$ and s.

To reach the R^k accessible model we follow the steps articulated above:

Step 1 We compute $\{v^{MP} \mid v \in P_{\geqslant}(w_1)\}$, which is $\{\{w_1\}, \{w_0, w_2\}\}$. Consequently, $\mathcal{C}(\{\{w_1\}, \{w_0, w_2\}\}) = \{\{w_1, w_0\}, \{w_1, w_2\}\}$. So $P^{MP}(w_1) = \{w_1, w_0\}$ or $P^{MP}(w_1) = \{w_1, w_2\}$

1 Suppose $P^{MP}(w_1) = \{w_1, w_0\}$:

Step 2 $W^{MP} = W$

Step 3 Since $w_2 \notin P^{MP}(w_1) \cup P_{\geqslant}(w_1)$, $ord^{MP}(w_2) = ord(w_2) = 2$. Next $w_1 \in P^{MP}(w_1)$ and $w_1 \in w_1^{MP}$ and $w_1 \in w^{MP}$, so $ord^{MP}(w_1) = ord(w_1) = 1$. Finally $w_0 \in P^{MP}(w_1)$ and $w_0 \in w^{MP}$, so $ord^{MP}(w_0) = ord(w_0) = 0$. The model transformation in this case does not alter the initial model. The cognitive capacity of the MP-accessible models is then reduced by the cognitive cost of applying MP, so that $cp = (12, 7)$.

2 Suppose now that $P^{MP}(w_1) = \{w_1, w_2\}$

Step 2 $W^{MP} = W \backslash \{u \in \{w_1, w_0\} \mid u \notin \{w_1, w_2\}\}\} = \{w_1, w_2\}$.

Step 3 $ord^{MP}(w_1) = ord(w_1) = 1$. Then, $w_2 \in P^{MP}(w_1)$ and, checking from which world(s) it originated in the particular choice, we find $w_2 \in w^{MP}$, so $ord^{MP}(w_2) = ord(w_0) = 0$. In this case we end up with the R^{MP}-accessible model in figure (D.2), whose new

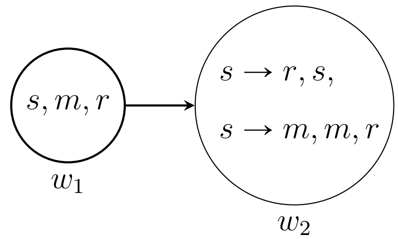


Figure D.2: Updated model

plausibility ordering is represented by the thick arrows and whose valuation is restricted to the worlds w_1, w_2 that have survived the application of MP:

In this case the MP-transformed model differs from the initial model, representing that the agent has thrown out the impossible world w_0 which satisfies $s \to m$, and s but which does not satisfy m. The cognitive capacity of the MP-accessible models is then reduced by the cognitive cost of applying MP, so that $cp = (12, 7)$.

E Anaphora in P-HYPE

P-HYPE is not only of interest in giving a solution to the problems of logical omniscience and granularity, but can also be combined with continuations to give a semantics for anaphora. Here we provide a mini case-study of how to combine P-HYPE with continuations, using the continuation monad (Shan 2002), (Barker and Shan 2014). This will enable us to capture certain cases of intra-sentential intensional anaphora. We leave for future work the question of how to extend our semantics of intensional anaphora to account 'Hob Nob' sentences (Van Rooij 2020, Geach 1967).

In **section** E.1, we introduce intensional anaphora, the State Set monad (**section** 8.27) (Unger 2011), (Charlow 2014) and the continuation monad and tower notation (**section** E.3) from (Barker and Shan 2014). Our emphasis will be on demonstrating how P-HYPE and the tower notation combine as a proof of concept, as opposed to developing a fully-fledged theory of anaphora in P-HYPE.

E.1 Intensional anaphora

In a sentence such as (10), the reflexive pronoun *himself* is obligatorily construed as referring to the same individual as *John*. *John* and *himself* are said to be *co-referring* and we say that *John* binds *himself* or that *himself* is anaphoric on *John*:

(10) John saw himself in the mirror.

Likewise, in a sentence such as (10) (below), the reciprocal pronoun *each other* is obligatorily construed as referring to the same group of individuals as is picked out by *We*. *We* and *each other* are said to be *co-referring* and we say that *We binds each other* or that *each other* is anaphoric on *We*:

(11) We saw each other in the seminar.

Reflexive pronouns and reciprocal pronouns are called *anaphors*. Anaphors are much studied in linguistics (Reinhart 1983), (Roberts 1989), (Heim 1994), (Heim 1998), (Asudeh 2012). We will follow the usual convention of annotating two sentence constituents with identical numerical indices whenever they co-refer given a certain construal of a sentence, and using bracketing to indicate the constituent to which the index attaches, as in (12a). If two expressions bear distinct indices, they do not co-refer on the intended construal of the sentence, as in (12b):

(12) a. [Every male resident in this town]$_i$ loves himself$_i$.

b. He$_i$ voted for him$_j$

When one expression binds another anaphoric pronoun within a sentence—as in (12a) — we call this *inter-sentential binding*; whereas, when one expression binds another anaphoric pronoun in a succeeding sentence—as in (13) —we call this *intra-sentential binding*.

(13) [John]$_i$ loves Mary. [He]$_i$ also loves Sue.

Intensional anaphora is anaphora in which the anaphoric pronouns are found within the scope of *intensional verbs* (Forbes 2020), where an intensional verb is a verb whose extension depends on the intension of its arguments, and not merely their extensions. Consider the reading of (14a) on which the anaphoric relationships are as in (14b):

(14) a. Mary Jane loves a man she believes is Peter Parker. She believes he isn't Spiderman.

b. Mary Jane loves [a man she believes is Peter Parker]$_i$. She believes he$_i$ isn't Spiderman.

c. *Scenario 1*: Mary Jane points at a man and says "that is the man I love. He is called Peter Parker." She then asserts that Peter Parker has many positive qualities that Spiderman doesn't have.

In this section we will give a compositional derivation of the reading described in *Scenario 1* in P-HYPE, using continuations. On this reading, we interpret both *Peter Parker* and *Spiderman* from Mary Jane's perspective; or rather, to be exact, we interpret both *Peter Parker* and *Spiderman* from what the utterer takes Mary

Jane's perspective to be, and from Mary Jane's perspective, both *Spiderman* and *Peter man* are different individuals.

We will show that P-HYPE can capture discourses like (14a), if we enrich it with the State Set monad for dynamic semantics, as has been done previously for non-intensional anaphora (Charlow 2014). Our aim is not to compare our semantics of anaphora with others, but to show that it is in principle possible to combine the tower notation of (Barker and Shan 2014) with P-HYPE, as a proof of concept.

E.2 The State.Set monad

Our analysis of anaphora follows (Charlow 2014) in using the State Set monad defined on stacks. Stacks are linear sequences of discourse referents (drefs) (Karttunen 1976) which serve as the items which anaphoric pronouns pick out in a discourse. We let r be the type of stacks and r, r' be variables over stacks. This ambiguity is harmless, since an occurrence of r is adequately disambiguated as either a variable or as a type in context. Stacks are constantly updated with discourse referents as a discourse proceeds. When stack r is extended with a new discourse referent x, we write $r\widehat{x}$ to indicate this. The last member of r is the most recently introduced dref, and is notated $r_\top$, such that $(r\widehat{a})_\top = a$. The State Set monad is defined as follows (for an y $\alpha, \beta \in TYPE$):

$$\diamondsuit\alpha \quad = \quad r \rightarrow (\alpha \times r) \rightarrow t \tag{8.24}$$

$$\eta_{\diamondsuit}\, a_\alpha \quad = \quad \lambda r_r, l_{\alpha\times r}.\ (\pi_1\ l) = a \wedge (\pi_2\ l) = r \tag{8.25}$$

$$m_{\diamondsuit\alpha} \bigstar f_{\alpha\rightarrow\diamondsuit\beta} \quad = \quad \lambda r_r, l_{\beta\times r}.\ \exists l'_{\alpha\times r}(m\, r\, l' \ \wedge\ (\ f\,(\pi_1\, l')\,(\pi_2\, l')\ l\ ) \tag{8.26}$$

However, in order to streamline our presentation with that of (Charlow 2014), throughout this chapter we use sets of pairs of terms of type $\alpha \times r$, for $\alpha \in TYPE$ as syntactic sugar for their characteristic functions, so that (8.25) and (8.26) are stated as follows:

$$\eta_{\diamond}\, a_\alpha = \lambda r_r.\{(a,r)\} \tag{8.27}$$

$$m_{\diamond\alpha} \bigstar f_{\alpha\to\diamond\beta} = \lambda r_r.\{(b,r') \mid (b,r') \in f\,a\,r'' \text{ and } (a,r'') \in m\,r, \text{ for some } a_\alpha \text{ and some } r''_r\,\} \tag{8.28}$$

$\diamond$ is a type constructor which applies to a type α to yield the type of functions from stacks to sets of pairs. The first members of these pairs are of type α and the second members are stacks. η applies to $a : \alpha$ and returns a function from a stack r to the the singleton set containing a single pair; to wit, the pair consisting of a as its first member and r as its second member. We can use η to lift proper names so that they can combine via $\bigstar$ with predicates of type $\diamond e \to \diamond t$ or $e \to \diamond t$. $\bigstar : \diamond\alpha \to (\alpha \to \diamond\beta) \to \diamond\beta$ acts on $m : \diamond\alpha$, a function from stacks to sets of pairs whose second members are stacks, and whose first member is type α, and on $\lambda v.\pi$, where $v : \alpha$ and $\pi : \diamond\beta$. Given a predicate $man^2 : e \to t$, an example of a term with type $\diamond e$ is given in (15):

(15) $a.man^2 =_{df} \lambda r.\{(x,\, r) \mid man^2\, x\} : \diamond e$

$a.man^2 : \diamond e$ denotes a function from stacks to sets of pairs whose first members are men and whose second members are stacks. $a.man$ can be derived via the renderings of a and man^2 which are below (from (Charlow 2014)):

$$a := \lambda c_{e\to\diamond t}, r.\,\{\,(\,x_e, r'\,) \mid (\top, r') \in c\, x_e\, r\} : (e \to \diamond t) \to \diamond e$$

$$man^2 := \lambda x_e, r'.\{(man^2\, x, r')\} : e \to \diamondsuit t$$

$$a.man^2 := \lambda r.\{(x_e, r') \mid (\top, r') \in \{(man^2 x, r)\}\} : \diamondsuit e$$

$$= \lambda r.\{(x,\, r) \mid man^2 x\} : \diamondsuit e$$

We can then use ★ to combine $a.man^2$ with $\lambda v.\eta(sing\, v)$:

(16) $$a.man^2_{\diamondsuit e} \bigstar \lambda v_e.\ \eta_{\diamondsuit}(sing_{e\to t}\, v\,)$$
$$=_{(df\, \eta_{\diamondsuit})}\ a.man^2 \bigstar \lambda v_e, r.\{(sing_{e\to t}\, v, r)\,)$$
$$=_{(df\, \bigstar)}\ \lambda r.\{(sing\, a\,, r') \mid man^2\ a \text{ and } r' \in (a.man^2)r\}$$
$$=_{(df\ a.man^2)}\ \lambda r.\{(sing\, a\,, r) \mid man^2\ a\}$$

We want a mechanism for allowing intra-sentential binding, so that indefinites like a *a man* can bind pronouns such as *he*, in discourses such as (17) (where the relevant binding is represented by two expressions sharing a subscripted index):

(17) A man$_i$ went to the pub. Some boring bloke spoke to him$_i$.

To do this, following (Charlow 2014), we define a special type preserving operation $(\cdot)^{\triangleright} : \diamondsuit\alpha \to \diamondsuit\alpha$:

$$m^{\triangleright} := m \bigstar \lambda v, r.\ (\,(\eta_{\diamondsuit}\, v)\, \widehat{rv}\,)$$
$$=_{df\, \eta_{\diamondsuit}} m \bigstar \lambda v, r.\ (\,(\lambda r'.\{(v, r')\})\, \widehat{rv}\,)$$
$$=_{\beta} m \bigstar \lambda v, r.\ \{(v, \widehat{rv})\}$$

$m^{\triangleright}$ is a way of updating a stack with a discourse referent, so that this discourse referent can be referred to by pronouns as the discourse expands. $a.man^{\triangleright}$ is a function from stacks to a set of pairs whose first members are discourse referents a which denote men and whose second member is the stack updated with a:

(18) $a.man^{2\rhd} := a.man^2 \bigstar \lambda v, r.\{(v, \widehat{rv})\}$

$= \lambda r.\{(b,\ r') \mid (b,\ r') \in \{(a, r''\widehat{\ }a)\}$ and $(a, r'') \in \{(x, r) \mid man^2\, x\}\}$

$= \lambda r.\{(a, \widehat{ra}) \mid man^2\ a\ \}$

Fact 1 will prove useful as a way of pushing discourse referents onto a stack and feeding them to a function:

Fact 1

$$
\begin{array}{llcl}
m^{\rhd}_{\diamondsuit\alpha} \bigstar \lambda v_\alpha.\ f_{\diamondsuit\beta} & =_{df\ \rhd} & (m & \bigstar\ \ \lambda u, r.\{(u, \widehat{ru})\}) \bigstar \lambda v.\ f_{\diamondsuit\beta} \\
& =_{Associativity} & m & \bigstar\ \ \lambda u(\lambda r.\{(u, \widehat{ru})\} \bigstar \lambda v.\ f_{\diamondsuit\beta}) \\
& =_{df\ \bigstar} & m & \bigstar\ \ \lambda u(\lambda r.\big\{(b,\ r') \mid (b,\ r') \\
& & & \quad \in f[a/v]\, r'' \text{ and} \\
& & & \quad (a,\ r'') \in \{(u,\ \widehat{r\ u})\}\ \big\}) \\
& =_{\substack{a=u,\\ r''=\widehat{r\ u}}} & m & \bigstar\ \ \lambda u(\lambda r.\{(b,\ r') \mid \\
& & & \quad (b,\ r') \in f\,[u/v]\,\widehat{r\ u}\,\}) \\
& = & m & \bigstar\ \ \lambda u, r.f\,[u/v]\,\widehat{r\ u} \\
& =_{\text{unifying free occurrences of } u \text{ with } v} & m & \bigstar\ \ \lambda v, r.f\ \widehat{r\ v}
\end{array}
$$

If the f contains r it is therefore updated with $\widehat{ru}$; otherwise, any stack variable remains untouched.

In addition, we adopt a lambda term $pro : \diamondsuit e$ for pronouns:

(19) $pro := \lambda r.\{(r_\top, r)\} : \diamondsuit e$

Note that $pro : \diamondsuit e$ has the following effect when fed to a function $f : e \to \diamondsuit\beta$ (for $\beta \in TYPE$):

$$pro \bigstar \lambda v.\ f = \lambda r.\ f\ [r_\top/v]\ r \tag{8.29}$$

We can now allow a $\rhd$-shifted m to bind a pronoun in its scope:

(20) $m^{\rhd} \bigstar \lambda v.pro \bigstar \lambda u.f = m^{\rhd} \bigstar \lambda v.f\ [v/u]$

Fact 2

$$\begin{aligned} m^{\rhd} \bigstar \lambda v.pro \bigstar \lambda u.\ f &=_{df\ \bigstar,(8.29)} m^{\rhd} \bigstar \lambda v, r.f\ [r_\top/u]\ r \\ &=_{\textbf{Fact 1}} m \bigstar \lambda v, r.\ f\ [\widehat{rv}_\top/u]\ \widehat{rv} \\ &=_{df\ \top} m \bigstar \lambda v, r.\ f\ [v/u]\ \widehat{rv} \\ &=_{\textbf{Fact 1}} m^{\rhd} \bigstar \lambda v.\ f[v/u] \end{aligned}$$

Using **Fact 1** and **Fact 2**, we can form simple discourses such as *A man sings. He is merry.*, given constants $sings, merry \in CON_{e \to t}$:

(21) $a.man^{2,\rhd} \bigstar \lambda x.\eta_{\diamondsuit}(sings) \bigstar \lambda f.\eta_{\diamondsuit}(and) \bigstar \lambda g.\ pro \bigstar \lambda y.\eta_{\diamondsuit}(merry)$
$\bigstar \lambda h.\eta_{\diamondsuit}(g\,(h\,y)(f\,x))$
$=_{df\ and}\ a.man^{2,\rhd} \bigstar \lambda x.\eta_{\diamondsuit}(sings) \bigstar \lambda f.\eta_{\diamondsuit}(\lambda q_t, p_t.p \wedge q) \bigstar \lambda g.\ pro \bigstar \lambda y.$
$\eta_{\diamondsuit}(merry) \bigstar \lambda h.\eta_{\diamondsuit}(g\,(h\,y)(f\,x))$
$=_{df\ LeftId,\beta}\ a.man^{2,\rhd} \bigstar \lambda x.pro \bigstar \lambda y.\eta_{\diamondsuit}(sings\,x \wedge merry\,y)$
$=_{\textbf{Fact 2}}\ a.man^{2\,\rhd} \bigstar \lambda x.\eta_{\diamondsuit}(sings\,x \wedge merry\,x)$
$=_{df\ \eta_{\diamondsuit}}\ a.man^{2\,\rhd} \bigstar \lambda x, r.\{(sings\,x \wedge merry\,x,\ \ r)\}$
$=_{\textbf{Fact 1}}\ a.man^{2} \bigstar \lambda x, r.\{(sings\,x \wedge merry\,x,\ \ \widehat{rx})\}$
$= \lambda r.\{(sings\,x \wedge merry\,x,\ \ \widehat{rx}) \mid man^2\,x\}$

We have formed a function from stacks to a set of pairs: the first member of each pair is the boolean true if there is some x who sings and is merry and is a man; the second member of each pair is the stack updated with this x. We want to capture a reading of the sentence (26)

(22) Mary Jane loves a man$_i$ she believes is Peter Parker. She believes he$_i$ isn't Spiderman.

We will treat this sentence as a function from stacks to sets of pairs: the first member of each pair will be the boolean true if there is some x such that Mary Jane loves whatever x denotes in her perspective, believes whatever x denotes in her perspective to be Peter Parker and believes that whatever x denotes in her perspective isn't Spiderman. As before,'whatever x denotes in her perspective' is really whatever the utterer of the sentence thinks x denotes in her perspective, given the conditions we laid down on π in previous chapters.

To arrive at something like these truth conditions, we need to define perspective-relative versions of $a.man^2$ and $a.man^{2\rhd}$. To do this, we need to use the predicate **man** : $\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond t$ (from the table of lexical entries in section (17)), and a suitable denotation for the indefinite determiner a, since $a : (e \rightarrow \diamondtimes t) \rightarrow \diamondtimes e$ cannot combine with **man** : $\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond t$.

We render the indefinite article a as the following lambda term:

(23) $\mathbf{a} := \lambda c_{\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond\diamondtimes t}, \delta_{\Diamond p}, r.\{\ (\eta_{\blacklozenge}\ (\ \lambda s'.x_{\blacklozenge\Diamond e}\,\delta\,s',\quad r')\mid(\top, r') \in c\,x\,\delta\,s'\,r\ \}$:
$(\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond\diamondtimes t) \rightarrow \blacklozenge\diamondtimes\ \blacklozenge\ \Diamond e$

Notice that $x_{\blacklozenge\Diamond e}$ is fed a variable $\delta_{\Diamond p}$ which is bound outside the set of pairs, and $\eta_{\blacklozenge}$ ensures that the first element of each pair has type $\blacklozenge\Diamond e$. The reason for this rendering become apparent later.

To combine **a** with **man** : $\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond t$ we need to lift **man** to a predicate of type ($\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond\otimes t$) using $\eta_{\otimes}$:

(24) $$\lambda x_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, s_s. \begin{cases} \eta_{\otimes}(\,man\,(x\,\delta\,s)\,(\delta\,s)\,s\,) & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\,s(\delta\,s) \\ \# & \text{if } \delta\,s \neq E \text{ and } C_{\mathbf{p}}\,s(\delta\,s) \neq \top \\ \eta_{\otimes}(\,man\,(x\,(\eta_s E)\,s)\,E\,s\,) & \text{if } \delta\,s = E \end{cases}$$

Please note that, from now on, in order to save space, we will only occasionally present the full braced lambda terms involving the conditions stated using 'if', as above.

We can now combine **a** with our shifted predicate of type $\blacklozenge\Diamond e \rightarrow \blacklozenge\Diamond\otimes t$, and they combine to form **a.man** : $\blacklozenge\otimes\blacklozenge\Diamond e$:

a.man

$= \textbf{a man}$

$=_{df\,\mathbf{a},df\,\mathbf{man}} (\lambda c, \delta, r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta\,s', r') \mid (\top, r') \in c\,a\,\delta\,s'\,r\})$
$\lambda x, \delta', s, r''.\{(man\,(x\,\delta'\,s)(\delta'\,s)\,s,\;\; r'')\}$

$=_{\beta} (\lambda\delta, r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta\,s', r') \mid (\top, r')$
$\in \lambda x, \delta', s, r''.\{(man\,(x\,\delta'\,s)(\delta'\,s)\,s,\;\; r'')\}\,a\,\delta\,s'\,r\})$

$=_{\beta} (\lambda\delta, r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta\,s', r') \mid (\top, r') \in \{(man\,(a\,\delta\,s')(\delta\,s')\,s',\;\; r)\}\;\;\})$

$=_{\beta} (\lambda\delta, r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta\,s',\;\; r) \mid man\,(a\,\delta\,s')(\delta\,s')\,s'\})$

Above we defined $\rhd : \diamondsuit\alpha \rightarrow \diamondsuit\alpha$ and in (17) applied it to $a.man^2$ to derive (8.30), a function from stacks to a set of pairs whose first member is a constant which will be such that $[\![man^2_{e\rightarrow t}]\!]^{\mathfrak{M},g}[\![a]\!]^{\mathfrak{M},g} = 1$ and whose second member is a stack updated with the constant a:

$$\lambda r.\{(a, r\widehat{\ }a) \mid man^2\ a\} \tag{8.30}$$

In what follows we will treat stacks as linear sequences of lambda terms of arbitrary type, instead of linear sequences of lambda terms of type e. We have found this generalisation necessary, in order to account for anaphora to expressions of type $\blacklozenge\lozenge e$.

One way of combining **a.man** with $\rhd$ would be to feed **a.man** an argument of type $\lozenge p$ (see (8.31)), and then combine the resulting lambda term of type $\diamondsuit\ \blacklozenge\ \lozenge e$ with $\rhd$ using the bind of the monad $(\diamondsuit, \eta_{\diamondsuit}, \bigstar)$, as in (8.32):

$$\begin{aligned}&(\lambda\delta, r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta\,s',\ \ r) \mid man\,(a\,\delta\,s')(\delta\,s')\,s'\})\ \delta^0_{\lozenge p}\\ &=_\beta\ \lambda r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta^0_{\lozenge p}\,s',\ \ r) \mid man\,(a\,\delta^0_{\lozenge p}\,s')(\delta^0_{\lozenge p}\,s')\,s'\}\end{aligned} \tag{8.31}$$

$$\begin{aligned}&\lambda r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta^0_{\lozenge p}\,s',\ \ r) \mid man\,(a\,\delta^0_{\lozenge p}\,s')(\delta^0_{\lozenge p}\,s')\,s'\}\bigstar\lambda v_{\blacklozenge\lozenge e}, r.\{\ (v, r\widehat{\ }v)\}\\ &=_\beta\ \lambda r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta^0\,s',\ \ r\widehat{\ }\eta_{\blacklozenge}\lambda s'.a\,\delta^0\,s')\} : \diamondsuit\ \blacklozenge\ \lozenge e\end{aligned} \tag{8.32}$$

We will need, however, to bind the term of type $\lozenge p$ which a applies to, from outside of the set, so that when we pass **a.man**$^{\rhd}$ to other functions the argument of type $\lozenge p$ is threaded through. For this reason, we want $\rhd$ to apply to **a.man** to produce (8.33):

$$\lambda\delta_{\lozenge p}, r.\{(\eta_{\blacklozenge}\lambda s^{'}_{s}.\ a\,\delta\,s', r\widehat{\ }(\eta_{\blacklozenge}\lambda s^{''}_{s}.a_{\blacklozenge\lozenge e}\,\delta\,s'')) \mid man\ (a\ \delta\ s')(\delta\,s')s'\} \tag{8.33}$$

In order to do this, we need to combine the monad $(\diamondsuit, \eta_{\diamondsuit}, \bigstar_{\diamondsuit})$ with the reader monad $(\blacklozenge, \eta_{\blacklozenge}, \bigstar_{\blacklozenge})$, where $\blacklozenge := (\Diamond p)$, just like the κ applicative discussed in 3.3. The reader monad $(\blacklozenge, \eta_{\blacklozenge}, \bigstar_{\blacklozenge})$ is defined as exactly as other reader monads are defined (see 3.3). Combining the reader monad with the monad $(\diamondsuit, \eta_{\diamondsuit}, \bigstar)$ produces the following monad $(\blacklozenge\diamondsuit, \eta_{\blacklozenge\diamondsuit}, \bigstar_{\blacklozenge\diamondsuit})$:

$$\blacklozenge \diamondsuit\alpha \quad = \quad \Diamond p \to r \to (\alpha \times r) \to t \tag{8.34}$$

$$\eta_{\blacklozenge\diamondsuit}\, a_\alpha \quad = \quad \lambda\delta_{\Diamond p}, r_r, l_{\alpha\times r}.\ (\pi_1\ l) = a \wedge (\pi_2\ l) = r \tag{8.35}$$

$$m_{\blacklozenge\diamondsuit\alpha} \bigstar_{\blacklozenge\diamondsuit} f_{\alpha\to\blacklozenge\diamondsuit\beta} \quad = \quad \lambda\delta_{\blacklozenge}, r_r, l_{\beta\times r}.\ \exists l'_{\alpha\times r}\ (\ m\,\delta\, r\, l' \wedge f\,(\pi_1\, l')\,\delta\,(\pi_2\, l')\ l\ ) \tag{8.36}$$

As previously, we will use set notation, so that (8.35) and (8.36) are as follows:

$$\eta_{\blacklozenge\diamondsuit}\, a_\alpha \quad = \quad \lambda\delta_{\Diamond p}, r.\ \{(a, r)\} \tag{8.37}$$

$$m_{\blacklozenge\diamondsuit\alpha} \bigstar_{\blacklozenge\diamondsuit} f_{\alpha\to\blacklozenge\diamondsuit\beta} \quad = \quad \lambda\delta_{\Diamond p}, r_r.\ \{(b_\beta, r') \mid (b, r') \in f\ a\ \delta\ r'' \text{ and } (a, r'') \in m\,\delta\, r, \text{ for some } a_\alpha \text{ and some } r''_r\} \tag{8.38}$$

With the monad $(\blacklozenge\diamondsuit,\ \eta_{\blacklozenge\diamondsuit},\ \bigstar_{\blacklozenge\diamondsuit})$ to hand, we form **a.man**$^{\rhd}$: $\blacklozenge\diamondsuit\ \blacklozenge\ \Diamond e$ as follows:

$$\textbf{a.man}_{\blacklozenge\diamondsuit\blacklozenge\Diamond e} \bigstar_{\blacklozenge\diamondsuit}\ \lambda u_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, r.\ \{(u, \widehat{ru})\} \tag{8.39}$$

$$=_{df\ \textbf{a.man}}\ (\lambda\delta'_{\Diamond p}, r'.\{(\eta_{\blacklozenge}\lambda s'.a_{\blacklozenge\Diamond e}\,\delta\, s',\ \ r') \mid man\,(a\,\delta'\, s')(\delta'\, s')\, s'\})_{\blacklozenge\diamondsuit\blacklozenge\Diamond e}$$
$$\bigstar_{\blacklozenge\diamondsuit}\ \lambda u_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, r.\ \{(u, \widehat{ru})\}$$

$=_{df\,\bigstar_{\blacklozenge\diamondsuit}} \lambda\delta_{\Diamond p}, r_r.\ \{(b_{\blacklozenge\Diamond e}, r') \mid (b, r') \in (\lambda u_{\blacklozenge\Diamond e}, \delta_{\Diamond p}, r.\ \{(u, \widehat{r u})\})\, c\, \delta\, r''$

and $(c, r'') \in (\lambda\delta', r'.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta\, s',\ \ r') \mid man\,(a\,\delta'\, s')(\delta'\, s')\, s'\})_{\blacklozenge\diamondsuit\blacklozenge\Diamond e}\, \delta\, r,$

for some $c_{\blacklozenge\Diamond e}$ and some $r''_r\}$

$=_{\beta}\ \lambda\delta_{\Diamond p}, r_r.\ \{(b_{\blacklozenge\Diamond e}, r') \mid (b, r') \in \{(c, \widehat{r'' c})\})$ and

$(c, r'') \in \{(\eta_{\blacklozenge}\lambda s'.a\,\delta\, s',\ \ r) \mid man\,(a\,\delta\, s')(\delta\, s')\, s'\},$ for some $c_{\blacklozenge\Diamond e}$ and some $r''_r\}$

$=_{\beta}\ \lambda\delta_{\Diamond p}, r_r.\ \{(\ \eta_{\blacklozenge}\lambda s'.a\,\delta\, s',\ \widehat{r(}\eta_{\blacklozenge}\lambda s'.a\,\delta\, s')\ ) \mid man\,(a\,\delta\, s')(\delta\, s')\, s'\}$

Fact 1$^{\blacklozenge\diamondsuit}$**—an analog of Fact 1—is then derived as follows:**

Fact 1$^{\blacklozenge\diamondsuit}$

$m^{\rhd}_{\blacklozenge\diamondsuit\alpha} \bigstar_{\blacklozenge\diamondsuit}\ \lambda v_{\alpha}.\ f_{\blacklozenge\diamondsuit\beta}$

$=_{df\,\rhd}\ (m_{\blacklozenge\diamondsuit\alpha}\bigstar_{\blacklozenge\diamondsuit}\lambda u_{\alpha}, \delta_{\Diamond p}, r.\{(u, \widehat{r u})\})\bigstar_{\blacklozenge\diamondsuit}\lambda v_{\alpha}.\ f$

$=_{Associativity}\ m_{\blacklozenge\diamondsuit\alpha}\bigstar_{\blacklozenge\diamondsuit}\lambda u_{\alpha}\ (\ \lambda\delta_{\Diamond p}, r.\{(u, \widehat{r u})\}\bigstar_{\blacklozenge\diamondsuit}\lambda v_{\alpha}.\ f\ )$

$=_{df\,\bigstar_{\blacklozenge\diamondsuit}}\ m_{\blacklozenge\diamondsuit\alpha}\bigstar_{\blacklozenge\diamondsuit}\lambda u_{\alpha}(\lambda\delta_{\Diamond p}, r.\big\{(b,\ r') \mid (b,\ r') \in f\,[a/v]\,\delta\, r''$ and

$(a,\ r'') \in \{(u,\ \widehat{r}\, u)\}\ \big\})$

$=_{\substack{a=u,\\ r''=\widehat{r}\,u}}\ m_{\blacklozenge\diamondsuit\alpha}\bigstar_{\blacklozenge\diamondsuit}\lambda u_{\alpha}(\lambda\delta_{\Diamond p}, r.\{(b,\ r') \mid$

$(b,\ r') \in f\,[u/v]\,\delta\,\widehat{r}\,u\,\})$

$=\ m_{\blacklozenge\diamondsuit\alpha}\bigstar_{\blacklozenge\diamondsuit}\lambda u_{\alpha}, \delta_{\Diamond p}, r.f\,[u/v]\,\delta\,\widehat{r}\,u$

$=_{\substack{\textit{unifying free occurrences of}\\ \textit{u with v}}}\ m_{\blacklozenge\diamondsuit\alpha}\bigstar_{\blacklozenge\diamondsuit}\lambda v_{\alpha}, \delta_{\Diamond p}, r.f\,\delta\,\widehat{r}\,v$

We will treat pronouns as stratified by type, such that, for the types τ in a given class γ, we have pronouns of type $\diamondsuit\tau$:

For $\tau \in \gamma :\ pro^{\tau} := \lambda r.\{(r_{\top_\tau},\ r)\} : \diamondplus\tau,$ where $r_{\top} : \tau$

We assume that $\blacklozenge\lozenge e \in \gamma$, so that we can account for anaphora to noun phrases of type $\blacklozenge\lozenge e$. The question of what other types besides e and $\blacklozenge\lozenge e$ are in γ is a matter of empirical research. As we have nothing to say about this matter, we will leave this question open.

In order to combine $pro^{\blacklozenge\lozenge e} : \diamondplus \blacklozenge \lozenge e$ with $\bigstar_{\blacklozenge\diamondplus}$ we need to lift it via $\eta_{\blacklozenge}$:

$$(\eta_{\blacklozenge}(pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\ ) : \blacklozenge\diamondplus \blacklozenge \lozenge e \tag{8.40}$$

Note that $(\eta_{\blacklozenge}(pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\ ) : \blacklozenge\diamondplus \blacklozenge \lozenge e$ has the following effect when fed to a function $\lambda v.\ f : \blacklozenge\lozenge e \rightarrow \blacklozenge\diamondplus\beta$ (for $\beta \in TYPE$):

$$(\eta_{\blacklozenge}(pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\ )_{\blacklozenge\diamondplus\blacklozenge\lozenge e} \bigstar_{\blacklozenge\diamondplus} \lambda v_{\blacklozenge\lozenge e}.\ f_{\blacklozenge\lozenge e \rightarrow \blacklozenge\diamondplus\beta} = \lambda\delta_{\lozenge p}, r.\ f\ [r_{\top}/v]\ \delta\ r \tag{8.41}$$

Fact 2$^{\blacklozenge\diamondplus}$ (see (25))—an analog of **Fact 2**—is then derived as in 8.42:

(25) $m^{\rhd}_{\blacklozenge\diamondplus\blacklozenge\lozenge e} \bigstar_{\blacklozenge\diamondplus}\ \lambda v_{\blacklozenge\lozenge e}.\ (\eta_{\blacklozenge}(pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\ )_{\blacklozenge\diamondplus\blacklozenge\lozenge e} \bigstar_{\blacklozenge\diamondplus}\ \lambda u_{\blacklozenge\lozenge e}.\ f_{\blacklozenge\diamondplus\beta}$
$= m^{\rhd} \bigstar_{\blacklozenge\diamondplus}\ \lambda v_{\blacklozenge\lozenge e}.\ f\ [v/u]$

Fact 2

$m^{\rhd} \bigstar_{\blacklozenge\diamondplus}\ \lambda v.(\eta_{\blacklozenge}(pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\ )_{\blacklozenge\diamondplus\blacklozenge\lozenge e} \bigstar_{\blacklozenge\diamondplus}\ \lambda u.\ f_{\blacklozenge\diamondplus\beta}$

$=_{df\ \bigstar,(8.41)}\ m^{\rhd} \bigstar_{\blacklozenge\diamondplus}\ \lambda v, \delta, r.f\ [r_{\top}/u]\ \delta\ r$

$=_{\textbf{Fact 1}}\ m \bigstar_{\blacklozenge\diamondplus}\ \lambda v, \delta, r.\ f\ [\widehat{rv}_{\top}/u]\ \delta\ \widehat{rv}$

$=_{df\ \top}\ m \bigstar_{\blacklozenge\diamondplus}\ \lambda v, \delta, r.\ f\ [v/u]\ \delta\ \widehat{rv}$

$=_{\textbf{Fact 1}^{\blacklozenge\diamondplus}}\ m^{\rhd} \bigstar_{\blacklozenge\diamondplus}\ \lambda v, \delta, r.\ f\ [v/u]\ \delta\ \widehat{rv}$

We have introduced the techniques above in order to treat anaphoric sentences such as (26):

(26) Mary Jane loves a man$_i$ she believes is Peter Parker. She believes he$_i$ isn't Spiderman.

We can now introduce the lambda term which we use to render (26). Using the abbreviations in section A.6, we aim to derive the lambda term (27):

(27)

$$\lambda r. \begin{cases} \Big\{ \Big(\Big[\forall s^1 \big[\pi \; (\kappa \, u \, s) \; s \; s^1 & \text{if } C_{\mathbf{p}} \, s \; (\kappa \, u \, s) = \top \\ \rightarrow love \; (\mathbf{b} \; (\kappa \, \mathbf{mj}) \, s^1) \; \mathbf{mj} \, (\kappa \, \mathbf{mj} \, s^1) \, s^1 \big] & \\ \wedge \big[\forall s^2 \big[\big[s \leqslant s^2 \wedge \pi \; (\kappa \, u \, s) \; s \; s^2 \big] & \\ \rightarrow \forall s^3 \big[DOX \; \mathbf{mj} \; s^2 \; s^3 & \\ \rightarrow is^1 (\mathbf{b} \; (\kappa \, \mathbf{mj}) \, s^3) \; (pp \; (\kappa \, \mathbf{mj} \, s^3) \; s^3)(\kappa \, \mathbf{mj} \, s^3) \; s^3 \big] \big] \big] & \\ \wedge \big[\forall s^4 \big[\big[s \leqslant s^4 \wedge \pi \; (\kappa \, u \, s) \; s \; s^4 \big] & \\ \rightarrow \forall s^5 \big[DOX \; \mathbf{mj} \; s^4 \; s^5 \rightarrow & \\ \forall s^6 (\; is^1 \; (\mathbf{b} \; (\kappa \, \mathbf{mj}) \, s^6) \; (\mathbf{sm} \; (\kappa \, \mathbf{mj} \, s^6) \; s^6) \; (\kappa \, \mathbf{mj} \, s^6) \, s^6 & \\ \rightarrow \; s^5 \perp s^6) \big] \big] \big] \Big], & \\ r \frown \mathbf{mj} \frown (\eta_{\blacklozenge} \lambda s'. \; \mathbf{b} \; (\kappa \, \mathbf{mj}) \; s') \frown \mathbf{pp} \frown \mathbf{sm} \Big) & \\ \mid man \; (\mathbf{b} \; (\kappa \, \mathbf{mj}) \; s^1) \; (\kappa \, \mathbf{mj} \, s^1) \; s^1 \text{ and} & \\ man \; (\mathbf{b} \; (\kappa \, \mathbf{mj}) \; s^3) \; (\kappa \, \mathbf{mj} \, s^3) \; s^3 & \\ \text{and } man \; (\mathbf{b} \; (\kappa \, \mathbf{mj}) \; s^5) \; (\kappa \, \mathbf{mj} \, s^5) \; s^5 \Big\} & \\ \# & \text{otherwise} \end{cases}$$

This is a function from stacks to sets of pairs, whose first members are booleans and whose second members update the stack with perspective relative, intensional discourse referents $(\eta_{\blacklozenge}\lambda s'.\ \mathbf{b}\ (\kappa\,\mathbf{mj})\ s')$, where $man\ (\mathbf{b}\ (\kappa\,\mathbf{mj})\ s')$ denotes a man relative to certain states s' and relative to $(\kappa\,\mathbf{mj})$. The first member of a given member of this set of pairs is $\top$, if in the π-accessible states s' of u, Mary Jane loves $(\mathbf{b}\ (\kappa\,\mathbf{mj})\,s^1)$, and in the states s'' which are doxastically accessible states according to Mary Jane from those states s' in the π set of u which are such that $s \leqslant s'$, $(\mathbf{b}\ (\kappa\,\mathbf{mj})\ s^3)$ is identical to $(\mathbf{pp}\ (\kappa\,\mathbf{mj}\,s^3)$ and $(\mathbf{a}\ (\kappa\,\mathbf{mj})\ s^5)$ is not identical to $(sm\ (\kappa\,\mathbf{mj}\,s^5)\ s^5)$.

Consider, however, (28):

(28)

$$\eta_{\blacklozenge}(\,(\eta_{\diamondplus}\mathbf{mj})^{\triangleright})\bigstar\lambda x.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar\lambda x'.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar\lambda x''.$$
$$\mathbf{a.man}^{\triangleright}\bigstar\lambda y.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar\lambda y'.\eta_{\blacklozenge}(\eta_{\diamondplus}\ \mathbf{pp})\bigstar\lambda z.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar$$
$$\lambda y''\eta_{\blacklozenge}(\eta_{\diamondplus}\ \mathbf{sm})\bigstar\lambda z'.$$
$$(\,\mathbf{love}\,y\,x\,(\kappa\,\mathbf{mj})\,(\kappa\,u)\,s)\,)\wedge(\,\mathbf{think}\,(\mathbf{is}\,z\,y')\,x'\,(\kappa\,u)\,s\,)$$
$$\wedge\ (\mathbf{think}(\eta_{\blacklozenge}((\mathbf{G\,not}^{2,H})\,\mathbf{is}\,z'\,y''\,(\kappa\,\mathbf{mj})\,)\,)\,x''\,(\kappa\,u)\,s\,)$$

Applying *LeftId* (28) reduces to (28):

(29)

$$\eta_{\blacklozenge}(\,(\eta_{\diamondplus}\mathbf{mj})^{\triangleright})\bigstar\lambda x.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar\lambda x'.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar\lambda x''.$$
$$\mathbf{a.man}^{\triangleright}\bigstar\lambda y.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar\lambda y'.(\eta_{\blacklozenge}pro^{\blacklozenge\lozenge e}_{\diamondplus\blacklozenge\lozenge e})\bigstar\lambda y''$$
$$\lambda\delta, r.\Bigg\{\Bigg(\ \ (\,\mathbf{love}\,y\,x\,(\kappa\,\mathbf{mj})\,(\kappa\,u)\,s\ )\wedge\ (\ \mathbf{think}\,(\mathbf{is}\,\mathbf{pp}\,y')\,x'\,(\kappa\,u)\,s\ )\ \wedge$$

$$(\textbf{think}((\textbf{G}\,\textbf{not}^{2,H})\,\textbf{is}\,\textbf{sm}\,y''\,(\kappa\,\textbf{mj})\,)\,x''\,(\kappa\,u)\,s\,), \quad r \Big) \Big\}$$

Given the definition of $\bigstar_{\blacklozenge\diamondsuit}$, and applying β-reduction to (28), we then arrive at (29):

(30)

$$\lambda\delta, r. \Big\{ \Big(\textbf{love}\ (\eta_{\blacklozenge}\lambda s'.\ a\ \delta\ s')\ \ \textbf{mj}\ (\kappa\textbf{mj})\,(\kappa u)\ s \wedge\ \textbf{think}\ (\textbf{is}\ \textbf{pp}(\eta_{\blacklozenge}\lambda s''.\ a\ \delta\ s'')\ )$$
$$\textbf{mj}\ (\kappa u)\ s\ \ \wedge \textbf{think}\ (\ \eta_{\blacklozenge}\ (\ (\textbf{G}\,\textbf{not}^{2,H})\ \textbf{is}\ \textbf{sm}\ (\eta_{\blacklozenge}\lambda s'''.\ a\ \delta\ s''')\ (\kappa\ \textbf{mj})\ )\ )\ \textbf{mj}\ (\kappa u)\ s,$$
$$r\widehat{\ }\textbf{mj}\widehat{\ }(\eta_{\blacklozenge}\lambda s'.\ a\ \delta\ s') \Big)\ \mid$$
$$\textbf{man}\ a\ \delta\ s',\ \textbf{man}\ a\ \delta\ s'',\ \textbf{man}\ a\ \delta\ s''' \Big\}$$

Then, applying (29) to κ **mj** we arrive at (30):

(31)

$$\lambda r. \Big\{ \Big(\textbf{love}\ (\eta_{\blacklozenge}\lambda s'.\ a\ (\kappa\ \textbf{mj})\ s')\ \ \textbf{mj}\ (\kappa\textbf{mj})\,(\kappa u)\ s\ \wedge$$
$$\textbf{think}\ (\textbf{is}\ \textbf{pp}(\eta_{\blacklozenge}\lambda s''.\ a\ (\kappa\ \textbf{mj})\ s'')\ )\ \textbf{mj}\ (\kappa u)\ s\ \wedge$$
$$\textbf{think}\ (\ \eta_{\blacklozenge}\ (\ (\textbf{G}\,\textbf{not}^{2,H})\ \textbf{is}\ \textbf{sm}\ (\eta_{\blacklozenge}\lambda s'''.\ a\ (\kappa\ \textbf{mj})\ s''')\ (\kappa\ \textbf{mj})\ )\ )\ \textbf{mj}\ (\kappa u)\ s,$$
$$r\widehat{\ }\textbf{mj}\widehat{\ }(\eta_{\blacklozenge}\lambda s'.\ a\ (\kappa\ \textbf{mj})\ s') \Big)\ \mid$$
$$\textbf{man}\ a\ (\kappa\ \textbf{mj})\ s',\ \textbf{man}\ a\ (\kappa\ \textbf{mj})\ s'',\ \textbf{man}\ a\ (\kappa\ \textbf{mj})\ s''' \Big\}$$

In turn, (30) β-reduces to (27), so that (27) is the normal form of (28).

If we can derive (28), then we can arrive at (27). However, as one can observe from the structure of (28), various pronouns scope below $\rhd$-lifted expressions, so that they will be bound to these expressions, thereby enabling anaphoric reference to them. But this property of (28) seems to bear no relation to the syntactic structure of (26). The question is inevitably raised, therefore, of how the lambda term (28) can be generated compositionally, in line with the structure of the two sentences which make up the discourse (26). It turns out that we can generate the lambda term (28) compositionally, by making use of continuations. To that end, we now introduce continuations and the tower notation which (Barker and Shan 2014) have introduced as a way of representing them.

We would like to note, however, that our treatment of anaphora will use HYPE negation and not dynamic negation, as in (Charlow 2014). Though HYPE negation is sufficient to capture the meaning of (26), it would not be sufficient, as it stands, to capture the fact that discourses such as (31) are infelicitous:

(32) No car arrived on time. It was waiting.

In order to rule out such examples we would have to merge HYPE negation with dynamic negation. However, we ignore this complication in the following and hope to return to the question of how best to integrate HYPE and dynamic negation in the future.

E.3 Continuations and the tower notation

Consider the addition function:

$$\lambda n, m.n + m.$$

We can rewrite this function using an extra parameter, as follows:

$$\lambda n, m, k.\ k(n + m)$$

The only difference between our lambda terms is that, in the second case, instead of returning a value, we pass the return value to the variable k. We can allow a function to be plugged in where k is. In this case, the function k is called a *continuation*. Continuations can be used to represent the future for a given computation (Kelsey R. et al. 1998: p.71). The continuation controls the future of the computation and can use the result in some further computation which itself contains a continuation.

Consider another example, this time from natural language: that a of a simple sentence consisting of a grammatical subject and a predicate. The grammatical subject of a sentence combines with a predicate to form a sentence. The predicate is the future for the (grammatical) computation which the grammatical subject will enter into in order to form a sentence. We can treat the predicate which combines with a grammatical subject as a continuation which is supplied to the grammatical subject. So, instead of treating a proper name such as *John* as a constant j_e, we can treat it as a function $\lambda P_{e \to t}.P\,j$, which takes its own continuation as argument. Generalising, if we have an expression p_α, we can form an expression $\lambda f_{\alpha \to \rho}.f\,p : (\alpha \to \rho) \to \rho$, where f is a continuation.

Continuations are useful in modelling scope-bearing expressions in natural language (Barker and Shan 2014), (Charlow 2014), such as quantifiers. The continuation monad provides a method of generating and combining expressions which contain continuations. It is defined as follows:

Definition 41 Given a return type $\rho \in TYPE$, and types $\alpha, \beta \in TYPE$, we define the continuation monad as follows:

$$\Diamond\alpha = (\alpha \to \rho) \to \rho$$

$$\eta\, a_\alpha = \lambda k_{\alpha\to\rho}.\, k\, a$$

$$m_{\Diamond\alpha} \bigstar \lambda v_\alpha.\, \pi_{\Diamond\beta} = \lambda k_{\beta\to\rho}.\, m\, (\lambda v_\alpha.\, \pi\, k)$$

$\Diamond$ applies to a type to form a higher-order function that takes a function of type $\alpha \to \rho$–called a *continuation*– and returns something of type ρ–where ρ is called the *return type*. η takes a lambda term and embeds it into the monad by forming a continuation that takes a as argument and returns some other type. Let us call terms of type $(\alpha \to \rho) \to \rho$ *continuised terms*. $\bigstar$ takes a continuised term m and feeds m its continuation $\lambda v.\, \pi\, k : \alpha \to \rho$. $\pi_{\Diamond\beta}$ is applied to $k : \beta \to \rho$ to produce a lambda term of type ρ. $\lambda k_{\beta\to\rho}.\, m\, (\lambda v_\alpha.\, \pi\, k)$ is therefore of type $(\beta \to \rho) \to \rho$.

Where $\Diamond\alpha = (\alpha \to \rho) \to \rho$, we will use **S** $m\ n$ to refer to the form of monadic function application induced by the continuation monad given the definition of $\mathbb{A}$ (given in ((3))), and where this is the case, we have that (E.3) holds, for any $\alpha, \beta \in TYPE$:

$$\mathbf{S}\; m_{\Diamond(\alpha\to\beta)}\; n_{\Diamond\alpha} \tag{8.42}$$

$$=_{df\, \mathbf{S}\, m\, n} \quad m_{\Diamond(\alpha\to\beta)} \bigstar \lambda x_{\alpha\to\beta}.\, n_{\Diamond\alpha} \bigstar \lambda y_\alpha.\, \eta\, (\, x\, y\,) : \Diamond\beta$$

$$= \quad \lambda k_{\beta\to\rho}.\, m_{((\alpha\to\beta)\to\rho)\to\rho}\; (\lambda x_{\alpha\to\beta}.\, n_{(\alpha\to\rho)\to\rho}(\lambda y_\alpha.\, k(\, x\, y\,))) : (\beta \to \rho) \to \rho$$

(E.3) allows us to apply a term of type $\Diamond(\alpha \to \beta)$ to a term of type $\Diamond\alpha$ to produce a term of type $\Diamond\beta$. Note that the continuation monad is not commutative, since scoping m over n in the above is not the same as scoping n over m, for example, when these lambda terms contain quantifiers.

It will be useful in the following to use the following trick, which allows us to form continuised terms in a given monad $(\Diamond, \eta, \bigstar)$ by applying $\bigstar$ to its first argument:

Definition 42 (Monadic lift) Given a monad $(\Diamond, \eta, \bigstar)$, and $m : \Diamond\alpha$, the monadic lift of m (notated $m^{\uparrow}$) is defined as follows:

$$(m_{\Diamond\alpha})^{\uparrow} :=$$
$$m\bigstar$$
$$= \lambda k_{\alpha\to\Diamond\beta}.\, (m \bigstar \lambda v_{\alpha}.\, k\, v) : (\alpha \to \Diamond\beta) \to \Diamond\beta$$

We write $m^{\bigstar}$ instead of $m \bigstar$

We are now going to present the tower notation of (Barker and Shan 2014). This notation is very useful in presenting continuations and functions on them in a user-friendly readable format. In this system, each expression of a language with a stacked triple. For example, the expression "John" would be associated with a stacked quadruple of the following kind:

$$\begin{array}{ll} \textit{Syntactic type}: & DP \\ \textit{Semantic type}: & e \\ \texttt{expression}: & \text{John} \\ \textit{lambda term}: & \text{j} \end{array} \tag{8.43}$$

This quadruple consists of a lambda term at the bottom, an expression in the middle layer, a semantic type (the type of the lambda term at the bottom of the tower) and a syntactic type. The semantic type and the lambda term layer of the tower can itself be a complex object consisting of various layers (as we shall shortly see). But in the simplest case all layers of the tower are presented on one line.

We can then combine the lambda terms via function application in the usual way, as the following example shows:

$$\begin{matrix} DP \\ \\ e \\ \\ \text{John} \\ \\ \mathrm{j} \end{matrix} \qquad \left(\begin{matrix} (DP\backslash S)/DP & DP \\ \\ e \to e \to t & e \\ \\ \text{saw} & \text{Mary} \\ \\ \mathrm{saw} & \mathrm{m} \end{matrix} \right) \tag{8.44}$$

$$= \qquad \begin{matrix} S \\ \\ t \\ \\ \text{John saw Mary} \\ \\ \mathrm{saw\,m\,j} \end{matrix}$$

From now on, we will suppress the syntactic type of our towers, as in (Charlow 2014) and we will use the word 'type' to talk about the semantic type of a lambda term, and not the syntactic type of a linguistic expression. A quantifier such as $\lambda k_{e \to t}.\forall x_e(man^2_{e \to t}\, x \to k\, x) : (e \to t) \to t$, contains a continuation $k : e \to t$. Towers are more complicated in the case of continuised terms, such as $\lambda k_{e \to t}.\forall x(man^2_{e \to t}\, x_e \to k\, x) : (e \to t) \to t$. We can represent the type of a continuised term graphically, as a tripartite structure of the following kind:

Definition 43 Tower semantic types

$$\frac{\rho \mid \rho}{\alpha} := (\alpha \to \rho) \to \rho \qquad \text{for all types } \alpha, \rho$$

Likewise, we can represent a lambda term of type $(\alpha \to \rho) \to \rho$ in the following way:

Definition 44 Tower lambda terms

$$\frac{f\,[\,]}{x} := \lambda k.f[kx] \qquad \text{where } \lambda k.f[kx] : (\alpha \to \rho) \to \rho,$$

$$\text{for all types } \alpha, \rho$$

$\lambda k_{e \to t}.\forall x(man^2_{e \to t}\, x_e \to k\, x) : (e \to t) \to t$ is therefore represented in tower notation as follows:

$$\textit{Semantic type}: \quad \frac{t \mid t}{e}$$

$$\texttt{expression}: \quad \texttt{every linguist} \tag{8.45}$$

$$\textit{lambda term}: \quad \frac{\text{ev. ling}(\lambda\text{x. [ ]})}{x}$$

Towers can then be combined, using the following schema:

Fact (combining):

$$\left(\begin{array}{cc} \dfrac{\rho \mid \rho}{\alpha} & \dfrac{\rho \mid \rho}{\alpha \rightarrow \beta} \\ \\ \texttt{left-exp} & \texttt{right-exp} \\ \\ \dfrac{\text{f[ ])}}{x} & \dfrac{\text{g[ ])}}{y} \end{array} \right) = \begin{array}{c} \dfrac{\rho \mid \rho}{\beta} \\ \\ \text{left-exp right-exp} \\ \\ \dfrac{\text{f[g[ ]])}}{x\ y} \end{array}$$

The top layers of the semantic type are composed together, and the bottom layers of the semantic types are combined using function application. The lambda terms at the bottom of the tower are simply composed together. For terms which are not continuised, semantic towers are of course not tripartite structures, and lambda terms are not layered, and the schema reduces to function application on the lambda terms (as in the example presented above).

Then the combination of these towers is in fact an instance of monadic scopal application **S** $m\ n$, as defined in (E.3), in which `left-exp` is rendered by a lambda term $\lambda k'_{\alpha\to\rho}.\, f_{\rho\to\rho}\,(k'\,x_\alpha) : (\alpha \to \rho) \to \rho$ and `right-exp` is rendered as a lambda term $\lambda k''_{(\alpha\to\beta)\to\rho}.\, g_{\rho\to\rho}\,(k''\,y_{\alpha\to\beta}) : ((\alpha \to \beta) \to \rho) \to \rho$:

$$\mathbf{S}\ \frac{\mathrm{f}[\,])}{x}\ \frac{\mathrm{g}[\,])}{y}$$

$$=_{\textbf{Definition}(\text{E.3})}\ \mathbf{S}\ \ (\lambda k''_{(\alpha\to\beta)\to\rho}.\, g_{\rho\to\rho}\,(k''\,y_{\alpha\to\beta})\ )_{(\alpha\to\beta)\to\rho\to\rho}$$
$$(\lambda k'_{\alpha\to\rho}.\, f_{\rho\to\rho}\,(k'\,x_\alpha)\ )_{(\alpha\to\rho)\to\rho}$$

$$=_{df\ \mathbf{S}}$$
$$\Big[\ (\lambda k''_{(\alpha\to\beta)\to\rho}.\, g_{\rho\to\rho}\,(k''\,y_{\alpha\to\beta})\ )_{(\alpha\to\beta)\to\rho\to\rho} \bigstar$$
$$\lambda y'_{\alpha\to\beta}.(\lambda k'_{\alpha\to\rho}.\, f_{\rho\to\rho}\,(k'\,x_\alpha)\ )_{(\alpha\to\rho)\to\rho}\ \bigstar\ \lambda x'_\alpha.\eta(y'\,x')\ \Big] : (\beta \to \rho) \to \rho$$

$$=_{df\,\bigstar}\ \Big[\ \lambda k_{\beta\to\rho}.\ (\lambda k''_{(\alpha\to\beta)\to\rho}.\, g_{\rho\to\rho}\,(k''\,y_{\alpha\to\beta})\ )_{(\alpha\to\beta)\to\rho\to\rho}$$
$$\lambda y'_{\alpha\to\beta}.(\lambda k'_{\alpha\to\rho}.\, f_{\rho\to\rho}\,(k'\,x_\alpha)\ )_{(\alpha\to\rho)\to\rho}\ \lambda x'_\alpha.\ k(y'\,x')\ \Big] : (\beta \to \rho) \to \rho$$

$$=_\beta\ \Big[\ \lambda k_{\beta\to\rho}.\ (\lambda k''_{(\alpha\to\beta)\to\rho}.\, g_{\rho\to\rho}\,(k''\,y_{\alpha\to\beta})\ )_{(\alpha\to\beta)\to\rho\to\rho}$$
$$\lambda y'_{\alpha\to\beta}.(\ f_{\rho\to\rho}\,(\,(\lambda x'_\alpha.\ k(y'\,x')\,)\,x_\alpha)\ )_\rho\ \Big] : (\beta \to \rho) \to \rho$$

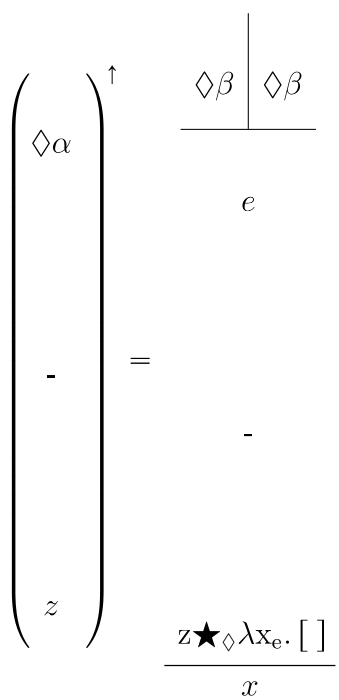
$$=_\beta \left[\lambda k_{\beta\to\rho}.\ (\lambda k''_{(\alpha\to\beta)\to\rho}.\ g_{\rho\to\rho}\,(k''\,y_{\alpha\to\beta})\ )_{(\alpha\to\beta)\to\rho\to\rho} \right.$$

$$\left. \lambda y'_{\alpha\to\beta}.(\ f_{\rho\to\rho}\,(\,k(y'\,x_\alpha)\,)\,)_\rho \right] : (\beta\to\rho)\to\rho$$

$$=_\beta \left[\lambda k_{\beta\to\rho}.\ g_{\rho\to\rho}\left(\ (\lambda y'_{\alpha\to\beta}.\ f_{\rho\to\rho}\,(\,k(y'\,x_\alpha)\,)\ )_{(\alpha\to\beta)\to\rho}\ \ y_{\alpha\to\beta}\right) \right] : (\beta\to\rho)\to\rho$$

$$=_\beta \left[\lambda k_{\beta\to\rho}.\ \left(\ g_{\rho\to\rho}\,(\ f_{\rho\to\rho}\,(\,k\ (y_{\alpha\to\beta}\,x_\alpha)\,)\ )_\rho\ \right)_\rho \right] : (\beta\to\rho)\to\rho$$

Monadic lifting (as defined in **Definition** (42)) can now be presented graphically as in the example below, where $\Diamond$ is any monadic modal operator:



By **Definition** (E.3), we have:

$$\frac{\mathrm{z}_{\Diamond\alpha} \bigstar \lambda \mathrm{x}_{\mathrm{e}}.[\,]}{x} \tag{8.46}$$

$$=_{\underset{\text{(E.3)}}{\textbf{Definition}}} (\lambda k_{\alpha\to\Diamond\beta}.\, z_{\Diamond\alpha} \bigstar \lambda x_{\alpha}.\, k\, x) : (\alpha \to \Diamond\beta) \to \Diamond\beta$$

Via monadic lifting we have turned $z : \Diamond\alpha$ into a term of type $(\alpha \to \Diamond\beta) \to \Diamond\beta$, i.e, the type of a quantifier.

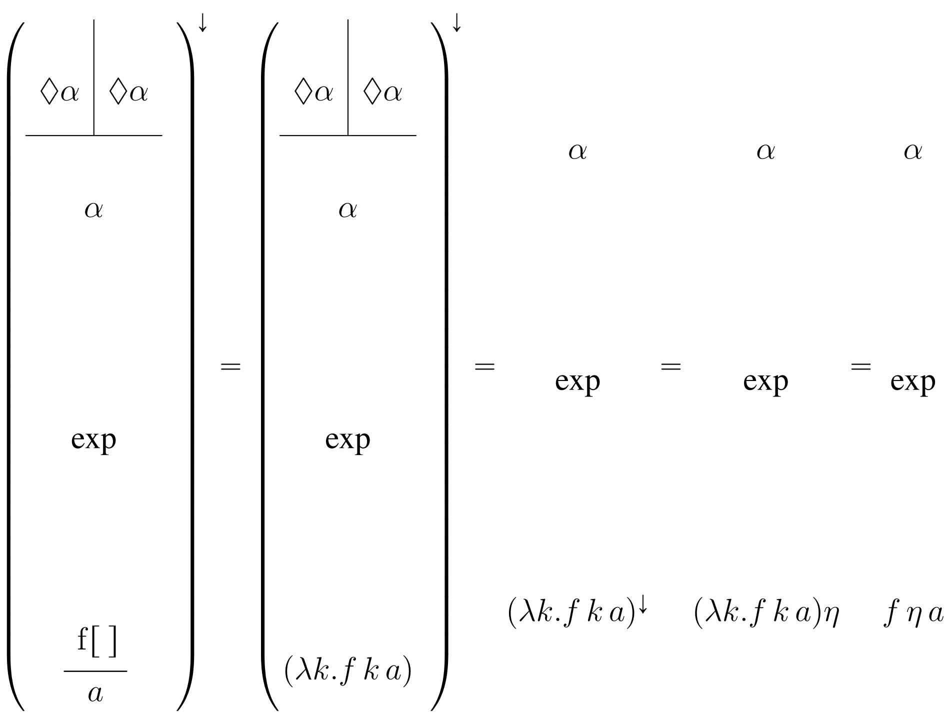
In order to collapse a tower, we apply η to it:

Definition 45 (Monadic lowering)

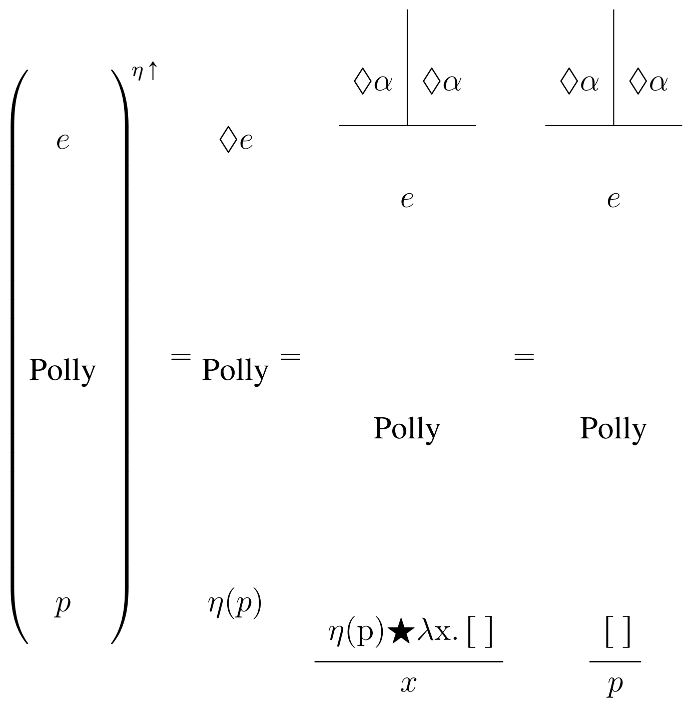
Given a monad $(\Diamond, \eta, \bigstar)$, and $m : (\alpha \to \Diamond\alpha) \to \Diamond\alpha$:

$m^{\downarrow} := m\, \eta : \Diamond\alpha$

We define the operation of $\downarrow$ on trees as follows:



For $\alpha, \beta \in TYPE$, using η we can now lift $m : \alpha$ to $\eta\, m : \Diamond\alpha$, and then we can lift $\eta\, m : \Diamond\alpha$ to $(\eta\, m)^{\uparrow} : (\alpha \rightarrow \Diamond\beta) \rightarrow \Diamond\beta$. This provides a general recipe for lifting non-monadic types into a monad and then making them have the type of a generalised quantifier ($(\alpha \rightarrow \rho) \rightarrow \rho$): we lift a non-monadic type α into a monadic type α using η, and then we use ★ to make our our expression into a generalised quantifier. Here is an example of this recipe at work, in which we lift a proper name of type e into a generalised quantifier of type $(e \rightarrow \Diamond\alpha) \rightarrow \Diamond\alpha$:



Suppose we want to lift the expression m in (33) to the type of the expression m' in (34):

(33) $m : (\alpha \rightarrow \Diamond\beta) \rightarrow \Diamond\beta$

(34) $m' : (((\alpha \rightarrow \Diamond\beta) \rightarrow \Diamond\beta) \rightarrow \Diamond\gamma) \rightarrow \Diamond\gamma.$

We first apply η to m so that it is of type $\Diamond((\alpha \rightarrow \Diamond\beta) \rightarrow \Diamond\beta)$, and then we apply

$\uparrow$.

For example, consider (35):

(35) $\lambda k_{\alpha \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(k\, a_\alpha) : (\alpha \to \Diamond\beta) \to \Diamond\beta$

We then apply η to (35)—as in (8.47)— and then apply $\uparrow$ to the result of this, as in (8.48):

$$(\eta(\lambda k_{\alpha \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(k\, a_\alpha)) : \Diamond((\alpha \to \Diamond\beta) \to \Diamond\beta) \tag{8.47}$$

$$\begin{aligned}&(\eta(\lambda k_{\alpha \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(k\, a_\alpha))^{\uparrow} \\ &=_{df\uparrow} \lambda c_{(\alpha \to \Diamond\beta) \to \Diamond\beta \to \Diamond\beta}(\eta(\lambda k_{\alpha \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(k\, a_\alpha)) \bigstar c \\ &=_{LeftId} \lambda c_{(\alpha \to \Diamond\beta) \to \Diamond\beta \to \Diamond\beta}.c\ (\ \lambda k_{\alpha \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(k\, a_\alpha)\ )\end{aligned} \tag{8.48}$$

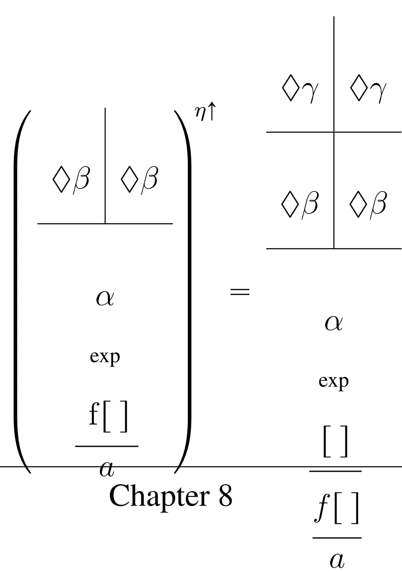
This is called *external lift* and can be represented as follows:

Definition 46 External lift

$(\ \lambda k_{\alpha \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(k\, a_\alpha) : ((\alpha \to \Diamond\beta) \to \Diamond\beta)\ )^{\uparrow\uparrow}$

$:= \lambda c_{((\alpha \to \Diamond\beta) \to \Diamond\beta) \to \Diamond\gamma}.\ c(\lambda k_{\alpha \to \Diamond\beta}.\ f_{\Diamond\beta \to \Diamond\beta}\,(k\, a_\alpha))\ :\ ((\alpha \to \Diamond\beta) \to \Diamond\beta) \to \Diamond\gamma \to \Diamond\gamma$

External lift



Alternatively, we apply lift by inserting a new scopal tier in the middle of the tower, so that the function f scopes higher in the lambda expression:

Definition 47 Internal lift

$(\lambda k_{\alpha \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(k\, a_\alpha) : ((\alpha \to \Diamond\beta) \to \Diamond\beta)\ )^{\uparrow\uparrow}$

$:= \lambda c_{((\alpha \to \Diamond\gamma) \to \Diamond\gamma) \to \Diamond\beta}.f_{\Diamond\beta \to \Diamond\beta}(c\,(\lambda k_{\alpha \to \Diamond\gamma}.k\, a_\alpha))\ :\ ((\alpha \to \Diamond\gamma) \to \Diamond\gamma) \to \Diamond\beta \to \Diamond\beta$

Internal lift

$$\left(\begin{array}{c} \begin{array}{c|c} \Diamond\beta & \Diamond\beta \end{array} \\ \hline \alpha \\ \text{exp} \\ \dfrac{\text{f}[\ ]}{a} \end{array} \right)^{\uparrow\uparrow} = \begin{array}{c} \begin{array}{c|c} \Diamond\beta & \Diamond\beta \end{array} \\ \hline \begin{array}{c|c} \Diamond\gamma & \Diamond\gamma \end{array} \\ \hline \alpha \\ \text{exp} \\ \dfrac{f[\ ]}{\dfrac{[\ ]}{a}} \end{array}$$

Three level towers can then be combined, using the following schema:

Fact (combining): Where $\{\gamma, \delta\} := \{\alpha, \alpha \to \beta\}$:

$$\left(\begin{array}{c|c} \sigma & \sigma \\ \hline \rho & \rho \\ \hline \multicolumn{2}{c}{\gamma} \\ \multicolumn{2}{c}{\text{left-exp}} \\ \multicolumn{2}{c}{\dfrac{f[\,]}{\dfrac{g[\,]}{x}}} \end{array} \quad \begin{array}{c|c} \sigma & \sigma \\ \hline \rho & \rho \\ \hline \multicolumn{2}{c}{\delta} \\ \multicolumn{2}{c}{\text{right-exp}} \\ \multicolumn{2}{c}{\dfrac{h[\,]}{\dfrac{i[\,]}{y}}} \end{array} \right) = \begin{array}{c|c} \sigma & \sigma \\ \hline \rho & \rho \\ \hline \multicolumn{2}{c}{\beta} \\ \multicolumn{2}{c}{\text{left-exp right-exp}} \\ \multicolumn{2}{c}{\dfrac{f[h[\,]]}{\dfrac{g[i[\,]]}{x\ y}}} \end{array}$$

Just as two-level towers can be combined using $\mathtt{A}$, three-level towers can be combined using $\mathcal{S}$, which is defined as follows (for any $\alpha, \beta, \gamma \in TYPE$):

$$\begin{aligned} &\mathcal{S}\ M_{(((\alpha\to\beta)\to\Diamond\beta)\to\Diamond\beta)\to\Diamond\gamma\to\Diamond\gamma}\ N_{((\alpha\to\Diamond\beta)\to\Diamond\beta)\to\Diamond\gamma\to\Diamond\gamma} \\ &:= \lambda c_{((\beta\to\Diamond\beta)\to\Diamond\beta)\to\Diamond\gamma} \\ &M(\lambda m_{((\alpha\to\beta)\to\Diamond\beta)\to\Diamond\beta}.\ N(\lambda n_{(\alpha\to\Diamond\beta)\to\Diamond\beta}.\ c\,(\mathbf{S}\ m\ n))) \end{aligned} \tag{8.49}$$

The terms that we can construct with external and internal lift are quite complex. Terms of type $((\alpha \to \beta) \to \beta) \to \gamma \to \gamma$ are continuised terms which take continuised terms of type $((\alpha \to \beta) \to \beta) \to \gamma$ as their continuations. A continuation that is itself a continuised term are called *higher-order continuations* (see (Barker and Shan 2014, Charlow 2014) for various applications of higher-level continuations).

Just as two-level towers can be flattened or lowered via **Definition** E.3, so can three-level towers:

Definition 48 (Monadic lowering) Given a three level tower $M : ((\alpha \to \Diamond\alpha) \to$

$\Diamond\alpha) \to \Diamond\alpha \to \Diamond\alpha$:

$$M^{\downarrow\downarrow} := M(\lambda m.m\,\eta) \qquad \text{where } m : ((\alpha \to \Diamond\alpha) \to \Diamond\alpha)$$

$$\left(\begin{array}{c} \begin{array}{c|c} \Diamond\alpha & \Diamond\alpha \\ \hline \Diamond\alpha & \Diamond\alpha \\ \hline \end{array} \\ \alpha \\ \text{exp} \\ \dfrac{g[\;]}{\dfrac{h[\;]}{a}} \end{array}\right)^{\downarrow\downarrow} = \begin{array}{c} \Diamond\alpha \\ \\ \text{exp} \\ \\ g[h[\eta\, a]] \end{array}$$

We are now in a position to show how we can use higher-order continuations in order to capture certain cases of intensional intra-sentential anaphora, such as (36):

(36) Mary Jane loves a man$_i$ she believes is Peter Parker. She believes he$_i$ isn't Spiderman.

Using higher-order continuations, binding is analysed via scope, and anaphoric pronouns live on different levels of towers.

E.4 The derivation

To recap, in (36), *a man*$_i$ binds he_i, and we interpret both *Peter Parker* and *Spiderman* from Mary Jane's perspective; or rather, to be exact, we interpret both *Peter*

Parker and *Spiderman* from what the utterer takes Mary Jane's perspective to be. To derive this reading compositionally we employ higher-order continuations, since depending on where a pronoun is located in a three-level tower, it will be caught by a bind-shifted expression on the same level of the tower when we eventually lower the tower. We can therefore lift all pronouns onto a given level of the tower and ensure that they are bound by an expression which we want them to be bound by. (Barker and Shan 2014) use this strategy to account for anaphoric reference to multiple pronouns. Let us set $\underline{\boldsymbol{\alpha}} := \blacklozenge \Diamond\, \alpha$.

An important point to bear in mind is that the bottom level of the lambda tower must be able to combine by function application. Thus $love : \blacklozenge\Diamond e \to e \to \blacklozenge\,\blacklozenge\,\Diamond t$ must combine with something of type $\blacklozenge\Diamond e$. For consider the following two towers (where X stands for some expression, and **?** indicates that we are unsure of the type of the lambda term associated with X):

$$\begin{array}{c|c} \blacklozenge\Diamond t & \blacklozenge\Diamond t \\ \hline \blacklozenge\Diamond t & \blacklozenge\Diamond t \\ \hline \multicolumn{2}{c}{\blacklozenge\Diamond e \to e \to \blacklozenge\,\blacklozenge\,\Diamond t} \end{array} \qquad \begin{array}{c|c} \blacklozenge\Diamond t & \blacklozenge\Diamond t \\ \hline \blacklozenge\Diamond t & \blacklozenge\Diamond t \\ \hline \multicolumn{2}{c}{\textbf{?}} \end{array}$$

$$\begin{array}{c} \text{loves} \\ \\ [\,] \\ \hline [\,] \\ \hline loves \end{array} \qquad\qquad \begin{array}{c} \text{X} \\ \\ [\,] \\ \hline t\bigstar\lambda y.[\,] \\ \hline y \end{array}$$

After combining and lowering these towers, we end up with the lambda term, at the top of this sequence, which reduces to the lambda term at the bottom of this sequence of lambda terms:

$$\eta_{\blacklozenge}\eta_{\lozenge}(love)\bigstar\lambda f, t\bigstar\lambda y.\eta_{\blacklozenge}\eta_{\lozenge}(f\, y\, x)$$
$$= t\bigstar\lambda y.\eta_{\blacklozenge}\eta_{\lozenge}(love\, y\, x)$$
$$= \lambda\delta,\ r\{(b,\ r') \mid (b,\ r') \in \{(love\, a\, x,\ r'')\} \text{ and } (a,\ r'') \in t\,\delta\, r\}$$

For the a in the last equation above to substitute for the y in the lambda term $\lambda y.\eta_{\blacklozenge}\eta_{\lozenge}(f\, y\, x)$, each a must be of type $\blacklozenge\Diamond e$, because of the typing of $love : \blacklozenge\Diamond e \rightarrow e \rightarrow \blacklozenge\Diamond t$. It is for this reason that we defined $a.man$ above as follows:

$$\lambda\delta, r.\{(\eta_{\blacklozenge}\lambda s'.a\,\delta\, s', \hat{r}(\eta_{\blacklozenge}\lambda s''.a\,\delta\, s'')) \mid man\ (a\ \delta\ s')(\delta\, s')s'\}$$

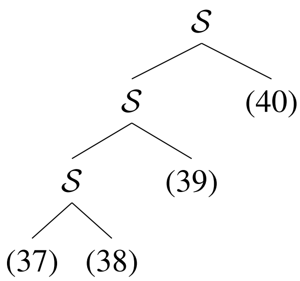
In this way, we can ensure that $love$ combines with $a.man$.

Now for the compositional derivation of (36) on the intended reading. Due to space constraints making it difficult to fit tower derivations on paper, we will present the derivation in stages. Furthermore, we will often represent that three-level towers combine via **Fact (combining)** (see above), by using more compact binary trees of the following form, where (4) and (5) refer to the example numbers where particular three-level towers are presented:

$\mathcal{S}$

(4) (5)

The following tree represents how the scope towers combine to form *Mary Jane* *loves a man*, with the example numbers referring to the towers below it:



(37)

$$\left(\begin{array}{c}\begin{array}{c|c}\blacklozenge\diamondtimes t & \blacklozenge\diamondtimes t \\ \hline \blacklozenge\diamondtimes t & \blacklozenge\diamondtimes t \\ \hline \end{array} \\ e \\ \text{Mary Jane} \\ \dfrac{\eta_{\blacklozenge}\,(\eta_{\diamondtimes}\,\mathbf{mj})^{\triangleright})\bigstar\lambda x.[\,]}{\dfrac{[\,]}{x}}\end{array}\left(\begin{array}{c}\begin{array}{c|c}\blacklozenge\diamondtimes t & \blacklozenge\diamondtimes t \\ \hline \blacklozenge\diamondtimes t & \blacklozenge\diamondtimes t \\ \hline \end{array} \\ \blacklozenge\lozenge e \to e \to \blacklozenge\,\blacklozenge\,\lozenge t \\ \textit{loves} \\ \dfrac{[\,]}{\dfrac{[\,]}{\mathbf{love}}}\end{array}\quad\begin{array}{c}\begin{array}{c|c}\blacklozenge\diamondtimes t & \blacklozenge\diamondtimes t \\ \hline \blacklozenge\diamondtimes t & \blacklozenge\diamondtimes t \\ \hline \end{array} \\ \blacklozenge\lozenge e \\ \text{a man} \\ \dfrac{[\,]}{\dfrac{\mathbf{a.man}^{\triangleright}\bigstar\lambda y.[\,]}{y}}\end{array}\right)\right)$$

(38)

$\blacklozenge\boxtimes t$	$\blacklozenge\boxtimes t$
$\blacklozenge\boxtimes t$	$\blacklozenge\boxtimes t$

$\Diamond p$

\-   \-

$$\frac{\frac{[\,]}{[\,]}}{\kappa\,\mathbf{mj}}$$

(39)

$\blacklozenge\boxtimes t$	$\blacklozenge\boxtimes t$
$\blacklozenge\boxtimes t$	$\blacklozenge\boxtimes t$

$\Diamond p$

\-

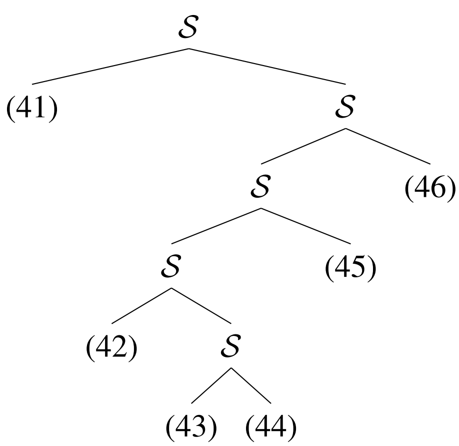
$$\frac{\frac{[\,]}{[\,]}}{\kappa\,u}$$

(40)

$$\begin{array}{r|l} \blacklozenge\boxtimes t & \blacklozenge\boxtimes t \\ \hline \blacklozenge\boxtimes t & \blacklozenge\boxtimes t \\ \hline \end{array}$$

$$\begin{array}{c} s \\ \_ \\ \frac{[\,]}{\frac{[\,]}{s}} \end{array}$$

The following tree represents the way that the scope towers directly below it combine to derive *who she thinks is Peter Parker*:



(41)

$\blacklozenge\Diamond t$	$\blacklozenge\Diamond t$
$\blacklozenge\Diamond t$	$\blacklozenge\Diamond t$

$$t \rightarrow t \rightarrow t$$

who

$$\frac{[\,]}{\frac{[\,]}{and}}$$

(42)

$\blacklozenge\Diamond t$	$\blacklozenge\Diamond t$
$\blacklozenge\Diamond t$	$\blacklozenge\Diamond t$

$$e$$

she

$$\frac{\eta_{\blacklozenge}(pro^{\blacklozenge\diamond e}_{\Diamond\blacklozenge\diamond e})\bigstar\lambda x'.[\,]}{\frac{[\,]}{x'}}$$

(43)

$$\begin{array}{c|c} \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \end{array}$$

$$\blacklozenge\Diamond t \to e \to \blacklozenge\Diamond t$$

thinks

$$\frac{[\ ]}{\frac{[\ ]}{\textbf{thinks}}}$$

(44)

$$\left(\begin{array}{c} \begin{array}{c|c} \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \end{array} \\ \blacklozenge\Diamond e \\ \text{GAP} \\ \dfrac{[\ ]}{\dfrac{\eta_{\blacklozenge}(pro^{\blacklozenge\Diamond e}_{\otimes\blacklozenge\Diamond e})\bigstar\lambda y'.}{y'}} \end{array} \left(\begin{array}{c} \begin{array}{c|c} \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \end{array} \\ {\scriptstyle \blacklozenge\Diamond e\to e} \\ {\scriptstyle \to\blacklozenge\Diamond t} \\ \text{is} \\ \dfrac{[\ ]}{\dfrac{[\ ]}{\textbf{is}}} \end{array} \begin{array}{c} \begin{array}{c|c} \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \blacklozenge\otimes t & \blacklozenge\otimes t \\ \hline \end{array} \\ \blacklozenge\Diamond e \\ \text{Peter Parker} \\ \dfrac{[\ ]}{\dfrac{\eta_{\blacklozenge}(\eta_{\otimes}\,\textbf{pp})\bigstar\lambda z.[\ ]}{z}} \end{array} \right) \right)$$

(45)

$\blacklozenge\otimes t$	$\blacklozenge\otimes t$
$\blacklozenge\otimes t$	$\blacklozenge\otimes t$

$$\lozenge p$$

$$\cfrac{[\,]}{\cfrac{[\,]}{\kappa\, u}}$$

(46)

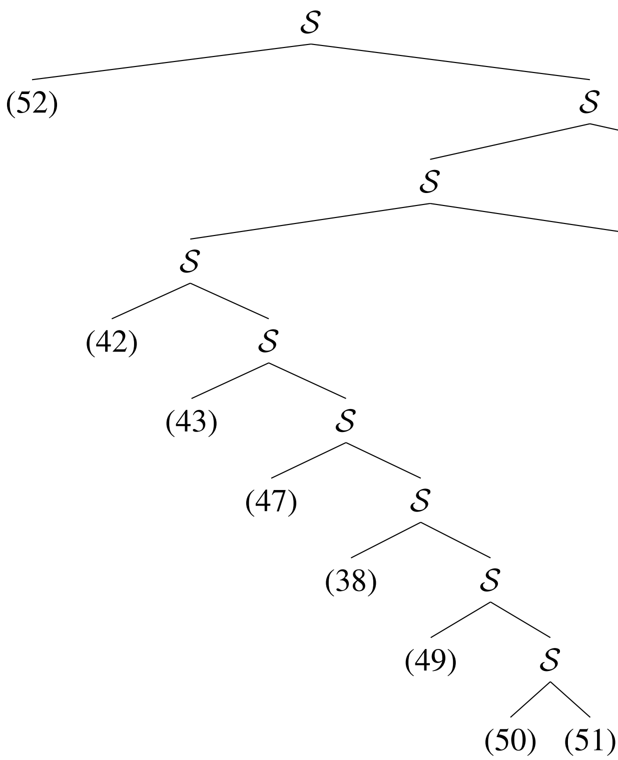
$\blacklozenge\otimes t$	$\blacklozenge\otimes t$
$\blacklozenge\otimes t$	$\blacklozenge\otimes t$

$$s$$

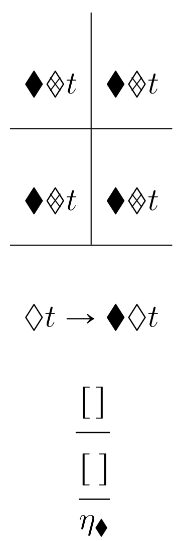
$$-$$

$$\cfrac{[\,]}{\cfrac{[\,]}{s}}$$

The following tree represents the way that the scope towers directly below it combine to derive *She thinks he isn't Spiderman*:



(47)



(48)

$\blacklozenge \diamondtimes t$	$\blacklozenge \diamondtimes t$
$\blacklozenge \diamondtimes t$	$\blacklozenge \diamondtimes t$

$\Diamond p$

$$\frac{[\,]}{\dfrac{[\,]}{\kappa\,\mathbf{mj}}}$$

(49)

$\blacklozenge \diamondtimes t$	$\blacklozenge \diamondtimes t$
$\blacklozenge \diamondtimes t$	$\blacklozenge \diamondtimes t$

$\blacklozenge \Diamond e$

he

$$\frac{[\,]}{\dfrac{\eta_{\blacklozenge}(pro^{\blacklozenge \Diamond e}_{\diamondtimes \blacklozenge \Diamond e}) \bigstar \lambda y''.}{y''}}$$

(50)

$$\left(\begin{array}{c}\begin{array}{c|c}\blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline \blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline\end{array}\\ \blacklozenge\Diamond e\to\blacklozenge\Diamond e\\ \to\blacklozenge\Diamond t\\ \text{is}\\ \dfrac{[\,]}{[\,]}\\ \hline \textbf{is}\end{array}\quad - \quad\left(\begin{array}{cc}\begin{array}{c|c}\blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline \blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline\end{array} & \begin{array}{c|c}\blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline \blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline\end{array}\\ \begin{array}{c}(\blacklozenge\Diamond t\to\blacklozenge\Diamond t)\\ (\blacklozenge\Diamond e\to\\ \blacklozenge\Diamond e\to\blacklozenge\Diamond t)\to\blacklozenge\Diamond e\\ \to\blacklozenge\Diamond e\\ \to\blacklozenge\Diamond t\end{array} & \blacklozenge\Diamond t\to\blacklozenge\Diamond t\\ - & \text{not}\\ \dfrac{[\,]}{[\,]} & \dfrac{[\,]}{[\,]}\\ \hline \textbf{G} & \textbf{not}^{2,H}\end{array}\right)\right)$$

(51)

$$\begin{array}{c}\begin{array}{c|c}\blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline \blacklozenge\lozenge t & \blacklozenge\lozenge t\\ \hline\end{array}\\ \\ \blacklozenge\Diamond e\\ \\ \text{Spiderman}\\ [\,]\\ \hline \eta_{\blacklozenge}(\eta_{\lozenge}\textbf{sm})\bigstar\lambda z'.[\,]\\ \hline z'\end{array}$$

The two sentences *Mary Jane loves a man who she thinks is Peter Parker. She thinks he isn't Spiderman.* are then derived by combining the towers for these sentences with (52):

(52)

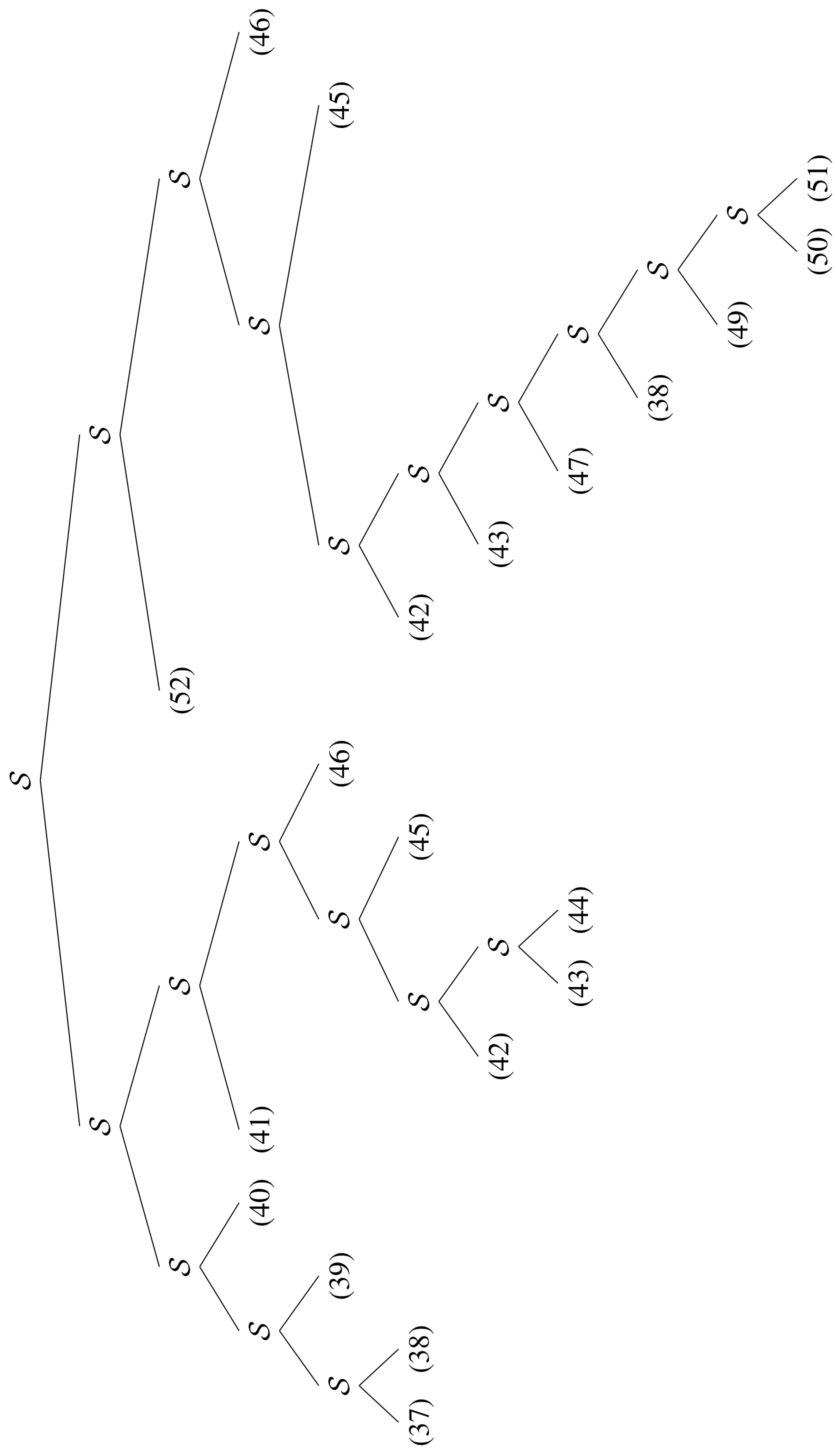
$$\begin{array}{c|c} \blacklozenge\lozenge t & \blacklozenge\lozenge t \\ \hline \blacklozenge\lozenge t & \blacklozenge\lozenge t \\ \hline \end{array}$$

$$t \to t \to t$$

-

$$\begin{array}{c} [\,] \\ \hline [\,] \\ \hline and \end{array}$$

The binary tree representing how all the towers combine to form the representation of them is as follows:


S
(37)
(38)
(39)
(40)
(41)
(42)
(43)
(44)
(45)
(46)
(52)
(42)
(43)
(47)
(38)
(49)
(50)
(51)
(45)
(46)

The root of the binary tree above is the following tower:

$$\begin{array}{c|c} \blacklozenge\lozenge t & \blacklozenge\lozenge t \\ \hline \blacklozenge\lozenge t & \blacklozenge\lozenge t \\ \hline \multicolumn{2}{c}{} \\ \multicolumn{2}{c}{t} \end{array}$$

Mary Jane loves a man she thinks is Peter Parker.

She thinks he isn't Spiderman.

$$\dfrac{\eta_{\blacklozenge}(\,(\eta_{\lozenge}\ \mathbf{mj})^{\rhd})\bigstar\lambda x.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda x'.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda x''.[\,]}{\dfrac{\mathbf{a.man}^{\rhd}\bigstar\lambda y.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda y'.\eta_{\blacklozenge}(\eta_{\lozenge}\ \mathbf{pp})\bigstar\lambda z.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda y''\eta_{\blacklozenge}(\eta_{\lozenge}\ \mathbf{sm})\bigstar\lambda z'.[\,]}{\begin{array}{c} and\,(and(\mathbf{think}\,(\mathbf{is}\,z\,y')\,x'\,(\kappa\,u)\,s\,)\,(\mathbf{love}\,y\,x\,(\kappa\,\mathbf{mj})\,(\kappa\,u)\,s)\,) \\ (\mathbf{think}(\eta_{\blacklozenge}((\mathbf{G}\ \mathbf{not}^{2,H})\ \mathbf{is}\ z'\,y''\,(\kappa\,\mathbf{mj})\,)\,)\,x''\,(\kappa\,u)\,s\,) \end{array}}}$$

After lowering the layered lambda term by applying $\downarrow\downarrow$ to the lambda term created by substituting the holes $[\,]$ with variables over continuations and abstracting over them, we arrive at the following lambda term, which β-reduces to the lambda term (27), which we aimed to compositionally derive:

$$=_{\substack{\textbf{Definition}\,(35),\\ df\ and}}$$

$$\eta_{\blacklozenge}(\,(\eta_{\lozenge}\mathbf{mj})^{\rhd})\bigstar\lambda x.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda x'.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda x''.$$

$$\mathbf{a.man}^{\rhd}\bigstar\lambda y.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda y'.\eta_{\blacklozenge}(\eta_{\lozenge}\ \mathbf{pp})\bigstar\lambda z.(\eta_{\blacklozenge} pro^{\blacklozenge\diamond e}_{\lozenge\blacklozenge\diamond e})\bigstar\lambda y''\eta_{\blacklozenge}(\eta_{\lozenge}\ \mathbf{sm})\bigstar\lambda z'.$$

$$(\,\mathbf{love}\,y\,x\,(\kappa\,\mathbf{mj})\,(\kappa\,u)\,s)\,)\wedge(\,\mathbf{think}\,(\mathbf{is}\,z\,y')\,x'\,(\kappa\,u)\,s\,)$$

$\wedge\ (\mathbf{think}(\eta_{\blacklozenge}((\mathbf{G}\,\mathbf{not}^{2,H})\,\mathbf{is}\,z'\,y''\,(\kappa\,\mathbf{mj})\,)\,)\,x''\,(\kappa\,u)\,s\,)$

$$
\lambda r. \begin{cases}
\Big\{ \Big(\Big[\forall s^1 \big[\pi\ (\kappa\, u\, s)\ s\ s^1 & \text{if } C_{\mathbf{p}}\, s\ (\kappa\, u\, s) = \top \\
\to love\ (\mathbf{b}\ (\kappa\, \mathbf{mj})\, s^1)\ \mathbf{mj}\, (\kappa\, \mathbf{mj}\, s^1)\, s^1 \big] & \\
\wedge \big[\forall s^2 \big[\big[s \leqslant s^2 \wedge \pi\ (\kappa\, u\, s)\ s\ s^2 \big] & \\
\to \forall s^3 \big[DOX\ \mathbf{mj}\ s^2\ s^3 & \\
\to is^1 (\mathbf{b}\ (\kappa\, \mathbf{mj})\, s^3)\ (pp\ (\kappa\, \mathbf{mj}\, s^3)\ s^3) (\kappa\, \mathbf{mj}\, s^3)\ s^3 \big] \big] \big] & \\
\wedge \big[\forall s^4 \big[\big[s \leqslant s^4 \wedge \pi\ (\kappa\, u\, s)\ s\ s^4 \big] & \\
\to \forall s^5 \big[DOX\ \mathbf{mj}\ s^4\ s^5 \to & \\
\forall s^6 (\ is^1\ (\mathbf{b}\ (\kappa\, \mathbf{mj})\, s^6)\ (\mathbf{sm}\ (\kappa\, \mathbf{mj}\, s^6)\ s^6)\ (\kappa\, \mathbf{mj}\, s^6)\, s^6 & \\
\to\ s^5 \perp s^6) \big] \big] \big] \Big], & \\
r \frown \mathbf{mj} \frown (\eta_{\blacklozenge} \lambda s'.\ \mathbf{b}\ (\kappa\, \mathbf{mj})\ s') \frown \mathbf{pp} \frown \mathbf{sm} \Big) & \\
\mid man\ (\mathbf{b}\ (\kappa\, \mathbf{mj})\ s^1)\ (\kappa\, \mathbf{mj}\, s^1)\ s^1 \text{ and} & \\
man\ (\mathbf{b}\ (\kappa\, \mathbf{mj})\ s^3)\ (\kappa\, \mathbf{mj}\, s^3)\ s^3 & \\
\text{and } man\ (\mathbf{b}\ (\kappa\, \mathbf{mj})\ s^5)\ (\kappa\, \mathbf{mj}\, s^5)\ s^5 \Big\} & \\
\#\!\# & \text{otherwise}
\end{cases}
$$

We have shown that in P-HYPE we can give a compositional analysis of certain cases of intensional anaphora using the operations of the continuation monad and the tower notation. In future work we hope to extend our analysis here.

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