Limitations of Portfolio Diversification through Fat Tails of the Return Distributions: Some Empirical Evidence

ABSTRACT

This study investigates the level of risk due to fat tails of the return distribution and the changes of tail fatness (**TF**) through portfolio diversification. **TF** is not eliminated through portfolio diversification, and, interestingly, the positive tail has declining fatness until a certain level is reached, while the negative tail has rising fatness. This indicates that fat tails are highly relevant to common factors on systematic risk and that the relevance of common factors is higher for the negative tail compared to the positive tail. In the portfolio diversification effect, the declining fatness of the positive tail further reduces risk, but the rising fatness of the negative tail does not contribute to this effect. The asymmetry between the fatness of the positive and negative tails in the return distribution corresponds to the asymmetry of the trade-off relationship between loss avoidance and profit sacrifice that is expected as a consequence of portfolio diversification. Investors use portfolio diversification to reduce their risk of suffering high losses, but following this strategy means sacrificing high-profit potential. Our study provides empirical confirmation for the practical limitation of portfolio diversification and explains why investors with diversified portfolios suffer high losses from market crashes. An examination of the Northeast Asian stock markets of China, Japan, Korea, and Taiwan show identical results.

Keywords: Fat tails; Portfolio diversification; Common factors; Principal components analysis; Random matrix theory; Singular value decomposition.
***JEL** classification*: G11, C30.

1. Introduction

Stock market crashes have cast doubt on the practical usefulness of modern portfolio theory. Even investors with a well-diversified portfolio based on the portfolio theory have still suffered high losses caused by market crashes, such as the 1997 Asian financial crisis and the 2008 global financial crisis, contrary to expectation. Portfolio diversification has a long history, even being mentioned in Shakespeare's *The Merchant of Venice*, as noted by Markowitz (1999) and Rubinstein (2002). Portfolio diversification is known as a useful investment tool that may effectively reduce the risk of future uncertainty in financial markets. Evans and Archer (1968) empirically show that portfolio risk is effectively reduced by increasing the number of stocks in a given portfolio. From the perspective of the risk and return relationship, investors with a portfolio investment are advised to sacrifice the opportunity for high profits in order to reduce the likelihood of high losses, compared to investing in individual stocks, i.e., there is a trade-off relationship between loss avoidance and profit sacrifice. However, the return distribution of a financial time series is known to have fat tails (Mandelbrot, 1963; Fama, 1965). Since large price fluctuations are located in the tail parts of the return distribution, the tail parts significantly affect the measurement of risk. The fat tails of the return distribution explain the changes in expected return (Kelly and Jiang, 2014) and are highly relevant to investor attention toward investment (Eom and Park, 2020). As noted in previous studies, determining the expected returns requires identifying whether the risk property included in the tail parts of the return distribution is systemic risk or unsystematic risk. Therefore, the purpose of this study is to empirically identify the risk included in the tails of the return distribution through portfolio diversification.

Among the stylized facts empirically observed in a financial time series (Cont, 2001), the existence of fat tails in the return distribution is widely known. Mandelbrot (1963) and Fama (1965) suggest that the empirical distribution of stock returns characteristically exhibits a more peaked central part and fatter tail parts compared to the normal distribution assumed by financial theories. They explain these distributional characteristics using the Stable distribution. Mantegna and Stanley (1995) show that the characteristics of the return distribution vary according to changes in the time-scale in converting price data into return data; that is, the distributions from high- (low-) frequency return data have much fatter (thinner) tails. Praetz (1972) and Blattberg and Gonedes (1974) suggest that Student's *t* distribution, in which the fat tails have a decreasing degree of freedom as a parameter, can explain the characteristics of the empirical return distributions. Peiro (1994) shows evidence that the empirical distributions from return data in developed countries such as the U.S., Japan, and Germany tend to be close to the properties of Student's *t* distribution. However, this distribution with a symmetric structure struggles to substantially explain the empirical return distribution with an asymmetric structure. Until recently, the characteristics of the tail fatness (**TF**) in the return distribution have not been sufficiently explained by theoretical distributions, possibly due to the failure of theoretical distributions to consider the economic implications included in return data generated by trading activities in stock markets, despite implementing the shape and characteristics of the distribution using key parameters. This research gap necessitates research aimed at identifying the economic implications of the tail parts of the return distribution.

The tail parts of the return distribution are highly related to both the risk measure of the expected return and the strength of investor attention on the expected returns in behavioral finance models. Return data in the tail parts of the distribution are created by large price fluctuations that occur infrequently. Most return data during market crashes are located in the negative tail parts of the return distribution. Hence, large price fluctuations are significant in measuring the risk. For example, value-at-risk (VaR) is a measure of the downside risk on the negative tail that is calculated using the statistical probability of high losses located in the area beyond a certain percentile (99%) in the cumulative distribution. Based on the Hill estimator (**HE**) of Hill (1975), Kelly and Jiang (2014) suggest that a factor of tail risk that is estimated from the past return distribution by the cross-sectional approach may explain the significant changes of expected returns in the future period. Eom and Park (2020) show evidence supporting the significant influence of fatness in the tails of the past return distribution on the prospect theory value (e.g., Tversky and Kahnamn, 1992; Barberis, Mukherjee, and Wang, 2016) that quantifies the degree of representativeness bias from investors. However, little research has attempted to directly identify whether the tail parts of the return distribution contain systematic risk or unsystematic risk, which are the two components of portfolio risk. Portfolio diversification effectively eliminates most unsystematic risk that is highly related to firm-specific factors but cannot reduce the systematic risk caused by common factors such as market, industry, and macroeconomic factors. The expected returns of a diversified portfolio may be explained by systematic risk under the risk and return relationship. Consequently, the method investigating the effect of portfolio diversification can be utilized to identify the risk property included in the tail parts by observing the changes of the fatness of the positive and negative tails in the return distribution.

This study empirically investigates whether the influence of the large price fluctuations included in the tail parts of the return distribution is reduced, along with the effect of risk reduction, through portfolio diversification. Verification of whether the fatness of the tail parts in the return distribution is eliminated by portfolio diversification requires that the risk property included in the tail parts is identified. In addition, the changes of fatness in each positive and negative tail of the return distribution are examined. To address the research goals of this study, the following empirical design is used.

First, the fatness of the positive and negative tails of the return distribution is employed as a measure of tail risk. Data located in the tail parts of the return distribution represent large price fluctuations that do not occur frequently. The fatness of the tail parts in the return distribution has a significant influence on the measurement of risk, because risk as the level of uncertainty is defined as the degree of deviation from the average value. Eom, Kaizoji, and Scalas (2019) measure the degree of the fatness of the positive and negative tails in the return distribution using the relative frequency calculated from the frequency distribution of return data. This study employs **TF** by the relative frequency in the distribution as a measure of tail risk in line with our research goal of investigating the changes of fatness in each positive and negative tail of the return distribution through portfolio diversification. To empirically verify the measure's accuracy for measuring the tail risks, we compare it with the measures of **HE** suggested by Hill (1975) and of tail risk (**TR**) suggested by Kelly and Jiang (2014) by using random data generated by Student's t distribution that has fat tails according to degrees of freedom.

Second, the risk property included in the tail parts of the return distribution and its economic implications are investigated based on the eigenvalues extracted from statistical methods that are generally utilized to explore the existence of common factors. In the field of finance, principal components analysis (**PCA**) has been a representative statistical method used to extract common factors from homogeneous properties among stock returns. Previous studies of King (1966) and Ross (1976) suggest that eigenvalues with high values (high-value eigenvalues) extracted from return data have the characteristics of common factors, and their findings greatly contribute to the development of asset pricing models. Eom, Jung, Kaizoji, and Kim (2009) report that the largest eigenvalue has the property of a market factor regardless of the number of stocks in a portfolio, and this eigenvalue is utilized in portfolio optimization and its investment strategies as a proxy of the market factor (Eom, Park, Kim and Kaizoji, 2015; Eom and Park, 2017; Eom, 2017; Eom and Park, 2018). As a result, high-value eigenvalues have economic implications related to common factors that commonly affect the changes in stock returns. This study investigates the relationship between the eigenvalues' magnitude and the TR measures in the distribution of an eigenvalue time series. A positive relationship means that the higher the eigenvalue, the more the eigenvalue distribution tends to have fatter tails. Since high-value eigenvalues are known to have economic implications related to common factors, the fat tails of the return distribution may be considered to include the properties of these eigenvalues; that is, the properties of common factors. To obtain reliable results, random data generated from each of the normal distribution, Student's t distribution, and the stable distribution are utilized in the same testing procedure. Theoretical distributions may implement the shape and characteristics of the return distribution using key parameters but do not include economic implications of price data created from trading activities in the stock market. Hence, even if eigenvalues extracted by the same statistical method are utilized, the results using random data based on theoretical distributions will differ from those using stock returns.

Third, the effect of common factors on the tail parts in the return distribution is investigated using two types of return data according to the properties of common factors; that is, return data both with and without the properties of common factors. If the tail parts of the return distribution have the properties of common factors, the results using the return data with common factors' properties will differ from the results using the return data without common factors' properties. Using the method devised in this study, we generate the two-type return data according to the properties of common factors and apply them to the testing procedure of portfolio diversification. The devised method combines the random matrix theory (**RMT**) and singular value decomposition (**SVD**). RMT and SVD utilize the same eigenvalues and eigenvectors with PCA. RMT identifies the number of common factors included in stock returns, and SVD generates the two-type return data with and without common factors' properties based on the number of common factors from RMT. Then, using two-type return data, the changes of the TF measures on the positive and negative tails of the return distribution are investigated via the testing procedure for portfolio diversification.

Fourth, the trade-off relationship between loss avoidance and profit sacrifice from portfolio diversification is quantitatively examined using the devised method. Investors usually employ portfolio diversification as an investment tool in order to reduce the likelihood of high losses occurring due to future uncertainty while accepting the reduced possibility of high profits based on the risk and return relationship. However, even with such a diversified portfolio, stock market crashes can cause high losses to investors, which casts doubt on the practical usefulness of portfolio diversification during crises. This study devises a measure, termed the profit/loss (P/L) ratio, to quantify the trade-off relationship between loss avoidance and profit sacrifice. The comparative criterion on loss and profit from a portfolio is the performance of the stocks that comprise the portfolio. A high loss from the investment in stocks corresponds to the effect of loss avoidance that can be obtained from portfolio diversification, while a high profit from the investment in a stock is the cost of profit sacrifice that must be given up from portfolio diversification. The L-ratio (P-ratio) that is a comparison between the high loss (profit) of a stock in a portfolio and the high loss (profit) of the portfolio is measured, and the P/L ratio is calculated by dividing the P-ratio by the L-ratio. This study employs the rolling-sample method that uses sub-periods for the whole period because of the trade-off relationship between loss avoidance and profit sacrifice based on performance changes over time.

The main results are as follows. The TF measure that is employed as a risk measure of the positive and negative tails of the return distribution is significantly and highly relevant with both the TR measure by Kelly and Jiang (2014) and the HE measure by Hill (1975). That is, the TR measure can act as a measure of the tail risk. The changes in the TF measure observed in portfolio diversification are very similar to the changing pattern of systematic risk due to common factors. That is, portfolio diversification does not eliminate the fatness of the tails of the return distribution. An interesting finding is an asymmetric relationship between the changing patterns of the TF measures observed from the positive and negative tails of the return distribution. As the number of stocks in a portfolio increases, the TF measure from the positive tail of the return distribution shows a declining trend until a certain level is reached, while the TF measure from the negative tail presents a rising trend until a certain level is reached. Using the devised methods, this study further explores these findings. According to the results using eigenvalues extracted from return data by PCA, the magnitude of eigenvalues has a positive relationship with the TF measures of the distribution using an eigenvalue time series; that is, the higher the eigenvalues, the fatter the tails of the eigenvalue distribution tend to be. Hence, since high-value eigenvalues extracted from stock returns have economic implications related to common factors, return data located in the tail parts of the return distribution may be regarded as containing the characteristics of high-value eigenvalues with common factors' properties. However, the results using random data generated from the theoretical distributions do not show a significant relationship between the eigenvalues' magnitude and the TF measures of the eigenvalue distribution. Therefore, eigenvalues extracted from stock returns have significant economic implications that cannot be contained in random data. Next, according to the results using two types of return data with and without common factors' properties through the devised method combining RMT and SVD, we verify that the tail parts of the return distribution are significantly and highly relevant to the properties of common factors. When using return data with the properties of common factors, the tail parts of the return distribution have a similar pattern of systematic risk, and the changes of the TF measures of the positive and negative tails are clearly asymmetric, as confirmed in the previous results. However, when using return data without the properties of common factors, the TF measures of both the positive and negative tails in the return distribution show a declining trend, and the difference between the TF measures of the positive and negative tails is not evident. Meanwhile, the asymmetric change of fatness in the positive and negative tails of the return distribution causes the asymmetric trade-off relationship between loss avoidance and profit sacrifice through portfolio diversification. According to the results using the P/L ratio as a devised measure, the cost of profit that investors are willing to sacrifice by utilizing portfolio diversification is much higher than the effect of loss avoidance that investors expect to gain from portfolio diversification. Therefore, these results provide empirical evidence supporting the practical limitations of portfolio diversification. Moreover, our finding that portfolio diversification cannot sufficiently reduce the fatness of the negative tail in the return distribution, contrary to expectation, may explain why investors who hold a diversified portfolio can nevertheless suffer high losses from market crashes. We expect future research to improve the practical usefulness of portfolio diversification in the stock market.

The remainder of this study is organized as follows. The next chapter describes the data and periods for the stock markets of four countries in Northeast Asia. Chapter 3 shows the results for the research objectives in detail. The final chapter summarizes the main findings and concludes.

2. Data and Periods

This study utilizes daily stock data (prices, P_t; returns, $R_t = {}^{P_t}/_{P_{t-1}} - 1$) from the Northeast Asia stock markets in

China, Japan, Korea, and Taiwan.[1] By comparing results observed from each country's stock data, we expect to enhance the data robustness using out-of-sample results. The data source is from Compustat Global provided by the Wharton Research Data Service. Data and periods of each country are as follows. First, stocks traded in the Shanghai exchange among 3,814 stocks in the Chinese stock markets are selected over the period from April 2001 to June 2018 (number of trading days, 4,573). Second, stocks traded in the Tokyo exchange among 5,581 stocks in the Japanese stock markets are selected over the period from January 1986 to June 2018 (7,983). Third stocks in the Korea exchange among 2,977 stocks in the Korean stock markets are selected over the period from January 1986 to June 2018 (7,980). Fourth, stocks in the Gretai exchange among 2,314 stocks in the Taiwanese stock markets are selected over the period from January 1993 to June 2018 (6,038). Following Fama and French (1992), stocks belonging to financial sectors are removed for each country. In the whole period of each country, stocks that have all price information within the sub-period divided according to research goals are utilized in the testing procedure. In particular, in order to control the influence of market crashes on the results, we test three-type sub-periods before and after the global financial crisis triggered by the U.S. credit crisis of September 2008. In addition, the time-varying performance according to market dynamics is investigated by using the rolling-sample method with divisions into various sub-periods.

Next, this study utilizes random data generated by the three-type theoretical distribution to analyze the robustness of the results using stock data; that is, the normal distribution that is generally employed in assumptions of financial theories, and Student's t distribution and the Stable distribution that is utilized to explain the characteristics of the empirical distribution in a financial time series. We briefly introduce the probability density function and its key parameters, as follows. The probability density function ($f(x|\mu,\sigma)$) of the normal distribution is defined as:

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

In Eq. (1), key parameters are mean (μ) and standard deviation (σ). In Student's t distribution, the probability density function ($f(x|\nu)$) is as follows (Blatterberg and Gonedes, 1974):

$$f(x|\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\frac{1}{(1+\frac{x^2}{\nu})^{\frac{\nu+1}{2}}} \tag{2}$$

Where, ν is degree of freedom as a key parameter, and $\Gamma(.)$ is the gamma function. Finally, the probability density function ($f(x|\alpha,\beta,\gamma,\delta)$) of the Stable distribution is introduced. This distribution does not have a close-form of the probability density function, and the characteristic function suggested by previous studies (e.g., Ponta, Trinh, Raberto, Scalas, and Cincotti, 2017; Nolan, 2018) is as follows:

$$\hat{f}(x|\alpha,\beta,\gamma,\delta) = E[\exp(i\kappa X)] = \begin{cases} \exp(-\gamma^{\alpha}|\kappa|^{\alpha}\left[1+i\beta(sign\ \kappa)\tan\frac{\pi\alpha}{2}((\gamma|\kappa|)^{1-\alpha}-1)\right]+i\delta\kappa) & for\ \alpha \neq 1 \\ \exp(-\gamma|\kappa|\left[1+i\beta(sign\ \kappa)\frac{2}{\pi}\ln(\gamma|\kappa|)\right]+i\delta\kappa) & for\ \alpha = 1 \end{cases} \tag{3}$$

In Eq. (3), this distribution consists of four key parameters to describe the distributional characteristics: α ($0 < \alpha \leq 2$) is for the tail index, β ($-1 \leq \beta \leq +1$) is for skewness, and γ ($0 < \gamma \leq +\infty$) and δ ($-\infty < \delta < +\infty$) are for scale and location, respectively. This study generates random data for each three-type theoretical distribution based on Eom et al. (2019). The key parameters of the theoretical distribution are directly estimated from stock returns traded in each stock market of the four countries,[2] and random data are generated

[1] This study is designed to concentrate on the risk property for fat tails in the return distribution and its relationship with portfolio diversification, rather than whether to be normality or not. Therefore, due to measuring the portfolio performance, we employ the method of discrete return, not logarithmic return that has measurement error when converting price into returns.

[2] This study employs Matlab's statistical toolbox (*fitdist* m-function) to estimate the key parameters of the theoretical

corresponding to the returns of each stock using estimated key parameters of each theoretical distribution. Consequently, we utilize random data having the same structure with stock returns in each stock market of the four countries, along with actual stock returns.

3. Results

3.1. Risk measures on the fat tails of the return distribution

This section verifies evidence that the TF measures on the positive and negative tails in the return distribution can act as a measure of tail risk. The tail parts of the return distribution have the risk property because large price fluctuations that occur infrequently are located in the tail parts of the distribution. The return distribution is well known to have fat tails (Mandelbrot, 1963; Fama, 1965; Mantegna and Stanley, 1995). Eom et al. (2019) identify the degree of fatness in the positive and negative tails of the return distribution using the relative frequency. Using standardized return data (x_z), return data are included from the positive and negative tail parts equivalent to each tail area of 0.5% ($x_z \leq -2.58$ and $x_z \geq +2.58$), separated from the central part in the return distribution. Here, the standardized return values are calculated by subtracting the mean of returns and dividing by the standard deviation of returns. The relative frequency is calculated by dividing the number of data ($f_x^{(-)}$, $f_x^{(+)}$) located in the 0.5% tail area of the distribution by the total number of data (f_T).

$$TF^{(-)} = \frac{f_x^{(-)}}{f_T}, \text{ where } f_x^{(-)} = f(x_z \leq -2.58) \quad (4)$$

$$TF^{(+)} = \frac{f_x^{(+)}}{f_T}, \text{ where } f_x^{(+)} = f(x_z \geq +2.58) \quad (5)$$

In the equations, the relative frequencies of the negative tail from Eq. (4) and the positive tail from Eq. (5) denote the TF measures for negative and positive tails, respectively. Whether the TF measure acts as the tail risk's measure is verified by comparing it with measures of tail risk reported in previous studies. We employ the HE measure suggested by Hill (1975) and the TR measure by Kelly and Jiang (2014). The HE estimator ($\widehat{\alpha_k}$) suggested by Hill (1975) is as follows:

$$\text{HE} \equiv \widehat{\alpha_k} = [\frac{1}{k}\sum_{i=1}^{k} \ln(\frac{X_{(i)}}{X_{(k)}})]^{-1}, \quad \text{where } X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(k_n+1)} \quad (6)$$

The TR measure suggested by Kelly and Jiang (2014) is defined by:

$$\text{TR} \equiv \alpha_t = \frac{1}{K_t}\sum_{k=1}^{K_t} \ln(\frac{R_{k,t}}{\upsilon_t}) \quad (7)$$

In equation (7), x and υ_t are critical values of -2.58 and +2.58, respectively, on the same basis for the TF measure in the standardized data. X_i and R_k are extreme values beyond the critical values of x and υ_t, respectively, and $k_n(\text{i} = 1,2,..k_n)$ and K_t $(\text{t} = 1,2,\ldots,K_t)$ are the total number of extreme values beyond the critical value. When the tail parts of the distribution get fatter, the TR measure has a higher value, the HE measure has a lower value, and the TF measure in this study has a higher value. Accordingly, the anticipated relationship among the three measures is as follows: a positive correlation between the TF and TR measures, a negative correlation between the TR and HE measures, and a negative correlation between the TF and HE measures. The results are presented in **Figure 1**. We utilize random data generated based on Student's t distribution according to the increasing degrees of freedom from 3 to 53 at increments of 0.5. This distribution has fatter (thinner) tails with decreasing (increasing) degrees of freedom. For obtaining robust results, random data with a sufficient sample size (100,000) are generated 100 times for each degree of freedom. The results are divided by the three measures.

distribution using stock returns.

The TF measures on the positive and negative tails are shown in Figures 1(a) and 1(d), and in Figures 1(b) and 1(e) for the TR measures, and in Figures 1(c) and 1(f) for the HE measures.[3] In the figures, the X-axis indicates the degree of freedom, and the Y-axis denotes the average values of 100 results repeated for each degree of freedom.

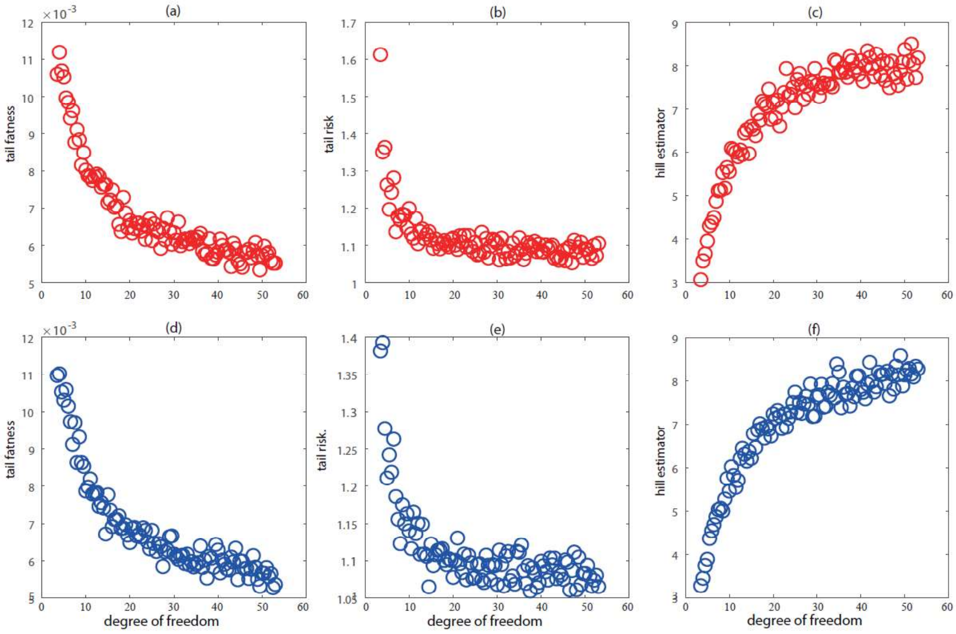

Fig. 1. Risk measures for the tails of the Student's t distribution. The figures show the changes in risk measures by the increasing degrees of freedom (DoF) in the Student's t distribution. The risk measures consist of tail fatness (TF; a, d), tail risk (TR; b, e) suggested by Kelly and Jiang (2014), and the Hill estimator (HE; c, f) suggested by Hill (1975). The positive and negative tails in the distribution are divided into upper (a, b, c) and lower (d, e, f) figures, respectively. The results are the average values of 100 risk measures repeatedly calculated using random data with a data number of 100,000 following Student's t distribution with each DoF within the range of 3 to 53 in 0.5 increments. The X-axis indicates the DoF, and the Y-axis indicates the values of risk measures.

According to the results, we determine that the TF measure can be utilized as a risk measure of the tail parts of the return distribution. Regardless of the positive and negative tails in the distribution, the TF measure is high (low) at low (high) degrees of freedom. The TR measure is high (low) at low (high) degrees of freedom, while the HE measure is low (high) at low (high) degrees of freedom. These results are consistent with the anticipated relationship among the three measures. Specifically, the correlation of the positive and negative tails between the TF and TR measures are 86.93% and 80.33%, between the TF and HE measures are -96.79% and -96.37%, and between the TR and HE measures are -84.19% and -82.19%, respectively. Based on these results, this study utilizes the TF measure to investigate the risk measure by the fatness of the positive and negative tails of the return distribution.

3.2. Portfolio diversification effect

[3] Student's t distribution has the characteristics of fat tails and a symmetric structure for negative and positive sides in the distribution. Although this study utilizes random data generated based on the characteristics of Student's t distribution, the distribution from random data does not show an exact symmetric structure between positive and negative sides. Therefore, Figure 1 shows all of the positive and negative tails for reliability.

This section presents the results for changes in the portfolio risk and the changes in the fatness of the tail parts of the return distribution through portfolio diversification. Here, portfolio risk is measured by the standard deviation of returns, and the fatness of the positive and negative tails is quantified by the TF measure. The portfolio diversification effect is the dramatic reduction in portfolio risk as the number of stocks in a portfolio increases. Portfolio risk consists of unsystematic risk related to firm-specific factors and systematic risk related to common factors. The effects of portfolio diversification are attributed to the unsystematic risk. From the perspective of performance, return data included in the positive tail of the return distribution correspond to high profits from large price fluctuations, while return data included in the negative tail represent high losses from large price fluctuations. Hence, the tail parts of the return distribution have a significant influence on measuring portfolio risk. By observing changes in the TF measure on the positive and negative tails of the return distribution, we investigate whether the influence of large price fluctuations in the tail parts of the return distribution is limited by the reduction of portfolio risk through portfolio diversification. We employ the method of Evans and Archer (1968), who empirically examine the effect of portfolio diversification based on the simulation using randomly selected stocks. For a portfolio constructed with the number of stocks ranging from 2 to 50, we calculate the standard deviation and the TF measures in the return distribution for the portfolio constructed using randomly selected stocks in each number of stocks.[4] This is repeated 100 times for each number of stocks. Portfolio returns are calculated using the equal-weighted method. The results are presented in **Figure 2**. In the figure, the X-axis indicates the number of stocks in a portfolio, the upper Y-axis denotes the average values of 100 standard deviations of returns, and the lower Y-axis represents the average values of 100 TF measures on the positive and negative tails in the return distribution. Figures are divided according to the stock markets of the four countries. Figure 2(a) represents the Chinese stock market over the period from July 2001 to June 2018, Figure 2(b) depicts the Japanese stock market, Figure 2(c) shows the Korean stock market, and Figure 2(d) represents the Taiwanese stock market over the period from July 2000 to June 2018.

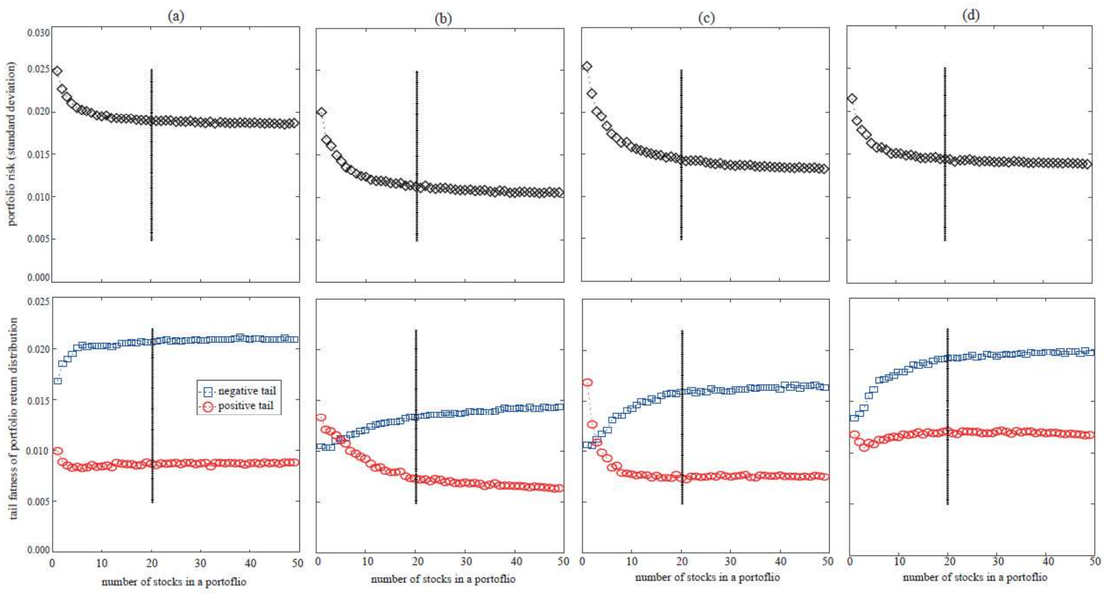

Fig. 2. Portfolio diversification effect by portfolio risk and tail fatness (TF) in the return distribution. The figures show the portfolio diversification effect according to an increasing number of stocks in a portfolio from the perspectives of portfolio

[4] Elton and Gruber (1977) suggest a total of 10~20 stocks in a portfolio is required to significantly reduce the unsystematic risk through portfolio diversification, and Statman (1987) mentions 30~40 stocks in a portfolio with the additional condition of borrowing and lending risk free rate. Accordingly, this study sets the number of stocks within the range from 2 to 50 in a portfolio, and specifies around 20 as the number of stocks that visually show the reduction of portfolio risk.

risk (upper figures) and tail fatness (TF) in the portfolio return distribution (lower figures). Portfolio risk is measured by the standard deviation of returns. The tail fatness is quantified by the TF measures on the positive and negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the distribution. The results consist of the four countries of China (a), Japan (b), Korea (c), and Taiwan (d). The X-axis indicates the number of stocks in a portfolio within the range of 2 to 50, and the Y-axis presents the average values of portfolio risk, and the TF measures calculated from 100 iterated simulations for each portfolio constructed by randomly selecting stocks in each stock market under the condition of sampling without replacement. The vertical line in the portfolio is comprised of 20 stocks based on Elton and Gruber (1975).

Figure 2 shows evidence supporting the effect of portfolio diversification regardless of stock markets. As previously reported, portfolio risk shows a declining trend with an increasing number of stocks in a portfolio, although this trend disappears at around 20 stocks and the trend plateaus at a certain level. However, the TF measures on the positive and negative tails of the return distribution show contrasting trends. As the number of stocks in a portfolio increases, the TF measures on the positive tail of the return distribution clearly decline, and the trend plateaus at around 20 stocks. However, the TF measures on the negative tail of the return distribution show a rising trend according to an increasing number of stocks in a portfolio, and the trend plateaus at around 20 stocks. The TF measures for the negative tail are higher values than those for the positive tail. These results are consistent for the four stock markets examined in this study. Based on the results displayed in **Figure 2**, the fatness of the tail parts in the return distribution is not eliminated through portfolio diversification. This means that the risk property included in the tail parts of the return distribution closely resembles the systematic risk that is attributed to common factors. An interesting finding is that the changes of the TF measures are asymmetric in the positive tail with a declining trend and in the negative tails with a rising trend. In the relationship between portfolio risk and the fatness of the tails in the return distribution, investors can benefit by reducing the magnitude of portfolio risk through portfolio diversification, but this reduction is highly related to the declining fatness of the positive tail rather than the rising fatness of the negative tail. Meanwhile, the high/low TF measures on the tail parts of the return distribution imply a large and small frequency for high profits and high losses located in the tail parts, respectively. Therefore, our finding suggests that, though portfolio diversification, the low TF measures observed from the positive tail in the return distribution indicate a decreasing frequency of high profits, while the high TF measures from the negative tail represent the increasing frequency of high losses, which corresponds to the asymmetric trade-off relationship between loss avoidance and profit sacrifice. We present further evidence supporting main findings observed in **Figure 2** in the next section. First, the risk property included in the tail parts of the return distribution is close to common factors. Second, the asymmetry of the TF measures on the positive and negative tails of the return distribution corresponds to the asymmetry of the trade-off relationship between loss avoidance and profit sacrifice.

3.3. Risk characteristics of fat tails using eigenvalues

This section presents the results using the eigenvalues extracted from the statistical method to determine whether the risk property included in the tail parts of the return distribution is close to the systematic risk due to common factors. Previous studies have reported that high-value eigenvalues have economic implications as related to common factors defined as the homogeneous property among stock returns, and these eigenvalues may be utilized as determinants in pricing models (King, 1966; Ross, 1976; Brown, 1989; Eom et al., 2009). We examine whether the relationship between the eigenvalues' magnitude and the fatness of the tail parts in the eigenvalue distribution is positive. A positive relationship means that high-value eigenvalues with the properties of common factors have a tendency to show fat tails in the distribution of eigenvalue time series. Then, the evidence implies that the fat tails of the return distribution contain the properties of high-value eigenvalues, and inferentially, suggests that the properties included in the tail parts of the return distribution may be related to common factors. Using PCA, K-eigenvalues ($E_k,\ k = 1,2,\dots,K$) are extracted from the return data ($R_j,\ j = 1,2,\dots,N$) of N-stocks. The number of eigenvalues is the same as the number of stocks, that is, $N \equiv K$. The time series corresponding to each eigenvalue is generated as follows:

$$R_{E(k),t} = \sum_{j=1}^{N} V_{k,j} R_{j,t} \quad (8)$$

In the equation, $V_{k,j}$ is an eigenvector of the k-th eigenvalue of stock j. This study investigates the relationship between the magnitude of K-eigenvalue (E_k) and the TF measures on the positive and negative tails of the distribution using the eigenvalue time series ($R_{E(k),t}$). The results are presented in **Table 1,** according to the stock markets of the four countries. In each country, the results using stock returns and random data from theoretical distributions are presented. The whole period (P1) is divided into before and after the global financial crisis (P3: 2007.07~2011.06) triggered by the 2008 U.S. credit crisis, i.e., four sub-periods of P1, P2, P3, and P4.[5] It is known that the largest eigenvalue has a much higher value than the second largest eigenvalue, so that including the largest eigenvalue may significantly affect the results. Hence, the table is divided according to the results using all eigenvalues, including the largest eigenvalue, and the results using all eigenvalues, except the largest eigenvalue.

Table 1.
Relationship between the eigenvalues' magnitude and tail fatness (TF) in the eigenvalue distribution

	# of stocks / duration	Negative tail		Positive tail	
		With 1st Eigenvalue	Without 1st Eigenvalue	With 1st Eigenvalue	Without 1st Eigenvalue
Panel A: the Chinese stock market					
▸ stock returns					
P1: 2001.07~2018.06	491 / 4,185	0.2996	0.5271	0.0401	0.5737
P2: 2001.07~2007.06	504 / 1,506	0.0944	0.4442	0.0739	0.5144
P3: 2007.07~2011.06	720 / 976	0.2586	0.4290	0.0228	0.4407
P4: 2011.07~2018.06	760 / 1,703	0.2016	0.5335	0.0815	0.6085
▸ random data using parameters estimated from stock returns					
Normal distribution	P1	0.0325	0.0354	0.0093	0.0133
Student' t distribution	P1	-0.1394	-0.1468	-0.1430	-0.1311
Stable distribution	P1	-0.2225	-0.2279	-0.2237	-0.2380
Panel B: the Japanese stock market					
▸ stock returns					
P1: 2000.07~2018.06	2,051 / 4,409	0.1899	0.5434	0.1141	0.6206
P2: 2000.07~2007.06	2,358 / 1,715	-	-	-	-
P3: 2007.07~2011.06	2,929 / 977	-	-	-	-
P4: 2011.07~2018.06	2,811 / 1,717	-	-	-	-
▸ random data using parameters estimated from stock returns					
Normal distribution	P1	0.0118	0.0036	-0.0118	-0.0047
Student' t distribution	P1	-0.0607	-0.0890	-0.0605	-0.0890
Stable distribution	P1	-0.2125	-0.2213	-0.2110	-0.2204
Panel C: the Korean stock market					
▸ stock returns					
P1: 2000.07~2018.06	416 / 4,440	0.3944	0.3233	0.0922	0.3926
P2: 2000.07~2007.06	492 / 1,719	0.3263	0.3954	0.1277	0.4795
P3: 2007.07~2011.06	622 / 999	0.2956	0.3727	0.1759	0.4209
P4: 2011.07~2018.06	717 / 1,722	0.3863	0.4316	0.2332	0.5589
▸ random data using parameters estimated from stock returns					
Normal distribution	P1	0.0341	0.0022	0.0539	0.0426
Student' t distribution	P1	-0.2987	-0.3474	-0.3001	-0.3612
Stable distribution	P1	-0.2632	-0.2684	-0.2694	-0.2731
Panel D: the Taiwanese stock market					
▸ stock returns					
P1: 2000.07~2018.06	276 / 4,288	0.4044	-0.0205	0.1187	-0.0223
P2: 2000.07~2007.06	285 / 1,674	0.2189	0.1554	0.1614	0.2281

[5] Results on the Japanese stock market are presented only for the whole period (2000.07 ~ 2018.06). In order to guarantee a stable set of eigenvalues, PCA requires a condition that the length of time series (T) should be greater than the number of stocks (N), that is, $T > N$. In the sub-periods of the Japanese stock market, the number of stocks is greater than the length of time series. Therefore, this study does not present results on the sub-periods of Japan.

P3: 2007.07~2011.06	668 / 962	0.2818	0.3762	0.0547	0.2610
P4: 2011.07~2018.06	729 / 1,652	0.2789	0.3089	0.1026	0.3474
▸ random data using parameters estimated from stock returns					
Normal distribution	P1	0.0025	0.0010	0.0518	0.0490
Student' t distribution	P1	-0.2257	-0.3636	-0.2327	-0.4094
Stable distribution	P1	-0.2268	-0.2814	-0.2329	-0.2928

Notes: The table shows the results on the relationship between the magnitude of eigenvalues and the tail fatness (TF) in the distribution using an eigenvalue time series. The relationship is quantified by the correlation coefficient. The tail fatness is quantified by the TF measures on the positive and negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the distribution. The Eigenvalue and its time-series data are generated from principal components analysis (PCA) using stock returns traded in each stock market of China (Panel A), Japan (Panel B), Korea (Panel C), and Taiwan (Panel D) for each sub-period (P1, P2, P3, and P4). Three types of random data from three theoretical distributions, the normal distribution, Student's t distribution, and the stable distribution, are generated using key parameters estimated from each stock traded in the stock markets of four countries over the whole period (P1). The results are separately presented for negative and positive tails using all eigenvalues with and without the largest eigenvalue.

Table 1 reports that the magnitude of the eigenvalue has a positive relationship with the TF measures of the positive and negative tails in the eigenvalue distribution, regardless of the stock market. That is, the higher the eigenvalue, the fatter the tail parts of the eigenvalue distribution tend to be. Compared to the results using all eigenvalues, including the largest eigenvalue, the results using all eigenvalues, excluding the largest eigenvalue, have a higher positive relationship, regardless of the positive and negative tails. This represents the greater influence of the largest eigenvalue on the results, compared to other eigenvalues. The scatter plot of **Figure 3** visually verifies the relationship between the eigenvalues and the TF measures of the eigenvalue distribution for the Japanese stock market. In the figures, the X-axis indicates the eigenvalues and the Y-axis denotes the TF measures. **Figure 3** shows the positive relationship between the eigenvalues' magnitude and the TF measures of the eigenvalue distribution. In addition, high-value eigenvalues tend to have much fatter tails in the eigenvalue distribution. As a result, the return data included in the tail parts of the return distribution may have properties that are included in the high-value eigenvalue. These results logically suggest that the tail parts of the return distribution have the properties of common factors contained in high-value eigenvalues.

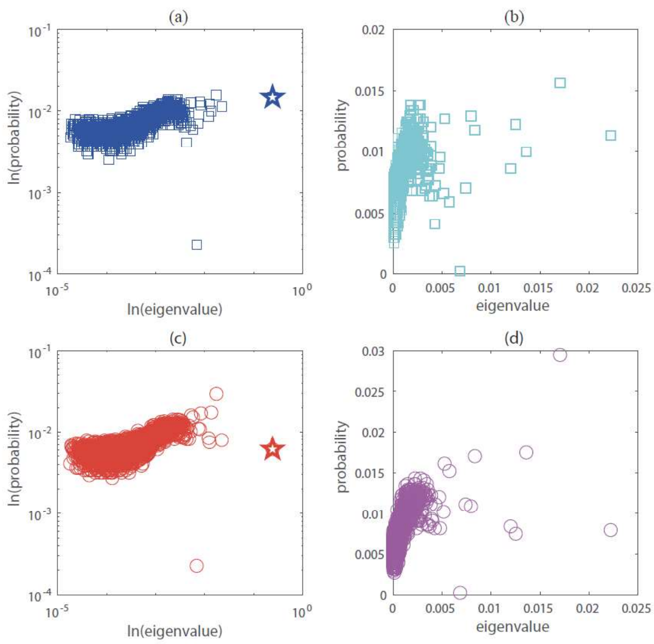

Fig. 3. Relationship between eigenvalues and tail fatness (TF) in the eigenvalue distribution (Japan). The figures show the results for the relationship between the magnitude of eigenvalues and the tail fatness (TF) in the distribution using an eigenvalue time series generated by principle components analysis (PCA) using 2,051 stocks traded in the Japanese stock market over the period from July 2000 to June 2018. The tail fatness is quantified by the TF measures on the positive and

negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the standard normal distribution. In the scatter-plot, the X-axis indicates the magnitude of the eigenvalues, and the Y-axis denotes the TF measures on the positive and negative tails of the eigenvalue distribution. Figures 3(a) and 3(b) show the negative tail, and Figures 3(c) and 3(d) show the positive tail. In addition, to control for the influence of the largest eigenvalue (☆) on the results, Figures 3(a) and 3(c) show the results for the case in which the largest eigenvalue is included, using a double-log plot; Figures 3(b) and 3(d) show the results for the case in which the largest eigenvalue is excluded.

For a deeper understanding, this study investigates whether eigenvalues extracted from random data that do not contain any economic factors may show the same results. If the same positive relationship is verified, it will be difficult to reliably interpret the economic implications based on results observed using eigenvalues extracted from stock returns because the results using eigenvalues from statistical methods are identical, regardless of the data types. On the contrary, if the relationship is different, this would point to a role played by economic factors. To this end, random data generated from each of three theoretical distributions (normal distribution, Student t-distribution, stable alpha distribution) introduced in Chapter 2 is utilized. We generate random returns using key parameters of the theoretical distribution that are directly estimated from each stock in the stock markets of the four countries. Hence, the structure of random data is the same as the actual stock returns in the number of stocks and length of the time series. According to the results presented in **Table 1**, the eigenvalues extracted from the random data mostly exhibit negative correlations with very small values. This demonstrates that random data without economic factors cannot generate a positive relationship between the eigenvalues' magnitude and the TF measures of the eigenvalue distribution. This result is also visually verified in **Figure 4,** which shows a scatter plot for the Japanese stock market on the relationship between eigenvalues in the X-axis and the TF measures of eigenvalue distribution in the Y-axis. Figure 4(a) shows the negative tail, and Figure 4(b) the positive tail.

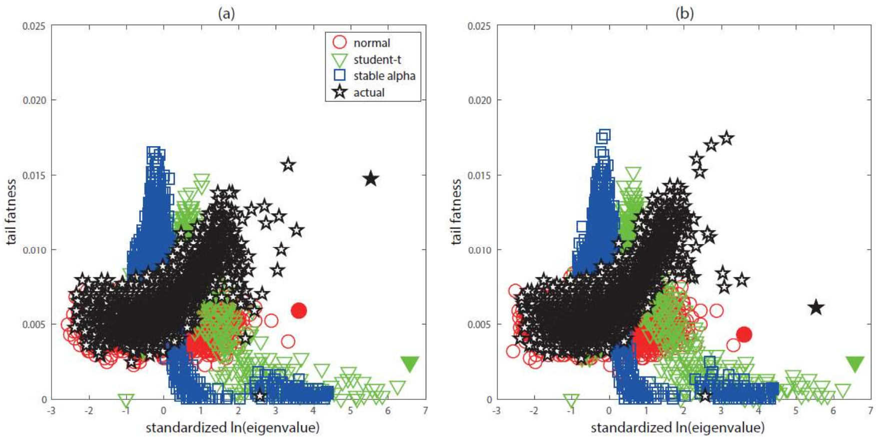

Fig. 4. Relationship between eigenvalues and tail fatness (TF) in the eigenvalue distribution (Random data). The figures show the results for the relationship between the magnitude of eigenvalues and the tail fatness (TF) in the distribution using an eigenvalue time series generated by principle components analysis (PCA) using each of the stock returns and theoretical random data. Stock data includes 2,051 stocks traded in the Japanese stock market over the period from July 2000 to June 2018. Three types of random data from three theoretical distributions, the normal distribution, Student's t distribution, and the stable distribution, are generated using key parameters that are directly estimated from each stock traded in the Japanese stock market over the same period. The tail fatness is quantified by the TF measures on the positive and negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the standard normal distribution. The results are divided into the negative tail part (a) and positive tail part (b). The X-axis indicates eigenvalues that are standardized by subtracting the average value and dividing the standard deviation, and the Y-axis indicates the TF measures on the positive and negative tails of the eigenvalue distribution. The markers indicate stock returns (★), random data from the normal distribution (○), Student's t distribution (▽) and the stable distribution (□), and the filled markers indicate the largest eigenvalue.

Figure 4 clearly reveals that the results using actual stock returns are different from those using random data, regardless of the theoretical distributions. Despite the eigenvalues from random data having high values like eigenvalues from stock returns, the tail parts of the eigenvalue distribution using random data do not have fat tails, unlike the eigenvalue distribution using stock returns. Therefore, in the relationship between the eigenvalues' magnitude and the TF measures of the eigenvalue distribution, the positive relationship observed from stock returns can be interpreted as exhibiting economic implications related to common factors, which cannot be identified using random data from theoretical distributions.

3.4. Effect of common factors on the fat tails in the return distribution

This section further investigates the changes of the TF measures on the tail parts of the return distribution using two types of return data according to whether the properties of common factors are included. Two types of return data are return data with common factors' properties and return data without common factors' properties. The same testing procedure for portfolio diversification is applied to each two-type return data, from which evidence supporting the previous results is expected. On the one hand, when using return data with the properties of common factors, results for the changes of the TF measures on the tail parts of the return distribution are consistent with the results observed in **Figure 2**. In particular, the TF measures on the negative tail show a higher value compared to the positive tail. On the other hand, when using return data without the properties of common factors, the results for the changes of the TF measures on the tail parts of the return distribution differ from those in **Figure 2**. In particular, the TF measures on the negative tail are more sensitive compared to the positive tail.

This study devises a method combining RMT and SVD methods in order to generate two-type return data according to common factors. RMT and SVD can generate data with specific properties using eigenvalues and eigenvectors, like PCA.[6] In the devised method, RMT is used to determine the number of common factors included in the return data, and SVD generates two-type return data with and without the properties of common factors using the number of common factors determined by RMT. RMT and SVD are briefly introduced as follows. In previous studies (e.g., Plerou et al., 2002), based on Sengupta and Mitra (1999), the probability density function ($P_{RM}(\lambda)$) of eigenvalues in random correlation matrices is presented in Eq. (9) under conditions of increasing the number of stocks ($N \to \infty$) and prolonging the length of the time series ($T \to \infty$) keeping the ratio Q=T/N constant, as follows:

$$P_{RM}(\lambda) = \frac{Q}{2\pi}\frac{\sqrt{(\lambda_{+}^{RM}-\lambda)(\lambda-\lambda_{-}^{RM})}}{\lambda}, \; (\lambda_{\pm}^{RM} = 1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}}, \text{ in here } Q \equiv \frac{T}{N}) \quad (9)$$

In the equation, eigenvalues to determine the random correlation matrix lie within the range of $\lambda_{+}^{RM} \geq \lambda \geq \lambda_{-}^{RM}$. Eigenvalues larger than the maximum value (λ_{+}^{RM}) in the random correlation matrix are assumed to have economic implications related to common factors that can explain the changes in expected stock returns. The number of eigenvalues within the range between the largest eigenvalue (λ^{MAX}) and λ_{+}^{RM} can be related to the L-number of common factors that are included in stock returns. Next, SVD generates two-type return data with and without the properties of common factors using L-number of common factors determined by RMT. In previous studies (e.g., Kleibergen and Paap, 2006), a $T \times N$ matrix of stock returns ($\boldsymbol{A}$-matrix) is divided into three-type matrix through SVD under the condition that the length of the time series (T) is greater than the number of stocks (N), as follows:

[6] Each method has a unique comparative advantage. RMT can mathematically define the eigenvalues that deviate from the range of eigenvalues having random properties in the distribution of eigenvalues estimated from the return data. In other words, the eigenvalues that have a higher value than an eigenvalue having the maximum value among the random eigenvalues are well known to have economic implications as common factors. SVD can generate the new return data that only have the properties of the eigenvalues included within the pre-specific range from the original return data. That is, this method can generate the return data having only the properties of the eigenvalues identified as common factors through RMT, and also the return data removing only the properties of the eigenvalues having the properties of the common factors. The specific details of each method are not presented here due to space considerations.

$$A = U \cdot S \cdot V' \tag{10}$$

In the equation, $\boldsymbol{U}$-matrix is a $T \times N$ orthogonal matrix ($U'U \equiv I_T$), and $\boldsymbol{S}$-matrix is the $N \times N$ diagonal matrix that has zero value for all elements of the matrix except elements located in diagonal part, and $\boldsymbol{V}$-matrix is $N \times N$ orthogonal matrix ($V'V \equiv I_N$), where, U' and V' indicate the transpose matrixes. In addition, as another expression, the linear combination form for stocks($j = 1,2,\ldots,N$) and the time series ($t = 1,2,\ldots,T$) are defined as:

$$a_{j,t} = \sum_{k=1}^{K} u_{t,k} s_k v_{j,k} \tag{11}$$

Eq. (11) represents the case using K-number of common factors ($k = 1,2,\ldots,K$) that are extracted using N-number of stocks ($j = 1,2,\ldots,N$) with T-length of the time series ($t = 1,2,\ldots,T$). Based on Eq. (9) by RMT, L-number of common factors is applied into Eq. (11) by SVD, and two-type return data are generated; that is, return data with the properties of common factor ($C_{j,t}$,$T \times N$) from Eq. (12) and return data without the properties of common factors ($F_{j,t}$,$T \times N$) from Eq. (13), as follows:

$$C_{j,t} = \sum_{k=1}^{L} u_{t,k} s_k v_{j,k} \tag{12}$$

$$F_{j,t} = \sum_{k=L+1}^{K} u_{t,k} s_k v_{j,k} \tag{13}$$

These results investigating the changes of the TF measures on the positive and negative tails in the distributions of two-type return data are separately presented in **Figures 5** and **6** for using return data with and without the properties of common factors, respectively. The testing procedure is the same as that of **Figure 2** for stock markets of the four countries over the same period. For the number of stocks within the range of 2 to 50 for constructing a portfolio, we calculate the standard deviation and the TF measures in the return distribution for the portfolio constructed using randomly selected stocks in each number of stocks. This is repeated 100 times in each number of stocks. In the figures, the X-axis indicates the number of stocks in a portfolio, and the Y-axis denotes the average values of 100 TF measures on the positive and negative tails in the return distribution. The figures are divided according to the stock markets of the four countries.

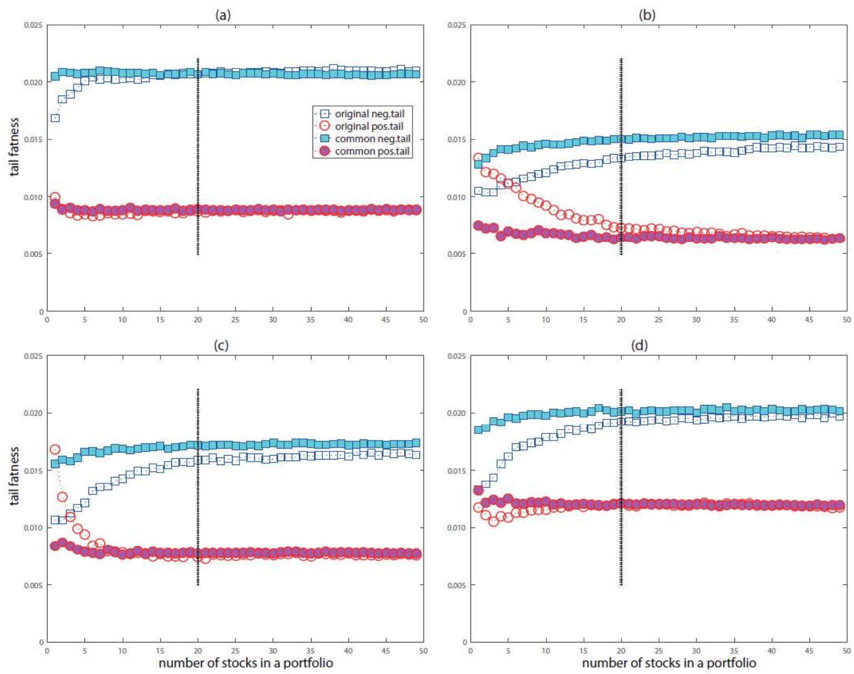

Fig. 5. Portfolio diversification by tail fatness (TF) from return data with common factors' properties. The figures show the results for the portfolio diversification effect according to increasing the number of stocks in a portfolio from the perspective of the tail fatness (TF) measured from the distribution using return data with the properties of common factors. The tail fatness is quantified by the TF measures on the positive and negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the distribution. The results are divided according to the two types of return data, i.e., original return data (□, ○ in Figure 2), and controlled return data (■, ●) having the properties of the common factors. The negative (□, ■) and positive (○, ●) tails in the portfolio return distribution are separately presented. The figures present the results of the four countries, China (a), Japan (b), Korea (c), and Taiwan (d), using all stocks over the whole period. The X-axis denotes the number of stocks in the portfolio from 2 to 50, and the Y-axis indicates the average values of the TF measures calculated from 100 simulations for each portfolio. The vertical line is comprised of 20 stocks based on Elton and Gruber (1975).

Figure 5 shows the results using return data with the properties of common factors. According to these results, the TF measures on the positive and negative tails are unaffected by the number of stocks in a portfolio. This is consistent with the results in **Figure 2**. Further, the TF measures on the negative tail have higher values than the TF measures on the positive tail. Notably, the plateau trend of the TF measures is evident in few stocks in the portfolio because random data with the properties of common factors are utilized. These results suggest that the tail parts of the return distribution are highly relevant to the properties of common factors. Among stock markets in the four countries, the Japanese and Chinese stock markets show the lowest and highest TF measures, respectively. Based on the maturity of the stock market, the Chinese stock market, which is classified as an emerging market, appears to have more frequent and larger changes in common factors compared to the Japanese stock market, which is classified as a developed market.

Next, **Figure 6** shows the results using return data without the properties of common factors. According to these results, the TF measures on the negative tail have declining trends as the number of stocks in a portfolio increases. This differs from the results observed in **Figure 2**. Moreover, the TF measures on the negative tail have mostly lower values than those on the positive tail. The changes of TF on both positive and negative tails' measures are similar to the declining trend of unsystematic risk through portfolio diversification because return data without the properties of common factors are utilized. These results suggest that the tail parts of the return distribution are highly related to the properties of common factors.

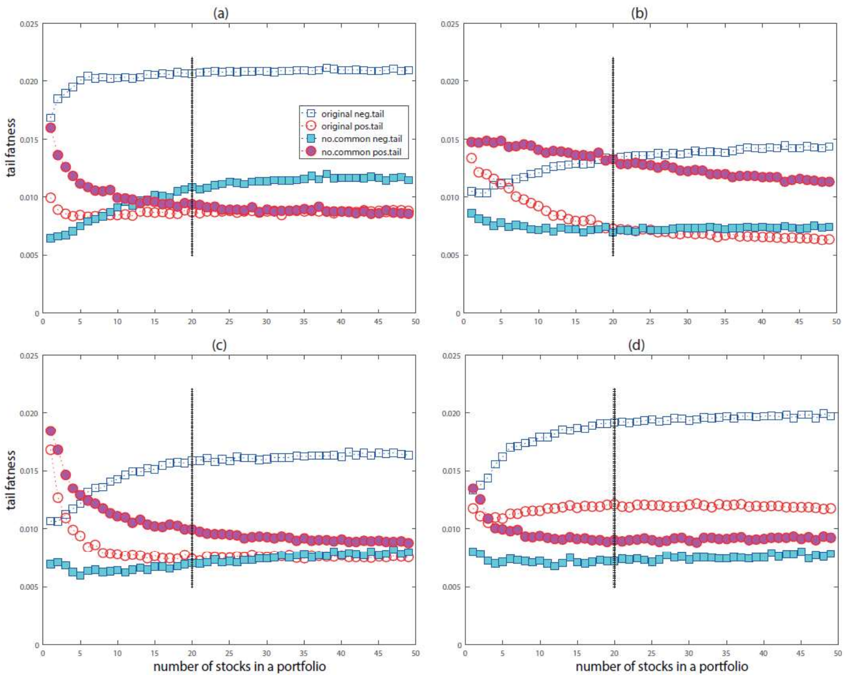

Fig. 6. Portfolio diversification by tail fatness (TF) from return data without common factors' properties. The figures show the results for the portfolio diversification effect according to increasing the number of stocks in a portfolio from the perspective of the tail fatness (TF) measured from the distribution using return data without the properties of common factors. The tail fatness is quantified by the TF measures on the positive and negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the distribution. The results are divided using the two types of return data, i.e., original return data (□, ○ in Figure 2) and controlled return data (■, ●) without the properties of the common factors. The negative (□, ■) and positive (○, ●) tails in the portfolio return distribution are separately presented. The figures present the results of the four countries, China (a), Japan (b), Korea (c), and Taiwan (d), using all stocks over the whole period. The X-axis denotes the number of stocks in the portfolio from 2 to 50, and the Y-axis indicates the average values of the TF measures calculated from 100 simulations for each portfolio. The vertical line is comprised of 20 stocks based on Elton and Gruber (1975).

Our results indicate that the tail parts of the return distribution through portfolio diversification are highly relevant to the properties of common factors and, moreover, the properties of common factors are more closely related to the negative tail than to the positive tail. On the one hand, from the perspective of risk reduction through portfolio diversification, the declining trend of the TF measure for the positive tail of the return distribution contributes to the reduction of portfolio risk, but the rising trend of the TF measures on the negative tail is unlikely to contribute to the reduction of portfolio risk. On the other hand, from the viewpoint of the trade-off relationship between loss avoidance and profit sacrifice, the declining trend of the TF measures on the positive tail indicates decreasing frequencies of high profits, but the rising trend of the TF measures on the negative tail indicates non-decreasing frequencies of high losses. Consequently, these results suggest that the asymmetry between the TF measures on negative and positive tails is linked to the asymmetry of the trade-off relationship between loss avoidance and profit sacrifice. Moreover, investors have to sacrifice a higher profit in order to gain a reduction in the risk of high losses through portfolio diversification. This is empirical evidence supporting the practical limitation of portfolio diversification and explaining why investors with diversified portfolios suffer high losses from market crashes. To further explore these findings, we directly investigate the quantitative degree of the asymmetric relationship between loss avoidance and profit sacrifice using the devised method in the next section.

3.5. Limitation of portfolio diversification: some empirical evidence

This section presents empirical evidence supporting the asymmetric relationship between loss avoidance and profit sacrifice that is caused by the practical limitation of portfolio diversification. In the portfolio diversification effect, loss avoidance indicates the effect of reducing the likelihood of high losses, and profit sacrifice represents the cost of giving up the possibility of earning high profits. The comparative criterion on loss and profit from a portfolio is the performance of the stocks that comprise the portfolio. In other words, high losses from stocks indicate the effect of loss avoidance that is expected from portfolio diversification, and high profits from stocks represent the profit that is sacrificed from portfolio diversification. This study devises the following measure of the P/L ratio to quantify the trade-off relationship between loss avoidance and profit sacrifice, as follows:

$$P/L = \frac{PR}{LR}\ \text{, where, } \mathrm{PR} = \frac{A_P - P_p}{A_P} \text{ and } \mathrm{LR} = \frac{A_L - P_L}{A_L} \tag{14}$$

In the equation, P/L is calculated by dividing the PR measure for profit sacrifice by the LR measure for loss avoidance. Compared to investing in stocks that comprise a portfolio, the LR measure indicates the ratio of high losses reduced from investment in the portfolio, and the PR measure represents the ratio of high profits sacrificed from investment in the portfolio. Accordingly, in the LR measures, A_L indicates the highest loss among all losses of stocks that comprise the portfolio, and P_L denotes the highest loss of the portfolio. In the PR measure, A_P indicates the highest profit among all profits of stocks that comprise the portfolio, and P_P denotes the highest profit of the portfolio. We employ two-type basis values, that is, first, a maximum value for the high profit and a minimum value for the high loss and, second, a value in the 95% percentile for the high profit and a value in the 5% percentile for the high loss. These percentile-based values are used to control the influence of extreme outliers on the results and to improve the reliability of the results. The criteria for evaluation using the devised P/L ratio are as follows. First, the case of $P/L > 1$ indicates the asymmetric relation between loss avoidance and profit sacrifice, i.e., the case of portfolio diversification due to a higher degree of profit sacrifice. Second, the case of $P/L < 1$ indicates the asymmetric relation between loss avoidance and profit sacrifice, i.e., the case of portfolio diversification due to a higher degree of loss avoidance. Third, the case of $P/L = 1$ indicates the symmetric relation between loss avoidance and profit sacrifice. For the criteria of evaluation, investors using portfolio diversification prefer $P/L < 1$ with a higher loss avoidance to $P/L > 1$ with a higher profit sacrifice.

Table 2.
Comparison on profit/loss (P/L) ratio in the portfolio and stocks

TF: portfolio, P		TF: stock, S		TF difference S-P		P/L ratio	
negative	positive	negative	positive	negative	positive	max/min	95%/5%
Panel A: the Chinese stock market							
0.020193[a]	0.007476[a]	0.014537[a]	0.015431[a]	-0.005655[a]	0.007956[a]	2.4590[a]	1.1544[a]
(12.74)	(12.32)	(14.26)	(21.32)	(-5.03)	(12.09)	(10.82)	(12.47)
Panel B: the Japanese stock market							
0.013930[a]	0.007333[a]	0.009928[a]	0.015004[a]	-0.004002[a]	0.007671[a]	2.2343[a]	0.8389[a]
(15.04)	(12.47)	(52.68)	(36.36)	(-4.53)	(12.62)	(12.62)	(11.37)
Panel C: the Korean stock market							
0.016399[a]	0.007635[a]	0.009367[a]	0.017672[a]	-0.007032[a]	0.010036[a]	3.5696[a]	1.4064[a]
(29.26)	(15.14)	(26.16)	(47.69)	(-17.90)	(4.43)	(18.17)	(15.08)
Panel D: the Taiwanese stock market							
0.017100[a]	0.009129[a]	0.008790[a]	0.012544[a]	-0.008311[a]	0.003416[b]	4.4288[a]	1.7144[a]
(14.72)	(15.17)	(14.84)	(11.78)	(-9.65)	(2.24)	(9.71)	(17.41)

Note: The table presents the results for tail fatness (TF) and the P/L ratio for China (Panel A), Japan (Panel B), Korea (Panel C), and Taiwan (Panel D), using all stocks over the whole period. TF is quantified by the TF measures on the positive and negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the distribution. The P/L ratio is calculated by dividing the PR measure for profit sacrifice by the LR measure for loss avoidance. Compared to investing in stocks that comprise a portfolio, the LR measure indicates the ratio of high losses reduced by investment in the portfolio, and the PR measure represents the ratio of high profits sacrificed from investment in the portfolio. The results are separately reported for the TF measures on the positive and negative tails in the return distribution of portfolio (P) and stock (S) returns, for the difference S-P between the TF measures of stock and portfolio distribution, and for the P/L ratio based on maximum and minimum (max/min) values and values located in the 95% and 5% percentiles (95%/5%). Values in parentheses are t-statistics. Significant results are presented as 'a,' 'b,' and 'c,' representing a significance level of 1%, 5%, and 10%, respectively.

Table 2 and **Figure 7** report the results. The rolling-sample method over the whole period is employed because the performance of the trade-off relationship between loss avoidance and profit sacrifice is affected by market dynamics. The main design is as follows. The number of stocks in a portfolio is fixed at 50, i.e., a well-diversified portfolio. In the whole period, the duration to construct the portfolio is 60 months and the period shift is 12 months. According to the rolling-sample method, the Chinese, Japanese, Korean, and Taiwanese stock markets have 13, 19, 18, and 9 sub-periods, respectively. The same testing procedure as for **Figure 2** is applied for each sub-period. That is, in each sub-period, the portfolio is constructed using randomly selected stocks, and this is repeated 100 times. **Table 2** reports the average values of results observed in all sub-periods, and **Figure 7** shows the distributions of results observed in all sub-periods using the box-plot method. According to the results, we attribute the asymmetric relationship between loss avoidance and profit sacrifice through portfolio diversification to a higher degree of profit sacrifice compared to loss avoidance. Specifically, the P/L ratio ranges from 2.23 to 4.42 based on the maximum and minimum values, and the P/L ratio ranges from 0.84 to 1.71 based on the 5% and 95% percentiles, respectively. These results suggest that investors have to sacrifice a greater opportunity to obtain high profits in order to reduce the likelihood of suffering high losses through portfolio diversification. Among the four countries, the Japanese stock market has the lowest P/L ratio, and the Taiwanese stock market featured the highest. Therefore, the application of the method devised herein empirically proves the quantitative degree of the asymmetric relationship between loss avoidance and profit sacrifice.

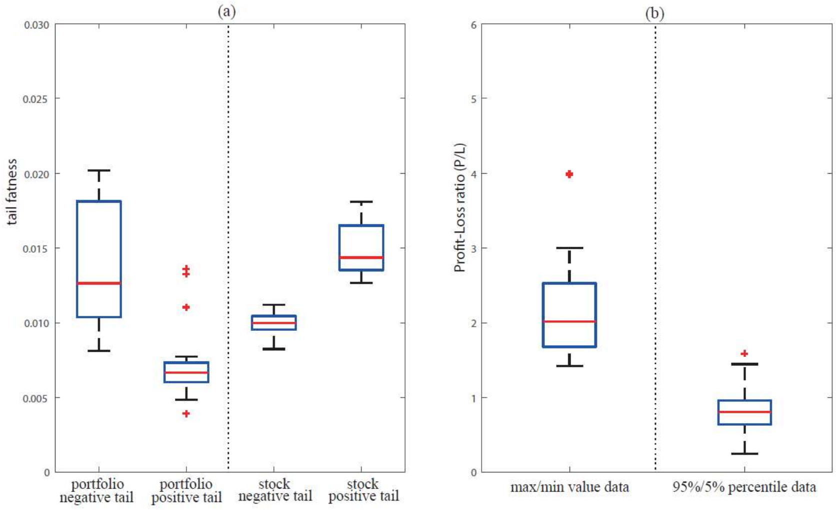

Fig. 7. Comparison of the distribution of the profit/loss (P/L) ratio in the portfolio and stocks. The figures show the results for the distribution of tail fatness (TF) in stock and portfolio return distributions and the P/L ratio through the box-plot method using stock data traded in the Japanese stock market from July 2000 to June 2018. The tail fatness is quantified by the TF measures on the positive and negative tails of the return distribution. Tails are defined as the area beyond the 99% central part of the distribution. The P/L ratio is calculated by dividing the PR measure for profit sacrifice by the LR measure for loss avoidance. Compared to investment in stocks that comprise a portfolio, the LR measure indicates the ratio of high losses reduced from investment in the portfolio, and the PR measure represents the ratio of high profits sacrificed from investment in the portfolio. The results are separately presented for the TF measure (a) and the P/L ratio (b). In Figure (a), the X-axis denotes the negative and positive tails for each portfolio and stock, and the Y-axis indicates the distribution of the TF measures for all sub-periods. In Figure (b), the X-axis denotes two-type comparative basis value of maximum and minimum (max/min) and 95% and 5% percentiles (95%/5%), and the Y-axis indicates the distribution of the P/L ratios for all sub-periods.

3.6. Discussion

According to the results of this study, tails in the return distribution verified by testing portfolio diversification have the property of systematic risk being sensitive to change in common factors. Interestingly, changes in the fatness of the positive and negative tails of the return distribution show contrasting behavior until a certain level; that is, increasing the number of stocks in a portfolio results in the positive tail showing a pattern of declining fatness and the negative tail shows the pattern of raising fatness. From the viewpoint of portfolio investment, this means that portfolio diversification does not give a balanced trade-off relationship between loss avoidance and profit sacrifice, contrary to the expectation of investors. In other words, the cost of the profit sacrifice is much higher than the benefit of loss avoidance. This is evidence supporting presenting the practical problem of portfolio diversification and provides a persuasive reason that investors who hold a portfolio according to modern portfolio theory still experience large losses during market crashes. In addition, the portfolio diversification using the equal-weighting method in this study based on Evans and Archer (1968) utilizes the naive 1/N portfolio investment that investors can easily adopt. In previous studies investigating the role of fat-tails in the return distribution on the portfolio diversification, Hwang, Xu, and In (2018) who focus on the role of the negative tail in the distribution using measures of value at risk (VaR), expected shortfall (ES) and tail risk (TR, Kelly and Jiang, 2014) present evidence similar to our findings. They suggest evidence supporting that the tail risk of naive 1/N portfolio increases as the number of stocks in a portfolio increase. This is consistent with our results that the fatness of the negative tail increases according to portfolio diversification. Consequently, it is evident that further works concerning the impact of fat tails in the distribution of portfolio performance need to explore methods to improve the practical usefulness of portfolio diversification.

In previous studies, the well-diversified mean-variance portfolio and utilization of the safe-haven assets are considered as a method to improve the practice of portfolio diversification. The mean-variance portfolio suggested by Markowitz (1952) has become a cornerstone in the modern portfolio theory. Hwang et al. (2018) show that portfolios using investment weights based on the mean-variance portfolio show a pattern of increasing risk in the negative tail of the distribution, similar to the naive 1/N portfolio. The magnitude of tail risk is lower than the naive 1/N portfolio. Of course, as DeMiguel, Garlappi, and Uppal (2009) argued, the naive 1/N portfolio outperforms the mean-variance portfolio from the perspective of investment performance. Eom and Park (2018) show that the mean-variance portfolio usually fails to construct a well-diversified portfolio, contrary to expectations. Using a method controlling for properties of correlation matrix among stocks in a portfolio, they prove empirically that a more diversified portfolio has greater performance and less risk, and moreover, has lower risk during market crashes. Next is safe-haven assets combining with the traditional portfolio constructed using financial assets such as stocks, bonds, and cash equivalents. Safe-haven assets usually have the characteristic of a non-correlation or negative correlation with the traditional financial portfolio. Many studies employ the safe-haven assets of gold, crude oil, real estate, commodities, foreign exchange currencies, and cryptocurrency (like bitcoin). Dimitriou, Kenourgios, and Simos (2020) suggest the effectiveness of gold, wine, commodities, crude oil, and shipping index for reducing the possibility of extreme losses from the traditional financial portfolio when occurring economic shocks from the 2007-2009 US credit crisis and the 2010-2012 European sovereign debt crisis. Meanwhile, Ji, Zhang, and Zhao (2020) present evidence supporting the utility of safe-haven assets of gold and soybean commodity futures, which effectively limit the magnitude of risk from the traditional financial portfolio in a global economic recession caused by non-economic shocks of the 2020 COVID-19 pandemic. Accordingly, safe-haven assets may be utilized as a tool overcoming the negative tail risk of portfolio investment when markets crash. However, the effects of safe-haven assets on reducing extreme losses in a traditional portfolio of financial assets tend to be dependent on the sources of market crashes.

As it is evident that the fatness of tails in the return distribution has a significant influence on portfolio investment under the risk-return domain, researchers will try to continuously improve the practical applicability of portfolio diversification. In addition to well-diversified portfolios and combinations with safe-haven assets, new methods to enhance the usefulness of portfolio investment are expected to be explored. Also, this study expects that future research related to portfolio diversification solve the asymmetry between loss avoidance in the negative tail and profit sacrifice in the positive tail. Finally, as noted in Goetzmann and Kumar (2008), individual investors find it difficult to construct a well-diversified portfolio like modern portfolio theory because of the lack of investment resources and information. Hence, from the perspective of whether to construct a well-diversified portfolio, the differentiated portfolio proposals for fund managers and individual investors are expected to be considered.

4. Conclusions

This study empirically investigates the risk property included in fat tails and the changes in fatness in the tail parts of the return distribution through portfolio diversification, using daily stock trade data in representative stock markets in the four Northeast Asian countries of China, Japan, Korea, and Taiwan. To accomplish the research goals, the following empirical design is utilized. First, the fatness of the positive and negative tails in the return distribution was defined as a measure of tail risk. Second, the changes in the fatness of the positive and negative tails are examined through portfolio diversification, along with the effect of risk reduction. Third, the risk property included in the positive and negative tails in the return distribution is identified using the devised method combining statistical methods. Finally, the trade-off relationship between loss avoidance and profit sacrifice is investigated using the devised method by comparing the performance of stocks and portfolios. The main results are summarized as follows. The tail parts of the return distribution through portfolio diversification are highly relevant to the properties of common factors, and these common factors have greater impacts on the negative tail than on the positive tail. Through portfolio diversification, the declining fatness of the positive tail of the return distribution helps to reduce the risk, but the rising fatness of the negative tail is unlikely to contribute to this risk reduction. The asymmetry between the fatness of the positive and negative tails in the return distribution represents the asymmetry of the trade-off relationship between loss avoidance and profit sacrifice that is expected by portfolio diversification. Investors aim to effectively reduce the likelihood of high losses through portfolio diversification, but, crucially, their potential for higher profits is sacrificed in order to ensure a reduction in the risk of high losses. Therefore, our findings provide empirical confirmation for the practical limitation of portfolio diversification and explain why investors with diversified portfolios suffer high losses from market crashes. In addition, this evidence denies the investment proverb that portfolio diversification is a free lunch for investors. We expect future research to examine how to enhance the practical applicability of portfolio diversification.

References

Barberis, N., A. Mukherjee, and B. Wang, (2016), "Prospect theory and stock returns: An empirical test," *Review of Financial Studies*, 29(11), 3068-3107.

Blattberg, R. and N. Gonedes, (1974), "A comparison of the stable and student distributions as statistical models for stock prices," *Journal of Business*, 47, 244-280.

Brown, S. T., (1989), "The number of factors in security returns," *Journal of Finance*, 44(5), 1247-1262.

Cont, R., (2001), "Empirical properties of asset returns: stylized facts and statistical issues," *Quantitative Finance*, 1, 223-236.

DeMiguel, V., L. Garlappi, and R. Uppal, (2009). "Optimal versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?", *Review of Financial Studies* 22, 1915–1953.

Dimitriou, D., D. Kenourgios, and T. Simos, (2020), "Are there any other safe haven assets? Evidence for "exotic" and alternative assets", *International Review of Economics and Finance*, 69, 614-628.

Elton, E. J, and M. J. Gruber (1977), "Risk reduction and portfolio size: an analytical solution" *Journal of Business*, 50(4), 415-437.

Eom, C., (2017), "Two-faced property of a market factor in asset pricing and diversification effect," *Physica A: Statistical Mechanics and its Applications*, 471, 190-199.

Eom, C., W. Jung, T. Kaizoji and S. Kim, (2009), "Effect of changing data size on eigenvalues in the Korean and Japanese stock markets," *Physica A: Statistical Mechanics and its Applications*, 388(22), 4780-4786.

Eom, C., T. Kaizoji, and E. Scalas, (2019), "Fat tails in financial return distributions revisited: Evidence from the Korean stock market," *Physica A: Statistical Mechanics and its Applications*, 526, 121055.

Eom, C., and J. W. Park, (2017), "Effects of common factors on stock correlation networks and portfolio diversification," *International Review of Financial Analysis*, 49, 1-11.

Eom, C., and J. W. Park, (2018), "A new method for better portfolio investment: A case of the Korean stock market," *Pacific-Basin Finance Journal* 49, 213-231.

Eom, C., and J. W. Park (2020), "Effects of the fat-tail distribution of the relationship between prospect theory value and expected return," *North American Journal of Economics and Finance*, 52, 101052.

Eom, C., J. W. Park, Y. H. Kim, and T. Kaizoji, (2015), "Effects of the market factor on portfolio diversification:

The case of market crashes," *Investment Analysts Journal*, 44(1), 71-83.

Evans, J. L. and S. H. Archer, (1968), "Diversification and the reduction of dispersion: An empirical analysis," *Journal of Finance*, 23(5), 761-767.

Fama, E., (1965), "The behavior of stock market prices," *Journal of Business*, 38, 34-105.

Fama, E. F., and K. R. French (1992), "The cross-section of expected stock returns" *Journal of Finance* 47, 427-465.

Getzmann, W. N., and Kumar, A., (2008), "Equity Portfolio Diversification", *Review of Finance*, 12, 433-463.

Hill, B. M., (1975), "A simple general approach to inference about the tail of a distribution," *Annals of Statistics*, 3(5), 1163-1174.

Hwang, I., S. Xu, and F. In, (2018), "Naive versus optimal diversification: Tail risk and performance", *European Journal of Operational Research*, 265, 372-388.

Jia, Q., D. Zhangb, and Y. Zhaoc, (2020), "Searching for safe-haven assets during the COVID-19 pandemic", *International Review of Financial Analysis*, 71, 101526.

Kelly, B., and H. Jiang, (2014), "Trail risk and asset prices," *Review of Financial Studies*, 27(10), 2841-2871.

King, B., (1966), "Market and industry factors in stock price behavior," *Journal of Business*, 39(1), 139-190.

Kleibergen, F., and R. Paap (2006), "Generalized reduced rank tests using the singular value decomposition" *Journal of Econometrics* 133, 97-126.

Mandelbrot, B. B., (1963), "The variation of certain speculative prices," *Journal of Business*, 36, 394-419.

Mantegna, R. N., and H. E. Stanley, (1995) "Scaling behaviour in the dynamics of and economic index," *Nature*, 376(6), 46-49.

Markowitz, H., (1952). "Portfolio Selection", *Journal of Finance*, 7(1), 77-91.

Markowitz, H., (1991), "Foundations of portfolio theory", *Les Prix Nobel 1990*, 292 (Nobel Foundation, Stockholm).

Nolan, J. P., (2018), "Stable distributions: Models for heavy tailed data," *working paper* available in website, http://fs2.american.edu/jpnolan/www/stable/chap1.pdf.

Plerou, V., P. Gopikrishnan, B. Rosenow, L. A. N. Amaral, T. Guhr, and H. E. Stanley (2002), "Random matrix approach to cross correlations in financial data" *Physical Review E* 65 066126.

Peiro, A., (1994), "The distribution of stock returns: International evidence," *Applied Financial Economics*, 4, 431-439.

Ponta, L., M. Trinh, M. Raberto, E. Scalas, and S. Cincotti, (2017), "Modeling non-stationarities in high-frequency financial time series," *Physica A: Statistical Mechanics and its Applications,* 521, 173-196.

Praetz, P. D., (1972), “The distribution of share price changes,” *Journal of Business*, 45(1), 49-55.

Ross, S., (1976), “The arbitrage theory of capital asset pricing,” *Journal of Economic Theory*, 13(3), 341-360.

Rubinstein, M., (2002), “Markowitz’s “portfolio selection”: A fifty-year retrospective” *Journal of Finance*, 57(3), 1041-1045.

Sengupta, A. M., and P. P. Mitra (1999), "Distributions of singular values for some random matrices" *Physical Review E* 60, 3389.

Statman, M., (1987), “How many stocks make a diversified portfolio?” *Journal of Financial and Quantitative Analysis*, 22(3), 353-363.

Tversky, A., and D. Kahneman, (1992), “Advances in prospect theory: Cumulative representation of uncertainty,” *Journal of Risk and Uncertainty* 5, 297–323.