Mathematics

Overview of the chapter

This chapter aims to provide the reader with a brief introduction to the origins of the various branches of mathematics. While tracing back these origins, an insight will be offered into how the mathematics as a discipline developed throughout many thousands of years and the variety of cultures. Key practices of the discipline of mathematics will be highlighted, followed by a discussion which argues in favour of incorporating these practices into the school mathematics.

But first, I will start with a personal account of what school mathematics was for me as a learner. Such experience informed my current view of mathematics, that of a powerful tool for making sense of the world; of an art with its aesthetic appeal; of a language with its syntax and syntactic rules that facilitate precise, concise and rigorous communication; of a poetry which I read and do for pure enjoyment and of a creative art, with its struggles, frustrations and elations.

A personal account

In school, I learned school mathematics, consisting of facts, results and procedures. I liked 'that' mathematics, as it enabled me to get the right answers to the questions I encountered. My mathematics education was further enhanced by opportunities I had to engage in an act of discovery and conjecture, intuition and inspiration. I have always felt alive doing mathematics, overwhelmed when an idea came to me at the right moment through, for example, noticing a relationship between the elements of a geometrical figure that went beyond just seeing it drawn on a page or by being in awe when a complicated looking algebraic expressions revealed itself to me in its simplest, powerful form. I found looking at mathematics expressions as fascinating as looking at a stereogram, where by focussing on a two-dimensional pattern one can also see a hidden three-dimensional image inside it; looking at an abstract piece of mathematics and being able to make sense of it in a way that went beyond simply decoding the written symbolic representation to the point where it 'spoke' to me. I take pleasure in looking at mathematical writing, with symbols and signs stringed together. They appeal to me aesthetically but also meaningfully, even when I cannot immediately make sense of what the mathematics is about, since I recognise it as a meaningful creation of a human mind, a story that was told and is out there, worth reading and listening to if I so wanted.

The Smith's (2004) inquiry into mathematics teaching post-14 affirmed the importance of studying mathematics as: mathematics for its own sake, mathematics for the knowledge economy, mathematics for science, technology and engineering, mathematics for the workplace and mathematics for the citizen. While each one of these arguments in itself is good enough for justifying its study, collectively they illustrate clearly why mathematics education is vital for our progress and development and it should thus be a compulsory aspect of one's education. To these aspects, I would add *mathematics for one's own sake*, as studying mathematics, when a pleasurable learning experience, is a meaningful human experience. Glimpses back into the history of mathematics help us in gaining an appreciation that mathematics is, historically, a relentless human endeavour with twists and turns, many lines of enquiry leading to knowledge development but also to dead-ends, and with resilience and determination in starting again.

Origins and Evolution of the various branches of the Discipline

The first abstraction in mathematics was very probably that of numbers, needed by prehistoric people not only for counting physical objects but also for counting abstract quantities, like time - days, seasons, years and moon cycles. Early humans used physical objects to represent and communicate their mathematical thinking, while amongst the very earliest evidence of mankind thinking about and recording numbers is from notched bones in Africa dating back to 35,000 to 20,000 years ago.

Humanity's later preoccupations with measuring land and performing calculations related to taxation and commerce signalled the beginning of what was to become one of the major areas of the discipline of mathematics. **Arithmetic** (from the Greek word 'arithmos', meaning 'number') is thus the oldest and the most elementary branch of mathematics, concerned with addition, subtraction, multiplication and division of numbers.

Geometry (from the Greek 'geo', meaning Earth and 'metron' meaning to measure) was introduced in relation to the division of land and measurements. For example, the clay tablets in the British Museum (dating from 1800 to 1600 BCE) provide evidence of the Babylonians' preoccupation with problems involving dividing up an area in to parts with different proportions. The methods for solving the 36 problems on the tablets are described entirely in words, as the Babylonians did not have any form of notation available to them. These problems, which would now be formulated as quadratic and cubic equations, provide evidence of early algebra work (Rooney 2009).

It wasn't until the middle of the 3rd century, when Diophantus (200-300 CE) produced his treatise *Arithmetica*, containing new methods of solving linear and quadratic equations; for his work, Diophantus became known as the 'father of **Algebra**'. The solutions he provided were cumbersome to read as a symbolic system was not in place yet: there was no symbol for the equal sign, if more than one solutions was yielded by whatever calculation, only the first one was considered, while the solution to the equation $4 = 4x + 20$ was called 'absurd' since, although known to Indian mathematicians in the 7th century, the concept of negative numbers was accepted by the Western mathematicians only as late as the 17th century (Burton, 2011, 220).

Just like the Egyptian and Babylonian mathematics, Diophantus was often concerned with solutions of specific, practical problems rather than general solutions of such equations. This did not happen until 500 years later when Muhamman ibn Musa al-Khwarizmi's (c. 780-850 BCE) wrote the treatise called *Al-Kitab al-Jabr wa'l Muquabala* (The Compendious Book on Calculations by Completion and Balancing). The treatise was concerned with algorithms of "balancing" equations, which the term *al-jabr* (algebra) originally referred to. He also developed quick methods for multiplying and dividing numbers, which are known as algorithms (the word being derived from his name).

While the early mathematics was mostly empirical, arrived at by trial and error, with little concern to the accuracy of the results and with no rigour or proofs given for the methods used, Al-Kwarizmi concentrated instead on developing procedures and rules for solving many types of problems in arithmetic. Unlike the Babylonian tablets or Diophantus's *Arithmetica*, his treatise was no longer concerned with a series of specific practical problems to be solved, but with clearly defined classes of problems to be solved for finding the values of their *objects of study* (what we would call today the 'unknowns'). From then on, algebra became an important part of the Arabic mathematics. It is worth noting that the problems and solutions continued to be written in words, as no symbolic notation was in place. Even the numbers were written out in full!

Although the Egyptians had some knowledge of calculating the slope of pyramids from the height and the base, by the 16th century, **Trigonometry**, the branch of mathematics concerned with calculating angles and lengths of sides of triangles, became an area of mathematics independent of geometry, despite relying on it (Rooney 2009).

A profound change occurred in the nature and approach to mathematics with the contributions of Greek scholars, as they made a distinction between the practical arithmetic of everyday life and the higher pursuit of mathematics and logic for solving purely abstract problems. The discovery of Pythagoras' theorem, for which the Greeks had a proof, led to the 'discovery' of irrational numbers when the theorem was applied to isosceles right -angled triangles. The Greeks themselves were quite displeased with their finding, given that they thought a number was 'the ratio of two whole numbers' (conceiving thus rational numbers as abstractions of proportions). Over time, some irrational numbers were accepted by the Greeks, as long as they were constructed with the basic instrument of a geometer (the straightedge and compass), such as square root of 2.

The greatest work of Greek Mathematicians however, remains Euclid's *Elements* (c.300 B.C.). Euclid presented five common notions and five axioms and deduced from them many theorems and results which were proved by using the principle of logical deduction. The effort to axiomatise geometry shows that mathematics never was a perfect or an exact science. Euclidean geometry was thus the first branch of mathematics to be systematically studied and placed on a firm logical foundation and it is still being studied in schools currently as a model of logical thought.

The concepts in Euclid's geometry remained unchallenged until the early 19th century when mathematicians realised that Euclid's geometry could not be used to describe all physical space and so other types of geometry emerged. Non-Euclidean geometry is an extension of Euclidean geometry and it arose from a purely intellectual effort of mathematicians to prove that the fifth postulate (the parallel axiom) could be derived from the other four. Lobachevsky, the founder of this new geometry, labelled his geometry "imaginary," since he could not see any application of it to the real world. The results of his geometry appeared to the majority of mathematicians to be not only ‘imaginary’ but absurd. Nevertheless, years later, the non-Euclidean geometry turned out to be an indispensable tool for Einstein’s revolutionary reinterpretation of the gravitational force, becoming the basis of the general theory of relativity.

In trying to improve the accuracy for the purpose of calculating the area of a circle through using ever-larger numbers of sides for the inscribed and circumscribed polygons, Archimedes (c. 287 B.C. – c. 212 B.C.) encountered two new concepts – that of limit and that of infinity. These new concepts were further applied by mathematicians of the 16th century for calculations of areas under curves. Isaac Newton and Gottfried Leibniz independently developed the foundations of **Calculus** (from the Latin ‘calculus’ meaning pebbles as used on abacus), by bringing together techniques together through the derivatives and integrals. Although considered the greatest tool ever invented for the mathematical formulation and solution of physical problems, during the 17th and 18th centuries Calculus was plagued by inconsistencies; the concepts of limit and infinity carried complex meanings, which were interpreted in inconsistent ways.

Throughout the 19th Century, mathematics in general became ever more complex and abstract. Whereas at first mathematics was created for the investigation of nature, by the 19th Century mathematics continued to develop through the pursuit of problems independent of science, losing grounding in reality. There was concern about the structure of mathematics and so there was a greater emphasis on mathematical rigour through a careful analysis of arguments put forward and formal proofs. One such attempt was that of Nicolas Bourbaki (a collective pseudonym for a group of mainly French 20th century mathematicians) who formulated mathematics on an extremely abstract and formal but self-contained basis, putting the basis of another branch of mathematics, namely **Analysis**.

The next major development in mathematics, one that unites arithmetic, geometry, algebra, and analysis is the notion of continuous function through its use in modelling physical and geometric situations, and its manipulations and analysis using algebra and arithmetic.

People always gambled, and luckily some of the mathematicians of the 17th century took an interest into these games. A gambler's dispute about a popular game of dice in 1654 led to the creation of a mathematical theory of probability, when two famous French mathematicians, Blaise Pascal and Pierre de Fermat were asked to look into an apparent contradiction concerning the dice game. Intrigued by the obvious observations they noted, the mathematicians set out to explain them rigorously and so a new area of mathematics was born, namely **Probability.** A mathematical theory of

Probability was not achieved until a sufficiently precise definition of probability in mathematics was put forward (which took almost three centuries) and, at the same time, being comprehensive enough to be applicable to a wide range of phenomena. The notion of chance events started being accepted by mathematicians, who until then mainly looked for regularity in mathematics. In 1933, in a monograph by a Russian mathematician A. Kolmogorov, a treatment of probability theory on an axiomatic basis was outlined. Further developments in this field and refinement of ideas lead to Probability theory being now part of a more general discipline known as Measure Theory.

Statistics had its origins in the analysis by John Graunt of weekly burial records in London, which he published in 1662. Although as a discipline **Statistics** uses mathematics and probability, there continues to be disputes over whether or not statistics is a sub-field of the discipline of mathematics (see Ben-Zvi & Garfield 2004 for an argument towards recognition that statistics, while a mathematical science, it is not a subfield of mathematics).
In addition to the standard fields already mentioned here: arithmetic, number theory, algebra, geometry, analysis (calculus), mathematical logic and set theory, and the more applied mathematics fields such as probability theory and statistics, an ever growing list of newer branches of mathematics could be produced. The discipline of mathematics now covers an ever increasing array of specialized fields of study, such as group theory, knot theory, topology, differential geometry, fractal geometry, to mention just a few.

This very brief overview of the origins of the various branches of mathematics, usually encountered through one's schooling, does much injustice to many other civilizations and mathematicians who made significant contributions to mathematics. In writing this overview I had no intention in favouring some peoples' ideas in the discourse of mathematics while denying others; any other selection would have inevitably favoured some peoples' ideas over others. Within the limited space of this chapter I wanted to portray a view of mathematics as a discipline in its own right, as a body of knowledge that evolved over time as a human activity, through cumulative contributions from many mathematicians all over the world, giving rise to mathematical developments which are now part of humanity's heritage.

Mathematics as a Discipline

A discipline is an organized, formal field of study which is defined by the types of problems it addresses, the methods it uses to address these problems, and the results it has achieved. The structure of the discipline is about how knowledge is organized and pursued in a particular subject area (Winch 2013).

The current abstract and highly specialised state of mathematics is the result of the evolution of the subject through human endeavour: from empirical mathematics that involved counting, calculations, measurements and the study of properties of shapes and motions of physical objects, to the more abstract ideas and problems which may or may not have roots in real, physical problems and whose solutions push the development of mathematical thinking, creating new areas of mathematical enquiry.

We have seen how the mathematicians became concerned that the structure of mathematics built over centuries did not have a solid foundation. On many such instances throughout history they showed resilience and started again, from the ground, looking for rigour, consistency and effective and unambiguous formalisms. Much of the structure of mathematics was strengthened over the years, despite the cracks that continued to appear. It was the goal of Hilbert's program in 1920s to put all of mathematics on a firm axiomatic basis, but we know now that there are propositions in mathematics which cannot be proved to be either true or false (Godel’s Incompletness Theorem, 1931), telling us that we cannot create an axiomatic system that is free from contradiction.

This however did not deter mathematicians in their quest for developing mathematics as an abstract intellectual pursuit (as a theoretical discipline), as well as a subject with real-life applications (as an applied discipline). Mathematicians’ main concern is with thought, abstractions and thinking about abstract ideas in seeking to solve problems that originate in the real world or problems whose solutions have no material consequences other than the advancement of mathematical knowledge *per se*; history tells us that very often knowledge, in the end, found real-life applications (for example, Mandelbrot’s fractal geometry remained ‘pure mathematics’ for much of his 35-year long career but became ‘applied mathematics’ in many fields such as statistical physics, meteorology, anatomy, taxonomy, neurology, to mention just a few).

Abstractions enable mathematicians to concentrate on some features of things, such as noticing a similarity between two or more objected or events. After abstractions have been made, mathematicians select some symbolic representations for their ideas such as numbers, letters, other marks, diagrams, geometrical constructions, or even words. Mathematical symbolism takes abstraction to another level. The symbolism of mathematics was needed in order to achieve complete precision in meaning and rigour in reasoning. Such symbols are more readily manipulated by mathematicians in reasoning than if they were to use symbols of common language. The symbols can be combined and recombined in various ways according to precisely defined rules. Manipulating the abstractions through deductive reasoning often results in the identification of new relationships, leading to the discovery of new knowledge and/or to testing for the validity of new ideas and/or to the discovery of ‘truth’. Mathematics does not express ‘true propositions’ in any absolute or empirical sense but rather, the truth in mathematics is achieved through logical reasoning within a particular axiomatic system.

The many axiomatic systems: for geometry (e.g., Euclid, Hilbert, Birkoff), for the natural numbers (Peano's axioms), the set theory (e.g. Zermelo-Fraenkel Set Theory) to mention just a few, show how mathematics has become increasingly independent of experience, and hence an abstract intellectual endeavour. However, mathematicians do not generate new knowledge by setting up axioms and using them in order to provide water-tight arguments. History tells us that mathematicians have always engaged imaginatively with problems that become of interest to them for one reason or another. We learn from the vast literature on the historical developments in mathematics that 'doing mathematics' has always been about the mathematicians' creativity, intuition, assumptions, conjecturing, generalising and abstracting, persisting, making links, arguing, justifying and proving, about conversations, debates, different points of view, struggles, dispelling paradoxes by reason, breakthroughs but also being ambiguous, reworking to find errors in arguments and pushing the boundaries.

These are important lessons about mathematics as a discipline that we learn from the past and inform what should be passed on to the new generations, when and how.

The discipline of mathematics reflected in the school subject

Since its origins, mathematics has evolved to become a discipline that is concerned not only with the development of *substantive knowledge* (the key facts, concepts, principles, structures and explanatory frameworks in a discipline) (Shulman & Grossman 1988), but also *syntactic knowledge*[1] (the rules of evidence and warrants of truth within that discipline, the nature of enquiry in the field, and how new knowledge is introduced and accepted into the community).

Inevitably, the school subject will be a 'simplified' form of the discipline and curriculum designers would take decisions as to how best to present a discipline to pupils. In Bernstein' s (2000) terms, school mathematics is a pedagogic discourse, formed by the recontextualisation of other discourses, including that of the discipline of mathematics but also other discourses such as, for example, theories of learning and teaching. Thus, in the case of school mathematics, its purposes and the interests of those participating in it are different from those of mathematicians. While this chapter is not concerned with the construction of a curriculum, it does put forward a view of the school mathematics that is different but related to the discipline of mathematics. They are related in that school mathematics too is concerned with *substantive knowledge* (learning mathematics) and *syntactic knowledge* (disciplinary practices). They are different since pupils should not be expected to learn the same substantive knowledge that concern mathematicians, but rather a breadth and depth of

[1] This latter type of knowledge is equivalent to *procedural knowledge*, term used throughout this book; however, mathematics education researchers usually define procedural knowledge in terms of knowledge type—as sequential or "step-by-step [prescriptions for] how to complete tasks" (Hiebert & Lefevre, 1986, p 6).

substantive mathematical knowledge that are accessible to them according to their experience. They are similar in that school pupils should be able to experience the *syntactic knowledge* that led to development of the discipline of mathematics, at a depth and breadth accessible to them according to their experience.

School mathematics and disciplinary practices

The insights into the chronological development of various branches of the discipline of mathematics throughout the history should reflect the content of mathematics pupils learn about at school. Pupils should become fluent in the various branches of the discipline of mathematics through development of a conceptual understanding and the ability to recall and apply knowledge as and when needed.

Fluency is an important aspect of studying school mathematics and it does involve practising various common problem solving techniques, memorising some formulae and important results and learning how to apply these concepts and skills to solve problems, all of which will give entry points in tackling new problems. However, there is a difference between 'fluent' performance and 'mechanical' performance. ‘Fluent performance is based on understanding of the routine which is being carried out; mechanical performance is performance by rote in which the necessary understanding is not present’ (Cockcroft, 1982: 70). To be mathematically fluent requires sufficient depth of conceptual understanding to be able to recognise when and how to apply existing knowledge. It also requires an understanding of how knowledge is connected, otherwise knowledge remains as fragmented, disparate, and not used unless in circumstances which clearly specify what knowledge is needed.

With a view that mathematics is more than a collection of disparate topics under broad headings such as number, algebra or geometry, Cuoco, Goldenburg & Mark (1996) proposed a ‘habits of minds curriculum’ which aims ‘to close the gap between what the users and makers of mathematics *do* and what they *say*’(Cuoco *et al.*, 1996: 2). Indeed, to do mathematics as mathematicians do it, pupils should have opportunities to learn how to bring together different aspects of their knowledge and how to apply their mathematical skills in tackling a variety of mathematics situations (routine and non-routine, within and outside mathematics). They will also need to learn how to proceed in attacking problems where there are more than one path leading to the solution, where paths they try will not always work, where different strategies might be needed before finding out what works and are able to reason mathematically, justifying why a line of enquiry is successful. In the re-contextualisation of the disciplinary knowledge into school knowledge (Bernstein 2000) the messiness and struggles of disciplinary debates and divides is often hidden. While we do want to present school children with a coherent picture of what mathematics is, there is much to be gained in acknowledging that doing mathematics is about being inquisitive, being resilient and persistent when ways forward are not clear, talking to others, refining explanations and solutions, listening to and learning from others’ insights.

For such communication to take place, pupils will need to acquire and become fluent in using the mathematical language, both written and spoken. Becoming fluent in using the mathematical language takes time and it requires practice in using the symbolic, formal and technical language and operations.

While pupils in schools learn about relatively simpler mathematical concepts and principles than those of the discipline of mathematics, they should have opportunities to learn and adopt some of the ways mathematicians do mathematics: through discovering patterns, formulating conjectures, making links, abstracting, generalising, presenting convincing arguments, justifying, and proving, thus helping them develop a conception of mathematics as an intellectually rewarding discipline. The next section thus exemplifies how some of these disciplinary practices could be made part pupils' learning of school mathematics.

Disciplinary practices in learning school mathematics

School mathematics introduces pupils to the various branches of the discipline of mathematics through concrete experiences such as counting and measuring. Pupils learn about numbers, introduced to them initially as mathematical objects based on the empirical idea of quantity, then as abstractions in an axiomatic system which are independent of the idea of quantity, namely the real (and in the later years of schooling the complex number) system with the real number properties, to include ideas about infinity and infinite and infinitesimal processes. Pupils at even a young age engage with the abstractness of mathematics and they will soon recall multiplication facts such as $3 \times 2 = 6$ as multiplications of abstract numbers, instead of the earlier concrete experience of calculating the number of apples eaten if 3 apples are eaten by each of the 2 pupils. Gradually, over the years, pupils' mathematical concepts will have less and less links to experience, becoming more abstract concepts to be operated on.

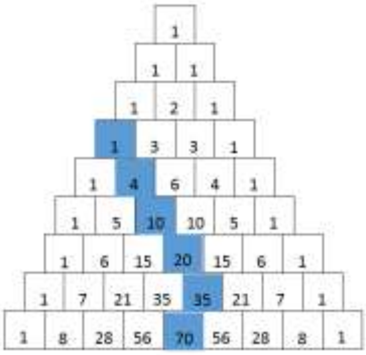
Geometry is another domain of mathematics where points and lines are used and thought of as abstract concepts, as idealised physical objects; points have no thickness, no size as such. Similarly, pupils develop concepts of a geometric figure as a result of abstraction from all the properties of actual objects, except their spatial forms and dimensions. In the early years of geometry education, the focus tends to be on shapes and solids, then it moves on to properties and relationships of shapes and solids. Pupils should have opportunities to engage with geometrical reasoning from a young age, by trying out different representations involving visualising, sketching, constructing accurate diagrams, building models, both physical and virtual, calculating and estimating lengths, areas, volumes and angles.

As abstract thinking progresses, geometry becomes much more about analysis and reasoning. Pupils will continue to develop their geometric reasoning skills by, for example, using a dynamic geometry environment to transform the image of mathematical objects and identifying what changes and what stays the same. Changing the size of a triangle by dragging its vertices, leads to noticing that the sum of the interior angles of each of the newly formed triangles equals 180 degrees. Pupils should be aware of the strength of empirical evidence and appreciate the difference between evidence and proof. Wondering if this relationship holds for any triangle 'out there' leads on to advancing a conjecture about the relationship between the sizes of the interior angles of *any* triangle, thus detaching their reasoning from the particular cases observed and moving towards developing a chains of reasoning to prove or disprove the conjecture advanced.

The development of these pupils' understanding of mathematical proof and deductive reasoning needs to be supported from early on in their school education. Empirical approaches to exploring mathematics encourage learners to develop an understanding of the need of a proof. In primary school, proofs could take the form of explanations of (mainly) number patterns, while at secondary schools pupils should be made aware of different types of proof methods (visual, algebraic, geometric) as method to certify not only that something is true but also why it is true.

Nowadays, geometry in most secondary school for most learners is mainly 'shape and space' without reason, deduction or proof, the focus instead being on calculations of lengths, perimeters, areas and volumes. Words such as assumption, axiom, given facts, conjecture, deduction, proposition, conclusion, statement, theorem are only briefly mentioned or not at all in mathematics textbooks. Pupils need opportunities to engage with proofs and the abstract. Proof is a fundamental component of the discipline of mathematics and so it should be part of mathematical education in schools. Polya suggested that Euclidean geometry was never on the curriculum for pupils to know about geometric facts themselves, but rather for pupils learn about and experience logical reasoning without which " he (sic) lacks a true understanding with which to compare alleged evidence of all sorts aimed at him in modern life" (Polya, 1990: 127). Each discipline has a different conception of what constitutes evidence or 'proof'. In the discipline of mathematics, it is not acceptable to justify a claim based solely on example data. Mathematicians want theorems to follow from axioms of a given system by means of logical deduction; when building a proof, the argument is clearly developed and each step is supported by a property, theorem, postulate or definition. Lewis Carroll, author of *Alice's Adventures in Wonderland* and mathematician said, 'The charm [of mathematics] lies chiefly in the absolute certainty of its results; for that is what, beyond all mental treasures, the human intellect craves for.'

The mathematical notation we use today was not invented until the 16th century. It came about from the realisation that mathematics requires more precision than use of everyday language and has since been continuously refined and further extended to accommodate the new developments. What is hidden-from sight and must be taught and learnt is an appreciation of ‘how empowering symbols can be in expressing generalities and justifications of arithmetical phenomena’ (Arcavi, 1994: 33). To exemplify, let’s consider the Hockey Stick Theorem which states that if a diagonal of numbers of any length is selected starting at any of the one's at the sides of Pascal’s triangle and ending on any number inside the triangle on that diagonal, the sum of the numbers inside the selection is equal to the number below the end of the selection that is not on the same diagonal itself. Here is an attempt to exemplify this theorem, assuming that the reader is already familiar with Pascal’s triangle.



Looking at the shaded numbers in Pascal’s triangle, notice that they create a *geometrical pattern* similar to a hockey stick, hence the name of the theorem. Also notice the *numerical relationship* within this hockey stick, namely 1 + 4 + 10 + 20 + 35 = 70. A fun fact to notice in this triangle!

Each number in Pascal’s triangle also has a symbolic notation assigned to it. For example, the notation $\binom{5}{2}$ is assigned to the shaded number 4, since this number is located on position 2 in row 5. Similarly, the shaded number 10, being on position 3 in row 6, could also be represented by $\binom{6}{3}$. Using this new symbolic notation, the *numerical relationship* in the shaded hockey stick above could now be expressed as $\binom{3}{0}+\binom{4}{1}+\binom{5}{2}+\binom{6}{3}+\binom{7}{4}=\binom{8}{4}$. But this is just an instance of all possible hockey sticks in Pascal’s triangle where similar numerical relationships hold true. Using *algebra* and the helpful notation introduced earlier (together with the sigma symbol to mean addition), this generalisation could be described as $\sum_{i=0}^{k}\binom{n+i}{i}=\binom{n+k+1}{k}$. Thus, the rather convoluted wordy description of the Hockey Stick Theorem above is now encapsulated in this concise format, enabled by the use of symbols and notations. Empowering and beautiful!

Algebra consists of learning to manipulate algebraic expressions, however, emphasis will be placed upon algebraic thinking and its power to generalise and abstract from particular cases. At the heart of algebra is generalizing mathematical ideas, representing and justifying generalizations in multiple ways, and reasoning with generalizations (Kaput 2008). Algebra is about solving problems employing rules and routines, which comes with an understanding of the rationale and deduction of those rules and it is not just about solving particular problems employing rules and routines.

A consistent finding of research in mathematics education is that the basis for using algebraic symbolisation successfully is not just learning the rules of the language, but also understanding the underlying operations and relations and being able to use symbolism correctly. When solving equations, e.g. $2x-3=149$, negative 3 is not moved over to the other side of the equation, changing the sign while doing so to give $2x = 152$. Terms just do not fly over the equal sign, changing the sign; one can do this because there is a mathematical reason behind it (adding the same quantity to both sides of the equation keeps both sides of the equation 'in balance') and thus pupils should be supported to develop an understanding of how the operations combine and relate to each other.

Mathematical language is more than a language, which facilitates expression and communication using written and spoken symbols. It uses everyday words, but not with their everyday meaning. For example, some mathematical words are shared with English and have comparable meanings: e.g., *difference* in mathematics means the answer to a subtraction problem, while in English *difference* is used as a general comparison.

Justification and argumentation are disciplinary practices because they are the means by which mathematicians validate new mathematics. Several authors emphasise the importance of learning to speak like a mathematician in order to take on the identity of a mathematician (Holland *et al*., 1998; Wenger, 1998). In school mathematics, written and oral argumentation and justification should be part of the learning mathematics because they have been shown to support pupils' understanding of mathematics and their proficiency at doing mathematics. Indeed, Wood, Staples, Larsen and Marrongelle propose that 'these practices are not just a desirable end productor outcome of a mathematics education; they are a means by which to learn and do mathematics' (2008: 1).

Pupils should have opportunities to develop a language with which to describe what they see and to explain their thinking. Thus I am in utmost agreement with Pimm's view that 'children need to learn how to mean mathematically, how to use mathematical language to create, control and express their own mathematical meanings as well as to interpret the mathematical language of others' (Pimm, 1995: 179).

The value of mathematics

As a 'tool' subject which equips pupils with the skills for solving problems is a good enough reason in itself for including mathematics in the school mathematics curriculum. This utilitarian view of mathematics (i.e., a view of mathematics as a practical and useful tool for everyday life; see Ernest, 1992 for a summary of how the aims for mathematics teaching and learning are reflected in the design of the National Curriculum) seems to have been better represented in the construction of the school curriculum for mathematics in England over the years. However, there is an imperative need of recognising mathematics as a school discipline in itself. By studying school mathematics pupils will be introduced to the great ideas and controversies in human thought and experience in the development of the discipline of mathematics. Adrian Smith succinctly and powerfully summarised the value of mathematics in one's education:

> 'Mathematics provides a powerful universal language and intellectual toolkit for abstraction, generalization and synthesis. It is the language of science and technology. It enables us to probe the natural universe and to develop new technologies that have helped us control and master our environment, and change societal expectations and standards of living. Mathematical skills are highly valued and sought after. Mathematical training disciplines the mind, develops logical and critical reasoning, and develops analytical and problem solving skills to a high degree.' (Smith, 2004: 11)

Teaching mathematics for its disciplinary and intellectual value aims at providing training to the mind of the learners and developing intellectual habits in them. Pupils will be empowered in expressing, justifying and arguing their views through logical arguments. Pupils will be able to construct arguments through the power of reason, developing themselves as liberal citizens.

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Some useful internet sites

Association of Teachers of Mathematics	http://www.atm.org.uk
Mathematical Association	http://www.m-a.org.uk/
NRICH Maths .	http://www.nrich.maths.org
National Centre for Excellence in Teaching Mathematics	Htpp://www.ncetm.org.uk