

Term Risk in Interest Rate Markets

Andrea Macrina^{†‡}, Obeid Mahomed[§]

[†]Department of Mathematics, University College London
London WC1E 6BT, United Kingdom

[‡]Department of Actuarial Science, University of Cape Town
Rondebosch 7701, South Africa

[§]African Institute of Financial Markets and Risk Management
University of Cape Town, Rondebosch 7701, South Africa

June 3, 2021

Abstract

Using a stylised financial system along with a systemic perspective thereof, the definition of an aggregated banking system that is default-free but vulnerable to liquidity risks is enabled. Within this setup, a consistent mathematical modelling framework for term interest rate systems is derived that enables the pricing and valuation of associated linear derivative instruments. It is then demonstrated that term rates may not be synthetically replicated, in general, which in turn enables the extraction and explanation of the genesis of term risk. These findings provide: (i) a rigorous understanding of the incomplete market paradigm that encapsulates inter-bank term rates and the risk management processes involved therein; and (ii) quantitative theoretical evidence against global interest rate reform proposals advocating for the replacement of term Libor (*London inter-bank offered rate*) reference rates with overnight rate-based alternatives.

Keywords: Term rates; inter-bank market; money market; interest rate derivatives; pricing kernel approach; liquidity risk; OIS and LIBOR; multi-curve term structures; LIBOR transition; benchmark reform.

Corresponding author: a.macrina@ucl.ac.uk

1 Introduction

A term rate is an interest rate that applies over a time period. In a fixed interest rate contract, both, the interest rate and the length of the period are typically fixed in advance. Attention to the usefulness of term interest rates has been rekindled by the transition from the *London inter-bank offered rate* (Libor) to overnight interest rate benchmarks, like the *Sterling overnight index average* (Sonia) or the *Secured overnight financing rate* (Sofr). With the termination of Libor, a global term interest rate system will be dismantled. At the same time, no suitable replacement is being pursued, since regulators believe that overnight rate benchmarks, also called risk-free rates, will reduce risk in fixed-income markets while increasing stability and transparency.

In this paper, as also in [1], [2] and [14], we investigate in detail term interest rate systems with two main aims: (i) The development of a consistent mathematical modelling framework for term interest rate systems that includes the pricing and valuation of associated linear derivative instruments. (ii) The demonstration that term rates may not be synthetically replicated, in general, thereby extracting and explaining the genesis of term risk. The second contribution is central to the understanding of why overnight interest rate benchmarks (and any instrument derived therefrom) do not offer a transfer mechanism for term risk, and in fact expose a borrower to maximum term risk when rolling over loans. This is the main reason for doubting that markets devoid of genuine term interest rate systems will be more stable and transparent. We show that the lack of *genuine term rate benchmarks*, prevents markets to transfer, value, and trade term risk. In the face of increased market incompleteness, due to the lack of term rate benchmarks, financial houses are likely to create over-the-counter products based on their in-house risk assessments and models. The consequence is a *less transparent* fixed-income market, most likely and often at the expense of funders and borrowers.

Term interest rates offer the certainty to a borrower that the interest rate on their loan does not change over the borrowing period. This certainty comes at a cost reflected in the spread between a term rate and a variable interest rate. The risk that commonly comes to mind when considering a variable interest rate is the so-called floating interest rate risk. This type of risk is due to the stochastic nature of interest rates. A term rate mitigates this kind of risk over the contractually specified time period it applies. Another way of insulating a loan against floating interest rate risk is to enter an interest rate swap contract. For example, the floating risk of an overnight interest rate on a loan lasting six months could be hedged by entering an overnight index swap (OIS). The swap applies over six months,

where the overnight interest rate is exchanged for a fixed one, that is, the so-called OIS fair rate. So, why would one care to have term interest rates when floating interest rate risk can be mitigated by swap contracts?

The key to the answer is that a term rate benchmark offers a mechanism to transfer refinancing risk to a lender. Refinancing risk—or roll-over risk, see, e.g., [1] and [2]—is the risk that when the loan period elapses, and the loan needs to be rolled over to the next period, the benchmark interest rate can no longer be accessed. In the example above, this is the risk that the borrower can no longer access the overnight rate on any of the days during the six-month loan period. In other words, the borrower is exposed daily to the risk that funds can no longer be raised to roll over their loan at the overnight benchmark rate (or the same spread to it) as the day before. Term rates, as for Libor, have been available for fixed time periods ranging from one day, one week and one month to twelve months. A six-month loan based on a six-month term rate is never rolled over within the six months (as opposed to a loan based on an overnight rate rolled over roughly 180 times), and so a borrower is not exposed to refinancing (roll-over) risk, regardless of the economic and financial environment, during the loan period.

The analysis undertaken to achieve the aims and objectives discussed thus far, first requires a careful description of the financial context and assumptions imposed thereon. This is described next followed by specific results and outcomes that support our theoretical findings, conjectures and conclusions.

The money market enables banks to source term funding from retail, corporate and public sector entities in the form of deposits¹. The inter-bank market plays a fundamental role within the banking system, enabling local banks to source funding from one another in order to facilitate financial intermediation and manage their own accounts² that are held with the respective local central or reserve bank. The settling of accounts may also be achieved via participation in the central bank's *repurchase operations*, which is a component of normal open market operations, or its *marginal lending facility*, however the latter is costly and saved for emergencies while the former requires possession of high quality collateral securities. Furthermore, these central bank facilities have very short-term maturities and are therefore generally inconsistent with longer-term financial intermediation requirements. Therefore, banks will generally access the money and inter-bank markets first and use central bank facilities as a last resort.

¹The canonical form of the deposit being a bearer financial instrument called a Negotiable Certificate of Deposit (NCD).

²These may be general reserve and capital requirement related accounts, and even accounts that are related to the local payment system.

There are two major sub-markets that constitute the inter-bank market: (i) the inter-bank cash market; and (ii) the inter-bank derivatives market. The inter-bank cash market involves term funding, or inter-bank lending, activity amongst major local banks within a given economy. In an emerging market, there are usually a small number of major banks with similar market share and credit-ratings, while developed economies usually have a large number of banking entities but major banks that constitute the inter-bank market are determined by market share and credit quality. Term funding involves Bank A borrowing local currency from Bank B over a specific term, or equivalently Bank B depositing local currency with Bank A over the same term³. Therefore, the primary market's microstructure may be fundamentally characterised by each participating bank offering deposit rates for various terms-to-maturity. This microstructure is essentially replicated in the money market, but for non-banking entities. For each bank, this term structure of offered deposit rates represents their cost of inter-bank funding, and therefore defines their *term funding curve*. An individual bank's term funding curve is not publicly observable, i.e., not accessible by the general public, but will be observable (at least partially) by inter-bank market participants. However, *inter-bank reference rates*, which are either aggregated indicative offered or transacted deposit rates, will provide the general public with a set of benchmark deposit rates for the inter-bank market as a whole, albeit limited with regard to the range of terms-to-maturity. Nonetheless, whether individual or aggregated, since each offered deposit rate applies to a distinct and unique term-to-maturity and the determination of each will involve a correspondingly unique set of non-fungible credit, funding and liquidity risk characteristics, the resultant term funding curve will constitute a *non-homogeneous* set of *term rates*.

Assuming the existence of the money and inter-bank cash markets only, as may be the case in an emerging market, this research paper considers the construction of the inter-bank linear derivatives market from first principles, i.e., via replication using inter-bank cash market instruments. The inter-bank cash market is therefore positioned as a primitive market, which is natural considering the discussion thus far. The inter-bank linear derivatives market may be fundamentally and completely characterised by *forward rate agreements* (FRAs) and *interest rate swaps* (IRSs) that reference inter-bank reference rates. IRSs are essentially portfolios of FRAs or equivalently, FRAs may be utilised to replicate IRSs completely. Therefore, the modelling of FRAs are considered to be the primary objective. In order to achieve this objective, the perspective of a linear derivatives market-maker is assumed, one who belongs to a banking entity but is a market-taker in the inter-bank cash market. Further,

³The cash flow structure of the deposit considered here will always have a capital payment at initiation and repayment at maturity, while interest rate cash flows may be structured in a bespoke manner.

in order to simplify the exposition and enhance tractability, a systemic perspective is postulated and assumed which, most importantly, precludes the need to consider idiosyncratic credit risks. The analysis within this systemic perspective yields a pricing kernel framework for the pricing and valuation of FRAs that is based on replication, which in turn provides the following results and outcomes:

- (i) modelled fair FRA rates are systematically lower than corresponding market forward rates implied from the systemic term funding curve, which may be attributed to potential funding-related liquidity costs within the postulated systemic setting;
- (ii) multiple inter-bank swap curves distinguished by the term of the underlying inter-bank reference rates, i.e., the underlying FRA or IRS instruments associated with each curve reference the same inter-bank reference rate;
- (iii) a mechanism to exchange fixed inter-bank interest rate risk across the systemic term funding and inter-bank swap curves; and
- (iv) a framework for consistent pricing and valuation of inter-bank instruments across curves, which also enables the fair early liquidation or settlement values.

For completeness, the systemic perspective is relaxed and idiosyncratic credit risk features are incorporated, and it is shown that the same results hold except that the discrepancy between fair FRA and forward rates may now also be attributed to potential default costs.

To be clear, each swap curve, mentioned in (ii), enables the exchange of floating interest rate risk associated with a specific inter-bank reference rate for equivalent fixed interest rate risk over various tenors. The mechanism from (iii) enables the exchange of fixed interest risk encoded in each swap curve and the systemic term funding curve. Further, within this emerging market context, the swap curve associated with the inter-bank reference rate with the shortest term-to-maturity, usually the overnight rate⁴, emerges as the best proxy for the *nominal risk-free zero curve*. However, this is shown to be problematic since each risk-free term rate is only accessible synthetically via: (1) a floating deposit (loan) dealt at the shortest term inter-bank reference rate that is continuously rolled over the term of the targeted risk-free rate; in combination with (2) a series of short (long) FRAs that reference the shortest term inter-bank reference rate and a combined tenor that matches the term of the targeted risk-free rate. While transaction (2) may be easily accessed via the inter-bank

⁴Practically and in reality, the overnight inter-bank reference rate has the least amount of credit risk exposure. However, one would still be exposed to funding-related liquidity costs. Therefore, this reference rate is not risk-free but the best proxy for a risk-free rate.

derivatives market, transaction (1) is subject to funding-related liquidity risks even if credit risks are negligible.

2 Axiomatic structure of interest rate system

This section provides a description of the financial system under consideration, an axiomatic construction of the interest rates therein and the modelling framework.

2.1 Stylised financial system

Figure 1 below provides a depiction of a stylised financial system within a generic economy, along with interactions between the constituent entities.

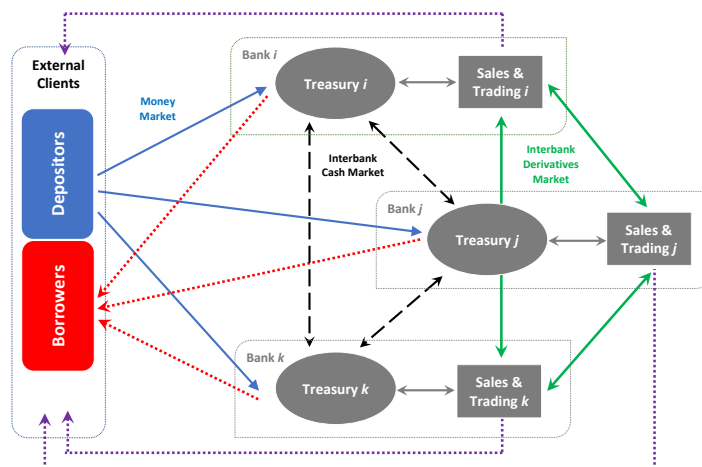


Figure 1: Stylised version of the financial system under consideration.

We draw attention to the *treasury* (TR) and *sales and trading unit* (ST) of each banking entity, highlighted in Figure 1. The TR is solely responsible for the sourcing of funding (or deposits) and lending, while the ST is responsible for engineering financial products. Interactions between TRs define the *inter-bank cash market*, which enables the transfer of surplus funds among banks. Each ST will borrow (or deposit) funds required for (generated through) creation of products, and for trading and hedging processes with their respective TR. Interactions among STs define the *inter-bank derivatives market*, which enables hedging, arbitrage and speculative strategies within the inter-bank system.

Individuals, corporations, and government entities constitute *external clients* (ECs) of the banking system. Functionally this group may be categorised further as *depositors* (EDs), *borrowers* (EBs) and *end-product users* (EPUs). In this setup, EDs and EBs interact with TRs while EPUs interact with STs. These categories are not mutually exclusive, in general. The interactions between the EDs and the TRs define the money market, an important source of term funding from outside the banking sector.

An important entity that is not explicitly shown in Figure 1 is the *central bank* (CB), which is the regulator and central entity for all banking activities and agents. Banks may engage in repurchase (repo) transactions with the CB to settle accounts that are in overdraft. The CB is therefore also considered as the lender of last resort. This facility will be utilised later on to argue against the possibility of a banking system default.

2.2 Interest rate system: an axiomatic construction

Let $A(u, u + n\delta)$ denote an arbitrary simple rate which applies over the term $[u, u + n\delta]$, where $\delta > 0$, $n \in \mathbb{N}$ and $u \in \mathbb{R}_{\geq 0}$ is a quote/mark-to-market/publishing time. In all that follows, the shorthand notation A_u^n will be used for such rates, such that an investment of 1 unit of currency at this rate will yield $(1 + n\delta A_u^n)$ at maturity time $u + n\delta$.

Axiom 2.1 (Central bank repos). *Central banks enable local banks to settle account deficits via a short-term (one week or less) repo facility, offering government bonds as collateral. The repo rate is set by the monetary policy committee (MPC) periodically in response to changes in inflation and economic growth expectations. Assuming that δ is representative of the tenor of these transactions, it is assumed that r_u^1 denotes the MPC's repo rate and a pure risk-free rate. This rate is only accessible by local banks.*

Axiom 2.2 (Government bond repos). *The secondary government bond market enables all participants to engage in repo, or buy-sell back, transactions. These are short-term (one year or less) collateralised loans, where the borrower offers government bonds as collateral. Suitably aggregating many such transactions, the effective simple rate for an $n\delta$ -term transaction is conjectured to be*

$$S_u^n := x_u^n + b_u^n + c_u^n, \quad (2.1)$$

and is also referred to as a *secured funding rate*⁵, where

⁵The Secured Overnight Financing Rate (SOFR), the benchmark rate for USD-denominated derivatives and loans is derived from the US Treasury repo market.

- $x_u^n := \frac{1}{n\delta} \left(\mathbb{E}_u \left[\prod_{i=0}^{n-1} (1 + \delta r_{u+i\delta}^1) \right] - 1 \right)$ is an $n\delta$ -term risk-free rate based on lenders' expectations for the evolution of the MPC's repo rate over this period given information available at time u , expressed here via the operator $\mathbb{E}_u[\cdot]$;
- b_u^n is a funding spread, which is term-dependent and may be less than or equal to zero when lenders have significant surplus funds available but is generally positive, and increases with term, since funding is generally limited;
- c_u^n is a spread for potential collateral-liability mismatch, which is non-negative and dependent on term, the level of x_u^n and b_u^n , government bond price/yield volatility and the initial loan-to-collateral value ratio.

Axiom 2.3 (Government bonds). *The local currency government bond market provides a benchmark for funded default-free term rates⁶. While cash flow structures and quoted yield conventions may be non-homogeneous, a set of consistent effective simple term rates*

$$G_u^n := x_u^n + f_u^n, \quad (2.2)$$

may be recovered from traded government bonds. This effective rate is conjectured to constitute a risk-free component, defined in Axiom 2.2, as well as a funding spread, f_u^n , akin to b_u^n from Axiom 2.2 but applied to the availability/supply of funding for local government debt.

Axiom 2.4 (Term funding & inter-bank lending). *The TR that forms part of the arbitrary banking entity i will quote a funded, credit-risky simple term rate*

$$R_{u,i}^n := G_u^n + d_{u,i}^n, \quad (2.3)$$

to internal (i.e., ST of bank i) and external clients (other TRs and EDs) for term deposits with a tenor of $n\delta$ at time u , where $d_{u,i}^n$ is a term-dependent debt premium offered as compensation for bearing the credit risk of bank i over the fixed term.

The pricing/quoting mechanism for inter-bank loans are also consistent with Axiom 2.4, since such a loan may be considered as one bank's TR (the borrower) quoting a deposit rate to another bank's TR (the lender), subject to the usual processes of negotiation involved in the general market-making process.

Axiom 2.5 (Funding-related liquidity). *External deposit (or money market) and inter-bank lending liquidity is not guaranteed. Assuming that some level of liquidity prevails, the TR of*

⁶Based on the assumption that a government will not default on debt issued in its local currency.

bank i may not, in general, be willing (or even able) to take deposits (borrow) at their fair rate, $R_{u,i}^n$. Ignoring profit margins, each bank will devise a demand schedule for deposits based on their specific term-based cash liquidity requirements. This may be quantified by introducing an additive, simple liquidity cost function $\ell_i(u, u + n\delta, N)$ that is both term- and nominal-dependent, with the latter denoted here by N . Entity i will then bid for deposits, with a nominal value of N , according to the rate

$$R_{u,i}^{n,N} := R_{u,i}^n + \ell_{u,i}^{n,N}, \quad (2.4)$$

where $\ell_{u,i}^{n,N}$ stand for $\ell_i(u, u + n\delta, N)$. If bank i has a high (low) demand for cash liquidity of nominal value N for a term of $n\delta$, then $\ell_{u,i}^{n,N}$ will be positive (negative).

With Axiom 2.5, it is now possible to define an individual bank's term funding curve. This is a term structure of non-homogeneous rates that is created, or market-made, by the respective bank's TR, as described and motivated in the introduction or Section 1.

Definition 2.1 (Idiosyncratic term funding curve). *Consider an arbitrary time u , funding term $n\delta$, and banking entity i . Let $\{N_{u,i,1}^n, N_{u,i,2}^n, \dots, N_{u,i,a}^n\}$ and $\{w_{u,i,1}^n, w_{u,i,2}^n, \dots, w_{u,i,a}^n\}$, where $a \in \mathbb{N}$, denote a set of nominals and corresponding weights that reflect the respective likelihoods of the TR, associated with bank i , securing funding at the respective nominal values for the $n\delta$ -term at time u , with $\sum_{j=1}^a w_{u,i,j}^n = 1$. Then a fair aggregated representation of the TR's $n\delta$ -term funding rate at time u is*

$$\begin{aligned} \bar{R}_{u,i}^n &:= R_{u,i}^n + \bar{\ell}_{u,i}^n \\ &= x_u^n + f_u^n + d_{u,i}^n + \bar{\ell}_{u,i}^n, \end{aligned} \quad (2.5)$$

where $\bar{\ell}_{u,i}^n := \sum_{j=1}^m w_{u,i,j}^n \ell_i(u, u + n\delta, N_{u,i,j}^n)$. Moreover, a fair aggregated representation of the TR's term funding curve at time u is then

$$\{\bar{R}_{u,i}^n; n \in \{1, 2, \dots, m\}\}, \quad (2.6)$$

with the $m\delta$ -term assumed to be the longest available funding tenor.

Thus far, the fair $n\delta$ -term rate, $R_{u,i}^n$, is assumed to be term-dependent only, with the liquidity cost function capturing all of the nominal value dependency. However, one should also consider f_u^n and $d_{u,i}^n$ as fair weighted average representations, constructed in the same way as $\bar{\ell}_{u,i}^n$. Being a risk-free term rate, x_u^n is not contingent on nominal value, by definition. Nonetheless, for all that follows, term dependency will be the only feature that matters.

Axiom 2.6 (Reference rates). Benchmark term rates R_u^n are constructed by specific aggregate functions⁷, which are applied to collated indicative quote or retrospective traded bank deposit or inter-bank lending data. Given the structure of such rates, as postulated in Axioms 2.4 and 2.5, the aggregated benchmark is conjectured to be

$$R_u^n := G_u^n + d_u^n + \ell_u^n, \quad (2.7)$$

where d_u^n and ℓ_u^n are now an aggregated debt premium and liquidity cost, respectively. Accordingly, R_u^n is also referred to as an inter-bank reference or market-based term rate.

2.3 Systemic perspective

Figure 1, Axiom 2.5 and Definition 2.1 demonstrates the idiosyncratic funding-related liquidity and credit risks that each TR is *directly* exposed to via the market-making process for term funding. Being an internal client to their respective TR, each ST is *indirectly* exposed to the same risks. These idiosyncratic exposures are difficult to model in a concise and consistent manner, so that one may make rigorous systemic inferences. Therefore, two simplifying assumptions and adjustments are effected to the stylised financial system depicted in Figure 1. These are shown in Figure 2 below.

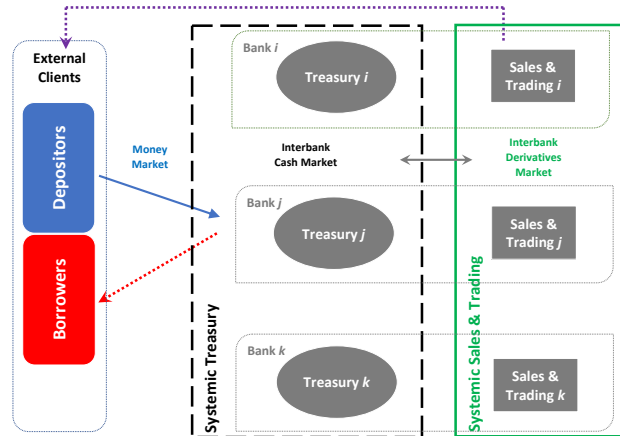


Figure 2: Stylised systemic version of the financial system under consideration.

In this setting, the banking system has a systemic representation and consists of a *systemic*

⁷For example, a trimmed median or average, which is utilised for rates such as JIBAR and LIBOR, or a volume-weighted average for reference rates such as the SAFEX overnight rate and SONIA (also trimmed).

treasury (STR) along with a systemic sales & trading unit (SST). The STR and SST are collections of the respective individual entities, and by definition and construction subsume the inter-bank cash and derivatives markets, respectively. This representation enables a systematic analysis of the risks underlying the market-making processes undertaken by the STR for term funding and the SST for general financial derivatives. Idiosyncratic credit risk only remains in loans offered by the STR to EBs, which is not the focus here. Thus, Axioms 2.4 and 2.5 may be replaced by the following single axiom.

Axiom 2.7 (Systemic term funding & liquidity). *The STR will quote to internal (SST) and external clients a funded, credit-risky and liquidity-cost-adjusted simple rate*

$$R_u^{n,N} = G_u^n + d_u^n + \ell_u^{n,N}, \quad (2.8)$$

for term deposits with a nominal value of N and a tenor $n\delta$ at time u . As before, d_u^n is a systemic term-dependent debt premium and $\ell_u^{n,N} := \ell(u, u + n\delta, N)$ is the STR's liquidity cost function. A quote (or liquidity) is not guaranteed for all nominal values and terms.

Remark 2.1 (Systemic debt premium and default paradox). *Since the STR is an aggregation of individual credit risky TRs, the systemic debt premium is also a suitable aggregation of each TR's idiosyncratic debt premiums, all of which are positive in general, indicating a chance for systemic default. However, in theory and reality, the CB will preclude the possibility of a systemic default, and hence the systemic banking entity (STR and SST) within this theoretical context. Hence the paradox, which is a natural artefact of the assumed systemic perspective.*

Axiom 2.6 still holds within this context, but now with the added feature that reference term rates are now a fair aggregated representation of the STR's term funding rates. This is by assumption and construction since the STR subsumes the inter-bank cash market. In addition, this enables a direct representation of the STR's term funding curve, which is provided in the next definition.

Definition 2.2 (Systemic term funding curve). *Analogous to Definition 2.1 in form, the fair aggregated representation of the STR's $n\delta$ -term funding rate at time u is given by the $n\delta$ -term reference rate, equation (2.7) from Axiom 2.6. The fair aggregated representation of the STR's term funding curve at time u is then*

$$\{R_u^n; n \in \{1, 2, \dots, m\}\}, \quad (2.9)$$

with the $m\delta$ -term again assumed to be the longest available funding tenor.

The systemic term funding curve is therefore a collection of non-homogenous reference rates, which, in the absence of other interest rate financial markets and information, forms the basis for fair valuation at the systemic level. The SST is assumed to be an internal client of the STR, therefore both entities exhibit the same level of credit risk. As a result, for the purpose of pricing in their market-making process, the SST need only consider: (i) the arbitrage-free dynamics of the set of reference term rates or, equivalently, the systemic term funding curve, along with (ii) the ad hoc cost of liquidity due to Axiom 2.7.

2.4 Mathematical framework for interest rate markets

The financial system depicted in Figure 2 forms the backdrop for the modelling framework, with the addition of the following assumptions:

- (i) The *money and inter-bank cash markets* are the *primitive* financial markets, with the only market-maker being the STR.
- (ii) The *inter-bank derivatives market* is the considered *derivative* financial market, with the SST and its constituents being market-makers.
- (iii) There are *no transaction costs, profit margins or taxes*.

It is assumed that this market system is *incomplete, arbitrage-free* and supported by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u \geq 0}, \mathbb{P})$ satisfying the *usual conditions*, where \mathbb{P} denotes the real-world probability measure, and where

$$\mathcal{F}_u := \mathcal{G}_u \vee \overline{\mathcal{L}}_u, \quad (2.10)$$

for $u \geq 0$. The filtration $(\mathcal{G}_u)_{u \geq 0}$ models information about all *tradable* variables, $(\overline{\mathcal{L}}_u)_{u \geq 0}$ models information about all *liquidity* variables associated with the money and inter-bank markets (i.e., items (i) and (ii)). The filtration $(\mathcal{L}_u)_{u \geq 0}$, where $\mathcal{L}_u \subset \overline{\mathcal{L}}_u$, models information about *liquidity* variables associated with the money and inter-bank cash markets only (i.e., item (i) only). Therefore, $(\mathcal{F}_u)_{u \geq 0}$ models information about all tradable variables and their liquidity characteristics.

It is assumed that the SST's starting point for modelling the systemic term funding curve or, equivalently, its set of non-homogeneous constituent rates is the specification of a corresponding set of statistically estimated stochastic discount factors (SDFs). The next definition describes one such SDF for an arbitrary $n\delta$ -term along with the calculation of estimated $n\delta$ -term rates, where $n \in \{1, 2, \dots, m\}$.

Definition 2.3 (Estimated $n\delta$ -Term SDF and Rates). *The estimated $n\delta$ -term SDF, $(\widehat{D}_u^n)_{u \geq 0}$, is assumed to be a $\{(\mathcal{G}_u), \mathbb{P}\}$ -continuous semimartingale. At any time u , it is possible to calculate an estimate for the $n\delta$ -term zero coupon bond (ZCB) price as follows*

$$\widehat{P}_{u, u+n\delta}^n := \frac{1}{\widehat{D}_u^n} \mathbb{E}^{\mathbb{P}} [\widehat{D}_{u+n\delta}^n \mid \mathcal{G}_u], \quad (2.11)$$

making use of the estimated $n\delta$ -term SDF, and

$$\widehat{R}_u^n := \frac{1}{n\delta} \left(\frac{1}{\widehat{P}_{u, u+n\delta}^n} - 1 \right), \quad (2.12)$$

is then the estimated $n\delta$ -term rate at time u , using the definition of a simple rate.

If the current time is t , then it is assumed that the parameters associated with the SDF are optimally estimated, with respect to \mathbb{P} , using the historical time series data for the $n\delta$ -term reference rate: $\{R_u^n; u \in \{t_0, t_1, \dots, t_k\}\} \subset \mathcal{G}_t$, where $\{t_0, t_1, \dots, t_k\}$ denote the set of trading days that lie within the interval $[0, t]$.

Having this setup, the objective now is to model the aforementioned interest rate markets in a systematic manner. The following scenarios are analysed for the money and inter-bank cash markets: (i) a single term rate with perfect liquidity; (ii) multiple term rates with perfect liquidity; and (iii) multiple term rates with illiquidity. Each of these market setups are considered in the three sections that follow, respectively.

3 Single term rate with perfect liquidity

The first system we consider is constituted by a money and inter-bank cash market with only a single reference term rate. This rate is assumed to be tradable, i.e., any of the entities within the systemic structure may deposit or borrow at this rate. Perfect liquidity also prevails, i.e., unlimited funding is available for all tenors via the use of this single rate. Without any loss of generality, the δ -term rate is chosen as this rate and is formally defined next.

Definition 3.1 (Perfectly liquid δ -term rate). *The δ -term rate is defined by Axiom 2.6 when $n = 1$, with the STR being the market-maker. Being tradable, R_u^1 is assumed to be \mathcal{G}_u -measurable, and it has the form*

$$R_u^1 = x_u^1 + d_u^1, \quad (3.1)$$

since perfect liquidity implies that $f_u^1 = \ell_u^1 = 0$.

Now, we assume that the SST's objective is to market-make interest rate derivatives based on the δ -term rate. With an estimated δ -term SDF, from Definition 2.3, the SST may now estimate a set of *zero coupon bond* (ZCB) prices, for various tenors but using the δ -term rate only. This procedure is described in the next definition.

Definition 3.2 (Estimated δ -term ZCB prices). *Assuming that the current time is t , then for $i, j \in \mathbb{N}_0$ with $i \leq j$, the expression*

$$\widehat{P}_{t+i\delta, t+j\delta}^1 := \frac{1}{\widehat{D}_{t+i\delta}^1} \mathbb{E}^{\mathbb{P}} \left[\widehat{D}_{t+j\delta}^1 \mid \mathcal{G}_{t+i\delta} \right], \quad (3.2)$$

is the estimated price at time $t + i\delta$ for a δ -term ZCB, with unit nominal, that matures at time $t + j\delta$, i.e., a ZCB with $(j-i)\delta$ -tenor that accrues interest via compounding $(j-i)$ fixed δ -term rates that are implied by the estimated δ -term SDF and information available at time $t + i\delta$. Shorthand notation: $\widehat{P}_{t,i,j}^1 := \widehat{P}_{t+i\delta, t+j\delta}^1$ for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $t \in \mathbb{R}_{\geq 0}$.

The SST's estimated δ -term ZCB term structure at time t is then given by $\{\widehat{P}_{t,0,j}^1; j \in \mathbb{N}_0\}$. These ZCB prices are completely model-dependent and are therefore not tradable, in general. Rather they may be utilised by the SST in the market-making process for such products. These ZCBs are equivalent to synthetic term rates with $j\delta$ -tenors, which accrue a compounded interest at a δ -term frequency. Later it will be shown how such ZCBs may be structured with forward rate agreements (FRA) or, equivalently, interest rate swaps (IRS). There is however one ZCB that is linked to the δ -term rate, and therefore tradable. This ZCB is introduced next.

Definition 3.3 (Tradable δ -term ZCB). *Assuming that the current time is t then, from Definition 3.1, the tradable δ -term rate is R_t^1 . The price at time t of a tradable δ -term ZCB, with unit nominal and δ -tenor, is given by*

$$P_{t,t+\delta}^1 := \frac{1}{1 + \delta R_t^1}. \quad (3.3)$$

Shorthand notation: $P_{t,i,1}^1 := P_{t+i\delta, t+\delta}^1$ for each $i \in \{0, 1\}$ and $t \in \mathbb{R}_{\geq 0}$.

The SST's estimated price $\widehat{P}_{t,0,1}^1$ for this ZCB will not be equal to $P_{t,0,1}^1$, in general. This discrepancy would expose the SST to potential arbitrage if their estimated model were used for pricing and valuation. Therefore, their estimated model must be adjusted to recover the price of the tradable ZCB. In the following lemma, we introduce the δ -term systemic pricing

measure⁸, denoted here by \mathbb{P}_1 , the *calibrated δ -term stochastic discount factor (SDF)*, and the *pricing kernel (PK)*.

Lemma 3.1 (Calibrated δ -term SDF and PK). *At time $t + \delta$, the \mathcal{G}_t -measurable SDF associated with the δ -term systemic pricing measure \mathbb{P}_1 is given by*

$$D_{t+\delta}^1 := \frac{1}{\Lambda_t^1 \widehat{D}_t^1} \mathbb{E}^{\mathbb{P}} \left[\Lambda_{t+\delta}^1 \widehat{D}_{t+\delta}^1 \mid \mathcal{G}_t \right] = \frac{1}{\widehat{D}_t^1} \mathbb{E}^{\mathbb{P}_1} \left[\widehat{D}_{t+\delta}^1 \mid \mathcal{G}_t \right]. \quad (3.4)$$

The $\{(\mathcal{G}_u)_{t \leq u \leq t+\delta}, \mathbb{P}\}$ -density martingale $(\Lambda_u^1)_{t \leq u \leq t+\delta}$, with $\Lambda_t^1 = 1$, induces a measure change $\mathbb{P} \rightarrow \mathbb{P}_1$ via the Radon-Nikodym derivative $\Lambda_u^1 = (d\mathbb{P}_1/d\mathbb{P})|_{\mathcal{G}_u}$. Moreover, $D_{t+\delta}^1 = P_{t,0,1}^1$. For $j \in \{0, 1\}$, the $\mathcal{G}_{t+j\delta}$ -measurable δ -term PK is thus given by

$$\pi_{t+j\delta}^1 := \Lambda_{t+j\delta}^1 D_{t+j\delta}^1, \quad (3.5)$$

where $D_t^1 = 1$, and where the PK is calibrated to the tradable δ -term ZCB.

Proof. The estimated δ -term SDF $\{\widehat{D}_{t+j\delta}^1; j \in \{0, 1\}\}$ is considered as an initial candidate for the calibrated δ -term SDF, which must be defined in discrete-time on the set $\{t, t + \delta\}$. Since $\widehat{D}_t^1 \widehat{P}_{t,0,1}^1 = \mathbb{E}^{\mathbb{P}}[\widehat{D}_{t+\delta}^1 \mid \mathcal{G}_t]$ and $\widehat{P}_{t,0,1}^1 \neq P_{t,0,1}^1$ in general, the estimated δ -term SDF and \mathbb{P} are not viable candidates for the calibrated SDF and pricing measure, respectively. Constructing and calibrating the change-of-measure $\{(\mathcal{G}_u), \mathbb{P}\}$ -martingale $(\Lambda_{t+\nu\delta}^1)_{0 \leq \nu \leq 1}$ such that equation (3.4) holds, with $D_{t+\delta}^1 = P_{t,0,1}^1$, yields the correct calibrated SDF specification. The correct and calibrated SDF model is obtained by introducing the $\{(\mathcal{G}_u), \mathbb{P}\}$ -measure-change martingale $(\Lambda_u^1)_{t \leq u \leq t+\delta}$, with $D_{t+\delta}^1 = P_{t,0,1}^1$, which leads to Eq. (3.4).

The δ -term PK specification, equation (3.5), follows trivially, from where it may be verified that

$$\frac{1}{\pi_{t+i\delta}^1} \mathbb{E}^{\mathbb{P}}[\pi_{t+\delta}^1 \mid \mathcal{G}_{t+i\delta}] = \frac{1}{D_{t+i\delta}^1} \mathbb{E}^{\mathbb{P}_1}[D_{t+\delta}^1 \mid \mathcal{G}_{t+i\delta}] = P_{t,i,1}^1,$$

for $i \in \{0, 1\}$, which concludes the proof. \square

Remark 3.1 (Single-period arbitrage-free model). *The availability of only one tradable term rate, and its associated ZCB, enables the SST to construct a single-period arbitrage-free model only, over $[t, t + \delta]$, with volatility estimated statistically since no derivative market exists.*

Remark 3.2 (Market price of systemic risk). *Under the real-world measure \mathbb{P} and with respect to the traded information filtration $(\mathcal{G}_u)_{u \geq t}$, the martingale $\{\Lambda_{t+j\delta}^1; j \in \{0, 1\}\}$ adjusts*

⁸If the δ -term rate is a perfectly liquid overnight reference rate, then d_u^1 is approximately zero, R_u^1 is risk-free for all practical purposes, and \mathbb{P}_1 is an approximation of the classical risk-neutral measure.

the real-world estimated δ -term ZCB price to the arbitrage-free tradable price, and therefore encodes the market price of δ -term interest rate systemic risk over $[t, t + \delta]$.

4 Multiple term rates with perfect liquidity

In this section, a second perfectly liquid reference term rate, the 2δ -term rate R_u^2 , is introduced. Its definition is analogous to that of the δ -term counterpart. As indicated later in Remark 4.2, the extension to a $n\delta$ -term rate system is produced by postulating R_u^n term rates as we do next for the 2δ -term rate R_u^2 . Using the estimated 2δ -term SDF, $(\widehat{D}_u^2)_{u \geq 0}$, the SST may estimate the 2δ -term ZCB-system $\{\widehat{P}_{t+2i\delta, t+2j\delta}^2; i, j \in \mathbb{N}_0, i \leq j\}$ at the current time t by directly replicating all of the results from the previous section. In what follows, we adopt the shorthand notation $\widehat{P}_{t, 2i, 2j}^2 := \widehat{P}_{t+2i\delta, t+2j\delta}^2$ for all $i, j \in \mathbb{N}_0$, where $i \leq j$ and $t \in \mathbb{R}_{\geq 0}$.

From Lemma 3.1 we obtain the $\mathcal{G}_{t+2j\delta}$ -measurable 2δ -term pricing kernel

$$\pi_{t+2j\delta}^2 := \Lambda_{t+2j\delta}^2 D_{t+2j\delta}^2, \quad (4.1)$$

where $j \in \{0, 1\}$. The $\{(\mathcal{G}_u), \mathbb{P}\}$ -martingale $(\Lambda_{t+2v\delta}^2)_{0 \leq v \leq 1}$, with $\Lambda_t^2 := 1$, enables the change-of-measure from \mathbb{P} to \mathbb{P}_2 , the 2δ -term systemic pricing measure on $\mathcal{G}_{t+2\delta}$. This PK is calibrated to

$$P_{t, t+2\delta}^2 := \frac{1}{1 + 2\delta R_t^2} = \frac{1}{\pi_t^2} \mathbb{E}^{\mathbb{P}}[\pi_{t+2\delta}^2 | \mathcal{G}_t], \quad (4.2)$$

which is the price of the tradable 2δ -term ZCB at the current time t . We bear in mind the notation $P_{t, 2i, 2}^2 := P_{t+2i\delta, t+2\delta}^2$ for each $i \in \{0, 1\}$ and $t \in \mathbb{R}_{\geq 0}$.

In general, the estimated δ - and 2δ -term SDFs are not equal, almost surely. Even if they are specified as the same model, their statistical estimation relies upon the historical time series of two non-homogeneous reference term rates. Each of these SDFs encodes different sources of floating interest rate risk. Accordingly, the estimated prices of δ - and 2δ -term ZCBs with tenor equal to 2δ will also not be equal in general, i.e.,

$$\widehat{P}_{t, 0, 2}^2 = \frac{1}{\widehat{D}_t^2} \mathbb{E}^{\mathbb{P}}[\widehat{D}_{t+2\delta}^2 | \mathcal{F}_t] \neq \frac{1}{\widehat{D}_t^1} \mathbb{E}^{\mathbb{P}}[\widehat{D}_{t+2\delta}^1 | \mathcal{F}_t] = \widehat{P}_{t, 0, 2}^1. \quad (4.3)$$

However, there are two relations between the tradable δ - and 2δ -term ZCBs that must hold to preclude arbitrage from the perspective of the SST. These relations, described in the lemmas below, allow the definition of the δ -term PK to be extended from $t + \delta$ to $t + 2\delta$.

Lemma 4.1 (Early liquidation enforced by replacement). *At time $t + \delta$, the fair early liqui-*

dation value of the tradable 2δ -term ZCB, issued at time t , is equal to

$$P_{t,1,2}^1 := \frac{1}{1 + \delta R_{t+\delta}^1} = \frac{1}{\pi_{t+\delta}^1} \mathbb{E}^{\mathbb{P}} [\pi_{t+2\delta}^1 \mid \mathcal{G}_{t+\delta}] = \frac{1}{D_{t+\delta}^1} \mathbb{E}^{\mathbb{P}^1} [D_{t+2\delta}^1 \mid \mathcal{G}_{t+\delta}], \quad (4.4)$$

which is the initial value of the tradable δ -term ZCB. Moreover, the calibrated δ -term SDE, defined in Lemma 3.1, may be specified at time $t + 2\delta$ as

$$D_{t+2\delta}^1 := \mathbb{E}^{\mathbb{P}} \left[\frac{\Lambda_{t+2\delta}^1}{\Lambda_{t+\delta}^1} \widehat{D}_{t+2\delta}^1 \mid \mathcal{G}_{t+\delta} \right] = \mathbb{E}^{\mathbb{P}^1} [\widehat{D}_{t+2\delta}^1 \mid \mathcal{G}_{t+\delta}], \quad (4.5)$$

with the definition of the $\{(\mathcal{G}_u), \mathbb{P}\}$ -martingale $(\Lambda_{t+v\delta}^1)_{0 \leq v \leq 2}$ extended to time $t + 2\delta$ with time-inhomogeneous parameters, such that $\frac{\Lambda_{t+2\delta}^1}{\Lambda_{t+\delta}^1} = \frac{d\mathbb{P}_1}{d\mathbb{P}} \mid_{\mathcal{G}_{t+2\delta}}$ and $D_{t+2\delta}^1 = D_{t+\delta}^1 P_{t,1,2}^1$.

Proof. See Appendix A.1 for the proof. \square

The previous lemma enabled the specification and calibration of the δ -term PK up to time $t + 2\delta$, from the vantage point of time $t + \delta$. The next result will provide information that will enable calibration up to the same time, but using information at the current time t .

Lemma 4.2 (Synthetic δ -term ZCB with 2δ -tenor). *At the current time t , the tradable δ - and 2δ -term ZCBs allow the SST to create a fair FRA, i.e., at zero cost, with payoff*

$$V_{t+2\delta} = \alpha N \left[\left(1 + \delta R_{t+\delta}^1 \right) - \frac{P_{t,0,1}^1}{P_{t,0,2}^2} \right], \quad (4.6)$$

at time $t + 2\delta$, where α is equal to 1 (-1) for a long (short) position and N denotes the nominal amount. This in turn enables the creation of $P_{t,t+2\delta}^1$, a tradable synthetic δ -term ZCB at time t with maturity time equal to $t + 2\delta$, with $P_{t,t+2\delta}^1 = P_{t,0,2}^2$.

Shorthand Notation: $P_{t,i,2}^1 := P_{t+i\delta,t+2\delta}^1$ for each $i \in \{0, 1, 2\}$ and $t \in \mathbb{R}_{\geq 0}$.

Proof. See Appendix A.2 for the proof. \square

With the result of Lemma 4.2 at hand, it is now possible to consider the calibration of the δ -term PK up to time $t + 2\delta$, using market information that is available at time t .

Theorem 4.1 (Initial calibration of the δ -term PK to $t + 2\delta$). *At the current time t , the tradable synthetic δ -term ZCB with 2δ -tenor may be modelled as*

$$P_{t,0,2}^1 := \mathbb{E}^{\mathbb{P}} \left[\frac{\Lambda_{t+2\delta}^1}{\Lambda_t^1} \widehat{D}_{t+2\delta}^1 \mid \mathcal{G}_t \right] = \mathbb{E}^{\mathbb{P}^1} [\widehat{D}_{t+2\delta}^1 \mid \mathcal{G}_t], \quad (4.7)$$

where the time-inhomogeneous $\{(\mathcal{G}_u), \mathbb{P}\}$ -martingale $(\Lambda_{t+v\delta}^1)_{0 \leq v \leq 2}$ enables the change-of-measure $\frac{\Lambda_{t+2\delta}^1}{\Lambda_t^1} = \frac{d\mathbb{P}_1}{d\mathbb{P}} \Big|_{\mathcal{G}_{t+2\delta}}$, and the free time-dependent parameter associated with $(\Lambda_{t+v\delta}^1)_{1 < v \leq 2}$ is calibrated such that $\mathbb{E}^{\mathbb{P}_1} [\widehat{D}_{t+2\delta}^1 \mid \mathcal{G}_t] = P_{t,0,2}^2 = P_{t,0,2}^1$.

Proof. The definitions of the calibrated δ -term SDF and PK are specified for maturity time $t + 2\delta$ with information at time $t + \delta$ in Lemma 4.1, viz., $\pi_{t+2\delta}^1 := \Lambda_{t+2\delta}^1 D_{t+2\delta}^1$. That structure is maintained here along with the observation that the no-arbitrage initial value of the tradable synthetic δ -term ZCB with tenor equal to 2δ must be

$$P_{t,0,2}^1 := \frac{1}{\pi_t^1} \mathbb{E}^{\mathbb{P}} [\pi_{t+2\delta}^1 \mid \mathcal{G}_t] = \frac{1}{D_t^1} \mathbb{E}^{\mathbb{P}_1} [D_{t+2\delta}^1 \mid \mathcal{G}_t],$$

where $D_t^1 := 1$ and $\Lambda_t^1 := 1 \Rightarrow \pi_t^1 = 1$. Substituting expression (4.5) into the right hand side of the above equation yields

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_1} [D_{t+2\delta}^1 \mid \mathcal{G}_t] &= \mathbb{E}^{\mathbb{P}_1} \left[\mathbb{E}^{\mathbb{P}_1} [\widehat{D}_{t+2\delta}^1 \mid \mathcal{G}_{t+\delta}] \Big| \mathcal{G}_t \right] \\ &= \mathbb{E}^{\mathbb{P}_1} [\widehat{D}_{t+2\delta}^1 \mid \mathcal{G}_t] \\ &= \mathbb{E}^{\mathbb{P}} \left[\frac{\Lambda_{t+2\delta}^1}{\Lambda_t^1} \widehat{D}_{t+2\delta}^1 \Big| \mathcal{G}_t \right], \end{aligned}$$

where the second line follows by the tower property of conditional expectations and the last line by definition. Lemma 4.2 advocates that $P_{t,0,2}^1$ must equal $P_{t,0,2}^2$. Since the time-dependent parameter associated with $(\Lambda_{t+v\delta}^1)_{1 < v \leq 2}$ is free to specify at time t , it is possible to calibrate this quantity such that $\mathbb{E}^{\mathbb{P}_1} [D_{t+2\delta}^1 \mid \mathcal{G}_t] = P_{t,0,2}^2$, which concludes the proof. \square

Remark 4.1 (Term-dependent market price of systemic risk). *Since the estimated δ -term SDF is used to model the tradable 2δ -term ZCB, it is conjectured that a time-inhomogeneous market price of risk structure is necessary. There are two notions of time in this framework: (i) universal calendar time defined by the variable u and a current time denoted by t ; and (ii) term and tenor times determined by natural number multiples of δ . Therefore, the process $\{\Lambda_{t+j\delta}^1; j \in \{0, 1, 2\}\}$ being time-inhomogeneous actually resolves to the framework advocating for term-dependent parameters, or a term-dependent market price of systemic risk construct.*

Remark 4.2 (Multiple term rates). *Theorem 4.1 may be iteratively repeated out to $t + j\delta$, for $j \in \{3, 4, 5, \dots, m\}$ if the corresponding reference term rates R_t^j exist, and are perfectly liquid. If one or a subset of these reference rates are not quoted, the result of the theorem still applies but there will now be a range of viable values for the missing rates for arbitrage-free calibration.*

Remark 4.3 (Term-specific SDFs but a unique term structure). *Each term has a distinct estimated SDF, which encodes floating interest rate risk, along with a unique single-period PK. The ability to replicate all tradable ZCBs via a system of FRAs leads to a single-curve interest rate term structure, and enables multi-period calibration for all PKs. Since the δ -term bears the least credit risk, its pricing measure is the closest to the classical risk-neutral measure.*

Theorem 4.1 relied on the SST's ability to create a FRA. The resultant δ -term PK therefore encodes the arbitrage-free mechanics to price and value such a product. Pricing this FRA, which is formalised in the next corollary, will reveal the fair FRA rate to be the simple forward rate that is constructed from the δ - and 2δ -term rates, which is defined next.

Definition 4.1 ($\delta \times 2\delta$ Forward rate). *At $u \in \mathbb{R}_{\geq 0}$, the $\delta \times 2\delta$ forward rate is a simple rate, denoted by $F(u; u + \delta, u + 2\delta)$, at which one may deposit (borrow) money over the future δ -term $[u + \delta, u + 2\delta]$. The net capital plus interest yield (cost) at time $u + 2\delta$ is*

$$1 + \delta F(u; u + \delta, u + 2\delta) := (1 + 2\delta R_u^2) / (1 + \delta R_u^1), \quad (4.8)$$

with $F(u; u + \delta, u + 2\delta)$ being \mathcal{G}_u -measurable. See Appendix A.3 for a construction strategy.

Shorthand Notation: In general the $j\delta \times (j + 1)\delta$ forward rate will be denoted as:

$$F_{u,i,j}^1 := F(u + i\delta; u + j\delta, u + (j + 1)\delta), \quad (4.9)$$

for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Corollary 4.1 ($\delta \times 2\delta$ FRA pricing). *The fair strike rate process for the general version of the $\delta \times 2\delta$ FRA defined in Lemma 4.2 is given by the $\mathcal{G}_{t+i\delta}$ -measurable process*

$$F_{t,i,1}^1 = \frac{1}{\delta} \left(\frac{P_{t,i,1}^1}{P_{t,i,2}^1} - 1 \right) = \frac{1}{\delta} \left(\frac{\mathbb{E}^{\mathbb{P}_1} [\widehat{D}_{t+\delta}^1 | \mathcal{G}_t]}{\mathbb{E}^{\mathbb{P}_1} [\widehat{D}_{t+2\delta}^1 | \mathcal{G}_{t+i\delta}]} - 1 \right), \quad (4.10)$$

for $i \in \{0, 1\}$, with $F_{t,1,1}^1 = R_{t+\delta}^1$, the δ -term rate at time $t + \delta$.

Proof. See Appendix A.4 for the proof. □

Remark 4.4 (Multi-period arbitrage-free models). *The availability of multiple tradable term rates, and their associated ZCBs, enables the SST to construct a multi-period arbitrage-free model, over $[t, t + m\delta]$ for $m \geq 2$. Equation (4.10) emphasises the fact that volatility is still statistically estimated, even with the existence of FRAs. Within this context, the SST may use the multi-period δ -term PK for general pricing and valuation, however to use the PK to risk manage derivatives written on rates other than the δ -term rate would be inconsistent.*

5 Multiple term rates with illiquidity

The multiple term rate system of the previous section is again considered here, except that the assumption of perfect liquidity is revoked. The definition of the respective term rates revert back to the general form in Axiom 2.6. From the perspective of the SST, the added illiquidity features proposed by Axiom 2.7 has to be modelled into the framework.

In Appendix B, one such model for a general $n\delta$ -term quote rate is provided, in Definition B.1, that incorporates term, nominal size, asymmetric liquidity spreads (due to loans and deposits), as well as specific SST and systemic illiquidity. Through fair valuation at the systemic level and suitable aggregation, Proposition B.1 and Corollary B.1 provide the necessary justification for a simpler symmetric model specification based on a *systemic liquidity indicator*. This construction is formalised in the next definition, followed by the definition of a potentially illiquid $n\delta$ -term rate, from the perspective of the SST. A more general 3-state costly systemic liquidity indicator is also presented in Appendix B, in Definition B.2, along with comparable results, in Lemma B.1 that are derived later in this section. In order to ease the exposition, the simpler 2-state specification is considered here. All of the results derived here still hold within the more general setting, modulo minor adjustments and assumptions.

Definition 5.1 (Systemic liquidity indicators). *At time $u \in \mathbb{R}_{\geq 0}$, the binary random variable L_u^n assumes a value of 1 if perfect systemic liquidity exists for the $n\delta$ -term rate, or 0 otherwise. If the current time is t , then the natural filtration associated with liquidity is*

$$\mathcal{L}_t := \sigma(\{L_u^1, L_u^2, \dots, L_u^m\}; u \in \{t_0, t_1, \dots, t_k\}), \quad (5.1)$$

where $\{t_0, t_1, \dots, t_k\}$ denotes the set of trading days that lie within the interval $[0, t]$. The systemic liquidity indicators are assumed to exhibit both serial and cross-sectional independence, or more formally:

$$\mathbb{E}^{\mathbb{P}}[L_u^n | \mathcal{L}_t \cap \sigma(\{L_u^n \notin \{0, 1\}\})] = \mathbb{E}^{\mathbb{P}}[L_u^n] = \mathbb{P}(L_u^n = 1) := q_u^n, \quad (5.2)$$

for all $t \leq u$, with $q_u^n := q(u, u + n\delta)$ being a deterministic function for the probability of perfect systemic $n\delta$ -term liquidity at time u .

Definition 5.2 (Potentially illiquid $n\delta$ -term rate). *At some arbitrary time $u < t$, the $n\delta$ -term rate*

$$\tilde{R}_t^n := R_t^n L_t^n, \quad (5.3)$$

is potentially illiquid from the vantage point of u if $L_t^n = 0$, i.e., it will not be possible for the SST to borrow from (or deposit with) the STR, at time t , for a tenor equal to $n\delta$.

These liquidity indicators enable the definition of various liquidity regimes, which in turn enables the definition of a set of term- and liquidity-dependent pricing kernels (LDPKs), all from the perspective of the SST. First, the various regimes of liquidity are defined.

Definition 5.3 (Liquidity regimes). *Let $i, j \in \mathbb{N}_0$, with $i < j$, and define the counting sets*

$$\begin{aligned}\mathbb{N}_{i,j} &:= \{i, i+1, \dots, j-1, j\} \text{ and} \\ \mathbb{N}_{i,j}^n &:= \{i, i+n, \dots, i+(k-1)n, i+kn\},\end{aligned}$$

where $k := \lfloor (j-i)/n \rfloor - 1$, $\mathbb{N}_{i,i} := \emptyset$ and $\mathbb{N}_{i,i}^n := \emptyset$. At time u , the following liquidity regimes are possible over the interval $[u, u+m\delta]$:

(i) **NPFL** - No present nor future liquidity exists on the set

$$\mathcal{L}_{u,u+m\delta}^{\text{NPFL}} := \sigma\left(\{L_{u+i\delta}^n = 0; n \in \mathbb{N}_{1,m}, i \in \mathbb{N}_{0,m}^n\}\right). \quad (5.4)$$

(ii) **NPL** - No present liquidity only exists on the set

$$\mathcal{L}_{u,u+m\delta}^{\text{NPL}} := \sigma\left(\{L_u^n = 0; n \in \mathbb{N}_{1,m}\}\right). \quad (5.5)$$

(iii) **PPL** - Only partial present liquidity exists on the set

$$\mathcal{L}_{u,u+m\delta}^{\text{PPL}} := \sigma\left(\{L_u^n = 1; n \in \mathbb{N}_{1,m}\}\right). \quad (5.6)$$

(iv) **CPL** - Complete present liquidity only exists on the set

$$\mathcal{L}_{u,u+m\delta}^{\text{CPL}} := \sigma\left(\{L_u^n = 1; n \in \mathbb{N}_{1,m}\}\right). \quad (5.7)$$

(v) **CPFL**: Complete present and future liquidity exists on the set

$$\begin{aligned}\mathcal{L}_{u,u+m\delta}^{\text{CPFL}} &:= \mathcal{L}_{u,u+m\delta}^1 \vee \mathcal{L}_{u,u+m\delta}^2 \vee \dots \vee \mathcal{L}_{u,u+m\delta}^m \\ &= \mathcal{L}_{u,u+m\delta}^{\text{CPL}} \vee \mathcal{L}_{u+\delta,u+m\delta}^1 \vee \mathcal{L}_{u+2\delta,u+m\delta}^2 \vee \dots \vee \mathcal{L}_{u+m\delta,u+m\delta}^m,\end{aligned} \quad (5.8)$$

where $\mathcal{L}_{u+i\delta,u+m\delta}^n := \sigma\left(\{L_{u+j\delta}^n = 1; j \in \mathbb{N}_{i,m}^n\}\right)$ models liquidity in the $n\delta$ -term rate over $[u+i\delta, u+m\delta]$ if $(m-i) \bmod n = 0$, or $[u+i\delta, u+m\delta]$ otherwise, for $i \in \mathbb{N}_{0,m}$.

Regimes (i) and (ii) are complements of (iv) and (v), respectively. Regimes (ii), (iii) and (iv) pose uncertain future liquidity, with (iii) also posing uncertain present liquidity for some terms.

Shorthand Notation: $\mathcal{L}_{u,i,j}^X := \mathcal{L}_{u+i\delta,u+j\delta}^X$ for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Remark 5.1 (Implications of the CPFL regime). *Under the CPFL regime all of the results from Sections 3 and 4 may be recovered, i.e., multi-period arbitrage-free term-dependent pricing kernels with associated term-dependent systemic pricing measures.*

In Appendix C, the definitions of an $n\delta$ -term LDPK is provided over a single period $[t, t+n\delta]$, and then over multiple periods $[t, t+i\delta]$, for $i \in \mathbb{N}_{0,m+n}^n$. Both Definitions C.1 and C.2 clearly reveal that the regime of liquidity has a significant impact on the form of the pricing kernel associated with each tradable term. The impact of present liquidity or illiquidity is fundamental, with the latter requiring the subjective process of market-making. Modelling the term rate market-making process of the STR is not an objective of this research, therefore the prevalence of the CPL regime will be a minimal assumption in all that follows. Then, from a practical perspective, in order to deal with potential future illiquidity, presently available liquidity must be fully exploited. This is achieved through the definition of the following *hybrid-term LDPK*.

Definition 5.4 (Hybrid-term LDPK). *At the current time t , the PK defined by*

$$\tilde{\pi}_{t+i\delta} = \begin{cases} \pi_{t+i\delta}^1, & \text{conditional on } \mathcal{L}_{t,0,i}^{\text{CPL}} \vee \mathcal{L}_{t,1,i}^1, & \text{for } i \in \mathbb{N}_{1,m+1}^1, \\ \pi_{t+i\delta}^2, & \text{conditional on } \mathcal{L}_{t,0,i}^{\text{CPL}} \vee \mathcal{L}_{t,2,i}^2, & \text{for } i \in \mathbb{N}_{2,m+2}^2, \\ \vdots, & \vdots, & \vdots, & \vdots, \\ \pi_{t+i\delta}^m, & \text{conditional on } \mathcal{L}_{t,0,i}^{\text{CPL}} \vee \mathcal{L}_{t,m,i}^m, & \text{for } i \in \mathbb{N}_{m,2m}^m, \end{cases} \quad (5.9)$$

with $\tilde{\pi}_t := 1$, is called the *hybrid-term liquidity-dependent pricing kernel*.

The hybrid-term LDPK is only well-defined at time t , and may be used to present value future cash flows due at time $t+i\delta$ back to time t only, for $i \in \mathbb{N}_{1,m}$. For the intertemporal valuation of the same cash flows back to a future time $t+h\delta$, for $h \in \mathbb{N}_{1,i-1}$, but from the vantage point of time t , one would require the prevalence of the CPFL regime at time t in order to have a well-defined PK over $[t+h\delta, t+i\delta]$. However, invoking the CPFL regime would recover the setup of the previous section, the PK would coincide with one of the term-dependent PKs defined in that section, and there'd be no need for the hybrid-term LDPK.

With the hybrid-term LDPK it is now possible to consider again the pricing (or market-making) of a $\delta \times 2\delta$ FRA, from the perspective of the SST, under potential future illiquidity.

Assuming that the CPL regime prevails at the time of pricing, the SST will have to subjectively specify the probability of future liquidity, articulated here via the liquidity indicators.

Lemma 5.1 ($\delta \times 2\delta$ FRA pricing under potential future illiquidity). *The fair strike rate process for the general version of the $\delta \times 2\delta$ FRA defined in Lemma 4.2 is*

$$\bar{F}(t+i\delta; t+\delta, t+2\delta) = \begin{cases} q_{t+\delta}^1 F_{t,0,1}^1, & i=0 \text{ and conditional on } \mathcal{L}_{t,0,2}^{\text{CPL}}, \\ F_{t,1,1}^1, & i=1 \text{ and conditional on } \mathcal{L}_{t,1,2}^{\text{CPL}}, \end{cases} \quad (5.10)$$

which is also $\mathcal{G}_{t+i\delta}$ -measurable.

Shorthand Notation: In general the $j\delta \times (j+1)\delta$ FRA rate will be denoted by:

$$\bar{F}_{u,i,j}^1 := \bar{F}(u+i\delta; u+j\delta, u+(j+1)\delta), \quad (5.11)$$

for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Proof. Assuming that $\mathcal{L}_t = \mathcal{L}_{t-} \vee \mathcal{L}_{t,0,2}^{\text{CPL}}$, the standard FRA replication strategy yields

$$\begin{aligned} \tilde{V}_{t+2\delta} &= \alpha N \left[(1 + \delta \tilde{R}_{t+\delta}^1) - (1 + \delta F_{t,0,1}^1) \right] \\ &= V_{t+2\delta} - \alpha N \delta (1 - L_{t+\delta}^1) R_{t+\delta}^1, \end{aligned}$$

where $F_{t,0,1}^1$ is defined in Corollary 4.1 and $V_{t+2\delta}$ is the payoff of a fair $\delta \times 2\delta$ FRA that is not exposed to liquidity risk. Let $\mathcal{M}_t := \mathcal{G}_t \vee \mathcal{L}_t$, then using the hybrid-term LDPK from Definition 5.4, the current value of the above payoff is

$$\begin{aligned} \tilde{\pi}_t \tilde{V}_t &= \mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t, L_{t+\delta}^1 \right] \mid \mathcal{M}_t \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t, L_{t+\delta}^1 = 1 \right] \mathbb{P}(L_{t+\delta}^1 = 1) \mid \mathcal{M}_t \right] \\ &\quad + \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t, L_{t+\delta}^1 = 0 \right] \mathbb{P}(L_{t+\delta}^1 = 0) \mid \mathcal{M}_t \right] \end{aligned}$$

which follows by the tower property of conditional expectations. Since $\tilde{\pi}_t := 1$ and observing that $\mathcal{L}_{t,0,2}^1 = \sigma(\{L_{t+\delta}^1 = 1\})$ and $\mathcal{L}_{t,0,2}^2 = \emptyset$, it follows that

$$\begin{aligned} \tilde{V}_t &= \mathbb{E}^{\mathbb{P}} \left[\pi_{t+2\delta}^1 V_{t+2\delta} \mid \mathcal{M}_t \right] q_{t+\delta}^1 - \mathbb{E}^{\mathbb{P}} \left[\pi_{t+2\delta}^2 \alpha N \delta F_{t,0,1}^1 \mid \mathcal{M}_t \right] (1 - q_{t+\delta}^1) \\ &= q_{t+\delta}^1 V_t - \alpha N (1 - q_{t+\delta}^1) \delta F_{t,0,1}^1 P_{t,0,2}^2, \end{aligned}$$

using the definition of the hybrid-term LDPK. V_t is the fair value of the FRA under perfect

liquidity, and therefore equal to 0. Trading this FRA with the strike rate equal to the fair FRA rate defined in the perfect liquidity setting therefore leads to an initial loss (gain) if the market-maker is long (short). The assumption here is that $V_{t+2\delta}$ will still be the FRA payoff even when $L_{t+\delta}^1 = 0$ and the strong case of no systemic liquidity is in effect. In reality, there will still be a reference δ -term rate that is contractually specified for such a case by the relevant FRA ISDA documentation.

Setting the FRA strike rate to an arbitrary value, $\bar{F}_{t,0,1}^1$, and pricing via the same process gives

$$\tilde{V}_t = \alpha q_{t+\delta}^1 N \left[P_{t,0,1}^1 - \left(1 + \delta \bar{F}_{t,0,1}^1 \right) P_{t,0,2}^1 \right] - \alpha N \left(1 - q_{t+\delta}^1 \right) \delta \bar{F}_{t,0,1}^1 P_{t,0,2}^2,$$

while setting $\tilde{V}_t = 0$, recalling that $P_{t,0,2}^2 = P_{t,0,2}^1$, and solving for the fair FRA strike rate yields $\bar{F}_{t,0,1}^1 = q_{t+\delta}^1 F_{t,0,1}^1$, as required. Repeating this pricing process at time $t + \delta$, for exactly the same contract and assuming that the CPL liquidity regime, $\mathcal{L}_{t,1,2}^{\text{CPL}}$, prevails at this time, it is trivial to show that $\bar{F}_{t,1,1}^1 = F_{t,1,1}^1$, which completes the proof. \square

Contingent on present liquidity, the pricing of a FRA still requires a subjective view on future liquidity. Therefore, the SST must be afforded some level of *risk appetite* in order to market-make such derivatives. This is in stark contrast with the perfect liquidity setting where the SST could replicate FRAs perfectly, and thereby required (nor deserved) any risk appetite. In general then, the SST will have the capacity for exposure to residual risk. This, combined with the *zero net supply*⁹ and *unfunded*¹⁰ nature of linear derivatives, such as FRAs, allows the SST substantial flexibility in their market-making process.

Remark 5.2 (FRA liquidity is not completely contingent on the CPL regime). *Even if the NPL regime were to prevail, the SST's capacity to carry residual risk and potentially hedge in the future, through offsetting positions, will still enable the pricing of FRAs. Practically, this decouples the theoretical contingency of the SST on the STR, or equivalently, interest rate derivatives on the set of primitive term rates. However, Lemma 5.1 reveals that there is structure to said decoupling with forward rates required to dominate corresponding FRA rates.*

Lemma 4.2, presented under the assumption of perfect liquidity, enforced the early liquidation value of a 2δ -term rate (or ZCB) by replacement. This result was used in conjunction with the replication of a $\delta \times 2\delta$ FRA to create a synthetic δ -term ZCB with 2δ -tenor. An analogous result is possible here, however it is contingent on $\delta \times 2\delta$ FRA and δ -term rate liquidity. Therefore, based on the discussion leading up to and including Remark 5.2, it

⁹A derivative transaction only exists once there is a willing buyer and seller - the market-maker may be either.

¹⁰In general, linear financial derivatives are exchange-traded and margined or over-the-counter and collateralised, subject to a zero-threshold credit support annex, and require no initial capital outlay.

is now assumed that a FRA market has been established within the inter-bank derivatives market. While the individual STs that constitute the SST would be responsible for the establishment of the FRA market through active market-making via model creation; here the SST is considered to be a separate entity that is observing this market at a systemic level and considering the problem of passive market-making via model calibration. Only the stylised problem from Lemma 4.2 and Theorem 4.1 is considered again here - the general version of this problem is considered in the next section.

Assumption 5.1 ($\delta \times 2\delta$ FRA market-making). *The individual STs have sufficient risk appetite to market-make and enable liquidity of the $\delta \times 2\delta$ FRA at time t . At this time, the fair or mid market FRA rate, denoted here by $\widehat{F}_{t,0,1}^1$, is used by the SST together with the result from Lemma 5.1 for the purpose of calibration.*

In particular, setting $\overline{F}_{t,0,1}^1 = \widehat{F}_{t,0,1}^1$ and assuming that $\mathcal{L}_t = \mathcal{L}_{t^-} \vee \mathcal{L}_{t,0,2}^{\text{CPL}}$ enables the SST to compute

$$q_{t+\delta}^1 = \frac{\overline{F}_{t,0,1}^1}{F_{t,0,1}^1}, \quad (5.12)$$

using equation (5.10), which is now the market-implied probability of perfect δ -term liquidity at time $t + \delta$ using information available at the current time t .

Definition 5.5 (Systemic $\delta \times 2\delta$ FRA liquidity indicator). *At time $t + i\delta$, for each $i \in \{0, 1\}$, the binary random variable $\overline{L}_{t,i,1}^1$ assumes a value of 1 if perfect systemic liquidity exists for the $\delta \times 2\delta$ FRA, or 0 otherwise. When $i = 0$, perfect systemic liquidity means that Assumption 5.1 holds, and it is assumed that*

$$\sigma\left(\{\overline{L}_{t,0,1}^1 = 1\}\right) \supset \mathcal{L}_{t,0,2}^{\text{CPL}}. \quad (5.13)$$

When $i = 1$, perfect systemic liquidity is equivalent to $L_{t+\delta}^1 = 1$, or

$$\sigma\left(\{\overline{L}_{t,1,1}^1 = 1\}\right) = \mathcal{L}_{t,1,2}^1. \quad (5.14)$$

If the current time is t , then the natural filtration associated with liquidity is now

$$\overline{\mathcal{L}}_t := \mathcal{L}_t \vee \sigma\left(\{\overline{L}_{u,0,1}^1; u \in \{t_0, t_1, \dots, t_k\}\}\right), \quad (5.15)$$

where $\{t_0, t_1, \dots, t_k\}$ denotes the set of trading days that lie within the interval $[0, t]$ and \mathcal{L}_t is defined in Definition 5.1, equation (5.1). Since the systemic FRA liquidity indicators will only be used to indicate regimes of liquidity and will not be used for pricing, the probabilistic structure of these are left unspecified.

Using the $\delta \times 2\delta$ FRA along with the δ -term rate, it is now possible to formulate the analog to Lemma 4.2 within this setting of potential illiquidity. The synthetic δ -term ZCB that is constructed here is referred to as a liquidity-contingent zero coupon bond (LCZCB), since its definition relies on the availability of liquidity in the aforementioned instruments.

Lemma 5.2 (Synthetic δ -term LCZCB with 2δ -tenor). *Assuming that $\bar{L}_{t,0,1}^{-1} = 1$, and setting $\bar{F}_{t,0,1}^{-1} = \widehat{F}_{t,0,1}^{-1}$, it is possible to replicate the following ZCB:*

$$\bar{P}_{t,i,2}^{-1} := \begin{cases} D_{t+\delta}^1 / (1 + \delta \bar{F}_{t,0,1}^{-1}) , & i = 0 , \\ P_{t,1,2}^1 , & i = 1 , \\ 1 , & i = 2 , \end{cases} \quad (5.16)$$

provided that $L_t^1 = L_{t+\delta}^1 = 1$, or equivalently that $\mathcal{L}_{t,0,2}^1$ holds.

Proof. Assuming that $L_t^1 = L_{t+\delta}^1 = 1$, it is possible to borrow (deposit) M units of currency at the δ -term rate at time t and refinance (re-deposit) the total cost (proceeds) thereof at time $t + \delta$, such that the cumulative cost (yield) is $M/D_{t+2\delta}^1$ at time $t + 2\delta$. Combining this loan (deposit) with a long (short) position in a fair $\delta \times 2\delta$ market FRA with strike rate $\bar{F}_{t,0,1}$ and $N = M/D_{t+\delta}^1$ will enable the conversion of the floating cost (yield) to a fixed cost (yield) equal to $M(1 + \delta \bar{F}_{t,0,1}^{-1})/D_{t+\delta}^1$ at time $t + 2\delta$. Setting $M = D_{t+\delta}^1 / (1 + \delta \bar{F}_{t,0,1}^{-1})$, enables the creation of the synthetic δ -term LCZCB, with 2δ -tenor, given by equation (5.16). Since it is assumed that $L_{t+\delta}^1 = 1$, it is clear that $\bar{P}_{t,1,2}^{-1} = P_{t,1,2}^1$, which completes the proof. \square

Lemma 5.1 and 5.2 provides the basis for the construction of a δ -term LCZCB system, one that is created by exchanging δ -term floating-for-fixed interest rate risk. It is possible to model this system via the definition of a liquidity-contingent pricing kernel (LCPK). The δ -term LCPK is defined over the interval $[t, t + 2\delta]$ in the next theorem.

Theorem 5.1 (δ -Term LCPK). *Contingent on $\bar{L}_{t,0,1}^{-1} = L_t^1 = L_{t+\delta}^1 = 1$, or equivalently*

$$\bar{\mathcal{L}}_t := \mathcal{L}_t \vee \sigma \left(\left\{ \bar{L}_{t,0,1}^{-1} = 1 \right\} \right) \vee \mathcal{L}_{t,0,2}^1 , \quad (5.17)$$

using the result from Lemma 5.2 and recalling that $\mathcal{F}_t := \mathcal{G}_t \vee \bar{\mathcal{L}}_t$, the δ -term LCPK may be defined as

$$\bar{\pi}_{t+j\delta}^1 := \pi_{t+j\delta}^1 \Theta_{t+j\delta}^1 , \quad (5.18)$$

for $j \in \{0, 1, 2\}$, where the time-inhomogeneous $\{(\mathcal{G}_u, \mathbb{P}_1)\}$ -martingale $(\Theta_{t+v\delta}^1)_{0 \leq v \leq 2}$, with $\Theta_t^1 := 1$, enables the change-of-measure from \mathbb{P}_1 to \mathbb{Q}_1 on $\mathcal{G}_{t+j\delta}$, i.e., $\frac{\Theta_{t+j\delta}^1}{\Theta_t^1} = \frac{d\mathbb{Q}_1}{d\mathbb{P}_1} \Big|_{\mathcal{G}_{t+j\delta}}$, such

that the $\mathbb{E}^{\mathbb{P}}[\bar{\pi}_{t+\delta}^1 | \mathcal{F}_t] = P_{t,0,1}^1$ and the $\mathbb{E}^{\mathbb{P}}[\bar{\pi}_{t+2\delta}^1 | \mathcal{F}_t] = \bar{P}_{t,0,2}^1$.

Proof. Since $\mathcal{L}_{t,0,2}^{\text{CPL}} \subset \sigma(\{\bar{L}_{t,0,1}^1 = 1\})$, by Definition 5.5, it follows that $\mathcal{L}_{t,0,2}^{\text{CPFL}} \subset \bar{\mathcal{L}}_t$ and that the δ -term PK is well defined over $[t, t+2\delta]$. Therefore, $\{\pi_{t+j\delta}^1, j \in \{0, 1, 2\}\}$ is a good initial candidate for the LCPK, however it does not recover the initial price of the synthetic δ -term LCZCB with 2δ -tenor. The definition of the $\{(\mathcal{G}_u), \mathbb{P}_1\}$ -martingale $(\Theta_{t+v\delta}^1)_{0 \leq v \leq 2}$ enables a change-of-measure such that

$$\mathbb{E}^{\mathbb{P}}[\Lambda_{t+2\delta}^1 \Theta_{t+2\delta}^1 D_{t+2\delta}^1 | \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}_1}[\Theta_{t+2\delta}^1 D_{t+2\delta}^1 | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}_1}[D_{t+2\delta}^1 | \mathcal{F}_t] := \bar{P}_{t,0,2}^1,$$

as required, recalling that $\Lambda_t^1 = \Theta_t^1 = 1$, i.e., the free time-dependent parameters associated with $(\Theta_{t+v\delta}^1)_{1 \leq v \leq 2}$ is free to specify at time t such that the expectation equals $\bar{P}_{t,0,2}^1$. Also, at the future time $t + \delta$, since $\bar{\mathcal{L}}_{t+\delta} \supset \bar{\mathcal{L}}_t \supset \mathcal{L}_{t,0,2}^1$ it follows that $L_{t+\delta}^1 = 1$ and recalling that $D_{t+2\delta}^1$ is $\mathcal{G}_{t+\delta}$ -measurable, then

$$\frac{1}{\bar{\pi}_{t+\delta}^1} \mathbb{E}^{\mathbb{P}}[\Lambda_{t+2\delta}^1 \Theta_{t+2\delta}^1 D_{t+2\delta}^1 | \mathcal{F}_{t+\delta}] = \frac{D_{t+2\delta}^1}{D_{t+\delta}^1 \Theta_{t+\delta}^1} \mathbb{E}^{\mathbb{P}_1}[\Theta_{t+2\delta}^1 | \mathcal{F}_{t+\delta}] = \frac{D_{t+2\delta}^1}{D_{t+\delta}^1} = P_{t,1,2}^1,$$

which shows that the value of the synthetic δ -term LCZCB, given by equation (5.16), is recovered by the δ -term LCPK. Since $D_{t+\delta}^1 = P_{t,0,1}^1$ is \mathcal{G}_t -measurable, it follows straightforwardly that

$$\mathbb{E}^{\mathbb{P}}[\Lambda_{t+\delta}^1 \Theta_{t+\delta}^1 D_{t+\delta}^1 | \mathcal{F}_t] = P_{t,0,1}^1 \mathbb{E}^{\mathbb{P}_1}[\Theta_{t+\delta}^1 | \mathcal{F}_t] = P_{t,0,1}^1,$$

which completes the proof, showing that: (i) the free time-dependent parameters associated with $(\Theta_{t+v\delta}^1)_{0 \leq v \leq 1}$ may be specified freely at time t ; and (ii) the δ -term LCPK is calibrated to the δ -term rate and the synthetic δ -term LCZCB with 2δ -tenor. \square

It may not be apparent but the definitions of the synthetic δ -term LCZCB and its associated LCPK, from Lemma 5.2 and Theorem 5.1, had two steps and associated contingencies:

- (i) the interval $[t, t + 2\delta]$, or more specifically the set $\{t, t + \delta, t + 2\delta\}$, requires that $\mathcal{L}_{t,0,2}^1$ holds, or equivalently that $L_t^1 = L_{t+\delta}^1 = 1$, and that $\bar{L}_{t,0,1}^1 = 1$; and
- (ii) the future interval $[t + \delta, t + 2\delta]$, or more specifically the set $\{t + \delta, t + 2\delta\}$, requires that $\mathcal{L}_{t,1,2}^1$ holds, or equivalently that $L_{t+\delta}^1 = 1$.

In general, to extend these definitions over the interval $[t, t + m\delta]$, would require m steps:

- at the current time t and over the set $\{t, t + \delta, \dots, t + m\delta\}$, one would require that $\mathcal{L}_{t,0,m}^1$ holds and that $\bar{L}_{t,0,j}^1 = 1$ for $j \in \mathbb{N}_{1,m-1}$;

- at each future time $t + i\delta$ and over the set $\{t + i\delta, t + (i + 1)\delta, \dots, t + m\delta\}$, for $i \in \mathbb{N}_{1, m-2}$, one would require $\mathcal{L}_{t, i, m}^1$ and that $\bar{L}_{t, i, j}^1 = 1$ for $j \in \mathbb{N}_{i+1, m-1}$; and
- at the future time $t + (m - 1)\delta$ and over the set $\{t + (m - 1)\delta, t + m\delta\}$, one would require that $\mathcal{L}_{t, m-1, m}^1$ holds.

This will form the basis for the reduced-form modelling approach that is developed in the next section. This section is concluded with a few remarks that aim to assist the reader to build intuition in relation to all of the theory that has been presented thus far.

Remark 5.3 (LCZCBs are not tradable). *The δ -term ZCB-system that was introduced in Section 4, assuming perfect liquidity, viz.,*

$$\{P_{t, i, j}^1; i \in \mathbb{N}_{0, j}, j \in \mathbb{N}_{0, m}\}, \quad (5.19)$$

denotes a set of tradable ZCBs whose tenors span the interval $[t, t + m\delta]$. Moreover, recall that under perfect liquidity all term ZCBs are replicated via the δ -term system, i.e., $P_{t, i, j}^1 = P_{t, i, j}^{j-i}$. Under potential illiquidity, the definition and tradability of the above set of ZCBs requires $\mathcal{L}_{t, 0, m}^{\text{CPFL}}$ to hold. When $\mathcal{L}_{t, 0, m}^{\text{CPL}}$ holds, this set reduces to

$$\{P_{t, 0, n}^n; n \in \mathbb{N}_{0, m}\}, \quad (5.20)$$

i.e., the set of ZCBs derived from the systemic term funding curve, defined in Definition 2.2. Then, the δ -term LCZCB-system, that was introduced in this section, is

$$\{\bar{P}_{t, i, j}^1; i \in \mathbb{N}_{0, j}, j \in \mathbb{N}_{0, m}\}, \quad (5.21)$$

and is contingent upon $\bar{L}_{t, i, j}^1 = 1$ for all $j \in \mathbb{N}_{i+1, m-1}$ and $\mathcal{L}_{t, i, m}^1$ holding for each $i \in \mathbb{N}_{0, m}$. Apart from $\bar{P}_{t, i, i+1}^1 = P_{t, i, i+1}^1$ which only requires present liquidity at $t + i\delta$, i.e., $L_{t+i\delta}^1 = 1$, for each $i \in \mathbb{N}_{0, m-1}$, the remainder of the set of LCZCBs are contingent on the availability of future liquidity and are therefore not tradable, in general¹¹.

Remark 5.4 (Interpretation of the δ -term LCPK). *Following the definition of a potentially illiquid $n\delta$ -term rate, in Definition 5.2, an intuitive approach might have been to:*

¹¹From the perspective of an EC, long positions in LCZCBs may be enabled by the SST offering FRA liquidity and the STR issuing floating rate notes (FRNs) that reference the δ -term rate. Short positions in LCZCBs would require the EC to secure bespoke variable rate loan agreements from the STR that reference the δ -term rate with zero additional spread for credit risk. This would not be possible however, unless the credit risk of the EC was comparable to that of constituent banking entities.

(i) specify a δ -term liquidity-cognisant SDF under \mathbb{P}_1 and over $[t, t + j\delta]$ as

$$\tilde{D}_{t+j\delta}^1 := \prod_{i=0}^{j-1} \frac{1}{1 + \delta \tilde{R}_{t+i\delta}^1} = \prod_{i=0}^{j-1} \frac{1}{1 + \delta L_{t+i\delta}^1 R_{t+i\delta}^1}, \quad (5.22)$$

for $j \in \mathbb{N}_{1,m}$, with $\tilde{D}_t^1 := 1$; and

(ii) proceed to directly calculate ZCB prices using the δ -term liquidity cognisant SDF

The SDF, from (i), and its associated bank account, $\tilde{B}_{t+j\delta}^1 := 1/\tilde{D}_{t+j\delta}^1$, are relevant from a practical perspective. The latter directly models the total proceeds (costs) of a deposit (loan) strategy that rolls over at the δ -term frequency. The unavailability of liquidity at any roll-over time will translate into an interest rate loss (gain) for the deposit (loan) strategy¹². It is also assumed that the depositor (borrower) will attempt again to re-invest (refinance) the total value of their investment (liability) at the next roll-over time.

While the ZCB prices, from (ii), may be calculated theoretically (with plausible and tractable model specifications), such ZCBs do not exist in reality. If these ZCBs existed, they would ensure multi-period δ -term funding at fixed rates while also ensuring early liquidation at the δ -term frequency, by definition. In other words, such ZCBs would immunise long (depositors) and short (borrowers) holders from all liquidity risks. In practice, there are no financial instruments that offer protection against liquidity risks, hence the approach that has been taken in this section which has culminated in the definition of LCZCBs and LCPKs.

The δ -term LCPK has the following form:

$$\bar{\pi}_{t+j\delta}^1 = \begin{cases} \Lambda_{t+j\delta}^1 D_{t+j\delta}^1 \Theta_{t+j\delta}^1, & \text{under } \mathbb{P}, \\ D_{t+j\delta}^1 \Theta_{t+j\delta}^1, & \text{under } \mathbb{P}_1, \\ D_{t+j\delta}^1, & \text{under } \mathbb{Q}_1, \end{cases}$$

is $\mathcal{F}_{t+j\delta}$ -measurable, for $j \in \mathbb{N}_{0,m}$, and is calibrated at the current time t such that it recovers the fair $i\delta \times (i+1)\delta$ FRA rates, for $i \in \mathbb{N}_{1,m-1}$. In other words, unlike the intuitive liquidity-cognisant δ -term SDF presented above, the δ -term LCPK is designed for the purpose of FRA pricing and valuation. The calibration process is enabled by the definition of the set of LCZCBs $\{\bar{P}_{t,0,j}^1; j \in \mathbb{N}_{0,m}\}$, and the $\{(\mathcal{G}_u), \mathbb{P}_1\}$ -martingale $(\Theta_{t+v\delta}^1)_{0 \leq v \leq m}$ which, via the fair

¹²The gain for the loan strategy comes at the cost of the borrower having to settle their total liability at the roll-over time, as opposed to deferring payment by refinancing at the δ -term rate at this time.

FRA rates, encodes the likelihood of δ -term liquidity ex-ante. In particular, this martingale process inflates (deflates) the δ -term SDF (bank account) based on the lack of liquidity. Hence $D_{t+j\delta}^1 \Theta_{t+j\delta}^1$, which is the δ -term LCPK under \mathbb{P}_1 , is an abstract representation of the δ -term liquidity cognisant SDF, equation (5.22), both ex-ante and ex-post.

Remark 5.5 (Liquidity-Contingent Term-Dependent Market Price of Systemic Risk). Under potential illiquidity, the δ -term market price of systemic risk modelled by $\{\Lambda_{t+j\delta}^1; j \in \mathbb{N}_{0,m}\}$ is effectively adjusted for the potential cost of illiquidity incurred when market-making FRAs through the process $\{\Theta_{t+j\delta}^1; j \in \mathbb{N}_{0,m}\}$. However, this is all strictly contingent on the availability of δ -term and FRA liquidity, as described above. Therefore, the product of the aforementioned processes $\{\Lambda_{t+j\delta}^1 \Theta_{t+j\delta}^1; j \in \mathbb{N}_{0,m}\}$ models the liquidity-contingent term-dependent market price of systemic risk associated with the δ -term LCPK.

Remark 5.6 (Multiple LCPKs and liquidity-contingent term structures). Each term, $n\delta$, will have a distinct LCPK, $\{\bar{\pi}_{t+j\delta}^n; j \in \mathbb{N}_{0,m}^n\}$, with an associated liquidity-contingent systemic pricing measure, \mathbb{Q}_n , modelled upon its perfect liquidity counterparts, $\{\pi_{t+j\delta}^n; j \in \mathbb{N}_{0,m}^n\}$ and \mathbb{P}_n . This will be further illustrated in the next section. The inability to replicate all tradable ZCBs via the system of FRAs, along with the contingency on future term rate liquidity, leads to liquidity-contingent multi-period calibration for each tradable term. This in turn leads to multiple liquidity-contingent term structures.

Remark 5.7 (Classical risk-neutral measure). Within this context, \mathbb{Q}_1 is the best proxy for the classical risk-neutral measure, however it is not clear that this measure produces consistent and coherent expectations of risk-free term rates considering the idiosyncratic and subjective market-making processes of the STR and the constituents of the SST, and the interactions thereof.

6 Reduced-form model development

In order to formalise the construction of an arbitrary $n\delta$ -term LCZCB-system and LCPK over an arbitrary horizon $[t, t + pn\delta]$, for $p \in \mathbb{N}$, it is useful to provide a definition for FRA liquidity regimes akin to the term loan and deposit regimes from Definition 5.3.

Definition 6.1 (FRA liquidity regimes). At an arbitrary time $u + i\delta$, complete $n\delta$ -term FRA liquidity over the interval $[u + i\delta, u + m\delta]$ exists on the set

$$\bar{\mathcal{L}}_{u+i\delta, u+m\delta}^n := \sigma(\{\bar{L}_{u,i,i+j}^n = 1; j \in \mathbb{N}_{n,m-i}^n\}), \quad (6.1)$$

where $i, m \in \mathbb{N}_0$ with $i \leq m$ and, as in Definition 5.5, the binary random variable $\bar{L}_{u,i,i+j}^n$ is equal to 1 if perfect systemic liquidity exists for the $j\delta \times (j+n)\delta$ FRA, or is equal to 0 otherwise. Also, it is assumed that

$$\sigma\left(\{\bar{L}_{u,i,i+j}^n = 1\}\right) \supset \sigma\left(\{L_{u+i\delta}^j = 1, L_{u+i\delta}^{j+n} = 1\}\right), \quad (6.2)$$

which is the analogous assumption to equation (??) from Definition 5.5.

Shorthand Notation: $\bar{\mathcal{L}}_{u,i,j}^n := \bar{\mathcal{L}}_{u+i\delta,u+j\delta}^n$ for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Analogous to the construction of the δ -term LCZCB-system and its associated LCPK that was described in the previous section, the construction of the comparable $n\delta$ -term quantities, viz., $\{\bar{P}_{t,in,jn}^n; i, j \in \mathbb{N}_{0,p}, i \leq j\}$ and $\{\bar{\pi}_{t+jn\delta}^n; j \in \mathbb{N}_{0,p}\}$, over the interval $[t, t + pn\delta]$, would require p steps:

- at the current time t and over the set $\{t, t + n\delta, t + 2n\delta, \dots, t + pn\delta\}$, one would require that $\mathcal{L}_{t,0,pn}^n$ and $\bar{\mathcal{L}}_{t,0,pn}^n$ holds;
- at each future time $t + in\delta$ and over the set $\{t + in\delta, t + (i+1)n\delta, \dots, t + pn\delta\}$, for $i \in \mathbb{N}_{1,p-2}$, one would require that $\mathcal{L}_{t,in,pn}^n$ and $\bar{\mathcal{L}}_{t,in,pn}^n$ holds; and
- at the future time $t + (p-1)n\delta$ and over the set $\{t + (p-1)n\delta, t + pn\delta\}$, one would require that $\mathcal{L}_{t,(p-1)n,pn}^n$ holds.

While the construction of such term-dependent and -consistent quantities is theoretically appealing, it is far too rigid for real-world pricing, valuation and risk management. This is practically demonstrated by the inability of the $n\delta$ -term LCPK to model the natural tenor transformation associated with fixed maturity financial instruments through the passage of time, as well as the asynchronicity between calendar (t), term ($n\delta$) and tenor ($pn\delta$) time. The objective of this section is to adapt the framework to cater for the aforementioned practical considerations - this is achieved through a *reduced-form modelling approach*.

To construct a reduced-form $n\delta$ -term LCZCB-system and LCPK over an arbitrary time interval $[t + i\delta, t + m\delta]$, where $n \leq m$ and $i \in \mathbb{N}_{0,m}$, the following assumptions are required.

Assumption 6.1 (CPL). *At time $t + i\delta$, the STR enables the CPL regime $\mathcal{L}_{t,i,m}^{\text{CPL}}$.*

Assumption 6.2 (FRA market-making). *At time $t + i\delta$, the individual STs have sufficient risk appetite to market-make and enable the liquidity of each $k\delta \times (k+n)\delta$ FRA with fair or mid*

market FRA rate $\widehat{F}_{t,i,i+k}^n$, for $k \in \mathbb{N}_{1,m-n-i}$. This FRA liquidity regime exists on the set

$$\overline{\mathcal{L}}_{u+i\delta,u+m\delta}^{(n)} := \sigma(\{L_{u,i,i+k}^n = 1; k \in \mathbb{N}_{1,m-n-i}\}) \supset \overline{\mathcal{L}}_{u+i\delta,u+m\delta}^n, \quad (6.3)$$

and is therefore a richer set than that defined in Definition 6.1. Using Lemma 5.1 within this context, the model fair $k\delta \times (k+n)\delta$ FRA rate is

$$\overline{F}_{t,i,i+k}^n := q_{t+(i+k)\delta}^n F_{t,i,i+k}^n,$$

which the SST may set equal to $\widehat{F}_{t,i,i+k}^n$ in order to calibrate $q_{t+(i+k)\delta}^n$, the probability of $n\delta$ -term liquidity at time $t + (i+k)\delta$. Assumption 6.1 enables the computation of $F_{t,i,i+k}^n$.

Assumption 6.3 (Future $n\delta$ -term liquidity). At time $t+i\delta$, future $n\delta$ -term liquidity according to the set

$$\mathcal{L}_{t,i,m}^{(n)} := \sigma(\{L_{t+(i+k)\delta}^n = 1; k \in \mathbb{N}_{1,m-n-i}\}) \supset \mathcal{L}_{t,i,m}^n,$$

is assumed to exist, with the definitions of the reduced-form $n\delta$ LCZCB-system and LCPK being contingent upon this assumption.

Assumption 6.4 (Reduced-form $n\delta$ -term PK). Using the estimated $n\delta$ -term SDF from Definition 2.3 and contingent on the CPFL regime, the calibrated reduced-form $n\delta$ -term SDF is

$$D_{t+j\delta}^{(n)} := \mathbb{E}^{\mathbb{P}} \left[\frac{\Lambda_{t+j\delta}^{(n)}}{\Lambda_{t+(j-1)\delta}^{(n)}} \widehat{D}_{t+j\delta}^n \mid \mathcal{G}_{t+(j-1)\delta} \right],$$

where the time-inhomogeneous process $(\Lambda_{t+\nu\delta}^{(n)})_{0 \leq \nu \leq m}$ is a $\{(\mathcal{G}_u), \mathbb{P}\}$ -martingale, with $\Lambda_t^{(n)} := 1$, that enables a change-of-measure from \mathbb{P} to $\mathbb{P}^{(n)}$, the reduced-form $n\delta$ -term systemic pricing measure. Then, commensurate with the δ -term, the calibrated reduced-form $n\delta$ -term PK is defined by $\pi_{t+j\delta}^{(n)} := \Lambda_{t+j\delta}^{(n)} D_{t+j\delta}^{(n)}$, with the time-inhomogeneous parameters associated with $\Lambda_{t+j\delta}^{(n)}$ chosen such that

$$P_{t,i,j}^{(n)} := \frac{1}{\pi_{t+i\delta}^{(n)}} \mathbb{E}^{\mathbb{P}} [\pi_{t+j\delta}^{(n)} \mid \mathcal{G}_{t+i\delta}] = \frac{1}{D_{t+i\delta}^{(n)}} \mathbb{E}^{\mathbb{P}^{(n)}} [D_{t+j\delta}^{(n)} \mid \mathcal{G}_{t+i\delta}] = P_{t,i,j}^{j-i},$$

which defines the reduced-form $n\delta$ -term ZCB-system, for $i, j \in \mathbb{N}_{0,m}$ with $i \leq j$. Finally, the

reduced-form $n\delta$ -term rate is defined by

$$R_{t+i\delta}^{(n)} := \frac{1}{n\delta} \left(\frac{1}{P_{t,i,j}^{(n)}} - 1 \right),$$

when $(j-i) = n$. For $n = 1$, the reduced-form δ -term PK is identical to its counterpart.

It is now possible to define the reduced-form synthetic $n\delta$ -term LCZCB-system, the intertemporal values of which will be used in the definition of the reduced-form $n\delta$ -term LCPK.

Lemma 6.1 (Reduced-form synthetic $n\delta$ -term LCZCB-system). *Given assumptions 6.1, 6.2, 6.3 and 6.4, the reduced-form synthetic $n\delta$ -term LCZCB system is defined by*

$$\bar{P}_{t,i,j}^{(n)} := \begin{cases} \frac{D_{t+(i+1)\delta}^1}{D_{t+i\delta}^1} \prod_{k=0}^{(j-i-n-1)/n} \left(1 + n\delta \bar{F}_{t,i,i+nk+1}^n \right)^{-1}, & \text{mod}(j-i, n) = 1, \\ \frac{D_{t+(i+2)\delta}^2}{D_{t+i\delta}^2} \prod_{k=0}^{(j-i-n-2)/n} \left(1 + n\delta \bar{F}_{t,i,i+nk+2}^n \right)^{-1}, & \text{mod}(j-i, n) = 2, \\ \vdots & \vdots, \\ \frac{D_{t+(i+n-1)\delta}^{n-1}}{D_{t+i\delta}^{n-1}} \prod_{k=0}^{(j-i-2n+1)/n} \left(1 + n\delta \bar{F}_{t,i,i+nk+n-1}^n \right)^{-1}, & \text{mod}(j-i, n) = n-1, \\ \frac{D_{t+(i+n)\delta}^n}{D_{t+i\delta}^n} \prod_{k=0}^{(j-i-2n)/n} \left(1 + n\delta \bar{F}_{t,i,i+n(k+1)}^n \right)^{-1}, & \text{mod}(j-i, n) = 0, \end{cases} \quad (6.4)$$

for $n < (j-i) \leq m$, while for $0 \leq (j-i) \leq n$ the definition resolves to

$$\bar{P}_{t,i,j}^{(n)} := \begin{cases} P_{t,j-n,j}^n, & i = j-n, \\ P_{t,j-n+1,j}^{n-1}, & i = j-(n-1), \\ \vdots, & \vdots, \\ P_{t,j-1,j}^1, & i = j-1, \\ 1, & i = j, \end{cases} \quad (6.5)$$

with $i, j \in \mathbb{N}_{0,m}$ and $i \leq j$.

Proof. See Appendix D.1 for the proof. □

Theorem 6.1 (Reduced-form $n\delta$ -term LCPK). *Maintaining the setup of Lemma 6.1, as well*

as the result thereof, a reduced-form $n\delta$ -term LCPK may be defined as

$$\bar{\pi}_{t+j\delta}^{(n)} := \pi_{t+j\delta}^{(n)} \Theta_{t+j\delta}^{(n)}, \quad (6.6)$$

for $j \in \mathbb{N}_{0,m}$, where $\Theta_t^{(n)} := 1$ and

$$\frac{\Theta_{t+j\delta}^{(n)}}{\Theta_{t+i\delta}^{(n)}} = \begin{cases} 1, & 0 \leq j-i \leq n, \\ \frac{X_{t+j\delta}^n}{X_{t+i\delta}^n}, & n < j-i \leq m. \end{cases}$$

The process $(X_{t+\nu\delta}^n)_{0 \leq \nu \leq m}$ is chosen to be a time-inhomogeneous $\{(\mathcal{G}_u), \mathbb{P}_{(n)}\}$ -martingale, with $X_t^n := 1$, that enables a change-of-measure from $\mathbb{P}_{(n)}$ to $\mathbb{Q}_{(n)}$ on $\mathcal{G}_{t+j\delta}$, i.e. $\frac{X_{t+j\delta}^n}{X_t^n} = \frac{d\mathbb{Q}_{(n)}}{d\mathbb{P}_{(n)}} \Big|_{\mathcal{G}_{t+j\delta}}$, such that

$$\bar{\pi}_{t+i\delta}^{(n)} P_{t,i,j}^{(n)} = \mathbb{E}^{\mathbb{P}} \left[\bar{\pi}_{t+j\delta}^{(n)} \mid \mathcal{G}_{t+i\delta} \right],$$

for all $i, j \in \mathbb{N}_{0,m}$ with $i \leq j$.

Proof. By Assumption 6.4 and construction, the reduced-form $n\delta$ -term PK recovers the reduced-form synthetic $n\delta$ -term LCZCB value for $0 \leq (j-i) \leq n$, i.e., $\bar{P}_{t,i,j}^{(n)} = P_{t,i,j}^{j-i} = P_{t,i,j}^{(n)}$. Therefore, the reduced-form $n\delta$ -term PK $\{\pi_{t+j\delta}^{(n)}; j \in \mathbb{N}_{0,m}\}$ is a good initial candidate for the reduced-form $n\delta$ -term LCPK. However, when $n < (j-i) \leq m$ then $P_{t,i,j}^{(n)} = P_{t,i,j}^{j-i} \neq \bar{P}_{t,i,j}^{(n)}$. The definition of the $\{(\mathcal{G}_u), \mathbb{P}_{(n)}\}$ -martingale $(X_{t+\nu\delta}^n)_{0 \leq \nu \leq m}$ enables a change-of-measure such that

$$\mathbb{E}^{\mathbb{P}} \left[\frac{\Lambda_{t+j\delta}^{(n)} \Theta_{t+j\delta}^{(n)} D_{t+j\delta}^{(n)}}{\Lambda_{t+i\delta}^{(n)} \Theta_{t+i\delta}^{(n)} D_{t+i\delta}^{(n)}} \mid \mathcal{G}_{t+i\delta} \right] = \mathbb{E}^{\mathbb{P}_{(n)}} \left[\frac{X_{t+j\delta}^n D_{t+j\delta}^{(n)}}{X_{t+i\delta}^n D_{t+i\delta}^{(n)}} \mid \mathcal{G}_{t+i\delta} \right] = \mathbb{E}^{\mathbb{Q}_{(n)}} \left[\frac{D_{t+j\delta}^{(n)}}{D_{t+i\delta}^{(n)}} \mid \mathcal{G}_{t+i\delta} \right],$$

may be set to the value of $\bar{P}_{t,i,j}^{(n)}$ by calibrating the free time-dependent parameters associated with $X_{t+j\delta}^n$. Finally, observe that

$$\frac{1}{\bar{\pi}_{t+i\delta}^{(n)}} \mathbb{E}^{\mathbb{P}} \left[\bar{\pi}_{t+j\delta}^{(n)} \mid \mathcal{G}_{t+i\delta} \right] = \begin{cases} 1, & j-i = 0, \\ P_{t,i,j}^{j-i}, & j-i \leq n, \\ \frac{1}{D_{t+i\delta}^{(n)}} \mathbb{E}^{\mathbb{Q}_{(n)}} \left[D_{t+j\delta}^{(n)} \mid \mathcal{G}_{t+i\delta} \right], & j-i > n, \end{cases}$$

for all $i, j \in \mathbb{N}_{0,m}$ with $i \leq j$, which is the required dynamics and completes the proof. \square

Acknowledgments. We are grateful to A. Backwell, E. Schlögl, and D. Skovmand for discussions on term structures of interest rates. We also thank seminar and conference participants at the SFRA Colloquium and Workshop, ICMS, Edinburgh, U.K. (February 2019), the Research in Options 2019 Conference, IMPA, Rio de Janeiro, Brazil (December 2019), at the Union Bank of Switzerland (UBS), London, U.K. (February 2020), and the Interest Rate Reform Conference, World Business Strategies (WBS), London, U.K. (March 2020) for comments and suggestions.

A Multiple term rates with perfect liquidity

A.1 Proof for Lemma 4.1

Proof. While the 2δ -term rate and its associated ZCB is a fixed-term product, the prevailing assumptions along with perfect liquidity enables early liquidation via replacement. The SST, as a market-taker, could easily terminate (redeem) the loan (deposit) at time $t + \delta$, by taking an opposite position using the tradable δ -term rate. Therefore, to preclude arbitrage, equation (4.4) must be the fair liquidation value of the tradable 2δ -term ZCB at time $t + \delta$. At time $t + 2\delta$, the calibrated δ -term SDF must have the representation

$$D_{t+2\delta}^1 := \frac{1}{(1 + \delta R_t^1)} \frac{1}{(1 + \delta R_{t+\delta}^1)},$$

to preclude arbitrage. Since R_u^1 is \mathcal{G}_u -measurable respectively, it should be clear that $D_{t+2\delta}^1$ is $\mathcal{G}_{t+\delta}$ -measurable. From equation (4.4), it then follows that $D_{t+2\delta}^1 = D_t^1 P_{t,1,2}^1$. Finally, equation (4.5) follows in a similar manner to Lemma 3.1 with the free time-dependent parameter associated with $(\Lambda_{t+v\delta}^1)_{1 < v \leq 2}$ enabling the calibration, which concludes the proof. \square

A.2 Proof for Lemma 4.2

Proof. The long (short) FRA payoff, equation (4.6), may be replicated, at zero cost (i.e. $V_t = 0$), by borrowing (depositing) $NP_{t,0,1}^1$ at the 2δ -term rate and simultaneously depositing (borrowing) $NP_{t,0,1}^1$ at the δ -term rate at time t , and depositing (borrowing) the proceeds thereof, which is the nominal amount N , again at the δ -term rate at time $t + \delta$.¹³ Using

¹³Depositing (borrowing) at one of the term rates is equivalent to buying (selling) the associated ZCB.

Lemma 4.1, it is straightforward to show that the fair value of the FRA at time $t + \delta$ is

$$V_{t+\delta} = \alpha N \left[1 - \frac{P_{t,0,1}^1}{P_{t,0,2}^2} P_{t,1,2}^1 \right]. \quad (\text{A.1})$$

Borrowing (Depositing) $P_{t,0,2}^2$ units of currency at the δ -term rate at time t and refinancing (redepositing) the total cost (proceeds) thereof at time $t + \delta$ would cumulatively cost (yield) $P_{t,0,2}^2 / D_{t+2\delta}^1$ at time $t + 2\delta$. Combining this position with a long (short) FRA, with $N = P_{t,0,2}^2 / D_{t+\delta}^1 = P_{t,0,2}^2 / P_{t,0,1}^1$, would result in a net cost (yield) of 1 unit of currency at time $t + 2\delta$, and a net cost (yield) of $P_{t,1,2}^1$ at time $t + \delta$, using equation (A.1). The combined strategy therefore has a value of $P_{t,0,2}^2$ at time t and replicates the value of the 2δ -term ZCB at times $t + \delta$ and $t + 2\delta$. Having the δ -term SDF as the key building block, this strategy creates a synthetic δ -term ZCB, $\{P_{t,i,2}^1; i \in \{0, 1, 2\}\}$, at time t with maturity time $t + 2\delta$, such that $P_{t,0,2}^1 = P_{t,0,2}^2$, $P_{t,2,2}^1 = P_{t,2,2}^2 = 1$ and $P_{t,1,2}^1$ is the interim fair value at time $t + \delta$, which concludes the proof. \square

A.3 Forward Rate Construction for Definition 4.1

Depositing (borrowing) one unit of currency at $F_{u,01}^1$ is achieved by:

- (i) borrowing (depositing) $P_{u,0,1}^1$ units of currency at the δ -term rate at time u ;
- (ii) simultaneously depositing (borrowing) the same amount at the 2δ -term rate; and
- (iii) depositing (borrowing) one unit of currency to settle transaction (i) at time $u + \delta$.

A.4 Proof for Corollary 4.1

Proof. The general version of the $\delta \times 2\delta$ FRA has the following terminal payoff

$$\begin{aligned} V_{t+2\delta} &= \alpha N \left[(1 + \delta R_{t+\delta}^1) - (1 + \delta K) \right] \\ &= \alpha N \left[\frac{1}{P_{t,1,2}^1} - (1 + \delta K) \right], \end{aligned}$$

where K is an arbitrary fixed rate specified at initiation of the contract, at time t . Using the calibrated δ -term PK from Theorem 4.1, the initial arbitrage-free value of the FRA is

$$\begin{aligned}
V_t &= \frac{1}{\pi_t^1} \mathbb{E}^{\mathbb{P}} [\pi_{t+2\delta}^1 V_{t+2\delta} \mid \mathcal{G}_t] \\
&= \frac{1}{D_t^1} \mathbb{E}^{\mathbb{P}^1} [D_{t+2\delta}^1 V_{t+2\delta} \mid \mathcal{G}_t] \\
&= \alpha \mathbb{E}^{\mathbb{P}^1} [D_{t+\delta}^1 \mid \mathcal{G}_t] - \alpha(1 + \delta K) \mathbb{E}^{\mathbb{P}^1} [D_{t+2\delta}^1 \mid \mathcal{G}_t] \\
&= \alpha P_{t,0,1}^1 - \alpha(1 + \delta K) P_{t,0,2}^1,
\end{aligned}$$

where the third line follows from Lemma 4.1, viz. $D_{t+2\delta}^1 = D_{t+\delta}^1 P_{t,1,2}^1$, and the last line by definition of the δ -term PK. Setting $V_t = 0$ and solving for the fair strike rate yields $K = (P_{t,0,1}^1 / P_{t,0,2}^1 - 1) / \delta$. Repeating this process at time $t + \delta$, for exactly the same contract, would then yield $K = (P_{t,1,1}^1 / P_{t,1,2}^1 - 1) / \delta$. Combining these two results leads to the definition of the fair strike rate process, equation (4.10), while observing that

$$P_{t,i,1}^1 = \frac{1}{D_{t+i\delta}^1} \mathbb{E}^{\mathbb{P}^1} [D_{t+\delta}^1 \mid \mathcal{G}_{t+i\delta}] = \frac{1}{D_{t+i\delta}^1} \mathbb{E}^{\mathbb{P}^1} [\widehat{D}_{t+\delta|t}^1 \mid \mathcal{G}_t],$$

for $i \in \{0, 1\}$, and that

$$P_{t,i,2}^1 = \frac{1}{D_{t+i\delta}^1} \mathbb{E}^{\mathbb{P}^1} [D_{t+2\delta}^1 \mid \mathcal{G}_{t+i\delta}] = \frac{1}{D_{t+i\delta}^1} \mathbb{E}^{\mathbb{P}^1} [\widehat{D}_{t+2\delta|t}^1 \mid \mathcal{G}_{t+j\delta}]$$

for $i \in \{0, 1, 2\}$ and $j = \min(i, 1)$, which follows by the tower property of conditional expectation, completes the proof. \square

B Systemic liquidity indicators

Definition B.1 (General $n\delta$ -Term Quoted Rate). *Assume that u and t are quoting and trading times respectively, with $u < t$. For a nominal amount N , the SST may model a future STR $n\delta$ -term deposit/loan quote rate as*

$$R_{t,\alpha,\beta}^{n,N} := R_t^n L_{t,\alpha,\beta}^{n,N} \quad (\text{B.1})$$

where α is equal to $\text{sgn}(1)$ for a deposit, $\text{sgn}(-1)$ for a loan; β is a state variable equal to 3 if the SST can source perfect liquidity, 2 if the SST can source costly liquidity, 1 if only the SST

can't source liquidity and 0 if there is no systemic liquidity; and the liquidity indicator

$$L_{t,\alpha,\beta}^{n,N} := \begin{cases} 1, & \text{if } \beta = 3, \text{ with probability } q_{t,\alpha,3}^{n,N}, \\ 1 - \alpha \Delta_{t,\alpha}^{n,N} / (n\delta R_t^n P_{t,0,n}^n), & \text{if } \beta = 2, \text{ with probability } q_{t,\alpha,2}^{n,N}, \\ 0, & \text{if } \beta = 1, \text{ with probability } q_{t,\alpha,1}^{n,N}, \\ 0, & \text{if } \beta = 0, \text{ with probability } q_{t,\alpha,0}^{n,N}; \end{cases} \quad (\text{B.2})$$

is \mathcal{L}_t -measurable, with $\Delta_{t,\alpha}^{n,N} := \Delta(t, t + n\delta, N, \alpha)$, a positive real-valued function which models the absolute future cost per unit nominal, and the probability $q_{t,\alpha,\beta}^{n,N} := q(t, t + n\delta, N, \alpha, \beta)$ both assumed to be deterministic functions. By the law of total probability $\sum_{\beta=0}^3 q_{t,\alpha,\beta}^{n,N} = 1$, while it must also hold that $L_{t,+0}^{n,N}(\omega) = L_{t,-0}^{n,N}(\omega)$ a.s., so that the likelihood of systemic illiquidity is equal for both loans and deposits, i.e., $q_{t,+0}^{n,N} = q_{t,-0}^{n,N}$.

Proposition B.1 (Expected Future Value and Cost of $n\delta$ -Term Liquidity). *Consider a set of nominals $\{N_{t,\alpha,1}^n, N_{t,\alpha,2}^n, \dots, N_{t,\alpha,b}^n\}$ with associated weights $\{w_{t,\alpha,1}^n, w_{t,\alpha,2}^n, \dots, w_{t,\alpha,b}^n\}$ that reflect the respective likelihood of the SST engaging in deposit and loan transactions at such nominals at time t for a term of $n\delta$, with $\sum_{i=1}^b w_{t,\alpha,i}^n = 1$, where $b \in \mathbb{N}$. To ease notation here, N_i and w_i are used to denote $N_{t,\alpha,i}^n$ and $w_{t,\alpha,i}^n$ respectively, for $i \in \{1, 2, \dots, b\}$. From the vantage point of the SST at time u , the weighted average future value at time t of an $n\delta$ -term deposit/loan with unit nominal is*

$$V_{t,\alpha}^n = \alpha \left[q_{t,\alpha,0}^n \widehat{P}_{t,0,n}^n + q_{t,\alpha,1}^n P_{t,0,n}^n + q_{t,\alpha,2}^n (1 - \alpha \Delta_{t,\alpha}^n) + q_{t,\alpha,3}^n \right], \quad (\text{B.3})$$

where the aggregated probabilities and cost function are respectively defined by

$$q_{t,\alpha,\beta}^n := \left(\sum_{i=1}^b w_i N_i q_{t,\alpha,\beta}^{n,N_i} \right) / \left(\sum_{i=1}^b w_i N_i \right), \text{ and} \quad (\text{B.4})$$

$$\Delta_{t,\alpha}^n := \left(\sum_{i=1}^b w_i N_i q_{t,\alpha,2}^{n,N_i} \Delta_{t,\alpha}^{n,N_i} \right) / \left(q_{t,\alpha,2}^n \sum_{i=1}^b w_i N_i \right). \quad (\text{B.5})$$

The expected future cost of $n\delta$ -term liquidity per unit nominal at time t is given by $(\alpha - V_{t,\alpha}^n)$.

Proof. At time t if $\beta = 3$ then perfect liquidity prevails, the reference $n\delta$ -term rate will exist and the fair value of the SST's deposit/loan will be

$$\frac{1}{\pi_t^n} \mathbb{E}^{\mathbb{P}} \left[\pi_{t+n\delta}^n \alpha N_i (1 + n\delta R_{t,\alpha,3}^{n,N_i}) \mid \mathcal{G}_t \right] = \alpha N_i P_{t,0,n}^n (1 + n\delta R_{t,\alpha,3}^{n,N_i}) = \alpha N_i. \quad (\text{B.6})$$

If $\beta = 2$, costly liquidity prevails, the reference $n\delta$ -term rate will exist and the fair value of the SST's deposit/loan will be

$$\frac{1}{\pi_t^n} \mathbb{E}^{\mathbb{P}} \left[\pi_{t+n\delta}^n \alpha N_i \left(1 + n\delta R_{t,\alpha,2}^{n,N_i} \right) \mid \mathcal{G}_t \right] = \alpha N_i P_{t,0,n}^n \left(1 + n\delta R_{t,\alpha,2}^{n,N_i} \right) = \alpha N_i - N_i \Delta_{t,\alpha}^{n,N_i}. \quad (\text{B.7})$$

If $\beta = 1$, only the SST can't access liquidity, the reference $n\delta$ -term rate will still exist and the fair value of the SST's position will now be

$$\frac{1}{\pi_t^n} \mathbb{E}^{\mathbb{P}} \left[\pi_{t+n\delta}^n \alpha N_i \left(1 + n\delta R_{t,\alpha,1}^{n,N_i} \right) \mid \mathcal{G}_t \right] = \alpha N_i P_{t,0,n}^n. \quad (\text{B.8})$$

In the case of a deposit, this represents the value foregone by not being able to access the $n\delta$ -term rate. For a loan, this represents the value gained by having to settle a liability early as opposed to deferring payment by accessing funding through the $n\delta$ -term rate.

When $\beta = 0$, there is no systemic liquidity and therefore no reference $n\delta$ -term rate. In this scenario, the SST may estimate the fair value of their position as

$$\frac{1}{\widehat{D}_t^n} \mathbb{E}^{\mathbb{P}} \left[\widehat{D}_{t+n\delta}^n \alpha N_i \left(1 + n\delta R_{t,\alpha,0}^{n,N_i} \right) \mid \mathcal{G}_t \right] = \alpha N_i \widehat{P}_{t,0,n}^n, \quad (\text{B.9})$$

by making use of the estimated $n\delta$ -term SDF. Then, the estimated value at time t is

$$V_{t,\alpha}^{n,N_i} = \alpha N_i \left[q_{t,\alpha,0}^{n,N_i} \widehat{P}_{t,0,n}^n + q_{t,\alpha,1}^{n,N_i} P_{t,0,n}^n + q_{t,\alpha,2}^{n,N_i} \left(1 - \alpha \Delta_{t,\alpha}^{n,N_i} \right) + q_{t,\alpha,3}^{n,N_i} \right], \quad (\text{B.10})$$

and the weighted average future value equation (B.3) is recovered by setting

$$V_{t,\alpha}^n := \left(\sum_{i=1}^b w_i V_{t,\alpha}^{n,N_i} \right) / \left(\sum_{i=1}^b w_i N_i \right), \quad (\text{B.11})$$

which holds for both $n\delta$ -term loans or deposits, and concludes the proof. \square

Corollary B.1 (Mid Expected Future Value and Cost of $n\delta$ -Term Liquidity). *From equation (B.3), it follows that the mid weighted average future value is*

$$V_t^n = \alpha \left[q_{t,0}^n \widehat{P}_{t,0,n}^n + q_{t,1}^n P_{t,0,n}^n + q_{t,2}^n \left(1 + \epsilon_t^n \right) + q_{t,3}^n \right], \quad (\text{B.12})$$

where the mid probabilities and cost function are respectively defined by

$$q_{t,\beta}^n := \frac{1}{2} (q_{t,+,\beta}^n + q_{t,-,\beta}^n), \text{ and} \quad (\text{B.13})$$

$$\epsilon_t^n := \frac{1}{2q_{t,2}^n} (q_{t,-,2}^n \Delta_{t,-}^n - q_{t,+,2}^n \Delta_{t,+}^n). \quad (\text{B.14})$$

with the expected future cost of $n\delta$ -term liquidity per unit nominal now given by $(\alpha - V_t^n)$.

Proof. Setting $V_t^n := \alpha (V_{t,+}^n - V_{t,-}^n) / 2$ yields the mid future value equation (B.12), with the expected future cost of liquidity result then following trivially. \square

Remark B.1 (The Spread Quantity ϵ_t^n). *Having constructed a mid value in Corollary B.1, the quantity ϵ_t^n may be interpreted as the mid value of the bid-offer spread associated with $n\delta$ -term deposit and loan liquidity at time t , suitably weighted by the probability of each transaction at specific nominals, from the perspective of the SST. The magnitude and sign of this mid spread depends on the funding market climate. In particular, one would expect:*

- $\epsilon_t^n > 0$, in a stressed market where the STR has difficulty sourcing term funding;
- $\epsilon_t^n \approx 0$, in a normal market where $\Delta_{t,+}^n$ and $\Delta_{t,-}^n$ may be attributed to profit margins;
- $\epsilon_t^n < 0$, in a stressed market where the STR has excess access to term funding, a scenario that is most likely to realise for near or shorter terms-to-maturity.

Remark B.2 (Systemic Liquidity Indicators). *Proposition B.1 and Corollary B.1 have enabled the aggregation of the nominal effect in the general $n\delta$ -term quoted rates, as well as the averaging of the spread asymmetry due to loans and deposits. The structure of the mid expected future value, equation B.12, indicates that a simpler symmetric and systemic specification for the liquidity indicator will suffice, especially under the assumption of a normal market, or $\epsilon_t^n \approx 0$. Therefore, in order to ease the exposition, the \mathcal{L}_t -measurable random variable*

$$L_t^n := \begin{cases} 1, & \text{perfect systemic liquidity with probability } q_t^n, \\ 0, & \text{no systemic liquidity with probability } 1 - q_t^n; \end{cases} \quad (\text{B.15})$$

is used to model systemic $n\delta$ -term liquidity at time t , with $q_t^n := q(t, t + n\delta)$ being a deterministic function that determines the probability of perfect liquidity or no systemic liquidity. With this indicator, states $\beta \in \{2, 3\}$ and $\beta \in \{0, 1\}$ of the general indicator are essentially combined, and provide a similar composite effect with $V_t^n = \alpha [(1 - q_t^n) \widehat{P}_{t,0,n}^n + q_t^n]$ now being the mid expected future value of $n\delta$ -term liquidity per unit nominal at time t .

A more general version of the systemic liquidity indicator, defined in Remark B.2, which incorporates the state of costly liquidity is considered next. Using the definition of this new liquidity indicator, the lemma below reveals the impact of costly liquidity on the fair $\delta \times 2\delta$ FRA rate that is derived in Lemma 5.1.

Definition B.2 (Costly Systemic Liquidity Indicators). *At time $u \in \mathbb{R}_{\geq 0}$, the random variable*

$$C_u^n := \begin{cases} 0, & \text{no systemic liquidity with probability } q_{u,0}^n, \\ 1, & \text{perfect systemic liquidity with probability } q_{u,1}^n, \\ 1 + \epsilon_u^n, & \text{costly systemic liquidity with probability } q_{u,2}^n; \end{cases} \quad (\text{B.16})$$

models $n\delta$ -term systemic liquidity, where $\epsilon_u^n \in \mathbb{R}$ is the deterministic spread quantity as defined in Corollary B.1 and described in Remark B.1. If the current time is t , then the natural filtration associated with liquidity is

$$\mathcal{L}_t := \sigma(\{C_u^1, C_u^2, \dots, C_u^m\}; u \in \{t_0, t_1, \dots, t_k\}), \quad (\text{B.17})$$

where $\{t_0, t_1, \dots, t_k\}$ denotes the set of trading days that lie within the interval $[0, t]$. These costly systemic liquidity indicators are assumed to exhibit both serial and cross-sectional independence, or more formally:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[C_u^n \mid \mathcal{L}_t \cap \sigma(\{C_u^n \notin \{0, 1, 1 + \epsilon_u^n\}\})] &= \mathbb{E}^{\mathbb{P}}[C_u^n] \\ &= \mathbb{P}[C_u^n = 1] + (1 + \epsilon_u^n) \mathbb{P}[C_u^n = 1 + \epsilon_u^n] \\ &= q_{u,1}^n + q_{u,2}^n (1 + \epsilon_u^n), \end{aligned}$$

for all $t \leq u$, with $q_{u,i}^n := q_i(u, u + n\delta)$ being a deterministic function for $i \in \{0, 1, 2\}$.

Lemma B.1 ($\delta \times 2\delta$ FRA Pricing under Potentially Costly Liquidity). *The fair strike rate process for the general version of the $\delta \times 2\delta$ FRA defined in Lemma 4.2 is*

$$\bar{F}_{t,i,1}^1 = \begin{cases} F_{t,0,1}^1 [q_{t+\delta,1}^1 + q_{t+\delta,2}^1 (1 + \epsilon_{t+\delta}^1)] , & i = 0 \text{ and conditional on } C_t^1 = C_t^2 = 1, \\ F_{t,1,1}^1 , & i = 1 \text{ and conditional on } C_{t+\delta}^1 = 1, \end{cases}$$

which is also $\mathcal{G}_{t+i\delta}$ -measurable.

Proof. Assuming that $\mathcal{L}_t = \mathcal{L}_{t^-} \vee \sigma(\{C_t^1 = 1, C_t^2 = 1\})$, the standard FRA replication strategy yields $\tilde{V}_{t+2\delta} = V_{t+2\delta} - \alpha N \delta (1 - C_{t+\delta}^1) R_{t+\delta}^1$, as was the case in Lemma 5.1.

Let $\mathcal{M}_t := \mathcal{G}_t \vee \mathcal{L}_t$, then using the hybrid-term LDPK from Definition 5.4, the current value

of the above payoff is

$$\begin{aligned}
\tilde{\pi}_t \tilde{V}_t &= \mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t \right] \\
&= \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t, C_{t+\delta}^1 \right] \mid \mathcal{M}_t \right] \\
&= \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t, C_{t+\delta}^1 = 0 \right] \mathbb{P}(C_{t+\delta}^1 = 0) \mid \mathcal{M}_t \right] \\
&\quad + \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t, C_{t+\delta}^1 = 1 \right] \mathbb{P}(C_{t+\delta}^1 = 1) \mid \mathcal{M}_t \right] \\
&\quad + \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\tilde{\pi}_{t+2\delta} \tilde{V}_{t+2\delta} \mid \mathcal{M}_t, C_{t+\delta}^1 = 1 + \epsilon_{t+\delta}^1 \right] \mathbb{P}(C_{t+\delta}^1 = 1 + \epsilon_{t+\delta}^1) \mid \mathcal{M}_t \right]
\end{aligned}$$

which follows by the tower property of conditional expectations. Then, it follows that

$$\begin{aligned}
\tilde{V}_t &= -\mathbb{E}^{\mathbb{P}} \left[\pi_{t+2\delta}^2 \alpha N \delta F_{t,0,1}^1 \mid \mathcal{M}_t \right] q_{t+\delta,0}^1 + \mathbb{E}^{\mathbb{P}} \left[\pi_{t+2\delta}^1 V_{t+2\delta} \mid \mathcal{M}_t \right] q_{t+\delta,1}^1 \\
&\quad + \mathbb{E}^{\mathbb{P}} \left[\pi_{t+2\delta}^1 (V_{t+2\delta} + \alpha N \delta \epsilon_{t+\delta}^1 R_{t+\delta}^1) \mid \mathcal{M}_t \right] q_{t+\delta,2}^1 \\
&= -\alpha N q_{t+\delta,0}^1 \delta F_{t,0,1}^1 P_{t,0,2}^2 + q_{t+\delta,1}^1 V_t + q_{t+\delta,2}^1 V_t + \alpha N q_{t+\delta,2}^1 \delta \epsilon_{t+\delta}^1 F_{t,0,1}^1 P_{t,0,2}^1 \\
&= (q_{t+\delta,1}^1 + q_{t+\delta,2}^1) V_t + \alpha N \delta F_{t,0,1}^1 P_{t,0,2}^1 (q_{t+\delta,2}^1 \epsilon_{t+\delta}^1 - q_{t+\delta,0}^1)
\end{aligned}$$

since $\tilde{\pi}_t := 1$, using the definition of the hybrid-term LDPK and taking note that the δ -term PK is well-defined when $C_{t+\delta}^1 \neq 0$. V_t is the fair value of the FRA under perfect liquidity, and therefore equal to 0. Trading this FRA with the strike rate equal to the fair FRA rate defined in the perfect liquidity setting therefore leads to an initial loss (gain) if the market-maker is long (short). As with Lemma 5.1, the assumption here is that $V_{t+2\delta}$ will still be the FRA payoff even when $C_{t+\delta}^1 = 0$ and the strong case of no systemic liquidity is in effect.

Setting the FRA rate to an arbitrary value, $\bar{F}_{t,0,1}^1$, and pricing via the same process gives

$$\begin{aligned}
\tilde{V}_t &= \alpha (q_{t+\delta,1}^1 + q_{t+\delta,2}^1) N \left[P_{t,0,1}^1 - (1 + \delta \bar{F}_{t,0,1}^1) P_{t,0,2}^1 \right] \\
&\quad - \alpha N q_{t+\delta,0}^1 \delta \bar{F}_{t,0,1}^1 P_{t,0,2}^2 + \alpha N q_{t+\delta,2}^1 \delta \epsilon_{t+\delta}^1 F_{t,0,1}^1 P_{t,0,2}^1 \\
&= \alpha N (1 - q_{t+\delta,0}^1) \delta F_{t,0,1}^1 P_{t,0,2}^1 + \alpha N q_{t+\delta,2}^1 \delta \epsilon_{t+\delta}^1 F_{t,0,1}^1 P_{t,0,2}^1 \\
&\quad - \alpha N (1 - q_{t+\delta,0}^1) \delta \bar{F}_{t,0,1}^1 P_{t,0,2}^1 - \alpha N q_{t+\delta,0}^1 \delta \bar{F}_{t,0,1}^1 P_{t,0,2}^1 \\
&= \alpha N \delta F_{t,0,1}^1 P_{t,0,2}^1 \left[(1 - q_{t+\delta,0}^1) + q_{t+\delta,2}^1 \epsilon_{t+\delta}^1 \right] - \alpha N \delta \bar{F}_{t,0,1}^1 P_{t,0,2}^1 \left[(1 - q_{t+\delta,0}^1) + q_{t+\delta,0}^1 \right]
\end{aligned}$$

while setting $\tilde{V}_t = 0$, recalling that $P_{t,0,2}^2 = P_{t,0,2}^1$, and solving for the fair FRA strike rate yields $\bar{F}_{t,0,1}^1 = F_{t,0,1}^1 \left[q_{t+\delta,1}^1 + q_{t+\delta,2}^1 (1 + \epsilon_{t+\delta}^1) \right]$, as required. Repeating this pricing process at time $t + \delta$, for exactly the same contract and assuming that $C_{t+\delta}^1 = 1$ at this time, it is trivial to show that $\bar{F}_{t,1,1}^1 = F_{t,1,1}^1$, which completes the proof. \square

C Liquidity-dependent pricing Kernels

Definition C.1 ($n\delta$ -Term LDPK over a Single Period). *At the current time t , under the CPL or CPFL liquidity regime the $n\delta$ -term LDPK is defined by*

$$\tilde{\pi}_{t+i\delta}^n := \pi_{t+i\delta}^n, \quad (\text{C.1})$$

for $i \in \{0, 1\}$, i.e., perfect liquidity enables the definition of the $n\delta$ -term PK. This scenario therefore allows a market-taker, such as the SST, to access $n\delta$ -term liquidity.

If the NPFL or NPL liquidity regime prevails then the $n\delta$ -term LDPK is given by:

$$\tilde{\pi}_{t+i\delta}^n := \widehat{D}_{t+i\delta}^n, \quad (\text{C.2})$$

for $i \in \{0, 1\}$, i.e., no liquidity requires the estimation of an $n\delta$ -term rate using the respective estimated SDF. This scenario therefore requires market-making to create liquidity.

Under the PPL liquidity regime with the $n\delta$ -term being illiquid but the $i\delta$ - and $j\delta$ -terms being liquid such that $j < n < k$, the SST may define the $n\delta$ -term LDPK by

$$\tilde{\pi}_{t+i\delta}^n := \Lambda_{t+i\delta}^n D_{t+i\delta}^n, \quad (\text{C.3})$$

where the $\{(\mathcal{G}_t), \mathbb{P}\}$ -martingale $(\Lambda_{t+v\delta}^n)_{0 \leq v \leq 1}$ must be chosen so that

$$\mathbb{E}^{\mathbb{P}^n} [\widehat{D}_{t+n\delta}^n \mid \mathcal{G}_t] = D_{t+n\delta}^n \in (D_{t+k\delta}^k, D_{t+j\delta}^j)$$

in order to ensure positive forward rates over $[t + j\delta, t + k\delta]$. Therefore, this scenario also requires market-making to create liquidity, however liquidity in the other adjacent terms provides information to create an arbitrage-free range for the calibrated SDF.

Definition C.2 ($n\delta$ -Term LDPK over Multiple Periods). *At the current time t , under the CPFL liquidity regime the $n\delta$ -term LDPK is defined by*

$$\tilde{\pi}_{t+i\delta}^n := \pi_{t+i\delta}^n, \quad (\text{C.4})$$

for $i \in \mathbb{N}_{0, m+n}^n$, i.e., perfect liquidity enables the definition of the $n\delta$ -term PK over the interval $[t, t + m\delta]$ if $m \bmod n = 0$, or $[t, t + m\delta)$ otherwise.

If the CPL or PPL (as defined in Definition C.1) liquidity regime prevails then

$$\tilde{\pi}_{t+i\delta}^n = \begin{cases} \pi_{t+i\delta}^n, & i \in \mathbb{N}_{0,2n}^n, \\ \widehat{D}_{t+i\delta}^n, & i \in \mathbb{N}_{2n,m+n}^n, \end{cases} \quad (\text{C.5})$$

i.e., potential future illiquidity requires market-making beyond the first period.

If the NPL or NPFL liquidity regime prevails then the $n\delta$ -term LDPK is defined by

$$\tilde{\pi}_{t+i\delta}^n := \widehat{D}_{t+i\delta}^n, \quad (\text{C.6})$$

for $i \in \mathbb{N}_{0,m+n}^n$, i.e., no present liquidity and no/uncertain future liquidity requires market-making to create liquidity for all periods.

D Reduced-form model development

D.1 Proof for Lemma 6.1

Proof. At time $t + i\delta$, the present value of 1 unit of currency due at time $t + j\delta$ is equal to

$$P_{t,i,j}^{j-i} = \frac{1}{1 + (j-i)\delta R_{t+i\delta}^{j-i}},$$

provided that $i \leq j \leq m$, according to Assumption 6.1. Therefore, considering a synthetic $n\delta$ -term LCZCB with tenor less than or equal to $n\delta$, i.e. $(j-i) \leq n$, it follows that $\overline{P}_{t,i,j}^{(n)} = P_{t,i,j}^{j-i}$, which is the result shown in equation (6.5).

For the case of $n < (j-i) \leq m$ and $\text{mod}(j-i, n) = h$, with $h \in \mathbb{N}_{0,n-1}$, a synthetic $n\delta$ -term LCZCB with $(j-i)\delta$ -tenor may be constructed using Assumptions 6.1, 6.2 and 6.3 as follows. At time $t + i\delta$, if $h > 0$:

(i) Borrow (Deposit) M units of currency at the $h\delta$ -term rate.

(0) Long (Short) the $h\delta \times (h+n)\delta$ fair FRA with nominal equal to $M \frac{D_{t+i\delta}^h}{D_{t+(i+h)\delta}^h}$.

(1) Long (Short) the $(h+n)\delta \times (h+2n)\delta$ fair FRA with nominal equal to

$$M \frac{D_{t+i\delta}^h}{D_{t+(i+h)\delta}^h} \left(1 + n\delta \overline{F}_{t,i,i+h}^n \right).$$

\vdots
 \vdots

(N) Long (Short) the $(j - n)\delta \times j\delta$ fair FRA with nominal equal to

$$M \frac{D_{t+i\delta}^h}{D_{t+(i+h)\delta}^h} \left(1 + n\delta \bar{F}_{t,i,i+h}^n\right) \left(1 + n\delta \bar{F}_{t,i,i+h+n}^n\right) \dots \left(1 + n\delta \bar{F}_{t,i,j-2n}^n\right).$$

At time $t + (i + h)\delta$, using Assumption 6.3:

- (i) The loan (deposit) matures which costs (yields): $M \frac{D_{t+i\delta}^h}{D_{t+(i+h)\delta}^h}$.
- (ii) Refinance (Re-deposit) the costs (proceeds) from (i) at the $n\delta$ -term rate.

At time $t + (i + h + n)\delta$, using Assumption 6.3:

- (ii) The loan (deposit) matures which costs (yields): $M \frac{D_{t+i\delta}^h}{D_{t+(i+h)\delta}^h} \frac{D_{t+(i+h)\delta}^n}{D_{t+(i+h+n)\delta}^n}$.
- (0) The long (short) FRA payoff: $(-)M \frac{D_{t+i\delta}^h}{D_{t+(i+h)\delta}^h} \left[\frac{D_{t+(i+h)\delta}^n}{D_{t+(i+h+n)\delta}^n} - \left(1 + n\delta \bar{F}_{t,i,i+h}^n\right) \right]$.
- (iii) Add (ii) and (0), and refinance (re-deposit) the costs (proceeds) at the $n\delta$ -term rate.

Repeating this process at each time $t + (i + h + nk)\delta$, for $k = 2, 3, \dots, (j - i - n - h)/n$, will eventually result in a total cost (yield) equal to

$$M \frac{D_{t+i\delta}^h}{D_{t+(i+h)\delta}^h} \left(1 + n\delta \bar{F}_{t,i,i+h}^n\right) \left(1 + n\delta \bar{F}_{t,i,i+h+n}^n\right) \dots \left(1 + n\delta \bar{F}_{t,i,j-n}^n\right).$$

at time $t + j\delta$, which is measurable at time $t + i\delta$. This strategy may then be used to create the synthetic $n\delta$ -term LCZCB with $(j - i)\delta$ -tenor by setting:

$$M := \frac{D_{t+(i+h)\delta}^h}{D_{t+i\delta}^h} \left(1 + n\delta \bar{F}_{t,i,i+h}^n\right)^{-1} \left(1 + n\delta \bar{F}_{t,i,i+h+n}^n\right)^{-1} \dots \left(1 + n\delta \bar{F}_{t,i,j-n}^n\right)^{-1},$$

and therefore $\bar{P}_{t,i,j}^{(n)} = M$, which is consistent with equation (6.4) for $h > 0$. If $h = 0$ then the relevant contracts are the $(k + n)\delta \times (k + 2n)\delta$ FRA contracts for $k = 0, n, 2n, \dots, (j - i - 2n)$. The same strategy may be employed for $h = 0$, which completes the proof. \square

References

- [1] M. Alfeus, M. Grasselli & E. Schlögl, (2020). A consistent stochastic model of the term structure of interest rates for multiple tenors. *Journal of Economic Dynamics and Control* 114, 103861.
- [2] A. Backwell, A. Macrina, E. Schlögl, D. Skovmand (2021). Term Rates, Multicurve Term Structures and Overnight Rate Benchmarks: a Roll-Over Risk Approach. <http://dx.doi.org/10.2139/ssrn.3399680>
- [3] J. Y. Campbell (1986). Bond and Stock Returns in a Simple Exchange Model. *The Quarterly Journal of Economics*, 101(4), 785-803.
- [4] J. Y. Campbell (1986). A Defense of Traditional Hypotheses About the Term Structure of Interest Rates. *The Journal of Finance*, 41(1), 183-193.
- [5] J. Y. Campbell (1995). Some Lessons from the Yield Curve. *The Journal of Economic Perspectives*, 9(3), 129-152.
- [6] J. H. Cochrane (2005). *Asset Pricing*. Princeton University Press.
- [7] G. M. Constantinides (1992). A Theory of the Nominal Term Structures of Interest Rates. *The Review of Financial Studies* 5, 531-552.
- [8] J. C. Cox, J. E. Ingersoll & S. A. Ross (1981). A Re-Examination of Traditional Hypotheses about the Term Structure of Interest Rates. *Journal of Finance*, 36(4), 769-799.
- [9] D. Duffie (2001). *Dynamic Asset Pricing Theory*. Princeton University Press.
- [10] D. Filipović & A. B. Trolle (2013). The Term Structure of Interbank Risk. *Journal of Financial Economics* 109, 707-733.
- [11] B. Guggenheim (2020). At the Crossroads in the Transition away from LIBOR: From Overnight to Term Rates. BIS Working Papers, No. 819. <https://www.bis.org/publ/work891.pdf>
- [12] J. R. Hicks (1939). *Value and Capital*. (Oxford University Press, U.K).
- [13] P. J. Hunt, J. E. Kennedy (2000) *Financial Derivatives in Theory and Practice*. Wiley.
- [14] S. Klingler & O. Syrstad (2021). Life after LIBOR. *Journal of Financial Economics*. <https://doi.org/10.1016/j.jfineco.2021.04.017>

- [15] A. F. Longstaff (2000). The Term Structure of Very Short-Term Rates: New Evidence for the Expectations Hypothesis. *The Journal of Financial Economics*, 58(3), 397-415.
- [16] F. A. Lutz (1940). The Structure of Interest Rates. *The Quarterly Journal of Economics*, 55(1), 36-63.
- [17] F. R. Macaulay (1938). Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856. NBER Working Paper Series, New York.
- [18] A. Macrina, O. Mahomed (2018). Consistent Valuation Across Curves using Pricing Kernels. *Risks*, 6(18), 1-32.
- [19] J. H. McCulloch (1993). A Re-Examination of Traditional Hypotheses about the Term Structure: A Comment. *The Journal of Finance*, 48(2), 779-789.