

Isomorphic Bisections of Cubic Graphs

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Abstract

Graph partitioning, or the dividing of a graph into two or more parts based on certain conditions, arises naturally throughout discrete mathematics, and problems of this kind have been studied extensively. In the 1990s, Ando conjectured that the vertices of every cubic graph can be partitioned into two parts that induce isomorphic subgraphs. Using probabilistic methods together with delicate recolouring arguments, we prove Ando's conjecture for large connected graphs.

Keywords: cubic graphs; clustered colouring.

1 Introduction

Graph theory enjoys applications to a wide range of disciplines because graphs are incredibly flexible mathematical structures, capable of modelling very complex systems. The study of complicated networks motivates the following important question: to what extent can a graph be decomposed into simpler subgraphs?

Spearheading this line of research is the classic problem of graph colouring, one of the oldest branches of graph theory. When colouring a graph, one seeks to partition the vertices into as few independent (edgeless) sets as possible. Indeed, subgraphs cannot get much simpler than independent sets, but such ambitious goals come at a cost. Not only is the determination of a graph's chromatic number a notoriously difficult problem, but even when dealing with sparse graph classes, one can require many colours. For instance, many d -regular graphs cannot be partitioned into much fewer than d independent sets.

Often one does not want to have such a large number of parts, and so it is natural to ask what can be achieved with fewer colours. An early result along these lines was provided by Lovász [14], who proved that, given a graph G of maximum degree d , some number of colours t , and a sequence d_1, d_2, \dots, d_t with $\sum_i d_i = d - t + 1$, one can t -colour the vertices of G such that the i th colour class induces a subgraph with maximum degree d_i . In particular, we can partition any graph into two subgraphs, each with half the maximum degree.

While reducing the maximum degree certainly simplifies graphs, it still allows for large connected subgraphs within the colour classes. A different objective, therefore, is to find clustered colourings, which are colourings where each monochromatic component is of bounded size. Alon, Ding, Oporowski and Vertigan [6] proved, among other results, that any graph with maximum degree four can be two-coloured such that the largest monochromatic components are of order at most 57. However, they also constructed six-regular graphs with arbitrarily large monochromatic components in every two-colouring. Answering one of their questions, Haxell, Szabó and Tardos [11] proved that clustered two-colourings of graphs of maximum degree five always exist. Aside from improving the bounds on the monochromatic component sizes, subsequent

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research has sought to explore which graph classes admit clustered colourings, and what is possible with more colours. A more general setting, in which the size of monochromatic components can grow with the size of graph, has also been studied. For example, Linial, Matoušek, Sheffet and Tardos [13] proved that any planar n -vertex graph has a two-colouring in which all monochromatic components have size at most $O(n^{2/3})$, and this is tight. For more results on vertex colourings with small monochromatic components, we refer the interested reader to the survey of Wood [17].

The above results show that when we restrict our attention to graphs of small maximum degree, there is less room for complexity, and therefore we can prove strong partitioning results. One might further try to achieve more than simply bounding the size of monochromatic components. Indeed, it appears that in the case of cubic graphs, it is also possible to control the structure of such components. An early result along these lines is due to Akiyama, Exoo and Harary [4], who proved that every cubic graph admits a two-*edge*-colouring in which every monochromatic component is a path, with a short proof of this result provided by Akiyama and Chvátal [3] soon after. Although these results allowed for paths of unbounded length, Bermond, Fouquet, Habib and Péroche [8] conjectured that paths of length at most five suffice, a result that would be best possible. Partial results, with larger but finite bounds on the path lengths, were obtained by Jackson and Wormald [12] and by Aldred and Wormald [5], before the conjecture was proven by Thomassen [16] in 1999.

Theorem 1.1. *The edges of any cubic graph can be two-coloured such that each monochromatic component is a path of length at most five.*

One can also try to find different structure in the partition than just finding path forests with small components. One attractive conjecture of Wormald [18] from 1987 asks whether we can colour the edges of every cubic graph with an even number of edges so that the red and blue subgraphs form *isomorphic* linear forests. This is known to hold for particular classes of graphs — it was proved for Jaeger graphs in the work of Bermond, Fouquet, Habib and Péroche [8] and Wormald [18], and for some further classes of cubic graphs by Fouquet, Thuillier, Vanherpe and Wojda [10]. Given Wormald’s conjecture, it is then natural to ask for analogues when colouring vertices rather than edges. In this direction, the following striking conjecture was made by Ando in the 1990s.

Conjecture 1.2. *The vertices of any cubic graph can be two-coloured such that the two colour classes induce isomorphic subgraphs.*

This conjecture was first mentioned in print by Abreu, Goedgebeur, Labbate and Mazzuoccolo [2], who drew connections between Conjecture 1.2, Wormald’s conjecture and a conjecture of Ban and Linial [7] (see Section 4 for discussion of the latter). They further obtained computational results verifying the conjecture for graphs on at most 32 vertices. Hence any counterexample to Conjecture 1.2 must have at least 34 vertices. Observe that any minimal counterexample must be connected, as the components of a graph can be coloured independently. In this paper we essentially resolve this problem and prove that large connected cubic graphs satisfy Ando’s conjecture. This shows that there can be at most finitely many minimal counterexamples.

Theorem 1.3. *Every sufficiently large connected cubic graph admits a two-vertex-colouring φ whose colour classes induce isomorphic subgraphs.*

The remainder of this paper is organized as follows. In Section 2, we present the proof of Theorem 1.3. The proof uses probabilistic methods together with delicate recolouring arguments. In the process of recolouring we use some gadgets which we call P_t -reducers. The existence of these structures is shown in Section 3. Finally, we make some concluding remarks and discuss open problems in Section 4.

Notation. We shall call the colours used in our two-colourings red and blue. To show that the red and blue subgraphs are isomorphic, we will need to keep track of the monochromatic components. Given a fixed graph H and a red-/blue-colouring φ of the vertices of a graph G , we denote by $r_H(G, \varphi)$ the number of red components of G under φ that are isomorphic to H , and define $b_H(G, \varphi)$ similarly for the blue components. Paths will play a significant role in our argument, and we write P_t for the path of length $t - 1$ on t vertices. Given a colouring of a graph, we call the colouring with colours reversed the *opposite* colouring. We will also often explore the neighbourhood of a vertex or a subset in a graph, and write $B_d(v)$ for the radius- d ball around a vertex v and write $B_d(X)$ for the set of all vertices of G within distance at most d from a subset X . We use $N^d(X)$ for the set of vertices at distance exactly d from X , and abbreviate $N^1(X)$ as $N(X)$ (so $N(X)$ is the set of external neighbours of X in G). Finally, all logarithms are to the base e .

2 Proving the theorem

Given a large connected cubic graph, our goal is to colour the vertices such that the colour classes induce isomorphic subgraphs, and we shall find this colouring in two stages. In the first, we take a semi-random vertex colouring, and show that this is very close to having the desired properties. Then, in the second stage, we make some deterministic local recolourings to balance the two subgraphs and ensure they are truly isomorphic.

2.1 A random colouring

We begin our search for the isomorphic bisection by showing that one may leverage Thomassen's result to define a random colouring that will produce monochromatic subgraphs that are nearly isomorphic. The following proposition collects the properties of this initial colouring.

Proposition 2.1. *For any $d \in \mathbb{N}$ and any sufficiently large cubic graph G , there is a red-/blue-colouring φ of the vertices for which the following hold.*

- (a) *Each monochromatic component is a path of length at most five.*
- (b) *For each $2 \leq t \leq 6$, $|r_{P_t}(G, \varphi) - b_{P_t}(G, \varphi)| \leq 3\sqrt{n \log n}$.*
- (c) *There are sequences of vertices $(u_i)_{i \in [s]}$ and $(w_i)_{i \in [s]}$, for some $s \geq 2^{-2d-5}n$, such that all the balls $B_d(u_i)$ and $B_d(w_i)$ are pairwise disjoint, and for each $i \in [s]$, $B_d(u_i)$ and $B_d(w_i)$ induce isomorphic subgraphs with opposite colourings.*
- (d) *The colouring φ is a bisection; that is, there are an equal number of red and blue vertices.*

In our proof, we will need to show that several random variables, each a function of the random colouring, lie close to their expected values. For this we employ McDiarmid's Inequality [15], which bounds large deviations in random variables defined on product probability spaces, provided they do not vary too much in response to changes in individual coordinates.

Theorem 2.2. *Let $X = (X_1, X_2, \dots, X_n)$ be a family of independent random variables, with X_k taking values in a set A_k for each $k \in [n]$. Suppose further that there is some real-valued function f defined on $\prod_{k \in [n]} A_k$ and some $c > 0$ such that $|f(x) - f(x')| \leq c$ whenever x and x' differ only in a single coordinate. Then, for any $m \geq 0$,*

$$\mathbb{P}(|f(X) - \mathbb{E}[f(X)]| \geq m) \leq 2 \exp\left(\frac{-2m^2}{c^2 n}\right).$$

Equipped with this tool, we can now proceed with our proof.

Proof of Proposition 2.1. To prove the proposition, we shall first define a random colouring φ' that satisfies properties (a), (b) and (c) while being very close to a bisection. We shall then make a few small changes to obtain a bona fide bisection φ , maintaining the other properties in the process.

To define φ' , we first apply Theorem 1.1 to our cubic graph G . This results in a partition of the edges of G into two spanning linear forests, F_1 and F_2 , neither of which contains a path of length six or more. We then take φ' to be a uniformly random proper colouring of F_1 . Observe that this is equivalent to selecting, independently for each path component of F_1 , one of its two proper colourings uniformly. Thus, our probability space is a product space, with each coordinate corresponding to a colouring of a path component of F_1 , of which there are at most n .

We now verify in turn that the first three properties hold. For (a), observe that since φ' is a proper colouring of F_1 , the only monochromatic components are subgraphs of F_2 . It thus follows immediately that each such component is a path of length at most five.

To establish (b), we seek to apply Theorem 2.2 to the random variable $r_{P_t}(G, \varphi')$. To do so, we must understand how changing the colouring of an individual path component of F_1 can affect $r_{P_t}(G, \varphi')$. Since the component in F_2 of each vertex is simply a path, changing the colouring of an individual vertex can affect at most two monochromatic copies of P_t . As each path component in F_1 is of length at most five, the colouring of each such path therefore affects at most 12 copies of P_t . Hence, we can take $c = 12$ in Theorem 2.2. It then follows that, for each $m > 0$, $\mathbb{P}(|r_{P_t}(G, \varphi') - \mathbb{E}[r_{P_t}(G, \varphi')]| \geq m) \leq 2 \exp(-m^2/(72n))$. In particular, the probability that $r_{P_t}(G, \varphi')$ deviates from its expectation by more than $\sqrt{n \log n}$ is at most $2n^{-1/72} = o(1)$.

The same argument holds for $b_{P_t}(G, \varphi')$, and, by symmetry, we have $\mathbb{E}[r_{P_t}(G, \varphi')] = \mathbb{E}[b_{P_t}(G, \varphi')]$. Hence, the probability that $|r_{P_t}(G, \varphi') - b_{P_t}(G, \varphi')| \geq 2\sqrt{n \log n}$ is $o(1)$. Taking a union bound over all choices of $2 \leq t \leq 6$, we find that with high probability we have $|r_{P_t}(G, \varphi') - b_{P_t}(G, \varphi')| \leq 2\sqrt{n \log n}$ for all $2 \leq t \leq 6$.

We next turn our attention to (c). To start, we wish to select a sequence of vertices whose pairwise distances are all at least $2d + 1$. Observe that, since G is cubic, for each vertex v we have $|B_{2d}(v)| \leq 1 + \sum_{i=1}^{2d} 3 \cdot 2^i < 3 \cdot 2^{2d+1}$. We can then greedily select our desired vertices. Indeed, after we add a vertex v_i to our sequence, we eliminate the vertices in $B_{2d}(v_i)$ from our consideration. This guarantees that we are now free to select any of the remaining vertices. This results in a sequence of vertices $v_1, v_2, \dots, v_{s'}$ for some $s' \geq \frac{1}{3}2^{-2d-1}n$, with each pair at distance $2d + 1$ or more. In particular, the balls $B_d(v_i)$ are pairwise disjoint.

Now consider the subgraphs induced by each ball $B_d(v_i)$. These are subcubic graphs on at most $3 \cdot 2^{d+1}$ vertices, and hence there are a finite number of possible isomorphism types. Furthermore, the subgraphs inherit a vertex-colouring from φ' . Since the number of vertices in the ball is bounded, there can be at most $2^{3 \cdot 2^{d+1}}$ different colourings. Therefore, considering the isomorphism type and colouring of each ball, there is some finite number $\kappa = \kappa(d)$ of different classes the balls can fall into.

For each $j \in [\kappa]$, we denote by Y_j the random variable counting the number of balls $B_d(v_i)$, $i \in [s']$, that belong to the j th class. Note that for each i , the distribution of the colouring of $B_d(v_i)$ depends on how the paths of F_1 appear in the ball. However, by symmetry, opposite colourings of $B_d(v_i)$ appear with equal probabilities. It follows that if j and \bar{j} represent opposite colourings of isomorphic balls, then $\mathbb{E}[Y_j] = \mathbb{E}[Y_{\bar{j}}]$.

Since the balls are pairwise disjoint, changing the colour of a single vertex can affect at most one ball $B_d(v_i)$. Hence, changing the colouring of a path component of F_1 can affect at most six balls. Applying Theorem 2.2 to the random variables Y_j , we can therefore take $c = 6$. Setting $m = \sqrt{n \log n}$, we find that with probability $1 - o(1)$ we have $|Y_j - \mathbb{E}[Y_j]| \leq \sqrt{n \log n}$ for each $j \in [\kappa]$. It then follows that for opposite

colourings j and \bar{j} we have $|Y_j - Y_{\bar{j}}| \leq 2\sqrt{n \log n}$. We can therefore match the balls $B_d(v_i)$ into isomorphic pairs with opposite colourings, with at most $2\sqrt{n \log n}$ unmatched balls for each of the κ isomorphism types. Provided n is suitably large, this is a total of at most $\frac{1}{2}s'$ unmatched balls, leaving us with at least $\frac{1}{4}s'$ matched pairs of balls.

This shows that the random colouring φ' is very likely to satisfy (a), (b) and (c). To finish the proof, we shall show that it is typically also close to being a bisection, and we can make it one without destroying the other properties.

Consider the random variable that is the difference between the numbers of red and blue vertices. Since every path component in F_1 of odd length contributes an equal number of red and blue vertices to the colouring, any discrepancy in the colour class sizes must come from the paths of even length. Furthermore, each such path contributes a difference in the colour class sizes of exactly one, with equal probability in either direction. Thus, the expected difference is zero, and recolouring a path component can affect the difference by at most two. We thus make a final appeal to Theorem 2.2, finding that with probability $1 - o(1)$, the difference in the two colour classes is of size at most $\frac{1}{20}\sqrt{n \log n}$.

In summary, with positive probability it holds that the colouring φ' satisfies properties (a), (b) and (c), and that the difference Δ between the number of red and blue vertices is at most $\frac{1}{20}\sqrt{n \log n}$. We can therefore obtain a bisection φ by taking φ' and replacing the colouring of Δ even paths in F_1 with their opposite colourings, thereby satisfying (d).

Note that φ remains a proper colouring of F_1 , and therefore property (a) is preserved. Moreover, we are changing the colours of at most 5Δ vertices. As previously discussed, changing the colour of a vertex can affect at most two monochromatic copies of P_t , and hence, for each $2 \leq t \leq 6$, the difference $|r_{P_t}(G, \varphi) - b_{P_t}(G, \varphi)|$ deviates from $|r_{P_t}(G, \varphi') - b_{P_t}(G, \varphi')|$ by at most $\sqrt{n \log n}$. In particular, we have $|r_{P_t}(G, \varphi) - b_{P_t}(G, \varphi)| \leq 3\sqrt{n \log n}$ for each $2 \leq t \leq 6$, establishing (b).

Similarly, under φ' we had at least $\frac{1}{4}s' \geq \frac{1}{3}2^{-2d-3}n$ pairs of isomorphic balls with opposite colourings. Since these balls are pairwise disjoint, each recoloured vertex can affect at most one such pair. Hence, under φ we still have at least $\frac{1}{4}s' - 5\Delta$ such pairs which, if n is large enough, is at least $2^{-2d-5}n$ pairs. Thus, (c) holds for φ as well, completing the proof. \square

2.2 Correcting the bisection

The colouring φ from Proposition 2.1 is close to being the bisection we need, but the number of short paths in the red and blue subgraphs can be a little off. In the second stage of our argument, we will make local changes to the colouring to correct these discrepancies. We do this via gadgets we call P_t -reducers. These are flexible subgraphs, in the sense that they admit two different colourings which have the same monochromatic subgraph counts, except for short paths. By choosing an appropriate colouring, then, we can adjust the values of $r_{P_t}(G, \varphi)$ and $b_{P_t}(G, \varphi)$, making them equal.

When doing so, though, we do need to take care that our changes do not leak out and affect subgraph counts elsewhere. We thus insulate the P_t -reducer by colouring its boundary with alternating colours, thus preventing any monochromatic components from extending outwards.

Definition 2.3 (P_t -reducer). Given some $t \geq 3$, an induced subgraph $R \subseteq G$ is a P_t -reducer if there are two vertex colourings ψ_1, ψ_2 of $B_2(R) = R \cup N(R) \cup N^2(R)$ such that:

- (i) the two colourings have the same number of red (and therefore also blue) vertices,
- (ii) in both ψ_1 and ψ_2 , all vertices of $N(R)$ are blue and all vertices of $N^2(R)$ are red,

- (iii) $r_H(B_2(R), \psi_1) = r_H(B_2(R), \psi_2)$ and $b_H(B_2(R), \psi_1) = b_H(B_2(R), \psi_2)$, unless $H = P_\ell$ for some $2 \leq \ell \leq t$, and
- (iv) $r_{P_t}(B_2(R), \psi_2) = r_{P_t}(B_2(R), \psi_1) - 1$ and $b_{P_t}(B_2(R), \psi_1) = b_{P_t}(B_2(R), \psi_2)$.

Of course, this definition is only useful if we can actually find P_t -reducers in our graph. Fortunately, they happen to be ubiquitous in cubic graphs.

Proposition 2.4. *Let G be a connected cubic graph on more than $3 \cdot 2^{50}$ vertices, and let $v \in V(G)$ be arbitrary. Then, for every $3 \leq t \leq 6$, there is a P_t -reducer in $B_{50}(v)$.*

We shall prove Proposition 2.4 in Section 3, but first we show how one can use P_t -reducers to obtain the desired isomorphic bisection.

Proof of Theorem 1.3. Given a sufficiently large connected cubic graph G , set $d = 57$ and let φ_0 be the bisection given by Proposition 2.1 with this d . We then have $r_H(G, \varphi_0) = b_H(G, \varphi_0) = 0$ for all H except $H = P_t$, $2 \leq t \leq 6$. For these paths, we have $|r_{P_t}(G, \varphi_0) - b_{P_t}(G, \varphi_0)| \leq 3\sqrt{n} \log n$, and we shall correct these imbalances one at a time, in decreasing order of path length.

We start with $t = 6$. Suppose, without loss of generality, that we have $r_{P_6}(G, \varphi_0) > b_{P_6}(G, \varphi_0)$. Take the first pair of isomorphic and oppositely-coloured balls, $B_{57}(u_1)$ and $B_{57}(w_1)$. By Proposition 2.4, we can find some P_6 -reducer $R \subseteq G[B_{50}(u_1)]$, and therefore we find a corresponding oppositely-coloured copy $\bar{R} \subseteq G[B_{50}(w_1)]$ as well. We then recolour $B_2(R)$ with ψ_2 , and colour $B_2(\bar{R})$ with $\bar{\psi}_1$, the opposite colouring of ψ_1 .

Let φ_1 be the resulting colouring. Note that this is still a bisection, since ψ_1 and ψ_2 have the same number of red vertices, and we have made symmetric changes in $B_2(R)$ and $B_2(\bar{R})$. We next claim that $r_H(G, \varphi_1) = b_H(G, \varphi_1)$ for all graphs except P_t , $2 \leq t \leq 6$, and that $r_{P_6}(G, \varphi_1) - b_{P_6}(G, \varphi_1) = r_{P_6}(G, \varphi_0) - b_{P_6}(G, \varphi_0) - 1$; that is, the difference in monochromatic copies of P_6 is reduced by one.

Observe that we only need to concern ourselves with monochromatic components contained within $B_{57}(u_1)$ and $B_{57}(w_1)$. Indeed, in φ_0 , all monochromatic components were paths of length at most five. As we only recoloured vertices in $B_2(R)$ and $B_2(\bar{R})$, which are contained in $B_{50}(u_1)$ and $B_{50}(w_1)$ respectively, any affected components do not reach outside the original balls $B_{57}(u_1)$ and $B_{57}(w_1)$. Also note that, since these balls had the opposite colourings, before the recolouring there was a one-to-one correspondence between red components in the first ball and blue components in the second ball and vice versa.

First we consider components not fully contained within $B_2(R)$ or $B_2(\bar{R})$. Recall that in ψ_2 , the vertices in $N(R)$ and $N^2(R)$ receive opposite colours, with the same being true of the colouring $\bar{\psi}_1$ of $B_2(\bar{R})$. Thus, any such component in $B_{57}(u_1)$ can only contain (red) vertices from $N^2(R)$, together with some vertices in $B_{57}(u_1) \setminus B_2(R)$. However, since $B_{57}(u_1)$ and $B_{57}(w_1)$ are isomorphic and oppositely-coloured, these components are in bijection with isomorphic blue components in $B_{57}(w_1)$, and hence no additional discrepancy is introduced through these components.

This leaves us with components fully within $B_2(R)$ and $B_2(\bar{R})$, where the properties of the P_6 -reducer come into play. For any component H that is not a path of length at most six, we have $r_H(B_2(R), \psi_2) = r_H(B_2(R), \psi_1) = b_H(B_2(\bar{R}), \bar{\psi}_1)$, and similarly $b_H(B_2(R), \psi_2) = r_H(B_2(\bar{R}), \bar{\psi}_1)$. Thus the monochromatic copies of these components remain balanced. As for paths of length six, we have

$$r_{P_6}(B_2(R), \psi_2) = r_{P_6}(B_2(R), \psi_1) - 1 = b_{P_6}(B_2(\bar{R}), \bar{\psi}_1) - 1,$$

while $b_{P_6}(B_2(R), \psi_2) = r_{P_6}(B_2(\bar{R}), \bar{\psi}_1)$, and so the difference between red and blue copies of P_6 is indeed reduced by one in φ_1 .

Although this recolouring could create monochromatic components that are not paths of length at most six, these must be fully contained within the balls $B_{57}(u_1)$ and $B_{57}(w_1)$, which are disjoint from all the other balls. We can thus repeat this process a further $r_{P_6}(G, \varphi_1) - b_{P_6}(G, \varphi_1) - 1$ times, using the next pairs of balls $B_{57}(u_i)$ and $B_{57}(w_i)$ in the sequence. This gives us a sequence of colourings, $\varphi_1, \varphi_2, \dots, \varphi_k$, the last of which will satisfy $r_{P_6}(G, \varphi_k) = b_{P_6}(G, \varphi_k)$.

Furthermore, since each P_6 -reducer is of constant size, every step of the process can only have created a constant number of monochromatic P_5 -components. Thus, we still have $|r_{P_5}(G, \varphi_k) - b_{P_5}(G, \varphi_k)| = O(\sqrt{n \log n})$. We can therefore use Proposition 2.4 to find P_5 -reducers in the next $O(\sqrt{n \log n})$ pairs of balls and balance the monochromatic P_5 counts. Once those are handled, we proceed to fixing the P_4 counts, and then finally the P_3 counts. Observe that we require a total of $O(\sqrt{n \log n})$ steps, and since Proposition 2.1 guarantees us $\Omega(n)$ pairs of isomorphic balls, we can see this process through to completion.

Let φ be the final colouring obtained. Following our corrections with the P_t -reducers, we know that for every $H \neq P_2$, we have $r_H(G, \varphi) = b_H(G, \varphi)$. However, since G is cubic and φ is a bisection, double-counting the edges between the colour classes shows that the total number of monochromatic red edges must equal the number of blue edges. It therefore follows that $r_{P_2}(G, \varphi) = b_{P_2}(G, \varphi)$ as well, and hence the subgraphs induced by the red and the blue vertices are isomorphic. \square

3 Constructing P_t -reducers

To complete the proof of Theorem 1.3, we need to prove Proposition 2.4, showing that we can find P_t -reducers in the local neighbourhoods of every vertex. We will first show a very simple construction that works in graphs of girth at least seven, thereby providing a short proof of Conjecture 1.2 for large cubic graphs without short cycles. The proof in the general case is a little more involved, requiring analysis of a few different cases, and can be found in Section 3.2.

3.1 Graphs of girth at least seven

Our construction is based around geodesics — that is, shortest paths in the graph between a pair of vertices. We start with a lemma about neighbourhoods of geodesics.

Lemma 3.1. *Let G be a cubic graph of girth at least seven, and let P be a geodesic in G . Then $N(P)$, the set of external neighbours of vertices on P , is an independent set.*

Proof. Let $u, w \in N(P)$ be arbitrary vertices in the neighbourhood of P , and suppose for contradiction we have an edge $\{u, w\}$. Let $u', w' \in P$ be neighbours of u and w respectively. If u' and w' are at distance at most three along the path P , then, together with the edges $\{u', u\}$, $\{u, w\}$ and $\{w, w'\}$, we would obtain a cycle of length at most six, contradicting our assumption on the girth of G .

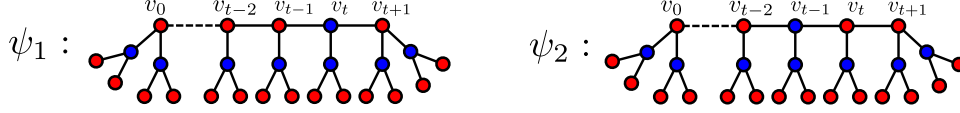
On the other hand, if u' and w' are at distance at least four on the path P , then we could shorten the path by rerouting it between u' and w' through u and w instead. This contradicts P being a geodesic.

Hence u and w cannot be adjacent, and thus it follows that $N(P)$ is an independent set. \square

This shows that the structure around a geodesic is particularly simple. As a result, we can easily find P_t -reducers, as we now show.

Proof of Proposition 2.4 (girth ≥ 7). Since there are at most $3 \cdot 2^{t+1} < 3 \cdot 2^{50}$ vertices at distance at most t from the vertex v , there must be a vertex w at distance exactly $t + 1$ from v . Let P be a geodesic from v to w , with vertices $v = v_0, v_1, v_2, \dots, v_t, v_{t+1} = w$. We will show that P is a P_t -reducer.

We now define the colourings ψ_1 and ψ_2 of $B_2(P) = P \cup N(P) \cup N^2(P)$. In both colourings, we colour all vertices in $N^2(P)$ red and all vertices in $N(P)$ blue. When colouring the path P , we make all vertices red, except in ψ_1 the vertex v_t is coloured blue, while in ψ_2 the vertex v_{t-1} is coloured blue instead.



Clearly both ψ_1 and ψ_2 have the same number of red vertices, and thus property (i) is satisfied. Property (ii) is satisfied by the definition of the colouring on $N(P)$ and $N^2(P)$.

Finally, we inspect the monochromatic components of $B_2(P)$ under ψ_1 and ψ_2 to verify properties (iii) and (iv). Let us start with the blue components. In the two colourings, the blue vertices are those in $N(P)$, together with one of v_{t-1} or v_t . By Lemma 3.1, the vertices in $N(P)$ form an independent set. As v_{t-1} and v_t are internal vertices on the path P , they have exactly one neighbour in $N(P)$. Thus in both ψ_1 and ψ_2 , the blue vertices induce one isolated edge and $|N(P)| - 1$ isolated vertices, and thus $b_H(B_2(P), \psi_1) = b_H(B_2(P), \psi_2)$ for every H .

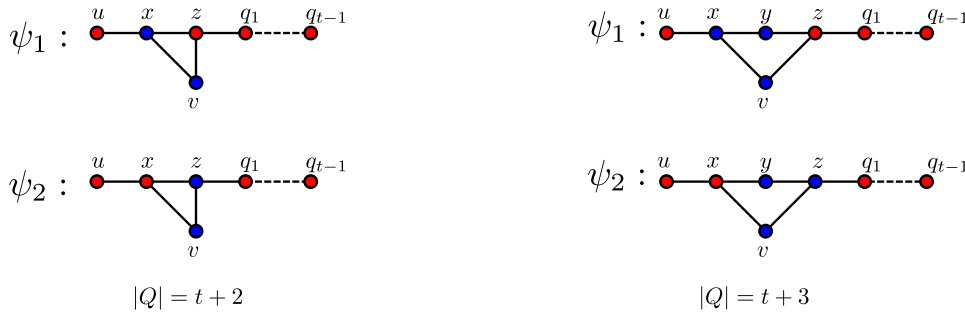
Since $N(P)$ is entirely blue, the red components of $B_2(P)$ are either wholly contained in $N^2(P)$ or in P . In the former case, since all vertices in $N^2(P)$ are red, both colourings have the same components. In the latter case, the path P is broken into two red paths by the sole blue vertex, which is v_t in ψ_1 and v_{t-1} in ψ_2 . Thus ψ_1 has a red P_t and a red P_1 , while ψ_2 has a red P_{t-1} and a red P_2 . This shows that (iii) and (iv) are indeed satisfied, and so the colourings ψ_1 and ψ_2 bear witness to P being a P_t -reducer. \square

3.2 The general case

In general, we cannot expect such simple structure around geodesic paths. However, induced paths will still play a key role in our construction of P_t -reducers, as shown by the following lemma.

Lemma 3.2. *In a cubic graph G , let Q be an induced path of length $t + 1$ or $t + 2$ with vertices labelled as $Q = (u, x, z, q_1, \dots, q_{t-1})$ or $Q = (u, x, y, z, q_1, \dots, q_{t-1})$ respectively. Let v be a vertex outside Q with the edges vx and vz present. Then $B_3(Q)$ contains a P_t -reducer.*

Proof. Depending on the length of Q , define ψ_1 and ψ_2 on $R := Q \cup \{v\}$ as follows:



Define both ψ_1 and ψ_2 to be blue on $N(R)$ and red on $N^2(R)$. We claim that this makes R a P_t -reducer. Properties (i) and (ii) are immediate by construction. Notice that $b_H(B_2(R), \psi_1) = b_H(B_2(R), \psi_2)$ for all H . Indeed the only blue component that changes between the two colourings is the one that contains v . However, since the graph is cubic, neither x nor z has any neighbours in $N(R)$, and therefore it is easy to see that this component is isomorphic between the two colourings. Finally notice that the only red components that change between the two colourings are the two red subpaths of Q . The colouring ψ_1 has a red P_t and

a red P_1 , while ψ_2 has a red P_{t-1} and a red P_2 . We also have $r_H(B_2(R), \psi_1) = r_H(B_2(R), \psi_2)$ for all other H . This completes the proof of (iii) and (iv). \square

Recall that a geodesic path Q is a shortest length path between its endpoints, and note that a length 14 geodesic path exists in $B_{14}(v)$ for every vertex v in a large connected cubic graph. An important consequence of the above lemma is that it reduces the problem to the case when every vertex outside a geodesic path Q has at most one neighbour on Q . Using this we can prove the following lemma.

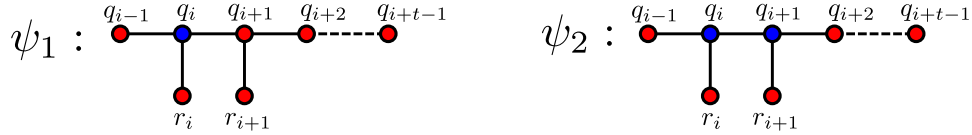
Lemma 3.3. *Let Q be a geodesic path of length 14 in a cubic graph G . Then $B_3(Q)$ either contains P_t -reducers for all $3 \leq t \leq 6$ or it contains, for each $3 \leq t \leq 6$, an induced subgraph R which has two colourings ψ_1 and ψ_2 of $B_2(R)$ that satisfy (ii), (iii) and (iv) of the definition of P_t -reducer and also:*

(i') ψ_1 has one more red vertex than ψ_2 .

Proof. Let $Q = (q_1, \dots, q_{15})$. For each $2 \leq i \leq 14$, let r_i be the neighbour of q_i outside Q . These are unique because the graph is cubic. If $r_i = r_j$ for some $2 \leq i < j \leq 14$, the fact that Q is a geodesic path forces $j = i + 1$ or $j = i + 2$. In either case, we find P_t -reducers for all $3 \leq t \leq 6$ using Lemma 3.2. Thus we can assume that r_2, \dots, r_{14} are all distinct.

Next, suppose that $r_i r_{i+1}$ is an edge for all $3 \leq i \leq 10$. Since G is cubic, this determines all the neighbours of r_i for $4 \leq i \leq 10$. It follows that for $t \in \{3, 4, 5, 6\}$, the path $Q' = (q_2, q_3, q_4, r_4, r_5, \dots, r_{t+4})$ is induced, has length $t + 3$, and the vertex $v := r_3$ has neighbours q_3 and r_4 . Thus Lemma 3.2 applies to give us P_t -reducers for all $3 \leq t \leq 6$.

Hence, we may assume that $r_i r_{i+1}$ is a non-edge for some $3 \leq i \leq 10$. Fix $t \in \{3, 4, 5, 6\}$. Define the subpath $Q' = (q_{i-1}, \dots, q_{i+t-1})$. Define ψ_1 and ψ_2 on $R := Q' \cup \{r_i, r_{i+1}\}$ as follows:



Extend both ψ_1 , and ψ_2 to be blue on $N(R)$ and red on $N^2(R)$. We claim that then these colourings of $B_2(R)$ satisfy (i'), (ii), (iii), and (iv). Properties (i') and (ii) are immediate by construction. For (iii) and (iv), notice that the only components that change between the two colourings are the blue component containing q_i (which, by virtue of the fact that all neighbours of q_i and q_{i+1} lie within R , is a P_1 in ψ_1 and a P_2 in ψ_2) and the red component containing q_{i+2} (which is a P_t in ψ_1 and splits into a P_1 and a P_{t-2} in ψ_2). \square

The above lemma implies Proposition 2.4 for $t \geq 4$.

Proof of Proposition 2.4 for $t \geq 4$. Note that within $B_{45}(v)$, we can find two geodesic paths of length 14, which we call Q and R , at distance 10 from each other. Thus the balls $B_3(Q)$ and $B_3(R)$ are disjoint. If either of these balls contains a P_t -reducer, we are done. Otherwise, by Lemma 3.3, we can assume the existence of some $S_1 \subseteq B_3(Q)$ with colourings $\psi_1^{S_1, t}$ and $\psi_2^{S_1, t}$ of $B_2(S_1)$ satisfying (ii), (iii), (iv) of the definition of a P_t -reducer and also (i') from Lemma 3.3. Similarly, we find $S_2 \subseteq B_3(R)$ with colourings $\psi_1^{S_2, t-1}$, $\psi_2^{S_2, t-1}$ of $B_2(S_2)$ satisfying (ii), (iii), (iv) of the definition of a P_{t-1} -reducer and also (i') above. We now construct colourings ψ_1 and ψ_2 of $B_{50}(v)$ satisfying the definition of a P_t -reducer.

- On $B_2(S_1)$, ψ_1 and ψ_2 agree with $\psi_1^{S_1, t}$ and $\psi_2^{S_1, t}$ respectively.
- On $B_2(S_2)$, ψ_1 and ψ_2 have the opposite colourings of $\psi_1^{S_2, t-1}$ and $\psi_2^{S_2, t-1}$.

- Outside $B_2(S_1)$ and $B_2(S_2)$, ψ_1 and ψ_2 are entirely blue, except on $N^{50}(v)$, where they are red.

We check that these colourings do indeed result in a P_t -reducer $B_{48}(v)$. Using property (i') of the pairs $\psi_1^{S_1,t}, \psi_2^{S_1,t}$ and $\psi_1^{S_2,t-1}, \psi_2^{S_2,t-1}$, and the fact that we use the opposite colourings on S_2 , we see that ψ_1 and ψ_2 have an equal number of red vertices in total, satisfying property (i) of a P_t -reducer. Property (ii) holds by the third bullet point of the construction. Property (iii) is immediate from it holding for the colourings $\psi_1^{S_1,t}, \psi_2^{S_1,t}, \psi_1^{S_2,t-1}$ and $\psi_2^{S_2,t-1}$, since the property (ii) for these colourings ensures that the monochromatic components involving vertices whose colours change are fully contained within $B_2(S_1)$ and $B_2(S_2)$ respectively. For property (iv), note we lose exactly one red copy of P_t when going from $\psi_1^{S_1,t}$ to $\psi_2^{S_1,t}$ on $B_2(S_1)$, while the number of blue copies of P_t stays the same. On the other hand, going from $\psi_1^{S_2,t-1}$ to $\psi_2^{S_2,t-1}$ on $B_2(S_2)$ does not affect the number of the monochromatic copies of P_t (by property (iii)). Thus in total exactly one red P_t is lost, as required. \square

The above proof doesn't work for $t = 3$ because our proof of Lemma 3.3 doesn't work for $t = 2$. Thus for $t = 3$, we need a different proof of the proposition.

Proof of Proposition 2.4 for $t = 3$. Let $Q = (q_0, \dots, q_{20})$ be a length 20 geodesic within distance 20 of v . For each $1 \leq i \leq 19$, let r_i be the (unique) neighbour of q_i . As in the proof of Lemma 3.3, if $r_i = r_j$ for some $1 \leq i < j \leq 19$, we must have $j = i + 1$ or $j = i + 2$, and then we find a P_3 -reducer using Lemma 3.2. Thus we may assume that r_1, \dots, r_{19} are all distinct.

Claim 3.4. *We have at least one of the following (see Figure 1):*

- (a) *For some $3 \leq i \leq 9$, none of the edges $r_i r_{i+1}, r_i r_{i+2}$ or $r_{i+1} r_{i+2}$ are present.*
- (b) *For some $3 \leq i \leq 10$, the edge $r_i r_{i+1}$ is present and either*
 - *the edges $r_i r_{i+2}$ and $r_i r_{i+3}$ are absent, or*
 - *the edges $r_{i-1} r_{i+1}$ and $r_{i-2} r_{i+1}$ are absent.*
- (c) *For some $3 \leq i \leq 8$, the edge $r_i r_{i+1}$ is present, the edges $r_{i+1} r_{i+2}, r_{i+1} r_{i+3}, r_{i-1} r_i$ and $r_{i-2} r_i$ are absent, and either $r_i r_{i+3}$ or $r_{i-2} r_{i+1}$ is present.*

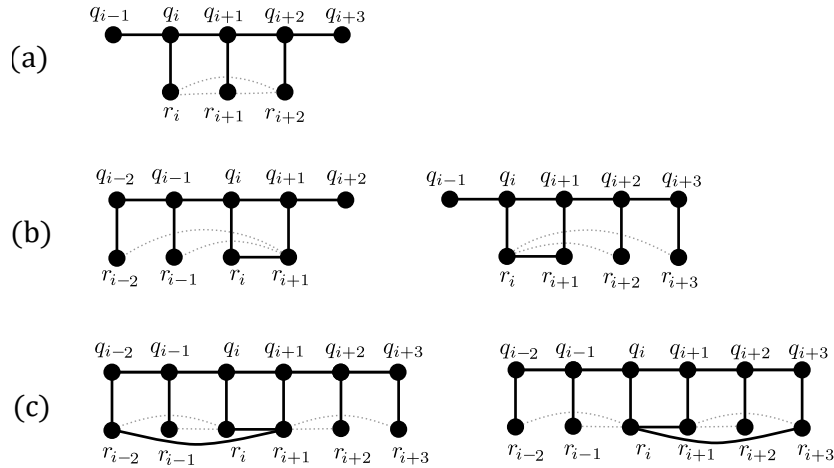


Figure 1: The three cases for Claim 3.4. Solid lines represent edges that are present while dashed lines represent edges which are absent (whereas unmarked edges may or may not be present).

Proof. First observe that the three edges r_3r_5, r_5r_7 and r_7r_9 cannot all be present, as that would contradict Q being a geodesic. Hence, if (a) does not hold, we must have the edge $r_i r_{i+1}$ for some $3 \leq i \leq 8$.

Next, suppose that $r_{i+1}r_{i+2}$ is also present. In this case, $N(r_{i+1}) = \{q_{i+1}, r_i, r_{i+2}\}$, which implies that $r_{i-1}r_{i+1}$ and $r_{i-2}r_{i+1}$ are both absent (since r_{i-1}, r_{i-2} are distinct from q_{i+1}, r_i, r_{i+1}). This leaves us in case (b). The same argument shows that we are in case (b) if any of the edges $r_{i+1}r_{i+3}, r_{i-1}r_i$ or $r_{i-2}r_i$ were present. Thus we can assume that $r_{i+1}r_{i+2}, r_{i+1}r_{i+3}, r_{i-1}r_i$ and $r_{i-2}r_i$ are absent.

If (c) doesn't hold, then both $r_i r_{i+3}$ and $r_{i-2}r_{i+1}$ must be absent. Then, to avoid case (b), both $r_i r_{i+2}$ and $r_{i-1}r_{i+1}$ must be present. Hence $N(r_i) = \{q_i, r_{i+1}, r_{i+2}\}$ and $N(r_{i+1}) = \{q_{i+1}, r_{i-1}, r_i\}$. Recalling that $r_{i+1}r_{i+2}$ and $r_{i+1}r_{i+3}$ are absent, note that $r_{i+2}r_{i+3}$ must be present, as otherwise we would be in case (a) for $i' = i + 1$. However, as $r_i r_{i+3}$ and $r_{i+1}r_{i+3}$ are both absent, we are then in case (b) with $i' = i + 2$. \square

We now find P_3 -reducers in each of the above three cases.

(a) Set $R = \{q_{i-1}, q_i, q_{i+1}, q_{i+2}, q_{i+3}, r_i, r_{i+1}, r_{i+2}\}$ and define colourings ψ_1, ψ_2 on R as follows:



Define both ψ_1 , and ψ_2 to be blue on $N(R)$ and red on $N^2(R)$. It is immediate that this colouring satisfies the definition of P_3 -reducer.

(b) Without loss of generality, we may suppose that $r_i r_{i+1}$ is present and the edges $r_i r_{i+2}, r_i r_{i+3}$ are absent (the other case is symmetric to this by reversing the order of the path Q). Let u be the third neighbour of r_i , i.e. $N(r_i) = \{q_i, r_{i+1}, u\}$. Consider the path $Q' = (u, r_i, q_i, q_{i+1}, q_{i+2}, q_{i+3})$. We claim that this is an induced path. Indeed the edges $r_i q_j$ are all absent for $j \neq i$, so the only way this could be non-induced is if $u = r_{i+1}, r_{i+2}$ or r_{i+3} . But u was the third neighbour of r_i so $u \neq r_{i+1}$ by definition, and $r_{i+2}, r_{i+3} \notin N(r_i)$ by the definition of case (b). Thus Q' is indeed induced, and by Lemma 3.2 (applied with $v = r_{i+1}$) there is a P_3 -reducer in $B_3(Q')$.

(c) Without loss of generality, we may suppose that $r_i r_{i+1}$ and $r_i r_{i+3}$ are present and $r_{i+1}r_{i+2}, r_{i+1}r_{i+3}, r_{i-1}r_i, r_{i-2}r_i$ are absent (the other case is symmetric). Set $R = \{q_{i-1}, q_i, q_{i+1}, q_{i+2}, q_{i+3}, r_i, r_{i+1}, r_{i+2}, r_{i+3}\}$ and define colourings ψ_1, ψ_2 on R as shown below. Extend both ψ_1 and ψ_2 to be blue on $N(R)$ and red on $N^2(R)$. Since q_i, r_i, q_{i+1} have no neighbours in $N(R)$, it is easy to check that this colouring satisfies the definition of P_3 -reducer.



\square

4 Concluding remarks

In this paper we have proven Ando's conjecture for all large connected cubic graphs. While this only leaves a finite number of graphs to be checked, and there is room to optimise the constants in our proof, there

will still be too many cases to be handled computationally. Hence, a complete resolution of Conjecture 1.2 will likely require some additional ideas. In this section we indicate some ways in which our proof could be modified, which might help make progress towards the full conjecture, and close with some related open problems.

A simpler starting block While our proof is relatively short, one could argue that it is not fully self-contained, as we use Thomassen’s theorem, which is already a very significant result. However, it is not crucial in our proof that, in the partition of the edges of G into linear forests F_1 and F_2 , the paths have length at most five. We could therefore replace Theorem 1.1 with one of its predecessors (the theorems of Aldred and Wormald [5] or Jackson and Wormald [12]), which have simpler proofs, but allow for longer paths. Although this comes at the cost of requiring P_t -reducers for larger values of t , our constructions readily generalise to longer paths. This is especially easy when we assume G has large girth, resulting in a truly short proof of this special case.

Fewer reducers Alternatively, one might seek to reduce the amount of work done in Section 3, when constructing the P_t -reducers. A potential route to simplification lies in the observation that, when using Theorem 1.1, it was not very important that the paths in F_1 were so strongly bounded in length. Indeed, we only used the lengths of the paths in F_1 to bound the Lipschitz constant c in our application of Theorem 2.2, and we can afford for this to be as large as $n^{o(1)}$. On the other hand, if we can limit the lengths of the paths in F_2 to some $\ell \leq 4$, then we would only need to construct P_t -reducers for $2 \leq t \leq \ell + 1$.

Question 4.1. *What is the smallest ℓ for which the edges of any connected cubic graph on n vertices can be partitioned into two spanning linear forests F_1 and F_2 , such that the paths in F_1 are of length $n^{o(1)}$, and the paths in F_2 are of length at most ℓ ?*

It is worth noting that, while the five in Theorem 1.1 is best possible, the examples of tightness given by Thomassen [16] are the two cubic graphs on six vertices. It would be interesting to know if there are arbitrarily large tight constructions, or if, when dealing with large connected graphs, one can achieve $\ell = 4$ even in the symmetric setting. Furthermore, we can weaken Question 4.1, as in our application F_1 does not have to be a linear forest, but rather a bipartite graph with bounded components.

Stronger conjectures While Theorem 1.3 sheds light on the structure of large cubic graphs, showing that they can be partitioned into isomorphic induced subgraphs, it does not directly address the motivating question raised in Section 1, as there are no guarantees that these subgraphs are simple. However, by analysing our proof, one can obtain some further information about the subgraphs obtained. As stated in Proposition 2.1, the only monochromatic components in the initial random colouring are paths of length at most five. When we then use the P_t -reducers to make the subgraphs isomorphic, we can introduce more complicated monochromatic components. However, since the P_t -reducers are all isolated within balls of bounded radius, it follows that the components in the isomorphic subgraphs are of bounded size.

In particular, if we assume that our connected graph has large girth,¹ it follows that the isomorphic subgraphs are forests. Moreover, we need never have vertices of degree three in the isomorphic subgraphs, as these can be recoloured (in pairs) to become isolated vertices of the opposite colour. Thus, we in fact partition large connected cubic graphs of large girth into isomorphic linear forests. It was conjectured by Abreu, Goedgebeur, Labbate and Mazzuocolos [2] that every cubic graph should admit such a partition; the challenge lies in removing the girth condition.

¹By carefully considering the P_t -reducers from Section 3.1, it suffices to assume girth at least 15.

Ban and Linial [7] went even further, conjecturing that much more should be true when we restrict our attention to two-edge-connected cubic graphs.

Conjecture 4.2. *The vertices of every bridgeless cubic graph, with the exception of the Petersen graph, can be two-coloured such that the two colour classes induce isomorphic matchings.*

The conjecture remains open, although it has been proven for three-edge-colourable graphs by Ban and Linial [7] and for claw-free graphs by Abreu, Goedgebeur, Labbate and Mazzuoccolo [1], while Esperet, Mazzuoccolo and Tarsi [9] proved that any counterexample must have circular flow number equal to five. It would be very interesting to see to what extent our methods can be applied to this conjecture, as well as to Wormald's conjecture on partitioning the edges of cubic graphs into isomorphic linear forests.

Acknowledgement

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