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Does the Helmholtz Boundary Element Method Suffer from the Pollution Effect?*

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Abstract. In d dimensions, accurately approximating an arbitrary function oscillating with frequency $\lesssim k$ requires $\sim k^d$ degrees of freedom. A numerical method for solving the Helmholtz equation (with wavenumber k and in d dimensions) suffers from the pollution effect if, as $k \to \infty$, the total number of degrees of freedom needed to maintain accuracy grows faster than this natural threshold (i.e., faster than k^d for domain-based formulations, such as finite element methods, and k^{d-1} for boundary-based formulations, such as boundary element methods).

It is well known that the h-version of the finite element method (FEM) (where accuracy is increased by decreasing the meshwidth h and keeping the polynomial degree p fixed) suffers from the pollution effect, and research over the last $\sim$30 years has resulted in a near-complete rigorous understanding of how quickly the number of degrees of freedom must grow with k to maintain accuracy (and how this depends on both p and properties of the scatterer).

In contrast to the h-FEM, at least empirically, the h-version of the boundary element method (BEM) does *not* suffer from the pollution effect (recall that in the boundary element method the scattering problem is reformulated as an integral equation on the boundary of the scatterer, with this integral equation then solved numerically using a finite element–type approximation space). However, the current best results in the literature on how quickly the number of degrees of freedom for the h-BEM must grow with k to maintain accuracy fall short of proving this.

In this paper, we prove that the h-version of the Galerkin method applied to the standard second-kind boundary integral equations for solving the Helmholtz exterior Dirichlet problem does not suffer from the pollution effect when the obstacle is nontrapping (i.e., does not trap geometric-optic rays). While the proof of this result relies on information about the large-k behavior of Helmholtz solution operators, we show in an appendix how the result can be proved using only Fourier series and asymptotics of Hankel and Bessel functions when the obstacle is a 2-d ball.

Key words. Helmholtz equation, scattering, high frequency, boundary integral equation, boundary element method, pollution effect

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1. Introduction.

The boundary element method is a popular way of computing approximations to solutions of scattering problems involving the Helmholtz equation. It has long been observed, but not yet proved, that this method does not suffer

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from the pollution effect (in contrast to the finite element method [5]). The main result of this paper is that the h-version of the Helmholtz boundary element method, using the standard second-kind boundary integral equations, does not suffer from the pollution effect when the obstacle has Dirichlet boundary conditions and is smooth and nontrapping; see Theorem 2.1 below.

In this introduction, we recap the concepts needed to understand this result, namely, the Helmholtz scattering problem and the concept of nontrapping (section 1.1), a precise definition of the pollution effect (section 1.2), our current understanding of the pollution effect for finite and boundary element methods (sections 1.3–1.4), and the definition of the boundary element method (sections 1.5–1.6). The main result is then stated in section 2, the ideas behind the result are discussed in section 3, and the result is proved in sections 4–6. In the special case when the obstacle is a 2-d ball, an alternative proof of the main result using only Fourier series and asymptotics of Hankel and Bessel functions is given in section A.

1.1. The Helmholtz Scattering Problem.

The Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1.1}$$

with wavenumber $k > 0$ is arguably the simplest possible model of wave propagation. For example, if we look for solutions of the wave equation

$$\partial_t^2 U - c^2 \Delta U = 0 \quad \text{in the form} \quad U(x,t) = u(x)\mathrm{e}^{\pm \mathrm{i}\omega t}, \tag{1.2}$$

then the function $u(x)$ satisfies the Helmholtz equation (1.1) with $k = \omega/c$ (where ω is the angular frequency and c is the wave speed).

Because the Helmholtz equation is at the heart of linear wave propagation, much effort has gone into both studying the properties of its solutions (for example, their asymptotic behavior as $k \to \infty$) and designing methods for computing the solutions efficiently; for the latter, see, e.g., the recent review articles [18, 32, 48, 49].

The main results of this paper concern the classic scattering problem of the Helmholtz equation posed in the exterior of an obstacle with Dirichlet boundary conditions. For simplicity, we state our results for plane-wave scattering by an obstacle with zero Dirichlet boundary conditions; see Remark 2.4 below for how they carry over to the general Dirichlet problem.

Let $\Omega^- \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set—the "scatterer" or "obstacle"—such that its open complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$ is connected. Let $\Gamma := \partial\Omega^-$; our main result requires that Γ is smooth (i.e., C^∞), although the scattering problem is well-defined for Lipschitz Γ. Let $H^1_{\mathrm{loc}}(\Omega^+)$ be the space of functions that are in $H^1(D)$ for every bounded $D \subset \Omega^+$.

Definition 1.1 (plane-wave sound-soft scattering problem). *Given $k > 0$ and the incident plane wave $u^I(x) := \exp(\mathrm{i}kx \cdot \widehat{a})$ for $\widehat{a} \in \mathbb{R}^d$ with $|\widehat{a}| = 1$, find the total field $u \in H^1_{\mathrm{loc}}(\Omega^+)$ satisfying*

$$\Delta u + k^2 u = 0 \ \text{ in } \ \Omega^+, \qquad u = 0 \ \text{ on } \Gamma,$$

and such that $u^S := u - u^I$ satisfies

$$\partial_r u^S - \mathrm{i}k u^S = o\big(r^{(1-d)/2}\big) \text{ as } r := |x| \to \infty, \text{ uniformly in } x/r. \tag{1.3}$$

It is well known that the solution of the sound-soft plane-wave scattering problem exists and is unique; see, e.g., [26, Theorem 3.13], [18, Theorem 2.12 and Corollary 2.13].

Fig. 1.1 *On the left, a nontrapping obstacle, and on the right, a trapping obstacle and one of its trapped rays.*

Condition (1.3) is the *Sommerfeld radiation condition*, and expresses mathematically that with the choice $e^{-i\omega t}$ in (1.2), the scattered wave moves away from the obstacle towards infinity; see, e.g., [58, section 1.1.2].

The key geometric condition that governs the behavior of Helmholtz solutions with k large is that of *trapping/nontrapping* (see, e.g., [67, Epilogue §1]).

Definition 1.2 (nontrapping). *The obstacle $\Omega^- \subset \mathbb{R}^d$ is* nontrapping *if Γ is C^∞ and, given R such that $\overline{\Omega^-} \subset B_R(0)$, there exists $T(R) < \infty$ such that all the billiard trajectories (a.k.a. geometric-optic rays) that start in $\Omega^+ \cap B_R(0)$ at time zero leave $\Omega^+ \cap B_R(0)$ by time $T(R)$.*

If Ω^- is C^∞ and not nontrapping, then we say that it is *trapping*; see Figure 1.1 for an example of a nontrapping obstacle and a trapping obstacle. The requirement that Γ is C^∞ is imposed so that when the billiard trajectories hit Γ, their reflection according to the law of geometric optics ("angle of incidence = angle of reflection") is well-defined (see [81]). There has been much rigorous study of the reflection of high-frequency waves from nonsmooth obstacles (see, e.g., [97, 82] and the references therein), but this does not impact the results of the present paper since we assume that Γ is smooth (see section 3 for a discussion of why we make this assumption).

Our main results are proved under the assumption that Ω^- is nontrapping; in section 3 we discuss how this assumption enters our arguments.

1.2. What Is the Pollution Effect?

Informal Definition. A numerical method for solving the Helmholtz equation (with wavenumber k) suffers from the pollution effect if, as $k \to \infty$, *the total number of degrees of freedom needed to maintain accuracy grows faster than k^n, where n is the dimension of the physical domain in which the problem is formulated.* Having the number of degrees of freedom growing like k^n is the natural threshold for the problem since an oscillatory function with frequency $\lesssim k$ can be accurately approximated by piecewise polynomials with k^n degrees of freedom; this is expected in one dimension from the Nyquist–Shannon–Whittaker sampling theorem [99, 91] (see, e.g., [6, Theorem 5.21.1]) and in arbitrary dimension from the Weyl law for the asymptotics of Laplace eigenvalues [98] (see section 5 for how the notion of the frequency of a function can be defined by Laplace eigenvalues).

Abstract Framework Covering Both BEM and FEM. Let V be a Hilbert space, and let $\mathcal{A} : V \to V'$ be a continuous, invertible linear operator, where V' is the dual space of V. Given $f \in V'$, let $v \in V$ be the solution of $\mathcal{A}v = f$; i.e., $v = \mathcal{A}^{-1}f$.

Let $(V_N)_{N>0}$ be an increasing sequence of finite-dimensional subspaces of V with dimension N (i.e., total number of degrees of freedom N), such that V_N are asymptotically dense in V, in the sense that, for all $w \in V$, the *best approximation error* $\min_{w_N \in V_N} \|w - w_N\|_V \to 0$ as $N \to \infty$.

Let v_N be the computed approximation in V_N to v; we write this as $v_N = (\mathcal{A}^{-1})_N f$, so that $(\mathcal{A}^{-1})_N : V' \to V_N$ is the approximation of the solution operator.

For the finite element method v is the restriction to the computational domain of the solution u of the sound-soft scattering problem (modulo any error incurred by this restriction), V is the space H^1, and $n = d$. For the boundary element methods we consider below, v is a function on Γ (possibly the normal derivative of u), V is $L^2(\Gamma)$ (i.e., square integrable functions on Γ), and $n = d-1$, since the boundary Γ is $(d-1)$-dimensional.

Quasi-optimality. A fundamental property one seeks to prove about a sequence of approximate solutions $(v_N)_{N>0}$ is that they are *asymptotically quasi-optimal*; i.e., there exist $N_0 > 0$ and $C_{\rm qo} > 0$ such that, for all $N \geq N_0$,

(1.4)
$$\|v - v_N\|_V \leq C_{\rm qo} \min_{w_N \in V_N} \|v - w_N\|_V , \quad \text{where } v = \mathcal{A}^{-1} f \text{ and } v_N = (\mathcal{A}^{-1})_N f.$$

The approximate solutions $(v_N)_{N>0}$ would be optimal if $\|v - v_N\|_V = \min_{w_N \in V_N} \|w - w_N\|_V$; "quasi-optimality" is then optimality up to a constant factor, and "asymptotically" refers to the fact that (1.4) holds for sufficiently large N.

The standard analysis of finite and boundary element methods for the Helmholtz equation proves that, for fixed k, the computed solutions are asymptotically quasi-optimal (see, e.g., [14] for FEM and [94, 89] for BEM), i.e., for each $k > 0$ there exists $N_0 = N_0(k)$, depending on k in some unspecified way, such that (1.4) holds.

Precise Definition of the Pollution Effect. The pollution effect is when there exist a choice of N larger than a constant multiple of k^n (i.e., $N \geq \Lambda k^n$ for some $\Lambda > 0$) and some choice of data ($f \in V'$) such that the smallest possible $C_{\rm qo}$ in (1.4) is unbounded in k, that is, if

$$\inf_{\Lambda>0} \limsup_{k\to\infty} \sup_{N \geq \Lambda k^n} \sup_{f \in V'} \inf \left\{ C_{\rm qo} : \left\|\mathcal{A}^{-1} f - (\mathcal{A}^{-1})_N f\right\|_V \leq C_{\rm qo} \min_{w_N \in V_N} \left\|\mathcal{A}^{-1} f - w_N\right\|_V \right\} = \infty; \tag{1.5}$$

see, e.g., [5, Definition 2.1]. Conversely, if the right-hand side of (1.5) is finite, then there exist k_0, Λ, and $C_{\rm qo}$ such that for all $k \geq k_0$, $N \geq \Lambda k^n$, and $f \in V'$,

$$\left\|\mathcal{A}^{-1} f - (\mathcal{A}^{-1})_N f\right\|_V \leq C_{\rm qo} \min_{w_N \in V_N} \left\|\mathcal{A}^{-1} f - w_N\right\|_V ;$$

i.e., k-uniform quasi-optimality is achieved (for all possible data) with a choice of N proportional to k^n.

When the meshes in the FEM or BEM are *quasi-uniform* (informally, all the mesh elements are of comparable size; see [89, Definition 4.1.13] for a precise definition), then the total number of degrees of freedom $N \sim (p/h)^n$, where h is the meshwidth and p the polynomial degree.

In the h-version of the FEM or BEM accuracy is increased by decreasing h and keeping p fixed, and thus $N \sim k^n$ corresponds to $hk \sim 1$. For these methods, the $\sup_{N \geq \Lambda k^n}$ in the definition of the pollution effect (1.5) can then be replaced by $\sup_{\Lambda \geq hk}$.

1.3. The Pollution Effect for Finite Element Methods Is Well Understood. Empirically, the h-version of the FEM applied to the Helmholtz equation suffers from the pollution effect. Furthermore [5] proved that in two or more dimensions the pollution effect is unavoidable for the h-FEM; more precisely, [5] worked in the framework of "generalized FEMs" introduced in [4] and proved that, in two or more dimensions, any method with fixed polynomial degree p (or, more generally, a fixed stencil) suffers from the pollution effect; see [5, Theorem 4.6].

Given that the h-FEM suffers from the pollution effect, two natural questions are the following.

Q1 How must h depend on k for the quasi-optimal error estimate (1.4) to hold with $C_{\rm qo}$ independent of k?

In engineering applications, the most commonly used measure of error is the *relative error*

$$\|v - v_N\|_V \,/\, \|v\|_V \,. \tag{1.6}$$

However, the relative error can only be small when restricting attention to a subclass of data. Indeed, since $\mathcal{A}$ is assumed to be invertible, given V_N, we can choose $v \in V$ orthogonal to V_N, let $f := \mathcal{A}v$, and let $v_N := (\mathcal{A}^{-1})_N f$. Then

$$\|v - v_N\|_V^2 = \|v\|_V^2 + \|v_N\|_V^2 \geq \|v\|_V^2 \,,$$

and thus the relative error cannot be small for all possible data.

Q2 For a physically relevant class of data $\widetilde{V}' \subset V'$ (such as that coming from an incident plane wave as in Definition 1.1), how must h depend on k for the relative error to be controllably small? That is, given $\varepsilon > 0$ and $\widetilde{V}'$, how must h depend on k and ε such that for all $f \in \widetilde{V}'$ the relative error (1.6) is $\leq \varepsilon$?

For the h-FEM applied to nontrapping problems, the answer to Q1 is that $h^p k^{p+1}$ must be sufficiently small, and the answer to Q2 is that $h^{2p}k^{2p+1}$ must be sufficiently small for data oscillating at scale k^{-1}.

These answers were first obtained for 1-d Helmholtz problems by [3, 61, 60] (see also [58, Chapter 4]). Obtaining the multidimensional analogues of these results for a range of different FEMs remains a very active research area; see the papers [76, 88] (the earliest multidimensional results), [35, 100, 101] (on discontinuous Galerkin and interior penalty methods), [24] (on Helmholtz problems on domains with corners), [9, 25, 45, 51] (on variable-coefficient Helmholtz problems), and [69, 46, 41] (on Helmholtz problems with perfectly matched layers).[1]

There has been much research on designing FEMs that mitigate against the pollution effect; four directions of this research are (i) high-order methods [102, 30, 23] and hp methods [79, 80, 33, 78, 65, 40], (ii) Trefftz methods (i.e., using basis functions that are locally solutions of $\Delta u + k^2 u = 0$); see, e.g., the review [56] (in particular [56, section 5], (iii) multiscale methods involving special precomputed test functions [47, 86, 15, 55, 37], and (iv) the so-called discontinuous Petrov Galerkin (DPG) method of [27] (which is a least-squares method in a nonstandard inner product).

[1] We note that the pollution effect for Helmholtz finite element and finite difference methods can also be heuristically studied via so-called dispersion analysis [54, 59, 61, 28, 1]. Here finite element or finite difference schemes are studied on an infinite uniform mesh for problems where an exact solution is $u(x) = \mathrm{e}^{\mathrm{i}kx}$, and one seeks the "discrete wavenumber" $\widetilde{k}$ such that a numerical solution is $u_N(x_j) = \mathrm{e}^{\mathrm{i}\widetilde{k}x_j}$, where x_j are the nodes. The condition "$h^{2p}k^{2p+1}$ sufficiently small" (i.e., the answer to Q2) arises as the condition for $|\widetilde{k} - k|$ to be controllably small; see [60, Theorem 3.2], [58, Theorem 4.22].

1.4. The Pollution Effect for Boundary Element Methods Is Not Yet Rigorously Understood. The situation for the BEM is well summarized by the following quotation from [8]:

> It is generally admitted that Boundary Integral Equations (BIE) lead to less "pollution effect" than FEMs even if to our knowledge, no formal study has confirmed such a property.

Indeed, it is completely standard in the numerical analysis and engineering communities to compute approximations to Helmholtz scattering problems via boundary integral equations (BIEs) using a fixed number of degrees of freedom per wavelength, i.e., $N \sim k^{d-1}$, for both Galerkin [36, 12] and collocation [71, 72] BEMs, and also for Nyström methods [16, 66, 53].[2]

Numerical experiments indicate that, at least for obstacles without strong trapping, the h-BEM is quasi-optimal (with constant independent of k) if hk is sufficiently small; see [70, section 4], [50, section 5]. However, in existing theoretical investigations [17, 7, 70, 77, 50, 43], the best result is that the h-BEM is quasi-optimal (with constant independent of k) for the standard second-kind BIEs for the exterior Dirichlet problem (defined in section 1.5) if $hk^{4/3}$ is sufficiently small and the scatterer is smooth and convex [43, Theorem 1.10(c)] (the current best results for more general domains, which are also in [43], involve higher powers of k).[3]

The results of [70, 77] show, for these same BIEs, that if Γ is analytic and the norm of the inverse of the boundary integral operator is bounded polynomially in k, then there exists $C_1, C_2 > 0$ such that the hp-BEM is quasi-optimal with $C_{\rm qo}$ independent of k if

$$\frac{hk}{p} \leq C_1 \quad \text{and} \quad p \geq C_2 \log k$$

(this is the analogous result to the hp-FEM results mentioned at the end of section 1.3). The abstract to [70] remarks that

> Numerical examples ... even suggest that in many cases quasi-optimality is given under the weaker condition that kh/p is sufficiently small [with p fixed].

In this paper we rigorously explain this observation when the obstacle is nontrapping, showing that in this case the h-BEM does not suffer from the pollution effect.

1.5. The Helmholtz Plane-Wave Sound-Soft Scattering Problem Solved via Boundary Integral Equations.

The Standard Second-Kind Boundary Integral Equations for Solving the Plane-Wave Sound-Soft Scattering Problem. In this section we recall how the solution of the plane-wave sound-soft scattering problem of Definition 1.1 can be expressed in terms of the solution of BIEs involving the operators

$$A_k := \frac{1}{2}I + D_k - \mathrm{i}kS_k \quad \text{and} \quad A'_k := \frac{1}{2}I + D'_k - \mathrm{i}kS_k, \tag{1.7}$$

[2]Intriguingly, however, [73, 11, 74] recently identified a loss of accuracy similar to the pollution effect in the collocation BEM applied to interior Helmholtz problems.

[3]The only rigorous result we know of that is (i) about the convergence of a boundary integral method applied to the Helmholtz equation and (ii) valid only when hk is small is that in [21]. Indeed, for the Helmholtz equation in an infinite half-plane with an impedance boundary condition solved using a collocation BEM and the finite-section method, [21] proved that the error is controllably small, relative to the data, if hk is sufficiently small.

where S_k, D_k, and D'_k are the single-, double-, and adjoint-double-layer operators defined in (1.8) and (1.9) below. The $'$ notation is used since A_k and A'_k are adjoint with respect to the real-valued $L^2(\Gamma)$ inner product.

There are a variety of spaces in which one can pose equations involving A_k and A'_k. The most natural space for solving such equations with the Galerkin method is $L^2(\Gamma)$ (since the inner product is local). When Γ is C^1, S_k, D_k, and D'_k are compact on $L^2(\Gamma)$, and thus A_k and A'_k are compact perturbations of a multiple of the identity. Such integral operators fall into the class of "second-kind" operators—see [2, section 1.1.4]—and the solvability of integral equations involving these operators is covered by Fredholm theory. One can then show that A_k and A'_k are bounded and invertible operators from $L^2(\Gamma)$ to itself when Γ is smooth [26, Theorem 3.33] (indeed, even when Γ is only Lipschitz; see [19, Theorem 2.7], [18, Theorem 2.27]).

How the Boundary Integral Equations (1.7) Are Obtained. Let $\Phi_k(x,y)$ be the fundamental solution of the Helmholtz equation

$$\Phi_k(x,y) := \frac{\mathrm{i}}{4}\left(\frac{k}{2\pi|x-y|}\right)^{(d-2)/2} H^{(1)}_{(d-2)/2}\big(k|x-y|\big) = \begin{cases} \dfrac{\mathrm{i}}{4} H_0^{(1)}\big(k|x-y|\big), & d=2,\\ \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, & d=3, \end{cases}$$

where $H^{(1)}_m$ denotes the Hankel function of the first kind of order m (see, e.g., [93, equation 5.118]). The single- and double-layer potentials, $\mathcal{S}_k$ and $\mathcal{D}_k$ respectively, are defined for $k \in \mathbb{C}$, $\phi \in L^2(\Gamma)$, and $x \in \mathbb{R}^d \setminus \Gamma$ by

$$\mathcal{S}_k\phi(x) = \int_\Gamma \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y) \quad \text{and} \quad \mathcal{D}_k\phi(x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial\nu(y)}\phi(y)\,\mathrm{d}s(y).$$

The standard single-layer, double-layer, and adjoint-double-layer operators are defined for $k \in \mathbb{C}$, $\phi \in L^2(\Gamma)$, and $x \in \Gamma$ by

$$S_k\phi(x) := \int_\Gamma \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \qquad D_k\phi(x) := \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial\nu(y)}\phi(y)\,\mathrm{d}s(y), \tag{1.8}$$

$$D'_k\phi(x) := \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial\nu(x)}\phi(y)\,\mathrm{d}s(y); \tag{1.9}$$

when Γ is C^2, the integrals defining S_k, D_k, and D'_k are all weakly singular; see, e.g., [26, page 6 and section 2.4].

THEOREM 1.3 (the plane-wave sound-soft scattering problem formulated in terms of BIEs).

(i) *If u is solution of the plane-wave sound-soft scattering problem of Definition* 1.1, *then*

$$A'_k\partial_\nu u = \partial_\nu u^I - \mathrm{i}k u^I \quad \textit{and} \quad u = u^I - \mathcal{S}_k(\partial_\nu u). \tag{1.10}$$

(ii) *If $v \in L^2(\Gamma)$ is the solution to*

$$A_k v = -u^I, \quad \textit{then} \quad u = u^I + (\mathcal{D}_k - \mathrm{i}k\mathcal{S}_k)v \tag{1.11}$$

is the solution of the plane-wave sound-soft scattering problem of Definition 1.1.

References for the proof and summary of the ideas. Part (i) is proved in, e.g., [18, Theorem 2.46]. Part (ii) is proved in, e.g., [18, equations 2.70–2.72]. Both parts use that $\mathcal{D}_k v$ and $\mathcal{S}_k v$ satisfy the Helmholtz equation away from Γ and satisfy the radiation condition (1.3). Part (i) uses that $u(x) = u^I(x) - \mathcal{S}_k(\partial_\nu u)(x)$ for $x \in \Omega^+$ by Green's integral representation theorem (applied to u^S in Ω^+ and u^I in Ω^-); this is the so-called *direct method*. Taking a linear combination of the limits of both this representation and its normal derivative as x approaches Γ from Ω^+, we obtain the integral equation in (1.10). This idea of taking a linear combination goes back to [13, 68, 85] and ensures that A'_k is invertible. Part (ii) poses the ansatz that $u^S(x) = (\mathcal{D}_k - \mathrm{i}k\mathcal{S}_k)v(x)$ for $x \in \Omega^+$ for some unknown density v; this is the so-called *indirect method*. Imposing the boundary condition that $u^S = -u^I$ on Γ, we obtain the integral equation (1.11). ∎

1.6. The Galerkin Method and Assumptions on the Boundary Element Space. We consider solving the BIE $\mathcal{A}v = f$ in $L^2(\Gamma)$ with the Galerkin method: given a finite-dimensional subspace $V_N \subset L^2(\Gamma)$,

(1.12)
$$\text{find } v_N \in V_N \text{ such that } \quad (\mathcal{A}v_N, w_N)_{L^2(\Gamma)} = (f, w_N)_{L^2(\Gamma)} \quad \text{for all } w_N \in V_N.$$

The abstract framework in section 1.2 involved the operator $(\mathcal{A}^{-1})_N$ mapping the data to the approximate solution; we show in section 4 below (see (4.3)) that, for the Galerkin method, $(\mathcal{A}^{-1})_N = (P_N\mathcal{A})^{-1}P_N$, where P_N is the orthogonal projection from V to V_N and $P_N\mathcal{A}$ is considered as an operator from V_N to V_N (after using the fact that V is a Hilbert space to identify V and V').

The h-version of the BEM uses a sequence of approximation spaces $(V_{N_h})_{h>0}$ given by piecewise polynomials of degree p for some fixed $p \geq 0$ on a sequence of meshes of diameter $h > 0$; for ease of notation we let $(V_h)_{h>0} := (V_{N_h})_{h>0}$. It is well known that when the meshes are additionally *shape-regular* (for each element, its width divided by the diameter of the largest inscribed ball is uniformly bounded; see [89, Definition 4.1.12]), these subspaces satisfy the following assumption.

ASSUMPTION 1.4. *$(V_h)_{h>0}$ is a sequence of finite-dimensional subspaces of $L^2(\Gamma)$, and there exists $C_{\rm approx} > 0$ such that for all $h > 0$*

$$\min_{w_h \in V_h} \|w - w_h\|_{L^2(\Gamma)} \leq C_{\rm approx} h \|w\|_{H^1(\Gamma)} \quad \text{for all } w \in H^1(\Gamma). \tag{1.13}$$

(Recall that $\|w\|^2_{H^1(\Gamma)} := \|\nabla_\Gamma w\|^2_{L^2(\Gamma)} + \|w\|^2_{L^2(\Gamma)}$, where ∇_Γ is the surface gradient operator, defined in terms of a parametrization of the boundary by, e.g., [18, equation A.14].)

Indeed, piecewise-polynomial subspaces satisfying Assumption 1.4 are described in [89, Chapter 4], with [89, Theorem 4.3.22] showing that the spaces of continuous boundary element functions denoted by $\mathcal{S}^{p,0}_{\mathcal{G}}$ [89, Definition 4.1.36] satisfy Assumption 1.4 and [89, Theorem 4.3.19] showing that the spaces of discontinuous boundary element functions denoted by $\mathcal{S}^{p,-1}_{\mathcal{G}}$ [89, Definition 4.1.17] satisfy Assumption 1.4. Note that in these cases, the constant $C_{\rm approx}$ depends on p.

We highlight that Assumption 1.4 is the only requirement on $(V_h)_{h>0}$ needed below. There are sequences $(V_h)_{h>0}$ arising from piecewise polynomials on non-quasi-uniform sequences of meshes that satisfy Assumption 1.4; however, as mentioned in section 1.2, quasi-uniformity is required for the total number of degrees of freedom to $\sim (p/h)^d$.

2. The Main Result: The h-BEM Does Not Suffer from the Pollution Effect.

THEOREM 2.1 (quasi-optimal error estimate for hk sufficiently small). *Suppose that Ω^- is nontrapping and $(V_h)_{h>0}$ satisfies Assumption* 1.4.

For all $k_0 > 0$, there exist $C_{\rm ppw} > 0$ and $C_{\rm qo} > 0$ such that if $\mathcal{A}$ is either A_k or A'_k,

$$hk \le C_{\rm ppw} \quad \text{and} \quad k \ge k_0, \tag{2.1}$$

then for all $f \in L^2(\Gamma)$, the Galerkin solution v_N to (1.12) *exists, is unique, and satisfies*

$$\|v - v_N\|_{L^2(\Gamma)} \le C_{\rm qo} \min_{w_N \in V_h} \|v - w_N\|_{L^2(\Gamma)}. \tag{2.2}$$

The order of the quantifiers in Theorem 2.1 (and also later results in the paper) dictates what the constants depend on; e.g., in Theorem 2.1, $C_{\rm ppw}$ and $C_{\rm qo}$ depend on Ω_-, the spaces $(V_h)_{h>0}$, and k_0, but are independent of k, h, and the choice of A_k or A'_k.

The subscript "ppw" on $C_{\rm ppw}$ indicates that, via (2.1), this constant controls the number of points per wavelength. If the spaces $(V_h)_{h>0}$ are quasi-uniform, then $N \sim h^{-d}$, and thus Theorem 2.1 shows that the Galerkin method is quasi-optimal (with constant independent of k) when the total number of degrees of freedom is a multiple of k^d; i.e., *the h-BEM does not suffer from the pollution effect.*

Theorem 2.1 covers the Galerkin method applied to $\mathcal{A}v = f$ for general $f \in L^2(\Gamma)$. We now restrict our attention to the case when the data comes from the plane-wave sound-soft scattering problem (i.e., the right-hand side f is as described in Theorem 1.3), and bound the relative error. To do this, we use in the bound (2.2) the bound (1.13) from Assumption 1.4 and the following lemma (proved in [44]), describing the oscillatory character of the solution v in this case.

LEMMA 2.2 (bound on the unknown v in the BIEs for the sound-soft scattering problem). *Given $k_0 > 0$ there exists $C_{\rm reg} > 0$ (with the subscript "*reg*" standing for "regularity") such that if $\mathcal{A}$ is one of A_k, A'_k and v is the solution to $\mathcal{A}v = f$ where the right-hand side f is as described in Theorem* 1.3*, then*

$$\|v\|_{H^1(\Gamma)} \le C_{\rm reg} k \|v\|_{L^2(\Gamma)} \quad \text{for all } k \ge k_0.$$

COROLLARY 2.3 (bound on the relative error for hk sufficiently small). *Suppose that Ω^- is nontrapping and $(V_h)_{h>0}$ satisfies Assumption* 1.4*. For all $k_0 > 0$, there exist $C_{\rm ppw} > 0$ and $C_{\rm qo} > 0$ such that if $\mathcal{A}$ is either A_k or A'_k and* (2.1) *holds, then for all data f coming from the plane-wave sound-soft scattering problem the Galerkin solution v_N to* (1.12) *exists, is unique, and satisfies*

$$\|v - v_N\|_{L^2(\Gamma)} \le C_{\rm qo} C_{\rm reg} hk \|v\|_{L^2(\Gamma)}. \tag{2.3}$$

The bound (2.3) shows that a prescribed relative error can be achieved with a choice of h such that $hk \sim 1$. Indeed, given $\varepsilon > 0$,

$$\text{if } \ hk \le \min\left\{\varepsilon (C_{\rm qo} C_{\rm reg})^{-1}, C_{\rm ppw}\right\}, \quad \text{then} \quad \|v - v_N\|_{L^2(\Gamma)} / \|v\|_{L^2(\Gamma)} \le \varepsilon.$$

REMARK 2.4 (general Dirichlet boundary conditions). *The general exterior Dirichlet problem is: given $k > 0$ and $g_D \in H^{1/2}(\Gamma)$, find $u^S \in H^1_{\rm loc}(\Omega^+)$ such that $\Delta u^S + k^2 u^S = 0$ in Ω^+, $u^S = g_D$ on Γ, and u^S satisfies the radiation condition* (1.3).

For the indirect method, we pose the ansatz $u^S(x) = (\mathcal{D}_k - \mathrm{i}k\mathcal{S}_k)v(x)$ *for* $x \in \Omega^+$ *and take the limit of this as* x *approaches* Γ *from* Ω^+ *to obtain the equation* $A_k v = g_D$. *Since* $g_D \in H^{1/2}(\Gamma)$, *this is a priori an equation in* $H^{1/2}(\Gamma)$; *however, since* A_k *is bounded and invertible as an operator from* $H^s(\Gamma)$ *to itself for* $0 \le s \le 1$ [18, Theorem 2.27], *and* $H^{1/2}(\Gamma) \subset L^2(\Gamma)$, *we can consider this equation in* $L^2(\Gamma)$, *and solve it using the Galerkin method as in section* 1.6. *In contrast, the exterior Dirichlet problem can only be solved by the direct method with the integral equation posed in* $L^2(\Gamma)$ *when* $g_D \in H^1(\Gamma)$; *see* [18, section 2.6].

3. Discussions of the Ideas behind the Proof of Theorem 2.1. The proof of Theorem 2.1 consists of three ingredients.

1. A slight modification of a standard condition for quasi-optimality of the Galerkin method applied to operators that are a perturbation of the identity (see (4.5) in Theorem 4.2 below), with this condition based on writing the Galerkin method as a projection method and using the result that if $\|T\| < 1$, then $I + T$ is invertible with $\|(I+T)^{-1}\| \le (1 - \|T\|)^{-1}$.
2. Bounds on the components of the boundary integral operators S_k, D_k, and D_k that have frequencies $> k$ (see Theorem 5.1), where the statement that a function has "frequencies $> k$" is understood by expanding the function in terms of eigenfunctions of the surface Laplacian on Γ (see section 5).

We see in section 6 that these two ingredients prove the following result.

LEMMA 3.1. *Suppose* $(V_h)_{h>0}$ *satisfies Assumption* 1.4. *For all* $k_0 > 0$, *there exists* $C_1 > 0$ *such that if* $k \ge k_0$, $\mathcal{A}$ *is either* A_k *or* A_k', *and*

$$hk\big(1 + \big\|\mathcal{A}^{-1}\big\|_{L^2(\Gamma)\to L^2(\Gamma)}\big) \le C_1, \tag{3.1}$$

then for all $f \in L^2(\Gamma)$, *the Galerkin solution* v_N *to* (1.12) *exists, is unique, and satisfies*

$$\|v - v_N\|_{L^2(\Gamma)} \le 2\,\big\|\mathcal{A}^{-1}\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \min_{w_N \in V_N} \|v - w_N\|_{L^2(\Gamma)}\,. \tag{3.2}$$

The result of Theorem 2.1 then follows from the third ingredient (note that this is the only place where our arguments use the nontrapping assumption).

3. If Ω^- is nontrapping, then, given $k_0 > 0$, there exists $C > 0$ such that

$$\|\mathcal{A}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C \qquad \text{for all } k \ge k_0. \tag{3.3}$$

Discussion of Point 1. It is perhaps surprising that the simple condition from Theorem 4.2, combined with points 2 and 3, gives a better result for the Galerkin method applied to $\mathcal{A}$ (at least when Ω^- is nontrapping) than more sophisticated conditions for quasi-optimality used in [17, 7, 70], which are all ultimately based on the ideas in the "Schatz argument" in the finite element setting; see [90, 88].

Discussion of Point 2. The bounds on the high-frequency components of S_k, D_k, and D_k' in Theorem 5.1 come from viewing these boundary integral operators as *semiclassical pseudodifferential operators*. We do not need any of the details of these operators in this paper, but it is instructive to discuss briefly here how, on the one hand, using pseudodifferential operators to study boundary integral equations is completely standard, but, on the other hand, the full potential of these operators for studying Helmholtz problems with large k has not been fully exploited.

Recall that the theory of standard pseudodifferential operators on a smooth surface Γ can be viewed as a generalization of Fourier analysis on the circle. The use of pseudodifferential properties in both the analysis and numerical analysis of boundary integral equations is both well established and current; see, e.g., the books [87, 57, 52].

A class of pseudodifferential operators exists that is tailor-made for studying problems where oscillations happen at a large frequency k; these are precisely *semiclassical pseudodifferential operators* [103], [31, Appendix E]. The adjective "semiclassical" essentially means "high frequency" and comes from the origin of this theory in the study of how classical dynamics arise from quantum mechanics in the high-energy limit (see, e.g., [103, section 1]).

Whereas S_k, D_k, and D'_k are standard pseudodifferential operators (of order -1; see, e.g., [57, section 9.2.2], [87, section 7], [95, Chapter 7, section 11]), they are *not* semiclassical pseudodifferential operators. Instead, each is the sum of a semiclassical pseudodifferential operator and an operator acting only on frequencies $\leq k$ that transports mass between points on the boundary connected by rays; this decomposition was recently established in [38, Chapter 4], with [38, Lemma 4.27] explicitly writing out the decomposition when Γ is curved. The estimates on boundary layer operators at high frequency in Theorem 5.1 were then proved using the ideas from [38, Chapter 4] in [42, Theorem 4.3].

Finally, we note that the assumption in section 1.1 that Γ is smooth is because the theory of pseudodifferential operators is simplest on smooth domains. In principle, Lemma 3.1 holds when Γ is C^M for some $M > 0$, and one could go through the arguments to determine a sufficiently large value of M; alternatively one could use more sophisticated pseudodifferential techniques to lower the regularity further; see, e.g., [96, Chapter 13].

Discussion of Point 3. The estimate (3.3) is proved in [10, Theorem 1.13] using the following decompositions of A_k^{-1} and $(A'_k)^{-1}$ [18, Theorem 2.33]:

$$A_k^{-1} = I - (\mathrm{ItD})^-\big[(\mathrm{DtN})^+ - \mathrm{i}k\big] \quad \text{and} \quad (A'_k)^{-1} = I - \big[(\mathrm{DtN})^+ - \mathrm{i}k\big](\mathrm{ItD})^-. \tag{3.4}$$

Here, $(\mathrm{DtN})^+$ is the Dirichlet-to-Neumann map for the Helmholtz equation $\Delta u^S + k^2 u^S = 0$ in Ω^+ satisfying the Sommerfeld radiation condition (1.3), and $(\mathrm{ItD})^-$ is the map $g \mapsto u|_\Gamma$ where, given $g \in L^2(\Gamma)$, $u \in H^1(\Omega^-)$ is the solution of the *interior impedance problem*

$$\Delta u + k^2 u = 0 \ \text{ in } \Omega^-, \quad \partial_\nu u - \mathrm{i}ku = g \ \text{ on } \Gamma. \tag{3.5}$$

The decompositions in (3.4) imply that bounds on A_k^{-1} and $(A'_k)^{-1}$ can be obtained from k-explicit bounds on $(\mathrm{DtN})^+$ and $(\mathrm{ItD})^-$. These estimates are obtained in [10] for nontrapping Ω^- (following the proof in [20, Theorem 4.3] of the analogous bounds for Ω^- that are star-shaped with respect to a ball).

The presence of $(\mathrm{DtN})^+$ in (3.4) is expected since $(\mathrm{DtN})^+$ is essentially the solution operator for the problem (and we are using the Galerkin method applied to A_k or A'_k to approximate this solution operator). The map $(\mathrm{ItD})^-$ appears in (3.4) since A_k and A'_k can also be used to solve the interior impedance problem; see, e.g., [18, Theorem 2.30].

We highlight that proving k-explicit bounds on exterior Helmholtz solution operators is a classic problem considered since the 1960s, with interest in the interior impedance problem (3.5) arising more recently *both* from this problem's role in determining the behavior of A_k and A'_k *and* because this problem is often used as a model problem in the numerical analysis of FEMs; see the literature reviews in [22], [64] (for exterior problems), [92, section 1.2], [10, section 1.2] (for both exterior and interior problems), and [39, sections 1.1 and 1.4] (for interior problems).

When Ω^- is trapping, $\|\mathcal{A}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ grows with k (see [22, 64]). Thus, although (3.1), (3.2) give a result about convergence of the h-BEM for Ω^- trapping, this result does not show that the h-BEM does not suffer from the pollution effect. The experiments in [50, Figure 2] indicate that at least for a certain form of mild trapping (so-called *parabolic trapping*), the h-BEM might not suffer from the pollution effect, although proving this remains open.

4. Formulation of the Galerkin Method as a Projection Method and an Abstract Condition for Quasi-optimality. As in section 1.2, V is a Hilbert space with dual V', and we let $B : V \to V'$ be a continuous, invertible, linear operator. Later we restrict attention to the case when B is a perturbation of the identity, i.e., $B = I+K$, and we apply these results with $B = 2\mathcal{A}$, with $\mathcal{A}$ one of A'_k and A_k (since A_k and A'_k are perturbations of $\frac{1}{2}I$ (1.7)).

Given $f \in V'$, let v be the solution of the variational problem

$$\text{find } v \in V \text{ such that} \quad \langle Bv, w\rangle_{V'\times V} = \langle f, w\rangle_{V'\times V} \quad \text{for all } w \in V, \tag{4.1}$$

i.e., $v = B^{-1}f$. Then, given $V_N \subset V$ closed, the Galerkin approximation to v with respect to V_N, $v_N =: (B^{-1})_N f$, is defined as the solution of the Galerkin equations

$$\text{find } v_N \in V_N \text{ such that} \quad \langle Bv_N, w_N\rangle_{V'\times V} = \langle f, w_N\rangle_{V'\times V} \quad \text{for all } w_N \in V_N. \tag{4.2}$$

We now rewrite equations (4.2) using the orthogonal projection operator $P_N : V \to V_N$. Then, $(I - P_N)$ is the orthogonal projection onto the orthogonal complement of V_N and, in particular,

$$\|(I - P_N)w\|_V = \min_{w_N \in V_N} \|w - w_N\|_V .$$

The Galerkin equations (4.2) are then equivalent to the operator equation

$$P_N B v_N = P_N f, \qquad v_N \in V_N, \tag{4.3}$$

where we have used that V is a Hilbert space to identify V and V' when applying P_N to B on the left. If $B = I + K$, then, since $v_N \in V_N$, (4.3) simplifies to

$$(I + P_N K)v_N = P_N f; \tag{4.4}$$

see, e.g., [2, section 3.1.3], [63, section 13.6]. Despite the fact that formally (4.4) is posed on V_N, the operator $I + P_N K$ as an operator on V maps $V_N \to V_N$ and hence we can study the operator $(I + P_N K)$ as a mapping $V \to V$.

LEMMA 4.1 (quasi-optimality in terms of the norm of the discrete inverse). *If $I + P_N K : V \to V$ is invertible, then the Galerkin solution, v_N, solving* (4.2) *exists, is unique, and satisfies*

$$\|v - v_N\|_V \leq \left\|(I + P_N K)^{-1}\right\|_{V\to V} \|(I - P_N)v\|_V .$$

Proof. Since $I + P_N K : V \to V$ is invertible and $I + P_N K : V_N \to V_N$, the solution v_N to (4.4) exists, lies in V_N, and is unique as an element of V. Then, by (4.1) and (4.2),

$$\begin{aligned}(I + P_N K)(v - v_N) &= (I + P_N K)v - P_N f \\ &= v + P_N K v - P_N\big((I + K)v\big) = (I - P_N)v. \qquad \square\end{aligned}$$

THEOREM 4.2 (sufficient condition for quasi-optimality). *Let $\delta > 0$. If $B = I+K$ and*

$$\big\|(I - P_N)K(I+K)^{-1}\big\|_{V\to V} \leq 1-\delta, \tag{4.5}$$

then the Galerkin solution v_N, solving (4.4)*, exists, is unique, and satisfies*

$$\|v - v_N\|_V \leq \delta^{-1}\big\|(I+K)^{-1}\big\|_{V\to V}\,\|(I-P_N)v\|_V\,. \tag{4.6}$$

Proof. The basis of the proof of (4.6) is Lemma 4.1 and the result that if $\|T\| < 1$, then $I + T$ is invertible with $\|(I+T)^{-1}\| \leq (1-\|T\|)^{-1}$. Indeed,

$$I + P_N K = I + K - (I-P_N)K = \Big(I - (I-P_N)K(I+K)^{-1}\Big)(I+K). \tag{4.7}$$

Therefore, if (4.5) holds, then

$$(I+P_NK)^{-1} = (I+K)^{-1}\Big(I - (I-P_N)K(I+K)^{-1}\Big)^{-1}.$$

Thus, by (4.5),

$$\|(I+P_NK^{-1})\|_{V\to V} \leq \delta^{-1}\|(I+K)^{-1}\|_{V\to V},$$

and the result (4.6) follows from applying Lemma 4.1 ∎

REMARK 4.3. *An analogous result to Theorem* 4.2 *under the condition*

$$\big\|(I+K)^{-1}(I-P_N)K\big\|_{V\to V} < 1 \tag{4.8}$$

is stated in, e.g., [63, Theorem 10.1], [2, Theorem 3.1.1]*; this result was used in the h-BEM context in* [50], [43, Lemma 3.3]. *Here we factor out* $(I+K)$ *from the right in* (4.7)*, rather than the left, leading to* (4.5) *rather than* (4.8).

5. The High-Frequency Behavior of the Boundary Integral Operators S_k, D_k, and D'_k.

Functions of the Surface Laplacian Defined via Eigenfunction Expansion. Let λ_j be the eigenvalues of the surface Laplacian (a.k.a. the Laplace–Beltrami operator) $-\Delta_\Gamma$, and let $\{u_{\lambda_j}\}_{j=1}^\infty$ be an orthonormal basis for $L^2(\Gamma)$ of eigenfunctions, i.e.,

$$(-\Delta_\Gamma - \lambda_j)u_{\lambda_j} = 0 \quad \text{and} \quad \big\|u_{\lambda_j}\big\|_{L^2(\Gamma)} = 1;$$

when Γ is the unit circle, $\{u_{\lambda_j}\}_{j=1}^\infty$ can be taken to be $\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}jt}\}_{j=-\infty}^\infty$; see section A.1 below.

We then define functions of $-\Delta_\Gamma$ using expansions in this basis. Precisely, for a function $f \in L^\infty(\mathbb{R})$ and $v \in L^2(\Gamma)$,

$$f(-\Delta_\Gamma)v := \sum_{j=1}^\infty f(\lambda_j)(v, u_{\lambda_j})_{L^2(\Gamma)}u_{\lambda_j}. \tag{5.1}$$

By taking norms and using orthonormality of the basis, we see that

$$\|f(-\Delta_\Gamma)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \|f\|_{L^\infty(\mathbb{R})}\,. \tag{5.2}$$

Frequency Cut-Offs Defined as Functions of the Surface Laplacian. With u_{λ_j} defined above, we say that "a function v has frequency $\geq M$" if, for some $a_{\lambda_j} \in \mathbb{C}$,

$$v = \sum_{\lambda_j \geq M^2} a_{\lambda_j} u_{\lambda_j}.$$

For $\chi \in C^\infty_{\rm comp}(\mathbb{R})$ with $\chi \equiv 1$ on $U \subset \mathbb{R}$, the operator $(I - \chi(-k^{-2}\Delta_\Gamma))$ therefore restricts to functions with frequencies outside the set kU. In particular, if $\chi \equiv 1$ in a neighborhood of $[-1,1]$, then $(I - \chi(-k^{-2}\Delta_\Gamma))$ restricts to functions with frequencies $> k$.

THEOREM 5.1 (the high-frequency behavior of S_k, D_k, and D'_k). *Suppose $\chi \in C^\infty_{\rm comp}(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of $[-1,1]$. Then for all $k_0 > 0$ there exists $C > 0$ such that for all $k \geq k_0$,*

(5.3)
$$\begin{gathered}\big\|(I - \chi(-k^{-2}\Delta_\Gamma))D_k\big\|_{L^2(\Gamma)\to H^1(\Gamma)} + \big\|(I - \chi(-k^{-2}\Delta_\Gamma))D'_k\big\|_{L^2(\Gamma)\to H^1(\Gamma)} \leq Ck, \\ \big\|(I - \chi(-k^{-2}\Delta_\Gamma))S_k\big\|_{L^2(\Gamma)\to H^1(\Gamma)} \leq C.\end{gathered}$$

By the discussion above, we see that the bounds in (5.3) are bounds on the outputs of D_k, D'_k, and S_k with frequencies $> k$.

References for the proof of Theorem 5.1. This is proved in [42, Theorem 4.4 and Remark 4.6]. We note that the key ingredient [42, Lemma 3.10] is a simplified version of [38, Lemma 4.27], and is the semiclassical analogue of [95, Chapter 7, section 11] and [57, Theorem 8.4.3]. ∎

LEMMA 5.2 (smoothing property of compactly supported functions of $-k^{-2}\Delta_\Gamma$). *Suppose that $f \in L^\infty_{\rm comp}(\mathbb{R})$. Then for all $s \geq 0$ there exists $C_{s,f} > 0$ such that*

$$\big\|f(-k^{-2}\Delta_\Gamma)\big\|_{L^2(\Gamma)\to H^s(\Gamma)} \leq C_{s,f} k^s \quad \textit{for all } k > 0. \tag{5.4}$$

Proof. By elliptic regularity, given $\ell > 0$ there exists C_ℓ such that for all v

$$\|v\|_{H^{2\ell}(\Gamma)} \leq C_\ell \Big(\big\|(-\Delta_\Gamma)^\ell v\big\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)} \Big);$$

this follows from interior regularity for second-order elliptic operators with variable coefficients; see, e.g., [34, section 6.3.1]. Thus

(5.5)
$$\big\|f(-k^{-2}\Delta_\Gamma)v\big\|_{H^{2\ell}(\Gamma)} \leq C_\ell \Big(\|(-\Delta_\Gamma)^\ell f(-k^{-2}\Delta_\Gamma)v\|_{L^2(\Gamma)} + \big\|f(-k^{-2}\Delta_\Gamma)v\big\|_{L^2(\Gamma)} \Big).$$

By (5.2), the last term on the right-hand side of (5.5) is bounded by $C\|v\|_{L^2(\Gamma)}$ for C depending on f but independent of k. For the first term on the right-hand side of (5.5) we use that fact that $s^\ell f(s) \in L^\infty$ (since f has compact support) to see that

$$\begin{aligned}\|(-\Delta_\Gamma)^\ell f(-k^{-2}\Delta_\Gamma)\|_{L^2(\Gamma)\to L^2(\Gamma)} &= k^{2\ell}\|(-k^{-2}\Delta_\Gamma)^\ell f(-k^{-2}\Delta_\Gamma)\|_{L^2(\Gamma)\to L^2(\Gamma)} \\ &\leq k^{2\ell}\|s^\ell f(s)\|_{L^\infty} \leq \widetilde{C}_\ell k^{2\ell}\end{aligned}$$

for some $\widetilde{C}_\ell > 0$. Using these bounds in (5.5) we obtain the bound (5.4) for even s. The bound for odd s then follows by interpolation (see, e.g., [75, Theorem B.2]) using the fact that $H^s(\Gamma)$ is an interpolation scale (see, e.g., [75, Theorem B.11]). ∎

6. Proof of Theorem 2.1. It is sufficient to prove Lemma 3.1, since Theorem 2.1 then follows from the bound (3.3).

As described in section 3, we use Theorems 4.2 and 5.1. We apply the former with $B = 2\mathcal{A}$, so that $K = 2\mathcal{A} - I$, and $\delta = 1/2$. Thus, we only need to prove that there exists $C_1 > 0$ (independent of h and k) such that if (3.1) holds, then

$$\left\|(I - P_N)(2\mathcal{A} - I)(2\mathcal{A})^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \tfrac{1}{2}.$$

By the bound (1.13) from Assumption 1.4, it is sufficient to show that there exists $C_1 > 0$ (independent of h and k) such that if (3.1) holds, then

$$hC_{\text{approx}} \left\|(2\mathcal{A} - I)(2\mathcal{A})^{-1}\right\|_{L^2(\Gamma)\to H^1(\Gamma)} \le \tfrac{1}{2}.$$

We therefore only need to show that

$$\left\|(2\mathcal{A} - I)(2\mathcal{A})^{-1}\right\|_{L^2(\Gamma)\to H^1(\Gamma)} \le C_2 k\big(1 + \left\|\mathcal{A}^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)}\big) \tag{6.1}$$

for some $C_2 > 0$ (independent of h and k), and then the result holds with $C_1 := (2C_{\text{approx}}C_2)^{-1}$.

To prove (6.1), let $\chi \in C^\infty_{\text{comp}}(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of $[-1, 1]$. Since $1 = \chi + (1 - \chi)$,

$$\begin{aligned}(2\mathcal{A} - I)(2\mathcal{A})^{-1} &= \chi(-k^{-2}\Delta_\Gamma)(2\mathcal{A} - I)(2\mathcal{A})^{-1} + \big(I - \chi(-k^{-2}\Delta_\Gamma)\big)(2\mathcal{A} - I)(2\mathcal{A})^{-1}\\ &= \chi(-k^{-2}\Delta_\Gamma)(I - (2\mathcal{A})^{-1}) + \big(I - \chi(-k^{-2}\Delta_\Gamma)\big)(2\mathcal{A} - I)(2\mathcal{A})^{-1}. \qquad (6.2)\end{aligned}$$

To deal with the first term on the right-hand side of (6.2), we use Lemma 5.2 applied with $f = \chi$ to find that

$$\begin{aligned}&\left\|\chi(-k^{-2}\Delta_\Gamma)(I - (2\mathcal{A})^{-1})\right\|_{L^2(\Gamma)\to H^1(\Gamma)}\\ &\qquad \le \left\|\chi(-k^{-2}\Delta_\Gamma)\right\|_{L^2(\Gamma)\to H^1(\Gamma)} \big(1 + \left\|(2\mathcal{A})^{-1}\right\|_{L^2\to L^2}\big)\\ &\qquad \le C_3 k(1 + \|\mathcal{A}^{-1}\|_{L^2\to L^2}) \qquad (6.3)\end{aligned}$$

for some $C_3 > 0$ (independent of h and k). We now consider the second term on the right-hand side of (6.2) when $\mathcal{A} = A_k$; the proof when $\mathcal{A} = A'_k$ follows in exactly the same way, just replacing D'_k by D_k. By the definition of A_k (1.7) and Theorem 5.1,

$$\begin{aligned}&\left\|(I - \chi(-k^{-2}\Delta_\Gamma))(2\mathcal{A} - I)(2\mathcal{A})^{-1}\right\|_{L^2(\Gamma)\to H^1(\Gamma)}\\ &\qquad = \left\|(I - \chi(-k^{-2}\Delta_\Gamma))(D_k - \mathrm{i}kS_k)(A_k)^{-1}\right\|_{L^2(\Gamma)\to H^1(\Gamma)}\\ &\qquad \le \left\|(I - \chi(-k^{-2}\Delta_\Gamma))(D_k - \mathrm{i}kS_k)\right\|_{L^2(\Gamma)\to H^1(\Gamma)} \left\|A_k^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)}\\ &\qquad \le C_4 k \left\|A_k^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)}\end{aligned} \tag{6.4}$$

for some $C_4 > 0$ (independent of h and k). Combining (6.3) and (6.4) we obtain (6.1), and the proof is complete.

Appendix A. A Simple Proof of Theorem 2.1 When Γ Is the Unit Circle. Ultimately, the most flexible tools to study the large-k behavior of Helmholtz boundary integral operators come from semiclassical analysis. Nevertheless, in the special case when Γ is the unit circle, Theorem 2.1 can be proved using only results about Fourier series and the asymptotics of Bessel and Hankel functions. The advantage of the latter proof is that it only uses classical tools of applied mathematics; furthermore, since we write this proof mirroring the general proof in section 6, we hope it makes the ideas in section 6 clearer.

A.1. Recap of Fourier-Series Results. Suppose Γ is the unit circle, with parametrization $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi)$. With this parametrization, $L^2(\Gamma)$ is isometrically isomorphic to $L^2(0, 2\pi)$. Given $v \in L^2(0, 2\pi)$, define the nth Fourier coefficient of v by

$$\widehat{v}_n := \frac{1}{\sqrt{2\pi}}\big(v, \mathrm{e}^{\mathrm{i}n\cdot}\big)_{L^2(0,2\pi)} = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nt} v(t)\,\mathrm{d}t, \quad \text{so that} \quad v(t) = \sum_{n=-\infty}^{\infty} \widehat{v}_n \frac{\mathrm{e}^{\mathrm{i}nt}}{\sqrt{2\pi}}$$

as an L^2 function. Parseval's theorem states that

$$\|v\|^2_{L^2(0,1)} = \sum_{n=-\infty}^{\infty} |\widehat{v}_n|^2 \quad \text{and thus} \quad \|v\|^2_{H^1(0,1)} := \sum_{m=-\infty}^{\infty} (1+m^2)|\widehat{v}_m|^2. \tag{A.1}$$

A.2. Results about the Eigenvalues of $2A_k$. When Γ is a circle, $A_k = A'_k$ since $D_k = D'_k$; this follows from the definitions of D_k and D'_k and the geometric property that $(x-y)\cdot\nu(y) = (x-y)\cdot\nu(x)$ for x, y on a circle.

LEMMA A.1 (expression for eigenvalues of $2A_k$ in terms of Bessel and Hankel functions). *If*

$$\lambda_m(k) := \pi k H^{(1)}_{|m|}(k)\Big(\mathrm{i}J'_{|m|}(k) + J_{|m|}(k)\Big), \tag{A.2}$$

then

$$(2A_k v)(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \lambda_m(k)\widehat{v}_m \mathrm{e}^{\mathrm{i}mt}. \tag{A.3}$$

References for the proof. See, e.g., [62, section 4 (in particular equation (4.4))] or [29, Lemma 4.1]. ∎

THEOREM A.2 (sign property of eigenvalues of $2A_k$ on unit circle). *If Γ is the unit circle, then there exists $k_0 > 0$ such that, for all m and for all $k \geq k_0$,*

$$\Re\lambda_m(k) \geq 1.$$

Reference for the proof. This is proved in [29, Theorem 4.2] using asymptotics of Bessel and Hankel functions. ∎

The only other rigorous result about the eigenvalues $\lambda_m(k)$ that we need is the following.

LEMMA A.3 (asymptotics of $\lambda_m(k)$ as $m \to \infty$ with $m > k$). *Let $z := k/m$. Then for all $\delta > 0$ there exists $C > 0$ such that for $0 < z < 1-\delta$,*

$$|\lambda_m(k) - 1| \leq Cz.$$

Proof. We first review some standard facts about uniform asymptotics for the Bessel functions $J_m(mz)$ and $H^{(1)}_m(mz)$ [84], [83, section 10.20], where $m \geq 0$ and $z < 1-\delta$. We define the decreasing bijection $(0,1) \ni z \mapsto \zeta(z) \in (0,\infty)$ by

$$\zeta := \frac{3}{2}\bigg(\int_z^1 t^{-1}(1-t^2)^{1/2}\mathrm{d}t\bigg)^{2/3},$$

and recall the definition of the Airy function, Ai,

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\mathrm{d}t.$$

By [83, section 9.7], for $|\arg(x)| < \pi - \delta$,

$$\begin{aligned} \mathrm{Ai}(x) &= \exp\big(-\tfrac{2}{3}x^{3/2}\big)\bigg(\frac{1}{2\sqrt{\pi}}x^{-1/4} + O(x^{-7/4})\bigg), \\ \mathrm{Ai}'(x) &= \exp\big(-\tfrac{2}{3}x^{3/2}\big)\bigg(-\frac{1}{2\sqrt{\pi}}x^{1/4} + O(x^{-5/4})\bigg), \end{aligned} \tag{A.4}$$

where the branch cut is taken on $x \in (-\infty, 0)$. Moreover, by [83, section 9.9], $|\mathrm{Ai}(x)|, |\mathrm{Ai}'(x)| > 0$ for $x \notin (-\infty, 0)$. Then, by [83, section 10.20], uniformly for $m \geq 1$ and $0 < z < 1$,

(A.5)
$$\begin{aligned} J_m(mz) &= \left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left(m^{-1/3}\,\mathrm{Ai}(m^{2/3}\zeta) + O\Big(m^{-5/3}\zeta^{-1/2}\,\mathrm{Ai}'(m^{2/3}\zeta)\Big)\right), \\ J_m'(mz) &= -\frac{2}{z}\left(\frac{1-z^2}{4\zeta}\right)^{1/4}\left(m^{-2/3}\,\mathrm{Ai}'(m^{2/3}\zeta) + O\Big(m^{-4/3}\zeta^{1/2}\,\mathrm{Ai}(m^{2/3}\zeta)\Big)\right), \\ H_m^{(1)}(mz) &= 2\mathrm{e}^{-\pi\mathrm{i}/3}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\Big(m^{-1/3}\,\mathrm{Ai}(\mathrm{e}^{2\pi\mathrm{i}/3}m^{2/3}\zeta) \\ &\qquad + O\Big(m^{-5/3}\zeta^{-1/2}\,\mathrm{Ai}'(\mathrm{e}^{2\pi\mathrm{i}/3}m^{2/3}\zeta)\Big)\Big). \end{aligned}$$

Next, note that when $0 < z < 1 - \delta$, there exists $c_\delta > 0$ such that $\zeta \geq c_\delta$ and thus we can use the asymptotics for Airy functions (A.4). Putting these asymptotics in (A.5) and using the definition of $\lambda_m(k)$ (A.2), we obtain that for any $\delta > 0$, there exists $C > 0$ such that

$$\Big|\lambda_m(k) - 1\Big| = \Big|\pi k H_{|m|}^{(1)}(k)\Big(\mathrm{i}J_{|m|}'(k) + J_{|m|}(k)\Big) - 1\Big| \leq C\frac{k}{m} \qquad \text{for } m > (1+\delta)k,$$

as claimed. ∎

A.3. Proof of Theorem 2.1 When Γ Is the Unit Circle. Observe that in the case of the circle, the functional calculus for the surface Laplacian reviewed in section 5 is simply the theory of Fourier multipliers; i.e., the collection $\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}mt}\}_{m=-\infty}^{\infty}$ is an orthonormal basis of eigenfunctions of $-\Delta_\Gamma$ satisfying

$$(-\Delta_\Gamma - m^2)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}mt} = 0, \qquad \left\|\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}mt}\right\|_{L^2(\Gamma)} = 1.$$

Thus (5.1) becomes

$$f(-\Delta_\Gamma)v := \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} f(m^2)\widehat{v}_m\mathrm{e}^{\mathrm{i}mt}.$$

To prove Theorem 2.1, we only need to check the conditions of Theorem 4.2 with $I + K = 2\mathcal{A} = 2A_k$. Using Assumption 1.4 as in the beginning of section 6, we see that we only need to prove the bound (6.1). The expansion (A.3) implies that

$$\big((2A_k)^{-1}v\big)(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \big(\lambda_m(k)\big)^{-1} \widehat{v}_m \mathrm{e}^{\mathrm{i}mt}.$$

By Theorem A.2, and the fact that $|\lambda_m| \geq |\Re\lambda_m| \geq 1$,

$$\sup_m |\lambda_m(k)|^{-1} \leq 1. \tag{A.6}$$

Therefore, by taking L^2 norms and using orthonormality (in a similar way to how (5.2) is obtained), we obtain the bound (3.3) in this setting:

$$\|(2A_k)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \sup_m |\lambda_m(k)|^{-1} \leq 1. \tag{A.7}$$

To prove the bound (6.1), we therefore only need to show that

$$\|(2A_k - I)(2A_k)^{-1}\|_{L^2(\Gamma)\to H^1(\Gamma)} \leq Ck; \tag{A.8}$$

we do this using the splitting (6.2) with $\chi \in C^\infty_{\text{comp}}(\mathbb{R};[0,1])$ with $\chi \equiv 1$ on $[-1-\varepsilon, 1+\varepsilon]$. To deal with the first term on the right-hand side of (6.2), we observe that, by (A.7),

$$\big\|I - (2A_k)^{-1}\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq 2. \tag{A.9}$$

The definition of the H^1 norm in (A.1), along with the compact support of χ and Parseval's theorem in (A.1), implies the following analogue of Lemma 5.2 with $s = 1$:

$$\big\|\chi(-k^{-2}\Delta_\Gamma)\big\|^2_{L^2(\Gamma)\to H^1(\Gamma)} \leq \sup_m \Big[(1+m^2)\big|\chi(k^{-2}m^2)\big|\Big] \leq Ck^2. \tag{A.10}$$

Combining (A.9) and (A.10), we obtain the following bound on the first term of the right-hand side of (6.2):

$$\big\|\chi(-k^{-2}\Delta_\Gamma)(I - (2A_k)^{-1})\big\|_{L^2(\Gamma)\to H^1(\Gamma)} \leq Ck. \tag{A.11}$$

To deal with the second term on the right-hand side of (6.2), we observe that

$$\big(I - \chi(-k^{-2}\Delta_\Gamma)\big)(2A_k - I)(2A_k)^{-1}\mathrm{e}^{\mathrm{i}mt} = \big(1 - \chi(-k^{-2}m^2)\big)\frac{\lambda_m(k) - 1}{\lambda_m(k)}\mathrm{e}^{\mathrm{i}mt}.$$

Thus, using the Fourier representation of $\big(1-\chi(-k^{-2}\Delta_\Gamma)\big)(2A_k - I)(2A_k)^{-1}$ and the definition of the $H^1(\Gamma)$ norm (A.1), we find that

$$\begin{aligned}\big\|\big(I - \chi(-k^{-2}\Delta_\Gamma)\big)&(2A_k - I)(2A_k)^{-1}\big\|^2_{L^2(\Gamma)\to H^1(\Gamma)}\\ &\leq \sup_m \left[(1+m^2)\big|\big(1-\chi(k^{-2}m^2)\big)\big|\frac{|\lambda_m(k) - 1|}{|\lambda_m(k)|}\right].\end{aligned}$$

By the definition of χ, $(1-\chi(k^{-2}m^2)) = 0$ when $m^2 \leq (1+\varepsilon)k^2$, and $(1-\chi(k^{-2}m^2)) \leq 1$ for all m; therefore,

$$\big\|\big(I - \chi(-k^{-2}\Delta_\Gamma)\big)(2A_k - I)(2A_k)^{-1}\big\|^2_{L^2(\Gamma)\to H^1(\Gamma)}$$

$$\leq \sup_{m^2 \geq (1+\varepsilon)k^2} \left[(1+m^2) \frac{|\lambda_m(k) - 1|}{|\lambda_m(k)|} \right].$$

Observe from (A.2) that $\lambda_m(k) = \lambda_{|m|}(k)$; the regime $m^2 \geq (1+\varepsilon)k^2$ is therefore exactly that covered by Lemma A.3 (with $(1+\varepsilon)^{-1/2} = 1-\delta$). Using Lemma A.3 along with (A.6), we obtain that

(A.12)

$$\begin{aligned} &\|(I - \chi(-k^{-2}\Delta_\Gamma))(2A_k - I)(2A_k)^{-1}\|^2_{L^2(\Gamma)\to H^1(\Gamma)} \\ &\leq \sup_{m^2 \geq (1+\delta)k^2} \left[(1+m^2) \frac{|\lambda_m(k) - 1|}{|\lambda_m(k)|} \right] \leq C^2 \sup_{m^2 \geq (1+\delta)k^2} \left[(1+m^2) \frac{k^2}{m^2} \right] \leq C' k^2. \end{aligned}$$

Combining the bounds (A.11) and (A.12), we obtain (A.8), and the proof is complete.

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