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Local Risk Minimization of Contingent Claims Simultaneously Exposed to Endogenous and Exogenous Default Times

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Abstract

We study the local risk minimization approach for contingent claims that might be simultaneously prone to both endogenous (or structural) and exogenous (or reduced form) default events. The exogenous default time is defined through a hazard rate process that can depend on both the underlying risky asset values and its running infimum process. On the other hand, the endogenous default time could be modeled by a first-passage-time approach. In particular, our framework provides a unification of structural and reduced form credit risk modeling. In our work, the evolution of the underlying risky asset values is modeled by an exponential Lévy process, for example exponential jump-diffusion models. Our aim is to determine locally risk minimizing hedging strategies of the contingent claims that are affected by both structural and reduced form default events, through solutions of either partial differential equations or partial-integro differential equations. Finally, we show that these solutions are numerically implementable, and we provide some numerical examples.

Keywords: Defaultable claims, Credit risk, FS decomposition, Local risk minimization, Running infimum process, Structural models, Reduced form models, Hazard rate

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1 Introduction

We analyse the local risk minimization (LRM) approach of certain contingent claims that can be prone to two types of default events. The first one is endogenous (or structural) defined by the first hitting time of the underlying risky asset values to a barrier, and the second one is caused exogenously which is modeled through a hazard rate process. The first type of default could happen due to a specific internal risk, and the second one may be due to a systematic risk.

In credit risk modeling, normally, the default time is either modeled by a predictable stopping time in structural models like first-passage-time models [9] and Merton's model [30], or it is completely unpredictable, modeled via a totally inaccessible stopping time such as hazard rate based models (also known as reduced form models), see for example [22] and [16]. Predictable default times are normally the result of modeling the underlying asset values with stochastic processes that admit continuous sample paths. It is also possible to model structural credit risk events using totally inaccessible stopping times, mainly in the context of jump processes of finite variation, for example, see [33].

However, in many interesting cases, the default time is neither predictable nor totally inaccessible. To understand this further, consider a structural credit risk model in which the default time is modeled by the first hitting time of an exponential jump-diffusion Lévy process to a certain barrier. This hitting time is a stopping time, but it is neither predictable nor totally inaccessible because the default can happen in two fashions, either through a sudden jump or by a continuous crossing of the barrier. Under exponential diffusion process, the stopping time is predictable and under certain exponential pure jump Lévy processes where default occurs by a sudden jump, it is totally inaccessible, but it is neither one under general exponential jump-diffusion Lévy processes. This intuitively indicates that in general, default as a stopping time is made of two other stopping times, one predictable and the other totally inaccessible. The precise statement of this is provided by Theorem III.4 of [34]. Hedging against the risk of this type of default events is particularly challenging which we have achieved in this paper through obtaining LRM hedging strategies in the context of a hybrid structural-reduced form modeling.

Structural and reduced form models have been studied and compared in the literature, we refer to [21] and many references therein. Note that the comparison of these models is not the subject of this research work, instead, our goal is to characterize certain orthogonal decompositions leading to a unification of structural and reduced form credit risk models. The unification between the two approaches has been studied in models known as "hybrid" solvency, see [11], [18], [10] and [26]. In [5], default times are modeled through stopping times having both predictable and totally inaccessible components. In [15], a structural model is constructed where the barrier is a random variable, and the evolution of the underlying asset values is modeled by a finite variation Lévy process. In our work, we obtain LRM hedging strategies of contingent claims, with focus on defaultable ones where there is a distinctive connection between the endogenous and exogenous default events through the running infimum process. In what follows, we first discuss main features of our model along with some literature review, and then we explain its contributions.

In our work, the evolution of the underlying risky asset values is modeled by an exponential Lévy process which can admit jumps. In addition, we also have the risk of default in the market. Hence, the market completeness is violated, i.e. the existence of a unique martingale measure (under which any contingent claim can be perfectly hedged and uniquely priced) is not guaranteed. For interesting discussions about market incompleteness, in particular in the context of derivative hedging, we refer to

[35].

Quadratic hedging approaches can be applied for hedging contingent claims in incomplete markets, and they are divided into two different categories. The first one is the LRM method where the cost process is minimized locally while making terminal payoffs attainable. The hedging strategies of this method are mean self-financing; meaning that the cost process is a martingale. The method is introduced in [17] where it is assumed that the underlying risky asset price process is a martingale. This approach is generalized later for local martingales (resp. semimartingales) through the Galtchouk-Kunita-Watanabe (GKW) decomposition (resp. the Föllmer-Schweizer (FS) decomposition), see for example [37].

The second approach of quadratic hedging is called mean-variance hedging (MVH), and in contrast to the local risk minimization, it determines hedging strategies that are self-financing. In simple terms, the MVH approach insists on building self-financing hedging portfolios where an optimal hedging strategy is the one that provides the best approximation of a given contingent claim in an L^2 sense. The strategies found via both methods coincide when the evolution of the underlying asset values is modeled by a local martingale. For a comparison of the two methods, we refer to [19].

We choose LRM approach to manage the risk for two main reasons; first, under some conditions, obtaining hedging strategies in this approach is equivalent to determining the FS decomposition which is mathematically tractable, and second the method has been applied in analyzing defaultable markets successfully, see for example, [6] and [7]. A popular approach in obtaining the FS decomposition is via determining an equivalent local martingale measure, called minimal equivalent local martingale measure. In particular, the method works well when underlying asset value processes admit continuous sample paths. Determining the FS decomposition leads to strategies that are called pseudo locally risk minimizing (PLRM); however, under some appropriate conditions, such as the structure condition (SC), the LRM approach is equivalent to the PLRM one, for more details, we refer to [37]. For the reader's convenience, a short review of the LRM approach within our setup is provided in Section 3.1. The concept of hedging against the risk of contingent claims using the LRM method when the underlying asset values are modelled through Lévy processes has been studied in [3] and [2]. More specifically, in [3], the representation of the FS-decomposition is determined explicitly using Malliavin calculus for call, Asian, and look-back options. Then in [2], they obtain the LRM representation for a call option numerically through the fast Fourier transform.

In [33], the LRM approach for a structural model in the context of finite variation Lévy process is studied, where the default time admits an intensity. Instead of using a minimal martingale measure, they obtain the LRM hedging strategies directly through solutions of partial-integro differential equations (PIDEs) by martingales and compensator techniques without the change of the underlying probability measure. In our work, we use this method in order to obtain the FS decompositions.

In credit risk modeling, the default time is not necessarily a stopping time under the reference filtration generated by the underlying asset values process. Hence, studying the LRM approach in the context of defaultable claims might require expanding the reference filtration to an enlarged one to make the default time a stopping time, see for example [24], [34], [27], and [23]. Roughly speaking, there are two main kinds of filtration enlargement: initial filtration and progressive filtration expansions. The initial filtration expansion is the right-continuous version of the filtration $\mathbb{G}$ generated by the original reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the sigma-algebra of the default time τ. On the contrary, the progressive filtration expansion of $\mathbb{F}$ by τ, which is used in our work, is the smallest right-continuous filtration $\mathbb{G}$ which includes $\mathbb{F}$ and the natural filtration generated by the default indicator process $(1_{\{\tau\leq t\}})_{t\geq 0}$.

Regarding hedging against the risk of contingent claims using the LRM approach, [6] and [7] analyze defaultable markets through the FS decomposition based on hazard rate models where the default intensity is modeled by a hazard rate process and the underlying risky asset prices are modeled by diffusion processes. They follow the approach of [8] and assume that hedging stops after default. Hence, they use stopped underlying asset value processes instead of the original price processes. We also use stopped price processes in our analysis, and hence hedging stops after default in our model as well.

Furthermore, the idea of pseudo stopping times, introduced by [32], and the Azéma supermartingale $Z = (Z_t)_{t\geq 0}$, $Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t)$ play a vital role in modelling default time through hazard rate approaches. Essentially, in [14], it is shown that if $(Z_t)_{t\geq 0}$ is decreasing and continuous then τ is a pseudo-stopping time that avoids all the $\mathbb{F}$-stopping times. This is an important result which we also use in our framework. Applications of progressive filtration expansion and pseudo stopping times in credit risk modeling can be found in [25], [29] and [14].

In [7], the case of defaultable claims with a recovery process is analyzed. Furthermore, [13] and [12] apply the LRM approach assuming that the evolution of the underlying risky asset values, which can be modeled through a jump diffusion process, is partially observed. However, none of these models can explain neither the simultaneous effect of endogenous (or internal) and exogenous (or external) default events nor they apply partial differential equation (PDE) or PIDE approaches.

In what follows, we specifically discuss the contribution of our work. Our main goal is to obtain semi-explicit solutions of hedging strategies (which are numerically implementable) in the context of the LRM method, when the defaultable claim is subject to both structural (caused specifically by the underlying risky asset) and exogenous (caused systematically by external risk factors) default events. In the context of the LRM method, this is an improvement over the existing credit risk models where the default time is linked to either a structural default event or a totally inaccessible one.

In order to consider both effects of internal and external default risks (which provides a unification of structural and reduced form credit risk modeling), we use the running infimum process (RIP) of the evolution of the underlying risky asset values as an auxiliary process. In other words, the strategies are determined by using not only the underlying risky asset price process but also the RIP. However, the strategies are still implemented using the underlying risky asset as the RIP is not a tradable one. As we mentioned earlier, we do not use an equivalent minimal local martingale measure, instead we determine LRM hedging strategies through solutions of either PDEs or PIDEs depending on whether or not the evolution of the underlying risky asset values is continuous. The inclusion of the RIP, introduces a new dimension to the PDEs and PIDEs, hence we work with three dimensional PDEs and PIDEs.

Although analysis and the numerical implementation could be more challenging due to the extra dimension, this extra dimension would allow us to manage the risk of more complicated defaultable claims. For instance, it would be then possible to extend structural credit risk models such as [33] in which the default time is totally inaccessible and the evolution of the underlying risky asset values is modeled by a finite variation Lévy process. More specifically, we improve their work, first by using an exponential Lévy process to model the evolution of the underlying risky asset values, and second by applying a more general hitting time to model the default event.

In addition, we allow for correlation between the default rate and historical asset values through the RIP. More precisely, in order to model the default event via τ, we assume that $\mathbb{P}(\tau > t \mid \mathcal{F}_t) = e^{-\int_0^t g(s,Y_s,\underline{Y}_s)ds}$, $t \geq 0$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the reference filtration, g is a non-negative continuous function, Y models the evolution of the underlying risky asset values, and $\underline{Y}_t = \inf_{0\leq s\leq t}(Y_t)$ is the RIP of Y. Note that the default rate is path-dependent, as it is a function of both the underlying risky asset values and the RIP. This would allow for a richer family of default rates. Several examples and models are provided in Sections 4 and 5.

Then, following a similar method in [33], under the progressive filtration expansion, the LRM hedging strategies for defaultable claims of the form $F(Y_T, \underline{Y}_T)1_{\{\tau>T\}}$, is obtained where T is the maturity time and F is a measurable function that is not necessarily continuous. Under this framework, the structural and exogenous default events are modeled respectively using $\underline{Y}$ and τ. We discuss some models in Sections 4 and 5 in which both types of defaults might be present. Furthermore, we provide applications and numerical examples by specifying function g explicitly.

The paper is structured as follows. In Section 2, the model and some preliminary results are introduced. The hedging strategy through the FS decomposition are obtained in Section 3. In Section 4, we focus on credit risk models based on diffusion processes, and Section 5 is devoted to credit risk models based on general Lévy processes such as jump-diffusions. In Section 6, we discuss the numerical implementation through solving PDEs. An important result regarding reflected Lévy processes is provided in Appendix A. Finally, some definitions and technical results are reviewed in Appendix B.

2 Model Description and Preliminary Results

Suppose that $(\Omega, \mathcal{G}, \mathbb{P})$ is a complete probability space equipped with filtration $\mathbb{F}$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ models the available information to investors generated by a one dimensional Lévy process X, i.e. $\mathcal{F}_t = \sigma(\{X_s; 0 \leq s \leq t\})$, for all $t \geq 0$. Without any loss in generality, we can assume that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{G}$, then Theorem I.31 of [34] shows that filtration $\mathbb{F}$ is right-continuous, i.e. this filtration satisfies the usual hypotheses.

It is assumed that the process $(X_t)_{t\geq 0}$ has the Lévy triplet (a, σ^2, ν), where $a \in \mathbb{R}$, $\sigma > 0$, and ν is a Lévy measure. Furthermore, $X_0 \in \mathbb{R}$ is deterministic, and the Lévy measure ν admits a continuous distribution. The continuity assumption of the Lévy measure is used in the proof of Proposition A.1 of Appendix A.

We suppose that the default-free market is composed of two assets, a risky asset where the evolution of its values is modeled by the stochastic process $Y = (Y_t)_{t\geq 0}$, defined by $Y_t = e^{X_t}$, $t \geq 0$, and a risk-free asset. We further assume that the interest rate is zero, and so the risk-free asset admits the value of one at all times.

Next, we need to specify a defaultable market (hence defaultable claims) within our setup. In order to do so, first, we define the running infimum process and then model the arrival rate of default as follows.

Definition 2.1. The running infimum process (RIP) of $Y = (Y_t)_{t\geq 0}$, $Y_t = e^{X_t}$, is shown by $\underline{Y}$ and defined by $\underline{Y}_t = \inf_{s\leq t} Y_s$, $t \geq 0$. A similar definition and notation hold for other processes.

Note that the sample paths of $\underline{Y}$ are decreasing, hence, $\underline{Y}$ is of finite variation even if Y is of unbounded variation. Also, for all $t \geq 0$, $\underline{Y}_t = e^{\underline{X}_t}$, where $\underline{X}_t = \inf_{s\leq t} X_s$, and so $\underline{Y}$ inherits some sample path properties of $\underline{X}$. Next, we define a default time that models the exogenous default event.

Assumption 2.2. *Suppose that τ is a non-negative $\mathcal{G}$-measurable random time modeling the default time of a firm (with the asset values Y) such that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$, for all $t \geq 0$. Furthermore, we assume that τ admits a hazard rate process λ given by*

$$\lambda_t = g(t, Y_t, \underline{Y}_t), \quad t \geq 0, \tag{2.1}$$

where $g : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ *is a continuous function, such that the survival process* $(Z_t)_{t\geq 0}$ *of* τ *with respect to* $\mathbb{F}$*, defined by* $Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t)$*,* $t \geq 0$*, satisfies*

$$Z_t = e^{-\int_0^t g(s,Y_s,\underline{Y}_s)\, ds}. \tag{2.2}$$

The survival process $(Z_t)_{t\geq 0}$ is known as the Azéma supermartingale, [4]. Regarding the hazard process λ, given by (2.1), among many others, one can consider the following choices:

(i) The simplest example is when g is a constant.

(ii) A company's asset values that are far from its historical infimum should be less prone to default. In fact, since $g(t, Y_t, \underline{Y}_t)$ models the instantaneous rate of default at $t \geq 0$, this rate should decrease as $Y_t - \underline{Y}_t$ increases, for instance, for $t \geq 0$ and $a > 0$, one can choose $g(t, Y_t, \underline{Y}_t) = e^{-a(Y_t - \underline{Y}_t)}$ or $g(t, Y_t, \underline{Y}_t) = \dfrac{1}{a(Y_t - \underline{Y}_t) + b}$, where a and b can be interpreted as the macroeconomic variables. Alternatively, one can consider $g(t, Y_t, \underline{Y}_t) = e^{-a(Y_t)}$, $a > 0$, which indicates that as asset values decrease the default rate increases as well.

(iii) Suppose that the default time is independent from the underlying risky asset values and admits a probability density function, then $g(t, x, y) = \dfrac{f_\tau(t)}{1 - F_\tau(t)}$, $t \geq 0$, $x \geq 0$, $y \geq 0$, where f_τ and F_τ are respectively the probability density and distribution function of τ.

We model a defaultable claim by the triplet (H, τ, T) where H is an $\mathcal{F}_T$-measurable random variable, τ is the default time as modeled in Assumption 2.2, and $T < \infty$ is the maturity of the claim. For simplicity, the recovery is considered to be zero. The non-zero recovery case could be investigated in future work, for instance based on the methodology of [7]. In fact, assuming that the recovery is settled at the maturity of the claim then an extension of our work should cover this case. To apply our methodology for this case, it will be required to obtain canonical decompositions of processes like $(f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau \leq t\}})_{t\geq 0}$, for smooth functions f, and actually we do this in the proof of Proposition 2.7. Using these canonical decompositions and following similar steps as our work, it should be possible to obtain LRM hedging strategies under this assumption for recovery.

Since the recovery rate is zero, the holder of this claim receives H, if $\tau > T$ and nothing otherwise, i.e. the payoff is $H1_{\{\tau > T\}}$. Furthermore, we let $H = F(Y_T, \underline{Y}_T)$, where $F : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ is a real-valued measurable function; hence, the defaultable claims admits the following form

$$F(Y_T, \underline{Y}_T)1_{\{\tau > T\}}. \tag{2.3}$$

Although we call (2.3) a defaultable claim, for $\tau = \infty$ and an appropriate choice of F, it can also cover the default-free derivatives as well.

In order to motivate our study furthermore, we have a closer look at (2.3) through providing an interesting example. More detailed examples and models are provided in Sections 4 and 5. Consider a firm whose equity is modeled by Y. This firm has just issued a debt modeled by $R(Y_T)$ where R is a real-valued measurable function, $R : \mathbb{R}^{\geq 0} \to \mathbb{R}$. The debt is subject to two types of default, one structural (which we also call endogenous or internal) and the other is exogenous caused by external factors. The structural default time can be modeled by $\zeta = \inf\{t : Y_t \leq b\}$ where $0 < b < Y_0$ is a pre-specified constant, i.e. it is assumed that if the firm's asset values fall under b, the equity holders liquidate the firm. Note that if the sample paths of Y are continuous then ζ is an $\mathbb{F}$-predictable stopping time.

The firm could be also subject to an exogenous default event which we model by τ, and unlike ζ, it might not even be a stopping time with respect to $\mathbb{F}$. Therefore, the debt will be settled if $\zeta > T$ and $\tau > T$, i.e. if the firm does not default by either endogenous and exogenous default events before the maturity. Since, $\{\zeta > T\} = \{\underline{Y}_T > b\}$, then the firm's debt is a special case of (2.3) for $F(x,y) = R(x)1_{\{y>b\}}$.

Since τ is not necessarily an $\mathbb{F}$-stopping time, an expansion of this reference filtration is required. Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the progressive filtration expansion of $\mathbb{F}$ by τ given by

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t, \quad t \geq 0, \tag{2.4}$$

where $\mathcal{N}_t$, $t \geq 0$ is the sigma-algebra generated by $1_{\{\tau \leq t\}}$, $t \geq 0$. It is easy to show that $\mathcal{N}_t = \sigma(\{\tau \leq u; u \leq t\})$, $t \geq 0$. If for all $t \geq 0$, $Z_t < 1$ (and so τ is not an $\mathbb{F}$-stopping time), then by Proposition 5.1.3 of [9], the process

$$\left(1_{\{\tau \leq t\}} - \int_0^{\tau \wedge t} \lambda_s ds\right)_{t \geq 0},$$

is a $\mathbb{G}$-local martingale. Hence in this case, τ is a $\mathbb{G}$-totally inaccessible stopping time.

Remark 2.3. As it is pointed out earlier, this framework could incorporate both features of structural and reduced form credit risk modeling. For example, if $\tau = \infty$, $F(x,y) = R(x)$, for all $x \geq 0$ and $y \geq 0$ and a real-value function R, then claim (2.3) is free from either structural or reduced form defaults.

If $F(x,y) = R(x)$ for all $x \geq 0$, $y \geq 0$ and $Z_t < 1$, for all $t \geq 0$, then we are in the context of a reduced form model. If F depends on both x and y variables, and g as defined in Assumption 2.2, is zero identically (by this we mean that $g(t,x,y) = 0$ for all $(t,x,y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$, and it is shown by $g \equiv 0$), then $Z_t = 1$ for all $t \geq 0$, $\tau = \infty$, and $\mathcal{N}$ is a trivial filtration, hence $\mathbb{G} = \mathbb{F}$, and in this case, the model reduces to a structural one.

On the other hand, if F depends on both x and y variables and $Z_t < 1$, for all $t \geq 0$, then we are in the context of a hybrid model incorporating both structural and exogenous default events. We provide examples of structural, reduced form, and hybrid credit risk models in Sections 4 and 5.

Remark 2.4. By Corollary B.3, since the Azéma supermartingale $(Z_t)_{t\geq 0}$ is continuous and decreasing, we have the following results:

(i) the random time τ is an $\mathbb{F}$-pseudo stopping time [32], i.e. for any bounded $\mathbb{F}$-martingale $\boldsymbol{M}$, we have that

$$\mathbb{E}[\boldsymbol{M}_\tau] = \mathbb{E}[\boldsymbol{M}_0], \tag{2.5}$$

(ii) the random time τ avoids all the $\mathbb{F}$-stopping times, i.e. $\mathbb{P}(\tau = \varsigma) = 0$, for all $\mathbb{F}$-stopping times ς.

Furthermore, since τ avoids all the $\mathbb{F}$-stopping times then $\Delta X_\tau = \Delta \underline{X}_\tau = \Delta Y_\tau = \Delta \underline{Y}_\tau = 0$.

2.1 Closed Form Formulas of Some Canonical Decompositions

In this section, for a given time horizon $0 < S < \infty$, we obtain explicit forms of certain canonical decompositions. These results are then applied in the next section for different time horizons. A function $f : [0,S] \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ is called $C^{1,2,1}$ on $[0,S] \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, and we write $f \in C^{1,2,1}([0,S] \times \mathbb{R}^{>0} \times \mathbb{R}^{>0})$, if it is $C^{1,2,1}$ on $(0,S) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$ and the indicated partial derivatives admit continuous extensions to $[0,S] \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$. Other relevant notation are interpreted similarly. For instance, $h \in C^{1,2,1}([0,S] \times$

$\mathbb{R}^{\geq 0} \times \mathbb{R}$) indicates that h is $C^{1,2,1}$ on $(0, S) \times \mathbb{R}^{>0} \times \mathbb{R}$ and the indicated partial derivatives admit continuous extensions to $[0, S] \times \mathbb{R}^{\geq 0} \times \mathbb{R}$.

In this section, given a function $f : [0, S] \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$, $f \in C^{1,2,1}([0, S] \times \mathbb{R}^{>0} \times \mathbb{R}^{>0})$, we provide a closed-form formula of the canonical decomposition of $(U_t)_{0\leq t\leq S}$, defined by $U_t = f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}}$, $0 \leq t \leq S$, in $\mathbb{G}$; we recall that $Y = e^X$. This canonical decomposition is applied in Section 3 to determine the hedging strategies using the LRM approach. To start with, we obtain the canonical decomposition of the stopped process

$$(f(\tau \wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t}))_{0\leq t\leq S},$$

in the augmented filtration $\mathbb{G}$. From a mathematical point of view, one advantage of using stopped processes is that their decomposition in $\mathbb{G}$ can be characterized by Theorem B.1 of Appendix B. This is clarified further in the following lemma.

Lemma 2.5. *Assume that* $0 < S < \infty$ *and* $f : [0, S]\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0} \to \mathbb{R}$ *belongs to* $C^{1,2,1}([0, S]\times\mathbb{R}^{>0}\times\mathbb{R}^{>0})$. *Furthermore, suppose that* f *satisfies the following integrability conditions*

$$\int_{-\ln(\frac{x}{y})}^{+\infty} |f(t, xe^z, y) - f(t, x, y) - zx\frac{\partial f}{\partial x}(t, x, y)|\nu(dz) < \infty, \tag{2.6}$$

and

$$\int_{-\infty}^{-\ln(\frac{x}{y})} |f(t, xe^z, xe^z) - f(t, x, y) - zx\frac{\partial f}{\partial x}(t, x, y)|\nu(dz) < \infty, \tag{2.7}$$

for all $0 \leq t \leq S$, $x > 0$, *and* $y > 0$. *Moreover, the external default time* τ *follows Assumption 2.2, and* $(X_t)_{t\geq 0}$ *is a Lévy process such that* $\mathbb{E}[|X_1|] < \infty$. *If the condition* $\frac{\partial f}{\partial y}(t, y, y) = 0$[1], *holds for all* $0 \leq t \leq S$, $y > 0$, *then the process*

$$\left(f(t \wedge \tau, Y_{t\wedge\tau}, \underline{Y}_{t\wedge\tau}) - f(0, Y_0, Y_0) - \int_0^{\tau\wedge t} \mathcal{L}f(s, Y_s, \underline{Y}_s)ds\right)_{0\leq t\leq S},$$

is a $\mathbb{G}$*-local martingale, where for all* $0 \leq t \leq S$, $x \geq y$, *and* $y > 0$, *the operator* $\mathcal{L}$ *at* f *is defined by*

$$\begin{aligned}\mathcal{L}f(t, x, y) \quad &= \quad \frac{\partial f}{\partial t}(t, x, y) + \beta x\frac{\partial f}{\partial x}(t, x, y) + \frac{\sigma^2}{2}(x^2\frac{\partial^2 f}{\partial x^2}(t, x, y) + x\frac{\partial f}{\partial x}(t, x, y)) \\ &+ \int_{-\ln(\frac{x}{y})}^{+\infty} (f(t, xe^z, y) - f(t, x, y) - zx\frac{\partial f}{\partial x}(t, x, y))\nu(dz) \\ &+ \int_{-\infty}^{-\ln(\frac{x}{y})} (f(t, xe^z, xe^z) - f(t, x, y) - zx\frac{\partial f}{\partial x}(t, x, y))\nu(dz),\end{aligned}$$

for $\beta = \mathbb{E}[X_1 - X_0]$.

Proof. Let function $h : [0, S] \times \mathbb{R}^{\geq 0} \times \mathbb{R} \to \mathbb{R}$ be defined by $h(t, x, y) = f(t, e^{x-y}, e^{-y})$. Then $h \in C^{1,2,1}([0, S] \times \mathbb{R}^{\geq 0} \times \mathbb{R})$ and for all $0 \leq t \leq S$ and $y \geq 0$, $h(t, X_t - \underline{X}_t, -\underline{X}_t) = f(t, Y_t, \underline{Y}_t)$ and $\frac{\partial h}{\partial x}(t, 0, y) + \frac{\partial h}{\partial y}(t, 0, y) = 0$. By Proposition A.1, the process

$$\left(h(t, X_t - \underline{X}_t, -\underline{X}_t) - h(0, 0, -X_0) - \int_0^t \mathcal{L}^*h(s, X_s - \underline{X}_s, -\underline{X}_s)ds\right)_{0\leq t\leq S},$$

is an $\mathbb{F}$-local martingale where the operator $\mathcal{L}^*$ is defined in Proposition A.1. After simplifying $\mathcal{L}^*h$, this leads to the $\mathbb{F}$-local martingale $(M_t)_{0\leq t\leq S}$, defined by

$$M_t = f(t, Y_t, \underline{Y}_t) - f(0, Y_0, Y_0) - \int_0^t \mathcal{L}f(s, Y_s, \underline{Y}_s)ds, \quad 0 \leq t \leq S.$$

[1]This means to take the partial derivative of $(t, x, y) \mapsto f(t, x, y)$ with respect to the third dimension y and then evaluate it at (t, y, y) for $0 \leq t \leq S$ and $y > 0$.

Based on Remark 2.4, since the Azéma supermartingale $Z = (Z_t)_{t\geq 0}$, $Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t)$, $t \geq 0$, is decreasing and continuous then τ is a pseudo stopping time. By applying part iv of Theorem B.1, $(M_{\tau\wedge t})_{0\leq t\leq S}$ is a $\mathbb{G}$-local martingale which proves the result. □

Remark 2.6. Note that if $\mathbb{E}[|X_1|] < \infty$, a direct application of Theorem 25.3 of [36] shows that for all $t \geq 0$, $\mathbb{E}[|X_t|] < \infty$ and this holds if and only if $\int_{|z|\geq 1} |z|\nu(dz) < \infty$.

Proposition 2.7. *Let $(X_t)_{t\geq 0}$ be a Lévy process, $0 < S < \infty$, and $\mathbb{E}[|X_1|] < \infty$. We also assume that $f : [0, S] \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ belongs to $C^{1,2,1}([0, S] \times \mathbb{R}^{>0} \times \mathbb{R}^{>0})$, it satisfies integrability conditions (2.6) and (2.7) for all $0 \leq t \leq S$, $x > 0$, $y > 0$, and the condition $\frac{\partial f}{\partial y}(t, y, y) = 0$, holds for all $0 \leq t \leq S$, $y > 0$. If the default time τ satisfies Assumption 2.2 and $Z_t < 1$ for all $t \geq 0$, then the process $(U_t)_{0\leq t\leq S}$ defined by $U_t = f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}}$, $0 \leq t \leq S$, admits the following canonical decomposition:*

$$U_t = U_0 + \int_0^{\tau\wedge t} \mathcal{A}f(s, Y_s, \underline{Y}_s)ds + O_t, \quad 0 \leq t \leq S,$$

where $O = (O_t)_{0\leq t\leq S}$ is a $\mathbb{G}$-local martingale, the operator $\mathcal{A}$ at f is given by

$$\mathcal{A}f(t, x, y) = \mathcal{L}f(t, x, y) - f(t, x, y)g(t, x, y), \text{ for all } (t, x, y) \in [0, S] \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}, x \geq y,$$

and the operator $\mathcal{L}$ is given in Lemma 2.5.

Proof. It is easy to see that $U_t = f(\tau \wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t}) - f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}}$, $0 \leq t \leq S$. By Lemma 2.5, the process $(M^f_{\tau\wedge t})_{0\leq t\leq S}$, where M^f_t, $0 \leq t \leq S$ is defined by

$$M^f_t = f(t, Y_t, \underline{Y}_t) - f(0, Y_0, \underline{Y}_0) - \int_0^t \mathcal{L}f(s, Y_s, \underline{Y}_s)ds, \quad 0 \leq t \leq S,$$

is a $\mathbb{G}$-local martingale. So the process $(f(\tau\wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t}))_{0\leq t\leq S}$ can be rewritten as $f(\tau\wedge\cdot, Y_{\tau\wedge\cdot}, \underline{Y}_{\tau\wedge\cdot}) = f(0, Y_0, \underline{Y}_0) + M^f_{\tau\wedge\cdot} + \Lambda^f_{\tau\wedge\cdot}$, where $(\Lambda^f_{\tau\wedge t})_{0\leq t\leq S}$ is a $\mathbb{G}$-predictable process given by

$$\Lambda^f_{\tau\wedge t} = \int_0^{\tau\wedge t} \mathcal{L}f(s, Y_s, \underline{Y}_s)ds, \; 0 \leq t \leq S.$$

Next, we proceed with the process $(f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}})_{0\leq t\leq S}$. Since f is continuous, and $(Z_t)_{t\geq 0}$ is continuous and decreasing, then by Corollary B.3, τ is a pseudo stopping time that avoids all the $\mathbb{F}$-stopping times. As a result, following Remark 2.4, we have that $\Delta f(\tau, Y_\tau, \underline{Y}_\tau) = 0$, and so

$$\int_0^t \Delta f(s, Y_s, \underline{Y}_s)d(1_{\{\tau\leq s\}}) = \Delta f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}} = 0, \quad 0 \leq t \leq S. \tag{2.8}$$

Since $Z_t < 1$ for all $t \geq 0$ and hence τ is not an $\mathbb{F}$-stopping time, the process $M^{(1)}$ defined by $M^{(1)}_t = 1_{\{\tau\leq t\}} - \int_0^{\tau\wedge t} \lambda_s ds$, $t \geq 0$, $\lambda_s = g(s, Y_s, \underline{Y}_s)$, is a $\mathbb{G}$-local martingale. So for $0 \leq t \leq S$, we get

$$\begin{aligned} f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}} &= \int_0^t f(s, Y_s, \underline{Y}_s)d(1_{\{\tau\leq s\}}) = \int_0^t f(s, Y_{s-}, \underline{Y}_{s-})d(1_{\{\tau\leq s\}}) \\ &= \int_0^t f(s, Y_{s-}, \underline{Y}_{s-})dM^{(1)}_s + \int_0^{\tau\wedge t} f(s, Y_s, \underline{Y}_s)g(s, Y_s, \underline{Y}_s)ds, \end{aligned}$$

where the second equality is due to (2.8). The result is then proved by defining the operator $\mathcal{A}$ by $\mathcal{A}f(t, x, y) = \mathcal{L}f(t, x, y) - f(t, x, y)g(t, x, y)$. □

3 LRM Hedging Strategies of Defaultable Claims Dependent on the RIP

In this section, we determine semi-closed-form formulas for the LRM hedging strategies of defaultable claims $F(Y_T, \underline{Y}_T)1_{\{\tau>T\}}$ where $F : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ is a real-valued measurable function. To achieve this, first, we obtain certain orthogonal decompositions, such as (3.5), of the defaultable claims similar to the FS decomposition. Note that these decompositions hold under much weaker integrability conditions than the FS decomposition. In Section 3.1, we then have a brief review of the LRM method and the FS decomposition and discuss when these orthogonal decompositions become the true FS decomposition and lead to the LRM hedging strategies.

Assumption 3.1. *Suppose that $\mathbb{E}[|X_1|] < \infty$, and the Lévy measure ν satisfies:*

$$\int_{-\infty}^{\infty} (e^z - z - 1)\, \nu(dz) < \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} (e^{2z} - 2z - 1)\, \nu(dz) < \infty.$$

In the case of finite variation Lévy processes, where $\int |z|\, \nu(dz) < \infty$, the conditions of Assumption 3.1 are equivalent to $\int_{|z|\leq 1}(e^z - 1)\, \nu(dz) < \infty$ and $\int_{|z|\leq 1}(e^{2z} - 1)\, \nu(dz) < \infty$. Assumption 3.1 is essential as otherwise decompositions such as Equation (3.1) in the following lemma would not be meaningful.

The following lemma determines the canonical decomposition of $(Y_{\tau\wedge t})_{0\leq t\leq S}$, $0 < S < \infty$, and its predictable quadratic covariation in $\mathbb{G}$.

Lemma 3.2. *Suppose that Assumption 3.1 holds. Then the stopped process $(Y_{\tau\wedge t})_{0\leq t\leq S}$, $0 < S < \infty$, is a $\mathbb{G}$-special semimartingale with the following canonical decomposition:*

$$Y_t^{\tau} = Y_0 + M_{\tau\wedge t}^Y + \Lambda_{\tau\wedge t}^Y,\ 0 \leq t \leq S, \tag{3.1}$$

where $\Lambda_t^Y = \alpha \int_0^t Y_s ds$, $\alpha = \beta + \dfrac{\sigma^2}{2} + \int_{-\infty}^{\infty}(e^z - z - 1)\, \nu(dz)$, $\beta = \mathbb{E}[X_1 - X_0]$, $M_t^Y = Y_t - Y_0 - \Lambda_t^Y$, and $(M_{\tau\wedge t}^Y)_{0\leq t\leq S}$ is a $\mathbb{G}$-local martingale. The predictable quadratic variation process $(\langle Y^\tau\rangle_t^{\mathbb{G}})_{0\leq t\leq S}$ is equal to

$$\langle Y^\tau\rangle_t^{\mathbb{G}} = \gamma \int_0^{\tau\wedge t} Y_s^2\, ds,\ 0 \leq t \leq S, \tag{3.2}$$

where $\gamma = 2(\beta - \alpha + \sigma^2) + \int_{-\infty}^{\infty}(e^{2z} - 2z - 1)\, \nu(dz)$.

Proof. Let $f_1 : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ be defined by $f_1(t, x, y) = x$. Because of Assumption 3.1, the integrability conditions (2.6) and (2.7) are satisfied for all $0 \leq t \leq S$, $x > 0$. Then $f_1(t, Y_t, \underline{Y}_t) = Y_t$, and by applying Lemma 2.5 for function f_1, we obtain

$$Y_{\tau\wedge t} = Y_0 + M_{\tau\wedge t}^Y + \int_0^{\tau\wedge t} \mathcal{L}f_1(s, Y_s, \underline{Y}_s)ds,\ 0 \leq t \leq S,$$

where $(M_{\tau\wedge t}^Y)_{0\leq t\leq S}$ is a $\mathbb{G}$-local martingale. The canonical decomposition of $(Y_{\tau\wedge t})_{0\leq t\leq S}$ is then followed by simplifying $\mathcal{L}f_1(s, Y_s, \underline{Y}_s)$, and letting $\Lambda_t^Y = \int_0^t \mathcal{L}f_1(s, Y_s, \underline{Y}_s)ds$, $0 \leq t \leq S$.

Next, we calculate its predictable quadratic variation process $(\langle Y^\tau\rangle_t^{\mathbb{G}})_{0\leq t\leq S}$. From the definition of quadratic variation, we know that $[Y^\tau] = Y_{\tau\wedge\cdot}^2 - 2\int_0^{\tau\wedge\cdot} Y_- dY$.

The canonical decomposition of the integral term in the last equation is easily obtained by noting that

$$\int_0^{\tau\wedge t} Y_{s-} dY_s = \int_0^{\tau\wedge t} Y_{s-} dM_s^Y + \alpha \int_0^{\tau\wedge t} Y_s^2 ds,\ 0 \leq t \leq S.$$

In order to obtain the decomposition of $(Y^2_{\tau\wedge t})_{0\le t\le S}$, let $f_2 := \mathbb{R}^{\ge 0}\times\mathbb{R}^{\ge 0}\times\mathbb{R}^{\ge 0}\to\mathbb{R}$, $f_2(t,x,y)=x^2$ then $f_2(t,Y_t,\underline{Y}_t)=Y_t^2$, $0\le t\le S$. By applying Lemma 2.5 one more time, it follows that the process $\hat{M}=(\hat{M}_{\tau\wedge t})_{0\le t\le S}$, where $\hat{M}_t = Y_t^2 - Y_0^2 - \int_0^t \mathcal{L}f_2(s,Y_s,\underline{Y}_s)ds$, $0\le t\le S$, is a $\mathbb{G}$-local martingale. So, we obtain

$$[Y^\tau]_t = Y_0^2 + \hat{M}_{\tau\wedge t} + \int_0^{\tau\wedge t}\mathcal{L}f_2(s,Y_s,\underline{Y}_s)ds - 2\int_0^{\tau\wedge t} Y_{s-}dM_s^Y - 2\alpha\int_0^{\tau\wedge t} Y_s^2 ds,\ 0\le t\le S. \tag{3.3}$$

The last equation implies that $(\hat{M}_{\tau\wedge t} - 2\int_0^{\tau\wedge t} Y_{s-}dM_s^Y)_{0\le t\le S}$ is of finite variation and so by Lemma I.3.11 of [20], it is locally of integrable variation. Hence, by (3.3) and Lemma I.3.10 of [20], it is easy to observe that $[Y^\tau]$ is locally of integrable variation, and so its compensator exists by Theorem I.3.18 of [20]. From Equation (3.3), this compensator is given by

$$\langle Y^\tau\rangle_t^{\mathbb{G}} = \int_0^{\tau\wedge t}\left(\mathcal{L}f_2(s,Y_s,\underline{Y}_s) - 2\alpha Y_s^2\right)ds,\ 0\le t\le S.$$

The result is then proved, once we simplify $\mathcal{L}f_2(t,Y_t,\underline{Y}_t)-2\alpha Y_t^2$, $0\le t\le S$. □

Our semi-closed-form solutions of LRM hedging strategies are based on solutions of PIDEs (or PDEs if there is no jump component) specified in the following definition.

Hypothesis-$\mathcal{P}$. Suppose that $f:[0,T)\times\mathbb{R}^{\ge 0}\times\mathbb{R}^{\ge 0}\to\mathbb{R}$ belongs to $C^{1,2,1}([0,T)\times\mathbb{R}^{>0}\times\mathbb{R}^{>0})$. We say that f satisfies Hypothesis-$\mathcal{P}$, if it is the solution of the following PIDE

$$\gamma x\mathcal{A}f(t,x,y) = (\mathcal{A}P(t,x,y) - x\mathcal{A}f(t,x,y) - f(t,x,y)x\alpha)\,\alpha,\ t\in[0,T),\ x\ge y,\ y>0, \tag{3.4}$$

where α and γ are defined in Lemma 3.2, for $(t,x,y)\in[0,T)\times\mathbb{R}^{>0}\times\mathbb{R}^{>0}$, $x\ge y$, $\mathcal{A}f$ is defined by

$$\mathcal{A}f(t,x,y) = \mathcal{L}f(t,x,y) - f(t,x,y)g(t,x,y),$$

the operator $\mathcal{L}$ is the same as in Lemma 2.5, $P(t,x,y)=xf(t,x,y)$, $\mathcal{A}P$ is defined like $\mathcal{A}f$, and the following conditions are satisfied:

(i) for all $0\le t<T$ and $y>0$, we have $\frac{\partial f}{\partial y}(t,y,y)=0$,

(ii)
$$f(t_n,x_n,y_n)\to F(x,y),\ \text{as } (t_n,x_n,y_n)\to(T,x,y),\ \text{point-wise for all } x\ge y \text{ and } y\ge 0,$$
where $0\le t_n\le T$, $x_n\ge y_n$, $y_n\ge 0$, $n\ge 1$ are any sequences such that $(t_n,x_n,y_n)\to(T,x,y)$ in Euclidean norm, and F is introduced in Equation (2.3).

Remark 3.3. The condition $\frac{\partial f}{\partial y}(t,y,y)=0$ is required in our analysis as otherwise Lemma 2.5 and Proposition 2.7 cannot be applied. Note that the x and y coordinates represent the asset values process and the RIP respectively. Hence, from a financial point of view, this condition states that the sensitivity of the value process of the hedging portfolio with respect to the RIP, before the default time, is zero when the underlying asset values process meets the RIP. This implies that for a short period of time, the RIP values are very close to the asset values after their interception. Hence, in this short period of time, the portfolio value process is completely driven by the asset values, and it should not be sensitive with respect to the RIP.

Remark 3.4. Suppose that $f : [0,T)\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0} \to \mathbb{R}$ satisfies Hypothesis-$\mathcal{P}$. Then f can be extended to the whole domain $[0,T]\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0}$ by defining $f(T,x,y) = \lim_{(t_n,x_n,y_n)\to(T,x,y)} f(t_n,x_n,y_n) = F(x,y)$, for all $x \geq y$, $y \geq 0$, and the limit is defined in Hypothesis-$\mathcal{P}$.

Note that if $(t,x,y) \mapsto f(t,x,y)$ is a continuous function for all $0 \leq t \leq T$, $x \geq y$, $y \geq 0$, the boundary condition of the PIDE in the previous hypothesis is simply $f(T,x,y) = F(x,y)$. However, $f(T,x,y)$ might not be even well-defined directly; for example, consider the pricing formula of a European Call option in the Black-Scholes model where f reduces to two dimensions. Hence, in general, $f(T,x,y)$ could be only meaningful in a limiting sense, and the extended version of f in Remark 3.4 is not necessarily $C^{1,2,1}$ on $[0,T]\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0}$ or $[0,T]\times\mathbb{R}^{>0}\times\mathbb{R}^{>0}$. This is also reflected in the proof of the next theorem which could have been shorten if f is a smooth function on $\{T\}\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0}$.

Theorem 3.5. *Suppose that $Z_t < 1$ for all $t \geq 0$, and Assumptions 2.2 and 3.1 are in force. We also assume that $f : [0,T)\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0} \to \mathbb{R}$ is a function that satisfies Hypothesis-$\mathcal{P}$ and the integrability conditions (2.6) and (2.7) for all $0 \leq t \leq T$, $x > 0$, $y > 0$. Also, f is extended to $[0,T]\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0}$ according to Remark 3.4, and $f(\cdot\wedge T, Y_{\cdot\wedge T}, \underline{Y}_{\cdot\wedge T})$ is locally integrable (see Definition B.6 of Appendix B). Then, we have the following two orthogonal decompositions:*

(a)

$$f(t,Y_t,\underline{Y}_t)1_{\{\tau>t\}} = f(0,Y_0,Y_0) + \int_0^t \theta_{s-}1_{\{s<T\}}dY_s^\tau + L_t$$
$$= f(0,Y_0,Y_0) + \int_0^t \theta_{s-}1_{\{s\leq\tau\}}1_{\{s<T\}}dY_s + L_t,\ 0\leq t\leq T,\ \mathbb{P}\text{-}a.s.,$$

where for $0 \leq t < T$, θ_t is given by

$$\theta_t = \frac{\mathcal{K}f(t,Y_t,\underline{Y}_t)}{\gamma Y_t^2},$$

the operator $\mathcal{K}$ at f for $(t,x,y) \in [0,T)\times\mathbb{R}^{>0}\times\mathbb{R}^{>0}$, $x \geq y$, is defined by

$$\mathcal{K}f(t,x,y) = \mathcal{A}P(t,x,y) - x\mathcal{A}f(t,x,y) - f(t,x,y)x\alpha,$$

$P(t,x,y) = xf(t,x,y)$, the operator $\mathcal{A}$ is defined in Proposition 2.7, and the process $(L_t)_{0\leq t\leq T}$ is a $\mathbb{G}$-local martingale orthogonal to the local martingale part of $(Y_t^\tau)_{0\leq t\leq T}$ (i.e. $(M^Y_{\tau\wedge t})_{0\leq t\leq T}$) in $\mathbb{G}$.

(b)

$$F(Y_T,\underline{Y}_T)1_{\{\tau>T\}} = f(0,Y_0,Y_0) + \int_0^T \theta_{s-}1_{\{s<T\}}dY_s^\tau + L_T,\ \mathbb{P}\text{-}a.s., \tag{3.5}$$

where $F(x,y)$ is the terminal condition of PIDE (3.4) and $(\theta_t)_{0\leq t\leq T}$ and $(L_t)_{0\leq t\leq T}$ are the same as part a.

Proof. Since f is not necessarily $C^{1,2,1}$ on $[0,T]\times\mathbb{R}^{>0}\times\mathbb{R}^{>0}$, the results of Section 2.1 cannot be used immediately on $[0,T]\times\mathbb{R}^{>0}\times\mathbb{R}^{>0}$. To overcome this challenge, we use an approximation technique. More precisely, fix an integer $n \geq 1$ and let $f^{(n)} : [0,a_n]\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0} \to \mathbb{R}$ be the restriction of f to the set $[0,a_n]\times\mathbb{R}^{\geq 0}\times\mathbb{R}^{\geq 0}$, where $a_n = T - \frac{1}{n}$. Note that $f^{(n)}$ is $C^{1,2,1}$ on $[0,a_n]\times\mathbb{R}^{>0}\times\mathbb{R}^{>0}$. Since the condition $\frac{\partial f}{\partial y}(t,y,y) = 0$, holds for all $0 \leq t \leq a_n$, $y > 0$, from Proposition 2.7 for $S = a_n$, we know that the processes $O^{(n)} = (O_t^{(n)})_{0\leq t\leq a_n}$ and $J^{(n)} = (J_t^{(n)})_{0\leq t\leq a_n}$ defined by

$$O_t^{(n)} = U_t^{(n)} - U_0^{(n)} - \int_0^{\tau\wedge t} \mathcal{A}f^{(n)}(s,Y_s,\underline{Y}_s)ds, \quad U_t^{(n)} = f^{(n)}(t,Y_t,\underline{Y}_t)1_{\{\tau>t\}},$$

and

$$J_t^{(n)} = P^{(n)}(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}} - P^{(n)}(0, Y_0, Y_0) - \int_0^{\tau\wedge t} \mathcal{A}P^{(n)}(s, Y_s, \underline{Y}_s)ds,$$

are $\mathbb{G}$-local martingale where $P^{(n)}(t, x, y) = xf^{(n)}(t, x, y)$, $0 \le t \le a_n$, $x \ge y$, and $y \ge 0$. For more clarity, we break down the rest of the proof into several steps.

Step.1. In the first step, we determine the KW decomposition (see Definition B.4 and for more details [1]) of $(O_t^{(n)})_{0\le t\le a_n}$ with respect to $(M^Y_{\tau\wedge t})_{0\le t\le a_n}$, see Lemma 3.2 for the definition of M_t^Y, $0 \le t \le a_n$. We show that $O_t^{(n)} = \int_0^t \theta^{(n)}_{s-}\, dM^Y_{\tau\wedge s} + L_s^{(n)}$, $0 \le t \le a_n$, where $(\theta_t^{(n)})_{0\le t\le a_n}$ is a $\mathbb{G}$-predictable process (for which, we determine its closed form) and $(L_t^{(n)})_{0\le t\le a_n}$, $L_0^{(n)} = 0$ is a $\mathbb{G}$-local martingale orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le a_n}$.

From the integration by parts formula, we have

$$U_t^{(n)}Y_t^\tau = U_0^{(n)}Y_0 + \int_0^t U_{s-}^{(n)}dY_s^\tau + \int_0^t Y_{s-}^\tau dU_s^{(n)} + [U^{(n)}, Y^\tau]_t,\ 0 \le t \le a_n,$$

where $U^{(n)} = (U_t^{(n)})_{t\ge0}$.

Let $F_t^{(n)} = \int_0^{\tau\wedge t} \mathcal{A}f^{(n)}(s, Y_s, \underline{Y}_s)ds$ and $D_t^{(n)} = \int_0^{\tau\wedge t} \mathcal{A}P^{(n)}(s, Y_s, \underline{Y}_s)ds$ for $0 \le t \le a_n$. From the above integration by parts formula, we obtain

$$\begin{aligned}&[U^{(n)}, Y^\tau]_t - \left(D_t^{(n)} - \int_0^{\tau\wedge t} U_s^{(n)}\mathcal{L}f_1(s, Y_s, \underline{Y}_s)ds - \int_0^{\tau\wedge t} Y_s dF_s^{(n)}\right)\\ &\quad= \left(J_t^{(n)} - \int_0^{\tau\wedge t} U_{s-}^{(n)}dM_s^Y - \int_0^{\tau\wedge t} Y_{s-}dO_s^{(n)}\right),\ 0 \le t \le a_n,\end{aligned} \tag{3.6}$$

where $f_1(t, x, y) = x$. Note that since all the stochastic processes involved in our work are càdlaĝ, their set of discontinuous points is countable and hence not charged by the Lebesgue measure. So Lebesgue integrals such as

$$\int_0^{\tau\wedge t} U_s^{(n)}\mathcal{L}f_1(s, Y_s, \underline{Y}_s)ds,\ t \ge 0,$$

can be also $\mathbb{P}$-a.s. written as $\int_0^{\tau\wedge t} U_{s-}^{(n)}\mathcal{L}f_1(s, Y_{s-}, \underline{Y}_{s-})ds$, $t \ge 0$. For consistency, we choose the former representation for the Lebesgue integrals.

The right-hand side of Equation (3.6) is locally of integrable variation by Lemma I.3.11 of [20]. The bracket on the left-hand side of (3.6) is also locally of integrable variation by Lemma I.3.10 of [20]. Therefore, $([U^{(n)}, Y^\tau]_t)_{0\le t\le a_n}$ is locally of integrable variation. By Theorem I.3.18 of [20], $([U^{(n)}, Y^\tau]_t)_{0\le t\le a_n}$ admits a compensator in $\mathbb{G}$ which is basically the predictable quadratic covariation process of $(O_t^{(n)})_{0\le t\le a_n}$ and $(M^Y_{\tau\wedge t})_{0\le t\le a_n}$ in $\mathbb{G}$. From (3.6), this compensator is given by

$$\langle O^{(n)}, M^Y_{\tau\wedge\cdot}\rangle_t^{\mathbb{G}} = \int_0^{\tau\wedge t}\left(\mathcal{A}P^{(n)}(s, Y_s, \underline{Y}_s) - f^{(n)}(s, Y_s, \underline{Y}_s)Y_s\alpha - Y_s\mathcal{A}f^{(n)}(s, Y_s, \underline{Y}_s)\right)ds,\ 0 \le t \le a_n.$$

From here, we observe that on $[0, a_n]$, the measure defined by $\langle O^{(n)}, M^Y_{\tau\wedge\cdot}\rangle^{\mathbb{G}}$ is absolutely continuous with respect to the measure defined by $\langle Y^\tau\rangle^{\mathbb{G}}$ with the Radon–Nikodym derivative $\theta_t^{(n)} = \frac{\mathcal{K}f^{(n)}(t,Y_t,\underline{Y}_t)}{\gamma Y_t^2}$, $0 \le t \le a_n$. Define the process $L^{(n)} = (L_t^{(n)})_{0\le t\le a_n}$ by $L_t^{(n)} = O_t^{(n)} - \int_0^t \theta_{s-}^{(n)}\, dM^Y_{\tau\wedge s}$, $0 \le t \le a_n$. Then this process is a $\mathbb{G}$-local martingale, $\langle L^{(n)}, M^Y_{\tau\wedge\cdot}\rangle_{0\le t\le a_n} = 0$ and $L^{(n)}$ is orthogonal to $M^Y_{\tau\wedge\cdot}$. Therefore, the KW decomposition of $(O_t^{(n)})_{0\le t\le a_n}$ with respect to $(M^Y_{\tau\wedge t})_{0\le t\le a_n}$ is equal to

$$O_t^{(n)} = \int_0^t \theta_{s-}^{(n)}\, dM^Y_{\tau\wedge s} + L_t^{(n)},\ 0 \le t \le a_n. \tag{3.7}$$

Step.2. In this step, we simplify KW decomposition (3.7) through applying Hypothesis-$\mathcal{P}$. More precisely, by breaking down KW decomposition (3.7), for $0 \le t \le a_n$, we obtain

$$U_t^{(n)} - \int_0^{\tau\wedge t} \mathcal{A}f^{(n)}(s, Y_s, \underline{Y}_s)\, ds = U_0^{(n)} + \int_0^{\tau\wedge t} \theta_{s-}^{(n)}\, dY_s - \int_0^{\tau\wedge t} \alpha\theta_s^{(n)} Y_s\, ds + L_t^{(n)}. \tag{3.8}$$

Consider the above decomposition at $s_n(t) = t \wedge a_n$ for $t \ge 0$ and $n \ge 1$. Since $f^{(n)}$ is the restriction of f on $[0, a_n] \times \mathbb{R}^{\ge 0} \times \mathbb{R}^{\ge 0}$, from (3.8), for $t \ge 0$, we obtain

$$f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)})1_{\{\tau > s_n(t)\}} - \int_0^{\tau\wedge s_n(t)} \mathcal{A}f(s, Y_s, \underline{Y}_s)\, ds =$$
$$f(0, Y_0, Y_0) + \int_0^{\tau\wedge s_n(t)} \theta_{s-}\, dY_s - \int_0^{\tau\wedge s_n(t)} \alpha\theta_s Y_s\, ds + L_{s_n(t)}^{(n)}.$$

However, since f satisfies Hypothesis-$\mathcal{P}$, we have $\mathcal{A}f(s, Y_s, \underline{Y}_s) = \alpha\theta_s Y_s$ for $s \in [0, \tau \wedge s_n(t)]$, hence we get

$$f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)})1_{\{\tau > s_n(t)\}} = f(0, Y_0, Y_0) + \int_0^{\tau\wedge s_n(t)} \theta_{s-}\, dY_s + L_{s_n(t)}^{(n)}, \quad t \ge 0. \tag{3.9}$$

Step.3. Now, we prove part a of the theorem by taking the limit of (3.9) as $n \to \infty$. We start with the left-hand side of (3.9). Remember that $a_n = T - \frac{1}{n}$, $n \ge 1$, and $s_n(t) = t \wedge a_n$, $t \ge 0$, then for all $0 \le u \le T$, we have

$$\begin{aligned} &\sup_{0\le t\le u} |f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)})1_{\{\tau > s_n(t)\}} - f(t, Y_t, \underline{Y}_t)1_{\{\tau > t\}}| \\ &\le \sup_{0\le t\le T} |f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)})1_{\{\tau > s_n(t)\}} - f(t, Y_t, \underline{Y}_t)1_{\{\tau > t\}}| \\ &= \sup_{a_n\le t\le T} |f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)})1_{\{\tau > s_n(t)\}} - f(t, Y_t, \underline{Y}_t)1_{\{\tau > t\}}| \\ &= \sup_{a_n\le t\le T} |f(a_n, Y_{a_n}, \underline{Y}_{a_n})1_{\{\tau > a_n\}} - f(t, Y_t, \underline{Y}_t)1_{\{\tau > t\}}|, \end{aligned} \tag{3.10}$$

where with some abuse of notation, for $t = T$, $f(T, \cdot, \cdot)$ is understood as $F(\cdot, \cdot)$. As $n \to \infty$, we have that $a_n \to T$, and because of the quasi-left continuity of Y, we get $Y_{a_n} \to Y_T$ and $\underline{Y}_{a_n} \to \underline{Y}_T$, $\mathbb{P}$-a.s.. Also, if $a_n \le t \le T$, as $n \to \infty$, we have $t \to T$. On the other hand, from Hypothesis-$\mathcal{P}$, $f(a_n, Y_{a_n}, \underline{Y}_{a_n}) \to f(T, Y_T, \underline{Y}_T)$ and $f(t, Y_t, \underline{Y}_t) \to f(T, Y_T, \underline{Y}_T)$ as $n \to \infty$ and $t \to T$ respectively. Therefore, as $n \to \infty$, the right-hand side of (3.10) goes to zero, and so $(f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)})1_{\{\tau > s_n(t)\}})_{t\ge 0}$ converges to $(f(t, Y_t, \underline{Y}_t)1_{\{\tau > t\}})_{0\le t\le T}$ uniformly on compacts in probability (abbreviated as ucp) as $n \to \infty$.

Since f is not necessarily partially differentiable on $\{T\} \times \mathbb{R}^{\ge 0} \times \mathbb{R}^{\ge 0}$, the process θ_- might not be well defined when $t = T$. So one should consider this when taking the limit of the integral term of (3.9). Obviously, we have $\int_0^{\tau\wedge s_n(t)} \theta_{s-}\, dY_s = \int_0^{\tau\wedge t} \theta_{s-}1_{\{s<T\}}1_{\{s\le T-\frac{1}{n}\}}\, dY_s$, and so by part iii of Theorem I.4.31 of [20], $(\int_0^{\tau\wedge s_n(t)} \theta_{s-}1_{\{s<T\}}\, dY_s)_{t\ge 0}$ converges to $(\int_0^{\tau\wedge t} \theta_{s-}1_{\{s<T\}}\, dY_s)_{0\le t\le T}$ in ucp. Therefore, $L_{s_n(\cdot)}^{(n)}$ converges in ucp to the process $(L_t)_{0\le t\le T}$ given by

$$L_t = f(t, Y_t, \underline{Y}_t)1_{\{\tau > t\}} - f(0, Y_0, Y_0) - \int_0^t \theta_{s-}1_{\{s<T\}} dY_s^\tau,\ 0 \le t \le T.$$

We show that $(L_t)_{0\le t\le T}$ is a $\mathbb{G}$-local martingale. Note that from (3.9), for all $t \ge 0$ we have

$$\begin{aligned} \sup_{u\le t} |\Delta L_{s_n(u)}^{(n)}| \le \sup_{u\le t} |\Delta f(u\wedge T, Y_{u\wedge T}, \underline{Y}_{u\wedge T})| &+ \sup_{u\le t} |f(u\wedge T, Y_{u\wedge T}, \underline{Y}_{u\wedge T})| \\ &+ \sup_{u\le t} |\theta_{(u\wedge T)-}||\Delta Y_{u\wedge T}^\tau|. \end{aligned} \tag{3.11}$$

The process $(L_t)_{0\le t\le T}$ is the limit of $\mathbb{G}$-local martingales, and it can be shown that $(L_t)_{0\le t\le T}$ is a $\mathbb{G}$-local martingale if $\sup_{n\ge1}\sup_{u\le t}|\Delta L^{(n)}_{s_n(u)}|$ is locally integrable. From (3.11), it is enough to show that the right-hand side of this equation is locally integrable.

Since $f(\cdot\wedge T, Y_{\cdot\wedge T}, \underline{Y}_{\cdot\wedge T})$ is locally integrable, the second term on the right-hand side of (3.11) is locally integrable. The first term on the right-hand side of (3.11) is also locally integrable because a càdlàg adapted process is locally integrable if and only if its jumps are locally integrable. Since $(\theta_{t-})_{0\le t\le T}$ is $\mathbb{G}$-predictable, càglàd, it is locally bounded and hence we may assume that it is uniformly bounded. Also, by Lemma 3.2 for $S=T$, $(Y^\tau_t)_{0\le t\le T}$ is a special semimartingale, and by Theorem III.32 of [34], $(\Delta Y^\tau_t)_{0\le t\le T}$ is locally integrable. So the third term on the right-hand side of (3.11) is also locally integrable. Therefore, the left-hand side of (3.11) and hence $\sup_{n\ge1}\sup_{u\le t}|\Delta L^{(n)}_{s_n(u)}|$ are locally integrable which concludes that L is a local martingale.

So, by taking the limit of (3.9) in ucp, we obtain

$$f(t,Y_t,\underline{Y}_t)1_{\{\tau>t\}} = f(0,Y_0,Y_0) + \int_0^t \theta_{s-}1_{\{s<T\}}\,dY^\tau_s + L_t,\ 0\le t\le T, \tag{3.12}$$

where $(L_t)_{0\le t\le T}$ is a $\mathbb{G}$-local martingale. We still need to show that $(L_t)_{0\le t\le T}$ is orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le T}$. Note that $L^{(n)}_{s_n(t)} = L_{s_n(t)} = L_t^{T-\frac{1}{n}}$, for $0\le t\le T$, and so in $\mathbb{G}$, we have that $\langle L, M^Y_{\tau\wedge\cdot}\rangle_t^{T-\frac{1}{n}} = \langle L^{(n)}_{s_n(\cdot)}, M^Y_{\tau\wedge\cdot}\rangle_t = 0$, $0\le t\le T$, because $(L^{(n)}_{s_n(t)})_{0\le t\le T}$ is orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le T}$. Also in $\mathbb{G}$, $\langle L, M^Y_{\tau\wedge\cdot}\rangle^{T-\frac{1}{n}}$ approaches to $\langle L, M^Y_{\tau\wedge\cdot}\rangle_{\cdot\wedge T}$ on $[0,T]$ as $n\to\infty$, hence $\langle L, M^Y_{\tau\wedge\cdot}\rangle_t = 0$, $0\le t\le T$, and so $(L_t)_{0\le t\le T}$ is orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le T}$. This proves part a of the theorem.

Step.4. Remember that f is extended to $[0,T]\times\mathbb{R}^{\ge0}\times\mathbb{R}^{\ge0}$ according to Remark 3.4 and $f(T,x,y)=F(x,y)$ for all $x\ge0$ and $y\ge0$. Finally, part b of the theorem can be proved by letting $t=T$ in (3.12). □

Remark 3.6. Note that in the non-martingale case, i.e. when α is non-zero, we have

$$\theta_t = \frac{\mathcal{A}f(t,Y_t,\underline{Y}_t)}{\alpha Y_t},\quad 0\le t<T,$$

which could be helpful in numerical implementations.

Corollary 3.7. *Suppose that $Z_t<1$ for all $t\ge0$, and Assumptions 2.2 and 3.1 are in force. In addition assume that $(Y^\tau_t)_{t\ge0}$ is a $\mathbb{G}$ local martingale. Suppose that $f:[0,T)\times\mathbb{R}^{\ge0}\times\mathbb{R}^{\ge0}\to\mathbb{R}$ belongs to $C^{1,2,1}([0,T)\times\mathbb{R}^{>0}\times\mathbb{R}^{>0})$, f is extended to $[0,T]\times\mathbb{R}^{\ge0}\times\mathbb{R}^{\ge0}$ according to Remark 3.4, $f(\cdot\wedge T, Y_{\cdot\wedge T}, \underline{Y}_{\cdot\wedge T})$ is locally integrable, and it satisfies the following PIDE:*

$$\mathcal{A}f(t,x,y)=0,\ \ 0\le t<T,\ x\ge y,\ y>0,$$

$f(t_n,x_n,y_n)\to F(x,y)$ point-wise as $(t_n,x_n,y_n)\to(T,x,y)$ in Euclidean norm, and $\frac{\partial f}{\partial y}(t,y,y)=0$ for all $0\le t<T$ and $y>0$. Then in this case, we have

$$F(Y_T,\underline{Y}_T)1_{\{\tau>T\}} = f(0,Y_0,Y_0) + \int_0^T \frac{\mathcal{A}P(s,Y_{s-},\underline{Y}_{s-})}{\gamma Y^2_{s-}}1_{\{s<T\}}dY^\tau_s + L_T,$$

where $(L_t)_{0\le t\le T}$ is a $\mathbb{G}$-local martingale orthogonal to $(Y^\tau_t)_{0\le t\le T}$, the operator $\mathcal{A}$ is introduced in Proposition 2.7, and $P(t,x,y)=xf(t,x,y)$.

Proof. Since Y^τ is a martingale, we have $\alpha=0$. It is easy to check that f is the solution of PIDE (3.4) (because $\mathcal{A}f(t,x,y)=0$), and the result follows easily from Theorem 3.5. □

In this section, up until now, we have assumed that $Z_t < 1$, for all $t \geq 0$ which puts us in the context of reduced form modeling. Consider a structural credit risk modeling in which g in Assumption 2.2 is identically zero which means that $Z_t = 1$ for all $t \geq 0$ and hence $\tau = \infty$. In order to obtain similar results for this structural modeling, we cannot simply let $g \equiv 0$ (i.e. $g(t, x, y) = 0$ for all $t \geq 0$, $x \geq 0$, $y \geq 0$) and for instance use Theorem 3.5 as this opposes the assumption $Z_t < 1$ for all $t \geq 0$. Nevertheless, starting from Proposition 2.7 and following the same arguments of this section (which we skip here), we can obtain similar results. Since $g \equiv 0$, the operator $\mathcal{A}$ is the same as $\mathcal{L}$, no filtration expansion is needed, i.e. $\mathbb{F} = \mathbb{G}$. We provide the following main result in the case of $g \equiv 0$. The proof is omitted as it is almost identical to the proof of Theorem 3.5.

Theorem 3.8. *Suppose that Assumption 3.1 holds and $g(t, x, y) = 0$ for all $t \geq 0$, $x \geq y$, $y \geq 0$. We also assume that, $f : [0, T) \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ satisfies Hypothesis-$\mathcal{P}$, f is extended to $[0, T] \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ according to Remark 3.4, and $f(\cdot \wedge T, Y_{\cdot\wedge T}, \underline{Y}_{\cdot\wedge T})$ is locally integrable. Then, we have the following decomposition*

$$F(Y_T, \underline{Y}_T) = f(0, Y_0, Y_0) + \int_0^T \theta_{s-} 1_{\{s<T\}} dY_s + L_T, \tag{3.13}$$

where for $0 \leq t < T$, the process θ is given by

$$\theta_t = \frac{\mathcal{K}^* f(t, Y_t, \underline{Y}_t)}{\gamma Y_t^2}, \; 0 \leq t < T,$$

the operator $\mathcal{K}^*$ is defined by

$$\mathcal{K}^* f(t, x, y) = \mathcal{L}P(t, x, y) - x\mathcal{L}f(t, x, y) - f(t, x, y)x\alpha, \; (t, x, y) \in [0, T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}, x \geq y$$

the operator $\mathcal{L}$ is given in Lemma 2.5, $P(t, x, y) = xf(t, x, y)$, and the process $(L_t)_{0\leq t\leq T}$ is an $\mathbb{F}$-local martingale orthogonal to the local martingale part of $(Y_t)_{0\leq t\leq T}$ in $\mathbb{F}$.

The orthogonal decompositions (3.5) and (3.13) are similar to FS decompositions as they admit the same orthogonal structure. Nevertheless, further regularity and integrability conditions are required to turn them into FS decompositions in which case $(\theta_t 1_{\{t<T\}})_{0\leq t\leq T}$ leads to the PLRM hedging strategies. In the next section, we have a brief review of the FS decomposition, PLRM and LRM hedging strategies, and their relations within our framework. We then analyze how they are obtained in our model.

3.1 LRM, PLRM, and the FS Decomposition

In this section, we present a short review of the LRM approach within our framework and its connection to the PLRM method and the FS decomposition; for details and discussions in a general set-up, we refer to [37].

Since the hedging is carried out on $[0, T]$, we restrict the underlying asset value process on this interval. Remember that from Equation (3.1), Y^τ admits the following decomposition in $\mathbb{G}$: $Y_t^\tau = X_0 + M_{\tau\wedge t}^Y + \Lambda_{\tau\wedge t}^Y$, $0 \leq t \leq T$. First, we introduce the spaces Ξ and L^2-strategies under $\mathbb{G}$.

Definition 3.9. We let Ξ denotes the space of all $\mathbb{G}$-predictable processes $(\xi_t)_{0\leq t\leq T}$ such that

$$\mathbb{E}\left[\int_0^T \xi_s^2 d[M_{\tau\wedge\cdot}^Y]_s\right] + \mathbb{E}\left[\int_0^T |\xi_s d\Lambda_{\tau\wedge s}^Y|^2\right] < \infty.$$

Definition 3.10. An L^2-strategy is a pair $\phi = (\xi_t, \eta_t)_{0\le t\le T}$ where $\xi = (\xi_t)_{0\le t\le T}$ is in Ξ, and $\eta = (\eta_t)_{0\le t\le T}$ is a real-valued $\mathbb{G}$-adapted process such that the value process $(V_t)_{0\le t\le T}$, defined by $V_t(\phi) = \xi_t Y_t^\tau + \eta_t$ is right continuous and square integrable for all $t \in [0, T]$.

In the above definition, ξ and η represent respectively the number of shares invested in the risky and the risk-free asset.

Following strategy $\phi = (\phi_t)_{0\le t\le T}$, the accumulated revenue (either profit or loss) from trading the underlying asset with values Y on $[0, t\wedge\tau]$, $0 \le t \le T$, is $\int_0^t \xi_s \, dY_s^\tau$, and therefore the accumulated cost at time $0 \le t \le T$ from holding the portfolio based on ϕ is $C_t(\phi) = V_t(\phi) - \int_0^t \xi_s \, dY_s^\tau$. A perfect strategy makes the claim attainable, i.e. $V_T(\phi) = H1_{\{\tau>T\}}$, $\mathbb{P}$-a.s., and its cost is constant through time. However, as markets, in particular defaultable ones, are incomplete, this is not possible and so instead, we look for those strategies that minimize the cost in some sense and lead to optimal strategies.

Suppose that an optimal hedging strategy is to cover a claim completely; though a constant cost process would be too much to ask for, it is reasonable to assume that the cost process of such optimal strategy should not deviate too much from its historical mean. This motivates the definition of a hedging strategy to be mean-self-financing if its cost process is a $\mathbb{G}$-martingale.

Definition 3.11. An L^2-strategy ϕ is called mean-self-financing if its cost process $(C_t(\phi))_{0\le t\le T}$, defined by $C_t(\phi) = V_t(\phi) - \int_0^t \xi_s \, dY_s^\tau$, $0 \le t \le T$, is a $\mathbb{G}$-martingale.

Definition 3.12. The risk process $\mathcal{R} = (\mathcal{R}_t(\phi))_{0\le t\le T}$, associated with the strategy ϕ and filtration $\mathbb{G}$, is defined by $\mathcal{R}_t(\phi) = \ E\left[(C_T(\phi) - C_t(\phi))^2 \,|\mathbb{G}_t\right]$, $0 \le t \le T$.

The main idea in risk-minimization is to minimize the risk process $\mathcal{R}$ in the following sense.

Definition 3.13. An L^2-strategy $\phi = (\theta, \eta)$ as in Definition 3.10 is called risk-minimizing if $V_T(\phi) = H1_{\{\tau>T\}}$, $\mathbb{P}$-a.s., and for any L^2-strategy ϕ' such that $V_T(\phi') = V_T(\phi)$, $\mathbb{P}$-a.s., we have $\mathcal{R}_t(\phi) \le \mathcal{R}_t(\phi')$, $\mathbb{P}$-a.s., for every $0 \le t \le T$.

If Y^τ is a $\mathbb{G}$-local martingale (i.e. $\Lambda^Y_{\tau\wedge\cdot} = 0$) then the existence of such risk-minimizing strategy (which is also mean-self-financing) is guaranteed. In this case the strategies are provided through the GKW decomposition of the claim which in our context states that the claim $H1_{\{\tau>T\}}$, $H = F(Y_T, \underline{Y}_T)$ can be uniquely represented as $H1_{\{\tau>T\}} = H_0 + \int_0^T \xi_s \, dY_s^\tau + L_T^H$, where $H_0 = \mathbb{E}[H1_{\{\tau>T\}}|\mathbb{G}_0]$, $\xi \in \Xi$, and $(L_t^H)_{0\le t\le T}$, $L_0^H = 0$, is a $\mathbb{G}$-square integrable martingale strongly orthogonal to $(\int_0^t \xi_s' dY_s^\tau)_{0\le t\le T}$ for all $(\xi_t')_{0\le t\le T}$ in Ξ which means that $(L_t^H \int_0^t \xi_s' dY_s^\tau)_{0\le t\le T}$ is a $\mathbb{G}$-martingale. Note that if $(\xi_t')_{0\le t\le T}$ is in Ξ then $(\int_0^t \xi_s' dY_s^\tau)_{0\le t\le T}$ is a $\mathbb{G}$-square integrable martingale.

However, for semimartingales a risk-minimizing optimal L^2-strategy in the above sense does not always exist, see [37] for a counterexample. In this case, risk minimization should be altered to LRM which is based on the idea that a strategy is optimal if its changes over a small interval of time should cause an increase of risk at least asymptotically. The exact definitions is rather technical which we skip here and can be found in [37]. Instead of LRM, we focus on PLRM which is a closely related concept and easier to understand. The relationship between the two approaches will be explained shortly.

Definition 3.14. Assume that $M^Y_{\tau\wedge\cdot}$ is a $\mathbb{G}$-square integrable martingale on $[0, T]$. An L^2-strategy ϕ with $V_T(\phi) = H1_{\{\tau>T\}}$, $\mathbb{P}$-a.s., is called PLRM if it is mean-self-financing and its cost process $(C_t(\phi))_{0\le t\le T}$ is strongly orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le T}$.

The advantage of PLRM is that it is equivalent to finding the FS decomposition. More precisely, assuming that $M^Y_{\tau\wedge\cdot}$ is a $\mathbb{G}$-square integrable martingale on $[0,T]$, by Proposition 3.4 of [37], the contingent claim $H1_{\{\tau>T\}}$ admits a PLRM hedging strategy ϕ if and only if $H1_{\{\tau>T\}}$ can be written as

$$H1_{\{\tau>T\}} = H_0 + \int_0^T \xi_s \, dY^\tau_s + L^H_T, \tag{3.14}$$

where the equality holds $\mathbb{P}$-a.s., $H_0 \in L^2(\Omega,\mathbb{G},\mathbb{P})$ is the initial cost to start the hedging process, the process ξ is $\mathbb{G}$-predictable and belongs to Ξ, and $(L^H_t)_{0\le t\le T}$, $L^H_0 = 0$, is a $\mathbb{G}$-square integrable martingale strongly orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le T}$ which means that $(L^H_t M^Y_{\tau\wedge t})_{0\le t\le T}$ is a $\mathbb{G}$-martingale. Orthogonal decomposition (3.14) is known as the FS decomposition.

In this case, the value process of the portfolio $(V_t(\phi))_{0\le t\le T}$ associated with the strategy ϕ is equal to

$$V_t(\phi) = H_0 + \int_0^t \xi_s \, dY^\tau_s + L^H_t, \quad 0 \le t \le T,$$

the number of shares to be invested in the risky and the risk-free asset are respectively equal to ξ and $\eta_t = V_t(\phi) - \xi_t Y^\tau_t$, $0 \le t \le T$, and finally the cost process is given by $C_t(\phi) = H_0 + L^H_t$, $0 \le t \le T$.

We keep in mind that the FS decomposition leads to just PLRM hedging strategies and not necessarily to LRM strategies. In what follows, we discuss conditions under which the two methods are equivalent, i.e. they lead to the same hedging strategies.

More precisely, first one needs to check that the so called structure condition (SC) is satisfied. The SC condition is satisfied if there exists a $\mathbb{G}$-predictable process $\zeta = (\zeta_t)_{0\le t\le T}$ such that $\Lambda^Y_t = \int_0^t \zeta_s \, d\langle M^Y_{\tau\wedge\cdot}\rangle^{\mathbb{G}}_s$, $0 \le t \le T$, and then to check the $\mathbb{P}$-a.s. finiteness of the process $\hat{K} = (\hat{K}_t)_{0\le t\le T}$ (called the mean-variance trade-off process) defined by $\hat{K}_t = \int_0^t \zeta^2_s \, d\langle M^Y_{\tau\wedge\cdot}\rangle^{\mathbb{G}}_s$, $0 \le t \le T$.

By Theorem 3.3 of [37], LRM and PLRM hedging strategies are the same if the following conditions are met: the SC condition is satisfied, $(M^Y_{\tau\wedge t})_{0\le t\le T}$ is a $\mathbb{G}$-square integrable martingale, $(\langle M^Y_{\tau\wedge\cdot}\rangle^{\mathbb{G}}_t)_{0\le t\le T}$ is $\mathbb{P}$-a.s., strictly increasing, and $\mathbb{E}[\hat{K}_T] < \infty$. Note that the equivalence between the two strategies (PLRM and LRM) holds in the one-dimensional case.

Now, we can discuss the connection between LRM, PLRM, and the FS decomposition. We just explained the circumstances under which LRM hedging strategies are the same as the PLRM ones, and earlier, we discussed that the latter exists if and only if the FS decomposition (3.14) exists.

The following theorem completes the above discussions by determining the LRM, PLRM strategies, the FS decomposition, and all the other components of the hedging strategy, i.e. the processes ξ, η, $V(\phi)$, and L^H in $\mathbb{G}$ that satisfy (3.14).

Theorem 3.15. *Suppose that the same assumptions as in Theorem 3.5 (resp. Theorem 3.8) hold. We further assume that $\gamma \neq 0$, $f(t, Y_t, \underline{Y}_t) \in \mathcal{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $0 \le t \le T$, $\mathbb{E}[|Y_1|^2] < \infty$, and $(\theta_{t-})_{0\le t\le T}$ (obtained from Theorem 3.5 (resp. Theorem 3.8)) belongs to Ξ in Definition 3.9. Then the LRM hedging strategy is the same as the PLRM hedging strategy $\phi_t = (\xi_t, \eta_t)$, $0 \le t \le T$, and the latter is determined as follows: $\xi_t = \theta_{t-}$ is the number of shares invested in the risky asset (with its values modeled by $(Y_t)_{t\ge 0}$), the value of the portfolio ϕ is given by*

$$V_t(\phi) = f(0, Y_0, Y_0) + \int_0^t \theta_{s-} 1_{\{s<T\}} dY^\tau_s + L_t, \; 0 \le t \le T,$$

function f is determined by Theorem 3.5 (resp. Theorem 3.8) and Hypothesis-$\mathcal{P}$, the process $(L_t)_{0\le t\le T}$

is a $\mathbb{G}$*-square integrable martingale given by*

$$L_t = f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}} - f(0, Y_0, Y_0) - \int_0^t \theta_{s-}1_{\{s<T\}}dY_s^\tau,\ 0 \le t \le T, \tag{3.15}$$

$(L_t)_{0\le t\le T}$ *is strongly orthogonal to the martingale part of* $(Y_t^\tau)_{0\le t\le T}$ *in* $\mathbb{G}$*, i.e.* $(L_t M_t^Y)_{0\le t\le T}$ *is a* $\mathbb{G}$*-martingale,*

$$\eta_t = V_t(\phi) - \theta_t Y_t^\tau,$$

and the cost process $(C_t)_{0\le t\le T}$ *is equal to*

$$C_t = f(0, Y_0, Y_0) + L_t.$$

Furthermore, the FS decomposition is equal to

$$H1_{\{\tau>T\}} = f(0, Y_0, Y_0) + \int_0^T \theta_{s-}\, dY_s^\tau + L_T,\ \mathbb{P}\text{-a.s.}.$$

Proof. We provide the proof under the assumptions of Theorem 3.5 as the argument under the assumptions of Theorem 3.8 is similar. In our model $\zeta = \frac{\alpha}{\gamma Y_-}$ and $\hat{K}_t = \frac{\alpha^2}{\gamma}t$, $t \ge 0$, which are well-defined and finite since $\gamma \ne 0$. Therefore, the SC condition is satisfied. By Lemma B.7, $M^Y_{\tau\wedge T\wedge\cdot}$ is a $\mathbb{G}$-square integrable martingale (since $\mathbb{E}[|Y_1|^2] < \infty$), $\langle M^Y\rangle^{\mathbb{G}}$ is $\mathbb{P}$-a.s., strictly increasing on $[0, T]$, $\mathbb{E}[\hat{K}_T] < \infty$, and so by Theorem 3.3 of [37], LRM and PLRM hedging strategies are the same. Furthermore, by Proposition 3.4 of [37], the latter is equivalent to FS decomposition (3.14). Hence, we need to determine this decomposition.

From part b of Theorem 3.5 we have

$$H1_{\{\tau>T\}} = f(0, Y_0, Y_0) + \int_0^T \theta_{s-}1_{\{s<T\}}dY_s^\tau + L_T,\ \mathbb{P}\text{-a.s.}, \tag{3.16}$$

where $(L_t)_{0\le t\le T}$, given by part a of Theorem 3.5, is a $\mathbb{G}$-local martingale orthogonal to the local martingale part of $(Y_t^\tau)_{0\le t\le T}$ (i.e. $(M^Y_{\tau\wedge t})_{0\le t\le T}$) in $\mathbb{G}$. Since X_0 (and hence Y_0) is deterministic, $f(0, Y_0, Y_0)$ belongs to $L^2(\Omega, \mathbb{G}, \mathbb{P})$. Because $(\theta_{t-})_{0\le t\le T} \in \Xi$ and $f(t, Y_t, \underline{Y}_t)$ is square integrable for all $0 \le t \le T$, then it is not difficult to show that $(L_t)_{0\le t\le T}$ is a $\mathbb{G}$-square integrable martingale. So in order to prove that Equation (3.16) is the FS decomposition, it only remains to prove that $(L_t)_{0\le t\le T}$ is strongly orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le T}$ in $\mathbb{G}$.

Since $(L_t)_{0\le t\le T}$ is orthogonal to $(M^Y_{\tau\wedge t})_{0\le t\le T}$, $\langle L, M^Y_{\tau\wedge\cdot}\rangle_t^{\mathbb{G}} = 0$, $0 \le t \le T$, then by Theorem I.4.2 of [20], $(LM^Y_{\tau\wedge t})_{0\le t\le T}$ is a uniformly integrable $\mathbb{G}$-martingale because both $(L_t)_{0\le t\le T}$ and $(M^Y_{\tau\wedge t})_{0\le t\le T}$ are $\mathbb{G}$-square integrable martingales. Hence (3.16) is the FS decomposition. Finally, for $0 \le t \le T$, $V_t(\phi)$, η_t, and C_t are determined directly from Proposition 3.4 of [37]. □

Remark 3.16. The closed form (3.15) is helpful in simulating the sample paths of $(L_t)_{0\le t\le T}$ and hence the cost process $(C_t)_{0\le t\le T}$.

In the next section, we apply Theorem 3.5 to obtain semi-closed form solutions for hedging strategies in our framework by specifying g explicitly.

4 Diffusion Models under Exogenous and Endogenous Default Times

In this section, we consider underlying risky asset values processes with continuous sample paths, i.e. $Y_t = e^{X_0+\mu t+\sigma B_t}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $t \ge 0$. Under the four cases, no default present, an endogenous default present and the exogenous one absent, an endogenous default absent and the exogenous one present, both types of default present, we obtain the FS decomposition with the understanding that under the hypotheses of Theorem 3.15, this orthogonal decomposition leads to LRM hedging strategies. Under the Black-Scholes framework, we also show that Delta and LRM hedging strategies coincide when we are in a default-free market.

We start with the main result of this section, which provides the optimal LRM hedging strategies.

Proposition 4.1. *Let the underlying process Y follow $Y_t = e^{X_0+\mu t+\sigma B_t}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $t \geq 0$. Suppose that Assumption 2.2 holds in which either $Z_t < 1$, for all $t \geq 0$, or g is identically zero, i.e. $g(t, x, y) = 0$ for all $(t, x, y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$, and there is a continuous function $f : [0, T) \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ such that for all $(t, x, y) \in [0, T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, $x \geq y$, f satisfies the following PDE*

$$\frac{\partial f}{\partial t}(t, x, y) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t, x, y) - f(t, x, y)g(t, x, y) = 0, \tag{4.1}$$

together with the conditions $f(t_n, x_n, y_n) \to F(x, y)$ point-wise as $(t_n, x_n, y_n) \to (T, x, y)$, $x_n \geq y_n \geq 0$, in Euclidean norm for all $(x, y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$, and $\dfrac{\partial f}{\partial y}(t, y, y) = 0$, for all $(t, y) \in [0, T) \times \mathbb{R}^{>0}$.

Also assume that f is extended to $[0, T] \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ according to Remark 3.4, and $f(\cdot \wedge T, Y_{\cdot\wedge T}, \underline{Y}_{\cdot\wedge T})$ is locally integrable. Then, we obtain

$$F(Y_T, \underline{Y}_T)1_{\{\tau>T\}} = f(0, Y_0, Y_0) + \int_0^T \theta_s 1_{\{s\leq\tau\}}1_{\{s<T\}}dY_s + L_T, \tag{4.2}$$

where for all $0 \leq t < T$, θ_t is given by

$$\theta_t = \frac{\partial f(t, Y_t, \underline{Y}_t)}{\partial x}.$$

Moreover, the process $(L_t)_{0\leq t\leq T}$ is a local martingale orthogonal to the local martingale part of $(Y^\tau_t)_{0\leq t\leq T}$ i.e. $(M^Y_{\tau\wedge t})_{0\leq t\leq T}$.

Furthermore, suppose that $f(t, Y_t, \underline{Y}_t)$ is in $\mathcal{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $0 \leq t \leq T$, and $(\theta_{t-})_{0\leq t\leq T}$ belongs to Ξ (see Definition 3.9). Then the FS decomposition of the claim $F(Y_T, \underline{Y}_T)1_{\{\tau>T\}}$ is given by (4.2).

Proof. Since there are no jumps in Y, all the integral terms related to the Lévy measure ν disappear, and from Lemma 3.2, we have that $\alpha = \beta + \frac{\sigma^2}{2}$ and $\gamma = \sigma^2$. Then PIDE (3.4) reduces to PDE (4.1). Since Y is a continuous process here, $f(\cdot \wedge T, Y_{\cdot\wedge T}, \underline{Y}_{\cdot\wedge T})$ is locally integrable, and Equation (4.2) follows from either Theorem 3.5 or Theorem 3.8 depending on whether or not g is identically zero. Finally, if $f(t, Y_t, \underline{Y}_t)$ is in $\mathcal{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $0 \leq t \leq T$, and $(\theta_{t-})_{0\leq t\leq T}$ belongs to Ξ (see Definition 3.9), Theorem 3.15 shows that (4.2) is the FS decomposition.

□

Remark 4.2. Under the assumptions of Proposition 4.1, by writing equation $f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}}$, $t \geq 0$, as $f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}} = -f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}} + \int_0^t 1_{\{s\leq\tau\}}\, df(s, Y_s, \underline{Y}_s) + f(0, Y_0, Y_0)$, applying Itô's formula to the process $(f(t, Y_t, \underline{Y}_t))_{t\geq 0}$, and simplifying the expression afterward, the following equivalent form of L can be obtained from Equation (3.15):

$$\begin{aligned} L_t &= -f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}} + \int_0^{t\wedge\tau} \left(\frac{\partial f}{\partial s}(s, Y_s, \underline{Y}_s) + \frac{\sigma^2}{2}Y_s^2\frac{\partial^2 f}{\partial x^2}(s, Y_s, \underline{Y}_s)\right) 1_{\{s<T\}}\, ds \\ &= -f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}} + \int_0^{t\wedge\tau} f(s, Y_s, \underline{Y}_s)g(s, Y_s, \underline{Y}_s)\, ds, \ 0 \leq t \leq T. \end{aligned} \tag{4.3}$$

From the last equation, the cost process $(C_t)_{0\leq t\leq T}$ can be written as

$$C_t = f(0, Y_0, Y_0) + \int_0^{t\wedge\tau} f(s, Y_s, \underline{Y}_s)g(s, Y_s, \underline{Y}_s)\, ds - f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}}. \tag{4.4}$$

Equation (4.4) provides a nice interpretation of the cost process. The initial cost to begin the hedging strategy is given by $f(0, Y_0, Y_0)$. Afterwards, at each instant of time $0 \leq s \leq \tau$, $g(s, Y_s, \underline{Y}_s)$ (the default rate) proportion of the value process, i.e. $f(s, Y_s, \underline{Y}_s)g(s, Y_s, \underline{Y}_s)$, is allocated for the hedging cost, and the accumulation of these proportions are given by the integral term. If default occurs before the maturity of the contract, the hedging process in our model ends, and there is no longer any requirement for allocating further capital. This releases the whole value process at the time of default which is represented by the last term of Equation (4.4).

Now suppose that we are in a default-free market, then g is identically zero (and so $\mathbb{F} = \mathbb{G}$), $\tau = \infty$, and from (4.3), we have $L_t = 0$ for all $0 \leq t \leq T$. As a result, the cost process $(C_t)_{0\leq t\leq T}$ is equal to the constant $f(0, Y_0, Y_0)$. Furthermore, the number of shares invested in the risky asset, is given by $\theta_t = \frac{\partial f(t,Y_t,\underline{Y}_t)}{\partial x}$. Therefore, in this case, the market is complete, and following Proposition 4.1, Equation (4.2) leads to a perfect replication of the claim $F(Y_T, \underline{Y}_T)$, and the LRM hedging strategy coincides with the Delta-hedging one.

Under the assumptions of Proposition 4.1 and along with Remark 4.2, we can discuss four cases:

(i) We start with the simplest case in which we want to hedge a claim H that only depends on Y, for instance $H = \max(Y_T - K, 0)$, $K > 0$. This claim is default free, but it can be still analyzed using the previous proposition by letting $g(t, x, y) = 0$, $F(x, y) = \max(x - K, 0)$, for all $t \geq 0$, $x \geq 0$, $y \geq 0$. Since $g \equiv 0$, then $\tau = \infty$, $1_{\{\tau>t\}} = 1$, for all $t \geq 0$, and so $\mathbb{F} = \mathbb{G}$. We are in fact in the setup of a default-free Black-Scholes model, the three dimensional PDE reduces to two dimensional, and the hedging strategy is obtained through solving the following PDE

$$\frac{\partial f}{\partial t}(t, x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t, x) = 0,\ 0 \leq t < T,\ x > 0,$$

together with the boundary conditions $f(t, x) \to \max(x - K, 0)$ as $t \to T$, for all $x \geq 0$. Note that since $\mathbb{F} = \mathbb{G}$, by Remark 4.2, the optimal hedging strategy is the same as the Delta-hedging one in the Black-Scholes model.

(ii) Next, we consider a structural credit risk model in which the default time $\mathcal{T}$ is defined by $\mathcal{T} = \inf\{t \geq 0; Y_t < b\}$ where $0 < b < Y_0$ is a pre-specified constant barrier. Note that $\mathcal{T}$ is a predictable time and does not admit an intensity. Suppose that we want to hedge the claim $\max(Y_T - K, 0)1_{\{\mathcal{T}>T\}}$, $K > 0$. We can use Proposition 4.1 for this case by letting $g \equiv 0$ (since there is no exogenous default) and $F(x, y) = \max(x - K, 0)1_{\{y\geq b\}}$. Then we can easily observe that $F(Y_T, \underline{Y}_T) = \max(Y_T - K, 0)1_{\{\mathcal{T}>T\}}$, and the hedging strategy is determined through solving the following PDE:

$$\frac{\partial f}{\partial t}(t, x, y) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t, x, y) = 0,\ 0 \leq t < T,\ x \geq y,\ y > 0,$$

together with the boundary conditions $f(t, x, y) \to F(x, y) = \max(x - K, 0)1_{\{y\geq b\}}$ as $t \to T$, for all $(x, y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$, and $\frac{\partial f}{\partial y}(t, y, y) = 0$ for all $0 \leq t < T$, $y > 0$. Similar to the previous case, due to the absence of an external default event, $\mathbb{F} = \mathbb{G}$, and therefore the optimal LRM strategy coincides with the Delta-hedging one.

(iii) We can study a reduced form credit risk model in which the default time τ follows Assumption 2.2 with an intensity $g(t, Y_t)$, $t \geq 0$. Assume that $Z_t < 1$ for all $t \geq 0$. Note that τ is a totally inaccessible stopping time. Suppose that we want to hedge the claim $\max(Y_T - K, 0)1_{\{\tau > T\}}$, $K > 0$. We can use Proposition 4.1 for this case by letting $F(x, y) = \max(x - K, 0)$. Then the three dimensional PDE reduces to two dimensions, and the hedging strategy is determined through solving the following PDE:

$$\frac{\partial f}{\partial t}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x) - f(t,x)g(t,x) = 0,\ 0 \leq t < T,\ x > 0,$$

together with the boundary condition $f(t, x) \to \max(x - K, 0)$, as $t \to T$, for all $x \geq 0$.

(iv) Finally, let us consider the most interesting case in which a claim is subject to both internal and exogenous default times (a double default model). This example unifies and incorporates the characteristics of both structural and reduced form models. More specifically, let say that the payoff $\max(Y_T - K, 0)$ is to be paid if $\mathcal{T} > T$ and $\tau > T$, where $\mathcal{T} = \inf\{t \geq 0; Y_t < b\}$ (for a fixed known barrier $0 < b < Y_0$) and τ satisfies Assumption 2.2 where $Z_t < 1$ for all $t \geq 0$, in other words, we want to hedge the defaultable claim $\max(Y_T - K, 0)1_{\{\mathcal{T} > T\}}1_{\{\tau > T\}}$.

Note that $\mathcal{T}$ is a $\mathbb{G}$-predictable structural default time while τ is a $\mathbb{G}$-totally inaccessible stopping time as in reduced form modeling. We can use Proposition 4.1 for this case by letting $F(x, y) = \max(x - K, 0)1_{\{y \geq b\}}$. Then we can easily observe that $F(Y_T, \underline{Y}_T) = \max(Y_T - K, 0)1_{\{\mathcal{T} > T\}}$, the claim to be hedged is equal to $F(Y_T, \underline{Y}_T)1_{\{\tau > T\}}$, and the hedging strategy is determined through solving the following PDE:

$$\frac{\partial f}{\partial t}(t,x,y) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x,y) - f(t,x,y)g(t,x,y) = 0,\ 0 \leq t < T,\ x \geq y,\ y > 0,$$

together with the boundary conditions $f(t, x, y) \to F(x, y) = \max(x - K, 0)1_{\{y \geq b\}}$ as $t \to T$, for all $(x, y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$, and $\frac{\partial f}{\partial y}(t, y, y) = 0$, for all $(t, x, y) \in [0, T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$.

For numerical implementation purposes, it might be easier to use the change of variable $h(t, x, y) = f(t, e^{x-y}, e^{-y})$. Then the PDE changes to

$$\frac{\partial h}{\partial t}(t,x,y) + \frac{\sigma^2}{2}(\frac{\partial^2 h}{\partial x^2}(t,x,y) - \frac{\partial h}{\partial x}(t,x,y)) - h(t,x,y)g(t,e^{x-y},e^{-y}) = 0, \tag{4.5}$$

where $(t, x, y) \in [0, T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$ with the conditions $\frac{\partial h}{\partial x}(t, 0, y) + \frac{\partial h}{\partial y}(t, 0, y) = 0$ for all $(t, y) \in [0, T) \times \mathbb{R}^{>0}$, and $h(t, x, y) \to \max(e^{x-y} - K, 0)1_{\{e^{-y} \geq b\}}$ as $t \to T$, for all $(x, y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$.

5 Jump-Diffusion Models and Running Infimum Process

In this section, we consider certain defaultable models, in which the underlying risky asset value process admits jumps. First, we consider a structural credit risk modeling.

Proposition 5.1. *Assume that Assumption 3.1 is in force. Suppose that $f : [0, T) \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ satisfies integrability conditions (2.6) and (2.7) for all $0 \leq t \leq T$, $x > 0$, $y > 0$, $f(\cdot \wedge T, Y_{\cdot \wedge T}, \underline{Y}_{\cdot \wedge T})$ is locally integrable, and f is the solution of the following PIDE:*

$$\gamma x \mathcal{L} f(t,x,y) = \left(\mathcal{L}P(t,x,y) - x\mathcal{L}f(t,x,y) - f(t,x,y)x\alpha\right)\alpha, \quad t \in [0,T),\ x \geq y,\ y > 0, \tag{5.1}$$

where α and γ are given in Lemma 3.2, the operator $\mathcal{L}$ is defined in Lemma 2.5, $P(t,x,y) = xf(t,x,y)$, and the following conditions are satisfied:

$$f(t_n, x_n, y_n) \to F(x,y) \text{ point-wise as } (t_n, x_n, y_n) \to (T, x, y),\ x_n \geq y_n \geq 0, \text{ in Euclidean norm,}$$

for all $(x,y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$, and

$$\frac{\partial f}{\partial y}(t,y,y) = 0, \text{ for all } 0 \leq t < T,\ y > 0.$$

Also assume that f is extended to $[0,T] \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ according to Remark 3.4. Then, we obtain

$$F(Y_T, \underline{Y}_T) = f(0, Y_0, Y_0) + \int_0^T \theta_{s-} 1_{\{s<T\}} dY_s + L_T, \tag{5.2}$$

$\theta_t = D(t, Y_t, \underline{Y}_t)$, where for all $(t,x,y) \in [0,T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, $D(t,x,y)$ is given by

$$\begin{aligned} &\frac{\sigma^2 \frac{\partial}{\partial x} f(t,x,y)}{\gamma} + \frac{1}{2} \frac{\left(\sigma^2 - 2\alpha + 2\beta\right) f(t,x,y)}{\gamma x} \\ &\qquad - \frac{\int_{-\ln\left(\frac{x}{y}\right)}^{\infty} \left(f(t,x,y)\, z - f(t, x\mathrm{e}^z, y)(\mathrm{e}^z - 1)\right) \nu(dz)}{\gamma x} \\ &\qquad - \frac{\int_{-\infty}^{-\ln\left(\frac{x}{y}\right)} \left(f(t,x,y)\, z - f(t, x\mathrm{e}^z, x\mathrm{e}^z)(e^z - 1)\right) \nu(dz)}{\gamma x}, \end{aligned} \tag{5.3}$$

and the process $(L_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale orthogonal to the martingale part of $(Y_t)_{t\geq 0}$ in $\mathbb{F}$.

Furthermore, suppose that $f(t, Y_t, \underline{Y}_t) \in \mathcal{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $0 \leq t \leq T$, $\mathbb{E}[|Y_1|^2] < \infty$, and $(\theta_{t-})_{0\leq t\leq T}$ belongs to Ξ (see Definition 3.9). Then the FS decomposition of the claim $F(Y_T, \underline{Y}_T) 1_{\{\tau > T\}}$ is given by (5.2).

Proof. Since the claim $F(Y_T, \underline{Y}_T)$ is default-free, we can assume that $g(t,x,y) = 0$, for all $(t,x,y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ which means that $\tau = \infty$ and so $1_{\{\tau > t\}} = 1$ for all $t \geq 0$. Also, the two operators $\mathcal{A}$ and $\mathcal{L}$ coincide, and so f satisfies the PIDE (3.4) . Then the results are direct applications of Theorem 3.8 and Theorem 3.15. □

Example 5.2. Suppose that we are in the setup of the previous proposition, and let $\zeta = \inf\{t : Y_t < b\}$ where $0 < b < Y_0$ is a fixed constant. Note that ζ is an $\mathbb{F}$-stopping time, but it is not necessarily an $\mathbb{F}$-predictable or an $\mathbb{F}$-totally inaccessible stopping time. Suppose that for example we want to find LRM hedging strategies of the claim $\max(Y_T - K, 0) 1_{\{\zeta > T\}}$, $K > 0$. Since $\{\zeta > T\} = \{\underline{Y} \geq b\}$, these hedging strategies can be found by using the previous proposition and setting $F(x,y) = \max(x - K, 0) 1_{\{y \geq b\}}$.

This improves the result of [33] in two main folds, first, the underlying risky asset values process can now be an exponential Lévy process rather than a finite variation one, second the internal default time ζ does not need to be totally inaccessible, and hence it does not need to admit an intensity.

Obtaining the hedging strategies depends on solving the PIDE in (5.1). For instance, if Y is an $\mathbb{F}$-martingale, then $\alpha = 0$, and PIDE (5.1) reduces to

$$\begin{aligned} &\frac{\partial f}{\partial t}(t,x,y) + x\left(\beta + \frac{\sigma^2}{2}\right) \frac{\partial f}{\partial x}(t,x,y) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t,x,y) \\ &\qquad + \int_{-\ln\left(\frac{x}{y}\right)}^{\infty} \left(f(t, xe^z, y) - f(t,x,y) - zx \frac{\partial f}{\partial x}(t,x,y)\right) \nu(dz) \\ &\qquad + \int_{-\infty}^{-\ln\left(\frac{x}{y}\right)} \left(f(t, xe^z, xe^z) - f(t,x,y) - zx \frac{\partial f}{\partial x}(t,x,y)\right) \nu(dz) = 0, \end{aligned} \tag{5.4}$$

where $(t, x, y) \in [0, T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$ with the condition $\frac{\partial f}{\partial y}(t, y, y) = 0$, for all $0 \leq t < T$, $y > 0$, and $f(t, x, y) \to \max(x - K, 0)1_{\{y \geq b\}}$ as $t \to T$, for all $x \geq 0$ and $y \geq 0$.

We also point out that if the payoff to hedge does not depend on $\underline{Y}$, for instance $\max(Y_T - K, 0)$, then the three dimensional PIDE reduces to two dimensions, and the condition $\frac{\partial f}{\partial y}(t, y, y) = 0$ is redundant. More precisely, in this case, we have $b = 0$ and $\zeta = \infty$, and hence $1_{\{y \geq b\}} = 1$.

For numerical implementations, it might be easier to use the transformation $h(t, x, y) = f(t, e^{x-y}, e^{-y})$ for Equation (5.4) and obtain:

$$\begin{aligned}
&\frac{\partial h}{\partial t}(t,x,y) + (\beta + \frac{\sigma^2}{2})\frac{\partial h}{\partial x}(t,x,y) + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}(t,x,y) \\
&\qquad + \int_{-x}^{\infty} (h(t, z+x, y) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\nu(dz) \\
&\qquad + \int_{-\infty}^{-x} (h(t, z+x, x+z) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\, \nu(dz) = 0,
\end{aligned}$$

where $(t, x, y) \in [0, T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$ with the conditions $\frac{\partial h}{\partial x}(t, 0, y) + \frac{\partial h}{\partial y}(t, 0, y) = 0$, for all $(t, y) \in [0, T) \times \mathbb{R}^{>0}$, and $h(t, x, y) \to \max(e^{x-y} - K, 0)1_{\{e^{-y} \geq b\}}$, as $t \to T$, for all $(x, y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$.

Example 5.3. Assume that $(X_t)_{t\geq 0}$ satisfies Assumption 3.1 and $\mathbb{E}[|Y_1|^2] < \infty$. Suppose that τ is independent of $\mathbb{F}$, and it follows Assumption 2.2; f satisfies integrability conditions (2.6) and (2.7) for all $0 \leq t \leq T$, $x > 0$, $y > 0$, and it is the solution of the PIDE (3.4) . We assume that τ admits a differentiable distribution F_τ, then the default rate g depends only on t and given by $g(t) = -\frac{-F_\tau^{'}(t)}{1 - F_\tau(t)}$.

Suppose that $F(x, y) = \max(x - K, 0)1_{\{y \geq b\}}$. Then for all $0 \leq t \leq T$, we have the following decomposition

$$F(Y_T, \underline{Y}_T)1_{\{\tau > T\}} = f(0, Y_0, Y_0) + \int_0^T \theta_{s-}1_{\{s \leq \tau\}}1_{\{s < T\}}dY_s + L_T,$$

where $F(x, y)$ is the PIDE's terminal condition, $\theta_t = D(t, Y_t, \underline{Y}_t)$, $D(t, x, y)$ is given by

$$\begin{aligned}
&\frac{\sigma^2 \frac{\partial}{\partial x} f\,(t,x,y)}{\gamma} + \frac{1}{2}\,\frac{\left(\sigma^2 - 2\,\alpha + 2\,\beta\right) f\,(t,x,y)}{\gamma\, x} \\
&\qquad - \frac{\int_{-\ln(\frac{x}{y})}^{\infty} (f\,(t,x,y)\,z - f\,(t, x\mathrm{e}^{z}, y)\,(\mathrm{e}^{z} - 1))\,\nu\,(dz)}{\gamma\, x} \\
&\qquad - \frac{\int_{-\infty}^{-\ln(\frac{x}{y})} (f\,(t,x,y)\,z - f\,(t, x\mathrm{e}^{z}, x\mathrm{e}^{z})\,(\mathrm{e}^{z} - 1))\,\nu\,(dz)}{\gamma\, x},
\end{aligned}$$

where $(t, x, y) \in [0, T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, $x \geq y > 0$, with the condition $\frac{\partial f}{\partial y}(t, y, y) = 0$, for all $0 \leq t < T$, $y > 0$, and $f(t, x, y) \to \max(x - K, 0)1_{\{y \geq b\}}$ as $t \to T$, for all $x \geq 0$ and $y \geq 0$.

Moreover, the process $(L_t)_{0\leq t\leq T}$ is a $\mathbb{G}$-local martingale orthogonal to the martingale part of $(Y_t^\tau)_{0\leq t\leq T}$ i.e. $(M^Y_{\tau\wedge t})_{0\leq t\leq T}$.

The PIDE to solve in this case is more complicated, but for example if Y is an $\mathbb{F}$-martingale, then through the transformation $h(t, x, y) = f(t, e^{x-y}, e^{-y})$, PIDE (3.4) reduces to

$$\begin{aligned}
&\frac{\partial h}{\partial t}(t,x,y) + (\beta + \frac{\sigma^2}{2})\frac{\partial h}{\partial x}(t,x,y) + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}(t,x,y) - h(t,x,y)g(t) \\
&\qquad + \int_{-x}^{\infty} (h(t, z+x, y) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\nu(dz) \\
&\qquad + \int_{-\infty}^{-x} (h(t, z+x, x+z) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\, \nu(dz) = 0,
\end{aligned}$$

where $(t,x,y) \in [0,T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$ with the conditions $\frac{\partial h}{\partial x}(t,0,y) + \frac{\partial h}{\partial y}(t,0,y) = 0$, for all $(t,y) \in [0,T) \times \mathbb{R}^{>0}$, and $h(t,x,y) \to \max(e^{x-y} - K, 0)1_{\{e^{-y} \geq b\}}$ as $t \to T$, for all $(x,y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$.

6 Numerical Implementation

In this section, we use our results and briefly explain a numerical example to show that the hedging strategies are actually implementable. Note that the main purpose of this paper, was to provide a theoretical framework of the LRM approach in defaultable markets using RIPs, and a thorough numerical analysis (which could be an interesting research work) is not the main focus here. We have not provided sufficient conditions under which the PIDEs admit solutions, nevertheless, we can still use finite difference method in our numerical implementation and observe the convergence of a solution. Although we only present the results for diffusion processes, the method works for the jump-diffusion processes as well.

Example 6.1. Let the underlying risky asset values process Y follow $Y_t = e^{X_0 + \mu t + \sigma B_t}$, $t \geq 0$, $T = 1$, and $\sigma = 0.15$ (the values of X_0 and μ are not required in this example). Suppose that we want to obtain the LRM hedging strategy for the claim $\max(Y_T - 6, 0)1_{\{\min(\tau,\zeta) > T\}}$, where τ is the reduced form default time defined by

$$\mathbb{P}(\tau > t | \mathcal{F}_t^Y) = e^{-\int_0^t g(s, Y_s, \underline{Y_s})\, ds},$$

$(\mathcal{F}_t^Y)_{t \geq 0}$ is the natural filtration generated by Y, $g(s, Y_s, \underline{Y_s}) = e^{-0.2(Y_s - \underline{Y}_s)}$, and $\zeta = \inf\{t > 0; Y_t \leq 2\}$ is the structural default time. Using Proposition 4.1, in order to obtain this hedging strategy, first, we need to obtain function $f : [0,T] \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ such that for all $(t,x,y) \in [0,T) \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, $x \geq y > 0$, f satisfies the following PDE

$$\frac{\partial f}{\partial t}(t,x,y) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t,x,y) - f(t,x,y) g(t,x,y) = 0, \tag{6.1}$$

together with the conditions $f(t,x,y) \to F(x,y) = \max(x-6,0)1_{\{y \geq 2\}}$ as $t \to T$, for all $(x,y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$, and $\dfrac{\partial f}{\partial y}(t,y,y) = 0$, for all $(t,y) \in [0,T) \times \mathbb{R}^{>0}$. In order to solve this PDE, we have first used the transformation $h(t,x,y) = f(t, e^{x-y}, e^{-y})$ to convert the PDE into Equation (4.5) and its boundary condition and then applied finite difference method to obtain h. Since this is a three dimensional PDE, we can only illustrate f for fixed values of t such as $t = 0$ as in Figure 1. Note that although here the PDE is solved on the domain $x > 0$, $y > 0$, we only require f in the part of its domain where $x \geq y > 0$.

In Figure 2, we have illustrated function $(x,y) \mapsto \frac{\partial f}{\partial x}(t,x,y)|_{t=0}$ which can then determine the number of shares, θ_0, to be invested in the risky asset (modeled by Y) at time $t = 0$ using the following formula

$$\theta_t = \frac{\partial f(t, Y_t, \underline{Y}_t)}{\partial x}, \quad \text{for all } 0 \leq t < T. \tag{6.2}$$

In Figures 3 and 4, we have respectively illustrated functions f and $(x,y) \mapsto \frac{\partial f}{\partial x}(t,x,y)|_{t=0.5}$. In general, given a sample path of $(Y_t)_{0 \leq t \leq T}$, we can determine the corresponding sample path of $(\underline{Y})_{0 \leq t \leq T}$. Then for a fixed $0 \leq t \leq T$, Equation (6.2) (after estimating $(x,y) \mapsto \frac{\partial f}{\partial x}(t,x,y)$ as in Figures 2 and 4) can be used to find the number of shares to be invested in the risky asset at time t if $\tau > t$. Finally, from Theorem 3.15, all the other characteristics of the hedging strategy, including the number of shares invested in the risk-free asset, can be determined.

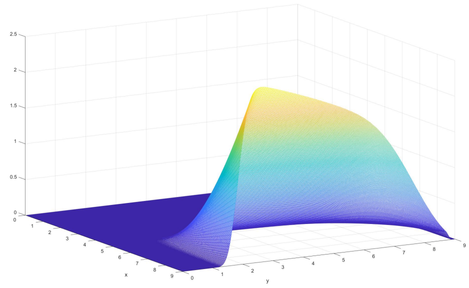

Figure 1: Function f at $t = 0$.

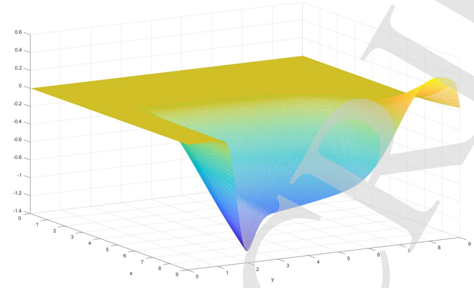

Figure 2: $\frac{\partial f}{\partial x}|_{t=0}$.

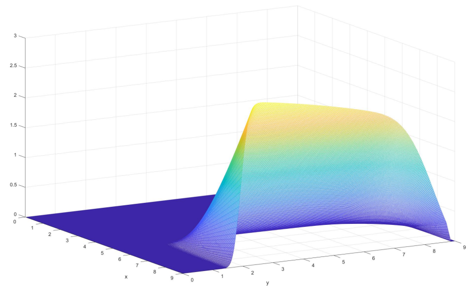

Figure 3: Function f at $t = 0.5$.

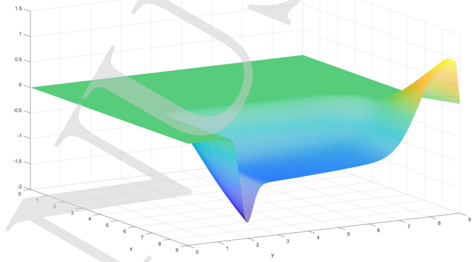

Figure 4: Function $\frac{\partial f}{\partial x}|_{t=0.5}$.

Conclusion

In this paper, we have applied the LRM method to analyze certain credit risk models by obtaining semi-closed form solutions of the LRM hedging strategies. By using the RIP process, KW decompositions, and FS decompositions, we have provided a unified framework for structural and reduced form credit risk modeling. The structural credit event can be modeled by a first-passage stopping time (which is not necessarily totally inaccessible) while the reduced form credit event is modeled through a hazard rate process. The hazard rate process could be dependent on the underlying risky asset values process and its RIP, and hence it can be path-dependent. Furthermore, the evolution of the underlying risky asset values can be modeled by an exponential Lévy process which is not necessarily a local martingale, and its trajectories may include jumps. The solutions of the hedging strategies are presented through the solution of three dimensional PDEs or PIDEs. Finally, we have provided some examples and applications in which we have shown that the strategies can be numerically implemented.

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A Martingales associated with reflected Lévy process

In this section, we present a martingale result related to the reflected Lévy process at infimum, under $\mathbb{F}$. The result and its proof are inspired by [31] that apply the Kennedy martingales (see [28]) for Lévy processes. More precisely, for $0 < S < \infty$ and a $C^{1,2,1}([0,S] \times \mathbb{R}^{\geq 0} \times \mathbb{R}$ function, under some circumstances, we obtain the canonical decomposition of $(f(t, X_t - \underline{X}_t, -\underline{X}_t))_{0\leq t\leq S}$ in $\mathbb{F}$, where $\underline{X}_t = \inf(X_s)_{0\leq s\leq t}$, $t \geq 0$. Note that $\underline{X}_0 = X_0$.

Proposition A.1. *Let $(X_t)_{t\geq 0}$ be a Lévy process with a Lévy measure ν that admits a continuous distribution and $\mathbb{E}[|X_1|] < \infty$. Let $0 < S < \infty$, and assume that $f : [0,S] \times \mathbb{R}^{\geq 0} \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2,1}$ (on its domain) that satisfies the following integrability conditions*

$$\int_{-x}^{+\infty} |(f(t, x+z, y) - f(t,x,y) - z\frac{\partial f}{\partial x}(t,x,y))|\nu(dz) < \infty,$$

and

$$\int_{-\infty}^{-x} |(f(t, 0, y-z-x) - f(t,x,y) - z\frac{\partial f}{\partial x}(t,x,y))|\nu(dz) < \infty,$$

for every $0 \leq t \leq S$, $x \geq 0$, and $y \in \mathbb{R}$. Let $\beta = \mathbb{E}[X_1 - X_0]$ and for all $0 \leq t \leq S$, $x \geq 0$, $y \in \mathbb{R}$, define the operator $\mathcal{L}^*$ by

$$\begin{aligned}\mathcal{L}^* f(t,x,y) =& \frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{-\infty}^{-x} (f(t,0,y-z-x) - f(t,x,y) - z\frac{\partial f}{\partial x}(t,x,y))\nu(dz) \\ &+ \int_{-x}^{+\infty} (f(t,x+z,y) - f(t,x,y) - z\frac{\partial f}{\partial x}(t,x,y))\nu(dz).\end{aligned}$$

Furthermore, suppose that for all $0 \leq t \leq S$ and $y \in \mathbb{R}$, the condition

$$\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0,$$

is satisfied, then the process

$$\left(f(t, X_t - \underline{X}_t, -\underline{X}_t) - f(0,0,-X_0) - \int_0^t \mathcal{L}^* f(s, X_s - \underline{X}_s, -\underline{X}_s)ds\right)_{0\leq t\leq S},$$

is an $\mathbb{F}-$local martingale.

Proof. To simplify notation, let $W_t = (t, X_t - \underline{X}_t, -\underline{X}_t)$, $t \geq 0$. By Theorem 2 of [38], we can smoothly extend f to $\mathbb{R}^{\geq 0} \times \mathbb{R} \times \mathbb{R}$. We then apply Itô's formula which for $0 \leq t \leq S$, it yields

$$\begin{aligned} f(W_t) =& f(W_0) + \int_0^t \frac{\partial f}{\partial s}(W_{s-})ds + \int_{0+}^t \frac{\partial f}{\partial x}(W_{s-})dX_s - \int_{0+}^t \frac{\partial f}{\partial x}(W_{s-})d\underline{X}_s \\ &- \int_{0+}^t \frac{\partial f}{\partial y}(W_{s-})d\underline{X}_s + \frac{1}{2}\int_{0+}^t \frac{\partial^2 f}{\partial x^2}(W_{s-})d[X-\underline{X}]_s^c \\ &+\frac{1}{2}\int_{0+}^t \frac{\partial^2 f}{\partial y^2}(W_s)d[\underline{X}]_s^c + \int_{0+}^t \frac{\partial^2 f}{\partial x \partial y}(W_{s-})d[X-\underline{X}, -\underline{X}]_s^c \\ &+\sum_{s\leq t}\Big\{f(W_s) - f(W_{s-}) + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(W_{s-})\Delta\underline{X}_s - \frac{\partial f}{\partial x}(W_{s-})\Delta X_s\Big\}, \end{aligned} \tag{A.1}$$

where $\Delta X_s = X_s - X_{s-}$, $\Delta\underline{X}_s = \underline{X}_s - \underline{X}_{s-}$ and$(X_t^c)_{t\geq 0}$, $(\underline{X}_t^c)_{t\geq 0}$ are respectively the continuous local martingale part of $(X_t)_{t\geq 0}$ and the path-by-path continuous part of $(\underline{X}_t)_{t\geq 0}$. Since $(\underline{X}_t)_{t\geq 0}$ is of finite variation then

$$[\underline{X}]_t^c = [\underline{X}^c]_t = [X-\underline{X}, -\underline{X}]_t^c = [(X-\underline{X})^c, -\underline{X}^c]_t = 0, \qquad t \geq 0.$$

On the other hand, for all $0 \leq t \leq S$,

$$\int_0^t \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(W_{s-})d\underline{X}_s = \int_0^t \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(W_{s-})d\underline{X}_s^c + \sum_{s\leq t}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(W_{s-})\Delta\underline{X}_s.$$

Using the support property, $X_{t-} = X_t = \underline{X}_{t-} = \underline{X}_t$, $t \geq 0$, on $supp(d\underline{X}^c)$, the term

$$\int_0^t \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(W_{s-})d\underline{X}_s^c = \int_0^t \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(s, 0, -\underline{X}_s)d(\underline{X})_s^c, \quad 0 \leq t \leq S,$$

is eliminated using the condition $\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0$. Based on this and since $[X]_t^c = \sigma^2$, $t \geq 0$, Equation (A.1) can be rewritten as follows

$$\begin{aligned} f(W_t) = f(W_0) &+ \int_0^t \frac{\partial f}{\partial s}(W_{s-})ds + \frac{\sigma^2}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(W_{s-})ds + \int_0^t \frac{\partial f}{\partial x}(W_{s-})dX_s \\ &+ \sum_{s\leq t}\Big\{f(W_s) - f(W_{s-}) - \frac{\partial f}{\partial x}(W_{s-})\Delta X_s\Big\}. \end{aligned}$$

We also have that $\mathbb{E}[X_t] = X_0 + \beta t$, $t \geq 0$, where $\beta = \mathbb{E}[X_1 - X_0]$, and so the last equation can be equivalently written as

$$\begin{aligned} f(W_t) = f(W_0) &+ \int_0^t \frac{\partial f}{\partial s}(W_{s-})ds + \frac{\sigma^2}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(W_{s-})ds + \int_0^t \frac{\partial f}{\partial x}(W_{s-})d(X_s - \mathbb{E}[X_s]) \\ &+ \beta\int_0^t \frac{\partial f}{\partial x}(W_{s-})ds \\ &+ \sum_{s\leq t}\Big\{f(W_s) - f(W_{s-}) - \frac{\partial f}{\partial x}(W_{s-})\Delta X_s\Big\}, \quad t \geq 0. \end{aligned}$$

Suppose that z denotes the jumps of $(X_t)_{t\geq 0}$ at a fixed time $t \geq 0$, i.e. $z = \Delta X_t$ and $R_t = \underline{X}_t - X_t$. We note that $\underline{X}_t$, $t \geq 0$, can be written as

$$\underline{X}_t = \min\left(X_t, \underline{X}_{t-}\right), \quad \text{for all } t \geq 0. \tag{A.2}$$

Let $t \geq 0$, and we investigate the following cases

(i) If $z \leq R_{t-}$ or $X_t - X_{t-} \leq \underline{X}_{t-} - X_{t-}$ then $X_t \leq \underline{X}_{t-}$ for all $t \geq 0$, and so from Equation (A.2), we have that $\underline{X_t} = X_t$, that is $R_t = 0$.

(ii) If $z \geq R_{t-}$ or $X_t - X_{t-} \geq \underline{X}_{t-} - X_{t-}$ then $X_t \geq \underline{X}_{t-}$ and so from Equation (A.2), we have that $\underline{X}_t = \underline{X}_{t-}$. In this case, R_t, $t \geq 0$ becomes $R_t = \underline{X}_{t-} - X_t$, and if we add and subtract X_{t-} then $R_t = \underline{X}_{t-} - X_{t-} - (X_t - X_{t-}) = R_{t-} - z$.

From the previous argument and using the jump measure $N(dt, dz)$, for all $0 \leq t \leq S$, we obtain

$$\begin{aligned}
f(W_t) = f(W_0) &+ \int_0^t \frac{\partial f}{\partial s}(W_{s-})ds + \frac{\sigma^2}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(W_{s-})ds \\
&+ \int_0^t \frac{\partial f}{\partial x}(W_{s-})d(X_s - \mathbb{E}[X_s]) + \beta \int_0^t \frac{\partial f}{\partial x}(W_{s-})ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} \left(f(s, 0, -X_{s-} - z) - f(W_{s-}) - z\frac{\partial f}{\partial x}(W_{s-}) \right) 1_{\{z \leq R_{s-}\}} N(ds, dz) \\
&+ \int_0^t \int_{-\infty}^{+\infty} \left(f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(W_{s-}) - z\frac{\partial f}{\partial x}(W_{s-}) \right) 1_{\{z > R_{s-}\}} N(ds, dz).
\end{aligned}$$

Assuming that $\tilde{N}(dt, dz)$ is the compensated jump measure of $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$, from the last equation, for all $0 \leq t \leq S$, we have

$$\begin{aligned}
f(W_t) = f(W_0) &+ \int_0^t \frac{\partial f}{\partial s}(W_{s-})ds + \frac{\sigma^2}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(W_{s-})ds \\
&+ \int_0^t \frac{\partial f}{\partial x}(W_{s-})d(X_s - \mathbb{E}[X_s]) + \beta \int_0^t \frac{\partial f}{\partial x}(W_{s-})ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} (f(s, 0, -X_{s-} - z) - f(W_{s-}) - z\frac{\partial f}{\partial x}(W_{s-}) 1_{\{z \leq R_{s-}\}}) \nu(dz)ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} \left(f(s, 0, -X_{s-} - z) - f(W_s) - z\frac{\partial f}{\partial x}(W_{s-}) \right) 1_{\{z \leq R_s\}} \tilde{N}(ds, dz) \\
&+ \int_0^t \int_{-\infty}^{+\infty} \Big(f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(W_{s-}) \\
&\qquad - z\frac{\partial f}{\partial x}(W_{s-}) \Big) 1_{\{z > R_{s-}\}} \nu(dz)ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} \Big(f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(W_{s-}) \\
&\qquad - z\frac{\partial f}{\partial x}(W_{s-}) \Big) 1_{\{z > R_{s-}\}} \tilde{N}(ds, dz).
\end{aligned}$$

Since the terms $\left(\int_0^t \int_{-\infty}^{+\infty} (f(s, 0, -X_{s-} - z) - f(W_{s-}) - z\frac{\partial f}{\partial x}(W_{s-})) 1_{\{z \leq R_{s-}\}} \tilde{N}(ds, dz)\right)_{0 \leq t \leq S}$ and

$$\left(\int_0^t \int_{-\infty}^{+\infty} (f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(W_{s-}) - z\frac{\partial f}{\partial x}(W_{s-})) 1_{\{z > R_{s-}\}} \tilde{N}(ds, dz) \right)_{0 \leq t \leq S},$$

are $\mathbb{F}$-local martingales, then by the continuity of the Lévy measure we obtain that

$$\left(f(W_t) - f(W_0) - \int_0^t \mathcal{L}^* f(W_s)ds \right)_{0 \leq t \leq S},$$

is an $\mathbb{F}$-local martingale, where for all $0 \leq t \leq S$, $x \geq 0$, and $y \in \mathbb{R}$,

$$\mathcal{L}^* f(t,x,y) = \frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{-\infty}^{-x} \left(f(t,0,y-z-x) - f(t,x,y) - z\frac{\partial f}{\partial x} \right) \nu(dz)$$
$$+ \int_{-x}^{+\infty} \left(f(t,x+z,y) - f(t,x,y) - z\frac{\partial f}{\partial x} \right) \nu(dz).$$

□

B Definitions and Technical Results

For the reader's convenience, we recall certain definitions and some fundamental properties of pseudo stopping times. We start by introducing some notation. The $\mathbb{F}$-dual optional projection of the indicator process $(1_{\{\tau \le t\}})_{t\ge 0}$ is denoted by A. Also, we introduce the cádlág martingale $(b_t)_{t\ge0}$ by $b_t = \mathbb{E}[A^\tau_\infty \mid \mathcal{F}_t] = A_t + Z_t$, $t \ge 0$, where $(Z_t)_{t\ge0}$ is the Azema supermartingale (2.2).

The following theorem characterizes pseudo stopping times under the progressive filtration expansion $\mathbb{G}$, for its proof, we refer to [32].

Theorem B.1. *The following properties are equivalent:*

(i) τ *is an* $\mathbb{F}$*-pseudo stopping time that is* (2.5) *is satisfied,*

(ii) $b_t = 1$, *a.s.,* $t \ge 0$,

(iii) $A_\infty = 1$, *a.s.,*

(iv) every $\mathbb{F}$*-local martingale* M *satisfies*

$$(M_{\tau\wedge t})_{t\ge0} \text{ is a } \mathbb{G}\text{-local martingale,}$$

where $\mathbb{G}$ *is the progressive filtration expansion of* $\mathbb{F}$ *by* τ*.*

Using Theorem B.1, the following important results are proved in [14].

Proposition B.2. *Let* τ *be a finite random time such that its associated Azéma supermartingale* $(Z_t)_{t\ge0}$ *is continuous. Then* τ *avoids all the* $\mathbb{F}$*-stopping times.*

Corollary B.3. *Let* τ *be a finite random time. Then the Azéma superamrtingale* $(Z_t)_{t\ge0}$ *is continuous and decreasing if and only if* τ *is a pseudo-stopping time (i.e.* (2.5) *is satisfied) that avoids all the* $\mathbb{F}$*-stopping times.*

Next, we provide the definition of KW decomposition, see [1] for more details.

Definition B.4. Assume that processes M and N are two local martingales. Then we say that N admits a KW decomposition versus M if there is a predictable process ξ such that the process $\int \xi^2 \, d\langle M\rangle^{\mathbb{F}}$ is locally integrable and the following decomposition holds:

$$N = N_0 + \int \xi \, dM + L,$$

where L with $L_0 = 0$, is a local martingale orthogonal to M.

Remark B.5. In [1], KW decomposition is discussed under four cases. In general, its existence is not guaranteed, however, if N and M are locally square integrable martingale or when M is continuous, this decomposition exists. We refer to the aforementioned reference for further discussions.

Definition B.6. A process $\mathcal{U}$ is called locally integrable if $(\sup_{0\le s\le t}|U_s|)_{t\ge 0}$ is locally in the class of integrable processes.

Finally, the following Lemma provides a simple condition under which the local martingale part of Y in $\mathbb{G}$ becomes a square integrable martingale.

Lemma B.7. *Suppose that* $\mathbb{E}[|Y_1|^2] < \infty$*, then* $M^Y_{\tau\wedge T\wedge\cdot}$ *is a* $\mathbb{G}$*-square integrable martingale.*

Proof. We have that $M^Y_{\tau\wedge T\wedge\cdot}$ is a $\mathbb{G}$-local martingale. By Corollary II.3 of [34], $M^Y_{\tau\wedge T\wedge\cdot}$ is a $\mathbb{G}$-martingale with $\mathbb{E}[(M^Y_{\tau\wedge T\wedge t})^2] < \infty$, for all $t \ge 0$, if and only if $\mathbb{E}[[M^Y_{\tau\wedge T\wedge\cdot}, M^Y_{\tau\wedge T\wedge\cdot}]_t] = \mathbb{E}[[M^Y_{\tau\wedge T\wedge\cdot}]_t] < \infty$ for all $t \ge 0$, in which case, we also have $\mathbb{E}[(M^Y_{\tau\wedge T\wedge t})^2] = \mathbb{E}[[M^Y_{\tau\wedge T\wedge\cdot}]_t]$. By Theorem II.23 of [34], we have that

$$\mathbb{E}[[M^Y_{\tau\wedge T\wedge\cdot}]_t] = \mathbb{E}[[Y^\tau]_{T\wedge t}],\ t \ge 0. \tag{B.1}$$

We know that $[Y^\tau] - \langle Y^\tau\rangle^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale; suppose that $\{T_n\}_{n\ge 1}$ is a localizing sequence for this local martingale. By using optional sampling theorem, $[Y^\tau]_{T_n\wedge T\wedge\cdot} - \langle Y^\tau\rangle^{\mathbb{G}}_{T_n\wedge T\wedge\cdot}$ is a $\mathbb{G}$-martingale. Therefore, we obtain

$$\begin{aligned}\mathbb{E}[[Y^\tau]_{T\wedge t}] &= \mathbb{E}[\lim_{n\to\infty}[Y^\tau]_{T_n\wedge T\wedge t}] = \lim_{n\to\infty}\mathbb{E}[[Y^\tau]_{T_n\wedge T\wedge t}] = \lim_{n\to\infty}\mathbb{E}[\langle Y^\tau\rangle^{\mathbb{G}}_{T_n\wedge T\wedge t}] \\ &= \mathbb{E}[\lim_{n\to\infty}\langle Y^\tau\rangle^{\mathbb{G}}_{T_n\wedge T\wedge t}] = \mathbb{E}[\langle Y^\tau\rangle^{\mathbb{G}}_{T\wedge t}],\end{aligned} \tag{B.2}$$

where we have used Lebesgue's dominated convergence theorem and the fact that $T_n \to \infty$ as $n \to \infty$.

On the other hand by Lemma 3.2 for $S = T$, and using Fubini's theorem, for all $t \ge 0$, we have $\mathbb{E}[\langle Y^\tau\rangle^{\mathbb{G}}_{T\wedge t}] = \mathbb{E}[\gamma\int_0^{\tau\wedge T\wedge t} Y_s^2\, ds] \le \gamma\int_0^{T\wedge t}\mathbb{E}[Y_s^2]\, ds$. Since $\mathbb{E}[|Y_1|^2] < \infty$, by using Theorem 25.17 of [36], we have that $\mathbb{E}[|Y_s|^2] < \infty$ for all $s \ge 0$ and furthermore $\mathbb{E}[Y_s^2]$ can be calculated explicitly. It turns out that $s \mapsto \mathbb{E}[Y_s^2]$ is a continuous map. Hence, $\sup_{t\ge 0}(\mathbb{E}[\langle Y^\tau\rangle^{\mathbb{G}}_{T\wedge t}]) < \infty$ and so $\sup_{t\ge 0}(\mathbb{E}[[Y^\tau]_{T\wedge t}]) < \infty$ by (B.2).

From (B.1) we have $\mathbb{E}[[M^Y_{\tau\wedge T\wedge\cdot}]] = \mathbb{E}[[Y^\tau]_{T\wedge\cdot}]$, so that $\sup_{t\ge 0}(\mathbb{E}[[M^Y_{\tau\wedge T\wedge\cdot}]_t]) < \infty$. This has two consequences, first of all, by Corollary II.3 of [34], $M^Y_{\tau\wedge T\wedge\cdot}$ is a $\mathbb{G}$-martingale and $\mathbb{E}[(M^Y_{\tau\wedge T\wedge t})^2] = \mathbb{E}[[M^Y_{\tau\wedge T\wedge\cdot}]_t]$, for all $t \ge 0$ which leads to $\sup_{t\ge 0}\mathbb{E}[(M^Y_{\tau\wedge T\wedge t})^2] < \infty$. Second, from Doob's inequality, we obtain $\mathbb{E}[(\sup_{t\ge 0}|M^Y_{\tau\wedge T\wedge t}|)^2] < \infty$, and therefore, $M^Y_{\tau\wedge T\wedge\cdot}$ is a $\mathbb{G}$-square integrable martingale. □